PRELIMINARY EDITION

BASIC CONCEPTS of MATHEMATICS

an INTRODUCTORY TEXT for TEACHERS

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Prepared
at the
1963 Entebbe Mathematics Workshop

Educational Services Incorporated
Watertown, Massachusetts, U.S.A.
NOTE

This book is the product of a study and writing workshop conducted during the summer of 1963 at Entebbe, Uganda, with more than fifty mathematicians and mathematics teachers from Africa, the United Kingdom, and the United States in attendance.

In order that it might be used experimentally in African training colleges as quickly as possible, the book was edited and produced with the utmost speed. As a consequence, both editing and production suffer from defects. More time might have made it possible to eliminate most of these, but it would also have made it impossible to try out this book during the 1963-64 academic year, and it was held that this latter need took priority over the former. The African Education Program can only hope that those who use this book will appreciate the circumstances under which it has been made available, and will be tolerant of its imperfections.
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ACKNOWLEDGEMENTS

This experimental teacher training text was prepared during the summer of 1963 at Entebbe, Uganda, as part of a program of curriculum revision being conducted by the African Education Program of Educational Services Incorporated.

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The 1963 Entebbe Mathematics Workshop was directed by Professor Walter Prenowitz of Brooklyn College, Brooklyn, New York, and Professor W. Ted Martin of Massachusetts Institute of Technology, Cambridge, Massachusetts.

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This experimental text has been produced by the 1963 Entebbe Mathematics Workshop for use in training colleges in English-speaking Africa. The Entebbe programme is a comprehensive one. Its general purpose is to write texts which reflect recent thinking about mathematical education and which adapt this thinking to African conditions.

Rapid progress in science and technology and in mathematics itself has stimulated major efforts to improve mathematical education elsewhere. Principal emphasis has been put on understanding the ideas that have unified and simplified modern mathematics. In addition, methods have been adopted which lead the student to discover things for himself. New facts are established either from first principles or from facts already known, so that undue reliance on rote learning is eliminated. A general discussion of these points will be found in the introductory section "Why Change Our Mathematics Teaching?"

The Entebbe Mathematics Workshop, comprising mathematicians and educators drawn from Africa, the United Kingdom and the United States of America, has produced experimental texts for use in Primary One and Two and in Secondary One and Two. These texts are being tried out in East and West Africa and will be revised in the light of experience in their use. Tests have also been devised by the Workshop to measure the effectiveness of the material under actual teaching conditions. Further texts are planned to follow those so far produced.
This experimental text is designed for use in training colleges for primary teachers. It aims to give teachers in training the kind of background of understanding which will help prepare them to teach the Primary texts produced by the Workshop, or other texts which are written to achieve the same purpose.

It is hoped that this text will be of interest to all those who are concerned with the new approaches to mathematics teaching.

This text is written so as to stimulate the discovery of central concepts by consideration of concrete examples. Problems are provided, both to deepen the understanding of the teacher and to assist in classroom teaching.

After the introductory section, the text is planned in two parts: "Structure of Arithmetic" and "Introduction to Geometry." At the 1963 Entebbe Mathematics Workshop the first four units out of a projected seven on "Structure of Arithmetic" were written by the Teacher Training Writing Group: basic concepts and language of sets, the whole numbers, the number line, and fractions. For expositions of negative numbers, the rational numbers, and the real numbers, which are the subjects of the units not yet written, appropriate portions of the Entebbe Secondary One and Two Student Texts and Teachers' Guides may be consulted. These presentations, however, are not necessarily organized in the same way as they may be written later for teachers. As for "Introduction to Geometry," indications of the directions the text for teacher training may take are to be found in Part 3: Content Outline.
for Primary I - III and the Appendix: Projected Content Outline for Primary IV - VI of the Entebbe Mathematics Teachers' Handbook prepared by the Primary Writing Group at the 1963 Entebbe Mathematics Workshop.

To enable both tutors and teachers in training to pursue the subject further, a bibliography has been provided at the end of the book. Tutors should encourage their students to read widely, to further their knowledge of the new approach to mathematics.
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INTRODUCTION: WHY CHANGE OUR MATHEMATICS TEACHING?

We live in a rapidly changing world. The younger nations, like the older ones, face challenges which call for greater knowledge and greater willingness to learn new ways. This is why education is more important than ever before.

Everyone agrees that mathematics is important and that it should be taught in our primary and secondary schools. A knowledge of arithmetic is necessary for everyone in the modern world. A much greater knowledge of mathematics is required for those who take an active part, for example, in harnessing water power to the needs of Africa or in handling the financial problems of a nation. So we must teach mathematics well in our schools whether our students are to become ordinary citizens or leaders of their countries. But do we need to change the kind of mathematics we teach or the way that we teach it?

Most of those concerned with the teaching of mathematics feel dissatisfied with the job which has been done in the past. In spite of the importance of the subject, very few students have thought of mathematics as alive, exciting and interesting. This is not true of other subjects. There are many people who read history for fun. How many read books on mathematics for fun? Not very many! Everyone will agree that mathematics has not been a very popular subject. This points to a weakness in the way in which it has been taught. In any case, the methods of the past do not meet the challenges of the present and future.

But something new has happened. It has been found that students can actually get excited about mathematics and enjoy it tremendously. This was a
fortunate discovery. If it is important to know mathematics, it is surely also important that students should find it interesting. The change which has come about is the result of a big effort to teach mathematics in a new way.

What is this new way? In brief, it is to get the student to understand why things work the way they do, to a much greater extent than has been customary. More important is the fact that to interest the student we must get him to take an active part in learning. He must be led to discover things for himself. This is true in all subjects; it is true in mathematics also. Mathematics is not a strange subject in which everything is different - where, for example, it is unnecessary to interest the student; where it is sufficient to drill him so that he always gets the right answer.

It is widely believed that there is only one right way to do a problem in mathematics, so that all we need to do is to show the student this right way and give him lots of practice in doing it. This idea leads the student to think that the only way he can be original is to be wrong. Of course, he is glad to oblige us! But, seriously, if we try to teach students to act like machines, we should not be surprised to find that those with independence and originality will rebel.

In fact, there are many ways to solve a mathematical problem correctly. Some may be shorter than others. Some may be longer but more illuminating. One way may seem more natural to a student than another. Not all students are alike. The important thing is that ways be found to solve problems which students can think out for themselves. In this case they will have a better chance to remember them. Moreover the subject will make sense to them and catch their interest.
Every teacher imagines, we suppose, that he teaches the student to understand the subject. In a way this is true. It all depends on what you mean by "understand." We can, if we wish, say that a student understands how to divide one fraction by another if he can always apply the "invert and multiply" rule when he is given two fractions to divide. Thus if we ask him to divide $\frac{2}{3}$ by $\frac{3}{4}$ he will invert the $\frac{3}{4}$ to $\frac{4}{3}$ and multiply $\frac{2}{3}$ by $\frac{4}{3}$ to get $\frac{8}{9}$. In the process he also applies correctly the rule for multiplying two fractions: "Multiply the numerators and multiply the denominators." To repeat, we can say that the student understands a rule if he knows how to apply it with confidence and success. In the same way, many of us can truly say that we understand how to drive a car. But few of us understand why it operates the way it does.

There is therefore a deeper meaning of "understand." Suppose that the student asks, "Why do we invert and multiply when we wish to divide one fraction by another?" To answer this question it is not enough to say, "That is the rule." What we have to do is to give a reason for the rule. We have to lead the student to see that the rule expresses what ought to be done. We have to explain the rule. An explanation always has to be given in terms of something else -- something which is already known.

Later we shall return to this example and show what kind of an explanation could be given. We are not quite ready for it now. The point is that mathematics hangs together. It has a plot like a novel. Knowledge has to be built up in stages. You have to know something about what happened in the earlier chapters. In this respect, mathematics is again like other subjects. You cannot understand modern physics without first learning something about earlier
discoveries and ideas, nor modern history without some knowledge of its background.

We have said that we should teach mathematics so that the student can discover things for himself and can understand why he does what he does. This is what makes learning exciting.

But there is another difference between the newer approach to the teaching of mathematics and the older one. This difference is particularly important for the students who will go on.

Mathematics itself has changed. It surprises many people to learn this. They know that science and technology do not stand still. They expect that a man will be sent to the moon one of these days. New discoveries are made and science grows and renews itself like a living thing. New knowledge cannot be added to the old like a new room to a house. The whole subject must be rebuilt from time to time to take account of new and better ways of thinking about old facts. New ways of talking about the older knowledge are invented which help us to understand it better and to connect it with recent discoveries.

Few people think of mathematics in the same way. Almost everyone tends to think of it as somehow finished and complete, so that what we know was discovered long, long ago and has not been added to since. This is simply not true. Mathematical knowledge is growing faster than ever before. New mathematics is constantly being created to answer questions that the scientist and engineer need to answer and to forge ahead in new directions.

The situation is exactly the same as with all other subjects of knowledge. There are many new things which are so important that they have to be added to
the old if we are to have the tools we need to solve our problems. There is only so much time. Something has to be done to make room for the new. There are two things which can be done: (1) Rework the older material so that it hangs together better, is more understandable and ties up with the new. (2) Leave out some things which no longer seem as important as they once did. Both of these things have been done. If we think about it, it seems obvious that this should be so. Mathematics is not something apart, without connection with other human concerns. Throughout history it has been developed to solve problems which mankind needed to solve. As human needs have changed with changing conditions, mathematics has spread out in new directions and set itself to new kinds of problems. Like everything alive, it must meet new challenges or die.

We have said that to make room for newer mathematics we have to look at the older mathematics in a different way. We arrange the knowledge in new patterns. It is like the situation in technology. We are continually finding new and more suitable ways to do old things. Africa is able to take advantage of the experience of the Western nations over the past century or more. It is unnecessary to go through all the stages over again. We can profit by the experience of others. In mathematics, too, we have learned by experience. Quicker and better ways of doing things have been found. The Greeks made wonderful discoveries in geometry 2,000 years ago but mankind has not been idle all these years: for example, the volume of a sphere can now be found by much simpler methods than they knew about. This does not mean that they were wrong. It does mean that a good deal has happened since their time. We do make progress.
It is like the opening up of a new territory. A fertile valley may have been reached by a very roundabout route. Once discovered it is possible to get there by a shorter route. Ultimately paved roads are built which help us to get where we want to go very quickly and comfortably. We do not have to follow the country roads. Mathematicians have been busy building wide straight roads so that they can get to the limits of the known fairly quickly. They are not worn out by the time they reach the frontier. This is lucky, because there is too much mathematics for anyone to know in detail.

The newer programmes of instruction have been worked out with the advice of mathematicians. They have tried to save time and labour for the students who are coming on, to make their paths easier and more comfortable. They have hoped that the student will reach places where he can look over the landscape and enjoy the view without losing himself in the bush.

What kind of understanding do we hope that the student will reach? What sort of views do we expect him to get on this journey through mathematics? We surely want him to think of mathematics as more than a collection of unrelated facts to learn by heart. We hope that he will see how the facts fall into patterns so that they make sense to him. For one thing, when he discovers these patterns he will not have to remember so much. If he should forget something, it is not lost forever. He can work out for himself what he needs to know. He will not be like a man lost in a rain forest.

To take a simple example, suppose that the pupil has forgotten how to add fractions, say to find $\frac{2}{5} + \frac{3}{2}$. This is a matter which puzzles many adults. If he multiplies $\frac{2}{5} + \frac{3}{2}$ by 10, thus $10 \times \left(\frac{2}{5} + \frac{3}{2}\right)$, he easily gets $10 \times \frac{2}{5} + 10 \times \frac{3}{2}$ which
is $4 + 15 = 19$. If 10 times the required answer is 19, that answer must be $\frac{19}{10}$.

Again if the pupil wishes to multiply 82 by 98 he can of course multiply in the well-known way. If, however, he notices that $98 = 100 - 2$, he can multiply 82 by 100, which is easy, then multiply 82 by 2 which is also easy and do a simple subtraction. This makes arithmetic more fun. It changes it from a dull routine into something more like a game. It gives the student a chance to use his mind instead of operating like a machine. A machine does not notice anything it has not been instructed to do. It does not discover patterns for itself.

But let us begin farther back. In saying that $3 + 4 = 7$ we are already stating a general fact. What is adding really? To what actual process does it correspond? It involves two separate sets, two piles of stones for example. We unite these piles into a single pile and find the number of stones in the combined pile. If the first pile contains 3 stones and the second pile contains 4 stones, we discover that their "union" contains 7 stones. The addition table is based on direct experience of this sort.

All of this is familiar. The only new thing is the use of the words "set" and "union." It has been found that the early use of these words makes things clearer. The idea of a set of things has turned out to be perhaps the most basic or fundamental one in all of mathematics.

The child learns quite early to add together any two numbers from 1 to 10. He learns the results by heart. At a later stage, he notices that $7 + 6$ and $6 + 7$
Introduction-8

are both equal to 13 and in general that the result of adding two numbers does not depend on the order in which they are added. This cuts the number of addition facts that he has to remember almost in half. He is beginning to discover some pattern. He is noticing something of the structure of arithmetic.

A little later the student can be led to discover, for example, that he can add 8, 7 and 3 in two ways: as \((8 + 7) + 3\) and as \(8 + (7 + 3)\), and that the results are the same. Thereafter he can choose the easier way to get the result, which, in this example, is the second way, because it uses the easily remembered fact that \(7 + 3 = 10\) and also the easy result that \(8 + 10 = 18\). The combinations that add up to 10 can be verified by looking at his two hands. If the three numbers happen to be given in the order 7, 8 and 3, the student will soon see that he can first add 7 and 3 and then add 8. Again he will get the correct result quickly and confidently.

These are examples of the kind of patterns which we hope that the student will learn. Of course we could just tell him some rules but it is surely much better if he can discover them for himself. We can get him to do this with a little guidance.

A very clever scheme has been invented to record patterns like those that we have mentioned. The idea is to use squares and triangles to mark places where numerals can be filled in. For example, we can write

\[
\square + \triangle = \triangle + \square
\]

The \(\square\) and \(\triangle\) do not need to be filled by the same numeral, but whatever numeral we put in the first \(\square\) must be put in the second \(\square\).
Similarly, the two triangles must be filled by the same numeral. This combination of symbols is a handy way of summarizing very briefly the truth of all possible statements like

\[ \square + \triangle = \triangle + \square \]

The expression is called an open sentence. We make real statements out of it by filling the spaces with particular numerals. The point is that all such statements are true.

The second general fact about addition that we mentioned -- the principle of grouping -- can be written in the form

\[ (\square + \triangle) + \diamond = \square + (\triangle + \diamond) \]

Later the student will write

\[ a + b = b + a \text{ (for all } a \text{ and } b) \]

and

\[ (a + b) + c = a + (b + c) \text{ (for all } a, b \text{ and } c) \]

using the "variables" \( a, b \) and \( c \). This is the language of algebra. Experience has shown that the squares, triangles and diamonds can be used successfully in Primary 4 or 5. They make a natural bridge to the language of algebra which, for many people, has been so much of a mystery.

Actually the boxes can be used in Primary 1 to help review simple addition
and subtraction facts. The pupil can be asked what he should put in □ to make 3 + 5 = □ a true statement or what he should use to make 3 + □ = 8 true. The boxes will be familiar by the time the teacher wants to bring out general principles which show the pattern of the subject.

In teaching arithmetic in the traditional way, little attempt is made to bring out such general principles. At the earliest, this is done in an algebra course, but even here it is unusual to do so. In algebra, too, it is usual to learn a set of rules for working with symbols like x and y. These rules often have little meaning to the student. As one adult said: "To me x was the unknown and so far as I was concerned, I was happy to leave it that way."

This is not the place to explain the full programme in detail. We would want to bring in 0, which has the strange property that 0 + □ = □ is a true statement whatever numeral is put in the box. We would also want to work out patterns for multiplication like those for addition. These patterns could be recorded in the open sentences

□ x △ = △ x □

and

(□ x △) x ◆ = □ x (△ x ◆)

which become true statements whatever numerals are put in □, △ and ◆. We would also like the student to discover that with the same understanding

□ x (△ + ◆) = (□ x △) + (□ x ◆).
This important rule connects multiplication with addition. It is a sort of principle of sharing the multiplier with each of the added numbers. The student would also discover the strange property of the number 1.

\[ 1 \times \square = \square \]

and the further property of 0,

\[ 0 \times \square = 0 \]

The open sentences which we have written bring out an important part of what we call the structure of arithmetic. They are discovered by the student from many examples of addition and multiplication. They are then used to simplify various numerical calculations and to make clear what goes on when we multiply \( 53 \times 78 \), for example, in the standard way. They make the facts of arithmetic understandable and interesting to the student. Finally, they make things easier for him later on. Algebra will mean much more to him because he already has met some of its more important principles.

In the study of mathematics, it is extremely important for the student to keep his feet firmly on the ground. There is a very real danger that mathematics will become a game with symbols. Unless the student can see what these symbols stand for in terms of his own experience, mathematics will mean little to him. The subject will seem to have no point. Moreover, when the student makes a mistake he will not see why he was wrong nor how he should correct it.

Let us take an example. A very common mistake in algebra is to change \( \frac{1}{x+y} \) to \( \frac{1}{x} + \frac{1}{y} \). How can a mistake like this be avoided? By replacing \( x \) and \( y \) by numerals we can check whether \( \frac{1}{x+y} \) and \( \frac{1}{x} + \frac{1}{y} \) have the same value. It is
very easy to see that they do not. For example, if we replace $x$ by 1 and $y$ by

$$\frac{1}{x+y}$$

becomes $\frac{1}{2}$, but $\frac{1}{x} + \frac{1}{y}$ becomes $1 + 1$, or 2.

It should, of course, be clear that we cannot prove that two expressions

are equivalent for all values of the variables, by verifying that they are equivalent for a few particular numbers. Our point is that if the expressions are not equivalent, this fact can usually be discovered by testing for a particular choice of numbers.

The letters $x$ and $y$ are more than marks on the paper or the blackboard. They stand for numbers. It is very important that students learn the habit of testing an equation with particular numbers. This habit can be taught by using boxes and triangles. If we write

$$\frac{1}{\Box} + \triangle$$

and ask whether it is equivalent to

$$\frac{1}{\Box} + \frac{1}{\triangle}$$

it is obvious that we should test by using particular numerals.

Our short account has been limited to the set of whole numbers $0, 1, 2, 3, \ldots$ and to the operations of addition and multiplication.

When subtraction and division are included, we soon discover that we need new kinds of numbers. We cannot subtract 5 from 3 and get a whole number for an answer. Fortunately there are other kinds of numbers, the negative integers, $-1, -2, -3, \ldots$, that will help us to do this. If we mark $0, 1, 2, 3, \ldots$ at equally spaced intervals along a "number line"
to form a scale, it is very natural to mark corresponding points to the left of 0
and use the labels -1, -2, -3, ... for them. On a thermometer, for example,
we can mark degrees "below zero" in this way. With negative integers we can
talk about subtracting 5 from 3. We write the answer -2. This result can be
interpreted as follows: if we start with the point marked 3 and count off 5 steps
to the left, we shall reach the point marked -2.

In a similar way we find that when we try to divide one whole number by
another, we do not ordinarily get a whole number for an answer. What is 2
divided by 3? There is no possible whole number answer. We need new kinds
of numbers. But we do not need to look very far to find the fractions. Fractions
were invented very early in history to represent the result of dividing things into
parts.

For example, on the number line which we used a few minutes ago, we
often need to locate points between those that we have marked. Thus if we cut
the interval from 0 to 1 into three parts of equal length and count off two of them
we reach a point which we mark $\frac{2}{3}$. This is the same as $2 \div 3$. To see this we
have to be clear about what division is. We are looking for a number which when
multiplied by 3 gives us 2. Does $\frac{2}{3}$ do this? Of course it does. If we use the
interval from 0 to $\frac{2}{3}$ as a measuring stick and lay it off three times, we do arrive
at the point 2. It is natural to include negative fractions as well as positive
ones. For example, $-\frac{2}{3}$ should correspond to a point on the opposite side of 0
from \( \frac{2}{3} \) and the same distance away.

When we have extended our system of whole numbers to include these new kinds, we have to ask ourselves how the new members of the club behave. How can we add, subtract, multiply and divide them with each other and the old members of the club?

The old club had certain rules of behaviour, like the rules we have mentioned. Suppose that we require that no new numbers can be admitted unless they obey these rules. Can we take in the negative integers and the positive or negative fractions or shall we have to say to them, "You cannot come in"? Must we form a new club to accommodate these new numbers?

It is a really surprising fact that we can admit all of them without any difficulty. They really belong. That is, if we take the open sentences and fill the blanks with fractions, the statements we get will all turn out to be true. Let us take as an example the positive fractions.

\[
\text{Is } \frac{1}{2} + \frac{2}{3} = \frac{2}{5} + \frac{1}{2} ? \\
\text{Is } \left( \frac{1}{2} + \frac{2}{3} \right) + \frac{1}{4} = \frac{1}{2} + \left( \frac{2}{3} + \frac{1}{4} \right) ? \\
\text{Is } \frac{1}{2} + \frac{2}{3} = \frac{2}{5} + \frac{1}{2} ?
\]

To answer these questions and others like them, we must make up our minds how we would go about adding and multiplying two fractions. If we look at the number line it will not take long to decide that at least the first two questions can be answered "Yes." Take the first one. To add \( \frac{2}{3} \) to \( \frac{1}{2} \) we should
begin with the point marked \( \frac{1}{2} \) and measure off to the right of it an interval equal to the interval from 0 to \( \frac{2}{3} \). Suppose that we do this in the other order. Do we come out at the same place? We certainly do. If we test the second question in the same way, we also will get the answer "Yes." Of course, we have tested our first question only with the particular fractions \( \frac{1}{2} \) and \( \frac{2}{3} \). This does not prove that the answer would be "Yes" for all fractions. We have merely tested the rule with special fractions. But the idea of adding on the number line is a general one and it is not hard to see that the answer will not depend on the order of adding or the method of grouping.

The open sentences that contain a multiplication sign are a little harder to test. We must first decide on a sensible way to multiply two fractions. One way to think of this is to remember that the multiplication of two numbers occurs naturally in connection with the area of a rectangle. The rectangle with sides 3 and 2 has the area \( 2 \times 3 = 6 \), as we see by counting squares.

What would be the area of a rectangle \( \frac{1}{2} \) by \( \frac{2}{3} \)? The answer should give a reasonable result for \( \frac{1}{2} \times \frac{2}{3} \). In the figure, if we cut the desired region (shown shaded) in half, we get a rectangle \( \frac{1}{2} \) by \( \frac{1}{3} \). It is easy to see that 6
rectangles $\frac{1}{2}$ by $\frac{1}{3}$ make up the square $1 \times 1$.

Then we are led to say that $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$, and so $\frac{1}{2} \times \frac{2}{3}$ should be $\frac{2}{6}$. In the same way we come to see that in general

$$\frac{a}{b} \times \frac{c}{d} \text{ should be } \frac{a \times c}{b \times d}.$$ 

Now that we know how to multiply fractions, we see that the order of multiplication does not matter so that

$$\square \times \triangle = \triangle \times \square$$

even when we fill the blanks with fractions. The other general rules of the club are obeyed by the new members, as can be shown without difficulty.

We are now able to fulfill a prc-nise. Why do we divide two fractions by inverting and multiplying? To take our previous example, why is $\frac{2}{3} \div \frac{3}{4}$ the same as $\frac{2}{3} \times \frac{4}{3}$? We remember what division means. We want to fill the square in

$$\frac{3}{4} \times \square = \frac{2}{3}$$

so that the two sides stand for the same number. We notice that if the $\frac{3}{4}$ is multiplied by $\frac{4}{3}$, the result will be 1. We use this fact to find the numeral to put in the box as follows:

$$\frac{4}{3} \times (\frac{3}{4} \times \square) = \frac{4}{3} \times \frac{2}{3}$$

The left side can be regrouped to give

$$(\frac{4}{3} \times \frac{3}{4}) \times \square$$

which is $1 \times \square$ or simply $\square$

So we have $\square = \frac{4}{3} \times \frac{2}{3}$ or what is the same, $\square = \frac{2}{3} \times \frac{4}{3}$. Thus $\frac{2}{3} \div \frac{3}{4}$ is indeed equal to $\frac{2}{3} \times \frac{4}{3}$.

To return to our main point, we have said that the new numbers can be
brought into the club as law-abiding members. When the student really sees this, he understands the structure of arithmetic. When he understands the way things tie together, mathematics means something to him. It makes sense. It also becomes part of him. He does not need to worry about forgetting it. With good luck he will see the beauty and simplicity of the subject and find it exciting. In any case he will find as he goes on that the kind of understanding which he has acquired will help him to take the next steps up the mathematical ladder.

It sometimes happens that students who have been encouraged to work things out for themselves discover new ways of doing things. For example, a class was asked to subtract 28 from 42. Of course, the standard way to do this is to say that 8 cannot

\[
\begin{array}{c}
28 \\
14
\end{array}
\]

be subtracted from 2. Therefore we must borrow from the next column and subtract 8 from 12 giving 4 in the unit's place and 1 in the ten's place. One student did this in quite a different way. The class had some familiarity with negative numbers. The boy said that 2 - 8 = -6 and 40 - 20 = 20. He reasoned that the answer must be the sum of 20 and -6, that is, 14. We do not say that this is a better way to do this exercise. But it is obvious that to be able to invent this new method is to add greatly to one's understanding. It also gives the student the very pleasant feeling of creating something new.

This is an important feeling for the student to have. It gives him confidence in using his own mind. In a changing world, we meet new situations. We
cannot meet all of our problems by following rules. We have to invent new ways of doing things. This means that we must look for new ways of thinking. The most important thing which any teacher can do is to encourage any sign of originality in his students. If one of them has a new way of doing a problem, do not tell him that he must do it in the textbook way. Let him discover for himself that the textbook way is better if it is. Meanwhile give him the pleasure of using his own mind.
UNIT I - BASIC CONCEPTS AND LANGUAGE OF SETS

CHAPTER 1

SETS AND SUBSETS

Purpose of unit

The purpose of this unit is to introduce you to some of the elementary ideas of "sets" and to begin to develop for you a language of sets. In recent years, the concept of set has become of great importance in the development of mathematics. In this and the other units which follow, you will study how to make use of sets to get a better understanding of arithmetic. For example, you will discover that numbers can be explained and understood in terms of sets and this helps children considerably in their learning. You will also learn how to compare two sets in order to find out whether one set has "just as many members as" another set, "fewer members than" a second set, or "more members than" a second set. You will find out how to match sets of objects with the counting sets and how to unite two sets to form a set with more members.

Some of the facts you will learn about sets in this unit will, no doubt, be new to you. Refer to the glossary at the end of the unit for an explanation of their meaning. Test your own understanding of each section with the exercises.

**************************
1 - 1 What is a set?

A set as a collection of things. In our daily life if we wanted to describe a collection of things we might talk of a bundle of sticks, a flock of sheep, a herd of cattle, a class of boys and girls, and so on. In mathematics, a collection of things is called a set. The words "flock", "herd", "class", "bundle" give the same idea as the word "set".

1 - 2 Examples of sets

We can talk of the set of books in the school library, the set of the colours of the rainbow, the set of the counting numbers less than ten, the set of all points on the line segment AB. Each of these sets is a set of similar things. A set can, however, contain any variety of objects. For example we may have, if we want, a set that is made up of the following things: a stick, a stone, a book and a banana.

1 - 3 Members of a set

It is important that a set be described so clearly that no-one can have any doubt about which objects are included in it. For example a set of numbers may be any one of many possible sets, but if we talk of the set of the counting numbers less than ten we have only one set in mind. This set consists of 1, 2, 3, 4, 5, 6, 7, 8, 9.

Each object in a set is called a MEMBER or element of the set. For instance, Uganda is a member of the set of independent African states; a cat is a member of the set of all domestic animals; a dog is also a member of this set. On the other hand, a lion is not a member of the set of all domestic animals, because it is not a domestic animal.
It is more usual to think of a collection of things which has two or more members, as a set. In mathematics, we use the term "set" to include a collection of things, which may have only one member or even no members at all. We can talk about the set of a book, the set whose member is a pencil, or say that the member of our set is a cat. Later we shall have more to say about a set which has no members at all.

Exercise 1 - 3a

1. Write down the members of the following sets:
   (a) the days of the week whose names begin with the letter S,
   (b) the even counting numbers less than 14,
   (c) all the cities in your country with a population greater than 60,000,
   (d) the months of the year whose names begin with the letter S,
   (e) all the cities in Africa with a population of over four million.

2. How would you describe the members of the set 1, 4, 9, 16, 25, 49, 64, 81, 100?

3. What are the members of the set of all the prime numbers which are less than twenty?

4. What are the members of the set of all two-digit whole numbers that are exactly divisible by five?

5. Write down the members of the set of all three-digit whole numbers for which the sum of the digits is three.

6. Here are some sets. Below each are phrases. Write the phrase which best describes each set.
1-4

(i) 0, 2, 4, 6, 8, 10.
   (a) the set of small even numbers;
   (b) the set of some even numbers;
   (c) the set of even whole numbers less than twelve.

(ii) 5, 10, 15, 20, 25, 30.
   (a) the set of counting numbers less than 35;
   (b) the set of the first six counting numbers that are exactly divisible by five;
   (c) the set of all numbers counting by fives.

(iii) January, June, July.
   (a) the set of the months of the year;
   (b) the set of the months whose names have more than three letters;
   (c) the set of the months whose names begin with the letter J.

7. Answer these questions about the members of a set.
   (a) Is a Nigerian a member of the set of all Africans?
   (b) Is an elephant a member of the set of all trees?
   (c) Is a square a member of the set of all four-sided figures?
   (d) Is "t" a member of the set of all vowels of the English alphabet?
   (e) Is 25 a member of the set of all squares of whole numbers?

1 - 4 Ways of describing a set: listing the members

We want to explain two ways of describing a set. One way is to list all the members of a set. For example, if the members of the set are all odd numbers less than 11, we list the members as follows:
We read this as "the set whose members are 1, 3, 5, 7 and 9."

Curly brackets are used to enclose the list of members and commas may be used to separate the members. Commas are not usually used to separate the members of a set when a picture of the members is drawn.

For example

\[
\{ \bigcirc, \star, \triangle, \bigcirc \} 
\]

is a picture of a set. The members of the set are a football, a star, a triangle and a man.

We may not always be able to list all the members in a set. There are several reasons for this. Let us discuss these one at a time.

First we shall look at the set of all odd numbers. No matter which odd number we think of there is always a larger one. We list a few of the odd numbers and then put three dots after the last listed member to indicate that the set of all odd numbers is unending. Thus we write

\[
\{ 1, 3, 5, 7, 9, \ldots \}
\]

Similarly the set of all counting numbers may be written

\[
\{ 1, 2, 3, 4, 5, \ldots \}
\]

Even if the set is finite we may not be able to list all the members of the set either because we have not enough time, or because we have not enough space. We could, if we wanted to, name all the members of the set of whole numbers less than 10,000 which are exactly divisible by 3. The
first one is 3, the second one is 6, the third one is 9, and the last one is 9,999. It would take a long time and a lot of paper to list all the members of this set. We might not even want to list all the members. Here again the dots are useful. Once the pattern is established and the last member of the set is known, we use three dots thus:

\[
\{3, 6, 12, 15, \ldots, 9,999\}
\]

This is read, "the set consisting of three, six, nine, twelve, fifteen and so on up to nine thousand, nine hundred, ninety-nine."

It is helpful to use a capital letter as a name for a set. For example,

Set \( A = \{3, 6, 9, \ldots, 9,999\} \)

or

\( A = \{3, 6, 9, \ldots, 9,999\} \)

\( A \) is then described as the set of all counting numbers less that 10,000 that are exactly divisible by 3.

1 - 5 Ways of describing a set: giving a word description

Sometimes we do not wish to list any of the members of a set at all; it may be easier to describe them. In this case we give a word description of the set. For example,

\( A = \) the set of all countries in the world.

\( B = \) the set whose members are all the even numbers.

\( C = \) the set made up of all the points lying in a certain circle of two inch radius.

\( D = \) the set consisting of a football, a star, a triangle and a man.

\( E = \) the set of numbers exactly divisibly by three and less than 10,000.
From the word description we can decide whether or not an object is a member of the set. Note that we do not use curly brackets when we have a word description of a set.

**Exercise 1 - 5a**

1. Describe the following sets in words:
   (a) \{ 1, 8, 27, 64, 125, 216 \}
   (b) \{ a, b, c, d, e, f, g, h \}
   (c) \{ Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday \}

2. List the members of the following sets:
   (a) the set of whole numbers which are greater than twenty,
   (b) the set of whole numbers less than ten,
   (c) the set of prime numbers between ten and thirty.

3. Let \( S \) be the set of whole numbers between 8 and 50 and exactly divisible by 3.
   (a) Describe \( S \) by listing its members.
   (b) Describe \( S \) in another way.

4. For which of the following sets is it possible to list all of the members?
   (a) The set of all living human beings.
   (b) The set of all counting numbers.
   (c) The set of all the counting numbers exactly divisible by four.
   (d) \{ 0, 2, 4, 6, \ldots \}\]
   (e) \{ 0, 2, 4, \ldots, 100,000,000 \}
5. Correct the mistakes in the following listings of sets:

(a) The set of even whole numbers less than ten: \[\{2, 4, 6, 8\}\]

(b) The set of all whole numbers: \[\{1, 2, 3, \ldots, 9, 999\}\]

(c) The set of odd numbers less than thirteen: \[\{0, 1, 3, 5, 7, 9, 11, 13\}\]

(d) The set of all the counting numbers which are exactly divisible by three: \[\{3, 6, 9, 12, 15\}\]

SPECIAL SETS

1 - 6 The empty set

What answer did you give to the question "List the members of all cities in Africa with a population of over four million? Did you say that this set had no members?

Let us try to discover the members of the following sets:

A = the set of all squares with five sides.

B = the set of all baby boys each of whom weighed 1000 pounds at birth.

C = the set of persons who are over 400 years old.

A has no members, B has no members and C has no members. The set which has no members is called the EMPTY set.

Here are some ways to picture the empty set for our pupils.

A way to represent the empty set is \[\{\}\]
Exercise 1 - 6a

Which of the following sets are empty?
(a) The set of even prime numbers between 20 and 30
(b) The set of all triangles with four sides
(c) The set of Presidents of African states
(d) The set of boys in a Primary One class who are fifty years old.

1 - 7 Sets with a single member

As we have said before, in mathematics we use the term "set" to include a collection of things which has only one member. For example, \( A = \{3\} \) is a set with only one member, the counting number 3. The set \( \{0\} \) has one member and is therefore different from the empty set. Can you make up some more examples of sets with only one member, or with no members at all, which you could use in your classroom to help your pupils to understand these ideas?

1 - 8 Subsets of a set: what is a subset?

We are going to discover some new facts about sets. Before we go on, let us look again at what we mean by the term "set." We have said that a set is a collection of things.

Now let us compare the pairs of sets listed below.

<table>
<thead>
<tr>
<th>List 1.</th>
<th>List 2.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) The set of players in your school football team.</td>
<td>(a) The set of forwards in your school football team.</td>
</tr>
</tbody>
</table>
(b) The set of all vegetables on a market stall.

(c) \( C = \{1, 2, 3, 4\} \)

(d) The set of letters in the English alphabet.

(e) \( K = \{\text{pin, tree, book}\} \)

(f) \( Y = \{\square, \blacksquare, \bigcirc, \blacktriangle, \blackdiamond\} \)

(g) The set of beans on the same market stall.

(c) \( D = \{2, 3\} \)

(d) The set of vowels in the English alphabet.

(e) \( N = \{\text{book}\} \)

(f) \( Z = \{\bigcirc, \blacktriangle, \blackdiamond, \square\} \)

In the first pair of sets did you find that each member of the set in List 2 was also a member of the corresponding set in List 1? Is this true for the other pairs of sets?

Did you discover in (c) that 2 and 3 are members of set \( D \) and that they are also members of set \( C \)? In (f) are the members of set \( Z \) also members of set \( Y \)? Have you found out that a book is a member of set \( N \) and also a member of set \( K \), but a pin and a tree are members of set \( K \) and not of set \( N \)? Each set in the second list is called a SUBSET of its corresponding set in the first list.

If each member of a set \( A \) is also a member of a set \( B \) then set \( A \) is called a \textit{subset} of set \( B \).
Exercise 1 - 8a

1. Copy the following statements and complete them.

(a) \{a, b\} is a _______ of \{a, b, c, d\}

(b) Set D = \{2, 4, 6, 8, 10\} Set E = \{4, 6, 8\} ______ is a
subset of _________.

2. Set Y = \{a house, a tree\}
Set V = \{a cat, a house, a tree\}
Which set is a subset of the other?

3. Is \{A, B, C, D, E\} a subset of \{A, B, C\}? Explain your answer.

4. Set X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} Use the information about set X
to fill in the blanks and answer the questions.

(a) Rewrite the statements below filling in the blanks.
Set K = \{1, 2, 6\} Set L = \{3, 11\} Set M = \{3, 5, 8, 9\}
Set K is ________ of set X
Set L is ________ of set X
Set M is ________ of set X

(b) How many subsets of set X did you find in (a)?

(c) Which members of set X can be divided exactly by 2? Write
these as a set. Is this set a subset of set X?

(d) Which members of set X can be divided exactly by 3? Write
these as a set. Is this set a subset of set X?

(e) Find three other subsets of set X.
1 - 9 Finding subsets of a set

Look again at Exercise 4. Count the number of subsets of set \( X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \) which you found. Can you find one more?

Now let us try to write down all the subsets of set \( A = \{1, 2, 3\} \). How many have you found? Six, seven, eight? Did you write
\[
\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\},
\]
or
\[
\{1\}, \{1, 2\}, \{3\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}, \{\}.
\]
Which of these answers is correct? You will notice that in listing the above subsets, \( \{1, 2\} \) and \( \{2, 1\} \) are regarded as being different ways of writing the same set. This will be explained fully later.

To help us decide which answers are correct let us consider again what we mean by a subset. Each member of a subset must also be a member of the given set. A subset cannot have any member which is not a member of the given set.

It is easy to see from this meaning of a subset that \( \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\} \) and \( \{3, 1\} \) are subsets of \( \{1, 2, 3\} \). Is \( \{1, 2, 3\} \) a subset of \( \{1, 2, 3\} \)? Is there any member of the set \( \{1, 2, 3\} \) that is not a member of the given set \( \{1, 2, 3\} \)? Because the answer to this last question is "No" we can say that \( \{1, 2, 3\} \) is a subset of \( \{1, 2, 3\} \). Now what about the empty set? If the empty set is a subset of \( \{1, 2, 3\} \) it cannot have any members which are not members of \( \{1, 2, 3\} \). The empty set has no members at all; therefore, it does not have any members which are not members of \( \{1, 2, 3\} \). Therefore the empty set is a subset of \( A \). A picture may help us to understand
this idea.

There are no members in the empty set. 1, 2, 3 has all the members that are in the empty set. Therefore, \( \emptyset \) is a subset of \( \{1, 2, 3\} \). We now know that there are eight subsets of the set \( A = \{1, 2, 3\} \) because we must include the empty set and \( \{1, 2, 3\} \).

**Exercise 1 - 9a**

1. Write down all the subsets of each of the following sets.
   
   (a) Set \( A = \{1, 2\} \)
   
   (b) Set \( B = \{a, b, c\} \)
   
   (c) Set \( C = \{x, y\} \)
   
   (d) Set \( D = \{1, 2, 3, 4\} \)
   
   (e) Set \( E = \{\triangle \square \circ\} \)

2. What have you discovered about the empty set from your answers to problem 1. above?

3. Have another look at your answers to problem 1. above. Did you find that \( \{1, 2\} \) was a subset of Set \( A \)? Was \( \{a, b, c\} \) a subset of set \( B \)? What about \( \{x, y\}, \{1, 2, 3, 4\} \) and \( \{\triangle \square \circ\} \) ? What conclusion
can you draw from these answers?

4. Which of the following statements is true?

(a) \( \{3, 5\} \) is a subset of \( \{3, 5, 7\} \)

(b) \( \{9, 6, 1\} \) is a subset of \( \{1, 9\} \)

(c) \( \{A, B, D\} \) is a subset of the set of letters which name the
vertices of the square \( ABCD \).

(d) The set of all birds in the world is a subset of the set of all
hens in the world.

(e) The set of names of all the days in the week is a subset of
\( \{\text{Sunday, Monday}\} \)

(f) \( \{-\} \) is a subset of \( \{a, b, c, d\} \)

(g) \( \{\text{red, orange, yellow, blue, green, indigo, violet}\} \) is a subset
of the set of the colours of the rainbow.

1-10 Picturing sets and subsets

We can draw pictures and diagrams to help us to understand subsets.

There are several ways in which we can do this.

First let us illustrate by means of a picture the fact that the set of
all forwards in the school football team is a subset of the set of all players
in the school football team.
Picture (a) shows the set of all players in the school football team. Picture (b) shows the set of all players in the school football team and the subset of all forwards in the same school football team.

Now let us show that the set of all countries in Africa is a subset of the set of all countries in the world.
Diagram (c) shows the set of all the countries in the world. Diagram (d) shows the set of all the countries in the world with the set of all the countries in Africa as a subset of the set of all the countries in the world.

As another example of the use of pictures and diagrams to illustrate sets and subsets, let us consider the set of all people in the world and two of its subsets.

Let \( U \) = the set of all people in the world,

\[ A = \text{the set of all people in Africa,} \]

\[ B = \text{the set of all people in the United States of America.} \]

Let us think of the set of all people in the world as the universal set. *

*What we take as the universal set will depend upon our interest. At one time we might use the pupils in a class as the universal set with subsets of girls, of boys, of pupils in the front row and so on. At another time, we might use the set of all the pupils in the school as the universal set.*
We can picture the universal set by a rectangle. All points inside the rectangle are members of the set \( U \). In the figure below we will represent the members of set \( A \) by the set of points within the circle \( A \); the points within the circle \( B \) will represent the members of set \( B \).

\[ U = \text{ALL THE PEOPLE IN THE WORLD} \]

We can see that all the members of both set \( A \) and set \( B \) are members of set \( U \). They are therefore represented by circles which lie entirely within the rectangle. The interior of the rectangle represents the set \( U \).

We also note that set \( A \) and set \( B \) have no members in common and so the circles representing them do not cut each other. We call two sets that have no members in common DISJOINT sets.

For our last example, let us take \( U \) to be again the set of all people in
the world. We shall look at two subsets of $U$, one of which is different from the subsets used in the last example.

Let $A =$ the set of all people in Africa,

and $C =$ the set of all people in Nigeria.

How shall we represent these sets in a diagram? First we note that both set $A$ and set $C$ are subsets of set $U$, the set of all people in the world.

Let us represent set $U$ by the inside of a rectangle. Then we can represent set $A$ and set $C$ by the insides of circles lying entirely within the rectangle.

Since Nigeria is a country in Africa, the set of all the people in Nigeria is a subset of the set of all the people in Africa. Therefore the circle for set $C$ must lie entirely within the circle for set $A$. (The two circles, however, do not need to have the same centre; also the sizes of the circles do not represent proportionally the number of people in Africa and the number of people in Nigeria. The choice of the shape of the figure used to represent a set is also one you can make for yourself.)

\[
\begin{align*}
U &= \text{the set of all the people in the world,} \\
A &= \text{the set of all the people in Africa,} \\
C &= \text{the set of all the people in Nigeria.}
\end{align*}
\]
Exercise 1 - 10a

1. Draw pictures that you could use in your classroom to show that the set of John, Mary and Kwame is a subset of the set of all pupils in a class. Make up other examples of such subsets and draw pictures to illustrate them.

2. Draw diagrams to show the set and subsets,
   
   \[
   A = \text{all the books in your classroom},
   
   B = \text{all the mathematics books in your classroom},
   
   C = \text{all the English books in your classroom}.
   \]

3. Copy this set.

Show each of the following subsets by drawing a shape on your copy.

(a) The set of all days of the week whose names begin with T,

(b) The set of all days of the week whose names begin with W,

(c) The set of all days of the week whose names begin with K.
THE IDEA OF EQUALITY

When you say that two plus two equals four, and write it down as

\[ 2 + 2 = 4 \]

you are thinking that 2 + 2 and 4 stand for the same number. In other words, what you write on one side of the sign \( = \) and what you write on the other side are just different names for the same number.

This is a point of view which we want to adopt. The statement \( a = b \) tells us that the thing named "a" and the thing named "b" are the same thing.

Here are some more examples:

<table>
<thead>
<tr>
<th>2</th>
<th>\quad = \quad \text{the smallest prime number}</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 5 + 3 \div 4 )</td>
<td>\quad = \quad 2</td>
</tr>
</tbody>
</table>

EQUAL SETS

Let us look carefully at the following sets:

<table>
<thead>
<tr>
<th>A</th>
<th>\quad = \quad { 1, 3, 4, 5, 2 }</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>\quad = \quad { 3, 4, 1, 5, 2 }</td>
</tr>
<tr>
<td>C</td>
<td>\quad = \quad \text{the set of the first five counting numbers}</td>
</tr>
</tbody>
</table>
Compare set A and set B. Did you discover that the members of set A were exactly the same as the members of set B?

The members of set A are 1, 2, 3, 4, 5.

The members of set B are 1, 2, 3, 4, 5.

The only difference between the members of set A and the members of set B is the order in which they are listed. We say that A and B are different names for the very same set. When we write $A = B$ in reference to sets, we mean that A is the same set as B.

Now look at set C. Is this set equal to set A and to set B? What are the members of set C? They are 1, 2, 3, 4, 5. The members of set C are the same as the members of set A and are the same as the members of set B. We say that $C = B$ and $C = A$. In general, sets with the same members are EQUAL.

It can happen that two sets may have some members which are the same and yet the two sets are not equal.

Consider the sets

$A = \{13, 5, 7, 9\} \quad B = \{1, 3, 5, 7, 9\}$

There are three members of set A which are also members of set B but by comparing members of both sets we find that they are not equal sets. This example shows the importance of putting commas in the proper places in listing sets.
Exercise 1 - 12a

1. Set $A = \{2, 4, 6, 8\}$ Set $B =$ the set of all even whole numbers less than 10. Is $A = B$?

2. Which of the following pairs of sets are equal?
   (a) $K = \{m, t, h, a, s\}$ $L = \{m, a, t, h, s\}$
   (b) The set of all toes on your left foot and $\{a, b, c, d, e\}$
   (c) The set of all eyes in the classroom and the set of pupils in the classroom.
   (d) $\{5, 10, 15, 20, 25\}$ and $\{5 \times 1, 5 \times 2, 5 \times 3, 5 \times 4, 5 \times 5\}$
   (e) $\{0\}$ and the empty set.
   (f) The set of all odd counting numbers less than 10 and the set $\{1, 5, 7, 8\}$
   (g) The set of letters in the word "bundle" and the set $\{n, d, l, e, b\}$.
   (h) The set of countries in East Africa and the set $\{\text{Kenya, Nigeria, Tanganyika}\}$.

3. Look at these sets carefully.
   $A = \{3, 5, 7, 9\}$ $D = \{9, 7, 2, 3\}$
   $B = \{5, 3, 9, 7\}$ $E = \{5, 7, 9, 2\}$
   $C = \{9, 7, 2, 5\}$ $F = \{7, 3, 9, 5\}$
   Which sets are not equal to set $A$? Write three sets equal to set $A$.

4. Set $X$ is the set of whole numbers less than 9. Set $Y$ is the set of counting numbers less than 9. Is set $X$ equal to set $Y$? Explain your answer.
MATCHING OF SETS

1 - 13 Matching of two sets

In some African countries children of pre-school age already have the idea of matching the objects of one set with the objects of another set. Thus, when a child is given a piece of meat, he sometimes asks for one more piece by holding out his other hand. He means that he wants one more piece corresponding to the other hand.

Again, when you ask a child his age he may hold up three fingers. Each one corresponds to one year of his age. When you ask a child how many noses he has, he often touches his nose with one finger. In these and other ways the child establishes the idea of an exact matching between the members of two sets.

Consider the sets

\[ A = \{ \text{book, ball, person, banana} \} \]
\[ B = \{ \triangle, \star, \Box, \text{tree} \} \]
\[ C = \{ \text{cup, bottle, fish, square} \} \]
\[ D = \{ a, b, c \} \]
All these sets at first glance appear to be different because they have different members. On closer examination, however, we see that set B and set C have something in common. We can find for each member of set B a partner from set C.

\[ B = \{ \triangle, \star, \Box, \text{tree} \} \]
\[ C = \{ \text{cup}, \text{bottle}, \text{fish}, \text{rectangle} \} \]

The double arrows show how each member of set B is matched with a member of set C, and how each member of set C is matched with a member of set B. This matching could be done in many different ways. Two other ways are shown below.

\[ B = \{ \triangle, \star, \Box, \text{tree} \} \]
\[ C = \{ \text{cup}, \text{bottle}, \text{fish}, \text{rectangle} \} \]

(i)
No matter which way we match the sets each member of set B is matched with exactly one member of set C and each member of set C is matched with exactly one member of set B.

If each member of set P is matched with exactly one member of a second set Q and each member of set Q is matched with exactly one member of set P, then we say that set P and set Q MATCH EXACTLY. Set P has "just as many members" as set Q. The sets are called EQUIVALENT sets.

We must be careful not to confuse equivalent sets with equal sets.

The set \{a, b, c, d\} and the set \{b, c, d, a\} are equal sets because they have the same members. The set \{a, b, c, d\} and the set \{p, q, t, s\} are equivalent sets because they match exactly. All equal sets are also equivalent sets but equivalent sets are not necessarily equal sets.

Look back at the list of sets at the beginning of this section. You can
see that set A and set B can be matched exactly. What happens when we try to match the members of set A with the members of set D.

\[ A = \{ \text{book, ball, person, banana} \} \]

\[ D = \{ d, b, c \} \]

The above is one way of trying to match the members of these sets. There are many other ways. No matter how we try to match the members of the sets there will always be at least one member of set A which will not have a partner in set D. We see that these sets do not match exactly. We say that set A has "more members" than set D or that set D has "fewer members" than set A.

1 - 14 Equivalence of more than two sets

You know that set B and set C are equivalent. Also set A and set B are equivalent. Does it follow that set A and set C are equivalent? Let us picture the matching.
When we remove the set B the picture becomes:

From this you can see that set A is equivalent to set C. Do you see that this result is true for any three sets? If we have three sets such that the first
is equivalent to the second, and the second is equivalent to the third, it
follows that the first is equivalent to the third.

1-15 Matching of sets with no last members

Can we match exactly sets like P, the set of all counting numbers and
Q, the set of all even counting numbers?

\[ P = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots \} \]
\[ Q = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, \ldots \} \]

Suppose we try to match set P with set Q.

\[ P : \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots \} \]
\[ Q : \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20, \ldots \} \]

How far can we continue this matching? Which member of set Q could be
matched with 107 of set P? Which member of set Q could be matched with
2046 of set P? Which member of set Q could be matched with 5,000,468
of set P? No matter which member of set P we choose we can always find
a partner for it in set Q and no matter which member of set Q we choose we
can always find a partner for it in set P. We see that set P and set Q match
exactly.

Exercise 1-15a

1. Which of the following pairs of sets can be matched exactly? Illustrate

how this is done in each case.

(a) \[ A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\} \]
\[ B = \{\text{Ghana, Guinea, Kenya, Liberia, Nigeria, N. Rhodesia,}
\text{Nyasaland, Sierra Leone, Uganda}\} \]
(b) \( X = \) the set of all even whole numbers less than 20.

\[ Y = \text{the set of all even numbers which are greater than 40 and which are also less than 60.} \]

(c) \( S = \{1, 2, 3, 4, 5, \ldots\} \)

\[ T = \{5, 10, 15, 20, \ldots, 50, 005\} \]

(d) \( K = \) the set of all counting numbers.

\( L = \) the set whose members are the cubes of all counting numbers.

(e) \( R = \) the set of all pupils in your class.

\( Q = \) the set of all feet in the same class.

2. Give examples of three pairs of sets which you could use with your pupils to show that the two sets in each pair can be matched exactly.

3. Is it possible to match exactly the set of all counting numbers and the set of all counting numbers greater than 200? If your answer is "yes," show how this could be done. If your answer is "no," explain your answer.

4. Give examples of three pairs of sets, that you could use in your classroom to show that the two sets in each pair do not match exactly.

5. Demonstrate with double arrows (\( \rightarrow \rightarrow \)) that the set \( \{a, b\} \) is equivalent to each of six subsets of the set \( \{p, q, r, s\} \).
CHAPTER 2
OPERATION ON SETS

2-1 Union of sets

We have learned some of the facts about sets. Now let us see how we can work with sets.

Study the following class list carefully. It could be for any class in any school.

<table>
<thead>
<tr>
<th>NAME</th>
<th>AGE</th>
<th>RELIGION</th>
<th>CLUBS / SOCIETIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>AGESA</td>
<td>11</td>
<td>CATHOLIC</td>
<td>✓</td>
</tr>
<tr>
<td>ALLI</td>
<td>11</td>
<td>MOSLEM</td>
<td>✓ ✓</td>
</tr>
<tr>
<td>JUMA</td>
<td>9</td>
<td>PROTESTANT</td>
<td>✓</td>
</tr>
<tr>
<td>KATC</td>
<td>11</td>
<td>MOSLEM</td>
<td>✓</td>
</tr>
<tr>
<td>KITTA</td>
<td>10</td>
<td>CATHOLIC</td>
<td></td>
</tr>
<tr>
<td>KIZZA</td>
<td>11</td>
<td>PROTESTANT</td>
<td>✓ ✓</td>
</tr>
<tr>
<td>KOFI</td>
<td>12</td>
<td>CATHOLIC</td>
<td>✓ ✓</td>
</tr>
<tr>
<td>LULE</td>
<td>10</td>
<td>PROTESTANT</td>
<td>✓</td>
</tr>
<tr>
<td>MUKASA</td>
<td>11</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NSIMBI</td>
<td>10</td>
<td>PROTESTANT</td>
<td>✓</td>
</tr>
<tr>
<td>OKOT</td>
<td>12</td>
<td>CATHOLIC</td>
<td>✓</td>
</tr>
<tr>
<td>ONGOM</td>
<td>9</td>
<td>CATHOLIC</td>
<td>✓</td>
</tr>
<tr>
<td>OPIO</td>
<td>10</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>OYELESE</td>
<td>9</td>
<td>PROTESTANT</td>
<td>✓</td>
</tr>
<tr>
<td>PATEL</td>
<td>12</td>
<td>HINDU</td>
<td></td>
</tr>
<tr>
<td>SENTAMU</td>
<td>12</td>
<td>CATHOLIC</td>
<td>✓</td>
</tr>
<tr>
<td>SENTEZA</td>
<td>10</td>
<td>PROTESTANT</td>
<td>✓</td>
</tr>
<tr>
<td>SINGH</td>
<td>11</td>
<td>SIKH</td>
<td></td>
</tr>
<tr>
<td>SOZI</td>
<td>10</td>
<td>CATHOLIC</td>
<td>✓ ✓ ✓</td>
</tr>
<tr>
<td>WASSWA</td>
<td>10</td>
<td>PROTESTANT</td>
<td></td>
</tr>
</tbody>
</table>
Let us write down the members of the various sets of boys in this class.

**Example A.** Suppose we list

1. the set of boys whose names begin with K.
2. the set of boys whose names begin with S.

Call the first set K and the second set S.

Have you written

1. \( K = \{ \text{Kato, Kitta, Kizza, Kofi} \} \)
2. \( S = \{ \text{Sentamu, Senteza, Singh, Sozi} \} \)?

What is the set whose members are in one or the other of the above sets? Call it \( R \).

Does your set \( R \) look like this?

3. \( R = \{ \text{Kato, Kitta, Kizza, Kofi, Sentamu, Senteza, Singh, Sozi} \} \)

Set \( R \) has each of the members of set \( K \) and each of the members of set \( S \) for its members. (Note that set \( K \) and set \( S \) are disjoint sets.)

**Example B.** Suppose we list

1. the set \( Q \) of boys who are ten years old.
2. the set \( L \) of boys who play football.

Have you written

1. \( Q = \{ \text{Kitta, Lule, Nsimbi, Opio, Senteza, Sozi, Wasswa} \} \)
2. \( L = \{ \text{Agesa, Alli, Juma, Kato, Kizza, Lule, Okot, Ongom, Opio, Sentamu, Sozi} \} \)?

Now list the new set whose members are in set \( Q \) or in set \( L \). Call it \( V \).

3. \( V = \{ \text{Kitta, Lule, Nsimbi, Opio, Senteza, Sozi, Wasswa, Agesa, Alli, Juma, Kato, Kizza, Okot, Ongom, Sentamu} \} \)
The members of set \( V \) are boys who are ten years old, or who play football, or who are ten years old and play football.

**Example C.** List the following sets.

1. \( X = \) the set of boys who take part in athletics
2. \( Y = \) the set of boys who are in the choir
3. \( N = \) the set of all the boys who take part in athletics or who are in the choir or both.

**Example D.** List the members of the following sets.

1. \( E = \) the set of boys who are eleven years old.
2. \( F = \) the set of boys whose names begin with the letter A.
3. \( M = \) the set of boys who are eleven years old or whose names begin with the letter A.

Are these your answers?

1. \( X = \{ \text{Alli, Kissa, Kofi, Nsimbi, Senteza, Sozi} \} \)
2. \( Y = \{ \text{Kofi, Oyelese, Sozi} \} \)
3. \( N = \{ \text{Alli, Kizza, Kofi, Nsimbi, Senteza, Sozi, Oyelese} \} \)

1. \( E = \{ \text{Agesa, Alli, Kato, Kizza, Mukasa, Singh} \} \)
2. \( F = \{ \text{Alli, Agesa} \} \)
3. \( M = \{ \text{Alli, Agesa, Kato, Kizza, Mukasa, Singh} \} \)

In example A, \( R \) is the set whose members are the members of \( K \) or the members of \( S \). It is called the "union" of \( K \) and \( S \). You obtained the set \( R \) by "uniting" the members of \( K \) with the members of \( S \). In the same way, in example B, the
set \( V \) is the union of \( Q \) and \( L \), because the members of \( V \) are all those boys who are members of \( Q \) or are members of \( L \).

For any two sets \( G \) and \( H \), the UNION of \( G \) and \( H \) is the set whose members are all the things which are members of \( G \) or are members of \( H \).

In symbols the union of \( G \) and \( H \) is \( G \cup H \) (which we may read as "\( G \) union \( H \)").

Referring again to the examples, you see that \( N \) is the union of \( X \) and \( Y \) and that \( M \) is the union of \( E \) and \( F \). In symbols \( R = K \cup S \), \( V = Q \cup L \), \( N = X \cup Y \) and \( M = E \cup F \).

There is something interesting to be discovered in trying to answer the following question about the union of two sets. Suppose you know how many members there are in each of the sets \( G \) and \( H \). Can you tell how many members there will be in the union of \( G \) and \( H \)? Take a look at example A. How many members are there in set \( K \); how many members are there in set \( S \)? Now how many members are there in set \( R \), the union of \( K \) and \( S \)? Do you recognize the familiar fact that \( 8 = 4 + 4 \)? Now what happens when you ask the same question in example B about the sets \( Q \) and \( L \) and their union, set \( V \)? Is it a familiar fact that \( 15 = 7 + 11 \)? How do you account for the difference between these two examples? Now try examples C and D to test your explanation. To take example D only, you find that Alli and Agesa are members of \( E \) and are also members of \( F \), but they appear only once as members of \( M \), the union of \( E \) and \( F \).

Now you can put the explanation in the formal language of sets: the
difference is that in example A the two sets K and S are disjoint, but in each of the other examples the two sets are not disjoint; that is they have some members in common.

**Exercise 2 - 1a**

1. Form the union of the following pairs of sets:

   Example: \( X = \{1, 2, 3,\} \)
   \( Y = \{2, 3, 4,\} \)

   \( X \cup Y = \{1, 2, 3, 4\} \)

   (a) \( R = \{5, 10, 15\} \)
   \( T = \{15, 20\} \)

   (b) \( B = \{\} \)
   \( D = \{\} \)

   (c) \( E = \{\} \)
   \( F = \{a, b, c, d\} \)

   (d) \( A = \{\} \)
   \( B = \{\} \)

   (e) \( S \) is the set of boys whose names begin with S.

   \( Y \) is the set of boys in the choir.

   (f) \( I \) is the set of boys who are Catholics.

   \( J \) is the set of boys who are Moslems.

2. Choose three other pairs of sets from the class list. List the members of the sets and the members of the union of each pair.

3. In the following sets, some members are missing. Complete them by putting in the missing members.

   (a) \( \{\triangle, \bigcirc\} \cup \{\square, \} = \{\triangle, \bigcirc, \square, \} \)

   (b) \( \{m, a\} \cup \{d, \} = \{m, a, d, g, o, n\} \)
2 - 2 Use of diagrams to show union of sets

You have seen how pictures and diagrams can help us to understand sets and subsets. Now let us see how diagrams and pictures can help us in understanding how we form the union of two sets.

Suppose you look again at the class list. List $F$, the set of boys whose names begin with the letter A, and $Y$, the set of boys who are in the choir.

$F = \{\text{Agesa, Alli}\}$

$Y = \{\text{Kofi, Oyelese, Sozi}\}$

$F \cup Y = \{\text{Agesa, Alli, Kofi, Oyelese, Sozi}\}$

The union of these two sets is illustrated in the diagram below.

(Note that these two sets have no members in common)

Is $F \cup Y$ the same as $Y \cup F$? Let us find out.
By comparing the members of the set $Y \cup F$ and the set $F \cup Y$ we find that the two sets have exactly the same members, so $Y \cup F = F \cup Y$.

Look back at example C where you wrote down

- $X =$ the set of boys who take part in athletics
  
  $= \{\text{Ali, Kizza, Kofi, Nsimbi, Senteza, Sozi}\}$

- $Y =$ the set of boys in the choir
  
  $= \{\text{Kofi, Oyelese, Sozi}\}$

As you have already found out set $X$ and set $Y$ have some members in common.

Which members are they? Let us draw diagrams to help us see this.

Since the union of two sets consists of all the members in one set or the other, the members they have in common do not appear twice in the union.

Can you see that $X \cup Y = Y \cup X$?

2-3 Intersection of sets

We have seen how to form the union of two sets. Let us look again at examples B, C, D and A (pages 2-2, 2-3).

Example B.

$Q =$ the set of boys who are ten years old

$= \{\text{Kitta, Lule, Nsimbi, Opio, Senteza, Sozi, Wasswa}\}$
L = the set of boys who play football

\[ \{ \text{Agesa, Alli, Juma, Kato, Kizza, Lule, Okot, Ongom, Opio,} \]
\[ \text{Sentamu, Sozi} \} \]

We found that the union of these two sets is

\[ Q \cup L = \{ \text{Agesa, Alli, Juma, Kato, Kizza, Lule, Okot, Ongom, Opio,} \]
\[ \text{Sentamu, Sozi, Kitta, Nsimbi, Senteza, Wasswa} \} \]

While you were forming the union did you notice that Lule, Opio and Sozi were members of Q and members of L. The set \( \{ \text{Lule, Opio, Sozi} \} \) is called the INTERSECTION of set Q and set L. It is written \( Q \cap L \) (and is read "Q intersection L").

In this case the intersection is the set of boys who play football and are 10 years old.

The INTERSECTION of G and H is the set of things which are in G and H. Thus \( G \cap H \) is the set of members common to G and H.

Example C.

X is the set of boys who take part in athletics.

\[ X = \{ \text{Alli, Kizza, Kofi, Nsimbi, Senteza, Sozi} \} \]

Y is the set of boys who are in the choir

\[ Y = \{ \text{Kofi, Oyelese, Sozi} \} \]

Therefore \( X \cap Y = \{ \text{Kofi, Sozi} \} \) because only Kofi and Sozi take part in athletics and are in the choir.

Example D.

E is the set of boys who are eleven years old

\[ E = \{ \text{Agesa, Alli, Kato, Kizza, Mukasa, Singh} \} \]
F is the set of boys whose names begin with A

\[ F = \{ \text{Agesa, Alli} \} \]

The intersection \( E \cap F \) is the set of boys who are 11 years old and whose names begin with the letter A. \( E \cap F = \{ \text{Alli, Agesa} \} \)

**Example A.**

\( K \) is the set of boys whose names begin with the letter K.

\[ K = \{ \text{Kato, Kitta, Kizza, Kofi} \} \]

\( S \) is the set of boys whose names begin with S.

\[ S = \{ \text{Sentamu, Senteza, Singh, Sozi} \} \]

In this case \( K \cap S = \{ \} \), the empty set, because the sets have no members in common. You can see that the intersection of any two disjoint sets is the empty set.

**Exercise 2 - 3a**

1. Find the intersection of the following pairs of sets. List the members.

Example \( X = \{ 1, 2, 3 \} \)

\[ Y = \{ 2, 3, 4 \} \]

\[ X \cap Y = \{ 2, 3 \} \]

(a) \( R = \{ 5, 10, 15 \} \) \hspace{1cm} \( T = \{ 15, 20 \} \)

(b) \( A = \{ \triangle, \square, \circ \} \) \hspace{1cm} \( B = \{ \star, \bigcirc, \triangle, \square \} \)

(c) \( E = \{ \} \) \hspace{1cm} \( F = \{ A, B, C, D \} \)

(d) \( B = \{ \bigcirc, \bigotimes \} \) \hspace{1cm} \( D = \{ \text{fish} \} \)

(e) \( S \) is the set of boys whose names begin with S.

\( Y \) is the set of boys in the choir.

(f) \( I \) is the set of boys who are Catholics.

\( J \) is the set of boys who are Moslems.
2. Choose three other pairs of sets from the class list. Find the members of the sets and the members of the intersection of each pair.

3. Start with a set $A$. Can you find a set $B$ so that $A \cup B = A \cap B$? Explain your answer.

4. Complete the following statements by filling in the blanks.

(a) $\{a, \} \cap \{b, c\} = \{b\}$

(b) $\{1, \ , \ 4\} \cap \{4, \ , 5, 6,\} = \{2, 3, 4\}$

2-4 Use of diagrams to show intersection of sets

You are now quite familiar with the use of pictures and diagrams to help you understand various facts about sets. Let us now use diagrams to illustrate the intersection of two sets.

First look back once more at the class list. Recall that $X$ is the set of boys who take part in athletics, and $Y$ is the set of boys who are in the choir.

$X = \{\text{Alli, Kizza, Kofi, Nsimbi, Senteza, Sozi}\}$

$Y = \{\text{Kofi, Oyelese, Sozi}\}$

How can we use diagrams to represent these sets? First you notice that Kofi and Sozi take part in athletics and are members of the choir. That is, they are the members which are common to both sets. They are, therefore, the members of the intersection of $X$ and $Y$. Then $X \cap Y = \{\text{Kofi, Sozi}\}$
This intersection is shown in the diagram below.

\[ X \cap Y \]

Can you draw a diagram to show that \( X \cap Y = Y \cap X \)? Now take another look at the class list. What is the set of boys whose names begin with the letter A? Set \( F = \{ \text{Alli, Agesa} \} \). The set of boys in the school choir is \( Y = \{ \text{Kofi, Oyelese, Sozi} \} \). How can we show the intersection of these sets by a diagram? Have they any members in common? No. Therefore we have to represent the sets by circles which do not have any members in common. That is, the intersection of the two sets is the empty set.
The intersection of set $F$ and set $Y$ is the empty set, because the two sets have no members in common. So $F \cap Y = \{\}$. 

2 - 5 Binary operations

It is important to note that the operation of union is defined as an operation on two sets. When we unite two sets $P$ and $Q$, we get a third set $R$. That is, $P \cup Q = R$. If we only said, however, "P union" or "the union of P and \( \cdot \), it would certainly not be clear what was to be done, until we were told the set with which $P$ should be united. In a similar way the operation of intersection is defined as an operation on two sets. Therefore, union and intersection are BINARY operations. (Binary means two at a time.) Thus, given any two sets there is a set which is their union. Also, given any two sets, there is a set which is their intersection. (Remember that the intersection of two disjoint sets is the empty set.)

Because the operation of union of sets is a binary operation, can we form the union of three sets? Strictly speaking, the answer is "No" since we can only form the union of two sets at a time.

Let us look at the following three sets.

$F$ is the set of boys whose names begin with the letter $A$.

\[ F = \{ \text{Agesa, Ali} \} \]

$K$ is the set of boys whose names begin with the letter $K$.

\[ K = \{ \text{Kato, Kitta, Kizza, Kofi} \} \]

and $W$ is the set of boys whose names begin with the letter $W$.

\[ W = \{ \text{Wasswa} \} \]
Can we unite these three sets? We can do this by taking them two at a time: \((F \cup K) \cup W\) or \(F \cup (K \cup W)\). Are these two new sets equal sets? Let us use pictures to help us answer the question. First we have:

\[
\begin{array}{ccc}
\text{Agesa} & \text{Kato, Kitta, Kizza, Kofi} & \text{Wasswa} \\
\text{Alli} & & \\
F & K & W \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Agesa} & \text{Kato} & \text{Kizza} \\
\text{Alli} & \text{Kitta} & \text{Kofi} \\
F & K & W \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Agesa} & \text{Kato} & \text{Kitta} \\
\text{Alli} & \text{Kizza} & \text{Kofi} \\
W & & \\
\end{array}
\]

\[
(F \cup K) \cup W
\]
Then

By comparing the members of the set \((F \cup K) \cup W\) and the set \(F \cup (K \cup W)\)

we find that the two sets have exactly the same members, so

\[(F \cup K) \cup W = F \cup (K \cup W).\]
SUMMARY OF UNIT I

A set is a collection of things.

Examples of ways of describing a set are as follows:

(i) \( T \) is the set of all trees in the garden

(ii) \( V = \{a, e, i, o, u\} \)

(iii) \( C = \{1, 2, 3, 4, \ldots\} \)

The empty set has no members and is represented as \( \{\} \)

If each of the members of set \( A \) is a member of set \( B \) then we say \( A \) is a subset of \( B \).

If \( C \) is a set then both \( C \) and the empty set are subsets of \( C \).

When we write \( A = B \) we mean that set \( A \) and set \( B \) have the same members. We say that \( A \) and \( B \) are the same set.

For example \( \{c, a, t\} = \{c, t, a\} \)

If two sets match exactly they are said to be equivalent sets.

For example

if \( X = \{1, 2, 3, 4\} \)

and \( Y = \{a, b, c, d\} \)

then \( X \) and \( Y \) are equivalent sets.

If set \( A \) is equivalent to set \( B \) and set \( B \) is equivalent to set \( C \) then it follows that set \( A \) is equivalent to set \( C \).
Summary - 2

It $S$ and $T$ are sets then $S \cup T$, the union of $S$ and $T$, is the set whose members are all the objects which are members of $S$ or members of $T$.

For example:

If $S = \{1, 2, 3, 4\}$ and $T = \{3, 4, 5\}$
then $S \cup T = \{1, 2, 3, 4, 5\}$
also $T \cup S = \{1, 2, 3, 4, 5\}$

If $K = \{1, 3, 5\}$ and $L = \{2, 4, 6\}$
then $K \cup L = \{1, 2, 3, 4, 5, 6\}$

If $S = \{1, 2, 3, 4\}$ and $E = \{\}$
then $S \cup E = \{1, 2, 3, 4\}$

If $A$ and $B$ are sets then $A \cap B$, the intersection of $A$ and $B$, is the set whose members are all the objects which are members of $A$ and are members of $B$.

For example:

if $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$
then $A \cap B = \{3, 4\}$
also $B \cap A = \{3, 4\}$

$A$ and $B$ are disjoint sets if they have no members in common, in other words if $A \cap B = \{\}$

The operation of union of sets and the operation of the intersection of sets are binary operations.
GLOSSARY

Counting numbers

When you first learned to count you began with 1 and counted 1, 2, 3, 4, 5 and so on. These numbers used in counting are called counting numbers. They are also called natural numbers. The set of counting numbers is the set \( \{1, 2, 3, 4, 5 \ldots \} \)

Whole numbers

In arithmetic the numbers 0, 1, 2, 3, 4 and so on are members of the set of whole numbers. The set of whole numbers is the set \( \{0, 1, 2, 3, 4, 5, \ldots \} \). The difference between the set of counting numbers and the set of whole numbers is that 0 is not a member of the set of counting numbers.

Even whole numbers

The set of numbers \( \{0, 2, 4, 6, 8, \ldots \} \) is called the set of even whole numbers. Note that the set of even counting numbers is the set \( \{2, 4, 6, \ldots \} \). An even number is a number exactly divisible by 2.

Odd whole numbers

The set \( \{1, 3, 5, 7, \ldots \} \) is called the set of odd numbers. An odd number is a number not divisible exactly by 2.

Prime numbers

A prime number is a whole number greater than 1 which is exactly divisible only by itself and 1. Note that 1 is not a prime number, and 2 is the only even prime number. The set of the first ten prime numbers is
The symbols ">" and "<"

The symbol ">" means "is greater than" and "<" means "is less than". Sometimes we also use the symbol "\geq" (or "\geq") to mean "is greater than or equal to" and "\leq" (or "\leq") to mean "is less than or equal to". For example "5 > 3" is read "5 is greater than 3". "3 < 5" is read "3 is less than 5".

The set of whole numbers less than ten is \(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}\) but the set of whole numbers less than or equal to ten is \(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\). The difference is that 10 is a member of the second set.

One-to-one correspondence

Another expression you may come across in other books is "one-to-one correspondence." When there is a matching between the members of one set and those of another so that each member of the first set is matched with exactly one member of the second set, and each member of the second set is matched with exactly one member of the first set, we say that the two sets are in one-to-one correspondence. For example, the set of heads of the pupils in a classroom is in one-to-one correspondence with the set of pupils in the same classroom.

Digit

In our number system we use the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, called digits, to represent numbers. For example, 343 is a numeral with three digits while 43 is a two-digit numeral. With the ten digits we can represent any number that we like. In a later chapter when you learn about place value you will realize that the first 3 in the numeral 343 represents 3 hundreds and
the second 3 represents three ones.

The union of two sets

Our definition of the union of two sets was the following:

"For any two sets G and H, the union of G and H is the set whose
members are all the things which are members of G or are members of H."

You may have some difficulty with the use of or in our definition because
it is different from its usual meaning. When we say "We will either go to
town or stay at home," we mean that we will do only one of these things but
not both. In mathematics, however, when we say "b is a member of X or a
member of Y" we mean one of the following:

(1) b is a member of X
(2) b is a member of Y
(3) b is a member of both X and Y.

For example if $X = \{1, 3, 5, 7\}$ and $Y = \{5, 7, 9, 11\}$, $X \cup Y = \{1, 3, 5, 7, 9, 11\}$. That is, $X \cup Y = \{1, 3, 5, 7, 9, 11\}$ consists of all the numbers
which belong to X or belong to Y.
ANSWERS TO UNIT I

CHAPTER ONE

Exercise 1 - 3a

1. (a) Sunday, Saturday.
   (b) 2, 4, 6, 8, 10, 12.
   (c) The answer to this question will vary from country to country.
   (d) September. (This is an example of a set with only one member.)
   (e) There are no cities in Africa with a population of over four million.

2. The set of squares of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10.
   or the set of squares of the first ten counting numbers.

3. 2, 3, 5, 7, 11, 13, 17, 19. (1 is not a prime number. Look at the glossary for an explanation of this.)

4. 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95. (Look at the glossary for explanation of a digit.)

5. 102, 111, 120, 201, 210, 300. These answers are correct because
   \(1 + 0 + 2 = 3 \quad 1 + 1 + 1 = 3 \) etc.

6. (i) The set of even whole numbers less than twelve.
   (ii) The set of the first six counting numbers that are exactly divisible by five.
   (iii) The set of the months whose names begin with the letter J.
   (The other descriptions are not good enough because they leave doubts about the set intended.)
Answers - 2

7.  (a) Yes          (d) No
(b) No          (e) Yes
(c) Yes

Exercise 1 - 5a

1.  (a) The set of the cubes of the first six counting numbers.
(b) The set whose members are the first eight letters of the English alphabet.
(c) The set of names of days of the week.

2.  (a) \[ \{21, 22, 23, 24, \ldots \} \]
(b) \[ \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \]
(c) \[ \{11, 13, 17, 19, 23, 29\} \]

3.  (a) \[ S = \{9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48\} \]
(b) \( S \) is the set of all multiples of 3 which are greater than six and less than 51.

4.  (a) and (e). (It is possible to list all the members of the set of living human beings but it would take a great amount of space and time and we would not describe this set by listing.)

5.  (a) \[ \{0, 2, 4, 6, 8\} \] (According to the definition in the glossary 0 is an even whole number.)
(b) \[ \{0, 1, 2, 3, \ldots \} \] (There are two errors -- 0 is a whole number and 9,999 is not the last whole number. The set of whole numbers has no last member.)
(c) \[ \{1, 3, 5, 7, 9, 11\} \] (0 is not an odd number. The numbers were less than 13, so 13 should not be included in the list.)
Answers - 3

(d) \{3, 6, 9, 12, 15, \ldots\} (This is a set without any last member so . . . must be inserted within the curly brackets to show this.)

Exercise 1 - 6a

1. (a), (b) and (d) are examples of the empty set.

Exercise 1 - 8a

1. (a) \{a, b\} is a subset of \{a, b, c, d\}
   (b) Set E is a subset of Set D.

2. Set Y is a subset of set V.

3. The set \{A, B, C, D, E\} is not a subset of the set \{A, B, C\} because the set \{A, B, C, D, E\} has D and E as members, which are not members of \{A, B, C\}. Note that the set \{A, B, C\} is, however, a subset of the set \{A, B, C, D, E\}.

4. (a) Set K is a subset of set X.
   Set L is not a subset of set X.
   Set M is a subset of set X.
   (b) Two
   (c) \{2, 4, 6, 8\} The set \{2, 4, 6, 8\} is a subset of set X.
   (d) \{3, 6, 9\} The set \{3, 6, 9\} is a subset of set X.
   (e) There are very many subsets of set X. Compare your answer to this question with the answers given by other students.

Exercise 1 - 9a

1. (a) \{\}, \{1\}, \{2\}, \{1, 2\}
   (b) \{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{c, a\}, \{a, b, c\}
(c) \[ \emptyset, \{ \gamma \}, \{ \gamma \}, \{ x, y \} \]

(d) \[
\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{2, 3\}, \\
\{3, 4\}, \{2, 4\}, \{1, 4\}, \{1, 3\}, \{1, 2, 3\}, \{2, 3, 4\}, \\
\{3, 4, 1\}, \{1, 2, 4\}, \{1, 2, 3, 4\} \]

(e) \[
\emptyset, \{\triangle\}, \{\square\}, \{\bigcirc\}, \{\triangle \square \bigcirc \}, \{\triangle \square \bigcirc \}, \\
\{\triangle \square \bigcirc \}, \{\triangle \square \bigcirc \}, \{\triangle \square \bigcirc \} \]

(Notice that it does not matter whether you write \{1, 2\} or \{2, 1\}, etc. This has been explained.)

2. The empty set is a subset of all the given sets.

3. \{1, 2\} is a subset of set A; \{a, b, c\} is a subset of set B; \{x, y\} is a subset of set C; then \{1, 2, 3, 4\} is a subset of \{1, 2, 3, 4\}.

\{\triangle \square \bigcirc \} is a subset of \{\triangle \square \bigcirc \}. The whole set is a subset of itself.

4. (a) True

(b) Not true

(c) True

(d) Not true

(e) Not true

(f) True

(g) True
Exercise 1 - 10a

1. This is just one way of picturing the class and Mary, John and Kwame. You will have other ideas.

2. A = all the books in the classroom
   
   all the Maths books in the classroom
   all the English books in the classroom

   This is one way of answering this question. There are many others.
Exercise 1 - 12a

1. The set of all even numbers less than ten is \( \{0, 2, 4, 6, 8\} \)
   Therefore A is not equal to B.

2. (a) and (d) are examples of pairs of equal sets.
   
   \[
   \text{Note in (d), (5) and (5 \times 1) are names for the same number}
   \]

3. C is not equal to A.
   D is not equal to A.
   E is not equal to A.
   \[
   \{3, 5, 7, 9\}, \{5, 3, 9, 7\}, \{7, 3, 9, 5\}
   \]
   All these sets are equal to A because they have the same members as A.

4. \( X = \{0, 1, 2, 3, 4, 5, 6, 7, 8\} \)
   \( Y = \{1, 2, 3, 4, 5, 6, 7, 8\} \)
   X is not equal to Y because X has one member that is 0 which is not a member of Y.
Exercise 1 - 15a

1. (a) \[ A = \{1, 2, 3, 4, 5, 6\} \]
\[ B = \{\text{Ghana, Guinea, Kenya, Liberia, Nigeria, N. Rhodesia, Nyasaland, Sierra Leone, Uganda}\} \]

A and B match exactly.

(b) \[ X = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18\} \]
\[ Y = \{42, 44, 46, 48, 50, 52, 54, 56, 58\} \]

X and Y do not match exactly. (18 in this case has no partner.)

(c) \[ S = \{1, 2, 3, 4, 5, \ldots\} \]
\[ T = \{5, 10, 15, 20, 25, 30, \ldots, 50, 005\} \]

S and T do not match exactly because we can match 50, 005 with a member of set S and then we can write at least one more member in set S.

(d) \[ K = \{1, 2, 3, 4, \ldots\} \]
\[ L = \{1, 8, 27, 64, \ldots\} \]

K and L match exactly because no matter what number of K we choose we can always find its cube in L, and every member of L is the cube of exactly one number in K.

(e) Each pupil has two feet. Therefore the set of all pupils does not match exactly with the set of feet of all the pupils.

2. Answers will vary.
3. \[
\{ 1, 2, 3, 4, 5, \ldots \}
\]
\[
\{ 201, 202, 203, 204, 205, \ldots \}
\]
It is possible to match these sets exactly.

4. Answers will vary

5. \[
\begin{align*}
\{ a, b \} & \quad \{ q, b \} & \quad \{ a, b \} & \quad \{ a, b \} & \quad \{ a, b \} \\
\{ p, q \} & \quad \{ q, r \} & \quad \{ r, s \} & \quad \{ s, p \} & \quad \{ s, q \} & \quad \{ p, r \}
\end{align*}
\]

ANSWERS TO UNIT I

CHAPTER TWO

Exercise 2 - 1a

1. (a) \( R \cup T = [5, 10, 15, 20] \)
(b) \( B \cup D = [\circlearrowleft, \circlearrowright, 20] \)
(c) \( E \cup F = \{a, b, c, d\} \)
(d) \( A \cup B = [\triangle, \square, \bigcirc, \star] \)
(e) \( S = \{\text{Sentamu, Senteza, Singh, Sozi}\} \)
\( Y = \{\text{Kofi, Oyelese, Sozi}\} \)
\( S \cup Y = \{\text{Sentamu, Senteza, Singh, Sozi, Oyelese, Kofi}\} \)
(f) \( I = \{\text{Agesa, Kitta, Kofi, Okot, Ongom, Sentamu, Sozi}\} \)
\( J = \{\text{Alli, Kato}\} \)
\( I \cup J = \{\text{Agesa, Alli, Kato, Kitta, Kofi, Okot, Ongom, Sentamu, Sozi}\} \)

2. Answers will vary.
3. (a) \[
\begin{align*}
\bigtriangleup & \bigcirc \bigcirc \\
\square & \bigdiamond \\
\end{align*}
\bigcup
\begin{align*}
\bigtriangleup & \bigcirc \square \bigdiamond \\
\end{align*}
\[
= \begin{align*}
\bigtriangleup & \bigcirc \square \bigdiamond \\
\end{align*}
\]
(b) \[
\{m, a, n\} \bigcup \{d, o, g\} = \{m, a, d, g, o, n\}
\]
This is an example of a correct answer.

Exercise 2 - 3a

1. (a) \(R \cap T = \{15\}\)
(b) \(A \cap B = \begin{align*}
\bigtriangleup & \square \bigcirc
\end{align*}\)
(c) \(E \cap F = \{\}\)
(d) \(B \cap D = \{\}\)
(e) \(S = \{Sentamu, Senteza, Singh, Sozi\}\)
\(Y = \{Kofi, Oyelese, Sozi\}\)
\(S \cap Y = \{Sozi\}\)
(f) \(I = \{Agesa, Kitta, Kofi, Okot, Ongom, Sentamu, Sozi\}\)
\(J = \{Alli, Kato\}\)
\(I \cap J = \{\}\)

2. Answers will vary.

3. A and B must be equal sets.

4. (a) \(\{a, b\} \cap \{b, c\} = \{b\}\)
(b) \(\{1, 2, 3, 4\} \cap \{4, 2, 5, 6, 3\} = \{2, 3, 4\}\)
This is a correct answer.
UNIT II - THE WHOLE NUMBERS

CHAPTER 3

NUMBER

In the previous unit we have considered some of the elementary facts about sets, and ways to use them. In this unit on whole numbers you are going to see how pupils first develop the idea of number from sets. We shall consider the representation of numbers. Finally, we shall study the arithmetic operations of addition, subtraction, multiplication and division, as well as the relations of order, in the set of whole numbers.

3 - 1 Comparison of sets by matching

It is interesting to consider the way in which pupils first develop the idea of number. Let us look at these sets of pawpaws and bananas.
We can match each banana with exactly one pawpaw and, in doing so, we also match each pawpaw with exactly one banana. You recall that the two sets are said to match exactly. We also say that they are equivalent sets. We conclude that there are just as many bananas as there are pawpaws. The set of pawpaws has JUST AS MANY MEMBERS as the set of bananas. Any other matching of the members of these sets will yield the same relationship: the sets would match exactly. We can show below some other matchings of these sets.
Exercise 3 - 1a

1. Show some other matchings of the given sets of pawpaws and bananas.

2. Take sets A and B shown below, and match their members in as many different ways as you can.

Now look at these sets:

Double arrows have been drawn to match each orange with exactly one pawpaw. Matching shows that there are some pawpaws left over. The set of pawpaws has MORE MEMBERS than the set of oranges. We may also say that the set of oranges has FEWER MEMBERS than the set of pawpaws.

As in the case of exact matching, we may show other ways of matching
pawpaws and oranges. In doing this we find that the relationship between
the sets does not change; the set of pawpaws has more members than the
set of oranges, and the set of oranges has fewer members than the set of
pawpaws. We show other matching arrangements of the sets:
From the preceding illustrations it is clear that when we match the members of a set A with the members of a set B, we discover that one and only one of the following relations holds:

Set A has **as many members** as set B.
Set A has **fewer members** than set B.
Set A has **more members** than set B.

**Exercise 3 - 1b**

1. Take various sets of objects and try to match their members. When you have compared the sets, make sentences about them using the phrases: "more members," "fewer members," or "as many members." (This matching should be done without counting.) Here are a few suggestions:

   (a) The set of coins in your pocket and the set of fingers on your right hand.

   (b) The set and the set of tutors in your classroom.

   (c) The set of Prime Ministers in Africa and the set of independent African states.

   (d) The set of Ministers in your Government and the set of all universities in your country.

**3 - 2 Sets in natural order**

Let us now compare the following sets of stickmen and boxes:
The set of stickmen has more members than the set of boxes. In fact, there is one stickman not matched with a box. This "one-more-than" relation provides a basis for placing sets in NATURAL ORDER.

Look at the following sets of triangles:
No set of triangles matches any other set exactly. However, we can arrange the set in a one-more-than relation, starting with the set that has fewest members. We note that there is an empty set. Because the empty set has the fewest members of all, it comes at the beginning of the new arrangement. The set consisting of a single triangle has one more member than the empty set, so it comes next. By continuing to choose the set which has one more member than the preceding set, we place the sets in natural order as shown below:
Exercise 3 - 2a

1. Place the following sets in natural order:

\[
\begin{align*}
\text{Set 1} &: \ldots, \\
\text{Set 2} &: \bigotimes, \bigcirc, \ldots, \\
\text{Set 3} &: \bigodot, \odot, \ldots, \\
\text{Set 4} &: \bigtriangleup, \bigcirc, \ldots, \\
\text{Set 5} &: \text{ABCDEF}, \\
\text{Set 6} &: \text{PQRSTU}, \\
\text{Set 7} &: \text{chickens}
\end{align*}
\]
3 - 3 Number as a property of sets

Suppose that we start with some set, for example, the set pictured here:

Think of sets which are equivalent to this set, that is, sets which can be matched exactly with it. Here are pictures of several such sets:
It is possible to form many other sets with exactly as many members as the original one. You can always test these sets by matching them with the set you started with. You can assemble in this way the sets equivalent to the original set.

**Exercise 3 - 3a**

1. Make up a few sets which match this set exactly.

2. Make up a few sets which match this set exactly.

Think about all the sets that are equivalent to the original one. Is there something which all these sets have in common? Of course you see what it is: each set has just as many members as any other. We say that all these sets have the same NUMBER of members. It is convenient to give names to numbers. The number we use to describe the set we began this section
with is named "four." We say that there are four members in this set. This number called "four" is attached to all the other sets we have pictured above. We speak of these sets as four arrows, four boxes, four stickmen, four bananas. This number four also characterizes every other set that is equivalent to the original set. All these sets have something in common: it is the property of "four-ness," the number four.

Every number has both a name and a symbol: the symbol "4" stands for the number whose name is "four."

We have now described several concepts for pupils to think about, understand, and learn in classes you will teach. There is the set of objects. Then there is the number telling how many members the set contains. Each number has a name. Each number is represented by a symbol, which is called a NUMERAL.

Exercise 3 - 3b

1. The number attached to this set

![Diagram of a set with three objects]

is called "three," and is represented by the symbol "3."

Picture other sets containing 3 members.

2. What is the name of the number attached to the set which in the natural order comes just after the set in problem 1?
3. Repeat this for any set which in natural order comes just before the set in problem 1. Picture a few such sets.
4-1 Identifying the whole numbers

You have seen how to place sets in natural order. Using this natural order, we give a name to the number of members in each set, and we write the numeral which represents the number.

<table>
<thead>
<tr>
<th>Set</th>
<th>Number</th>
<th>Numeral</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>zero</td>
<td>0</td>
</tr>
<tr>
<td>△</td>
<td>one</td>
<td>1</td>
</tr>
<tr>
<td>△△</td>
<td>two</td>
<td>2</td>
</tr>
<tr>
<td>△△△</td>
<td>three</td>
<td>3</td>
</tr>
<tr>
<td>△△△△</td>
<td>four</td>
<td>4</td>
</tr>
<tr>
<td>△△△△△</td>
<td>five</td>
<td>5</td>
</tr>
<tr>
<td>△△△△△△</td>
<td>six</td>
<td>6</td>
</tr>
<tr>
<td>△△△△△△△</td>
<td>seven</td>
<td>7</td>
</tr>
<tr>
<td>△△△△△△△△</td>
<td>eight</td>
<td>8</td>
</tr>
<tr>
<td>△△△△△△△△△△</td>
<td>nine</td>
<td>9</td>
</tr>
</tbody>
</table>
The numbers 0, 1, 2, 3, 4, 5 and so on are called WHOLE NUMBERS.
The set \( \{0, 1, 2, 3, 4, 5, \ldots\} \) is the set of whole numbers. We give these numbers a special name because as you will see later there are other kinds of numbers.

4-2 **Order in the set of whole numbers**

When sets are placed in natural order -- as above -- we also say that the corresponding whole numbers are themselves in natural order. Thus 0, 1, 2, 3 are in natural order. On the other hand 3, 6, 5, 4 are not in natural order; the natural order is 3, 4, 5, 6.

**Exercise 4-2a**

Here are some sets:

\[
\begin{align*}
\{ & A, B, C, E \} \\
\{ & B, C \} \\
\{ & A, B, D \} \\
\{ & E \}
\end{align*}
\]

Find the whole numbers attached to each of these sets. Place the sets in natural order, and the corresponding numbers also in natural order.

We have already seen how a set of objects has a whole number attached to it. Suppose the number of members in set A is \( m \), and the number of members in set B is \( n \):

(i) If A has **just as many** members as B, the number of members in A equals the number of members in B. We express this fact:

\[ m = n. \]
(ii) If A has fewer members than B -- and therefore B has more members than A -- we say that the number of members in A IS LESS THAN the number of members in B. We express this fact:

\[ m < n. \]

(iii) If A has more members than B -- and therefore B has fewer members than A -- we say that the number of members in A IS GREATER THAN the number of members in B. We express this fact:

\[ m > n. \]

From the preceding paragraphs two things should be clear. The statement \( 5 > 2 \) amounts to the same thing as \( 2 < 5 \); similarly, for any whole numbers \( a \) and \( b \), if \( a > b \) then also \( b < a \) and if \( b > a \), then also \( a < b \). Furthermore to say that \( a > b \) amounts to the same thing as saying that the whole number \( a \) comes after the whole number \( b \) in the natural order. Since 3, 4, 5, 6 are in natural order, 6 > 5, 6 > 4, 6 > 3, 5 > 4, 5 > 3, 4 > 3.

Exercise 4 - 2b

Here are four sets:

\[
\begin{align*}
\{A, B, C, D\} \\
\{□, ○, △\} \\
\{\} \\
\{B, C, E, F\}
\end{align*}
\]

Find the number of members in each. Now write all the statements you can, using these numbers and \(<, =, >\).
**4-3 Counting and the counting numbers**

In section 4-1, diagrams show sets of triangles in natural order. Each set has one more member than the preceding set. Just as we placed these sets of triangles in natural order, we can also form sets of whole numbers and place these sets in natural order. \( \{1\} \) is a set whose one member is the number 1. \( \{1, 2\} \) is a set of the two numbers 1 and 2 in their natural order. \( \{1, 2, 3\} \) is a set of the three numbers 1, 2, and 3 in natural order, and so on. We come in this way, for example, to the set \( \{1, 2, 3, \ldots, 27\} \), which is a set of the twenty-seven numbers 1, 2, 3, and so on up to 27, in natural order. The last number in the set is 27, and 27 is the number of members in the set. Similarly, the set \( \{1, 2, 3, \ldots, n\} \) is the set of \( n \) whole numbers 1, 2, 3, and so on up to \( n \) in natural order. The last number in the set is \( n \), and \( n \) is the number of members in the set. These sets of numbers are called **COUNTING SETS**. The numbers in each counting set are counting numbers in natural order starting from 1. The complete set of counting numbers is the set \( \{1, 2, 3, 4, 5, \ldots\} \). Counting sets are represented below.

<table>
<thead>
<tr>
<th>Counting sets</th>
<th>Number of members</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1}</td>
<td>1</td>
</tr>
<tr>
<td>{1, 2}</td>
<td>2</td>
</tr>
<tr>
<td>{1, 2, 3}</td>
<td>3</td>
</tr>
<tr>
<td>{1, 2, 3, 4}</td>
<td>4</td>
</tr>
<tr>
<td>{1, 2, 3, 4, 5}</td>
<td>5</td>
</tr>
<tr>
<td>{1, 2, 3, 4, 5, 6}</td>
<td>6</td>
</tr>
<tr>
<td>{1, 2, 3, 4, 5, 6, 7}</td>
<td>7</td>
</tr>
</tbody>
</table>
Corresponding to each counting set is the number of members in that set.
The last number in a counting set also tells the number of members in that counting set.

4-4 Finding the number of members in a set

When we use the counting sets we follow the natural order of numbers; that is, we count: "one, two, three, ...". Now you may ask, "How do we find the number of members in any given set?" For example, take the set of fish shown below.

\[
\left\{ 1, \ 2, \ 3, \ 4 \right\}
\]

Match the fish with the numbers in a counting set starting with 1, proceeding in natural order. The set of fish is exhausted when the last fish is matched with the last number in a counting set. This number tells us the number of fish. It is the largest number in the counting set.
When we teach children how to count, we teach them to use a set which they describe with the spoken words "one," "two," "three," "four," and so on. It is important to note that as they say "one," they are matching the number 1 with an object; as they say "two," they are matching the number 2 with another object; etc. The last number they name is matched with the last remaining object in the set. This last number tells the number of objects in the set.

In counting, it is important to attach just one number to each member of a set. In the preceding example of counting fish, the order in which we take the fish is not important. What is important is that each fish is counted just once. When we count the members of a set, no matter how we do the counting, we always find the same number of members. When you count the sides of a square, for example, you always get 4.

Exercise 4 - 4a

1. Count the set of chairs in your classroom.
2. How many walls are in your classroom?
3. How many brothers and sisters do you have?
4. How many palm trees are there in your training college compound?

4-5 Equivalence using counting sets

You can use counting sets to tell whether two sets are equivalent. Recall that earlier in this unit we stated that two sets A and B are equivalent when there is an exact matching of the members of set A with the members of set B. We can now restate the idea of equivalence of sets using counting sets.
Let us suppose we have a set $A$ of fish and a set $B$ of oranges. In order to tell whether the two sets are equivalent, find the counting set which matches exactly the set $A$. See whether the set $B$ also matches exactly the same counting set. If it does, then we can conclude that because set $A$ is equivalent to the counting set, and set $B$ is equivalent to the counting set, then set $A$ is equivalent to set $B$.

\[
\{\text{fish}, \text{fish}, \text{fish}, \text{fish}\} \\
\{1, 2, 3, 4\} \\
\{\text{orange}, \text{orange}, \text{orange}, \text{orange}\}
\]

In this case we see that set $A$ is equivalent to the counting set $\{1, 2, 3, 4\}$, and set $B$ is equivalent to the counting set $\{1, 2, 3, 4\}$. So $A$ and $B$ are equivalent.

**Exercise 4 - 5a**

Find, by counting, whether set $A$ and set $B$ are equivalent sets.

1. Set $A$: \[\square \bigcirc \rightarrow \triangle \heartsuit \square\]
   Set $B$: the set of fingers on your left hand.

2. Set $A$: the set of consonants in the word *relation*.
   Set $B$: the set of sides in the figure \[
   \square
   \]
CHAPTER 5

REPRESENTATION OF NUMBER

5-1 Forming the number concept. Abstraction

Children are not born with the idea of number. It takes them some years to learn it.

We say that a set of objects like

![Diagram of objects](image)

has three members. But at first a child sees only the objects or notices that they are together on the table. Even after he has learned the number names: one, two, three, four, ..., he may not connect the word "three" with the set shown. Before he can do this, he must see many sets of three objects and come to notice what these sets have in common. He must see them as sets of three things. To do this he must, as we say, abstract from the nature of the objects, that is, learn not to pay attention to what the things are. For example, he must forget or not pay attention to the fact that

![Diagram of objects](image)

is a set of pencils

and

![Diagram of objects](image)

is a set of bananas.
These are the most obvious things he notices about these sets. They are hard to push out of his mind so that he sees something alike about the two sets even if their members are so different. These sets are alike in a way that the two sets are not alike, even though these two sets have elements of the same kind.

In learning to see the likeness of the set of pencils and the set of bananas (in spite of their obvious difference) he is learning to abstract. At an earlier age, he has already made a beginning in this process of abstraction. He has learned for example to use the word "banana" for any one of a great many objects which look somewhat alike but not exactly alike. Some of these objects may be green and some may be yellow! He has learned that this does not matter. They are bananas just the same.

The abstraction needed for number is more difficult than that needed in naming simple objects. Much more has to be taken away from what is seen to leave what is wanted. Usually a child has to be about 6 years old before he can understand what number words mean in his own experience. Teaching this understanding takes great patience. As we have seen it can be done by showing the student sets of many sorts, matching them with each other, and learning the proper number words and numerals.
5-2 The need for symbols

The number four cannot be seen like a stone or a banana. It is an idea. To make it real we must represent it by a word or a symbol. Thus we can use the word "four" or the symbol "4" which we call a numeral. Just as there are different words for the number four in different languages, there are different systems of numerals which have been used. The ancient Egyptians would have written \( I\ II \) instead of 4. The Romans would have written IV.

While numbers cannot be seen, numerals can be. They can be written on paper or on the chalk board. They seem more like real things. This is good. However, there is a danger. Numerals can become just marks on the paper which the student learns to write without connecting them with sets of objects. A little further on he may learn to put down 5 when he sees \( 2 + 3 \) without having any idea why he can do this or what \( 2 + 3 \) means in his experience. Then arithmetic becomes a meaningless game with symbols.

The facts of arithmetic do not depend on the names which we give to numbers. For example

\[ 2 + 3 \text{ is } 5 \]

says the same thing as

\[ II + III \text{ is } V \]

The first statement is written in Hindu-Arabic numerals; the second in Roman numerals. They mean the same thing.
Ways of representing numbers (Egyptian, Babylonian)

Probably among the earliest symbols in use were model matching sets such as stroke (I) for one, the wings of a bird (urovision) for two, three leafed clover (३) for three, the four legs of an animal (४) for four, and so on. This principle was adopted by the early Egyptians around 3,500 B.C. possibly as an extension of a system of tallying as follows:

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & \ldots & 9 \\
\text{I} & \text{II} & \text{III} & \text{IV} & \ldots & \text{IX} \\
\end{array}
\]

Clearly such systems are of little use for larger numbers. We cannot keep on making up and remembering new symbols and names for numbers. Some kind of grouping becomes necessary with special symbols to represent the groups and groups of groups. The early Egyptians built up their system in groups of ten. Their single numerals were the following:

<table>
<thead>
<tr>
<th>NUMBER</th>
<th>EGYPTIAN NUMERAL</th>
<th>OBJECT REPRESENTED BY THE NUMERAL</th>
</tr>
</thead>
<tbody>
<tr>
<td>ONE</td>
<td>I</td>
<td>STROKE OR STAFF</td>
</tr>
<tr>
<td>TEN</td>
<td>C</td>
<td>HEEL BONE</td>
</tr>
<tr>
<td>ONE HUNDRED</td>
<td>@</td>
<td>SCROLL OR COILED ROPE</td>
</tr>
<tr>
<td>ONE THOUSAND</td>
<td>🌸</td>
<td>LOTUS FLOWER</td>
</tr>
<tr>
<td>TEN THOUSAND</td>
<td>🌸</td>
<td>POINTING FINGER</td>
</tr>
<tr>
<td>HUNDRED THOUSAND</td>
<td>🌸</td>
<td>POLLIWOG</td>
</tr>
<tr>
<td>ONE MILLION</td>
<td>🌸</td>
<td>ASTONISHED MAN</td>
</tr>
</tbody>
</table>
In the Egyptian system, the order in which the symbols are arranged does not matter. Thus $\text{II}, \text{II} \text{I}, \text{I} \text{II}, \text{II} \text{I}$ are all numerals for twenty-one. Other examples of Egyptian numerals are as follows:

<table>
<thead>
<tr>
<th></th>
<th>7</th>
<th>23</th>
<th>456</th>
<th>1821</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>EGYPTIAN NUMERALS</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Exercise 5-3a**

1. Write Egyptian numerals for the following: six, fourteen, three hundred and fifty-six, three thousand and twenty.

2. What numbers are represented by the following Egyptian numerals:

   $\text{II}, \text{II} \text{I}, \text{I} \text{II}, \text{II} \text{I}, \text{I} \text{II}, \text{II} \text{I}, \text{I} \text{II}, \text{II} \text{I}$

In the Babylonian system which was used about 4,000 years ago, there are only two symbols, $\text{V}$ and $\text{n}$. $\text{V}$ was used to represent one and $\text{n}$ to stand for ten. These two symbols were repeated to write numeral from 1 to 59. Examples of Babylonian numerals are as follows:

<table>
<thead>
<tr>
<th></th>
<th>6</th>
<th>23</th>
<th>39</th>
<th>45</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>BABYLONIAN NUMERALS</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[\text{112}\]
In the Babylonian system, unlike the Egyptian system, a symbol for a large number always precedes a symbol for a smaller number. The same two symbols were used to represent numbers greater than 59, but the method was confusing. For example, △ could represent either sixty or three hundred and sixty, as well as one.

**5-4 The Hindu-Arabic system of numeration**

Nowadays we use a system of numerals which is called the Hindu-Arabic system. This system has ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 which are called digits. We are able to represent any number, however large, by using these digits and an especially clever idea, that of **place value**. As you know, in the Hindu-Arabic system, we write digits in the ones place, the tens place, the hundreds place, the thousands place, and so on. The value of any digit depends both on the digit and on the place it occupies in the row of digits.

Thus, in the numeral 3234, the first 3 represents three thousands and the second 3 stands for three tens.

This number would be written in the Egyptian system as

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You will notice that both systems work with groupings of ten. We are told that early man chose this particular grouping because he used his ten fingers for counting. When counting or matching sets of more than ten objects, his ten fingers would soon be used up. He would have to record this in some way or another. Perhaps he put a stone in his pocket if he had one, and then he could continue using his fingers again. Each stone would then stand for ten. If he got a large number of stones in his pocket, he could replace each ten of them by a larger stone, again using ten as the natural group. Each larger stone would then stand for ten tens, i.e. one hundred or \(10 \times 10\) written as \(10^2\) for short. And so he could continue. Ten of these larger stones would form a new group of one thousand, i.e. \(10 \times 10 \times 10\) written as \(10^3\) for short. The next group would consist of ten thousands or \(10 \times 10 \times 10 \times 10 = 10^4\). When we write numerals in this way, the raised numeral (the 4 in \(10^4\), for example) is called an index or power. We speak of 4 as the power to which 10 must be raised to give 10,000, or as the index of 10.

5-5 Grouping

The idea of grouping is fundamental in recording numbers and we must see that our pupils understand it thoroughly. They must realise the necessity for grouping. Our position is similar to that of a messenger boy of long ago who had to report to his chief how many people there were in a village. For each person he put a small stone in his sack. He intended to carry the sack to the chief and say, "Behold! There are as many people as stones." This would have been a good method if there had been only a few people in the village, but many
stones would make the sack too heavy to carry. The messenger, therefore, used two sizes of stones, with the larger representing ten of the smaller ones but weighing much less. He made the sack lighter by taking out ten small stones and putting in one large stone. He used ten because this is the number of his fingers. He could easily tell whether a set of small stones corresponded to a large stone by matching them with his fingers. He now said to his chief. "Behold! For each large stone there are as many people as fingers on my hands. For each small stone there is one person." The messenger needed no more than nine small stones. Instead of ten small stones, he could use one large stone.

In the above story the messenger was matching and not counting. Before people had developed the idea of counting and systems of numeration, they were able to keep track of large numbers of people or cattle by tallying, which is a process of matching equivalent sets. If the messenger had cut notches in a stick to match the number of people, he could have cut a deeper notch to represent each group of ten and so simplified his task. Children need to be given much practice in this kind of work, putting sticks into bundles of ten, arranging beans in groups of ten and so on. Some people carry out the process by writing nine strokes and a final one which crosses all the others (|||||||) for each ten. Sometimes people work in fives in this way. It makes the final counting of a large number of strokes much easier than it would be otherwise. Matching sets of things with sets of strokes or notches is sometimes called tallying.
5-6  A symbol for zero

When you were comparing the Egyptian system with the Hindu-Arabic system, you would realise that the Egyptian does not need a symbol for zero. Can you explain why this is so? Why, then, does the Hindu-Arabic system need such a symbol? If we could manage without it, the children would have much less trouble. However, the symbol for zero is absolutely essential to keep the digits in their correct position when there happen to be empty places. For example, if there are no groups of ten in a particular numeral, we must have a symbol to show this. Take the numeral 306. Without the symbol 0, the 3 or the 6 could easily be written in the tens place and the numeral would look like 36. The symbol 0 shows that the tens place in 306 is empty. Thus we could not use place value without a symbol for an empty place. The symbol for zero was invented by the Hindus. It is said to be one of man's greatest inventions because it made possible our system of numeration.

5-7  Representing numbers on the abacus

It has taken the human race thousands of years to develop and accept the Hindu-Arabic system of writing numbers. It is not easy to understand and many people who have studied mathematics at school may still not understand fully how it works. One way of giving your pupils experience in building numbers and then writing them is to use an abacus. This is really an aid in counting and is simply a set of sticks on which beads are put to stand for numbers. The sticks are mounted on a stand so that they can be used more easily. There are several ways of making an abacus. Here are two for you to try. Make one of them to
use for the exercises which follow.

Method 1. Use a piece of wood about 9" long, 3" wide and 1" thick. Mark 4 points along the middle of the wood about 2" apart. Knock a very long nail right through the wood at each of the points until it cannot go any further. File the ends of the nails if they are sharp. Your abacus is now made.

Method 2. You will need 4 reels or spools such as are used for thread for sewing. Push a stick or pencil or used ball point pen into each hole. Now you need some way to keep the 4 reels together in line. You can put them into a box of suitable size, or glue them to a piece of cardboard, or nail a piece of wood onto their bases. Your abacus should look like this.

Now you will need some beads, or rings to slip over the rods to show the numbers. These should look like this or this and can be made from rings of grass stems, or twisted grass or bamboo slices, or cardboard from
a circular sweet packet like this.

![Image of a circular sweet packet]

Cut this along the dotted lines. It must be possible to put 10 rings on a rod.

To show numbers from 1 to 9 you put rings on the first rod from the right hand side. This is the ones rod and every ring on the ones rod stands for one.

![Diagram of rings on rods]

Your pupils can use an abacus for scoring games. Sooner or later they will want to show a score of ten. This is where they must learn the rule for using an abacus.

**Rule of Procedure.** Whenever there are 10 rings on 1 rod you must take them off and replace them by 1 ring on the next rod to the left.

So every ring on the second rod from the right represents ten and it is called the tens rod. Here is thirteen shown on the abacus.

![Diagram of rings on rods]

Thirteen is 1-ten and 3-ones and so we have 1 ring on the tens rod and 3 rings on the ones rod.

**Exercise 5-7a**

Make your abacus look like each of these in turn and write the numbers which are represented on them.
Here is the first one written out fully for you: there are 5 tens and 2 ones; this is $50 + 2$ which is 52. Look at the last picture. Suppose we add another ring to the tens rod. This will make 10 rings and so by the rule we must take them off and replace them by 1 ring on the next rod to the left. This ring will represent 10 tens which is 100. So the rings on the third rod are each worth 100 and we call this the hundreds rod.

\[
\begin{array}{c}
\text{TENS} & \text{ONES} \\
\hline
\text{TENS} & \text{ONES} \\
\hline
\text{TENS} & \text{ONES} \\
\hline
\text{TENS} & \text{ONES}
\end{array}
\]

\[
\begin{array}{c}
\end{array}
\]

This abacus shows 3 hundreds, 4 tens and 2 ones. This is $300 + 40 + 2$, which is 342.
Exercises 5-7b

Explain these in the same way.

In the last example there is a ring on the fourth rod from the right; what is it worth? It is worth 10 of the rings on the rod to the right of it, so it is worth 10 hundreds. You know that 10 hundreds are 1,000 and so the fourth rod is the thousands rod. Every ring on this rod is worth 1,000. So the last number is 1 thousand, 2 hundreds, 5 tens and 5 ones which is the same as \(1,000 + 200 + 50 + 5\) which is 1,255.

Exercises 5-7c

Show each of these numbers on your abacus and make drawings of them.

1. (a) 6,324  
   (b) 7,562  
   (c) 6,666  
   (d) 3,427

2. Write each of the numbers represented on these abaci in (a) Hindu-Arabic numerals and (b) Egyptian numerals.

3. If you had an abacus with eight rods, what would a ring on
   (a) the 5th rod represent?  
   (b) the 6th rod?  
   (c) the 7th rod?  
   (d) the 8th rod?

Draw a large abacus with 8 rods and label each rod.
4. Draw a picture of an abacus which shows a million.

5. Draw an abacus with 8 rods and then mark some rings on each rod. Write underneath this the number it represents. Make 4 more numbers in this way.

6. Draw a picture of this number represented on an abacus: 2 ten millions, 5 millions, 3 hundred thousands, 6 ten thousands, 1 thousand, 8 hundreds, 8 tens and 5 ones. This is the number: 25,361,885.

7. Write the numbers which are represented on these abaci, first in symbols, then in words.

5-8 The empty rod on the abacus

Some kind of abacus has been known for a very long time. It has been used in some form by most races of people from the ancient Egyptians to the present-day Russians and Japanese. The abacus is first used to make a record of a count. Its other use, in calculation, you will read about later in this book. Suppose that you have counted all the people in a town and have made a record of your count on an abacus like this:
Now the question arises, "How shall we record in writing the number represented above?" What symbols shall we use? Different races of people have solved this problem in different ways. You have already heard about the ancient Egyptian way, and the Hindu-Arabic way we use today, and you will hear soon about the Ancient Roman way.

**Exercise 5-8a**

Write the number represented on the abaci below in

1. **Egyptian numerals**
2. **Hindu-Arabic numerals**

How did you show the empty rod?

In the Egyptian numerals there is no symbol for an empty rod but you will remember that we do not need one. An Egyptian symbol tells us which power of ten it represents, but it does not tell us how many there are. To show thirty, that is 3 tens, we write \( \text{\textcircled{3}} \). This is like writing ten ten ten. We do not need a zero. We just leave out the symbol which is used for the rings on that particular rod.

In the Hindu-Arabic system the symbol does not tell us "how big" but only "how many". We can only tell how big a number is by its place value, that is,
by its position in the line of digits. The tens place is the second from the right, and so to show 3 tens we put a 3 into the second place. But if we just write 3 we do not know to which of the rods it refers and so we use a zero to show the empty rod to its right. Try to write the answers to questions 2, 3, and 4 above without a zero. This is what they will be like: 314, 314, 314. You can see that it would be easy to read these wrongly. We need the zero to keep the symbols in their right places. Then we can write the answers as 3140, 3104, 3014. We can think of zero as standing for the empty rod in the abacus. This fits in with what you have learned about sets. The set of rings on the empty rod is the empty set. The number of members in the empty set is zero and is written 0.

Exercise 5-8b

Draw abaci to show these numbers:

(a) 3,052  
(b) 2,308  
(c) 31,450  
(d) 700,103  
(e) 6,500  
(f) 5,000  
(g) 7,602,019  
(h) 9,999

Exercise 5-8c

Add one to the number in (h), and draw the result.

Exercise 5-8d

Write the numbers represented on these abaci in

(a) Egyptian numerals  
(b) Hindu-Arabic numerals

1.  

2.  

3.  

123
5-9 The Roman system of numeration. Comparison with Hindu-Arabic system

The Hindu-Arabic system of numeration replaced the Roman system in European countries. The Roman system apparently came into use in its original form about 300 B.C. It took several centuries for people to give up the Roman system for the other and even today we still use Roman numerals on clock faces and in other ways. Why did the Roman system persist for so long? Why do we now prefer the Hindu-Arabic system?

The symbols are as follows:

<table>
<thead>
<tr>
<th>Roman Numerals</th>
<th>I</th>
<th>V</th>
<th>X</th>
<th>L</th>
<th>C</th>
<th>D</th>
<th>M</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>500</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1,000</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

By combining these, new numerals are formed, for example:

<table>
<thead>
<tr>
<th>Roman Numerals</th>
<th>4</th>
<th>6</th>
<th>54</th>
<th>45</th>
<th>90</th>
<th>110</th>
<th>1900</th>
<th>2100</th>
<th>1778</th>
</tr>
</thead>
<tbody>
<tr>
<td>IV</td>
<td></td>
<td></td>
<td>LIV</td>
<td>XLV</td>
<td>XC</td>
<td>CX</td>
<td>MCM</td>
<td>MMC</td>
<td>MDCCCLXXVIII</td>
</tr>
</tbody>
</table>

Note that both an additive and a subtractive method is used in forming numerals. Thus XI has the stroke I written after X showing the addition of one to ten to name eleven, while IX has the stroke I written before X showing the subtraction of one from ten to name nine. Similarly the symbol for four is IV, for ninety XC, and for
Exercise 5-9a

1. Write Roman numerals for the following:
   14, 19, 23, 468, 1964, 44, 82.

2. What numbers are named by the following Roman numerals:
   XXIV, XXVI, LXIV, XLVI, CCXXVIII, MMCMLXXIX, MDCCLII.

3. Show how both the addition and subtraction methods are used to write:
   XLVI, LXIX, LIV, MCM.

We have systems of numeration for two purposes, (a) in order to be able to name numbers and (b) so that we can calculate with numbers. Which system do you think is better, Roman or the Hindu-Arabic for naming numbers? In the Roman system, the five symbols I, V, X, L and C will take us up to four hundred and ninety-nine and there is no zero to trouble us, whereas in the other system all nine symbols are required to take us to ten. Young children find the learning of ten symbols burdensome, as you know, and it seems that the Roman system may be easier for smaller numbers than the Hindu-Arabic system. The Roman system is not so useful with larger numbers, however, especially when we reach thousands and over.
5-10 Calculation with Roman numerals*

Can we calculate with Roman numerals?

Addition:

Addition is easy in the Roman system for we simply collect all the symbols together, combine them where necessary and write them in order. For example:

1. MCL + CCXII = MCLCCXII = MCCCLXII
2. XXV + LXIV = XXVLI = LXXXIX (because VIV = IX).

This work can be carried out quickly with practice.

Perhaps the working of these examples is easier to understand if we use a Roman abacus. The rods on a Roman abacus would have to represent ones (I), fives (V), tens (X), fifties (L), hundreds (C), five hundreds (D) and thousands (M), as in the diagram.

Roman Abacus

*If desired, sections 10 and 11 can be omitted until after addition and multiplication have been studied.
5 rings on the I-rod are equivalent to 1 ring on the V-rod. 2 rings on the V-rod are equivalent to 1 ring on the X-rod, and so on. How many L-rings are equivalent to a C-ring? How many C-rings to a D-ring? Consider question 2 above, \( XXV + LXIV \).

Place rings on the abacus as shown, one set for each numeral.

Putting the sets together, we can read off the answer as

\[
\begin{array}{ccccccc}
\text{LXXXVIV or LXXXIX} & & & & & & \\
\text{(LXIV)} & & & & & & \\
\text{(XXV)} & & & & & & \\
\end{array}
\]

Work out the remaining examples for yourselves. Remember that when a symbol is subtractive (for example, the I in IX, or the X in XC) remove a ring from the appropriate rod on the abacus when this is possible. If not, as in the example above, IV becomes 4 rings on the I-rod.

**Subtraction:**

1. \( XXXIII - XXII \)
   
   This means that we have to take away XXII from XXXIII.

   Write \( XXXIII \) and remove the symbols in the circles.

   We have XI left. \( XXXIII - XXII = XI \).

2. \( L - XXX \)

   This time we have to subtract XXX from L. To do this we have to write L as XXXXX. Then it is easy. \( L = XXXX \). Take away (subtract) XXX. We are left with X

   \( L - XXX = XX \).
3. \( \text{XXXIV - XXV} \)

\[ \text{XXXIV - XXV} = \text{XIV} - \text{V} = \text{VIV} - \text{V} = \text{IX} \]

**Multiplication:**

For multiplication in the Roman system, we could use a multiplication table as follows:

\[
\begin{array}{cccccc}
I & V & X & L & C & M \\
I & V & X & L & C & M \\
V & V & \text{XXV} & L & & \\
X & X & L & C & & \\
L & & & & & \\
C & & & & & \\
M & & & & & \\
\end{array}
\]

This is obtained by multiplying each number represented in the top row in the left hand column by each number represented. Each result is entered in its appropriate place in the table.

Thus \( X \) (left hand column) multiplied by \( V \) (top row) = \( L \) (entered directly opposite \( X \) and below \( V \)). \( X \) (left hand column) multiplied by \( X \) (top row) = \( C \) entered directly opposite one \( X \) and below the other).

Use the table to multiply \( \text{XXV} \) by \( \text{XIV} \).

\[ \text{XXV} \times \text{XIV} = (\text{XXV} \times X) - (\text{XXV} \times I) + (\text{XXV} \times V) \]

But \( \text{XXV} \times X = \text{CCL} \) (using the multiplication table)

\[ \begin{align*}
\text{XXV} \times I &= \text{XXV} \\
\text{XXV} \times V &= \text{LLXXV}
\end{align*} \]

Altogether we have \( \text{XXV} \times \text{XIV} = \text{CCL} + \text{LLXXV} - \text{XXV} \)

\[ = \text{CCLLLXXV} - \text{XXV} \]

\[ = \text{CCLLL} = \text{CCCL} \]
The abacus could be used to help if necessary.

**Division:**

Division problems may take a long time to work out in the Roman system. Do not try to master the process unless you wish to. Apparatus is often necessary. Consider, for example, CCCLXIX ÷ XXIV. We could obtain CCCLXIX stones and count how many times we could remove XXIV stones, perhaps using an abacus to help. Alternatively, we might notice that XXIV divides IV times into C with the remainder IV. How many times will XXIV divide into CCC and what is the remainder? Can you now solve the problem?

Generally speaking, division is not a straightforward process. The notation serves the purpose of stating the problem and recording the result when it has been obtained. But it does not permit easy written calculation by rule.

**Exercise 5-10a**

Work the following examples in any way you can but without using the Hindu-Arabic system:

(a) with an abacus
(b) without an abacus.

1. (a) CCLXV + DCCLVIII  (b) MDXCLX + MCCXLIV
2. (a) LXXVII - XLIII  (b) CCLXIV - CLXIX
3. Complete the multiplication chart begun above and use it in the following:
   (a) LXXVIII × XXI  (b) CCLXIV × CLXIX
4. See what you can do with the following:
   (a) CCLXXIV ÷ XXV  (b) MCXCVIII ÷ LI
Use any method you wish. Note the difficulty we get into with 3 (b) because there are no symbols for numbers greater than M.

5-11 Calculation in the Hindu-Arabic system

Addition and subtraction in the Hindu-Arabic system present little difficulty even with large numbers if we have understood the rules. Even the division example above, $369 \div 24$, can be worked quite easily. To find the answer, we do not look for stones but for pencil and paper. With practice, we can proceed mechanically. It is as if the notation does our thinking for us and is its own reckoning instrument. The Hindu-Arabic system has a power which the Roman system lacks. The secret is in place value and zero. The Hindu-Arabic system of numeration solved the problem of reckoning for man and is an example of a good mathematical notation.

When we are teaching arithmetic to children, we show them how to use the Hindu-Arabic system as a calculating tool. As we do this, we are giving the children their first taste of mathematical language as an aid to thinking and we must be careful to ensure that they know what they are doing. At first, they will use the notation simply to record the results of problems solved by other means, maybe counters, beads, imitation money, abacus, and other kinds of apparatus. They will then gradually learn to change the method of calculation. They will rely less and less on the apparatus and more and more on the notation until the notation becomes their calculating instrument and a genuine aid to thinking.

Exercise 5-11a

1. Add 74, 362, 57 and 138.
2. Subtract 172 from 634.

3. (a) $149 \times 7$  (b) $36 \times 9$  (c) $128 \times 8$  (d) $392 \times 6$

4. (a) $217 \div 4$  (b) $395 \div 5$  (c) $207 \div 2$  (d) $309 \div 6$

5. (a) $1493 \times 57$  (b) $608 \times 78$  (c) $297 \times 97$

6. (a) $2432 \div 19$  (b) $1006 \div 31$  (c) $74002 \div 74$

7. My typewriter had a missing key and typed the following addition. What key was missing?

```
  32
+ 53
---
  87
+ 9
---
```

8. Find the missing digits

(a) $7\underline{1} \times 8 \times \underline{1} \times$

\[
\begin{array}{c}
19830 \\
\text{rem. 3.}
\end{array}
\]

(b) $6 \times 4 \times 2$

\[
\begin{array}{c}
4 \\
\times 6172
\end{array}
\]
6-1 Numeration in base five

We saw earlier that because man possesses ten fingers, ten became his natural counting group. For this reason, he chose to work in units, tens (10), hundreds (tens of tens = $10 \times 10 = 10^2$), thousands (tens of tens of tens = $10 \times 10 \times 10 = 10^3$) and so on. Some people, the Romans and certain Liberians for example, have chosen five, the number of fingers on one hand, as a natural counting group. In practice, they do not all count alike but some count as follows: one, two, three, four, 1-five, 1-five and one, 1-five and 2-ones, 1-five and 3-ones, 1-five and 4-ones, 2-fives, 2-fives and one, and so on, as in the following table:

<table>
<thead>
<tr>
<th>Counting numbers in groups of five</th>
<th>Numerals for counting numbers in base five</th>
</tr>
</thead>
<tbody>
<tr>
<td>one</td>
<td>1</td>
</tr>
<tr>
<td>two</td>
<td>2</td>
</tr>
<tr>
<td>three</td>
<td>3</td>
</tr>
<tr>
<td>four</td>
<td>4</td>
</tr>
<tr>
<td>1-five</td>
<td>10</td>
</tr>
<tr>
<td>1-five and one</td>
<td>11</td>
</tr>
<tr>
<td>1-five and 2-ones</td>
<td>12</td>
</tr>
<tr>
<td>1-five and 3-ones</td>
<td>13</td>
</tr>
<tr>
<td>1-five and 4-ones</td>
<td>14</td>
</tr>
<tr>
<td>2-fives</td>
<td>20</td>
</tr>
<tr>
<td>2-fives and one</td>
<td>21</td>
</tr>
<tr>
<td>2-fives and 2-ones</td>
<td>22</td>
</tr>
</tbody>
</table>
Counting numbers in groups of five | Numerals for counting numbers in base five
---|---
2-fives and 3-ones | 23
2-fives and 4-ones | 24
3-fives | 30
 | 
 | 
4-fives and 3-ones | 43
4-fives and 4-ones | 44
1-twenty-five (5²) | 100 (1 \times 5^2 + 0 \times 5 + 0 \times 1)
1-twenty-five and one | 101 (1 \times 5^2 + 0 \times 5 + 1 \times 1)
 | 
 | 

We read 10 as one-zero and 12 as one-two. Notice that we need only five digits 0, 1, 2, 3, and 4 when counting in groups of five. When we are counting in groups of five as in the above table, we say we are using the five-system of enumeration or that we are working in base five. We refer to the numerals as base five numerals.

Exercise 6-1a

1. Copy the chart below for counting numbers from one to one hundred, and complete the chart with numerals in base five:
2. Use your chart to answer the following question. What are the largest numbers in base ten which are represented as one-digit, two-digit and three-digit numbers in your chart?

3. In the United States of America and in some other countries such as Liberia, the smallest unit of money is 1 cent. 5 cents make 1 nickel and 5 nickels make 1 quarter.

   (a) Express the following amounts in quarters, nickels and cents:

   (1) 6 cents  (3) 26 cents  (5) 33 cents
   (2) 10 cents  (4) 46 cents

   (b) How many cents are there in:

   (1) 2 quarters, 1 nickel and 2 cents?
   (2) 3 quarters, 4 nickels and 3 cents?
   (3) 4 nickels, 4 cents?
   (4) 2 quarters, 2 nickels and 2 cents?
   (5) 8 quarters and 6 nickels?

Notice that in question 3, we are dealing in the five system of enumeration.
6-2 Calculation in base five (addition and subtraction)

If we wish, we may do our reckoning with base five numerals instead of the base ten numerals we are used to. We could set up a base five abacus to help us. The first rod would register ones as usual but whenever we have 5 rings on it we should remove them and put 1 ring on the second rod.

Thus 1 ring on the second rod is equivalent to 5 rings on the first rod. Similarly, 1 ring on the third rod represents 5 rings on the second rod which represents 25 rings on the first rod or 25 ones. A ring on the fourth rod represents 5 rings on the third rod or 125 ones in all. In the five system, as you remember, we work in ones, fives (5), twenty-fives (5²), one hundred and twenty-fives (5³) and so on.

Example:

Find the value of $4_{five} + 3_{five}$.

(Note: we indicate that a numeral is a base five numeral by means of the subscript five. If there is no subscript, we intend a base ten numeral.)
Place 4 rings and 3 rings on the first rod of a base five abacus. Remove 5 rings and place 1 ring on the second rod, leaving 2 rings on the first rod.

Thus $4_{\text{five}} + 3_{\text{five}} = 12_{\text{five}}$.

Example:

$23_{\text{five}} + 44_{\text{five}}$

Step 1. Replace 5 rings on the first rod by 1 ring on the second rod (diagram 2).

Step 2. Replace 5 rings on the second rod by 1 ring on the third rod (diagram 3).

Thus $23_{\text{five}} + 44_{\text{five}} = 122_{\text{five}}$.

Subtraction may be worked similarly.

Example: $34_{\text{five}} - 21_{\text{five}}$

Method 1.
Set up $34_{\text{five}}$ on the abacus. In order to subtract $21_{\text{five}}$, we remove one ring from the 1-rod and two rings from the 5-rod. This leaves three rings on the 1-rod and one ring on the 5-rod. Hence $34_{\text{five}} - 21_{\text{five}} = 13_{\text{five}}$.

**Method 2.**

Here we need two abaci and use the method of complementary addition. Set up $34_{\text{five}}$ on one abacus and $21_{\text{five}}$ on the other. We then match the rings on the second abacus to those on the first abacus by adding extra rings. We add three rings to the 1-rod and one ring to the 5-rod. The extra rings added are equivalent to $13_{\text{five}}$. Hence $34_{\text{five}} - 21_{\text{five}} = 13_{\text{five}}$.

**Example: $42_{\text{five}} - 24_{\text{five}}$**

**Method 1.** Set up $42_{\text{five}}$ on the abacus (diagram 1). We must now remove four rings from the 1-rod which is impossible. What can we do? One ring on the 5-rod is worth five rings on the 1-rod. We take a 5-ring off and replace it by five rings on the 1-rod (diagram 2). The subtraction
can now proceed. Remove four rings from the 1-rod and two rings from the 5-rod (diagram 3). This leaves one ring on the 5-rod and three rings on the 1-rod.

\[
\text{Hence } 42_{\text{five}} - 24_{\text{five}} = 13_{\text{five}}.
\]

**Method 2.** Work this out for yourselves as in the previous example.

**Exercise 6-2a**

Work the following on a base five abacus.

1. \(3_{\text{five}} + 3_{\text{five}}\)
2. \(13_{\text{five}} + 2_{\text{five}}\)
3. \(11_{\text{five}} + 31_{\text{five}}\)
4. \(12_{\text{five}} - 2_{\text{five}}\)
5. \(12_{\text{five}} - 4_{\text{five}}\)
6. \(142_{\text{five}} - 21_{\text{five}}\)
7. \(113_{\text{five}} - 31_{\text{five}}\)

Perhaps now you will be able to add and subtract base five numerals without using an abacus.

**Example:** Find the sum of \(242_{\text{five}}, 101_{\text{five}}\) and \(34_{\text{five}}\).

Set this down in the usual way.

1. \(4 + 1 + 2 = 1 \text{ five and 2 ones.}\)
   Write down 2 and carry 1.
2. \(1 + 3 + 0 + 4 = 1 \text{ five and 3 ones.}\)
   Write 3 down and carry 1.
3. \(1 + 1 + 2 = 4. \text{ Write down 4.}\)

**Example:** Subtract \(24_{\text{five}}\) from \(241_{\text{five}}\)

**Method 1.**

\(241 \quad (1) \quad 4 \text{ from } 1, \text{ I can't. Borrow 1 from the 4 leaving 3.}\)
\(24 \quad (2) \quad 4 \text{ from } 11_{\text{five}} = 2. \text{ Write down 2.}\)
(3) 2 from 3 leaves 1. Write down 1.

(4) 0 from 2 leaves 2. Write down 2.

Hence \(241_{\text{five}} - 24_{\text{five}} = 212_{\text{five}}\)

Check the procedure and result on the abacus.

Method 2.

\[
\begin{array}{c|c|c|c|c|c|c}
241 & 1 & 4 & = & 4 & \downarrow \text{from 1, I can't. Give it } 10_{\text{five}}. \\
24 & 4 & 0 & = & 4 & \downarrow \text{ from } 11_{\text{five}} = 2. \text{ Write down 2.} \\
212 & 4 & 0 & 4 & 0 & \downarrow \text{ from 4 leaves 1. Write down 1.} \\
212 & 4 & 0 & 4 & 0 & \downarrow \text{ from 2 leaves 2. Write down 2.} \\
\end{array}
\]

Check the procedure with an abacus.

Notice that the method used is identical with that for base ten numerals, as you would expect.

**Exercise 6-2b**

1. Complete the following addition table for base five numerals:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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</table>

2. Without using an abacus, work the following:

(a) \(1_{\text{five}} + 4_{\text{five}}\)  
(b) \(12_{\text{five}} + 4_{\text{five}}\)  
(c) \(33_{\text{five}} + 42_{\text{five}}\)  
(d) \(14_{\text{five}} - 3_{\text{five}}\)  
(e) \(13_{\text{five}} - 4_{\text{five}}\)  
(f) \(34_{\text{five}} - 21_{\text{five}}\)  
(g) \(324_{\text{five}} - 142_{\text{five}}\)  
(h) \(222_{\text{five}} - 133_{\text{five}}\)
Check your working on an abacus.

Working with base five numerals can be fun. Can you understand the following statements written in a secret code?

1. The enemy is fast approaching. They outnumber us 2 to 1 because they have 33 men against our 14. Send help quickly, please. And the 102 men will bring our strength up to 121 when we ourselves would outnumber the enemy by 2 to 1 and spring on him a surprise.

2. I have two brothers, Kwame and Kobi. Kobi is aged 100 years and is the eldest. My age is 31 years for I am 14 years younger than Kobi. Kwame is exactly half my age for he is 13. (What are our three ages?)

6-3 Changing the base of a numeral

Changing base five numerals to base ten and vice versa can often be done by inspection as follows:

\[
\begin{align*}
24_{\text{five}} & = 2 \times 5 + 4 = 10 + 4 = 14_{\text{ten}} \\
24_{\text{ten}} & = 2 \times 10 + 4 = 4 \times 5 + 4 \text{ or } 4 \text{ fives and } 4 = 44_{\text{five}}
\end{align*}
\]

(a) From base ten to base five

With larger numbers, we need a rule. Consider expressing \(234_{\text{ten}}\) as a base five numeral. How should we do it? We have to divide the numeral up into ones, fives, twenty-fives, one hundred and twenty-fives and so on. The way to do this is to keep dividing by 5 as follows:

\[
5 \left| \begin{array}{c} \ 234 \\ 46 \ \text{fives and 4 ones left over} \end{array} \right.
\]
Thus on the base five abacus, we should put 4 rings on the first rod and 46 rings on the second rod. Every 5 rings on the second rod could be replaced by 1 ring on the third rod. Dividing 46 by 5 shows that we would be left with 1 ring on the second rod and there would be 9 rings on the third rod. 5 rings from the third rod would make 1 ring on the fourth rod leaving 4 rings on the third rod.

\[
5 \overset{}{\overline{\underline{46}}} \\
9 \text{ fives and one left over.}
\]

Thus \( 234_{\text{ten}} = 1414_{\text{five}} \)

The division could be set out as follows:

\[
\begin{array}{c|c}
5 & 234 \\
\hline
5 & 46 \text{ fives and 4 ones left over} \\
\hline
5 & 9 \text{ twenty-fives and 1 five left over} \\
\hline
1 & \text{One-hundred and twenty-five and 4 twenty-fives left over}
\end{array}
\]
Hence \(234_{\text{ten}} = 1414_{\text{five}}\).

Thus \(234_{\text{ten}} = 2 \times 10^2 + 3 \times 10 + 4\)

\[= 1 \times 5^3 + 4 \times 5^2 + 1 \times 5 + 4\]

\[= 1414_{\text{five}}\]

Note that the successive remainders and the final quotient give the digits.

Study this carefully until you understand it.

(b) From base five to base ten

Changing from a base five numeral to a base ten numeral is rather easier.

Example: Express \(2304_{\text{five}}\) as a base ten numeral.

\[2304_{\text{five}} = 2 \times 5^3 + 3 \times 5^2 + 0 \times 5 + 4\]

\[= 2 \times 125 + 3 \times 25 + 0 \times 5 + 4\]

\[= 250 + 75 + 0 + 4 = 329_{\text{ten}}\]

\(2304_{\text{five}} = 329_{\text{ten}}\)

Exercise 6-3a

1. Change the following to base ten numerals:

(a) \(11_{\text{five}}\)  
(b) \(21_{\text{five}}\)  
(c) \(34_{\text{five}}\)  
(d) \(124_{\text{five}}\)  
(e) \(331_{\text{five}}\)  
(f) \(4201_{\text{five}}\)
2. Change the following to base five numerals:
   (a) 7
   (b) 9
   (c) 16
   (d) 27
   (e) 264
   (f) 819
   (g) 490

3. Perform the following additions:
   (a) \[ \begin{array}{c}
   213_{\text{five}} \\
   + 420_{\text{five}} \\
   \hline
   633_{\text{five}}
   \end{array} \]
   (b) \[ \begin{array}{c}
   104_{\text{five}} \\
   + 241_{\text{five}} \\
   \hline
   345_{\text{five}}
   \end{array} \]
   (c) \[ \begin{array}{c}
   4213_{\text{five}} \\
   + 1234_{\text{five}} \\
   \hline
   5447_{\text{five}}
   \end{array} \]

   Check your answer by reworking in the ten system.

4. Perform the following subtractions:
   (a) \[ \begin{array}{c}
   213_{\text{five}} \\
   - 20_{\text{five}} \\
   \hline
   203_{\text{five}}
   \end{array} \]
   (b) \[ \begin{array}{c}
   132_{\text{five}} \\
   - 41_{\text{five}} \\
   \hline
   91_{\text{five}}
   \end{array} \]
   (c) \[ \begin{array}{c}
   402_{\text{five}} \\
   - 134_{\text{five}} \\
   \hline
   268_{\text{five}}
   \end{array} \]

   Check your answer by working in the ten system.

6-4 Multiplication and division in base five

We see that we can add and subtract with base five numerals. Can we also multiply and divide? Try some examples.

Exercise 6-4a

1. (a) \[ 4_{\text{five}} \times 1_{\text{five}} \]
   (b) \[ 3_{\text{five}} \times 3_{\text{five}} \]
   (c) \[ 2_{\text{five}} \times 2_{\text{five}} \]
   (d) \[ 4_{\text{five}} \times 3_{\text{five}} \]

2. (a) \[ 33_{\text{five}} \div 3_{\text{five}} \]
   (b) \[ 31_{\text{five}} \div 4_{\text{five}} \]
   (c) \[ 102_{\text{five}} \div 3_{\text{five}} \]
   (d) \[ 310_{\text{five}} \div 10_{\text{five}} \]

If you are not sure how to work these out, get a base five abacus to help.

regarding multiplication as repeated addition and division as repeated
subtraction. Check your answer by changing to base ten numerals and reworking the exercise.

When dealing with large numbers we cannot proceed by inspection. We need a rule and this operates as in the ten system. We make use of the multiplication tables. Consider, for instance, \(433_{\text{five}} \times 32_{\text{five}}\). Let us make out the table of twos and the table of threes.

\[
\begin{array}{c c c c}
1 \times 2 &=& 2 & 1 \times 3 &=& 3 \\
2 \times 2 &=& 4 & 2 \times 3 &=& 11 \\
3 \times 2 &=& 11 & 3 \times 3 &=& 14 \\
4 \times 2 &=& 13 & 4 \times 3 &=& 22 \\
10 \times 2 &=& 20 & 10 \times 3 &=& 30 \\
\end{array}
\]

The multiplication can then proceed in the usual way as follows:

\[
\begin{array}{c c c c}
433_{\text{five}} & \times & 32_{\text{five}} \\
\hline
\underline{24040} & & & \\
\underline{1421} & & & \\
\underline{31011} & & & \text{five}
\end{array}
\]

The first partial product, 24040, is obtained by multiplying \(433_{\text{five}}\) by \(3_{\text{five}}\) using the 3 times table.

We say:

\[
\begin{align*}
3 \times 3 &= 14; \text{ 4 down, carry 1.} \\
3 \times 3 &= 14; \text{ 1 to carry makes 20; 0 down, carry 2.} \\
3 \times 4 &= 22; \text{ 2 to carry makes 24; 24 down.}
\end{align*}
\]
For the second partial product, 1421, we have:

\[ 2 \times 3 = 11; \text{ 1 down, carry 1.} \]
\[ 2 \times 3 = 11; \text{ 1 to carry makes 12; 2 down, carry 1.} \]
\[ 2 \times 4 = 13; \text{ 1 to carry makes 14; 14 down.} \]

Check the addition for yourself, and finally rework the example in base ten numerals as a check.

\[
433_{\text{five}} = 4 \times 5^2 + 3 \times 5 + 3 = 118_{\text{ten}}
\]
\[
32_{\text{five}} = 3 \times 5 + 2 = 17_{\text{ten}}
\]
\[
31011_{\text{five}} = 3 \times 5^4 + 1 \times 5^3 + 0 \times 5^2 + 1 \times 5 + 1
\]
\[
= 1875 + 125 + 5 + 1
\]
\[
= 2006_{\text{ten}}
\]

\[
\begin{array}{c}
118 \\
\underline{17} \\
1180 \\
\underline{826} \\
2006
\end{array}
\]

This checks.

Notice that multiplication tables in the five-system are shorter and there are fewer of them to learn. This is an advantage but on the other hand, a long line of digits is needed to express quite small numbers and this is inconvenient.

Exercise 6-4b

1. \[ 123_{\text{five}} \times 23_{\text{five}} \]
2. \[ 104_{\text{five}} \times 21_{\text{five}} \]
3. \[ 240_{\text{five}} \times 43_{\text{five}} \]
4. \[ 1043_{\text{five}} \times 24_{\text{five}} \]

Rework each example in base ten as a check.
Division with base five numerals proceeds as with base ten numerals.

### Example

\[
31014_{\text{five}} \div 32_{\text{five}}
\]

\[
\begin{array}{c}
\text{32)31014} \quad \text{We say:} \\
\text{233} \\
\text{221} \\
\text{201} \\
\text{204} \\
\text{201} \\
\text{3 remainder}
\end{array}
\]

We say:
- 32 into 3 won't go.
- 32 into 31 won't go.
- 32 into 310 goes 4 times. (The 201 is found by trial. Clearly 32 goes into 310 nearly 10 times and we try the next figure below 10 which is 4). \(4 \times 32 = 233\) which we write down and subtract from 310, leaving 22. Bring down the 1. Check the working as it continues for yours.

### Check in base ten

\[
\begin{align*}
32_{\text{five}} &= 3 \times 5 + 2 = 17 \\
31014_{\text{five}} &= 3 \times 5^4 + 1 \times 5^3 + 0 \times 5^2 + 1 \times 5 + 4 \\
&= 1875 + 125 + 5 + 4 = 2009
\end{align*}
\]

\[
\begin{array}{c}
\text{17)2009} \quad 433_{\text{five}} = 4 \times 5^2 + 3 \times 5 + 3 \\
\text{17} \\
\text{30} \\
\text{17} \\
\text{139} \\
\text{136} \\
\text{3 remainder}
\end{array}
\]

This checks.

### Exercise 6-4c

1. \(2431_{\text{five}} \div 22_{\text{five}}\)
2. \(3004_{\text{five}} \div 14_{\text{five}}\)
3. \(4231_{\text{five}} \div 42_{\text{five}}\)

Rework each example in base ten as a check.
We see that reckoning with base five numerals is possible and that it takes place exactly as in the decimal system. The work may have seemed difficult because it is unfamiliar but it is very valuable. It makes you think more deeply about the Hindu-Arabic system of numeration and makes the understanding of place value much clearer to you.

If you have found the work difficult, how much more so must young children when they are learning to use base ten numerals for the first time. It is not surprising that they sometimes get lost. At this stage, they are still learning the shapes of the symbols and how to write properly. You must be especially patient with them in your teaching in these early stages. Time is needed for children to understand numbers and how they are written before they do calculations with them. When they have begun calculation, give them plenty of easy examples carefully graded in order of difficulty so that they can build up confidence. Let each child proceed at his own pace and remember that praise and encouragement are much appreciated. If your teaching fails in these early stages, it may cause some children to dislike mathematics and give up its study and this at a time when your country needs all the mathematicians it can get.

6-5 Other bases, particularly base seven

There is an important discovery to be made in the above work which perhaps you have already made for yourself. We know now how to reckon with base five and base ten numerals. Has it occurred to you that numerals may be
written in other bases, base two, base three, base four, ... base twelve, ...
base twenty, ... and that we can calculate with these as with base ten numerals? In other words, there is nothing special about the decimal system.
If you have realized this, you have done very well indeed. You have made a generalization of the kind which mathematicians frequently have to make and which is fundamental to the growth of mathematics. As your studies progress, you will realize that generalization is at the heart of mathematical thinking.
Help your pupils to develop its power.

Check that you have made the generalization by working the following examples which use base seven numerals. The first three questions illustrate a practical illustration of grouping in sevens.

**Exercise 6-5a**

1. Make up a number chart for base seven numerals for numbers from one to one hundred.
   How many symbols do base seven numerals require?

2. Complete the following addition table for base seven numerals:

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<th></th>
<th>0</th>
<th>1</th>
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<th>3</th>
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</table>
3. Complete the following multiplication table for base seven numerals:

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4. What numbers are named by the numerals $11_{seven}$, $16_{seven}$, $34_{seven}$, $246_{seven}$, and $125_{seven}$? Express these in base ten numerals.

5. Express $32$, $56$, $129$, and $421$ as base seven numerals.

6. Find answers to the following:
   
   (a) $12_{seven} + 24_{seven}$  
   (b) $24_{seven} + 23_{seven}$  
   (c) $43_{seven} + 35_{seven}$  
   (d) $45_{seven} - 15_{seven}$  
   (e) $44_{seven} - 26_{seven}$  
   (f) $30_{seven} - 15_{seven}$  

   Express your answers as base seven and as base ten numerals.

7. Work out:
   
   (a) $405_{seven} \times 24_{seven}$  
   (b) $4362_{seven} - 21_{seven}$

   Write out any necessary tables and rework the problems in the base ten system as a check.

6-6 Grouping in twelves

Now that you can work in base seven, base five and base ten, you will find it easy to work in any base. The base twelve system is worth special
mention because we often group in twelves in everyday life. We often work with dozens for instance. We may buy a dozen eggs and this is grouping in twelves. Your school may buy pencils and exercise books by the gross where a gross consists of twelve dozen or twelve twelves or \(12^2\). Twelve inches make one foot and twelve months make one year. There are twelve hours marked on the face of the clock and in many countries, there are 12 pennies to a shilling. Here are some examples in grouping by twelves:

**Exercise 6-6a**

(In these examples, work in base ten and then regroup into twelves.)

1. **Add**
   
   (a) s. d.  
   4 3  
   2 7  
   1 6  
   4 2 

   (b) s. d.  
   5 1  
   17 3  
   15 9  
   7 6 

2. **Subtract**
   
   (a) s. d.  
   7 9  
   4 7

   (b) s. d.  
   15 3  
   7 8

3. (a) Multiply 2s. 7d. by 12  
     (b) Multiply 16s. 6d. by 7

4. (a) Divide 19s. 4d. by 8  
     (b) Divide 12s. 4d. by 4

5. If a carpenter is paid 10s. 6d. a day, how much does he earn in 4 days, 7 days, 9 days?

6. Change to pence : 4s., 3s. 9d., 2s. 6d., 9s. 10d.

7. **Add**  
   
   feet  
   1  
   9

   inches  
   4

   8
8. What is the total length in feet and inches of 5 sticks if each is 6 inches long?

9. Add 3 dozen and 5, 2 dozen and 7, 5 dozen and 0, and 10.

10. Divide 5 dozen and 8 pencils exactly between 2 classes. How many will each receive?

When working with base twelve numerals, we need 12 symbols, which is two more than in the base ten system. The extra ones needed are for ten and eleven which we may denote by t and e respectively. Counting then proceeds: 1, 2, 3, 4, 5, 6, 7, 8, 9, t, e, 10, 11, 12, ... 19, lt, le, 20, 21 ... 10_{twelve} represents 1-twelve, 11_{twelve} represents 1-twelve and one, and so on.

**Exercise 6-6b**

1. Copy and complete this number chart for base 12 numerals for numbers from one to one hundred and forty-four.
2. From the chart say what numbers are named by $2_{\text{twelve}}$, $3_{\text{twelve}}$, $4_{\text{twelve}}$, $5_{\text{twelve}}$, $6_{\text{twelve}}$, $7_{\text{twelve}}$. 

3. Complete the following addition table for base 12 numerals.

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</table>

4. Complete the following multiplication table for base 12 numerals.

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</tbody>
</table>
6-7 Grouping in twenties

Grouping by twenties is also commonly practised. Twenty shillings make one pound and twenty hundredweights make one ton. In some African countries counting is done in groups of twenty (the total number of a person's fingers and toes). In the Ibo country of Eastern Nigeria, twenty is called ogu. Forty, sixty and eighty are referred to as ogu abua, ogu abo and ogu ano meaning 2-twenties, 3-twenties, and 4-twenties. Do you have any words like these in your own language? The word score in English means twenty.

6-8 Grouping and calculation in base two

Base two numerals are especially interesting because they employ only two symbols 0 and 1. Counting in the base two system proceeds:

1, 10, 11, 100, 101, 110, 111, 1000, ....

The places represent units, twos, fours (2^2), eights (2^3) and so on. The only tables needed for calculation are:

0 + 0 = 0; 0 + 1 = 0; 1 + 0 = 1; 1 + 1 = 10; 0 x 0 = 0; 0 x 1 = 0; 1 x 0 = 0;

1 x 1 = 1.

Here is a multiplication example:

\[
\begin{array}{c}
11010_{\text{two}} \\
 \underline{\times 101_{\text{two}}} \\
11010_{\text{two}} \\
10000010_{\text{two}}
\end{array}
\]

Now \(11010_{\text{two}} = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2 + 0\)

\[= 16 + 8 + 0 + 2 + 0 = 26_{\text{ten}}\]

\(101_{\text{two}} = 1 \times 2^2 + 1 = 4 + 1 = 5_{\text{ten}}\)

\(10000010_{\text{two}} = 2^7 + 2 = 128 + 2 = 130_{\text{ten}}\)
Hence the multiplication expressed in base ten numerals is

\[ 26 \times 5 = 130 \] and the two results check.

Calculation is very easy with base two numerals and the tables are easy to learn. What is the drawback to using the base two (binary) system, do you think?

The binary system does have a very important use, however, in electronic calculating machines. These machines calculate at lightning speed. They take only seconds to work problems which would take a man weeks and months to do. Binary arithmetic is used because it needs only two symbols, 0 and 1. These can correspond to current switched on and current switched off, or a long and short buzz. Binary numerals can also be recorded on tape by means of a punched hole for the symbol 1 and no hole for the symbol 0.

**Exercise 6-8a**

1. Copy and complete the chart below for base 2 numerals from one to one hundred.

<table>
<thead>
<tr>
<th></th>
<th>10</th>
<th>11</th>
<th>100</th>
<th>101</th>
<th>110</th>
<th>111</th>
<th>1000</th>
<th>1001</th>
<th>1010</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
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</tbody>
</table>
2. Construct the addition and multiplication tables for base two numerals.

3. Rework the example $65_{\text{ten}} \times 17_{\text{ten}}$ with base ten and base two numerals and check that your two answers are equivalent.

You will find further examples dealing with numerals in bases other than ten in Chapters 14 and 15.
CHAPTER 7

ADDITION

7-1  Reminder of union of sets

Let us go back to what we have learned about sets, particularly about
the union of sets and about counting sets. We learned that from the set
\[ \{ \bigcirc \bigtriangleup \square \} \] and the set \[ \{ \star + \bigtriangleup \bigdiamond \} \] we can get a third set
\[ \{ \bigcirc \bigtriangleup \square \bigtriangleup + \star \bigdiamond \} \] which we call the union of the two
original sets. This third set contains all the members of the two previous sets.

We express the result thus:
\[ \{ \bigcirc \bigtriangleup \square \} \cup \{ \star + \bigtriangleup \bigdiamond \} = \{ \bigcirc \bigtriangleup \square \bigtriangleup + \star \bigdiamond \} \]

Your pupils have learned how to form the union of disjoint sets. For example,
they have made a set of a pencil, a rubber and a book; a second set of a piece of
chalk and a pen; and have put these two sets together to make their union.

7-2  Reminder of counting sets

You will remember that we saw how in counting we match the members of a
set with the members of a counting set and so find the number of members in the set.

Sets of objects \[ \{ \bigcirc \bigtriangleup \square \} \cup \{ \star \bigdiamond \bigtriangleup + \} = \{ \square \bigtriangleup \star \bigcirc \bigdiamond + \bigtriangleup \} \]

Counting sets \[ \{ 1 \ 2 \ 3 \} \cup \{ 1 \ 2 \ 3 \ 4 \} \cup \{ 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \} \]
We recognise that there are 3 members in the first set, 4 members in the second set and 7 members in their union.

Addition of numbers

We write what we found in the previous section as

\[ 3 + 4 = 7. \]

This statement expresses the fact that the sum of the numbers of members of the two sets is the same as the number of members in their union. Here is another example:

\[
\begin{align*}
\{A, B, C, D, E\} \cup \{X, Y, Z, P\} &= \{A, B, C, D, E, X, Y, Z, P\} \\
\{1, 2, 3, 4, 5\} \cup \{1, 2, 3, 4\} &= \{1, 2, 3, 4, 5, 6, 7, 8, 9\}
\end{align*}
\]

So we can write \( 5 + 4 = 9. \)

When we find, as in these examples, the number of members in the union of two disjoint sets we are doing ADDITION. If there are 12 boys in the classroom and 8 girls come in (these are disjoint sets), there will be 12 + 8 pupils in the classroom altogether. The sign + tells us to add 8 to 12. We read it as "plus." (12 + 8) is called the "sum" of 12 and 8.

If we have to find how many chairs we need to seat three boys and four girls, we want to know what number is represented by 3 + 4; that is, what is the sum (3 + 4). We can write this as 3 + 4 = \( \square \) and we want to find the numeral to put into the box to make a true statement. You will be able to help your pupils to work this out by using sets. They will have beans or stones or sticks and will put out a set of three things and a set of four other things.
They will then put them together and count the number of things in their union.

The numeral to be put into the box is of course 7 and we write

\[ 3 + 4 = \boxed{7} \]

**Exercise 7-3a**

Draw pictures to show how you would help children to find the numerals to put into the boxes to make these equations true.

1. \[ 4 + 3 = \square \]
2. \[ 7 + 1 = \square \]
3. \[ 2 + 8 = \square \]
4. \[ 5 + 5 = \square \]

**Exercise 7-3b**

Make up a word problem about each equation in the preceding exercise.

**Exercise 7-3c**

In the discussion in the text we have seen in the explanation of addition that if \( A \) and \( B \) are disjoint sets, then the number of elements in the union of \( A \) and \( B \) is the number of elements in \( A \) plus the number of elements in \( B \). Can you see what happens when sets \( A \) and \( B \) are not disjoint? Make up several examples and so decide what the result should be.

7-4  *Many names for one number*

You will have noticed that the answers to the last two problems were the same.

\[ 2 + 8 = 10 \quad \text{and} \quad 5 + 5 = 10 \]
We can say that the sum of the number of members in the two sets is \((2 + 8)\) and that 10 is the number of members in their union. We therefore have two ways of naming the number, as 10 and as \(2 + 8\). Also \(5 + 5\) gives us another name for 10. \(2 + 8\), 10 and \(5 + 5\) are symbols which represent the same number. So do \(3 + 7\), \(7 + 3\) and \(9 + 1\).

**Exercise 7-4a**

Write four different symbols which name each of these numbers: 6, 13, 22, 8.

**7-5 Working with the empty set**

If there is a set of 5 boys and 0 girls we can represent the union of these sets in a picture.

\[
\{ \text{boy} \} \cup \{ \text{ } \} = \{ \text{boy, girl} \}
\]

If we form the union of a set with the empty set, the resulting set is the same as the first set. This will help your pupils to understand how to add 0. They will put out five stones to represent the set of five boys, and no stones to represent the empty set of girls.

**Exercise 7-5a**

Explain how you would help your pupils to work out

\[
\begin{align*}
(1) & \quad 4 + 0 \\
(2) & \quad 9 + 0 \\
(3) & \quad 0 + 2 \\
(4) & \quad 0 + 5 \\
(5) & \quad 0 + 0
\end{align*}
\]
Exercise 7-5b

1. Make up story problems for each one of the problems of the preceding exercise.

Exercise 7-5c

1. Form the union of the two sets in a and in b below:
   a. \[ \text{\{ } \text{\}} \]
   b. \[ \text{\{ } \text{\}} \]

2. Match each of the sets in a and b with a counting set:
   a. \( \{ P, Q, R, S, T, O, W \} \)
   b. \( \{ \triangle, \bigtriangleup, \bigtriangledown, \bigtriangleup \} \)

3. In a and b explain each statement fully in terms of numbers of members in
   the sets to show the relation which two sets bear to their union:
   a. \( 1 + 9 = 10 \)
   b. \( 0 + 9 = 9 \)

4. Give 4 different names for each of the numbers named by these symbols:
   a. \( 2 + 6 \)
   b. \( 25 \)
   c. \( 7 + 1 \)
   d. \( 17 \)

5. In the following fill in the missing member or members as necessary in one
   or more of the sets in each statement to make the statement complete and
   correct:

   a. \[ \{ X, Y \} \cup \{ \bigtriangleup, \bigodot \} = \{ X, Y, \bigodot \} \]

   b. \[ \{ X, \bigtriangleup \} \cup \{ V, W, X, Y, Z \} = \{ \} \]

   c. \[ \{ \bigtriangleup, \bigtriangleup \} \cup \{ Y, \bigodot \} = \{ \bigtriangleup, \bigtriangleup, Y, \bigodot \} \]
The use of the box (□)

In problem 5 of the preceding exercises, to answer correctly you had to examine the members of all the sets in each statement. You did this to find which members were missing. In a., for example, you see that the missing member or members are in the union. You know this because a space has been left to hold a place for them. You match the members of the two sets with the members of their union and find that the symbol △ has been left out. △, then, is the missing member and you put it into the place left for it.

When we work with numbers we sometimes have to find a missing numeral in a statement. You worked one example earlier. It was $3 + 4 = □$. The box holds the place for the numeral which represents the sum $3 + 4$. We use the box to remind us that we have a missing number. The box is holding the place for it. You know already how to help your pupils to find the number which is needed by using sets of stones or sticks.

Missing numbers in addition equations

We can use a box in any position in a number statement. We can write $4 + □ = 7$. The box is holding a place for a numeral which gives the number of members in a set which must be added to a set of 4 members to give 7 members in their union. This is not the kind of addition we have been doing but it is very useful for pupils. It helps them to understand how one number can be made up by adding many different pairs of numbers. Therefore it helps them to realise that one number can have many names. We can write statements about the number 4 like this:
7-7

\[ \square + 4 = 4, \quad \square + 2 = 4, \quad \square + 1 = 4, \quad 3 + \square = 4 \]

and so on. We can fill in the missing numerals and write the true statements

\[ 0 + 4 = 4, \quad 2 + 2 = 4, \quad 3 + 1 = 4, \quad 3 + 1 = 4. \]

**Exercise 7-7a**

Find the numerals to put into the boxes to make each of the following equations true:

1. \[ \square + 2 = 4 \]
2. \[ 3 + \square = 10 \]
3. \[ 4 + 7 = \square \]
4. \[ 0 + \square = 8 \]
5. \[ 4 + 3 = \square + 5 \]

Now you may be wondering how your pupils will be able to find the missing numerals. They will be able to work them out with sets. Perhaps they have to find the missing numeral for this statement: \[ \square + 7 = 9. \] They can use a single bar abacus with nine beads or they can use a set of nine objects. If they use an abacus they arrange seven of the beads on the right hand side. Then the number of beads on the left hand side will be the number we need to make the statement true. The number of course is two.

![Abacus representation]  

Similarly, if the statement is \[ 6 + \square = 9, \]
the abacus would look like this:

![Abacus representation]  

so that we find that the box is holding a place for the numeral 3.

What numeral should be put into the box in the statement below to make it true?

\[ \square + 9 = 9. \]
You know that it is 0, but your pupils may have difficulty with this and so it can be shown on the abacus:

\[
\begin{array}{ccccccccccc}
\Box & \Box & \Box & \Box & \Box & \Box & \Box & \Box & \Box & \Box & \Box \\
\end{array}
\]

There is no bead on the extreme left of the abacus and so the box is holding the place for the numeral 0. So we write

\[0 + 9 = 9\]

Using the abacus in this way it is easy to discover each of these true statements:

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<tr>
<th>Sum</th>
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<th>Sum</th>
<th>Sum</th>
<th>Sum</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 + 9 = 9</td>
<td>5 + 4 = 9</td>
<td>1 + 8 = 9</td>
<td>6 + 3 = 9</td>
<td>2 + 7 = 9</td>
<td>7 + 2 = 9</td>
</tr>
<tr>
<td>3 + 6 = 9</td>
<td>8 + 1 = 9</td>
<td>4 + 5 = 9</td>
<td>9 + 0 = 9</td>
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</tr>
</tbody>
</table>

Thus the sums 0 + 9, 1 + 8, 2 + 7, etc., are all the same number 9.

**Exercise 7-7b**

Find, by use of the abacus or otherwise, all the sums of pairs of numbers that give 5. Do the same for 7 and for 8.

**Exercise 7-7c**

1. I am thinking of a number. This number plus 5 is 11. What is the number I am thinking of?

2. Each of the 12 children in a class is assigned a number from 1 to 12 so that all these numbers are assigned. If some of the students form pairs so that the sum of the numbers assigned to the pairs are 13, which children are left without partners? (Same question with 13 replaced by 14, 15, 16.)
**7-9**  Addition table

When your pupils thoroughly understand how to find the sum of two numbers by forming the union of disjoint sets, they can build an addition table.

First make a square with 121 small squares in it. That means that there will be 11 rows with 11 squares in each row. In the top left hand square put the + sign to show that you are adding. Follow this by the symbols 0, 1, 2, ..., 9, one in each square along the top row. Similarly, write these symbols 0, 1, 2, ..., 9 in the squares down the left hand side. You now have a square like this:

```
+ 0 1 2 3 4 5 6 7 8 9
|+|
|0|
|1|
|2|
|3|
|4|
|5|
|6| 11
|7|
|8|
|9|
```

Begin with the first empty square on the second row. Put into this square the numeral which represents the sum of the numbers you have already shown in the outside spaces at the left end of the row and the top of the column. This sum is 0 + 0 so we put 0 into the first empty space. The space next to this on the
right is for the sum of $0 + 1$. So 1 is put into this space. The next square along the second row has 0 and 2 at the ends of its row and column and so its numeral stands for $0 + 2 = 2$.

Where do we put $6 + 5$, that is, 11? We find the row which begins with 6, then the column which begins with 5. Then we find the square which is their intersection and in this square we put 11.

Exercise 7-8a

1. Prepare a table similar to the one described and fill in all the sums for pairs of numbers from $0 + 0$ to $9 + 9$.

2. Use the table to find six pairs of numbers whose sum is 8, and write the addition equations which show these facts.

3. Mary and her sister went to market. Mary bought six pineapples and her sister bought eight coconuts. How many fruits did they buy altogether?

4. Yabu and Konteh went to their farm to collect oranges. They collected twenty oranges, but while Yabu was busy packing oranges in the basket she brought with her, Konteh went away to look at a bird's nest. When Yabu had put fifteen oranges into her own basket, Konteh came back. How many oranges were left for Konteh to take home? Write an equation for this problem using numerals and a box.

5. Thirteen children were picked by a teacher to play a game of "pairs." Each child was given a card on which was printed one of the numerals from 0 to 12. When the teacher called a number, each child was to find a partner so that the sum of the numbers marked on their cards was the number called by the teacher. If the teacher called "ten," how many pairs could be formed?
Write them down. Which children had no partners?

6. If the teacher called "eight," how many children would have no partner? Which cards were they holding? Write down the pairs of numbers held by the children who found partners.

7. Imagine that you are given two dice. Each dice has 6 sides which each show one of the numerals 1, 2, 3, 4, 5 or 6. Throw both of them on a table at the same time. Make a table showing the pairs of numbers that could turn up and their sums.

8. Find the numerals to put into the boxes to make each of the following equations true:

(a) \[ \square + 9 = 9 \]
(b) \[ 8 + \square = 8 \]
(c) \[ 6 + \square = 6 \]

(d) \[ \square + 5 = 5 \]
(e) \[ \square + 4 = 4 \]
(f) \[ \square + 1 = 1 \]

7-9 Property of zero in addition

In the last problem of the preceding exercise you found that in each case the box must be filled by 0 to make the equation true.

Consider the following true equations

\[
\begin{align*}
0 + 1 & = 1 \\
0 + 2 & = 2 \\
0 + 3 & = 3 \\
0 + 4 & = 4 \\
0 + 5 & = 5 \\
\text{(and so on)} & \quad \text{(and so on)}
\end{align*}
\]

\[
\begin{align*}
1 + 0 & = 1 \\
2 + 0 & = 2 \\
3 + 0 & = 3 \\
4 + 0 & = 4 \\
5 + 0 & = 5 \\
\text{(and so on)} & \quad \text{(and so on)}
\end{align*}
\]
Do you notice the effect of adding 0 to a number? It leaves the number the same as it was before zero was added to it. You can see that this is always true. Zero is the number of members of the empty set. If we take the union of the empty set with any set we get just that set.

We can write this property of zero as

\[ 0 + \square = \square \]

which is true no matter what numeral is put in the boxes. You can choose any numeral you please (so long as both boxes are filled in the same way). Similarly

\[ \square + 0 = \square \]

is true no matter what numeral is put in the boxes.

**Exercise 7-9a**

Put 6 different numerals in turn into the boxes in the equations which show the property of zero in addition. Here is one: \[ 3 + 0 = 0 + 3 = 3 \].

**Exercise 7-9b**

Explain how you would help your pupils to understand this property of zero.
8-1 **Commutative property of union of sets**

When you form the union of two sets of things you know that it does not matter which set you take first. The union of a set of 3 bananas and a set of 2 oranges is the same set as the union of the set of 2 oranges and the set of 3 bananas. The order does not make any difference to the result.

We can draw a picture to show this property of the union of sets.

A is a set of 3 bananas, B is a set of 2 oranges.

![Diagram showing sets A and B and their union](image)

We can make an exact matching between the members of the united sets. We see that the sets $A \cup B$ and $B \cup A$ contain the same members and so are equal sets: $A \cup B$ and $B \cup A$ are names for the same set and so we write

$$A \cup B = B \cup A.$$
8-2

**Exercise 8-1a**

Make up some illustrations of the commutative property of the union of sets and draw pictures which show them.

**8-2  Commutative property of addition**

Is this property of the union of sets also a property of addition of numbers?

You know that it is. We write the number facts for the union of the set of 3 bananas and the set of 2 oranges like this:

\[
3 + 2 = 5 \\
2 + 3 = 5
\]

and therefore \(3 + 2 = 2 + 3\).

Experience of uniting sets suggests that this works for any two numbers we choose. This property of addition is known as the **commutative property of addition**. The children you will teach will have had many experiences of making the union of two sets of objects. This will help them to discover this property for themselves. When they have found the member which is represented by \(8 + 4\), they should realize that it is also represented by \(4 + 8\).

You will help them to realize this by encouraging them to find the "twin facts" such as \(3 + 2 = 5\) and \(2 + 3 = 5\), \(6 + 4 = 10\) and \(4 + 6 = 10\).

**Exercise 8-2a**

Find numerals to put into the boxes to make the following statements true:
8-3

1. \[
\phantom{\square} + 3 = 3 + 2
\]
2. \[
11 + \phantom{\square} = 7 + 11
\]
3. \[
7 + 9 = \phantom{\square} + 7
\]
4. \[
\phantom{\square} + 3 = 3 + \phantom{\square}
\]

8-3 General notion of a variable

In Exercise 8-2a, problem 4 is very different from problems 1, 2, and 3. Do you see why? For how many numerals is it true that \[
\phantom{\square} + 3 = 3 + 2?
\]
Only for the single numeral "2". For how many numerals is it true that
\[
\phantom{\square} + 3 = 3 + \phantom{\square}?
\]
For all numerals! Of course we must agree that whatever numeral is put in the left-hand box must also be put in the right-hand box.

Thus
\[
1 + 3 = 3 + 1
\]
\[
2 + 3 = 3 + 2
\]
and so on and so on.

We speak of
\[
\phantom{\square} + 3 = 3 + \phantom{\square}
\]
as an identity. An identity is an equation which is true for all the things under consideration -- here for all whole numbers.

We can go further than this actually
\[
\phantom{\square} + 1 = 1 + \phantom{\square}
\]
\[
\phantom{\square} + 2 = 2 + \phantom{\square}
\]
\[
\phantom{\square} + 3 = 3 + \phantom{\square}
\]
\[
\phantom{\square} + 4 = 4 + \phantom{\square}
\]
and so on.
All of these identities can be put together in the single one
\[
\Box + \triangle = \triangle + \Box
\]
In writing this it is important to understand that the two \(\Box\)'s must be filled in the same way and the two \(\triangle\)'s must also be filled in the same way (which may or may not be the same as the one used for the \(\Box\)'.s).

For example,
\[
3 + 3 = 3 + 3
\]
is true.

So is
\[
3 + 4 = 4 + 3
\]
For all numerals, it is true that
\[
\Box + \triangle = \triangle + \Box
\]
The order in which any two whole numbers are added does not matter.
\[
\Box + \triangle = \triangle + \Box
\]
expresses what we call the **commutative property of addition**.

**Exercise 8-3a**

1. Fill the boxes in 6 different ways so that
\[
\Box + \triangle = \triangle + \Box
\]
becomes a true statement.

2. In
\[
\Box + \Box + \triangle + \Box + \triangle = \Box
\]
put 3 in the squares and 9 in the triangles. What must we put in the circle to get a true statement?
When we go on in mathematics we use letters instead of boxes to express identities. For example, we can write the commutative property of the addition of whole numbers as follows:

\[ a + b = b + a \]

for all whole numbers \( a \) and \( b \).

**Exercise 8-3b**

What does the commutative property tell us if

1. \( a = 3, b = 1 \)  
   Answer: \( 3 + 1 = 1 + 3 \)

2. \( a = 5, b = 2 \)

3. \( a = 4, b = 9 \)

4. \( a = 6, b = 4 \)

A letter which is used to stand for any one of a set of numbers is called a variable. We can use variables in this way to help us to write down and remember more easily such properties of numbers as the commutative property of addition.

**8-4 Associative property of union of sets**

What do we mean by \( 1 + 2 + 3 \)? Here we are asked to add three numbers and we have so far talked only about adding two numbers. The only way we can add three numbers is to add two of them and then add the remaining one to the result. Does it matter which pair of numbers we add first? Let us try both ways. To show which pair of numbers to add first we put brackets (or parentheses) around them

\[
(1 + 2) + 3 = 3 + 3 = 6
\]

\[
1 + (2 + 3) = 1 + 5 = 6.
\]
So it does not matter how we group these numbers.

We can show this with sets also. If you have a set of 3 bananas and a set of 2 oranges in a basket you have a set of 5 fruits. If someone then gives you a set of 4 pawpaws you will now have a set of 9 fruits in your basket. If you have a set of 3 bananas first and then someone brings you a basket holding a set of 2 oranges and a set of 4 pawpaws you will, again, have a set of 9 altogether.

We can draw a picture of this.

You can see that the combined sets \((A \cup B) \cup C\) and \(A \cup (B \cup C)\) can be exactly matched with each other. They contain the same members and so are equal sets. Therefore we can write

\[(A \cup B) \cup C = A \cup (B \cup C)\]

So you see it does not matter how we group sets when we form their union.

8-5 **Associative property of addition**

As you know already you can write the number facts which go with the example above as

\[(3 + 2) + 4 = 5 + 4 = 9\]

\[3 + (2 + 4) = 3 + 6 = 9\]

and therefore

\[(3 + 2) + 4 = 3 + (2 + 4)\]
So you see that if you have to find the sum of these three numbers it does not matter how you group them in pairs to add them. Experience in forming unions of sets suggests that this will work for any numbers we choose, and so we write (for all whole numbers \(a, b\) and \(c\))

\[(a + b) + c = a + (b + c).\]

This is called the **associative property of addition**.

**Exercise 8-5a**

What does the associative property of addition tell us if

1. \(a = 1, \ b = 2, \ c = 1\)  
   Answer: \((1 + 2) + 1 = 1 + (2 + 1)\)

2. \(a = 4, \ b = 3, \ c = 0\)

3. \(a = 5, \ b = 5, \ c = 2\)

4. \(a = 3, \ b = 7, \ c = 6\)

5. \(a = 0, \ b = 8, \ c = 6\)

**Exercise 8-5b**

1. Write down 6 examples of the associative property of addition.

2. Give three problems you could use to help your pupils to understand this property.

This is a useful property because it sometimes makes addition easier for us. If we have \(19 + 6 + 4\) we might find \(19 + 6\) first and then add 4. When we know the associative property we can first find \(6 + 4 = 10\) and then \(19 + 10 = 29\). This is easier. We write this

\[19 + (6 + 4) = 19 + 10 = 29.\]

**Exercise 8-5c**

Put brackets (parentheses) in these sums to show how you would group them for adding. Then find the answers.
1. $9 + 7 + 3$
2. $6 + 4 + 7$
3. $4 + 16 + 7$
4. $9 + 5 + 15$
5. $7 + 4 + 6$
6. $7 + 3 + 3$

Suppose that you wish to add 4 and 7 and 6. We could write

$$4 + (7 + 6).$$

But by the commutative property we can replace $7 + 6$ by $6 + 7$ to get

$$4 + (6 + 7) = (4 + 6) + 7 \quad \text{(associative property)} = 10 + 7 = 17.$$

In this case we have changed both the order and the grouping to take advantage of the fact that $4 + 6 = 10$.

**Exercise 8-5d**

Make up 5 more problems like those above. Put brackets around pairs of numerals to show which sums you would find first.

**Exercise 8-5e**

Find the numeral to put into the box to make the following sentence true:

$$5 + (4 + \square) = (4 + 5) + 6.$$

Make up other similar problems to illustrate the associative property of addition.
8-6 **Generalized properties of addition**

You will see from these exercises that you can add numbers in any order and in any grouping that you like. This means that if you have to find the sum of several numbers you can begin with pairs whose sum is easy to find, and that you can re-arrange the numbers to bring the pairs together.

**Exercise 8-6a**

1. Name the property of addition illustrated below when each sum is stated to be equal to the following one:

   \[(4 + 3) + 2 = 4 + (3 + 2) = 4 + (2 + 3) = (2 + 3) + 4 = 2 + (3 + 4)\]

2. The associative property of addition makes it possible, in adding three numbers, to group them however you please. If you add four numbers, can you group them in any way you please? Explain. Is this also true for more numbers?

3. How many different combinations of five or fewer different numbers from 1 to 10 add up to 15? (For example, 10 and 5 is the same combination as 5 and 10. Also 3 + 3 + 9 uses the number 3 twice and this is not allowed.)

4. The game of "31" is played by two players. The first player announces a number from 1 to 5. The players then alternate, each adding a number from 1 to 5 to the previous result and announcing the new result. The player who announces 31 is the winner. Explain how the player who goes first can be sure of winning if he knows the secret. (You can find the secret at the end of the second section of the next chapter.)
CHAPTER 9

SUBTRACTION

9-1 Reminder of addition

You learned how to add when you were a school child yourself. At that time you did not give much thought to it, but you just learned all the different sums by heart. In this course, you have studied the meaning of addition in terms of the union of sets. You know that it is based on putting together the members of two disjoint sets into one set and counting the members in that union. In this section, you will study subtraction in the same way, using the idea of sets, and discover that it means the inverse of addition. In subtraction, instead of putting sets together you separate a set into parts called subsets. For example, you know that if you have a set of 2 members and a set of 3 members these when put together give a set of 5 members. On the other hand, if you have a set of 5 members and remove a subset of 3 members, you have left a subset of two members. You know that if you add 3 to 2 you get 5; if you subtract 3 from 5 you get 2. In this section you will find why this is so.

When you learned to add, you saw that you could build tables of sums of pairs of numbers. For example, you found that $6 + 0 = 6$, $5 + 1 = 6$, $4 + 2 = 6$, $3 + 3 = 6$, $2 + 4 = 6$, $1 + 5 = 6$ and $0 + 6 = 6$. This is the family of all the pairs of whole numbers whose sum is 6. Each of the sums on the left hand sides of the statements is a different way of representing the number 6.
9-2

Exercise 9-1a

Write similar tables for the pairs of numbers with sums 5, 8 and 9.

9-2 Tables of constant sums

Just as you found that there are many ways of selecting pairs of sets
to make a given third set, so you can see that there are many ways to
separate the members of a given set into subsets. For example, begin with
a set of numbers and separate the numbers to form two sets in as many ways
as possible. If you look at your addition table, you will find that these
separations suggest the following pairs for 7: 0 and 7, 1 and 6, 2 and 5,
3 and 4, 4 and 3, 5 and 2, 6 and 1, 7 and 0.

Exercise 9-2a

Use a single bar abacus with beads and separate the beads to show
each of these statements in turn. For example:

```
[Diagram: Bar abacus showing 7 beads being separated]
```

This shows that $7 = 4 + 3$. Do the others in the same way.

Exercise 9-2b

Repeat the above exercise, using an abacus with 6 beads to show the
different pairs of subsets into which a set of 6 things can be separated.

Exercise 9-2c

You can make up many problems using the idea of a number being
the sum of different pairs of numbers. Here is one: Suppose you had
9 girls in your class and you wanted some of them to work on the blackboard,
while others worked at their desks. You could show all the different ways
of separating the class. Tell all the different ways you can separate the
9 girls in this problem.

In the statement $2 + 3 = 5$, 2 and 3 are called addends. Sometimes the sum and only one of the addends may be already known. In such cases, the other addend can be chosen in only one way. In the problem above, suppose that you had space for only 3 girls at the blackboard and that you wanted to use all the blackboard space. Then you know that of your 9 girls, 3 must be at the blackboard and the others at their desks. You would look at your addition table and find that the only pair of numbers in which one of the addends is 3 and the sum 9 is the pair 3 and 6. And so you would know that there had to be 6 girls at their desks because $9 = 3 + 6$.

Exercise 9-2d

Make up a problem of this type which you would use with your pupils.

Work this problem in the way shown above.

Exercise 9-2e

The figure above shows a clock face with the hands locked so that they always point in opposite directions; at what numbers do the hands point when the sum of these numbers is 16?

Note: The secret for problem 4 of Exercise 8-6a in the preceding chapter is for the first player to announce the numbers 1, 7, 13, 19, 25 and 31.
9-3  Missing addends in addition problems

You can write problems like this in a special way. In the example above, there are 9 girls and 3 will work at the blackboard. You want to know how many will work at their desks. This is the missing addend. Thus you can write $9 = 3 + \square$ where the box shows that some numeral is missing or not known. The box is holding the place for some numeral you want to find.

Another problem of this kind is the following: you want 6 slates for your class, but you have only 4. How many more do you need? You can write the sentence like this

$$6 = 4 + \square$$

You all know of course that if you have 2 more slates, you will have enough. So you can put 2 in the box and have the true statement $6 = 4 + 2$.

Exercise 9-3a

Kafi is looking for his eight chicks and he has found five. How many more must be found? Write $8 = 5 + \square$. What numeral must you put into the box to make this statement true? Now write the answer to the question in this way: Kafi must put 3 in the box to make the statement true.

Exercise 9-3b

Find the numerals which make these statements true:

(a) $3 + \square = 4$  (c) $3 + \square = 6$  (e) $3 + \square = 9$
(b) $7 = 5 + \square$  (d) $7 = 7 + \square$  (f) $8 = 1 + \square$

You can look at problems involving missing addends in another way. Suppose you had some money, and a man gave you $5 more, so that you now
had £8. How much money did you have at the start? You could express this as follows:

\[ \square + 3 = 8 \]

You know, of course, that you had £5 to start with, and so you should put the numeral 5 in the box. You can see now that it does not matter whether the box, which holds the place for the missing numeral, comes first or second.

**Exercise 9-3c**

Find the numerals which make these equations true:

1. (a) \[ \square + 5 = 5 \]
2. (b) \[ 5 = 3 + \square \]
3. (c) \[ \square + 2 = 7 \]
4. (d) \[ 8 = \square + 6 \]
5. (e) \[ 4 + \square = 8 \]
6. (f) \[ 9 + \square = 9 \]

**Exercise 9-3d**

For each of the problems in the exercise above, make up a word problem suitable for a class of children.

**9-4 Subtraction as finding the missing addend**

When you find the missing addend in an addition equation you should see that you are really doing SUBTRACTION. In the example above, when someone gave you £3, so that you had £8, you could have found how much you started with by subtracting 3 from 8. In fact, this is what you really did, because subtraction means finding a missing addend in an addition equation. The sign for subtraction is written "-" and is read "minus." In this example we wrote the addition equation

\[ \square + 3 = 8 \]
and so now we write the subtraction equation

\[ 8 - 3 = \square \]

where we will, of course, put 5 in the box.

**Exercise 9-4a**

For each of the addition equations below write a corresponding subtraction equation:

<table>
<thead>
<tr>
<th>Addition Equation</th>
<th>Subtraction Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) (4 + \square = 6)</td>
<td>Answer (6 - 4 = \square)</td>
</tr>
<tr>
<td>(b) (\square + 7 = 7)</td>
<td></td>
</tr>
<tr>
<td>(c) (4 = 3 + \square)</td>
<td></td>
</tr>
<tr>
<td>(d) (8 + \square = 9)</td>
<td></td>
</tr>
<tr>
<td>(e) (\square + 6 = 6)</td>
<td></td>
</tr>
<tr>
<td>(f) (7 = \square + 2)</td>
<td></td>
</tr>
</tbody>
</table>

**Exercise 9-4b**

Make up more problems of this type, writing both addition and subtraction equations.

**Exercise 9-4c**

Write word problems suitable for your pupils for examples (d), (e), and (f) in Exercise 9-4a.

**Exercise 9-4d**

For the following subtraction equations write the corresponding addition equations:
9-7

Subtraction Equation  Addition Equation

(a) $8 - 6 = $ \boxed{}  \text{ Answer } 6 + \boxed{} = 8

(b) $\boxed{} = 4 - 4$

(c) $9 - 2 = $ \boxed{}

(d) $\boxed{} = 8 - 7$

In each of the problems above the answer is very easy to find. All you have to do is to look up your addition table or to separate a set of things into two subsets. You can use an abacus or a set of stones, to help teach children how to find answers to these problems. In the last problem above, the children could take a set of 8 stones, and then remove 7 stones. They should then see immediately that a subset of 1 stone remained and that $8 - 7 = 1$.

**Exercise 9-4e**

Find the numbers that make the subtraction equations true in the exercises above.

**Exercise 9-4f**

1. One tree is 90 feet high and another is 70 feet high. How much higher is the first tree than the second?

2. Find two numerals to put into the boxes to make $3 + (5 + \boxed{})$

   $= 2 + (4 + \triangle)$, a true sentence. If you put the same numerals into the same boxes in $\triangle - \boxed{}$, what result do you get?

**9-5 Subtraction problems which cannot be solved with whole numbers**

Sometimes you will find that you cannot do what a problem tells you to do. For instance, if someone tells you to separate a set of 8 stones into a subset of 9 stones and another unknown subset, you cannot do it. You can write the addition and subtraction equations as follows:
\[ 8 = 9 + \square, \quad \quad \quad 8 - 9 = \square \]

But you know that you cannot find any whole number which makes the equations true.

**Exercise 9-5a**

In the following, find the missing numeral that would make each equation true, whenever possible to do so:

<table>
<thead>
<tr>
<th>Addition Equations</th>
<th>Subtraction Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a) ( 7 + \square = 7 )</td>
<td>(d) ( 9 - 2 = \square )</td>
</tr>
<tr>
<td>(b) ( 5 = \square + 8 )</td>
<td>(e) ( \square = 6 - 6 )</td>
</tr>
<tr>
<td>(c) ( \square + 2 = 3 )</td>
<td>(f) ( \square = 1 - 8 )</td>
</tr>
</tbody>
</table>

**9-6 Subtraction facts from addition facts**

Think about the statement \( 4 + 2 = 6 \). What subtraction facts do you think you can make from it? You can cover up the 4 and have as a result the addition equation \( \square + 2 = 6 \); or you can cover up the 2 and have, as a result, the addition equation \( 4 + \square = 6 \). From these you can get the subtraction equations \( 6 - 2 = \square \) and \( 6 - 4 = \square \) and finally the subtraction facts: \( 6 - 2 = 4 \) and \( 6 - 4 = 2 \).

**Exercise 9-6a**

Make up two subtraction facts from each of the addition facts:

(a) \( 3 + 2 = 5 \); \quad (b) \( 6 = 6 + 0 \); \quad (c) \( 7 + 1 = 8 \); \quad (d) \( 2 + 5 = 7 \).

**Exercise 9-6b**

Make up word problems from the subtraction facts you wrote in Exercise 9-6a which you can use with children.
9-7 Separation of sets into subsets

Now you can see that if you separate a given set into two subsets and you know the number of members in one subset you subtract to find the number of members in the other subset. For example, you could tell your pupils that father caught 5 fishes, of which 3 are big and the others little. To find the number of little fishes, you write the addition equation first, $5 = 3 + \square$ and then the subtraction equation $5 - 3 = \square$. You can in this way guide the class to give the correct answer.

Exercise 9-7a

Write the addition and subtraction equations for each of the following problems and then write the answers:

1. There are 9 people in a room, 6 of them are seated and the rest are standing. How many are standing?
2. Kwame found 8 eggs, but 2 were broken. How many were not broken?
3. Esi puts 7 cups on a table. Only one cup is large and the rest are small. How many are small?
4. Aba picks 10 flowers, 5 are red and the rest are blue. How many blue flowers did Aba pick?

Exercise 9-7b

If $A$ and $B$ are sets, we let $A - B$ be the set of members of $A$ which are not members of $B$. Find $\{a, b, c, d, e, f, g\} - \{b, d, f\}$. Interpret this in terms of the number of members in each set.
Comparing sets

You can use this approach in comparing two sets with different numbers of members. Show your pupils how many more or fewer members the first set has than the second. For example, you might show your class 6 red flowers and 4 white flowers and ask them how many more red flowers than white flowers there are. You can write the addition equation $4 + \square = 6$, from which the subtraction equation $6 - 4 = \square$ follows. Another way to approach this problem would be to make a matching between the members of the two sets as follows:

![Diagram of matching red flowers and white flowers]

In this way you compare the two sets. This shows a remainder in the larger set after matching each member in the one set with one member of the other set. In the problem above you have matched 4 flowers in the upper set with 4 flowers in the lower set, so that 2 remain. 2 is called the DIFFERENCE between the numbers 6 and 4.

Exercise 9-8a

1. For each of the following sentences make a picture of the two sets and make a matching between the members in order to show which has more members and find how many more members it has. Then write the addition and subtraction equation for each, and a sentence describing the result.
Example:

Ama has two oranges and Araba has seven oranges. We match the sets of oranges in this way:

![Diagram showing oranges]

The addition and subtraction equations are

\[
7 = 2 + \square \\
7 - 2 = \square
\]

and we state the difference in the form

\[
7 - 2 = 5
\]

and say that Araba has five more oranges than Ama.

Now work the rest:

(a) There are 7 good eggs and 2 broken ones.

(b) Kofi caught 2 fish and Kwesi caught 2 fish.

(c) There are 3 houses here and 5 houses across the road.

2. Try these next:

(a) I have six counters. My sister has 9 counters. Find the difference between the number of counters my sister has and the number I have.

(b) There are 7 buttons on Lucy's dress and 6 buttons on mine. How many more buttons has Lucy?
(c) There are 8 birds on a tree and 4 chickens on the ground. How many more birds are there than chickens?

(d) We have 7 pots and 2 spoons. What is the difference between the number of pots and the number of spoons?

(e) Ama has 2 oranges. She wants 6 to make juice. How many more does she need?

(f) Araba has 6 pawpaws and Lucy has 8. How many more did Lucy have than Araba?

9-9 Subtraction problems

There are many kinds of problems which give rise to a subtraction equation. Here are some of these problems, all of which give the same subtraction equation

\[ 5 - 3 = \square \]

The addition equation for each problem is stated.

(a) What whole number must be added to 3 to give the sum 5?

\[ 3 + \square = 5. \]

(b) If a set of 5 members is separated into two subsets, one of which has 3 members, how many members has the other subset?

\[ 5 = 3 + \square \]

(c) If the members of a set of 5 are compared with the members of a set of 3, find the difference between the number of members in the first and second sets.

\[ 3 + \square = 5. \]

(d) If 3 members are removed from a set of 5 members, how many members remain?

\[ 5 = 3 + \square \]
Exercise 9-9a

Below are some subtraction problems. For each one write the addition equation as was done in -- 9(a), 9(b), 9(c) or 9(d). Then write its subtraction equation.

(a) There were 5 empty chairs in a room. 9 people came in and 5 sat in the chairs. How many people were left standing?

(b) Mary picked 3 flowers to put into a jar. Her mother wanted 9 flowers, so how many more must Mary pick?

(c) One tree has 9 mangoes ready to eat. A second tree has only 2 mangoes ready to eat. How many more are ready to eat on the first tree?

(d) We have 4 hens and 8 chickens. Find the difference between the number of chickens and the number of hens.

(e) We have 7 pots and we need 9 pots. How many more pots do we need?

(f) Kofi and Kwesi went fishing. Kofi caught 5 fish and Kwesi caught 3. How many more did Kofi catch than Kwesi?

(g) Efua has 2 mangoes in her hand and 4 in her basket. How many more mangoes are in the basket than in her hand?

(h) I had 7 balls in a basket and my friend took 4 of them. How many were left in the basket?

(i) Kiajuma was putting spoons on the table. There were places for eight spoons and she has put four spoons out already. How many more must she place?
Araba has 8 pawpaws and she gives Lucy 2 pawpaws. How many does Araba have left?

You can see by now that there are many ways of talking about the same sets. Think of a set of 5 bananas, of which 3 are green and 2 are yellow.

You can write four statements about these bananas:

\[ 3 + 2 = 5 \quad 5 - 2 = 3 \]
\[ 2 + 3 = 5 \quad 5 - 3 = 2 \]

Exercise 9-9b

Make up sets of four statements like those above, which connect the three numbers in each of the following:

1. 7, 8, 15 4. 1, 19, 20
2. 3, 11, 14 5. 12, 2, 10
3. 15, 5, 20 6. 6, 6, 12

9-10 Subtraction as the inverse of addition

Suppose you have a set of 3 books. Now put 2 more with them. From this new set remove 2 of the books. How many books remain? Of course you have just 3. You can state this as follows:

\[ (3 + 2) - 2 = 3 \]

Again, suppose you start with a set of 3 coins. Now remove 2 coins. Then put in 2 coins again. How many coins are there? Once again you have just 3. You can state this:

\[ (3 - 2) + 2 = 3 \]
Exercise 9-10a

Show these two patterns with each of the pairs of members below:

1. 15, 10
2. 8, 3
3. 12, 0
4. 9, 9

You ought by now to be saying to yourself that addition and subtraction each undoes the work that the other has done. We thus talk of them in mathematical language as "inverse" operations. Subtraction is the inverse of addition and addition is the inverse of subtraction.

Thus we can summarize:

\[(a + b) - b = a\]
\[(a - b) + b = a\]

(Naturally we have to be sure that \((a - b)\) is a whole number.)

As one more example, when \(a = 25\) and \(b = 13\), these become

\[(25 + 13) - 13 = 25\]
\[(25 - 13) + 13 = 25\]

Exercise 9-10b

1. Write these statements for \(a = 12\) and \(b = 6\) and \(a = 5\) and \(b = 0\).

2. Write a word problem for your pupils using \(a = 15, b = 3\).

Exercise 9-10c

A bag contains 6 oranges but will hold 10 oranges when it is full. If John takes 2 oranges out of the bag, how many oranges will it take to fill it?
CHAPTER 10

MULTIPLICATION

10-1  Reminder of addition

You know that addition is based on counting the number of things in a union of disjoint sets. For example, you know that, if three boys and four girls get perfect papers on your arithmetic test, there are seven children who get perfect papers. You have learned the addition table, and you know why each of the sums works out as it does. You also know how to teach addition to the children in your school so that they too will never forget what it means.

10-2  Repeated addition

Now you will do problems in which addition is repeated several times. If a man earns 5 shs a day, and works for 7 days, you can find how much he earns by taking 7 sets of 5 shs each and putting them together. You get an addition equation like this:

\[5 + 5 + 5 + 5 + 5 + 5 + 5 = \square\]

You know how to find the numeral to put in the box, because you know how to add. Of course, you find the answer to be 35. Thus, since the sum of seven 5's is 35, the man earns 35 shs for his 7 days work.
Exercise 10-2a

Find the answers to the following:

1. \(3 + 3 + 3 + 3 + 3 + 3 = \) □
2. \(7 + 7 + 7 = \) □
3. \(4 = \) □
4. \(9 + 9 + 9 + 9 + 9 + 9 + 9 + 9 = \) □
5. \(8 + 8 + 8 + 8 = \) □

Exercise 10-2b

Make up word problems to fit each of the problems in the above exercise. These word problems should be ones which you could use with primary school children who are just learning about repeated addition. Show the way in which you would present the solution to each of these problems to your class.

10-3 Ways of writing repeated addition

If your children have trouble with such problems, you can always use sets to help them. For instance, in problem 5, above, you can take 4 sets of 8 stones each, and lay them on the table as follows:

```
  o o o o o o o o
  o o o o o o o o
  o o o o o o o o
  o o o o o o o o
  o o o o o o o o
```

You may need to remind the children that adding is based on counting. If so, you can ask the children to count the number of stones in the union of the four sets. In this way they will see that the answer is 32. Draw a picture
of the 4 sets of 8 stones on the blackboard, and then write the following statements underneath the picture.

\[ 8 + 8 + 8 + 8 = 32 \]

The union of 4 sets of 8 things has 32 members.

8 added 4 times = 32.

4 8's are 32.

**Exercise 10-3a**

Draw pictures for each of the problems in exercise 10-2a above. Then write under each picture the four sentences as in the example above.

**10-4 Multiplication as repeated addition**

You can see what this is leading to. And if you have done this correctly in the classroom, your children will see where you are leading them. They -- and you -- will want to write this in an easier way. Repeated additions, based on repeated unions of equivalent sets, are so common that mathematicians give them a special name. You know that name, of course. It is multiplication, sometimes called finding the product. Instead of writing any of the four sentences which were given in section 3 above, you write the one short and easy sentence,

\[ 4 \times 8 = 32. \]

Point out to your pupils that "\( \times \)" is read "times," so that this statement is read "4 times 8 equals 32." This means concretely that you take the union of 4 disjoint sets of 8 things each, and count the resulting set. This is the same as repeated addition, and is the same as adding the number 8 four times. So you have the following five statements, all of which say the same thing in
different ways.

\[ 8 + 8 + 8 + 8 = 32 \]

The union of 4 sets of 8 things has 32 members.

8 added 4 times = 32

4 8's are 32

\[ 4 \times 8 = 32 \]

**Exercise 10-4a**

Draw sets and write the five statements to show the meaning of the following multiplication equations. Find the numerals to put in the boxes.

1. \[ 9 \times 3 = \_ \]  
2. \[ 2 \times 5 = \_ \]  
3. \[ 1 \times 7 = \_ \]  
4. \[ 7 \times 1 = \_ \]  
5. \[ 5 \times 0 = \_ \]

**Exercise 10-4b**

Write word problems to go with each of the problems in the above exercise. These word problems should be suitable for primary students just learning how to multiply.

**10-5 Problems of the type a \times 0**

You may have had some trouble with problem 5, in the exercise above. The first step in thinking about such a problem is to think what multiplication means. You know that it goes back to addition. So if you multiply \( 3 \times 0 \), you know that it means the same as

\[ 0 + 0 + 0 = \_ \]

And if you remember that each of these 0's is the number of the empty set, you will know that this means
the union of 3 sets of 0 things.

And of course that union has no members in it. It is itself the empty set.

And so you can write

0 added 3 times = 0.

And so you can see easily that the last two lines of the work would be as follows:

3 0's are 0

3 \times 0 = 0

**Exercise 10-5a**

Draw pictures and write the five statements which show the meaning of the following multiplication equations. Fill in the boxes correctly.

1. \(4 \times 0 = \square\)

2. \(7 \times 0 = \square\)

3. \(2 \times 0 = \square\)

**10-6 Problems of the type 0 \times a**

Another problem which will give you trouble is one like \(0 \times 4\). You have to think what is really means. Again you go back to addition. So to find \(0 \times 4\), you take a set of 4 things a certain number of times. How many times? Think of \(2 \times 4\) and \(1 \times 4\). In the case of \(2 \times 4\), you take 2 sets of 4 things. In the case of \(1 \times 4\), you take 1 set of 4 things. In this case, then, you take the union of 0 sets of 4 things.

What does that mean? Draw a picture showing \(2 \times 4\), and a picture showing \(1 \times 4\). They look like this:
Thus a picture showing $0 \times 4$ would have 0 sets of 4 things and would look like this.

And so you can see that this union has 0 members, and is itself the empty set.

And so you can write

$$4 \text{ taken } 0 \text{ times} = 0$$

$$0 \text{ 4's are } 0$$

$$0 \times 4 = 0$$

**Exercise .0-6a**

Tell in words the meaning of the following multiplication equations in the way described above.

1. $0 \times 1 = \square$
2. $0 \times 5 = \square$
3. $0 \times 0 = \square$

**10-7 Multiplication in terms of arrays of dots**

You may have noticed another way to think about multiplication.

Remember that you can show $4 \times 8$ as follows:

```
 o o o o o o o o
 o o o o o o o o
 o o o o o o o o
 o o o o o o o o
```

In this way you show the product as a rectangular array, with four sets of eight members each. You can think of that array as having a set of eight
members on one side and a set of four members on the other side. You can show any product in this way, using an array of dots. If you show \(3 \times 9\), for instance, you would draw 3 dots in each column, and nine dots in each row, as follows:

\[
\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}
\]

Exercise 10-7a

Show these products as arrays of dots.

1. \(2 \times 9\) 2. \(4 \times 3\) 3. \(9 \times 7\) 4. \(1 \times 6\)

10-8 Multiplication as mixing sets

An interesting way to look at this is in terms of a problem like the following.

Flumo and Sumo enter a restaurant, which is serving jolof rice, fufu and soup, or rice and palm butter. What are the different possible ways of pairing boys and foods? You could make a chart showing this as follows.

<table>
<thead>
<tr>
<th></th>
<th>jolof rice</th>
<th>fufu and soup</th>
<th>rice and palm butter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flumo</td>
<td>(Flumo, jolof rice)</td>
<td>(Flumo, fufu and soup)</td>
<td>(Flumo, rice and palm butter)</td>
</tr>
<tr>
<td>Sumo</td>
<td>(Sumo, jolof rice)</td>
<td>(Sumo, fufu and soup)</td>
<td>(Sumo, rice and palm butter)</td>
</tr>
</tbody>
</table>

This chart can be made simpler in this way,

\[
\begin{array}{ccc}
\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
\end{array}
\]

Thus you can easily see that it is equivalent to an array with 2 dots in each column and 3 dots in each row.
What you are doing in problems like this is mixing one set with another set, so that you get a new set whose members are pairs of members, one from each of the original sets. Moreover, such sets of pairs are equivalent to arrays of dots. In this problem the two sets are the set of boys and the set of foods. The members of the new set are all the possible combinations of boys and goods. You can see that in this case the first boy has three choices and the second boy has the same three choices. Thus there are 2 sets of 3 possible choices, so that you can write the equation for this as

\[ 3 + 3 = \square \]

You can see that this gives rise to the same type of multiplication equation you found before, in the same way,

\[ 2 \times 3 = \square \]

It is easy to see that the number of pairs in such a set is found by multiplying the number of members in the first set and the number of members in the second set. You can see this by thinking of the equivalent array of dots. Sometimes it is easier to think of multiplication in terms of repeated union of sets, and at other times it is easier to think of it in terms of mixing sets. The two ways give the same result, and so you should learn both. And, what is more, you can use both in teaching children.

**Exercise 10-8a**

Show each of the following multiplication equations by drawing two sets and mixing them as shown above. The first one is done for you.
1. \(2 \times 4 = 8\)

2. \(5 \times 7 = 35\)

3. \(1 \times 8 = 8\)

4. \(5 \times 3 = 15\)

5. \(2 \times 0 = 0\)

6. \(7 \times 1 = 7\)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>((a, 1))</td>
<td>((a, 2))</td>
<td>((a, 3))</td>
<td>((a, 4))</td>
</tr>
<tr>
<td>b</td>
<td>((b, 1))</td>
<td>((b, 2))</td>
<td>((b, 3))</td>
<td>((b, 4))</td>
</tr>
</tbody>
</table>

Here \(a\) and \(b\) name the members of the first set, and 1, 2, 3, 4 name the members of the second set.

Exercise 10-8b

For each of the above problems, make up a story which shows the idea of mixing sets in a way which can be taught to primary children.

Exercise 10-8c

Every road from Aras to Cona passes through Buka. If there are 5 routes from Aras to Buka and 7 routes from Buka to Cona, how many routes are there from Aras to Cona?

10-9 Multiplication table

You are now ready to work out the whole multiplication table for yourself. At first, take only numbers from 0 to 9. You will have a chance, later on, to work with products of numbers which are themselves greater than 9. But this needs special methods, which are better learned by themselves. You already know the multiplication table, since you learned it when you were a school child yourself. Maybe you never thought much about it at the time, since your teacher might not have told you what it meant and how to work it out. But now that you know the way to work it out, you should go through the whole thing for yourself, find each answer, and put it in the proper place in the table. Remember that there are several ways of finding the product of two
numbers. Repeated union of sets, repeated addition of numbers, arrays of dots, and mixing of sets, are some of these. Any way you do it is all right, just so long as you satisfy yourself that the answers you learned as a child are right. In this way, you will be getting yourself ready to teach children what multiplication really means. Thus your classes will never sing the multiplication tables without knowing what the words mean! In doing this work, set up a table like the one below. One example -- to find $6 \times 4$ -- is given to you already, where the 6 in the column at the left is multiplied by the 4 in the row at the top. In every case, when you are multiplying $a$ by $b$, find $a$ in the left column and $b$ in the top row, and find the place where the row and column beginning at those numbers meet. Put the answer in that spot. In the example given, $a = 6$ and $b = 4$, and so 24 is put in the place where the proper row and column meet each other.

**Exercise 10-9a**

Complete the following multiplication table, using in turn all of the methods given in this section.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>24</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
10-10 Property of 1

You can see from this table that, for instance, $1 \times 3 = 3$ and $7 \times 1 = 7$. In general, $1 \times a = a$ and $b \times 1 = b$, where $a$ and $b$ are any whole numbers. You can see this by thinking of multiplication as repeated addition or in terms of mixing sets or in terms of arrays of dots. For instance, 1 set of 2 members has 2 members. Likewise, 2 sets of 1 member have 2 members.

Exercise 10-10a

Show by pictures these products.

1. $1 \times 9$
2. $16 \times 1$
3. $1 \times 45$
4. $10 \times 1$
CHAPTER 11

PROPERTIES OF MULTIPLICATION

11-1 A reminder of multiplication

You learned above that you can show children the meaning of multiplication in several ways. You can take the union of several disjoint sets, each with the same number of members. In this case, the product of the numbers is found by counting the union of the sets. This in turn gives the same result as adding the same number to itself several times. Another way of explaining multiplication is through the mixing of sets. The result is the same as in a repeated union, but the concrete interpretation is a bit different. Both ways are useful, and both ways help children really to understand multiplication.

Exercise 11-1a

1. If a pattern of dots consists of 3 rows of 6 dots each, how many dots are there in the pattern?

2. The sum of two numbers is 10 and their product is 24. What are the numbers?

3. Ama has 3 shillings and Jacob has 4 times as many. How many more shillings than Ama does Jacob have?
4. Find the difference between the product of the numbers 6 and 8 and their sum.

5. A merchant sold articles for 5 shillings which had cost him 3 shillings. If he sold 7 of these articles in one sale what would be his profit on this sale?

11-2 Introduction to Commutative property

You will use these ways of looking at multiplication to find out some important facts about products. Look back at the multiplication table you made, and think about it. Do you see anything special? Does one part of the table look the same as some other part? Look at the products 5 \times 7 and 7 \times 5, for instance. Do you see something special there? Of course you do. You see that they have the same product (35). Look at 2 \times 9 and 9 \times 2. You find the same thing again, i.e., they have the same product (18).

Exercise 11-2a

1. Go through the multiplication table and find all the pairs of numbers with the same product. Some of these pairs are like 5 \times 7 and 7 \times 5, and others are not. Mark pairs like 5 \times 7 and 7 \times 5 in a special way, and list them.

2. Go back to the section on addition, and find a property that you used there, which gave results very much like those in the exercise above. Give the name of that property, and write down a similar property for multiplication.
### 11-3 Generalized form of commutative property

If you did your work right in the last exercise, you used the commutative property of addition. You found a similar property, the commutative property of multiplication, which states that any pair like \(5 \times 7\) and \(7 \times 5\) gives the same result. You can write this, as you did for addition, as

\[a \times b = b \times a\]

where \(a\) and \(b\) are any whole numbers.

**Exercise 11-3a**

What does the commutative property of multiplication say when:

1. \(a = 3\) and \(b = 9\)
2. \(a = 5\) and \(b = 5\)
3. \(a = 0\) and \(b = 1\)
4. \(a = n\) and \(b = 3\)
5. \(a = n\) and \(b = m\)

### 11-4 Commutative property in terms of repeated union

You can see from the table the commutative property of multiplication for any one-digit number (that is, numbers from 0 to 9). But you would like to know it for all whole numbers. Of course, it must always be true, but it is not fair in mathematics to believe something without trying to think of how it can be understood. Think of two sets, each with a goat, a chicken and a cow. Then think of three sets, each with a goat and a chicken. Do the unions
in each case have the same number of members? You know that they do, and you know how to find out. Two sets have the same number of members if you can pair each member of either set with one and only one member of the other set. To start this problem, line the sets up like this:

```
<table>
<thead>
<tr>
<th>goat</th>
<th>goat</th>
</tr>
</thead>
<tbody>
<tr>
<td>chicken</td>
<td>chicken</td>
</tr>
<tr>
<td>cow</td>
<td>cow</td>
</tr>
</tbody>
</table>
```

```
<table>
<thead>
<tr>
<th>goat</th>
<th>goat</th>
<th>goat</th>
</tr>
</thead>
<tbody>
<tr>
<td>chicken</td>
<td>chicken</td>
<td>chicken</td>
</tr>
</tbody>
</table>
```

It is easy to pair the members of the first set with the members of the second set. Probably the best way to do it is to pair the two goats of the first set with the first goat and chicken of the second set, the two chickens of the first set with the second goat and chicken of the second set, and the two cows of the first set with the third goat and chicken of the second set. Your pairing would look like this.

```
<table>
<thead>
<tr>
<th>goat</th>
<th>goat</th>
</tr>
</thead>
<tbody>
<tr>
<td>chicken</td>
<td>chicken</td>
</tr>
<tr>
<td>cow</td>
<td>cow</td>
</tr>
</tbody>
</table>
```

```
<table>
<thead>
<tr>
<th>goat</th>
<th>goat</th>
<th>goat</th>
</tr>
</thead>
<tbody>
<tr>
<td>chicken</td>
<td>chicken</td>
<td>chicken</td>
</tr>
</tbody>
</table>
```

**Exercise 11-4a**

Show in the same way, using a drawing, that the union of 5 sets of 7 houses has the same number of members as the union of 7 sets of 5 houses.

Now, if you think about it, you can see why \( a \times b = b \times a \) is true for any whole numbers \( a \) and \( b \).
Exercise 11-4b

Show in this same way that these statements are true:

1. \[3 \times 5 = 5 \times 3\]
2. \[6 \times 9 = 9 \times 6\]
3. \[1 \times 8 = 8 \times 1\]

11-5 Commutative property in terms of mixing sets

You can see the same thing by using mixing of sets instead of the repeated union of sets. In this case it is even easier to see the argument.

For instance, mix a set of two boys and a set of three girls. This is the set of pairs of boys and girls, as follows:

Now take the product the other way, that is, the product of a set of three girls and a set of two boys. You can draw it this way.
You can see that the pairs in the first product have the boy first and the girl second, and that the pairs in the second product have the girl first and the boy second. Otherwise, except for their place in the picture, the pairs are exactly alike. This is just as true in the case where the two sets have $a$ and $b$ members. The pairs in the set product which gives $a \times b$ will be just the same as the pairs in the set product which gives $b \times a$, except that they will be turned around. So you can see in this way that $a \times b = b \times a$.

Exercise 11-5a

Three Nigerians, three Ghanaians, three Liberians and three Sierra Leoneans met. Show this as the mixing of a set of the four countries and a set of three persons. Hint: use $N$, $G$, $L$ and $S$ as letters for the countries, and 1, 2 and 3 as numbers for the persons in each set. Show that the result is the same if three sets, each containing a Nigerian, a Ghanaian, a Liberian,
and a Sierra Leonean, meet. Show in this way, following the method given above for mixing sets, that $4 \times 3 = 3 \times 4$.

11-6 Commutative property in pictures

You can help children to understand the commutative law of multiplication by using objects. You can bring to your class 3 sets of 5 blocks, and put them together to look like this:

Now take 5 sets of 3 blocks, and put them together in the other way, to look like this:
Now you can see by simple pairing that the two figures have the same number of blocks. This method will always work, as you can plainly see.

Exercise 11-6a

Make a drawing, showing the same thing for $2 \times 9$ and $9 \times 2$. Give other examples useful for explaining the commutative property of multiplication to children.

11-7 Introduction to associative property

Now look back again to the section on the properties of addition. Do you find another property there which, like the commutative property, might suggest a similar property for multiplication? Yes, you do, if you look hard. You find the associative property of addition, which says that:

$$(a + b) + c = a + (b + c) \text{ (for all whole numbers } a, b \text{ and } c.)$$

You remember that this property tells you that if you add three numbers, adding first either the first two or the last two you will get the same result. You remember that addition combines only two numbers at a time. So this is a very useful property, since it extends addition to three numbers by starting with two and then adding the third to the result in the proper order. Now see what the same property would be for multiplication. It would read as follows:

$$(a \times b) \times c = a \times (b \times c) \text{ (for all whole numbers } a, b \text{ and } c.)$$

You should check some special cases to see if this property may hold. Let $a = 2$, $b = 3$ and $c = 2$. You will get

$$(2 \times 3) \times 2 = 2 \times (3 \times 2)$$
If you check it, you will find that this is true.

**Exercise 11-7a**

Check the following statements, to see if both sides give the same results:

1. \((1 \times 2) \times 4 = 1 \times (2 \times 4)\)

2. \((3 \times 0) \times 5 = 3 \times (0 \times 5)\)

3. \((4 \times 2) \times 4 = 4 \times (2 \times 4)\)

**Exercise 11-7b**

Make up word problems for your primary class to show the meaning of the products in the exercise above. For example, \((1 \times 2) \times 4\) can be thought of as \((1 \times 2)\) sets of four things, thinking of multiplication as repeated union. \((1 \times 2)\) can itself be thought of as one set of two things. This problem can be illustrated through the following example. A man is building a wall with concrete blocks. Each section is four blocks long, and is built of the number of such four-block units given by the fact that the wall is one block deep and two blocks high. Thus each section of the wall has \((1 \times 2)\) four block units, and thus has \((1 \times 2) \times 4\) blocks in it. Give word problems for each of the other products. Here is a picture of a section of this wall.
11-8 Example of associative property

You have found that this property holds in each case you have tried, but you do not yet know whether it always holds. You would like to know this fact, but it might seem hard for you to see it. Actually it is easy, but it takes a lot of words. You can do it in either of the two ways you understood the commutative property for multiplication. You can think of multiplication as repeated addition, or you can think of it as coming from the mixing of sets. Both ways are shown by the example in the exercise above of the wall built of sections made of $(1 \times 2) \times 4$ concrete blocks. You can think of that wall as built of sections of $1 \times (2 \times 4)$ blocks. In this case the $(2 \times 4)$ means that there are two sets of four blocks each, or the product of two blocks by four blocks. Then taking $1 \times (2 \times 4)$ means either to take one set of $(2 \times 4)$ blocks or to mix the set of one block and the set of $(2 \times 4)$ blocks. In either way you will find that $(1 \times 2) \times 4$ blocks is the same as $1 \times (2 \times 4)$ blocks.
Exercise 11-8a

Show in the ways given above that the following statements are correct:

1. \((3 \times 1) \times 5 = 3 \times (1 \times 5)\)

2. \((4 \times 2) \times 4 = 4 \times (2 \times 4)\)

3. \((7 \times 1) \times 8 = 7 \times (1 \times 8)\)

Exercise 11-8b

Try to give a general argument for the associative property of multiplication, in either of the two ways suggested above.

Exercise 11-8c

1. Give three whole numbers whose sum is 6 and whose product is 6.
   Can you think of any other numbers for which this can be done?
   (You can find the answer at the end of the next section.)

2. Give four whole numbers whose sum is 8 and whose product is 8 (they need not all be different numbers.) Can you think of any other numbers for which this can be done? (You can find the answer at the end of the next section.)

11-9 Associative property in pictures

You can draw pictures to show the associative property. For instance, you can draw the following picture of \((3 \times 2) \times 4\), where \((3 \times 2)\) is shaded:
and the following picture of $3 \times (2 \times 4)$, where $(2 \times 4)$ is shaded:

![Diagram of 3x2x4 blocks with the 2x4 part shaded]

It is easy to see that both pictures have the same number of blocks. You can use sets of blocks in this way in teaching children.

**Exercise 11-9a**

Draw pictures like those above for the products given in Exercise 11-8a of the preceding section.

**NOTE:** Answers for Exercise 8c: (1) 1, 2, 3; no others possible. (2) 1, 1, 2, 4; no others possible.

**11-10 Introduction to distributive property**

There is one last property, which combines multiplication and addition. Think of a class which has 3 girls and 4 boys. Then think of 2 such classes, each with 3 girls and 4 boys. How many children would there be altogether in the 2 classes? You could do it in two different ways. You could find the total
number of girls (which is 6) and the total number of boys (which is 8) and 
add them to get 14 children. Or you could find the total number of children 
in one class (which is 7) and multiply it by 2 to get the same 14.

**Exercise 11-10a**

Find out by working the problems if you get the same answer on both 
the right and the left hand sides.

1. \(1 \times (3 + 2) = (1 \times 3) + (1 \times 2)\)
2. \((4 \times 1) + (4 \times 2) = 4 \times (1 + 2)\)

**Exercise 11-10b**

Write word problems which you could use with primary school children 
for each of the problems in the exercise above.

**11-11 Distributive property in pictures**

You can understand this property in the same two ways as you under­
stood the commutative property for multiplication. However, again it would 
take many words. If you would like to exercise your brain, it would be good 
for you to try to write out an argument. It's not too hard. But, whether or 
not you try it, you can see that it is true with particular numbers by drawing 
a picture. For example, you can show \((4 \times 1) + (4 \times 2)\) as follows:
You can also show $4 \times (1 + 2)$ as follows:

It is easy to match the two sets of dots and see that they have the same number of members.

It is also useful in your class to show this property by using blocks, as in these two pictures.
11-15

Exercise 11-11a

Draw arrays of dots and pictures for these problems:

1. \(1 \times (3 + 2) = (1 \times 3) + (1 \times 2)\)
2. \(3 \times (3 + 3) = (3 \times 3) + (3 \times 3)\)
3. \((4 \times 2) + (4 \times 0) = 4 \times (2 + 0)\)

11-12 Generalized form of distributive property

This property can be written as follows:

\(a \times (b + c) = (a \times b) + (a \times c)\) (for all whole numbers \(a, b\), and \(c\)).

It is called the distributive property, and says that if you multiply one number and the sum of two other numbers, you get the same result as if you multiplied that number by each of the other two numbers, and then added the two results.

Exercise 11-12a

Tell what the distributive property says in the following cases:

1. \(a = 1, \; b = 2, \; c = 3\)
2. \(a = 0, \; b = 5, \; c = 4\)
3. \(a = x, \; b = 2, \; c = 3\)

Exercise 11-12b

1. Is \((2 + 3) \times (2 + 3)\) equal to \((2 \times 2) + (3 \times 3)\)? Explain.
2. What numeral must be put into each box to make \(3 \times [\quad] + 2 \times [\quad] = 35\) into a true statement?
3. A newspaper boy sold on a certain day 30 newspapers and on another day 60. If the cost of a newspaper is 2 pence, which of the following statements can be used to determine his total sales on the two days?

- $60 \times 4$ pence
- $30 \times 4$ pence
- $(30 \times 2) + (60 \times 2)$ pence
- $(30 + 60) \times 4$ pence
- $(30 + 60) \times 2$ pence
- $2 \times (60 + 30)$ pence

4. A group of children were divided into 4 teams of 3 children each. Each person on each team was assigned one of the numbers 1, 2, and 3 so that each of these numbers appeared on each team. What was the total sum of all the numbers assigned? Can you think of several ways to work this problem?

5. Towns A, B and C lie on a straight road with B between the other two. The distance from A to B is 5 miles and the distance from B to C is 4 miles. The distance from another town D to town A is 3 times the distance from town A to town C. What is the distance between town A and D? Illustrate the distances mentioned by means of a picture.

6. Find the number $(2 + 3) \times (4 + 5)$. Can you think of three or more different ways to do this? In each case state the properties of the operations you use.
7. If a set with 15 members is separated into 3 disjoint subsets, so that each successive subset has one more member than the preceding subset, what is the number of members in each of these subsets?

(Make up another exercise like this one which you could use with pupils in your classes, in which the number of members in the total set is different from 15.)
CHAPTER 12

DIVISION

You remember from your own school days that division and multiplication are closely related. Maybe you never knew why, but you must have noticed that if you multiplied, for instance, $2 \times 3$ to get 6, then you could divide the result 6 by 2 and get back to 3 again. There is a reason for this and you are going to look at the reason.

12-1 Reminder of multiplication in terms of union of equivalent sets

First think of what you do when you multiply. For example, you can take a set of two members and then take another set of two members and then a third set of two members and form their union by putting them all together. At first you thought of this as repeated addition and wrote it as $2 + 2 + 2 = 6$ but later learned to write it more briefly as $3 \times 2 = 6$ and to call this multiplication. Similarly $4 + 4 + 4$, which you first knew as the union of 3 sets of 4 things, was later written as $3 \times 4$.

If you forget a multiplication fact such as $6 \times 9 = 54$ you can work it out because you know that $6 \times 9 = 9 + 9 + 9 + 9 + 9 + 9$ and you can add the nines
to find their sum. When we multiply two numbers we find their product. The numbers which are multiplied are called factors of the product.

**Exercise 12-1a**

Write the products in the boxes in these equations. Show them also as addition equations.

1. $3 \times 5 = \underline{15}$
2. $4 \times 3 = \underline{12}$
3. $9 \times 6 = \underline{54}$
4. $6 \times 9 = \underline{54}$
5. $7 \times 8 = \underline{56}$
6. $8 \times 5 = \underline{40}$
7. $9 \times 8 = \underline{72}$
8. $8 \times 9 = \underline{72}$
9. $7 \times 7 = \underline{49}$

**12-2 Tables of multiplication facts**

You have memorized many multiplication facts and so do not need to work them out by repeated addition. It is necessary for your pupils to know these facts very thoroughly in order to be able to do arithmetic quickly. It is even more important that they shall understand how these multiplication tables are built up and so be able to work them out for themselves. Each table can be written in two ways:

<table>
<thead>
<tr>
<th>The table of &quot;twos&quot;</th>
<th>The &quot;2-times&quot; table</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \times 2 = 2$</td>
<td>$2 \times 1 = 2$</td>
</tr>
<tr>
<td>$2 \times 2 = 4$</td>
<td>$2 \times 2 = 4$</td>
</tr>
<tr>
<td>$3 \times 2 = 6$</td>
<td>$2 \times 3 = 6$</td>
</tr>
</tbody>
</table>
Exercise 12-2a

Write out the "fives" and "5-times" tables for the numbers from 0 to 9.

12-3 Tables of constant products

Now we will look at a different way of grouping these tables. Can you find the set of all the multiplication facts where the product is 8? Here is one such product: \( 2 \times 4 = 8 \). You can do this in two ways:

**Method 1. Using sets of things.**

Make up a set of things whose number is the product you want, for example 8 stones. Arrange them on the table in rows having the same number of stones in each row. You might have any of these arrangements:

- 4 rows with 2 stones in each
  
  0 0 0 0
  
  0 0 0 0
  
  0 0 0 0
  
  0 0 0 0
  
  \( 4 \times 2 \)

- 2 rows with 4 stones in each
  
  0 0 0 0
  
  0 0 0 0
  
  \( 2 \times 4 \)

- 1 row with 8 stones
  
  0 0 0 0 0 0 0 0
  
  \( 1 \times 8 \)

- 8 rows with 1 stone in each
  
  0
  
  0
  
  0
  
  0
  
  0
  
  0
  
  0
  
  \( 8 \times 1 \)
So we can write the table which shows the pairs of numbers whose product is 8: \(8 = 1 \times 8; 8 = 2 \times 4; 8 = 4 \times 2; 8 = 8 \times 1\). Note that in this list the first factors in each pair are in order of increasing size.

**Exercise 12-3a**

Using sets of objects, arrange them in rows to show all the pairs of factors which give as their product 6, 7, 9, 12, 20. Write out the table for each product as we did for 8.

**Method 2. Using multiplication facts.**

You do not need to work out all these pairs with sets. You know them already. If you want to be quite sure that you have written every one of them you can look at the multiplication table you made out in a big square. Write down the multiplication facts where the product is 6. You will have, \(6 = 1 \times 6, 6 = 2 \times 3, 6 = 3 \times 2, 6 = 6 \times 1\).

**Exercise 12-3b**

Write down the table for all pairs of factors whose product is 16, 18, 23, 36, 40, 56.

There are many problems which show this kind of multiplication. You might, for example, have a class of 27 children and you might want to arrange the desks in your classroom so that you have them in rows of equal length. If you think about it you will see that one way is to have 3 rows with 9 desks in each row,
because \( 3 \times 9 \) is one of the pairs which make 27. You could have worked out this problem by arranging 27 stones in 3 rows, or you could have looked at your multiplication table to find pairs of numbers whose product is 27.

**Exercise 12-3c**

If there are 18 children in the Primary I class at your school, and you want them to march in a parade, in how many different ways can you arrange them in rows of equal length? Write 18 as a product of factors to show all these different ways.

**12-4 Division as finding the missing factor**

It may be that in some cases one of the pair of factors in a certain product is already fixed. For example, there might have to be 6 children in each row in the parade above. We can make the equation like this: \( \square \times 6 = 18 \) and we know that 3 is the numeral to be put into the box. So we must make 3 rows of children with 6 in each row.

Here is another example: Think of a hand of bananas with 14 bananas on it. You want to give each man who is working on your farm 2 bananas. In this case each subset must have 2 bananas. You would like to know the number of such subsets you can have, so you can know the number of men to whom you can give 2 bananas each. You write the equation \( \square \times 2 = 14 \) and find that 7 men can each receive 2 bananas.
In these problems we have to find the missing factor. You will remember that when we have to find a missing addend in an addition equation we are doing subtraction and we write the equation to show this. For example: \(3 + \square = 5\) becomes \(5 - 3 = \square\). When we find the missing factor in a multiplication equation, we are doing DIVISION. The sign for division is "\(\div\)" and is read "divided by". \(\square \times 2 = 14\) may be written as \(14 \div 2 = \square\).

In this example the missing factor is, of course 7. We call 7 the QUOTIENT of \(14 \div 2\). In other words, the quotient of \(a \div b\) is the number which makes \(\square \times b = a\) into a true equation.

**Exercise 12-4a**

1. I am thinking of a number. Three times this number is 18. What is the number?
2. What is the number which, when it is divided by 5 and multiplied by 3, gives 6?
3. What is the number which, when divided by 3 three times in succession, gives 2?

**12-5 Division as grouping or as sharing**

Division can be thought of in the following way. Suppose you have a set
of things you wish to break up into subsets each with the same number of members. How many such subsets can you get? To put this in another way, if there are \( n \) members of a set and \( m \) members in each subset, how many such subsets can be found? This can be written \( \square \times m = n \) or \( n \div m = \square \). This kind of division problem is sometimes called "grouping" because we group the members of the set into equivalent subsets.

**Exercise 12-5a**

Write each of these multiplication equations as a division equation. Then find each quotient.

Example: \( 24 = \square \times 4 \)

<table>
<thead>
<tr>
<th>Example</th>
<th>Equation</th>
<th>Quotient</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. 72 = \square \times 8</td>
<td>5. ( 8 = \square \times 1 )</td>
<td></td>
</tr>
<tr>
<td>2. 56 = \square \times 7</td>
<td>6. ( 28 = \square \times 4 )</td>
<td></td>
</tr>
<tr>
<td>3. 42 = \square \times 6</td>
<td>7. ( 18 = \square \times 18 )</td>
<td></td>
</tr>
<tr>
<td>4. 21 = \square \times 3</td>
<td>8. ( 45 = \square \times 5 )</td>
<td></td>
</tr>
</tbody>
</table>

This way of thinking of division helps young children to understand it and gives them a way to work out the answer. For example if a child wants to find an answer to \( 42 \div 6 = \square \), he can change it to \( \square \times 6 = 42 \) and he then has to find how many subsets with 6 members each he can make from 42. If he knows the multiplication table of sixes he knows that \( 7 \times 6 = 42 \) and gets the quotient 7 quickly. But if he does not know this then he can take 42 stones and put them into subsets of 6 to find how many subsets can be made.
Exercise 12-5b

1. A man had 15 bags of rice on his farm, which was away from the motor road. He had 5 workers who could each carry one bag of rice. How many trips would the 5 workers have to make to get all the rice to the road?

2. If each man can head-load four cement blocks, how many men are needed to head-load 32 cement blocks to a construction job in the bush?

Division can also arise in the following way. You have 21 copybooks for your family of 7 children. How many can they each have if the copybooks are shared equally? Here we break up a set of objects into a given number of equivalent subsets and have to find how many members there are in a subset.

In this case each of the 7 children will receive 3 copybooks.

\[
21 = 7 \times 3 \\
21 \div 7 = 3
\]

One practical way to find the answer to this problem is to hand the copybooks out to the children, one by one, until they are finished. You will go around the circle of children 3 times. You will "share" the copybooks among the seven children. This is why this kind of division problem is sometimes called "sharing."

Exercise 12-5c

If you have 24 oranges to share equally among 6 people how many oranges will they each receive? Write the multiplication and division equations, and write the answer in words in a sentence.
In such cases division can be described in this way. If you have a set of $n$ members which must be broken up into $m$ equivalent subsets, how many members are there in each subset? The equation can be written $n = m \times \square$ or $n \div m = \square$

It is good for young children to do division in this way also. If they do not see how to find the answer to the problem about the oranges they can take 24 stones and put them out into 6 piles, one by one.

Both these ways of division give the same answer. If we have 6 bananas we can divide them in 2 ways.

1. How many children can have 2 bananas each?

   The answer is 3 children.

   \[ 6 \div 2 = 3 \]

2. Share 6 bananas equally between 2 children.

   The answer is 3 bananas each.

   \[ 6 \div 2 = 3 \]
Both these ways of division give the same number as the answer which can be written \( n \div m = \square \) and this equation can have come from either of two equations

\[
\begin{align*}
  n &= \square \times m \\
  n &= m \times \square
\end{align*}
\]

**Exercise 12-5d**

1. If 30 pieces of candy are to be shared equally among 6 children, what is each child's share?
2. A farmer has 20 acres of crops to be harvested and he and his helpers have allowed 5 days to do the harvesting. If the farmer and his helpers work equally hard each day, how many acres should he plan to harvest each day?
3. The cost of sending a telegram is 1 shilling for the first 9 words or part thereof, and 1 penny for each additional word. How much will it cost a man to send a telegram of
   \[a. \text{18 words?} \]
   \[b. \text{12 words?} \]
   \[c. \text{8 words?} \]
   \[d. \text{x words?} \]

**12-6 Remainders**

Sometimes we have a division problem which cannot be answered with a whole number. For example, if you wish to arrange 27 children in rows of 8
you cannot do it. There will be 3 rows of 8 children and 3 children left over. You remember that we call this a remainder of 3. A set of 27 members can be separated into 3 subsets of 8 and a subset of 3. We can write this as

\[ 27 = (3 \times 8) + 3. \]

The remainder is 3. Your pupils should answer the problem in words as follows: There will be 3 rows of 8 children and 3 children left over. There is no answer to \( 27 \div 8 \) in the set of whole numbers.

Again we might have the problem: How many oranges will there be for each of 4 people if there are 25 oranges to share among them? We can see that there will be 6 each and 1 orange left over. Again we could write this

\[ 25 = (4 \times 6) + 1 \]

but also as

\[ 25 = (6 \times 4) + 1. \]

Your pupils should write the answer in a sentence explaining that there will be 1 orange left over.

In these two examples we speak of division with a remainder, since there is no whole number which is the quotient of \( 27 \div 4 \). The remainder in the first example is 3 because \( 27 = (3 \times 8) + 3 \), and in the second the remainder is 1 because \( 25 = (6 \times 4) + 1 \).

If there is no whole number which will make the equation \( n \div m = \) true, we say that there is no quotient in the set of whole numbers.
Exercise 12-6a

Show each of the following division equations as multiplication equations and give if possible the whole number quotient to each.

1. \(48 \div 8 = \square\)  
2. \(65 \div 9 = \square\)  
3. \(47 \div 5 = \square\)  
4. \(63 \div 9 = \square\)  
5. \(63 \div 8 = \square\)

Exercise 12-6b

Bankole has 45 shillings which he wants to share among his five brothers. How much will each receive?

Exercise 12-6c

Bankole has 28 shillings to buy petrol. How many gallons can he buy if petrol costs 4 shillings a gallon?

Exercise 12-6d

Solve these problems, if possible, by finding a whole number which makes the equation true.

1. \(54 \div 9 = \square\)  
2. \(3 \times \square = 24\)  
3. \(\square \div 8 = 35\)  
4. \(23 \div 7 = \square\)  
5. \(\square \times 4 = 36\)  
6. \(6 \times \square = 16\)

Exercise 12-6e

1. If a number is divided by 12 the quotient is 6 and the remainder is 3, what is the number?
2. John has 23 shillings and toy dogs cost 5 shillings each. If John buys as many toy dogs as he can, how many does he buy and how many shillings does he have remaining?

12-7 Division as inverse of multiplication

It should be clear to you now that multiplication and division are very closely related to each other. The division equation $n \div m = \square$ corresponds to either of the two multiplication equations.

\[
\begin{align*}
    n &= \square \times m \\
    n &= m \times \square.
\end{align*}
\]

Thus division finds one of the factors in a product if the product and the other factor are known. It does it by dividing the product by the known factor giving the missing factor as the answer. The missing factor is thus the quotient.

You should be saying to yourself by this time that division is just the opposite of multiplication. Multiplication builds up a product from two numbers. Division breaks down that product again.

For instance we know that $7 \times 9 = 63$, because 7 sets each with 9 objects make a total of 63 objects. We also know that if we divide a set of 63 objects into sets of nine objects each, we will get 7 such sets. From this situation we can see the following facts. If we begin with 7, multiply by 9 and then divide the product by 9, we get back to 7 again. We can write this

\[
\begin{align*}
    7 \times 9 &= 63 \\
    63 \div 9 &= 7 \\
    (\text{therefore}) \quad (7 \times 9) \div 9 &= 7
\end{align*}
\]
so you can see that division undoes what multiplication does. We say that division is the inverse operation to multiplication. On the other hand, if we begin with 63, first divide by 9, and then multiply the quotient by 9, we again get back to the number with which we began, in this case 63.

\[(63 \div 9) \times 9 = 63\]

So multiplication builds up what division has broken down and we can also think of multiplication as the inverse operation to division.

Whatever pairs of factors we choose we can always make similar multiplication and division statements. We can show this using \(m\) and \(n\):

\[(m \times n) \div n = m\]
\[(m \div n) \times n = m\]

From these equations you can see that multiplication and division do opposite things. This is why each is said to be the inverse of the other.

**Exercise 12-7a**

Make sets of equations, similar to those above, with:

1. \(m = 3\) \(n = 5\)
2. \(m = 7\) \(n = 8\)
3. \(m = 10\) \(n = 4\)
4. \(m = 9\) \(n = 6\)

**12-8 Division by zero**

Suppose you were asked to solve the division problem \(2 \div 0 = \square\)

What would you do? You would write the corresponding multiplication equation:

\[\square \times 0 = 2\]
But there is no numeral which can be put in the box to make this equation true. Why not? Because \( \square \times 0 = 0 \) is an identity (true for any numerals put in the box) and \( \square \times 0 \) cannot be both 2 and 0.

**Exercise 12-8a**

Show in the same way that it is impossible to solve any of the following division problems.

\[
1 \div 0 \quad 3 \div 0 \quad 4 \div 0 \quad 5 \div 0
\]

Could we perhaps find \( 0 \div 0 \)? That is, could we fill the box in

\[
0 \div 0 = \square
\]

by a numeral so as to make the equation true? The corresponding multiplication equation is:

\[
0 = \square \times 0
\]

But this is true for all numerals. To be able to get every answer is almost as bad as to get no answer at all. If we use \( 0 \div 0 \) we would like to have it stand for some particular number.

In mathematics we agree to say that division by zero is impossible.
CHAPTER 13

SUMMARY OF PROPERTIES OF ADDITION AND MULTIPLICATION

13-1 Commutative property of addition and multiplication

You found out when you were doing addition that if you had to add any two whole numbers, the order in which the operation is done does not affect the answer. For example, $8 + 3 = 3 + 8$. This illustrates the commutative property of addition, which, put in generalized form and using $a$ and $b$ for any whole numbers, reads thus:

$$a + b = b + a.$$ 

You will remember that multiplication also has this same property; that the order in which any two whole numbers are multiplied does not affect the product. For example, $5 \times 7 = 7 \times 5$. Using $a$ and $b$ for any whole numbers:

$$a \times b = b \times a.$$ 

From the above, this question follows naturally. "Do subtraction and division have this commutative property?" You can easily answer this for yourself by using any two different numbers you wish. For example, is $6 - 2 = 2 - 6$ or is $6 \div 2 = 2 \div 6$? Take some more whole numbers and try them and then state your conclusion in a sentence. You will notice that $a - b = b - a$ if $a = b$. For example, $6 - 6 = 6 - 6$. We do not on this account say that subtraction is commutative. If subtraction were commutative, we should have $a - b = b - a$.


for all pairs of whole numbers and this is not the case. In fact if \( a \neq b \) (\( a \) not equal to \( b \)) only one of \( a - b \) and \( b - a \) is a whole number. Similarly for division, if \( a = b \), \( a \div b = b \div a \). This does not make division commutative. If division were commutative we would require that \( a \div b = b \div a \) for every pair of whole numbers. For \( b \neq a \) only one of \( a \div b \) and \( b \div a \) is a whole number.

13-2  **Associative property of addition and multiplication**

You have discovered that in adding any three whole numbers the manner of grouping the numbers does not affect that sum. For example, if we have to add 2, 3 and 5, then we know that \( 2 + (3 + 5) = (2 + 3) + 5 \). This illustrates the associative property of addition. Put in general form it reads thus:

\[
a + (b + c) = (a + b) + c
\]

where \( a, b \) and \( c \) represent any whole numbers.

You ought to remember that multiplication also has this same property, namely, that no matter how you group three whole numbers which are to be multiplied together, the product will be the same. For example \( 2 \times (5 \times 3) = (2 \times 5) \times 3 \).

Using \( a, b \) and \( c \) to represent any whole numbers, this property of multiplication can be stated thus:

\[
a \times (b \times c) = (a \times b) \times c.
\]

You will also be interested to know whether this associative property holds for subtraction and division. This you can discover for yourself by using
three whole numbers.

For example: Is it true that \((8 - 4) - 2 = 8 - (4 - 2)\) and 
\((8 \div 4) \div 2 = 8 \div (4 \div 2)\)?

Obviously neither of these is true, and we conclude that subtraction and division do not have the associative property.

**Exercise 13-2a**

Give the whole number answer, if any, in each of the sentences below. In each pair, what evidence do the answers give about the associative property for the operation involved?

(a) \(3 + (2 + 4) = \) \( (3 + 2) + 4 = \)
(b) \(5 \times (4 \times 2) = \) \( (5 \times 4) \times 2 = \)
(c) \(4 \div (2 \div 1) = \) \( (4 \div 2) \div 1 = \)
(d) \(8 - (7 - 1) = \) \( (8 - 7) - 1 = \)
(e) \(7 + (6 + 5) = \) \( (7 + 6) + 5 = \)
(f) \(12 \div (4 \div 2) = \) \( (12 \div 4) \div 2 = \)
(g) \(10 - (5 - 3) = \) \( (10 - 5) - 3 = \)
(h) \((8 - 5) - 2 = \) \( 8 - (5 - 2) = \)
(i) \((12 \div 6) \div 2 = \) \( 12 \div (6 \div 2) = \)

**13-3 The distributive property**

This property states in general form that if \(a, b\) and \(c\) are whole numbers then 
\[ a \times (b + c) = (a \times b) + (a \times c). \]
For example, when \( a = 5, \ b = 4 \) and \( c = 2 \) this becomes

\[
5 \times (4 + 2) = 5 \times 4 + 5 \times 2.
\]

Multiplication is said to be distributive over addition. Is multiplication also distributive over subtraction?

For example: Is it true that \( 5 \times (4 - 2) = 5 \times 4 - 5 \times 2 \)? Yes, it is.

Examples like this lead us to believe that multiplication is indeed distributive over subtraction; in other words that

\[
a \times (b - c) = (a \times b) - (a \times c)
\]

for whole numbers \( a, b \) and \( c \). (Of course, we must assume also that \( c \) is not greater than \( b \), to make sure that \( b - c \) is a whole number.) We can see that this statement holds if we can show that the number \( a \times (b - c) \) makes the sentence

\[
\square + a \times c = a \times b
\]

true. That means we want to see whether it is true that

\[
\left[ a \times (b - c) \right] + a \times c = a \times b.
\]

The left-hand side can be written

\[
a \times \left[ (b - c) + c \right]
\]

(using the distributive property of multiplication over addition) and this is equal to

\[
a \times b.
\]

Therefore, we see that \( a \times (b - c) \) does make our sentence true, and so it is the case, as we guessed, that \( a \times (b - c) = (a \times b) - (a \times c) \).

Multiplication is therefore distributive over subtraction. Is it also
distributive over multiplication? Is $5 \times (4 \times 2) = (5 \times 4) \times (5 \times 2)$? Obviously not. Is multiplication distributive over division?

Is $5 \times (4 \div 2) = (5 \times 4) \div (5 \times 2)$?

Again obviously not. So multiplication is distributive over addition and subtraction only. These two distributive properties can be combined in the form:

$$a \times (b \pm c) = (a \times b) \pm (a \times c).$$

**Exercise 13-3a**

Give the whole number answer, if any, in each of the sentences below.

(a) $6 \times (3 + 2) = \square \quad 6 \times 3 + 6 \times 2 = \square$

(b) $8 \times (6 - 2) = \square \quad 8 \times 6 - 8 \times 2 = \square$

(c) $9 \times (6 \div 1) = \square \quad (9 \times 6) \div (9 \times 1) = \square$

(d) $12 \times (4 - 0) = \square \quad 12 \times 4 - 12 \times 0 = \square$

(e) $5 \times (5 + 5) = \square \quad 5 \times 5 + 5 \times 5 = \square$

(f) $5 \times (5 - 5) = \square \quad 5 \times 5 - 5 \times 5 = \square$

(g) $5 \times (5 \div 5) = \square \quad (5 \times 5) \div (5 \times 5) = \square$

(h) $5 \times (5 \times 5) = \square \quad (5 \times 5) \times (5 \times 5) = \square$

(i) $20 \times (8 \div 4) = \square \quad (20 \times 8) \div (20 \times 4) = \square$

Next you will now wonder whether division like multiplication is distributive over addition and subtraction, but not over multiplication and division. You can find these out yourself by considering each of these statements:
To test whether division is distributive over addition, test this statement:

\[20 \div (5 + 4) = (20 \div 5) + (20 \div 4)\]

To test whether division is distributive over subtraction, test this statement:

\[20 \div (5 - 4) = (20 \div 5) - (20 \div 4)\]

Similarly to test whether division is distributive over multiplication, test this:

\[20 \div (5 \times 4) = (20 \div 5) \times (20 \div 4)\]

Test whether \(20 \div (5 \div 4) = (20 \div 5) \div (20 \div 4)\) to test whether division is distributive over division.

You discover that each of the four statements above is false. Therefore, division is not distributive over addition, subtraction, multiplication, or division.

13-4 Properties of zero and one

You recall that 0 has the property that when it is added to any whole number, the sum is such as to leave the whole number unchanged. For example, \(0 + 5 = 5\) and \(5 + 0 = 5\). In fact, \(0 + a = a\) and \(a + 0 = a\), where \(a\) is any whole number. The similar property for multiplication is that when any whole number is multiplied by 1, the product is such as to leave the whole number unchanged. Thus \(1 \times a = a\) and \(a \times 1 = a\), where \(a\) is any whole number. To summarize these two properties, it is said that 0 is the identity element for addition and 1 is the identity element for multiplication.
This statement means that the result you obtain after adding 0 or multiplying by 1 is identically the same as the whole number you started with. Let's put these findings all together:

1. \(0 + a = a\) \(\quad a + 0 = a\)
2. \(1 \times a = a\) \(\quad a \times 1 = a\)

where \(a\) is any whole number.

You will now be wondering whether 0 and 1 act as identity elements for subtraction and division. Let us try to test this:

\[3 - 0 = 3 \text{ but } 0 - 3 \neq 3\]
\[\text{also } 3 - 1 \neq 3 \text{ and } 1 - 3 \neq 3\]

so neither 0 nor 1 is an identity element for subtraction. Again

\[3 \div 1 = 3 \text{ but } 1 \div 3 \neq 3\]

so 1 is therefore not an identity element for division.

There is another important property of 0. The product of any whole number and 0 is always 0; that is, if \(a\) is any whole number then \(a \times 0 = 0\) and \(0 \times a = 0\). In multiplication, therefore, 0 is as different from the identity element as possible.

Now let us consider the effect of 0 in division. What meaning can we attach to \(0 \div 3\) and \(3 \div 0\)? We will think of the answers as missing factors in multiplication equations.

Thus \(0 \div 3 = \square\) becomes \(0 = \square \times 3\) and we know that 0 is the only
number which we can put in the box to make this statement true. Therefore $0 = \square \times 3$ and so $0 \div 3 = 0$. Now $3 \div 0 = \square$ may be written as $3 = 0 \times \square$. What number can we put into the box to make the statement true? We know that the product of 0 and any number is 0, and so there is no whole number which will make this statement true. So we do not divide a number by 0 because as with $3 \div 0$ there is no whole number answer.

You see now that 0 does not act as identity element for division and so we can say that neither 0 nor 1 is an identity element for division. In fact, subtraction and division do not have any identity elements.

**Exercise 3-4a**

Complete the sentences by putting a whole number into each box, if possible.

1. $3 + 0 = \square$  
2. $3 \times 0 = \square$  
3. $3 - 0 = \square$  
4. $3 \div 0 = \square$  
5. $0 \times 5 = \square$  
6. $0 \div 5 = \square$  
7. $0 - 10 = \square$  
8. $100 - 0 = \square$  
9. $2 \times (16 \times 0) = \square$  
10. $(12 \div 3) \div 0 = \square$

**13-5 Properties summarized**

Here $a$, $b$, and $c$ stand for any whole numbers

**Commutative property of Addition**  
$a + b = b + a$

**Commutative property of Multiplication**  
$a \times b = b \times a$

**Associative Property of Addition**  
$a + (b + c) = (a + b) + c$

**Associative property of Multiplication**  
$a \times (b \times c) = (a \times b) \times c$
Distributive Property

\[ a \times (b + c) = (a \times b) + (a \times c) \]

**Additive Property of Zero**

- \[ a + 0 = a \]
- \[ 0 + a = a \]

**Multiplication Property of One**

- \[ a \times 1 = a \]
- \[ 1 \times a = a \]

**Multiplication Property of Zero**

- \[ a \times 0 = 0 \]
- \[ 0 \times a = 0 \]
14-1 Basic addition facts and related facts in column addition

In earlier work attention was given to the meaning of addition as an operation. Two whole numbers were selected and an operation was performed on them, and that operation yielded a third whole number, called the sum of the two numbers. The sum was shown in terms of an operation on sets. For example, the sum of 7 and 2 was found by choosing two disjoint sets, a set C of 7 objects and a set K of 2 objects, and finding their union. The sum of 7 and 2 is the number of objects in the union of sets C and K.

Addition is a commutative operation:

\[ 5 + 4 = 4 + 5, \]

and so on. In general, the order in which two numbers are added does not affect the sum. We say \( a + b = b + a \).

Addition is an associative operation. \( (97 + 75) + 25 = 97 + (75 + 25) \).

In general, the grouping of numbers in addition may be changed without changing the sum. We say \( (a + b) + c = a + (b + c) \).

The above discussion is concerned with the operation of addition. A clear distinction needs to be made between the operation and the various procedures for performing addition which you will study in this chapter. However, you will see that these procedures rely on your understanding of the number system to rename numbers and your ability to use such commutative and associative properties.
You have dealt with the basic facts of addition. An understanding of the
relations which exist among these basic facts should aid in learning new facts.
For example, when the child has mastered the basic fact \(5 + 3 = 8\), he can see
how this fact is related to other facts such as
\[
\begin{align*}
15 + 3 &= 18 \\
25 + 3 &= 28 \\
35 + 3 &= 38 \\
&\quad \vdots \\
95 + 3 &= 98 \\
&\quad \vdots \\
145 + 3 &= 148 \\
&\quad \vdots
\end{align*}
\]
and so on.

The mathematics is merely the application of the associative property as shown below:
\[
15 + 3 = (10 + 5) + 3 \\
= 10 + (5 + 3) \\
= 10 + 8 \\
= 18.
\]

It is important that the meaning be made clear. It is also necessary that
with much meaningful practice, the child should gain the ability to give auto-
matic responses to such facts as \(15 + 3 = 18\). Important applications of these
facts are found in column addition. If the child is adding \(8 + 7 + 3\) he needs to
be able to give automatic responses to the basic fact \(8 + 7 = 15\), and then to the
fact that $15 + 3 = 18$ which is related to the basic fact $5 + 3 = 8$.

Similarly, the basic facts with sums greater than 10 are useful in learning new facts. For example, when the child has mastered the basic fact $8 + 6 = 14$, he can see how this is related to other facts such as

\[
18 + 6 = 24 \\
28 + 6 = 34 \\
38 + 6 = 44 \\
\vdots \\
78 + 6 = 84 \\
\vdots \\
288 + 6 = 294 \\
\text{and so on.}
\]

Again, we see an application of the associative property

\[
18 + 6 = (10 + 8) + 6 \\
= 10 + (8 + 6) \\
= 10 + 14 \\
= 10 + (10 + 4) \\
= (10 + 10) + 4 \\
= 20 + 4 \\
= 24.
\]

It is important to note that the example above is not taught to children in the same way as the addition of two numbers each represented by a two-digit numeral. Rather we show children that the sum of $18 + 6$ will end in 4 because of the basic fact $8 + 6 = 14$. Next, because children understand the number
system, they can see that the sum \(18 + 6\) will be in the twenties. Therefore, the sum is 24.

It is essential that the meaning be made clear. It is also necessary that with much meaningful practice, the child should gain the ability to give automatic responses to such facts as \(18 + 6 = 24\). Unless he can do this, he is not ready for examples in column addition requiring this ability.

Look at this column addition: 

\[
\begin{array}{c}
9 \\
9 \\
6
\end{array}
\]

If a child adds downwards, he should give an automatic response to the basic fact \(9 + 9 = 18\), and then to the fact \(18 + 6 = 24\), which is related to the basic fact \(8 + 6 = 14\). When performing the addition, he will think: "9, 18, 24".

Although the examples illustrate addition downwards, the direction is not significant. In column addition it is desirable that children learn to add in both directions to check their work. Regardless of the direction in which pupils add, they will need automatic responses to the basic facts of addition and the related facts.

14-2 Addition of multiples of 10

In this unit we are mainly concerned with the addition of two numbers each of which is represented by a two-digit numeral. Let us first consider the addition of two numbers each of which is represented by a two-digit numeral ending in zero. Let us take the example \(30 + 20\) which may appear in vertical form thus:

\[
\begin{array}{c}
30 \\
+20
\end{array}
\]
Class activity will include revision of the meaning of numbers represented by two-digit numerals. The child should be able to count by tens with understanding. He represents 30 as 3 bundles of sticks with 10 sticks in each bundle. He checks by counting 10, 20, 30, which gives him 3 tens. Similarly, he represents 20 as 2 tens. With bundles of sticks the children discover that:

3 bundles of ten + 2 bundles of ten = 5 bundles of ten

3 tens + 2 tens = 5 tens

30 + 20 = 50.

It is important that children understand that:

3 tens is the same as thirty (30)

2 tens is the same as twenty (20)

5 tens is the same as fifty (50).

The problem should also appear in vertical form. This emphasizes the place value in the numeration system.

<table>
<thead>
<tr>
<th></th>
<th>30</th>
<th>3 tens</th>
</tr>
</thead>
<tbody>
<tr>
<td>+20</td>
<td>50</td>
<td>5 tens</td>
</tr>
</tbody>
</table>

**Exercise 14-2a**

Get a collection of single sticks. Form bundles of ten from the large collection. Use these bundles in the same way as you will teach children to use them. Solve the following examples:

(a) \( 40 + 10 = \)  
(b) \( 60 + 30 = \)
14-3  Renaming numbers in the decimal system

The previous work leads to the addition of two numbers represented by two-digit numerals ending in non-zero digits. In this type of example, considerable emphasis is placed on the way in which we name numbers.

There are many ways of naming a number. The idea of different names for the same number is basic for learning the procedures of addition and subtraction. For example, 46 may be renamed as $48 - 2$, $46 + 0$, $47 - 1$, $14 + 32$, $20 + 26$, $92 \div 2\sqrt{2116}$, ............. It is clear that a number can be renamed in a very great variety of ways. But here we are concerned with using names which are particularly helpful in addition and subtraction. For example we name 46 as $40 + 6$, or $(4 \times 10) + (6 \times 1)$, or $4$ tens $+6$ ones. Sometimes it is useful to name 46 as $30 + 16$, or $(3 \times 10) + (16 \times 1)$ or $3$ tens $+16$ ones. In each situation, we are expressing the number as (so many) tens $+$ (so many) ones.

We may extend this to larger numbers. For instance, 376 may be written as follows:

\[
\begin{align*}
300 & + 70 & + & 6 \\
3 \text{ hundreds} & + 7 \text{ tens} & + & 6 \text{ ones} \\
(3 \times 100) & + (7 \times 10) & + & (6 \times 1) \\
(3 \times 10 \times 10) & + (7 \times 10) & + & (6 \times 1) \\
(3 \times 10^2) & + (7 \times 10) & + & (6 \times 1)
\end{align*}
\]

These representations show that the grouping is by ten or that the base of the number system is ten. The system is called a decimal system from the
Latin word "decem" which means ten. In the operations of addition and subtraction, it is sometimes necessary to use slight modifications of the representations shown above. These modifications are often referred to as regrouping. For example, we may use the following names for 376:

<table>
<thead>
<tr>
<th>300 + 60 + 16</th>
<th>(3 × 100) + (6 × 10) + (16 × 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 hundreds + 6 tens + 16 ones</td>
<td>(3 × 10 × 10) + (6 × 10) + (16 × 1)</td>
</tr>
<tr>
<td></td>
<td>(3 × 10^2) + (6 × 10) + (16 × 1)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>200 + 170 + 6</th>
<th>(2 × 100) + (17 × 10) + (6 × 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 hundreds + 17 tens + 6 ones</td>
<td>(2 × 10 × 10) + (17 × 10) + (6 × 1)</td>
</tr>
<tr>
<td></td>
<td>(2 × 10^2) + (17 × 10) + (6 × 1)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>200 + 160 + 16</th>
<th>(2 × 100) + (16 × 10) + (16 × 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 hundreds + 16 tens + 16 ones</td>
<td>(2 × 10 × 10) + (16 × 10) + (16 × 1)</td>
</tr>
</tbody>
</table>

Besides using bundles of sticks it is also possible to show representations such as these on an **abacus**. The abacus pictured on the left shows 32 as 3 tens + 2 ones and the abacus on the right shows 32 as 2 tens + 12 ones.

There is a distinct difference between representations of numbers on the abacus and representations of numbers with bundles of sticks. With the bundles we actually use ten sticks tied together to form 1 ten. 3 bundles of 10 sticks
each would represent thirty (30). It is physically possible for the child to count one, two, three --- to thirty and actually see thirty single sticks in these bundles. At this stage of development, the child should count by tens to be convinced that he has thirty sticks. On the abacus, however, we do not have thirty beads. The position of the wires on the abacus shows whether a bead represents 1 bead or 10 beads or \((10 \times 10)\) beads and so on.

14-4   Addition without regrouping

In the work that follows we will try to show the transition from the stage in which the child deals with three bundles of ten to the stage where he sees three beads in the tens position or on the tens' wire.

Let us take an example requiring the addition of numbers represented by two-digit numerals. How do we add 46 and 12? What is the sum of 46 and 12? Because you already have skill with the basic operations of arithmetic, you follow the systematic procedure you have learned and you arrive at a sum of 58. You are so well acquainted with the procedure, that you follow it automatically. Your work is almost mechanical. It requires very little thought. This is an advantage to you, of course, but when you teach children you must realize that the mechanical procedure you use as an adult is the result of a long period of development. It is the final stage in a carefully planned sequence to teach children the meaning of each step along the way. As teachers, we should strive to teach children in such a manner that they will discover basic meanings and important relationships which lead to a greater understanding of mathematics and its applications. In teaching children how to add 46 and 12, a particularly bad beginning would be to write on the blackboard \(46 + 12\) and then proceed to
say mechanically: "6 + 2 = 8. Write 8 under 6 and 2. 4 + 1 = 5. Write 5 under 4 and 1." This is a poor way to begin teaching children to add. It merely tells children what to do, but not why they do it. Our job is to help children discover why certain procedures will lead them to correct answers. If children really understand the number system and the important properties which govern operations, they will be more likely to enjoy mathematics, to remember what they have learned, and to excel in applying it. With this goal in mind, we proceed with the example of 46 + 12 to show how children should learn the procedure in the very beginning. Then from this introductory stage we show a development through an intermediate stage to a final and more mature form of the procedures for performing the operation.

Example: What numeral goes in the box to make the following sentence true?

\[46 + 12 = \square\]

We shall solve this example by showing three stages in the learning process. The introductory stage will be referred to as Stage I. In this stage we will emphasize place value using physical objects; namely, bundles of sticks.

Stage I. (An introductory stage in learning)

46 will be shown as 4 bundles of 10 sticks + 6 loose sticks. We shall call the loose sticks 6 ones. Similarly, 12 will be shown as 1 bundle of 10 + 2 ones.
46 can be renamed as $40 + 6$

12 can be renamed as $10 + 2$

Next, show the result of adding the ones.

Show the result of adding the tens.

$40 + 6$

$10 + 2$

$50 + 8$

$50 + 8$ is ordinarily expressed in the decimal system as 58.

$40 + 6$

$10 + 2$

$50 + 8 = 58$. 

Exercise 14-4a (Stage I)

Get a collection of single sticks. Form bundles of ten from the large collection. Solve the following examples by using these bundles in the same way you will teach children to use them.
In preceding work, we found the number which made the statement \( 46 + 12 = \) true by using bundles of sticks. If we were to solve the same example on the abacus, we would proceed as follows.

**Stage II. (An intermediate state in learning)**

First, place beads on the abacus to show each addend separately. Emphasize that 1 bead on the tens' wire represents 10 beads on the ones' wire.

Next, show the result of adding the ones.

Show the result of adding the tens.

The abacus below shows the two addends combined, giving 58, which is their sum.

The solution which we have shown on the abacus can be recorded in two ways as shown below:
4 tens + 6 ones  \[ (4 \times 10) + (6 \times 1) \]
1 ten + 2 ones  \[ (1 \times 10) + (2 \times 1) \]
5 tens + 8 ones  \[ (4 \times 10) + (1 \times 10) + (6 \times 1) + (2 \times 1) \]

\[ (4 + 1) \times 10 + (6 + 2) \times 1 \text{ Note the use of the distributive property.} \]

\[ (5 \times 10) + (8 \times 1) \]
50 + 8
58

**Exercise 14-4b (Stage II)**

Solve the following problems using the abacus or using the notation as indicated.

(a) 31 + 47 = 78
Use the abacus.

(b) 53 + 15 = 68
Represent 53 as 5 tens and 3 ones, etc.

(c) 29 + 60 = 89
Represent 29 as \((2 \times 10) + (9 \times 12)\), etc.

The previous work provides the kind of meaningful practice which is essential for children in learning addition. The child so far has represented 46 as 4 tens + 6 ones or \((4 \times 10) + (6 \times 1)\). The abacus was used to strengthen this meaning. With sufficient practice, the child masters these ideas and is ready to record them in a shorter form as shown below.

**State III (A final stage in learning).**

Solve 46 + 12 = 58

<table>
<thead>
<tr>
<th>Tens</th>
<th>Ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>8</td>
</tr>
</tbody>
</table>

We think in this way:
Ones: \[ 6 + 2 = 8 \]
Write 8 in ones' place.

Tens: \[ 4 + 1 = 5 \]
Write 5 in tens' place.

**Exercise 14-4c (Stage III)**

Solve the following problems:
What numerals go into the boxes to make the following sentences true?

\[
\begin{align*}
(a) \quad 44 + 33 &= \boxed{} \\
(b) \quad 72 + 27 &= \boxed{} \\
(c) \quad \boxed{} &= 51 + 100
\end{align*}
\]

Stage I with bundles is really basic to the process of adding. All three stages are familiar to us as teachers, but it is very important that we take the child gradually through each of them so that he understands how we arrive at the final stage. The last stage has no magic behind it; rather it is the process which we learn to use and which we understand better when we realize that it is an outgrowth of the preceding stages.

**14-5 Addition with regrouping**

Now let us move to examples which require regrouping.

Example:

What is the sum of 35 and 27?

OR

What numeral goes in the box to make the following sentence true?

\[ 35 + 27 = \boxed{} \]
Use bundles as demonstrated in Stage I.

\[35 = 30 + 5\]
\[27 = 20 + 7\]

Next, show the result of adding the ones.

Show the result of adding the tens.

\[30 + 5\]
\[20 + 7\]
\[50 + 12 = 50 + (10 + 2)\] Note that 12 loose form 1 bundle of 10 and 2 ones.
\[= (50 + 10) + 2\] Note the use of the associative property.
\[= 60 + 2\]
\[= 62.\]

Exercise 14-5a (Stage II)

Use an abacus to solve the example:

\[35 + 27 = \square\]

Note that the 12 beads on the ones' wire are replaced by 1 bead on the tens' wire and 2 beads on the ones' wire.

Do you agree that the solution on the abacus could be recorded in the two ways shown on the next page?
14-15

\[35 = 3 \text{ tens} + 5 \text{ ones}\]

\[27 = 2 \text{ tens} + 7 \text{ ones}\]

\[5 \text{ tens} + 12 \text{ ones} = 5 \text{ tens} + (1 \text{ ten} + 2 \text{ ones})
= (5 \text{ tens} + 1 \text{ ten}) + 2 \text{ ones}
= 6 \text{ tens} + 2 \text{ ones}
= 62\]

\[(3 \times 10) + (5 \times 1)\]

\[(2 \times 10) + (7 \times 1)\]

\[= [(3 + 2) \times 10] + [(5 + 7) \times 1] \text{ (Why?)}\]

\[= (5 \times 10) + (12 \times 1)\]

\[= (5 \times 10) + [(10 + 2) \times 1] \text{ (Why?)}\]

\[= (5 \times 10) + [(10 \times 1) + (2 \times 1)] \text{ (Why?)}\]

\[= [(5 \times 10) + (1 \times 10)] + (2 \times 1) \text{ (Why?)}\]

\[= [(5 + 1) \times 10] + (2 \times 1) \text{ (Why?)}\]

\[= (6 \times 10) + (2 \times 1)\]

\[= 60 + 2\]

\[= 62\]

**Exercise 14-5b (Stage II)**

Solve the examples stated below. You need not use the abacus, but record your solution in the two ways shown above. The regrouping which makes use of the distributive property is particularly important.

(a) \[36 + 58 = \square\]  
(b) \[45 + 45 = \square\]  
(c) \[\square = 29 + 17\]

**Stage III.**

\[
\begin{array}{c|c|c}
\text{Tens} & \text{Ones} \\
3 & 5 \\
2 & 7 \\
6 & 2
\end{array}
\]
We think in this way:

Ones: \[ 5 + 7 = 12 \]

Think of 12 as 1 ten and 2 ones.

Record 2 in the ones' column and remember that 1 ten must be added to the tens' column. The most efficient practice is to begin the addition in the tens' column with the 1 ten.

Tens: \[ 1 + 3 + 2 = 6 \]

Record 6 in the tens' column.

**Exercise 14-5c**

Solve the following problems using the procedure demonstrated in Stage III.

(a) \[ 64 + 26 = \] 
(b) \[ 74 + 19 = \] 
(c) \[ 46 + 29 = \]

In the exercises above, you were required to regroup 10 ones as 1 ten. It is sometimes necessary to regroup further using names shown below:

1 hundred in place of 10 tens

1 thousand in place of 10 hundreds,

and so on.

**Example:**

\[ 848 + 537 + 192 = \]
Stage I.

(It may not be possible for you to use bundles of sticks to solve the problem below. Nevertheless, you should think of 800 as 8 bundles of 100 with each 100 formed by 10 bundles of 10.)

\[
\begin{align*}
848 &= 800 + 40 + 8 \\
537 &= 500 + 30 + 7 \\
192 &= 100 + 90 + 2 \\
1400 + 160 + 17 &= (1000 + 400) + (100 + 60) + (10 + 7) \\
&= 1000 + (400 + 100) + (60 + 10) + 7 \\
&= 1000 + 500 + 70 + 7 \\
&= 1577
\end{align*}
\]

Stage II.

\[
\begin{align*}
848 &= 8 \text{ hundreds} + 4 \text{ tens} + 8 \text{ ones} \\
537 &= 5 \text{ hundreds} + 3 \text{ tens} + 7 \text{ ones} \\
192 &= 1 \text{ hundred} + 9 \text{ tens} + 2 \text{ ones} \\
14 \text{ hundreds} + 16 \text{ tens} + 17 \text{ ones} &= (10 \text{ hundreds} + 4 \text{ hundreds}) + (10 \text{ tens} + 6 \text{ tens}) + (10 \text{ ones} + 7 \text{ ones}) \\
&= 1 \text{ thousand} + (4 \text{ hundreds} + 1 \text{ hundred}) + (6 \text{ tens} + 1 \text{ ten}) + 7 \text{ ones} \\
&= 1 \text{ thousand} + 5 \text{ hundreds} + 7 \text{ tens} + 7 \text{ ones} \\
&= 1000 + 500 + 70 + 7 \\
&= 1577
\end{align*}
\]
Exercise 14-5d

Use the procedure demonstrated in Stage II.

(a) \(563 + 77 = \) 

(b) \(641 + 279 = \) 

(c) \(= 179 + 658\) 

(d) \(888 + 135 + 38 = \) 

Stage III (Final Stage).

\(848 + 537 + 192 = \) 

We think in this way:

**Ones:** \(8 + 7 + 2 = 17\)

Think of 17 as 1 ten and 7 ones.

Record the 7 in the ones' column and remember 1 ten must be added in the tens' column.

**Tens:** \(1 + 4 + 3 + 9 = 17\)

Think of 17 tens as 1 hundred and 7 tens.

Record 7 in the tens' column, and remember 1 hundred must be added in the hundreds' column.
Hundreds: $1 + 8 + 5 + 1 = 15$

Think of 15 hundreds as 1 thousand and 5 hundreds.

Record 5 in the hundreds' column.

Record 1 in the thousands' column.

**Exercise 14-5c**

Use the procedure demonstrated in Stage III to solve the following problems:

(a) $325 + 146 + 946 = \square$
(b) $808 + 136 + 376 = \square$
(c) $127 + 376 + 298 = \square$
(d) $528 + 643 + 729 = \square$

14-6 Addition in other bases

Just as we added numbers in base ten or the decimal notation, we can use the same principles to add numbers expressed in other bases. In the examples which follow, you will use the addition table, base five, which is already familiar to you.

**Addition Table Base Five**

<table>
<thead>
<tr>
<th>+</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>10</td>
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<td>13</td>
</tr>
</tbody>
</table>
Example:

\[321_{\text{five}} + 122_{\text{five}} = \Box_{\text{five}}\]

Stage II.

You may enjoy using an abacus to solve this problem. Remember that the wires, moving from right to left, represent the ones' column, the fives' column, the five times fives' column and so on.

\[321_{\text{five}} = 3 \times 25 + 2 \times 5 + 1 \times 1 \]
\[122_{\text{five}} = 1 \times 25 + 2 \times 5 + 2 \times 1\]

\[= 4 \times 25 + 4 \times 5 + 3 \times 1\]
\[= 443_{\text{five}}\]

Note: All numerals are expressed base five.

Exercise 14-6a (Stage II)

(a) \[123_{\text{five}} + 201_{\text{five}} = \Box_{\text{five}}\]

(b) \[\Box_{\text{five}} = 101_{\text{five}} + 122_{\text{five}}\]

Let us use the procedure demonstrated in Stage II to work an example which requires regrouping.
Example:

\[\begin{array}{c}
\text{Stage II,} \\
432_{\text{five}} = 4 \text{twenty-fives} + 3 \text{fives} + 2 \text{ones} \\
414_{\text{five}} = 4 \text{twenty-fives} + 1 \text{five} + 4 \text{ones} \\
\hline
13 \text{twenty-fives} + 4 \text{fives} + 11 \text{ones}
\end{array}\]

\[= (1 \text{one hundred twenty-five} + 3 \text{twenty-fives}) + 4 \text{fives} + (1 \text{five} + 1 \text{one})\]

\[= 1 \text{one hundred twenty-five} + 3 \text{twenty-fives} + (4 \text{fives} + 1 \text{five}) + 1 \text{one}\]

\[= 1 \text{one hundred twenty-five} + 3 \text{twenty-fives} + 10 \text{fives} + 1 \text{one}\]

\[= 1 \text{one hundred twenty-five} + (3 \text{twenty-fives} + 1 \text{twenty-five}) + 0 \text{fives} + 1 \text{one}\]

\[= 1 \text{one hundred twenty five} + 4 \text{twenty-fives} + 0 \text{fives} + 1 \text{one}\]

\[= 1401_{\text{five}}\]

We can solve the above example in brief form using the procedure demonstrated in Stage III.

Example:

\[\begin{array}{c}
432_{\text{five}} + 414_{\text{five}} = \boxed{1401_{\text{five}}}
\end{array}\]
We think in this way:

<table>
<thead>
<tr>
<th>Ones:</th>
<th>2 + 4 = 11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Think of 11 as 1 five and 1 one</td>
<td></td>
</tr>
<tr>
<td>Record 1 in the ones' column, and remember 1 five must be added to the fives' column.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Fives:</th>
<th>1 + 3 + 1 = 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Think of 10 as 1 twenty-five and 0 fives.</td>
<td></td>
</tr>
<tr>
<td>Record 0 in the fives column and remember 1 twenty-five must be added to the twenty-fives' column.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Twenty-fives:</th>
<th>1 + 4 + 4 = 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Think of 14 as 1 one hundred twenty-five and 4 twenty-fives.</td>
<td></td>
</tr>
<tr>
<td>Record 4 in the twenty-fives column.</td>
<td></td>
</tr>
<tr>
<td>Record 1 in the one hundred twenty-fives column.</td>
<td></td>
</tr>
</tbody>
</table>
Exercises 14-6b (Stage III)

(a) $324 + 122 = \boxed{555}$

(b) $4321 + 314 = \boxed{555}$

(c) $4322 + 320 + 432 = \boxed{555}$

(d) $2001 + 212 + 433 + 23 = \boxed{555}$

We have worked problems in the addition of numbers expressed in base ten and base five. The purpose of these exercises is to emphasize that the particular choice of ten or five for the base does not change the underlying principles of addition. These principles of place value and regrouping remain the same regardless of the choice of a base. To further emphasize that these principles remain unchanged in the addition of numbers, let us work a few examples comparing addition in various bases. Addition tables in base two and base twelve are included for your reference.
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<td>16</td>
<td>17</td>
<td>18</td>
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<td>1T</td>
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</tbody>
</table>
Example: i) \(19_{\text{ten}} + 23_{\text{ten}} = \square_{\text{ten}}\)

ii) Express \(19_{\text{ten}}\) and \(23_{\text{ten}}\) in base two, in base five, and in base twelve, and find their sum working in the different bases. Use the procedure demonstrated in Stage III.

| \(19_{\text{ten}}\) | \(10011_{\text{two}}\) | \(34_{\text{five}}\) | \(17_{\text{twelve}}\) |
| \(23_{\text{ten}}\) | \(10111_{\text{two}}\) | \(43_{\text{five}}\) | \(1E_{\text{twelve}}\) |
| \(42_{\text{ten}}\) = | \(101010_{\text{two}}\) | \(132_{\text{five}}\) = | \(36_{\text{twelve}}\) |

Each of these numerals represents the sum of the same two numbers even though these numbers are expressed in different bases and different numerals.

Exercise 14-6c

Find the numerals which make the following sentences true:

(a) \(70_{\text{ten}} + 35_{\text{ten}} + 76_{\text{ten}} = \square_{\text{ten}}\)

(b) \(240_{\text{five}} + 120_{\text{five}} + 301_{\text{five}} = \square_{\text{five}}\)

(c) \(5T_{\text{twelve}} + 2E_{\text{twelve}} + 64_{\text{twelve}} = \square_{\text{twelve}}\)

The following two addition problems illustrate the fact that the same numerals represent different numbers in different bases.

Example: \(968_{\text{ten}}\) and \(968_{\text{twelve}}\)

\(47_{\text{ten}}\) and \(47_{\text{twelve}}\)

\(1015_{\text{ten}}\) ≠ \(9E3_{\text{twelve}}\)
since \(968_{\text{ten}} \neq 968_{\text{twelve}}\)

and \(47_{\text{ten}} \neq 47_{\text{twelve}}\)

\[
968_{\text{ten}} = (9 \times 10 \times 10) + (6 \times 10) + (8 \times 1)
\]

\[
968_{\text{twelve}} = (9 \times 12 \times 12) + (6 \times 12) + (8 \times 1) = 1376_{\text{ten}}
\]

and

\[
47_{\text{ten}} = (4 \times 10) + (7 \times 1)
\]

\[
47_{\text{twelve}} = (4 \times 12) + (7 \times 1) = 55_{\text{ten}}
\]

14-7 Subtraction in relation to addition

In earlier work you have seen subtraction discussed as the inverse operation to addition. You saw for instance that the addition problem

\[46 + 12 = \square\]

may be stated as a subtraction problem thus:

\[46 + \square = 58,\]

in which we now look for the missing addend. This basic relationship between addition and subtraction will be helpful in this section on subtraction. It is essential that pupils master the basic subtraction facts, and further understand that place value and regrouping will be as essential in finding the missing addend in subtraction problems as in finding the sum in addition problems. We shall proceed systematically from easy subtraction problems to more difficult problems just as in the case of addition.

14-8 Subtraction without regrouping

Take this example requiring the subtraction of numbers represented by two-digit numerals. Subtract 34 from 97. This may be restated as: what
number would make the following statement true?

\[ 97 - 34 = \Box \]

Furthermore, we have learned to phrase the statement as \( \Box + 34 = 97 \) in which we think of subtraction as the inverse operation to addition. In future we shall make use of any of these forms to help increase our familiarity with them.

As in the case of addition, let us use the numeration system (and regrouping where necessary) to solve problems. Thus \( 97 - 34 = \Box \) may be treated in this manner.

Stage I.

You will recall that in the procedure demonstrated in Stage I in addition, numbers were represented by loose sticks, bundles of 10 sticks, bundles of \((10 \times 10)\) sticks and so on. Similarly, we shall represent 97 as 9 bundles of 10 sticks and 7 loose sticks (ones). When we subtract 34 from 97, we merely remove 4 loose sticks (ones) and 3 bundles of 10 sticks. It is easy to subtract 4 ones from 7 ones. The result is 3 ones. In the same way, we can remove 3 bundles of 10 (30) from 9 bundles of 10 (90). The result is 6 bundles of 10 (60).

We may state our solution in this way:

\[
\begin{align*}
97 &= 90 + 7 \\
34 &= 30 + 4 \\
60 + 3 &= 63
\end{align*}
\]

In the above example we have subtracted the ones separately, and the subtracted the tens separately. No regrouping was necessary. The general
form of this solution is stated as follows:

For a pair of whole numbers like 97 and 34 which have the form

\[(a + b)\) and \((c + d)\) where \(a + b\) is greater than or equal to \(c + d\), and \(a\) is greater than or equal to \(c\), and \(b\) greater than or equal to \(d\), it is true that

\[(a + b) - (c + d) = (a - c) + (b - d)\]

In vertical form we may state this as follows:

Subtract:

\[
\begin{array}{c}
| a + b \\
- | c + d \\
\hline
| (a - c) + (b - d)
\end{array}
\]

For the particular problem which we started with we would have:

Subtract:

\[
\begin{array}{c}
| 97 \\
- | 34 \\
\hline
| 63
\end{array}
\]

Exercise 14-8a (Stage I)

Get a collection of single sticks. Form bundles of 10 from the large collection. Solve the following examples by using these bundles in the same way you will teach children to use them.

(a) \(86 - 52 = \) [ ]  

(b) \(74 - 31 = \) [ ]
The work above leads easily to the solution of the problem in Stage II.

Stage II. \[ 97 - 34 = \Box \]

Using an abacus, we can solve the problem in the following way:

In the abacus on the left, 97 is shown as 9 tens and 7 ones. In the abacus on the right, 4 ones have been separated from the 7 ones leaving 3 ones. Similarly, 3 tens have been separated from 9 tens leaving 6 tens. The result is 63, as shown on the lower part of the abacus.

The solution shown on the abacus can be recorded in either of the forms stated below:

\[
\begin{align*}
&\text{9 tens} + \text{7 ones} \\
= \quad \text{3 tens} + \text{4 ones} \\
\equiv \quad \text{(9 tens } - \text{3 tens)} + \text{(7 ones } - \text{4 ones)} \\
= \quad \text{6 tens} + \text{3 ones} \\
= \quad 60 + 3 \\
= \quad 63
\end{align*}
\]

\[
\begin{align*}
&\text{(9 X 10) + (7 X 1)} \\
= \quad \text{(3 X 10) + (4 X 1)} \\
\equiv \quad \text{[(9 X 10) - (3 X 10)] + [(7 X 1) - (4 X 1)]} \\
= \quad \text{[(9 - 3) X 10] + [(7 - 4) X 1]} \text{ (Why ?)} \\
= \quad \text{(6 X 10) + (3 X 1)} \\
= \quad 60 + 3 \\
= \quad 63
\end{align*}
\]
Exercise 14-8b (Stage II)

Solve the following problems using the abacus or using the notations as indicated:

(a) \( 64 - 64 = \) Use the abacus.
(b) \( 74 + \square = 99 \) Represent 99 as 9 tens and 9 ones etc.
(c) \( 95 - 62 = \) Represent 95 as \( (9 \times 10) + (5 \times 1) \) etc.

In previous work in addition, we gradually proceeded to a point where it was no longer necessary to write out fully the meaning of every number. It was possible for us at that stage to add our separate units and regroup (where necessary), mentally and quickly. This was the procedure in Stage III.

In much the same way we may now write out the subtraction procedure in Stage III.

\( 97 - 34 = \)

We think in this way:

- **Ones:** \( 7 - 4 = 3 \). (4 from 7 = 3)
  - Write 3 in ones' column
- **Tens:** \( 9 - 3 = 6 \). (3 from 9 = 6)
  - Write 6 in tens' column.
Exercise 14-8c (Stage III)

Solve the following problems using the procedure demonstrated in Stage III.

(a) \[ 57 - \_ \_ \_ = 47 \]
(b) \[ 943 - 803 = \_ \_ \_ \]
(c) \[ 649 - \_ \_ \_ = 508 \]
(d) \[ \_ \_ \_ + 875 = 886 \]
(e) \[ \_ \_ \_ + 35 = 68 \]
(f) \[ 205 - 201 = \_ \_ \_ \]

14-9 Subtraction with regrouping

You will have noticed that regrouping has not been necessary in our work thus far. Let us take examples where regrouping is a necessary step in finding the missing addend.

Example: \[ 97 - 79 = \_ \_ \_ \]

What number makes the above sentence true?

Stage I.

Solving this problem by the procedure demonstrated in Stage I, 97 is represented as 9 bundles of 10 sticks and 7 loose sticks (ones). We are required to subtract 7 bundles of 10 sticks and 9 loose sticks (ones). Since we cannot take 9 loose sticks from 7 loose sticks, we must regroup 97 in order to provide more loose sticks. Therefore, we represent 97 as 8 bundles of 10 sticks and 17 loose sticks.

Exercise 14-9a (Stage I)

Study the explanation above and complete the solution using sticks and bundles of sticks.
The procedure in Stage I is particularly important because it gives children the physical interpretation of the need to regroup. Whereas in addition the children discover that they had enough ones to make more bundles of 10, in subtraction they see the need to do the reverse; namely, to break up one bundle of 10 sticks into 10 loose sticks. With larger numbers it is sometimes necessary to break a bundle of \((10 \times 10)\) sticks into 10 bundles of 10 sticks and so on.

**Exercise 14b (Stage I)**

Using sticks and bundles of sticks, solve the following problems.

(a) \(43 - 25 = \) □
(b) \(127 - 93 = \) □
(c) \(100 - 78 = \) □
(d) \(111 - 56 = \) □

**Stage II.**

Suppose we attempt to solve the problem \(97 - 79 = \) □ using the abacus. We represent 97 with 9 beads on the tens' wire and 7 beads on the ones' wire. (See A below). We are required to subtract 7 tens and 9 ones. We see the need to regroup 97 as 8 beads on the tens' wire and 17 beads on the ones' wire. (See B below). The important step from A to B is the replacement of one bead on the tens' wire by 10 beads on the ones' wire. Diagram C below shows the result of subtracting 7 tens and 9 ones. We have left 1 ten and 8 ones, 18.
Stage II.

The solution demonstrated on the abacus may be stated in either of the forms shown below.

\[ 97 = 9 \text{ tens} + 7 \text{ ones} \]
\[ -79 = 7 \text{ tens} + 9 \text{ ones} \]

\[ (9 \text{ tens} - 7 \text{ tens}) + (7 \text{ ones} - 9 \text{ ones}) \]

We should notice immediately that we have more ones in the given addend (79), than in the sum (97). This calls for regrouping.

\[ 97 = 8 \text{ tens} + 1 \text{ ten} + 7 \text{ ones} = 8 \text{ tens} + 17 \text{ ones} \]

\[ 97 = (9 \times 10) + (7 \times 1) \]
\[ -79 = (7 \times 10) + (9 \times 1) \]

\[ = [(8 + 1) \times 10 - (7 \times 10)] + [(7 \times 1) - (9 \times 1)] \]
\[ = [(8 \times 10) + (1 \times 10) - (7 \times 10)] + [(7 \times 1) - (9 \times 1)] \]
\[ = [(8 \times 10) - (7 \times 10) + (1 \times 10)] + [(7 \times 1) - (9 \times 1)] \]
\[ = [(8 \times 10) - (7 \times 10) + (10 \times 1)] + [(7 \times 1) - (9 \times 1)] \]
\[ = [(8 - 7) \times 10] + [(10 + 7) \times 1 - (9 \times 1)] \]
We may now state the problem thus:

\[ 97 = 8 \text{ tens } + 17 \text{ ones} \]
\[ -79 = 7 \text{ tens } + 9 \text{ ones} \]

\[ (9 \text{ tens } - 7 \text{ tens}) + (17 \text{ ones } - 9 \text{ ones}) \]
\[ = 1 \text{ ten } + 8 \text{ ones} \]
\[ = 18 \]

The solution on the right is very lengthy. Nevertheless, it is valuable for teachers in training because it provides a revision of basic mathematical principles.

**Exercise 14-9c (Stage II)**

Solve the following problems using the abacus or using the notation as indicated:

(a) \(38 + \Box = 93\). Use the abacus.

(b) \(82 - 65 = \Box\) Represent 82 as \((8 \times 10) + (2 \times 1)\) etc.

(c) \(817 - 748 = \Box\) Represent 817 as 8 hundreds and 1 ten and 7 ones.

When understanding has been developed in Stage II, we may proceed to the final stage in which we record our work in brief form as shown below:

**Stage III.**

\[
\begin{array}{c|c|c}
\text{Tens} & \text{Ones} \\
9 & 7 \\
7 & 9 \\
1 & 8 \\
\end{array}
\]
We think in this way:

Ones: We need more ones in the sum, 97. Mentally, we rename 97 as 8 tens and 17 ones. Then 17 - 9 = 8 ("9 from 17 equals 8"). Write 8 in ones' column.

Tens: 8 - 7 = 1 ("7 from 8 equals 1")

Write 1 in the tens' column.

Exercise 14-9d (Stage III)

Solve the following problems.

(a) 87 - 29 =

(b) 736 - 495 =

(c) Subtract 398 from 576.

(d) Find the result:

8021

-5167

14-10 Subtraction operations in other bases

In order to emphasize place value and regrouping in subtraction, let us take examples where the bases are different from ten. The solutions can be demonstrated in the introductory and intermediate stages, but it is assumed here that the teacher in training can work through Stages I and II for himself. The solution will be stated in Stage III.
Stage III.
Example: Find the missing addend.

\[ 30_5 - 143_5 = \square_5 \]

\[
\begin{array}{c}
30_5 \\
-143_5 \\
104_5
\end{array}
\]

We think in this way.

Ones: We need more ones in the sum, \( 302_5 \).

Fives: We also need more fives (10's) in the sum. Mentally we rename \( 302_5 \) as 2 twenty-fives, 4 fives, 12 ones.

Then,

Ones: \( 12_5 - 3_5 = 4_5 \) (3 from 12 in base five = 4 in base five)

Fives: \( 4_5 - 4_5 = 0_5 \)

Twenty-fives: \( 2_5 - 1_5 = 1_5 \)

Stage III.

Example: \( 10336_{12} - 2ET7_{12} = \square_{12} \)

We observe that regrouping is essential in each place. Quickly we rename \( 10336_{12} \) as follows:

16 ones
12 twelves
12 (twelve \( \times \) twelve)
5 (twelve \( \times \) twelve \( \times \) twelve)
Then subtract.

Ones: \[16_{\text{twelve}} - 7_{\text{twelve}} = E_{\text{twelve}}\]

Twelves: \[12_{\text{twelve}} - T_{\text{twelve}} = 4_{\text{twelve}}\]

Twelve \times Twelve: \[12_{\text{twelve}} - E_{\text{twelve}} = 3_{\text{twelve}}\]

Twelve \times Twelve \times Twelve: \[E_{\text{twelve}} - 2_{\text{twelve}} = 9_{\text{twelve}}\]

Therefore \[10336_{\text{twelve}}\]

\[2ET7_{\text{twelve}}\]

\[934E_{\text{twelve}}\]

**Exercise 14-10a**

Use any method we have studied to solve the following problems:

(a) \[4321_{\text{five}} - 1234_{\text{five}} = \]

(b) In base ten we could state problem (a) as follows:

\[586 - 194 = \]

Write the answer in base ten. Note that the numerals in these two problems are different, but the numbers remain the same. Express these numbers in base twelve and subtract. Compare your answer with the results obtained in base five and base ten.

(c) \[67E8_{\text{twelve}} - 4TE9_{\text{twelve}} = \]

Repeat the exercise required in the above problem.

State these numbers in base 10 and solve.
Answers to chapter 14

Exercise 14-2a:
(a) 50 (b) 90

Exercise 14-4a:
(a) 89 (b) 76 (c) 48

Exercise 14-4b:
(a) 78 (b) 68 (c) 89

Exercise 14-4c:
(a) 77 (b) 99 (c) 61

Exercise 14-5b:
(a) 94 (b) 90 (c) 46

Exercise 14-5c:
(a) 90 (b) 93 (c) 75

Exercise 14-5d:
(a) 640 (b) 920 (c) 837 (d) 1,061

Exercise 14-5e:
(a) 1419 (b) 980 (c) 801 (d) 1,900

Exercise 14-6a:
(a) $324_{\text{five}}$ (b) $223_{\text{five}}$

Exercise 14-6b:
(a) $1,001_{\text{five}}$ (b) $10,140_{\text{five}}$ (c) $11,124_{\text{five}}$ (d) $3,224_{\text{five}}$
Exercise 14-6c:
(a) \(181\text{ ten}\)  (b) \(1,211\text{ five}\)  (c) \(131\text{ twelve}\)

Exercise 14-8a:
(a) 34  (b) 43

Exercise 14-8b:
(a) 21  (b) 25  (c) 33

Exercise 14-8c:
(a) 10  (b) 140  (c) 141  
(d) 11  (e) 33  (f) 4

Exercise 14-9b:
(a) 18  (b) 34  (c) 22  (d) 55

Exercise 14-9c:
(a) 55  (b) 17  (c) 69

Exercise 14-9d:
(a) 58  (b) 241  (c) 178  (d) 2,854

Exercise 14-10a:
(a) \(3,032\text{ five}\)

(b) \(392\text{ ten}\)

\(40T\text{ twelve} - 142\text{ twelve} - 288\text{ twelve}\)

(c) \(67E8\text{ twelve} - 4TE9\text{ twelve} - 18EE\text{ twelve}\)

\(11516\text{ ten} - 8493\text{ ten} - 3023\text{ ten}\)
15-1  Review of multiplication as repeated addition

Multiplication of whole numbers may be thought of as repeated addition. We say that $4 \times 3$ means $3 + 3 + 3 + 3$, thus we have $4 \times 3 = 12$. But when we set out to solve more difficult examples such as $14 \times 39$, the product is not so obvious. However we know how to find the product because we have learned a few rules and have practised them. We cannot say that we really understand rules merely because we can apply them successfully. It is not sufficient to say "Those are the rules." We want to be able to give reasons for rules.

To understand why $14 \times 39 = 546$ is more difficult than to perform the multiplication $14 \times 39$. Can you actually explain why the rules we apply give that product? If you think you really understand the procedure, then try to work the example $14_{\text{twelve}} \times 39_{\text{twelve}}$.

You will discover that in solving such problems you have depended on principles of the notational system, the commutative and associative properties of addition and multiplication, the properties of zero and one, and the distributive property of multiplication over addition. Therefore, let us examine the use of these properties in the procedure of multiplication. Let us start with the product of 10 and a number expressed by a one-digit numeral.

15-2  Multiplication by 10 and by multiples of 10

Example: $8 \times 10 = □$

This may be written as $8 \times 1$ ten or 8 tens, which is 80.
15-2

\[
\begin{array}{c@{}c@{}c}
   1 & \text{ten} & 10 \\
   \times 8 & \text{tens} & \times 8 \\
   8 & \text{tens} & 80 \\
\end{array}
\]

All we have used here is the **principle of place value**. To write the product we have merely recorded 8 in the tens column. Using the same idea, we may write the following statements:

\[
\begin{align*}
8 \times 10 &= 8 \times 1 \text{ ten} = 8 \text{ tens} = 80 \\
9 \times 10 &= 9 \times 1 \text{ ten} = 9 \text{ tens} = 90 \\
10 \times 10 &= 10 \times 1 \text{ ten} = 10 \text{ tens} = 100 \\
\end{align*}
\]

Note that 100 can be named 10 tens or 1 hundred.

Similarly,

\[
\begin{align*}
11 \times 10 &= 11 \times 1 \text{ ten} = 11 \text{ tens} = 110 \\
26 \times 10 &= 26 \times 1 \text{ ten} = 26 \text{ tens} = 260 \\
\end{align*}
\]

Do you agree that 110 can be named 11 tens as well as 1 hundred plus 1 ten?

**Exercise 15-2a**

Write the numerals for the following:

(a) 12 tens  
(b) 14 tens  
(c) 19 tens  
(d) 37 tens

The ideas presented above will enable you to write quickly the following products:

\[
\begin{align*}
13 \times 10 &= \\
15 \times 10 &= \\
48 \times 10 &= \\
\end{align*}
\]

To arrive at the answers, did you think in this manner:

\[
13 \times 10 = 13 \text{ tens} = 130
\]
15-3

\[ 15 \times 10 = 15 \text{ tens} = 150 \]
\[ 48 \times 10 = 48 \text{ tens} = 480 \]

480 may be grouped as 4 hundreds + 8 tens or it may be regrouped as 48 tens.

**Example:** If the following are grouped as all tens, how many tens do you have in each?

(a) 160
(b) 210
(c) 350
(d) 740
(e) 1250

Your answers in this example can be shown as:

(a) 160 = 16 tens = 16 \times 10
(b) 210 = 21 tens = 21 \times 10
(c) 350 = 35 tens = 35 \times 10
(d) 740 = 74 tens = 74 \times 10
(e) 1250 = 125 tens = 125 \times 10

When we teach children, we give them many exercises similar to those shown above. We gradually lead pupils to discover that to multiply any number by 10, we write the numeral which states how many tens are in the number.

**Exercise 15-2b**

Write the following products:

(a) 14 \times 10
(b) 60 \times 10
(c) 10 \times 53
(d) 248 \times 10
(e) 10 \times 965
(f) 730 \times 10

We have shown how to multiply two numbers when one of the factors is 10. Now we want to learn how to multiply two numbers when one of them is a multiple of 10.
Example: $3 \times 20 = \square$

$$3 \times 20 = 3 \times (2 \times 10) \quad \text{Why?}$$

$$= (3 \times 2) \times 10 \quad \text{Why?}$$

$$= 6 \times 10 \quad \text{Why?}$$

$$= 60$$

This is not a difficult example. Nevertheless, you recognize the use of place value when 20 is expressed as $2 \times 10$. In the second step, observe the use of the associative property of multiplication, $3 \times (2 \times 10) = (3 \times 2) \times 10$ which leads to $6 \times 10$. Again the use of place value enables us to write $6 \times 10 = 60$.

**Exercise 15-2c**

Show how you arrive at the products below. List in order the properties used:

(a) $6 \times 30 = \square$

(b) $\square = 70 \times 7$

(c) $\square = 50 \times 8$

(d) $270 = \square \times 10$

(e) $3900 = 390 \times \square$

(f) $4200 = \square \times 20$

**15-3 Multiplication by a number represented by a single-digit numeral**

You should now be able to tell the products in the last exercise by using what we have learned about multiplying by 10 and its multiples. We shall proceed to the multiplication of two numbers, neither of which is a multiple of 10. Take the case where one of these factors is represented by a single-digit numeral, and the other factor represented by a numeral of two or more digits.

**Example:** What would make the following statement true?

$$7 \times 56 = \square$$

You are aware that 56 may be named as $50 + 6$; this is the expanded numeral for
56. We may therefore write $7 \times 56 = 7 \times (50 + 6)$ and proceed as follows:

**Stage I.**

$$7 \times 56 = 7 \times (50 + 6)$$

$$= (7 \times 50) + (7 \times 6)$$

$$= 350 + 42$$

$$= 392$$

Below is the solution demonstrated in Stage I presented in diagram form.

**Explanation of diagram:** Use one of the factors as the starting point of multiplication. Write the other factor (preferably in this case the larger number) in decimal notation along the lines as shown. At each arrowhead write the product of the number at the starting point and the number along that line. Add the separate products.

We may develop this diagram to find the product of two numbers, one of which is represented by a single-digit numeral and the other by a three-digit numeral.

$$6 \times 429 = \boxed{\text{ }}$$

$$6 \times 429 = 6 \times (400 + 20 + 9)$$
Exercise 15-3a

Solve each of the following problems, using the diagram or the expanded numeral form as indicated:

(a) $31 \times 6$ (Expanded numeral form)

(b) $\square = 5 \times 77$ (Diagram)

(c) $1302 \times 4 = \square$ (Expanded numeral form)

(d) $4 \times 80 = \square$ (Diagram)

(e) $6 \times 324 = \square$ (Diagram)

The solutions presented in Stage I above would be long and laborious if both factors were large numbers. However, there is value in revising these procedures because they demonstrate the processes which guide the brief form of multiplication which we use. In the brief form which we shall now describe, many of the steps which we have learned in the preceding work will be used; but often they are not explicitly stated. In the following worked examples, try to find and identify these steps.
Stage II.  

Example: 7 \times 56 = \underline{392}

Lead up

\[
\begin{array}{c@{}c}
56 & \\
\times & 7 \\
\hline
42 & \\
350 & \\
\hline
392 & \\
\end{array}
\]

When you studied the brief form of multiplication above, did you find any hidden steps? They are shown below:

\[
\begin{array}{c@{}c}
56 & \\
\times & 7 \\
\hline
42 & (7 \times 6) \\
350 & (7 \times 50) \\
\hline
392 & \\
\end{array}
\]

But we ordinarily work the problem in a form that is even briefer:

Stage III.

\[
\begin{array}{c@{}c}
56 & \\
\times & 7 \\
\hline
392 & \\
\end{array}
\]

We think in this manner:

Ones: 7 \times 6 = 42. Record 2 in the one's column and remember 4 tens for the ten's column.

Tens: 7 \times 5 = 35. 35 + 4 = 39. Record 39 tens or 9 tens and 3 hundreds.

Exercise 15-3b

Solve the following problems using the stages indicated.

(a) 82 \times 7 = \underline{574}  
Stage II

(b) 586 \times 9 = \underline{5274}  
Stage II

(c) 97 \times 4 = \underline{388}  
Stage III

(d) 451 \times 3 = \underline{1353}  
Stage III
15-4  Multiplication in which each factor is represented by a numeral of two
or more digits.

Now we are ready to find the product of two numbers, both of which are
represented by numerals of two or more digits.

Example:  27 \times 34 =

Let us solve the problem in each of the three stages learned thus far.

Stage I.

\[ 27 \times 34 = (20 + 7) \times (30 + 4) \]
\[ = [20 \times (30 + 4)] + [7 \times (30 + 4)] \quad \text{Why?} \]
\[ = (20 \times 30) + (20 \times 4) + (7 \times 30) + (7 \times 4) \quad \text{Why?} \]
\[ = 600 + 80 + 210 + 28 \]
\[ = 918 \]

Stage I. (Diagram)
**Explanation of Diagram:** Note that we have used 1 as the starting point and we have stated 27 as the product of \((20 \times 1)\) and \((7 \times 1)\). Briefly, \(27 = (20 \times 1) + (7 \times 1)\). The rest is repeated application of the procedure already explained.

**Stage II.**

\[
27 \times 34 = \boxed{} \\
\begin{array}{c}
\text{Lead up} \\
\times \\
27 \\
\hline \\
28 \quad (7 \times 4) \\
210 \quad (7 \times 30) \\
80 \quad (20 \times 4) \\
600 \quad (20 \times 30) \\
918 \\
\end{array}
\]

thus we have

\[
\begin{array}{c}
34 \\
\times \\
27 \\
\hline \\
238 \quad (7 \times 34) \\
680 \quad (20 \times 34) \\
918 \\
\end{array}
\]

**Stage III.** The intermediate steps recorded in Stage II are not only shortened but now we just make mental notes of the various products, stating only those necessary.

\[
\begin{array}{c}
34 \\
\times \\
27 \\
\hline \\
238 \\
680 \\
918 \\
\end{array}
\]

**Exercise 15-4a**

Find the following products, using the stage indicated:

(a) \(57 \times 86 = \boxed{\quad} \) Stage II

(b) \(26 \times 143 = \boxed{\quad} \) Stage III

(c) \(42 \times 95 = \boxed{\quad} \) Stage III
15-10

(d) $73 \times 502 = \square$ Stage II
(e) $703 \times 502 = \square$ Stage III

15-5  Review of multiplication and division as inverse operations

The idea of division as the inverse of multiplication has been discussed in earlier work. We have seen that these two statements

\[ 6 \times \square = 18 \quad \text{and} \quad 18 \div 6 = \square \]

are both true if 3 is placed in the box. We have emphasized that in order to divide 18 by 6, we must find the number which multiplied by 6, gives 18.

Division has been defined in terms of multiplication in this way:

If \(a\) is a counting number, and \(b\) is a whole number, then the numeral put in the box to make the statement \(b \div a = \square\) true is the same as the numeral put in the box to make the statement \(a \times \square = b\) true.

In the division problem, \(b \div a = \square\), the numeral in the box \((\square)\) names the quotient.

We must keep in mind that \(a \times \square = b\) does not always have a solution in the set of whole numbers. For example, can you find a whole number which will make the following statements true?

\[ 6 \times \square = 17 \quad \text{and} \quad 17 \div 6 = \square \]

But \(17 \div 6\) is not a whole number. In this chapter we are concerned only with those statements \(b \div a = \square\) where the solution is a whole number.

Just as the procedure for subtraction was shown in terms of addition, so the procedure for division is shown in terms of multiplication. For example, if pupils are asked to find \(36 \div 4\), they must find the number to be multiplied by 4 to give 36. For what numeral in the box is it true that \(4 \times \square = 36\)? The
solution can be reached by distributing 36 objects into 4 rows with the same number of objects in each row. By counting, the pupils find that there are 9 objects in each row. They would conclude that 9 is the number to be multiplied by 4 to give 36.

\[ 4 \times 9 = 36 \quad \text{and} \quad 36 \div 4 = 9 \]

describe a basic fact in multiplication and division. The basic multiplication facts have been summarized in the multiplication table, which is familiar to you.

Exercise 15-5a

Use the multiplication table to find the solutions:

(a) \( 8 \times [\ ] = 56 \)  
(b) \( 63 \div 9 = [\ ] \)  
(c) \( 54 \div [\ ] = 6 \)  
(d) \( [\ ] \div 3 = 7 \)  
(e) \( 5 \times [\ ] = 0 \)  
(f) \( [\ ] \div [\ ] = 1 \)

Before we proceed to more difficult work on division, let us underline these two facts:

i) The meaning of division as the inverse of multiplication is basic to this study and must be clearly grasped.

ii) The basic facts of multiplication and division should be understood and the facts eventually memorized.

15-6 Division by a number represented by a single-digit numeral: no regrouping

With this background we may proceed to the division of numbers where the quotient is not stated in the multiplication table. Let us start with an example where we already know the answer but which may illustrate the ideas we wish to master.
Example:

Stage I.

Solve: $48 \div 2 = \square$ or $2 \times \square = 48$. Replace the $\square$ in the problem stated above by ( ) for convenience. We may now write:

$48 \div 2 = ( )$ or $2 \times ( ) = 48$.

$2 \times ( ) = 48$ may be written as $2 \times ( ) = 40 + 8$.

Former work with the expanded form of numerals coupled with the immediate object suggests such an expansion. Thus, we now have $2 \times ( ) = 40 + 8$.

We see that $2 \times 4$ gives 8, and $2 \times 20$ gives 40. Therefore,

$2 \times (20 + 4) = 40 + 8$

or $2 \times 24 = 48$

or $48 \div 2 = 24$.

This form of solution uses the expanded form of the numeral shown in Multiplication Stage I. We may therefore label this as Division Stage I.

Exercise 15-6a

Find the quotients in the following problems using the procedure demonstrated in Stage I:

(a) $963 \div 3 = \square$

(b) $770 \div 7 = \square$

(c) $484 \div 4 = \square$

(d) $105 \div 5 = \square$ (Remember 1 hundred = 10 tens).

Stage II. We can also write the problem out in the following manner

$48 \div 2$ as $\underline{2/4} \text{ tens} + 8 \text{ ones}$

or $\underline{2/40 + 8}$
As before, 2 is multiplied by 2 tens to give 4 tens.

2 is multiplied by 4 ones to give 8 ones.

We record this as follows

\[
\frac{2 \text{ tens} + 4 \text{ ones}}{2/4 \text{ tens} + 8 \text{ ones}} \quad \text{or} \quad \frac{20 + 4}{2/40 + 8}
\]

Stage III. Using place value, the division procedure may be further abbreviated.

We no longer write out the full meaning of numbers, we record numerals in the appropriate columns.

\[248\]

and this may be further abbreviated thus: \(2/48\)

In the solutions presented in Stages II and III, the quotient is the sum of the separate quotients obtained by dividing 40 by 2 and then 8 by 2. This is an application of the distributive property.

**Exercise 15-6b**

Find the quotients in the following problems, using the stages indicated.

\[840 \div 4 = \quad \text{Stage II}\]
\[909 \div 9 = \quad \text{Stage II}\]
\[2468 \div 2 = \quad \text{Stage III}\]

15-7 Division by a number represented by a single-digit numeral: regrouping necessary

Thus far it has been possible to complete the division examples by using the expanded decimal notation form of the numeral. But if we attempted to find the quotient in the problem \(48 \div 3 = \quad \), it will be necessary to regroup 48 in a different manner, since \((40 + 8)\) would not yield the desired
results. Let us solve the problem $48 \div 3 = \square$ in the three stages demonstrated thus far.

**Stage I.**

$48 \div 3 = \square$ may be stated as $3 \times \square = 48$.

$48 \div 3 = (\square) \text{ may be stated as } 3 \times (\square) = 48$.

$3 \times (\square) = 48 \text{ may be expressed as } 3 \times (\square) = 40 + 8$

or $3 \times (\square) = (4 \times 10) + 8$

$(4 \times 10)$ is not a multiple of 3 but $4 = 3 + 1$.

Therefore

$(4 \times 10)$ may be expressed as $(3 + 1) \times 10$ or $(3 \times 10) + (1 \times 10)$. Why?

We may now write $3 \times (\square) = 48$ as

$3 \times (\square) = (4 \times 10) + 8$

$= [(3 + 1) \times 10] + 8$

$= (3 \times 10) + (1 \times 10) + 8 \text{ Why?}$

$= (3 \times 10) + (10 + 8)$

$= (3 \times 10) + 18$

$= (3 \times 10) + (3 \times 6) \text{ Why?}$

3 must be multiplied by 6 to give 18, and 3 must be multiplied by 10 to give 30.

It is important (and it tends to simplify the solution) to choose the highest possible factor in each place.

We may continue the solution as follows:

$3 \times (\square) = (3 \times 10) + (3 \times 6)$

$= 3 \times (10 + 6) \text{ Why?}$

$= 3 \times 16$
We see that

\[ 48 \div 3 = (\quad) \text{ or } 3 \times (\quad) = 48 \]

can be solved when we put 16 in the brackets or box.

**Stage II.**

\[ 48 \div 3 = \underline{\quad} \]

We may write 48 as \((4 \text{ tens } + 8 \text{ ones})\) or as \((40 + 8)\).

Thus we would have

\[ 3\sqrt{4 \text{ tens } + 8 \text{ ones}} \quad \text{or} \quad 3\sqrt{40 + 8} \]

which can be reduced to

\[ 3\sqrt{3 \text{ tens } + 18 \text{ ones}} \quad \text{or} \quad 3\sqrt{30 + 18} \]

and which give the solutions

\[ \frac{1 \text{ ten } + 6 \text{ ones}}{3\sqrt{3 \text{ tens } + 18 \text{ ones}}} \quad \text{or} \quad \frac{10 + 6}{3\sqrt{30 + 18}} \]

The answer to a problem such as the above division is easy to establish.

With more difficult problems, it may not be easy to find the suitable multiples.

Therefore we show a detailed solution below.

**Stage III.**

\[ 48 \div 3 = \underline{\quad} \]

**Lead up**

\[
\begin{array}{ccc}
3 & \sqrt{48} & 6 \\
\text{subtract} & 30 & 10 \\
& 18 & 6 \\
\end{array}
\]

These two solutions differ only in the manner in which the quotient is recorded. 48 has been regrouped as \((30 + 18)\) as in Stage II. However, the computation appears in vertical form and this is more convenient in later work.
We may further shorten the solution on the right-hand side. Instead of writing 10 in the quotient, merely record 1 in the tens column, and later record 6 in the ones column as follows:

\[
\begin{array}{c|c|c}
\text{Tens} & \text{Ones} \\
\hline
1 & 6 \\
3 & 43 \\
-30 & 18 \\
-18 & \\
\hline
\end{array}
\]

We have only regrouped one time in the given example. When we are working with larger numbers it is often necessary to repeat the regrouping.

Study the example below:

\[
1935 \div 9 = \underline{ } \underline{ } \underline{ } \underline{ } \underline{ } \underline{ } \underline{ }
\]

**Stage III.**

\[
\begin{array}{c|c}
5 & 215 \\
10 & \\
200 & \\
9 & 1935 \\
\hline
1800 & (9 \times 200) \\
135 & \\
-90 & (9 \times 10) \\
45 & \\
-45 & (9 \times 5) \\
\hline
\end{array}
\]

In the solution at the left, note that 1935 has been regrouped as (1800 + 90 + 45). In each place, we have used the largest possible multiple of 9.

Suppose we had the example 3123 \div 9 = \underline{ } \underline{ } \underline{ } \underline{ } \underline{ } . Which of the regroupings shown below would give the "easier" solution?

- \((3000 + 100 + 20 + 3)\)
- \((2700 + 360 + 63)\)

**Exercise 15-7a**

Use the procedure demonstrated in Stage III to find the following quotients:

(Regroup carefully, taking the largest possible multiple for each place.)
15-8 Division by multiples of 10

So far we have divided by numbers represented by one-digit numerals. We can also divide by numbers represented by two-digit numerals. The same principles which we have followed thus far will still be used. We shall regroup the number to be divided, keeping in mind the largest multiples of the divisor in each place. As in previous work, let us start with an easy case, $480 \div 20 = \underline{?}$

In order to regroup 480, keeping in mind multiples of 20, we may regroup it as $(400 + 80)$. In horizontal form, the solution appears:

$$20 \begin{array}{r}400 + 80 \\ 20 + 4 \end{array} = 24$$

In vertical form, we have either of the solutions shown below:

**Stage III.**

```
20 \begin{array}{r}480 \underline{-400} \\ \underline{-80} \end{array} \quad 20 \begin{array}{r}4 \underline{-400} \\ \underline{-80} \end{array} = 24
```

Only when the ideas above are completely understood do we go to the short form:

```
\begin{array}{c|c}
\text{Tens} & \text{Ones} \\
\hline
2 & 4 \\
\hline
20 \underline{-400} & \underline{-40} \\
\underline{-80} & \underline{-80}
\end{array}
```

(a) $340 \div 4 = \underline{?}$  
(b) $1435 \div 5 = \underline{?}$  
(c) $504 \div \underline{?} = 7$  
(d) $1056 \div 8 = \underline{?}$
Exercise 15-8a

(a) \(960 \div 30 = \Box\) 
(b) \(\Box = 840 \div 20\) 
(c) \(460 \div \Box = 20\) 
(d) \(2060 \div 20 = \Box\) 
(e) \(\Box = 1350 \div 10\) 
(f) \(1560 \div \Box = 30\) 
(g) \(41,200 \div 40 = \Box\) 
(h) \(50,250 \div 50 = \Box\)

15-9 Division by a number represented by a numeral of two or more digits not a multiple of 10

In light of this approach, consider the example \(18,102 \div 21 = \Box\)

First we look upon \(18,102\) as 1 ten thousand + 8 thousands + 1 hundred + 0 tens + 2 ones. Since we do not have a multiple of 21 in the ten-thousands place, we use the associative property to regroup \(18,102\) as

\(18\) thousands + 1 hundred + 0 tens + 2 ones.

Again, we do not have a multiple of 21 in the thousands' place. Therefore, we regroup again and get:

\(181\) hundreds + 0 tens + 2 ones.

We notice that 181 in the hundreds' place includes a multiple of 21, since

\(21 \times 8\) hundreds = 168 hundreds.

So we begin with the regrouping

\(168\) hundreds + 13 hundreds + 0 tens + 2 ones.

Just as we took care to find the largest possible multiple of 21 in hundreds' place, we shall do the same in tens' place and in ones' place successively.

This solution can be written in the vertical form of Stage III as follows:
In the solution above, we have found the numeral which makes the following statement true.

\[ 18,102 \div 21 = \square \]

Do you recall that the same numeral makes this sentence true?

\[ 21 \times \square = 18,102 \]

In order to check the solution by division, we may perform the multiplication operation

\[ 21 \times 862. \]

If you perform this multiplication, you should obtain

\[ 21 \times 862 = 18,102. \]

In division examples involving large numbers, it is useful to check the accuracy of the division procedure by performing the multiplication operation.

The solution below shows the sequential steps taken separately. Step (c) is the complete form of the division problem where we record the numerals in the appropriate place-value columns without writing out the full meaning of the separate quotients.
Stage III.

We encourage pupils to check division by multiplication in order to emphasize the meaning of division and multiplication as mutually inverse operations and to provide revision of multiplication where children actually see the need to multiply.

Exercise 15-9a

Solve the following problems. Check your answers by multiplication:

(a) \( \frac{6,496}{\phantom{0000}} = 32 \)  
(b) \( \frac{14,605}{23} = \phantom{0000} \)  
(c) \( \frac{323}{\phantom{0000}} = 17 \)  
(d) \( \frac{9,204}{13} = \phantom{0000} \)  
(e) \( \frac{1,888}{59} = \phantom{0000} \)
### Answers to Chapter 15

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<td>(d) 320</td>
<td>(e) 1944</td>
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<td>(b) 140</td>
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<td>(c) 190</td>
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<td>(d) 370</td>
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<td>(b) 5274</td>
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<td>(e) 10</td>
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<td>(f) 210</td>
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<td>(c) 9</td>
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<td>(d) 21</td>
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<td>(e) 0</td>
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<td>(b) 110</td>
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Exercise 6c:
(a) 210
(b) 101
(c) 1234

Exercise 7a:
(a) 85
(b) 287
(c) 72
(d) 132

Exercise 8a:
(a) 32
(b) 42
(c) 23
(d) 103
(e) 135
(f) 52
(g) 1030
(h) 1005

Exercise 9a:
(a) 203
(b) 635
16-1 Sets in natural order

You learned before how to put sets in natural order. For example, the following sets are in natural order:

- 
- 
- 
- 
- 
- (and so on)

Each of the sets has one more member than the set above it. There is, of course, no end to the sets that you can build in this way.

Exercise 16-1a

Use matches to form sets in natural order. Make sure that the matches are all of the same size. Arrange them in sets as shown below:
Exercise 16-1b

You know that each set has one more member than the set just before it.

What set is just before

\{ \cdots \} ?

What set is just after

\{ \cdots \} ?

What set is just after

\{ \cdots \} ?

16-2 Counting sets and sets in natural order

You also learned how to match a set with a counting set. You can thus tell the number of things in the set you started with. The sets of matches pictured above can be matched with counting sets as follows:
In this way the sets are assigned numbers, and we say we have counted the sets.

**Exercise 16-2a**

In the exercise above you arranged sets of matches end-to-end so that the sets were in natural order. Now match these sets of matches with counting sets, as we did above. An easy way to make your counting sets is as follows. Take a set of matches, and lay a strip of paper beside it as below:

```
---|---|---|---|---|---
```

At the left-hand end of the strip mark 0 for the number of matches in the empty set. At the right-hand end of each match mark the proper numeral on the strip of paper so that the strip is a counting set for that set of matches. Below you can see the example finished.
16 - 3 Using counting sets

What would you do if you wished to make more sets of matches like those you made above, but you had no more matches? One answer would be to use the counting sets, which you made on strips of paper in the preceding exercise, to help you cut from a stick of wood more pieces of the same size as the matches. If you had a thin piece of bamboo or long twig from a tree, you could cut it into pieces of the same size as the matches, by laying one of your counting sets against the bamboo or the twig. If you had a paper counting set long enough, you could tell how many match-lengths you could get from any stick. An example is drawn below:

There are 5 match-lengths in this stick.
Exercise 16-3a

1. Use your paper counting sets to find the number of match-lengths in pencils, sticks, desk-tops and other objects which you find in the classroom. There may be something left over. For example, a stick could be longer than 5 match-lengths and shorter than 6 match-lengths.

2. Make up a new collection of paper counting sets. This time use nails of the same size. With these new paper counting sets find the number of nail-lengths in the same objects used in problem 1.

16-4 Counting sets used to build the number line

By now you have already seen that these numbered strips of paper, these sets which help count match-lengths in things, are very much like rulers. You are really using them to "measure" things when you count the number of match-lengths. If you have a long enough strip of paper, with enough numbers on it, you can use it to measure very many things. In this way you can tell the number of pieces of a given size that are in any of these things.

There is no limit to the length of such a strip of paper, since if you have enough paper and enough time you can make one as long as you wish. And so in your mind you can think of a paper counting set which is longer than anything you can make. The edge of such a paper counting set illustrates a number line. All the whole numbers are on it in order, and you can use the number line to tell the number of pieces of the size you chose, in any object.
Exercise 16-4a

You cannot draw the whole number line, since there is no end to it. But you can draw a piece of it. Make such a piece of the number line on a straight-edged piece of cardboard or flat stick, and use it to count the number of lengths (of the size you chose) in different things in the classroom.

16-5 Size of the unit piece on the number line

In the preceding exercise you made a part of the number line, and used it for counting the number of lengths of a given size in things. In making it, you did one thing which is very important, and if you did not do it, you must work the exercise again. You should have labeled your line in such a way that each numeral was the same distance from the next numeral. When you arranged your sets of matches, you had to do this, because the matches were all the same size. And when you made your counting sets on strips of paper, you had to number your line in this way, because these strips were paired with the sets of matches. You did the same thing when you arranged your sets of nails. You must always do this in making a number line, since only in this way can you count pieces of a given size.

To build a number line, you can use any size piece as your unit piece. But you must keep the same unit size for as much of the number line as you build.
Exercise 16-5a

Make two different number lines on pieces of cardboard or flat sticks. One should have small unit pieces, and the other should have larger unit pieces. Use the two number lines to count the number of unit lengths in different objects in the classroom. Compare the results you get using the two number lines on each object.

16-6 Drawing the number line

You saw in the last exercise that you can put many numerals on a section of the line when each unit piece is small. You probably wondered how many numerals can go on a number line. The answer is easy. The line can be extended further than any part of it you can draw. Someone else can extend your section of line, and you can then go on adding pieces to the end of his line. In fact, pieces can be added to either end of your section of line, so that the line can be extended in either direction.

When you draw a line without any numbers on it, you need to show that it can be extended in both directions. Therefore, you put arrows at both ends of your line. Your line will look like this:

To make a number line out of a given straight line, you mark some point as the starting point, or 0 point. You choose some length as your unit length, and then you keep putting down succeeding numerals as you attach more and more of these units to each other end-to-end. You could add unit pieces in either direction.
from the 0 point, but only the pieces going to the right from the 0 point are numbered. Thus you will end up with a line which looks like this:

![Number Line Diagram]

You can put your 0 point anywhere, and you can make your unit length any length you wish. But once you pick this, and decide to go to the right, you can draw your number line in only one way.

On a number line there is a point for every whole number. Between the points for one whole number and the next whole number, there is one unit of length.

**Exercise 16-6a**

1. Draw a number line on a piece of paper as carefully as you can. Find these numerals on it: 7, 3, 0, 12, 1, 10.

2. On the number line, what number is four units to the right of 9?
   What number is four units to the left of 9?

3. On a number line, mark the whole numbers 8 through 14 only in red. How many whole numbers are marked in red?

4. If a match stick is taken to represent the unit length on a number line, exactly how many match sticks could be placed end-to-end between the points marked 81 and 111?

5. A man takes 6 steps east, 3 steps west, and then 4 more steps east. How many steps is he away from his starting point? Also show the steps on the number line.
6. A man takes 3 steps forward, 1 step backward, 5 steps forward, another 2 steps forward, and then 4 steps backward. How many steps must he take to return to his starting point? Also mark the steps on a number line.

7. Suppose you start at the point 2 on the number line and in each successive step move 3 units to the right. What point do you reach

(a) after 2 steps?

(b) after 7 steps?

(c) after n steps?
17-1 "Less than" on the number line

When you put counting sets in natural order, a counting set with fewer members comes before a counting set with more members. For example, compare the counting sets for 3 and 5, as shown below:

\{1, 2, 3\}
\{1, 2, 3, 4, 5\}

The counting set for 3 has fewer members than the counting set for 5, because matching the sets always leaves members in the second set. In this case, you write

3 is less than 5,

or, in symbols,

3 < 5.

You remember that the number line is built up out of successive counting sets. Look at this number line:

The section of the number line representing the counting set for 5 goes all the way from 0 to 5. The section of the number line representing the counting set for 3 goes from 0 to 3, and thus stops before 5. You see one more important
fact here. The numeral 3 is to the left of the numeral 5. Since the counting set for 3 is completed before the counting set for 5, 3 is to the left of 5.

**Exercise 17-1a**

Make a general statement for whole numbers, so that you can tell from their places on the number line which of two numbers is less.

**17-2  "Greater than" on the number line**

In the same way you can see when one whole number is greater than another by looking at their places on the number line. You know that 9 is greater than 4. You write this as follows:

9 is greater than 4

or, in symbols,

9 > 4.

On the number line 9 is to the right of 4.

If you did the exercise above correctly, you learned that a number m is less than another number n if m is to the left of n on the number line. Likewise, you can see that the number p is greater than the number q, if p is to the right of q on the number line. Look at this number line:

```
0 1 2 3 4 5 6 7 8 9 10 11 12
```

You can see that 12 is to the right of 10, and thus 12 is greater than 10. Also 7 is to the left of 10, and so 7 is less than 10. You can write these two facts in this way:
12 > 10  
12 is greater than 10  
7 < 10  
7 is less than 10  

Exercise 17-2a

Draw a number line and show by a picture which of the following statements are true:

1. $5 < 9$
2. $0 > 1$
3. $12 < 11$
4. $45 < 54$
5. $27 < 20$

Exercise 17-2b

1. If $a > 5$ and $5 < c$, can you find numbers $a$ and $c$ such that:
   (a) $a < c$?
   (b) $a = c$?
   (c) $a > c$?
2. Arrange the numbers $a, b, c$ and $d$ on the number line so that the following relations are satisfied:
   $a < b$
   $c < d$
   $c < a$
   $b < d$
3. Arrange the numbers d, f, n, p and r on the number line using the following relations:
   \[ n < p \]
   \[ f < r \]
   \[ d < f \]
   \[ p < d \]
   \[ d < r \]

4. Arrange the following numbers on the number line using the given relations:
   a > b
   n < b
   p > q
   p > a
   q > a

From the number line find the relation between the following pairs of numbers:
q, a
b, p
b, n

17-3 "Between" on the number line

You can say one more thing about the order of whole numbers on the number line. Go back to the three numbers 10, 12 and 7. You can say which one number lies "between" the other two numbers. 10 is between 7 and 12 be-
cause it lies to the right of 7 and to the left of 12. Or we can say that 10 is greater than 7 and less than 12.

**Exercise 17-3a**

Using the number line, put the three numbers in each of the following sets into natural order, and tell which number is between the other two:

1. 1, 0, 3
2. 5, 7, 8
3. 6, 2, 3
4. 0, 1, 9
5. 11, 16, 14
6. 25, 14, 13

**Exercise 17-3b**

How many whole numbers on the number line lie between:

1. 3 and 7?
2. 30 and 39?
3. a and b when a and b are whole numbers such that a < b?

**17-4 Order and addition and subtraction**

Let us locate 1 and 5 on the number line.
Now let us add 2 to each of them. Each point has been replaced by one 2 units to the right. Since 1 is to the left of 5, $1 + 2$ is to the left of $5 + 2$. In terms of inequalities we can write

$$1 < 5$$

$$1 + 2 < 5 + 2$$

$$3 < 7$$

From the true sentence $1 < 5$, we get the new true sentence $3 < 7$ by adding the same number 2 to both sides.

In general, let $a$ and $b$ be two whole numbers with $a < b$. Let us add the whole number $c$ to both $a$ and $b$. We get $a + c$ and $b + c$. Now $a + c$ is $c$ units to the right of $a$ on the number line and $b + c$ is $c$ units to the right of $b$. We have moved equal distances to the right of $a$ and $b$. This does not change the order. So if

$$a < b$$

is true

$$a + c < b + c$$

is true.

For example, from

$$4 < 6$$

we conclude that

$$8 < 10.$$
same whole number \( c \) from both \( a \) and \( b \). This means that we have moved \( c \) units to the left on the number line.

\[
\begin{array}{c}
0 & \quad a \quad & \quad b \quad & \quad (a-c) \quad & \quad (b-c)
\end{array}
\]

Again the order has not been changed so that

\[ a - c < b - c \]

must be true.

For example, from

\[
4 < 6
\]

it follows that

\[
4 - 3 < 6 - 3,
\]

that is

\[
1 < 3.
\]

Of course, if \( c \) is too large, \( a - c \) will not be a whole number. For example, if we tried to subtract 5 from both sides of \( 4 < 6 \), we would discover that \( 4 - 5 \) was not a whole number.

**Exercise 17-4**

1. We know that it is true that \( 0 < 2 \). By adding \( c = 1, 2, 3, 4 \) and 5 to both sides of this inequality write five new inequalities which are true.

2. Given \( 4 < 9 \) show that by subtracting a proper number from both sides we can get a new inequality with 0 on the left side. Show what you have done by using the number line and arrows.

3. Show that if \( a \) and \( b \) are whole numbers and \( a < b \) then \( 0 < b - a \).

Show also that if \( 0 < b - a \) then \( a < b \).
CHAPTER 18

OPERATIONS ON THE NUMBER LINE

18-1 Comparing sections of the number line

You may have thought of a new problem about the number line. How can you compare two sections or "segments" of the number line which do not begin at zero? All you have done so far is to compare segments beginning at the 0 point, telling which one is bigger and which one is smaller. Look at the number line below and think how you would compare the segment from 3 to 5 with the segment from 0 to 2.

\[ \begin{align*}
0 & \quad 1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 \\
\end{align*} \]

Do they look in some way alike? Yes. Is the number of unit pieces the same in each? Yes. Do they seem to be the same length? Yes.

Exercise 18-1a

How many units are there on the segment from 0 to 2? How many units are there on the segment from 3 to 5?

18-2 Addition

The segment from 0 to 5 is made up of two segments, one from 0 to 3,
and the other from 3 to 5. Thus the segment consisting of 5 unit pieces can be broken down into two pieces, one of 3 units, and one of 2 units. You can see what you are leading up to in this way -- and if you guide children correctly they will see it too. You are leading up to addition, shown on the number line.

The sum $3 + 2$ on the number line thus looks like the following:

![](image)

$3 + 2$

Any addition problem can be shown in this way, as you can plainly see.

In general we may picture $(a + b)$ on the number line as follows:

![](image)

When you add on the number line you move to the right.

**Exercise 18-2a**

Show the following additions on the number line:

1. $2 + 4$
2. $3 + 3$
3. $4 + 2$
4. $5 + 1$
5. $6 + 0$
6. $12 + 7$
7. $19 + 0$
8. $6 + 11$
9. $11 + 6$
10. $0 + 6$
Exercise 18-2b

While Bandele sat at his desk looking at the number line in his arithmetic book, a cricket jumped on the book, landing at 0. First it jumped to 7 and then it jumped on to 11. This was a very smart cricket and was really doing an addition problem. What was that problem?

Exercise 18-2c

Tell how you can use examples of the kind in the exercise above to help children understand the number line. Make up games and stories to explain each of the addition problems given above.

Exercise 18-2d

1. If \( x \) is a whole number between 3 and 6 and \( y \) is a whole number between 2 and 7, illustrate on the number line the set of all possible points corresponding to \( x + y \).

2. Starting from the point 0 on the number line a boy lays match sticks in the following order:
   1 stick, then 2 sticks, next 3 sticks and so on. How many sticks would he have used after laying the fifth set of sticks?

3. A child jumping along a straight track marked in feet, made jumps of 3 feet, 4 feet, 3 feet, 4 feet, and so on. If he started at the point 0 with a jump of 3 feet, after what jump would he land on the point marked 31 feet?
18-3 Commutative and associative properties of addition

You can see from the number line that such sums as $3 + 2$ and $2 + 3$ give the same number.

In illustrating the operation of addition it does not matter which of the moves on the number line is taken first.

Exercise 18-3a

1. What is the name of this property of addition, and what is the most general way to state it? If you do not remember, refer to the section on properties of addition.

2. Prepare word problems which will help children to understand this property through use of the number line.

3. Name another property of addition. Give examples on the number line to help children understand the property.

4. Fill in the missing blanks.
(a) The above diagram on the number line shows that \[\_\_\_\_ = \_\_\_\_\].

(b) The diagram illustrates the \_\_\_\_ property and also the \_\_\_\_ property of addition.

### 18-4 Subtraction

It is not always necessary to go in one direction on the number line.

The cricket who landed on Bandele's book might have jumped from 0 to 7 and then jumped back 2 units. In this case he would have landed at 5. You can draw his moves on the number line below:

![Number Line Diagram](image)

#### Exercise 18-4a

Draw on a number line the moves of a frog making the following jumps in the order given:

1. 0 to 3
2. 3 to 7
3. 7 to 5
4. 5 to 11
5. 11 to 4

#### Exercise 18-4b

What operation do you think the frog is performing when he jumps to the left on the line? What operation when he jumps to the right? What name was given in an earlier section to the relation between these two operations?

This second operation, that of jumping to the left on the number line, is of course subtraction. In the drawing above, the final resting place of the cricket is at 5, which is 7-2. You can also think of subtraction on the number
line as finding a missing jump to the right: if the cricket jumped from 0 to 3 on his first jump, and then wanted to land at 8, how many units would he have to jump to the right? Of course, the answer is 5, since $8 - 3 = 5$.

**Exercise 18-4c**

Show the following subtractions on the number line, both as jumps to the left, and as missing jumps to the right:

1. $4 - 2$
2. $5 - 3$
3. $6 - 4$
4. $2 - 0$
5. $17 - 8$
6. $9 - 0$
7. $15 - 14$
8. $12 - 12$

**Exercise 18-4d**

1. If $x$ is a whole number between 8 and 12, and $y$ is a whole number between 3 and 7, illustrate on the number line the set of all possible points corresponding to whole numbers $x - y$.
2. If you begin with 12 and repeatedly subtract 2, after how many steps will the process end?
3. Starting from the point marked 35 on the number line, every third point to the left is marked with a cross. What is the last point marked with a cross? How many points are marked with a cross?
Exercise 18-4e

1. You have learned that addition is commutative. Do you think that subtraction is commutative? Hint: On the number line, do the subtraction 5 - 3. Can you do the subtraction 3 - 5?

2. Determine the points 4 - (3 - 1) and (4 - 3) - 1 on the number line. What does this show about the associative property for subtraction?

18-5 Repeated addition

Imagine now that you have a cricket land at 0 on your book, and then start to jump by 3's to the right along the number line. He will start at 0 and then land at 3, 6, 9, 12, 15 and so on. His jumps will look like this:

This method of jumping illustrates repeated addition of successive 3's. It would be possible to think of other ways of jumping -- by 2's or 7's or by any other number.

18-6 Multiplication as repeated addition

What operation is related to repeated addition? What other way did you find to think of this operation? This operation is, of course, multiplication, which can be shown on the number line as repeated addition. Thus the stopping points of the cricket in the previous section could be written as $1 \times 3$, $2 \times 3$, $3 \times 3$, $4 \times 3$, and so on. The first number shows the number of jumps it makes, and
the second number shows how many units it covers in each jump.

**Exercise 18-6a**

Show on the number line the following products:

1. $6 \times 2$
2. $7 \times 1$
3. $1 \times 7$
4. $5 \times 3$
5. $2 \times 6$
6. $6 \times 2$

**Exercise 18-6b**

Make up word problems to go with each of the above products, which will help children see how they can use the number line to find the answers.

**18-7 Commutative property of multiplication**

You have noticed by now that, for example, $2 \times 6$ and $6 \times 2$, will take you to the same point on the number line. On the number line, you can illustrate that the product of two numbers takes you to the same point, in whichever order they are multiplied. We may say that $a \times b = b \times a$.

**Exercise 18-7a**

Illustrate that when $3 \times 5$ and $5 \times 3$ are shown on the number line, they give the same point. Does this give you any idea how you might illustrate that $a \times b$ gives you the same point as $b \times a$ on the number line? Hint: Look back at the section under multiplication to see how the commutative property was demonstrated. Try to do the same thing on the number line, using an
18-8 Associative and distributive properties of multiplication

You remember that there are two other properties of multiplication -- the associative and distributive properties. An illustration of the associative property on the number line is as follows:

\[(2 \times 2) \times 3\]

Thus, \((2 \times 2) \times 3 = 2 \times (2 \times 3)\).

An illustration of the distributive property on the number line is as follows:

\[3 \times (2 + 4)\]

\[(3 \times 2) + (3 \times 4)\]

Thus, \(3 \times (2 + 4) = (3 \times 2) + (3 \times 4)\)

Exercise 18-8a

Make up other sample problems for the associative and distributive
properties and show them on the number line.

18-9 Division

Suppose that the cricket landed on your number line at 0, and that he could jump only 3 spaces at a time. How many jumps would he need to get to 12? You could draw it this way:

0 1 2 3 4 5 6 7 8 9 10 11 12

You can see that this is really a division problem since he makes 4 jumps of 3 units each to get to 12. You know this because $12 \div 3 = 4$.

Exercise 18-9a

1. Kofi had 10 shs., of which he spent 2 shs. every day. How many days did his money last? Show the result on the number line.
2. Make up other word problems to help children understand division on the number line.
3. You were asked in a previous exercise to tell the name for the relation between addition and subtraction, since each "undoes" what the other "does." What is the name for the relation between multiplication and division? Give an example on the number line.

Exercise 18-9b

1. You can make a very simple adding machine in the following way. Take
a piece of paper and draw a thick straight line. Construct a portion of the number line on this straight line, and mark the points heavily, placing the numerals both above and below the line as follows:

```
0 1 2 3 4 5 6 7 8
0 1 2 3 4 5 6 7 8
```

Now cut the paper along this line. You should now have two copies of the number line which fit together. To add two numbers, simply place the 0 on the upper line over the first of the numbers on the lower line, and read the answer on the lower line below the second number on the upper line. For example, the following shows how to find 2 + 3 and obtain 5.

```
0 1 2 3 4 5 6 7 8
0 1 2 3 4 5 6 7 8
```

Notice that we can read off the answer for 2 plus any whole number with this setting. Do several additions in this way.

2. Devise a simple machine using two number lines which perform subtractions.

3. Devise a simple machine which can be used to multiply by 3, using two number lines. Two possibilities are illustrated below:
UNIT IV - FRACTIONS

CHAPTER 19

INTRODUCTION TO FRACTIONS

19-1 A reminder of division

You can think of division in several ways. If you have a set of objects, division can tell you how many members are in each of a given number of equivalent disjoint subsets into which the set is shared. Or you can reverse the conditions and use division to tell you how many disjoint subsets, each with a given number of members, can be formed from the set. Either way, division helps you work out a multiplication problem with a missing factor.

You learned how to solve such problems as the following:

\[
24 = 6 \times [ ]
\]

\[
132 = [ ] \times 11.
\]

In the first problem, you want to know how many members there are in each of 6 equivalent disjoint subsets of a set of 24 members. In the second problem, you want to know the number of disjoint subsets of 11 members each that can be formed from a set of 132 members. The answers to these problems can be written using division, as follows:

\[
[ ] = 24 \div 6 = 4
\]

\[
[ ] = 132 \div 11 = 12.
\]

In both the answer is a whole number. You know that for many problems like these the answers are whole numbers. Such problems cause you no real trouble.

Exercise 19-1a

Which of these division problems have whole-number answers?
There is real trouble, however, when you cannot find a whole-number answer to a division problem. If you divide 81 by 8, you find that there are ten 8's, with 1 remaining. Thus, if you have 81 candies and wish to share them equally among 8 children, each can have 10 candies, but there will remain 1 candy which is not shared. It is hard to know how to speak of this remaining 1. In this chapter you will learn how to take care of such problems with fractions.

**Exercise 19-1b**

Write out some word problems involving division with remainders, which you can use with children in primary classes.

**19-2 Names for fractions**

Problems like that of sharing 81 candies among 8 children are not new problems, of course. Nor is the way of solving the problem new. The 1 remaining candy can be broken into 8 parts of equal size, and each child given 1 of these 8 parts. The more difficult problem is how to speak of what each child gets. For this purpose, special words have been made up, telling what each receives in the sharing. In English, you say that each child has received 1 eighth part of the left-over piece of candy. Thus the answer to the problem of sharing 81 candies among 8 children is that each child receives 10 and 1 eighth candies.

You can speak in the same way in English of the result in any sharing problem. For example, if 1 banana is shared between 2 people, each is said to...
receive 1 half of that banana, as in the following picture:

If it is shared among 3 people, each receives 1 third of it, as in the following picture:

If there are 4 persons, each receives 1 fourth, and so on.

Moreover, you know how to speak in English of what happens when 2 or more things are shared among several people. For example, if 2 bananas are shared among 3 people, each receives 2 thirds, as in the following picture:

If 5 things are shared among 8 people each receives 5 eighths. In a more difficult problem, if 13 oranges are shared among 5 people, you can think of the number each gets as follows. Clearly each will get 2 whole oranges, and then the remaining 3 oranges will be shared so that each gets 3 fifths. Thus each gets 2 and 3 fifths oranges. You can show this in a picture as follows:

Words describing results of such sharing problems are found in most languages, although in some cases the system is not so complete as in English.

Exercise 19-2a

If you speak a language other than English, find what words are used for
naming fractions. Write them in a list and compare them with the English words.

What use do you think you can make of this list in teaching fractions to children who speak your language?

19-3 Symbols for fractions

As you can see, the idea of a fraction is really very easy. The fraction words simply name what part is received when the corresponding division problem has a remainder.

It is more difficult to choose good symbols for fractions. One such system of symbols is that invented in Europe a few hundred years ago. This system is related to the Hindu-Arabic system of symbols for whole numbers in a direct way. The fraction names, such as 2 thirds, are simply written as pairs of numerals separated by a stroke. Thus you have learned to write 2-thirds as \( \frac{2}{3} \) or \( \frac{2}{3} \). It does not matter whether the stroke is a slanting line separating two numerals on the same line, or a horizontal line separating one numeral above another. The numeral above the line is called the NUMERATOR, and the numeral below the line is called the DENOMINATOR. The fraction \( \frac{2}{3} \), to use the example given above, tells what part each gets when 2 bananas are shared among 3 people. The result can be shown in this picture:

\[
\begin{align*}
\text{\( \frac{2}{3} \)} & \quad \text{\( \frac{2}{3} \)} & \quad \text{\( \frac{2}{3} \)} \\
\end{align*}
\]

The symbol for a fraction is useful since it tells into how many parts each whole thing is divided, and how many of these parts each person receives.

Thus, in the example of 2 bananas shared among 3 persons, the fraction \( \frac{2}{3} \)
19-5

tells that each banana is divided into 3 parts, and each person receives 2 of these parts. Each of these parts is 1 third of a whole banana, and each person receives 2 of these third parts. Thus you read $\frac{2}{3}$ as "2 thirds." You can see, therefore, that $\frac{2}{3}$ can be thought of either as the result of sharing 2 things into 3 by breaking the things into 3 parts each, or as 2 of the third parts of a given type of thing.

**Exercise 19-3a**

State in words the meanings of the following fractions:

1. $\frac{1}{4}$
2. $\frac{5}{8}$
3. $\frac{2}{10}$
4. $\frac{3}{2}$
5. $\frac{1}{100}$
6. $\frac{2}{1}$

19-4 Proper and improper fractions and mixed numbers

Some of the problems above may have given you trouble. Number 4 is one to think about. What does it mean to write $\frac{3}{2}$? An example which might help you is the following. Suppose you had 3 bananas to share between 2 people. Perhaps the fairest way to share them would be to cut each of them in half, and then give each person half of every banana. In this way each person gets 3-halves of bananas, and this can be written $\frac{3}{2}$. Such a fraction is called an improper fraction. Actually there is nothing at all improper about it. It is just as good a fraction as $\frac{1}{2}$, which you learned to call a proper fraction. The reason why people have used the word improper was that they could not, obviously, divide one thing into more than 2 halves. But there is no reason why you cannot divide 3 whole things into halves and count the halves you get.
There is another way of writing an improper fraction, and that is as a mixed number. Take the same problem of 3 bananas to be divided between 2 people. If the bananas all looked the same, the people might each take 1 banana and then cut the remaining banana in half. In this way each person receives 1 banana and 1 half banana. This can be written in short form, of course, as $1 \frac{1}{2}$ bananas. This kind of fraction you learned to call a mixed number because it has in it both the whole number 1 and the fraction $\frac{1}{2}$. Of course, it means the same thing as the improper fraction $\frac{3}{2}$.

Exercise 19-4a

Make up word problems which show the meaning of each of the following fractions:

1. $\frac{2}{5}$
2. $\frac{9}{8}$
3. $3 \frac{1}{2}$
4. $2 \frac{2}{3}$
5. $\frac{17}{2}$
6. $5 \frac{5}{6}$

Exercise 19-4b

Find the improper fractions which are equal to each of the mixed numbers in the problems of the preceding exercise. Find the mixed numbers which are equal to each of the improper fractions in the problems of the preceding exercise.

19-5 The relation of fractions to division

Fractions can give answers to problems which arise in division. Thus we can relate fractions to division in an easy way. Take the fraction $\frac{4}{2}$. This means, as you know, 4 halves, and may be seen to come from sharing 4 whole
things between 2 people so that each person gets half of each of the things.
But you can also think of sharing 4 things between 2 people in terms of the divi-
sion equation $2 \times \square = 4$. Both $\frac{4}{2}$ and $4 \div 2$ can be put into the box to make
the sentence true, and so you can write

$$\frac{4}{2} = 4 \div 2.$$  

Similarly, you can think of any fraction as answering a division problem. For
example, the fraction $\frac{2}{5}$ can be thought of as the answer to the division
equation

$$\square = 2 \div 5.$$  

In this way you see that fractions help you answer division problems which
previously you could not do. In the problem $2 = 5 \times \square$, you wish to know how
much is in each of the 5 parts into which 2 whole objects are divided. From
division, you can write $\square = 2 \div 5$, and you know that the answer is not a
whole number. But from your study of fractions you know that $\square = \frac{2}{5}$. Thus
you can write

$$2 \div 5 = \frac{2}{5},$$  

which gives you the answer to the division problem you could not do before.

In general, if you have the problem

$$\square \times a = b,$$

you can see that the answer, by division, is to be written

$$\square = b \div a.$$

Now you know that it is possible to show $b \div a$ as a fraction and get the
answer

$$\square = \frac{b}{a},$$
so that you can write

\[ b \div a = \frac{b}{a}. \]

**Exercise 19-5a**

Solve the following division problems, using both improper fractions and mixed numbers in your answers.

1. \( 72 \div 7 \)
2. \( 13 \div 4 \)
3. \( 96 \div 6 \)
4. \( 147 \div 24 \)
5. \( 2176 \div 322 \)
6. \( 625 \div 10 \)
7. \( 83 \div 83 \)
8. \( 1215 \div 100 \)

**19-6 Decimal fractions**

There is a way to write the special mixed numbers which are answers to such problems as 6. and 8. in the previous exercise. Instead of writing

\[ 625 \div 10 = 62 \frac{5}{10} \]

you can write

\[ 625 \div 10 = 62.5. \]

In this statement, the number 62.5 is called a decimal fraction. It is a short way of writing \( 62 \frac{5}{10} \). The decimal point separates the whole number from the fractional part. The one-digit numeral after the decimal point tells how many tenths there are. Thus, for example, the decimal fraction 2.3 means \( 2 \frac{3}{10} \).

In the same way, you can see that the decimal fraction .7 means simply \( \frac{7}{10} \).

Problem 8. in the exercise above is another problem of this same kind.

In it you were asked to divide 1215 by 100. In this case, you write

\[ 1215 \div 100 = 12.15 \]
which is a short way of writing

\[ 1215 \div 100 = 12 \frac{15}{100} . \]

The two-digit numeral after the decimal point tells how many hundredths there are, just as in the former case the one-digit numeral after the decimal point tells how many tenths there are.

**Exercise 19-6a**

Write the answers to the following division problems as decimal fractions:

1. \( 23 \div 10 \)
2. \( 7 \div 10 \)
3. \( 813 \div 100 \)
4. \( 2 \div 100 \)

**Exercise 19-6b**

State in words the numbers represented by the following decimal fraction:

1. 61.7
2. 8.81
3. .54
4. 1.06

**19-7 Pictures to represent fractions**

One easy way to show the meaning of a fraction by a picture is to use pieces of paper cut or folded into parts. For example, the fraction \( \frac{1}{2} \) can be shown as in the following picture:

\[ \frac{1}{2} \]

Then to show the fraction \( \frac{1}{4} \), each of the parts in the above picture can be cut or folded in half. The result looks like this:
If you want to help young children understand fractions, you can let them fold or cut pieces of paper to look like these pictures. It is useful to fold a piece of paper and then have the child name the parts, as in this picture, where sixths are shown.

To show an improper fraction, more than one such piece of paper can be used. For instance, the fraction 7/4 can be shown as follows, where the shaded part shows the fraction:

If the first rectangle were not folded into fourths, the same picture drawn as follows would show the mixed fraction 1 3/4.
Exercise 19-7a

Make pictures for the following fractions by using folded pieces of paper. Explain how this method is useful for teaching fractions to children.

1. \( \frac{3}{8} \)  
4. \( \frac{7}{3} \)
2. \( \frac{2}{3} \)  
5. \( \frac{1}{8} \)
3. \( 1 \frac{1}{2} \)  
6. \( 2 \frac{1}{8} \)

Another common way to show fractions in a picture is by cutting up a circle of paper into pieces. For example, you can show \( \frac{1}{4} \) this way:

Or you can show \( \frac{3}{5} \) this way:

In the same manner as above, you can show mixed and improper fractions by using circles cut into pieces. The fraction \( \frac{9}{4} \) can be shown this way:
Exercise 19-7b

Draw circle pictures to show these fractions:

1. \( \frac{2}{4} \)

2. \( \frac{9}{8} \)

3. \( 2\frac{1}{4} \)

4. \( \frac{10}{5} \)

Exercise 19-7c

Describe some concrete examples for fractions as parts of circles.

It is also useful to picture a fraction in terms of a subset of a set. For example, you might have 35 children in your class, and 23 of them might be boys. In this case you can say that \( \frac{23}{35} \) of the class are boys. This means, of course, that there are 35 members, or parts, in the class, and of those 35 parts, 23 are boys. For another example, we can say that \( \frac{1}{32} \) of the independent African states have emperors, since only Ethiopia has an emperor.

You can show fractions as subsets by drawing pictures of sets. If you have a set with 7 members, and subset with 2 members, as drawn below, the subset can be said to have \( \frac{2}{7} \) of the members of the set.

Exercise 19-7d

1. What fraction of all the countries in the United Nations are African countries?

2. What fraction of the trainees in your class have names beginning with A?
Exercise 19-7e

Write word problems using sets and subsets to show the meaning of these fractions for school children:

1. \( \frac{3}{5} \)
2. \( \frac{9}{11} \)
3. \( \frac{7}{7} \)
4. \( .8 \)

One further common way in which sets often appear is as arrays of dots. For instance, a set of 20 dots can be drawn as follows:

```
  0 0 0 0 0
  0 0 0 0 0
  0 0 0 0 0
  0 0 0 0 0
```

You can show the fraction \( \frac{4}{20} \) as the left-hand column of dots in the array, as in this picture:

```
  0 0 0 0 0
  0 0 0 0 0
  0 0 0 0 0
  0 0 0 0
```

Exercise 19-7f

Show the following fractions by using arrays of dots:

1. \( \frac{3}{9} \)
2. \( \frac{2}{6} \)
3. \( \frac{4}{10} \)
4. \( \frac{9}{15} \)
5. \( \frac{2}{4} \)
6. \( \frac{15}{45} \)
19-8 Fractions on the number line

If you look back to the sections on the number line, you will see that you thought of the number line as growing out of successive counting sets. The ruler represents part of such a number line, and can be used to tell the number of pieces of a certain size in a portion of a line. If the ruler is placed against an object having a straight edge, it can be used to tell, for instance, the number of inch-lengths in that object, as in the following picture:

As you know, very often the thing to be measured with a ruler does not come out just at an exact inch mark. You might get a situation like this:

In this case, the unit length of one inch is too large, because the stick being measured does not come out exactly in inches.

In order to speak of the length of the stick, you have to divide the unit into smaller pieces. In the case shown above, it is enough to break each unit piece into two smaller equal-sized pieces. Each of those pieces is, of course, half an inch, and thus the length of the stick is $5 \frac{1}{2}$ inches. In this way, you can see how to show fractions on the number line. You break up the unit piece on the number line into the right number of equal-sized parts, and name these parts as fractions.

Here is a number line with the unit pieces broken into four parts, and with the
correct fractions attached to each point.

![Number line diagram]

**Exercise 19-8a**

Draw carefully a number line showing each unit piece divided into tenths. Label each point with the proper fraction.

**Exercise 19-8b**

Place the following fractions at the proper points on that number line you drew in the last exercise:

1. \( \frac{2}{5} \)
2. \( \frac{5}{2} \)
3. \( 7 \frac{3}{4} \)
4. \( 3.8 \)

You can see by looking at the number line that if you have a good enough pencil and a sharp enough eye, you can mark a point showing any fraction. And between any two points you can always put another point, marking another fraction between those two fractions. Thus there is no end to fractions.

**19-9 Order properties**

When you learned about the number line before, you saw that you could put the whole numbers in order, from 0 and on to the right. You saw that the fact that \( 5 < 8 \) means that 5 is to the left of 8 on the number line. Another reason that \( 5 < 8 \) is, of course, that a set of 5 things has fewer members than a set of 8 things.
You can do the same sort of thing with fractions. Look at the number line drawn below, where each unit piece is divided into three equal-sized parts, and the fractions are put at the proper points.

\[ \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \frac{8}{3}, 3, \frac{10}{3}, \frac{11}{3}, 4, \frac{13}{3}, \frac{14}{3}, 5, \frac{16}{3}, \frac{17}{3}, 6, \frac{19}{3}, \frac{20}{3} \]

You can say from the number line that \( \frac{1}{3} < \frac{2}{3} \), because \( \frac{1}{3} \) is to the left of \( \frac{2}{3} \) on the line. That \( \frac{1}{3} < \frac{2}{3} \) can be pictured in terms of sets. If a thing is divided into thirds, the set of 1 of those thirds has fewer members than the set of 2 of those thirds.

In a similar way you can say that one fraction is greater than another. On the number line you can see that \( \frac{7}{3} > \frac{4}{3} \), because \( \frac{7}{3} \) is to the right of \( \frac{4}{3} \).

Moreover, by looking at the number line, you can see that 5 is greater than 4 and less than \( \frac{16}{3} \), and thus lies between 4 and \( \frac{16}{3} \).

**Exercise 19-9a**

By drawing a number line and carefully labeling the points, find which one of the two fractions in each pair is greater.

1. \( \frac{3}{4} \) and \( \frac{3}{5} \)
2. 1.2 and 1.1
3. \( \frac{5}{6} \) and \( \frac{3}{4} \)
4. \( 2\frac{1}{2} \) and \( 3\frac{1}{2} \)
5. \( \frac{9}{5} \) and \( 1\frac{2}{3} \)
6. 3.2 and \( \frac{17}{5} \)

**Exercise 19-9b**

Tell how you would make a number line which you could use to demonstrate fractions in a classroom with no blackboard.
CHAPTER 20

PROPERTIES OF FRACTIONS

20-1 Equal fractions: parts of a whole

You probably noticed in some of the problems you have done that there seemed to be different ways of writing the same fraction.

Take this problem, for example. A man had a piece of sugar cane, which he cut into 4 equal pieces for his children. But only 2 of the children came to get some. Each of the 2 children thus got 2 pieces of sugar cane. You can write this fraction then as $\frac{2}{4}$, since each got 2 fourth parts. You can draw the picture as follows:

When you look at the picture, you see that each child would get the same amount of sugar cane as he would if the man cut his piece into only 2 equal parts and gave each child 1 part. In this case you can write the fraction as $\frac{1}{2}$, since each got 1 part out of 2. The picture looks like this:
You can see from this example that the fraction \( \frac{1}{2} \) and the fraction \( \frac{2}{4} \) give the same thing. Thus you can write
\[
\frac{1}{2} = \frac{2}{4}
\]
If you think about fractions in this way, you can see that there are many equal pairs of fractions. Think about the fraction \( \frac{6}{5} \) for example. One way to show it is in terms of circles:

Here the circles are each cut into 5 parts, and 6 of these fifth parts are shaded. Now if each of the 5 parts is again cut into 2 parts, the picture looks like this:

You can see that the first picture shows \( \frac{6}{5} \). The second picture shows \( \frac{12}{10} \), since each circle is divided into 10 parts, and 12 of these tenth parts are shaded. Thus you can write
\[
\frac{6}{5} = \frac{12}{10}
\]
You remember also from earlier work that
\[
\frac{12}{10} = 1.2
\]
and thus you can see that
\[ \frac{6}{5} = 1.2 \]

Another way to write the same fraction is
\[ \frac{6}{5} = \frac{11}{5} \]

Thus you can see the following:
\[ \frac{11}{5} = \frac{6}{5} = \frac{12}{10} = \frac{12}{10} = 1.2 \]

**Exercise 20-1a**

Write each fraction in five different ways, and show a picture for each way of writing it.

1. \( \frac{2}{8} \)
2. 1.7
3. \( \frac{32}{3} \)
4. .7
5. 1.25
6. \( \frac{16}{4} \)

**Exercise 20-1b**

Write word problems suitable for use with young children to help them understand different ways of writing fractions.

**20-2 Fractions on the number line**

You can see these same things about fractions by using the number line.

For example, you can divide each unit length on the number line into 2 equal parts, as in this picture:
If you then divide each of those parts again into 3 parts, you get this picture.

From these two pictures of the number line, you can see that there can be many different ways of naming the same fraction. For instance, you can see that the point which was before numbered \( \frac{1}{2} \) can now be numbered \( \frac{3}{6} \), since \( \frac{1}{2} \) and \( \frac{3}{6} \) come at the same point on the line. This is easy to see also in terms of parts of wholes, since 1 part out of 2 can be thought of as the same as 3 parts out of 6. For another example, you can see that the point marked \( \frac{1}{2} \) can now be given the name \( \frac{9}{6} \). Of course it could also be given the name \( \frac{3}{2} \). So you get the following:

\[
\frac{1}{2} = \frac{3}{6} = \frac{9}{6} = \frac{13}{6}
\]

You know that you can keep on dividing the number line up into smaller and smaller pieces, and so you see that you can get many different fraction names for the same fraction.

**Exercise 20-2a**

Draw a number line as carefully as you can on a large sheet of paper. Mark it with numbers from 0 to 3 and show that the following strings of facts hold for that number line.

1. \[
\frac{1}{2} = \frac{2}{4} = \frac{4}{8} = \frac{8}{16}
\]

2. \[
\frac{1}{3} = \frac{4}{12} = \frac{12}{36} = \frac{13}{39} = \frac{12}{9}
\]
3. \[ \frac{22}{5} = \frac{12}{5} = 2.4 = \frac{24}{10} = 2 \frac{4}{10} \]

20-3 Many names for one number

Each time you have thought about numbers you have learned that they could be named in many different ways. For example, when you were studying whole numbers, you learned that 2 could be written in all the ways shown in this string:

\[ 2 = 1 + 1 = 5 - 3 = 2 \times 1 = 6 \div 3 \]

There is no end to the number of ways you can write names for 2 using the four operations. But now you have learned some more ways to write names for numbers, this time the fractions. For example, you can write the following string:

\[ \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{20}{10} \]

or the following:

\[ \frac{13}{4} = \frac{7}{4} = \frac{14}{8} = \frac{16}{8} \]

Just as you yourself may be called by more than one name, so every number has many names. If you are going to understand numbers well, you must know how to find their names.

20-4 The secret

There is an easy way to tell when two names represent the same fraction. You know how to do this already, because someone taught it to you as a schoolchild. But when you enter a classroom to teach, you will find that you have to teach the easy way to children. There are two ways in which you can
do this: tell it to them, and let them memorize it; or let them find it out for themselves. Surely the better way is to let the children find it out for themselves, since then they can have the joy of discovery. Ask them "What is the secret?", and they will soon know it and never forget it!

You probably already know the secret, but if you don't, you should think about it. Are the following fractions all equal?

$$\frac{2}{5} = \frac{4}{10} = \frac{6}{20} = \frac{8}{17} = \frac{10}{25}$$

If you start with $\frac{2}{5}$, some of the rest are names for the same fraction, but not all of them are. Which ones are not $\frac{2}{5}$? If you draw a number line, you can see that $\frac{6}{30}$ and $\frac{8}{17}$ are wrong, but that $\frac{4}{10}$ and $\frac{10}{25}$ are right. But do you have to draw a number line or some other picture? No, you don't.

What is the secret?

To find the secret, try to think, without drawing any picture, of more names for the fraction $\frac{2}{5}$. Some of them are $\frac{4}{10}$, $\frac{6}{15}$, $\frac{8}{20}$, $\frac{10}{25}$, $\frac{12}{30}$. How can you find more? How can you find as many as you want? Can you find 100 different names for the same fraction in an easy way? Yes, you can, if you know the secret. And so can your pupils if they also know it. But do not tell them the secret. They should find it out for themselves. And if you help them in the way shown in this section, they will find out, and they will understand it.

If they -- and you -- still do not know the secret, you can think of the reason why $\frac{2}{5}$ and $\frac{4}{10}$ are names for the same fraction. You remember that you can take a circle divided into 5 parts, and then divide each part into 2 parts. In this way 2 fifth parts are the same as 4 tenth parts. Or you can
divide each part into 3 parts, and find that 2 fifth parts are the same as 6 fifteenth parts. Or you can divide each part into 43 parts, and find that 2 fifth parts are the same as 86 215th parts. You get this string of facts:

\[
\frac{2}{5} = \frac{4}{10} = \frac{6}{15} = \frac{86}{215}
\]

Now what is the secret?

You will learn the secret here, because you will be a teacher, and you should know how to say it in an easy way. But when you are in a class you should not tell it, until all the children have figured it out for themselves.

The secret can be put this way:

MULTIPLY TOP AND BOTTOM BY THE SAME NUMBER

You get a new name which represents the same fraction as \(\frac{2}{5}\) this way:

multiply the top (the numerator) and bottom (the denominator) of \(\frac{2}{5}\) by 7. You get \(\frac{14}{35}\), which you can see is another name for \(\frac{2}{5}\).

\[
\frac{2}{5} = \frac{2 \times 7}{5 \times 7} = \frac{14}{35}
\]

The secret works the other way around, too. Start with \(\frac{6}{15}\). You get this fraction if you multiply the top and bottom of \(\frac{2}{5}\) by 3. So if you divide the numerator and denominator of \(\frac{6}{15}\) by 3 you get \(\frac{2}{5}\), and you see that \(\frac{6}{15}\) is another name for \(\frac{2}{5}\).

\[
\frac{6}{15} = \frac{2 \times 3}{5 \times 3} = \frac{2}{5}
\]

A useful way to think of the secret in a case like this is:

DIVIDE TOP AND BOTTOM BY THE SAME NUMBER

We can put the property of equal fractions, which is expressed in the
secret we have found as follows: if $\frac{a}{b}$ is a fraction and $c$ is not 0, then

$$\frac{a}{b} = \frac{a \times c}{b \times c}$$

If we think of this property of equal fractions as going from left to right, it "fattens up" the fraction $\frac{a}{b}$ to $\frac{(a \times c)}{(b \times c)}$. If we think of this property as going from right to left, it "slims down" the fraction $\frac{(a \times c)}{(b \times c)}$ to $\frac{a}{b}$.

Now you know the secret. Don't tell your pupils the secret. Let them find it out, but help them in every way you can to find it out.

**Exercise 20-4a**

Use this property of fractions to find ten other names for each of these fractions.

1. $\frac{2}{3}$
2. $\frac{5}{2}$
3. $\frac{3}{4}$

4. $\frac{5}{6}$
5. $\frac{2}{5}$
6. $\frac{2}{2}$

**Exercise 20-4b**

Use the secret to find numbers which make each of the following sentences true:

1. $\frac{2}{4} = \frac{20}{\square}$
2. $\frac{7}{3} = \frac{\square}{63}$
3. $\frac{a}{5} = \frac{39}{15}$

4. $\frac{7}{b} = \frac{2}{6}$
5. $\frac{x}{220} = \frac{5}{11}$
6. $3.2 = \frac{64}{n}$

**Exercise 20-4c**

Write out a story from your pupils' experience which will help them find the secret.
Exercise 20-4d

For each fraction find the fraction with the smallest numerator and denominator which is equal to it:

1. \( \frac{12}{15} \) 
2. \( \frac{24}{36} \)
3. \( \frac{72}{28} \)
4. \( \frac{40}{50} \)
5. \( \frac{27}{18} \)

20-5 Whole numbers and fractions

In the preceding section you thought about this string of facts:

\( \frac{2}{1} = \frac{4}{2} = \frac{6}{3} = \frac{20}{10} \)

You know what the last three fractions mean. \( \frac{4}{2} \) means to divide wholes into halves and take 4 such halves. \( \frac{6}{3} \) means to divide wholes into thirds and take 6 such thirds. \( \frac{20}{10} \) means to divide wholes into tenths and take 20 such tenths. But what does \( \frac{2}{1} \) mean? On this same basis it would mean to divide wholes into 1 part each, and take 2 such parts.

Another way to think about such fractions is to take the fractions \( \frac{3}{4}, \frac{3}{3}, \frac{3}{2}, \) and \( \frac{3}{1} \). The first one means to divide things into fourths and to take 3 such fourths. The second one means to divide things into thirds, and take 3 such thirds. The third one means to divide things into halves and take 3 such halves. The fourth one means to divide things into 1 part, and take 3 such parts. Using rectangles to show fractions, these look like this:
Exercise 20-5a

Draw pictures, using rectangles, circles and the number line, to show these fractions.

1. \[ \frac{5}{1} \]  
2. \[ \frac{3}{1} \]  
3. \[ \frac{1}{1} \]  
4. \[ \frac{10}{1} \]

You should be saying by this time that such fractions are not at all different from the whole numbers related to them. For example, in the picture above, \( \frac{3}{1} \) is clearly shown as 3 whole things, as a set of 3 things. \( \frac{3}{1} \) can be considered the same as the whole number 3. You can write

\[ \frac{3}{1} = 3. \]

It is clear that this is so no matter what whole number you take, so that you can have the general equation

\[ \frac{a}{1} = a, \]

where \( a \) is any whole number.

From this you can learn some more facts. For example, you know from the property of fractions you discovered above that

\[ \frac{3}{1} = \frac{6}{2} = \frac{9}{3}, \]

and so on. Thus you can have the following string of facts:

\[ 3 = \frac{3}{1} = \frac{6}{2} = \frac{9}{3}, \]

and so on. You see that there are many more ways of writing the same whole number than you knew before. Some of the ways of writing the number 5 would be
\[
5 = 2 + 3 = 5 - 0 = 1 \times 5 = 15 \div 3 = \frac{5}{1} = \frac{10}{2} = \frac{20}{4}
\]

**Exercise 20-5b**

Find five different ways to write each of the following whole numbers. Draw a suitable picture (rectangles, circles, sets and subsets, number line) for each way.

1. 3
2. 10
3. 7
4. 1

**20-6 The special question of zero**

Just as you thought about the special problem of multiplying a whole number by 0, so with fractions there are special problems with 0. Think of the following string: \(\frac{3}{2}, \frac{2}{2}, \frac{1}{2}, \frac{0}{2}\). The first one is 3 halves, the second 2 halves, the third 1 half, and the last -- what? Of course, it is 0 halves. But to take 0 halves is the same as to take 0 thirds or 0 fourths, or 0 anything. The answer is the same in every case -- it is 0. Just as

\[0 \times a = 0\]

so you can see that if \(a\) is not 0,

\[\frac{0}{a} = 0\]

You can tell stories about such problems which will help explain this answer to children. For instance, you can tell about the man who hired 5 men to work for him. They worked for a month and then went to get their pay. But they were told that the employer has no money at all. Thus they had 0 to divide up among them. Each one got paid \(\frac{0}{5}\), when 0 was divided 5 ways. But this is clearly 0.
Exercise 20-6a

Make up some different stories which help explain to children why \( \frac{0}{2} = 0 \).

A harder problem is to decide what to do with something like \( \frac{3}{0} \).

You will find that such things are not even fractions. Think of the string:

\[
\frac{3}{4} \quad \frac{3}{3} \quad \frac{3}{2} \quad \frac{3}{1} \quad \frac{3}{0}.
\]

The first one means to take 3 fourths parts of whole things.

The second one means to take 3 thirds. The third means to take 3 halves.

The fourth means to take 3 whole things, broken into 1 part each, and this means to take 3 wholes. But what can \( \frac{3}{0} \) mean? We learned in the chapter on division that division by zero is impossible. There is no answer for \( \frac{3}{0} \).

If you think you have an answer for \( \frac{3}{0} \), where do you think the point is for it on the number line?

20-7 Multiplication equations and fractions: relation to division

You remember from the earlier work that whole numbers are often found as solutions of multiplication equations. For example, you see that 9 is a solution of the multiplication equation

\[
12 \times \_\_\_ = 108,
\]

because putting the numeral 9 into the box in the sentence makes a true statement.

But you also know that whole number solutions cannot always be found for multiplication equations. For example, this equation has no whole number answer.

\[
101 = \_\_\_ \times 7
\]
But now you know that you can find a fraction which is a solution of this equation, namely, \( \frac{101}{7} \).

Of course, the earlier problem had a fractional answer too, but that fraction you can show to be the same as the whole number answer. For that earlier problem, you can write

\[
\frac{108}{12} = 9
\]

because \( \frac{108}{12} = \frac{(9 \times 12)}{(1 \times 12)} = \frac{9}{1} \)

**Exercise 20-7a**

Find fractions which are solutions of each of the following multiplication equations:

1. \( 13 \times \Box = 182 \)
2. \( a \times 22 = 451 \)
3. \( 625 = 25 \times \Box \)
4. \( 2154 = 17 \times y \)
5. \( 9 \times y = 828 \)
6. \( \Box \times 15 = 1253 \)

**Exercise 20-7b**

Find six other names for the fractions which are the answers in the preceding exercises.

**Exercise 20-7c**

For which of the problems above are there whole number answers?

What are these answers?

How did you find answers to the problems in the exercises above? You used division, because you remember that a fraction \( \frac{a}{b} \) can be shown in terms of division as \( a \div b \), and any answer to a division problem can be shown as a fraction. For instance, the answer to problem 1. in the first exercise can be shown as \( 182 \div 13 \), which is the same as \( \frac{182}{13} \).
In both cases you got 14 as the answer. In other cases, such as problem 2 above, you cannot get a whole number answer, but a fraction.

In this particular problem your answer turns out to be

\[
451 \div 22 = \frac{451}{22}
\]

If you worked out the division problem by the procedures shown in the chapter on division of whole numbers, you found the answer as follows:

\[
\begin{align*}
22 & \overline{)451} \\
\quad & \underline{44} \\
\quad & 11 \\
\quad & 0
\end{align*}
\]

Thus \(451 \div 22\) gives you 20 with a remainder of 11, that is, \(451 = 20 + 11\) \(22 \times 20 + 11\). Using fractions, you can write this as

\[
451 \div 22 = \frac{20\frac{11}{22}}{}
\]

Of course, \(20\frac{11}{22}\) is just another way of writing \(\frac{451}{22}\). Another even simpler way of writing this fraction, using the property of equal fractions is:

\[
\frac{451}{22} = 20\frac{11}{22} = 20\frac{1}{2}
\]

Exercise 20-7d

Do the following division problems, giving your answers as fractions.

1. \(655 \div 41\)  
2. \(1349 \div 21\)  
3. \(2526 \div 111\)  
4. \(5482 \div 84\)

Exercise 20-7e

Write word problems suitable for primary children to show the relation between long division and fractions.
CHAPTER 21

OPERATIONS ON FRACTIONS

21-1 Addition: its meaning in pictures and on the number line

If fractions are to act as numbers, you must be able to do the things with them that you can do with whole numbers. You want to be able to add, subtract, multiply and divide them. Thus, the first thing to think about is what it means to add fractions.

You know what addition means for whole numbers. If you want to understand addition for fractions, you must build on what addition is for whole numbers. Addition of whole numbers is based on union of sets. Because you can show fractions in terms of pictures and sets, you can try to build up the meaning of addition of fractions through pictures and sets.

Thus you can begin by looking at a picture. For instance, think of rectangles divided into parts, as in the following picture:

![Picture of rectangles]

The first rectangle is divided into 5 parts, and the fraction shown by the shading is \( \frac{3}{5} \). The second rectangle is divided into 4 parts, and the fraction shown by the shading is \( \frac{1}{4} \). The total shaded parts of the two rectangles would be one way of giving the meaning of the sum of \( \frac{3}{5} \) and \( \frac{1}{4} \), since it shows how much of the two rectangles is shaded. This is the same sort of operation as addition of whole numbers, because it is very much like taking a union of sets. In fact, to take a slightly different example, if each rectangle is only divided
into 1 part, and that part of each shaded, as in the picture below, the sum of the two fractions, $\frac{1}{1}$ and $\frac{1}{1}$, represented by the shaded portions is the same as the sum of the whole numbers 1 and 1. That sum is found, you remember, by taking the union of the two sets of rectangles.

From this you can see that

$$\frac{1}{1} + \frac{1}{1} = \frac{2}{1}$$

which is the same addition fact as

$$1 + 1 = 2.$$  

You must not fall into the trap of thinking that you can do this same type of thing for any pair of fractions. In the pictures of $\frac{3}{5}$ and $\frac{1}{4}$ above, you cannot just put together 3 parts and 1 part and get 4 parts and still think of them as fractions. It is true, of course, that there are 4 parts shaded in the picture, but to think of them this way is not to think of them as fractional parts of certain given wholes.

Look at another picture of fractions, this time on the number line. Here you see the points $1 \frac{3}{4}$ and $2 \frac{1}{4}$. From the point $1 \frac{3}{4}$ to the point $2 \frac{1}{4}$, you see an additional $\frac{2}{4}$.

You remember that you can show addition of whole numbers on the number line by marking off one distance from 0, and then marking off the second distance to the right after it. Thus to explain addition of fractions in the same way, you could
say that the picture here shows the sum of \( \frac{3}{4} \) and \( \frac{2}{4} \). (You remember the story of the cricket that landed on the number line on a book, and then did addition problems for the owner of the book. You can use that story with your children to show addition of fractions as well as addition of whole numbers.)

**Exercise 21-1a**

Draw pictures, using rectangles, circles and the number line, which show the following additions. Do not give the answers. Just show what the answers look like with pictures.

1. \( \frac{2}{5} + \frac{1}{2} \)
2. \( \frac{5}{3} + \frac{2}{3} \)
3. \( 1.2 + 2.3 \)
4. \( 2 \frac{1}{2} + \frac{3}{4} \)

**Exercise 21-1b**

Write some word problems which help to introduce addition of fractions to children.

**21-2 Addition: \( \frac{a}{b} + \frac{c}{b} \)**

You probably began to see ways to find the fractions which must be the sum of the two fractions in some of the problems in the previous section. For instance, you can show a picture for problem 2, as follows:

If you actually count the total number of thirds shown in the picture, you will see that there are 7. Thus you can say, from this picture, that
\[ \frac{5}{3} + \frac{2}{3} = \frac{7}{3}. \]

Another problem is the one shown above on the number line, where \( \frac{2}{4} \) is added to \( 1 \frac{3}{4} \), bringing you to the point \( 2 \frac{1}{4} \). Can you work this problem the same way as the problem \( \frac{5}{3} + \frac{2}{3} \)? Yes, you can, if you change the mixed numbers to improper fractions. You know that \( 1 \frac{3}{4} \) is the same as \( \frac{7}{4} \), and that \( 2 \frac{1}{4} \) is the same as \( \frac{9}{4} \). Thus you get the statement

\[ \frac{7}{4} + \frac{2}{4} = \frac{9}{4}. \]

You should by now see what addition of fractions must mean, when the fractions have the same denominator. In such a situation they both refer to the same kind of parts of a whole. Thus when you add \( \frac{7}{4} \) and \( \frac{2}{4} \), you get \( \frac{9}{4} \), because all the fractions have to do with fourth parts.

**Exercise 21-2a**

Find the answers to the following addition problems, using pictures and the number line to help.

1. \( \frac{2}{3} + \frac{7}{3} = \)

2. \( 1 \frac{1}{5} + \frac{13}{5} = \)

3. \( \frac{9}{4} + \frac{2}{4} = \)

4. \( 7.1 + \frac{9}{10} = \)

Did problem 4, in the exercise give you trouble? If it did, you should remember that a decimal fraction like 7.1 is a short way to write fractions with denominator 10. Thus 7.1 is the same as \( \frac{71}{10} \), and \( \frac{9}{10} \) is the same as .9. So you can write this problem either as

\[ \frac{71}{10} + \frac{9}{10} = \]

or as

\[ 7.1 + .9 = \]

In either way you get the same answer, which can be written in many different
forms:

\[ \frac{71}{10} + \frac{9}{10} = \frac{71}{10} + \frac{9}{10} = \frac{80}{10} = 8.0 = \frac{8}{1} = 8. \]

**Exercise 21-2b**

If you had a classroom with no blackboard, what kinds of things could you make which would help you to teach children how to add fractions with the same denominator, as in the above problems?

You now see from the problems above what it means to add any two fractions with the same denominator. In this way, you can find the missing fraction in this sentence.

\[ \frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} \]

You get the answer in an easy way, as follows:

\[ \frac{a}{b} + \frac{c}{b} = \frac{(a + c)}{b} . \]

You say that the sum of two fractions with the same denominator is a fraction with that same denominator whose numerator is the sum of the two numerators.

**21-3 Addition: \( \frac{a}{b} + \frac{c}{d} \).**

It was easy to decide how to add the fractions in the previous section because they were of the same kind, that is, they had the same denominator. It is harder to deal with a problem like:

\[ \frac{1}{2} + \frac{3}{4} . \]

It helps to look at it on the number line. First mark the point \( \frac{1}{2} \). Then go \( \frac{3}{4} \) beyond that point, as shown below.
You can see from the number line that the second jump brings you to the point $1 \frac{1}{4}$, which can also be written $\frac{5}{4}$. Thus you get the following:

$$\frac{1}{2} + \frac{3}{4} = \frac{5}{4}$$

Is there an easy way to see why this must be so from what has already been decided about addition of fractions with the same denominator?

Yes, there is, if you remember the secret of equality of fractions. There are many names for one number, and you should think of another name for $\frac{1}{2}$ which will help you work this problem. The number $\frac{1}{2}$ has to do with halves and the number $\frac{3}{4}$ has to do with fourths. Is there any way to bring them together?

Yes, by remembering that each of the halves can be cut in half again to find fourths. Thus you can write

$$\frac{1}{2} = \frac{2}{4},$$

which leads you to the statement:

$$\frac{2}{4} + \frac{3}{4} = \frac{5}{4}.$$  

What works is to make both fractions of the same kind, with the same denominator. In the problem above, you renamed $\frac{1}{2}$ as $\frac{2}{4}$, so that you could easily add $\frac{2}{4}$ and $\frac{3}{4}$ to get $\frac{5}{4}$. Thus, $\frac{1}{2} + \frac{3}{4} = \frac{5}{4}$. Is it always possible to do this for any pair of fractions, say, $\frac{3}{5}$ and $\frac{1}{3}$? Look at them as pictured below, and you will think of a way to do it.
You see that the rectangle showing $\frac{3}{5}$ is divided by lines which run from top to bottom, and the rectangle showing $\frac{1}{3}$ is divided by lines which run from side to side. If you look long and hard at that picture, you will see that one way to have both fractions refer to the same parts of a whole is to subdivide each rectangle in the same way that the other has already been divided as in this picture.

Into how many parts has the first rectangle now been divided? Into 15, of course. And into how many parts has the second rectangle now been divided? Into 15, also. How many parts of the first rectangle are shaded? The answer is 9. How many parts of the second rectangle are shaded? The answer is 5.

So how many parts—fifteenth parts, that is—of the two rectangles are shaded? The answer is 14. Thus you can write for the sum of these two fractions the following:

$$\frac{3}{5} + \frac{1}{3} = \frac{9}{15} + \frac{5}{15} = \frac{14}{15}.$$  

This is so because $\frac{3}{5} = \frac{9}{15}$ and $\frac{1}{3} = \frac{5}{15}$. You saw those facts from the picture but the property of equal fractions which you learned earlier tells you them also.

$\frac{3}{5} = \frac{9}{15}$ because $\frac{(3 \times 3)}{(5 \times 3)} = \frac{9}{15}$. If you use the secret of equal fractions again you can also see that $\frac{1}{3} = \frac{(1 \times 5)}{(3 \times 5)} = \frac{5}{15}$.

Clearly this plan will always work. For another example, take the sum $\frac{2}{3} + \frac{5}{4}$. $\frac{2}{3}$ can be shown as 2 third parts of a rectangle. $\frac{5}{4}$ can be shown as 5 fourth parts of two rectangles. Break up the parts of the first rectangle into 4
parts each and the parts of the second pair of rectangles into 3 parts each. In this way, each rectangle will be broken into twelfth parts, and you can then add the fractions easily. The picture looks like this:

\[
\begin{array}{c}
\frac{2}{3} \\
\frac{5}{4}
\end{array}
\]

If you count the number of twelfth parts in the three rectangles, you can see that there are 23 of them. Thus you can write

\[
\frac{2}{3} + \frac{5}{4} = \frac{8}{12} + \frac{15}{12} = \frac{23}{12}
\]

But this method would not be very useful if you always had to draw rectangles and count parts. Clearly, there is an easier way. In the case of \(\frac{3}{5} + \frac{1}{3}\), you finally found the answer in terms of fifteenth parts. In the case of \(\frac{2}{3} + \frac{5}{4}\), you finally found the answer in terms of twelfths. What do you think, without drawing pictures, you can do with the sum \(\frac{3}{7} + \frac{9}{8}\)? You will find the answer in terms of 56th parts. This is true because your rectangles would be divided into sevenths along one side and into eighths along the other side. And you know from what has been done before that an array like that with 7 on one side and 8 on the other will have \(7 \times 8 = 56\) individual parts in it. Thus you see that you can always find an answer by multiplying the denominators of the two fractions, and then finding new names for the two original fractions in terms of this new denominator. To finish this problem started above, namely, \(\frac{3}{7} + \frac{9}{8}\), you would have

\[
\frac{3}{7} = \frac{(3 \times 8)}{(7 \times 8)} = \frac{24}{56}
\]
and \[
\frac{9}{8} = \frac{(9 \times 7)}{(8 \times 7)} = \frac{63}{56}.
\]

Thus your sum is

\[
\frac{3}{7} + \frac{9}{8} = \frac{24}{56} + \frac{63}{56} = \frac{87}{56}.
\]

Now we can say that for two fractions \(\frac{a}{b}\) and \(\frac{c}{d}\)

\[
\frac{a}{b} + \frac{c}{d} = \frac{(a \times d)}{(b \times d)} + \frac{(b \times c)}{(b \times d)} = \frac{(a \times d) + (b \times c)}{(b \times d)}.
\]

(Or we could abbreviate this as \(\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}\).

We have finally explained the meaning of addition for fractions.

**Exercise 21-3a**

Find the answers to the following addition problems.

1. \(\frac{3}{7} + \frac{9}{14}\)
2. \(\frac{2}{3} + \frac{5}{2}\)
3. \(\frac{7}{6} + \frac{3}{4}\)
4. \(1.3 + \frac{2}{3}\)
5. \(\frac{7}{10} + \frac{3}{5}\)
6. \(1\frac{7}{8} + 3\frac{2}{3}\)

You may have noticed that in some of the problems in the exercise there are two ways of finding the answer. In problem 1., for instance, you can replace \(\frac{3}{7}\) by \(\frac{6}{14}\) and add, getting \(\frac{15}{14}\). Or you can change both denominators to 98, and get the answer as follows:

\[
\frac{3}{7} + \frac{9}{14} = \frac{42}{98} + \frac{63}{98} = \frac{105}{98}.
\]

But use of the property of equal fractions will tell you that

\[
\frac{105}{98} = \frac{(15 \times 7)}{(14 \times 7)} = \frac{15}{14},
\]

so that the answers are the same fraction. The easier way is to think, by any means possible, of some common denominator, and then rewrite both fractions to have that denominator. But sometimes, as in problem 6., you cannot use this easier way and you have to use the more difficult way.
Exercise 21-3b

In Exercise 3a, do the problems in both ways where possible, and show that the answers found in the two ways are equal. Write each answer in three other ways.

Exercise 21-3c

Describe how you would teach this lesson on fractions in a classroom with no blackboard.

Exercise 21-3d

Outline the steps of a classroom presentation which you can use to lead your pupils to find and understand the way to do addition of the type \( \frac{a}{b} + \frac{c}{d} \).

21-4 Commutative property of addition

If you look back to the section on addition of whole numbers, you will find certain properties of addition of whole numbers. Do these same basic properties hold for addition of fractions? Look at the following statement:

\[
\frac{3}{4} + \frac{5}{6} = \frac{5}{6} + \frac{3}{4}.
\]

Do you agree with it? If you draw a picture, you can easily see that you get the same answer both ways.

You can write

\[
\frac{3}{4} + \frac{5}{6} = \frac{18}{24} + \frac{20}{24} = \frac{38}{24}.
\]
and you can write

\[ \frac{5}{6} + \frac{3}{4} = \frac{20}{24} + \frac{18}{24} = \frac{38}{24}, \]

so that both sums give the same fraction.

You can see the same thing if you show the two fractions added on the number line, first in one way, and then in the other.

It does not matter, in the picture, which arrow comes first, since, in each case the two put together reach the same point.

You can also see the commutative property of addition for fractions from the general equation for addition of fractions, which we have found. You know that (remember the abbreviations \(ad\) for \(a \times d\), and so forth)

\[ \frac{a}{b} + \frac{c}{d} = \frac{(ad + bc)}{bd} \]

and that

\[ \frac{c}{d} + \frac{a}{b} = \frac{(cb + da)}{db} \]

You know that

\[ cb = bc \quad (\text{why?}) \]
\[ da = ad \quad (\text{why?}) \]
\[ db = bd \quad (\text{why?}) \]

Also you know that

\[ cb + da = da + cb \quad (\text{why?}) \]

So you can write...
\[
\frac{c}{d} + \frac{a}{b} = \frac{(cb + da)}{db} = \frac{(ad + bc)}{bd}
\]

which is the same as \(\frac{a}{b} + \frac{c}{d}\). Thus you can see that
\[
\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}.
\]

**Exercise 21-4a**

Show that the following statements hold and explain them.

1. \(\frac{7}{10} + \frac{5}{4} = \frac{5}{4} + \frac{7}{10}\)
2. \(1\frac{7}{8} + 9\frac{2}{3} = 9\frac{2}{3} + 1\frac{7}{8}\)
3. \(2.3 + \frac{15}{2} = \frac{15}{2} + 2.3\)
4. \(\frac{a}{2} + \frac{3}{d} = \frac{3}{d} + \frac{a}{2}\)

**21-5 Associative property of addition**

Another property of addition of whole numbers is the associative property. If you think about it, you can see that this property holds for fractions. For instance, you can see easily that
\[
(\frac{1}{2} + \frac{1}{4}) + \frac{1}{8} = \frac{1}{2} + (\frac{1}{4} + \frac{1}{8}).
\]
By adding the fractions in parentheses, you get \(\frac{3}{4} + \frac{1}{8}\) on the left and \(\frac{1}{2} + \frac{3}{8}\) on the right.
Then adding these fractions you get \(\frac{7}{8}\) on both sides. Thus,
\[
(\frac{1}{2} + \frac{1}{4}) + \frac{1}{8} = \frac{1}{2} + (\frac{1}{4} + \frac{1}{8}).
\]
You will find in general that
\[
(\frac{a}{b} + \frac{c}{d}) + \frac{e}{f} = \frac{a}{b} + (\frac{c}{d} + \frac{e}{f}).
\]
It is not useful here to work out all the steps to show this result, but it will give you good practice to try it for yourself.
Exercise 21-5a

Show the associative property of addition of fractions in the following cases:

1. \( \frac{3}{2} + (\frac{2}{3} + \frac{5}{8}) = (\frac{3}{2} + \frac{2}{3}) + \frac{5}{8} \)

2. \((1.3 + \frac{2}{5}) + 2.7 = 1.3 + (\frac{2}{5} + 2.7)\)

3. \((1 \frac{1}{8} + \frac{2}{2}) + \frac{1}{5} = 1 \frac{1}{8} + (\frac{2}{2} + \frac{1}{5})\)

21-6 Property of zero

Another fact about addition of whole numbers that you have discovered was the property of 0 that

\[ 0 + a = a \text{ and } a + 0 = a. \]

Do you think this result holds for fractions, that

\[ 0 + \frac{a}{b} = \frac{a}{b}? \]

Think for example, about \( 0 + \frac{1}{2} \). First of all, since 0 is a whole number, you know that \( 0 = \frac{0}{1} \). But you also know that

\[ \frac{0}{1} = \frac{(0 \times 2)}{(1 \times 2)} = \frac{0}{2}. \] (Why?)

And so

\[ 0 = \frac{0}{2}. \]

Thus you can write

\[ 0 + \frac{1}{2} = \frac{0}{2} + \frac{1}{2} = \frac{(0 + 1)}{2} = \frac{1}{2}. \]

Using the commutative property of addition, you can see that

\[ \frac{1}{2} + 0 = \frac{1}{2}. \]

You can do the general problem in the same way. You can write
Thus you can write

\[ 0 + \frac{a}{b} = \frac{0 + a}{b} = \frac{a}{b}. \]

Again using the commutative property of addition, you can see that

\[ \frac{a}{b} + 0 = \frac{a}{b}. \]

**Exercise 21-6a**

Show that the following statements are correct.

1. \( \frac{0}{4} + \frac{5}{3} = \frac{5}{3} \)
2. \( \frac{2}{9} + \frac{0}{15} = \frac{2}{9} \)

**21-7 Subtraction**

You will not have trouble with subtraction of fractions if you remember what addition means for fractions, and how to do it. To add two fractions, you write both as fractions with the same denominator. Thus to add fractions:

\[ \frac{5}{7} + \frac{13}{10} = \]

you first find a common denominator which in this case is 70.

The same method works for subtraction of fractions. You remember that subtraction is the inverse of addition. For example, the missing addend in the whole number equation:

\[ 35 + \square = 63 \]

is found by subtracting, and the equation can be rewritten as follows:

\[ \square = 63 - 35. \]

You have learned how to solve such problems, and now you will see that we can use almost the same methods to solve such problems as this:
Both the left-hand side and the right-hand side are written in terms of thirds. The missing addend is the fraction which must be added to \( \frac{5}{3} \) to get \( \frac{9}{3} \). If you think of this problem in the same way you thought of similar whole number problems, for example, \( 5 + \square = 9 \), you see that \( \frac{4}{3} \) must be added to \( \frac{5}{3} \) to get \( \frac{9}{3} \), just as 4 must be added to 5 to get 9. Hence you can say that the missing addend in the first sentence is \( \frac{4}{3} \) and get the true statement:

\[
\frac{5}{3} + \frac{4}{3} = \frac{9}{3}.
\]

Thus we write for these fractions:

\[
\frac{9}{3} - \frac{5}{3} = \frac{4}{3}
\]

in the way that you write \( 9 - 5 = 4 \) for the similar whole number problem. You see in this way that if missing-addend problems have only fractions with the same denominator, you can find the missing addend by using the same approach as in whole number problems. Subtraction means finding the missing addend.

Now let us look at a more complicated missing-addend problem for fractions:

\[
\frac{5}{4} + \square = \frac{8}{3}.
\]

To find the fraction to fill the box you can begin in the same way as you did in adding fractions. You can see this in a picture:
You must find what fractional parts of rectangles to adjoin to the picture on the left to get the picture on the right. Clearly, the best way to do this is by breaking up all rectangles so that each has the same number of parts. You can do this in the same way as for addition. The picture will look like this:

From this picture you can easily see that an additional 17 twelfth parts are needed to make the picture on the left into the same as the picture on the right. Thus you can write:

$$\frac{5}{4} + \square = \frac{8}{3}$$

as:

$$\frac{15}{12} + \square = \frac{32}{12}$$.

Just as before you can see that the missing addend can be found by subtraction, so that:

$$\square = \frac{32}{12} - \frac{15}{12}$$.

Thus you know by subtraction that the missing addend is $$\frac{17}{12}$$.

Of course you do not actually need to use the pictures to solve such problems. You can just rewrite the fractions so that they have the same denominator. Then treat the problem like a whole number subtraction problem for the numerators. In the example above you can write:

$$\frac{8}{3} - \frac{5}{4} = \frac{32}{12} - \frac{15}{12} = \frac{(32 - 15)}{12} = \frac{17}{12}$$.
When both fractions have the same denominator, then subtraction is easy. And if the denominators are not the same, then rewrite the fractions so that both have the same denominator. For any fractions you can write it this way:

\[
\frac{a}{b} - \frac{c}{d} = \frac{(ad - bc)}{bd}
\]

This is very similar to the statement for addition of fractions. It is similar, of course, because addition and subtraction are inverses.

**Exercise 21-7a**

Find the missing addends in the following sentences. Rewrite the equations as subtraction equations.

1. \(\frac{2}{3} + \square = \frac{5}{3}\) 
2. \(x + 7 = 15\)
3. \(1\frac{5}{8} = \square + \frac{3}{4}\)
4. \(1.7 + n = 5.5\)
5. \(\square + 2.3 = \frac{16}{5}\)
6. \(\frac{23}{7} + x = \frac{41}{4}\)
7. \(3\frac{9}{10} + \square = 7\frac{3}{4}\)
8. \(c + \frac{7}{8} = \frac{14}{16}\)

**Exercise 21-7b**

Write word problems useful for teaching children how to subtract fractions.

**21-8 Subtraction as inverse of addition**

You can see from the above that for fractions addition and subtraction are inverses of each other in the same way that for whole numbers addition and subtraction are inverses of each other. For example, think of the addition:

\[
\frac{3}{4} + \frac{1}{8} = \frac{7}{8}.
\]

If you go one step further, and subtract \(\frac{1}{8}\) from each side, you get

\[
\left(\frac{3}{4} + \frac{1}{8}\right) - \frac{1}{8} = \frac{3}{4}.
\]
In the same way you can start with the subtraction:

\[
\frac{7}{8} - \frac{1}{8} = \frac{3}{4}.
\]

If you then add \(\frac{1}{8}\) to each side, you get

\[
(\frac{7}{8} - \frac{1}{8}) + \frac{1}{8} = \frac{7}{8}.
\]

If you try any pair of subtraction and addition facts using the same fractions, you will find that this works. Thus, for fractions as well as for whole numbers, addition and subtraction are inverses of each other.

**Exercise 21-8a**

1. Make up some examples showing that subtraction and addition are inverses of each other.

2. Describe pictures which you can draw on the blackboard to show primary school children that addition and subtraction of fractions are inverses.

**21-9 Subtraction problems which cannot be solved with fractions or whole numbers**

Think of the following subtraction problem:

\[
\square = \frac{5}{3} - \frac{7}{2}.
\]

If you try to do it in the way you learned above, you get:

\[
\square = \frac{10}{6} - \frac{21}{6} = \frac{(10 - 21)}{6}.
\]

But you do not yet know how to solve such subtraction problems as 10 - 21, since there is no whole number which you can add to 21 to get 10. Thus there is no fraction which will make the sentence

\[
\frac{7}{2} + \square = \frac{5}{3}
\]
true, nor any whole number either.

You can determine when you can or cannot solve a subtraction problem as follows: rewrite the fractions as fractions with the same denominator; then try to solve the subtraction problem given by the numerators. Later you will see how to solve such problems.

**Exercise 21-9a**

Tell which of the following subtraction problems you can solve and which you cannot solve:

1. $\frac{32}{5} - \frac{16}{2}$
2. $\frac{3}{1} - \frac{6}{2}$
3. $1\frac{7}{9} - 2\frac{1}{2}$
4. $3.2 - 2.7$
5. $\frac{81}{17} - \frac{41}{8}$
6. $\frac{25}{2} - \frac{2}{25}$

**21-10 Multiplication: $a \times \left( \frac{m}{n} \right)$**

If you think back to multiplication of whole numbers, you will remember that you learned about it first in terms of repeated addition, which itself was based on successive unions of equivalent sets. It is easy to see that this method will help you on the meaning of multiplication of fractions by whole numbers. Think of this sum:

$$\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3}.$$

You can picture this sum as follows:
It is clear from the picture that:

\[
\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = \frac{8}{3},
\]

which can also be written:

\[
\frac{(2 + 2 + 2 + 2)}{3} = \frac{8}{3}.
\]

This in turn can be written:

\[
\frac{(4 \times 2)}{3} = \frac{8}{3}.
\]

But in the same way as in multiplication of whole numbers we would write:

\[
\frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} = 4 \times \left(\frac{2}{3}\right).
\]

Thus it is easy to decide what meaning to give to multiplication here:

\[
4 \times \left(\frac{2}{3}\right) = \frac{(4 \times 2)}{3}.
\]

For any whole number \(a\) and fraction \(\frac{m}{n}\) we say:

\[
a \times \left(\frac{m}{n}\right) = \frac{(a \times m)}{n}.
\]

You can say this in words very simply. To multiply a fraction by a whole number, multiply the numerator of the fraction by that whole number, and do not change the denominator. For another example, take the product

\[
7 \times (3 \frac{3}{8}).
\]

In this problem, of course, you first rewrite the mixed number as the improper fraction \(\frac{27}{8}\). Then you have

\[
7 \times (3 \frac{3}{8}) = 7 \times \frac{27}{8} = \frac{(7 \times 27)}{8} = \frac{189}{8}.
\]

Another example is

\[
6 \times 3.4.
\]

In this case, you rewrite 3.4 as an improper fraction \(\frac{34}{10}\). Then you have
\[6 \times 3.4 = 6 \times \frac{34}{10} = \frac{204}{10} = 20.4.\]

**Exercise 21-10a**

Find the answers to the following multiplication problems:

1. \(7 \times \frac{5}{6}\)
2. \(4 \times \frac{3}{4}\)
3. \(13 \times 3 \frac{2}{3}\)
4. \(5 \times 7.2\)
5. \(83 \times \frac{27}{4}\)
6. \(32 \times \frac{5}{8}\)

**Exercise 21-10b**

Write word problems which will illustrate the multiplication problems above.

### 21-11 Making multiplication simple

You probably noticed that you can make your work easier in some of the problems in Exercise 10a. For example, take problem 6. You can write:

\[32 \times \frac{5}{8} = \frac{(32 \times 5)}{8} = \frac{160}{8}.\]

But, if you use the secret of equal fractions for finding new names for the same fraction, you will see that

\[\frac{160}{8} = \frac{20}{1} = 20.\]

You can get this answer also by seeing that \(\frac{32}{8} = \frac{4}{1} = 4.\) You can look for common factors in the numerator and denominator of a fraction and divide both numbers by these factors. This is just the opposite of what you did when you learned to multiply numerator and denominator of a fraction by the same factor to find another name for the fraction. The reason is similar in both cases: the property of equal fractions.
For another example, think of the product:

$$16 \times \frac{5}{12} = \frac{(16 \times 5)}{12} = \frac{80}{12}.$$ 

You know that $80 = 4 \times 20$, and $12 = 4 \times 3$, and so you can write:

$$\frac{80}{12} = \frac{(4 \times 20)}{(4 \times 3)} = \frac{20}{3}.$$ 

You can get this answer also by noticing that

$$\frac{16}{12} = \frac{4}{3}$$

for the same reason, and working the problem from there.

**Exercise 21-11a**

Work these multiplication problems, first using the long method, and then using the shortcut mentioned above.

1. $15 \times \frac{3}{5}$
2. $18 \times \frac{3}{16}$
3. $25 \times 7 \frac{1}{5}$
4. $30 \times 5.7$
5. $14 \times 3 \frac{2}{7}$
6. $2 \times 5.6$

**Exercise 21-11b**

Draw pictures (using rectangles, circles or the number line) illustrating the two ways of finding the answers for problems 1. and 6. in the preceding exercise.

**21-12 Multiplication:** $\frac{1}{b} \times \frac{m}{n}$

It is harder to think what a product such as

$$\frac{1}{2} \times \frac{3}{4}$$
can mean. But if you begin by remembering what such products as $3 \times \frac{3}{4}$, $2 \times \frac{3}{4}$ and $1 \times \frac{3}{4}$ mean, you will see what to do. But, even before that, it is useful to remember what it means to take 3 or 2 or 1 or any number of things. Three sets of 4 stars look like this:

Likewise, 2 sets of 4 stars look like this:

And 1 set of 4 stars looks like this:

You learned that to find for each picture the number of stars in the union of the sets, you must multiply the number of stars in each set by the total number of sets. Thus you get these statements -- $3 \times 4 = 12$, $2 \times 4 = 8$, $1 \times 4 = 4$ -- which tell you the number of stars in the successive pictures.

In the same way you can understand $\times \frac{3}{4}$, $2 \times \frac{3}{4}$ and $1 \times \frac{3}{4}$, as telling you how many fourth parts are in the following pictures:
You can see from the pictures that $3 \times \frac{3}{4} = \frac{9}{4}$, $2 \times \frac{3}{4} = \frac{6}{4}$, and $1 \times \frac{3}{4} = \frac{3}{4}$.

Think again of the sets of stars. You understand what 3 sets of 4 stars, 2 sets of 4 stars, and 1 set of 4 stars all mean. And you know that you find the numbers of stars in every case by multiplying. Likewise, you know what $\frac{1}{2}$ set of 4 stars means. You can see it in the following picture, where the $\frac{1}{2}$ set of 4 stars is circled as a subset. You remember from before that $\frac{1}{2}$ means 1 part out of 2 equal parts of a whole.

If you look at the picture, you will see that $\frac{1}{2}$ set of 4 stars contains 2 stars.

You found the number of stars in 3 sets of 4 stars, 2 sets of 4 stars, and 1 set of 4 stars by multiplying, and now you can decide that the product $\frac{1}{2} \times 4$ must be the number of stars in $\frac{1}{2}$ set of 4 stars. Thus you get

$$\frac{1}{2} \times 4 = 2.$$  

In the same way, you know what $3 \times \frac{3}{4}$, $2 \times \frac{3}{4}$, and $1 \times \frac{3}{4}$ all mean.

Now $\frac{1}{2}$ set of 3 fourth parts should look like the double-shaded part of the
Thus you can see what you want $\frac{1}{2} \times \frac{3}{4}$ to mean. It must be the parts of the whole in the picture drawn above. Thus you must have

$$\frac{1}{2} \times \frac{3}{4} = \frac{3}{8},$$

or $\frac{1}{2}$ set of 3 fourth parts must contain 3 eighth parts.

This means that multiplication of a number by $\frac{1}{2}$ and taking $\frac{1}{2}$ a given set are related in the same way as multiplication of a number by 2 and taking 2 of a given set. Thus, for another example,

$$\frac{1}{2} \times \frac{6}{5} = \frac{6}{10},$$

which can be drawn as follows:

![Diagram 1](image1)

![Diagram 2](image2)

You probably have noticed that this can also be written:

$$\frac{1}{2} \times \frac{6}{5} = \frac{3}{5},$$

by drawing the picture as follows:

![Diagram 3](image3)

![Diagram 4](image4)

You know that $\frac{3}{5}$ and $\frac{6}{10}$ are different names for the same fraction, and thus the answer is the same in each case.

You can do this with other fractions. For example,

$$\frac{1}{5} \times \frac{15}{4}$$
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can be drawn as follows:

![Diagram of fractions]

From the picture you can see that
\[
\frac{1}{5} \times \frac{15}{4} = \frac{15}{20} = \frac{3}{4}.
\]

You can see the general law at this point: If you multiply a fraction \( \frac{m}{n} \) by a fraction of the form \( \frac{1}{b} \), you get a new fraction whose denominator is \( b \) times that of the first fraction and whose numerator is not changed. In other words,
\[
\left( \frac{1}{b} \right) \times \left( \frac{m}{n} \right) = \frac{m}{(b \times n)}
\]

Of course, in some cases you can write the answer in simpler form, as in the above example. But the fact does not change the basic method.

**Exercise 21-12a**

Draw pictures which show the answers to these multi:

1. \( \frac{1}{5} \times \frac{2}{3} \)
2. \( \frac{1}{3} \times 1.2 \)
3. \( .1 \times \frac{3}{2} \)
4. \( \frac{1}{4} \times 2 \frac{1}{2} \)

**Exercise 21-12b**

Make up word problems which you can use to explain such problems as those above to primary school children.

**21-13 Multiplication:** \( \frac{a}{b} \times \frac{m}{n} \)

From the preceding sections we can now decide what such products as
\[
\frac{3}{4} \times \frac{2}{5}
\]

mean. First of all,
\[
\frac{3}{4} = 3 \times \left(\frac{1}{4}\right),
\]
so that we must have:
\[
\frac{3}{4} \times \frac{2}{5} = \left[ 3 \times \left(\frac{1}{4}\right) \right] \times \frac{2}{5}
\]

Now, whatever meaning multiplication has for fractions, we certainly want the associative property to hold. In order to do that, therefore, we must have:
\[
\frac{3}{4} \times \frac{2}{5} = \left[ 3 \times \left(\frac{1}{4}\right) \right] \times \frac{2}{5} = 3 \times \frac{1}{4} \times \frac{2}{5}.
\]

Now what do you decide? You can write
\[
\frac{1}{4} \times \frac{2}{5} = \frac{2}{20},
\]
so then
\[
3 \times \left[ \frac{1}{4} \times \frac{2}{5} \right] = 3 \times \left(\frac{2}{20}\right).
\]
But then you can write
\[
3 \times \left(\frac{2}{20}\right) = \frac{6}{20}.
\]

Finally, therefore, we decide on the meaning of multiplication:
\[
\frac{3}{4} \times \frac{2}{5} = \frac{6}{20}.
\]
You can draw this as follows:

For any fractions, you can write:
\[
\frac{a}{b} \times \frac{m}{n} = \frac{(a \times m)}{(b \times n)}.
\]
Exercise 21-13a

Find the following products:

1. \( \frac{2}{5} \times \frac{9}{4} \)
2. \( \frac{5}{3} \times \frac{1}{8} \)
3. \( 1 \frac{3}{4} \times \frac{19}{5} \)
4. \( 1.7 \times \frac{5}{8} \)
5. \( 5 \frac{1}{8} \times \frac{16}{5} \)
6. \( 9.2 \times 3.7 \)

Exercise 21-13b

Describe teaching aids you can use to teach the multiplication of fractions.

These aids should be ones you can make out of materials available at a rural school.

21-14 Multiplication on the number line

Multiplication of fractions may also be illustrated using the number line.

Think of the product \( 3 \times \frac{3}{4} \) on the number line. It must mean 3 jumps of \( \frac{3}{4} \), as in this picture:

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Any product of the form \( a \times \left( \frac{m}{n} \right) \) can be shown in this way, by finding the point \( \frac{m}{n} \) on the line, and jumping a distance \( \frac{m}{n} \) from 0 a total of \( a \) times.

Likewise, you can see what a fraction of the form \( \frac{1}{b} \times \frac{m}{n} \) means on the number line. For instance, \( \frac{1}{3} \times \frac{7}{5} \) can be shown as follows:

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In general, you must divide each of the nth parts of the line itself into b parts, and show the result as m of these \((b \times n)\)th parts. In the case above, you have 7 fifteenth parts.

Finally, to show a fraction of the form \(\frac{a}{b} \times \frac{m}{n}\) you must take a such parts of size \(\frac{m}{(b \times n)}\). For example, \(\frac{5}{2} \times \frac{3}{4}\) can be drawn as follows:

\[
\begin{array}{ccccccccccccc}
0 & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{1}{2} & \frac{5}{8} & \frac{3}{4} & \frac{7}{8} & 1 & \frac{9}{8} & \frac{5}{4} & \frac{11}{8} & \frac{3}{2} & \frac{13}{8} & \frac{7}{4} & \frac{15}{8} & 2
\end{array}
\]

Exercise 21-14a

Show the following products on the number line:

1. \(7 \times \frac{3}{4}\)
2. \(5 \times 2.4\)
3. \(\frac{1}{3} \times \frac{7}{2}\)
4. \(.1 \times 2 \frac{1}{2}\)
5. \(\frac{3}{8} \times \frac{5}{3}\)
6. \(2.5 \times \frac{5}{2}\)

21-15 Commutative property of multiplication

You can see that multiplication of fractions has the commutative property.

For instance, you found in the preceding section that

\[
\frac{3}{4} \times \frac{2}{5} = \frac{6}{20} = \frac{3}{10},
\]

and you drew a picture showing the product:

If you take the product in reverse order, you get the same answer:
Moreover, the picture showing this product is similar to the picture showing 
\[
\frac{3}{4} \times \frac{2}{5}:
\]

![Diagram showing the product of \(\frac{3}{4} \times \frac{2}{5}\)]

It is possible to see the commutative property directly by using the equation for multiplication of fractions,

\[
\frac{a}{b} \times \frac{m}{n} = \frac{(a \times m)}{(b \times n)}.
\]

If you reverse the order, you get

\[
\frac{m}{n} \times \frac{a}{b} = \frac{(m \times a)}{(n \times b)}.
\]

But you know that

\[
m \times a = a \times m \text{ (why?)}
\]

\[
n \times b = b \times n \text{ (why?)}
\]

and thus you can see that

\[
\frac{m}{n} \times \frac{3}{b} = \frac{a}{b} \times \frac{m}{n}.
\]

**Exercise 21-15a**

Check the commutative property of multiplication for the following pairs of products:

1. \(\frac{5}{3} \times \frac{2}{7} = \frac{2}{7} \times \frac{5}{3}\)

2. \(1\frac{3}{8} \times 3\frac{1}{2} = 3\frac{1}{2} \times 1\frac{3}{8}\)

3. \(5.4 \times 13.1 = 13.1 \times 5.4\)
21-16  Associative property of multiplication

It is also possible to see the associative property of multiplication of fractions. In fact, we have already used it in deciding on the meaning of multiplication. Think of the following product:

\[(\frac{1}{2} \times \frac{2}{3}) \times \frac{3}{4} = \frac{1}{3} \times \frac{3}{4} = \frac{1}{4}\] (why?).

Grouping in the other way, you can write

\[\frac{1}{2} \times (\frac{2}{3} \times \frac{3}{4}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}\] (why?).

You can see easily that for any three fractions you get the same result no matter which pair of fractions you group for the first multiplication.

**Exercise 21-16a**

Check the following products to see that the associative property of multiplication holds:

1. \[(\frac{7}{3} \times \frac{15}{13}) \times \frac{16}{9} = \frac{7}{3} \times (\frac{15}{13} \times \frac{16}{9})\]

2. \[(1\frac{3}{4} \times 7.3) \times \frac{4}{4} = 1\frac{3}{4} \times (7.3 \times \frac{4}{4})\]

**Exercise 21-16b**

The general statement of the associative property for multiplication of any whole numbers is: \((a \times b) \times c = a \times (b \times c)\). Give the general statement for fractions, and give examples of its use.

21-17  Distributive property

You remember from the unit on whole numbers that there is a property combining addition and multiplication. For any whole numbers, it is stated as follows:

\[a \times (b + c) = (a \times b) + (a \times c)\].
In words, when multiplying a sum of two whole numbers by a third whole number, you can multiply each of the two numbers by the third number and add the results.

Think of the following statement using fractions, to see whether the distributive property holds for fractions as well:

\[
\frac{1}{2} \times \left( \frac{2}{3} + \frac{3}{4} \right) = \left( \frac{1}{2} \times \frac{2}{3} \right) + \left( \frac{1}{2} \times \frac{3}{4} \right)
\]

On the left-hand side you get:

\[
\frac{1}{2} \times \left( \frac{2}{3} + \frac{3}{4} \right) = \frac{1}{2} \times \frac{17}{12} \quad \text{(why?)}
\]

On the right-hand side you get:

\[
\left( \frac{1}{2} \times \frac{2}{3} \right) + \left( \frac{1}{2} \times \frac{3}{4} \right) = \frac{2}{6} + \frac{3}{8} \quad \text{(why?)}
\]

Completing the work on each side you get, on the left:

\[
\frac{1}{2} \times \frac{17}{12} = \frac{17}{24}
\]

and, on the right:

\[
\frac{1}{3} + \frac{3}{8} = \frac{17}{24}.
\]

The two results are the same.

You can write, for any three fractions, the distributive property as follows:

\[
\frac{a}{b} \times \left( \frac{c}{d} + \frac{e}{f} \right) = \left( \frac{a}{b} \times \frac{c}{d} \right) + \left( \frac{a}{b} \times \frac{e}{f} \right).
\]

**Exercise 21-17a**

Check that the distributive property holds for the following:

1. \( a = 5, b = 3, c = 7, d = 2, e = 4, f = 9 \)
2. \( a = 1, b = 10, c = 4, d = 5, e = 6, f = 7 \)

**Exercise 21-17b**

Try to check the general statement of the distributive property given
above, to see whether the left-hand side results in the same fraction as the right-hand side.

**Exercise 21-17c**

Make up word problems showing the use of the distributive property.

**21-18 Properties of one and zero**

You saw earlier that if you multiply any whole number by 1 or multiply by any whole number, you do not change that whole number. This property of 1 for whole numbers as follows:

\[ a \times 1 = 1 \times a = a. \]

Of course, this property also holds for multiplication of fractions. You can write it as follows:

\[ \left( \frac{a}{b} \right) \times 1 = 1 \times \left( \frac{a}{b} \right) = \frac{a}{b}. \]

For example, you have:

\[ \left( \frac{3}{5} \right) \times 1 = \frac{3}{5} \times \frac{1}{1} \quad \text{(why?)} \]

\[ = \frac{3 \times 1}{5 \times 1} \quad \text{(why?)} \]

\[ = \frac{3}{5} \quad \text{(why?).} \]

Likewise, you will find that

\[ 1 \times \left( \frac{3}{5} \right) = \frac{3}{5}. \]

For another example, if you take the fraction \(1.8\), you get

\[ 1.8 \times 1 = \left( \frac{18}{10} \right) \times 1 \quad \text{(why?)} \]

\[ = \frac{18}{10} \times \frac{1}{1} \quad \text{(why?)} \]

\[ = \frac{18 \times 1}{10 \times 1} \quad \text{(why?)}. \]
In the same way, you will find that
\[ 1 \times 1.8 = 1.8. \]

**Exercise 21-18a**
Show that the fractions you get by doing the multiplications on the left-hand side and the right-hand side of the following equation are equal:
\[ \left( \frac{a}{b} \right) \times 1 = 1 \times \left( \frac{a}{b} \right). \]

**Exercise 21-18b**
Give a classroom procedure to explain this property of 1 to a class of primary school children.

**Exercise 21-18c**
One further property of multiplication for whole numbers is the property that
\[ a \times 0 = 0 \times a = 0. \]
State this property for fractions, give examples and give reasons why it holds.

**21-19 Reciprocals**
Think of the two fractions \( \frac{3}{8} \) and \( \frac{8}{3} \). They look very much like each other, except that one is upside down. They give a very interesting result when multiplied together, as follows:
\[ \frac{3}{8} \times \frac{8}{3} = \frac{(3 \times 8)}{(8 \times 3)} = \frac{24}{24} = \frac{1}{1} = 1 \text{ (why?).} \]
Fractions like \( \frac{3}{8} \) and \( \frac{8}{3} \), whose product is 1, are called RECIPROCALS.

You can see that for any two non-zero fractions \( \frac{a}{b} \) and \( \frac{b}{a} \) the product of reciprocals is 1. Thus you can write, if neither \( a \) nor \( b \) is 0:

\[
\frac{a}{b} \times \frac{b}{a} = \frac{(a \times b)}{(b \times a)} \quad \text{(why?)}
\]

\[
= \frac{(a \times b)}{(a \times b)} \quad \text{(why?)}
\]

\[
= 1 \quad \text{(why?)�}.
\]

To find the reciprocal of a whole number, such as 3, remember that 3 = \( \frac{3}{1} \), so that the reciprocal of 3 is \( \frac{1}{3} \).

**Exercise 21-19a**

Find the reciprocals of these fractions, and show that the product of each fraction and its reciprocal is 1.

1. \( \frac{5}{7} \)
2. \( \frac{3}{20} \)
3. \( 3 \frac{1}{2} \)
4. \( \frac{1}{10} \)
5. 3.7
6. \( \frac{4}{4} \)

**Exercise 21-19b**

Write word problems which illustrate this property of reciprocals for children.

**21-20 Division as finding the missing factor**

After you learned to multiply whole numbers, you found that you could solve problems like the following by division:

\[
144 = 9 \times \square
\]

The missing factor, of course, is

\[
\square = 144 \div 9 = 16.
\]
You found the answer by thinking that 144 is 9 times something, and then realizing that that something must be 16. You found that number by using division.

You can do problems whose answers are not whole numbers in the same way. For a simple example, think of this problem:

\[ 28 = 13 \times \square \]

Your answer is found in the same way, by thinking what numeral must be put into the box to make the sentence true. But clearly, using what you have learned about multiplication of fractions, you can see that

\[ 13 \times \left( \frac{28}{13} \right) = 28. \]

Thus it is obvious that \( \frac{28}{13} \) must be the number which makes the sentence true, so that

\[ 28 \div 13 = \frac{28}{13}. \]

It is a more difficult problem to find what to put into the box in the following sentence:

\[ \frac{1}{2} = 3 \times \square \]

The question can be stated in words in such a way as to suggest the answer. You can see that \( \frac{1}{2} \) is 3 times something, and the problem is to find that something. But clearly \( \frac{1}{2} = 3 \times \left( \frac{1}{3} \times \frac{1}{2} \right) \). Thus \( \frac{1}{6} \) is the fraction which must be put into the box.

A slightly more difficult example is the following:

\[ \frac{5}{3} = 4 \times \square \]

You see that \( \frac{5}{3} \) is 4 times something. But clearly \( \frac{5}{3} = 4 \times \left( \frac{1}{4} \times \frac{5}{3} \right) \). Thus \( \frac{5}{12} \) is the fraction to be put into the box.
A still more difficult problem is the following:

\[
\frac{1}{3} \times \underline{\quad} = \frac{5}{7}.
\]

In this case, the fraction \(\frac{5}{7}\) is \(\frac{1}{3}\) of the missing factor you want. That number must be 3 times \(\frac{5}{7}\), in order for \(\frac{5}{7}\) to be \(\frac{1}{3}\) of it. Thus you can see that the fraction

\[3 \times \frac{5}{7} = \frac{15}{7}\]

completes the sentence given above.

The last type of problem to think of is shown by the following:

\[
\frac{2}{5} = \frac{3}{4} \times \underline{\quad}.
\]

You can clearly combine the methods you used in each of the previous problems to solve this one. You know that \(\frac{3}{4}\) times the missing factor is \(\frac{2}{5}\). Thus you can figure out that \(\frac{2}{5}\) is \(\frac{3}{4} \times \left(\frac{4}{3} \times \frac{2}{5}\right)\). The missing factor itself must therefore be \(\frac{4}{3} \times \frac{2}{5}\) or just \(\frac{8}{15}\).

You have now solved several problems involving missing fractional factors. The division operation on whole numbers was based on problems where the missing factors were whole numbers. Thus, if division of fractions is to be related to division of whole numbers in the same way that the other operations on fractions were related to their corresponding operations on whole numbers, the answers to these problems with missing fractional factors must be found by division. Thus you can write the following statements based on the problems you did above.

\[
\frac{1}{6} = \frac{1}{2} \div 3 = \frac{1}{2} \div \frac{3}{1} \quad \text{(why?)}
\]

\[
\frac{5}{12} = \frac{5}{3} \div 4 = \frac{5}{3} \div \frac{4}{1}
\]
Exercise 21-20a

Find answers to the following problems. Write your answers using division equations, as shown above.

1. $\frac{15}{7} = \frac{5}{7} \div \frac{3}{1}$
2. $\frac{8}{15} = \frac{2}{5} \div \frac{3}{4}$

Exercise 21-20a

Find answers to the following problems. Write your answers using division equations, as shown above.

1. $5 \times \square = \frac{9}{4}$
2. $\frac{7}{3} = \square \times 4$
3. $\frac{5}{6} = \frac{1}{3} \times \square$
4. $\frac{5}{2} \times \square = \frac{7}{3}$
5. $4.4 = \square \times 3.3$
6. $\frac{17}{3} = \frac{14}{7} \times y$

21-21 Another secret

There is an easy way to put all these answers to division problems into simple form. Perhaps you have found the rule by now. Perhaps you can remember it from your school days. If you do not know it yet, you can discover it. But remember that your pupils only know what they learn with and from their teachers. You must help them discover this secret. It is much better for them if they find it for themselves, than if you have to tell it to them, since then it is truly their own.

Look once again at the results of the problems which were worked out in the previous section. They are

$$\frac{1}{2} \div \frac{3}{1} = \frac{1}{6}$$
$$\frac{5}{3} \div \frac{4}{1} = \frac{5}{12}$$
$$\frac{5}{7} \div \frac{1}{3} = \frac{15}{7}$$
Do you see any pattern as you look at these statements? If you do, then perhaps you know the secret. But, if you do not yet see the pattern, you can rewrite the statements as follows:

\[
\frac{1}{2} \div \frac{3}{1} = \frac{(1 \times 1)}{(2 \times 3)}
\]

\[
\frac{5}{3} \div \frac{4}{1} = \frac{(5 \times 1)}{(3 \times 4)}
\]

\[
\frac{5}{7} \div \frac{1}{3} = \frac{(5 \times 3)}{(7 \times 1)}
\]

\[
\frac{2}{5} \div \frac{3}{4} = \frac{(2 \times 4)}{(5 \times 3)}
\]

If you check these statements, you will see that they are merely new ways of writing the statements given above. But what is special about the second way of writing them? Do you see the secret?

If you have looked closely, you see that in each case you have found an answer whose numerator is the product of the numerator of the first and denominator of the second of the two fractions, and whose denominator is the product of the denominator of the first and numerator of the second. You can write it this way:

\[
\frac{a}{b} \div \frac{c}{d} = \frac{(a \times d)}{(b \times c)}
\]

But this is the same number you would find if you multiplied the first fraction by the reciprocal \(\frac{d}{c}\) of the second fraction, as follows:

\[
\frac{a}{b} \times \frac{d}{c} = \frac{(a \times d)}{(b \times c)}
\]

Thus you can see that to divide one fraction by another, you simply multiply the first by the reciprocal of the second. For short, to remember it more easily, this is sometimes said in the form: "invert and multiply". But don't
teach it that way. Your children can learn it for themselves, and they deserve
to have the chance to do so. Children can discover much for themselves, and
they will be better thinkers if you let them do it.

Exercise 21-21a

Solve the following division problems:

1. \( \frac{3}{5} \div 7 = \) \[\_\]
2. \( \frac{5}{8} \times \) \[\_\] = \( \frac{9}{3} \)
3. \( 1.6 \div a = 9.3 \)

Exercise 21-21b

Try to give a general argument for the rule that to divide \( \frac{a}{b} \) by \( \frac{c}{d} \) you
multiply \( \frac{a}{b} \) by the reciprocal of \( \frac{c}{d} \). Show that this is the missing factor in the
sentence \( \frac{c}{d} \times \) \[\_\] = \( \frac{a}{b} \). Good luck!

21-22 Division as inverse of multiplication

You remember that division is the inverse of multiplication for whole
numbers. The same thing is true for fractions. For instance, thin
multiplication facts:

\[
\frac{2}{5} \times \frac{9}{4} = \frac{(2 \times 9)}{(5 \times 4)} = \frac{18}{20} \text{ (why?) .}
\]

Then you can divide \( \frac{18}{20} \) by \( \frac{9}{4} \) and get

\[
\frac{18}{20} \div \frac{9}{4} = \frac{18}{20} \times \frac{4}{9} = \frac{72}{180} \text{ (why?) .}
\]
Thus you can write

\[
(\frac{2}{5} \times \frac{9}{4}) \div \frac{9}{4} = \frac{2}{5}.
\]

Now instead begin with the division fact:

\[
\frac{2}{5} \div \frac{9}{4} = \frac{2}{5} \times \frac{4}{9} = \frac{(2 \times 4)}{(5 \times 9)} = \frac{8}{45}.
\]

If you multiply \(\frac{8}{45}\) by \(\frac{9}{4}\), you will get back the original \(\frac{2}{5}\), as follows:

\[
\frac{9}{4} \times \frac{8}{45} = \frac{(9 \times 8)}{(4 \times 45)} = \frac{2}{5}.
\]

Thus you can write

\[
(\frac{2}{5} \div \frac{9}{4}) \times \frac{9}{4} = \frac{2}{5}.
\]

In general, you can write

\[
\left(\frac{a}{b} \div \frac{c}{d}\right) \times \frac{c}{d} = \frac{a}{b} = \left(\frac{a}{b} \times \frac{c}{d}\right) \div \frac{c}{d} = \frac{a}{b}.
\]

Exercise 21-22a

Verify the following statements

1. \( (1 \frac{1}{3} \times \frac{7}{8}) \div \frac{7}{8} = 1 \frac{1}{3} \)
2. \( (1.7 \times 3.4) \div 3.4 = 1.7 \)
3. \( (\frac{19}{5} \div \frac{17}{8}) \times \frac{17}{8} = \frac{19}{5} \)
4. \( (5.8 \times 5 \frac{2}{3}) \div 5 \frac{2}{3} = 5.8 \)
The question of zero again

What do you think \( \frac{3}{0} \) could mean? A way to think about this question is to think about the sentence:

\[
0 \times \frac{a}{b} = 3.
\]

Is there any fraction which can be put into the box to make this sentence true? You have learned that

\[
0 \times \frac{a}{b} = 0,
\]

where \( \frac{a}{b} \) is any fraction. Thus no matter what fraction you put into the box, you get 0 as a result on the left-hand side and never 3. Thus there is no fraction which can be multiplied by 0 to get 3. You can see also that \( \frac{2}{0} \) or \( \frac{5}{0} \) or anything else like that will result in the same difficulty. The only exception is \( \frac{0}{0} \). In the sentence \( 0 \times \square = 0 \) what fraction can be put in the box to make the sentence true? The answer is any fraction at all. This is almost as bad as no answer because \( \frac{0}{0} \) can be given no one definite meaning. As was said in the chapter on division of whole numbers, division by zero is not an allowable operation.
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Watertown, Mass., U.S.A.
This supplementary booklet has been prepared for use with the text Basic Concepts of Mathematics, an Introductory Text for Teachers. In it will be found a glossary and index for the text and the answers to almost all of the exercises, except the answers for Chapters 1, 2, 6, 14, 15. The answers to those five chapters were included at the end of the chapters in the text.

The answers are not meant to "give" you the answers to the exercises. On the contrary, use them to check the answers to an exercise only after you have thoughtfully worked them out for yourself. However, if after trying, you are unable to work a problem, looking at the answer might give you a helpful clue to a solution. Do not be discouraged if your answers often differ from those given here. Very often an answer in this booklet may be simply a different form for your answer. Try to see why your answer and the answer in the booklet are both correct. If the answer given here is not equivalent to your answer, check your work again; you may have made a mistake somewhere or misunderstood the problem.

This supplementary booklet was prepared by Robert B. Brown of the University of California, Berkeley, California.
CHAPTER THREE

Exercise 3-1a

1.

Here are three other matchings of the sets of pawpaws and bananas. These three and the three shown at the beginning of Chapter 3 are the only possible matchings.

2.

This is just one way of matching sets A and B.

In all there are twenty-four different ways.
Exercise 3-1b

Answers will vary.

Exercise 3-2a

1.

- Hexagon
- Square
- Circle
- Triangle
- Arrow
- Chicken
- P Q R S T U V
- A B C D E F G H
- Diamond
Chapter 3

Exercise 3-3a

1. This is just one set matching the given set. You will have other ideas.

2. This is just one set matching the given set. You will have other ideas.

Exercise 3-3b

1. Any set which you used to answer question 2 in Exercise 3-3a contains 3 members.

2. Four

3. Two

Here is one such set. You will have other ideas.
CHAPTER FOUR

Exercise 4-2a

Set | Number | Numeral
---|---|---
{ E } | one | 1
{ B, C } | two | 2
{ A, B, D } | three | 3
{ A, B, C, E } | four | 4

Exercise 4-2b

Set | Number | Numeral
---|---|---
{ A, B, C, D } | four | 4
{ □ ○ △ } | three | 3
{ } | zero | 0
{ B, C, E, F } | four | 4

4 > 3, 3 < 4, 4 > 0, 0 < 4, 3 > 0, 0 < 3,
4 = 4, 3 = 3, 0 = 0.

Exercise 4-4a

Answers will vary.

Exercise 4-5a

1. Sets A and B are not equivalent sets. Set A matches exactly the counting set \{1, 2, 3, 4, 5, 6\}, but the set B does not.

2. Set A = \{ r, l, t, n \}. Sets A and B are equivalent sets. Each matches the counting set \{1, 2, 3, 4\}.
CHAPTER FIVE

Exercise 5-3a

1. Six \[\boxed{1111}\]
   Fourteen \[\boxed{1111}\]
   Three hundred and fifty-six \[\boxed{11111}\]
   Three thousand and twenty \[\boxed{11111}\]

2. 3, 22, 402, 1113, 20211.

Exercise 5-7a

1. 52
2. 36
3. 71
4. 95

Exercise 5-7b

1. 352
2. 555
3. 1255

Exercise 5-7c

1. (a) \[\boxed{1111}\]
   (b) \[\boxed{1111}\]
   (c) \[\boxed{1111}\]
   (d) \[\boxed{1111}\]

2. (a) 2134, 5213, 3048
   (b) \[\boxed{11111111}\]
3.  (a) Ten thousand  
(b) One hundred thousand  
(c) One million  
(d) Ten million

---

4.  

---

5.  

32,915,374  
3 ten millions, 2 millions, 9 hundred thousands, 1 ten thousand, 5 thousands, 3 hundreds, 7 tens and 4 ones.

This is just one example. Answers will vary.
6.

7. (a) 276,133
   Two hundred thousands, seven ten thousands, six thousands, one hundred, three tens and three ones

(b) 7,132,432
   Seven million, one hundred thousand, three ten thousands, two thousands, four hundreds, three tens and two ones

(c) 44,444,444
   Four ten millions, four millions, four hundred thousands, four ten thousands, four thousands, four hundreds, four tens and four ones.

(d) 12,345,678
   One ten million, two millions, three hundred thousands, four ten thousands, five thousands, six hundreds, seven tens and eight ones.

Exercise 5-8a

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<th>Hindu-Arabic numerals</th>
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<td>4.  u s u e n I I I I</td>
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</table>
Exercise 5-8b

(a)  

(b)  

(c)  

(d)  

(e)  

(f)  

(g)  

(h)  

Exercise 5-8c
Exercise 5-8d

(a) Egyptian numerals

1. Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ 111

2. Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ

3. Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ

4. Ⲝ Ⲝ

5. Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ

6. Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ Ⲝ

(b) Hindu-Arabic numerals

1. 241,063

2. 60,510

3. 10,601

4. 1,010

5. 48,124

6. 30,527

Exercise 5-9a

1. XIV, XIX, XXIII, CCCCLXVIII, MCMLXIV, XLIV, LXXXII.

2. 24, 26, 64, 46, 228, 2979, 1752.

3. XLVI: L - X = XL, V + I = VI, XL + VI = XLVI.

   LXIX: L + X = LX, X - I = IX, LX + IX = LXIX.

   LIV: V - I = IV, L + IV = LIV.

   MCM: M - C = CM, M + CM = MCM.

Exercise 5-10a

1. (a) MXXIII (b) MMDCCXXCIV

2. (a) XXXIV (b) XCV
### Exercise 5-11a

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</table>

3. (a) MDCXXXVIII
   (b) MMMMMMMMMMMM
       MMMMMMMMMMMM
       MMMMMMMMMMMM
       MMMMMMMMMMMM
       MMMMMMMMD

4. (a) X rem. XXIV
   (b) XXIII rem. XXV

Exercise 5-11a

1. 631
2. 462
3. (a) 1043  (b) 324  (c) 1024  (d) 2352
4. (a) 54 rem. 1  (b) 79  (c) 103 rem. 1  (d) 51 rem. 3
5. (a) 85101  (b) 47424  (c) 28809
6. (a) 128  (b) 32 rem. 14  (c) 1000 rem. 2
7. The missing key was '1'. The problem was

\[
\begin{align*}
321 \\
53 \\
817 \\
\hline
1191
\end{align*}
\]

8. (a) \( \boxed{7 \quad 138813} \)

\[
19830 \text{ rem. } 3
\]

(b) \( \boxed{65432} \)

\[
261728
\]
CHAPTER SEVEN

Exercise 7-3a

(1) 7  (2) 8  (3) 10  (4) 10

Pictures will vary.

Exercise 7-3b

Answers will vary.

Exercise 7-3c

If A and B are not disjoint sets, then the number of elements in the union of A and B is the number of elements in A plus the number of elements in B minus the number of elements that A and B have in common.

Exercise 7-4a

A few of the many possibilities are given here.

\[
\begin{align*}
6 &= 5 + 1 = 4 + 2 = 3 + 3 \\
13 &= 12 + 1 = 11 + 2 = 10 + 3 = 9 + 4 \\
22 &= 10 + 12 = 13 + 9 = 16 + 6 = 17 + 5 = 21 + 1 \\
8 &= 6 + 2 = 5 + 3 = 4 + 4 = 1 + 7
\end{align*}
\]

Exercise 7-5a

Answers will vary.

Exercise 7-5b

Answers will vary.

Exercise 7-5c

1. (a) \[
\begin{array}{c}
\text{\includegraphics[width=1in]{image1}}
\end{array}
\]

(b) \[
\begin{array}{c}
\text{\includegraphics[width=1in]{image2}}
\end{array}
\]
2. \( \{ P, Q, R, S, T, O, W \} \)
\[ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \]
\[ \{ 1, 2, 3, 4, 5, 6, 7 \} \]
\[ \{ \bigtriangleup \ \square \ \bigtriangledown \ \bigodot \ \bigtriangleup \ \bigodot \ \bigtriangleup \} \]

3. The sets involved will vary. Here is one possibility.

(a) \( \{ \bigodot \} \cup \{ \bigtriangleup \ \square \ \bigtriangledown + \times \ \bigodot \ \bigtriangleup \} \)

\[ = \{ \bigodot \ \bigtriangleup \ \square \ \bigtriangledown + \times \ \bigodot \ \bigtriangleup \} \]

(b) \( \{ \} \cup \{ \bigtriangleup \ \square \ \bigtriangledown + \times \ \bigodot \ \bigtriangleup \} \)

\[ = \{ \bigtriangleup \ \square \ \bigtriangledown + \times \ \bigodot \ \bigtriangleup \} \]

4. A few of the many possibilities are given here.

(a) \( 2 + 6 = 5 + 3 = 1 + 7 = 4 + 4 = 8 \)
(b) \( 25 = 24 + 1 = 22 + 3 = 20 + 5 = 18 + 7 \)
(c) \( 7 + 1 = 2 + 6 = 5 + 3 = 4 + 4 = 8 \)
(d) \( 17 = 16 + 1 = 15 + 2 = 14 + 3 = 12 + 5 \)

5. \( \{ \bigcirc \ \times \ \bigvee \} \cup \{ \bigtriangleup \ \bigodot \} = \{ \bigcirc \ \times \ \bigvee \ \bigodot \ \bigtriangleup \} \)
Exercise 7-7a

1. 2  
2. 7  
3. 11  
4. 8  
5. 2

Exercise 7-7b

5 = 0 + 5 = 1 + 4 = 2 + 3 = 3 + 2 = 4 + 1 = 5 + 0  
7 = 0 + 7 = 1 + 6 = 2 + 5 = 3 + 4 = 4 + 3 = 5 + 2  
= 6 + 1 = 7 + 0  
8 = 0 + 8 = 1 + 7 = 2 + 6 = 3 + 5 = 4 + 4 = 5 + 3  
= 6 + 2 = 7 + 1 = 8 + 0

Exercise 7-7c

1. 6  
2. If the sum of the numbers assigned to the pairs is 13, no children are left without partners. If the sum is 14, the children assigned 1 and 7 are without partners. If the sum is 15, the children assigned 1 and 2 are without partners. If the sum is 16, the children assigned 1, 2, 3 and 8 are without partners.
**Exercise 7-8a**

1. 

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2. 

\[0 + 8 = 8, \ 1 + 7 = 8, \ 2 + 6 = 8, \ 3 + 5 = 8, \ 4 + 4 = 8,\]
\[5 + 3 = 8, \ 6 + 2 = 8, \ 7 + 1 = 8, \ 8 + 0 = 8\]

3. 14

4. 5, 15 + \[\square\] = 20.

5. The following five pairs could be formed.

\[0 + 10, \ 1 + 9, \ 2 + 8, \ 3 + 7, \ 4 + 6.\]

The children with the numbers 5, 11 and 12 would be without partners.

6. The following pairs could be formed.

\[0 + 8, \ 1 + 7, \ 2 + 6, \ 3 + 5.\]

The children with the numbers 4, 9, 10, 11 and 12 would be without partners.
### Exercise 7-9a

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**Exercise 7-9b**

Answers will vary.
CHAPTER EIGHT

Exercise 8-1a

Answers will vary.

Exercise 8-2a

1. 2
2. 7
3. 9
4. Any number may be put into the boxes, provided that the same number is put into both boxes.

Exercise 8-3a

1. Answers will vary.
2. \[3 + 3 + 9 + 3 + 9 = 27\]

Exercise 8-3b

1. \[3 + 1 = 1 + 3\]
2. \[5 + 2 = 2 + 5\]
3. \[4 + 9 = 9 + 4\]
4. \[6 + 4 = 4 + 6\]

Exercise 8-5a

1. \[(1 + 2) + 1 = 1 + (2 + 1)\]
2. \[(4 + 3) + 0 = 4 + (3 + 0)\]
3. \[(5 + 5) + 2 = 5 + (5 + 2)\]
4. \[(3 + 7) + 6 = 3 + (7 + 6)\]
5. \[(0 + 8) + 6 = 0 + (8 + 6)\]

Exercise 8-5b

1. Answers will vary.
2. Answers will vary.

Exercise 8-5c

1. \[9 + (7 + 3) = 9 + 10 = 19\]
2. \[(6 + 4) + 7 = 10 + 7 = 17\]
3. \[(4 + 16) + 7 = 20 + 7 = 27\]
4. \[9 + (5 + 15) = 9 + 20 = 29\]
5. \[7 + (4 + 6) = 7 + 10 = 17\]
6. \[(7 + 3) + 3 = 10 + 3 = 13\]

**Exercise 8-5d**

Answers will vary.

**Exercise 8-5e**

\[5 + (4 + \boxed{6}) = (4 + 5) + 6\]

**Exercise 8-6a**

1. \[(4 + 3) + 2 = 4 + (3 + 2)\] associative law
   \[4 + (3 + 2) = 4 + (2 + 3)\] commutative law
   \[4 + (2 + 3) = (2 + 3) + 4\] commutative law
   \[(2 + 3) + 4 = 2 + (3 + 4)\] associative law

2. Four numbers can be grouped in any manner when added. The same is true of more than four numbers.

3. \[
    \begin{array}{ccc}
    1 + 14 & 1 + 4 + 10 & 4 + 5 + 6 \\
    2 + 13 & 1 + 5 + 9 & 1 + 2 + 3 + 9 \\
    3 + 12 & 1 + 6 + 8 & 1 + 2 + 4 + 8 \\
    4 + 11 & 2 + 3 + 10 & 1 + 2 + 5 + 7 \\
    5 + 10 & 2 + 4 + 9 & 1 + 3 + 4 + 7 \\
    6 + 9 & 2 + 5 + 8 & 1 + 3 + 5 + 6 \\
    7 + 8 & 2 + 6 + 7 & 2 + 3 + 4 + 6 \\
    1 + 2 + 12 & 3 + 4 + 8 & 1 + 2 + 3 + 4 + 5 \\
    1 + 3 + 11 & 3 + 5 + 7 & \\
    \end{array}
    \]

4. The first player announces the number 1, 7, 13, 19, 25 and 31
CHAPTER NINE

Exercise 9-1a

5 + 0 = 5,  4 + 1 = 5,  3 + 2 = 5,  2 + 3 = 5,
1 + 4 = 5,  0 + 5 = 5.
8 + 0 = 8,  7 + 1 = 8,  6 + 2 = 8,  5 + 3 = 8,
4 + 4 = 8,  3 + 5 = 8,  2 + 6 = 8,  1 + 7 = 8,
0 + 8 = 8.
9 + 0 = 9,  8 + 1 = 9,  7 + 2 = 9,  6 + 3 = 9,
5 + 4 = 9,  4 + 5 = 9,  3 + 6 = 9,  2 + 7 = 9,
1 + 8 = 9,  0 + 9 = 9.

Exercise 9-2a

7 = 0 + 7
7 = 1 + 6
7 = 2 + 5
7 = 3 + 4
7 = 4 + 3
7 = 5 + 2
7 = 6 + 1
7 = 7 + 0
Exercise 9-2b

6 = 0 + 6
6 = 1 + 5
6 = 2 + 4
6 = 3 + 3
6 = 4 + 2
6 = 5 + 1
6 = 6 + 0

Exercise 9-2c

0 and 9, 1 and 8, 2 and 7, 3 and 6, 4 and 5, 5 and 4,
6 and 3, 7 and 2, 8 and 1, 9 and 0.

Exercise 9-2d

Answers will vary.

Exercise 9-2e

11 and 5

Exercise 9-3a

3

Exercise 9-3b

(a) 1 (c) 3 (e) 6
(b) 2 (d) 0 (f) 7

Exercise 9-3c

(a) 0 (c) 5 (e) 4
(b) 2 (d) 2 (f) 0

Exercise 9-3d

Answers will vary.
Exercise 9-4a

(a) \(6 - 4 = \)_
(b) \(7 - 7 = \)_
(c) \(4 - 3 = \)_
(d) \(9 - 8 = \)_
(e) \(6 - 6 = \)_
(f) \(7 - 2 = \)_

Exercise 9-4b

Answers will vary.

Exercise 9-4c

Answers will vary.

Exercise 9-4d

(a) \(6 + \_ = 8\)
(b) \(\_ + 4 = 4\)
(c) \(2 + \_ = 9\)
(d) \(\_ + 7 = 8\)

Exercise 9-4e

In Exercise 9-4a the answers are (a) 2, (b) 0, (c) 1, (d) 1, (e) 0, (f) 5.
In Exercise 9-4d the answers are (a) 2, (b) 0, (c) 7, (d) 1.

Exercise 9-4f

1. 20 feet

2. There are many possible answers; for example,
\[ 3 + (5 + 1) = 2 + (4 + 3) \] or \[ 3 + (5 + 4) = 2 + (4 + 6). \]
Exercise 9-5a

(a) 0  
(b) no answer  
(c) 1  
(d) 7  
(e) 0  
(f) no answer

Exercise 9-6a

(a) 5 - 2 = 3, 5 - 3 = 2  
(b) 6 - 6 = 0, 6 - 0 = 6  
(c) 8 - 7 = 1, 8 - 1 = 7  
(d) 7 - 5 = 2, 7 - 2 = 5

Exercise 9-6b

Answers will vary.

Exercise 9-7a

1. \[6 + \square = 9, \quad 9 - 6 = \square\] \[\text{Answer: 3.}\]  
2. \[2 + \square = 8, \quad 8 - 2 = \square\] \[\text{Answer: 6.}\]  
3. \[\square + 1 = 7, \quad \square = 7 - 1.\] \[\text{Answer: 6.}\]  
4. \[5 + \square = 10, \quad 10 - 5 = \square\] \[\text{Answer: 5.}\]

Exercise 9-7b

\[\{a, b, c, d, e, f, g\} - \{b, d, f\} = \{a, c, e, g\}.\]

There are 7 members in \(\{a, b, c, d, e, f, g\}\) and 3 members in \(\{b, d, f\}\).  
There are \(7 - 3 = 4\) members in \(\{a, c, e, g\}\)
Exercise 9-8a

1. (a)  

\[
\begin{align*}
7 &= 2 + \square \\
7 - 2 &= \square \\
7 - 2 &= 5
\end{align*}
\]

There are 5 more good eggs than broken eggs.

(b)  

\[
\begin{align*}
2 &= 2 + \square \\
2 - 2 &= \square \\
2 - 2 &= 0
\end{align*}
\]

Kofi and Kwesi caught the same number of fish.

(c)  

\[
\begin{align*}
5 &= 3 + \square \\
5 - 3 &= \square \\
5 - 3 &= 2
\end{align*}
\]

There are 2 more houses across the road than there are here.
2. (a) 

\[9 = 6 + \square\]
\[9 - 6 = \square\]
\[9 - 6 = 3\]

My sister has 3 more counters than I have.

(b) 

\[7 = 6 + \square\]
\[7 - 6 = \square\]
\[7 - 6 = 1\]

Lucy has one more button than I have.

(c) 

\[8 = 4 + \square\]
\[8 - 4 = \square\]
\[8 - 4 = 4\]

There are 4 more birds than chickens.
There are 5 more pots than spoons.

Ama needs 4 more oranges.

Lucy has 2 more pawpaws than Araba.
Exercise 9-9a

(a) \( 9 = 5 + \square \), \( 9 - 5 = \square \)  
Answer: 4

(b) \( 9 = 3 + \square \), \( 9 - 3 = \square \)  
Answer: 6

(c) \( 2 + \square = 9 \), \( \square = 9 - 2 \)  
Answer: 7

(d) \( 8 = 4 + \square \), \( 8 - 4 = \square \)  
Answer: 4

(e) \( 7 + \square = 9 \), \( \square = 9 - 7 \)  
Answer: 2

(f) \( \square + 3 = 5 \), \( \square = 5 - 3 \)  
Answer: 2

(g) \( 4 = \square + 2 \), \( 4 - 2 = \square \)  
Answer: 2

(h) \( 7 = 4 + \square \), \( 7 - 4 = \square \)  
Answer: 3

(i) \( 8 = 4 + \square \), \( 8 - 4 = \square \)  
Answer: 4

(j) \( 2 + \square = 8 \), \( \square = 8 - 2 \)  
Answer: 6

Exercise 9-9b

1. \( 7 + 8 = 15 \) \( \quad 15 - 8 = 7 \)  
\( 8 + 7 = 15 \) \( \quad 15 - 7 = 8 \)

2. \( 3 + 11 = 14 \) \( \quad 14 - 11 = 3 \)  
\( 11 + 3 = 14 \) \( \quad 14 - 3 = 11 \)

3. \( 15 + 5 = 20 \) \( \quad 20 - 5 = 15 \)  
\( 5 + 15 = 20 \) \( \quad 20 - 15 = 5 \)

4. \( 1 + 19 = 20 \) \( \quad 20 - 19 = 1 \)  
\( 19 + 1 = 20 \) \( \quad 20 - 1 = 19 \)

5. \( 2 + 10 = 12 \) \( \quad 12 - 10 = 2 \)  
\( 10 + 2 = 12 \) \( \quad 12 - 2 = 10 \)

6. Only two different statements are possible.  
\( 6 + 6 = 12 \) \( \quad 12 - 6 = 6 \)
Exercise 9-10a

1. \((15 + 10) - 10 = 15, \quad (15 - 10) + 10 = 15\)
2. \((8 + 3) - 3 = 8, \quad (8 - 3) + 3 = 8\)
3. \((12 + 0) - 0 = 12, \quad (12 - 0) + 0 = 12\)
4. \((9 + 9) - 9 = 9, \quad (9 - 9) + 9 = 9\)

Exercise 9-10b

1. \((12 + 6) - 6 = 12\)
   \((12 - 6) + 6 = 12\)
   \((5 + 0) - 0 = 5\)
   \((5 - 0) + 0 = 5\)

2. Answers will vary.

Exercise 9-10c

6
CHAPTER TEN

Exercise 10-2a

1. 18 4. 72
2. 21 5. 32
3. 4

Exercise 10-2b

Answers will vary.

Exercise 10-3a

1.  
   
   
   
   

   3 + 3 + 3 + 3 + 3 + 3 = 18
   The union of 6 sets of 3 things has 18 members.
   3 added 6 times = 18
   6 3's are 18

2.  
   
   
   

   7 + 7 + 7 = 21
   The union of 3 sets of 7 things has 21 members.
   7 added 3 times = 21
   3 7's are 21

3.  

   4 = 4
   The union of 1 set of 4 things has 4 members.
   4 added 1 time = 4
   1 4 is 4
4. 

\[ 9 + 9 + 9 + 9 + 9 + 9 + 9 + 9 + 9 = 72 \]

The union of 8 sets of 9 things has 72 members.
9 added 8 times = 72
8 9's are 72

**Exercise 10-4a**

1. 

\[ 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 27 \]

The union of 9 sets of 3 things has 27 members.
3 added 9 times = 27
9 3's are 27
9 \times 3 = 27

2. 

\[ 5 + 5 = 10 \]

The union of 2 sets of 5 things has 10 members.
5 added 2 times = 10
2 5's are 10
2 × 5 = 10

3. 

The union of 1 set of 7 things has 7 members.
7 added 1 time = 7
1 7 is 7
1 × 7 = 7

4. 

The union of 7 sets of 1 thing has 7 members.
1 added 7 times = 7
7 1's are 7
7 × 1 = 7

5. 
0 + 0 + 0 + 0 + 0 = 0
The union of 5 sets of 0 things has 0 members.
0 added 5 times = 0
5 0's are 0
5 × 0 = 0

Exercise 10-4b

Answers will vary.
Chapter 10

Exercise 10-5a

1. 
\[ 0 + 0 + 0 + 0 = 0 \]
The union of 4 sets of 0 things has 0 members.
0 added 4 times = 0
4 0's are 0
4 \times 0 = 0

2. 
\[ 0 + 0 + 0 + 0 + 0 + 0 + 0 = 0 \]
The union of 7 sets of 0 things has 0 members.
0 added 7 times = 0
7 0's are 0
7 \times 0 = 0

3. 
\[ 0 + 0 = 0 \]
The union of 2 sets of 0 things has 0 members.
0 added 2 times = 0
2 0's are 0
2 \times 0 = 0

Exercise 10-6a

1. 1 taken 0 times = 0
0 1's are 0
0 \times 1 = 0

2. 5 taken 0 times = 0
0 5's are 0
0 \times 0 = 0

3. 0 taken 0 times = 0
0 0's are 0
0 \times 0 = 0
Exercise 10-7a

1. 

2. 

3. 

4. 

Exercise 10-8a

The sets used for mixing may vary. Some of the possible answers are given here.

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Exercise 10-8b

Answers will vary

Exercise 10-8c

35
### Exercise 10-9a

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<td>36</td>
<td>45</td>
<td>54</td>
<td>63</td>
<td>72</td>
<td>81</td>
</tr>
</tbody>
</table>

### Exercise 10-10a

1. 
   ![Diagram](image1)

2. 
   ![Diagram](image2)

3. 
   ![Diagram](image3)

4. 
   ![Diagram](image4)
CHAPTER ELEVEN

Exercise 11-1a

1. 18
2. 6 and 4
3. 9 shillings
4. 34
5. 14 shillings

Exercise 11-2a

1. There are many such pairs. A few of them are

   \[ 0 \times 9 = 9 \times 0 \]
   \[ 1 \times 9 = 9 \times 1 \]
   \[ 2 \times 7 = 7 \times 2 \]
   \[ 5 \times 7 = 7 \times 5 \]
   \[ 6 \times 5 = 5 \times 6 \]
   \[ 3 \times 8 = 8 \times 3 \], and so on.

2. Commutative property of addition: \( a + b = b + a \).

   Commutative property of multiplication: \( a \times b = b \times a \).

Exercise 11-3a

1. \( 3 \times 9 = 9 \times 3 \)
2. \( 5 \times 5 = 5 \times 5 \)
3. \( 0 \times 1 = 1 \times 0 \)
4. \( n \times 3 = 3 \times n \)
5. \( n \times m = m \times n \)
Exercise 11-4a
Exercise 11-4b

1.

2.

Row 1 〇 〇 〇 〇 〇 〇 〇 〇
Row 2 〇 〇 〇 〇 〇 〇 〇 〇
Row 3 〇 〇 〇 〇 〇 〇 〇 〇
Row 4 〇 〇 〇 〇 〇 〇 〇 〇
Row 5 〇 〇 〇 〇 〇 〇 〇 〇
Row 6 〇 〇 〇 〇 〇 〇 〇 〇

Match the objects in row 1 with the objects in column 1. Match the objects in row 2 with the objects in column 2, and so on.
3.

Exercise 11-5a

\[
\begin{array}{cccc}
N & G & L & S \\
1 & (1,N) & (1,G) & (1,L) & (1,S) \\
2 & (2,N) & (2,G) & (2,L) & (2,S) \\
3 & (3,N) & (3,G) & (3,L) & (3,S) \\
\end{array}
\]

Exercise 11-6a
Exercise 11-7a

1. \((1 \times 2) \times 4 = 8, 1 \times (2 \times 4) = 8\)
2. \((3 \times 0) \times 5 = 0, 3 \times (0 \times 5) = 0\)
3. \((4 \times 2) \times 4 = 32, 4 \times (2 \times 4) = 32\)

Exercise 11-7b

Answers will vary.

Exercise 11-8a

1. \((3 \times 1) \times 5 = 3 \times (1 \times 5) = 15\)
2. \((4 \times 2) \times 4 = 4 \times (2 \times 4) = 32\)
3. \((7 \times 1) \times 8 = 7 \times (1 \times 8) = 56\)

Explanations will vary.

Exercise 11-8b

Answers will vary.

Exercise 11-8c

1. 1, 2, 3; no others possible.
2. 1, 1, 2, 4; no others possible.

Exercise 11-9a

1. 

![Diagram of blocks]

Explanations will vary.
Chapter 11

Exercise 11-10a

1. \(1 \times (3 + 2) = 5, (1 \times 3) + (1 \times 2) = 5\)

2. \((4 \times 1) + (4 \times 2) = 12, 4 \times (1 + 2) = 12\)

Exercise 11-10b

Answers will vary.

Exercise 11-11a

1. \[\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{array}\]

2. \[\begin{array}{cccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\]
Exercise 11-12a

1.  \( 1 \times (2 + 3) = (1 \times 2) + (1 \times 3) \)
2.  \( 0 \times (5 + 4) = (0 \times 5) + (0 \times 4) \)
3.  \( x \times (2 + 3) = (x \times 2) + (x \times 3) \)

Exercise 11-12b

1.  \((2 + 3) \times (2 + 3) = 25, (2 \times 2) + (3 \times 3) = 13 \)
   \((2 + 3) \times (2 + 3) = (2 \times 2) + (3 \times 3) \) is not a case of the distributive law.
2.  7; \( (3 \times 7) + (2 \times 7) = 35 \)
3.  \((30 \times 2) + (60 \times 2) \) pence
   \((30 + 60) \times 2 \) pence
   \(2 \times (60 + 30) \) pence
4.  24
5.  27

Either arrangement is possible.

6.  45
7.  4, 5, 6; 4 + 5 + 6 = 15
    5, 6, 7; 5 + 6 + 7 = 18
    6, 7, 8; 6 + 7 + 8 = 21, and so on.
CHAPTER TWELVE

Exercise 12-1a

1. 15; 3 \times 5 = 5 + 5 + 5
2. 12; 4 \times 3 = 3 + 3 + 3 + 3
3. 54; 9 \times 6 = 6 + 6 + 6 + 6 + 6 + 6 + 6 + 6
4. 54; 6 \times 9 = 9 + 9 + 9 + 9 + 9 + 9
5. 56; 7 \times 8 = 8 + 8 + 8 + 8 + 8 + 8 + 8
6. 40; 8 \times 5 = 5 + 5 + 5 + 5 + 5 + 5 + 5 + 5
7. 72; 9 \times 8 = 8 + 8 + 8 + 8 + 8 + 8 + 8 + 8 + 8
8. 72; 8 \times 9 = 9 + 9 + 9 + 9 + 9 + 9 + 9 + 9 + 9
9. 49; 7 \times 7 = 7 + 7 + 7 + 7 + 7 + 7 + 7

Exercise 12-2a

The table of "fives"

| 0 \times 5 | = 0 |
| 1 \times 5 | = 5 |
| 2 \times 5 | = 10 |
| 3 \times 5 | = 15 |
| 4 \times 5 | = 20 |
| 5 \times 5 | = 25 |
| 6 \times 5 | = 30 |
| 7 \times 5 | = 35 |
| 8 \times 5 | = 40 |
| 9 \times 5 | = 45 |

The "5-times" table

| 5 \times 0 | = 0 |
| 5 \times 1 | = 5 |
| 5 \times 2 | = 10 |
| 5 \times 3 | = 15 |
| 5 \times 4 | = 20 |
| 5 \times 5 | = 25 |
| 5 \times 6 | = 30 |
| 5 \times 7 | = 35 |
| 5 \times 8 | = 40 |
| 5 \times 9 | = 45 |
Exercise 12-3a

6: 〇〇〇〇〇〇 1 × 6
   〇〇
   〇〇
   〇〇

   〇〇
   〇〇
   〇〇

2 × 3

7: 〇〇〇〇〇〇 1 × 7
   〇〇
   〇〇
   〇〇

   〇〇
   〇〇
   〇〇

7 × 1

9: 〇〇〇〇〇〇〇〇〇〇〇 1 × 9
   〇〇
   〇〇
   〇〇

   〇〇
   〇〇
   〇〇

3 × 3

9 × 1
Exercise 12-3b

16 = 1 \times 16; \ 16 = 2 \times 8; \ 16 = 4 \times 4; \ 16 = 8 \times 2; \ 16 = 16 \times 1.

18 = 1 \times 18; \ 18 = 2 \times 9; \ 18 = 3 \times 6; \ 18 = 6 \times 3; \ 18 = 9 \times 2;
18 = 18 \times 1.

23 = 1 \times 23; \ 23 = 23 \times 1.

36 = 1 \times 36; \ 36 = 2 \times 18; \ 36 = 3 \times 12; \ 36 = 4 \times 9; \ 36 = 6 \times 6;
36 = 9 \times 4; \ 36 = 12 \times 3; \ 36 = 18 \times 2; \ 36 = 36 \times 1.

40 = 1 \times 40; \ 40 = 2 \times 20; \ 40 = 4 \times 10; \ 40 = 5 \times 8; \ 40 = 8 \times 5;
40 = 10 \times 4; \ 40 = 20 \times 2; \ 40 = 40 \times 1.

56 = 1 \times 56; \ 56 = 2 \times 28; \ 56 = 4 \times 14; \ 56 = 7 \times 8; \ 56 = 8 \times 7;
56 = 14 \times 4; \ 56 = 28 \times 2; \ 56 = 56 \times 1.
Chapter 12

Exercise 12-3c

- $1 \times 18$
- $2 \times 9$
- $3 \times 6$
- $6 \times 3$

Exercise 12-4a

1. 6
2. 10
3. 54

Exercise 12-5a

1. $72 \div 8 = \square$  
   Quotient is 9.
2. $56 \div 7 = \square$  
   Quotient is 8.
3. $42 \div 6 = \square$  
   Quotient is 7.
4. $21 \div 3 = \square$  
   Quotient is 7.
5. $8 \div 1 = \square$  
   Quotient is 8.
6. $28 \div 4 = \square$  
   Quotient is 7.
7. $18 \div 18 = \square$  
   Quotient is 1.
8. $45 \div 5 = \square$  
   Quotient is 9.

Exercise 12-5b

1. 3
2. 8
Exercise 12-5c

\[
24 = 6 \times 4 \quad 24 \div 6 = 4
\]

Each person will receive 4 oranges.

Exercise 12-5d

1. 5 pieces
2. 4 acres
3. a. 1 shilling 9 pence
   b. 1 shilling 3 pence
   c. 1 shilling
   d. 1 shilling and \((x - 9)\) pence

if \(x\) is greater than 9; 1 shilling if \(x\) is less than 9 or equal to 9.

Exercise 12-6a

1. \(48 = 8 \times \square\) \text{ quotient is } 6.
2. \(65 = 9 \times \square\) \text{ no whole number quotient.}
3. \(47 = \square \times 5\) \text{ no whole number quotient.}
4. \(63 = 9 \times \square\) \text{ quotient is } 7
5. \(63 = \square \times 8\) \text{ no whole number quotient.}

Exercise 12-6b

9 shillings

Exercise 12-6c

7 gallons

Exercise 12-6d

1. 6 \quad 4. no whole number answer.
2. 8 \quad 5. 9
3. 280 \quad 6. no whole number answer.

Exercise 12-6e

1. 75
2. John buys 4 toy dogs and has 3 shillings remaining.
Exercise 12-7a

1. \((3 \times 5) \div 5 = 3\) \hspace{1cm} (15 \div 5) \times 5 = 15
2. \((7 \times 8) \div 8 = 7\) \hspace{1cm} (56 \div 8) \times 8 = 56
3. \((10 \times 4) \div 4 = 10\) \hspace{1cm} (40 \div 4) \times 4 = 40
4. \((9 \times 6) \div 6 = 9\) \hspace{1cm} (54 \div 6) \times 6 = 54

Exercise 12-8a

\[\square \times 0 = 1\] \hspace{1cm} \[\square \times 0 = 4\]
\[\square \times 0 = 3\] \hspace{1cm} \[\square \times 0 = 5\]

Numerals cannot be put into any of the above boxes to make the equations true.
CHAPTER THIRTEEN

Exercise 13-2a

(a) 9, 9
(b) 40, 40
(c) 2, 2
(d) 2, 0
(e) 18, 18
(f) 6, no whole number answer
(g) 8, 2
(h) 1, 5
(i) 1, 4

Answers (a) and (e) are evidence that the associative property might be true for addition. Answers (d), (g), and (h) show that the associative property is not true for subtraction. Answer (b) is evidence that the associative property might be true for multiplication. Although answer (c) suggests that the associative property might be true for division, answers (f) and (i) show that it is, in fact, not true.

Exercise 13-3a

(a) 30, 30
(b) 32, 32
(c) 54, 6
(d) 48, 48
(e) 50, 50

(f) 0, 0
(g) 5, 1
(h) 125, 625
(i) 40, 2

Exercise 13-4a

1. 3
2. 0
3. 3
4. no whole number answer
5. 0
6. 0
7. no whole number answer
8. 100
9. 0
10. no whole number answer
CHAPTER SIXTEEN

Exercise 16-1b

The set just before \[ \{ \quad \} \] is \[ \{ \quad \} \]

The set just after \[ \{ \quad \} \] is \[ \{ \quad \} \]

The set just after \[ \{ \quad \} \] is \[ \{ \quad \} \]

Exercise 16-5a

There will be more small unit lengths in a given object than there will be larger unit lengths.

Exercise 16-6a

1. 

2. 13, 5

3. 7 whole numbers are marked in red.

4. 30

5. 7 steps away

6. He can return in 5 steps to his starting point.
7. (a) 8
(b) 23
(c) $2 + 3n$
CHAPTER SEVENTEEN

Exercise 17-1a

Of two whole numbers on the number lines, the number to the left of the other number is less.

Exercise 17-2a

Only statements 1 and 4 are true.

Exercise 17-2b

1. There are many possible answers. For example,
   
   (a) \( a = 6, \ c = 7 \)

   (b) \( a = 6, \ c = 6 \)

   (c) \( a = 7, \ c = 6 \)

2. 

3. 

4. 

\[ q > a, \ b < p, \ b > n. \]
Exercise 17-3a

1. 1 is between 0 and 3.

2. 7 is between 5 and 8.

3. 3 is between 2 and 6.

4. 1 is between 0 and 9.

5. 14 is between 11 and 16.

6. 14 is between 13 and 25.

Exercise 17-3b

1. 3

2. 8

3. (b-a) - 1
Exercise 17-4

1. $1 < 3$, $2 < 4$, $3 < 5$, $4 < 6$, $5 < 7$.

2. Subtract 4. $0 < 5$.

3. To obtain $0 < b - a$, subtract $a$ from both sides of the inequality $a < b$. To obtain $a < b$, add $a$ to both sides of the inequality $0 < b - a$. 
CHAPTER EIGHTEEN

Exercise 18-1a

Exercise 18-2a

1.

2.

3.

4.

5.

6.

7.

8.
Exercise 18-2b

\[ 7 + 4 = 11 \]

Exercise 18-2c

Answers will vary.

Exercise 18-2d

1. Possible \( x + y \)

2. 15 sticks

Exercise 18-3a

1. The commutative property of addition.
   \[ a + b = b + a \]
2. Answers will vary.
3. The associative property of addition, and the additive property of zero. Examples will vary.

4. (a) \((2 + 3) + 4 = (4 + 3) + 2\)
   
   (b) associative, commutative

**Exercise 18-4a**

**Exercise 18-4b**

The frog is performing subtraction when he jumps to the left and addition when he jumps to the right. Addition and subtraction are inverse operations (see section 9-10).

**Exercise 18-4c**
Exercise 18-4d

1. 

2. After six stops you will arrive at 0.

3. 2; 12 points are marked.

Exercise 18-4e

1. Subtraction is not commutative.

possible x - y

0 1 2 3 4 5 6 7 8 9 10
2. \[ 4 - (3 - 1) = 2, (4 - 3) - 1 = 0. \]

Subtraction is not associative.

**Exercise 18-6a**

1. 

   ![Diagram](image)

2. 

   ![Diagram](image)

3. 

   ![Diagram](image)

4. 

   ![Diagram](image)

5. 

   ![Diagram](image)

6. 

   ![Diagram](image)

**Exercise 18-6b**

Answers will vary.

**Exercise 18-7a**

![Diagram](image)
Exercise 18-8a

Answers will vary.

Exercise 18-9a

1. 5 days

2. Answers will vary.

3. Multiplication and division are inverse operations.
   Examples will vary.

Exercise 18-9b

2. One possibility is shown here. Make two number lines with the numbers increasing from right to left on the upper one. To subtract \( b \) from \( a \), place the 0 of the upper line over the number \( a \) on the lower line, and read the answer on the lower line below the number \( b \) on the upper line. The above illustration shows how to find \( 9 - 2 \) and obtain 7. Notice that with this setting we can read off the answer for 9 minus any whole number less than 10.
CHAPTER NINETEEN

Exercise 19-la

1. no whole number answer 4. 87
2. 17 5. no whole number answer
3. no whole number answer 6. 25

Exercise 19-1b

Answers will vary.

Exercise 19-2a

Answers will vary.

Exercise 19-3a

1. \(\frac{1}{4}\) can be thought of as the result of sharing 1 object among 4 people, or as 1 of the fourth parts of an object.
2. \(\frac{5}{8}\) can be thought of as the result of sharing 5 objects among 8 people, or as 5 of the eighth parts of an object.
3. \(\frac{2}{10}\) can be thought of as the result of sharing 2 objects among 10 people, or as 2 of the tenth parts of an object.
4. \(\frac{3}{2}\) can be thought of as the result of sharing 3 objects among 2 people, or as 3 half parts of 2 objects each broken into halves.
5. \(\frac{1}{100}\) can be thought of as the result of sharing 1 object among 100 people, or as 1 of the hundredth parts of an object.
6. \(\frac{2}{1}\) can be thought of as the result of sharing 2 objects among 1 person, or as 2 whole objects.

Exercise 19-4a

Answers will vary.
Chapter 19

Exercise 19-4b

1. \( \frac{2}{5} \)  
2. \( 1 \frac{1}{8} \)  
3. \( \frac{7}{2} \)  
4. \( \frac{8}{3} \)  
5. \( 8 \frac{1}{2} \)  
6. \( \frac{35}{6} \)

Exercise 19-5a

1. \( \frac{72}{7} \) or \( 10 \frac{2}{7} \)  
2. \( \frac{13}{4} \) or \( 3 \frac{1}{4} \)  
3. \( \frac{96}{6} \) or \( 16 \)  
4. \( \frac{147}{24} \) or \( 6 \frac{3}{24} \)  
5. \( \frac{2176}{322} \) or \( 6 \frac{244}{322} \)  
6. \( \frac{625}{10} \) or \( 62 \frac{5}{10} \)  
7. \( \frac{83}{83} \) or \( 1 \)  
8. \( \frac{1215}{100} \) or \( 12 \frac{15}{100} \)

Exercise 19-6a

1. \( 2.3 \)  
2. \( .7 \)  
3. \( 8.13 \)  
4. \( .22 \)

Exercise 19-6b

1. Sixty-one and seven tenths  
2. Eight and eighty-one hundredths  
3. Fifty-four hundredths  
4. One and six hundredths
Exercise 19-7a

1.

2.

3.

4.
5.

6.

Exercise 19-7c

Answers will vary.

Exercise 19-7d

1. \[
\frac{35}{113}
\]

2. Answers will vary from class to class.

Exercise 19-7e

There are many possibilities. For example:

1. ·

2. ·
Exercise 19-7f

1. 

2. 

3. 

4. 

5. 

6.
Exercise 19-8a and 19-8b
Exercise 19-9a

1. 

2. 

3. 

4. 

5. 

6.
Exercise 19-9b

Answers will vary.
CHAPTER TWENTY

Exercise 20-1a

1. 

2. 

1.7

\[
\frac{17}{10}
\]

\[
\frac{14}{20}
\]
Chapter 20

3.

\[3 \frac{2}{3}, \quad \frac{22}{6}, \quad \frac{11}{3}, \quad \frac{34}{6}, \quad \frac{33}{9}\]

4.

\[\frac{7}{10}, \quad \frac{14}{20}, \quad \frac{21}{30}, \quad \frac{28}{40}\]
5.

\[ \frac{5}{4} \]

\[ \frac{10}{8} \]

6.

\[ \frac{16}{4} \]

\[ \frac{4}{1} \]

\[ \frac{8}{2} \]

\[ \frac{12}{3} \]
Exercise 20-1b

Answers will vary.

Exercise 20-2a

1. 

2. 

3.
Exercise 20-4a

There are many possibilities. For example:

1. \( \frac{2}{3} = \frac{4}{6} = \frac{6}{9} = \frac{8}{12} = \frac{10}{15} = \frac{12}{18} = \frac{14}{21} = \frac{16}{24} = \frac{18}{27} = \frac{20}{30} \)

2. \( \frac{5}{2} = \frac{10}{4} = \frac{15}{6} = \frac{20}{8} = \frac{25}{10} = \frac{30}{12} = \frac{35}{14} = \frac{40}{16} = \frac{45}{18} = \frac{50}{20} \)

3. \( \frac{3}{4} = \frac{6}{8} = \frac{9}{12} = \frac{12}{16} = \frac{15}{20} = \frac{18}{24} = \frac{21}{28} = \frac{24}{32} = \frac{27}{36} = \frac{30}{40} \)

4. \( \frac{5}{6} = \frac{10}{12} = \frac{15}{18} = \frac{20}{24} = \frac{25}{30} = \frac{30}{36} = \frac{35}{42} = \frac{40}{48} = \frac{45}{54} = \frac{50}{60} \)

5. \( \frac{12}{5} = \frac{14}{10} = \frac{16}{15} = \frac{18}{20} = \frac{20}{25} = \frac{22}{30} = \frac{24}{35} = \frac{26}{40} = \frac{28}{50} = \frac{30}{60} \)

6. \( \frac{22}{2} = \frac{2}{1} = \frac{1}{5} = \frac{2}{4} = \frac{3}{15} = \frac{2}{5} = \frac{5}{25} = \frac{6}{30} = \frac{7}{35} \)

Exercise 20-4b

1. 10

2. 27

3. \( a = 13 \)

4. \( b = 3 \)

5. \( x = 100 \)

6. \( n = 20 \)

Exercise 20-4c

Answers will vary.
Exercise 20-4d
1. \(\frac{4}{5}\)
2. \(\frac{2}{3}\)
3. \(\frac{18}{7}\)
4. \(\frac{4}{5}\)
5. \(\frac{3}{2}\)

Exercise 20-5a
1. [Diagram of a number line with fractions labeled: 0, \(\frac{1}{1}\), \(\frac{2}{1}\), \(\frac{3}{1}\), \(\frac{4}{1}\), \(\frac{5}{1}\).]

2. [Three shaded circles with fractions: \(\frac{3}{1}\).]

3. [A shaded rectangle with a fraction \(\frac{1}{1}\).]

4. [A set of shaded rectangles with a fraction \(\frac{10}{1}\).]
Exercise 20-5b  There are many possibilities. For example,

1.

\[
\begin{array}{c}
\text{3} \\
\text{3} \\
\text{6} \\
\text{1} \\
\text{2} \\
\text{3} \\
\text{4} \\
\text{5} \\
\text{6} \\
\text{0} \\
\text{2} \\
\text{2} \\
\text{2} \\
\text{12} \\
\text{10} \\
\text{~E} \\
\text{9} \\
\text{M} \\
\text{M} \\
\text{10} \\
\text{11} \\
\text{12} \\
\text{13} \\
\text{14} \\
\text{15} \\
\text{16} \\
\text{17} \\
\text{18} \\
\text{19} \\
\text{20} \\
\text{21} \\
\text{22} \\
\text{23} \\
\text{24} \\
\text{25} \\
\text{26} \\
\text{27} \\
\text{28} \\
\text{29} \\
\text{30} \\
\end{array}
\]
3.
Exercise 20-6a

Answers will vary.

Exercise 20-7a

1. \( \frac{182}{13} \)

2. \( a = \frac{451}{22} \)

3. \( \frac{625}{25} \)

4. \( y = \frac{2154}{17} \)

5. \( y = \frac{828}{9} \)

Exercise 20-7b

1. \( \frac{182}{13} = 14 = \frac{14}{1} = \frac{28}{2} = \frac{42}{3} = \frac{56}{4} = \frac{70}{5} \)

2. \( \frac{451}{22} = \frac{41}{2} = \frac{82}{4} = \frac{123}{6} = \frac{164}{8} = \frac{205}{10} = \frac{246}{12} \)

3. \( \frac{625}{25} = 25 = \frac{25}{1} = \frac{50}{2} = \frac{75}{3} = \frac{100}{4} = \frac{125}{5} \)

4. \( \frac{2154}{17} = \frac{4308}{34} = \frac{126}{17} = \frac{126}{17} = \frac{24}{34} = \frac{36}{51} \)

5. \( \frac{828}{9} = 92 = \frac{92}{1} = \frac{184}{2} = \frac{276}{3} = \frac{368}{4} = \frac{460}{5} \)

6. \( \frac{1253}{15} = \frac{2506}{30} = \frac{3759}{45} = \frac{83}{15} = \frac{83}{15} = \frac{83}{30} = \frac{83}{45} = \frac{83}{60} \)

Exercise 20-7c

The answers to problems 1, 3, and 5 are whole number answers: 14, 25, and 92.
Exercise 20-7d

1.  $15 \frac{40}{41}$

2.  $64 \frac{5}{21}$

3.  $229 \frac{7}{11}$

4.  $65 \frac{11}{42}$

Exercise 20-7e

Answers will vary.
CHAPTER TWENTY-ONE

Exercise 21-la

1.

2.

\[ \frac{2}{5} \quad \frac{1}{2} \]
3.

4.

Exercise 21-1b
Answers will vary.

Exercise 21-2a
1. $\frac{9}{3}$ or 3
2. $\frac{19}{5}$ or $3 \frac{4}{5}$
3. $\frac{11}{4}$ or $2 \frac{3}{4}$
4. 8

Exercise 21-2b
Answers will vary.
Exercises 21-3a and 21-3b

1. \(\frac{105}{98}\) or \(\frac{7}{98}\) or \(\frac{1}{14}\) or \(\frac{15}{14}\) or \(\frac{30}{28}\)

2. \(\frac{19}{6}\) or \(\frac{3}{6}\) or \(\frac{1}{2}\) or \(\frac{2}{12}\) or \(\frac{3}{12}\) or \(\frac{4}{24}\)

3. \(\frac{46}{24}\) or \(\frac{22}{24}\) or \(\frac{11}{12}\) or \(\frac{23}{12}\) or \(\frac{69}{36}\)

4. \(\frac{59}{30}\) or \(\frac{29}{30}\) or \(\frac{58}{60}\) or \(\frac{87}{90}\)

5. \(\frac{65}{50}\) or \(\frac{15}{50}\) or \(\frac{3}{10}\) or \(\frac{13}{10}\) or \(\frac{26}{20}\)

6. \(\frac{133}{24}\) or \(\frac{13}{24}\) or \(\frac{266}{48}\) or \(\frac{26}{48}\)

Exercise 21-3c

Answers will vary.

Exercises 21-3d

Answers will vary.

Exercise 21-4a

Each statement illustrates the commutative property of addition.

1. \(\frac{7}{10} + \frac{5}{4} = \frac{39}{20}\); \(\frac{5}{4} + \frac{7}{10} = \frac{39}{20}\)

2. \(1\frac{7}{8} + 9\frac{2}{3} = \frac{277}{24}\); \(9\frac{2}{3} + 1\frac{7}{8} = \frac{277}{24}\)

3. \(2.3 + \frac{15}{2} = \frac{49}{5}\); \(\frac{15}{2} + 2.3 = \frac{49}{5}\)
4. \( \frac{a}{2} + \frac{3}{d} = \frac{ad + 6}{2d} \); \( \frac{3}{d} + \frac{a}{2} = \frac{ad + 6}{2d} \)

Exercise 21-5a

1. \( \frac{3}{2} + \left( \frac{2}{3} + \frac{5}{8} \right) = \frac{67}{24} \); \( \frac{3}{2} + \frac{2}{3} + \frac{5}{8} = \frac{67}{24} \)

2. \( (1.3 + \frac{2}{5}) + 2.7 = 4.4 \); \( 1.3 + \left( \frac{2}{5} + 2.7 \right) = 4.4 \)

3. \( \left( \frac{1}{8} + \frac{2}{2} \right) + \frac{1}{5} = \frac{93}{40} \); \( \frac{1}{8} + \left( \frac{2}{2} + \frac{1}{5} \right) = \frac{93}{40} \)

Exercise 21-6a

1. \( \frac{0}{4} + \frac{5}{3} = \frac{(0 \times 3)}{(4 \times 3)} + \frac{(5 \times 4)}{(3 \times 4)} = \frac{0}{12} + \frac{20}{12} = \frac{20}{12} = \frac{(5 \times 4)}{(3 \times 4)} = \frac{5}{3} \)

2. \( \frac{2}{9} + \frac{0}{15} = \frac{(2 \times 15)}{(9 \times 15)} + \frac{(0 \times 9)}{(15 \times 9)} = \frac{30}{135} + \frac{0}{135} = \frac{30}{135} = \frac{(2 \times 15)}{(9 \times 15)} = \frac{2}{9} \)

Exercise 21-7a

1. \( \frac{3}{3} \) or 1; \( \square = \frac{5}{3} - \frac{2}{3} \)

2. \( x = 8 \); \( x = 15 - 7 \)

3. \( \frac{7}{8} \); \( \square = \frac{15}{8} - \frac{3}{4} \)

4. \( n = 3.8 \); \( n = 5.5 - 1.7 \)

Exercise 21-7b

Answers will vary.

Exercise 21-8a

Answers will vary.
Exercise 21-9a

1. No fraction or whole number answer
2. \( \frac{3}{1} - \frac{6}{2} = 0 \)
3. No fraction or whole number answer
4. \( 3.2 - 2.7 = .5 \)
5. No fraction or whole number answer
6. \( \frac{25}{2} - \frac{2}{25} = \frac{621}{50} \)

Exercise 21-10a

1. \( \frac{35}{8} \) or \( 4\frac{3}{6} \)
2. \( \frac{12}{4} \) or 3
3. \( \frac{143}{3} \) or \( 47\frac{2}{3} \)
4. 36
5. \( \frac{2241}{4} \) or \( 560\frac{1}{4} \)
6. \( \frac{160}{8} \) or 20

Exercise 21-10b

Answers will vary.

Exercise 21-11a

1. 9
2. \( \frac{117}{8} \) or \( 14\frac{5}{8} \)
3. 180
4. 171
5. 46
6. 11.2
Exercise 21-11b

1.

\[ 15 \times \frac{3}{5} = \frac{45}{5} \]

\[ 15 \times \frac{3}{5} = \frac{15 \times 3}{5} = \frac{15}{5} \times 3 = 3 \times 3 = 9 \]
\[ 2 \times 5.6 = 2 \times \frac{56}{10} = \frac{2 \times 56}{10} = \frac{112}{10} = 11 \frac{2}{10} = 11.2 \]

\[ 2 \times 5.6 = 2 \times \frac{\cancel{56}}{\cancel{10}} = 2 \times \frac{28}{\cancel{5}} = \frac{2 \times 28}{\cancel{5}} = \frac{56}{5} = 11 \frac{1}{5} \]
Exercise 21-12a

1. \[ \frac{2}{15} \]

3. 

4. 

\[ \frac{12}{30} \]
Exercise 21-12b

Answers will vary.

Exercise 21-13a

1. \( \frac{18}{20} \) or \( \frac{9}{10} \)
2. \( \frac{5}{24} \)
3. \( \frac{133}{20} \) or \( 6 \frac{13}{20} \)

Exercise 21-13b

Answers will vary.

Exercise 21-14a

1. \( 0 \frac{1}{4} \frac{2}{4} \frac{3}{4} 1 \frac{5}{4} \frac{6}{4} \frac{7}{4} 2 \frac{9}{4} \frac{10}{4} \frac{11}{4} 3 \frac{13}{4} \frac{14}{4} \frac{15}{4} \frac{17}{4} \frac{18}{4} \frac{19}{4} 5 \frac{21}{4} \frac{22}{4} \)

2. \( 0 0.5 1 1.5 2 2.5 3 3.5 4 4.5 5 5.5 6 6.5 7 7.5 8 8.5 9 9.5 10 10.5 11 11.5 12 \)

3. \( 0 \frac{1}{8} \frac{2}{8} \frac{1}{2} \frac{4}{8} \frac{5}{8} 1 \frac{7}{8} \frac{8}{8} \frac{3}{2} \frac{10}{8} \frac{11}{8} 2 \frac{13}{8} \frac{14}{8} \frac{5}{2} \frac{16}{8} \frac{17}{8} 3 \frac{19}{8} \frac{20}{8} \frac{7}{2} \)
Exercise 21-15a

1. \( \frac{5}{3} \times \frac{2}{7} = \frac{10}{21} \); \( \frac{2}{7} \times \frac{5}{3} = \frac{10}{21} \)

2. \( 1 \frac{3}{8} \times 3 \frac{1}{2} = \frac{77}{16} \); \( 3 \frac{1}{2} \times 1 \frac{3}{8} = \frac{77}{16} \)

3. \( 5.4 \times 13.1 = 70.74 \); \( 13.1 \times 5.4 = 70.74 \)

Exercise 21-16a

1. \( \left( \frac{7}{3} \times \frac{15}{13} \right) \times \frac{16}{9} = \frac{1680}{351} \); \( \frac{7}{3} \times \left( \frac{15}{13} \times \frac{16}{9} \right) = \frac{1680}{351} \)

2. \( \left( 1 \frac{3}{4} \times 7.3 \right) \times \frac{4}{4} = \frac{2044}{160} \); \( 1 \frac{3}{4} \times \left( 7.3 \times \frac{4}{4} \right) = \frac{2044}{160} \)

Exercise 21-16b

\( \left( \frac{a}{b} \times \frac{c}{d} \right) \times \frac{e}{f} = \frac{a}{b} \times \left( \frac{c}{d} \times \frac{e}{f} \right) \). Examples will vary.

Exercise 21-17a

1. \( \frac{5}{3} \times \left( \frac{7}{2} + \frac{4}{9} \right) = \frac{5}{3} \times \frac{71}{18} = \frac{355}{54} \); \( \left( \frac{5}{3} \times \frac{7}{2} \right) + \left( \frac{5}{3} \times \frac{4}{9} \right) \)

\[ = \frac{35}{6} + \frac{20}{27} = \frac{315}{54} + \frac{40}{54} = \frac{355}{54} \]

2. \( \frac{1}{10} \times \left( \frac{4}{5} + \frac{6}{7} \right) = \frac{1}{10} \times \frac{58}{35} = \frac{58}{350} \); \( \left( \frac{1}{10} \times \frac{4}{5} \right) + \left( \frac{1}{10} \times \frac{6}{7} \right) \)

\[ = \frac{4}{50} + \frac{6}{70} = \frac{28}{350} + \frac{30}{350} = \frac{58}{350} \]

Exercise 21-17b

Left-hand side: \( \frac{a}{b} \times \left( \frac{c}{d} + \frac{e}{f} \right) = \frac{a}{b} \times \frac{cf + de}{df} = \frac{a \times (cf + de)}{b \times df} \)

\[ = \frac{(a \times cf) + (a \times de)}{b \times df} \]
Right-hand side: 
\[ \frac{a}{b} \times \frac{c}{d} + \frac{a}{b} \times \frac{e}{f} = \frac{(a \times c)}{(b \times d)} + \frac{(a \times e)}{(b \times f)} \]

\[ = \frac{(a \times c) \times f}{(b \times d) \times f} + \frac{(a \times e) \times d}{(b \times f) \times d} \]

Using the commutative and associative properties of multiplication of whole numbers, we can see that \((b \times d) \times f = (b \times f) \times d = b \times df\).

Therefore, the right-hand side is
\[ \frac{(a \times c) \times f}{b \times df} + \frac{(a \times e) \times d}{b \times df} = \left[ \frac{(a \times c) \times f}{b \times df} \right] + \left[ \frac{(a \times e) \times d}{b \times df} \right] \]

Using the commutative and associative properties of multiplication of whole numbers again we can see that
\[ \left[ \frac{(a \times c) \times f}{b \times df} \right] + \left[ \frac{(a \times e) \times d}{b \times df} \right] = \frac{(a \times cf) + (a \times de)}{b \times df} \]

Therefore, the left-hand and right-hand sides are equal.

Exercise 21-17c

Answers will vary.

Exercise 21-18a

\[ \left( \frac{a}{b} \right) \times 1 = \left( \frac{a}{b} \right) \times \left( \frac{1}{1} \right) = \frac{(a \times 1)}{(b \times 1)} = \frac{a}{b} \]

\[ 1 \times \left( \frac{a}{b} \right) = \left( \frac{1}{1} \right) \times \left( \frac{a}{b} \right) = \frac{(1 \times a)}{(1 \times b)} = \frac{a}{b} \]

Exercise 21-18b

Answers will vary.

Exercise 21-18c

Property of zero for fractions: \( \left( \frac{a}{b} \right) \times 0 = 0 \times \left( \frac{a}{b} \right) = 0 \)

Examples will vary.
\[ \left( \frac{a}{b} \right) \times 0 = \left( \frac{a}{b} \right) \times \left( \frac{0}{1} \right) = \frac{a \times 0}{b \times 1} = \frac{0}{b} = 0 \]

\[ 0 \times \left( \frac{a}{b} \right) = \left( \frac{0}{1} \right) \times \left( \frac{a}{b} \right) = \frac{0 \times a}{1 \times b} = \frac{0}{b} = 0 \]

**Exercise 21-19a**

1. \[ \frac{7}{5} ; \frac{5}{7} \times \frac{7}{5} = \frac{\left( \frac{5 \times 7}{7 \times 5} \right)}{35} = 1 \]

2. \[ \frac{20}{3} ; \frac{3}{20} \times \frac{20}{3} = \frac{\left( \frac{3 \times 20}{20 \times 3} \right)}{60} = 1 \]

3. \[ \frac{2}{7} ; \frac{1}{2} \times \frac{2}{7} = \frac{7}{2} \times \frac{2}{7} = \frac{14}{14} = 1 \]

4. \[ \frac{10}{1} ; \frac{1}{10} \times 10 = \frac{10}{10} \times \frac{10}{1} = \frac{10}{10} = 1 \]

5. \[ \frac{10}{37} ; \frac{3.7}{10} \times \frac{10}{37} = \frac{37}{10} \times \frac{10}{37} = \frac{370}{370} = 1 \]

6. \[ \frac{4}{4} ; \frac{4}{4} \times \frac{4}{4} = \frac{16}{16} = 1 \]

**Exercise 21-19b**

Answers will vary.

**Exercise 21-20a**

1. \[ \frac{9}{20} = \frac{9}{4} \div 5 \]

2. \[ \frac{7}{12} = \frac{7}{3} \div 4 \]

3. \[ \frac{15}{6} = \frac{5}{6} \div \frac{1}{3} \]

4. \[ \frac{14}{15} = \frac{7}{3} \div \frac{5}{2} \]

5. \[ \frac{4}{3} = 4.4 \div 3.3 \]

6. \[ \frac{119}{42} = \frac{17}{3} \div \frac{14}{7} \]
Chapter 21

Exercise 21-21a

1. \[ \frac{3}{35} \]

2. \[ \frac{72}{15} \text{ or } \frac{24}{5} \]

3. \[ a = \frac{16}{93} \]

4. \[ \frac{77}{116} \]

5. 2

6. \[ d = \frac{153}{13} \]

Exercise 21-21b

Explanations will vary.

Exercise 21-22a

1. \[ \left( \frac{1\frac{1}{3}}{3} \times \frac{7}{8} \right) \div \frac{7}{8} = \frac{7}{6} \div \frac{7}{8} = \frac{7}{6} \times \frac{8}{7} = \frac{8}{6} = 1 \frac{2}{6} = 1 \frac{1}{3} \]

2. \[ (1.7 \times 3.4) \div 3.4 = 5.78 \div 3.4 = 1.7 \]

3. \[ \left( \frac{19}{5} \div \frac{17}{8} \right) \times \frac{17}{8} = \frac{152}{85} \times \frac{17}{8} = \frac{2584}{680} = \left( \frac{19 \times 136}{5 \times 136} \right) = \frac{19}{5} \]

4. \[ (5.8 \times 5 \frac{2}{3}) \div 5 \frac{2}{3} = \frac{986}{30} \div \frac{5 \frac{2}{3}}{3} = \frac{986}{30} \times \frac{3}{17} = \frac{2958}{510} = \left( \frac{58 \times 51}{10 \times 51} \right) = \frac{58}{10} = 5.8 \]
Addition

Addition is an operation which we have defined on whole numbers and fractions. To two such numbers a and b addition assigns their sum. The two numbers being added are called addends.

Addition has several important properties. One is the commutative property of addition: \( a + b = b + a \) for any two numbers a and b. The associative property of addition is also true: \((a + b) + c = a + (b + c)\) for any three numbers a, b and c. Zero is a number which behaves in a special way in addition: \(a + 0 = 0 + a = a\) for any number a. This is the addition property of zero.

Base five

There are systems other than the decimal system for enumerating the whole numbers. One is by use of base five numerals. In the base five system the whole numbers in natural order are written as follows:

\[
0, 1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, \ldots 43, 44, 100, 101
\]

and so on. We are counting in groups of five and only the digits 0, 1, 2, 3 and 4 are used. As in the decimal system, the digit 0 is used as a place holder.

Decimal system

A system in wide use among English-speaking people for enumerating the whole numbers is the decimal system or base ten system. The whole numbers are enumerated in groupings by ten. In their natural order they are written

\[
0, 1, 2, 3, \ldots 9, 10, 11, 12, \ldots 99, 100, 101, 102
\]

and so on. The digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are used, the digit 0 being used in empty places to keep the digits in their correct positions.

Distributive property

Addition and multiplication satisfy the distributive property:

\[
a \times (b + c) = (a \times b) + (a \times c)\]

for any three numbers a, b and c.
Division

Division is an operation which we have defined on whole numbers and fractions. To two such numbers \( a \) and \( b \) division assigns their quotient \( a \div b \). The quotient is the number which makes the following sentence true: \( b \times \square = a \). In case \( b = 0 \), no quotient is assigned, and \( a \div 0 \) is meaningless.

Equation

We often deal with sentences of the form \( 5 + \square = 7 \) or \( 8 \times \square = 16 - 10 \), which are often called equations. They are commonly written with letters instead of boxes: \( 5 + x = 7 \) or \( 8 \times p = 16 - 10 \). The letters in an equation are called variables. Not all equations are as simple as these—they may be of great complexity. Usually just one (occasionally more) numeral may be substituted into the box to make the sentence into a true statement. This number is called a solution of the equation. The remaining numbers make the sentence into a false statement when they are substituted into the box.

Fraction

The attempt to solve any equation \( b \times \square = a \) in which \( b \) is not zero leads to the idea of fractions. If we take a whole object, break it into \( b \) equal parts and take \( a \) of those parts, the number of objects we have is the number which makes the above sentence true.

A numeral which we use for the above fraction is \( \frac{a}{b} \) or \( \frac{a}{b} \). The numeral above the line is called the numerator, and the numeral below the line is called the denominator. Other common numerals for fractions are mixed numbers, which consist of a whole number and a fraction written together. For example, \( 7 \frac{1}{2} \) is a mixed number. It represents the sum of \( 7 + \frac{1}{2} \) or \( \frac{15}{2} \). The fractions represented by \( \frac{a}{b} \) and \( \frac{b}{a} \) are called reciprocal fractions. The product of a pair of reciprocal fractions is always one.

A special way of writing fractions and mixed numbers whose fractional parts have 10, 100, 1000 and so on, as their denominator, is as decimal fractions. For example, the decimal fraction for \( \frac{3}{10} \) is 0.3, for \( \frac{37}{100} \) is 0.37, for \( \frac{28}{1000} \) is 0.028 and so on.
Identity

We often deal with sentences of the form \( a + b = b + a \) or \( (b \times c) \times a = b \times (a \times c) \) which are true no matter what numbers are substituted for the variables \( a, b \) and \( c \), just so that the same number is substituted for the same variable each time it occurs. Such sentences are called identities.

Intersection of two sets

For any two sets \( G \) and \( H \), the intersection \( G \cap H \) of \( G \) and \( H \) is the set of all things which are at the same time in \( G \) and \( H \).

Multiplication

Multiplication is an operation which we have defined on whole numbers and fractions. To two such numbers multiplication assigns their product. The two numbers being multiplied are called factors of the product.

Multiplication has several important properties. One is the commutative property of multiplication: \( a \times b = b \times a \) for any two numbers \( a \) and \( b \). The associative property of multiplication is also true: \( (a \times b) \times c = a \times (b \times c) \) for any three numbers \( a, b \) and \( c \).

One is a number which behaves in a special way in multiplication: \( a \times 1 = 1 \times a = a \) for any number \( a \). This is the multiplication property of one.

Natural order

Sets which contain no members, one member, two members and so on, can be placed in an order so that each set has one more member than the preceding set. The sets are then in natural order. We also say that their corresponding whole numbers are in natural order. The natural order for whole numbers is \( 0, 1, 2, 3, 4, 5, 6 \) and so on.

Number

The abstract idea of number is developed to tell how many members are in a set. All sets which can be matched exactly are said to have the same number of
members. The number named zero tells how many members are in the empty set.

Number line

We can geometrically picture the set of whole numbers and fractions on the number line. The unit piece is the segment from 0 to 1,

\[0 \quad 1 \quad \frac{5}{3} \quad 2 \quad \frac{12}{5} \quad 3 \quad 3\frac{1}{2} \quad 4\]

and the distance between points representing two successive whole numbers is always equal to the unit piece. A number \(a\) is less than a number \(b\) if the point for \(a\) is to the left of the point for \(b\) on the number line. And \(a\) is greater than \(b\) if the point for \(a\) is to the right of the point for \(b\) on the number line. The fraction represented by \(\frac{a}{b}\) may be located by breaking the unit piece into \(b\) equal-sized pieces and measuring \(a\) of these pieces to the right of zero.

Numeral

Each number is represented by symbols called numerals. For example, the simplest numeral for the number one is 1. However, there is an unlimited number of numerals which represent the number one (or any other number):

\[3 - 2, \quad \frac{4}{4}, \quad (5 \times 3) - (2 \times 7) \text{ and so on.}\]

Operation

If we have a set, we often have a rule which assigns a member of the set to two members of the set that we may choose. For example, if the set is all whole numbers, addition and multiplication are two different such rules. To any two numbers which we choose, the rule of addition assigns their sum, while the rule of multiplication assigns their product. Any such rule on a set is called an operation on the set, or a binary operation because it yields a member of the set for any two members that we choose.

Set

Any collection of objects may be called a set. The objects in a set are the members or elements of the set. To specify a set we list its members between
braces. Thus, the set of counting members is \( \{1, 2, 3, 4, 5, \ldots \} \). A collection without any objects at all in it is also regarded as a set and is called the empty set \( \{ \} \). If A and B are two sets and every member of set A is also a member of set B, then set A is called a subset of set B. In contrast, set A and set B may have no elements at all in common. Then they are called disjoint sets.

**Subtraction**

Subtraction is an operation we have defined on whole numbers and fractions. To two such numbers \( a \) and \( b \) subtraction assigns their difference \( a - b \). The difference \( a - b \) is the number which makes the following sentence into a true statement: \( b + [\quad] = a \). Unlike addition and multiplication, subtraction is not commutative: \( a - b \) is usually not equal to \( b - a \). Furthermore, if \( a \) is less than \( b \), then we have not defined \( a - b \) as a whole number or fraction.
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PRELIMINARY EDITION

BASIC CONCEPTS of MATHEMATICS

ERRATA

Prepared at the 1963 Entebbe Mathematics Workshop

Educational Services Incorporated
Watertown, Massachusetts, U.S.A.
Compiled herein are corrections for *Basic Concepts of Mathematics*. Some known errors of a typographical or otherwise minor nature are not included. Those errors which affect the meaning or clarity of the text have been included. It is extremely important, therefore, that these changes be incorporated into this text. Without them certain problems and passages will not be clear.

Printed in red is the page number and location on the page where the error occurs. The corrected text is in black ink. Corrections may be made in one of two ways. For simple corrections it is easier to cross out the error and write the correction above. For others it might be better to cut the correction (black printing) from this booklet and paste it over the error in the textbook itself.

The editors regret that this inconvenience is necessary. It is hoped that the user of this text will agree the inconvenience is not too great a price to pay for the dispatch with which the book was produced.
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Chapter 2 - Operations On Sets

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Unit III - The Number Line

Introduction - 14, examples of positive fractions

Is \( \frac{1}{2} + \frac{2}{3} = \frac{5}{6} + \frac{1}{2} \) ?

Is \( (\frac{1}{2} + \frac{2}{3}) + \frac{1}{4} = \frac{1}{2} + (\frac{2}{3} + \frac{1}{4}) \)?

Is \( \frac{1}{2} \times \frac{2}{3} = \frac{2}{3} \times \frac{1}{2} \) ?
Introduction - 15, bottom figure and last line

(shown shaded) in half, we get a rectangle \( \frac{1}{2} \) by \( \frac{1}{3} \). It is easy to see that 6

Page 1-8, first paragraph of SPECIAL SETS

1 - 6 The empty set

What answer did you give to the question "List the members of all cities in Africa with a population of over four million?" Did you say that this set had no members?

Page 1-11, third line

(b) Set \( D = \{2, 4, 6, 8, 10\} \). Set \( E = \{4, 6, 8\} \). \( D \) is a.

Page 1-13, first line under figures

There are no members in the empty set. \( \{1, 2, 3\} \) has all the members

Page 1-22, "(g)" at middle of page

(g) The set of letters in the word "bundle" and the set \{n, d, l, e, b\}. 
Page 1-25, Illustration at top should have labels "B" and "C"

Page 2-1, chapter title

OPERATIONS ON SETS

Page 2-5, last line

(b) \{m, a\} \cup \{d, \ , \} = \{m, a, d, g, o, n\}

Page Answers - 5, bottom figure, "B" and "C" should appear

\[ A = \text{all the books in the classroom} \]

\[ B = \text{all the Maths books in the classroom} \]

\[ C = \text{all the English books in the classroom} \]
Errata - 5

Page 5-21, first sentence after figure should read

This is obtained by multiplying each number represented in the top row by each number represented in the left hand column. Each result is entered in its

Page 6-10, the division example should be just before the last sentence of the paragraph

1 ring on the second rod and there would be 9 rings on the third rod.

\[
\begin{array}{c}
5 \underline{46} \\
9 \text{ fives and one left over. 5 rings from the third rod would make 1 ring on the fourth rod leaving 4 rings on the third rod.}
\end{array}
\]

Page 6-18, problem 7(b) is division

(b) \[
\begin{array}{c}
4362 \div 21 \\
\text{seven} \div \text{seven}
\end{array}
\]

Page 6-23, chart

\[
\begin{array}{cccccccc}
1 & 10 & II & 100 & 101 & 110 & 111 & 1000 & 1001 & 1010 \\
10 & 11 & 1 & 10 & 101 & 110 & 111 & 1000 & 1001 & 1010 \\
1011 & & & & & & & & & \\
1100 & & & & & & & & & \\
1100100 & & & & & & & & & \\
\end{array}
\]
Page 8-6, illustration of union, delete one banana

Page 9-7, delete last sentence prior to section 9-5 (Exercise 2, 9-4f)

Page 10-6, change illustration at top of page

Page 10-8, line 6

combinations of boys and foods. You can see that in this case the first

Page 11-15, add brackets in last problem on page

2. What numeral must be put into each box to make \((3 \times \square) + (2 \times \square) = 35\)

into a true statement?

Page 14-12, problem (c) at middle of page

\(29 + 60 = \square\)

Represent 29 as \((2 \times 10) + (9 \times 1)\), etc.

Page 14-13, problem (c) near middle of page

\(\square = 51 + 10\)
Page 14-14, add word "sticks" to note near middle of the page

Note that 12 loose sticks form 1 bundle of 10 and 2 ones.

Page 14-19, change exercise number

Exercise 14-5e

Page 14-22, fourth line from bottom

Twenty-fives: \( 1 + 4 + 4 = 14 \)

Page 14-32, change exercise number

Exercise 14-9b (Stage I)

Page 14-34, fourth line, left-hand side

\((8 \text{ tens} - 7 \text{ tens}) + (17 \text{ ones} - 9 \text{ ones})\)

Page 14-38, answer "(a)". Exercise 14-5e

(a) 1417

Page 14-39, last three lines, add equal signs

\(40_{\text{twelve}} - 142_{\text{twelve}} = 288_{\text{twelve}}\)

\(67E8_{\text{twelve}} - 4TE9_{\text{twelve}} = 18E9_{\text{twelve}}\)

\(11516_{\text{ten}} - 8493_{\text{ten}} = 3023_{\text{ten}}\)
Page 15-1, change title of first section

15-1 Reminder of multiplication as repeated addition

Page 15-2, products near bottom

\[
\begin{align*}
13 \times 10 &= \Box \\
15 \times 10 &= \Box \\
48 \times 10 &= \Box
\end{align*}
\]

Page 15-3, last sentence before Exercise 15-2b should be deleted

Page 15-10, change title of section 15-5

15-5 Reminder of multiplication and division as inverse operations

Page 18-8, problem 6 of Exercise 18-6a

6. \(3 \times 5\)

Page 19-11, next to the last sentence on the page

In the same manner as above, you can show mixed numbers and improper fractions by using circles cut into pieces. The fraction \(\frac{9}{4}\) can be shown this way:

Page 19-15, Exercise 19-8a, use word "appropriate" rather than "proper"

Draw carefully a number line showing each unit piece divided into tenths.

Label each point with the appropriate fraction.
Page 20-14, lines six and seven, near middle of page

Thus \(451 \div 22\) gives you 20 with a remainder of 11, that is, \(451 = 22 \times 20 + 11\).

Using fractions, you can write this as

Page 21-27, fourth multiplication example

\[
\frac{3}{4} \times \frac{2}{5} = \left(3 \times \left(\frac{1}{4}\right)\right) \times \frac{2}{5} = 3 \times \left(\frac{1}{4} \times \frac{2}{5}\right).
\]

Page 21-42, the two examples are in error

21-23 The question of zero again

What do you think \(\frac{3}{0}\) could mean? A way to think about this question is to think about the sentence:

\[
0 \times \square = 3
\]

Is there any fraction which can be put into the box to make this sentence true? You have learned that

\[
0 \times \frac{a}{b} = 0
\]

where \(\frac{a}{b}\) is any fraction. Thus no matter what fraction you put into the box, you get 0 as a result on the left-hand side and never 3. Thus there is no fraction which can be multiplied by 0 to get 3. You can see also that \(\frac{2}{0}\) or \(\frac{5}{0}\) or anything else like that will result in the same difficulty. The only exception is \(\frac{0}{0}\). In the sentence \(0 \times \square = 0\) what fraction can be put in the box to make the sentence true? The answer is any fraction at all. This is almost as bad as no answer because \(\frac{0}{0}\) can be given no one definite meaning. As was said in the chapter on division of whole numbers, division by zero is not an allowable operation.
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Entebbe Mathematics Teachers' Handbook, Primary I-III, Preliminary Edition

Secondary I  . Preliminary Edition
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Teachers' Guide: Three volumes

Secondary II  . Preliminary Edition
Student Text: Three volumes
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Secondary III  . Preliminary Edition
Student Text: Algebra-One volume
Geometry-One volume
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Geometry-One volume

Basic Concepts of Mathematics  . Preliminary Edition
An Introductory Text for Teachers
Volume 1 - Structure of Arithmetic - Chapters 1-21
Volume 2 - Structure of Arithmetic - Chapters 22-46
Volume 3 - Foundations of Geometry - Chapters 47-57
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CONCEPTS
OF
MATHEMATICS

Volume II
An Introductory Text for Teachers

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In this volume of *Basic Concepts of Mathematics*, the Teacher Training Writing Group at the 1964 Entebbe Mathematics Workshop has completed "Structure of Arithmetic", the first part of an experimental text to be used by primary teachers in Training Colleges. The text was planned at the 1963 Entebbe Mathematics Workshop, and four units were written then and published in a preliminary edition. The units now added treat integers, rational numbers, real numbers, and approximations. In these units, the number system of arithmetic is further enlarged. The operations of arithmetic are studied with particular attention to problems arising in everyday situations and to the systematic underlying properties of the operations.

As in the earlier units, the exercises have two purposes: to develop and extend the understanding of the mathematical content presented in the text, and to suggest by example kinds of exercises the trainee could create for use in his own classes when he becomes a teacher. Answers for the more difficult exercises in this volume will be found at the end of this book.

The preliminary edition has been produced under pressure of time, and there is still much to be done by way of improving exposition and organization as well as adding to the stock of exercises. To all users, therefore, the Teacher Training Writing Group directs an earnest request for comments and suggestions which can contribute to the work of preparing a more finished text. Reports from experimental use of the preliminary edition are a source of ideas which will make the next edition of greater value to mathematics education.

The succeeding volume of *Basic Concepts of Mathematics*, also prepared by the Teacher Training Writing Group at the 1964 Entebbe Mathematics Workshop, is devoted to "Introduction to Geometry", the second part of the text planned in 1963.
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ORDER PROPERTIES
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22-1 Revision of order

When you learned about fractions on the number line, you found a way to find out when
one fraction is greater than another fraction. If you draw a number line showing \( \frac{2}{3} \) and \( 1\frac{1}{6} \).

\[
\begin{array}{cccccccc}
0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{5}{6} & 1 & 1\frac{1}{6} & 1\frac{1}{3} & 1\frac{1}{2} & 1\frac{2}{3} & 1\frac{5}{6} & 2
\end{array}
\]

you will see that \( \frac{2}{3} \) is less than \( 1\frac{1}{6} \) because \( \frac{2}{3} \) is to the left of \( 1\frac{1}{6} \) on the number line. In symbols \( \frac{2}{3} < 1\frac{1}{6} \). Another way of saying the same thing is \( 1\frac{1}{6} \) is greater than \( \frac{2}{3} \), or, in symbols, \( 1\frac{1}{6} > \frac{2}{3} \). You see that \( 1\frac{1}{6} \) is to the right of \( \frac{2}{3} \) on the number line.

EXERCISE 22-1A

1. Draw a number line and locate each pair of points on it. Which fraction in each pair is
greater?
   a. \( 2\frac{1}{3} \) and \( \frac{5}{3} \)  
   b. \( \frac{2}{5} \) and \( \frac{1}{3} \)  
   c. \( 1\frac{1}{2} \) and \( \frac{5}{3} \)  
   d. \( \frac{5}{4} \) and \( 1\frac{1}{6} \)

2. Write fractions less than each of the following fractions: \( \frac{5}{6}, \frac{9}{4}, \frac{1}{6}, \frac{8}{7} \).
   Do you recall that if you have three different numbers on the number line, one will be
   between the other two? Of the three numbers \( \frac{1}{2}, 1 \) and \( 1\frac{1}{2} \), 1 is between \( \frac{1}{2} \) and \( 1\frac{1}{2} \).

\[
\begin{array}{cccc}
0 & \frac{1}{2} & 1 & 1\frac{1}{2} & 2
\end{array}
\]

In symbols, you can write \( \frac{1}{2} < 1 < 1\frac{1}{2} \). This means the same thing as writing

\[ \frac{1}{2} < 1 \text{ and } 1 < 1\frac{1}{2}. \]
What do you see about $\frac{1}{2}$ and $1\frac{1}{2}$, the two outside points? You see that $\frac{1}{2} < 1\frac{1}{2}$ because $\frac{1}{2}$ is to the left of $1\frac{1}{2}$.

**EXERCISE 22-1B**

1. Locate the three fractions in each of the following sets on the number line, and say which one is between the other two. Also say which of the outside two fractions is the greater.

   a. $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}$
   
   b. $\frac{3}{2}, \frac{1}{2}, 1\frac{1}{2}$

   c. $\frac{2}{3}, \frac{5}{6}, 1$

   d. $\frac{7}{5}, \frac{1}{2}, \frac{4}{3}$

2. Make up three sets, each containing three different fractions, and show each set on a number line.

### 22-2 Order and addition and subtraction

Some properties of order which we know already for whole numbers are also true for fractions. Let us start by locating $\frac{1}{2}$ and $\frac{7}{2}$ on the number line. We see that $\frac{1}{2}$ is to the left of $\frac{7}{2}$.

That is, $\frac{1}{2} < \frac{7}{2}$. If we add the same fraction $\frac{3}{2}$ to both $\frac{1}{2}$ and $\frac{7}{2}$, we move each to the right a distance $\frac{3}{2}$.

![Number line with fractions located and marked]

The sums are 2 and 5, and we notice that $2 < 5$ because 2 is to the left of 5. The sums are in the same order as the original numbers $\frac{1}{2}$ and $\frac{7}{2}$. We have

$$\frac{1}{2} < \frac{7}{2},$$
$$\frac{1}{2} + \frac{3}{2} < \frac{7}{2} + \frac{3}{2},$$
$$2 < 5.$$

Now think of $\frac{a}{b}$ and $\frac{c}{d}$ as any two fractions with $\frac{a}{b} < \frac{c}{d}$. Then $\frac{a}{b}$ will be to the left of $\frac{c}{d}$ on the number line. If we add the fraction $\frac{p}{q}$ to $\frac{a}{b}$ and also to $\frac{c}{d}$, the point for $\frac{a}{b}$ is replaced by a point $\frac{p}{q}$ units to the right of $\frac{a}{b}$, and the point for $\frac{c}{d}$ is replaced by a point $\frac{p}{q}$ units to the right of $\frac{c}{d}$. The point $\frac{a}{b} + \frac{p}{q}$ is still to the left of $\frac{c}{d} + \frac{p}{q}$. 

2
Thus:
If \( \frac{a}{b} < \frac{c}{d} \),
then \( \frac{a}{b} + \frac{p}{q} < \frac{c}{d} + \frac{p}{q} \).

Do you recognize this as one of the properties of order which we knew before only for whole numbers?

Again, let \( a \) and \( b \) be whole numbers with \( a < b \). We have already seen that if \( c \) is any whole number which can be subtracted from \( a \) to make \( a - c \) a whole number, then\[ a - c < b - c. \]
The same property is true of fractions.

For example, from both sides of the true inequality
\[ \frac{4}{5} < 1\frac{3}{5}, \]
we can subtract \( \frac{3}{5} \) to obtain
\[ \frac{4}{5} - \frac{3}{5} < 1\frac{3}{5} - \frac{3}{5}, \]
that is,
\[ \frac{1}{5} < 1. \]

We still have to be careful that the number we subtract is not too large. For example, we can not yet subtract \( 1\frac{1}{5} \) from \( \frac{4}{5} \).

The subtraction property can be stated as follows:

If \( \frac{a}{b} < \frac{c}{d} \),
then \( \frac{a}{b} - \frac{p}{q} < \frac{c}{d} - \frac{p}{q} \).

**EXERCISE 22-2A**

1. Is it true that \( \frac{5}{3} < \frac{9}{2} \)?
2. By adding \( \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1 \) to both sides of \( \frac{5}{3} < \frac{9}{2} \), find four more true inequalities.
3. By subtracting $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{3}$ from both sides, find four more true inequalities.

4. Can you subtract $\frac{9}{5}$ from both sides of the original inequality?

22-3 Generalized addition property

Let us start once again from the true inequality $\frac{1}{2} < \frac{7}{2}$. In the last section, we added $\frac{3}{2}$ to both sides to obtain $2 < 5$. What will happen if we add a larger number to the right side $\frac{7}{2}$ than to the $\frac{1}{2}$ on the left? For example, since $\frac{3}{2} < \frac{5}{2}$, let us add $\frac{3}{2}$ to the $\frac{1}{2}$ on the left and the larger number $\frac{5}{2}$ to the $\frac{7}{2}$ on the right. Then we obtain the inequality $2 < 6$, which is also true. That is, we start with

$$\frac{1}{2} < \frac{7}{2} \text{ and } \frac{3}{2} < \frac{5}{2}.$$ 

After adding, we see that the sum of the right-hand sides is greater than the sum of the left-hand sides:

$$\frac{1}{2} + \frac{3}{2} < \frac{7}{2} + \frac{5}{2}$$

or

$$2 < 6.$$ 

Will this always be the case? Let us start with

$$\frac{3}{4} < \frac{4}{2} \text{ and } \frac{2}{5} < \frac{1}{2}.$$ 

Is the following true?

$$\frac{3}{4} + \frac{2}{5} < \frac{4}{2} + \frac{1}{2}$$

Yes, because $\frac{3}{4} + \frac{2}{5} = \frac{23}{20}$ and $\frac{4}{2} + \frac{1}{2} = \frac{13}{10} = \frac{26}{20}$ and

$$\frac{23}{20} < \frac{26}{20}.$$ 

The generalized property that these examples illustrate is the generalized addition property:

If

$$\frac{a}{b} < \frac{c}{d} \text{ and } \frac{p}{q} < \frac{r}{s},$$

then

$$\frac{a + p}{b + q} < \frac{c + r}{d + s}.$$ 

To see that this is really so, we first add $\frac{p}{q}$ to both sides of

$$\frac{a}{b} < \frac{c}{d}.$$
and obtain
\[ \frac{a}{b} + \frac{p}{q} < \frac{c}{d} + \frac{r}{s}. \]

We can get another true inequality by adding \( \frac{c}{d} \) to both sides of \( \frac{p}{q} < \frac{r}{s} \).

This gives us
\[ \frac{p}{q} + \frac{c}{d} < \frac{r}{s} + \frac{c}{d}, \]
or, by using the commutative property of addition,
\[ \frac{c}{d} + \frac{p}{q} < \frac{c}{d} + \frac{r}{s}. \]

Looking at the second and fourth inequalities in this paragraph, we see that the order of
\[ \frac{a}{b} + \frac{p}{q}, \frac{c}{d} + \frac{p}{q}, \frac{c}{d} + \frac{r}{s} \]
along the number line must be as shown here.

[Diagram showing the order of the inequalities]

Of course, you see straightaway that
\[ \frac{a}{b} + \frac{p}{q} < \frac{c}{d} + \frac{r}{s}. \]

**EXERCISE 22-3A**

What true inequality do you obtain by using the generalized addition property with each of the following pairs of inequalities?

1. \( \frac{1}{2} < \frac{3}{4} \) and \( 1\frac{1}{4} < 1\frac{1}{2} \)
2. \( 1\frac{1}{2} < 3 \) and \( \frac{2}{3} < \frac{5}{6} \)
3. \( \frac{3}{5} < \frac{5}{6} \) and \( \frac{1}{2} < \frac{2}{3} \)
4. \( \frac{5}{3} < \frac{9}{4} \) and \( \frac{1}{8} < \frac{1}{7} \)

**22-4 Order and multiplication by \( \frac{1}{n} \)**

You know that \( \frac{1}{3} < \frac{2}{3} \). What can you say about \( \frac{1}{4} \times \frac{1}{3} \) and \( \frac{1}{4} \times \frac{2}{3} \)? Which is greater? To find the answer, let us use rectangles to show the original fractions.

![Rectangle diagrams showing fractions]
The set of one of three equal parts is smaller than a set of two of the three equal parts. So \( \frac{1}{3} < \frac{2}{3} \). To picture \( \frac{1}{4} \times \frac{1}{3} \), we divide \( \frac{1}{3} \) by lines running from side to side. We can picture \( \frac{1}{4} \times \frac{2}{3} \) in the same way.

![Diagram showing \( \frac{1}{4} \times \frac{1}{3} \) and \( \frac{1}{4} \times \frac{2}{3} \).]

Each rectangle is now cut into 12 equal parts. How many twelfth parts is \( \frac{1}{4} \times \frac{1}{3} \)? How many twelfth parts is \( \frac{1}{4} \times \frac{2}{3} \)? You see that from

\[
\frac{1}{3} < \frac{2}{3}
\]

you can conclude that

\[
\frac{1}{4} \times \frac{1}{3} < \frac{1}{4} \times \frac{2}{3}.
\]

That is,

\[
\frac{1}{12} < \frac{2}{12}.
\]

We can show this conclusion on the number line by first locating \( \frac{1}{3} \) and \( \frac{2}{3} \).

![Number line showing \( \frac{1}{3} \) and \( \frac{2}{3} \).]

It is farther to go from 0 to \( \frac{2}{3} \) than it is to go from 0 to \( \frac{1}{3} \). So \( \frac{2}{3} \) must be farther to the right on the number line than \( \frac{1}{3} \). If you only go one-fourth of the way to \( \frac{2}{3} \), that is still farther than going...
one-fourth of the way to \( \frac{1}{3} \). So \( \frac{1}{4} \times \frac{2}{3} \) must be farther to the right than \( \frac{1}{4} \times \frac{1}{3} \). That is

\[
\frac{1}{4} \times \frac{1}{3} < \frac{1}{4} \times \frac{2}{3}.
\]

Now you can see the general rule. If \( \frac{a}{b} < \frac{c}{d} \) is true and you multiply both sides of the inequality by \( \frac{1}{n} \), then you get another true inequality,

\[
\frac{1}{n} \times \frac{a}{b} < \frac{1}{n} \times \frac{c}{d}.
\]

It is not difficult to see that this general rule must be true. If \( \frac{a}{b} \) is less than \( \frac{c}{d} \) and we divide both \( \frac{a}{b} \) and \( \frac{c}{d} \) into \( n \) equal parts, one of the parts of \( \frac{a}{b} \) will be smaller than one of the parts of \( \frac{c}{d} \). That is to say,

\[
\frac{1}{n} \times \frac{a}{b} < \frac{1}{n} \times \frac{c}{d}.
\]

**EXERCISE 22-4A**

From the inequality \( \frac{2}{3} < \frac{3}{4} \), by drawing rectangles and number lines show that the following inequalities are true.

1. \( \frac{1}{3} < \frac{3}{8} \)
2. \( \frac{2}{9} < \frac{1}{4} \)
3. \( \frac{1}{6} < \frac{3}{16} \)

**22-5 Order and multiplication by \( m \)**

What can we say about multiplying both sides of the inequality \( \frac{1}{2} < \frac{3}{4} \) by 5? If we locate \( \frac{1}{2} \) and \( \frac{3}{4} \) on the number line, we see that the jump from 0 to \( \frac{3}{4} \) is greater than the jump from 0 to \( \frac{1}{2} \).

Then five jumps of the size from 0 to \( \frac{3}{4} \) will carry you farther to the right on the number line than five jumps of the size 0 to \( \frac{1}{2} \). So \( 5 \times \frac{1}{2} < 5 \times \frac{3}{4} \). From the true inequality

\[
\frac{1}{2} < \frac{3}{4}.
\]

we can conclude that \( 5 \times \frac{1}{2} < 5 \times \frac{3}{4} \).
You can guess what the general statement must be. If we know that
\[
\frac{a}{b} < \frac{c}{d},
\]
and \(m\) is a counting number, then we can conclude that
\[
m \times \frac{a}{b} < m \times \frac{c}{d}.
\]

Let us see if this guess is true. Imagine jumps of distances \(\frac{a}{b}\) and \(\frac{c}{d}\) on the number line. The jump for \(\frac{c}{d}\) is longer than the jump for \(\frac{a}{b}\). Then \(m\) jumps of distance \(\frac{c}{d}\) will certainly be longer than \(m\) jumps of distance \(\frac{a}{b}\). That is,
\[
m \times \frac{a}{b} < m \times \frac{c}{d}.
\]

The general statement is true.

**EXERCISE 22-5A**

1. What new inequalities do you get if you multiply both sides of the inequality \(\frac{3}{4} < \frac{4}{3}\) by the following whole numbers?
   - 2, 3, 4, 5, 6, 12

2. Explain how you can show that \(2 \times \frac{2}{3} < 2 \times \frac{5}{4}\) by using rectangles.

**22-6 Order and multiplication by \(\frac{m}{n}\)**

Suppose we multiply both sides of the inequality \(\frac{2}{3} < \frac{4}{5}\) by \(\frac{2}{7}\). What new inequality do you think will be true? Of course, you think \(\frac{2}{7} \times \frac{2}{3} < \frac{2}{7} \times \frac{4}{5}\) might be true. But how can you see that it is true? One way, of course, is to multiply the fractions and compare the two sides of the inequality. You will get \(\frac{4}{21} < \frac{8}{35}\), which is true.

There is another way. Suppose you begin by writing \(\frac{2}{7} = 2 \times \frac{1}{7}\). Since it is true that
\[
\frac{2}{3} < \frac{4}{5},
\]
you can conclude that
\[
\frac{1}{7} \times \frac{2}{3} < \frac{1}{7} \times \frac{4}{5}.
\]
After further multiplying by the whole number 2, you can conclude that
\[ 2 \times \left( \frac{1}{2} \times \frac{2}{3} \right) < 2 \times \left( \frac{1}{2} \times \frac{4}{5} \right), \]
or
\[ \left( 2 \times \frac{1}{2} \right) \times \frac{2}{3} < \left( 2 \times \frac{1}{2} \right) \times \frac{4}{5}. \]

Then
\[ \frac{2}{7} \times \frac{2}{3} < \frac{2}{7} \times \frac{4}{5}. \]

(What property of multiplication did you use to go from the first inequality on this page to the second inequality?

The result about order and multiplication that you now suspect is true is the following:

If \[ \frac{a}{b} < \frac{c}{d}, \]

then we can conclude that
\[ \frac{m}{n} \times \frac{a}{b} < \frac{m}{n} \times \frac{c}{d}. \]

To see this we write down the following list of inequalities.

\[ \frac{a}{b} < \frac{c}{d}, \]
\[ \frac{1}{n} \times \frac{a}{b} < \frac{1}{n} \times \frac{c}{d}, \]
\[ m \times \left( \frac{1}{n} \times \frac{a}{b} \right) < m \times \left( \frac{1}{n} \times \frac{c}{d} \right), \]
\[ \left( m \times \frac{1}{n} \right) \times \frac{a}{b} < \left( m \times \frac{1}{n} \right) \times \frac{c}{d}, \]
\[ \frac{m}{n} \times \frac{a}{b} < \frac{m}{n} \times \frac{c}{d}. \]

**EXERCISE 22-6A**

1. Obtain new inequalities by multiplying both sides of the inequality \( \frac{5}{4} < \frac{3}{2} \) by \( \frac{2}{15}, \frac{1}{3}, \frac{4}{5}, 2 \).

2. From the inequality \( \frac{3}{4} < \frac{5}{6} \), how can you conclude that \( 1 < \frac{10}{9} \)? Can you conclude that \( 9 < 10 \)?

3. Give the properties of fractions that you use to go from one inequality to the next one in showing the general property in this section that \( \frac{m}{n} \times \frac{a}{b} < \frac{m}{n} \times \frac{c}{d} \).

4. Say how you might convince pupils that if \( \frac{a}{b} < \frac{c}{d} \) is true, then
\[ \frac{a}{b} \div \frac{m}{n} < \frac{c}{d} \div \frac{m}{n} \]
is also true (unless \( \frac{m}{n} = 0 \)).
22-7 Generalized multiplication property

In the last section, we started with the inequality \( \frac{2}{3} < \frac{4}{5} \). Let us do so again, but this time multiply the larger number \( \frac{4}{5} \) by a greater number than the one we use to multiply \( \frac{2}{3} \). Let us multiply \( \frac{2}{3} \) by \( \frac{3}{5} \), but multiply \( \frac{4}{5} \) by \( \frac{4}{3} \), which is greater than \( \frac{3}{5} \).

From

\[
\frac{2}{3} < \frac{4}{5}
\]

and

\[
\frac{3}{5} < \frac{4}{3},
\]

we get

\[
\frac{3}{5} \times \frac{2}{3} < \frac{4}{5} \times \frac{4}{3}
\]

which is a true inequality. The product of the left-hand sides is less than the product of the right-hand sides.

We can do this problem in another way to see that the conclusion must follow. Since \( \frac{3}{5} < \frac{4}{3} \), we can conclude that \( \frac{4}{5} \times \frac{3}{5} < \frac{4}{5} \times \frac{4}{3} \). That is,

\[
\frac{12}{25} < \frac{16}{15}.
\]

From the true inequality \( \frac{2}{3} < \frac{4}{5} \), we can conclude that \( \frac{3}{5} \times \frac{2}{3} < \frac{3}{5} \times \frac{4}{5} \). That is,

\[
\frac{6}{15} < \frac{12}{25}.
\]

The last inequality tells us that \( \frac{12}{25} \) is to the right of \( \frac{6}{15} \) on the number line.

And \( \frac{12}{25} < \frac{16}{15} \) tells us that \( \frac{16}{15} \) is to the right of \( \frac{12}{25} \). So \( \frac{16}{15} \) must be to the right of \( \frac{6}{15} \) and \( \frac{6}{15} < \frac{16}{15} \). 

Thus, \( \frac{3}{5} \times \frac{2}{3} < \frac{4}{3} \times \frac{4}{5} \).

The general property that we would like to establish is this generalized multiplication property:

\[
\text{If} \quad \frac{a}{b} < \frac{c}{d} \quad \text{and} \quad \frac{m}{n} < \frac{p}{q},
\]

\[
\text{then} \quad \frac{a}{b} \times \frac{m}{n} < \frac{c}{d} \times \frac{p}{q}.
\]
We will follow the plan we have just used in the example. Since \( \frac{m}{n} < \frac{p}{q} \), we can conclude that

\[
\frac{c}{d} \times \frac{m}{n} < \frac{c}{d} \times \frac{p}{q}.
\]

From the inequality \( \frac{a}{b} < \frac{c}{d} \), we can conclude that

\[
\frac{m}{n} \times \frac{a}{b} < \frac{m}{n} \times \frac{c}{d} \quad \text{or} \quad \frac{a}{b} \times \frac{m}{n} < \frac{c}{d} \times \frac{m}{n}.
\]

(Where did we use the commutative property of multiplication?) From \( \frac{a}{b} \times \frac{m}{n} < \frac{c}{d} \times \frac{m}{n} \), we know that \( \frac{c}{d} \times \frac{m}{n} \) is to the right of \( \frac{a}{b} \times \frac{m}{n} \) on the number line. Also \( \frac{a}{b} \times \frac{m}{n} < \frac{c}{d} \times \frac{p}{q} \) tells us that \( \frac{c}{d} \times \frac{p}{q} \) is to the right of \( \frac{c}{d} \times \frac{m}{n} \). Then \( \frac{c}{d} \times \frac{p}{q} \) must also be to the right of \( \frac{a}{b} \times \frac{m}{n} \).

![Number line diagram](image)

That is,

\[
\frac{a}{b} \times \frac{m}{n} < \frac{c}{d} \times \frac{p}{q}.
\]

**EXERCISE 22-7A**

1. Using the same plan as the example, show that \( \frac{5}{7} < \frac{7}{8} \) by starting from the inequalities

\[
\frac{3}{7} < \frac{1}{2} \quad \text{and} \quad \frac{5}{7} < \frac{7}{8}.
\]

2. Using the generalized multiplication property, what inequality do you get from each of the following pairs of inequalities?

\[
\frac{1}{2} < \frac{3}{4} \quad \text{and} \quad \frac{2}{3} < \frac{5}{6}
\]

\[
\frac{3}{5} < \frac{5}{6} \quad \text{and} \quad \frac{1}{2} < \frac{2}{3}
\]

\[
\frac{5}{3} < \frac{9}{4} \quad \text{and} \quad \frac{1}{8} < \frac{1}{7}
\]
Chapter 23

DECIMAL FRACTIONS

23-1 Revision of decimal numeration

You will remember that at school you learned another way of writing fractions, that is, as decimal fractions. These are the same kind of numbers as the fractions that you have just been studying, but they are written in a different way. Just as whole numbers may be written as Hindu-Arabic numerals, or as Roman numerals or in the way the ancient Egyptians wrote them, so the parts of a whole may be written in two ways. You have already studied ways of introducing your pupils to fractions which are written with a numerator and a denominator, such as \( \frac{1}{4} \). We will, in future, refer to fractions written this way as fractions in common form or common fractions. We do this to distinguish them from fractions which are written as an extension of the decimal notation for whole numbers, such as \( .25 \). We will refer to fractions written this way as fractions in decimal form or decimal fractions. Each common fraction has its decimal fraction equivalent but not every decimal fraction has an equivalent common fraction, as you will see in Unit VII.

You know already that the notation for decimal fractions was invented as an extension of the way we write whole numbers. It uses the ideas of base and place value which are the basis of the Hindu-Arabic notation system. If you will always remember this fact, it will be no harder to understand decimal fractions than it is to understand the decimal notation for whole numbers.

The idea of a base

We will begin by recalling what you know about the decimal notation for whole numbers. You will remember that the notation for whole numbers is based on counting. When large numbers of objects have to be counted, it is simpler to group them into equivalent sets, that is, into sets with the same number of members. They can be grouped into sets of 2 or 3 or 4 members, or any number of members you choose. If you group the objects in threes, then you are using base three. If you group them in sixes, then you are using base six. Traditionally, we group our numbers in tens; that is, we use base ten. This is why our system of notation is called the decimal system, from the Latin word *decem* which means "ten". When the number of the set to be counted is large, it is necessary to put groups together to make larger groups, still using the same base number. If 5 ones are grouped together, then 5 fives will make the next larger group. This grouping of groups to make larger groups is continued as far as is necessary. In the deci-
mal system the groups have special names. They are ones, tens, hundreds, thousands, ten-thousands, hundred-thousands, millions and so on.

You will remember that you can help your pupils to understand the idea of grouping by letting them practice grouping a set of sticks, using several different bases. Here is an example you could use with them.

Put out a long row of sticks. Choose a base. We will use base three. First group the sticks in threes, beginning from the left.

```
\[ HI IHIHHH HH HIT I T \]
```

Then group the threes into larger groups each of 3 threes.

```
\[ LW \]
```

Continue in this way until you have no more than 2 of any kind of group. Here is the last picture. You will see that there is 1 group of 3 groups of 3 threes.

```
\[ 1 \text{ group of 3 groups of 3 threes} + 1 \text{ group of 3 threes} + 2 \text{ groups of three} + 1. \]
```

Now you ask your pupils which is the biggest group they have made? It is 1 group of 3 groups of 3 threes. It can be written as $3 \times 3 \times 3$.

Then ask about the next largest group, the group of 3 threes. There is only 1 of this group also. Then there are 2 groups of three and 1 one. Put these together and you have 1 group of $3 \times 3 \times 3$ and 1 group of $3 \times 1$, and 2 groups of three and 1 one. Do you remember the short way of writing this? You use the index notation and show $3 \times 3 \times 3$ as $3^3$. It is $1 \times 3^3 + 1 \times 3^2 + 2 \times 3 + 1 \times 1$.

**EXERCISE 23-1A**

1. Set out 43 sticks or draw 43 strokes. Group them in base six and write the result, using index notation showing powers of 6.
2. Using an equivalent set of sticks, group them in base ten and write the result, using index notation.
3. A set of sticks has been grouped in fours and the result written as

   $1 \times 4^3 + 2 \times 4^2 + 3 \times 4 + 0 \times 1$.

   Draw a picture to show this grouping.
**Place value**

What does the numeral $3243_{\text{ten}}$ mean? How is it different from $3243_{\text{five}}$? It is helpful to your pupils to make a number chart to show the value of each of the digits in a numeral. Here are two number charts, one for base ten and one for base five.

<table>
<thead>
<tr>
<th>Thousands</th>
<th>Hundreds</th>
<th>Tens</th>
<th>Ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^3$</td>
<td>$10^2$</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Hundred twenty-fives</th>
<th>Twenty-fives</th>
<th>Fives</th>
<th>Ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>$5^3$</td>
<td>$5^2$</td>
<td>5</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

The numeral $3243_{\text{ten}}$ can be read from this number chart as three thousand, two hundred and forty-three. Without the number chart, how do you know how to read the number? You know the value of a digit by its position in the row. The place value of the 4 is ten because the 4 is in the tens place. The place value of the 3 on the left is a thousand because it is in the fourth place from the right, the thousands place. Your pupils should be familiar with the value of these places in the decimal system and should be able to read a number easily.

You will remember from your earlier work how to read a number in another base, such as $3243_{\text{five}}$. You will remember that we read this as “three two four three base five” and do not attempt to give names to the places. (Why not?)

**Change of base**

You should remember also how to change a numeral from one base to another. If you have difficulty with the following exercise, look back to Chapter 6.

**EXERCISE 23-1B**

1. Write the following numerals in the expanded form, using the index notation.
   a. $1213_{\text{four}}$
   b. $2375_{\text{eight}}$
   c. $121304_{\text{five}}$

2. Rewrite each of the numbers in Question 1 in the decimal notation.

3. Write the following numbers in base seven.
   a. $65_{\text{ten}}$
   b. $77_{\text{ten}}$
   c. $36490_{\text{ten}}$

4. Make a number chart for base six and show on it the following numbers.
   a. $35002_{\text{six}}$
   b. $2020_{\text{six}}$
   c. $13452_{\text{six}}$

5. What is the value in decimal notation of each 2 in the numerals in the last question?

6. Tell in what base each of the following equations is written.
   a. $2 + 1 = 10$
   b. $13 - 4 = 4$
   c. $14 \times 4 = 104$
   d. $26 \div 4 = 5$
7. Write the following numbers in the decimal notation.
   a. Five hundred thousand, eight hundred and seventeen.
   b. Six million, ninety-two thousand and twenty-three.
   c. Nineteen thousand, nine hundred and nine.

Relationship between the digits in a numeral in the decimal system

Before we introduce the notation for decimal fractions, it is useful to think again about the relationship between one, ten, hundred and so on. Pictures can help your pupils to understand and remember what they learn. One and ten are easy numbers to understand. Ten ones make ten and we can see a set of 10 members by looking at our fingers or toes. One hundred is harder. Where do you see a hundred? A thousand is a big number. Do you ever see a thousand? And so you can think also of a million. How many thousands do you need to make a million? Here is one way to help your pupils to gain some idea of the relative sizes of numbers. If you have graph paper with small squares whose sides are, for example, each \( \frac{1}{10} \) inch, use it. If you have not, then use, or make, paper with small squares on it. If you use graph paper, then you will be able to show easily how a thousand is built up from 10 hundreds, how a hundred is built up from 10 tens and how a ten is built up from 10 ones. You are about to make pictures of one, of ten, of a hundred and of a thousand.

Draw round 1 small square. This is 1 one.
Draw round 10 small squares. This is a ten-strip.
So 1 ten = 10 ones.

\[
\begin{align*}
1 & \quad 10
\end{align*}
\]

Next, draw round 10 ten-strips.
This is a hundred-square.
So 1 hundred = 10 tens = 100 ones.

\[
100
\]

Now your pupils can build up a thousand-strip by drawing round ten of these hundred-squares. What can they write about this big strip? It is a thousand.

So 1 thousand = 10 hundreds = 100 tens = 1000 ones.

You could continue further and build up a ten-thousand-square, a hundred-thousand-strip and even a million-square. How many thousand-strips would be needed for a million-square?

A ten-thousand-square needs 10 thousand-strips.
A hundred-thousand-strip needs 100 thousand-strips.
So a million-square needs 1000 thousand-strips.
Can you write a set of equalities for a million as we did for a thousand? Here it is. Were you right?

\[
\begin{align*}
1 \text{ million} &= 10 \text{ hundred thousands} \\
&= 100 \text{ ten thousands} \\
&= 1,000 \text{ thousands} \\
&= 10,000 \text{ hundreds} \\
&= 100,000 \text{ tens} \\
&= 1,000,000 \text{ ones}
\end{align*}
\]

If you let your pupils draw pictures like these above, it helps them to see the relationship between each digit in a numeral. Think of the number 11,111. Each 1 is ten times the value of the digit to the right of it, or 100 times the value of the digit two places to the right of it. You can think of the 3 in 3641 as

- 3 thousands or
- 30 hundreds or
- 300 tens or
- 3,000 ones.

**EXERCISE 23-1C**

1. Using squares and the method shown above, draw pictures to show the value of 1111.
2. Give the value of each of the digits underlined in three different ways.
   a. 32,541  
   b. 5,578  
   c. 10,327
3. a. How many tens are there in 362?
   b. How many tens are there in 5,362?
   c. How many hundreds are there in 37,140?

**23-2 Decimal fractions**

Do you remember how to represent numbers on an abacus? Here is an abacus with four rods. We label the rods as we label the places in a numeral.

![Abacus diagram]

The bead on the tens rod is worth 10 of the beads on the ones rod. The bead on the hundreds rod is worth 10 of the beads on the tens rod or 100 of the beads on the ones rod.

Thus each bead is worth 10 of the beads on the rod next to it on the right.
Shown above are three numbers, represented on the abacus. Can you write them down? Use the index notation first, and then write them in the usual way. Here they are:

(a) $3 \times 10^3 + 5 \times 10^2 + 6 \times 10 + 3 \times 1 = 3563$
(b) $2 \times 10^3 + 0 \times 10^2 + 4 \times 10 + 5 \times 1 = 2045$
(c) $5 \times 10^3 + 2 \times 10^2 + 4 \times 10 + 3 \times 1 = 5243$

It will be useful to us now to think of the relationship between the beads the other way round. To make it easier to write, we will call the rods A, B, C and D.

Ten beads on rod B are worth 1 bead on rod A. So a bead on rod B is worth $\frac{1}{10}$ of a bead on rod A. The same relationship, "$\frac{1}{10}$ of", will be true for each bead and a bead on the rod immediately to the left of it.

A bead on rod C is worth $\frac{1}{10}$ of a bead on rod B.

A bead on rod D is worth $\frac{1}{10}$ of a bead on rod C.

If we place another rod, E, to the right of rod D, we can say that a bead on rod E is worth $\frac{1}{10}$ of a bead on rod D. But a bead on rod D is worth one, and so a bead on rod E is worth $\frac{1}{10}$ of one, that is, 1 tenth. So we can name rod E the "tenths rod" just as rod D is named the "ones rod" and rod B is named the "hundreds rod". You will note that a space has been left between rod D and rod E. This is to remind us that the whole numbers end with rod D. After rod D we have tenths, which are fractions, decimal fractions.

We will now add a rod F to the right of rod E. A bead on rod F must be worth $\frac{1}{10}$ of a bead on rod E. But a bead on rod E is worth $\frac{1}{10}$ of one. So a bead on rod F is worth $\frac{1}{10}$ of $\frac{1}{10}$ of 1. In the chapter on multiplication of fractions, you learned how to find that this is $\frac{1}{100}$. But you can help your pupils to understand this more fully by working it out on the abacus.
How many beads on rod F make 1 bead on rod E? 10.
How many beads on rod E make 1 bead on rod D? 10.
So how many F beads make 1 bead on rod D? 100. So an F bead is \(\frac{1}{100}\) of a D bead. But a D bead is worth one, so an F bead is worth \(\frac{1}{100}\) of 1 = \(\frac{1}{100}\). So now you can name the F rod the "hundredths rod".

**Place value of decimal fractions**

You know that the value of a digit in a numeral is shown by its position in the row of digits. Each position has a value \(\frac{1}{10}\) of the position next to it on the left. Using the place-value system, we can write the number shown on the abacus above as 1111 11. The last 1 represents 1 hundredth and the 1 to the left of it represents 1 tenth. You can see that you could easily mistake this number for 1111,11, which has no fractional part. You need some way of telling which digits represent whole numbers and which digits represent fractions. That is why we use a dot called the decimal point. The number shown on the abacus is then written 1,111.11. This, you remember, is spoken as “one thousand, one hundred and eleven point one one”. We do not usually say the value of the decimal fraction when we read it. For another example, we will read 362.54. It is read as “three hundred and sixty-two point five four”.

It will be helpful to your pupils to make a number chart which includes decimal fractions. Here is one.

<table>
<thead>
<tr>
<th>Decimal Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>Millions</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>(a)</td>
</tr>
<tr>
<td>(b)</td>
</tr>
<tr>
<td>(c)</td>
</tr>
<tr>
<td>(d)</td>
</tr>
<tr>
<td>(e)</td>
</tr>
<tr>
<td>(f)</td>
</tr>
</tbody>
</table>

You will notice that the columns have been extended farther to the right to include thousandths, ten-thousandths, hundred-thousandths and millionths. The place value of each column is \(\frac{1}{10}\) of the value of the column next to it on the left. The column on the right of the hundredths column is worth \(\frac{1}{100}\) of \(\frac{1}{1000}\) and this is \(\frac{1}{10000}\) or 1 thousandth. Can you show the value of each of the remaining three columns in the same way?
EXERCISE 23-2A

1. Write in full, as was done above for 362.54, the numbers shown on the chart above.
2. Write as common fractions or as whole numbers the digits underlined on the chart.
3. Write each of the numbers (a), (b), (c) and (d) shown above in the expanded form. (For example, 23·67 would be $2 \times 10 + 3 \times 1 + 6 \times \frac{1}{10} + 7 \times \frac{1}{100}$)

Pictures of decimal fractions

You can show decimal fractions in pictures, using squares as you used them to show whole numbers. Last time you used one small square for 1 one. This time you will need one large square to represent 1 one. Make a large square with each side $2\frac{1}{2}$ inches. Now divide it into strips each $\frac{1}{4}$ inch wide and $2\frac{1}{2}$ inches long. There will be ten of these strips, so each one represents $\frac{1}{10}$ or .1. How many strips will represent .2? .6?

(a)

Now you want to show .01. This is $\frac{1}{100}$, and $\frac{1}{100}$ is $\frac{1}{10}$ of $\frac{1}{10}$. So each strip must be divided into 10 small squares. One small square represents .01, or $\frac{1}{100}$, as there are 100 small squares.

(b)

10 strips, each $\frac{1}{10}$ of 1;
1 one = 10 tenths.

1 one = 10 tenths
= 100 hundredths.
If you now divide each small square into 10 small strips, each small strip will be \( \frac{1}{10} \) of \( \frac{1}{100} = \frac{1}{1000} \), or 1 thousandth. An alternative way of writing a decimal fraction which is less than 1 is to put a 0 in the ones place; for example, 0.2 or 0.35. It is thought that the 0 helps to draw attention to the decimal point.

**Regrouping of decimal fractions**

Pictures like those above can help you to see how many tenths or hundredths there are in a number. You can think of 630 as 63 tens and in the same way you can think of .63 as 63 hundredths. If you look at picture (b), you can see that this is so; .6 is 6 strips and each strip has 10 hundredths in it. So altogether there are \([6 \times 10] + 3\) hundredths.

So .63 = 63 hundredths.

You can also think of this number in tenths. There are 6 tenths and 3 hundredths. One hundredth is \( \frac{1}{10} \) of 1 tenth, so you can write .63 as \( 6 + \left(3 \times \frac{1}{10}\right) \) tenths.

So .63 = 6.3 tenths.

In the same way, 630 can be written as 63 tens or as 6.3 hundreds or as 6300 tenths.

You will have found this way of writing numbers in newspapers. Instead of 3,650,000, you will see 3.65 millions. It is shorter and easier to read. Or you may see 6.5 thousands. This, in full, would be 6,500. You will read more about this way of writing numbers when you come to the unit on approximations.

**EXERCISE 23-2B**

1. Draw pictures to show these decimal fractions.
   a. .3  b. .7  c. .02  d. .07  e. .72  f. .88

2. What decimal fractions are represented by the shaded parts of these pictures if the big square represents 1 one?
3. Write each of the numbers in Question 2 as tenths and then as hundredths.

4. Write each of these numbers as a decimal fraction of the unit mentioned.
   a. 325 as tens
   b. 325 as tenths
   c. 27.56 as tenths
   d. 394.61 as hundredths
   e. 3,620,000 as millions
   f. 7,200 as thousands
   g. 0.037 as thousandths
   h. 2.37 as tens

5. Make a number chart stretching from hundreds to thousandths. Then write these numbers as decimal fractions on your chart.
   a. \(\frac{2}{10}\)
   b. \(\frac{3}{100}\)
   c. \(\frac{15}{100}\)
   d. \(\frac{15}{10}\)
   e. \(\frac{269}{100}\)
   f. \(\frac{269}{10}\)
   g. \(\frac{7}{100}\)
   h. \(\frac{29}{100}\)
   i. Forty-two tenths
   j. 29 hundreds
   k. 327 thousandths
   l. Twenty-five, five tenths and three thousandths

No improper decimal fractions

There are proper fractions, such as \(0.36\), and mixed numbers, such as \(2.3\), in decimal fractions, also, but there are no improper fractions. Try to write an improper fraction such as \(\frac{15}{10}\) as a decimal fraction and you find that you get a mixed number \(1.5\). This is because the principle of place value makes it impossible to write more than one digit in one column.

Common fractions as decimal fractions

You already know how to write some fractions as decimal fractions. A fraction whose denominator is 10 or 100 or some power of 10 is easily written as a decimal fraction. Examples are \(\frac{23}{100} = 0.23\), \(\frac{5718}{1000} = 5.718\). Some other fractions have denominators which are easily converted into powers of 10 by using the procedure you learned for finding fractions equal to a given fraction.
Do you remember how to rewrite $\frac{2}{3}$ in sixths? Why is it true that $\frac{2}{3} = \frac{4}{6}$? Look back to Chapter 20 if you have forgotten, because you will need this procedure in the next paragraph.

Here are some fractions which have been rewritten with denominators as powers of 10. What can you say about all the denominators of the first fractions of all the sets?

\[
\frac{4}{5} = \frac{8}{10} = .8; \quad \frac{1}{4} = \frac{25}{100} = .25; \quad \frac{1}{8} = \frac{125}{1000} = .125.
\]

The description is very simple. The denominators are 5, 4 and 8, and these numbers are all products of powers of 2 or 5. Numbers which are products of powers of 2 or 5 are factors of a power of 10 and so can be multiplied by a whole number to give 10.

**Exercise 23-2C**

1. Write each of these fractions as an equal fraction with its denominator a power of 10. Then write the corresponding decimal fraction.

   a. $\frac{3}{5}$  
   b. $\frac{1}{2}$  
   c. $\frac{3}{4}$  
   d. $\frac{5}{8}$  
   e. $\frac{3}{20}$  
   f. $\frac{21}{25}$

   g. $\frac{31}{50}$  
   h. $\frac{31}{500}$  
   i. $\frac{17}{5}$  
   j. $\frac{5}{2}$  
   k. $\frac{27}{4}$  
   l. $\frac{9}{8}$

   m. $\frac{45}{20}$  
   n. $\frac{201}{25}$  
   o. $\frac{921}{50}$  
   p. $\frac{17}{16}$

There is another way to find the decimal fraction equivalent of a common fraction. This is by division. Think of $\frac{1}{2}$. You first met $\frac{1}{2}$ as the missing factor in the multiplication equation.

\[2 \times \square = 1.\]

This equation corresponds to the division equation $1 \div 2 = \square$. So we can divide 1 by 2 and know that the quotient will be one-half. Let us divide 1 by 2 and see what quotient we can find. You will remember how to set this down as a division exercise.

To find $1 \div 2$:

**Ones:** There is no whole-number answer to $1 \div 2$, so we re-group 1 one as 10 tenths.

This we do by simply putting a decimal point after the 1 and adding a 0 in the tenths column.

**Tenths:** $10 \div 2 = 5$. We write 5 in the tenths column.

So we have shown that $\frac{1}{2} = .5$.

It is useful also to work this division in common fractions.

\[
1 \div 2 = \left(10 \times \frac{1}{10}\right) \div 2
\]

\[
= \frac{10}{10} \div 2 = \frac{5}{10} = .5
\]
Can you find \( \frac{1}{4} \) as a decimal fraction, using common fractions in this way? \( \frac{1}{4} = \frac{1}{4} \).

\[
1 \div 4 = \frac{10}{10} \div 4 = \left( \frac{8}{10} + \frac{2}{10} \right) \div 4 \\
= \left( \frac{8}{10} + \frac{2}{10} \right) \div 4 = \frac{2}{10} + \left( \frac{2}{10} \div 4 \right)
\]

We now have a digit for the tenths column, so our answer is \( .2 + \frac{1}{4} \). This fraction is \( \frac{2}{10} \div 4 = \frac{20}{100} \div 4 = \frac{5}{100} \). So the decimal fraction which is equal to \( \frac{1}{4} \) is \( .25 \). You can put all this together and write it as follows:

\[
1 \div 4 = \left( \frac{10}{10} \div 4 \right) = \left( \frac{8}{10} + \frac{2}{10} \right) \div 4 = \frac{2}{10} + \left( \frac{2}{10} \div 4 \right)
\]

\[
= \frac{2}{10} + \left( \frac{20}{100} \div 4 \right) = \frac{2}{10} + \frac{5}{100} = .25
\]

The usual way of working this is very much shorter than this method and when your pupils really understand what they are doing, they may use the shorter method. Here it is.

<table>
<thead>
<tr>
<th>Ones</th>
<th>Tenths</th>
<th>Hundredths</th>
</tr>
</thead>
<tbody>
<tr>
<td>( .2 )</td>
<td>( 4 )</td>
<td>( 1 \cdot 0 )</td>
</tr>
</tbody>
</table>

Ones: \( 1 \div 4 \) is not a whole number. 
Regroup 1 one as 10 tenths.

Tenths: \( 10 \div 4 = 2 + 2 \) tenths over.
Write 2 tenths. Regroup 2 tenths as 20 hundredths.

So \( \frac{1}{4} = .25 \).

You will know this short way. You keep on adding zeros until you find that you have no remainder. Here is \( \frac{7}{8} \) worked out in this way. \( \frac{7}{8} = 7 \div 8 \).

\[
\begin{array}{c|cccc}
\text{Ones} & \text{Tenths} & \text{Hundredths} & \text{Thousandths} \\
\hline
8 & 75 & 40 & 0 \\
8 & 70 & 0 & 0 \\
\end{array}
\]

So \( \frac{7}{8} = .875 \).

Now try \( \frac{1}{3} \). Each time you divide you have a remainder of 1 and you cannot find a whole number which is \( \frac{10}{3} \), or \( \frac{100}{3} \), or \( \frac{1000}{3} \). The calculation could go on forever.
We cannot write all these 3's and so we will agree for the present not to write any digit which has a value less than 1 thousandth and we will put three dots (\ldots) after the last digit to show that the decimal fraction is unending. We will use three decimal places only.

1.4 has one decimal place.
1.67 has two decimal places.
1.125 has three decimal places.

So we write \( \frac{1}{3} = \ldots333 \); the three dots show that there are other digits which we have not written down.

Decimal fractions which are found this way from common fractions often have a pattern of recurring digits after the decimal point. You will meet several of these in this chapter. They are called recurring decimal fractions. There is an alternative way of writing a recurring decimal. You place a dot over the digits which recur with the exception that if more than two digits recur the dots are placed over the first and last recurring digits only. Thus \( \frac{1}{3} = \overline{3}, \frac{1}{6} = \overline{16} \). We will use here the three dots notation and show the alternative notation in brackets afterward.

You see that \( \frac{1}{3} \) is not exactly \( \ldots333 \). In fact, \( \frac{1}{3} = \ldots333 + \text{a fraction} \). Can you work out what common fraction you could write instead of the three dots?

Work 1 \( \div 3 \) the long way,

\[
1 \div 3 = \frac{10}{10} \div 3 = \left(\frac{9}{10} + \frac{1}{10}\right) \div 3 = \frac{3}{10} + \left(\frac{1}{10} \div 3\right)
\]

\[
= \frac{3}{10} + \left(\frac{100}{100} \div 3\right) = \frac{3}{10} + \left(\frac{9}{100} + \frac{1}{100}\right) \div 3
\]

\[
= \frac{3}{10} + \frac{3}{100} + \left(\frac{1000}{1000} + \frac{1}{1000}\right) \div 3
\]

\[
= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \left(\frac{1}{1000} \div 3\right)
\]

So we have \( \frac{1}{3} = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{1}{3000} \)

\[
= \ldots333 + \frac{1}{3000}
\]

Instead of writing the \( \frac{1}{3000} \), we write \( \frac{1}{3} = \ldots333 \).

**EXERCISE 23-2D**

1. Find the decimal fraction equivalent of these common fractions by division.
   
   a. \( \frac{1}{2} \)  
   b. \( \frac{1}{4} \)  
   c. \( \frac{3}{4} \)  
   d. \( \frac{3}{5} \)  
   e. \( \frac{7}{10} \)  
   f. \( \frac{5}{8} \)  
   g. \( \frac{5}{16} \)

2. These common fractions that follow all have recurring decimal fraction equivalents. Work each division far enough to be sure you have found the pattern of recurring digits.
Put dots over the digits to show the recurring digits.

\[ \begin{align*}
\text{a. } \frac{1}{6} & \quad \text{b. } \frac{1}{7} \\
\text{c. } \frac{1}{9} & \quad \text{d. } \frac{1}{11} \\
\text{e. } \frac{2}{3} & \quad \text{f. } \frac{5}{6} \\
\text{g. } \frac{7}{13}
\end{align*} \]

You will often need to change a common fraction to a decimal fraction and vice versa. It is a good thing to remember these equivalent forms in pairs. Your pupils should also remember them, but be sure that they understand first how to work them out and why they are equivalent. It will help your pupils to draw pictures using the big square of a hundred small squares so that they can see that the number represented by a common fraction and by its decimal equivalent is the same number.

**EXERCISE 23-2E**

1. **Fill in the gaps in this table. Do not use more than three places in decimal fractions.**

<table>
<thead>
<tr>
<th>Common Fraction</th>
<th>1/2</th>
<th>1/5</th>
<th>3/4</th>
<th>6/8</th>
<th>2/5</th>
<th>1/10</th>
<th>37/100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Decimal Fraction</td>
<td>.25</td>
<td>.2</td>
<td>.2</td>
<td>.07</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. **Draw pictures to show that each of the pairs of numerals above represents the same number.**

3. **Give three common fraction names and three decimal fraction names for each of these numbers.**

   a. \( \frac{2}{3} \)  
   b. 31 tens  
   c. 31 tenths  
   d. 5.7  
   e. \( \frac{16}{32} \)  
   f. \( \frac{3}{8} \)

**23-3 Fractions in other bases**

You know how to write whole numbers in bases other than ten, and you know how to write fractions in base ten. In exactly the same way, you can write fractions in bases other than ten. For example, the base six notation can be extended to include fractions formed as a result of dividing by 6.

\[
243.12_{\text{ten}} = (2 \times 10^2) + (4 \times 10) + (3 \times 1)
\]

\[
+ \left( 1 \times \frac{1}{10} \right) + \left( 2 \times \frac{1}{10^2} \right)_{\text{ten}}
\]

And so:

\[
243.12_{\text{six}} = (2 \times 6^2) + (4 \times 6) + (3 \times 1)
\]

\[
+ \left( 1 \times \frac{1}{6} \right) + \left( 2 \times \frac{1}{6^2} \right)_{\text{ten}}
\]
EXERCISE 23-3A

1. Write each of these numbers in the expanded form in base ten.
   a. $162.5_{seven}$  
   b. $325.321_{six}$  
   c. $110.111_{two}$

2. Write each of these numbers as a common fraction (base ten) with the lowest possible numerator and the lowest denominator.
   a. $0.375_{ten}$  
   b. $0.4_{eight}$  
   c. $0.02_{six}$  
   d. $0.1_{two}$
   e. $1.22_{three}$  
   f. $2.4_{twelve}$  
   g. $3.2_{five}$  
   h. $1.02_{four}$

3. Find the numeral equivalent to $\frac{1}{2}$ in each of the following bases. Use three decimal places only. Use three dots to show a non-ending decimal fraction.
   a. base two  
   b. base three  
   c. base four  
   d. base five  
   e. base six  
   f. base seven

4. Make a third row to the table in Exercise 23-2E, Question 1. Fill this row with the base six numerals for these fractions.

5. Can a non-ending decimal in base ten be an ending numeral in a different base? Give six examples to explain your answer to this question, using the results of previous exercises.

Now that you have worked through this chapter on decimal fractions and fractions in other bases, you will see that there is nothing mysterious about them. They are simply other names for the common fractions. You have seen, too, that the fraction shown shaded in the picture can be called by many different names. Here are some of them.

\[
\begin{array}{cccccccc}
\frac{1}{2} & \frac{2}{4} & \frac{10}{20} & \cdot\frac{5}{ten} & \cdot\overline{111}_{three} & \cdot\overline{1}_{two} \\
\cdot\frac{4}{eight} & \cdot\overline{222}_{four} & \cdot\overline{222}_{five} & \cdot\overline{3}_{six} \\
\cdot\frac{8}{sixteen} \\
\end{array}
\]

One-half may be written either as an ending numeral (bases 2, 4, 6, 8...) or as a non-ending numeral (bases 3, 5, 7, 9...). So there is nothing difficult about a non-ending decimal fraction. It is the notation which makes it appear more complicated. What could be simpler than the idea of one-half? Yet in base three it is $\cdot\overline{111}$... While one-third in base ten is the non-ending numeral $\cdot\overline{333}$..., in base three it is $\cdot\overline{1}$. 

26
Chapter 24
OPERATIONS IN DECIMAL NOTATION

24-1 Addition and subtraction

When your pupils thoroughly understand decimal fractions, they should be able to do problems in addition and subtraction with very little difficulty. The methods are exactly the same as for addition and subtraction of whole numbers.

Addition

You will remember that when you add two whole numbers you take each column in turn and add first the ones, then the tens, then the hundreds and so on. If any total is greater than nine, it has to be regrouped. In exactly the same way, you regroup 10 hundredths to give 1 tenth or 10 tenths to give 1 one. Here are two examples of addition; one is addition of whole numbers, and the other is addition of decimal fractions.

\[ \begin{array}{c|c|c}
\text{Hundreds} & \text{Tens} & \text{Ones} \\
\hline
1 & 6 & 7 \\
2 & 9 & 8 \\
\hline
4 & 6 & 5 \\
\end{array} \]

Ones: \( 7 + 8 = 15 = 1 \text{ ten} + 5 \text{ ones. Write 5 ones.} \)

Tens: \( 6 + 9 + 1 = 16 = 1 \text{ hundred} + 6 \text{ tens. Write 6 tens.} \)

Hundreds: \( 1 + 2 + 1 = 4. \) \( \text{Write 4 hundreds.} \)

\[ \begin{array}{c|c|c|c|c}
\text{Tens} & \text{Ones} & \text{Tenths} & \text{Hundredths} \\
\hline
1 & 6 & 7 & 6 \\
2 & 1 & 8 & 9 \\
\hline
 & 3 & 8 & 6 & 5 \\
\end{array} \]

Hundredths: \( 6 + 9 = 15 = 1 \text{ tenth} + 5 \text{ hundredths.} \) \( \text{Write 5 hundredths.} \)

Tenths: \( 7 + 8 + 1 = 16 = 1 \text{ one} + 6 \text{ tenths.} \) \( \text{Write 6 tenths.} \)

Ones: \( 6 + 1 + 1 = 8. \) \( \text{Write 8 ones.} \)

Tens: \( 1 + 2 = 3. \) \( \text{Write 3 tens.} \)

If your pupils are to work a problem in which the addends are written horizontally, you will need to remind them to think of the value of each digit and then rearrange the addends vertically. For example, suppose you give them this problem: \( 3 + 1.4 + 0.0016 + 29 = \) \( \text{square.} \) The numbers should be rearranged with the decimal points vertically in line. The first number is
3 tenths and so must be written in the tenths column. Here is the setting out of this calculation and the thinking which should go with it.

\[
\begin{array}{c|c|c|c|c}
\hline
& \text{Tens} & \text{Ones} & \text{H'd'reds} & \text{Th'ths} \\
\hline
\cdot3 & 3 \text{ tenths} & & & \\
1\cdot4 & 1 \text{ one and 4 tenths} & & & \\
\cdot0016 & 0 \text{ tenths, 0 h'd'reds} & & & \\
& 1 \text{ th' th} \text{ and 6 t-th' th's.} & & & \\
29 & 2 \text{ tens and 9 ones} & & & \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
\hline
\text{Tens} & \text{Ones} & \text{H'd'reds} & \text{Th'ths} & \text{Ten-Th' ths} \\
\hline
\cdot3 & & & & \\
1 & 4 & & & \\
0 & 0 & 6 & & \\
2 & 9 & & & \\
3 & 0 & 7 & 0 & 1 & 6 \\
\hline
\end{array}
\]

Pupils who have practiced writing decimal fractions on a number chart should not have any difficulty in doing this kind of problem correctly.

**EXERCISE 24-1A**

1. Find the sums of these sets of numbers, setting out the explanation at the side as you would do for your pupils.
   a. 294       b. 217       c. 2987       d. 1815
   \[
   \begin{align*}
   519 & \\
   135 & \\
   967 & \\
   96 & \\
   \end{align*}
   \]

2. Make up four examples of addition written horizontally and show how you would expect your pupils to work these problems.

**Subtraction**

Subtraction without regrouping is very simple. Here are two examples. One is using whole numbers and the other is using decimal fractions.

\[
\begin{array}{c|c|c|c|c|c}
\hline
& \text{Tens} & \text{Ones} & \text{H'd'reds} & \text{Th'ths} & \text{Ten-Th' ths} \\
\hline
3 & 6 & 5 & & & \\
2 & 2 & 1 & & & \\
1 & 4 & 4 & & & \\
\hline
\end{array}
\]

\[
\begin{array}{c|c}
\text{Ones:} & 5 - 1 = 4 \\
\text{Tens:} & 6 - 2 = 4 \\
\text{Hundreds:} & 3 - 2 = 1 \\
\text{Hundredths:} & 9 - 7 = 2 \\
\text{Tenths:} & 2 - 0 = 2 \\
\text{Ones:} & 3 - 1 = 2 \\
\end{array}
\]

If you have the problem 26 - 18 = □, you will see that there are not enough ones in the first number 26 to enable you to take 8 from them. 6 - 8 is not a whole number. So you have to
regroup the tens to have sufficient ones. 26 can be regrouped. Instead of 2 tens and 6 ones you write 1 ten and 16 ones. Now you can find 16 – 8. Here is the calculation and the thinking which goes with it.

\[
\begin{array}{c|c}
\text{Tens} & \text{Ones} \\
\hline
2 & 6 \\
1 & 8
\end{array}
\]

Ones: 6 – 8 is not a whole number. Regroup 1 ten and 16 ones: 16 – 8 = 8

Tens: 1 – 1 = 0

8

The procedure is similar for decimal fractions. Suppose you have to find 3.21 – .08. Here is the thinking.

\[
\begin{array}{c|c|c}
\text{Hundredths:} & 1 & 8 \\
\hline
\text{Subtract 7. So regroup 5 tenths as 4 tenths} & 3 & 2 \quad \text{Tenths} \\
\hline
\text{This can be regrouped as 1 tenth + 11 hundredths} & 0 & 8 \quad \text{Hundredths}
\end{array}
\]

Tenths: 4 – 7 is not a whole number. Regroup 6 ones and 4 tenths as 5 ones and 14 tenths.

Ones: 5 – 3 = 2. Write 2 ones.

Because of the relationship between a column and the column next to it on the left you can always regroup a decimal numeral to make subtraction possible, provided that the first number is greater than the second. Here is another example. Find 6.5 – 3.77. We first rewrite this in the vertical form.

\[
\begin{array}{c|c|c}
\text{Hundredths:} & 5 \quad & 7 \\
\text{Subtract 7. So regroup 5 tenths as 4 tenths} & 3 \quad & 7 \quad \text{Tenths} \\
\hline
\text{This can be regrouped as 1 tenth + 11 hundredths} & 0 \quad & 8 \quad \text{Hundredths}
\end{array}
\]

\[
\begin{array}{c|c|c}
\text{Tenths:} & 4 \quad & 7 \\
\text{Regroup 6 ones and 4 tenths as 5 ones and 14 tenths} & 3 \quad & 7 \quad \text{Tenths} \\
\hline
\text{14 – 7 = 7. Write 7 tenths.} & 2 \quad & 7 \quad \text{Tenths}
\end{array}
\]

Ones: 5 – 3 = 2. Write 2 ones.

**EXERCISE 24-1B**

1. Explain how to work these problems as you would explain them to your pupils.
   a. 231 – 98 
   b. 87.65 – 64.32 
   c. 2.35 – 1.79 
   d. 76.84 – 18.92

2. Which number in each of the following pairs of numbers is the greater and by how much is it greater?
   a. 2.3 and 1.59 
   b. 87.32 and 24.118 
   c. .0017 and .12 
   d. .3 and .168

3. The rainfall in Freetown on a certain week was recorded in inches as follows: Sunday, 1.40 inches; Monday, 3.20 inches; Tuesday, 3.70 inches; Wednesday, 3.21 inches; Thursday, 0.80 inches; Friday, 0.10 inches; Saturday, 0.01 inches.
a. How many inches of rain fell in Freetown that week?
b. How many more inches of rainfall were there on Wednesday than on Thursday?

4. Decimal fractions are used in many ways but chiefly to show the results of measurement. Look at a daily newspaper and make a list of the ways in which decimal fractions are used there. Use this list to make up some story problems for your pupils in addition and subtraction. Question 3 shows two kinds of questions you can ask them.

24-2 Multiplication and division by powers of 10

Whole numbers

You remember that multiplication by 10 or 100 is very easy. Perhaps you have learned that to multiply by 10 you add a zero? This is what it looks like at first, but now that you understand place value and decimal fractions you will know that it is misleading to say "To multiply by 10, add a zero after the number". If you add 0 to 23, it means that you add zero tenths and write it 23-0. This is just the same as 23. So you must think carefully about what really happens when you multiply by 10. Look at the following equation:

\[ 23 \times 10 = 230 \]

In the product 230, in which place is the 0? In which place is the 3 and in which place is the 2?

If you make two number charts like these for your pupils to see, they will be able to understand what has happened. The digits of the number 23 have each moved one place to the left. Each digit has a value 10 times as great as it had.

If you multiply by 100, the digits move two places to the left. (Why?) Each digit has become worth 100 times as much as it was.

What task is performed by the zeros in this equation? They show the empty set of ones, or the empty set of tens. They are needed here to keep the other digits in their correct places.

Think again about division by 10. Think of 50 \div 10; of 560 \div 10. You know already that this is the opposite process to multiplication by 10. You remember that we call division the inverse of multiplication. Multiplication by 10 moves the digits one place to the left. Division by 10 moves the digits one place to the right. Every digit becomes 10 times as small in value. You can show this on a pair of number charts.
EXERCISE 24-2A

1. Make number charts and use them to show what happens in these problems.
   a. 7 × 10       b. 56 × 100       c. 30 × 10
   d. 70 ÷ 10      e. 700 ÷ 100      f. 560 ÷ 10

2. Make up six more examples like those above and show them on a pair of number charts.

Decimal fractions

Now that you think of multiplication by a power of 10 as moving digits to the left, you will very quickly see how to multiply decimal fractions by powers of 10. It works exactly the same way, because decimal fractions are written in the same way as the whole numbers: Each place in a decimal fraction is worth 10 times as much as the place immediately to the right. Here are some examples written on number charts.

<table>
<thead>
<tr>
<th>H'heads</th>
<th>Tens</th>
<th>Ones</th>
<th>Tens</th>
<th>Ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0</td>
<td>÷ 10 = 5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>6</td>
<td>÷ 10 = 5 6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>÷ 100 = 5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

You can also work this out by using the expanded form of a decimal fraction:

\[ \cdot 5 \times 10 = \left(5 \times \frac{1}{10}\right) \times 10 = 5 \times \frac{10}{10} = 5 \times 1 = 5 \]

Or, much more briefly, as:

\[ \cdot 5 \times 10 = \left(5 \times \frac{1}{10}\right) \times 10 = 5 \times 1 = 5 \]

You can show your pupils how to work it out this way with a longer number also. Here is the last one on the chart.

\[ 63.17 \times 10 = \left(6 \times 10 + 3 \times 1 + \left(1 \times \frac{1}{10}\right) + \left(7 \times \frac{1}{100}\right)\right) \times 10 \]
\[ = (6 \times 10) \times 10 + (3 \times 1) \times 10 + \left(1 \times \frac{1}{10}\right) \times 10 + \left(7 \times \frac{1}{100}\right) \times 10 \]
(Note use of distributive property.)

\[
631.7 = (6 \times 10^2) + (3 \times 10) + (1 \times 1) + (7 \times \frac{1}{10})
\]

Did you notice on the last line how every digit in the number 63.17 became worth 10 times as much and so was moved one place to the left?

If you multiply by 100, the digits are moved two places to the left, and so 2.3 \times 100 = 230. You can show this in two stages.

\[
2.3 \times 100 = 2.3 \times (10 \times 10) = (2.3 \times 10) \times 10
\]

\[
= 23 \times 10 = 230
\]

Now you will see that division by a power of 10 presents no difficulty whatever. Just move the digits one place to the right for every 10 by which you divide.

\[
73 \div 10 = (7 \times 10 + 3 \times 1) \div 10 = (7 \times 10) \div 10 + (3 \times 1) \div 10
\]

\[
= 70 + 3 \times \frac{1}{10} = 7.3
\]

\[
73 \div 100 = 73 \div (10 \times 10) = (73 \div 10) \div 10 = 7.3 \div 10
\]

\[
= .73
\]

You can do the last problem in one step by moving two places to the right straightaway.

\[
73 \div 100 = .73
\]

Your pupils should work these problems in two stages at first and should use a pair of number charts until they really understand what they are doing. Here are some division problems.

<table>
<thead>
<tr>
<th>Hundreds</th>
<th>Tens</th>
<th>Ones</th>
<th>Tenths</th>
<th>H.dreds</th>
<th>Tens</th>
<th>Ones</th>
<th>Tenths</th>
<th>H.dreds</th>
<th>T.h.dths</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 3 \cdot</td>
<td>\div 10 =</td>
<td>7 \cdot 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9 8 \cdot</td>
<td>\div 100 =</td>
<td>9 \cdot 8</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1 6 \cdot 5</td>
<td>\div 10 =</td>
<td>1 \cdot 6 \cdot 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2 3 7 \cdot 7 1</td>
<td>\div 100 =</td>
<td>2 \cdot 7 \cdot 7 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**EXERCISE 24-2B**

1. Find the following by moving the digits.
   a. \( 7 \times 10 \)
   b. \( .09 \times 10 \)
   c. \( .32 \times 10 \)
   d. \( 5.72 \times 10 \)
   e. \( .7 \div 10 \)
   f. \( 3.09 \div 10 \)
   g. \( 97.32 \div 10 \)
   h. \( 105.72 \div 10 \)

2. Find the following by moving the digits.
   a. \( 32 \times 100 \)
   b. \( 7.5 \times 100 \)
   c. \( .012 \times 1000 \)
   d. \( 19.401 \times 100 \)
   e. \( 763 \div 100 \)
   f. \( 76.2 \div 100 \)
   g. \( 79.321 \div 1000 \)
   h. \( 1 \div 100 \)
Choose four different types of problems from Questions 1 and 2 and tell how you would help pupils understand how to find the answer to them. You will of course have noticed that instead of moving the digits of a number one place to the left when multiplying by 10, you can instead move the decimal point one place to the right. You can see this in this equation:

\[ 24.81 \times 10 = 248.1 \]

The point has "jumped" over one digit, the 8. Similarly, when you divide by 100, you can move the decimal point two places to the left instead of moving the digits of the number two places to the right. In the equation

\[ 526.43 \div 100 = 5.2643 \]

you see that the point has jumped over two digits, the 2 and the 5.

When your pupils thoroughly understand multiplication and division by a power of 10, they will find moving the decimal point a quicker method.

24-3 Multiplication and division by whole numbers less than 10

**Multiplication**

You will remember that multiplication is very similar to addition. Whenever you have more than 9 in any column, you regroup to make 1 or more for the next column. You multiply 17 \times 8 as follows.

| Ones: 7 \times 8 = 56 = 5 \text{ tens and 6 ones.} & \text{Tens: } 1 \times 8 = 8, 8 + 5 = 13 |
| --- & --- |
| Write 6 ones. & = 1 \text{ hundred and 3 tens.} |
| Tens: 1 \times 8 = 8, 8 + 5 = 13 & Write 3 tens and 1 hundred. |

You will see that tenths may be multiplied in a similar way.

\[ \frac{7}{10} \times 8 = \frac{56}{10} = 5 \frac{6}{10} = (5 \times 1) + \left(6 \times \frac{1}{10}\right) \]

\[ = 5.6 \]

You need not think this out in this long way, because in the last chapter you practised writing fractions such as \(\frac{56}{10}\) as decimal fractions. You can say that \(\frac{56}{10} = 5.6\) straightaway. Your pupils, however, will need to think about what they are doing more carefully and will need to write their multiplications in a chart at first. Here is \(0.76 \times 4\) worked out.
Hundredths: $6 \times 4 = 24$. Regroup as 2 tenths and 4 hundredths. Write 4 hundredths.

Tenths: $7 \times 4 = 28$, $28 + 2 = 30$. Regroup as 3 ones and 0 tenths. Write 0 tenths and 3 ones.

\[
\begin{array}{c|c|c}
\text{Ones} & \text{Tenths} & \text{Hundredths} \\
\hline
7 & 6 & \times 4 \\
\hline
3 & 0 & 4 \\
\hline
\end{array}
\]

**Division**

Division by a whole number may give a whole-number answer or it may give a fractional answer. If the answer is not a whole number, then it is often very useful to write the fractional part as a decimal fraction. Decimal fractions can be compared for size more easily than common fractions can be compared and so are more often used in a practical situation such as measuring.

You remember how to do division to obtain a decimal fraction from a common fraction. You worked many examples in Chapter 23. First, we will work two examples without remainders. Here they are.

*First example*: $357 \div 3$.

- **Hundreds**: $3 \div 3 = 1$. Write 1 hundred.
- **Tens**: $5 \div 3 = 1$ ten and 2 tens over. Write 1 ten. Regroup the 2 tens as 20 ones.
- **Ones**: $20 + 7 = 27$, $27 \div 3 = 9$. Write 9 ones.

So $357 \div 3 = 119$.

\[
\begin{array}{c|c|c|c}
\text{H} & \text{D} & \text{T} & \text{H} \\
\hline
3 & 5 & 7 & \div 3 \\
\hline
1 & 1 & 9 & . \\
\hline
3 & 6 & 7 & . \\
\hline
0 & 5 & 3 & 7 \\
\hline
2 & 7 & 2 & 7 \\
\end{array}
\]

*Second example*: $27.95 \div 5$.

- **Tens**: $2 \div 5$ is not a whole number. Regroup as 20 ones.
- **Ones**: $7 + 20 = 27$, $27 \div 5 = 5$ and 2 over. Write 5 ones. Regroup 2 ones as 20 tenths.
- **Tenths**: $9 + 20 = 29$, $29 \div 5 = 5$ and 4 over. Write 5 tenths. Regroup 4 tenths as 40 hundredths.
- **Hundredths**: $5 + 40 = 45$, $45 \div 5 = 9$.

So $27.95 \div 5 = 5.59$.

\[
\begin{array}{c|c|c|c|c}
\text{T} & \text{O} & \text{T} & \text{H} \\
\hline
2 & 7 & 5 & 9 \\
\hline
5 & 3 & 9 & 2 \\
\hline
5 & 2 & 7 & 2 \\
\hline
0 & 5 & 3 & 7 \\
\hline
2 & 7 & 2 & 7 \\
\end{array}
\]

In each problem there was no remainder, but in the second problem there were decimal fractions to be divided. You will see that the problem is no different here. It is better to let your pupils work problems like these first, before they tackle problems which do not come out exactly, that is, which have a remainder.
EXERCISE 24-3A

1. Work these problems.
   a. \(7.4 \times 6\)  
   b. \(91.23\times 8\)  
   c. \(1.005 \times 2\)  
   d. \(11.642 \times 5\)  
   e. \(1.274 \times 10\)  
   f. \(1.25 \times 8\)

2. Find the numbers to put into the boxes to make these statements true.
   a. \(5 \times \square = 1\)  
   b. \(1 \times \square = .1\)  
   c. \(.1 \times \square = .8\)  
   d. \(7 \times \square = 7\)  
   e. \(.5 \times \square = .25\)  
   f. \(.3 \times \square = 2.4\)  
   g. \(.8 \times \square = 0\)  
   h. \(1.2 \times \square = 3.6\)  
   i. \(1.1 \times \square = 4.4\)

3. Work these problems.
   a. \(336 \div 8\)  
   b. \(26.35 \div 5\)  
   c. \(2900 \div 100\)  
   d. \(39.83 \div 7\)  
   e. \(1.01208 \div 4\)  
   f. \(20.712 \div 3\)

4. Explain how you would help your pupils understand how to find the answer to Question 3(b), 26.35 \div 5.

5. Make up four division problems which have no remainders.

The two division problems which were worked before the last exercise were shown with the subtraction set down. Many pupils will, of course, be able to find these differences without writing them down, and this shorter method of working is shown in the division problems which follow.

Division with remainders

When you have a problem such as \(25 \div 2 = \square\), you know that there is no whole-number answer but that you can find a fraction which will make the equation \(25 \div 2 = \square\) true. It is \(25\), or \(12\frac{1}{2}\). Now you have another way to work this problem. You can regroup the remainder of 1 one as 10 tenths and get the number 12.5 to make the equation true.

\[25 \div 2 = 12.5\]

Here is a longer example, written out in full to show the thinking which goes with it.

\[67 \div 5 = \square\]

Tens: \(6 \div 5 = 1\) and 1 over. Write 1 ten.  
Regroup 1 ten as 10 ones.

Ones: \(7 \div 10 = 17, 17 \div 5 = 3\) and 2 over. 
Write 3 ones. Regroup 2 over as 20 tenths.  
Insert decimal points.

Tenths: \(0 + 20 = 20, 20 \div 5 = 4\). Write 4 tenths.

So \(67 \div 5 = 13.4\).

If there are decimal fractions in the number to be divided, there is no difference in the method. They are added to the regrouped units from the next column on the left in exactly the same way as before. Here is an example.

\[27.1 \div 4\]
Tens: 2 ÷ 4 is not a whole number.
Regroup 2 tens as 20 ones.

Ones: 7 + 20 = 27, 27 ÷ 4 = 6 and 3 over. Write 6 ones.
Insert decimal point. Regroup 3 ones as 30 tenths.

Tenths: 1 + 30 = 31, 31 ÷ 4 = 7 and 3 over.
Write 7 tenths. Regroup 3 tenths as 30 hundredths.

Hundredths: 0 + 30 = 30, 30 ÷ 4 = 7 and 2 over. Write 7 hundredths.
Regroup 2 hundredths as 20 thousandths.

Thousandths: 0 + 20 = 20, 20 ÷ 4 = 5. Write 5 thousandths.

So 27.1 4 = 6.775.

Sometimes you will find that however long you continue to regroup, you never come to a whole-number answer. This means that the quotient is a non-ending decimal. You met these non-ending decimals before when you tried to find a decimal fraction for \( \frac{1}{3} \). It was \( 0.333 \ldots \), and the three dots show that it goes on forever. You will remember that this can also be written as \( \frac{1}{3} \). We could agree to stop after only one decimal place or we could work to four or more decimal places. There is no special reason to choose any particular decimal place as the last one. When your pupils first meet non-ending decimals as quotients in division problems, you should always tell them how many decimal places to use. Do not leave them to try to find an answer to \( 3 ÷ 7 \) or they will fill pages and pages with a very long division problem and perhaps be very unhappy because it never seems to end. You will know that \( 0.428 \) is not exactly the same as \( \frac{1}{3} \) and later you will explain to your pupils how to deal with these non-ending decimals to get answers as nearly correct as they need. At this stage, it is sufficient for them to work to as many decimal places as you require. Here is the working and thinking for \( 3 ÷ 7 \).

Ones: 3 ÷ 7 is not a whole number. Put in decimal points.
Regroup 3 ones as 30 tenths.

Tenths: 0 + 30 = 30, 30 ÷ 7 = 4 and 2 over.
Write 4 tenths. Regroup 2 tenths as 20 hundredths.

Hundredths: 0 + 20 = 20, 20 ÷ 7 = 2 and 6 over.
Write 2 hundredths. Regroup 6 hundredths as 60 thousandths.

Thousandths: 0 + 60 = 60, 60 ÷ 7 = 8 and 4 over.
Write 8 thousandths.

There are three decimal places wanted in the answer, so stop and add three dots to the quotient.

\[ 3 ÷ 7 = 0.428 \ldots \]

What is the exact answer to the division problem? You have so far arrived at

\[ 3 ÷ 7 = 0.428 + \text{a fraction.} \]

This fraction is the remainder divided by 7. The remainder from the thousandths column was 2 thousandths. So you have 4 thousandths ÷ 7. This we can write as \( \frac{4}{1000} ÷ 7 = \frac{4}{7000} \). So the correct answer to \( 3 ÷ 7 \) is \( 0.428 + \frac{4}{7000} \). Of course we do not write an answer like this. We do not mix common fractions and decimal fractions, but it is useful to think about the error which occurs when a quotient is cut off after a certain number of decimal places.
EXERCISE 24-3B

1. Make up three problems of division of a decimal fraction by a whole number less than 10 and describe how you would help your pupils to think the working of each problem.

2. Work each problem given below to three decimal places.
   a. 34.621 ÷ 4
   b. 5.317 ÷ 3
   c. 6 ÷ 7

3. Work each problem to the number of places given after it.
   a. 5.276 ÷ 8 (to four decimal places)
   b. 37.01 ÷ 3 (to three decimal places)
   c. \( \frac{7}{9} \) (to two decimal places)

4. In each problem in Question 3, what is the fraction omitted from the quotient?

24-4 Multiplication in decimal notation

You know that 2 × 3 = 6, but what is 2 × \( .3 \)? You can work this out in two ways.

1. Since fractions have the commutative property of multiplication, 2 × \( .3 \) = \( .3 \) × 2, and you already know that this is \( .6 \). (3 tenths × 2 = 6 tenths.)

2. Rewrite \( .3 \) in the expanded form as \( 3 \times \frac{1}{10} \). Then:

   \[
   2 \times .3 = 2 \times \left(3 \times \frac{1}{10}\right) = (2 \times 3) \times \frac{1}{10}
   \]

   (Note use of associative property.)

   \[
   = 6 \times \frac{1}{10} = .6
   \]

This second method is more useful, because it can be used to show how to multiply any two decimal fractions. Let us see whether we can find a rule which works for any two decimal fractions we choose. Here are some examples. Can you see what is happening each time?

1. \( .3 \times .2 = \left(3 \times \frac{1}{10}\right) \times \left(2 \times \frac{1}{10}\right) = (3 \times 2) \times \left(\frac{1}{10} \times \frac{1}{10}\right) \)

   (Note use of commutative property.)

   \[
   = 6 \times \frac{1}{10^2} = .06 \quad \therefore .3 \times .2 = .06
   \]

2. \( .5 \times .7 = \left(5 \times \frac{1}{10}\right) \times \left(7 \times \frac{1}{10}\right) = (5 \times 7) \times \left(\frac{1}{10} \times \frac{1}{10}\right) \)

   \[
   = 35 \times \frac{1}{10^2} = .35 \quad \therefore .5 \times .7 = .35
   \]

3. \( .5 \times .02 = \left(5 \times \frac{1}{10}\right) \times \left(2 \times \frac{1}{10^2}\right) = (5 \times 2) \times \left(\frac{1}{10} \times \frac{1}{10^2}\right) \)

   \[
   = 10 \times \frac{1}{10^3} = .010 \quad \therefore .5 \times .02 = .010
   \]

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4. \(0.02 \times 0.03 = (2 \times \frac{1}{10^2}) \times (3 \times \frac{1}{10^3}) = (2 \times 3) \times \left(\frac{1}{10^2} \times \frac{1}{10^3}\right)\)

\[= 6 \times \frac{1}{10^5} = 0.00006\quad 0.02 \times 0.03 = 0.00006\]

Did you notice that in every problem you separated the digits in the two numbers from the powers of 10 which show their value?

\[0.02 \times 0.03 = (2 \times 3) \times \left(\frac{1}{10^2} \times \frac{1}{10^3}\right)\] (omitting one step)

Then you found the products of each pair of factors.

\[= 6 \times \frac{1}{10^5}\]

This gave you the product of the digits in the original numbers divided by a power of 10 which tells you the place value of the 6.

\[= 0.00006\]

To find where to write the 6, you can think of \(6 \times \frac{1}{10^5}\) as \(6 \times 10^{-5}\) and you will remember that this can be worked out with a number chart. It means that 6 is moved five places to the right and becomes 0.00006. Now where does this five come from? It is the power of 10 which comes from multiplying together the \(\frac{1}{10^2}\) and \(\frac{1}{10^3}\). But \(\frac{1}{10^2}\) tells you that the first number has two places of decimals and the \(\frac{1}{10^3}\) tells you that the second number has three places of decimals. So you can add the number of places in the two factors, that is, in the two numbers you multiplied.

This is a very important result and you can help your pupils see this pattern by making a table of numbers and their products. Your pupils can first work out such numbers as \(0.3 \times 7\) and \(1.42 \times 3\), which they already know how to compute. Thus, they can work out by using fractions such numbers as \(0.3 \times 0.4\), \(5 \times 0.01\), \(0.13 \times 0.04\). They then tabulate these results as follows:

<table>
<thead>
<tr>
<th>Equation</th>
<th>Number of decimal places in first number</th>
<th>Number of decimal places in second number</th>
<th>Number of decimal places in product</th>
</tr>
</thead>
<tbody>
<tr>
<td>(.3 \times 7 = 2)</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(1.42 \times 3 = 4.26)</td>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(.13 \times .04 = .0052)</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

When your pupils have looked at a table like this and thought about the way they find a product using fractions, they will understand the rule. Here it is ..

To multiply two decimal fractions: First multiply them without the decimal points. Then make as many decimal places in your product as the sum of the number of decimal places in the
two decimal fractions. Here is an example.

\[ 27.312 \times 1.25 \]

\[
\begin{array}{c|c}
27312 & \text{Number of decimal places in the two decimal fractions} \\
125 & \text{is } 3 + 2 = 5. \text{ Therefore the number of decimal places in} \\
136560 & \text{the product must be five.} \\
546240 & 27.312 \times 1.25 = 34.14000 \\
2731200 & = 34.14 \\
3414000 &
\end{array}
\]

You will notice that it is very important to keep any zeros in the product until the value of the product is decided. Once the decimal point has been put in, the zeros which are not needed can be omitted.

**EXERCISE 24-4A**

1. Find the product of each pair of numbers.
   a. \(2 \times 0.3\)  
   b. \(0.7 \times 8\)  
   c. \(0.3 \times 0.1\)  
   d. \(0.1 \times 0.1\)  
   e. \(2 \times 0.1\)  
   f. \(9 \times 1.1\)  
   g. \(7.9 \times 10\)  
   h. \(1.2 \times 1.2\)  
   i. \(0.1 \times 0.05\)  
   j. \(0.6 \times 0.25\)  
   k. \(1.02 \times 0.7\)  
   l. \(12.5 \times 0.8\)

2. Find the product of each pair of numbers.
   a. \(1.142 \times 7.3\)  
   b. \(23.121 \times 0.005\)  
   c. \(23.54 \times 21.5\)  
   d. \(0.087 \times 0.0014\)

3. A pupil writes that \(0.1 \times 0.1 = 0.1\). Explain how you would help him to see his error.

4. Show by working in fractions why \(0.13 \times 0.04 = 0.0052\).

5. Make a table like the one on page 38 and show on it six equations such as you could use with your pupils to help them discover the rule for multiplication with decimal fractions. How would you make sure that they understood why the rule works?

**24-5 Division in decimal notation**

We have already discussed the division of decimal fractions by whole numbers less than 10. Division by whole numbers greater than 10 is very similar. Your pupils will probably be able to tell you how to do it without any further teaching. Ask them a problem like this one.

\[ 144 \div 16 = \square \]

They will work it like this.

Hundreds: \(1 \div 16\) is not a whole number.
Regroup as 10 tens.

<table>
<thead>
<tr>
<th>Tens</th>
<th>Ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 4 4</td>
<td></td>
</tr>
</tbody>
</table>

Tens: \(10 \div 4 = 14\), \(14 \div 16\) is not a whole number.
Regroup as 140 ones.

<table>
<thead>
<tr>
<th>Tens</th>
<th>Ones</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 4 4</td>
<td></td>
</tr>
</tbody>
</table>

Ones: \(140 \div 4 = 144\). \(144 \div 16 = 9\). Write 9 ones.

Then ask the problem

\[ 14.4 \div 16 = \square \]

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and help them to set it down in the same way but showing tens, ones and tenths this time. They will readily see that the answer is 10 times as small and is 0.9. So the pattern of working is exactly similar to the pattern of division by a whole number less than 10. The only difference is that we need to set down the subtraction instead of working it mentally.

Division of a decimal fraction by a whole number is quite straightforward and so we turn all division by a decimal fraction into division by a whole number. How do we do this? Can you remember? If you cannot remember, can you work it out? Suppose you want to find 24 ÷ 0.3. You want to have 3 instead of 0.3. What must we use to make 0.3 \times \square = 3? We must multiply by 10. 0.3 \times 10 = 3. But can we multiply 0.3 by 10 without altering the result? Of course not, it will be 10 times too small. So we must also multiply the 24 by 10. So instead of 24 ÷ 0.3, we have 240 ÷ 3. This is 80.

You can check to see whether this gives the same quotient by thinking of a real situation. Let us imagine that you want to share 24 oranges among 3 children. How many oranges will each child receive? Here is the equation.

\[ \frac{24}{3} = 8 \]

Now suppose you have 30 children, that is, 10 times as many children. How many oranges will you need in order to give each child the same share as before, that is, 8 oranges? You know that you would need 10 times as many oranges for 10 times as many children. Here are the equations.

\[ \square ÷ 30 = 8 \]

\[ 24 \times 10 ÷ 30 = 8 \]

You can see the pattern of the working if you show both the multiplications by 10.

\[ (24 \times 10) ÷ (3 \times 10) = 8 \]

Examples like this will help your pupils to understand that the quotient of a division problem is not altered if the two numbers are each multiplied by the same factor.

Another way to make the divisor a whole number is to think of 24 ÷ 0.3 as a fraction.

\[ 24 ÷ 0.3 = \frac{24}{0.3} = \frac{24 \times 10}{0.3 \times 10} = \frac{240}{3} = 80 \]

(To make an equal fraction)

So \( 24 ÷ 0.3 = 80 \).

Here is another example. Find 7.2 ÷ 0.09.

\[ 7.2 ÷ 0.09 = \frac{7.2}{0.09} = \frac{7.2 \times 100}{0.09 \times 100} = \frac{720}{9} = 80 \]

\[ 7.2 ÷ 0.09 = 80 \]

Your pupils will probably need some practise in deciding what they must do to a decimal fraction to make it a whole number. They will need problems such as these:
\[
\begin{align*}
-31 \times \square &= 31 \\
-002 \times \square &= 2 \\
73.46 \times \square &= 7346
\end{align*}
\]

And these:

What must you do to \( \cdot7 \) to make it a whole number?
What must you do to \( 1.32 \) to make it a whole number?

This can be a game in which pupils make up questions such as these and ask each other for answers.

**EXERCISE 24-5A**

1. Make up problems like those above to ask your pupils.
2. Find these quotients:
   
   a. \( 16 \div .4 \)  
   b. \( 21 \div .3 \)  
   c. \( 30 \div .05 \)  
   d. \( 5.6 \div .7 \)  
   e. \( 64 \div .08 \)  
   f. \( .032 \div .4 \)  
   g. \( 55 \div 1.1 \)  
   h. \( 121 \div .11 \)  
   i. \( 1.32 \div .04 \)
3. Find these quotients. Do not work more than two decimal places.
   
   a. \( 33.79 \div 23 \)  
   b. \( .17 \div .8 \)  
   c. \( 31.012 \div .56 \)  
   d. \( 3.653 \div 3.7 \)

**24-6 Percentage**

There is another way of writing a decimal fraction and that is as a percentage. \( 23\% \), read 23 per cent, means 23 things out of every 100 things. If there are 100 boys in a school, then 23% of the boys of the school is 23 boys: 23 out of 100. You will remember that when you learned to think of fractions in terms of sets you said that if a set had 5 members, then the fraction of this set represented by 3 of its members is \( \frac{3}{5} \). One member is \( \frac{1}{5} \) and three members are \( \frac{3}{5} \). If the set has 100 members, then 1 member is \( \frac{1}{100} \) of the total number of the set and 23 members are \( \frac{23}{100} \) of the number of the set. So \( 23\% \) and \( \frac{23}{100} \) are both names for the same number.

What decimal fraction can also be used to name the number? \( \frac{23}{100} \) is \( 23 \times \frac{1}{\text{100}} \), which is written as \( .23 \). So we have the relationships

\[
23\% = \frac{23}{100} = .23,
\]

and

\[
37\% = \frac{37}{100} = .37,
\]

and

\[
9\% = \frac{9}{100} = .09.
\]

Percentages are frequently used in everyday affairs, in shops, in factories and in government. A 10% discount may be allowed off the prices of articles bought in a certain shop. Money may be invested and earn interest of 2%. A firm of building contractors may decide that
it must make a profit of 50% in order to pay its workers. The final profit will be much less. These are some of the uses of percentages, and you should look for more examples and use them to make problems for your pupils.

These percentages mentioned above are ones which can be written as very simple fractions. What fractions name the same numbers as 10%, 2% and 50%? It is easy to find out.

\[ 10\% = \frac{10}{100} = \frac{1}{10} \]
\[ 2\% = \frac{2}{100} = \frac{1}{50} \]
\[ 50\% = \frac{50}{100} = \frac{1}{2} \]
\[ \frac{3\frac{1}{2}}{100} = \frac{7}{200} \left( \frac{3\frac{1}{2}}{2} \text{ out of 100 = 7 out of 200} \right) \]

Percentages must be rewritten as common or decimal fractions before they can be used in calculations. The common fraction is generally more convenient for this purpose, but you should always consider whether using the decimal fraction might reduce the amount of work. Here is an example worked in both ways.

Amodu buys some books and his bill is 40 shs. If he is allowed 10% discount off his bill how much must he pay?

By common fractions
\[ 10\% \text{ of } 40 \text{ shs} = \frac{1}{10} \times 40 \text{ shs} = 4 \text{ shs} \]

So Amodu must pay 40 shs - 4 shs = 36 shs.

Your pupils should be able to work simple problems like this in their heads. For this, they should have a sound understanding of percentages and their equivalent fractions.

**EXERCISE 24-6A**

1. Fill in the gaps in this table.

<table>
<thead>
<tr>
<th>Percentage</th>
<th>50%</th>
<th>25%</th>
<th>12(\frac{1}{2})%</th>
<th>75%</th>
<th>20%</th>
<th>10%</th>
<th>5%</th>
<th>2(\frac{1}{2})%</th>
<th>33(\frac{1}{3})%</th>
<th>60%</th>
<th>35%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Common Fraction</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Decimal Fraction</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Work out each of the following percentages in two ways.
   a. 5\(\frac{1}{2}\)% of £75
   b. 3\(\frac{1}{2}\)% of £120
   c. 7% of 720 shs
   d. 4% of $312.00
In all the problems so far, you were given a percentage and asked to find the corresponding fraction. Often we need to know what percentage of a total is represented by a certain amount. For example, what percentage of the whole class of 40 is a group of 35? Now that you know that a percentage is a fraction, you can see that the first step will be to find what fraction the group is of the whole class. The group is 35 out of 40, so this is $\frac{35}{40}$ of the whole class. We want this as a fraction with a denominator of 100; that is, we want it in hundredths. How can we do this? First we can make it simpler. $\frac{35}{40} = \frac{7}{8}$.

Now we want to know how many hundredths there are in $\frac{7}{8}$. So we find the quotient $\frac{7}{8} = 0.875$, and this is 0.875 hundredths. Can you explain why 0.875 is 87.5 hundredths? If you have forgotten, you can make a number chart and write 0.875 on it and you will see that you have

\[
8 \text{ tenths } + \ 	ext{7 hundredths } + \ 5 \text{ thousandths}
\]

\[
= 87 \text{ hundredths } + \ 5 \text{ thousandths}
\]

\[
= 87.5 \text{ hundredths}.
\]

Do you see what we have done?

87.5 is $\frac{7}{8} \times 100$ and 0.875 is $\frac{7}{8}$.

So 87.5 is $\frac{7}{8} \times 100$. So instead of dividing by 8 and then reading this as hundredths, we can straightaway multiply our fraction by 100. We need not simplify the fraction first. Here is the working to find what percentage of 40 is represented by 35.

\[
35 \text{ is } \frac{35}{40} \text{ of } 40.
\]

\[
\frac{35}{40} \times 100 = \frac{175}{2} = 87.5
\]

So 35 is 87.5% of 40.

Here is a problem.

In an examination there were 70 problems. A boy had 55 right. All problems had equal marks. What percentage did the boy have right? Work to one decimal place only.

\[
\text{Fraction right } = \frac{55}{70}
\]

\[
\text{Percentage right } = \frac{55}{70} \times 100 = \frac{550}{7} = 78.5\%
\]

Another kind of problem is one where we know the percentage and what it represents but we do not know the total.

In a certain town, 78% of the electorate voted in an election. There were 5,600 ballot papers. How many people were entitled to vote? (That is, how many people were in the electorate?) We can make an equation. The number of the electorate is to be put into the box.

\[
\frac{78}{100} \times \square = 5,600
\]

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This is an equation with a missing factor and you know what this means. It means division. It can be rewritten as

\[ 5,600 \div \frac{78}{100} = \square. \]

This is division by a fraction and to divide by a fraction we multiply by its reciprocal. The reciprocal of \( \frac{78}{100} \) is \( \frac{100}{78} \). (Why? Because \( \frac{78}{100} \times \frac{100}{78} = 1 \))

So the equation becomes

\[ 5,600 \times \frac{100}{78} = \frac{560,000}{78} = 7179. \]

We did not work this problem further than the ones. (Why not?) So the total electorate is 7179.

We can fit most problems about percentages to one equation. Sometimes we have to find one part of the equation and sometimes another part. Let us think about an equation where we know everything.

25% of 44 = 11

This is written as

\[ \frac{25}{100} \text{ of } 44 = 11. \]

The three problems that can be asked about the situation are:

1. What is 25% of 44?

\[ \frac{25}{100} \text{ of } 44 = \square \]

2. What percentage of 44 is 11?

\[ \frac{\square}{100} \text{ of } 44 = 11 \text{ or } \frac{\square}{100} \times 44 = 11, \]

which can be written as division as

\[ \frac{\square}{100} = 11 \div 44. \]

This we saw was \( \square = \frac{11}{44} \times 100. \)

3. 11 is 25% of what number?

\[ \frac{25}{100} \text{ of } \square = 11, \]

which can be written as division as

\[ 11 \div \frac{25}{100} = \square \]

As long as you understand the meaning of percentage, you can work out any problem by using this equation.
EXERCISE 24-6B

1. Write each of these percentages as a common fraction.
   a. \( \frac{31}{2}\% \)  
   b. \( 33\frac{1}{3}\% \)  
   c. 65%  
   d. 5·5%  
   e. 17%  
   f. 16\frac{1}{4}%

2. Write each of these fractions as a percentage.
   a. \( \frac{2}{3} \)  
   b. -0·4  
   c. 4  
   d. \( \frac{4}{5} \)  
   e. -0·32  
   f. -0·01  
   g. \( \frac{7}{100} \)  
   h. \( \frac{8}{13} \)

3. Find each percentage.
   a. 5% of 1 hour, in minutes  
   b. 25% of a year, in days  
   c. 16% of 200 shs

4. Find what percentage the first of each pair is of the second of each pair. Work to two decimal places only.
   a. 33, 300  
   b. 100, 200  
   c. 300, 100  
   d. 4, 5  
   e. 45 shs, 500 shs  
   f. 55 marks out of 80 marks

5. Find the total amount in each case. Do not work to more than one place of decimals.
   a. When 25% is 4  
   b. When 20% is 36  
   c. When 65% is 1,500  
   d. When 32% is £565
UNIT V • Integers

Chapter 25
INTRODUCTION TO INTEGERS

25-1 A reminder of subtraction

You can think of subtraction in several ways. If you have a set of objects, you can separate it into two subsets. If you remove one of them, subtraction tells you how many are left. For example, Kwame and his sister Araba have 6 bananas. Kwame takes 4 bananas. How many are left for Araba?

Then you can use subtraction to compare two sets to find out which has more members. To find out how much taller the red lily is than the yellow lily, you can subtract the height of the yellow lily (4 feet) from the height of the red lily (5 feet) and get the equation 5 - 4 = 1.

Or you can think of subtraction on the number line and work out 8 - 5. You will find out where your pencil point will be if you start at 8 and then subtract 5; that is, go back from 8 to the left 5 steps.

Another kind of problem which we solve by using subtraction is one with a missing addend. Mary has saved 5 shillings and wants to buy some sandals that cost 9 shillings. How much more must she save? You write the equation 5 + □ = 9 and know that the answer is 4. The equation could have been written as 9 - 5 = □.

Sometimes the answer to a subtraction problem is 0. If Mary’s family has 40 bananas and they eat 40 bananas, they have 0 bananas left.

If Kofe is 40 inches tall and Kwame is 40 inches tall, then Kofe is 0 inches taller than Kwame and Kwame is 0 inches taller than Kofe. In each case, we can write the subtraction equation

40 - 40 = 0.

All these problems have whole-number answers, but sometimes you have a problem which does not have a whole-number answer. In each of the problems above you needed to subtract a whole number from a larger number or from an equal number and this is always possible, but can you find a whole-number answer to 3 - 5 = □? Look at this next problem.

Kwafi owes Kwesi 5 shs and only has 3 shs with which to pay his debt. How much money has Kwafi? He gives Kwesi the 3 shillings but he still owes some money. He has (3 - 5) shillings. You cannot say that this is the same as having 2 shillings. The equation 3 - 5 = □ can be rewritten as an addition equation 5 + □ = 3. You can see that there is no whole number which can be added to 5 to give 3.

Work the following problems if it is possible to find a whole-number answer. Which problems have no whole-number answer?
EXERCISE 25-1A

1. \( 7 + \square = 10 \)  
2. \( 3 + \square = 1 \)  
3. \( 11 - 4 = \square \)  
4. \( \square + 2 = 9 \)  
5. \( 4 - 8 = \square \)  
6. \( 2 - 11 = \square \)  
7. \( 13 + \square = 13 \)

You will have found that Questions 2, 5 and 6 have no whole-number answers. In mathematics we do not like to have problems without any answers. You will remember that at first you had no answer to the problem \( 8 \div 5 = \square \), but that by using fractions you could give the answer \( \frac{8}{5} \). In this chapter, you will learn about another kind of number which will make it possible to find an answer to \( 3 - 5 = \square \).

25-2 Physical models

You have learned in geography how the positions of places on the earth are described using latitude and longitude. Latitude is measured in degrees (\( ^\circ \)) north and south of the equator and longitude in degrees east and west of the meridian of longitude which passes through Greenwich. On a map of the world you will see, along the edges of the page, lines which are marked to show latitude and longitude.

Latitude is marked on a line segment down each side of the map. Here is part of one such line segment. You can tell which side is north and which side is south by the letter N or S marked before the number of degrees.

\[ \text{Latitude} \]

\[ \begin{array}{c}
\text{N30}^\circ \\
\text{N20}^\circ \\
\text{N10}^\circ \\
\text{0}^\circ \text{ at equator} \\
\text{S10}^\circ \\
\text{S20}^\circ \\
\text{S30}^\circ \\
\end{array} \]

Longitude is marked on a line segment along the top and bottom of the map. Here is part of one of them.

\[ \text{Longitude} \]

\[ \begin{array}{cccccccc}
\text{W40}^\circ & \text{W30}^\circ & \text{W20}^\circ & \text{W10}^\circ & \text{0}^\circ & \text{E10}^\circ & \text{E20}^\circ & \text{E30}^\circ \\
\end{array} \]

Again, you can tell which is east and which is west by the letters E or W written before the number of degrees.
You will have noticed that in each case we measure in two opposite directions from a central point which is called zero. Zero degree latitude is on the equator and zero degree longitude is on the meridian through Greenwich.

There are many other things which are measured in two opposite directions. Here are some of them: the height of land above sea level and the depth of the ocean below sea level; the number of years A.D. (anno domini), and the number of years B.C. (Before Christ).

Time before the hour and time past the hour. Here is a clock-face on which are marked the times before the hour and after the hour. These are the times which are shown by the minute hand.

\[ \text{HOUR} \]

\[ \begin{array}{c}
5 \text{ mins. to} & 5 \text{ mins. past} \\
10 \text{ mins. to} & 10 \text{ mins. past} \\
15 \text{ mins. to} & 15 \text{ mins. past} \\
20 \text{ mins. to} & 20 \text{ mins. past} \\
25 \text{ mins. to} & 25 \text{ mins. past} \\
\end{array} \]

\[ \frac{1}{2} \text{ past} = 30 \text{ mins. past} \]

You will see that these times are measured in opposite directions from the hour. There are the times past the hour and the times to the hour. Both sets of times are measured from the hour. We will draw a line picture for these times. What time shall we choose as zero time? This must be the hour, because the times are measured from the hour. At the hour there are 0 minutes past or 0 minutes before the hour. On one side of 0, we will mark the times past the hour. On the other side of 0, we will mark the times before the hour, backward (to the left) from 0.

To show which side is past the hour and which side is to the hour we have marked each number either p (for past) or t (for to). Can you see that our line is like the line around the edge of the clock-face? If you cut the line around the edge of the clock-face and straighten it out, where would you make the cut? You want it to look like the number line we have just drawn. You will have to cut the line at half-past.

When the time is 20 past the hour, where will the minute hand be on the number line? At 20p. What will the time be at 10t? 10 to the hour.

The seconds before and after firing off a rocket are also measured in this way. This is called "countdown" and is spoken, "Ten, nine, eight, seven, six, five, four, three, two, one,
At zero the rocket is fired. The count goes on after the rocket is fired, "One, two, three" and so on.

You can think of gains and losses in a similar way. If you win 6 shs at a game of cards, this is not the same thing as losing 6 shs. If you win 6 shs, 2 shs, 3 shs and 1 sh, your wins can be shown by dots on a line as shown in the picture. You must first decide where to put the dot for a result when you neither win nor lose. This will be the zero point. Then you decide on which side of 0 you will mark the line for wins. We usually use the right-hand side for this. Then we can mark losses in the opposite direction.

The dots show the wins 1 sh, 2 shs, 3 shs and 6 shs. What losses are shown? 1 sh, 2 shs, 4 shs and 5 shs.

In each of these cases you will see that you can make pictures of the measurements by representing them as points on a line. This is how you do it.
1. Draw a line.
2. Mark a zero point with 0.
3. Decide what the zero point represents.
4. Mark the scale for the measurements in one direction.
5. Mark the scale for the measurements in the opposite direction.
6. Give the points on each side of 0 a letter or symbol to enable you to distinguish between them.

**EXERCISE 25-2A**

Each of the sets of measurements described below can be shown on a line. For each one (a) draw the line, (b) mark the zero point and say what it represents, and (c) mark in the measurements.

1. **Longitude east and west of the Greenwich meridian.** Mark two points on this line to show the longitude of the most easterly and most westerly parts of the African coast line. Find the longitude of six African towns and mark them also.
2. **A hole has been dug in the ground for a mine shaft 100 feet deep.** Above it has been built a tower 60 feet high. The tower has platforms at 20 feet and 50 feet. Mark these on the line. Mark also a platform in the mine which is as far below ground as the first platform is above ground.
3. A Centigrade thermometer measures the temperature from 40° below zero to 70° above zero. Mark on the line a temperature of 30° above zero.
4. A shopkeeper has debts of 20 shs, 50 shs, 65 shs and 25 shs. He has credits of 15 shs, 25 shs, 50 shs, 70 shs and 35 shs. Show these amounts on a line.
5. Some boys are to run a race. As some of them are taller and older and some of them are shorter and younger, they are to start from different places behind or ahead of the starting point. Four young boys stand 3 ft., 5 ft., 2 ft. and 7 ft. in front of the line. Seven boys stand at the starting point and three big boys stand behind the starting point at distances of 3 ft., 6 ft. and 8 ft. Show these boys as dots waiting to start their race. (You will need fourteen lines, side by side, for the fourteen boys to stand on.)
Chapter 26

THE NUMBER LINE AS A PICTURE FOR INTEGERS

26-1 Naming the new numbers

You will have realized while making the lines in the last exercise that it was very like making a number line. When you made a number line before, in Chapter 16, you first chose a point to be zero and marked it 0, and then measured out equal steps to the right. You then used the counting numbers to name the points at the ends of these steps. But when you made the line picture for longitude east and west, you used the counting numbers twice. You used them to the right and to the left of 0. When you made the line picture for the mine shaft and its tower, you drew the line going upwards instead of across the page. But you still used the counting numbers twice, once on each side of the zero point. So you will see that we want a new kind of number line as a picture of any of these things which are measured in two directions.

Look at the number line below.

```
|   |   |   |   |   |
0 1 2 3 4
```

You can see that we must have some way to distinguish between the number 3 to the right of 0 and the number 3 to the left of 0. We want names for the two sets of numbers. We want to make a number line which we can use to show all these situations: right and left, north and south of the equator, east and west of the meridian through Greenwich, ahead of and behind the starting point, above and below sea level, gain and loss, credit and debit, surplus and shortage, after and before and many more. What we need is a way to tell apart numbers on opposite sides of zero. In many of these cases, descriptions such as ahead, above, gain and credit seem to suggest "having something" while their opposites below, behind, loss and debit may seem to suggest "lacking something". So we distinguish the first as being positive and the second as being negative. On the new number line, we label positive numbers on one side of 0 and label negative numbers on the other side of 0. The number 0 at the starting point is neither positive nor negative. The positive numbers are conventionally shown to the right of 0 and negative numbers to the left of 0, but we could put the positive numbers above 0 and the negative numbers below 0. This whole set of numbers—positive, negative and zero—we call the SET OF INTEGERS.

The numbers to the left of zero are negative integers and so we will call the 3 on this side neg 3 for short. The numbers to the right of zero are the positive integers and so we will call the 3 on this side pos 3 for short.
Here is a picture of the number line showing some of the negative integers, zero and some of the positive integers.

26-2 Zero

The number zero has a special position in the middle between these two sets of numbers. You will remember that you first heard of zero, in Chapter 1, as the number of the empty set. You used 0 then to show an empty set. For example, in 202, the number which is equal to \(2 \times 10^2 + 0 \times 10 + 2 \times 1\), the set of tens is empty, and in the number which is equal to \(3 \times 10^2 + 7 \times 10 + 0 \times 1\), 370, the set of ones is empty. You also used 0 to mark the point on the number line from which you started to measure out unit lengths. So in this way you can once again think of zero as the number of the empty set, the empty set of unit lengths. This is the way we think of zero with the integers. When we think about latitude, 0° is at the equator. For minutes past and to the hour we chose 0 to be the hour and in the problem about the mine, 0 was at ground level. Zero has a very important part to play on this number line. It separates the positive integers from the negative integers and later on we shall see how it helps us to do addition and subtraction.

26-3 Opposites

You will see on the number line that the integers can be matched in pairs. We can match any integer and its OPPOSITE. Here are some pairs. They are shown on the line also.

\[
\begin{align*}
\text{neg 3} & \quad \text{and} \quad \text{pos 3} \\
\text{pos 2} & \quad \text{and} \quad \text{neg 2} \\
\text{neg 2} & \quad \text{and} \quad \text{pos 2}
\end{align*}
\]

Such pairs are called "opposites". You will see that the two members of a pair are the same distance from 0, but they are on opposite sides of it. To find the opposite of an integer, we look for the integer which is on the opposite side of 0 and at the same distance from 0. We can write this in another way.

The opposite of \(\text{neg 3}\) is \(\text{pos 3}\).
The opposite of \(\text{pos 2}\) is \(\text{neg 2}\).
The opposite of \(\text{pos 15}\) is \(\text{neg 15}\).
The opposite of \(\text{neg 52}\) is \(\text{pos 52}\).
The opposite of 0 is itself, because 0 is neither positive nor negative. What is the opposite of the opposite of an integer? We can work this out in three stages.

The opp. of the opp. of pos 6 = the opp. of (the opp. of pos 6) = the opp. of neg 6 = pos 6.

So the opposite of the opposite of pos 6 is pos 6 itself. Is this true if we begin with a negative number? You can see that it must be so. Can you see this for yourself? Begin with the opposite of the opposite of neg 3 and work it out in the same way. You will find that the opposite of the opposite of neg 3 is neg 3 itself. You can see that whether you begin with a positive number or a negative number or 0, the opposite of the opposite will be the number itself, the number you began with.

You can prove this for yourself on the number line also. If you want to find the opposite of any number you begin with that number, you jump to 0 and then you make another jump of the same size and land on the opposite. To find the opposite of this second number, you simply jump back again the way you came. So to find the opposite of the opposite you jump there and back again. So you need not jump at all; you are at the answer already.

**EXERCISE 26-3A**

1. What are the opposites of the following measurements?
   a. Latitude N30°
   b. Longitude W45°
   c. Temperature 15° above zero
   d. 10 minutes past the hour
   e. A win of 7 shs
   f. A debt of 50 shs

2. What are the following?
   a. the opposite of neg 1
   b. the opposite of pos 11
   c. the opposite of pos 17
   d. the opposite of neg 73
   e. the opposite of pos 129
   f. the opp. of the opp. of pos 8
   g. the opp. of the opp. of neg 42
   h. the opp. of the opp. of neg 9
   i. the opp. of the opp. of the opp. of pos 23
   j. the opp. of the opp. of pos 14

   k. the opp. of the opp. of 0

26-4 Order properties

We now have to decide which of two integers is the greater or the less. You know already that a win of 3 shs is less than a win of 6 shs and that any positive integer is less than any other positive integer to the right of it on the number line. You will remember that this is the way to decide the order of two whole numbers: the greater is to the right. Here are some examples of inequalities between positive integers. You remember that the sign ">" means greater than and the sign "<" means less than.

pos 3 < pos 6
pos 100 < pos 200
pos 1 > 0

You will see that 0 must be less than any positive integer, because all the positive integers are to the right of 0.

neg 4  neg 3  neg 2  neg 1  0  pos 1  pos 2  pos 3  pos 4

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Now we want to decide the order of any two integers. Which is greater: \(-2\) or \(-3\)?

We can decide this in two ways. We already have a rule which says that any positive integer is less than any other positive integer to the right of it. It would be useful to have the same rule for negative integers. \(-1\) is to the left of \(0\), so by this rule \(-1\) is less than \(0\): \(-1 < 0\).

Similarly, \(-2\) is to the left of \(-1\) and so \(-2 < -1\), and \(-3\) is to the left of \(-2\) and so \(-3 < -2\). You can see that, by this rule, any negative integer is less than \(0\).

You can check that this result is reasonable by using gains and losses. Think about the money you gain or lose if you have a market stall. If you lose 6 shs, you have less money than if you lose 1 sh. Using \(L\) for loss and \(G\) for gain, you can write this \(L6 < L1\). If you lose 4 shs, you have less money than if you gain 3 shs: \(L4 < G3\). If you have neither gains nor losses, you have more money than if you had made a loss but less money than if you made a gain. So every loss is less than every gain, and every gain is greater than every loss. With a loss of 100 you have less money than with a loss of 1. So you see that it works out sensibly if you use the same scheme you used before: on the number line any integer is less than any integer to the right of it.

To sum up: If \(a\) and \(b\) are any two integers, then \(a < b\) means that \(a\) is to the left of \(b\) on the number line.

**EXERCISE 26-4A**

1. Put in the inequality sign, \(<\) or \(>\), to make each of the following into true statements.
   a. \(\text{pos } 6 \quad \text{pos } 10\)
   b. \(\text{neg } 6 \quad \text{neg } 10\)
   c. \(\text{pos } 15 \quad \text{neg } 15\)
   d. \(\text{neg } 15 \quad \text{pos } 15\)
   e. \(\text{neg } 200 \quad \text{neg } 1,000\)
   f. \(0 \quad \text{neg } 3\)
   g. \(0 \quad \text{pos } 8\)
   h. \(\text{neg } 11 \quad 0\)

2. Put in an integer to make each of these inequalities true.
   a. \(\text{pos } 1 \quad \Box \quad \text{pos } 1\)
   b. \(\text{pos } 1 \quad \Box \quad \text{neg } 3\)
   c. \(\text{neg } 5 \quad \Box \quad \text{neg } 5\)
   d. \(\text{neg } 5 \quad \Box \quad \text{neg } 3\)
   e. \(\text{pos } 100 \quad \Box \quad \text{neg } 100\)
   f. \(\text{neg } 100 \quad \Box \quad \text{neg } 100\)
   g. \(0 \quad \Box \quad \text{neg } 11\)
   h. \(0 \quad \Box \quad \text{neg } 11\)

Can you attach a meaning to the order of numbers in the physical models of integers? Can you say that one measurement is greater than another? Will this statement always have meaning? There must be such an order because the number line can be used for all such physical models, but are the words "greater than" and "less than" the best ones to use? How would you compare the latitude of a place \(A\) at \(\text{W}27^\circ\) with the latitude of a place \(B\) at \(\text{E}19^\circ\)? You would make one of two statements.

\[A\text{ is to the west of }B,\]
\[B\text{ is to the east of }A.\]

You notice that there is no mention of greater than or less than. Here are some more comparisons.

The top of a mountain 1,000 feet high is higher than the top of a house 10 feet high. The top of a house 10 feet high is higher than the bottom of a lake by which it stands.

The Great Pyramid was built about 1800 B.C. and the ancient kingdom of Ghana flourished about A.D. 900. How would you compare these two events? You would say the Great Pyramid was built before ancient Ghana flourished, or that ancient Ghana flourished after the Great Pyramid was built.
EXERCISE 26-4B

1. Make two comparative statements about each of the following pairs.
   a. Longitude N29° and Longitude S50°
   b. The time 10 past 3 P.M. by your watch and the time 5 to 3 P.M. by your friend's watch
   c. A man on a platform in the mine shaft 53 feet underground and a man who is on the ground
   d. The temperature at noon today, which is 95° Fahrenheit, and the temperature yesterday, which was 102° Fahrenheit
   e. A boy Kofi who starts the race 5 feet ahead of the starting point and a boy Kwesi who starts the race 5 feet behind the starting point

Now we must think about the order of the opposites of integers. You know that pos 8 > pos 3. What is the order of the opposites of pos 8 and pos 3? We can rewrite the opposites as neg 8 and neg 3 and see that neg 8 < neg 3. You can check this on the number line. So the order relation between two positive integers is reversed between their opposites. You can see this is true for other integers also by working through the next exercise.

EXERCISE 26-4C

Write down the order relationship between each of these pairs followed by the order relationship between their opposites. The first one is done for you.

1. \( \text{neg 3 < pos 2, pos 3 > neg 2} \)
2. \( \text{neg 5, neg 8} \)
3. \( \text{pos 2, pos 11} \)
4. \( \text{pos 2, neg 11} \)
5. \( \text{pos 7, 0} \)
6. \( \text{0, neg 2} \)
7. \( \text{neg 6, neg 1} \)
8. \( \text{pos 10, 0} \)

26-5 "Between"

What do we mean when we say that a whole number is between two other whole numbers? For example, suppose I say that I am thinking of a whole number between 3 and 7. You know that this whole number must be a member of the set of numbers which are greater than 3 and less than 7. It must be a member of the set \( 14, 5, 6 \). You can see that this way of finding the set of numbers between two numbers will work for the integers also. An integer between pos 3 and pos 7 must be greater than pos 3 and less than pos 7. It must be a member of the set \( 1 \text{pos 4, pos 5, pos 6} \). Can you write the members of the set of integers between neg 3 and pos 1? Each of these integers must be greater than neg 3 and less than pos 1. The set is \( \text{lneg 2, neg 1, 0} \). You can check each of these examples by looking at the number line.
**EXERCISE 26-5A**

1. Write the set of integers between these pairs.
   a. pos 20 and pos 25  
   b. pos 2 and neg 2  
   c. neg 3 and 0  
   d. neg 5 and neg 7  
   e. pos 8 and pos 9  
   f. neg 3 and neg 4  
   g. pos 2 and 0.

2. Write a description of the following sets using the idea of "between".
   a. lneg 2, neg 11  
   b. lpos 19, pos 20, pos 21, pos 22l  
   c. lpos 1, 0, neg 11l  
   d. 10l  
   e. 1 l

3. Find how many integers there are in the following sets.
   a. lintegers between neg 6 and pos 6 l  
   b. lintegers between pos 5 and neg 11 l

4. Name a set of points which is between each of the pairs of points identified in Exercise 26-4b.

**26-6 Summary**

You have been introduced to a new set of numbers, the integers. This is made up of the negative integers, zero and the positive integers. Integers can be represented on a number line and can be given an order of greatness. You will have noticed that the least number on any number line section you have drawn is always the number on the extreme left. After this, the numbers become greater as you move along the number line from left to right. Then you reach 0 and after 0 the numbers continue to increase. The greatest number on any section of the number line you have drawn is always the number on the extreme right. You are also able to find the set of integers between any two integers by using what you know about order on the line. You have also seen how to find the opposite of an integer and have discovered that to find the opposite of the opposite of a number is to leave the number unchanged.

You will now be wondering whether you can find ways to add, subtract, multiply and divide using these new numbers, and in the next chapters you will see that you can, in fact, do this.
Chapter 27

OPERATIONS ON INTEGERS

27-1 Addition

The positive integers pictured on a number line are so like the whole numbers pictured on a number line that you will wonder whether you can add integers in the same way as you added whole numbers on the number line. Before answering this question, we shall first have to think again about what we mean by these integers. In every case, you measured from 0 a number of units along the line.

To find latitude N23° you measured 23° from 0° upward.
To find longitude W19° you measured 19° to the left of 0°.
To find 100 feet below sea level you would measure 100 feet downward.
To find pos 3 you measured 3 steps of 1 unit each to the right of 0.
To find neg 3 you measured 3 steps of 1 unit each to the left of 0.

So you can think of the sum of two integers as the result of moving twice along the number line. You can find pos 3 + ... 3 steps from 0 to the right followed by 5 steps from pos 3 to the right. This will bring you to pos 8. You can write

\[ \text{pos } 3 + \text{pos } 5 = \text{pos } 8. \]

Now you have found the sum of two positive integers. You can see that it was found in just the same way that you found 3 + 5, the sum of two whole numbers, on the number line.

You can check the answer by thinking of gains and losses. A gain of 3 shs followed by a gain of 5 shs gives a gain of 8 shs. As gains may be thought of as positive integers you will see that this is another way of thinking of pos 3 + pos 5 and that this is the same as pos 8.

You can also work out the result of addition and subtraction of integers by using a slide rule. This is how to make it.

You need two strips of ruled paper. Fold each strip into a long strip with the ruled lines across the width like this:

```
[| | | | |]
```

Along the lower edge of the first strip mark the integers as if the edge were a number line.
Now mark, in the same way, the upper edge of the second strip. The picture shows you how your two strips should look.

```
   neg 5  neg 4  neg 3  neg 2  neg 1  0  pos 1  pos 2  pos 3  pos 4  pos 5
   neg 5  neg 4  neg 3  neg 2  neg 1  0  pos 1  pos 2  pos 3  pos 4  pos 5
```

This is a slide rule, but yours should have many more numbers than there are in the picture. The more integers you write, the more use you will find for your slide rule.

Now you are ready to find \( \text{pos} 2 + \text{pos} 3 \) on your slide rule. Find \( \text{pos} 2 \) on the lower strip. Now slide the upper strip along to the right until the zero point of the upper strip is exactly above the point \( \text{pos} 2 \). Your slide rule should look like this:

```
   neg 2  neg 1  0  pos 1  pos 2  pos 3  pos 4
```

From the lower zero point, you have moved to the lower \( \text{pos} 2 \) point. Now you want to add \( \text{pos} 3 \) to \( \text{pos} 2 \). To do this, you move 3 more units to the right from \( \text{pos} 2 \) and you do this with the upper strip. The upper zero point is at \( \text{pos} 2 \) and to add \( \text{pos} 3 \) you move along the upper strip 3 units to the right, that is, to the \( \text{pos} 3 \) point. Now you have added \( \text{pos} 3 \) to \( \text{pos} 2 \) and the answer will be on the lower strip below the \( \text{pos} 3 \). What is the answer? You see that it is \( \text{pos} 5 \). So you have found by using your slide rule that

\[ \text{pos} 2 + \text{pos} 3 = \text{pos} 5. \]

**EXERCISE 27-1A**

1. Without moving your slide rule from the position shown above, find the answers to these problems:
   a. \( \text{pos} 2 + \text{pos} 4 \)
   b. \( \text{pos} 2 + \text{pos} 5 \)
   c. \( \text{pos} 2 + \text{pos} 2 \)
   d. \( \text{pos} 2 + \text{pos} 1 \)
   e. \( \text{pos} 2 + 0 \)
   f. \( \text{pos} 2 + \text{pos} 1 \)

2. Now use your slide rule to find the answers to these problems:
   a. \( \text{pos} 3 + \text{pos} 1 \)
   b. \( \text{pos} 1 + \text{pos} 3 \)
   c. \( \text{pos} 4 + 0 \)
   d. \( \text{pos} 4 + \text{pos} 2 \)
   e. \( 0 + \text{pos} 6 \)
   f. \( 0 + 0 \)

3. Make up eight addition problems like these but use greater integers. Check that your slide rule gives you the correct answers.
You now know a way to add two positive integers. Can you add two negative integers in the same way? First think of the question in terms of losses and then use your slide rule to see if it gives the same answer. If you have a loss of 3 shs followed by a loss of 2 shs, you have lost the same amount as if you had one loss of 5 shs. Thinking of losses as negative integers, you can write this as $\text{neg } 3 + \text{neg } 2 = \text{neg } 5$.

What do we mean by addition of negative integers? If it is to have the same meaning as addition of positive integers, it must mean one movement followed by another movement. You can find the sum $\text{neg } 3 + \text{neg } 2$ by using your slide rule. Where will you put the 0 of the upper strip? Above the first number, that is, above $\text{neg } 3$. You now have moved 3 steps, from 0 to $\text{neg } 3$. Look along the upper strip from 0 until you find $\text{neg } 2$. You now have added $\text{neg } 2$. What number is in the answer place below $\text{neg } 2$? It will be $\text{neg } 5$. So with the slide rule, also, you have found that

$$\text{neg } 3 + \text{neg } 2 = \text{neg } 5.$$  

You found this sum by moving 3 steps to the left and then another 2 steps farther to the left.

You now know how to add two positive integers or two negative integers. Can you work out how to add one negative integer and one positive integer? You should be able to work this out for yourself in the next exercise.

**EXERCISE 27-1B**

For each of these problems make up a story about gains and losses and then check your answer by using the slide rule.

1. $\text{neg } 2 + \text{pos } 5$
2. $\text{neg } 5 + \text{pos } 3$
3. $\text{neg } 3 + 0$
4. $\text{pos } 2 + \text{neg } 5$
5. $\text{pos } 5 + \text{neg } 2$
6. $0 + \text{neg } 4$
7. $\text{pos } 3 + \text{neg } 3$
8. $\text{neg } 3 + \text{pos } 3$
9. opposite of $\text{neg } 3 + \text{neg } 3$
10. $\text{pos } 3 + \text{opposite of pos } 3$

Did you need to use a slide rule all the time? Did you discover that to add a positive integer you move your finger on the number line to the right, and to add a negative integer you move your finger to the left? So that instead of moving your slide rule, you can use one strip only as a number line and count along it. Here is an example. To find $\text{neg } 3 + \text{pos } 2$, you will first move to $\text{neg } 3$ on the line and then move 2 steps to the right to add $\text{pos } 2$. This will bring you to $\text{neg } 1$. Here is a picture to show what you have done.

```
<-H-- 1 0 pos 1 pos 2
```

Work the next exercise using a number line only.
1. Make a picture to show how you find each of these sums.
   a. pos 2 + pos 4  
   b. pos 3 + neg 1  
   c. pos 5 + neg 8  
   d. pos 1 + neg 1  
   e. pos 4 + 0  
   f. 0 + pos 2  
   g. neg 2 + neg 6  
   h. neg 3 + pos 2  
   i. neg 4 + pos 5  
   j. neg 4 + pos 4  
   k. neg 1 + 0  
   l. 0 + neg 1.

2. Use a number line to find the following.
   a. (pos 3 + neg 5) + pos 4  
   b. (pos 4 + neg 4) + neg 4  
   c. neg 5 + (neg 1 + neg 2)  
   d. (neg 3 + neg 3) + pos 4  
   e. neg 3 + (neg 3 + pos 4)

   What do you notice about the answers to d. and e.? Of what property of whole numbers does this remind you?

3. An aeroplane is flying across Africa. The pilot finds that he is at a position whose longitude is E18°. He then flies 25° to the west. Draw a number line and mark his positions on it. Then make an addition equation to show what he did and where he was finally.

   On another day the pilot starts from a place W5° and flies east for 14° of longitude. Mark his journey on the number line. Write an addition equation to show what he did. Make up two more problems about the pilot and his aeroplane.

**Addition of opposites**

When you worked these problems, did you notice something about the result when you added a pair of opposites? What is neg 3 + pos 3? pos 2 + neg 2? pos 100 + neg 100? In each case the answer is 0. You can see that this must be so by looking at a picture of the addition of a pair of opposites. Think of neg 2 + pos 2.

Neg 2 is 2 steps to the left from 0. Pos 2 is 2 steps to the right. 2 steps to the left followed by 2 steps to the right brings you back to where you started. So neg 2 + pos 2 = 0.

You will find this property of opposites very useful later on. It is very important and so we will write it here.

*The sum of an integer and its opposite is zero.*

OR

*An integer added to its opposite is zero.*
**EXERCISE 27-1D**

For each question, draw the number line and show the addition on it. Write also the addition equation.

1. Arua in Uganda has a latitude of N3°. Lubushi in Zambia is $10\frac{1}{4}$° of latitude due south of Arua. (Due south means that the two towns are on the same meridian of longitude—they both have longitude E31°.) What is the latitude of Lubushi?

2. Make up some problems like this about your country and places north and south of it.

3. An aeroplane flying at a height of 3,000 feet above the sea drops a heavy weight which falls through 3,300 feet to the bottom of the sea. How deep is the sea at that point?

4. The minute hand of the clock points to 23 minutes to the hour (4 P.M.) What time will it be in 35 minutes?

5. Kwasi and Kofe were playing a game in which 10 seeds were worth 1 cent and 10 cents were worth 1 sh. Their wins and losses are given below. How much had each of them won at the end of the game?

   Kofe: Win 8, Lose 7, Win 23, Win 2, Lose 15.

6. Make up an addition problem, suitable for your pupils, about each of the situations described in Exercise 25-2A.

**27-2 Subtraction**

Now you must think about subtraction of integers. You will remember that you have learned to think about subtraction of numbers as finding the missing addend in an addition equation. For example, to find $11 - 5$ you would find the missing addend in the equation

$$11 = 5 + \square.$$

So once you can add two integers, you should also be able to subtract one from the other. If you have the problem $\text{pos } 11 - \text{pos } 5 = \square$, you can rewrite the equation as

$$\text{pos } 11 = \text{pos } 5 + \square.$$

You will probably know at once that the missing addend is $\text{pos } 6$, and so you can write

$$\text{pos } 11 - \text{pos } 5 = \text{pos } 6.$$

For harder problems, the slide rule is very useful so let us see how to use it first on the easy problem we just did. Can you do it yourself? Try first and then read what follows here.

Find the addend you know, $\text{pos } 5$, on the lower strip and move the upper strip so that the zero point is above $\text{pos } 5$. Your slide rule will look like the next picture.
Now you want to know what you must add (on the upper strip) to pos 5 to get pos 11 on the lower strip. So find pos 11 on the lower strip. What number is above it? pos 6. So pos 6 is added to pos 5 to give pos 11.

What is the answer to pos 3 - pos 3? You will know that this is 0 and can check that your slide rule also gives this answer. Can you write the equation showing this problem as finding the missing addend? It will be

$$\text{pos } 3 + \text{[ ]} = \text{pos } 3.$$ 

You know that 0 is the only number which will make the equation true.

Now we will do subtraction with two negative integers. Think of neg 5 - neg 2. You probably know that neg 5 can be found by taking a step of neg 2 followed by a step of neg 3. Can you write this as a missing addend problem? What must be added to neg 2 to give neg 5?

Can you work this out with your slide rule? Here is the picture to help you.

<Diagram>

The zero point of the upper strip is above neg 2 on the lower strip. What number on the upper strip will give you neg 5 on the lower strip? You see that it is neg 3.

$$\text{neg } 5 = \text{neg } 2 + \frac{\text{neg } 3}{\text{[ ]}},$$

or

$$\text{neg } 5 - \text{neg } 2 = \frac{\text{neg } 3}{\text{[ ]}}.$$ 

You can see that these answers are reasonable by thinking of some real problems. Suppose you have a credit of 5 shs at a shop and you want to buy a pair of sandals which cost 1. shs. How much more money do you need? This is a missing addend problem. We can use positive integers for credits and write

$$\text{pos } 11 = \text{pos } 5 + \text{[ ]}.$$ 

You know that you need another 6 shs and so pos 6 will be put into the box:

$$\text{pos } 11 = \text{pos } 5 + \frac{\text{pos } 6}{\text{[ ]}}.$$ 

The corresponding subtraction equation is

$$\text{pos } 11 - \text{pos } 5 = \frac{\text{pos } 6}{\text{[ ]}}.$$ 

Now suppose instead of a credit of 5 shs you have a debt of 16 shs at the shop. The shopkeeper will not allow you to have so large a debt any longer. He says you must reduce it to 6 shs only. What must you give him?

You can write negative integers for debts and so the missing addend equation will be

$$\text{neg } 16 + \text{[ ]} = \text{neg } 6.$$ 

To reduce your debt from 16 shs to 6 shs, you must give the shopkeeper 10 shs. That is, you add a credit of 10 shs to your account at the shop. Therefore, we have

$$\text{neg } 16 + \boxed{\text{pos } 10} = \text{neg } 6.$$ 

The corresponding subtraction equation is

$$\text{neg } 6 - \text{neg } 16 = \boxed{\text{pos } 10}.$$ 

When you studied how to help your pupils understand subtraction, you found that there were several different types of problems which subtraction could solve.

One problem was to separate a set into two subsets, to remove one subset, and then to find how many members there were in the remaining subset. You "took away" one subset from the whole set. This way of looking at subtraction is useful to help young children understand what they are doing, but it is not very useful for problems about integers. For instance, it is awkward even to think about a positive subset of a negative set. So we will not use this idea of subtraction. We will think instead of subtraction as comparing two sets or two measurements. Think of the two problems about credits and debits. You found the answer by thinking of these as missing addend problems, but they are also comparison problems. In a comparison problem, you find the difference between two numbers.

In the first example, you found the difference between two successive credits at the shop:

$$\text{pos } 11 - \text{pos } 5 = \boxed{\text{pos } 6}.$$ 

In the second example, you found the difference between two successive debits at the shop:

$$\text{neg } 6 - \text{neg } 16 = \boxed{\text{pos } 10}.$$ 

In each case you wrote the missing addend equation and then used your slide rule to find the required difference. This is one way to find the difference between two integers. Later you will find a quicker way.

**EXERCISE 27-2A**

1. Rewrite each of the following subtraction equations as a missing addend equation and find the missing integer.
   a. $$\text{pos } 7 - \text{pos } 5 = \boxed{\text{pos } x}$$
   b. $$\text{pos } 3 - \text{pos } 1 = \boxed{\text{pos } y}$$
   c. $$\text{pos } 3 - \text{pos } 3 = \boxed{\text{pos } z}$$
   d. $$\text{pos } 2 - 0 = \boxed{\text{pos } w}$$
   e. $$\text{neg } 7 - \text{neg } 7 = \boxed{\text{neg } x}$$
   f. $$\text{neg } 4 - \text{neg } 2 = \boxed{\text{neg } y}$$
   g. $$\text{neg } 3 - \text{neg } 3 = \boxed{\text{neg } z}$$
   h. $$\text{neg } 2 - 0 = \boxed{\text{neg } w}$$
   i. $$\text{neg } 3 - \text{neg } 1 = \boxed{\text{neg } x}$$

2. A mine shaft 100 feet deep has above it on the ground a tower 60 feet high. A ladder goes from the bottom to the top. Write equations in positive or negative integers which you use to find the answers to the following problems.
   a. A man climbs from the ground to a platform 20 feet high and then climbs another 15 feet. How high is he now?
   b. Above his head there is another platform 50 feet from the ground. How much higher is this platform than the first platform?
   c. Another man is 25 feet down the mine. How much farther has he to go to reach the bottom?
3. Make up some problems about gains and losses to fit these equations.
   a. pos 2 + pos 4 = pos 6     b. pos 4 - pos 3 = pos 1
   c. neg 4 - neg 1 = neg 3     d. neg 2 - neg 2 = 0

Now we come to harder problems. How can we subtract a positive integer from a negative integer, and vice versa? Think of the equation

pos 2 - neg 1 = □.

If you think of this as comparing pos 2 and neg 1 then you can see that you are asking the question "How much greater is pos 2 than neg 1?" This is a missing addend problem again and you can write it

pos 2 = neg 1 + □.

How will you find the answer? You may see at once, by thinking of the position of pos 2 and the position of neg 1 on the number line that the missing addend is pos 3. (Your pupils will need to work it out with their slide rules and should not be urged to use only the number line just yet. Let them see why pos 3 is the only number which will make the equation true.) Here are the equations.

and so

pos 2 - neg 1 = □ + □.

You have now subtracted a negative integer from a positive integer. You could equally well use the same method to subtract a positive integer from a negative integer. Suppose you have

neg 5 - pos 3 = □.

This can be written as

neg 5 = pos 3 + □.

By using your slide rule, you will discover that neg 8 is needed to make this equation true.

neg 5 = pos 3 + □.

so

neg 5 - pos 3 = □.

You can check these answers by thinking about debits and credits. If you have a credit of £2 at the shop and your friend has a debt of £1 at the same shop, then you can compare your credit of £2 with your friend's debt of £1 and say that in that shop you are £3 richer than your friend. You can use positive integers for credits and negative integers for debts and write

pos 2 - neg 1 = pos 3.

Perhaps on another occasion you have a debt of £5 at the shop and your friend has a credit of £3 in the same shop. You can compare your debt of £5 with your friend's credit of £3 and write

neg 5 - pos 3 = neg 8.

The result tells you that in that shop you are £8 poorer than your friend.

You will be realizing now that it seems as though subtraction is always possible with positive and negative integers. From your knowledge of the order of integers, you will
have noticed that you can subtract a greater integer from a smaller integer and have an answer. You will remember

\[ \text{neg} \ 5 - \text{pos} \ 3 = \text{neg} \ 8. \]

Neg 5 is less than pos 3 because it is to the left on the number line. Again, you cannot find a whole-number answer to 3 - 5, but what about pos 3 - pos 5? Try it.

\[ \text{pos} \ 3 - \text{pos} \ 5 = \]
\[ \text{pos} \ 3 = \text{pos} \ 5 + \]

What must be added to pos 5 to get pos 3? It must be neg 2.

\[ \text{pos} \ 3 = \text{pos} \ 5 + [\text{neg} \ 2], \]

so

\[ \text{pos} \ 3 - \text{pos} \ 5 = \text{neg} \ 2. \]

It looks as if these new numbers will be very useful to us. We can do any subtraction problem with them. We can find an answer when we subtract a larger number from a smaller number and also when we subtract a positive number from a negative number.

You know something important now about the operation of subtraction in the set of integers. No matter what two integers \(a\) and \(b\) you select, it is always true that there is an integer which is the difference \(a - b\). We say that the set of integers is CLOSED under subtraction.

What set of numbers is closed under addition? The set of whole numbers is, because there is always a whole number which is the sum \(a + b\), whichever whole numbers \(a\) and \(b\) we select. But the set of integers is also closed under addition because we can always find \(a + b\), no matter what integers \(a\) and \(b\) we select.

(Which set of numbers is closed under division? Is there always a quotient in the set of fractions? If we choose any two fractions \(\frac{a}{b}\) and \(\frac{c}{d}\), is there always a fraction \(\frac{a}{b} \div \frac{c}{d}\)?

We must be careful here, because we cannot divide by zero.

\[ \frac{\frac{a}{b} \div \frac{c}{d}}{\frac{ad}{bc}} \]

This quotient \(\frac{ad}{bc}\) can always be found provided that neither \(b\) nor \(c\) is 0. With this condition, the set of fractions is closed under division.)

**EXERCISE 27-2B**

1. Use your slide rule to find these differences.
   a. \(\text{pos} \ 5 - \text{pos} \ 2\)
   b. \(\text{pos} \ 5 - \text{pos} \ 6\)
   c. \(\text{pos} \ 2 - \text{pos} \ 2\)
   d. \(\text{pos} \ 5 - \text{neg} \ 1\)
   e. \(0 - \text{neg} \ 3\)
   f. \(0 - \text{pos} \ 2\)
   g. \(\text{neg} \ 5 - \text{pos} \ 3\)
   h. \(\text{neg} \ 1 - \text{neg} \ 7\)
   i. \(\text{neg} \ 3 - \text{neg} \ 3\)
   j. \(\text{neg} \ 3 - \text{pos} \ 3\)

2. Explain how you could use a slide rule to find \(\text{pos} \ 1 - \text{neg} \ 4\).

3. Make up a story problem about wins and losses in a game in which the following equation occurs.
pos 3 = pos 7 = neg 4

Let us think again about comparing two integers. Think about debits and credits. If you have a debt of £3 in a shop and your friend has a credit of £2, then you can compare your financial standing in that shop in two ways. You will be £5 poorer in that shop than your friend, and your friend will be £5 richer than you. Can you write the two equations?

\[ \text{neg 3} - \text{pos 2} = \text{neg 5}, \]

and

\[ \text{pos 2} - \text{neg 3} = \text{pos 5}. \]

These two equations are very closely related. The numerical parts of the differences are the same, but one difference is neg 5 and the other difference is pos 5. Here pos 5 represents "richer by £5" and neg 5 represents "poorer by £5".

These results fit in with the way we usually compare two measurements \( A \) and \( B \). We either say that one, \( A \), is greater than the other, \( B \), or that \( B \) is less than \( A \). Instead of greater than, we may have several different comparisons, such as higher than, later than, richer than, to the north of and in front of. For less than, the corresponding comparatives are lower than, earlier than, poorer than, to the south of and behind.

In any particular situation, there is no ambiguity. You always know which town of two is to the east by looking at their latitudes. You know which of two wins is greater by comparing the two amounts on a number line. There is a convention—an agreement—about the operation of finding the difference between two numbers. The number which is mentioned second is the number which must be subtracted. Find the difference between neg 1 and pos 2 means find neg 1 - pos 2. Find the difference between pos 2 and neg 1 means find pos 2 - neg 1.

We can write this convention as a general rule using \( a \) and \( b \) to stand for any integers we select.

*The difference between \( a \) and \( b \) is \( a - b \).*

In comparing the two integers we will, of course, say that

- \( a \) is greater than \( b \) if the difference is positive,

or

- \( a \) is less than \( b \) if the difference is negative.

You will notice that the first integer in the difference \( a - b \) is written first in the sentence making the comparison.

Here is an example. Find the difference between pos 3 and pos 11. The difference is pos 3 - pos 11.

\[ \text{pos 3} = \text{pos 11} + \square \]

What must be added to pos 11 to make pos 3? It must be a negative integer, so it will be neg 8.

\[ \text{pos 3} = \text{pos 11} + \text{neg 8}. \]

So we say that pos 3 - pos 11 = neg 8, or the difference between pos 3 and pos 11 is neg 8. Now you know that pos 3 is less than pos 11. If we had asked for the difference between pos 11 and pos 3, we would have

\[ \text{pos 11} - \text{pos 3} = \text{pos 8}. \]
You know that pos 11 is greater than pos 3.

So if the first integer is greater than the second integer, the difference is a positive integer, and if the first integer is less than the second integer, the difference is a negative integer.

This property of the difference of two integers gives you a new way to work out the answer to a subtraction problem. Here it is.

Find the distance on the number line between the two points which correspond to the integers. Then if the first integer is greater than the second, make the answer a positive integer. If the first integer is less than the second, make the answer a negative integer.

For example, suppose you want to find the difference between neg 4 and neg 7, that is, neg 4 – neg 7. The distance between the points on the number line which correspond to these integers is 3 steps. So the answer will be either pos 3, or neg 3. Look at the numbers. Which is the greater? Which number is to the right on the number line? Neg 4 is to the right and so neg 4 is greater than neg 7. So now you can write

\[ \text{neg } 4 - \text{neg } 7 = \text{pos } 3. \]

Check this by writing it as a missing addend problem:

\[ \text{neg } 7 + \text{[pos } 3\text{]} = \text{neg } 4 \]

Similarly, the difference between neg 7 and neg 4, that is, neg 7 – neg 4, would be neg 3:

\[ \text{neg } 7 - \text{neg } 4 = \text{neg } 3 \]

It is useful at first for your pupils to work subtraction problems in this way also, because it helps them to understand subtraction of integers and the meaning of a negative integer result.

**EXERCISE 27-2C**

1. Use the missing addend method to find the answers to the following problems.
   a. pos 3 – pos 8
   b. neg 3 – neg 8
   c. pos 1 – neg 1
   d. pos 4 – neg 2
   e. neg 3 – pos 5
   f. 0 – neg 4
   g. pos 2 – pos 1
   h. neg 4 – neg 1

   Write a sentence to describe and explain each result.

2. Find the difference between the numbers in each of the following pairs by using a number line, first finding the distance between the points corresponding to the numbers, and then writing pos or neg to show whether the first number is greater than or less than the second number. Write your result first in an equation and secondly in a sentence.
   a. pos 2, pos 1
   b. pos 3, neg 5
   c. 0, neg 4
   d. neg 1, neg 5
   e. pos 3, 0
   f. pos 1, pos 7
   g. neg 3, pos 3
   h. neg 11, neg 2

3. Look at the questions in Exercise 26-4B. You were asked to compare pairs of measurements by saying which position was higher than, or to the east of, the other and so on. Now find the difference between each of the numbers in these pairs. Show the subtraction equation and the corresponding addition equation. Write each answer in a sentence comparing the two measurements.
27-3 Relation between addition and subtraction

Now that you know how to do addition and subtraction with these new numbers, the integers, you see how they are more useful than the whole numbers alone. With the whole numbers, you can find an answer to $5 + 3$, and $5 - 3$, but not to $3 - 5$. With the integers, you can find the sum of any two integers, and you can also find the difference between any two integers. For example, you can find pos $5 -$ pos $3$ and, also, pos $3 -$ pos $5$. With whole numbers you can always do addition, and sometimes subtraction. But with the integers you can always do addition and always do subtraction.

Do you remember the mathematical way of writing these properties of the set of whole numbers and the set of integers? The set of whole numbers is closed under addition, because whatever whole numbers $a$ and $b$ we select, a whole number can always be found which is the sum $a + b$. The set of integers is closed under addition and subtraction.

**EXERCISE 27-3A**

1. Write a sentence similar to the above about:
   a. the set of integers and addition
   b. the set of integers and subtraction

2. What set of numbers is closed under division? How would you explain this to your pupils?

There is another very useful property of addition and subtraction of integers. Exercise 27-3C will help you to discover this property, but first work the short Exercise 27-3B to remind yourself about some very useful pairs of integers.

**EXERCISE 27-3B**

1. Put an integer into each of these boxes to make the equation true.
   a. pos $3 + [ ] = 0$
   b. neg $3 + [ ] = 0$
   c. $[ ] +$ pos $1 = 0$
   d. $[ ] +$ neg $5 = 0$
   e. $[ ] +$ neg $1 = 0$
   f. $[ ] +$ 0 = 0

Now answer these questions.
   g. What are these pairs of integers called?
   h. Draw a number line to show the pair in Question 1b.
   i. What is the special property of these pairs of integers?
   j. Write three more pairs.

Now that you are sure that you remember opposites and their properties, do the next exercise and see what you can discover about their use in addition and subtraction.

**EXERCISE 27-3C**

1. Put an integer into each box to make the equations true.
   a. pos $3 -$ pos $1 = [ ]$
   b. pos $3 +$ neg $1 = [ ]$
   c. pos $3 -$ pos $1 =$ pos $3 + [ ]$
Do you see here a connection between addition and subtraction? Describe it in a sentence.

What did you discover in the last exercise? Did you find these two properties of addition and subtraction of integers?

\[ \begin{align*}
    b. \quad & neg 3 + neg 1 = \quad \square \\
    & neg 3 - pos 1 = \quad \square \\
    & neg 3 + neg 1 = neg 3 - \quad \square \\
    c. \quad & pos 2 - pos 6 = \quad \square \\
    & pos 2 + neg 6 = \quad \square \\
    & pos 2 - pos 6 = pos 2 + \quad \square \\
    d. \quad & 0 - pos 7 = \quad \square \\
    & 0 + neg 7 = \quad \square \\
    & 0 - pos 7 = 0 + \quad \square \\
    e. \quad & 0 - neg 7 = \quad \square \\
    & 0 + \quad \square = pos 7 \\
    & 0 - neg 7 = 0 + \quad \square \\
\end{align*} \]

Do you see here a connection between addition and subtraction? Describe it in a sentence.

What did you discover in the last exercise? Did you find these two properties of addition and subtraction of integers?

**Discovery 1.** To subtract an integer from another integer you can add its opposite. This is shown in the equations below.

\[
\begin{align*}
    & pos 3 - pos 5 = neg 2 \\
    & pos 3 + (opp. of pos 5) = pos 3 + neg 5 \\
    & \quad = neg 2
\end{align*}
\]

Therefore,

\[ pos 3 - pos 5 = pos 3 + (opp. of pos 5). \]

**Discovery 2.** To add an integer to another integer you can subtract its opposite.

\[
\begin{align*}
    & neg 3 + pos 4 = pos 1 \\
    & neg 3 - (opp. of pos 4) = neg 3 - neg 4 \\
    & \quad = pos 1
\end{align*}
\]

Therefore,

\[ neg 3 + pos 4 = neg 3 - (opp. of pos 4). \]

These two properties, linking addition and subtraction of integers, are really only one property: To add an integer to another integer you subtract its opposite, and to subtract an integer from another integer you add its opposite. You will see in the next chapter how this property is used to make addition and subtraction of negative integers much easier.

Now let us think about subtraction on the number line. We have not yet found what movement on the number line corresponds to subtraction. We already have movements for addition on the number line:

To add a positive integer to another integer, move to the right.

To add a negative integer to another integer, move to the left.

Can you draw the pictures for the two additions \( neg 1 + neg 4 \) and \( neg 2 + pos 3 \)?
Here they are.

\[
\begin{align*}
\text{neg} 1 + \text{neg} 4 &= \text{neg} 5 \\
\text{neg} 2 + \text{pos} 3 &= \text{pos} 1
\end{align*}
\]

Each of these equations can be rewritten showing subtraction of the opposite. Take the first equation.

\[
\text{neg} 1 + \text{neg} 4 = \text{neg} 5
\]

becomes

\[
\text{neg} 1 - (\text{opposite of} \ \text{neg} 4) = \text{neg} 5
\]
or

\[
\text{neg} 1 - \text{pos} 4 = \text{neg} 5.
\]

So the movement to the left which represents + neg 4 also represents − pos 4. So we have found a meaning for subtraction of a positive integer in terms of a movement on the number line.

To subtract a positive integer move to the left. Here is a picture.

\[
\begin{align*}
\text{neg} 1 - \text{pos} 4 &= \text{neg} 5 \\
\end{align*}
\]

You will see that this is the same as the first picture above except that − pos 4 has replaced + neg 4.

Now consider the second equation,

\[
\text{neg} 2 + \text{pos} 3 = \text{pos} 1,
\]

which becomes

\[
\text{neg} 2 - (\text{opp. of} \ \text{pos} 3) = \text{pos} 1
\]
or

\[
\text{neg} 2 - \text{neg} 3 = \text{pos} 1.
\]
So the movement to the right which represents $+ \text{pos } 3$ also represents $- \text{neg } 3$. Now we have a movement for subtraction of a negative integer.

To subtract a negative integer move to the right. Here is a picture.

\[
\text{neg } 2 - \text{neg } 3 = \text{pos } 1
\]

You will see that this is the same as the second picture above except that $- \text{neg } 3$ has replaced $+ \text{pos } 3$.

Put these two discoveries together and you have the movements for subtraction on the number line.

- To subtract a positive integer from another integer, move to the left.
- To subtract a negative integer from another integer, move to the right.

To summarize what we now know about addition and subtraction as movements we will write:

- A move to the right is caused by the
  1. addition of a positive integer,
  2. subtraction of a negative integer.

- A move to the left is caused by the
  1. addition of a negative integer,
  2. subtraction of a positive integer.

You now have three ways of finding the difference between two integers $a$ and $b$:

\[ a - b = \square. \]

**Method 1.** Rewrite the subtraction equation as an addition equation and find the missing addend $c$ by using a slide rule or the number line:

\[ a = b + \square \]

**Method 2.** Find the distance $d$ in units on the number line between the two points. Decide whether $a > b$ or $b > a$.

If $a > b$, the difference will be positive:

\[ a - b = d > 0 \]

If $a < b$, the difference will be negative:

\[ a - b = d < 0. \]

**Method 3.** Add the opposite of the second integer in the equation $a - b = \square$:

\[ a + (\text{opp. of } b) = \square \]
**EXERCISE 27-3D**

1. Work the following problems in subtraction of integers by adding their opposites in each case.
   a. pos 3 - pos 8
   b. pos 12 - pos 7
   c. neg 5 - neg 9
   d. neg 4 - neg 1
   e. neg 3 - pos 7
   f. neg 5 - pos 2
   g. pos 3 - pos 3
   h. neg 5 - pos 5
   i. pos 7 - neg 10
Chapter 28

RELATION OF INTEGERS TO WHOLE NUMBERS

In the last three chapters you have learned much about the integers. You know how they are used to show measurement in opposite directions; you know how to find which of two integers is the greater and how to do addition and subtraction with integers. Now you will want to know how they fit in with what you have learned before that. How are the integers related to the whole numbers?

28-1 The positive integers

Think first about the set of positive integers and the set of counting numbers. You will remember that the counting numbers do not include zero. Each set has a least member, pos 1 or 1, and each of these can be matched with the other. All numbers greater than these can be matched in pairs in order of size beginning with pos 2 and 2. Here is a picture of this matching.

<table>
<thead>
<tr>
<th>Counting Numbers</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive Integers</td>
<td>pos 1</td>
<td>pos 2</td>
<td>pos 3</td>
<td>pos 4</td>
<td>pos 5</td>
<td>pos 6</td>
</tr>
</tbody>
</table>

The difference between any two successive counting numbers and the difference between any two successive positive integers is the same, one unit. Each number in either set is one unit greater than the number to the left of it and one unit less than the number to the right of it. If we now consider the set consisting of the counting numbers and zero—that is, the set of whole numbers—and compare it with the set consisting of the positive integers and zero, we can take the comparison further. The number one unit less than pos 1 is 0, and the number one unit less than 1 is 0. So you can see that there is a very close resemblance between the set of whole numbers and the set made up of zero and the positive integers. It is such a close resemblance that you can show them both on the same number line.

<table>
<thead>
<tr>
<th>Whole numbers</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero and pos. integers</td>
<td>0</td>
<td>pos 1</td>
<td>pos 2</td>
<td>pos 3</td>
<td>pos 4</td>
<td>pos 5</td>
</tr>
</tbody>
</table>

Now we must ask, "Do the numbers of both sets behave in the same way when you add or subtract with them?" Do they give corresponding results? Is the answer to 2 + 4 the whole
number which corresponds to the positive integer which is the answer to pos 2 + pos 4? The answers are 6 and pos 6 and these are corresponding numbers. You can see that this relationship will always be true for addition, but will it be true for subtraction? Work through the next exercise and make a note of any cases where there is not a correspondence between the answer in whole numbers and the answer in positive integers with zero.

**EXERCISE 28-1A**

1. Find the answers to these problems.
   a. $23 + 65 = \_\_\_\_$  
      pos $23 + \text{pos } 65 = \_\_\_\_$
   b. $203 - 129 = \_\_\_\_$  
      pos $203 - \text{pos } 129 = \_\_\_\_$
   c. $24 - 79 = \_\_\_\_$  
      pos $24 - \text{pos } 79 = \_\_\_\_$
   d. $17 - 25 = \_\_\_\_$  
      pos $17 - \text{pos } 25 = \_\_\_\_$
   e. $94 - 37 = \_\_\_\_$  
      pos $94 - \text{pos } 37 = \_\_\_\_$

Have you found that the whole numbers and the positive integers with zero give corresponding results for subtraction as long as the first number in the subtraction equation is not less than the second number? $25 - 17$ has a result 8, and pos $25 - \text{pos } 17$ has a result pos 8. If you have a subtraction equation where the first number is less than the second number, then there is a result when you work with integers, but this result is a negative integer. For example, pos $17 - \text{pos } 25 = \text{neg } 8$. There is no corresponding answer to the whole-number equation. That is, $17 - 25$ has no whole-number answer. So if you work with the positive integers and zero only, you can see that they behave in exactly the same way as the whole numbers. Because of this resemblance, we will not continue to give the positive integers their special label “pos”. We say that pos 3 behaves exactly like 3, and so we will write 3 for pos 3.

Instead of pos 1, we write 1.
Instead of pos 2, we write 2.
Instead of pos 3, we write 3.
Instead of pos 100, we write 100.

Here are some equations. They are written first in the old notation and second in the new notation for positive integers.

- pos 3 + pos 2 = pos 5 becomes $3 + 2 = 5$.
- pos 5 + neg 3 = pos 2 becomes $5 + (-3) = 2$.
- pos 5 - pos 2 = pos 3 becomes $5 - 2 = 3$.
- pos 3 + 0 = pos 3 becomes $3 + 0 = 3$.
- 0 + pos 1 = pos 1 becomes $0 + 1 = 1$.
- pos 3 - pos 5 = neg 2 becomes $3 - 5 = -2$.

**EXERCISE 28-1B**

1. Find the answers to the following problems. Then rewrite them, putting whole numbers in place of the corresponding positive integers.
   a. pos $15 + \text{pos } 14$  
      b. $0 - \text{pos } 3$  
      c. pos $1 - \text{pos } 3$
   d. pos $23 - \text{pos } 7$  
      e. pos $17 + \text{neg } 9$  
      f. pos $3 - \text{neg } 2$

2. Make up four problems like the ones you have just done.
28-2 The negative integers

Draw a number line for the integers and write the counting numbers where you formerly wrote the positive integers.

You have a number line along which you have measured in two opposite directions. You can still know which 3 is on the right of 0 and which 3 is on the left of 0, because the one on the right is written 3 and the one on the left is written neg 3. You will ask whether there is a simpler way of writing neg 3. What do you know about neg 3? You know that neg 3 names the point you reach by taking three steps to the left from 0. You can also think of this movement as adding neg 3 to 0. The equation is

\[0 + \text{neg} 3 = \text{neg} 3.\]

You have also learned that you can replace \(+\) neg 3 by \(-\) pos 3, and in the new notation \(-\) pos 3 will be \(-\ 3\). Instead of adding neg 3, you can subtract the opposite of neg 3; that is, you can subtract 3. So we can now have two equations

\[0 + \text{neg} 3 = \text{neg} 3\]

and

\[0 - 3 = \text{neg} 3.\]

The difference \((0 - 3)\) and the sum \((0 + \text{neg} 3)\) are the same number, \text{neg} 3:

\[0 + \text{neg} 3 = 0 - 3 = \text{neg} 3\]

Because of this equality, we agree to write "\(\text{neg} 3\)" for "\(-3\)". The minus sign now has two meanings instead of only one. From now on, every minus sign you meet can have two possible meanings and you should know which one is meant. In \(0 - 3\) the minus sign means the operation of subtraction. In \(-3\) (for \text{neg} 3) it means negative. It must mean negative here because no other integer is in front of it from which you can subtract it. It is useful to write \(-3\), when it is a negative integer, with brackets enclosing it as \((-3)\) to remind pupils that it does not mean subtract 3 but the integer \text{neg} 3.

Instead of \text{neg} 1 we now write \((-1)\).

\text{neg} 2 becomes \((-2)\).

\text{neg} 3 becomes \((-3)\).

\text{neg} 251 becomes \((-251)\).

Each integer except 0 has a new name in the new notation, but the set of integers is the same as before: It consists of the positive integers, zero and the negative integers. The old counting numbers are included in the set of integers as the positive integers. The old whole numbers are now the set of positive integers with zero. Here is the number line labelled with the new notation.

Now you can rewrite equations in the new notation. Here are some examples.

\[\text{pos} 3 + \text{neg} 1 = \text{pos} 2 \quad \text{becomes} \quad 3 + (-1) = 2.\]

\[\text{neg} 5 + \text{neg} 8 = \text{neg} 13 \quad \text{becomes} \quad (-5) + (-8) = (-13).\]

\[\text{neg} 5 - \text{neg} 7 = \text{pos} 2 \quad \text{becomes} \quad (-5) - (-7) = 2.\]

\[\text{pos} 3 - \text{pos} 12 = \text{neg} 9 \quad \text{becomes} \quad 3 - 12 = (-9).\]
pos 5 - pos 2 = pos 3 becomes 5 - 2 = 3.
0 + neg 5 = neg 5 becomes 0 + (-5) = (-5).
0 - pos 2 = neg 2 becomes 0 - 2 = (-28).

Each time the minus sign appears in the second column of equations above, it means either subtract or negative. Can you say which it is in every case? Here are three equations. In the first equation the minus means negative:

\[ 3 + (-1) = 2 \]

In the second equation the minus means subtract:

\[ 5 - 2 = 3 \]

In the third equation there are two minus signs. The first means subtract and the second means negative:

\[ 4 - (-3) = 7 \]

**EXERCISE 28-2A**

1. Find the answer to each of these problems and then write the complete equation in the new notation.
   a. neg 13 + neg 71
   b. neg 21 - pos 11
   c. neg 3 - 0
   d. neg 21 - neg 15
   e. pos 13 + neg 20
   f. 0 + neg 4
   g. pos 14 - pos 32
   h. pos 3 - neg 7
   i. 0 - neg 17

2. In each of the following equations, decide which of the two possible meanings for the minus sign is meant and write the equation in words. Here is an example:

   \[ (-7) - 1 = (-8) \]

   in words is “neg 7 subtract 1 equals neg 8”.
   a. \((-3) + 2 = \)
   b. \((-3) - 2 = \)
   c. \(4 - (-5) = \)
   d. \(4 - 5 = \)
   e. \((-4) - 0 = \)
   f. \((-3) + (-5) = \)
   g. \((-3) - (-5) = \)
   h. \(0 - 2 = \)
   i. \(0 - (-2) = \)
   j. \((-4) + 2 = \)

3. Draw a number line showing the integers written in the new notation. Mark on it the points represented by each of the following integers and its opposite. For each pair, make an equation; for example, opp. of 3 = .
   a. 3
   b. -2
   c. 7
   d. pos 9
   e. (opposite of 10)
   f. -5
   g. 0
   h. [opposite of (-6)]
   i. neg 4

**28-3 Opposites**

What can we discover about an integer and its opposite in the new notation? Look at these equations.
opp. of 3 = neg 3 becomes opp. of 3 = (-3).
opp. of neg 5 = pos 5 becomes opp. of (-5) = 5.

In each pair of opposites, one integer is positive and the other integer is negative. Can you see that here the minus sign can be thought of in a third way, as saying "the opposite of"?

(-3) is the opposite of 3.
-(neg 4) is the opposite of neg 4.
-(4) is the opposite of (-4).

**EXERCISE 28-3A**

1. For each of the following, first find the integer to make the equation true, then rewrite the equation in the new notation for "negative" and for "the opposite of".

Example:
opp. of neg 3 = opp. of neg 3
= -(-3):

a. opp. of pos 2 = 

b. opp. of neg 5 = 

c. opp. of 0 = 

d. opp. of opp. of pos 1 = 

e. opp. of opp. of neg 3 = 

2. Find the integer which is the simplest way of writing each of the following.

a. (-4) 

b. [(-2)] 

c. (-0) 

d. [(-6)]

**28-4 Addition and subtraction in the new notation**

You will be wondering how to perform addition and subtraction in the new notation. Perhaps you already know some rules for adding and subtracting integers, but do you understand how they work? Can you explain them to your pupils?

It will help to think of addition and subtraction on the number line. Think about adding or subtracting a positive integer. The problems are given first in the old notation and then in the new notation.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Old Notation</th>
<th>New Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>pos 6 + pos 2</td>
<td>6 + 2 steps to right</td>
<td>8</td>
</tr>
<tr>
<td>neg 2 + pos 4</td>
<td>(-2) + 4 steps to right</td>
<td>2</td>
</tr>
<tr>
<td>pos 6 - pos 4</td>
<td>6 - 4 steps to left</td>
<td>2</td>
</tr>
<tr>
<td>neg 2 - pos 4</td>
<td>(-2) + 4 steps to left</td>
<td>(-6)</td>
</tr>
</tbody>
</table>

**EXERCISE 28-4A**

1. Draw a number line to show each of the following and then write the complete equation.

a. 2 + 3 

b. 2 - 3 

c. 0 - 2 

d. (-2) + 1 

e. (-3) + 5 

f. (-4) + 6

You see that the addition and the subtraction of positive integers from other integers
present no difficulty at all. And in the last chapter we learned that to add an integer we
can subtract its opposite, and to subtract an integer we can add its opposite. Here are
two examples showing these procedures in the new notation.
In the old notation we have

\[
\text{pos } 3 \cdot \text{neg } 5 = \text{pos } 3 - (\text{opp. of neg } 5) \\
= \text{pos } 3 - \text{pos } 5 = \text{neg } 2.
\]

In the new notation we have

\[
3 + (-5) = 3 - \text{opp. of } (-5) \\
= 3 - 5 \\
3 + (-5) = (-2).
\]

Now we will work another problem, first in the old notation:

\[
\text{neg } 2 - \text{neg } 3 = \text{neg } 2 + (\text{opp. of neg } 3) \\
= \text{neg } 2 + \text{pos } 3 \\
= \text{pos } 1
\]

In the new notation this becomes

\[
(-2) - (-3) = (-2) + [\text{opp. of } (-3)] \\
= (-2) + 3 \\
(-2) - (-3) = 1.
\]

**EXERCISE 28-4B**

1. Work these problems in the way shown above, first in the old notation and then in the
new notation.
   a. pos 3 + neg 4
   b. neg 3 + neg 3
   c. 0 + neg 1
   d. 0 - neg 4
   e. pos 3 - neg 2
   f. neg 5 - neg 2
   g. neg 2 - neg 2
   h. neg 7 + neg 3

2. Work these problems using any method you choose, but explain your method as you would
explain it to your pupils.
   a. neg 2 + pos 1
   b. neg 6 + neg 3
   c. pos 1 - pos 7
   d. pos 3 - neg 3

3. Think of the physical models of the integers and make up some problems about them
which need addition and subtraction of integers.

**28-5 Some properties of integers**

Do you remember that when you studied addition of whole numbers and of fractions you
found that zero played a particular role in addition? It is an identity element. If 0 is added to
a number, that number is unchanged. For example, 9 + 0 = 9 and 0 + 9 = 9. When you tested to
see if zero was an identity element for subtraction also, you found that it only "half
worked". 3 - 0 = 3, but 0 - 3 had no answer. Now you know the answer to 0 - 3: it is (-3),
neg 3. Try the question again beginning with (-3): (-3) - 0 = (-3), and 0 - (-3) = 3. So zero
does not act as an identity element for subtraction. Do you know why not? It is because of an­
other property which addition possesses and which subtraction does not.
You know that $3 + 5 = 8$ and $5 + 3 = 8$ and also that $5 - 3 = 2$ but $3 - 5 \neq 2$. This shows the commutative property of addition and it shows that subtraction does not possess this property.

In general you know that, for all integers $a$ and $b$, $a + b = b + a$ but that $a - b \neq b - a$ unless $a = b$. The commutative property is not a property of subtraction.

There is, however, a special relationship between $a - b$ and $b - a$. Can you see what it is? Look at these results.

| $5 - 3 = 2$ | $3 - 5 = (-2)$ |
| $12 - 4 = 8$ | $4 - 12 = (-8)$ |
| $16 - 19 = (-3)$ | $19 - 16 = 3$ |

In each pair of equations the differences are opposites, 2 and $(-2)$, 8 and $(-8)$ and $(-3)$ and 3.

So $(a - b)$ and $(b - a)$ are opposites.
So $(a - b) = \text{opp.} (b - a)$,
$(a - b) = -(b - a)$.

This is a very useful relationship. If you have, for example, $8 - (3 - 4)$ in an equation, you can replace it by $8 + (4 - 3)$ which is easier to find.

**EXERCISE 28-5A**

1. Work these problems
   a. $(\text{pos} 6 + \text{pos} 2) - \text{neg} 3$
   b. $\text{neg} 3 + (\text{neg} 2 + \text{pos} 1)$
   c. $[7 + (-2)] + 8$
   d. $(-4) + [(-2) + (-1)]$
   e. $[(3) - (-5)] - (-3)$
   f. $(-8) - [(-4) + (-2)]$

2. Work these problems in subtraction by "adding the opposite".
   a. $(2) - (3 - 8)$
   b. $6 - (2 - 1)$
   c. $(-4) - [(-2) - 6]$
Chapter 29
OPERATIONS ON INTEGERS, CONTINUED

29-1 Addition, using the new notation

You have learned about new numbers, called the integers. You know that they include the counting numbers, zero, and the negatives, which are the opposites of the counting numbers. You have learned what it means to add and subtract these numbers, and you have seen that these operations act the same way as addition and subtraction of counting numbers.

You have also learned an easy way to write these numbers. Instead of writing pos 3, for example, you learned that you could write simply 3. The familiar counting numbers are, in fact, the positive numbers. But when you use them, you must remember, of course, that they are positive, and that they behave as opposites of the negative numbers. And you learned also that instead of neg 3, for example, you could simply write (-3). The symbol (-3) seems to ask you to subtract 3—and there is a very good reason for that, namely, that subtraction of 3 is equivalent to addition of (-3). But the number (-3) is as good a number in its own right as 3. Both are equally important as numbers and deserve your equal respect.

It is useful now to recall some of the ways integers act when you add them, and to use the new way of writing integers in doing so. Thus, you learned to write, for example,

\[ \text{pos} \ 3 + \text{pos} \ 4 = \text{pos} \ 7. \]

But now you can write simply

\[ 3 + 4 = 7. \]

This new equation means the same thing as the one before it, but it is easier to write. It hides the fact that the numbers are really positive integers, but you should be able to keep that fact in mind by now. You can think of this addition exercise in terms of two successive increases of money in your pocket, or two successive jumps to the right on the number line, as in the following pictures.

You also saw the same pattern for negative numbers. Instead of

\[ \text{neg} \ 3 + \text{neg} \ 4 = \text{neg} \ 7, \]
you can now write

\((-3) + (-4) = (-7),\)

where the parentheses around \((-4)\) remind you that the minus sign, "\(-\)", in \((-4)\) is not a subtraction sign but rather a marker showing that the number is the opposite of whatever follows the sign. Thus, in this case the "\(-\)" changes positive 4 to its opposite, neg 4. You can also write this as in the previous chapter

\((-3) - 4 = (-7),\)

since addition of \((-4)\) is the same as subtraction of 4. In practice, whenever you add a negative number you do it by subtracting its opposite, which is positive. We will write it both ways.

The sum of two negative integers can be represented in two ways: as the result of successive losses, or as the result of two jumps to the left on the number line. You remember that you learned to think of addition of negative numbers as successive losses or as leftward movements on the number line. We simply reverse our pictures of the addition of positive integers. Thus, we can represent the addition of negative integers in these pictures:

\[\text{DEBT} \quad 3 \text{ SHS.} + \quad \text{DEBT} \quad 4 \text{ SHS.}\]

It is a bit harder to understand the addition of two numbers of opposite sign. Recall that you add positive integers by moving to the right on the number line or by considering gains and that you add negative integers by moving to the left or by considering losses. Thus, you can easily see that these equations are correct:

\[\text{neg 3} + \text{pos 4} = \text{pos 1}\]

and

\[\text{pos 3} + \text{neg 4} = \text{neg 1},\]

which can be written more simply as

\[(-3) + 4 = 1\]

and

\[3 + (-4) = (-1),\]

which can be written also as

\[3 - 4 = (-1).\]

These can be drawn as follows.
EXERCISE 29-1A

Draw on the number line and use gains and losses to illustrate the following addition problems:

1. $5 + (-8)$
2. $(-2) + 7$
3. $(3 + (-2)) - 4$
4. $(-7) + 1[(-2) + (-4)] + 8!$
5. $(\text{pos} 2 + \text{neg} 4) + [(\text{neg} 1) + \text{pos} 1]$
6. $1[(\text{pos} 3 + \text{neg} 2) + \text{pos} 5] + (-\text{C})$

Remember to do what is required inside the innermost brackets first and then move to the outer brackets.

EXERCISE 29-1B

A pupil complains to you, his teacher, that he does not believe you can add negative integers to anything, because adding makes a number bigger. What would you tell this pupil?

29-2 Subtraction, using the new notation

You saw that subtraction of integers is closely related to subtraction of the counting numbers. Thus, if you consider the problem

$$\text{pos} 5 - \text{pos} 2 = \text{pos} 3,$$

it can be rewritten more simply, using counting numbers, as

$$5 - 2 = 3.$$

You learned also how to solve the problem, which in a previous section of this book remained unsolved, of subtracting a larger counting number from a smaller counting number, as in this example:

$$5 - 7 = \square$$

This problem can also be considered as a missing addend problem:

$$7 + \square = 5$$

It is impossible to solve this problem using only positive integers, but, using both positive and negative integers, you can say

$$\text{pos} 5 - \text{pos} 7 = \text{neg} 2,$$

which can be rewritten

$$5 - 7 = (-2).$$

Such a problem can be pictured as follows:
In the same way, it is easy to see that
\[ \text{neg } 3 - \text{pos } 5 = \text{neg } 8, \]
which can be rewritten
\[ (-3) - 5 = (-8) \]
and which can be shown, using your slide rule made from two rulers sliding one on the other, as in the following picture.

![Slide rule diagram](image)

It is harder to see how to subtract a negative number. This problem can best be understood by looking for a missing addend. Thus, you can write
\[ \text{pos } 3 - \text{neg } 6 = \square \]
so that
\[ \text{pos } 3 = \text{neg } 6 + \square. \]
Clearly, the result you want to put in the box is pos 9, because
\[ \text{pos } 3 = \text{neg } 6 + \text{pos } 9 \]
and, thus,
\[ \text{pos } 3 - \text{neg } 6 = \text{pos } 9. \]
Using the new notation, you can write
\[ 3 - (-6) = 9 \]
\[ 3 = (-6) + 9. \]

The approach remains the same in the case where both numbers in the subtraction problem are negative integers, as in the following example.
\[ (-4) - (-7) = 3, \]
which is a simplification of the expression
\[ \text{neg } 4 - \text{neg } 7 = \text{pos } 3. \]

You learned, moreover, that subtraction of one integer from another simply means addition of the opposite integer. This result made life much easier for you, so that you could write
\[ \text{neg } 4 - \text{neg } 7 = \text{neg } 4 + \text{pos } 7 = \text{pos } 3, \]
which could be written more simply as
\[ (-4) - (-7) = (-4) + 7 = 3. \]

This equation can be illustrated as follows, using your slide rule:
Because subtraction and addition are inverse operations, it is possible to subtract any number by adding its opposite, and to add any number by subtracting its opposite.

**EXERCISE 29-2A**

Simplify each of the following expressions and find the result.

1. \(4 - (-7)\)  
2. \((-5) - (-2)\)  
3. \((-3) - 14 + [(-3) - (-4)]\)  
4. \(\text{pos} \ 6 + (\text{neg} \ 3 - \text{neg} \ 2)\)  
5. \([- (\text{neg} \ 6) + \text{pos} \ 6] - [(-2) + 3]\)  
6. \(\text{neg} \ l[((-2) + (-3)) + [(-2) - \text{neg} \ 6]]\)

**EXERCISE 29-2B**

1. A pupil tells you his father has less than nothing in the bank, because the bank put on his account a service charge which was greater than his bank balance. Can you explain this situation for the class?
2. Two men argue as to who is better off: the one who received two gifts of £10 each or the one who had two debts of £10 each cancelled. Each started with the same amount of money. What do you think?
3. Make up similar word problems for the use of your primary school class.

29-3 Multiplication of integers

When you studied the addition of counting numbers, you met problems where the same number was added to itself several times. You found that such addition problems could be solved easily by using a new operation, called multiplication. Thus, you learned to replace

\[3 + 3 + 3 + 3 = 15\]

by the multiplication equation

\[5 \times 3 = 15.\]

You saw that such an equation could be pictured by an array of objects or by repeated motions on the number line, as follows:
You can do much the same thing with the integers. Repeated addition of positive integers is the same as the problem given above. Thus,

\[ \text{pos } 3 + \text{pos } 3 + \text{pos } 3 + \text{pos } 3 + \text{pos } 3 = \text{pos } 15 \]

can be written

\[ 5 \times \text{pos } 3 = \text{pos } 15. \]

In the same way you can do repeated addition of negative integers. In the following series of equations, the reasons for the successive steps should be clear to you.

\[ \text{neg } 4 + \text{neg } 4 + \text{neg } 4 = \text{neg } 12 \]
\[ 3 \times \text{neg } 4 = \text{neg } 12 \]
\[ 3 \times (-4) = (-12) \]

This problem can be pictured on the number line as follows:

\[ \text{numbers on the number line} \]

In this way, you can see that if you multiply successive integers by a positive integer, the answers form a pattern. Look at the following examples:

\[ 4 \times 3 = 12 \]
\[ 4 \times 2 = 8 \]
\[ 4 \times 1 = 4 \]
\[ 4 \times 0 = 0 \]
\[ 4 \times (-1) = (-4) \]
\[ 4 \times (-2) = (-8) \]
\[ 4 \times (-3) = (-12) \]

It will make things clearer both to you and to your pupils if you look at multiplication by integers in this way. There are many patterns in mathematics, and you and your pupils should look for these patterns.

A pattern like this one can help you understand multiplication by negative integers. Look at the following statements:

\[ 4 \times 3 = 12 \]
\[ 3 \times 3 = 9 \]
\[ 2 \times 3 = 6 \]
\[ 1 \times 3 = 3 \]
\[ 0 \times 3 = 0 \]

Each result is 3 less than the one preceding it. What do you think should come next? These products can be shown on the number line as follows:
It seems clear, doesn’t it, that the next products should be

\[
\begin{align*}
(-1) \times 3 &= (-3) \\
(-2) \times 3 &= (-6) \\
(-3) \times 3 &= (-9) \\
(-4) \times 3 &= (-12),
\end{align*}
\]

which can be shown on the number line as follows:

You should try to think what such a sequence of products might mean, starting from \(4 \times 3 = 12\) and going to \((-4) \times 3 = (-12)\). Imagine you are a shopkeeper and have a book in which you keep a record of your cash on hand, your earnings and your expenses. You record £3 in the book by writing 3 in the proper place. If you make 2 sales of £3 each, you write \(2 \times 3 = 6\) in the book, and for 3 sales of £3 each, you write \(3 \times 3 = 9\). As a shopkeeper, you must buy goods to keep up your stock. If you buy something for £3, you write \((-3)\) in the book, since you have spent your money. Two such purchases of £3 each would mean you must write \(2 \times (-3) = (-6)\).

Imagine now that a customer returns an item for which he paid £3. You must return his money and cancel the £3 entry you made in your book. You are back where you were before you sold that item. You have removed a £3 entry, and thus you write \((-3)\) in the book. If two customers each return a £3 item, you must repay them. You have moved back two steps, and thus you must write \((-2) \times 3 = (-6)\) in your book. In the same way, three items returned at £3 each mean you must write \((-3) \times 3 = (-9)\) in the book.

This gives you a real-life situation which can help you interpret the product of a negative number and a positive number, in that order. Let us continue the story of your life as a shopkeeper. In order to have enough stock in your shop, you must buy new goods. If you buy something for £3 you must write \((-3)\) in the book, since you have spent your money. Two such purchases of £3 each would mean you must write \(2 \times (-3) = (-6)\). Three £3 purchases would be written \(3 \times (-3) = (-9)\). No such purchases, of course, would be 0. Thus, you can make a sensible interpretation of these products.

\[
\begin{align*}
3 \times (-3) &= (-9) \\
2 \times (-3) &= (-6) \\
1 \times (-3) &= (-3) \\
0 \times (-3) &= 0.
\end{align*}
\]

What do you think comes next? Picturing the products on the number line is always useful in such a problem:
clearly, the next set of products should be

\[
\begin{align*}
(-1) \times (-3) &= 3, \\
(-2) \times (-3) &= 6, \\
(-3) \times (-3) &= 9.
\end{align*}
\]

We think of these products, using the same story of your life as a shopkeeper. Imagine that you made a £3 purchase for your store, but you found it was no good. You returned it and got your money back. You are now in the same position as before you bought the item. Thus, you write 3 in your book to cancel the (-3) from before. Suppose you returned two £3 purchases. Then you would write \((-2) \times (-3) = 6\) in your book, cancelling the two £3 purchases, which you had recorded before as \(2 \times (-3) = (-6)\). In the same way, returning three £3 purchases would be written \((-3) \times (-3) = 9\).

Look now at all the cases we have considered in this section. They can be summarized as follows:

\[
\begin{align*}
2 \times 3 &= 6 \\
2 \times (-3) &= (-6) \\
(-2) \times 3 &= (-6) \\
(-2) \times (-3) &= 6
\end{align*}
\]

You should be able to give a general rule by this time. But the general rule has to come after you have thought about these special cases, rather than before. If you try other cases, you find similar results, all of which lead you to want to make certain rules. What do you think they are?

Perhaps as a child you were taught to multiply negative and positive numbers by using rules about the sign of the product. You memorized these rules, without being one bit wiser about the integers. But if you look at all the products in these last pages, and try a few more for yourself, you will come to see that these rules must be true. Here they are:

- The product of two numbers of the same sign is positive.
- The product of two numbers of opposite sign is negative.

You know the rules, and you understand them now. But when you teach, be sure you lead your pupils to discover them, just as you have been led. Do not just give them the rules and make them memorize. If you do that, you might spoil whatever mathematical promise a child has.

Let it not be said of you as a teacher that you kept a child from learning by forcing him to remember that which he does not understand!

**EXERCISE 29-3A**

1. Perform the indicated operations in each of the following, and in at least one case give a detailed description, using the number line as well as gains and losses of money, of how you got the answer.
   a. \((-5) \times 2\)  
   b. \((-6) \times (-4)\)  
   c. \(3 \times (-7)\)  
   d. \((-4) \times 13 - [4 - (-3)]\)  
   e. \(1(-2) + |(-3) \times (4 - 1)|\)  
   f. \((8 - 14 \times |(-2) + 6|) \times ((-5) + |(-3) \times l(-4) + 8|)\)

Remember to do the work inside the innermost brackets first.

2. Outline a procedure which would help your pupils discover the rules given above for the multiplication of integers.
3. Suppose a pupil in your class suggested that it is easy to see that \((-3) \times 4 = (-12)\), giving as his reason that it must be the same as \(4 \times (-3)\), which he knows gives \((-12)\), through repeated addition. You, as his teacher, want to make the best use of his suggestion, because you know that he will learn better whatever he has discovered for himself. What property of multiplication of integers has the pupil used, and how would you explain to the rest of the class how he got his answer? Would this perhaps be as good a way for teaching multiplication by a negative integer as that given in the text? Remember that you still have to justify using this property of multiplication. Explain your answer.

4. Outline a procedure for explaining multiplication of integers, using arrays of stones which you increase and decrease according to the numbers given in problems.

### 29-4 Division of integers

You remember from a previous chapter that division can be considered in a number of different ways. You thought of it as sharing a collection of objects among several persons; as finding the number of sets of a given size that can be taken from a given set; as breaking up a segment on the number line into a certain number of equal portions; and as finding a missing factor in a multiplication equation. It should be clear to you, as you think about it, that you can think about division of integers in the same ways. The only complication is the question of “sign”, and this is not a serious complication. It merely means that you cannot interpret every problem in every one of these ways.

It is easy to see how to divide any integer by a positive integer. Thus, the problem

\[
4 ÷ 2 = \square
\]

you did before when you studied counting numbers. Moreover, the problem

\[
(-4) ÷ 2 = \square
\]

can also be thought of in some of the same ways as before. Assume that a debt of 4 shs must be paid by 2 boys. They agree to share the debt evenly, and so each pays 2 shs as his share of the debt, which can be shown this way:

\[
(-4) ÷ 2 = (-2)
\]

This problem can also be understood by using a multiplication equation with a missing factor. The 2 boys must each pay some number of shillings in order to make up the total debt of 4 shs. Thus, we can write

\[
2 \times \square = (-4),
\]

and the answer is clearly \((-2)\) because

\[
2 \times (-2) = (-4),
\]

and thus we can write

\[
(-4) ÷ 2 = (-2).
\]

It is a bit harder to understand division by a negative integer. Take the following problem, for example:

\[
4 ÷ (-2) = \square
\]

What could it possibly mean to divide 4 by \((-2)\)? It makes the problem easier to think of it as
a multiplication equation with a missing factor. Thus, in the above case we can write

\[ (-2) \times \square = 4. \]

You remember from before that multiplying by \((-2)\) means to take away 2 of whatever the second factor shows. You remember that if you cancel 2 purchases of £2 each, this means a positive gain of £4, since you have back in your possession money you had previously paid out. Thus, the answer to this problem is \((-2)\), because

\[ (-2) \times (-2) = 4. \]

So you can write

\[ 4 \div (-2) = (-2). \]

Problems such as this may be confusing—to the teacher as well as to the pupil—unless they are considered in terms of multiplication problems with a missing factor. If you do that, then neither you nor your pupils should get into trouble. Look at this next one, for instance:

\[ (-4) \div (-2) = \square \]

Rewrite it as a multiplication equation, thus getting the following:

\[ (-4) = \square \times (-2) \]

In this form the problem seems much easier. How many \((-2)\)s are necessary to make \((-4)\)? To put it in a practical situation, how many boys must pay 2 shs each in order to make up a loss of 4 shs? The answer is easy: 2 boys. Thus you get the result which you expected: 2 must go in the box to make the equation true.

\[ (-4) = 2 \times (-2) \]

Thus, we write

\[ (-4) \div (-2) = 2. \]

Now you can make a summary of the possibilities in division problems just as you did before for multiplication problems. You find four different cases, as follows:

\[ 4 \div 2 = 2 \]
\[ (-4) \div 2 = (-2) \]
\[ 4 \div (-2) = (-2) \]
\[ (-4) \div (-2) = 2 \]

Clearly the results are the same as in the case of multiplication. If you think about it, they have to be, because division is simply the inverse of multiplication. You can give the following rules, which you—and thus also your pupils—can discover just as we did above.

The quotient of two numbers of the same sign is positive.

AND

The quotient of two numbers of opposite sign is negative.

Once again, it is very important that you do not teach these rules to the pupils as mysterious, incomprehensible laws that they must simply memorize at the pain of failure. Lead them to discover them, just as you discovered them in this section. The way may seem tricky at times, but don't be discouraged, since it will pay you and your pupils great dividends. Good luck!
EXERCISE 29-4 A

1. Perform the indicated operations to get the answer in the following problems, and in
at least one case, give a practical situation in which the problem might have a meaning.

a. \((-6) - 3\)

b. \(12 \div (-2)\)

c. \((-15) \div (-5)\)

d. \(6 \div (-1)\)

e. \(|(-3) + 5| \times [2 + (-8)] + [(-2) + 2]|

f. \(14 \div [(-1) + 3] \times [(-3) \times 2] \div [4 + (-2)]|

g. \(|(-3) + [(-2) \times (-6)]| \div [(-4) \times (-1)] - (3 + 4)|

EXERCISE 29-4 B

1. A pupil tells you that in a multiplication or division pro
to forget about the signs and do the problem as he learned it for whole numbers, and then
count the number of negative signs in the problem. If the number is odd the answer is
negative, and otherwise the answer is positive. Is the pupil correct? Why? What if there
were addition and subtraction in the problem also?

2. Make up several word problems suitable for use in an elementary school class, showing
the basic principles of multiplication and division of integers.

EXERCISE 29-4 C

1. Do these problems, and see what pattern appears in the answers. Remember what you
learned before about inverse operations.

a. \([4 \times (-2)] \div (-2)\)

b. \([(2) \times 3)] \div 3\)

c. \([(-2) \times (-5)] \div (-5)\)

d. \([8 \times (-1)] \div (-1)\)

e. \([(-8) \div 4] \times 4\)

f. \([(15 \div (-3)] \times (-3)\)

g. \([(-12) \div (-2)] \times (-2)\)

h. \([1 \div (-1)] \times (-1)\)

With parentheses

\((-4) - 2 = (-2)\)

\(4 \div (-2) = (-2)\)

\((-5) \times 2 = (-10)\)

\(2 \times (-5) = (-10)\)

\((-4) - (-7) = 3\)

Without parentheses

\(-4 - 2 = -2\)

\(4 \div (-2) = -2\)

\(-5 \times 2 = -10\)

\(2 \times (-5) = -10\)

\(-4 - (-7) = 3\)
UNIT VI · The Rational Numbers

Chapter 30
NEGATIVE FRACTIONS

30-1 Division of integers

The problems we did in the last chapter when we divided one integer by another were easy to understand. We saw how to interpret the answers to all the problems, but we also made our work easier by only working on problems where the division had a whole-number answer. You might ask now, what about problems like the following?

\((-7) \div 3 = \square\)

We have not done such problems before, and now we must try to find a way of solving them. We know what \((-6) \div 3\) means, and we can give \(-2\) as the answer. What about \((-7) \div 3\)?

It is always a good idea in doing mathematical problems to look back to easier and somewhat similar problems which you have solved before. In this case, you should think back to problems of this kind which came up in the discussion of the counting numbers. There we faced the difficulty that the problem

\(7 \div 3 = \square\)

has no answer among the counting numbers. And you remember that we had to find a new kind of number to solve this problem. That number was called a fraction, and was used to name parts of a whole. We found that these new numbers gave us answers to all such otherwise unsolvable division problems.

We will find in the following sections of this chapter that no really new problems arise in dividing one integer by another. If we use what we have already learned about fractions and about integers, we will find that the answer is right in front of us. This is what mathematicians always do when they try to solve a new problem. They look at similar problems they have done before, and if they are lucky, the answer will be there.

EXERCISE 30-1A

Which of the following problems can be solved using only integers, and which require new numbers?

1. \(18 \div (-3)\)
2. \(16 \div (-5)\)
3. \((-9) \div (-3)\)
4. \(2 \div (-2)\)
5. \(17 \div (-3 - 5)\)
6. \([-8 + (-2) \times 3] \div [3 \times (-3) - (-2)]\)
7. \([(-8) \div (-2 - 4)] - [3 \div (-3) \times 2]\)
8. \([(7) \times (-5) - 35] \div [(-16) - (4 \times (-2))]\)
EXERCISE 30-1B

When you tell your class that there are some numbers they can't yet divide, a pupil tells you that with a sharp enough knife he can divide anything. What do you think you should say to this pupil?

30-2 Division as multiplication with a missing factor

When you tried to solve such problems as \(7 \div 3\), you found that it helped to change them into multiplication equations with missing factors. Thus, you wrote

\[3 \times \square = 7,\]

and you found the fraction \(\frac{7}{3}\) to be the answer. This fraction is, of course, still the answer when you think of the counting numbers as positive integers. But the problem is not so easy when you have negative integers. For example, take the division problem

\[(-7) \div 3 = \square,\]

which gives rise to the multiplication equation with missing factor

\[3 \times \square = -7.\]

You can guess what the answer to this problem ought to look like if you remember how you solved the two problems

\[3 \times \square = 6\]

and

\[3 \times \square = -6.\]

In the one case you can put \(6 \div 3 = \frac{6}{3} = 2\) in the box, because

\[3 \times \left(\frac{6}{3}\right) = 6,\]

and in the other case you would want to put \((-6) \div 3 = \frac{-6}{3} = -2\) in the box, because you would feel that

\[3 \times \left[\frac{(-6)}{3}\right] = -6.\]

Thus, if the answer to the problem \(7 \div 3 = \square\) is

\[7 \div 3 = \frac{7}{3},\]

then the answer to the problem \((-7) \div 3 = \square\) ought to appear in the form

\[(-7) \div 3 = \frac{(-7)}{3},\]

so that

\[3 \times \left[\frac{(-7)}{3}\right] = -7.\]

In the same way we can think of the other two cases which might come up, as in these examples:

\[(-3) \times \square = 7\]
\[(-3) \times \square = -7\]
As in the previous case, we can solve the equations

\[ (-3) \times \boxed{} = 6 \]
and
\[ (-3) \times \boxed{} = -6, \]
and obtain

\[ 6 \div (-3) = \frac{6}{(-3)} = -2 \]
\[ (-6) \div (-3) = \frac{(-6)}{(-3)} = 2. \]

Thus, the answer to our new problems ought to be given in the same way, as

\[ 7 \div (-3) = \frac{7}{(-3)} \]
and
\[ (-7) \div (-3) = \frac{(-7)}{(-3)}. \]

We have used the phrases "ought to be" and "ought to appear". You might ask what kind of mathematics that is. Mathematics, you might say, should tell you what is, not what ought to be. But there is a reason for what we have done. All you have seen so far is a juggling of numerals. We showed you some problems you have already done, and then showed you some more problems you have not done. And we asked you to guess only what the answer might look like. In the next section of this chapter, however, you will see that what has been done here does have a real meaning, after all, and that these numbers, like \( \frac{(-7)}{3} \), refer to real-life situations.

**EXERCISE 30-2A**

Look back at those questions in Exercise 30-1A which did not have integers as answers, and tell what the answers "ought to be".

**EXERCISE 30-2B**

Make up questions like those in Exercise 30-1A which "ought to give" the following answers.

1. \( \frac{15}{8} \)
2. \( \frac{(-7)}{2} \)
3. \( \frac{(-8)}{(-4)} \)
4. \( \frac{12}{(-9)} \)
5. \( \frac{0}{(-1)} \)
6. \( \frac{(-5)}{5} \)

**30-3 Interpreting our new numbers on the number line**

The problem that faces us now is to decide what a number like \( \frac{(-7)}{3} \) might mean. We know what \(-7\) is, and we know how to interpret it. For example, we can place it on the number line as follows:
But if we remember that \( \frac{7}{3} \) is \( \frac{1}{3} \) of the way from 0 to 7, then perhaps we can think of \( \frac{-7}{3} \) as \( \frac{1}{3} \) of the way from 0 to \(-7\). Is there a point which is \( \frac{1}{3} \) of the way from 0 to \(-7\)? Of course there is. We just have to take that segment of the number line and break it into three equal parts, and mark the first such point to the left of 0 as \( \frac{1}{3} \) of the way from 0 to \(-7\). On the number line it looks like this:

\[
\begin{array}{cccccccc}
-8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
-7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
\end{array}
\]

It is just as good a point as \( \frac{7}{3} \), and thus \( \frac{-7}{3} \) deserves its place in our collection of numbers.

Clearly this number \( \frac{-7}{3} \) is opposite to \( \frac{7}{3} \), which is a point to the right of 0 on the number line. In fact, every point to the right of 0 has an opposite point to the left of 0. There is no reason why we should have opposites only for the integers. We can write this opposite as \( -\frac{7}{3} \), and we can thus see that \( -\frac{7}{3} \) and \( \frac{-7}{3} \) should mean the same thing. The numbers 7 and \(-7\) are opposites, and thus the numbers \( \frac{7}{3} \) and \( \frac{-7}{3} \), which show \( \frac{1}{3} \) part of that which is represented by 7 and \(-7\) respectively, must be opposites. But the opposite of any number \( x \) is written \(-x\).

Thus the opposite of \( \frac{7}{3} \) is \( -\frac{7}{3} \), and \( \frac{-7}{3} \) and \( -\frac{7}{3} \) are two names for the same number. Thus, we can begin to fill in the number line in the following way:

\[
\begin{array}{cccccccc}
-8 & -7 & -6 & -5 & -4 & -3 & -2 & -1 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
-3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
\end{array}
\]

This picture shows that each fraction to the right of 0 has its opposite to the left of 0, and thus we want to call these new numbers *negative fractions*.

The only remaining problems concern \( \frac{7}{3} (-3) \) and \( \frac{-7}{3} (-3) \). Before trying to give an answer, we should think where these symbols, which "ought to" represent numbers, came from. The problem \( (-3) \times \square = 7 \) gave rise to the possible answer \( \frac{7}{(-3)} \), and thus we should think what the problem \( (-3) \times \square = 7 \) means. This problem is similar to the problem

\[
(-3) \times \square = 6.
\]

We know the answer to this familiar problem, namely, \(-2\). Think back to the problems which arose when you imagined yourself to be a shopkeeper. To multiply \( (-3) \times (-2) \) meant to return three items which you had purchased for £2 apiece. You went back to your previous financial position, and thus gained £6. In the problem \( (-3) \times \square = 7 \), you are returning 3 items for a total of £7. Thus, for each, you receive back £\( \frac{7}{3} \), and you record in your book...
\((-3) \times \left( -\frac{7}{3} \right) = 7.\)

Thus, the result which was suggested by the equation, namely \(\frac{7}{(-3)}\), must have the same meaning as \(-\frac{7}{3}\). We cannot give a direct real-life meaning to \(\frac{7}{(-3)}\) as it is written, but we know it is another name for \(-\frac{7}{3}\), which we understand to be the fraction opposite to \(\frac{7}{3}\), or \(\frac{1}{3}\) part of \(-7\).

Finally, we have to think about \(\left( -\frac{7}{3} \right) \). Again we should look at the multiplication equation
\[(-3) \times \square = -7.\]
You already know how to do the very similar problem
\[(-3) \times \square = -6\]
which has as its answer
\[2 = (-6) \div (-3).\]
Thus, in the original problem we see that
\[(-3) \times \frac{2}{3} = -7,\]
so that
\[(-7) \div (-3) = \left( -\frac{7}{3} \right) = \frac{7}{3}.\]

To consider your shop once again, this means that you record in your book that you refunded the money for 3 items which were returned to your shop and which you had originally sold for £\(\frac{7}{3}\) apiece.

Thus, we can see that the answer suggested, namely \(\left( -\frac{7}{3} \right) \), has the same meaning as \(\frac{7}{3}\).

Once again, we cannot give a direct real-life meaning to \(\left( -\frac{7}{3} \right) \) as it is written, but we know it is another name for \(\frac{7}{3}\), with which we are already familiar.

**EXERCISE 30-3A**

Locate each of the fractions given in Exercise 30-2B on the number line, and give a practical meaning for each of them.

**EXERCISE 30-3B**

If you look back through this section, you will find that there are three ways of writing a given negative fraction and two ways of writing a given positive fraction. Look for each of these ways, and write the fractions in Exercise 30-2B in each of the ways possible for it. What special fact do you notice in Question 5 of that exercise?
EXERCISE 30-3C

In Chapter 25, some situations were given in which numbers on both sides of 0 were given physical meanings. Read that chapter over again, and state for each of those situations what negative fractions mean.

30-4 Interpreting our new numbers with gains and losses

You know now that each of these numbers which we invented to fill a box in a multiplication equation actually has a real-life meaning. Thus, they are not merely the result of juggling with numerals. And, if you apply what you learned in the last section, you can think of them in other ways as well.

We showed, before, that a negative integer represented the loss of some whole number of objects, if we were thinking in terms of sets. But there is no reason why we must gain or lose only whole numbers. We have found fractions on both sides of 0 on the number line, and clearly there are situations in life where these fractions make sense. Think of the case where three men who live in a village must jointly pay a tax of £4. They would share the debt, and thus each man's share would be

\[ (-4) \div 3 = \frac{-4}{3} = -\frac{4}{3}. \]

This is clearly the opposite case to that where the three men share a £4 gift which someone has given them, which would be written as

\[ 4 \div 3 = \frac{4}{3}. \]

If the one makes sense, so does the other.

EXERCISE 30-4A

Five men receive a gift of £7 through the mail. They share the money together. Then one leaves for a foreign country and cannot be reached, and a letter arrives stating that the gift was a mistake, and they must return the whole £7. Thus, the four remaining men divide the debt among them and pay it. Write equations showing each step of this process, being careful to use the correct sign at every point. Give a number to show each man's final gain or loss.

EXERCISE 30-4B

Draw up a series of test questions designed to find out whether your pupils understand positive and negative fractions and the relations between them.
**EXERCISE 30-4C**

Find the opposite of each of the following fractions. Put your answer in "simplest" terms.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>( \frac{7}{3} )</td>
<td>2.</td>
</tr>
<tr>
<td>3.</td>
<td>( \frac{-5}{4} )</td>
<td>4.</td>
</tr>
<tr>
<td>5.</td>
<td>( \frac{-2}{3} )</td>
<td>6.</td>
</tr>
<tr>
<td>7.</td>
<td>( \frac{-1}{(-4)} )</td>
<td>8.</td>
</tr>
</tbody>
</table>
Chapter 31  
THE SET OF RATIONAL NUMBERS

31-1 Fractions, positive and negative

We should stop briefly now and look at what we have done. You learned about fractions in an earlier chapter, and now you have found that these fractions have opposites. And if you look at the number line, you can see now that it is, or at least seems to be, as crowded as you can make it. Actually, you will find later that there are more numbers to be put on the number line. But let us be content for now with what we have, and try to understand these numbers fully.

We call the fractions we originally had positive fractions, and we call our new numbers negative fractions. There are many ways of writing any given fraction, of course, just as there are many ways of writing every number. Take the positive fraction \( \frac{3}{5} \) and its opposite, the negative fraction \( -\frac{3}{5} \), for example. Here are some other ways of writing each of these fractions:

\[
\begin{align*}
\frac{3}{5} &= \frac{6}{10} = \frac{150}{250} = \frac{\left(-\frac{3}{5}\right)}{\left(-\frac{120}{5}\right)} \\
-\frac{3}{5} &= \frac{-3}{5} = \frac{-90}{150} = \frac{-\frac{33}{5}}{\frac{3000}{5000}}
\end{align*}
\]

Do you remember how you could prove that all these fractions are names for the same number? If not, check back to the chapter on fractions and find the secret. And remember also that one minus sign in the fraction makes the whole fraction a negative fraction, and two make it a positive fraction. In every case, the simplest way to write a fraction is the first way given in each of the series of equal fractions above. Write it as a positive fraction, without minus signs, or as the opposite of such a positive fraction. We will call this the standard form, for example, \( \frac{2}{3} \) and \( -\frac{2}{3} \). Of course, the fraction may arise in a problem in one of the other forms. If that happens, it is useful to reduce numerator and denominator to the smallest whole numbers possible, using the secret you found in a previous chapter. Then put the minus sign, if there remains one, in front.

If you write a fraction in standard form, it is easy to see where it belongs on the number line. All positive fractions are, of course, to the right of 0, where they represent gains, or steps in the forward direction or increases. Negative fractions, on the other hand, are to the left of 0, where they represent losses, or steps in the backward direction or decreases. Every fraction thus has a place on the number line, and although not every point on the number line
indicates a fraction (We will say more about that later!), there is always a fraction as close as you would like to every point on the number line.

You may be thinking at this stage that there are points on the number line which are not fractions, points you already know about. You may say that we have not mentioned the counting numbers and their opposites and 0. After a whole series of chapters in which you studied those numbers, which you learned to call the integers, we seem to have forgotten them again. That is a good question—but if you think about it, you will see that the integers are still here and that they can be written as fractions also. If you look back to Chapter 20, you will see that the counting numbers were written as special fractions, as in the following example:

\[
\frac{6}{3} = \frac{2}{1} = 2
\]

The fraction which has 1 as its denominator or whose numerator is equal to the product of the denominator with some counting number was shown to be simply another name for one of the counting numbers. (That is true, of course, because to divide a thing into one part or into some number of equal parts, each of which contains a whole number of members, is the same as division of whole numbers where there is no remainder.)

The same fact is true for negative fractions as well. It is easy to see, for instance, that

\[
\frac{-6}{3} = \frac{-2}{1} = -2.
\]

If -6 is broken into 3 equal parts, each will contain -2. For a practical example, think of a debt of £6 shared among 3 people. Clearly, each will pay £2. More complicated examples can be worked in the same way, always remembering first to put the fraction in standard form. Thus, we have

\[
-\frac{(-10)}{(-2)} = -\frac{10}{2} = -5.
\]

Thus, the fractions, positive and negative, include all the numbers we have used up to this point and can even be extended to include 0, which can be written, for instance, as

\[
0 = \frac{0}{7}.
\]

These numbers are an interesting set, and we will think much more about their properties in the following sections of this chapter, as well as in the next few chapters.

**EXERCISE 31-1A**

Find each fraction in standard form, and four other fractions equal to each of the following. Locate each on the number line.

1. \(\frac{-32}{14}\)  
2. \(\frac{7}{5}\)

3. \(\frac{-20}{17}\)  
4. \(-\frac{13}{-13}\)

5. \(\frac{0}{-(-11)}\)  
6. \(-15.3\)

Remember to do the work within brackets first. Thus, for example,
EXERCISE 31-1B

1. Outline a classroom procedure for teaching that it is often important and useful to reduce fractions to standard form and lowest terms when working with them. Your procedure should show them how to make this reduction.

2. Prepare word problems which require students to make use of the fact that a given fraction can be named in several ways. Include both positive and negative fractions.

3. Prepare a classroom demonstration designed to show that some fractions, for example, \(\frac{4}{2}\) or \(-\frac{6}{2}\), are simply other names for positive or negative integers.

31-2 Definition of the rational numbers

The set of numbers which is made up of the positive fractions, zero and the negative fractions has a special name. We call it the set of RATIONAL NUMBERS. You can think of the set of rational numbers as a large set with three subsets, namely, the positive fractions (which include the positive integers), zero, and the negative fractions (which include the negative integers), as in the following diagram.

![Diagram showing subsets of rational numbers]

Each of the two large subsets of fractions thus can be thought of as containing a subset. The positive fractions contain the positive integers, which are the same as the counting numbers, as a subset. And the negative fractions contain the negative integers as a subset. Thus, our diagram can be completed in the following way.

![Completed diagram]

To summarize, the rational numbers contain the positive fractions, zero and the negative fractions. The positive and negative fractions and zero, together, contain the integers. And the integers contain the counting numbers, which is where we began this course in basic concepts of mathematics.

The rational numbers are not only a set of numbers—theY are an ordered set. The most obvious fact about their order is shown in the diagrams above. The negative fractions are all less than the positive fractions, and the rational numbers to the left of a number on the number line are all less than that number. In a later chapter, you will study the idea of order more closely, just as you will study all the other properties of the rational numbers. For now, it is enough simply to state that it is possible to determine of any two fractions which is greater and which is lesser.
For example, consider the two fractions $-\frac{3}{4}$ and $\frac{7}{4}$. This case is obvious. The first fraction is less than 0 and the second is greater than 0. In fact, any negative number $x$ is less than 0, not only because it is to the left of 0 on the number line, but because the difference $0 - x$ is positive. Look at this example on the number line.

\[
\begin{array}{cccccccccccc}
-2 & -\frac{7}{4} & -\frac{6}{4} & -\frac{5}{4} & -1 & -\frac{3}{4} & -\frac{2}{4} & -\frac{1}{4} & 0 & \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 & \frac{5}{4} & \frac{6}{4} & \frac{7}{4} & 2
\end{array}
\]

Take another example, $\frac{13}{5}$ and $\frac{17}{6}$. Which is lesser and which is greater? Here is one way in which you can tell. You can draw the number line, place both fractions there and compare them, as follows.

\[
\begin{array}{cccccccccccc}
2 & \frac{11}{5} & \frac{12}{5} & \frac{13}{5} & \frac{14}{5} & \frac{15}{5} & \frac{16}{5} & \frac{17}{5} & 3
\end{array}
\]

Obviously, $\frac{17}{5}$ is to the right of $\frac{13}{5}$ and is, thus, the greater of the two.

Take a final example: $-\frac{4}{3}$ and $-\frac{2}{3}$. You can show these on the number line as follows.

\[
\begin{array}{cccccccc}
-2 & -\frac{5}{3} & -\frac{4}{3} & -1 & -\frac{2}{3} & -\frac{1}{3} & 0
\end{array}
\]

Obviously $-\frac{4}{3}$ is to the left of $-\frac{2}{3}$, and thus,

$-\frac{4}{3} < -\frac{2}{3}$.

You remember, of course, the meaning of the two symbols "<" and ">". The first means "less than" and the second means "greater than".

**EXERCISE 31-2A**

Put the following set of fractions in order, from least to greatest, and show their positions on the number line. First simplify each and put it in standard form. Then compare them by pairs, and place each on the number line.

\[
\frac{4}{7}, \frac{-3}{13}, -2\cdot\frac{7}{6}, \frac{(-20)}{(-2)}, \frac{(-3)}{(-4)}, \frac{0}{(-4)}
\]
**EXERCISE 31-2B**

Draw a large poster useful for picturing the set of rational numbers and all the subsets which we have discussed, as on the previous page.

**NOTE.** We have used the word *fraction* to denote sometimes a number and sometimes the numeral which is its name. Perhaps, strictly speaking, we should not do this. However, it is hard to maintain the distinction in a consistent way, and it seems better not to insist on distinctions which we cannot keep up in practice. It should be clear from the context whether the word *fraction* means a number or a numeral.

The term *rational number*, of course, always denotes a number and not a numeral.
Chapter 32
OPERATIONS ON RATIONAL NUMBERS

32-1 Addition

It is not really necessary at this point to say anything new about the meanings of the operations. You should be quite familiar not only with their meanings but with how to teach them to young children. Very briefly, addition can be thought of in at least two ways: successive motions on the number line, to the right for positive numbers and to the left for negative numbers; and successive change of the number of objects in a set, increasing it for addition of positive numbers and decreasing it for addition of negative numbers. You have carried out a thorough and detailed study of this operation for counting numbers, for fractions and for integers. It clearly does not change the picture to include the whole set of rational numbers in this discussion. The only difficult thing at this point—both for you and for the pupils you teach—is to become quick at finding the correct answer when you are faced with a problem.

Let us look at an example, using only positive fractions, to remind ourselves of the methods we learned before. Take the problem

\[
\frac{3}{5} + \frac{7}{3}.
\]

The way you did such a problem before was to draw a picture showing parts of rectangles for each of the fractions, as follows:

![Diagram](3/5 + 7/3)

Then you divided each part into smaller pieces so that the resulting small pieces were of the same size. By counting the number of small pieces for each fraction, you could rename both fractions so that they had the same denominator. This was shown by dividing up each rectangle in the second direction, using the denominator of the other fraction to tell the number of pieces. For the example given above, the picture becomes as follows:
You count the number of small pieces and find the answer:
\[
\frac{9}{13} + \frac{35}{15} = \frac{44}{15}
\]

You learned in the study of fractions that there is a short way to carry through this process that saves you having to draw pictures and cut up rectangles. You discovered a rule which did the same thing for the general pair of fractions \( \frac{a}{b} \) and \( \frac{c}{d} \). The rule was stated as follows:
\[
\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}
\]

The question at hand is whether this rule works for all rational numbers, including negative fractions. Let us look at a couple of examples and then come back to the rule. But as you think about the examples, keep the rule in mind. First take this problem:
\[
\frac{4}{3} + \left( -\frac{2}{3} \right) = \frac{4}{3} + \left( -\frac{2}{3} \right)
\]

This addition can be shown on the number line as follows:

Clearly the answer is \( \frac{2}{3} \). If you think back to the problem of cutting bananas into parts, this problem speaks of 4 one-third pieces and tells you that 2 of them are removed, leaving 2 one-third pieces, or \( \frac{2}{3} \) of a banana.

This was an easy problem. Take a somewhat more difficult one; for example,
\[
-\frac{3}{5} + \frac{3}{2} = \frac{(-3)}{5} + \frac{3}{2}
\]

You can look at it on your slide rule in this way:
You set your 0 mark on the upper strip to the $-\frac{3}{5}$ mark on the lower. And you read your answer on the lower strip below the $\frac{3}{2}$ mark on the upper strip. That answer is, of course, $\frac{9}{10}$.

You can also think of this problem in terms of rectangles. The fraction $\frac{3}{2}$ can be drawn as follows:

![Diagram of rectangles](image)

To add $-\frac{3}{5}$ means to remove $\frac{3}{5}$ of a rectangle. Thus, you can redivide each rectangle into 5 parts in the usual way, and remove 3 of those parts. You then count the remaining pieces. You can picture it as follows:

![Diagram of rectangles](image)

You can see that 9 one-tenth pieces remain, so that the answer is $\frac{9}{10}$.

Where the answer is likely to be a negative fraction, it is not useful to think of dividing rectangles, but you can show such a problem on the number line or the slide rule. Think of the example

$$-\frac{3}{2} + \left(-\frac{5}{3}\right) = \frac{(-3)}{2} + \frac{(-5)}{3}.$$ 

Look at it on the number line.

![Number line diagram](image)

You can see that the result is $-\frac{19}{6}$ by counting to the left of 0, so that

$$-\frac{3}{2} + \left(-\frac{5}{3}\right) = \frac{(-3)}{2} + \frac{(-5)}{3} = -\frac{19}{6}.$$
Let us look for the pattern shown by these examples. In the first case, $\frac{4}{3} + \frac{(-2)}{3} = \frac{2}{3}$, the denominator 3 remained the same, and the resulting numerator was the sum of the two original numerators. In the second case, $\frac{(-3)}{5} + \frac{3}{2} = \frac{9}{10}$, the denominator of the result was 10. If you rewrite each of $\frac{(-3)}{5}$ and $\frac{3}{2}$ with denominator 10, you get $\frac{(-6)}{10}$ and $\frac{15}{10}$. Their sum is, of course, $\frac{9}{10}$. But $-6 = (-3) \times 2$ and $15 = 5 \times 3$. Thus, you get $\frac{(-3)}{5} + \frac{3}{2} = \frac{[(-3) \times 2] + [5 \times 3]}{5 \times 2} = \frac{9}{10}$.

In the same way, you get $\frac{(-3)}{2} + \frac{(-5)}{3} = \frac{(-9)}{6} + \frac{(-10)}{6} = \frac{[(-3) \times 3] + [2 \times (-5)]}{2 \times 3} = \frac{(-1)}{6}$.

But these are obviously examples of the rule for adding positive fractions which is extended to include negative fractions as well. If you think about it, you will see that this must always be true, so that the rule $\frac{a}{b} + \frac{c}{d} = \frac{(ad + bc)}{bd}$ must be true for all rational numbers.

Thus, the same rule you used for the positive fractions can be extended to include all fractions, both positive and negative. But don't forget the principle of good teaching: the pupils must learn the general rule only after they have tried out special examples. Then they can see that their own work fits the rule, and they will understand it and be able to use it much better. Never force anything on your pupils blindly and mechanically, because a bad mathematician is likely to be the product!

**EXERCISE 32-1A**

Find the sums of the following pairs of fractions, illustrating at least one of them using the number line, the slide rule and rectangles.

1. $\frac{4}{3} + \frac{(-3)}{5}$
2. $\frac{2}{(-3)} + \frac{(-6)}{(-9)}$
3. $-\frac{8}{5} + 1 \cdot 3$
4. $\frac{(-7)}{6} + \frac{0}{(-7)}$
5. $-\frac{3}{5} + \frac{6}{10}$
6. $\frac{(-8)}{(-4)} + \frac{21}{(-7)}$

**EXERCISE 32-1B**

What fractions suggest the use of money as examples in addition problems? What about fractions like $\frac{1}{3}$? What about fractions like $\frac{1}{5}$?
EXERCISE 32-1C

In Exercise 32-1A Question 2 presents an important special case. You should, of course, have found the answer 0. This means that the two numbers are opposites, and they cancel each other out. Such pairs of numbers, which have the sum 0, are called additive inverses. Outline a procedure for teaching the meaning of additive inverses to your pupils.

EXERCISE 32-1D

In Exercise 32-1A, Question 4 presents another important special case. The answer is $\frac{7}{6}$, the same as the first number in the sum. When you add 0 to any number, you do not change that number. The number 0 is called the additive identity element. Outline a procedure for teaching the meaning of the additive identity to your pupils.

32-2 Subtraction

You remember that subtraction is simply the inverse of addition. To subtract a positive number from some other number, you move to the left on the number line; and to subtract a negative number from some other number, you move to the right. Subtracting a positive number decreases the total, while subtracting a negative number is like removing a debt from your account books, thus increasing your total. You can think of subtraction in terms of finding the missing addend in an addition equation. And you can finally think of subtraction as addition of the opposite. You have done many exercises and read many pages on these interpretations of subtraction. Here you need only see that you can understand subtraction of any rational number from any rational number in the same way.

Not only can we understand the meaning of subtraction of rational numbers in terms of the fractions and the integers, but also we can see how to perform such subtraction problems by remembering what we did with integers. There we learned that subtraction of integers meant the addition of the opposite. Thus when you subtract $-5$ from 3, it is the same as adding 5 to 3. You write this as follows:

$$3 - (-5) = 3 + 5 = 8$$

Can you think of rational numbers in the same way? Look at this problem:

$$\frac{3}{5} - \left(-\frac{7}{3}\right) = \square$$

The answer is the missing addend in this equation:

$$\square + \left(-\frac{7}{3}\right) = \frac{3}{5}$$

Clearly, if you put $\frac{3}{5} + \frac{7}{3}$ in the box, the equation will be true. Thus, you get

$$\frac{3}{5} - \left(-\frac{7}{3}\right) = \frac{3}{5} + \frac{7}{3}$$
\[
\frac{(3 \times 3) + (5 \times 7)}{5 \times 3} = \frac{9 + 35}{15} = \frac{44}{15}.
\]

(Why?)

For another example, take the problem

\[
\frac{4}{3} - \frac{6}{5} = \square.
\]

The answer must be the solution to the equation \[
\square + \frac{6}{5} = \frac{4}{3},
\]
which is clearly

\[
\frac{4}{3} + \left(-\frac{6}{5}\right) = \frac{4}{3} + \left(-\frac{6}{5}\right) = \frac{[4 \times 5] + [3 \times (-6)]}{3 \times 5} = \frac{20 - 18}{15} = \frac{2}{15}.
\]

Once again you added the opposite to find the result, and you used the rule for addition of rational numbers to do so.

The procedure is always the same. Reduce the rational numbers to the simplest form, change the second of the pair into its opposite, and add. You understand this now—be don't forget this is not an easy trick to understand. When you teach it to your pupils, remember their difficulty, and help them in every possible way to understand it.

**EXERCISE 32-2A**

Perform the following subtractions, and for one of the problems show the meaning with the number line and the slide rule.

1. \[
\frac{3}{(-7)} - \left(-\frac{5}{3}\right)
\]

2. \[
\frac{(-6)}{(-3)} - \frac{2}{(-3)}
\]

3. \[
\frac{0}{9} - \frac{(-4)}{(-6)}
\]

4. \[
1.83 - \frac{47}{50}
\]

**EXERCISE 32-2B**

A pupil tells you that 0 is the identity element for subtraction too. Is he correct? Why? How would you explain it to your class?

**32-3 Multiplication**

You should have no difficulty at this point understanding multiplication of rational numbers, since it follows the same pattern as did the other operations. You remember both how to multiply integers and how to multiply fractions. And if you put this knowledge together, you will see how to multiply rational numbers.
In the first place, if you multiply two numbers of the same sign, your result will be positive, and if you multiply two numbers of opposite sign, your result will be negative. This is as true for fractions as it is for integers.

Think back to your experience as a shopkeeper. If you planned to sell an item for £3 but sold it for half price, you would write \( \frac{1}{2} \times 3 = \frac{3}{2} \) in your book. If a customer returned a £3 item and you refunded half this money, you would write \( \left(-\frac{1}{2}\right) \times 3 = -\frac{3}{2} \) in your book. If you bought a £3 item for half price, you would write \( \frac{1}{2} \times (-3) = -\frac{3}{2} \) in your book. And if you returned an item which you had purchased for £3 and got half the purchase price back, you would write \( \left(-\frac{1}{2}\right) \times (-3) = \frac{3}{2} \) in your book. The argument given in this example can be applied for any rational number times any rational number.

The actual result obtained for any product of rational numbers can be found by using the procedure you previously learned for fractions, and then using the rule about signs to find the correct sign for your answer. Take the two problems

\[
\frac{1}{2} \times \frac{4}{3} \quad \text{and} \quad \frac{1}{2} \times \left(-\frac{4}{3}\right).
\]

The results are clearly

\[
\frac{1}{2} \times \frac{4}{3} = \frac{4}{6} = \frac{2}{3},
\]

and

\[
\frac{1}{2} \times \left(-\frac{4}{3}\right) = -\frac{4}{6} = -\frac{2}{3}.
\]

These two results can be pictured on the number line as follows:

\[
\frac{1}{2} \times \left(-\frac{4}{3}\right) \quad \frac{1}{2} \times \frac{4}{3}
\]

In the same way, it is possible to obtain answers for the two problems which use the opposites to these fractions.

\[
\left(-\frac{1}{2}\right) \times \frac{4}{3} = -\frac{4}{6} = -\frac{2}{3},
\]

\[
\left(-\frac{1}{2}\right) \times \left(-\frac{4}{3}\right) = \frac{4}{6} = \frac{2}{3}.
\]

There are several important and interesting facts worth noting about multiplication of rational numbers. The first is that the rule for multiplication of fractions is applied in exactly the same form to rational numbers. You remember that rule:

\[
a \times \frac{c}{d} = \frac{ac}{bd}
\]
If you will look at the examples just given, you will see that each one of them fits this rule perfectly, both in terms of the sign and in terms of the numbers. Thus, for example, take the last case and apply the rule:

\[
\left( \frac{-1}{2} \right) \times \left( -\frac{4}{3} \right) = \frac{(-1) \times (-4)}{2 \times 3}
\]

\[
= \frac{4}{6} = \frac{2}{3}
\]

You need to change each of the rational numbers into an equivalent form and then perform the multiplication, but you get the same answer.

The second important fact concerns the number 0. You learned before that if you multiply any fraction by 0 the result is again 0. The same thing is true for rational numbers, as you can easily see. You know already that you can write 0 as a rational number, for instance, \(\frac{0}{1}\). Thus, you can write

\[
0 \times \frac{(-3)}{2} = \frac{0}{1} \times \frac{(-3)}{2}
\]

\[
= 0 \times \frac{(-3)}{1 \times 2} = 0.
\]

The number 0 has the property that its product with any rational number is 0.

The third important fact concerns the number 1. Again recall what you learned before. If you multiply any fraction by 1, the number remains the same. Clearly you can get the same result for rational numbers, since you can write \(\frac{1}{1}\). Thus, for example,

\[
1 \times \frac{(-3)}{2} = \frac{1}{1} \times \frac{(-3)}{2}
\]

\[
= \frac{1 \times (-3)}{1 \times 2} = \frac{(-3)}{2}.
\]

A fourth important fact concerns the product of a number with itself. This product is called the square of that number, and can be written as follows:

\[
\left( \frac{-2}{3} \right) \times \left( -\frac{2}{3} \right) = \left[ \frac{-2}{3} \right]^2
\]

with a 2 as a right-hand superscript indicating that the number is multiplied by itself. The square of any number \(a\) is \(a \times a\) and can be written \(a^2\). In the example, the result is clearly

\[
\left[ \frac{-2}{3} \right]^2 = \frac{4}{9},
\]

which is positive. In fact, the square of any rational number is positive. (If you have trouble seeing that fact, look at Exercise 32-3D.)

The fifth important point concerns what was previously called the reciprocal of a fraction. If you don't remember what a reciprocal is, look back to Section 21-19 in the chapter on fractions. Thus the reciprocal of the fraction \(\frac{2}{3}\) is the fraction \(\frac{3}{2}\), where the numerator of the first becomes the denominator of the second and vice versa. Question: Does the fraction
0 have a reciprocal? Further question: Is there such a fraction as \( \frac{3}{0} \)? The first question should be enough if you remember your earlier work, but the second question is put in to help your memory! Consider once again a fraction and its reciprocal and look at the product of \( \frac{2}{3} \) and \( \frac{3}{2} \):

\[
\frac{2}{3} \times \frac{3}{2} = \frac{6}{6} = 1
\]

You found earlier that it is always true that the product of a fraction and its reciprocal is the number 1. Is it always true for any rational number also (except, of course, 0)? Look at this example:

\[
\frac{-2}{3} \times \frac{3}{-2} = \frac{-6}{-6} = 1.
\]

In general, you can write, where neither \( a \) nor \( b \) is 0,

\[
\frac{a}{b} \times \frac{b}{a} = \frac{ab}{ba} = 1.
\]

Thus, it is always true that the product of a non-zero fraction and its reciprocal is the number 1. This should remind you of addition, where the sum of a number and its opposite is the number 0.

**EXERCISE 32-3A**

Find the following products of rational numbers and show at least one of them using the example of the shopkeeper and also the number line.

1. \( \frac{3}{2} \times \left( -\frac{1}{4} \right) \)
2. \( \frac{2}{(3)} \times 6 \)
3. \( \left[ \frac{2}{5} + \frac{-6}{8} \right] \times \left[ \frac{7}{4} + \frac{-3}{(4)} \right] \)
4. \( \left[ \frac{-5}{7} + 4 - \frac{3}{(8)} \right] \times \left[ \frac{4}{3} + \frac{1}{(2)} \right] \)
5. \( \left[ \frac{3}{4} - \frac{-6}{8} \right] \times \frac{2}{5} \)
6. \( \frac{(-6)}{5} \times \frac{(-5)}{6} \)

**EXERCISE 32-3B**

Give three examples of each of the following.

1. A number and its reciprocal
2. A rational number and its square
3. A number and its opposite

**EXERCISE 32-3C**

Find the squares of the following numbers, and verify that these squares are positive.

1. \( \frac{3}{5} \)
2. \( -\frac{4}{9} \)
3. \( \frac{(-1)}{2} \)
4. \( \frac{3}{(-2)} \)
The final topic in this chapter is division, which you remember is the inverse of multiplication. If you think of division in these terms, you should have no difficulty understanding how to divide numbers. In the first place, you learned that the same rule of signs applies as for multiplication, since division can be understood in terms of multiplication problems with missing factors. Thus, if you divide two rational numbers of the same sign, the result is positive, and if you divide two rational numbers of opposite sign, the result is negative.

In the second place, you learned to divide fractions by multiplying the first number by the reciprocal of the second, assuming, of course, that the second is not 0. This same rule also applies to rational numbers. Thus, for example, to divide \( \frac{3}{4} \) by \( \frac{1}{2} \) is to solve the problem

\[
\frac{3}{4} = \square \times \frac{1}{2},
\]

and the result is clearly

\[
\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \times 2 = \frac{6}{4} = \frac{3}{2}.
\]

Do you see that \( \frac{3}{4} = \frac{3}{2} \times \frac{1}{2} \)?

The same reasoning applies to the other possible variations in sign.

\[
\frac{3}{4} = \square \times \frac{(-1)}{2}
\]

\[
\frac{(-3)}{4} = \square \times \frac{1}{2}
\]

\[
\frac{(-3)}{4} = \square \times \frac{(-1)}{2}
\]

The results are clearly obtained the same way:

\[
\frac{3}{4} \div \frac{(-1)}{2} = \frac{3}{4} \times \frac{2}{(-1)} = \frac{6}{(-4)} = -\frac{3}{2}
\]

\[
\frac{(-3)}{4} \div \frac{1}{2} = \frac{(-3)}{4} \times \frac{2}{1} = \frac{(-6)}{4} = -\frac{3}{2}
\]

\[
\frac{(-3)}{4} \div \frac{(-1)}{2} = \frac{(-3)}{4} \times \frac{2}{(-1)} = \frac{(-6)}{(-4)} = \frac{3}{2}
\]

Only in the case where the second rational number is 0 is this procedure impossible. You remember, of course, that you cannot divide by 0—and if you don't remember why you can't divide by 0, look back to Section 12-8 in the first chapter on division.

**EXERCISE 32-4A**

Work out the following division problems. Try to understand at least one problem in terms of a physical situation.

1. \( \frac{(-3)}{4} \div \left[ \frac{2}{3} + \frac{(-3)}{10} \right] \)  
2. \( \frac{5}{8} \div \left( -\frac{5}{2} \right) \)
EXERCISE 32-4B

Prepare a set of revision exercises which will test your pupils’ understanding of the material in this chapter. Include both word problems and strictly numerical problems.

EXERCISE 32-4C

Prepare an examination which covers the material in this unit on the rational numbers and which would enable you to see how much time you should spend on revision.

EXERCISE 32-4D

If one of your fellow students told you that this chapter didn’t really teach him anything new but that he had learned it all before in earlier parts of the book, would you agree with him? Why?

EXERCISE 32-4E

Find answers to the following problems, and discuss the relation between multiplication and division which these answers show.

1. \[
\left[ \frac{2}{3} \div \left( -\frac{4}{5} \right) \right] \times \frac{(-4)}{5}
\]

2. \[
\left[ \left( -\frac{1}{5} \right) \div \frac{2 \times 4}{1-6} \right] \times \frac{8}{(-5)}
\]

3. \[
\left[ 1.7 \times \left( -\frac{4}{5} \right) \right] \div \left[ \left( -\frac{2}{3} \right)^2 \right]
\]

4. \[
\left[ (3 - 5) \times 1.1 \right] \div \frac{11}{10}
\]
Chapter 33

REVISION OF NUMBERS

33-1 Introduction

During this course, we have studied about numbers. We have learned about many different kinds of numbers. And we are not finished yet. Let us look back over the path we have travelled, and see what the important milestones we have passed have been.

33-2 Sets and counting numbers

We started our study by talking about sets of things. We observed that sometimes the members of two sets could be placed in one-to-one correspondence with each other. We called such sets equivalent. Sometimes it was not possible to place the members of one set in one-to-one correspondence with the members of another set. Such sets are not equivalent. We agreed to say that any two sets which were equivalent to each other had the same number of members, and that any two sets which were not equivalent had different numbers of members. We agreed that if a set A was a subset of a set B and not equivalent to B, then A had a smaller number of members than B and B had a larger number of members than A. So every set has a number of members which is equal to the number of members of every set equivalent to the given set.

These numbers associated with sets were called the counting numbers. Because they were ordered, we were able to picture them on a line by marking equally spaced points, and we agreed that if a number a was less than a number b, the point marked a was to the left of the point marked b.

We learned to add numbers by forming the union of two disjoint sets, whose members we had previously counted, and counting the members of this union.

\[
\begin{align*}
3 & \quad \cup \quad 4 \\
\quad & \quad = \quad 7
\end{align*}
\]

And we saw that our picture of the number line could be useful in this addition, since we could get the sum of two numbers by stepping along to the right of the number line.
33-3 The empty set and zero

The counting numbers were sufficient to take care of our number needs as long as the answers to our problems could be represented by sets of things. But if a man has a set of 3 shillings and pays 3 shs for a basket, the money he has left can hardly be described as a set of shillings. We found it convenient to describe the money he has left as the empty set. And then we invented the number 0 to describe the number of members of the empty set. We then had two sets of numbers: the set of counting numbers, and the set whose only member is zero. The union of these two sets we gave the name of whole numbers.

On the number line, the point labelled 0 was our starting point and was to the left of all the points for the counting numbers. And we found that we could still get the sum of two numbers by stepping to the right on the number line, with the understanding that if we were adding 0, we would step 0 units to the right; that is, we would not take a step.

The set of counting numbers is a subset of the set of whole numbers, and we found that all the problems involving counting numbers could still be done by working with the set of whole numbers; also, we could work problems involving the empty set. The extension from counting numbers to whole numbers was not a large one—the whole numbers include only one new member—but it is an important one.

33-4 Fractions

We learned to multiply the whole numbers, and interpreted multiplication as repeated addition. We then defined division in terms of multiplication. We said that if \( 3 \times \frac{1}{3} = 6 \), \( \frac{1}{3} \times 6 = 2 \), and more generally, if \( a \times \frac{1}{a} = b \), \( \frac{1}{a} = b \div a \). But sometimes there is no whole-number answer. For example, in the problem \( 3 \times \frac{1}{3} = 7 \), there is no whole-number solution. In such cases, we agreed to invent new numbers, called fractions, which would have the required properties. In the example \( 3 \times \frac{1}{3} = 7 \), \( \frac{1}{3} = 7 \div 3 \), and we agreed to call this answer the fraction \( \frac{7}{3} \). So \( 3 \times \frac{7}{3} = 7 \). More generally, if \( a \times \frac{1}{a} = b \), \( \frac{1}{a} = b \div a = \frac{b}{a} \), and \( a \times \frac{b}{a} = b \). These new numbers, the fractions, included the whole numbers. For if \( a \) is a whole number, \( 1 \times \frac{1}{a} = a \) has the solution \( \frac{1}{1} = a \). It also has the solution \( a \). So \( \frac{a}{1} = a \), and every whole number is a fraction.

The set of whole numbers is thus a subset of the set of fractions. Since some fractions are not whole numbers, the set of fractions is an extension of the set of whole numbers.

We saw that we could not assign a meaning to division by 0, and so we were not able to assign a meaning to the fraction \( \frac{a}{0} \) if \( b = 0 \). In the fraction \( \frac{a}{b} \), \( a \) can represent any member of the set of whole numbers and \( b \) any member of the set of counting numbers.

But we have also seen that not all such fractions are different. In fact, \( \frac{a}{b} = \frac{ka}{kb} \) for any counting number \( k \).

We learned to add any two fractions, obtaining \( \frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd} \). 

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We were able to assign an order to the fractions, and we were able to picture the fractions on the number line, with points between the whole numbers along with the whole numbers themselves. Again, $\frac{a}{b} < \frac{c}{d}$ if $\frac{a}{b}$ is to the left of $\frac{c}{d}$.

We learned to multiply any two fractions, and saw that $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. And we learned that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

### 33-5 Integers

Just as the process of division led to the extension of the whole numbers to the fractions, so the process of subtraction also led to an extension of the whole numbers to the integers. If $a$ represents a whole number, we defined a new number, called neg $a$, and later introduced the notation $-a$ to represent it, which had the property that $a + (-a) = 0$. Since $a + \square = 0$ has, by the definition of subtraction, the solution $\square = 0 - a$, $-a$ is the number obtained by subtracting $a$ from 0.

These new numbers were called the negative integers. The counting numbers were relabeled the positive integers. The complete set of integers is the union of the negative integers, the positive integers and zero.

We were able to establish an order for the integers. We were able to picture the integers on the number line by extending the number line to the left of 0. On the extended line, the smaller of two numbers was still pictured to the left of the larger. We were able to use the line for adding positive integers, as before, by stepping to the right. But we found that stepping to the left on the number line was required for adding negative integers. Since this was the same as subtracting positive integers, we concluded that $a + (-b) = a - b$.

Since negative integers may be added on the number line by stepping to the left of 0 in the same manner that positive integers may be added on the right of 0, we concluded that $(-a) + (-b) = -(a + b)$.

We saw that the opposite of a negative integer was a positive integer.

The statement $a + (-b) = a - b$ assures us that every subtraction problem of the form $a - b$ can be changed to an addition problem of the form $a + (-b)$.

We learned to multiply the integers, and discovered that $a \times (-b) = -(ab)$, $(-a) \times b = -(ab)$ and $(-a) \times (-b) = ab$.

### 33-6 Rational numbers

Just as the fractions permitted the naming of some of the points between the whole numbers to the right of 0 on the number line, so we needed to name points between the negative integers to the left of 0. These new numbers, the opposites of the non-zero fractions, we called the negative fractions, and, together with the fractions, they make the set of rational numbers.
33-7 Summary

We now have a very large set of numbers to work with—the set of rational numbers. In the next chapter, we shall revise the properties of operations on these numbers. We have seen that the set of rational numbers is composed of several important subsets. Sometimes we do not need this huge set of numbers when we are solving our problems—sometimes one of the subsets is big enough to take care of what we need to do. But it is comforting to know that with this large set of rational numbers, we can now perform any of the four basic operations on any numbers with just one exception: we are not permitted to divide by 0.

In Unit VII we are going to expand our number system again and for the last time in this course. But we shall still be unable to give meaning to division by 0.

EXERCISE 33-7A

List as many subsets of the set of rational numbers which we have studied and named as you can.
Chapter 34

REVISION OF PROPERTIES
OF OPERATIONS

In Chapter 13 we gave a summary of the properties of operations when the operations
were performed with whole numbers. Since that chapter, we have studied operations on numbers
other than whole numbers; for instance, on integers and rational numbers. We are now, there­
fore, in a position to expand our summary to include properties of operations on integers and
rational numbers.

34-1 Closure under addition and multiplication

A set of numbers is closed under addition if the sum of any two of the members is also a
member of the set. It is closed under multiplication if the product of any two of the members is
a member of the set.

1. Since \( a + b \) is a whole number if \( a \) and \( b \) are whole numbers, the set of whole num­
bers is closed under addition. This we know already.

Since \( a + b \) is an integer if \( a \) and \( b \) are integers, the set of integers is closed under
addition. For example, \((-8) + 12\) is an integer. Are the rational numbers closed under addition?
Is \( \frac{p}{q} + \frac{m}{n} \) a rational number whenever \( \frac{p}{q} \) and \( \frac{m}{n} \) are rational numbers? You see this is so,
because

\[
\frac{p}{q} + \frac{m}{n} = \frac{pn + qm}{qn} = \frac{\text{an integer}}{\text{another integer}} = \text{a rational number.}
\]

(Of course, we must be sure that the denominator \( qn \) cannot be zero. How do we know that \( q \neq 0 \)
and \( n \neq 0 \)? Could \( qn \) be \( 0 \)?)

2. Since \( a \times b \) is a whole number whenever \( a \) and \( b \) are whole numbers, the set of whole
numbers is closed under multiplication.

Since \( a \times b \) is an integer whenever \( a \) and \( b \) are integers, the set of integers is closed
under multiplication.

For example, \((-8) \times 12 = -96\), which is an integer.

Is the set of rational numbers closed under multiplication? Is \( \frac{p}{q} \times \frac{m}{n} \) a rational number if
\( \frac{p}{q} \) and \( \frac{m}{n} \) are? Yes, because

\[
\frac{p}{q} \times \frac{m}{n} = \frac{pm}{qn} = \frac{\text{an integer}}{\text{another integer}} = \text{a rational number, since } qn \neq 0.
\]

3. The set of whole numbers is not closed under subtraction, since, for example, \( 2 - 7 \) is
not a whole number.

The set of integers is closed under subtraction. So is the set of rational numbers. Can
you give examples illustrating this?
4. The set of whole numbers is not closed under division. For example, $2 \div 7$ is not a whole number.

Is the set of integers closed under division? Is $(-2) \div 7$ an integer, for example?

If we divide any rational number $\frac{p}{q}$ by any non-zero rational number $\frac{m}{n}$, will we always get a rational number? Let us see.

\[
\frac{p}{q} \div \frac{m}{n} = \frac{pn}{qm} = \text{an integer}
\]

Could $qm = 0$? Only if $q = 0$ or $m = 0$. But $q = 0$ is not allowed and if $\frac{m}{n} \neq 0$, $m$ cannot be 0 either. We conclude, then, that the set of all rational numbers except zero is closed under division.

**EXERCISE 34-1A**

1. State under which operations (addition, subtraction, multiplication, division), if any, each of the following sets is closed.
   a. $12, 4, 6, 8$
   b. The set of all rational numbers
   c. $112, 14, 16, 18, \ldots$
   d. $\{1, 2, 3, 4, 5, \ldots\}$
   e. The set of all integers
   f. The set of all odd numbers

**34-2 Commutative property of addition and multiplication**

1. The order in which two whole numbers are added will not affect their sum. That is, $a + b = b + a$ is true when $a$ and $b$ are whole numbers. Is it true when $a$ and $b$ are integers? Yes, it is. For example, $-8 + 3 = 3 + (-8)$, each side being equal to $-5$. This is not new. Is this true for rational numbers? Is it true that

\[
\frac{p}{q} + \frac{m}{n} = \frac{m}{n} + \frac{p}{q},
\]

that is, that

\[
\frac{pn + qm}{qn} = \frac{mq + np}{nq}?
\]

Remember that $p, q, m$ and $n$ are integers. (Which of them cannot be 0?) Then $qn = nq$, Why?

And $pn + qm = mq + np$. Why?

So we see that the commutative property of addition holds for rational numbers.

2. The order in which two whole numbers are multiplied does not affect the product. That is, $a \times b = b \times a$ when $a$ and $b$ are whole numbers. It is true also for integers, as it is for rational numbers. For example, $(-8) \times 3 = 3 \times (-8)$ since each side is equal to $-24$, and $\left(-\frac{1}{8}\right) \times \frac{2}{3} = \frac{2}{3} \times \left(-\frac{1}{8}\right)$ since each side is equal to $-\frac{1}{12}$. Can you show that $\frac{p}{q} \times \frac{m}{n} = \frac{m}{n} \times \frac{p}{q}$ in all cases?

We found that subtraction and division do not have the commutative property. Check these statements of subtraction and division to see that neither operation has this property.
34-3 Associative property of addition and multiplication

1. \((a + b) + c = a + (b + c)\) is true for whole numbers. Is it true for integers? Suppose you want to add \(-8, 7\) and \(-6\). How would you proceed? You could add \(-8\) and 7 and then add \(-6\) to the sum; or you could add to \(-8\) the sum of 7 and \(-6\). But you know you would get the same sum for each: \((-8 + 7) + (-6) = -8 + [7 + (-6)]\). Of course, you would do the same if the given numbers had been rational numbers. So \((a + b) + c = a + (b + c)\) if \(a, b,\) and \(c\) are rational numbers.

2. \(a \times (b \times c) = (a \times b) \times c\) is true for whole numbers. Is it true for integers? Is \((-8) \times (5 \times 11) = [(-8) \times 5] \times 11\)? Yes, since each side is equal to \(-440\).

Is it true for rational numbers? Is \(\frac{5}{8} \times \left(\frac{4}{3} \times \frac{2}{3}\right)\) the same as \(\left(\frac{5}{8} \times \frac{4}{3}\right) \times \frac{2}{3}\)? Yes, for each is equal to \(\frac{5}{6}\).

It is not difficult to show that for rational numbers,
\[
\left(\frac{p}{q} \times \frac{m}{n}\right) \times \frac{r}{s} = \frac{p}{q} \times \left(\frac{m}{n} \times \frac{r}{s}\right).
\]
In fact, each is equal to \(\frac{pmr}{qns}\).

Subtraction and division do not have this property as you can see by answering the questions below:

Is \((-8) - [3 - (-2)] = [(-8) - 3] - (-2)\)?

Is \(\frac{1}{4} - \left(\frac{3}{4} - \frac{1}{2}\right) = \left(\frac{1}{4} - 3\right) - \frac{1}{2}\)?

Is \(\left[\frac{9}{8} \times \left(-\frac{1}{2}\right)\right] \div 4 = \frac{9}{8} \div \left[\left(-\frac{1}{2}\right) \div 4\right]\)?

Is \(2 \div \left[\left(-\frac{1}{2}\right) \div \frac{1}{8}\right] = \left[2 \div \left(-\frac{1}{2}\right)\right] \div \frac{1}{8}\)?

34-4 Distributive property

1. \(a \times (b + c) = (a \times b) + (a \times c)\) is true when \(a, b\) and \(c\) are whole numbers. Is this same distributive property true for integers and rational numbers? Check these statements, which illustrate the distributive property of multiplication over addition:

\((-8) \times [5 + (-2)] = [(-8) \times 5] + [(-8) \times (-2)]\)

\[
\frac{2}{3} \times \left[\frac{5}{8} + \left(-\frac{1}{2}\right)\right] = \left(\frac{2}{3} \times \frac{5}{8}\right) + \left[\frac{2}{3} \times \left(-\frac{1}{2}\right)\right]
\]

In the first, each side equals \(-24\), so the statement is true. In the second, each side equals \(\frac{1}{12}\), so the statement is true.
2. \( a \times (b - c) = (a \times b) - (a \times c) \) is known to be true for whole numbers. Is it true for integers and rational numbers? Check these statements:

\[
\frac{2}{3} \times \left[ \frac{5}{8} - \left(-\frac{1}{2}\right) \right] = \left(\frac{2}{3} \times \frac{5}{8}\right) - \left(\frac{2}{3} \times \left(-\frac{1}{2}\right)\right)
\]

In the first, each side equals \(-56\), so the statement is true. In the second, each side equals \(\frac{3}{4}\), so this statement is also true.

Suppose the statement about multiplication and subtraction had been written in general form. We would like to show that

\[
\frac{p}{q} \times \left(\frac{m}{n} - \frac{r}{s}\right) = \left(\frac{p}{q} \times \frac{m}{n}\right) - \left(\frac{p}{q} \times \frac{r}{s}\right).
\]

If we work out both sides, we find that the left-hand and right-hand sides are, respectively,

\[
\frac{pms - pnr}{qns} \quad \text{and} \quad \frac{pmgs - qnpr}{quqs}.
\]

Do you see that these expressions represent the same rational number? Thus the statement is true. In a quite similar way, we could prove that the statement of the distributive property is true.

### 3.4.5 Properties of zero and one

We have learned that 0 has the property that \(0 + a = a + 0 = a\), when \(a\) is any whole number, and that 1 has the property that \(1 \times a = a \times 1 = a\), when \(a\) is any whole number.

Do 0 and 1 have these properties when \(a\) is any integer or any rational number? Yes, they do. We have met these operations when working on integers and rational numbers. For example,

\[
0 + (-8) = (-8) + 0 = -8,
\]

\[
0 + \frac{1}{2} = \frac{1}{2} + 0 = \frac{1}{2},\]

\[
1 \times (-8) = (-8) \times 1 = -8,
\]

\[
1 \times \frac{2}{3} = \frac{2}{3} \times 1 = \frac{2}{3}.
\]

In general,

\[
\frac{p}{q} + 0 = 0 + \frac{p}{q} = \frac{0 + p}{1 + q} = \frac{(0 \times q) + (1 \times p)}{1 \times q} = \frac{p}{q},
\]

and

\[
\frac{p}{q} \times 1 = 1 \times \frac{p}{q} = \frac{1 \times p}{1 \times q} = \frac{p}{q}.
\]

So 0 is called the identity element for addition, and 1 is called the identity element for multiplication. (Recall that we have seen that 0 is not an identity element for subtraction, nor is 1 an identity element for division.)

There is another important property of 0. We know that the product of any whole number and 0 is 0. Is this true when the other factor is an integer or a rational number? Are \(0 \times (-8) = 0\)
and \( 0 + \left( -\frac{1}{4} \right) = 0 \) true? From our study of integers and rational numbers, this statement is true: Any number multiplied by 0 will give 0 as answer.

This property of 0 may be written as follows: If \( a \) is any rational number, then

\[
a \times 0 = 0 \times a = 0.
\]

### 34-6 Opposites and reciprocals

Much work has been done with integers and rational numbers on the number line, and you should by now be familiar with such pairs of numbers on the number line as \(-3\) and 3, \(-\frac{1}{2}\) and \(\frac{1}{2}\), \(-a\) and \(a\). One number in each pair is to the right and the other to the left of the zero point—unless they are both 0—and both are at equal distances from it. The sum of the numbers in each pair is zero. We have called the numbers in such a pair opposites.

Given a rational number which is not 0, we can find its **reciprocal**. For example, the reciprocal of \(7\) is \(\frac{1}{7}\), of \(-\frac{8}{5}\) is \(-\frac{5}{8}\), of \(\frac{1}{9}\) is 9. The most important property which we discovered about a number and its reciprocal is that the product of the two is always 1. Always keep in mind that the number 0 has no reciprocal.

### 34-7 Inverse operations

In Chapters 9 and 12, it was mentioned that addition and subtraction are inverses of each other and that multiplication and division have a similar relation to each other. This can be shown to be true for integers and rational numbers as it was found to be true for whole numbers. Here are the statements expressing these relations:

\[
(a - b) + b = a \quad \text{and} \quad (a + b) - b = a
\]

if \(a\) and \(b\) are any rational numbers, and

\[
(a \times b) : b = a \quad \text{and} \quad (a : b) \times b = a
\]

if \(a\) and \(b\) are any rational numbers (with \(b \neq 0\)).

### 34-8 Properties summarized

In this summary, the letters \(a\), \(b\) and \(c\) represent any rational numbers. Since an integer is a particular kind of rational number, the properties listed will also apply to integers.

(a) **Closure properties**

- \(a + b\) is a rational number.
- \(a \times b\) is a rational number.

(b) **Commutative properties of addition and multiplication**

- \(a + b = b + a\).
- \(a \times b = b \times a\).
(c) **Associative properties of addition and multiplication**
\[ a + (b + c) = (a + b) + c. \]
\[ a \times (b \times c) = (a \times b) \times c. \]

(d) **Distributive properties**
\[ a \times (b + c) = (a \times b) + (a \times c). \]
\[ a \times (b - c) = (a \times b) - (a \times c). \]

(e) **Properties of zero**
\[ a + 0 = a. \]
\[ a \times 0 = 0. \]

(f) **Property of one**
\[ a \times 1 = a. \]

(g) **Opposites and reciprocals**
\[ a + (-a) = 0. \]
\[ a \times \frac{1}{a} = 1 \quad \text{(if } a \neq 0). \]

(h) **Inverse operations**
\[ (a - b) + b = a. \]
\[ (a + b) - b = a. \]
\[ (a \times b) \div b = a \quad \text{(if } b \neq 0). \]
\[ (a \div b) \times b = a \quad \text{(if } b \neq 0). \]
Chapter 35

ORDER FOR RATIONAL NUMBERS

35-1 Order on the number line for positive rational numbers

Do you remember how we described "order" for the set of whole numbers in Chapter 17 and "order" for fractions in Chapter 22? Let us see. Here is a number line.

You will remember that we said that a number \( a \) is "greater than" a number \( b \) if \( a \) is to the right of \( b \) on the number line. For example, 5 is greater than 3 and we see on the number line that 5 is to the right of 3. Similarly, \( \frac{11}{2} \) is to the right of 4 and so \( \frac{11}{2} \) is greater than 4.

In symbols, we write

\[ a > b, \quad 5 > 3, \quad \frac{11}{2} > 4. \]

In a similar way we described the idea of "less than" by saying that a number \( b \) is less than a number \( a \), if \( b \) is to the left of \( a \) on the number line. Thus 3 is less than 5, because it is to the left of 5 on the number line. Similarly 4 is to the left of \( \frac{11}{2} \) on the number line and so 4 is less than \( \frac{11}{2} \). In symbols, we write

\[ b < a, \quad 3 < 5, \quad 4 < \frac{11}{2}. \]

You see that \( a > b \) and \( b < a \) really mean exactly the same thing. Both inequalities say that \( a \) is to the right of \( b \) on the number line, and this is the same as saying that \( b \) is to the left of \( a \) on the number line.

**EXERCISE 35-1A**

1. Draw a number line and locate on it by dots the following:
   a. six consecutive whole numbers greater than 8
b. four even whole numbers less than 16
c. the three smallest fractions greater than \( \frac{3}{3} \) and having 3 for a denominator

2.
a. Find the smallest whole number greater than 6.
b. Find the greatest whole number less than 10.
c. Find the three largest and the three smallest whole numbers less than \( \frac{22}{3} \).

In Chapters 19 and 22, you learned to compare two fractions by putting them on the number line. It is not always easy to locate two fractions accurately on the number line when they are very close together. We therefore have to learn another way of finding out which of two fractions is the greater. Suppose we wish to find out which of \( \frac{2}{3} \) and \( \frac{3}{4} \) is the greater fraction without locating them on the number line. How shall we do it? One way would be to draw a diagram to represent each fraction and then compare the two diagrams. The two fractions \( \frac{2}{3} \) and \( \frac{3}{4} \) are shown in the diagrams below.

![Diagram showing \( \frac{2}{3} \) and \( \frac{3}{4} \)]

Clearly the second picture represents the larger fraction, and so we say that \( \frac{3}{4} \) is greater than \( \frac{2}{3} \); that is, \( \frac{3}{4} > \frac{2}{3} \).

Another method of comparing two fractions is to express both of them with the same denominator and then compare their numerators. Consider the above example where we compared \( \frac{2}{3} \) and \( \frac{3}{4} \). We may write \( \frac{2}{3} = \frac{8}{12} \) and \( \frac{3}{4} = \frac{9}{12} \). Hence, by comparing the numerators we see that \( 9 > 8 \) and, therefore, that \( \frac{3}{4} > \frac{2}{3} \). Of course we could also have arrived at the same result by taking the difference between \( \frac{3}{4} \) and \( \frac{2}{3} \). That is \( \frac{3}{4} - \frac{2}{3} = \frac{9}{12} - \frac{8}{12} = \frac{1}{12} \), which is positive.

**EXERCISE 35-1B**

1. For each pair of rational numbers, determine their order by
   (i) locating them on the number line;
   (ii) writing each pair with the same denominator;
   (iii) subtracting one from the other.
In each case, say which is larger.

a. \( \frac{4}{5} \) and \( \frac{8}{10} \)
   b. \( \frac{3}{5} \) and \( \frac{3}{4} \)
   c. \( \frac{9}{2} \) and 5
   d. \( \frac{5}{3} \) and \( \frac{4}{3} \)

2. Arrange the following rational numbers in order, starting with the least and ending with the greatest in each case.

a. \( \frac{2}{3}, \frac{1}{2}, \frac{3}{4}, \frac{2}{5} \)
   b. \( \frac{5}{3}, \frac{3}{2}, \frac{25}{3}, 3, \frac{11}{4} \)
   c. \( 3 \frac{1}{2} + 2 \frac{3}{4}; 2 \frac{1}{2} + 3 \frac{2}{3}; 1 \frac{1}{6} + 4 \frac{1}{2}; 5 \frac{3}{4} + \frac{1}{8} \)

Order for decimal fractions

When we wanted to find out which of two fractions was greater, we expressed both fractions as new fractions with the same denominator and then compared their numerators. A similar method is found useful with decimals. For example, 0.5 may be written as \( \frac{5}{10} \) and 0.05 as \( \frac{5}{100} \).

In order to write \( \frac{5}{10} \) as a fraction with 100 as a denominator, we have to multiply both numerator and denominator by 10. This gives \( \frac{5}{10} = \frac{5 \times 10}{10 \times 10} = \frac{50}{100} \). This means that \( \frac{50}{100} > \frac{5}{100} \); that is, 0.5 > 0.05.

Here is another example. Which is greater, 0.6 or 0.55? We write 0.6 = \( \frac{6}{10} \) and 0.55 as \( \frac{55}{100} \). Since 6 has 10 for denominator, we write \( \frac{6}{10} = \frac{6 \times 10}{10 \times 10} = \frac{60}{100} \). Hence, \( \frac{60}{100} > \frac{55}{100} \); that is 0.6 > 0.55.

EXERCISE 35.1C

1. Arrange the following numbers in order, starting with the least and ending with the greatest.
   0.35, 1.35, 3.5, 1.035, 10.35, 0.035, 17.5, 2.25, 2.

2. Insert the correct inequality sign in each box below.

   a. \( \frac{5}{2} \square 3.5 \)
   b. \( 1.7 \square 1.75 \)
   c. 0 \( \square 2.5 \)
   d. 3.8 \( \square 0.38 \)
   e. 0.75 \( \square 0.7 \)
   f. 0.75 \( \square 0.7 \)
   g. 2.5 \( \square 4 \)
   h. 5 \( \square 4.8 \)
   i. \( \frac{22}{5} \square 4.2 \)
35-2 Order on the number line for the set of rational numbers

Just as we did for whole numbers and for positive fractions, we use the term *is greater than* with rational numbers to mean "is to the right of" on the number line. If \( a \) and \( b \) are rational numbers, "\( a \) is greater than \( b \)" written \( a > b \), is interpreted to mean that \( a \) is "to the right of" \( b \) on the number line.

In a similar way, we use *is less than* for the set of all the rational numbers to mean "is to the left of" on the number line. If \( a \) and \( b \) are rational numbers, "\( b \) is less than \( a \)" is written \( b < a \).

We may restate the above by saying that if \( a > b \), then on the number line we have to move to the right from the point which represents \( b \) to get to the point which represents \( a \). This represents the addition of a positive number \( p \) to \( b \) to get \( a \). We may write,

\[
\text{if } a > b, \quad \text{then } a = b + p \quad \text{and} \quad a - b = p
\]

where \( p \) is a positive number.

In a similar way, if \( a < b \), then we have to move to the left from the point which represents \( b \) to the point which represents \( a \). That is,

\[
\text{if } a < b, \quad \text{then } a = b + q \quad \text{and} \quad a - b = q
\]

where \( q \) is a negative number.

Here is a number line with some rational numbers represented on it.

![Number Line]

We easily see that 3 is to the right of -1: 3 > -1 and 3 = -1 + 4. Similarly, -3·5 > -5 and -3·5 = -5 + 1·5, but \( \frac{7}{2} < \frac{11}{2} \) and \( \frac{7}{2} = \frac{11}{2} + (-2) \).

On a number line, all negative rational numbers are to the left of 0, and so for any negative number \( k \) we may write \( k < 0 \). Also, all positive rational numbers are to the right of 0, and so for any positive number \( m \) we may write \( m > 0 \).

**EXERCISE 35-2A**

1. Determine which of the following statements are true and which are false. For the true statements of the form \( a > b \), find a positive rational number \( p \) such that \( a = b + p \). For the true statements of the form \( a < b \), find a negative rational number \( q \) such that \( a = b + q \).

   a. \( 16 < -32 \)  
   b. \( -16 < 32 \)
   c. \( -1 < -3 \)  
   d. \( 1·6 < -3·2 \)
   e. \( -1·6 > 3·2 \)  
   f. \( 0 > 1 \)
   g. \( 25 < -26 \)  
   h. \( -25 < 26 \)
   i. \( 36 > 0 \)  
   j. \( -36 > 0 \)
   k. \( -3 > -6 \)  
   l. \( 0 < -\frac{3}{2} \)
   m. \( -\frac{3}{2} < 0 \)  
   n. \( -4 + (-2) < -2 + 2 \)
   o. \( -5 + 1 > 2 + 1 \)
2. Insert the correct inequality sign in the following statements.
   a. \(4 + (-3) \quad \square \quad -4 + 8\)
   b. \(4 + 8 \quad \square \quad -4 + (-8)\)
   c. \(6 + (-1) \quad \square \quad 6 + 2\)
   d. \(-\frac{3}{2} + \left(-\frac{5}{2}\right) \quad \square \quad -\frac{3}{2} + \frac{5}{2}\)
   e. \(-6 + (-1) \quad \square \quad -6 + 2\)
   f. \(-3.5 + 3.5 \quad \square \quad 3.5 + 3.5\)
   g. \(-\frac{1}{4} \quad \square \quad \left(-\frac{1}{3}\right)\)

3. For each pair of numbers, determine their order. Write a statement involving the sign > for each pair. Then for each statement \(a > b\), find a positive rational number \(p\) which makes the statement \(a = b + p\) true.
   a. \(\frac{4}{5} \quad \square \quad -\frac{8}{10}\)
   b. \(2.5 \quad \square \quad -5.5\)
   c. \(-\frac{3}{5} \quad \square \quad -\left(-\frac{3}{4}\right)\)
   d. \(-\frac{5}{3} \quad \square \quad -\frac{4}{3}\)
   e. \(-\frac{9}{2} \quad \square \quad -5\)
   f. \(2.25 \quad \square \quad 3.75\)

4. Locate on different number lines
   a. four negative integers greater than \(-6\),
   b. five negative integers less than \(-5\),
   c. the six greatest integers less than \(4\),
   d. four negative integers less than \(0\).

5. From each of the following statements about equality, deduce the corresponding statements about order, using first the sign < and then the sign >. For example, from \(-3 = -5 + 2\), we see first that \(-5 < -3\), and then that \(-3 > -5\).
   a. \(6 = -3 + 9\)
   b. \(-2 = -8 + 6\)
   c. \(0 = -4 + 4\)
   d. \(12 = 9 + 3\)
   e. \(3.25 = -3 + 6.25\)
   f. \(\frac{3}{4} = \frac{2}{3} + \frac{1}{12}\)
   g. \(-9 = -12 + 3\)
   h. \(2 = -8 + 10\)

35-3 "Between" for rational numbers

In Chapter 17 we saw that given any three whole numbers \(a\), \(b\) and \(c\), we can say which ones lies "between" the other two numbers. For example, on a number line we see that \(4\) is to the left of \(7\) and so \(4 < 7\); also, \(10\) is to the right of \(7\) and so \(10 > 7\). We can write \(4 < 7 < 10\).

What do we mean by "between" for rational numbers? A number line will help us. This number line shows that

\[\begin{array}{ccccccccccccc}
-7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
\hline
-3.5 & -1.5 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
\hline
\end{array}\]

\(-6 < -4\) and \(-4 < -2\). We therefore say that \(-4\) is between \(-6\) and \(-2\), and we write \(-6 < -4 < -2\). In a similar way, we see from the number line that \(\frac{5}{2}\) is between \(0\) and \(5\), and we write \(0 < \frac{5}{2} < 5\).
EXERCISE 35-3A

1. Arrange the rational numbers \( a, b, c \) and \( d \) on the number line, on the assumption that \( a \) is between \( b \) and \( c \), and \( b \) is between \( a \) and \( d \). Note that there are two possible arrangements.

2. Write the set whose members are
   a. the integers between -5 and 5,
   b. the negative integers greater than -4,
   c. the positive integers less than 7,
   d. the integers between 0 and -4,
   e. all the fractions between 1 and 3 with a denominator of 5.

3. Find a rational number between each of the following pairs of rational numbers.
   a. \( \frac{1}{3} \) and 1
   b. \( \frac{1}{3} \) and \(-1\)
   c. \( \frac{1}{5} \) and \( -\frac{1}{7} \)
   d. \( \frac{2}{5} \) and \( -\frac{1}{2} \)
   e. \( \frac{5}{2} \) and \( \frac{3}{4} \)
   f. \( \frac{9}{2} \) and \( -\frac{8}{3} \)

From your answers to Question 3, it will be clear to you that you can find more than one rational number between any pair of rational numbers. For example, what was the rational number you found between \( \frac{1}{3} \) and 1? One possibility is \( \frac{2}{3} \). What about \( \frac{3}{4} \) or \( \frac{1}{2} \) or \( \frac{3}{5} \)? Can you find any more rational numbers between \( \frac{1}{3} \) and 1?

We have seen that given any integer we can always tell which integer precedes it or comes after it. Given any three integers we can always tell which is between the other two. The example above seems to suggest that we cannot talk of the rational number between \( \frac{1}{3} \) and 1, because there are many such rational numbers.

Can we talk of the rational number which follows a rational number? For example, can we find the next rational number after \( \frac{1}{3} \)? Suppose we take \( \frac{3}{5} \), one of the answers we suggested above. We represent both these points on the number line.

```
0 1
\frac{1}{3}  \frac{2}{3}  \frac{11}{15}  \frac{6}{15}  \frac{3}{5}
```

We observe that \( \frac{1}{3} = \frac{5}{15} \) and \( \frac{3}{5} = \frac{9}{15} \), so that \( \frac{6}{15} \), \( \frac{7}{15} \), \( \frac{8}{15} \) are all closer to \( \frac{1}{3} \) than \( \frac{3}{5} \). If we choose \( \frac{6}{15} \) as the next rational number after \( \frac{1}{3} \), we note that \( \frac{1}{3} = \frac{10}{30} \) and \( \frac{6}{15} = \frac{12}{30} \) and so \( \frac{11}{30} \) is nearer to \( \frac{1}{3} \) than \( \frac{6}{15} \). Once more we observe that \( \frac{1}{3} = \frac{20}{60} \) and \( \frac{11}{30} = \frac{22}{60} \), so \( \frac{21}{60} \) is nearer \( \frac{1}{3} \) than \( \frac{11}{30} \). We may continue this process as long as we wish, so that there is no rational number "next" after a given rational number. A similar argument would show that we cannot identify the rational number "just" before a given...
rational number. All this suggests to us that between any two rational numbers, no matter how close together they are, there is always a third rational number.

**EXERCISE 35-3B**

Find four rational numbers between each pair of rational numbers in Question 3 of Exercise 35-3A.

**35-4 Two basic properties of order for rational numbers**

1. **The Comparison (or Trichotomy) Property**

   If \( a \) and \( b \) are any two rational numbers, then one and only one of the following is true:
   
   \[ a > b, \quad a = b, \quad a < b. \]

   This property is one which requires no proof. For, given any two integers, we can locate them on a number line and see whether one is to the right or left of the other. That is, we can say which is greater than the other. In the case \( a = b \), we have two different names for the same number. For example, \( 2 \times 5, 10, \frac{20}{2}, 9 + 1 \) are all different names for the same number.

**EXERCISE 35-4A**

Put the correct inequality or equality sign into the boxes to make each statement true.

1. \[ -\frac{6}{5} \square \frac{7}{2} \]
2. \[ -\frac{30}{10} \square \frac{(-9)}{3} \]
3. \[ 5 + \frac{1}{2} \square 5 - \frac{1}{2} \]
4. \[ \frac{2}{3} - \frac{1}{4} \square \frac{1}{12} + \frac{1}{6} \]
5. \[ -7 \square -\frac{14}{4} \]
6. \[ 0 \square -\frac{1}{-1} \]
7. \[ 8 \square 2 - (-6) \]
8. \[ -6 \square -\frac{19}{3} \]

2. **The Transitive Property**

   Let us look again at the relation \( 4 < 7 \) and \( 7 < 10 \), which we discussed earlier. What conclusion can we draw as to the relation between \( 4 \) and \( 10 \)? Do you see that \( 4 \) is less than \( 10 \)? You can verify easily that \( 4 < 10 \) by looking at the number line.

   Here is another example. We know that \(-6 < 2 \) and \( 2 < 6 \). Of course, you see straightaway that \(-6 < 6 \), and again you can check the conclusion by looking at the number line.

   As another example, we have \( -\frac{1}{2} < \frac{1}{4} \) and \( \frac{1}{4} < \frac{1}{2} \). We thus see that \(-\frac{1}{2} < \frac{1}{2} \).

   Now choose any three rational numbers \( a, b \) and \( c \) on the number line in such a way that \( a < b \) and \( b < c \). Find several sets of rational numbers which satisfy these two relations. What can we say as to the relation between \( a \) and \( c \)? You can see straightaway that \( a < c \). The property that we have just established we will call the transitive property of order. It states,
if \(a, b\) and \(c\) are any three rational numbers and if \(a < b\) and \(b < c\), then \(a < c\).

**EXERCISE 35-4B**

1. In each of the following groups of rational numbers, determine their order.
   
   a. \(2.5, -5.2, 0\)
   
   b. \(-\frac{1}{4}, -\frac{1}{2}, -\frac{1}{3}\)
   
   c. \(\frac{1}{4}, \frac{1}{2}, \frac{1}{3}\)
   
   d. \(25, 2.05, 2.25\)
   
   e. \(5, -6, 4\)
   
   f. \(\frac{4}{3}, -\frac{3}{4}, \frac{3}{4}\)
   
   g. \(-\frac{9}{5}, -2, -\frac{12}{5}\)

The transitive property often makes it easy for us to compare some pairs of fractions.

Suppose we wish to find out which of the two fractions \(\frac{79}{17}\) and \(\frac{97}{19}\) is the greater. By the method described earlier in this chapter, we first write the fractions with the same denominator and compare the numerators. Thus, \(\frac{79}{17} = \frac{1501}{323}\) and \(\frac{97}{19} = \frac{1649}{323}\). By comparing the numerators 1649 and 1501 we see that \(\frac{97}{19} > \frac{79}{17}\). You of course know how we arrived at this result. It has involved the finding of the three products \(19 \times 17, 97 \times 17\) and \(79 \times 19\), and you will agree that this is a lot of work.

Can the transitive property help us? If we can find a rational number which is between \(\frac{79}{17}\) and \(\frac{97}{19}\), then we can easily compare the two fractions. We note that \(\frac{79}{17} < \frac{85}{17}\); that is, \(\frac{79}{17} < 5\). Also \(\frac{97}{19} > \frac{95}{19}\); that is, \(\frac{97}{19} > 5\). If in the transitive property we take \(a = \frac{79}{17}\), \(b = 5\) and \(c = \frac{97}{19}\), we see that \(a < c\); that is, \(\frac{79}{17} < \frac{97}{19}\).

**EXERCISE 35-4C**

1. Determine the order of the following fractions.
   
   a. \(\frac{287}{29}\) and \(\frac{340}{31}\)
   
   b. \(-\frac{79}{17}\) and \(-\frac{97}{19}\)
   
   c. \(\frac{16}{3}\) and \(\frac{19}{5}\)
   
   d. \(\frac{111}{7}\) and \(\frac{190}{11}\)

2. Sometimes it is not so easy to find the order of two fractions by using the transitive property. Try to find the order of \(\frac{3}{7}\) and \(\frac{8}{19}\) by using the transitive property.

We shall now sketch a proof of the transitive property, if \(a, b\) and \(c\) are rational numbers and if \(a < b\) and \(b < c\), then \(a < c\).

It was shown in Section 35-2 that

\[a < b\] means \(a = b + q_1\) where \(q_1 < 0\);
also,

\[ b < c \text{ means } b = c + q_2, \text{ where } q_2 < 0. \]

Hence,

\[
a = b + q_1 = (c + q_2) + q_1, \text{ by writing } b = c + q_2
\]
\[
\because c + (q_2 + q_1), \text{ by the associative law of addition}
\]
\[
= c + q, \text{ where } q = q_2 + q_1.
\]

But we know that the sum of two negative numbers is another negative number, so \( q \) is a negative number.

Hence,

\[
a = c + q, \text{ where } q \text{ is a negative number,}
\]

which means that

\[
a < c.
\]

**EXERCISE 35-4D**

If \( a, \ b \) and \( c \) are rational numbers and if \( a > b \) and \( b > c \), then \( a > c \). Prove this property as above, giving a reason for each step in your proof.

**35-5 Addition property of order**

We have already considered the addition property of order for whole numbers and for fractions in Chapters 17 and 22. Let us now see whether the property is true for the set of all rational numbers.

It will help us to picture addition and order on the number line. Let us first choose two rational numbers \( a \) and \( b \) on the number line with \( a < b \). We remember that the addition of a positive rational number to both \( a \) and \( b \) means moving to the right, while the addition of a negative rational number means moving to the left.

If we now add the same positive rational number \( c \) to \( a \) and to \( b \), we see that the point for \( a + c \) is \( c \) units to the right of the point for \( a \) and the point for \( b + c \) is also \( c \) units to the right of the point for \( b \). The number line indicates that \( a + c < b + c \) (see Fig. 1, below).

If we add a negative rational number \( c \) to \( a \) and to \( b \), the points for \( a + c \) and \( b + c \) will each be the same distance to the left of the points \( a \) and \( b \) respectively. The number line still suggests that \( a + c < b + c \) (Fig. 2).

![Fig. 1](image_url)
Let us see if we can prove the addition property of order. Starting from \( a < b \), we know that \( a = b + q \), where \( q \) is some negative rational number. We add \( c \) to both sides of the equality obtaining \( a + c = (b + q) + c \), or \( a + c = (b + c) + q \). (Why?) But the last equality tells us that

\[
\begin{align*}
a + c &< b + c.
\end{align*}
\]

The order relation between the numbers is preserved. Hence, we have the addition property of order:

If \( a, b \) and \( c \) are rational numbers and if \( a < b \),

then

\[
a + c < b + c.
\]

**EXERCISE 35-5A**

1. Formulate an addition property of order for the relation "" and write out a proof of it.
2. Illustrate the truth of the addition property of order by taking \( b = -\frac{3}{2} \), \( a = -\frac{5}{2} \), and

with \( c \) having successively the values \(-\frac{3}{2}, 6, 0, -\frac{5}{2}, -4\).

**35-6 Generalized addition property of order**

In the addition property of order considered above, we saw that the addition of the same rational number to both sides of an inequality preserves the order relation. Naturally we may want to know whether an order relation is still preserved when the two rational numbers added to both sides are not equal but have the same order relation between them that the original rational numbers have.

Let us illustrate this with an example on the number line. We know that \( 2 < 6 \) and that \(-3 < -2\). What relation holds between \( 2 + (-3) \) and \( 6 + (-2) \)? Add 2 to \((-3)\); this gives \(-1\). The addition of 6 and \(-2\) gives 4.

What relation exists between \(-1\) and 4? We see that \(-1 < 4\). From this we may write,

\[
\begin{align*}
\text{if } & 2 < 6 \text{ and } -3 < -2, \\
\text{then } & 2 + (-3) < 6 + (-2).
\end{align*}
\]
That is,

\[-1 < 4,\]

which is true.

Now choose any four rational numbers \(a, b, c\) and \(d\) with \(a < b\) and \(c < d\). Add \(a\) to \(c\) and \(b\) to \(d\). Do this with two or more sets of rational numbers. Write in each case the relation which exists between \(a + c\) and \(b + d\). Your answers will lead you to the generalized addition property of order which states,

\[\text{if } a, b, c \text{ and } d \text{ are rational numbers such that } a < b \text{ and } c < d, \text{ then } a + c < b + d.\]

Let us now give a proof of this property.

If \(a < b\), then \(a + c < b + c\) (by the addition property of order).

If \(c < d\), then \(b + c < b + d\). (Why?)

Hence, using the transitive property of order, we have

\[a + c < b + d.\]

**EXERCISE 35-6A**

Write the generalized addition property of order for the relation "\(<\)". Try to make a proof for it.

35-7 Numbers and their opposites

The addition property of order often helps us to see the truth of some properties of numbers which are not at once apparent. One such property is the relation between two rational numbers and their opposites.

By considering a few numerical examples, we can easily see that if \(a\) and \(b\) are rational numbers and if \(a < b\), then \(-b < -a\). Try to convince yourself of the truth of this property by drawing a number line and by locating several pairs of rational numbers and their opposites on it.

Let us see now whether the addition property of order can help us to give a proof of the relation which exists between the opposites of a pair of rational numbers. Let \(a\) and \(b\) be rational numbers with \(a < b\). Then adding \(-a\) to both sides of the inequality, we have

\[a + (-a) < b + (-a)\] (by the addition property of order).

That is,

\[0 < b + (-a)\] (by the property of opposites).

Again, by adding \(-b\) to both sides of the inequality, we get

\[0 + (-b) < b + (-a) + (-b).\] (Why?)

So

\[-c < -a.\] (Why?)

Thus, the relation is proved.
35-8 Multiplication property of order

You will remember from your earlier study of the rational numbers that if \( a \) and \( b \) are positive rational numbers, then \( a \times b \) is a positive rational number and \( a \times (-b) \) is a negative rational number. (In fact, it is the opposite of \( a \times b \).) That is, the product of two positive rational numbers is always a positive rational number, while the product of a negative rational number and a positive rational number is always a negative rational number.

Consider the set of rational numbers \( \{ -\frac{3}{2}, -4, 3, \frac{5}{2}, 6 \} \) represented on the number line below. Now multiply each element

\[ \frac{-3}{2}, 5, 2, -4 \]

of the set by 2, a positive number. We obtain the set \( \{-3, -8, 6, 5, 12\} \). These numbers are represented on the number line below. On the new number line the points which

\[ -12, -11, -10, -9, -8, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7 \]

correspond to \(-4\) and \(-\frac{3}{2}\) are \(-8\) and \(-3\). The relation between \(-4\) and \(-\frac{3}{2}\) is \(-4 < -\frac{3}{2}\), while the relation between \(-8\) and \(-3\) is also \(-8 < -3\). That is, the order relation is preserved.

Similarly,

since \( \frac{5}{2} < 3 \), then \( 2 \times \frac{5}{2} < 2 \times 3 \);

since \( 3 < 6 \), then \( 2 \times 3 < 2 \times 6 \);

since \( -\frac{3}{2} < 3 \), then \( 2 \times (-\frac{3}{2}) < 2 \times 3 \).

These examples suggest that if \( a \) and \( b \) are rational numbers with \( a < b \), then if \( c \) is a positive whole number, \( a \times c < b \times c \).

Now let \( c = \frac{1}{2} \). Multiplying the set of rational numbers \( \{-\frac{3}{2}, -4, 3, \frac{5}{2}, 6\} \) by \( \frac{1}{2} \), which is a positive rational number, we get the set \( \{-\frac{3}{4}, -2, \frac{3}{2}, \frac{5}{4}, 3\} \). From the number line we see that \(-4 < -\frac{3}{2}\) and that \(-2 < -\frac{3}{4}\). That is, since \(-4 < -\frac{3}{2}\), then \( \frac{1}{2} \times (-4) < \frac{1}{2} \times (-\frac{3}{2}) \).

Similarly, since \(-4 < 6\), then \( \frac{1}{2} \times (-4) < \frac{1}{2} \times 6 \). This all suggests the multiplication property of order:

If \( a, b \) and \( c \) are rational numbers and if \( a < b \) and \( c \) is positive, then

\[ a \times c < b \times c. \]

A proof of this property will be given in the next section.
**EXERCISE 35-8A**

1. Complete the following statements using one of the symbols > or <.
   a. If $9 > 7$, then $9 \times 3 \_ _ _ _ 7 \times 3$.
   b. If $-5 < -3$, then $-5 \times 3 \_ _ _ _ -3 \times 3$.
   c. If $a > b$ and $c > 0$, then $a \times c \_ _ _ _ b \times c$.
   d. If $7 > -9$, then $7 \times 4 \_ _ _ _ -9 \times 4$.
   e. If $3 > 0$, then $3 \times 6 \_ _ _ _ 0 \times 6$.
   f. If $a < b$ and $c > 0$, then $a \times c \_ _ _ _ b \times c$.
   g. If $-5 < 5$, then $-5 \times 4 \_ _ _ _ 5 \times 4$.
   h. If $0 < 8$, then $0 \times 2 \_ _ _ _ 8 \times 2$.

**35-9 Proof of the multiplication property of order**

The property states that if $a$, $b$ and $c$ are rational numbers and if $a < b$ and $0 < c$, then $a \times c < b \times c$. If $a < b$, then $b = a + p$, where $p > 0$. Multiplying by $c$ we obtain

$$b \times c = (a + p) \times c = (a \times c) + (p \times c).$$

Now $p \times c$ is a positive rational number, because it is the product of two positive rational numbers.

Therefore

$$b \times c = (a \times c) + \text{positive number}.$$

Hence

$$a \times c < b \times c.$$

**EXERCISE 35-9A**

1. Prove that if $a > b$ and $c > 0$, then $a \times c > b \times c$.
2. Choose any three rational numbers $a$, $b$ and $c$ with $a < b$ and $c$ negative. Find the products $a \times c$ and $b \times c$. What is the relation between $a \times c$ and $b \times c$?
3. Repeat Question 2 with three different sets of numbers $a$, $b$ and $c$: (i) choose $a$ and $b$ to be positive and $c$ negative, (ii) choose $b$ to be positive and $a$ and $c$ negative, (iii) choose all three numbers negative. Write down the product $a \times c$ and $b \times c$ in each case and state the order relation between them.
4. Can you deduce a new multiplication property of order from the answers to Questions 2 and 3?

Your answers to Questions 2 and 3 will have shown you that if $a$, $b$ and $c$ are rational numbers and if $a < b$ and $c < 0$, then $a \times c > b \times c$.

**35-10 Generalized multiplication property of order**

It was established in Chapter 22 for positive fractions that if

$$\frac{a}{b} < \frac{c}{d} \text{ and } \frac{m}{n} < \frac{p}{q},$$

then

$$\frac{a}{b} \times \frac{m}{n} < \frac{c}{d} \times \frac{p}{q}.$$
This was proved by using the number line. This property is not true for rational numbers if we allow some of the four numbers to be negative. However, we do have the following property:

If \( a, b, c \) and \( d \) are rational numbers, with \( a < b \) and \( c < d \), and \( a, b, c \) and \( d \) are positive, then \( a \times c < b \times d \).

We shall give a proof which uses the multiplication property of order.

If \( 0 < a < b \) and \( 0 < c < d \),
then \( a \times c < b \times c \) and \( b \times c < b \times d \). (Why?)

Hence, \( a \times c < b \times d \).

**EXERCISE 35-10A**

1. Find three sets, each consisting of four positive rational numbers \( a, b, c \) and \( d \) satisfying \( a < b \) and \( c < d \). Verify in each case that the generalized multiplication property of order is satisfied.
2. Rewrite the generalized multiplication property of order given above using \( a > b \) and \( c > d \). What is the relation between \( a \times c \) and \( b \times d \) ?

**35-11 Summary of properties of order**

In the properties of order given below the sign \( > \) may replace the sign \( < \) as appropriate.

1. *Comparison Property of Order*
   For any rational numbers \( a \) and \( b \), one and only one of the following is true:
   \( a < b, a = b, a > b \).

2. *Transitive Property of Order*
   For any rational numbers \( a, b \) and \( c \), if \( a < b \) and \( b < c \), then \( a < c \).

3. *Addition Property of Order*
   For any rational numbers \( a, b \) and \( c \), if \( a < b \), then \( a + c < b + c \). (NOTE: \( c \) may be positive, negative or zero.)

4. *Order Property of Opposites*
   For any rational numbers \( a \) and \( b \), if \( a < b \), then \( -b < -a \).

5. *Multiplication Property of Order*
   a. For any rational numbers \( a, b \) and \( c \), if \( a < b \) and \( c \) is positive, then \( a \times c < b \times c \).
   b. If \( a < b \) and \( c \) is negative, then \( a \times c > b \times c \).
   c. For any positive rational numbers \( a, b, c \) and \( d \), if \( a < b \) and \( c < d \), then \( a \times c < b \times d \).
36-1 Introduction

We have used the decimal fraction .3 as a different way of writing \( \frac{3}{10} \). In the same way, 

\[ .2 = \frac{2}{10} \]

We can write an equal fraction with the smallest possible numerator and denominator. We then say that the fraction is written in lowest terms. In lowest terms, \( \frac{2}{10} = \frac{1}{5} \), so that \( .2 = \frac{1}{5} \).

These are examples of one-place decimal fractions. The same idea applies if there are two, three or even more decimal places. Thus,

\[ .25 = \frac{25}{100} = \frac{1}{4} \]

\[ .05 = \frac{5}{100} = \frac{1}{20} \]

\[ .125 = \frac{125}{1000} = \frac{1}{8} \]

In each case, we have written the fraction in lowest terms.

If we begin with fractions like \( \frac{1}{5} \), \( \frac{1}{4} \), \( \frac{7}{20} \) and \( \frac{3}{8} \), it is easy to go in the other direction and express them in decimal form. For these examples, we write

\[ \frac{1}{5} = \frac{2}{10} = .2, \]

\[ \frac{1}{4} = \frac{25 \times 1}{25 \times 4} = \frac{25}{100} = .25, \]

\[ \frac{7}{20} = \frac{5 \times 7}{5 \times 20} = \frac{35}{100} = .35, \]

\[ \frac{3}{8} = \frac{125 \times 3}{125 \times 8} = \frac{375}{1000} = .375. \]
Can we find a finite decimal form for any given fraction? It is not hard to see that the answer is "No".

If a fraction \( \frac{a}{b} \) is to be written in finite decimal form, it must be possible to "fatten it up" so that \( b \) becomes 10 or 100 or 1000 or some higher power of 10.

Except for 1, which is not interesting, the only numbers which divide 10 are 10, 5 and 2. Any fraction which is equal to a one-place decimal must, therefore, have a denominator which is 10 or 5 or 2. In fact:

\[
\begin{align*}
\cdot 1 & = \frac{1}{10} & \cdot 4 & = \frac{2}{5} & \cdot 7 & = \frac{7}{10} \\
\cdot 2 & = \frac{2}{10} = \frac{1}{5} & \cdot 5 & = \frac{5}{10} = \frac{1}{2} & \cdot 8 & = \frac{4}{5} \\
\cdot 3 & = \frac{3}{10} & \cdot 6 & = \frac{3}{5} & \cdot 9 & = \frac{9}{10}
\end{align*}
\]

Any fraction which is equal to a two-place decimal fraction must have a denominator which divides 100. What are the possibilities? The only ones are 100, 50, 25, 20, 10, 5, 4 and 2. What are the possible denominators for fractions that are equal to a three-place decimal fraction?

Suppose that we have a fraction whose denominator does not divide any power of 10.

The fraction \( \frac{1}{3} \) is the simplest example. Clearly 3 does not divide 10 or 100 or 1000 or any other power of 10. What can we do?

If we try to write \( \frac{1}{3} \) as a one-place decimal fraction, we find that \( \cdot 3 \) is too small and \( \cdot 4 \) too large, because \( 3(\cdot 3) = \cdot 9 \) and \( 3(\cdot 4) = 1.2 \). We say that \( \frac{1}{3} \) is between \( \cdot 3 \) and \( \cdot 4 \) and write

\[
\cdot 3 < \frac{1}{3} < \cdot 4.
\]

If we try to write \( \frac{1}{3} \) as a two-place decimal fraction, we soon find that \( \cdot 33 \) is too small and \( \cdot 34 \) is too large. In fact, \( 3(\cdot 33) = \cdot 99 \) and \( 3(\cdot 34) = 1.02 \). So

\[
\cdot 33 < \frac{1}{3} < \cdot 34.
\]

In the same way, it turns out that

\[
\cdot 333 < \frac{1}{3} < \cdot 334,
\]

\[
\cdot 3333 < \frac{1}{3} < \cdot 3334,
\]

and so on forever. The best that we can do is to write

\[
\frac{1}{3} = \cdot 3333\ldots,
\]

where the three dots are meant to show that the 3's go on without end. We call the right side of the equal sign an unending decimal fraction.
Let us take another example, \( \frac{2}{11} \). Since 11 does not divide any power of 10, we know that \( \frac{2}{11} \) cannot be written as a decimal fraction. We expect to find that it can be written as an unending decimal fraction. In fact,

\[
\frac{2}{11} = .181818\ldots,
\]

where the digits 1 and 8 repeat endlessly.

Let us verify this by doing a long division.

\[
\begin{align*}
11 & \overline{)18} \\
\underline{-11} & \\
7 & \\
\underline{-7} & \\
10 & \\
\underline{-9} & \\
1 & \\
\underline{-1} & \\
0 & \\
\end{align*}
\]

We actually went further with the work than we needed to. When we got the remainder 2 after two divisions (shown by a circle), we were in the same situation as we were when we started. So we know that the later results of division will repeat the earlier ones.

Let us look at one more example. What decimal fraction is \( \frac{5}{6} \) equal to? Does 6 divide any power of 10? Since it does not, we must get an unending decimal fraction. Let us see what it is:

\[
\begin{align*}
6 & \overline{)5.00} \\
\underline{-4.8} & \\
\underline{-0.20} \\
\underline{-1.8} & \\
\underline{-1.8} & \\
0 & \\
\end{align*}
\]

Do you see that because the remainders in the two circles are the same, the 3's in the answer must go on endlessly? We write

\[
\frac{5}{6} = .8333\ldots.
\]

**EXERCISE 36-2A**

1. Find unending decimal fractions for each of the following fractions.
   a. \( \frac{1}{9} \)  
   b. \( \frac{1}{99} \)  
   c. \( \frac{1}{999} \)  
   d. \( \frac{5}{11} \)  
   e. \( \frac{1}{7} \)  
   f. \( \frac{2}{7} \)  
   g. \( \frac{1}{37} \)

36-3 Changing unending decimals to common fractions

You see from our work that some fractions are equal to decimal fractions which end and others to decimal fractions which do not end. How did you learn to tell without dividing out whether a fraction is of the first kind or the second?
We discovered something else. When the decimal is unending, the digits repeat, at least after a while. For example,

\[
\frac{2}{11} = 0.1818... \quad \text{(repeating 18)}
\]

\[
\frac{5}{6} = 0.833... \quad \text{(repeating 3 after passing the 8)}.
\]

It is fairly easy to see why the digits must repeat. When dividing by 6, for example, there can be no more than 5 remainders different from 0. (What would happen if the remainder were 0?) Then if we keep on dividing, we must eventually get a remainder that appeared before, and then the digits in the answer start repeating.

Can we go the other way? That is, if we have an unending decimal fraction that repeats, can we find the fraction that it is equal to? Let us see.

What is a common fraction which is equal to 0.3939...?

Let us write

\[ D' = 0.3939... \]

Notice that two digits repeat. Now if we multiply by 100, we must move the decimal point two places to the right. This gives

\[ 100 \times D' = 100 \times (0.3939...) = 39.3939... \]

The unending digits on the right of the decimal point are exactly the same in both cases. (Remember that the 39's go on forever!) So if we subtract, we get

\[ 99 \times D' = 39 \] (exactly)

and therefore the fraction which goes in the box is

\[ \frac{39}{99} = \frac{13}{33}, \]

You can easily verify that \(13 \div 33 = 0.3939...\).

Let us take another example. We would like to know a common fraction for 0.027027.

We write

\[ R = 0.027027... \]

We multiply by 1000 (why not 100?). We get

\[ 1000 \times R = 27.027027... \]

\[ 999 \times R = 27, \]

\[ R = \frac{27}{999} = \frac{3}{111} = \frac{1}{37}, \]

where the fraction on the right has been written in lowest terms. Again, you should check that

\[ \frac{1}{37} = 0.027027... \]

As a final example, let us take an unending decimal for which the digits do not repeat from the beginning.

If \[R = 0.1333... \] (repeating 3's), what fraction goes into the box? We multiply by 10.

Then

\[ 10 \times R = 1.333... \]
and subtracting

\[ 9 \times \square = 1.2, \]

\[ \square = \frac{1}{9} \times \frac{12}{10} = \frac{12}{90} = \frac{2}{15}. \]

**EXERCISE 36-3A**

1. Write each of the following unending decimal fractions as a common fraction:
   a. \( \cdot 222. \ldots \)  
   b. \( \cdot 2323. \ldots \)
   c. \( \cdot 234234. \ldots \)  
   d. \( \cdot 1111. \ldots \)
   e. \( \cdot 0101. \ldots \)  
   f. \( \cdot 001001. \ldots \)
   g. \( \cdot 1666. \ldots \)  
   h. \( \cdot 11010101. \ldots \)

2. Show how you can use the results of parts d, e and f of Question 1 to find the answers to parts a, b and c without working with the box \( \square \).

**36-4 Irrational numbers and real numbers**

We have learned that common fractions can be written as decimal fractions which either end or repeat. We also learned that any decimal fraction which ends or repeats is another name for a common fraction. We can show this in a diagram:

![Diagram showing common fractions and decimal fractions which end or repeat]

Have we left out any possibilities? Could we have a decimal fraction that is unending but not repeating? Yes, we could.

An example is

\( \cdot 1010010001. \ldots \)

where each time we move along from a 1 we put in an extra 0 before the next 1. In this example, there is no block of digits which repeats.

Another example is

\( \cdot 1234567891011. \ldots \),

where the scheme of writing digits should be clear. Again there is no repetition in the unending digits.

These decimal fractions cannot represent common fractions. They must correspond to a new kind of number, which is called an **IRRATIONAL NUMBER**.

So far we have talked only about positive decimals. For every positive decimal, ending or unending, there is an opposite written with a minus sign in front of it. Thus, the opposite of \( \cdot 101001 \ldots \) is \( -\cdot 101001. \ldots \). The numbers that we represent in this way are called negative numbers. When we include them and the number 0 we have the following scheme:

![Diagram showing rational numbers and decimal fractions that end or repeat (whether positive, negative or 0)]
irrational numbers \[\iff\] decimal fractions that do not end and do not repeat (whether positive or negative)

If we put the rational numbers and the irrational numbers together, we get the numbers which correspond to all decimal fractions. These numbers are called REAL NUMBERS.

real numbers \[\{\]
    rational numbers
    irrational numbers
\[

EXERCISE 36-4A

Invent some more unending, non-repeating decimal fractions.
Chapter 37
A GEOMETRY PROBLEM

37-1 Introduction

Let ABCD be a square 2 inches on a side. Suppose that E, F, G and H are the midpoints of its sides. If E is joined to G and F to H, ABCD is divided into four squares, all alike, each 1 inch on a side. We therefore see that the area of ABCD is 4 square inches.

We notice that $4 = 2 \times 2$. In general, the number of square inches in a square will be the number of inches on its side multiplied by itself. Show that this is true for squares having 3, 4 and 5 inches for their sides.

Let us now join E, F, G and H. Each of the one-inch squares is cut into two congruent triangles. We see that EFGH is a square consisting of four of these triangles. The area of EFGH is therefore 2 square inches.

How long is a side of EFGH, for example EF? If $S$ is the length of side EF, it must be true that $S \times S$ is 2.

Could $S$ be a whole number? Certainly not, because $1 \times 1 = 1$ is too small and $2 \times 2 = 4$ is too large.

Is $S$ equal to some fraction? If so, the fraction must be between 1 and 2. Remember that we want

$$S \times S = 2.$$  

A good guess is

$$S = \frac{7}{5}(= 1 \frac{2}{5}).$$

Now

$$\frac{7}{5} \times \frac{7}{5} = \frac{49}{25},$$

Because $2 = \frac{50}{25}$, we see that $\frac{7}{5}$ is a little too small.

Can we do better? Let us divide 2 by $\frac{7}{5}$. We get
\[
2 \div \frac{7}{3} = \frac{10}{7}.
\]

From the meaning of division, this means that
\[
\frac{7}{3} \times \frac{10}{7} = 2.
\]

Remember that \(\frac{7}{3}\) was too small.

How about \(\frac{10}{7}\)? Of course
\[
\frac{10}{7} \times \frac{10}{7} = \frac{100}{49}.
\]

We want
\[
S \times S = 2 = \frac{98}{49},
\]
so \(\frac{10}{7}\) is too large. Could we have seen this without multiplying \(\frac{10}{7}\) by \(\frac{10}{7}\)?

Clearly we could. We know that
\[
\frac{7}{3} \times \frac{10}{7} = 2.
\]

If \(\frac{7}{3}\) is too small, \(\frac{10}{7}\) must be too large.

What do we know? We know that \(S\) is between \(\frac{7}{3}\) and \(\frac{10}{7}\). How can we do better? We can average these fractions; that is, add them and divide by 2. In this way we get a new fraction, which is between \(\frac{7}{3}\) and \(\frac{10}{7}\).

\[
\frac{\frac{7}{3} + \frac{10}{7}}{2} = \frac{\frac{49 + 50}{21}}{2} = \frac{99}{70}.
\]

This fraction is \(\frac{99}{70}\), so it's too large. Could we have seen this without multiplying \(\frac{10}{7}\) by \(\frac{10}{7}\)?

Clearly we could. We know that
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This fraction is \(\frac{99}{70}\), so it's too large. Could we have seen this without multiplying \(\frac{10}{7}\) by \(\frac{10}{7}\)?

Clearly we could. We know that
\[
\frac{7}{3} \times \frac{10}{7} = 2.
\]

If \(\frac{7}{3}\) is too small, \(\frac{10}{7}\) must be too large.
\[
\frac{140}{99} \times \frac{140}{99} = \frac{19600}{9801}
\]

If \( \frac{140}{99} \) had been the required value of \( S \), we should have gotten \( \frac{19602}{9801} \) when we multiplied \( \frac{140}{99} \) by itself. We got a number a trifle smaller. So now we know that \( S \) must be between \( \frac{140}{99} \) and \( \frac{99}{70} \).

We could average these two results. You should do this and find out whether we have finally succeeded in finding a fractional value for \( S \). You will discover that the answer is "No".

A new question comes to mind. Suppose that we kept on in this way, would we ever get an absolutely correct answer? It is very surprising that the answer is "No". We shall show this in the next section.

Meanwhile it will be interesting to change our fractions to decimals. In this way, we can see how close we are getting to the desired result.

First we found \( S \) between

\[
\frac{7}{5} = 1.4 \text{ and } \frac{10}{7} = \frac{3}{17} = 1.428571\ldots
\]

Next we located \( S \) between

\[
\frac{140}{99} = 1.414141\ldots \text{ and } \frac{99}{70} = 1.4142857\ldots
\]

What better result were you able to find?

**Exercise 37-1A**

Suppose that \( S \) is the side of a square with area 3 so that \( S \times S = 3 \). Use the scheme of this section to get better fractions which approximate \( S \).

37-2 \( S \times S = 2 \) has no solution in common fractions

We have tried some fractions \( \frac{a}{b} \) to see if we could find the side of a square with area 2 square inches. We did not succeed. Were we unlucky? Or were we lacking in patience? The answer to both questions is "No". There is no fraction \( \frac{a}{b} \) which makes

\[
\frac{a}{b} \times \frac{a}{b} = 2
\]

This is one of the most famous discoveries in mathematics. The discovery was made by a Greek, a follower of Pythagoras, who lived approximately 600 years before Christ. It produced a crisis in the history of mathematics. Later we shall explain why this was true.

But first let us prove this very surprising result. Suppose that there is a fraction \( \frac{a}{b} \) for which \( \frac{a}{b} \times \frac{a}{b} = 2 \), so that
\[ S \times S = 2 \]

with \( S = \frac{a}{b} \).

Now if \( S \times S = 2 \),
\[ S = \frac{2}{S}. \]

This means that if \( S = \frac{a}{b} \), then \( \frac{a}{b} = 2 \div \frac{a}{b} \) or
\[ \frac{a}{b} = \frac{2b}{a}. \]

We must show that this is impossible no matter how we choose the whole numbers \( a \) and \( b \).

We may surely assume that the fraction which solves our problem (if there is one) is written in lowest terms, because if it were not in lowest terms we could replace it by a fraction that was. For example, if \( \frac{14}{10} \) were an answer (it isn't, of course), then \( \frac{7}{5} \) would have to be an answer also.

Now we suppose that \( \frac{a}{b} = \frac{2b}{a} \) in the lowest terms. \( \frac{2b}{a} \) is supposed to be equal to \( \frac{a}{b} \).

When are two fractions equal?

Take a definite fraction in lowest terms, say \( \frac{10}{7} \). What fractions are equal to \( \frac{10}{7} \)?

The possibilities are
\[ \frac{10}{7}, \frac{2 \times 10}{14}, \frac{3 \times 10}{21}, \ldots \]

and so on. It is enough to look at the possible denominators. They are
\[ 7, 14, 21 \text{ and so on.} \]

Now if \( \frac{a}{b} \) is any fraction written in lowest terms, the fractions which are equal to it must have one of the denominators
\[ b, 2b, 3b, 4b \text{ and so on.} \]

This means that if \( \frac{2b}{a} = \frac{a}{b} \), \( a \) must be one of the numbers \( b, 2b, 3b, 4b \) and so on; so that \( \frac{a}{b} \) is one of the numbers \( 1, 2, 3, 4 \) and so on. That is, \( \frac{a}{b} \) must be a whole number.

But \( S \times S = 2 \) has no whole-number solution. So no matter what fraction \( \frac{a}{b} \) we choose,
\[ \frac{a}{b} \times \frac{a}{b} \text{ cannot be equal to } 2. \]

**EXERCISE 37-2A**

Show that it is not possible to find a fraction \( \frac{a}{b} \) so that \( \frac{a}{b} \times \frac{a}{b} = 3. \)
We know how to locate fractions on the number line. For example, we can easily find points to show $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{5}{4}$. Can we locate the number $S$, the side of a square of area 2? We can certainly do this geometrically. We showed at the beginning of this chapter that $S$ is the length of the diagonal of a square of side 1. Let a number line be drawn along the base of the square with 0 at the left end and 1 at the right end. You can take 0 to be the center of a circle of radius $S$. The circle will intersect the number line at a point $P$ between 1 and 2. So we know on the picture how to locate the point $P$ which corresponds to the number $S$.

But we know that $S$ is not a rational number. Of course $S$ is between $\frac{7}{5}$ and $\frac{10}{7}$. We can locate both of these numbers by points on the number line, and $P$ will lie between these points.

We can do better. $P$ must lie between the points which show $\frac{140}{99}$ and $\frac{99}{70}$.

These points are much closer together than the previous pair. The interval between them is shorter. Also $\frac{140}{99}$ is to the right of $\frac{7}{5}$ and $\frac{99}{70}$ to the left of $\frac{10}{7}$. So our new interval is inside the earlier one.

The important thing to notice is this. If we continue to get new fractions by the same scheme, we can locate $P$ within intervals as short as we please but we never reach $P$ itself. There are points like $P$ on the number line which do not correspond to common fractions. We may call them *irrational points* because they show irrational numbers.

Let us say this in a different way.

Between 1 and 2 there is a point which divides the interval in 2 equal pieces. There are 2 points which separate it into 3 equal pieces, 4 points that separate it into 5 equal pieces and so on. There are 99 points which divide the interval into 100 equal pieces, 999 points which divide it into 1000 equal pieces. But no matter what measuring stick we choose that divides the interval from 1 to 2 into a number of equal pieces, we cannot use this measuring stick to measure exactly the number whose square is 2. We can say that $S$ and 1 are *incommensurable*. This means that $S$ and 1 have no common measure. This was a discovery that shocked the Greeks when it was first discovered. It showed that the number line as we have met it up to now is full of holes. To say it another way, it shows the need of new kinds of numbers, which we call irrational numbers.
Let us talk about these numbers in terms of decimal fractions. As we learned, rational numbers correspond to decimal fractions which either end or repeat, while irrational numbers correspond to decimal fractions which neither end nor repeat. We saw an example of a decimal of this second sort,

\[ \cdot 01001001001 \ldots \]

Our number \( S \) must be of this non-ending, non-repeating kind. We found that \( S \) was between

\[ 1.414141 \ldots \quad \text{and} \quad 1.4142857 \ldots \]

We can write \( 1.4141 < S < 1.4143 \).

We have located \( S \) within \( 0.0002 \). Of course with patience we could do still better. We know that \( S \) must be represented by an unending decimal that never endlessly repeats.

Can these unending, non-repeating decimals be treated like the rational numbers? Can they be ordered? Can they be added, subtracted, multiplied and divided? If so, do addition, subtraction, multiplication and division have the properties with which we are familiar? The answers to all of these questions are "Yes".

It would take a long discussion to prove in detail that this is true. For our purposes it is sufficient to give some idea of how it could be done.

Which of the decimal fractions \( \cdot 13275 \ldots \) or \( \cdot 13268 \ldots \) represents the larger number?

The decimal fractions agree in the first three digits 1, 3 and 2. They differ in the next decimal place. Since 7 is greater than 6, the first number is greater than the second. Do you see that \( \cdot 13275 \ldots \) locates a point on the number line to the right of that for \( \cdot 13268 \ldots \) ? Is it also clear to you that if \( a \) and \( b \) are any real numbers, there are just three possibilities:

\[ a = b, \quad a < b \quad \text{and} \quad a > b ? \]

Could you explain why this must be true?

Do you think that \( a + b = b + a \) is true for all real numbers \( a \) and \( b \)? Suppose not. Then for some \( a \) and \( b \), it must be true that

\[ a + b > b + a \quad \text{or else} \quad a + b < b + a. \]

Each of these possibilities will now be shown to lead to a contradiction.  

If \( a + b > b + a \) is true, then \( a + b = (b + a) + p \) where \( p \) is a positive number. Now \( p \) itself must be representable as a decimal fraction, maybe a small one, say \( \cdot 0000012 \ldots \).

Let us imagine that \( a \) and \( b \) are written as unending decimal fractions. Let us break off each of them after \( n \) decimal places. Suppose that we add the corresponding rational numbers. The result does not depend on the order in which we add. Why? It is because the commutative property holds for rational numbers.

Now look at the equation

\[ a + b = (b + a) + p. \]

The first \( n \) decimal places of \( a + b \) agree with the first \( n \) decimal places of \( b + a \). Then the first \( n \) decimal places of \( p \) must be 0. Since this must be true no matter how large \( n \) is—that is, for as many decimal places as we like—all of the decimal places of \( p \) must be 0. This contradicts the assumption that \( p \) is positive. So it is impossible that \( a + b > b + a \) be true.

In the same way, it can be seen that it is impossible for \( a + b < b + a \) to be true. Consequently we know that \( a + b = b + a \) is true.

In a quite similar manner we could show that all the properties of addition and multiplication for rational numbers hold for real numbers as well. Also, the order properties of the rational numbers are properties of the real numbers.
Chapter 38

THE VIEW FROM THE TOP

38-1 Looking backward

We have come a long way from our starting point. We began with numbers which could be
used to count a set of objects like a herd of cattle. These numbers were known to early man.
They appear in the oldest records. Since then man has travelled a long, long road. The idea of
number has grown and grown. It is one of the most important ideas that mankind has ever had
and one of the most successful. In this book we have tried to show how new kinds of numbers
have been invented. We learned about the importance of zero, about the uses of fractions so
that numbers could be used not merely to count but to measure. We learned about negative num­
bers, which help us to include not merely the idea of how many or how much but also the idea
of direction, right or left, up or down. We have just extended the idea of number once again to
include unending decimals which do not repeat. We have seen that if we want to measure the
diagonal of a square of side 1 we need a number of this new kind. Here we have reached the
end of our journey. (If you go further in mathematics, you will find that this is not really the
end but that there are still new kinds of numbers which man has invented later.) The time has
come to look back over the road which we have followed.

We have travelled slowly and patiently. The road has sometimes been dusty and the
journey may have been tiring. But we have come to the top of a mountain. We should stop and
enjoy the view.

At each stage of our journey we have learned to arrange the numbers in order of lesser
or greater. And we have learned to add them, to subtract them, to multiply them and divide them.
Let us forget for the moment just how we did this at each stage. These are details—important
details but still details. Let us ask ourselves what has been accomplished by bringing in new
kinds of numbers and learning to work with them. Man has invented zero, the fractions, the
negative numbers, and the irrational numbers. What for?

At each stage, man has found himself stopped by a difficulty. He wanted to be able to do
something which he could not do with the numbers that he already had. There was a roadblock
which stood in the way of going ahead. When old ideas fail or do not help, we seek to invent
something new. "Necessity is the mother of invention."

For example, we cannot divide 4 by 5 if we have only counting numbers to work with.
After fractions were invented, we could divide 4 by 5. We cannot subtract 5 from 3 if we have
only counting numbers to work with. Negative numbers allow us to do so. After the new kinds
of numbers have been invented, we have more freedom. We can remove restrictions.

But a very remarkable thing happens. It could be true that the new numbers behave in
quite a different way from the old ones. If this were true we should always be having to remem­
ber what kinds of numbers we were working with, so that we could know what properties of
addition, subtraction, multiplication and division to apply. By good luck it turns out that the properties are the same for counting numbers, for integers, for rational numbers or for real numbers. We do not have to keep learning new principles. This makes things much easier.

At this point it will be useful to reread the introduction, in particular the latter part about the patterns which it was hoped would be discovered. It will be remembered that we thought of the whole numbers as belonging to a club with certain rules. The new kinds of numbers could be admitted to this club because they were able to obey the rules.

38-2 The "club rules" for addition and multiplication

What are the club rules that all of our members are required to obey? First there are the properties of addition and multiplication. These properties were summarized very briefly at the end of Chapter 13, where \( a, b \) and \( c \) stood for any whole numbers. These same properties appeared in Chapter 34 for rational numbers. Now we shall use \( a, b \) and \( c \) to stand for any numbers at all, that is, for any real numbers whether rational or irrational. Remember that the set of real numbers includes all the numbers that we have talked about. Here then are the club rules. First are three for addition:

**The Commutative Property of Addition (CA)**
\[ a + b = b + a \]

**The Associative Property of Addition (AA)**
\[ a + (b + c) = (a + b) + c \]

**The Addition Property of Zero (AO)**
\[ a + 0 = a \]

Then there are three corresponding rules for multiplication:

**The Commutative Property of Multiplication (CM)**
\[ a \times b = b \times a \]

**The Associative Property of Multiplication (AM)**
\[ a \times (b \times c) = (a \times b) \times c \]

**The Multiplication Property of One (M1)**
\[ a \times 1 = a \]

Notice that these three rules can be found from those for addition simply by changing \(+\) to \(\times\) and \(0\) to \(1\). Can you see that \(1\) behaves as a factor the same way that \(0\) does as an addend?

There is another rule that connects multiplication and addition:

**The Distributive Property (D)**
\[ a \times (b + c) = (a \times b) + (a \times c) \]

Finally, we had

**The Multiplication Property of Zero (MO)**
\[ a \times 0 = 0 \]

In all, we have eight properties which we can think of as club rules for numbers.

Let us look at these rules as requirements that any proposed new members of the number club must obey. For example, suppose we propose the negative integers for membership in the club consisting of \(0, 1, 2, 3, \ldots\). How must \(-1\) behave if we are going to admit it?
What must \((-1) \times 1\) be equal to?

Rule M1 says that
\[ a \times 1 = a. \]

If \(-1\) is to be a good club member, it must therefore be true that
\[ (-1) \times 1 = -1. \]

That is, we must be able to use \(-1\) as a particular value of \(a\).

What must \(1 \times (-1)\) be equal to? CM says that any members \(a\) and \(b\) must obey the rule
\[ a \times b = b \times a. \]

Then \(1 \times (-1)\) must be equal to
\[ (-1) \times 1 \]
which we know is \(-1\). So we must require that
\[ 1 \times (-1) = -1. \]

A harder question is to find what \((-1) \times (-1)\) must be. When we introduced \(-1\), we thought of it as the opposite of 1, so that
\[ 1 + (-1) = 0. \]

So \(1 + (-1)\) and \(0\) are two names for the same number. Then
\[ (-1) \times [1 + (-1)] = (-1) \times 0. \]

Rule D says that the left side is
\[ [(-1) \times 1] + [(-1) \times (-1)]. \]

and rule M0 says that the right side \((-1) \times 0\) is 0. So we must require that
\[ [(-1) \times 1] + [(-1) \times (-1)] = 0. \]

But we know that \((-1) \times 1 = -1\), so it must be true that
\[ -1 + [(-1) \times (-1)] = 0. \]

Then \((-1) \times (-1)\) must be the opposite of \(-1\), that is 1. So finally we have the requirement
\[ (-1) \times (-1) = 1, \]
if \(-1\) is to be allowed in the club.

We know of course that \(-1\) does indeed pass all these tests. In fact,
\[ 1 \times (-1) = -1, \]
\[ (-1) \times 1 = -1, \]
and \[ (-1) \times (-1) = 1, \]
as we saw earlier in the book.
**EXERCISE 38-2A**

1. Show from the rules that
   
   $$(-3) \times a = -(3 \times a)$$
   
   must be true. (HINT: Write $$[(-3) + 3] \times a$$ in two ways.)

2. Show that if we follow the rules, $$\frac{7}{(-3)}$$ must be equal to $$\frac{(-7)}{3}$$. (HINT: If $$\frac{7}{(-3)} = \square$$, then $$3 \times \square = -7$$.)

3. Show from A0 that
   
   $$0 + 0 = 0$$.

**38-3 Simplifying the rules**

We have listed some rules that we require numbers to obey to become members of the number club. Can we perhaps simplify these rules? For example, can we make a shorter list that would really say the same thing? The answer is "Yes". In fact, we have already shortened the list from the one that was given in Chapter 13.

There we included $$0 + a = a$$, $$1 \times a = a$$ and $$0 \times a = 0$$. Can you see why it is not necessary to include them in our present list? Can you see for example that $$0 + a = a$$ follows from $$a + 0 = a$$ by using CA?

We shall now show that we can also leave out M0, which reads

$$a \times 0 = 0$$.

We show that this rule must hold if the other seven rules hold.

According to rule D

$$a \times (b + c) = (a \times b) + (a \times c)$$.

If $$b = 0$$ and $$c = 0$$, we have

$$a \times (0 + 0) = (a \times 0) + (a \times 0)$$.

But we know that $$0 + 0 = 0$$, so we require that

$$a \times 0 = (a \times 0) + (a \times 0)$$.

To simplify the writing, let us call $$a \times 0$$ by the new name $$b$$. Then we must have

$$b = b + b$$.

We hope to show that $$b$$ must be 0.

Rule AA tells us that

$$(-b + b) + b = -b + (b + b)$$.

Now $$-b + b = 0$$, since $$-b$$ and $$b$$ are opposites. Also $$b + b = b$$. Therefore, we have

$$0 + b = -b + b$$.
We know that \( 0 + b = b \) and that \(-b + b = 0\). So finally
\[ b = 0. \]
That is,
\[ a \times 0 = 0, \]
which is rule MO.
So in applying tests for new members, it is not necessary to require MO if we have already satisfied ourselves about the other rules.

### 38-4 The rules of order

We first met also some properties of order for the counting numbers:

01 if \( a \) and \( b \) are counting numbers, there are only three possibilities:
\[ a = b, \quad a < b, \quad a > b. \]

Again, if \( a, b \) and \( c \) are any counting numbers:

02 if \( a < b \) and \( b < c \), then \( a < c \),

03 if \( a < b \), then \( a + c < b + c \)
and

04 if \( a < b \) then \( a \times c < b \times c \).
These same rules now apply if \( a, b \) and \( c \) are any real numbers, except that in 04 we must require that \( c > 0 \) (which is automatically true when \( c \) is a counting number). Again, the real numbers are good club members.

We have learned that 04 can be supplemented by:

05 if \( a < b \) and \( c < 0 \), then \( a \times c > b \times c \).
There is of course no occasion for this rule with counting numbers, because a counting number is never less than 0.

These rules too can be simplified. If we say that
\[ a < b \]
means that \( b = a + p \) where \( p \) is a positive number—that is, \( p > 0 \)—we can replace all the rules of order by a new list.

**New Order Rules**

0'1 If \( a > 0 \) and \( b > 0 \), then \( a + b > 0 \).

0'2 If \( a > 0 \) and \( b > 0 \), then \( a \times b > 0 \).

0'3 For any real number \( a \), there are exactly three possibilities:
\[ a = 0, \quad a > 0, \quad a < 0. \]

For example, let us show that 03 follows from our new rules. 03 says that
\[ \text{if } a < b, \text{ then } a + c < b + c. \]
If \( a < b \), we can write
\[ b = a + p \quad (p > 0). \]
Then \[ b + c = (a + p) + c \]
\[ = a + (p + c) \] (Why?)
\[ = a + (c + p) \] (Why?)
\[ = (a + c) + p. \] (Why?)

But then finally \( a + c < b + c \). 

**EXERCISE 38-4A**

1. Prove 04 from 0'1, 0'2 and 0'3.
2. Prove 02 from 0'1, 0'2 and 0'3.
3. Prove 01 from 0'1, 0'2 and 0'3.

**38-5 Summing up**

What we have done in the last two sections is not easy. It is harder than the rest of the book. We have given some examples of the way in which mathematical proofs are constructed. The elementary teacher will not use proofs like this in his own classes. But the teacher should have an idea of what lies ahead for some of his pupils—those who go on in mathematics.

When we continue the study of mathematics we find that more and more simplifications occur. The facts that we know about numbers are connected with each other in surprising ways. The simplifications make mathematics more beautiful and more powerful. But we have to pay a price. The price is that we have to be prepared to think deeply about our experience. We must not be satisfied with knowing how to get answers in routine ways. We must be willing to ask ourselves "Why?" again and again.

The knowledge that is power is the fruit of our unceasing effort to understand more clearly, more fully and more deeply.
Chapter 39

APPROXIMATIONS AS RESULTS OF COUNTING

39-1 Introduction

Once our pupils held a party and we tried to find out how many attended the party. This was very difficult because when we were counting, several pupils had already left the party, others came after we counted, while many of the people present were moving around. After counting, we got the result 137 pupils. Do you think that this was the exact number of pupils who attended the party? Is counting really always easy and simple?

If you are asked to measure the length of your classroom with a foot ruler and you get the answer 30 feet, can you be sure that this is the exact length? Or could it be 29 feet and some inches, or even 30 feet and some inches? Does measuring give the exact number?

Suppose a tailor needs 3 yards of fabric to make a dress. How many dresses can he make from a piece of 40 yards of fabric? Dividing 40 by 3, you obtain $13\frac{1}{3}$. It is clear that your answer would not be $13\frac{1}{3}$ dresses. You will say that the tailor can make 13 dresses. Such rounding off is often used in everyday life.

These three examples have something in common. What is it? We are now going to consider in detail the use of numbers in instances such as given above. We shall discuss what are commonly called "approximate numbers" or "approximations".

39-2 Approximations in Counting

We have all learned how to count and how to make use of the set of counting numbers. We also know the set of whole numbers, which is the set of counting numbers and zero.

Suppose you ask one of your pupils to count the number of pupils in your class or the shillings in his pocket. No doubt his answer will be correct, and he will give you the exact number. He will count the pupils or shillings one by one or in groups. This will be easy if the number of pupils is rather small. If you asked him to count the number of windows in a very big building, then he can get the exact number if he counts carefully. You could be quite sure about the answer if some other pupils counted the windows and got the same number. But counting the number of members in a set becomes harder as the set becomes larger. Even so, it is possible in many instances to obtain accurate counts of large sets. Sometimes it is quite necessary to obtain accurate counts. For example, a bank teller must count the exact amount of money at the end of the day. There are, however, instances when counting the number of members in a set is extremely hard or even impossible.
Would it be easy or even possible for the government to count the exact number of people in your country? The number of people in your country does not stay the same even for one day. For many purposes, however, the government has to know how many people live in various regions and in the whole country. Of course, it is practically impossible to count all of these people. Besides, does the government really need to know the exact number of people?

There is another example in which it is very difficult to count the number of members in a certain set. Suppose you ask a pupil to find the number of trees in a certain park or piece of land. If he counts and gives the answer 563, do you think his answer is exact? It probably is not exact for the following reasons. First, it was rather inconvenient and hard to count such a large number of trees scattered about without recounting some and without missing others.

Next, it was probably difficult for the pupil to decide whether the dead trees or some larger bushes should be counted or not. In other words, it was difficult for him to determine exactly what things were members of the set of trees. If some other pupils count the number of trees in the park, they will probably obtain different answers, perhaps 559, or 550 or 571. In fact, it would be interesting to see whether the pupil who first counted would get his original answer if he counted again. In situations like these we are usually quite satisfied with approximate, rather than precise, results. The numbers in the statements below are certainly not exact.

We will call them approximate numbers.

- The population of Uganda is 6,780,000.
- There were two thousand people present at the lecture.
- Our college library contains 6,700 books.
- Ali has 300 chickens at home.

**EXERCISE 39-2A**

1. Ask your pupils to try to count how many people are in your school in one day. What makes it hard to obtain an exact number? Do you think that the number they count would change from time to time during the day? Suppose instead you asked them to count how many different people in all were in your school on a certain day. Would they still have difficulties?

2. Ask your pupils to count some of the sets in the examples we have given.

3. Find other situations you can use with your pupils to show that the results of counting are not always exact.

**39-3 Averages**

By using many examples, you can convince your pupils that we can often obtain only approximate numbers in counting certain large sets. Of course, you will want them to obtain the most accurate approximations that they can. We will now see how to make sure that the approximate answers are rather accurate.

If four pupils tried to find out how many chickens Ali has at home, each pupil would probably get a different result by counting. Why is it so? Suppose the first pupil counts 295, the second 305, the third 304 and the fourth 297. Which answer do you think would be the best? You might think that a good answer would be one between 295 and 305. In order to find a good answer between 295 and 305 for the number of chickens, we can proceed as follows: we find what we call the arithmetic mean or average of all results of counting. We first find the sum of all the results. Then we divide the sum by the number of terms in the sum. The quotient
obtained is called the arithmetic mean or average of the numbers that we started with. For example, the average of the numbers 17 and 25 is \((17 + 25) \div 2 = 21\).

Going back to Ali’s chickens, the sum of the counting results is

\[295 + 305 + 304 + 297 = 1201.\]

After dividing 1201 by 4 (the number of counts), we get the average 300.25. Taking into account the objects we are dealing with, we may say 300 chickens is the final result. Remember that this result is only approximate. We do not claim that it is precise, but it is certainly more precise and reliable than any one of the four individual counts. Finding the average is a good way of obtaining an approximate answer when repeated counting gives different numbers as results.

**EXERCISE 39-3A**

1. Find the average (arithmetic mean) of each of the following sets of numbers. If the quotient is an unending decimal, write the answer to one decimal place and then write three dots, ..., to show that the answer is unending.

   a. 18, 22, 23
   b. 20, 23, 26, 29
   c. 101, 102, 105, 108
   d. 248, 251, 252, 267
   e. 61, 63, 64
   f. 248, 251, 257, 267

2. Using fractions, write the averages that were decimals in Question 1.

3. Make up examples to use with your class of situations in which you would want to find averages.

**39-4 Deviation**

We have said that the average of several counts of a large set can be taken as a good answer for the number of members in the set. However, this number may look somewhat artificial to your pupils, and they may raise questions such as: How does this number correspond to the reality? How reliable is it as a solution to our problem?

Let us use the following example to try to see how to answer these questions. Suppose you ask your class to determine the number of grains of rice in one ounce of rice. Let five pupils weigh five separate heaps of one ounce of rice each, and count the number of grains in each heap. Suppose they got the following numbers:

308, 332, 328, 342, 307.

The average of these numbers is \(1617 \div 5 = 323.4\). The digit "3" for the hundreds appears in each of the five counts, and therefore we may conclude that we can rely on the number of hundreds. Thus, we say that the digit 3 is reliable. The digit "2" for the tens in the average is questionable, because in the five countings we got various digits in the tens place, namely

0, 3, 2, 4, 0.

The digit "3" in the ones place in the average is clearly not reliable at all and, thus, worth-
Therefore, that digit as well as the digit in the tenths place (.4) ought to be rejected in the final result.

Since the right-hand two digits (3.4) in the average are worthless, the answer 320 would be just as good an answer. Therefore, we will say that the number of grains of rice in a heap of one ounce is approximately 320. We can be quite sure about the first digit of this number, which indicates the hundreds of grains. In the second digit (2), which expresses the number of tens, there may be a small inaccuracy. About the remaining digits, we just cannot say anything.

We may summarize our procedure as follows:
1. Find the average.
2. Compare the average with each separate count.
3. The digits which are the same in every count are reliable and are to be kept in the final result.
4. Take the next digit in the average even though it is questionable.
5. Replace all remaining digits by zeros, since they are worthless. (More definite instructions about replacing rejected digits by zeros will be given in Chapter 40).

**EXERCISE 39-4A**

1. Suppose five pupils in a class counted the number of books in the school library. The results of their five counts were

   275, 274, 278, 279, 271.

   Find the average of the five counts.

2. Which digits in the average are reliable, which are questionable and which are worthless?

3. What would you say is the final number of books?

   We have used the words reliable, questionable and worthless. You may feel that they have not been explained sufficiently; perhaps there are still questions in your mind about the procedure. Let us now discuss more carefully how we can tell whether a given digit is reliable enough to keep.

   We have seen that the numbers obtained in the separate counts of grains of rice are different: 308, 332, 328, 342, 307. Each of these numbers is different also from the calculated average 323.4. Suppose we find now how much each count differs from the average. We will call these differences the deviations from the average. In our example of rice, they are:

   \[
   \begin{align*}
   323.4 - 308 &= 15.4 \\
   332 - 323.4 &= 8.6 \\
   328 - 323.4 &= 4.6 \\
   342 - 323.4 &= 18.6 \\
   323.4 - 307 &= 16.4
   \end{align*}
   \]

   (NOTE To find the deviation from the average we subtract the smaller number from the larger.)

   Now we find the average of these deviations by adding them and dividing their sum by 5: 63.6 ÷ 5 = 12.72. This quotient, 12.72, is called the average deviation.

   In our example the left-most digit (1) in the average deviation 12.72 is in the tens place. Therefore, the digit in the tens place (2) in the average 323.4 we will call
questionable. We keep the digit “2” for tens in the average 323.4 as the first questionable digit. We replace all the digits to the right of the tens place by zeros. As the final result, we get 320. In order to avoid misunderstandings, it is sometimes convenient to underline the “2” as the questionable digit in the final result 320. This method can be used in any problem, not keeping any digits beyond the left-most place in the average deviation.

We can set out the whole problem as follows:

<table>
<thead>
<tr>
<th>One ounce each</th>
<th>Number of grains</th>
<th>Deviation from the average</th>
</tr>
</thead>
<tbody>
<tr>
<td>First counting</td>
<td>308</td>
<td>15.4</td>
</tr>
<tr>
<td>Second counting</td>
<td>332</td>
<td>8.6</td>
</tr>
<tr>
<td>Third counting</td>
<td>328</td>
<td>4.6</td>
</tr>
<tr>
<td>Fourth counting</td>
<td>342</td>
<td>18.6</td>
</tr>
<tr>
<td>Fifth counting</td>
<td>307</td>
<td>16.4</td>
</tr>
<tr>
<td>Sum</td>
<td>1617</td>
<td>Average 63.6</td>
</tr>
<tr>
<td>Average</td>
<td>323.4</td>
<td>deviation 12.72</td>
</tr>
</tbody>
</table>

The average number of grains in one ounce is 320.

**EXERCISE 39-4B**

1. Indicate whether the number appearing in each of the following statements is exact, or approximate.
   a. According to the class registers, the school has 387 pupils.
   b. The town has 14,700 inhabitants.
   c. John received 125 shs for the work done.
   d. During the month, Ali worked 6 days overtime.
   e. The train had been on its way for $3\frac{1}{2}$ days.
   f. The sum of the ages of father, mother and son is 112 years.
   g. The store sold 463 pairs of shoes in a week.
   h. 6200 people visited the museum in a month.
   i. The theatre sold 527 tickets yesterday.
   j. The dairy farm produces 430 quarts of milk a day.
   k. The machine weighs 1325 pounds.
   l. The room is 12 yards, 5 inches long.
   m. The flight lasted 1 hour and 17 minutes.

2. You have probably noticed that the average is always between the smallest and the largest of the numbers that you start with. Explain how you might convince a class that this is always so.

3. On five different walks, a pupil counted the number of steps he made in 100 metres, and obtained the following numbers:
   132, 150, 138, 147, 143.
   What is his average number of steps in 100 metres?

4. Suppose you count the number of people watching a football match. You count six times and get a different result each time. Your counts were
   574, 562, 573, 567, 580, 571.
a. Why do you think you got different results?
b. Find the average of your six counts. Also find the deviations from the average and the average deviation. Indicate in the average of your counts the reliable, questionable and worthless digits.
c. What answer will you finally give for the number of people watching the football match?

5. Make up more problems of this type for your pupils to work out.
Chapter 40
APPROXIMATIONS IN MEASURING

40-1 Approximate measurements

You saw in the previous chapter that the results of counting the number of members in sets are sometimes exact but often they are only approximate numbers. Let us now consider what happens when we measure lengths and weights of objects or periods of time.

What do you think will be the standard unit for measuring lengths of main roads and railways? If a chart indicates that the distance from Dar es Salaam to Nairobi is 498 miles, does it mean that this is an exact number or perhaps that it may be 498.5 or even 497.5 miles? When measuring such great distances, we usually disregard a difference amounting to less than a mile in the final results. This means that for our purpose we are quite satisfied if we find the approximate number of miles, with a precision to one mile. Parts of a mile are in practice neglected.

However, when measuring material for dresses or curtains, we do realize that a difference of even one inch or half an inch is important and has to be taken into account. In such cases, tenths of an inch only can be neglected.

What would you as a teacher say if a pupil was told to draw in his notebook a line segment 2.3 inches long, and his segment was only 2.1 inches long? Would you say that the pupil has done it correctly, because a few tenths of an inch do not matter? In such cases it does matter, because you required him to be precise to the nearest tenth of an inch—the pupil's segment 2.1 inches long is not correct. There are even instances where more precision is important. For example, those who design precision instruments, such as wrist watches, require precision to lengths so small that we cannot observe them with our eyes.

From the several examples above, the following conclusion is easily reached: when we consider measuring lengths in practical life, we see that in each case there is some desired unit of length used, while smaller units are ignored. Measurement of length always gives us an approximate number.

Measurement of time also has different degrees of precision. When an adult is asked to give his age, he will do it in terms of whole years. A mother expresses the age of her small child in terms of years and months, neglecting days. The length of a class lesson or of a football match is usually given in hours and minutes, ignoring seconds. However, in such sports as running or swimming, seconds and even tenths of seconds are counted.

EXERCISE 40-1A
1. In a way similar to our discussion of measuring length and time, explain how approximate numbers are obtained when weighing various objects.
2. What unit of weight is usually used in each of the following cases? What units can be neglected in each case?
   a. A shopkeeper weighing sugar
   b. A postman weighing letters
   c. A nurse weighing a new-born baby
   d. A doctor weighing an adult

3. What is the degree of precision used in railway and airline time-tables?

We have shown how approximate numbers are obtained when we measure quantities—lengths, weights and periods of time—and how in each case the appropriate unit of measurement is chosen according to the need. (In this unit, we will use the term "quantity" rather informally to denote things that can be measured.)

On the other hand, it is also very important to understand that we can never get an exact number from any of these measurements of quantities. Among the essential reasons for this impossibility are:

(a.) the inaccuracy of measuring instruments,
and
(b.) the inaccuracy of human senses.

In some cases, repeated measurements of the same quantity could not give even the same approximate number because of

(c.) the changing conditions under which the successive measurements are made.

You should discuss reasons (a) and (b), and give examples. To discuss (c), think, for example, of the influence of the temperature on the length of an object, or of the evaporation of a liquid whose weight is to be found.

Conclusions

Every measurement gives only an approximate value of what is measured, and it is carried out with a certain definite precision. When we record the result of measuring, we show which units have been considered and which ignored.

(REMINDER The results of measuring are always approximate numbers. As we have seen in the previous chapter, the numbers obtained as results of counting are sometimes exact numbers and sometimes approximate numbers.)

EXERCISE 40-1B

1. What definite standard unit is used in each of the following, in order to get reasonable measurements? What units can be neglected in each case?
   a. An architect designing a house
   b. A surveyor mapping a city
   c. A shoemaker taking the size for a pair of shoes

40-2 Basic agreement for recording approximate numbers

There is a method of recording the results of counting and measuring, showing clearly the precision of these results. This method will be applied in the following example.
Problem. To measure with a foot ruler, having marks of tenths of an inch, the length of the diagonal of a square whose side is 4 inches. We use the symbol < for "is less than", and the symbol \( \approx \) for "is approximately equal to", to write the result as follows.

![Diagram of a square with diagonal to be measured]({})

<table>
<thead>
<tr>
<th>Ruler</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>4 inches</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Diagonal to be measured</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. 5 in. \( < x < 6 \) in.
2. \( x = 6 \) in.
3. 5.6 in. \( < x < 5.7 \) in.
4. \( x = 5.65 \) in.

Explanations.

1. \( x \), the length to be found, is between 5 and 6 inches to the nearest inch.
2. Measured to the nearest inch, \( x \) appears to be nearer to the 6-inch mark than to the 5-inch mark.
3. Measuring to the nearest tenth of an inch, we see from the figure that \( x \) lies between 5.6 inches and 5.7 inches.
4. From the figure, we are unable to decide whether \( x \) is nearer to 5.6 inches or 5.7 inches and so we may conclude that \( x = 5.65 \) inches, taking the average of 5.6 and 5.7.

In our final result, we have two reliable digits: namely, the digit for ones (5) and the digit for tenths (6). The third digit (5) for hundredths is questionable. The final result of our measure-
ment does not allow us to say that length $x$ contains exactly $5$ hundredths of an inch beyond $5.6$ inches.

(Actually, if we had made a very precise drawing and had used a more accurate and precise ruler, we might have found that the length $x$, with a precision to the nearest thousandth of an inch, is equal to $5.657$ inches. Therefore, taking a precision to one hundredth of an inch, it is correct to write $x = 5.66$ inches.)

Let us consider next an example of measuring and recording temperature. What does the recording "$T = 37^\circ C$" mean? It says that we measured with a precision to one degree. On the other hand, if with a more precise thermometer we record "$T = 37.0^\circ C$", the "0" indicates that we measured with a precision to one tenth of a degree.

The examples above of recording approximate numbers are based on the following agreement.

**BASIC AGREEMENT**

An approximate result should be recorded in such a way that its last digit to the right indicates its precision. All digits, except the last, ought to be reliable. Only the last digit is questionable and may be slightly inaccurate.

### 40.3 Repeated measurements

As we have mentioned before, it often happens that when we measure the same quantity again we get a somewhat different result, even though we use the same instrument each time. This happens frequently in measuring long distances. In such situations, we obtain the most precise result by finding the average of all the results of the repeated measurements. (This we do in the same way as when dealing with several counts of a large set.) The average is then rewritten, preserving all reliable digits and only one questionable digit. In order to know which digits in the average are to be kept, it is useful to find the average deviation. In the previous chapter, we studied how this is done. Let us illustrate the method by an example, measuring length in metres.

**Example.** We are to measure the length $x$ of a building, using a metric ruler marked for centimetres. The results of six successive measurements are as follows.

<table>
<thead>
<tr>
<th>Measuring length $x$</th>
<th>Result in metres</th>
<th>Deviation from average in metres</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. 1</td>
<td>51.63</td>
<td>0.352</td>
</tr>
<tr>
<td>No. 2</td>
<td>52.12</td>
<td>0.138</td>
</tr>
<tr>
<td>No. 3</td>
<td>52.20</td>
<td>0.218</td>
</tr>
<tr>
<td>No. 4</td>
<td>51.87</td>
<td>0.112</td>
</tr>
<tr>
<td>No. 5</td>
<td>51.91</td>
<td>0.072</td>
</tr>
<tr>
<td>No. 6</td>
<td>52.16</td>
<td>0.178</td>
</tr>
<tr>
<td><strong>Sum</strong></td>
<td><strong>311.89</strong></td>
<td><strong>1.070</strong></td>
</tr>
<tr>
<td><strong>Average</strong></td>
<td>$311.89 : 6 = 51.982$</td>
<td>$0.178$</td>
</tr>
</tbody>
</table>

The highest order digit of the average deviation is the tenths place. We thus conclude that the tenths digit in the number $51.982$ is questionable, and therefore the digits of hundredths and of thousandths are to be rejected as worthless. Moreover,
51.9 < x < 52.0.

Of these two numbers, 52.0 is closer to the average that we calculated, so we accept it as the final result:

\[ x = 52.0 \text{ m}. \]

**EXERCISE 40-3A**

1. Indicate the reliable, questionable and worthless digits in each of the following approximate numbers. Write down each of these numbers according to the basic agreement.
   a. 254.3 with a precision to the nearest one
   b. .2502 with a precision to the nearest hundredth
   c. 52.03 with a precision to the nearest tenth

2. Five weighings of the same object gave the following results (in pounds):
   2.834, 2.832, 2.837, 2.833, 2.835
   a. Find the average weight.
   b. Indicate in the average the reliable, questionable and worthless digits.
   c. Write the final result according to the basic agreement.

3. Four measurements of the same distance have given the following results (in yards):
   2648, 2656, 2663, 2678
   a. Find the average of the four numbers.
   b. Find the deviations from the average and the average deviation.
   c. Indicate in the average the reliable, questionable and worthless digits, and write the final result according to the basic agreement.
Chapter 41

ROUNDING OFF

41-1 Introduction

We have previously discussed two situations in which we obtain approximate numbers: counting and measuring. We will consider now a third way of getting such numbers.

You have already done some problems in arithmetic in which you had to record the resulting answer "to the nearest ten" or "to the nearest unit". What you actually had to do was to replace your answer (a natural number, decimal fraction or unending decimal) with a simpler number close to it. The simpler number was to have fewer non-zero digits. Such replacement is called rounding off.

The following examples illustrate the process of rounding off.

Example 1. The census shows that a certain city has 246,143 inhabitants. Suppose a friend of yours asks you how many people live in that city. If you know that he does not need a very precise answer, would you say to him 246,143 people? Of course not. You would probably simply answer 246 thousand.

Example 2. There are certainly cases when results ought to be expressed to the highest degree of precision possible. For example, the assets of a bank must be recorded in the yearly report to the nearest pound. For general information, however, it is sufficient to know that the assets of a bank are 57 million pounds rather than 56,967,146 pounds.

In the two examples, certain numbers were rounded off. The results of rounding off are clearly approximate numbers.

41-2 Rounding up and rounding down

Rounding off can be done in two ways: we can "round off upwards" (Example 2) or "round off downwards" (Example 1). To avoid long phrases, we shall call these two ways "rounding up" and "rounding down", respectively.

Rounding off numbers is easy. To round down a number to a digit in a certain place, we replace all the digits of the number written to the right of that place by zeros. For example, 274 rounded down to tens is 270, 27.4 rounded down to ones is 27, 27.4 rounded down to tens is 20.

To round up a number to a certain place, we add one to the digit in that place and replace all digits to the right of it by zeros. For example,

- 274 rounded up to tens is 280,
- 27.4 rounded up to ones is 28,
- 27.4 rounded up to tens is 30.
Consider this complete example of rounding off the number 217.5073:

<table>
<thead>
<tr>
<th>Number</th>
<th>Rounded Down to Hundreds</th>
<th>Rounded Up to Hundreds</th>
<th>Rounded Down to Tens</th>
<th>Rounded Up to Tens</th>
<th>Rounded Down to Ones</th>
<th>Rounded Up to Ones</th>
<th>Rounded Down to Tenths</th>
<th>Rounded Up to Tenths</th>
<th>Rounded Down to Hundredths</th>
<th>Rounded Up to Hundredths</th>
<th>Rounded Down to Thousandths</th>
<th>Rounded Up to Thousandths</th>
</tr>
</thead>
<tbody>
<tr>
<td>217.5073</td>
<td>200</td>
<td>300</td>
<td>210</td>
<td>220</td>
<td>217</td>
<td>218</td>
<td>217.5</td>
<td>217.6</td>
<td>217.50</td>
<td>217.51</td>
<td>217.507</td>
<td>217.508</td>
</tr>
</tbody>
</table>

It is clear that in all these cases the original number is increased by rounding up and decreased by rounding down.

You may ask when we apply rounding up, or rounding down. The answer to this question is often suggested by the situation we are dealing with or by the conditions of the problem.

Example 3. We want to divide 50 shs equally among 6 people. How much will each get?

To get the answer, you must divide 50 by 6. But the quotient of 50 : 6 results in the unending decimal 8.33... . Rounding down to the order of ones gives us 8 shs and rounding down to the order of hundredths gives us 8.33 shs. If everyone gets 8 shs, we remain with 2 shs. If everyone gets 8.33 shs, there will be 2 cents left over because 8.33 × 6 = 49.98 shs. The latter is no doubt the best we can do, because a cent is the smallest coin. Rounding down is certainly the only appropriate procedure here, because if we round up 8.33... to hundredths we get 8.34 shs. We cannot give everyone 8.34 shs, because 8.34 × 6 = 50.04 shs and there are only 50 shs to be shared.

Example 4. A group of 14 pupils decided to collect 100 pounds of oranges for the children of an orphanage. How many pounds should each pupil collect?

100 : 14 = 7.14285714... . Rounding down is not applicable here because less would be collected than aimed for. In this case, it is necessary to round up to get at least 100 pounds of oranges. Rounding up to ones, we get 8. So if each pupil collects 8 pounds, together they get 8 × 14 = 112 pounds, which is substantially more than wanted. Rounding up to tenths gives 7.2 pounds for each pupil, and in all they collect 7.2 × 14 = 100.8 pounds, which is quite close to the desired 100 pounds and reasonable from the point of view of weighing oranges. A practical answer to our problem is that each pupil should collect at least 7.2 pounds of oranges. If 100 shs (and not 100 pounds of oranges) were to be collected by the students, it would have been proper to round up to hundredths to obtain 7.15 shs for each student. Altogether they would then collect 7.15 × 14 = 100.10 shs.

41-3 Fundamental rules for rounding off

The examples in the last section illustrate two cases when the conditions of the problem actually show whether a given number is to be rounded down or rounded up. It was also clear to what place the rounding off should be made. The question naturally arises: What kind of rounding off is to be applied when there is no indication what to do?

As you have seen, rounding down replaces the given number by a second number which
is smaller than the given number, while rounding up replaces the given number by a number which is larger. When a given number is to be rounded off and there is no special indication whether it should be up or down, it is reasonable to round off so that the number obtained differs as little as possible from the original number. For example, the number 17.384 rounded down to ones is 17, which is .384 less than the given number. On the other hand, 17.384 rounded up to ones gives 18, which is .616 greater than the original number. Certainly 17 is closer than 18 to the original number. So here it is better to round down.

Suppose now we want to round off to tenths instead of ones. You see that 17.384 rounded down to tenths is 17.3, which is .084 less than the given number. But rounding up to tenths gives 17.4, which is only .016 greater than the given number. Therefore, the better result in rounding off to tenths is obtained by rounding up.

We see that if it is permissible either to round up or to round down a given number, it is better to round down when the first rejected digit is less than 5 and to round up if the first rejected digit is greater than 5. In each of these cases, we will obtain a closer approximation; that is, the rounded-off number is closer to the original number.

Suppose you want to round off .2604 to tenths. It is better to round up to .27, because that differs from .2604 by .0096. Rounding down results in .26, which differs by .0604 from .2604. If, however, we have to round off the same number .2604 to hundredths, we should round down since

\[ .2604 - .26 = .0004 \]
\[ .27 - .2604 = .0096. \]

You may notice that we have not said how to round off numbers in which the first rejected digit is 5. We consider here the following two cases.

1. The first rejected digit is 5 and it is followed by digits some of which are non-zero digits. For example,

   \[ \text{round off 43,503 to thousands} \]
   \[ \text{and} \]
   \[ \text{round off } .257 \text{ to tenths}. \]

   It is easy to see that here we get a closer approximation by rounding up. Show that this is so. Thus, to round off 43,503 to thousands, we round up and obtain 44,000. To round off .257 to tenths, we round up and get .3. Therefore, if the first rejected digit is 5 and is followed by digits, some of which are non-zero digits, then we round up.

2. The first rejected digit is 5 which is followed by zeros only, or the first rejected digit is 5 and it is the last digit in our number. For example,

   \[ \text{round off 43,500 to thousands}, \]
   \[ \text{round off } 45 \text{ to tens} \]

   and

   \[ \text{round off } 7.5 \text{ to ones}. \]

   If we round down 43,500 to thousands, we obtain 43,000; if we round up, we get 44,000. Each of these rounded-off numbers differs from the original number by 500. We may say that they are "equally close" approximations. The same remark applies to rounding off the other two numbers. In cases like these, we simply agree to round up. Therefore,

   \[ 43,500 \text{ rounded off to thousands is } 44,000, \]
   \[ 45 \text{ rounded off to tens is } 50, \]
   \[ 7.5 \text{ rounded off to ones is } 8. \]
(In some treatments of approximate numbers, the following agreement is made, which we will not use in this text.

If the first rejected digit is 5 which is followed by zeros only, or if the first rejected digit is 5 and it is the last digit, then we round down if the digit before 5 is even, and round up if the digit before 5 is odd.)

Here is our fundamental rule for rounding off numbers. We will always apply it if there are no special reasons to either round down or round up.

If it is permissible either to round up or to round down a given number, we round it down when the first rejected digit is 0, 1, 2, 3 or 4, and round it up if the first rejected digit is 5, 6, 7, 8 or 9.

You should have already seen that the result of rounding off is always a number which represents an approximate value of the given number. It is an approximate number. The difference between the given number and the rounded-off number depends entirely on the way the rounding off is done. If a given number is to be rounded off to a certain place and if it is known that we have to round down or have to round up, then the difference between the result and the given number does not exceed but may come close to one unit in the last place preserved. If, however, the problem does not show us which way to round off, we will use the fundamental rule. Then the difference between the given number and the result will never be more than one-half of the unit in the last place kept. If an approximate value of a quantity differs from its exact value by not more than one-half of a unit in the last place kept, then we say that all digits of the approximation are accurate. Therefore, if we obtain an approximation by applying the fundamental rule for rounding off, then all the digits in the approximation are accurate. For example: if we rounded off the number \(2 \frac{5}{7} = 2.7142\ldots\) to hundredths, we would get the approximate number 2.71 with all digits accurate.

**EXERCISE 41-3A**

1. Round down to tens each of the following numbers and find the error of rounding down (the difference between the number and the rounded-down number).
   - 503; 817; 4,305; 21,658; 12,814; 17,15

2. Round off to tens each of the numbers in Question 1 and find the error of rounding off (from the larger number, subtract the smaller number).

3. Round up to thousands each of the following numbers and find the error of rounding up.
   - 23,458; 17,501; 13,709; 60,500; 100,998; 365,651; 1,349,673

4. Round off to thousands each of the numbers in Question 3 and find the error of rounding off.

5. Round off to ones each of the following numbers and find the error of rounding off.
   - 0.8; 2.55; 3.7; 15.5; 41.4; 0.379; 0.49; 1.813

6. Round off to tenths each of the following numbers.
   - 0.512; 11.395; 4.03; 6.15; 4.08; 6.17; 10.0098

7. Round off to hundredths each of the following numbers.
   - 9.647; 12.784; 0.231; 1.054; 19.6723; 0.455

8. a. Indicate the reliable, questionable and worthless digits in each of the following approximate numbers.
   - b. Round off each number to the place of its questionable digit.
   - c. Write down each number according to the basic agreement of Chapter 40.
343 with a precision to the nearest ten;
6750 with a precision to the nearest hundred;
47.0983 with a precision to the nearest hundredth;
9.0015 with a precision to the nearest thousandth.

9. Three experiments to find the weight in grams of 1 cubic centimetre of the same piece of iron gave the following results: 7.62, 7.80, 7.64. Find the average. Indicate in it the reliable, questionable and worthless digits. Round off the average to the place of the questionable digit. Write down the final result according to the basic agreement.
Chapter 42
MAXIMUM ERROR AND RELATIVE ERROR

42-1 Maximum error—precision

We have seen that every measurement of length, weight, time and so on can be made only approximately and the result is an approximate number. Even when it is possible to find an exact number (in counting the members of a set, for example), it is sometimes sufficient to know only its approximate value.

Here is an example. A pupil worked after school and saved money for a holiday. The exact amount was 101.30 shs. When asked how much money he had saved, he answered "about 100 shs". It is clear that the exact number representing his savings and the approximate number he gave are different. The pupil got the approximate number by rounding off. Similarly, the exact value of a measured quantity and the result of measuring are different.

The difference between the exact value of a measured or counted quantity and its approximate value is called the maximum or absolute error. In the quoted example, the maximum error is equal to 1.30 shs.

You know already that exact values are known only very rarely, for example, in some cases of counting. This means, of course, that the actual value of the maximum error can very seldom be found exactly. However, in carrying out various measurements we can usually give the bounds or limits of the maximum error. In other words, we can expect to find out that the maximum error does not exceed a definite number.

For example, if you weigh an object on a shop scale, the maximum error will usually not be more than one ounce. But on laboratory scales, you can weigh an object so that the maximum error is no more than one-half of one hundredth of an ounce (that is, $\frac{1}{200}$ of an ounce).

42-2 Relative error—accuracy

We must, however, realize that the maximum error does not give us an idea of the quality or accuracy of the measurement. In other words, the maximum error does not indicate how accurately the measuring has been done. The maximum error tells us only about the precision of the measurement. For example, if the maximum error in a measurement is $\frac{1}{2}$ yard, then the measurement is made with a precision to the nearest yard. Conversely, if a weighing is made with a precision to the nearest pound, then the maximum error in the weighing is $\frac{1}{2}$ pound.
Suppose we made two measurements with a maximum error of $\frac{1}{2}$ inch: the first was of the length of a room, and we obtained 20 yards; the second was of the length of a book, and we obtained 12 inches. We say that each measurement was made with a precision to the nearest inch.

It is clear that the first measurement was done very carefully and is of high quality, but the second measurement is quite rough and unsatisfactory.

The same can be said about weighing. An error of one ounce in 50 pounds is usually not important. But an error of one ounce in $\frac{1}{2}$ pound can seldom be allowed. You can now see that to evaluate the quality of a measurement, it is not the maximum error that is important. Instead, it is how the maximum error compares with the measured value itself. In other words, we would like to know what part of the measured quantity the maximum error represents. Let us go back to the measurements of the lengths of the room and the book.

In measuring 20 yards, an error of $\frac{1}{2}$ inch is only $\frac{1}{1440}$ part of the length. However, in measuring the length of the book an error of $\frac{1}{2}$ inch is $\frac{1}{24}$ part of the measured quantity. The fraction obtained by dividing the maximum error by the measured value is called the relative error. As we have seen before, we do not know how good or accurate a measurement is by knowing the maximum error alone. It is the relative error that tells how accurate the measurement is. For example, we can compare the relative errors when measuring the room and the length of the book. These are the numbers $\frac{1}{1440}$ and $\frac{1}{24}$, respectively. The first fraction is 60 times smaller than the second fraction, so the accuracy of the first measurement is much higher than the accuracy of the second.

It is usual to express the relative errors as percentages. Then it is easy to compare the accuracy of two different measurements. The relative error in measuring the length of the room is $\frac{1}{1440} \times 100 = 0.0695\%$, and the relative error in measuring the length of the book is $\frac{1}{24} \times 100 = 4.17\%$. Certainly, a measurement with a relative error of 0.0695% is much more accurate than a measurement with a relative error of 4.17%.

A special notation is often used to show the precision of a measurement. Suppose we measure a certain length $d$ with a precision to the nearest inch, and obtain the result 132 inches. This result is then written in the form

$$d = 132 \ (\pm \cdot 5) \ \text{inches},$$

since $\cdot 5$ inch is the maximum error here.

**EXERCISE 42-2A**

1. Find the maximum error of the approximate number $\cdot 66$, if its exact value is $\frac{2}{3}$.

2. Find the maximum error for each of the fractions

$$\frac{2}{7}, \frac{5}{13}, \frac{4}{19},$$

expressed by the approximations

$\cdot 28, \cdot 384, \cdot 2105,$

respectively.
3. Find the relative error (in percentage) of the approximate number 5.47, if its maximum error is .005.

4. Express the number $\frac{52}{7}$ by an approximate decimal fraction with a precision to the nearest hundredth. Find the maximum error and the relative error (in percentage) of the approximate number.

5. The width of a narrow street measured with a precision to the nearest ten centimetres is 7.6 metres. The length of the street measured with a precision to the nearest metre is 76 metres. Which of these measurements is more precise? Is one of the measurements more accurate than the other?

6. Measuring a segment of length 8.75 centimetres, we made an error of .25 centimetres. Measuring another segment of length 10.5 metres, we made an error of 25 centimetres. Which of the two measurements is more accurate?

7. Measuring a segment 25 inches long, a student obtained 25.2 inches. Find the relative error of the measurement.

8. The volume of a container is 25 cubic inches. A pupil, however, computed the volume as 24.6 cubic inches. Find his maximum and relative errors.

9. Using the formula on the relationship between an approximate number, the maximum error and the relative error, complete the table.

<table>
<thead>
<tr>
<th>Approximate Number</th>
<th>Maximum Error</th>
<th>Relative Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>.05</td>
<td></td>
</tr>
<tr>
<td>.654</td>
<td>.001</td>
<td></td>
</tr>
<tr>
<td>48.4</td>
<td></td>
<td>.1%</td>
</tr>
<tr>
<td>.348</td>
<td></td>
<td>.5%</td>
</tr>
<tr>
<td>260</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3.40</td>
<td></td>
<td>.5%</td>
</tr>
</tbody>
</table>
Chapter 43
DECIMAL PLACES AND SIGNIFICANT DIGITS

43-1 Decimal places

We have learned how approximate numbers are obtained from counting, measuring and rounding off. We have seen that counting sometimes gives exact values, while measuring and rounding off always give approximate numbers.

In order to be able to discuss and understand operations such as addition and multiplication of approximate numbers, we have to study in more detail the notion of the precision of an approximate number. From one point of view this was done in Chapter 42. We will now consider two new ideas which are also closely related to the notion of precision: decimal places and significant digits.

Do you remember what the decimal places of a number are? You studied them in Chapter 23. All digits of a number written to the right of the decimal point are called the decimal places of the number. For example, the numbers 7.2, 6.03, .417 have one, two and three decimal places, respectively. The number 46 has no (or zero) decimal places.

43-2 Significant digits

The concept of significant digits is a harder one, and we will have to develop it in several successive stages. In dealing with exact numbers, you may have heard the term significant digit. First of all, any digit from 1 to 9 appearing in a number are significant, because each of these digits shows a definite number of units in the place where that digit appears. For example, in the number 56.71, there are four significant digits: 5, 6, 7 and 1. To see this, $56.71 = 5 \times 10 + 6 \times 1 + 7 \times \frac{1}{10} + 1 \times \frac{1}{100}$, because the 5 shows that the number contains 5 tens, the 6 shows that the number contains 6 ones, and similarly the 7 and the 1 show 7 tenths and 1 hundredth.

In the same way each digit zero that is between digits from 1 to 9 is also called significant. In the number 603, the digit 0 is a significant digit because it indicates that there are no, or zero, tens in the number which has 6 hundreds and 3 ones.

On the other hand, the number 0.23 has only two significant digits, 2 and 3. You see that this number has 2 hundredths and 3 thousandths, and this completely describes it. The digit 0 here is not a significant digit, because it is only used to locate the decimal point. It performs the role of a place (position) holder only, and we will not regard it as a significant digit. In a decimal fraction, all 0 digits to the left of the first non-zero digit are not significant digits. The numbers 0.001, 0.25, 0.0305 have one, two and three significant digits, respectively.
Let us now consider the digit 0 and its meaning when it is written at the end of a decimal fraction. Here it is important to know whether the decimal fraction is an exact number, or an approximate number.

If the decimal fraction is an exact number, the digits 0 written at the end do not have any significance. The decimal fractions 3.8, 3.80, 3.8000 represent the same number. Therefore, a digit 0 when written at the end of an exact decimal fraction is not significant, and it does not make any difference whether we omit the zero or write it.

The situation is completely different when the digit 0 is written at the end of a decimal fraction which represents an approximate number. We will show that the 0 has in this case a definite meaning. Consider, for example, the two approximate numbers 3.8 and 3.80, differing only by the digit 0 at the end. These two decimal fractions represent two different approximate numbers for the following reason.

The approximate number 3.8 could have been obtained from rounding off to tenths such numbers as 3.81, 3.82, 3.83, 3.84, or 3.75, 3.76, 3.77, 3.78, 3.79. This means that originally in our number there might have been hundredths or thousandths, but the number was rounded off to tenths. Suppose the approximate number 3.8 was obtained by measuring. Then the digit on the right (8) is in the tenths place and shows that the measurement was made with a precision to the nearest tenth.

If, however, an approximate number is written as 3.80, it means that the ones (3), the tenths (8) and the hundredths (0) are known to us. If 3.80 were obtained by rounding off to hundredths, the original number might have had thousandths. Suppose 3.80 is obtained by measuring. Then the digit on the right (0) is in the hundredths place and shows that the measurement was made with a precision to the nearest hundredth.

We see that the digit 0 appearing at the end of an approximate decimal fraction has a definite meaning, and is therefore to be considered as a significant digit.

We will now give special attention to approximate values written as whole numbers. An approximate whole number may contain zeros at the right-hand end. Such a zero is a significant digit if it shows the absence of units in its place. But often a zero at the end replaces a worthless or unknown digit. Then the zero is not a significant digit.

Let us look at an example. Suppose the approximate value of a weight is 14.7 kilograms. This number has three significant digits. If we express this approximate value in grams, we get the number 14700, because there are 1000 grams in a kilogram. This number also has only three significant digits, because the two 0 digits at the end replace unknown digits.

If however the approximate number 14700 grams was obtained by using a more precise scale, which weighs with a precision to the nearest gram, then this approximate number 14700 has five significant digits. To express this approximate number in kilograms, we would have to write it as 14.700 kilograms. The last zero at the end is written in the thousandths place. This says that the measurement was made with a precision to the nearest thousandth of a kilogram; that is, to the nearest gram.

Thus, there is a difficulty in reading an approximate whole number ending in zeros. We know that the number of significant digits in an approximate whole number with digits 0 at the end depends on the precision of that number. For example, if we look at the approximate number 2400, as it is written down, we cannot decide in which of the following three ways it was obtained.

1. The number 2400 may be the result of rounding off to the nearest hundred or of measuring with a maximum error of 50. Then neither zero is a significant digit.
2. It may be the result of rounding off to the nearest ten or of measuring with a maximum error of 5. This makes the first zero a significant digit, and the second zero not a significant digit.
3. It may be the result of rounding off to the nearest one or of measuring with a maximum error of $\frac{1}{2}$. Then both zeros are significant digits.

We summarize our discussion in the following detailed statement of the meaning of a significant digit in an approximate number.

If an approximate number is written according to the basic agreement in Chapter 40, then:

1. Any of the digits 1, 2, ..., 9 are significant.
2. Any digit 0 at the right-hand end of an approximate decimal fraction is significant.
3. Any digit 0 at the right-hand end of an approximate whole number in the place showing the precision of the approximate number (or of the measurement) is significant.
4. Any digit 0 between significant digits is significant.

43-3 Examples

The approximate decimal fractions 8.2, 7.06, .1230, .061 have two, three, four and two significant digits, respectively.

250 precise to the nearest one has three significant digits, because the 0 is in the ones place showing the precision.

2500 precise to the nearest ten has three significant digits. The first zero from the left is a significant digit, since it is in the place showing the precision. The last zero (in the ones place) is not significant.

2500 precise to the nearest one has four significant digits, because the last zero on the right is in the place showing the precision; and the zero in the tens place is significant, because it is between two significant digits, the 5 and the 0 at the end.

2050 precise to the nearest one has four significant digits.

Let us look at some other examples of rounded-off numbers. If we round off the number 2803 to tens, we obtain the approximate number 2800 with three significant digits. If 2803 is rounded off to hundreds, we also obtain the approximate number 2800, however with only two significant digits.

Consider a rod measured to be 124 millimetres long with a precision to the nearest millimetre (10 millimetres make a centimetre). The number 124 has three significant digits. If we round it off to tens, we obtain the approximate number 120, containing only two significant digits.

In order to avoid any misunderstanding concerning the digits 0 at the end of approximate whole numbers, it is better to leave out the 0 digits which replace worthless or rejected digits (that is, which are not significant) and to change to larger units. For example, when rounding off the number 83,542 to hundreds, it is better to write 83,500 rather than 83,500. If the three zeros at the end of 3,569,000 square metres are not significant, it would be better to write 3.569 square kilometres (1 square kilometre equals 1,000,000 square metres).

Sometimes it is not convenient to write an approximate number in larger units and drop the non-significant 0 digits. Then it would be important to say which of the zeros are worthless. One way of doing this is to underline the questionable digit, as we have done in Chapter 39. For example,

if \( x \approx 36 \) kilometres, then \( x \approx 36,000 \) metres
\[(\text{two significant digits});\]

if \( y \approx 8.4 \) metres, then \( y \approx 840 \) centimetres
\[(\text{two significant digits});\]
or \( y = 8400 \text{ millimetres} \)

(two significant digits).

We would be allowed to write 36 kilometres = 36,000 metres only if we measured the distance with a precision to the nearest metre.

For a final review of the notions of decimal places and significant digits, look at this list of approximate numbers:

- 7, one significant digit, no decimal places
- \( \cdot 7 \), one significant digit, one decimal place
- 0.07, one significant digit, two decimal places
- \( \cdot 070 \), two significant digits, three decimal places
- \( \cdot 37 \), two significant digits, two decimal places
- 2.037, four significant digits, three decimal places
- \( \cdot 0370 \), five significant digits, four decimal places.

43-4 Comparison of approximate numbers

We should now point out that in order to give an idea of the precision and accuracy of an approximate number, we can tell the number of its decimal places or the number of its significant digits.

The method of counting the number of decimal places is recommended, when we compare approximate values of the same quantity. For example, the first weighing of an object is 14.7 grams, and the second weighing, using a more precise scale, is 14.684 grams. The second approximate value is clearly more accurate and more precise than the first, because it has three decimal places, whereas the first only has one decimal place.

In changing from one unit of measurement to another in the metric system, the number of decimal places changes. But the number of significant digits remains unchanged. For example, 254 centimetres = 2.54 metres. The number of significant digits in each number is three. But the first number has no decimal places, while the other has two. For this reason, it is a good idea to compare the accuracy of various approximate numbers by counting the number of their significant digits. For example, if measuring a segment resulted in the number 6.3 centimetres, and measuring the length of a field gave the number 254 metres, we must admit that the second approximate number is more accurate that the first, since the second has three significant digits and the first only two.

43-5 Exact whole numbers

The major part of the discussion in this chapter was devoted to the meaning of significant digits in approximate numbers. For the sake of completeness, we give now a rather simple statement on the meaning of significant digits in exact numbers.

In exact whole numbers, all the digits are significant.

By the significant digits of an exact decimal fraction, we mean all its digits except zeros written to the left of its first non-zero digit and zeros written at the right-hand end.

The exact whole numbers 45, 305, 27108, 560,000 have two, three, five and six significant digits, respectively.

The exact decimal fractions \( 8.2, 7.06, \cdot 1230, \cdot 61 \) have two, three, three and two significant digits, respectively.
EXERCISE 43-5A

1. How many significant digits has each of the following approximate numbers, given with a precision to the nearest ten?
   230, 480; 2,080; 81,050; 70,190; 13,700; 12,000; 201,000

2. How many significant digits has each of the following exact numbers?
   230; 480; 2,080; 81,050; 70,190; 13,700; 12,000; 201,000

3. How many significant digits has each of the following approximate numbers, given with a precision to the nearest hundred?
   32,400; 70,300; 190,100; 149,000; 10,050,000

4. How many decimal places and how many significant digits has each of the following approximate decimal fractions?
   8.5; .42; .703; 6.05; 1.003; 201.03; .03; .004; .0005; 2.60; 8.240; 8.040; .070; .2080; .300; 2.500; 603.100; 2004.50

5. How many decimal places and how many significant digits has each of the numbers listed in Question 4, if they are given as exact decimal fractions?

6. Explain the difference between the two recordings:
   "the length of the segment is 12 inches"
   and "the length of the segment is 12.0 inches".

7. Recalling that a kilogram has 1000 grams, express each of the following in kilograms if the 0 digits at the end are significant.
   2,860 grams; 8,700 grams; 250 grams; 23,400 grams

8. Express in kilograms each of the numbers in Question 7 if the 0 digits at the end are non-significant.
Chapter 44
ADDITION AND SUBTRACTION
OF APPROXIMATE NUMBERS

44-1 Introduction

Let us look at a simple problem from everyday life. Find the length of a fence around a rectangular field. To solve this problem, we must first measure the length and the width of the rectangle. Suppose we obtain the approximate numbers 225 yards and 112 yards. To find the answer to our problem, the perimeter of the rectangle, we must add the lengths of the four sides, which are these approximate numbers:

\[ 225 + 225 + 112 + 112 = 674. \]

In this way we are led to perform the operation of addition on approximate numbers. We clearly obtain the approximate number 674 yards.

If we had to find the area of the same field, we would get it by multiplying the two approximate numbers:

\[ 225 \times 112 = 25,200. \]

Thus, in the second case we must perform the operation of multiplication on approximate numbers. The result of 25,200 square yards is clearly also an approximate number.

The question naturally arises, what kind of approximate number this area is. In other words, which digits of the approximate number 25,200 are reliable and which are not? We ask the same question about the length we found for the fence.

We must thus discuss computations with approximate numbers. The results of such operations are also approximate numbers.

As we have seen before, approximate numbers are obtained from counting, measuring and rounding off. We see now that besides these three sources, there is still a fourth source for obtaining approximate numbers, namely from computations or operations. Whenever we calculate with numbers, one or more of which is approximate, the result of the calculation is an approximate number.

We will discover some rules which tell us which digits of such a sum or product are reliable, and how to record the answer according to our basic agreement.

We will study addition and subtraction of approximate numbers in two stages:
1. Addition and subtraction of approximate whole numbers.
2. Addition and subtraction of approximate decimal fractions.
Addition and subtraction of approximate whole numbers

In the previous example of the rectangular field whose length and width are 225 and 112 yards, respectively, we saw that the fence had to be 674 yards long. However, we need to find out how reliable the digits of 674 are. We note that the length 225 yards and the width 112 yards are approximate numbers with a precision to the nearest yard. Thus, the ones digits 5, 5, 2, 2 of the terms in the sum

\[ 225 + 225 + 112 + 112 \]

are questionable, which leads us to believe that certainly the ones digit 4 in the answer 674 is questionable.

Let us now consider a slightly harder problem. In a certain region there are a town with 720 people (counted with a precision to the nearest ten), two villages with 234 and 88 people and farm land with a population of 4,300 people (counted with a precision to the nearest hundred). Find the total population of the region.

Adding the four numbers in the usual way, we obtain:

\[
\begin{array}{c}
4,300 \\
720 \\
234 \\
88 \\
\hline
5,342
\end{array}
\]

Since the terms are approximate numbers obtained by counting various large sets, the number 5,342 is clearly also an approximate number. The question is which digits of this sum are to be kept in the final answer.

In the first number, 4,300, precise to the nearest hundred, the tens digit and the ones digit are unknown to us. We do know the tens digits and the ones digit in the number 88. However, when we add these to unknown digits in the number 4,300, the tens digit and the ones digit in the sum 5,342 remain unknown. We simply have to disregard them in the final result. When we first add the terms, we will take into account the tens digits and the ones digits in those terms in which they are known. But then we will round off the sum obtained to get the final result.

Let us write down the problem as follows. In place of digits unknown to us we will write the letter "U" for "unknown".

\[
\begin{array}{c}
43UU \\
72U \\
234 \\
88 \\
\hline
52UU
\end{array}
\]

Rounding off the sum 5,342 (obtained in the usual way), we reject the worthless digits of tens and of ones and obtain

\[ 5,342 \approx 5,300 \] for the final answer.

We see that in the final result we rejected—that is, replaced by zeros—the digits in those places in the sum for which the digits in even one of the addends are unknown.

We can write down what we have discussed as the following rule:

In adding approximate whole numbers, we reject in the final result (according to the fundamental rule for rounding off) digits in those places in the sum for which the digits...
are unknown even in one of the approximate terms. (This rule will be included in a more general rule later.)

We use this rule even if there are one or more exact numbers among the terms. (In the example above, 88 was an exact number.)

Consider now the following simple problem. From a stock of 480 pounds of sugar, 117 pounds were sold in one day. How much sugar remained?

Here we have to subtract approximate numbers.

\[
\begin{array}{c}
\text{(a)} & 480 \\
-117 & \hline
363
\end{array}
\]

\[
\begin{array}{c}
\text{(b)} & 480 \\
-117 & \hline
373
\end{array}
\]

Taking into consideration the difference 363 obtained in the usual way (a), and the unknown digit in the ones place (b), we round off 363 to tens to obtain the final result, 360 pounds.

It is not hard to see that our rule for adding approximate whole numbers will also apply to subtraction.

44.3 Addition and subtraction of approximate decimal fractions

Suppose two or more approximate numbers, written according to our basic agreement, have the same number of decimal places. Then all but the last digit of each number is reliable, the last digit of each being questionable. We will see that if we simply add the numbers, then all digits in the sum except the last are reliable and the last is questionable. The sum obtained is thus automatically recorded according to our basic agreement. For example:

\[
\begin{array}{c}
18.6 \\
+ 23.9 \\
\hline
42.5
\end{array}
\]

We will verify that this procedure gives the correct results by using some rather extensive reasoning based on the meaning of approximate numbers and on properties of inequalities.

From our work on order properties, we know the following statement to be always true:

If \(a, b, c, d, e, f\) are any numbers such that

\[
c < a < d
\]

and \(e < b < f\),

then \(c + e < a + b < d + f\).

You may read, "If \(a\) is between \(c\) and \(d\), and \(b\) is between \(e\) and \(f\), then \(a + b\) is between \(c + e\) and \(d + f\)." In other words, if two double inequalities hold, then the double inequality obtained by adding the corresponding terms also holds.

Return now to our example. Let \(a\) and \(b\) be the approximate numbers that we are representing by 18.6 and 23.9, respectively.

\[
\begin{array}{c}
18.55 < a < 18.65 \\
\text{and, similarly,} \\
23.85 < b < 23.95
\end{array}
\]

We add and obtain \(42.40 < a + b < 42.60\).

Therefore, \(a + b = 18.6 + 23.9 = 42.5\).

We move now to the case when the approximate terms to be added have different num-
bers of decimal places. We have to be careful here, because the sum obtained in the usual way will contain worthless digits and will have to be rounded off. Consider the following example.

Let a machine weigh $3.507$ kilograms, and let a wooden box, in which the machine is placed, weigh $2.8$ kilograms. What is the total weight of the box with the machine inside?

To find the answer we proceed as follows.

\[
\begin{align*}
(a) & \quad 3.507 \\
+ & \quad 2.8 \\
\hline
6.307
\end{align*}
\]

We will now explain what we have done. In (a) we added the decimal fractions in the usual way, treating the terms as if they were exact, and not approximate, numbers. In (b) we show that such simplified addition of approximate numbers with different numbers of decimal places is inappropriate. In the sum in (a), there are worthless digits. Therefore, the answer must not be written according to our basic agreement. Actually, in the first term we know the ones, tenths, hundredths and thousandths. In the second term we know only ones and tenths, and nothing about the further decimal places. It is, therefore, clear that in the sum the hundredths digit (0) and the thousandths digit (7) do not deserve any confidence at all. They are worthless and ought to be rejected. The addition of the approximate numbers should be done as shown in (c).

Using double inequalities as we did before, we can verify that method (c) gives the correct result. Let $a$ and $b$ be the approximate numbers that we are representing by $3.507$ and $2.8$, respectively.

\[
\begin{align*}
3.5065 < a & < 3.5075 \\
2.75 < b & < 2.85 \\
6.2565 & < a + b < 6.3575 \\
6.2 & < a + b < 6.4 \\
a + b & = 3.507 + 2.8 = 6.3
\end{align*}
\]

Notice that the left-hand sum, $6.2565$, is not rounded off according to the fundamental rule, but is rounded down. The reason is this. If $6.2 < 6.2565$ and $6.2565 < a + b$, then it follows, by transitivity of "less than", that $6.2 < a + b$. But it does not follow that $6.3 < a + b$. Similarly, the right-hand sum is rounded up.

Subtraction can be treated in exactly the same way. For example, let us subtract the approximate number $14.2714$ from the approximate number $42.7$.

\[
\begin{align*}
42.7 \\
-14.2714 \\
\hline
28.4286
\end{align*}
\]

Again the correct way of recording the subtraction is the third one.

We will verify that this procedure for subtracting approximate numbers is correct by again using the meaning of approximate numbers and properties of inequalities. To do it, we first have to obtain some more facts about inequalities.

We saw that if $c < a < d$ and $e < b < f$ are true statements, then $c + e < a + b < d + f$. Do you think the corresponding subtraction inequalities are true? Is $c - e < a - b < d - f$? In fact, this is not true. To show that the statement is false, it is enough to give one instance when it is false. Suppose $a$ and $b$ are such that
2 < a < 3,
and 1 < b < 3.

Then the subtraction inequalities would be

1 < a - b < 0.

But the difference a - b cannot at the same time be greater than 1 and less than 0. So our statement is false.

Is there not another way we can use inequalities in subtraction? It is not hard to see that the following statement does hold:

If c < a and e > b,
then c - e < a - b.

The proof is very simple. c < a and e > b mean the same thing as a < c and e < b. We can add the last two inequalities and obtain

c + (−e) < a + (−b),
that is, c - e < a - b.

Check this statement by substituting any numbers for a, b, c, e. When the last statement is extended to double inequalities, we obtain the following true statement.

If a, b, c, d, e, f, are any numbers such that

c < a < d
and e > b > f
then c - e < a - b < d - f.

On the left, from a number less than a, we subtract a number greater than b.

On the right, from a number greater than a, we subtract a number less than b.

With the help of our new double subtraction inequality, we can see that our method of subtracting approximate numbers is correct. Here is the previous problem.

\[
\begin{align*}
42.65 &< a < 42.75 \\
14.27145 &> b > 14.27135 \\
28.37855 &< a - b < 28.47865 \\
28.3 &< a - b < 28.5 \\
a - b &= 28.4
\end{align*}
\]

From the discussion of the examples on addition and subtraction of approximate numbers, it is clear that even one unknown digit in any place makes the digit in that place in the answer worthless. Therefore, in addition and subtraction of approximate decimal fractions we will use the following rule.

Addition and Subtraction Rule

When adding or subtracting approximate decimal fractions, we preserve only as many decimal places in the result as there are in the approximate term with the least number of decimal places.

We clearly see that our addition and subtraction rule is based on the idea of decimal places.
The addition rule for approximate whole numbers that we have already discovered is really contained here. To see this, let us return to the problem in Section 44-2 about the population of a region. This time let us use a larger unit, say hundreds, to represent the counts. We then have:

- Population of the farm land, 43 hundreds
- Population of the town, 72 hundreds
- Population of the first village, 234 hundreds
- Population of the second village, 88 hundreds

The problem is now reduced to adding approximate decimal fractions, and we apply our addition and subtraction rule.

\[
\begin{array}{ccc}
\text{(a)} & 43 & \text{(b)} 43.\text{UU} & \text{(c)} 43 \\
7.2 & 7.2U & 7.2 \\
2.34 & 2.34 & 2.34 \\
.88 & .88 & .88 \\
53.42 & 52.\text{UU} & 53.42 \\
\end{array}
\]

We round off the sum in (c) to ones because the term 43 has no decimal places.

Thus, we see that to apply our addition and subtraction rule to approximate whole numbers, we only have to avoid zeros which replace unknown or rejected digits and express the approximate terms in larger units.

**EXERCISE 44-3A**

1. Find the sum of the approximate numbers.
   a. .52 + .038 = 0.558
   b. 2130 + 420 = 2550
   c. 2.725 + .6482 = 3.3732
   d. 35200 + .01686 = 35216.86

2. For Question 1a, carry out the complete analysis using double inequalities.

3. Find the differences of the approximate numbers.
   a. 1430 — 510 = 820
   b. 8.53 — .282 = 8.248
   c. 8.53 — .0065 = 8.5235

4. For Question 3b, carry out the complete analysis using double inequalities.

5. A rectangular field has length 1240 yards and width 136 yards. Find the perimeter of the field.

6. A park had 7300 trees. In one year 860 trees were cut down. How many trees remained in the park?

7. A wire was cut up into four parts of lengths 3.54 yards, .756 yards, 8.49 yards, 1.138 yards. Find the original length of the wire.

8. A bottle of milk weighs 2.42 pounds. The weight of the bottle is .543 pounds. What is the weight of the milk?
Chapter 45
MULTIPLICATION AND DIVISION
OF APPROXIMATE NUMBERS

45-1 Introduction

In this chapter we will discuss the remaining two operations on approximate numbers—
multiplication and division. In Chapter 44 we developed a procedure for adding and subtracting
approximate numbers in two stages—first for approximate whole numbers, then for approximate
decimal fractions. At the end of the chapter, however, we showed that addition and subtraction of approximate whole numbers can easily be reduced to the same operations on approximate decimal fractions. We will find the same situation in multiplication and division.
As a matter of fact, we will not even find it important to distinguish between the two types of
approximate numbers. Instead, we will simply discuss a single procedure for multiplying and
dividing approximate numbers.

45-2 Multiplication

Consider a very simple problem of the same kind as at the beginning of Chapter 44.
Suppose the sides of a rectangular field are 254 yards and 194 yards long, measured with a
precision to the nearest yard. To find the area of the field, we multiply 254 and 194 and obtain
49,276 square yards. Since the measures of the sides are approximate numbers, it is clear that
their product only approximately gives the area and probably has to be rounded off. Again,
we must ask which digits of this product should be retained in the final result.
Because of the precision of the measurement, we know that the unknown exact values
of the length and of the width of the field are greater than or equal to 253.5 yards and 193.5
yards, respectively. And we know they are less than 254.5 yards and 194.5 yards, respec-
tively. Therefore, the rectangular area is greater than 49,052.25 square yards (253.5 \times 193.5),
but less than 49,500.25 square yards (254.5 \times 194.5).
As you can see, we have just used the following true statement about inequalities:
If \( a, b, c, d, e, f \) are any non-negative numbers such that
\[
  c < a < d \\
  e < b < f
\]
then \( c \times e < a \times b < d \times f \).
The assumption that all the numbers involved are non-negative is essential. Can you
give an example that shows that the statement is not necessarily true if you allow some of
the numbers to be negative?
We can now set down the solution to our problem as follows

\[
253.5 < a < 254.5 \\
193.5 < b < 194.5
\]

Therefore,

\[
49,052.25 < a \times b < 49,500.25 \\
49,000 < a \times b < 49,500
\]

\[a \times b = 49,300.\]

This longer procedure shows us that the first two digits of our original product 49,276 for the area are reliable (4 and 9), and the third digit (2) is questionable. According to our basic agreement, only these first three digits ought to be preserved. This means that we have to round off the product 49,276 to hundreds to obtain the final result of 49,300 square yards.

This problem showed us that if we multiply two approximate numbers, each having three significant digits, their product is an approximate number also containing three significant digits.

The use of "U" in place of unknown digits leads to the same result:

<table>
<thead>
<tr>
<th></th>
<th>(a) 254</th>
<th>(b) 254U</th>
<th>(c) 254</th>
</tr>
</thead>
<tbody>
<tr>
<td>194</td>
<td>194</td>
<td>1U</td>
<td></td>
</tr>
<tr>
<td>1016</td>
<td>1016U</td>
<td>1016</td>
<td></td>
</tr>
<tr>
<td>2286</td>
<td>2286U</td>
<td>2286</td>
<td></td>
</tr>
<tr>
<td>254</td>
<td>254</td>
<td>254</td>
<td></td>
</tr>
<tr>
<td>49276</td>
<td>492UUUU</td>
<td>49276</td>
<td></td>
</tr>
</tbody>
</table>

Rounding off the product 49,276 to three significant digits as required by (b), we obtain the final answer 49,300 (c).

In a similar way we can show that if we multiply two approximate numbers, one with three significant digits and the other with two significant digits, the product is an approximate number with two significant digits. In general, the product of two approximate numbers will be an approximate number with as many significant digits as the number of significant digits in the factor with the lesser number of significant digits. As we will illustrate later, exactly the same can be said about division.

**Multiplication and Division Rule**

When multiplying or dividing approximate numbers, in the result we preserve as many significant digits as there are in the original approximate number with the lesser number of significant digits.

We see now clearly that our multiplication and division rule is based on the idea of significant digits. On the other hand, our rule for addition and subtraction is based on the idea of decimal places.

Here is an additional illustration of the satisfactory results obtained using our multiplication and division rule. We will consider a problem using common fractions represented by decimal fractions.

Find the product of \(\frac{10}{11}\) and \(\frac{5}{6}\), following the instructions below:

a. Represent the common fractions as decimal fractions.

b. Round off the decimal representing \(\frac{10}{11}\) to four significant digits.

c. Round off the decimal representing \(\frac{5}{6}\) to three significant digits.
\[
\frac{10}{11} = 1.9090 \ldots \approx 1.909 \\
\frac{5}{6} = 1.8333 \ldots = 1.83 \\
1.909 \times 1.83 = 3.49347 = 3.49.
\]

The result is rounded off to three significant digits, as many as there are in the factor with the lesser number of significant digits.

Let us now compare the approximate result with the exact value of the product of the two fractions.

\[
\frac{10}{11} \times \frac{5}{6} = \frac{21}{11} \times \frac{11}{6} = \frac{21}{6} = \frac{7}{2} = 3.50
\]

Our approximate result 3.49 differs from the exact product 3.50 by only one unit in the third significant digit. In fact, we expect the last significant digit of an approximate number to be questionable. Therefore, the result we obtained by rounding off the product to three significant digits, according to our rule for multiplication and division, is satisfactory.

**45-3 Division**

We now consider a problem leading to division of approximate numbers. Measurements show that the weight of a piece of iron is 491 grams and that its volume is 63 cubic centimetres. Find the weight of one cubic centimetre of this iron.

\[
491 \div 63 = 7.79 \ldots = 7.8.
\]

Here the dividend is an approximate number with three significant digits, the divisor an approximate number with only two significant digits. Our rule says that we should preserve two significant digits in the quotient.

Let us verify our solution by assuming that the unknown exact weight of the piece of iron is greater than or equal to 490.5 grams and less than 491.5 grams, and that its unknown exact volume is greater than or equal to 62.5 cubic centimetres and less than 63.5 cubic centimetres.

We know that any quotient of positive numbers decreases if we decrease the dividend or increase the divisor. Also, a quotient of positive numbers increases if we increase the dividend or decrease the divisor.

It follows that:

\[
490.5 \div 63.5 = 7.724 \ldots \text{ is less than the quotient}; \\
491.5 \div 62.5 = 7.8064 \ldots \text{ is greater than the quotient}.
\]

Let us make a general statement about the division of double inequalities. If \(a, b, c, d, e, f\) are any positive numbers such that

\[
e < a < d \\
\text{and } f > b > e \\
\text{then } c : f < a : b < d : e
\]

A number less than an \(a\) is divided by a number greater than \(b\). A number greater than \(a\) is divided by a number less than \(b\).
Do you see a similarity between this statement and the statement involving subtraction of double inequalities?

Can you illustrate by an example that this is not necessarily true if any of the numbers involved are negative?

Here is a solution of our problem using the statement about division of double inequalities.

\[ 490.5 < a < 491.5 \]
\[ 63.5 > b > 62.5 \]

Therefore, \[ 490.5 \div 63.5 < a \div b < 491.5 \div 62.5; \]

that is,

\[ 7.724\ldots < a \div b < 7.864\ldots, \]
\[ 7.7 < a \div b < 7.9, \]
\[ a \div b < 7.8. \]

We see that in the quotient \( 491 \div 63 = 7.79\ldots \), obtained at the beginning of our problem, the first digit is reliable, the second digit is questionable and the remaining digits are worthless. We conclude therefore that our rounding off to two significant digits (7.8) is correct, and in accord with our basic agreement for recording approximate numbers.

We obtain the same result, if we write "U" instead of the unknown digits in the division.

\[
\begin{array}{c}
7.8U \\
63\cdot U \\
\hline
491\cdotUU \\
441 U \\
50\ UU \\
50\ 4U \\
UU \\
\end{array}
\]

45.4 Operations with one approximate number and one exact number

We remark that our multiplication and division rule can also be applied in multiplication or division when one number is an approximate number and the other is an exact number. For example, multiply an approximate number with four significant digits by an exact number. A satisfactory answer is obtained by preserving in the product four significant digits, the same number of significant digits as in the approximate factor. The number of significant digits in the exact number is simply not taken into account.

Multiply the approximate number 24.3 by the exact number 34. In the product 826.2 we keep three significant digits, because the approximate factor 24.3 has three significant digits. Therefore, the answer is

\[ 24.3 \times 34 = 826. \]

We conclude with the following rule.

When applying our multiplication and division rule to multiplication or division of an approximate number by an exact number, we keep in the final answer as many significant digits as in the approximate number (and disregard the number of significant digits of the exact number).
EXERCISE 45-4A

1. Find the products of the approximate numbers.
   a. .53   b. 4800   c. 1928

   .06   .523   .00552

2. For Question 1a, carry out the complete analysis using double inequalities.

3. Find the product of the approximate number .431 and the exact number 54.

4. A shop received 183 boxes. Each box contained 24 pounds of oranges. Find the weight of the oranges received.

5. Find the quotients of the approximate numbers.
   a. .06 ÷ 2.3
   b. 800 ÷ 35
   c. .385 ÷ 27

6. Find the quotient of the approximate number 2,600 divided by the exact number 165.

7. Find the product \( \frac{5}{16} \times \frac{8}{23} \) in two ways.
   a. Represent the common fractions as decimal fractions and take each of them with a precision to the nearest thousandth.
   b. Multiply the common fractions and represent the result as a decimal fraction. Then take it with a precision to the nearest thousandth.

8. Find the quotient \( \frac{16}{3} ÷ \frac{45}{4} \) with a precision to the nearest hundredth in two ways.
   a. Represent each common fraction as a decimal fraction with two decimal places. Divide the decimal fractions.
   b. Divide the common fractions and represent the quotient as a decimal fraction with two decimal places.
Chapter 46

COMBINED OPERATIONS ON APPROXIMATE NUMBERS

46-1 Introduction

In the previous two chapters we learned how to add, subtract, multiply and divide approximate numbers. Many simple problems, however, require a combination of these operations.

For example, suppose measurement of the lengths, width and height of a parallelepiped gives the following approximate numbers:

- length, \( l = 34.7 \) cm.
- width, \( w = 26.8 \) cm.
- height, \( h = 42.1 \) cm.

To find the volume, \( V \), of the parallelepiped we use the known formula

\[
V = l \times w \times h = 34.7 \times 26.8 \times 42.1 \text{ cubic cm.}
\]

We have to perform two multiplications using our rule for multiplying or adding approximate numbers. First we find \( 34.7 \times 26.8 \). Then we multiply the product, an intermediate result, by 42.1 to get the final result. Note the distinction we make between the final result, obtained after the last operation is performed, and the intermediate results obtained at earlier stages.

If we are asked to find the total surface area of the same solid we have

\[
S = (l \times w) + (l \times h) + (l \times h) + (w \times h) + (w \times h)
\]

\[
= (2 \times l \times w) + (2 \times l \times h) + (2 \times w \times h)
\]

\[
= (2 \times 34.7 \times 26.8) + (2 \times 34.7 \times 42.1) + (2 \times 26.8 \times 42.1).
\]

Here we have to multiply six times and add twice.

The rules concerning the order of performing combined operations on exact numbers are also valid for approximate numbers. Thus to find the surface area, \( S \), we proceed in the following order. First we perform all multiplications, and then add the three terms of the sum.

How many intermediate results will we have?

Could we solve problems involving more than one operation on approximate numbers by rounding off the intermediate results by our rules? If we did this we would increase the errors of our approximate numbers even more by rounding off. The accumulation of rounding-off errors could substantially influence the final result. It turns out that in many cases of combined operations, this influence is greatly reduced in the final result if in each of the intermediate results we preserve one more digit than our rules for operations on approximate numbers tell.
us to preserve. Let us underline these extra digits. In the final result, however, we will reject the extra digit. Here is the rule we will use.

Intermediate Result Rule

_In solving problems which involve more than one operation on approximate numbers, we preserve in the intermediate results one more digit than recommended by our rules for operations on approximate numbers. In determining the number of significant digits in an intermediate result, the extra digit is not counted, according to our basic agreement for rounding off approximate numbers._

46-2 An example

We will now illustrate this procedure for combined operations. Compute the value of the quotient

\[
\frac{2 \frac{13}{16} \times 1 \frac{7}{18}}{(2 \frac{13}{16} - 1 \frac{7}{18}) \times \frac{2}{7}}
\]

Let us first represent the given common fractions as decimal fractions and then round them off to hundredths. We have

\[
\frac{13}{16} = 2.8125\ldots = 2.81,
\]
\[
\frac{7}{18} = 1.3888\ldots = 1.39,
\]
\[
\frac{2}{7} = 0.2857\ldots = .29.
\]

Now we must compute

\[
\frac{2.81 \times 1.39}{(2.81 - 1.39) \times .29}.
\]

(1) Intermediate result:

\[
\begin{array}{c}
2.81 \\
1.39 \\
2529 \\
843 \\
281 \\
39059 \\
3.906
\end{array}
\]

(2) Intermediate result:

\[
\begin{array}{c}
2.81 \\
1.39 \\
2.81 \\
1.42 \\
.29 \\
12.78 \\
28.4 \\
.4118 \\
.412
\end{array}
\]

(3) Intermediate result:

\[
\begin{array}{c}
1.42 \\
12.78 \\
28.4 \\
.4118 \\
.412
\end{array}
\]

(4) Final result:

\[
3.906 \div .412 = 9.48 \approx 9.5
\]

Final answer \( \approx 9.5 \)
As the first operation, we multiplied two approximate numbers with three significant digits each. But we rounded off the product to four significant digits, underlining the last digit (6) as the extra digit.

The second operation gave as a difference an approximate number having two decimal places and three significant digits. Note that there was no extra digit to preserve.

As the third operation, we multiplied approximate numbers with three and two significant digits. The product was rounded off to three significant digits, the third digit (2) being marked as an extra digit.

The fourth operation consisted in dividing an approximate number with three significant digits by an approximate number with two significant digits (the extra digits do not enter into this count). This gives the final result and therefore we reject the extra digit, preserving only two significant digits.

Let us check our approximate answer by calculating the exact answer using the original common fractions. We obtain

\[
\frac{99}{164} = 9.603\ldots
\]

We see, therefore, that in the approximate value 9.5 only the last digit was questionable. This is in accordance with our basic agreement for recording approximate numbers.

**EXERCISE 46-2A**

1. Perform the operations on the approximate numbers.
   a. \(2.98 - (1.4 + .387)\)
   b. \(23000 - (2645 + 15300 - 1639)\)
   c. \((562 \div 87) \times 7\)
   d. \(2.75 + (1.2 - .30103)\)
   e. \(36408 \div (236 \times 28)\)
   f. \(5325 + [(832860 \div 211) \times 37]\)
   g. \([4.5 - (.03 \times 1.5)] \div 7.8\)

2. Find the volume, \(V\), of the parallelepiped with length, \(l = 34.7\) cm., width, \(w = 26.8\) cm., and height, \(h = 42.1\) cm.

3. Find the total surface area of the parallelepiped in Question 2.
ANSWERS

Structure
of Arithmetic

Only answers for the more difficult problems are given.

Chapter Twenty-two

EXERCISE 22-2A

4. You cannot subtract $\frac{9}{3}$ from both sides of $\frac{5}{3} < \frac{9}{2}$, because $\frac{5}{3} - \frac{9}{3}$ is not yet defined.

EXERCISE 22-6A

4. A somewhat formal way you could use would be this. Since $\frac{a}{b} + \frac{m}{n} = \frac{a}{b} \times \frac{n}{m}$ (if $\frac{m}{n} \neq 0$), and $\frac{c}{d} = \frac{m}{n} = \frac{c}{d} \times \frac{n}{m}$, the inequality

$$\frac{a}{b} + \frac{m}{n} < \frac{c}{d} + \frac{m}{n}$$

is the same as the inequality

$$\frac{a}{b} \times \frac{n}{m} < \frac{c}{d} \times \frac{n}{m}.$$

Chapter Twenty-three

EXERCISE 23-1B

2. a. 103  
   b. 10217  
   c. 4579

3. a. 122 seven  
   b. 140 seven  
   c. 211246 seven

6. a. base three  
   b. base five  
   c. base six  
   d. base seven
**EXERCISE 23-2C**

1. a. .6  
   b. .5  
   c. .75  
   d. .625  
   e. .15  
   f. .84  
   g. .62  
   h. .062  
   i. 3.4  
   j. 2.5  
   k. 6.75  
   l. 1.125  
   m. 2.25  
   n. 8.04  
   o. 18.42  
   p. 1.0625

**EXERCISE 23-2D**

1. a. .5  
   b. .25  
   c. .75  
   d. .6  
   e. .7  
   f. .625  
   g. .3125  
   h. .062  
   i. 1.125  
   j. 2.5  
   k. 6.75  
   l. 8.04  
   m. 8.04  
   n. 5.38461

**EXERCISE 23-3A**

2. a. $\frac{3}{8}$  
   b. $\frac{1}{2}$  
   c. $\frac{1}{18}$  
   d. $\frac{1}{2}$  
   e. $\frac{17}{9}$  
   f. $\frac{7}{3}$  
   g. $\frac{17}{5}$  
   h. $\frac{9}{8}$  
3. a. .1  
   b. .111...  
   c. .2  
   d. .222...  
   e. .3  
   f. .333...  
   g. .4  
5. Yes

Chapter Twenty-four

**EXERCISE 24-1A**

1. a. 813  
   b. 44.8  
   c. 126.62  
   d. 23.458

**EXERCISE 24-1B**

2. a. 2.3 is greater by 0.71  
   b. 87.32 is greater by 63.252  
   c. 0.12 is greater by 0.1183  
   d. 0.3 is greater by 0.132  
   e. 2.41 inches

3. a. 12.42 inches

**EXERCISE 24-2B**

1. a. 7  
   b. 0.9  
   c. 3.2  
   d. 57.2  
   e. 0.07  
   f. 0.309  
   g. 9.732  
   h. 10.572
2. a. 3200 b. 750 c. 12 d. 1940.1 e. 7.63 
g. 79.321 h. 0.01

EXERCISE 24-3A

1. a. 44.4 b. 729.84 c. 2.01 d. 58.21 e. 12.74 f. 1
2. a. 3 b. 1 c. g. 0 h. 3 i. 40
3. a. 42 b. 5.27 c. 29 d. 5.69 e. 0.25302 f. 6.904

EXERCISE 24-3B

2. a. 8.655 b. 1.772 c. 0.857
3. a. 0.6595 b. 12.336 c. 0.77

EXERCISE 24-4A

1. a. 0.6 b. 5.6 c. 0.03 d. 0.01 e. 0.2 f. 9.9
g. 79 h. 1.44 i. 0.005 j. 0.15 k. 0.714 l. 10
2. a. 8.3366 b. 0.115605 c. 506.11 d. 0.0001218

EXERCISE 24-5A

2. a. 40 b. 70 c. 600 d. 8 e. 800 f. 0.08
g. 50 h. 1100 i. 33
3. a. 1.46 b. 0.21 c. 55.37 d. 0.98

EXERCISE 24-6A

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<th>50%</th>
<th>25%</th>
<th>12 2%</th>
<th>75%</th>
<th>20%</th>
<th>10%</th>
<th>5%</th>
</tr>
</thead>
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<td>1/2</td>
<td>1/4</td>
<td>1/8</td>
<td>3/4</td>
<td>1</td>
<td>1/10</td>
<td>1/20</td>
</tr>
<tr>
<td>Decimal fraction</td>
<td>0.5</td>
<td>0.25</td>
<td>0.125</td>
<td>0.75</td>
<td>0.2</td>
<td>0.1</td>
<td>0.05</td>
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</tbody>
</table>

<table>
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<th>33 1/3%</th>
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<th>35%</th>
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<tbody>
<tr>
<td>Common fraction</td>
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<td>1/3</td>
<td>3/5</td>
<td>7/20</td>
</tr>
<tr>
<td>Decimal fraction</td>
<td>0.025</td>
<td>0.33</td>
<td>0.6</td>
<td>0.35</td>
</tr>
</tbody>
</table>
EXERCISE 24-6B

1. a. \( \frac{7}{200} \)  b. \( \frac{1}{3} \)  c. \( \frac{13}{20} \)  d. \( \frac{11}{200} \)  e. \( \frac{17}{100} \)  f. \( \frac{13}{80} \)
2. a. 66\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%\%}\n
Chapter Twenty-five

EXERCISE 25-1A

1. 3
2. No whole-number answer
3. 7'
4. 7
5. No whole-number answer
6. No whole-number answer
7. 0

Chapter Twenty-six

EXERCISE 25-3A

1. a. Latitude S30°
   b. Longitude E45°
   c. Temperature 15° below zero
   d. 10 minutes before the hour
   e. A loss of 7 shs
   f. A gain of 50 shs
2. a. pos 1
   b. neg 11
   c. neg 17
   d. pos 73
   e. neg 129
   f. pos 8
   g. neg 42
   h. neg 9
   i. neg 23
   j. pos 14
   k. zero
EXERCISE 26-4A

1. a. pos 6 < pos 10  
   c. pos 15 > neg 15  
   e. neg 200 > neg 1000  
   g. 0 < pos 8  
   b. neg 6 > neg 10  
   d. neg 15 < pos 15  
   f. 0 > neg 3  
   h. neg 11 < 0

EXERCISE 26-4B

1. a. The first place is north of second place.  
   The second place is south of first place.  
   b. My watch is ahead of my friend's watch.  
   My friend's watch is behind my watch.  
   c. The man on the platform is below the man on the ground.  
   The man on the ground is above the man on the platform.  
   d. Noon today is colder than noon yesterday.  
   Noon yesterday was hotter than noon today.  
   e. Kofi starts ahead of Kwesi.  
   Kwesi starts behind Kofi.

EXERCISE 26-4C

2. neg 5 > neg 8;  pos 5 < pos 8  
3. pos 2 < pos 11;  neg 2 > neg 11  
4. pos 2 > neg 11;  neg 2 < pos 11  
5. pos 7 > 0;  neg 7 < 0  
6. 0 > neg 2;  0 < pos 2  
7. neg 6 < neg 1;  pos 6 > pos 1  
8. pos 10 > 0;  neg 10 < 0

EXERCISE 26-5A

1. a. |pos 21, pos 22, pos 23, pos 24|  
   b. |pos 1, 0, neg 1|  
   c. |neg 2, neg 1|  
   d. |neg 6|  
   e. ||  
   f. ||  
   g. |pos 1|  

2. a. Set of integers between neg 3 and 0  
   b. Set of integers between pos 18 and pos 23  
   c. Set of integers between pos 2 and neg 2  
   d. Set of integers between neg 1 and pos 1  
   e. Set of integers between pos 101 and pos 102
Chapter Twenty-seven

**EXERCISE 27-1A**

1. a. pos 6  
   b. pos 7  
   c. pos 4  
   d. pos 3  
   e. pos 2  
   f. pos 8  

2. a. pos 4  
   b. pos 4  
   c. pos 4  
   d. pos 6  
   e. pos 6  
   f. 0  

**EXERCISE 27-2B**

1. a. pos 3  
   b. neg 1  
   c. 0  
   d. pos 6  
   e. pos 3  
   f. neg 2  
   g. pos 3  
   h. pos 6  
   i. 0  
   j. neg 6

2. a. pos 2 - pos 1 = 0; pos 2 - pos 1 = pos 1  
   b. pos 3 - neg 5 = 0; neg 5 + pos 8 = pos 3  
   c. 0 - neg 4 = 0; neg 4 + pos 4 = 0  
   d. neg 1 - neg 5 = 0; neg 5 - neg 4 = neg 1  
   e. pos 3 - 0 = 0; 0 + pos 3 = pos 3  
   f. pos 1 - pos 7 = 0; pos 7 - pos 1 = 0  
   g. neg 3 - pos 3 = 0; pos 3 - pos 6 = neg 3  
   h. neg 11 - neg 9 = neg 2; neg 2 + neg 9 = neg 11

3. a. N29° - S50° = 0; S50° + = N29°; S50° + N79° = N29°  
   b. 3.10 P.M. - 2.55 P.M. = 0.15 min.; 2.55 P.M. + = 3.10 P.M.; 2.55 P.M. + 15 min. = 3.10 P.M.  
   c. -53 ft. - 0 = 0; 0 + = -53 ft.; 0 - 53 ft. = -53 ft.  
   d. 95° - 102° = 0; 102° + = 95°; 102° - 7° = 95°  
   e. 5 ft. ahead - 5 ft. behind = 0  
   f. 5 ft. behind + 5 ft. ahead; 5 ft. ahead + 5 ft. ahead

**EXERCISE 27-3A**

1. a. The set of integers is closed under addition.  
   b. The set of integers is closed under subtraction.

2. The set of fractions, excepting 0, is closed under division.

**EXERCISE 27-3B**

1. a. \[
    \text{pos 3 + neg 3} = 0
    \]
   b. \[
    \text{neg 3 + pos 3} = 0
    \]
   c. \[
    \text{neg 1 + pos 1} = 0
    \]
   d. \[
    \text{pos 5 + neg 5} = 0
    \]
   e. \[
    \text{pos 9 + neg 9} = 0
    \]
   f. \[
    0 + 0 = 0
    \]
   g. opposites

2. h. neg 3 neg 2 neg 1 0 pos 1 pos 2 pos 3  
   i. Their sum is 0.

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EXERCISE 27-3C

1. a. \( \text{pos } 3 - \text{pos } 1 = \text{pos } 2 \)
   \( \text{pos } 3 + \text{neg } 1 = \text{pos } 2 \)
   \( \text{pos } 3 - \text{pos } 1 = \text{pos } 3 + \text{neg } 1 \)

   b. \( \text{neg } 3 + \text{neg } 1 = \text{neg } 4 \)
   \( \text{neg } 3 - \text{pos } 1 = \text{neg } 4 \)
   \( \text{neg } 3 + \text{neg } 1 = \text{neg } 3 - \text{pos } 1 \)

   c. \( \text{pos } 2 - \text{pos } 6 = \text{neg } 4 \)
   \( \text{pos } 2 + \text{neg } 6 = \text{neg } 4 \)
   \( \text{pos } 2 - \text{pos } 6 = \text{pos } 2 + \text{neg } 6 \)

   d. \( 0 - \text{pos } 7 = \text{neg } 7 \)
   \( 0 + \text{neg } 7 = \text{neg } 7 \)
   \( 0 - \text{pos } 7 = 0 + \text{neg } 7 \)

   e. \( 0 - \text{neg } 7 = \text{pos } 7 \)
   \( 0 + \text{pos } 7 = \text{pos } 7 \)
   \( 0 - \text{neg } 7 = 0 + \text{pos } 7 \)

Chapter Twenty-eight

EXERCISE 28-3A

2. a. \(-4\)   b. \(2\)   c. \(0\)   d. \((-6)\)

EXERCISE 28-4B

1. a. \( \text{pos } 3 + \text{neg } 4 = \text{pos } 3 - (\text{opp. of neg } 4) \)
   \( = \text{pos } 3 - \text{pos } 4 \)
   \( = \text{neg } 1 \)
   \( \text{or } 3 + (-4) = 3 - [\text{opp. of } (-4)] \)
   \( = 3 - 4 = (-1) \)

   b. \( \text{neg } 3 + \text{neg } 3 = \text{neg } 3 - (\text{opp. of neg } 3) \)
   \( = \text{neg } 3 - \text{pos } 3 \)
   \( = \text{neg } 6 \)
   \( \text{or } (-3) + (-3) = (-3) - [\text{opp. of } (-3)] \)
   \( = (-3) - 3 \)
   \( = (-6) \)

   c. \( 0 + \text{neg } 1 = 0 - [\text{opp. of neg } 1] \)
   \( = 0 - \{\text{pos } 1\} \)
   \( = \text{neg } 1 \)
   \( \text{or } 0 + (-1) = 0 - [\text{opp. of } (-1)] \)
   \( = 0 - 1 \)
   \( = (-1) \)
d. \[0 - \text{neg} 4 = 0 + \text{[opp. of neg 4]}\]
   \[= 0 + \text{[pos 4]}\]
   \[= \text{pos 4}\]

   or \[0 - (-4) = 0 + \text{[opp. of (-4)]}\]
   \[= 0 + 4\]
   \[= 4\]

e. \[\text{pos 3} - \text{neg 2} = \text{pos 3} + \text{[opp. of neg 2]}\]
   \[= \text{pos 3} + \text{[pos 2]}\]
   \[= \text{pos 5}\]

   or \[3 - (-2) = 3 + \text{[opp. of (-2)]}\]
   \[= 3 + 2\]
   \[= 5\]

f. \[\text{neg 5} - \text{neg 2} = \text{neg 5} + \text{[opp. of neg 2]}\]
   \[= \text{neg 5} + \text{pos 2}\]
   \[= \text{neg 3}\]

   or \[\text{neg} (-5) - (-2) = \text{[opp. of (-2)]}\]
   \[= (-5) + 2\]
   \[= (-3)\]

g. \[\text{neg 2} - \text{neg 2} = \text{neg 2} + \text{[opp. of neg 2]}\]
   \[= \text{neg 2} + \text{pos 2}\]
   \[= 0\]

   or \[(-2) - (-2) = (-2) + \text{[opp. of (-2)]}\]
   \[= (-2) + 2\]
   \[= 0\]

h. \[\text{neg 7} + \text{neg 3} = \text{neg 7} + \text{[opp. of neg 3]}\]
   \[= \text{neg 7} + \text{[pos 3]}\]
   \[= \text{neg 7} - \text{pos 3}\]
   \[= \text{neg 10}\]

   or \[(-7) + (-3) = (-7) + \text{[opp. of (-3)]}\]
   \[= (-7) - 3\]
   \[= -10\]

**EXERCISE 28-5A**

1. a. \[\text{pos 8} - \text{neg 3} = 8 - (-3) = 8 + 3 = 11\]

   b. \[\text{neg 3} + \text{(neg 2} + \text{pos 1)} = \text{neg 3} + \text{neg 1} = \text{neg 4} = (-4)\]

   c. \[\text{[7} + (-2)] + 8 = [7 - 2] + 8 = 5 + 8 = 13\]

   d. \[(-4) + \text{[(-2} + (-1)] = (-4) + \text{[(-2} - 1]} = (-4) + (-3) = (-7)\]

   e. \[\text{[(-3} - (-5)] - (-3} = \text{[(-3} + 5]} + 3 = 2 + 3 = 5\]

   f. \[\text{(-8} - \text{[(-4} + (-2)] = (-8) - \text{[((-4} - 2]} = (-8) - (-6) = (-8) + 6 = (-2)\]

2. a. \[\text{(-2} - (3 - 8) = (-2} + 8 - 3} = (-2} + 5 = 3\]

   b. \[6 - (2 - i) = 6 + (1 - 2) = 6 - 1 = 5\]

   c. \[\text{(-4} - \text{[(-2} - 6]} = \text{(-4} + \text{[6} - (-2)]\]

   \[= (-4} + [6 + 2] = (-4} + 8 = 4\]

   d. \[8 - \text{[5} - (-4)] = 8 + \text{[(-4} - 5]} = 8 + (-9) = (-1)\]
Chapter Twenty-nine

**EXERCISE 29-2A**

1. 11  
2. (−3)  
3. (−8)  
4. 5  
5. 11  
6. 1

**EXERCISE 29-3A**

a. (−10)  
b. 24  
c. (−21)  
d. 18  
e. (−99)  
f. 136

**EXERCISE 29-4A**

a. (−2)  
b. (−6)  
c. 3  
d. (−6)  
e. 2  
f. (−6)  
g. −3

Chapter Thirty

**EXERCISE 30-1A**

Those using integers: 1, 3, 4, 6, 8.  
Those needing new numbers: 2, 5, 7.

**EXERCISE 30-4A**

Each of the five men received £7/5.  
Each of the remaining four men had to pay £7/4.  
Each of the remaining four men finally lost £7/20 = £7/5 − £7/4.

Chapter Thirty-one

**EXERCISE 31-1A**

1. \(-\frac{16}{7}\)  
2. \(\frac{38}{5}\)  
3. \(-\frac{20}{17}\)  
4. −1  
5. 0  
6. \(-\frac{153}{10}\)

**EXERCISE 31-2A**

−\(\frac{27}{10}\), −\(\frac{3}{4}\), −\(\frac{3}{13}\), 0, \(\frac{4}{7}\), \(\frac{55}{8}\), 10

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EXERCISE 32-1A

1. \( \frac{11}{15} \) 2. 0 3. \( \frac{3}{10} \) 4. \( -\frac{7}{6} \) 5. 0 6. -1

EXERCISE 32-2A

1. \( \frac{26}{21} \) 2. \( \frac{8}{3} \) 3. \( -\frac{2}{3} \) 4. \( \frac{89}{100} \)

EXERCISE 32-3A

1. \( -\frac{3}{8} \) 2. \( -\frac{16}{15} \) 3. \( -\frac{7}{8} \) 4. \( -\frac{35}{528} \) 5. 0 6. 1

EXERCISE 32-3C

1. \( \frac{9}{25} \) 2. \( \frac{16}{81} \) 3. \( \frac{1}{4} \) 4. \( \frac{9}{4} \)

EXERCISE 32-4A

1. \( -\frac{45}{22} \) 2. -1 3. \( -\frac{2}{3} \) 4. \( -\frac{12}{5} \) 5. -24 6. 1

EXERCISE 32-4E

1. \( \frac{2}{3} \) 2. -1 3. 1.7 4. -2

Chapter Thirty-four

EXERCISE 34-1A

1. a. Not closed under any
   b. Under addition, subtraction and multiplication
   c. Under addition and multiplication
   d. Not close under any
   e. Under addition, subtraction and multiplication
   f. Closed under multiplication
Chapter Thirty-five

**EXERCISE 35-1B**

1. (ii) a. \(\frac{8}{10}\) and \(\frac{8}{10}\) are the same number.
   
   b. \(\frac{12}{20}\) and \(\frac{15}{20}, \frac{3}{4}\) is larger.
   
   c. \(\frac{9}{2}\) and \(\frac{10}{2}, 5\) is larger.
   
   d. \(\frac{5}{3}\) and \(\frac{4}{3}, \frac{5}{3}\) is larger.

(iii) a. 0; b. \(\frac{3}{20}\); c. \(\frac{1}{2}\); d. \(\frac{1}{3}\).

2. a. \(\frac{2}{5}; \frac{1}{2}; \frac{2}{3}; \frac{3}{4}, \frac{5}{6}\).

   b. \(\frac{3}{2}; \frac{5}{3}; \frac{2}{2}; \frac{1}{4}; 3\).

   c. \(1\frac{1}{6} + 4\frac{1}{2}; 5\frac{3}{4} + 1\frac{1}{8}; 2\frac{1}{2} + 3\frac{2}{3}; 3\frac{1}{2} + 2\frac{3}{4}\).

**EXERCISE 35-1C**

1. 0.035, 0.35, 1.035, 1.35, 2, 2.25, 3.5, 10.35, 17.5

**EXERCISE 35-2A**

1. a. False
   
   b. True \(-16 = 32 + (-48)\)
   
   c. False
   
   d. False
   
   e. False
   
   f. False
   
   g. False
   
   h. True \(-25 = 26 + (-51)\)
   
   i. True \(36 = 0 + 36\)
   
   j. False
   
   k. True \(-3 = -6 + 3\)
   
   l. False
   
   m. True \(-\frac{3}{2} = 0 + \left(-\frac{3}{2}\right)\)
   
   n. True \([-4 + (-2)] = [-2 + 2] + (-6)\)
   
   o. False
   
   3. a. \(\frac{4}{5} \cdot \frac{8}{10}; \frac{4}{5} \cdot \frac{8}{10} = \frac{8}{5}\)

   b. \(2.5 \cdot -5.5; 2.5 = -5.5 \cdot 8\)

   c. \(-\frac{3}{5} \cdot -\frac{3}{4}; -\frac{3}{5} = -\frac{3}{4} \cdot \frac{3}{20}\)
5.  
   d. \(-\frac{4}{3} > -\frac{5}{3}; \frac{4}{3} = -\frac{5}{3} + \frac{1}{3}\)
   e. \(-\frac{9}{2} > -5; \frac{9}{2} = -5 + \frac{1}{2}\)
   f. \(3.75 > 2.25; 3.75 = 2.25 + 1.5\)

   a. \(6 > -3; -3 < 6\)
   b. \(-2 > -8; -8 < -2\)
   c. \(0 > 4; -4 < 0\)
   d. \(12 > 9; 9 < 12\)
   e. \(3.25 > -3; -3 < 3.25\)
   f. \(\frac{3}{4} > \frac{2}{3}; \frac{2}{3} < \frac{3}{4}\)
   g. \(-9 > -12; -12 < -9\)
   h. \(2 > -8; -8 < 2\)

**EXERCISE 35-4A**

1. < 2. = 3. > 4. > 5. < 6. < 7. = 8. >

**EXERCISE 35-4B**

1. a. \(-5.2 < 0 < 2.5\)
   b. \(-\frac{1}{2} < -\frac{1}{3} < -\frac{1}{4}\)
   c. \(\frac{1}{4} < \frac{1}{3} < \frac{1}{2}\)
   d. \(2.05 < 2.25 < 25\)
   e. \(-6 < 4 < 5\)
   f. \(-\frac{3}{4} < \frac{3}{4} < \frac{3}{5}\)

**EXERCISE 35-4C**

1. a. \(\frac{287}{29} < \frac{340}{31}\)
   b. \(-\frac{79}{17} > -\frac{97}{19}\)
   c. \(\frac{16}{3} > \frac{19}{5}\)
   d. \(\frac{111}{7} < \frac{190}{11}\)

**EXERCISE 35-8A**

1. a. > b. < c. > d. > e. > f. < g. < h. <

Chapter Thirty-six

**EXERCISE 36-2A**

1. a. \(\cdot11111\ldots\) b. \(\cdot010101\ldots\) c. \(\cdot001001\ldots\) d. \(\cdot4545\ldots\)
e. \( 0.142857142857 \ldots \)

f. \( 0.285714285714 \ldots \)

g. \( 0.027027 \ldots \)

**EXERCISE 36-3A**

1. 
   a. \( \frac{2}{9} \)
   b. \( \frac{23}{99} \)
   c. \( \frac{234}{999} = \frac{26}{111} \)
   d. \( \frac{1}{9} \)
   e. \( \frac{1}{99} \)
   f. \( \frac{1}{999} \)
   g. \( \frac{1}{6} \)
   h. \( \frac{109}{990} \)

2. 
   a. \( -22.2 = 2 \times (-11.1) \)
   
   \[ = 2 \times \frac{1}{9} = \frac{2}{9} \]
   
   b. \( -232.3 = 23 \times (-0.0101) \)
   
   \[ = 23 \times \left( \frac{1}{99} \right) = \frac{23}{99} \]
   
   c. \( -234234 = 234 \times (-0.001001) \)
   
   \[ = 234 \times \frac{1}{999} = \frac{234}{999} = \frac{26}{111} \]

**EXERCISE 36-4A**

Some possible answers are:
   a. \( -2020020002 \ldots \)
   b. \( -010110111011111 \ldots \)
   c. \( -030330333 \ldots \)
   d. \( -1001000010000000100 \ldots \)
   
   (ones at 1st, 4th, 9th, 16th places and so on)
   
   e. The same as d. with 1's replaced by 2, 3, 4, \ldots or 9.
   
   f. \( -234567891011 \ldots \)
   
   g. \( -34567891011 \ldots \)

Chapter Thirty-seven

**EXERCISE 37-1A**

Successive trials might be \( \frac{7}{4}, \frac{12}{7}, \frac{12}{7} + \frac{7}{2} = \frac{97}{56}, \frac{168}{97} \).

**EXERCISE 37-2A**

If \( \frac{a}{b} \times \frac{a}{b} = 3 \), then \( \frac{a}{b} = \frac{3b}{a} \). Also, \( 1 < \frac{a}{b} < 2 \).

Let \( \frac{a}{b} \) be in lowest terms. Then \( a \), the denominator on the right, must be one of \( b, 2b, 3b, 4b \) and so on.
That is, \( \frac{a}{b} \) must equal one of \( 1, 2, 3, 4, \ldots \). This is impossible since \( \frac{a}{b} \) must lie between 1 and 2.

Chapter Thirty-eight

**EXERCISE 38-2A**

1. \[\begin{align*}
((-3) + 3) \times a &= [(-3) \times a] + (3 \times a) \\
0 \times a &= [(-3) \times a] + (3 \times a) \\
0 &= [(-3) \times a] + (3 \times a)
\end{align*}\]

Hence, \( (-3) \times a \) is the opposite of \( 3 \times a \), that is,
\[(-3) \times a = -(3 \times a).\]

2. By Question 1,
\[7 = (-3) \times \frac{7}{(-3)} = -\left(3 \times \frac{7}{(-3)}\right)\]

Hence,
\[3 \times \frac{7}{(-3)} = -7\]

and
\[\frac{7}{(-3)} = \frac{(-7)}{3}.\]

3. Since \((-1) + 1 = 0\), the left side is
\[(-1) \times (-1) + 0 = (-1) \times (-1).\]

On the right side,
\[\begin{align*}
[(-1) \times (-1)] + (-1) &= [(-1) \times (-1)] + (-1) (1) \\
&= (-1) \times [(-1) + 1] \\
&= -1 \times 0 = 0.
\end{align*}\]

Hence, the right side becomes 1.

4. \(a + 0 = a\) (AO)

This holds for all values of \(a\), in particular for \(a = 0\).

Hence, \(0 + 0 = 0\).

**EXERCISE 38-4A**

1. If \(a < b\),
\[b = a + p\quad \text{where} \quad p > 0.\]

Then
\[b \times c = a \times c + p \times c\quad [D]\]

since \(p > 0\) and \(c > 0\), \(p \times c > 0\) \(\{0'2\}\).

Then by definition,
\[a \times c < b \times c.\]
2. If \( a < b \)
\[ b = a + p, \quad p > 0. \]
If \( b < c \)
\[ c = b + p', \quad p' > 0. \]
Then \( c = (a + p) + p' = a + (p + p') \).
Since \( p > 0 \) and \( p' > 0 \),
\[ p + p' > 0 \] \( [0'11] \).
Hence, by definition,
\( a < c \), as required.

3. Given \( a \) and \( b \), any two real numbers, consider the real number \( c = a - b \).
By [0'31], there are three possibilities:
\[ c = 0 \]
\[ c > 0 \]
\[ or \quad c < 0. \]
That is,
\[ (1) \quad a - b = 0 \]
\[ (2) \quad a - b > 0 \]
\[ or \quad (3) \quad a - b < 0. \]
Now \( a = (a - b) + b. \)
\[ (1) \quad If \quad a - b = 0, \quad a = b. \]
\[ (2) \quad If \quad a - b > 0, \quad a - b \quad is \quad a \quad positive \quad number \quad p \]
so that \( a = p + b. \)
This means by definition that \( a > b. \)
\[ (3) \quad If \quad a - b < 0, \quad b - a > 0. \]
Since \( (b - a) + a = b, \)
\( b = a + a \quad positive \quad number. \)
By definition then,
\( a < b. \)

Chapter Thirty-nine

**EXERCISE 39-3A**

1. a. 21  
   b. 25  
   c. 104  
   d. 254.5  
   e. 62.6...  
   f. 255.7...

2. d. \( 254 \frac{1}{2} \)  
   e. \( 62 \frac{2}{3} \)  
   f. \( 255 \frac{3}{4} \)

**EXERCISE 39-4A**

1. 275.4
2. The (2) and (7) are reliable. The (5) is questionable, and the (4) is worthless.
3. 275 books
EXERCISE 39-4B

E means exact, A means approximate.

   h. A  i. E  j. A  k. A  l. A  m. A

3. 140 steps

4. b. 

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<tr>
<td>562</td>
<td>9.2</td>
</tr>
<tr>
<td>573</td>
<td>1.8</td>
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<tr>
<td>567</td>
<td>4.2</td>
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<tr>
<td>580</td>
<td>8.8</td>
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<tr>
<td>571</td>
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</table>

Average 571.2  Average deviation 4.5

Reliable digit: 5  Questionable: 7  Worthless: 1.2

Chapter Forty

EXERCISE 40-3A

1. a. 254  b. -25  c. 52.0

2. a. 2.8342  c. 2.834 pounds

3. a. 2661.25  c. 2660

Chapter Forty-one

EXERCISE 41-3A

8. c. 340, 6800, 47.10, 9.002

9. 7.7

Chapter Forty-two

EXERCISE 42-2A

1. .007  2. .006, .0007, .00003  3. .1%
      4. 5.29, .005, .09%

5. The same  6. The second  7. .8%
      8. 1.6%

Chapter Forty-three

EXERCISE 43-5A

1. 2, 2, 3, 4, 4, 4, 4, 5  2. 3, 3, 4, 5, 5, 5, 6
      3. 3, 3, 4, 4, 6
4. Decimal places 1, 2, 3, 2, 3, 2, 2, 3, 4, 2, 3, 3,
   Significant digits 2, 2, 3, 3, 4, 5, 1, 1, 3, 4, 4,
   Decimal places 3, 4, 3, 3, 2
   Significant digits 2, 4, 3, 4, 6, 6
5. Decimal places 1, 2, 3, 2, 3, 2, 2, 3, 4, 1, 2, 2,
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   Decimal places 2, 3, 1, 1, 1
   Significant digits 1, 3, 1, 2, 4, 5

Chapter Forty-four

**EXERCISE 44-3A**

1. a. .56  
   b. 37800  
   c. 3.390
3. a. 1160  
   b. 4.82  
   c. 8.52
5. 2750 yards
6. 6400
7. 13.92
8. 1.88

Chapter Forty-five

**EXERCISE 45-4A**

1. a. .03  
   b. 2,500,000  
   c. 10.6
3. 23.3
4. 4400 pounds
5. a. .03  
   b. 20
   c. .014
6. 16
7. 1.848
8. .47

Chapter Forty-six

**EXERCISE 46-2A**

1. a. 1.2  
   b. 7000  
   c. 50  
   d. 3.6
   e. 5.5  
   f. 150,000  
   g. .57
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An Introductory Text for Teachers

Prepared at the 1964 Entebbe Mathematics Workshop

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This experimental teacher training text was prepared during the summer of 1964 at Entebbe, Uganda, as part of a program of curriculum revision being conducted by the African Education Program of Educational Services Incorporated.

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The 1964 Entebbe Mathematics Workshop was directed by Professor W. T. Martin of the Massachusetts Institute of Technology, Mr. John O. Oyelese of the University of Ibadan, and Professor Donald E. Richmond of Williams College.

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In this volume, the Teacher Training Writing Group at the Entebbe Mathematics Workshop presents a preliminary edition of "Introduction to Geometry". This is the second part of Basic Concepts of Mathematics, an experimental text to be used in Training Colleges for primary teachers. (The first part, "Structure of Arithmetic", has already been published in a preliminary edition in two preceding volumes.) The treatment of geometry is rather novel. There is an unusual emphasis on the basis of geometric concepts in experience and, following that, a transition to more abstract ideas of point, line and plane. The standard constructions which can be used by the teacher in the classroom are first explained in an intuitive way, and then made the object of a modest theory which can serve as a suitable introduction to a deductive system.

As in the earlier volumes, the exercises have two purposes: to develop and extend the understanding of the mathematical content presented in the text, and to suggest by example kinds of exercises a trainee could later create as a teacher for use in his own classes. Answers for the more difficult exercises in this volume will be found at the end of this book.

The whole of Basic Concepts of Mathematics has been produced under pressure of time, and there is still much to be done by way of improving exposition and organization as well as adding to the stock of exercises. To all users, therefore, the Teacher Training Writing Group directs an earnest request for comments and suggestions which can contribute to the work of preparing a more finished text. Reports from experimental use of the preliminary edition are a source of ideas which will make the next edition of greater value to mathematics education.

At the end of this book is a glossary of terms for this and the preceding volume.
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**FOUNDATIONS OF GEOMETRY**

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GLOSSARY

ANSWERS
Chapter 47

SOME ELEMENTARY IDEAS AND FIGURES

47-1 Introduction

Our world is full of physical objects: sticks and stones, ant hills and trees, bananas and oranges, tins and bottles, boards and boxes, huts and houses. All through our lives we learn about such objects by observing and touching them, by using them, and by making some of them. We learn that these objects have parts with names such as "ends", "corners", "edges", "sides", "tops", "bottoms" and "insides". Some of these parts are straight, some are flat, some are rounded. At play and in school we also learn how to make pictures of certain objects, using sticks and strings and paper and pencils and chalk.

Touching and seeing bring these parts to our minds because of what we learn through the muscles of our hands and eyes. If we move a finger along an edge of a book to a corner, our touch tells us we must change the direction if we are to keep moving our finger past the corner to follow another edge. The same is true of our eyes if we follow the edges with our sight. So, too, if we move our fingers along one side of a box to an edge and then cross the edge to another side, we are aware of the flatness of the first side, the feeling of "dropping off" at the edge and of the change of muscle action needed to begin feeling the second side.

It is in the same way that we learn first to tell apart the objects in a set and then to count the objects. Children, given a set of two oranges, first learn that the oranges are distinct by observing differences in their positions, by touching them and moving them and by noticing differences in how they look and feel. Later they count the oranges by moving them, one movement for each orange, and counting the movements. Thus, both counting, the earliest part of arithmetic, and simple figures, the earliest part of geometry, arise from "handling" physical objects with arm and eye muscles and from the impressions our minds get from these actions. So in this discussion of geometry we shall try to develop geometric figures as ideas that come from physical objects that we find and make. These ideas begin when, through our observing and handling objects, our minds become aware of certain aspects of the objects. We then make drawings or images of these aspects, getting figures that we call "point", "line segment", "triangle", "circular region", "pyramid" and so on. Finally, through long study of the objects and the drawings, there form in our minds the general ideas that are called "geometric figures" and have the same names.

When we said "aspects", we meant certain parts of some objects, such as the point of a pin, a side of a house, a corner of a door, the skin of an orange, an edge of this book and so forth, and the shapes and sizes of objects and parts of objects; for example, the roundness of the moon, the smallness of a pin point, the straightness of an edge of a table, the flatness of a floor, the squareness of a table leg, the largeness of the earth.
When we think about these things, we notice that some seem to go together: the tip of a leaf, the point of a needle, a speck of dust, the end of a mosquito leg—all seem very small and quite different from a tree, a bottle, an elephant. For another, the moon, a shilling piece and a wheel seem to share something; their shapes have something in common. Besides these two groups there are many others.

In this first unit of geometry, we shall talk about some of these groups. The first group is made up of those things that, like the point of a needle, are so small we call them "points". The next group is of those things that are "straight" and the next those that are "flat". From these three groups we get the figures that we call "points", "line segments", "straight lines", "triangles" and "planes". Then with bricks and coins we discover certain common shapes and figures, such as rectangles and circles.

The second chapter of this first unit discusses these figures in more detail and introduces some other figures, such as "quadrilateral", "polygon" and "prism". It includes the ideas of straight lines being parallel and being perpendicular and of polygons being congruent (fitting exactly on each other). It contains constructions of perpendicular lines, bisecting lines and rays, congruent figures and so on.

The third chapter discusses measuring line segments, angles, plane regions and solid figures; that is, length, area and volume.

The fourth chapter is a brief introduction to geometric figures as ideas, or how we try to make points, line segments, triangles and so on in our minds.

47-2 Points

Let us begin with the very first and simplest figure in geometry—a point! One of the commonest parts of your experience, as well as the experience of the child, is to see and feel (when possible) things which suggest a point: the tip of a leaf or spear or pin or needle, the point of a sharpened stick or pencil, the tip of a corner of a door or of a piece of paper, a star as seen in the sky, a grain of sand. There are many such objects in the world, and they all have something in common: they look very small, so small they seem almost to be "places" or "spots" rather than things. So we call them "points". Sometimes, moreover, we give a special point a special name. We might call it, for example, the tip of Flomo's yellow pencil; or the star Arcturus; or, if we wanted to be very brief, we might call it simply "the point A". There could hardly be a shorter or simpler name than that for one of the smallest and simplest objects we can find!

The points we have just mentioned are physical objects; we can see them or feel them. And we not only can find them; we can make some, for example, by sharpening a stick and by folding paper. Even better for some purposes, we can make points by drawing or tracing other points. If you take a corner of a matchbox (or even one of the harder corners of this book) and press it onto a piece of paper, it will make a small dent, a sort of picture, in the paper, of the tip of the corner. This dent is itself a kind of point. For another example, if you put the tip of a sharpened pencil down on paper, it will make a small mark, or "dot". This dot is a point that is a drawing of the pencil tip. It is a point that we see rather than feel; it is a physical object but its dimensions are too small to be felt. As a last example, if you use the point of a pin to punch a tiny hole in a piece of paper, the hole is a picture of the pin point, and is itself a point.
Using pencils or pins or even sharpened sticks, we can make as many new points as we wish by drawing on paper or the ground or by punching holes; and we can make them almost wherever we wish. All the ones we make will look alike and will almost fit each other. You could try fitting these to each other: the tip of a sharpened pencil, the tip of a corner of a matchbox, a dot on paper made by a sharp pencil, a pin hole in paper. An exact fit would be two pin holes made by the same pin!

Thus, we have started with common things such as the sharp ends of leaves and pencils and pins, and corner tips of doors and boxes and books. From these, and from the dents, dots and holes that are images of them, we have made new points and have developed the idea of "points" as very small things, so small they seem almost to be places rather than measurable objects. However made, these points are very much alike and resemble pin points and small pencil dots.

**EXERCISE 47-2A**

1. Find some points you can see and feel. Name some that you cannot see or feel at the moment you read this sentence.

2. How many points are at the tip of one corner of this book? How many points are at the tips of all the corners of this book? How many points do you think you could find or make on the top cover of this book? How many on an edge of the top cover?

47-3 Straightness

Now let us move on to another idea we get from physical objects—that of "straightness". What objects would you call "straight"? An edge of this book, the path of a stone you let fall, the place where one wall of a room meets another, the beam of an electric torch at night, your line of sight to a star—would these be straight? They have something which separates them from, say, the edge of a banana leaf, a bend in a path, the trail of a snail. To try to become clear as to what this might be, let us go back to the ideas of feeling and seeing.

"Feeling straightness" is hard to describe and to test, and pencil drawings are difficult to feel. Perhaps we can do better with "looking straight". The most natural way of deciding whether something "looks straight" is to sight along it. If any of us looked down a string that had hanging at the other end a heavy, still object, he would say that the string looked straight. And if we stretched a string tightly between two nails (metal nails or thumbnails) and you and I took turns sighting along it, we would agree that it was straight. If the nails were steady and we marked on them the spots where the string was tied, we could remove the string and still have a "line of sight" between the two spots. The line of sight is our sight directed from one spot to the other. We place our eye so that the two nail points coincide in our vision—one hides the other.

We thought the stretched string was straight because it fitted the line of sight exactly. That is, every part of the string was on the line of sight, and every part of the line of sight...
between the nails was filled by string. So if we have an object such as a string, an edge or a drawing, we say it is straight if it exactly fits the line of sight between its two ends. In this sense, we are now sure that a stretched string is straight, and that the edges of this book are straight (or almost so!)

Thus, we have one test for straightness—line of sight. But just as important is the fact that we can test an object for straightness by fitting, from one of its ends to the other, another object we already know to be straight (a stretched string or a straight-edge). If the straight object fits all along the object being tested, the tested object is straight. Here are pictures of some tests.

In the third picture from the left, the test is not being made correctly; the straight-edge was not fitted at the ends of the drawing. In the remaining three the test is made correctly, and only the first drawing is straight.

If we have to test a long object with only a short string we can use another fact about straightness: a string or edge or drawing is straight just in case every part of it is straight. This means we can test by sliding our straight-edge or string steadily along the whole length of what we are testing.

To sum up: if \( A \) and \( B \) are the points at the ends of an object such as a string or edge or drawing, the object is straight if and only if it fits exactly the line of sight between \( A \) and \( B \). Any line of sight is straight and so is any tightly stretched string. An object is straight if and only if whenever we fit next to it something we know to be straight (line of sight, stretched string, straight-edge) the two fit each other exactly wherever they overlap.

Suppose we have no string and no straight-edge. How can we make something that is straight or has a straight edge? One easy and useful way is to find a piece of paper, no matter how ragged at the edges, hold it at two opposite places on the edges,

fold the paper by bringing these two places together and then press the paper flat by smoothing out toward the fold. The result will be a folded paper showing a straight edge.

(If you have a piece of string, you can stretch it and test the edge.)
Such paper straight-edges are useful, as we shall see: They are particularly useful in making paper square corners. To make such a corner, put a dot on the paper straight-edge somewhere about midway, dividing the edge into a left-hand part and a right-hand one. Fold the left-hand part of the edge over to fit along the right-hand part, and smooth the fold; the corner you have just made is called a square corner.

**EXERCISE 47-3A**

1. Name five objects, besides those mentioned in the text, that you would call "straight".
2. Find three things near you that you believe to be straight. Test them, first by sighting and then by using a stretched piece of string.
3. Test two edges of this book for straightness.
4. Find or make something you think is not straight. Prove that it is not straight.
5. Here are some drawings. Judge them for straightness, first by only glancing at them, second by sighting, third by using a stretched string.

6. Find or make something that is almost straight. Why do you think it is almost straight but not really straight?
7. Find a ragged piece of wrapping paper or newspaper and make a paper straight-edge by the method in the text. Test it with a stretched string (the thinner the string, the better the test). If your paper edge is straight, use it to test the figures in Question 5, above. If it is not straight, make a new one that is straight.
8. Find or make a cardboard straight-edge. Test the edge to be sure it is straight.
9. Below are two points, \( J \) and \( K \). By sighting from \( J \) to \( K \), mark two dots on the line of sight from \( J \) to \( K \) and name them \( M \) and \( N \). Cover \( J \) and \( K \) with small pieces of paper. On the line of sight from \( M \) to \( N \) mark another dot and name it \( P \). Do you think \( P \) is on the line of sight from \( J \) to \( K \)? (Take away the two pieces of paper and see what you think.)
47-4 Line segments

If you take straight-edges and trace them, you will make drawings that may look like these:

Straight drawings like these we will call line segments. We can be sure a drawing is straight if we made it by tracing a straight-edge, or if we test it successfully using a stretched string or a straight-edge or line of sight.

Each line segment has a point at each end. These two points are called its endpoints, and we say the line segment is between these points. On the other hand, if we choose any two points, there is a line of sight between them. If we put a straight-edge to these two points and trace along the edge from either point to the other, we have traced the line of sight between the two points. The resulting drawing is a line segment whose endpoints are the two given points. We knew that any line segment has two endpoints; now we know that between any two points there is exactly one line segment. If the two points have been named, say $P$ and $R$, we use the name "$\overline{PR}\$" or "$\overline{RP}\$" for the line segment having these endpoints.

A line segment contains lots of other points besides its endpoints. Any dot you make on a line segment shows a point, and there are many dots that can be made. Another way to show the points that are on a line segment is to place a paper straight-edge along the line segment and then fold the paper over to make a square corner. The tip of the corner marks a point on the line segment. Each different fold of the paper will mark a different point on the line segment.

Suppose we began with a line segment $\overline{AB}$ and marked two more points, $C$ and $D$, on $\overline{AB}$, as shown:
Both $C$ and $D$ are on $AB$. We therefore say that each of them is between $A$ and $B$. But we also notice that $D$ is on the line segment $AC$. If we sight from $A$ to $C$, $D$ hides $C$. So $D$ is between $A$ and $C$. In the same way, $C$ is between $D$ and $B$. And $D$ is not between $B$ and $C$.

**EXERCISE 47-4A**

1. In the last figure above there are two points that the point $C$ is not between. What are they?
2. Take the paper straight-edge you made in Question 7 of Exercise 47-3A and fold it to make a square corner. Then unfold it so that the paper straight-edge has come back. How many line segments can you see on your paper?
3. Make another paper straight-edge. Place it edge to edge on top of your first paper straight-edge. Do the edges fit each other wherever they are together? Now change each straight-edge into a square corner. Put the two square corners together, one on top of the other, with the edges together. Do the edges of one square corner fit exactly on the edges of the other as far as they go?
4. For each of the following drawings, decide whether or not it is a line segment.

![Diagrams]

5. For a point $X$ to be between two other points $Y$ and $Z$, it must be a point on the line segment $YZ$. Below are seven points. Write down which points are between which other points. (For example, $T$ is between $Q$ and $V$.)

```
P   S   U
  Q
  R
```

6. Here is a drawing made of four line segments.

```
B
A
D
C
```

How many of these line segments have one endpoint at $A$? How many have one endpoint at $B$? at $C$? How many have one endpoint at $D$ and the other endpoint at $C$?
7. Here is a line segment and four of its points.

P S R Q

How many line segments have each endpoint at one of these four points?

8. Below are three points. Name the line segments that have their endpoints among these three, and draw them.

Y Z X

9. Here are four points and two line segments. Each of these two line segments connects two points, that is, has its endpoints at two of the four given points. Draw all possible line segments of this kind.

10. In the figure below, count the line segments that have their endpoints among the points H, I, J, K, L. (IK is such a line segment, for example.)

H K

I J L

11. Here are two points C and D.

C D

On the line segment CD, mark with a dot any third point you choose and name it E. Cover E with a small piece of paper. On the line segment CD choose first some point F between C and E and then some point G between E and D. Do you think E lies on the line segment FG? (Remove the piece of paper and test to see what the answer is.)

At night when looking at a star, we sometimes think of how far the star's light has come to meet our eyes. It has travelled over a very long line segment! Sometimes, too, we imagine how, if the earth had not been in the way, the light might have kept on going through space, on and on without end. That would be along a line segment that stayed the same at one end but at
the other end kept growing and never stopped! To show this idea we can make a drawing like this,

```
Light path
Star
Where our eye would have been
```

the arrowhead reminding us that the light goes on and on without end.

Given any two points $A$ and $B$ in space, we can think of having a star at either point and our eye at the other; then we can take our eye away from the point and let the light shine past endlessly.

```
A
Star
B
Eye
```

If now we also take away the star from the point where it is, we are left with

```
A
B
```

a straight figure that has only one endpoint, the arrow at the other end indicating that the figure continues indefinitely in that direction and is always straight. (Our paper stops at its edge, so we have to stop drawing; the arrow is just to show what we would like to draw if only we could!) Such a figure we call a ray. In this case, it is the ray from $A$ through $B$, or $AB$ for short (or $BA$).

We could have thought of the star being at $B$ and our eye at $A$; then we would have finished with this drawing,

```
A
B
```

showing the ray from $B$ through $A$, or $BA$.

Thus, any two points $A$ and $B$ in space determine not only one line segment but also two rays, the one from $A$ through $B$ and the one from $B$ through $A$.

Looking at a star, we might wonder whether there is someone there looking towards us and seeing the Earth as a shining point in his sky. If so, we would have two rays:

```
T
Their eyes
O
Our eyes
```

and

```
T
Their eyes
O
Our eyes
```

Since both may be happening at the same time, we draw this figure,

```
T
O
```

The other end kept growing and never stopped! To show this idea we can make a drawing like this,

```
Light path
Star
Where our eye would have been
```

the arrowhead reminding us that the light goes on and on without end.

Given any two points $A$ and $B$ in space, we can think of having a star at either point and our eye at the other; then we can take our eye away from the point and let the light shine past endlessly.

```
A
Star
B
Eye
```

If now we also take away the star from the point where it is, we are left with

```
A
B
```

a straight figure that has only one endpoint, the arrow at the other end indicating that the figure continues indefinitely in that direction and is always straight. (Our paper stops at its edge, so we have to stop drawing; the arrow is just to show what we would like to draw if only we could!) Such a figure we call a ray. In this case, it is the ray from $A$ through $B$, or $AB$ for short (or $BA$).

We could have thought of the star being at $B$ and our eye at $A$; then we would have finished with this drawing,

```
A
B
```

showing the ray from $B$ through $A$, or $BA$.

Thus, any two points $A$ and $B$ in space determine not only one line segment but also two rays, the one from $A$ through $B$ and the one from $B$ through $A$.

Looking at a star, we might wonder whether there is someone there looking towards us and seeing the Earth as a shining point in his sky. If so, we would have two rays:

```
T
Their eyes
O
Our eyes
```

and

```
T
Their eyes
O
Our eyes
```

Since both may be happening at the same time, we draw this figure,
and call it a straight line. It is the straight line through \( T \) and \( O \), or, more briefly, \( \overrightarrow{TO} \). It can be thought of as the union, or combination, of the two rays \( \overrightarrow{TO} \) and \( \overrightarrow{OT} \).

We can now say that given any two points \( A \) and \( B \), there is exactly one line segment between them, there is exactly one ray through \( B \) having \( A \) as endpoint and exactly one ray through \( A \) having \( B \) as endpoint, and there is precisely one straight line containing the points \( A \) and \( B \). The symbols for these are respectively \( AB, \overrightarrow{AB} \) (or \( \overrightarrow{BA} \)), \( BA \) (or \( \overrightarrow{BA} \)), and \( \overrightarrow{AB} \); the first and last can also be written \( BA \) and \( AB \). The ray \( \overrightarrow{AB} \) can be thought of as the combination of all line segments that contain \( B \) and have \( A \) as one endpoint, and the straight line \( AB \) as combining all line segments that contain both \( A \) and \( B \).

**Exercise 47-4B**

1. Here are three rays. Give the full name and a short name of each.

   \[ \text{\includegraphics[width=\textwidth]{rays.png}} \]

2. For each of the three pairs of points below, make a drawing that shows the straight line determined by them and name the straight line.

   \[ \text{\includegraphics[width=\textwidth]{straight_lines.png}} \]

3. Each of the following four figures shows either a line segment, a ray or a straight line; say what each figure shows. Each figure contains a named point; on each figure choose another point and name it, and make a name for the figure using the names of the two points.

   \[ \text{\includegraphics[width=\textwidth]{figures.png}} \]
4. Here is a ray.

What line segments can you combine to make the whole ray?

5. Make a paper straight-edge and call the endpoints of the edge \( A \) and \( B \). Then fold the paper over so that \( A \) fits exactly on \( B \). Smooth down the fold. Now unfold the paper so that your straight-edge has come back; there will be a straight crease down the middle. Mark the point where the crease meets the straight-edge and call it \( M \). Your paper should look like this:

Now trace the straight-edge, marking a line segment; on the line segment, mark the points that correspond to \( A \), \( M \) and \( B \), like this:

Now move the paper to the right so that \( M \) is at the right-hand endpoint of the line segment you just drew and the straight-edge still fits the line segment.

Trace the rest of the straight-edge, getting this picture:

Now move the straight-edge to the right again as you did before,

and trace again. If you kept this up indefinitely, always moving to the right, what old acquaintance would you have made?
If we are asked to name some things that we think are flat, we would be likely to mention such man-made objects as table tops, walls and book covers, and as a natural object, the surface of still water. What do these have in common that makes us say they feel flat and look flat? And why do we think the skin on a banana or an orange is not flat and neither is a crumpled piece of paper or cloth?

One way of trying to answer such a question is to ask this: can I make something that I think is flat? If I work at this, perhaps I will be clearer about what "flat" means.

How can we make something that we think is flat? As a starter, perhaps we should go back to something that we consider to be flat, such as a table top or a book cover. What do we see that makes this look flat? Puzzlement. Perhaps we had better contrast it with something that doesn’t look flat. Try an orange. Why does an orange look not flat? Hmmm. Perhaps it is because the orange bends or curves, but the table top does not. If something bends or curves, it lacks straightness somewhere, and if it doesn’t bend or curve, it is because it is full of straightness.

So now we want to make something that is “full of straightness”. This reminds us of how we made straight things; we made our first one by taking string or thread and stretching it. Maybe a stretched string is “full of straightness”. Yes, but not full enough; it doesn’t seem flat. It doesn’t seem flat because it doesn’t “spread”. So now we see that we need something “full of straightness” that also “spreads”. “Spread”, when we think about it, seems to mean “stretching more than one way”. Now it appears we want something that stretches more than one way and is full of straightness. We stretched a string, but in only one direction, and got straightness: is there something we could stretch in more than one direction and get flatness? We would have to find something that could be stretched in more than one direction and would be full of straightness. If string and thread stretch in one direction, what could we stretch in two directions? Cloth, perhaps; it’s made of thread going in two directions. How can we stretch it to make something we feel sure is flat? If we try to stretch a piece of cloth with our hands we have difficulty seeing how we might find an answer. But if we continue trying, it soon appears that if we could stretch a cloth tightly over the ends of three sticks we would have something that spreads and is close to being flat. (Two sticks are not enough; they don’t spread the cloth. And four sticks are troublesome. Three sticks are just right.) The cloth is not quite flat because it curves a little between each pair of stick-ends. We can correct this if we put straight-edges between the stick-ends. So we now take three straight-edges and fasten them end to end like this,

then cover them with a cloth and pull the edges down so that the part of the cloth covering the frame is stretched tightly. The cloth and frame will look something like this:
Now do we have something that spreads and is full of straightness?

It certainly spreads. Is it full of straightness, as the table top seemed to be? Perhaps now is the time to see what "full of straightness" could mean. How and where can we see straightness in the cloth? "How?" is answered by sighting or by comparing with a stretched string. As to "Where?", the answer is "wherever we look along the cloth". Whenever we put our eyes at some point on the edge of the stretched cloth and look directly across to any opposite edge-point, we find our line of sight running entirely and exactly along the cloth. This happens no matter in what direction we look across the cloth, and no matter from what edge-point we look. This is the sense in which the cloth is full of straightness. So now we can say what it means for a surface to be full of straightness: no matter what two points we may choose on the surface, the line segment between them lies entirely on the surface. The test is to put both ends of a straight-edge on the surface at all sorts of places and particularly where we think not all of the edge would fit the surface. If no matter where we put the two ends on the surface all of the edge fits the surface, then the surface is full of straightness. If there is one position in which part of the edge does not exactly fit the surface, this is enough for us to say the surface is not full of straightness. This way we may test any table top and any stretched cloth.

Now that we have something we know is flat and have a test for a surface being "full of straightness", we can find or make other flat things. If your table top passes our full-of-straightness test, it is certainly flat, and to make something else that is flat you need only to copy the table top—for example, by laying a piece of paper evenly on the top. Another way is to cut a piece of paper or finely woven cloth into a shape like this,

![Diagram](image1)

with three straight edges. If the three corners are each pulled away from the other two, the paper or cloth will be stretched flat.

We have made a surface that spreads and is full of straightness. We also remember that when we were working with straight things, we went on to discuss straight lines: these are straight, and each is unbounded in its two directions. Can we invent a surface that spreads, is full of straightness, and is unbounded in all of its directions? We remember that straight lines can be made from line segments and our flat cloth or paper surface is full of line segments. Perhaps if we made straight lines out of all the line segments in our surface, the new figure (made of all these lines) might be flat. (It certainly spreads and it is certainly unbounded in all its directions!) So suppose we start with a figure showing our cloth or paper stretched flat:
Then draw some of these lines:

![Diagram of lines](image)

Do you think, if we could draw all such lines, the surface we would get would be full of straightness, and would pass the straight-edge test everywhere? The answer turns out to be "Yes". Even better, this surface would pass any test using an "unbounded straight-edge"; this means it is "full of straight lines", in the sense that for any two points $A$ and $B$ in the surface, the straight line $AB$ will lie entirely in the surface. So our surface not only spreads and is full of straightness, it is full of straight lines. A last observation: it cannot be all of space, as you no doubt can see easily. It is what is called a plane: any surface that spreads (contains three points that are not on any straight line), and is not all of space, and has the property that for any two points $A$ and $B$ on the surface the line $AB$ lies entirely on the surface is a plane.

If we start with any three points that do not lie together on any straight line and draw the three line segments connecting them, we will make a figure like this,

![Triangle](image)

called a triangle. It is like the frame on which we stretched the cloth. Suppose we next draw line segments with endpoints on the triangle; here are six examples.

![More line segments](image)

If we could draw all such line segments we would get this figure,

![Surface](image)

da surface that spreads and is full of straightness and looks like the stretched cloth. It is flat, and has the name triangular region. If every line segment in the triangular region is extended to a straight line, the collection of all these lines together makes a plane.

Thus, three points in space that do not lie on any straight line determine three figures: the triangle and the triangular region shown above and the plane just described. The triangular
region is full of straightness and the plane is full of straight lines; both spread, but neither fills all of space. Both are flat.

We can now say what a plane figure is. Any figure, or collection of points, is a plane figure if there is some plane that contains it. Thus, any three points in space form a plane figure. So do any line segment and any straight line, and so do any triangle and any triangular region. But a house is not a plane figure, and neither is this book. A plane region is any plane figure that contains somewhere in itself some triangular region. Here are two plane figures, \( \square \) and \( \triangle \). Do you think they are plane regions? Why?

**EXERCISE 47-5A**

1. Choose several surfaces and test them for being full of straightness.
2. Can a straight line be a plane? Why?
3. Can a triangular region be a plane? Can a plane figure be a plane?
4. In each of the following, find out whether the space figure described is a plane figure or not. Answer "Always", or "Sometimes, but not always", or "Never", giving reasons each time.
   a. a point b. two points c. four points
d. two lines e. a circle f. a seed

Is any one of these a plane region?

5. Space is a figure that is full of straight lines but not a plane. Can you name another figure for which these two statements are true?

6. On a flat table top, Abu and Ben lay down two pieces of paper looking like these

   \[ \begin{array}{c}
   \text{M} \\
   \text{N}
   \end{array} \]

   They agree that the two paper surfaces are plane figures and therefore should be called flat. They also think both surfaces are plane regions: do you agree? They decide that the first figure is full of straightness but cannot decide this for the second. Can you?

7. Here are some figures in the plane of the surface of this page. In each case, answer these questions:
   a. Is it a plane region?
   b. Is it full of straightness?
   c. Is it full of straight lines?

\( \square \) (1) \( \bigcirc \) (2) P Q \( \bigcup \) (3) (4) (5) \( \times \) (6)

**47-6 How figures can come from physical objects**

Now let us look at the shapes of some physical objects with which most of us are familiar. One group consists of rounded objects, such as eggs, seeds, oranges, clay balls and so on. These have no straight or flat parts. In this group, also, are certain man-made objects that are
perfectly rounded and are called *balls* (footballs and cricket balls, for example). A second group is composed of objects that are partly rounded and partly straight or flat; tree trunks, reeds, stretched vines, leaf spines, fingers, tins, round sticks and poles, coins and bottles are examples. A third group is made up of those objects that have no roundness; they are all straightness and flatness. Most of these are made by men.

In this last group, there is one particular kind of object that we shall call a *brick*. In ordinary language a brick is a solid object made of baked clay and looking like this:

![Image of a brick]

Any solid object like this we shall call a *brick*, whether made of clay or not. For example, a shoe box with no lid but filled with sand or earth is a brick. If you close this book and put it flat on the table, it becomes a brick. The outside, or "skin", of a brick, all of it that we can see and feel, is called its *surface*. This surface has six parts, as you can see from the illustration above. These are called the *faces* of the brick. (For this book, they are the front-cover face, the back-cover face, and the four side faces between the two covers.) A brick has eight corners, and a certain number of edges (how many?). You can see that each face is flat and each edge is straight.

We can make drawings by tracing parts of any brick. We could press one corner into a piece of paper and make a dent. When we mark the dent with a pencilled dot, we have a point. If we place any brick (this book, for example) so that one face is flat on a piece of paper, there are four edges of the brick resting on the paper. If we choose an edge and trace along it with a pencil, the resulting drawing is a line segment.

![Image of a brick drawn on paper]

If we traced all around the face of the brick, we would draw four line segments and our drawing would look like this from directly above:

![Image of a rectangle drawn on paper]

Such a drawing we call a *rectangle*. The four line segments that compose the drawing are called the *sides* or *edges* of the rectangle, and the four points that are endpoints of the sides...
are the vertices of the rectangle. The rectangle can be thought of as a picture of the edges of the face of the brick. If we fill in the rectangle,

we have a drawing called a rectangular region. It is a picture of the face whose edges we traced. The region includes the rectangle, and the edges and vertices of the rectangle are also considered to be the edges and vertices of the region.

Some bricks have faces that look like this:

A figure such as this is called a square region and the four line segments at the edges of the figure make a special kind of rectangle called a square.

Suppose we take a brick and copy onto paper each of its six faces. Suppose we cut out these six pieces of paper and then paste them together in the same way that the faces are connected on the brick. It is as if we had wrapped the brick in paper and then by magic removed the brick without disturbing the paper, leaving only a paper shell. This figure we call a box. It can be thought of as a picture of the surface of a brick, a picture made of six rectangular regions.

Any object in the second group, being partly flat and partly round, can be placed with a flat face down on paper, and the edge of the flat face can be traced. For example, a round coin such as a shilling piece can be traced to produce a drawing like this:

This is called a circle, and if we fill in the inside of the circle we get a circular region, like this:

We could do the same using the bottom or top face of a round tin, or the bottom of a bottle. If we had an empty round tin with a flat bottom and a flat top, we would have what is called a circular cylinder. If the tin were filled, it and its contents, together, would be a solid circular cylinder.

Trying to trace a rounded object is difficult. When we put a ball onto a piece of paper, it touches the paper only at a very small spot, and if we trace this we get only a point, which is a picture of only a very tiny part of the surface of the ball. But from feeling and seeing the
surface of a ball, we can imagine what the surface would look like if we could remove all that is inside it (like removing the inside of an egg leaving only the shell). The shape of what would be left is called a sphere.

**EXERCISE 47-6A**

1. How many edges does a brick have?
2. How many edges does a rectangle have?
3. How many edges would you say a circle has?
4. Do you think a circular region has any vertices? If so, how many do you think it has?
5. How many vertices does a box have? How many does a brick have?
6. What would you call the "faces" of a circular cylinder? How many of them are there?
   Answer the same two questions, first with the word "faces" replaced by "edges", and then by "vertices".
7. A ball has how many faces? edges? vertices? If these questions were asked about a sphere, would your answers be the same?

By folding paper or by cutting cardboard, we can make flat objects that have not just one or two straight edges but three or four or five or more. When we trace with pencil or chalk along such an object, we have a plane figure called a polygon. If such an object has all its edges straight and we trace all around it, we get a closed polygon. Thus we might get figures like these:

```
Line segment  2-sided polygon  3-sided polygon  4-sided polygon  5-sided closed polygon
```

We have already seen that a three-sided closed polygon, like this,

```
\[ \text{Line segment} \]
```

has a shorter name, triangle, and a triangle, together with its inside,

```
\[ \text{3-sided polygon} \]
```

is called a triangular region.

There are three special kinds of triangles that will be discussed in the next chapter: right-angled triangles, isosceles triangles and equilateral triangles.
A four-sided closed polygon that does not cross itself is called a quadrilateral.

Quadrilateral  Not a quadrilateral  Not a quadrilateral

Rectangles are a special kind of quadrilateral, and so are squares. There are other special kinds that will be mentioned later. A quadrilateral combined with its inside is a quadrilateral region.

There are, as you probably know, other ways of drawing geometrical figures besides tracing certain physical objects. Circles, for example, can be drawn using pencil, a drawing pin and either string or a piece of cardboard, or a specially constructed instrument called a compass.

So far, we have talked about tracing bricks, coins, bottles, cutouts and so on, and about getting special drawings called line segments, angles, polygons, circles and so on. If we think about it, we can see that anything that can be traced on paper will give a drawing that can be looked at as a geometrical figure—the tracing of your foot or your shoe or of a ragged piece of paper or cloth is a geometric figure. You may draw one that does not have such a standard name as "quadrilateral" or "oval". So we have a general name for all drawings made by running a pencil point over paper without making the pencil jump; the name is path. If the pencil comes back to its starting point, the figure is a closed path. A path that never crosses itself is a simple path. Thus, any circle is a path that is closed and simple. So is this figure.

And a polygon is a special kind of path. Simply drawing your pencil over paper without jumping makes a path. You may do it any way you wish.

A little earlier we traced a shilling piece and made a figure called a "circle". Other objects such as bottles, rounded tins and bottle caps can be found and traced to give the same
sort of figure. Can we make an object that can be traced to make a circle? This isn't easy to do. But it is easy to make a compass, which will itself trace a circle.

As children, many of us made and used our first compass when we fixed one heel on the ground, turned around on the heel and traced a rough circle in the dust with the end of our big toe. Later we found we could make more and better circles, holding pointed sticks. If we want to make circular drawings on paper or blackboard, we use the same idea, but in place of big toes or stick-ends we use pencils or chalk; in place of our heel, pointed metal or wood; and in place of a foot or a stick, we use cardboard, wood, metal or stretched string.

A simple cardboard compass can be made from a narrow piece of cardboard, a tack or a pin, and a pencil, by sticking the tack or pin through the cardboard near one end and the pencil point through a small hole near the other end. If you then stick the pin into a piece of paper and move the pencil point on the paper all the way around the pin, the pencil point will make a circle on the paper. The hole made in the paper by the pin point is a point called the centre of the circle. If we draw on the cardboard the line segment between the pin hole and the pencil hole, the length of this line segment is the radius of the circle. After the circle has been drawn, any line segment with one endpoint at the centre and the other endpoint on the circle is called a radial segment of the circle. The length of each of these radial segments is the same as the radius.

Another kind of compass, called a "string compass", has string in place of cardboard. One end of the string is looped around the tack or pin and the other end is looped around the pencil point. A circle with a different radius can be made by changing the place of the pencil on the string. For blackboard work, you can make a rough string compass by using chalk instead of a pencil, and your thumb in place of a pin. If you have no string, a piece of cloth will sometimes do.

More accurate and convenient is the manufactured adjustable compass that many of us have seen and used. It is made of two metal or wooden arms hinged together at one end, just as a forefinger and a middle finger are hinged together at the knuckles, so that the amount the arms are open can be varied. One arm has a sharp metal point at one end, and the other arm has a pencil or chalk point at one end: we shall call these the "pin arm" and the "pencil arm".
A compass can be used to copy old circles, and to make new ones. If you are given a point $P$ and a line segment $AB$

![Diagram of compass use](image)

and are asked to make a circle with centre at $P$ and radius the same as the length of $AB$, you can do so as follows. Put the point of one compass arm at $A$, and adjust the opening of the arms so that the point on the other arm fits at $B$. (We call this fitting the compass points to $A$ and $B$, or giving the compass the radius $AB$.) Then stick the point of the pin arm through the paper at $P$ and keep it there. Put the point of the pencil arm on the paper. Move the pencil point over the paper until it has gone all around $P$; this will trace the circle you want.

**EXERCISE 47-6B**

1. Make a cardboard compass. Use it to draw two circles with the same centre but different radii. (Note: You can do this by making two different pencil holes in the cardboard.)
2. Make a string compass and draw three circles. Try to draw two of them to be just touching each other.
3. Using a compass and thin paper, make a copy of this circle.

![Circle diagram](image)

Fit your copy over this circle to see how closely the two fit.

4. On a piece of paper, mark a point. Draw a circle that has its centre at this point and has its radius equal to the length of the following line segment.

47-7 **Review exercises**

1. In a few of your own words, say what you think is the difference between a circular cylinder and a solid circular cylinder. Give an example of each in real life.
2. Leaving aside this book, try to say in your own words what each of the following is: polygon, closed polygon, triangle, quadrilateral, path, triangular region, simple path.

3. Give a name for each of the following figures.

   a. 
   b. 
   c. 
   d. 
   e. 
   f. 
   g. 
   h. 
   i. 
   j. 
   k. 
   l. 
   m. 
   n. 
   o. 

4. In Question 3, which figures are full of straightness?

5. Draw on your paper a figure which will represent each of the following.
   a. Circle
   b. Triangle
   c. Triangular region
   d. Path
   e. Simple path
   f. Closed path
   g. Simple closed path
   h. Square
   i. Polygon
   j. Closed simple polygon
   k. Circular region
   l. Quadrilateral
   m. Rectangle
   n. Line segment
   o. Rectangular region

6. Does the edge of the top of a drum remind you of a certain plane figure? Which one? Do you think the top of a drum is a plane region? How would you test one to find out? Suppose you had a drum top that by test was a plane region; how would you describe how to make the plane that it determines?

7. Give answers, and reasons for your answers, to the following questions.
   a. Is every square a quadrilateral?
   b. Is every square a rectangle?
   c. Is every quadrilateral a rectangle?
Chapter 48

PLANE FIGURES
AND SPACE FIGURES

48-1 Introduction

In Chapter 47 certain figures were briefly introduced—among them such plane figures as points, line segments, straight lines, planes, triangles and rectangular regions, and some figures in space, such as bricks, boxes and balls, that are not plane figures. In this chapter, we want to discuss more fully these figures and some others such as angles and prisms. In particular, we want to look at various types of angles, triangles and plane quadrilaterals and see how to construct and copy them. For a pair of line segments, a pair of angles, a pair of triangles and a pair of quadrilaterals, there will come in the important ideas of congruence (meaning "the same size and shape") and of similarity (meaning "the same shape"). We shall also learn the constructions of line segments, angles and triangles that are congruent, and of triangles that are similar. Brief treatments of plane paths, including circles, and of plane regions end the discussion of plane figures in this chapter.

The rest of the chapter contains a short discussion of certain surfaces and solids in space and ends with a view of how lines and planes in space intersect each other and how they may be parallel or perpendicular.

48-2 Plane figures

In this part we shall talk only about figures that lie in a plane, the plane of any flat sheet of paper such as a page of this book or a piece of your drawing paper.

In the last chapter when we talked about straightness, we discussed line segments, rays and straight lines. Now we want to talk about pairs of line segments, pairs of rays and pairs of straight lines. Please bear in mind that we are thinking now only of a pair lying in one plane. The case, for example, of two lines not in the same plane will be looked at later.

48-3 Pairs of line segments

Suppose we have a pair of straight-edges; for example, two pieces of cardboard.

First straight-edge

Second straight-edge
If we played with these like children, we might put the two straight-edges together and notice that where they meet they fit closely with no gaps or holes. In other words, there is no difference in their straightness. But if we slide them along until one end of one fits at an end of the other, the second straight-edge would have a part not fitted to any of the first straight-edge, so the two edges are different in some respect.

We say that the second straight-edge is longer than the first, the first is shorter than the second, and the two straight-edges do not fit exactly.

If two straight-edges can be fitted together so that endpoints fit endpoints, as in this figure,

then the straight-edges are said to fit exactly, or to be congruent. In the same way, we say that two line segments are congruent, or fit together exactly, if they can be fitted together so that an endpoint of one segment fits on an endpoint of the other segment and the remaining endpoint of the first segment fits on the remaining endpoint of the other segment. Here, for example, are two pairs of congruent line segments.

This page can be folded over so that the segments in the right-hand pair fit each other; for the other pair, the page would have to be cut before the line segments could be fitted. This can be avoided; there is a way to compare the two segments while leaving the page and the segments undisturbed, and that is to find or make a straight-edge that exactly fits one of the two line segments and then compare it with the other line segment. For example, let us see if these two line segments are congruent:

We fit a straight-edge to $\overline{AB}$, the left end of the edge fitting at $A$, and mark the point on the edge that fits $B$; let's call it $B'$, and use $A'$ for the left endpoint of the straight-edge.
The part of the edge that runs from the left endpoint $A'$ to the marked point $B'$ is a copy of $AB$. If it fits $CD$, the two segments are congruent; if it does not, then they are not congruent. So we try to fit the edge to $CD$.

\[ \text{A' \quad B'} \quad \text{C \quad D} \]

In this position the left endpoints of these two fit, but the right endpoints, $B'$ and $D$, do not; the part $A'B'$ of the paper straight-edge fits the segment $AB$ but not $CD$. So the two segments are not congruent.

We now know how to test two line segments for congruence when the segments are given. Can we make a pair of congruent line segments? Quite easily—we can take any straight-edge and trace it twice. (We can, in fact, regard any congruent pair as having been made this way.)

That was easy. Let us try a problem that looks a little more difficult. We start with a line segment,

\[ \text{T \quad X} \]

and try to make another line segment that is congruent to it. One approach to solving problems of this sort is to think backwards, as follows. If we already had such a segment, what would we know about it? In this case, we could think of it as another tracing of some straight-edge that made $FT$. It is easy to make such a straight-edge. We saw how to do it when we learned just now how to test two line segments for congruence. Take a sufficiently long straight-edge, fit the left end to $T$ and mark on the straight-edge the point that fits at $X$. Then move the edge to another position and trace that part of the edge between the left end and the point just marked; the line segment so made is congruent to $TX$.

Let us ask one more question about congruent line segments. Given a line segment and a ray, such as these,

\[ \text{Q \quad A} \]

\[ \text{P \quad A} \]

can we find a point $B$ on the ray so that $AB$ is congruent to $PQ$? Yes, fit a straight-edge to $PQ$ just as we did to $TX$, by fitting one end at $P$ and marking on the straight-edge the point that fits $Q$. Then fit the straight-edge to the ray, so that the marked point falls at $A$ and the edge fits along the ray. The left end of the edge is at some point on the ray; if we mark it and call it $B$, we shall have found a line segment $AB$ that is congruent to $PQ$.
A compass also could be used to find $B$. If we fitted the two compass points to $P$ and $Q$ and then centred the compass at $A$ and drew an arc cutting the ray, the resulting point of intersection would be $B$.

At the beginning of this discussion we compared two straight-edges. Now let us compare two line segments in the same fashion, by fitting them together at one end. If they also fit at the other end, we know they are congruent. If they don't fit at the other end, we have something like this.

The two right ends do not fit when the left ends do. In this case the top segment is longer than the bottom one, and the bottom segment is shorter than the top one.

Suppose we have two segments that for some reason cannot be brought together. In this case, we make a copy of one and carry it to the other to compare. (We know we can make copies because we have just learned how to construct line segments congruent to a given segment; congruent means "exact copy"). For example, it is practically impossible to fold this page to fit these two segments together:

But if on some straight-edge we copy $\overline{HK}$ and compare the copy with $\overline{VW}$, we will know how the two original segments compare.

Thus, $\overline{HK}$ is a little shorter than $\overline{VW}$.

Sometimes it is easier to make a copy of each and compare the two copies. We could do this for the two segments above by making a third segment and on it drawing copies of the first two segments, in this fashion:
This is essentially what we do when we use a ruler to compare lengths.

For a final manoeuvre with line segments, see if you can divide any line segment into two line segments that are congruent. This means that given a line segment

\[ \overline{AB} \]

you wish to make two line segments that fit end to end, are congruent, and together make the line segment \( \overline{AB} \). The simplest way is to find on \( \overline{AR} \) a point \( C \) between \( A \) and \( B \) such that \( \overline{AC} \) is congruent to \( \overline{CB} \). You can do it in one way that is easy to understand. You can take the paper on which the line segment is drawn and fold the paper so that the two endpoints are on top of each other. The fold in the paper cuts the segment \( \overline{AB} \) at a point. If you call this point \( C \), you have two congruent segments \( \overline{AC} \) and \( \overline{BC} \). That they are congruent is obvious, since they match each other in the folding. And if you look at the piece of paper for a minute, you can see a way to get the same result without folding. The crease of the fold makes a line segment that cuts \( \overline{AR} \) at the point you want; we have just seen that it does. How can you make this line segment without folding?

To do this you can proceed as follows. Using first \( A \), then \( B \), as centres, draw two arcs which meet each other at two points. Call these points \( D \) and \( E \), and draw the segment \( \overline{DE} \). The line segments \( \overline{DE} \) and \( \overline{AB} \) will intersect at some point; call it \( C \).

You can see from the figure that if you folded the paper along the segment \( \overline{DE} \), the two sides of the drawing would fit on each other exactly. In particular, \( \overline{BC} \) and \( \overline{AC} \) would fit exactly; they are congruent.

Since \( \overline{AC} \) and \( \overline{CB} \) are congruent, we can think of \( C \) as being halfway between \( A \) and \( B \). It is therefore called the midpoint of \( \overline{AB} \). And since \( \overline{DE} \) cuts \( \overline{AB} \) at this midpoint, it is a bisector of \( \overline{AB} \).

The upper half of the drawing is like a paper straight-edge, the straight-edge of the paper being \( \overline{AB} \). If the paper were folded, the folded upper half would become a paper square corner, since the part \( \overline{DCB} \) would fit exactly on the part \( \overline{DC} \). (If you have a square corner handy, you can test the drawing now to see if \( \overline{AC} \) and \( \overline{CD} \) make a square corner and if \( \overline{BC} \) and \( \overline{CD} \) do.) For this reason, we call \( \overline{DE} \) a perpendicular bisector of \( \overline{AB} \); it cuts \( \overline{AB} \) at the midpoint of \( \overline{AB} \), and each of the pairs \( \overline{AC} \) and \( \overline{CD} \), and \( \overline{BC} \) and \( \overline{DC} \), makes a square corner.

To help develop the idea of one line segment being a perpendicular bisector of another, consider these examples:
In a, $\overline{GH}$ is a bisector of $\overline{AB}$ but is not a perpendicular bisector; and $\overline{AB}$ is a bisector of $\overline{GH}$ but is not a perpendicular bisector of $\overline{GH}$. In b, $\overline{GH}$ is a perpendicular bisector of $\overline{AB}$; and $\overline{AB}$ does not bisect $\overline{GH}$ but does make square corners with $\overline{GH}$. In c, each of the two line segments is a perpendicular bisector of the other.

**EXERCISE 48-3A**

1. Suppose we have traced a straight-edge and made the following line segment.

$$\overline{AB}$$

If we choose another straight-edge and trace it on the same paper, we shall have a second line segment like this:

$$\overline{CD}$$

There are many kinds of positions in which $\overline{CD}$ might be drawn. One kind is that in which the two line segments have no point in common; they are non-intersecting.

$$\overline{AB}$$

$$\overline{CD}$$

Then there are those positions in which the two line segments have one point in common. Here are some examples.

- Crossing
- Touching
- Connected end to end
Finally, there are those positions in which they have more than one point in common.

2. In terms of where the points of intersection occur, try to describe accurately in words the differences between the three cases "crossing", "touching" and "connected end to end". Are "aligned end to end" and "making a square corner" special cases of "connected end to end"?

3. Here are some pairs of line segments. In each case, use the phrase from Question 1 that most accurately describes the relationship of the pair.

4. Test each pair in Question 3 for congruence.

5. Here is a ray with endpoint $T$. 

\[ \text{T} \]
On the ray draw three line segments, with endpoints at $T$, that are congruent to these three line segments.

Which of these three given segments is the longest? Which is the shortest? (Do not use a ruler.)

6. In the following picture, try to find a line segment that is a perpendicular bisector of $XY$; of $DE$; of $FG$.

7. Make a paper straight-edge and bisect it by folding. Bisect each resulting half of the straight-edge in the same way.

8. Copy each of these line segments and use a ruler and compass to construct a perpendicular bisector of each copy.

9. Among these line segments, which pairs are congruent pairs?
10. Without using a ruler, compare these line segments for length. Say which is the longest, the next longest and so forth.

11. Copy each of these line segments and find the midpoint of your copy:
   a. 
   b. 
   c. 

12. According to the construction in the text, if \( A \) and \( B \) are any two points whatever, the line segment \( AB \) contains a third point \( C \) midway between \( A \) and \( B \). Suppose we start with two points \( A \) and \( B \) and find the midpoint \( C \) of \( AB \), getting this:

   \[ \overline{AB} \]

   Suppose as a next step we use our construction to find the midpoint \( D \) of \( AC \) and the midpoint \( E \) of \( CB \).

   \[ \overline{AC} \overline{CD} \overline{CE} \overline{EB} \]

   Pressing on, suppose we find and name the midpoints of \( AD, \overline{DC}, \overline{CE} \) and \( E \overline{B} \).

   \[ \overline{AC} \overline{CD} \overline{CE} \overline{EB} \overline{AF} \overline{DF} \overline{DG} \overline{GC} \overline{CH} \overline{HE} \overline{EI} \overline{IB} \]

   We could continue by finding the midpoints of the eight successive segments appearing above. These new points together with previous ones would divide \( AB \) into sixteen segments. At the next step we would run out of alphabet letters when we named the new bunch of midpoints. But if we could keep making new names for new points, could we keep up this process for as many steps as we wish? If we could, how many different points on the line segment \( \overline{AB} \) could we find?
48-4 Pairs of straight lines

With line segments there are many types of intersections possible, as we saw in Question 1 of Exercise 48-3A. But with straight lines the situation is much simpler (as frequently happens when we idealize). Two straight lines either have no point in common, or have one point in common, or else are the same straight line. To see this, suppose two straight lines have more than one point in common. Choose two of these common points; then each of our lines goes through these two points. But through two points goes just one straight line. So the two straight lines must be the same.

After this, when we say "two straight lines" we shall mean two that are not the same line. We can now say that given any two lines in a plane, either they do not intersect or they intersect at just one point (called the point of intersection). Note that among the points on a straight line no one is more distinctive than the others, in contrast to the endpoints of rays and line segments.

If two straight lines intersect, we can try to fit a square corner at the intersection. Here are three cases:

In each case, we put the tip of the square corner at the point of intersection and fit one straight-edge of the corner to one of the straight lines. We then see whether the other straight-edge fits the other straight line. If it does, we call the two straight lines perpendicular. In the right-hand case above, the two lines are perpendicular; in the other cases, they are not. So two straight lines are perpendicular if, first, they intersect and, second, a square corner fits at the intersection in the way just described. If they are perpendicular, and if their point of intersection has been named, say, "Q", we say they are perpendicular at Q.

Now let us ask a type of question we have already asked several times—can we make a pair of perpendicular lines? We would have to make a pair at whose intersection a square corner would fit. Let us try working this backwards. Could we start with a square corner and find two lines that fit it? Here is a square corner. See if you can fit two straight lines to it.

Now suppose we are not free to draw two perpendicular lines. Suppose we are given one of the lines and are asked to find a second line that is perpendicular to the given one. Here is a line; can you use a square corner to draw another line that is perpendicular to this one? Draw it anywhere, as long as it is perpendicular to this one.
There is another problem that is a little harder than that one. Here is a line and a point chosen on the line.

Can you draw a line that is perpendicular to this line at this point? If you don’t see how to do this, would placing a square corner in this position help you to see what you could draw?

The last problem along these lines is this. Given a line and a point off the line, draw another line that is perpendicular to the given line and goes through the given point.

The solution to this is not difficult if a square corner is used. We fit one edge of the square corner to the given line, like this,

and then slide the square corner along, keeping it fitted to the line, until the other straight edge of the square corner comes to the given point.

Here we stop and trace the edge that is at the point.
This makes a line segment, and the line segment determines a straight line that solves our problem. What are the two reasons why the line solves the problem?

Work done using a square corner may sometimes not be accurate enough. In the constructions we just did, perhaps we could use a compass to get better figures. We try with the last two constructions.

In one we were given a line and a point C on the line. We were to draw through C another line perpendicular to the given line. This is fairly simple to do with a compass. (The idea is a variation of the one we used to bisect a segment.) Put the compass point at C and draw two arcs that cut the line on opposite sides of C. Let A and B be the names of the points at which these arcs cut the line.

Now open the compass some more and with A and B as centres draw two arcs (just as we did in bisecting a line segment). They intersect at two points, D and E.

The straight line through D and E is perpendicular to \( \overline{AB} \), the given line. For if we folded the figure along the line \( \overline{DE} \), the two sides of the drawing would fit each other exactly, as you can see. This tells us that the upper half of the drawing is like a paper straight-edge folded to make a square corner! So we don't need to fit a square corner at the intersection of the lines; it is already there in the paper, and nothing could fit better!

When the given point C is not on the given straight line, you can do as follows. With C as centre, use your compass to draw an arc that cuts the line in two places; name these A and B.
Fit the two compass points to $A$ and $C$, and with first $A$, then $B$, as centre draw two arcs that intersect at two points, one of which is $C$. If you draw the line through these two points, you will have a line through $C$ that is perpendicular to the given line.

There is another important relationship that a pair of lines may have—that of being parallel. Two straight lines are parallel if there is another straight line that is perpendicular to each of them. Since we know how to construct perpendicular lines, it is easy to construct parallel ones. For example, to draw a pair of lines that are parallel we can start with any line we wish,

![Diagram](image)

construct a second line that is perpendicular to it by one of the methods we have just learned,

![Diagram](image)

and then by the same method draw a third line that is perpendicular to the second line.
The first and third lines are parallel since the second is perpendicular to each of them.

We can do even more; we can be given any straight line and any point not on it and proceed to find another line that is parallel to the given line and goes through the given point. We start by constructing through the given point a second line that is perpendicular to the given line, using one of the methods for doing this that we have recently learned.

Then we construct a third line that is perpendicular to the second line at the given point; this, too, we have learned. The result is this,

and a little thought shows that the third line and the given line are parallel.

**EXERCISE 48-4A**

1. First by sight, second using a square corner, test each of the following pairs of straight lines for being perpendicular and for being parallel.
2. In Question 1, which pairs of lines do you think are intersecting? (For example, a is an intersecting pair.) Remember that we can never draw all of any straight line; so two straight lines may intersect although the parts of them shown in our drawings may not intersect.

3. It is a fact that two straight lines in a plane that are parallel cannot intersect. This being so, try to answer these two questions.
   a. Can two parallel lines in a plane be perpendicular?
   b. Can two perpendicular lines in a plane be parallel?

4. Draw a line segment $\overline{ST}$ about half as long as your hand. Choose a point $C$ not on the line $\overline{ST}$. Through $C$ draw a line $\overline{CD}$ that is perpendicular to $\overline{ST}$. Draw a line $\overline{CE}$ through $C$ that is parallel to $\overline{ST}$. What relation do these last two lines have? Through $E$ draw a line $\overline{EF}$ that is perpendicular to $\overline{CE}$. What relation does $\overline{EF}$ have with $\overline{ST}$?

5. a. Which pairs of straight lines in the following figure are pairs of perpendicular lines?
b. Which are pairs of parallel lines?

48-5 Rays and angles

Now let us move on to pairs of rays in a plane. There are several ways in which a pair of rays may intersect; these are taken up in an exercise below. Very important in mathematics is the one in which the two rays have the same endpoint but no other point in common. When this happens, we say that the two rays are an angle. That is, an angle consists of, or is a combination of, two rays that have the same endpoint but no other point in common. Here are three pairs of rays; only the middle pair is an angle.
If two rays are an angle, we shall also say that they are connected end to end. The common endpoint is called the vertex of the angle, and the two rays are the sides or edges of the angle.

There are two special cases—the one in which a square corner fits the sides and the vertex,

and the one in which the two rays not only make an angle but also make a straight line.

The first of these is called a right angle and the second, a straight angle.

In considering angles, we should also think how we should name them. Suppose we have two rays that are an angle and the rays are already named (if they are not, we can name them). The vertex has a letter name, perhaps $E$, and the sides must bear names such as $\overrightarrow{ET}$ and $\overrightarrow{EG}$. We could then say "the angle $\overrightarrow{ET}$ and $\overrightarrow{EG}$". This is quite all right, but can we be briefer? Can we make one symbol? How about "$\overrightarrow{ET} - \overrightarrow{EG}$"? This is all right, but can it too be shortened? Would "$\overrightarrow{GET}$" do, as a combination of $\overrightarrow{GE}$ and $\overrightarrow{ET}$? It is a shorter symbol, but it suggests that the three points $G$, $E$, and $T$ lie on a line and that it is a name for that line. If $G$, $E$, and $T$ do not lie on a line, as they very well may not, this symbol would be misleading. Could we avoid this? Yes—just bend the double-headed arrow! This gives us "$\overrightarrow{GET}$", the top of which even looks like an angle! But alas, we don't stop here; mathematicians cut it down a little more, to "$\overrightarrow{GTE}$" (being kindly folk, they have omitted the barbs). So if you see the symbol "$\overrightarrow{ZQQ}$", you will know it stands for an angle that consists of the two rays $\overrightarrow{XZ}$ and $\overrightarrow{XQ}$.

Just as triangles and rectangles have inside regions, so does any angle that is not a straight angle. Let us look at these three figures and ask what we should regard as the "inside" of the one that is an angle.

There are only two candidates, namely:
As you may have guessed, it has been agreed that the right-hand one should be the inside. Does this choice have anything in common with the inside of a triangle and the inside of a rectangle, even though the latter two are "closed" figures and an angle is "open"? All three have sides, a fact which gives us a common way of forming the inside: draw all line segments that have their endpoints on the sides of the figure. When we do this we will, in all three cases, have drawn the inside region of the figure.

Now that we have some ideas about individual angles, we might ask, out of sheer curiosity, what could we do with a pair of angles? Can we combine them to make something, as sometimes two rays can combine to make an angle? Can we combine two angles to make a third? Yes, if we can put them together. If we have two angles

and we move one until it fits next to the other, like this, with \( \overrightarrow{T} \) fitting exactly on \( \overrightarrow{AC} \),

clearly we can regard the two rays \( \overrightarrow{AB} \) and \( \overrightarrow{OR} \) as forming a third angle. More precisely, we say that two angles are adjoining if they have a common side but nothing else common to their insides. Here are examples of adjoining, and not adjoining, angles:
If two angles are adjoining and we take the two sides that are not common, as in the left-hand example just shown, we have a third angle that can be thought of as the *join* of the two adjoining angles:

Thus, if two angles are adjoining, we can make from them a third angle, called their *join*.

Given two angles, we can also try to compare them. The way to compare is to put them together by fitting one side of one angle on one side of the other so that the inside of one angle fits on top of the inside of the other. If we try this with two angles—for example, these shown here—

they will fit together in either of these two ways.

In either case, it is clear that the inside of $\hat{EFG}$ is contained in the inside of $\hat{XYZ}$, and it is clear that we should consider $\hat{EFG}$ as being *smaller than* $\hat{XYZ}$, and $\hat{XYZ}$ as being *larger than* $\hat{EFG}$. In case the fit is exact—that is, if we can place one angle on top of the other so that each side of one angle coincides exactly with a side of the other angle—then we say the angles
are congruent. As an example, any two right angles are congruent, as we can easily see; and any two straight angles are congruent.

Suppose you wish to compare two angles that can't be fitted together. How can you compare them? What did we do when this problem came up with line segments? We made copies and compared them. Trying to do this with angles raises a new problem. How can you copy a given angle? One way, of course, is to put thin paper over the angle and copy the angle onto the paper by tracing. Another way would be to cut a model out of a sheet of paper. But neither of these is satisfactory for our purposes. We would like to be able to copy on any sheet of paper any angle given to us.

Let us suppose we have been given an angle on one sheet of paper,

\[ \angle BAC \]

and on another sheet of paper a ray,

\[ \text{ray } DE \]

and we want to construct on the second sheet an angle which is congruent to \( \angle BAC \) and has the ray \( \overline{DE} \) as one side. Take a compass and with vertex \( A \) as centre draw an arc of a circle. Be sure this arc cuts both rays \( \overline{AB} \) and \( \overline{AC} \). If you now name the points where your arc cuts the rays, your drawing should be as follows:

\[ \text{Leaving your compass unchanged, move it to the second sheet of paper, put the compass point at } D \text{ and draw a circular arc that cuts the ray } \overline{DE} \text{ and looks about as long and in about the same position as the arc you drew on the first sheet. The second sheet should now look like this if you gave the name } R \text{ to the point at which the arc cuts the ray } \overline{DE}. \]

\[ \text{Now go back to the first sheet, put your compass point at } P, \text{ and adjust the other arm of the compass so that it fits exactly at } Q. \text{ Move your compass to the second sheet, put the point at } R \text{ and draw an arc that cuts the first arc you drew. Now draw the ray that starts at } D \text{ and goes through the point, } S, \text{ at which the two arcs cross. The rays } \overline{DE} \text{ and } \overline{DS} \text{ form an angle } \angle SDE \text{ that is congruent to } \angle BAC. \]
There is another important construction with angles, called *bisection*. This means dividing an angle into two adjoining angles that are congruent, just as bisecting a line segment means dividing it into two end-to-end segments that are congruent. Can any angle be divided in this way? That is, if we have any angle \( \angle BAC \),

![Diagram of angle bisection]

...can we find a third ray \( \overline{AM} \) so that the two angles \( \angle BAM \) and \( \angle CAM \) are adjoining and congruent and have the original angle, \( \angle BAC \), as their join? The answer is "Yes", and the method is this. With the point \( A \) as centre draw an arc that cuts both rays \( \overline{AB} \) and \( \overline{AC} \), and give the names \( F \) and \( G \) to the points at which the rays are cut by the arc.

![Diagram showing the construction]

Adjust your compass to be open about three-quarters of the way from \( F \) to \( G \), and with \( F \) and \( G \) as centres draw two arcs that intersect away from \( A \), such as these.

![Diagram showing the construction]

Let \( M \) be the name of the point of intersection of these two arcs, and draw the ray \( \overline{AM} \).
If you folded the paper along the ray $\overrightarrow{AM}$, you would find that $\overrightarrow{AC}$ would fit exactly on $\overrightarrow{AB}$, and since the two angles $B\hat{A}M$ and $C\hat{A}M$ have the other side $\overrightarrow{AM}$ in common, clearly the two angles are congruent. They are a bisection of $B\hat{A}C$.

Let us briefly revise line segments. If two line segments have a common endpoint, the pair is not an angle, because an angle is a pair of rays with common endpoint. But given two line segments $\overrightarrow{AB}$ and $\overrightarrow{AC}$ with a common endpoint,

we can make an angle from them by drawing the two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

This is called the angle made by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Each line segment lies on just one line, so any pair of line segments determines a pair of straight lines. If this pair of straight lines are perpendicular (or parallel), we say the pair of line segments are perpendicular (or parallel). Here are some examples.
**EXERCISE 48-5A**

1. Draw, if you can, a pair of rays that:
   a. do not intersect;
   b. do not intersect and are not parallel;
   c. intersect at a single point;
   d. intersect at a single point and are not an angle;
   e. intersect at two points only;
   f. have in common a line segment;
   g. are a right angle;
   h. have in common a ray;
   i. are an angle but do not intersect.

   If in any of these cases, you think there is no such pair, give your reasons for thinking so.

2. Here are three rays, \( \overrightarrow{YX}, \overrightarrow{YZ}, \overrightarrow{YW} \). How many angles can you make from them? How many right angles can you make from them? Among these angles list all pairs that are adjoining.

3. If in Question 2 the bisector of \( \angle XYZ \) were added to the other rays, out of these four rays how many right angles and how many straight angles could you make?

4. Here are seven rays. How many angles do these rays make? Of these angles, how many pairs are adjoining?
5. Draw the inside of $MKL$ and of $POQ$, below.

6. Copy the following angle, $BAC$, by tracing it on a piece of thin paper laid over it. On another piece of paper, draw a ray with endpoint $D$ and use a compass to construct an angle congruent to $BAC$ and having your ray as one side. Put your thin paper copy over the compass copy and compare them.

7. Judging only by sight, list the angles in Question 4 in decreasing order of size, starting with the largest.

8. Draw a ray and on it construct an angle congruent to $BAC$ of Question 4. On it construct also another angle, this one congruent to $\hat{LHJ}$ of Question 4. Which of these two angles in Question 4 is the larger?

9. In each of the following, say how you could use paper folding to determine whether or not $\hat{ATB}$ and $\hat{CTB}$ are congruent, and if they are not, which is larger.
10. On paper draw three angles and bisect each. Test each bisection by paper folding.

11. Here are some line segments. Which pairs are perpendicular and which are parallel?

48-6 Polygons

In Chapter 47 we mentioned polygons and showed some examples, and also talked about the special kinds of polygons called "triangles" and "quadrilaterals". Now we will look at these more closely.

If we have drawn two line segments, \( \overline{AB} \) and \( \overline{BC} \), that are connected end to end,

we can make another step and draw a third line segment that is connected end to end with these two, by placing a straight-edge with one end at \( A \) or one end at \( C \) and tracing the straight-edge. Here are some ways of doing this.

In each case, we have three line segments. In the first two cases, the segments are \( \overline{DA}, \overline{AB}, \overline{BC} \); clearly, \( \overline{DA} \) and \( \overline{AB} \) fit end to end at \( A \), and \( \overline{AB} \) and \( \overline{BC} \) fit end to end at \( B \). What, in the last two cases, are the three line segments? Where do they fit end to end?

When we have drawn two line segments end to end, we say we have a two-sided polygon, and when we have drawn three line segments end to end, as in the figures just above, we have three-sided polygons. We can keep this up and make four-sided polygons, like these:
or five-sided ones like these:

We can go on adding more sides as long as we wish, making forty-sided polygons, thousand-sided ones and so on, as long as we have time, energy and patience. This gives us the general idea of a polygon: a *polygon* is a collection of two or more line segments drawn end to end in succession: if the number of line segments in the collection is \( k \), we call the polygon a *\( k \)-sided polygon*. The line segments that form a polygon are called the *sides of the polygon*. Their endpoints are called the *vertices of the polygon*.

Let us look at our original two-sided polygon.

This is not an angle because it consists of two line segments instead of two rays. But as we have seen we can make an angle from it. All we need do is to draw the ray \( \overrightarrow{BA} \), which we can make from the segment \( \overline{BA} \), and the ray \( \overrightarrow{BC} \), which we can get from \( \overline{BC} \); we then have this,

which is an angle, the *angle made by the two-sided polygon \( ABC \)*. We remember that the name for this angle is \( ABC \). In the same way, this three-sided polygon
makes two angles, \( \triangle ABC \) and \( \triangle DAB \), and this four-sided polygon makes four angles. Can you name them? The angles made this way by a polygon are called the angles of the polygon.

**EXERCISE 48-6A**

1. Determine which of the following figures are polygons.
2. For each figure in Question 1 that is a polygon, name the sides, the angles and the vertices, and say whether the polygon is simple or not, and whether it is closed or not.

3. In Question 1 there are five figures that are not polygons. Choose two of them and say in each case what could be added to or taken away from the figure that would change the figure to a polygon.

48-7 Triangles

When we had our two-sided polygon and added a third side, we fitted the straight-edge so that one end was either at \( A \) or at \( C \). Suppose instead of doing that we had fitted a straight-edge at both \( A \) and \( C \) and traced along the edge from \( A \) to \( C \): we would have drawn a line segment that had one endpoint at \( A \) and one at \( C \). The resulting figure

\[ \begin{array}{c}
\text{A} \\
\text{B} \\
\text{C}
\end{array} \]

is closed: we can view ourselves as having come back, in our tracing, to the starting point \( A \). From Chapter 47 we recall that such a figure is a closed three-sided polygon, and that the special name for it is triangle. This name was chosen long ago because from the figure we can make three angles, one of which, for example, is this, \( \angle BAC \):

\[ \begin{array}{c}
\text{A} \\
\text{B} \\
\text{C}
\end{array} \]

The three line segments of a triangle are called the sides of the triangle, and the three endpoints of these sides are the vertices of the triangle. In the triangle above, the sides are
\( \overline{AB}, \overline{AC} \text{ and } \overline{BC} \); the vertices are \( A, B \) and \( C \). There are a number of important special types of triangles. One type is that in which all three segments are congruent. We call such a triangle \textit{equilateral}. It is easy to construct such a triangle even when all its sides must be congruent to a given line segment, for instance, \( \overline{AB} \) in the following picture. Using your compass and taking \( A \) as the centre, and the length of \( \overline{AB} \) as the radius, construct an arc as shown. Then using \( B \) as the centre, and with the same radius, construct another arc to intersect the first arc at point \( C \). Draw the segments \( \overline{AC} \text{ and } \overline{BC} \). Clearly the three segments \( \overline{AB}, \overline{AC} \text{ and } \overline{BC} \) form a triangle all of whose sides are congruent to \( \overline{AB} \) and therefore are congruent to each other. Thus, \( \triangle ABC \) is an equilateral triangle.

\[ \text{\includegraphics{triangle.png}} \]

Another type of triangle is one in which two sides are congruent and the third side may be of any size whatever; perhaps it is congruent to each of the other two, perhaps not. We call such a triangle \textit{isosceles} and the third side the \textit{base}. Clearly an equilateral triangle is a special kind of isosceles triangle. The construction of an isosceles triangle is just as easy as that of an equilateral triangle. Again, as in the following picture, construct a segment \( \overline{AB} \). Now choose any radius greater than half the length of \( \overline{AB} \), and draw two arcs with that radius, one with \( A \) as the centre and one with \( B \) as the centre. Make sure that you draw two arcs that intersect (why was it necessary to require that the radius be greater than half the length of \( \overline{AB} \)) and mark the intersection point \( C \). The two segments \( \overline{AC} \text{ and } \overline{BC} \) are congruent, and the triangle \( \triangle ABC \) is isosceles.

\[ \text{\includegraphics{isosceles_triangle.png}} \]

A third important kind of triangle is one which has a right angle for one of its angles. Such triangles are called \textit{right-angled triangles}. It is easy to construct right-angled triangles. Draw a segment \( \overline{AB} \), and then construct another segment \( \overline{AC} \) perpendicular to it at \( A \). Now draw the segment \( \overline{BC} \). The triangle composed of the three segments \( \overline{AB}, \overline{AC} \text{ and } \overline{BC} \) is clearly a right-angled triangle.

\[ \text{\includegraphics{right_angle_triangle.png}} \]
There are other special names which we can give to triangles, but we will not deal with them all. In further study of mathematics, you will learn many more properties of triangles as well as the relations between the sides and angles of triangles.

One important thing remains for us to do, however, and that concerns the relationships which might exist between two triangles. In the first place, we can have triangles which are congruent, that is, triangles which fit each other perfectly. How would you construct a triangle congruent to a given triangle? One way, of course, would be to cut out the first triangle and trace around its edge to form a second triangle. Or you could put tracing paper over the triangle and copy it that way. But we would like to find a way which is like the other constructions we have done, using the drawing tools and methods with which we have become familiar.

There are actually several ways of constructing a triangle congruent to a given triangle. Perhaps the easiest is done in the following way. Let $ABC$ denote the given triangle. Construct a line segment $DE$ congruent to $AB$, as we did in Chapter 47. With point $D$ as centre, construct an arc with radius the length of $AC$. With point $E$ as centre, construct an arc with radius the length of $BC$ so that it intersects the first arc at point $F$.

You can see that the segment $DE$ is congruent to the segment $AB$, that the segment $DF$ is congruent to the segment $AC$ and that the segment $EF$ is congruent to the segment $BC$. Moreover, if you try to fit the two triangles together, you will find that the angles also are congruent. The two triangles thus fit perfectly and are congruent.

Another way is to take the given triangle $ABC$, as shown in the following picture. Construct segment $DE$ congruent to $AB$. Now copy the angle $BAC$ by drawing a segment $DG$ so that $DG$ is congruent to $BAC$. Then lay off a segment $DF$ on $DG$ congruent to $AC$. Join points $E$ and $F$ to form segment $EF$. The triangle $DEF$ looks congruent to the triangle $ABC$. If you traced one of the triangles, the tracing would fit exactly on top of the other.

A final construction is that shown in the following picture. Again take triangle $ABC$. Construct segment $DE$ congruent to $AB$. Construct an angle $CDE$ at $D$ congruent to $CAB$, and construct an angle $DEH$ at $E$ congruent to $ABC$. Call $F$ the point where the rays $DG$ and $EH$ intersect each other. The triangle $DEF$ is congruent to the triangle $ABC$. 
You have now considered three cases. In the first, you constructed a triangle congruent to a given triangle by copying the three sides of the first. In the second case, you constructed the new triangle by copying two sides of the first and the angle between them. In the third, you used two angles of the first triangle and the side between them. Now think what might happen if you made a new triangle by copying the three angles of the given triangle. Could you make a new triangle this way, and would it be congruent to the given triangle?

Let us try such a construction, shown in the following words and picture. Let \( \triangle ABC \) be any given triangle. Copy the angle \( \angle BAC \) and call it \( \angle IGD \). On the ray \( \overrightarrow{DE} \), choose any point \( E \) and construct an angle congruent to \( \angle ABC \), using the ray \( \overrightarrow{DE} \) as one side, so that the second side is the ray \( \overrightarrow{EI} \). The rays \( \overrightarrow{EI} \) and \( \overrightarrow{DG} \) intersect at a point \( F \). Look at the triangle \( \triangle DEF \). You know that \( \triangle EDF \) is congruent to \( \triangle BAC \) and that \( \triangle DEF \) is congruent to \( \triangle ABC \). What about the relation between \( \angle ACB \) and \( \angle DPE \)? You can find that these two angles are in fact congruent, either by fitting them together or by using the construction for copying angles.

But you will notice by looking at the picture that triangle \( \triangle DEF \) is much bigger than triangle \( \triangle ABC \). So the two triangles are not congruent even though their angles are congruent. We say in this case that the triangles are similar, meaning that their angles are congruent, even though their sides may not be congruent. But it is worth checking the relation between the sides of one triangle and the sides of the other triangle. If you do so, you will find in this particular case that the length of each side in triangle \( \triangle DEF \) is exactly twice the length of the corresponding side in triangle \( \triangle ABC \). This should suggest to you that there might exist some such relation for every pair of similar triangles.

You can check for the possibility of such a relation in the following way. Take a triangle \( \triangle ABC \), as drawn in the following picture. Construct a segment \( \overrightarrow{DE} \) which has a length three times the length of \( \overrightarrow{AB} \). Now construct an angle \( \angle GDE \) which is congruent to \( \angle CAB \), and an angle \( \angle HED \) which is congruent to \( \angle CBA \). The rays \( \overrightarrow{DG} \) and \( \overrightarrow{EH} \) intersect in a point \( F \). You can see that the angle \( \angle DFE \) is congruent to \( \angle ACB \). And if you check the length, you will find that the length of \( \overrightarrow{DF} \) is three times the length of \( \overrightarrow{AC} \), and that of \( \overrightarrow{EF} \) is three times that of \( \overrightarrow{BC} \).
In general you will find that such a relation exists between the lengths of sides of similar triangles. If all the angles of one triangle are congruent to all the angles of a second triangle, and the length of one side of the first triangle is some number times the length of the corresponding side of the second triangle, then the length of each of the other two sides of the first triangle is that same number times the length of the corresponding side of the second triangle.

**EXERCISE 48-7A**

1. Any equilateral triangle is isosceles. As an isosceles triangle, how many bases does an equilateral triangle have?

2. Draw line segments \( \overline{AB} \) and \( \overline{XY} \), with the length of \( \overline{XY} \) greater than half the length of \( \overline{AB} \). Construct a triangle having one side congruent to \( \overline{AB} \) and each of the other two sides congruent to \( \overline{XY} \). What kind of triangle have you constructed? What kind would it be if \( \overline{AB} \) and \( \overline{XY} \) were congruent?

3. Finish the construction of these two incomplete right-angled triangles.

4. Construct a right-angled triangle that has one side congruent to the following line segment.

5. Here is a right angle.

\[ \text{a. Draw a right-angled triangle that has a side on each of these two rays.} \]
b. How many such triangles can be drawn?
c. Can you draw an isosceles right-angled triangle that has a side on each of these two
rays?
d. How many of these could be drawn?

6. Construct, if you can, a triangle with two of its angles right angles; if you cannot, try
to explain why you cannot.

7. Can you draw a triangle that is right-angled and equilateral?

8. In Question 4 you drew a triangle that had two sides perpendicular. Can you draw a tri-
angle with two sides parallel? If not, give reasons why not.

9. The construction of a triangle congruent to a given triangle, using two angles and the
included side of the given triangle, was not given completely in the text. Complete that
construction in all details.

10. Construct a triangle whose angles are congruent to those of a given triangle, and whose
sides are four times as long as the corresponding sides of the given triangle. Then con-
struct a triangle whose angles are congruent to those of a given triangle, and whose
sides are \(1\frac{1}{2}\) times as long as those of the given triangle.

11. Construct a triangle similar to a given triangle, but do not fix in advance the relation
between the lengths of corresponding sides. Find whether or not the relation is the same
for each pair of corresponding sides.

48-8 Quadrilaterals

Do you remember the special kind of polygon called a quadrilateral? Which of these fig-
ures is a quadrilateral?

We remember that a quadrilateral is a four-sided, closed, simple polygon; that is, it is
made of four line segments that are connected end to end, that intersect only at endpoints, and
are such that each endpoint is the end of two of the segments. It can be thought of as four line
segments drawn one after the other, the second beginning where the first one ended, the third
beginning where the second one ended and not intersecting the first; the fourth starting where
the third ended and ending where the first began and not intersecting the second. So these are
quadrilaterals,

and these are not.
Since a quadrilateral is a polygon, it has sides, angles and vertices. It is easy to see that it has four of each. Any two sides that have a common endpoint are called adjacent sides. Any two sides that are not adjacent are called opposite. In the quadrilateral

\[\text{A} \quad \text{B} \quad \text{C} \quad \text{D}\]

there are four pairs of adjacent sides (\(\overline{AB}\) and \(\overline{BC}\), \(\overline{BC}\) and \(\overline{CD}\), \(\overline{CD}\) and \(\overline{DA}\), \(\overline{DA}\) and \(\overline{AB}\)) and two pairs of opposite sides (\(\overline{AB}\) and \(\overline{CD}\), \(\overline{BC}\) and \(\overline{AD}\)). Two angles that have a common side are called adjacent angles. Two angles that are not adjacent are called opposite. The quadrilateral above, like all other quadrilaterals, has four pairs of adjacent angles (can you name them?) and two pairs of opposite angles (one pair are \(\angle DAB\) and \(\angle BCD\); can you name the other pair of opposite angles?).

In Chapter 47 we saw that if we laid this book on its back on a flat piece of paper and traced it, we would get a figure like this,

\[\text{A} \quad \text{C} \quad \text{B} \quad \text{D}\]

and we said it was called a rectangle. Let us be more precise now about what is meant by "rectangle". First, it is a quadrilateral. Second, the four angles are all right angles. This is enough. But we can also see in the figure that \(\overline{AB}\) and \(\overline{CD}\) are parallel (because \(\overline{AC}\) is perpendicular to both) and they look congruent (but we are not sure—they just look that way). Similarly, \(\overline{AD}\) and \(\overline{BC}\) are parallel and look as if they may be congruent. (If you take a piece of paper that has four square corners and you fold it over so that one edge fits on the edge opposite to it, do the two edges fit exactly end to end?

So a rectangle is a quadrilateral whose four angles are all right angles. Opposite sides of a rectangle are parallel, and from paper folding, or comparison by string, we find that opposite sides are also congruent.

To construct a rectangle, begin by constructing a line segment. Then at each endpoint of the segment, erect a ray perpendicular to the segment. At some point on one of the perpendicular rays construct a line perpendicular to that ray, and find the point where this line meets the second perpendicular ray. The rectangle made by this construction is shown in the following figure, where \(\overline{AB}\) is the original segment, \(\overline{AE}\) and \(\overline{BF}\) are the rays erected perpendicular to \(\overline{AB}\), \(C\) is a point different from \(A\) chosen on \(\overline{AE}\), \(\overline{CG}\) is the line drawn perpendicular to \(\overline{AE}\) at \(C\), and \(D\) is the point where the line \(\overline{CG}\) meets \(\overline{BF}\). The rectangle is \(\overline{ABDC}\).
In Chapter 47 we said that a rectangle like this

is called a *square*. From looking at the sides of this figure you can easily guess what a square really is— it is a rectangle with all four sides congruent to each other. To see how to construct a square, go back to the method just given for constructing a rectangle. If you follow that method, but choose $C$ so that $\overline{AC}$ is congruent to $\overline{AB}$, you will make a rectangle that has all four sides congruent to $\overline{AB}$. It must be a square.

We know that a rectangle is a quadrilateral for which each pair of opposite sides are parallel. Let us give a name to such figures: a quadrilateral for which each pair of opposite sides are parallel is a *parallelogram*. So every rectangle is a parallelogram. Is there any parallelogram that is not a rectangle? This is the same as asking if there is a quadrilateral with opposite sides parallel but with some angle that is not a right angle. Try the following construction. Draw two line segments $\overline{AB}$ and $\overline{AC}$ that meet at $A$ but are not perpendicular. At $C$ construct a line $\overline{CE}$ parallel to $\overline{AB}$. At $B$ construct a line $\overline{BF}$ parallel to $\overline{AC}$. The lines $\overline{CE}$ and $\overline{BF}$ will meet at a point $D$. The quadrilateral $ABDC$ is a parallelogram that is not a rectangle.

It is a quadrilateral with angle $\angle BDA$ not a right angle. The opposite sides $\overline{CD}$ and $\overline{AB}$ are parallel because $\overline{CE}$ was drawn parallel to $\overline{AB}$, and the opposite sides $\overline{BD}$ and $\overline{AC}$ are parallel since $\overline{BF}$ was drawn parallel to $\overline{AC}$. These facts show that $ABDC$ is not a rectangle but is a parallelogram.

We have so far mentioned parallelograms, rectangles and squares as interesting special cases of quadrilaterals. We have also answered the question "Is there a parallelogram that is not a rectangle?" There are other questions of the same kind. You can look back at some of our figures and easily answer the following questions; for example, "Is there a quadrilateral that has all four angles congruent but does not have all four sides congruent?"
EXERCISES 48-8A

1. Which of these figures are quadrilaterals?

![Quadrilaterals Diagram]

2. In Question 1, which figures are squares? Which are parallelograms? Which are rectangles?

3. Is every square a parallelogram? Give reasons for your answer.

4. Give examples of quadrilaterals from ordinary life. Divide those you find into four groups: squares, rectangles that are not squares, parallelograms that are not rectangles, and quadrilaterals that are not parallelograms.

5. Abu has a secret rectangle he will not show to his friend Ben. Ben thinks it is a square but is not sure, and Abu will not tell him. But Ben gets from Abu the fact that two adjacent sides are congruent. Is this enough for Ben to be sure it is a square?

6. A square is a quadrilateral that has all four sides congruent to each other. But a quadrilateral with all four sides congruent to each other may not be a square. Can you find such a quadrilateral? If you can, try to construct one using a straight-edge and compass.

7. Draw a rectangle on paper by the method in the text. Cut out the piece of paper bounded by the rectangle and fold it to show that opposite sides of a rectangle are congruent. Note that when opposite sides are fitted together, the fold is smooth and the paper lies flat.

8. We have seen that certain quadrilaterals have all four angles forming right angles; these are the rectangles. Can a quadrilateral have exactly one right angle? exactly two? exactly three?

9. In a rectangle, opposite sides are parallel. Let us turn this around and write, "If opposite sides are parallel, we must have a rectangle". Is this true? Is it possible to draw a quadrilateral which has opposite sides parallel but which is not a rectangle?

10. Draw a paper parallelogram by the method in the text, and cut it out. Try to fold the paper to show that opposite sides of your parallelogram are congruent. Do you think every parallelogram has each pair of opposite sides congruent? Do you think opposite angles are congruent? Can you fold your paper to find out?

11. In a rectangle, opposite sides are congruent. Let us turn this around and say, "If
opposite sides of a quadrilateral are congruent, the quadrilateral must be a rectangle". Is this statement true? Why?

There are other kinds of quadrilaterals besides parallelograms, rectangles and squares; and there are many other questions that could be asked about quadrilaterals. We have the ideas of opposite sides and of adjacent sides. In the same way, there are two pairs of opposite angles and four pairs of adjacent angles. For a pair of line segments (such as the sides of a quadrilateral), we have the relations of being perpendicular, parallel and congruent. For a single angle, we have the idea of being a right angle and that of being a straight angle. And for a pair of angles, we have the congruence relation. Using these ideas we make a "question-machine" as follows: we write down,

"Is there a quadrilateral with"

and then we choose one part in each of these columns:

$$\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
\end{array} \begin{array}{c}
\text{pair(s) of} \\
\text{opposite} \\
\text{adjacent} \\
\text{angles} \\
\end{array} \begin{array}{c}
\text{that are} \\
\text{parallel} \\
\text{perpendicular} \\
\text{congruent} \\
\end{array} \begin{array}{c}
\text{of} \\
\text{oppo} \\
\text{pair(s}} \\
\text{perpendicular} \\
\text{congruent} \\
\text{right angles} \\
\text{straight angles} \\
\text{congruent} \\
\end{array}$$

For example, "Is there a quadrilateral with one pair of adjacent angles that are right angles?" Another is, "Is there a quadrilateral with two pairs of opposite sides that are parallel?" To reply "Yes" to one of these questions, we have to draw a quadrilateral that has what the question calls for. Some of these questions are easily and correctly answered "Yes"; others are more difficult, including some that have the answer "No".

There are other questions we can ask: for example, if all four sides are congruent, what can we say about the angles? Must all four angles be congruent? three of them? two of them? How many pairs of angles must be congruent? any at all?

Another is: if all four angles are congruent, what can we say about the sides? Can we say anything else about the angles?

There are many questions you yourself can ask and answer. (Do not be surprised if you find one you can ask but cannot answer—this happens to all of us at one time or another.)

Just as with line segments, angles and triangles, we can define congruence for quadrilaterals: two quadrilaterals are congruent if one can be fitted exactly on top of the other. One way to make two congruent quadrilaterals is to take or make a cardboard cutout having four edges, all of them straight, and trace it twice. Just as with triangles, if we have two congruent quadrilaterals, then when we fit one exactly on the other, the fitting makes a correspondence between the sides of one and the sides of the other. To each side of the lower quadrilateral, there corresponds the side of the upper quadrilateral that is fitting exactly on it (these two sides are congruent line segments). Similarly, to each angle of the bottom quadrilateral there corresponds the angle of the top one that fits on it exactly; each of these pairs of angles is a congruent pair. The study of the congruence of quadrilaterals is more complicated than that of triangles and will be omitted here except for a few exercises below.
EXERCISES 48-8B

1. If you are asked, "Is there a quadrilateral with one pair of adjacent angles that are right angles?" you would probably answer immediately, "Yes, any rectangle". But can you tell how to draw such a quadrilateral that is not a rectangle?

2. Is there a quadrilateral with one pair of adjacent angles that are straight angles?

3. Is there a quadrilateral with a pair of opposite sides that are perpendicular?

4. Draw a quadrilateral that has one pair of adjacent angles congruent and the other two angles, also, a congruent pair. From your figure can you draw any conclusions about the sides?

5. If all four sides of a quadrilateral are congruent, must all four angles be congruent? If not, are there any pairs of angles that are congruent?

6. Try to construct a quadrilateral with all four angles congruent. If you do construct one, what can you say about the angles and sides of your drawing?

7. Construct, using compass and straight-edge, another square congruent to this one.

8. Construct, using compass and straight-edge, a quadrilateral congruent to the following quadrilateral.

9. Construct, using a straight-edge and compass, a quadrilateral congruent to the following one.

10. Tell in your own words how you would construct a quadrilateral congruent to this one.
48-9 Plane paths, circles and ellipses

We have been talking about polygons, particularly the special cases of one-, two-, three-, and four-sided ones. The three-sided closed polygons were called triangles, and the four-sided closed ones that did not cross themselves were called quadrilaterals. But polygons are special cases of what in Chapter 47 we called "paths". A path is a drawing made by running a pencil or chalk over paper or blackboard without letting the pencil (or chalk) jump or skip. If the drawing ends back at its starting point, the path is closed. If a path never crosses itself, it is a simple path. So "quadrilateral" is a short (!) name for a simple closed four-sided polygon. Triangles and quadrilaterals are simple closed polygons and simple closed paths. Here are other examples; some are paths, some are not.

Simple path

Simple closed path

Closed path

Path

Simple closed path (circle)

Not a path

Not a path

It is now time that we became clear as to what a circle is, just as we became clear about a rectangle being a quadrilateral with all four angles right angles. At the end of Chapter 47, we used a compass to draw a figure we called a "circle". Let us see what we can find about this figure and the way it was made and what it is made of.

To start with, one compass point stayed fixed. The point on the paper where the compass point was fixed we call the centre of the figure and give the temporary name $C$. The other point of the compass moved around; if $A$ is any point on the figure, this compass point has moved over it. This tells us that $CA$ is congruent to the line segment that has the compass points as endpoints. So for any point $A$ on the figure, the line segment $CA$ is congruent to this compass-points line segment. The other way around happens to be true, also; if $A$ is any point on the paper such that $CA$ is congruent to the compass-points line segment, then the moving compass point must have passed through the point and the point $A$ must be on the figure.

This gives us our general definition of a circle, as follows. Let $C$ be a point and $XY$ any line segment (such as that between the compass points). Lay off all possible line segments that are congruent to $XY$ and have one endpoint at $C$. Draw through the endpoints of all these line segments; the figure that results is the circle with centre at $C$ and radius the length of $XY$. A circle is any figure obtained in this way; it has a centre and a radius. Any line segment that has one endpoint at the centre of the circle and the other endpoint on the circle is a radial segment of the circle. The radius of the circle is the length common to all the radial segments. Any line segment that has both endpoints on the circle is a chord of the circle; a chord through the centre of the circle is a central chord. All central chords are congruent, since each consists of two adjoined radial segments. Their common length, which is twice the radius of the circle, is called the diameter of the circle.
There is one other simple closed path worth mentioning here, namely, the **ellipse**. The ellipse is a round, but somewhat flattened, figure, rather like the circle.

The method for constructing an ellipse is actually quite simple. Mark out two points on paper, and take a piece of string somewhat longer than the segment joining those two points, which are called the **foci**. Put a loop at each end of the string and insert a pin through each loop. Then push the two pins into the paper at the two foci. This connects the ends of the string to these two points. Stretch this string with a pencil; keeping it stretched, put the pencil point on the paper and trace all around the two foci, as shown in the following picture.

If you have done this carefully, your drawing will look somewhat like a flattened circle, or as if it were traced from an egg which is the same at both ends. Instead of having a diameter, as a circle has, it has two axes: one of these is the longest line segment across the ellipse, and the other is the shortest line segment across the ellipse. You will notice that the longer axis (called the **major axis**) goes through the two foci, while the shorter axis (called the **minor axis**) is the perpendicular bisector of the longer axis. This picture shows the ellipse, with its foci and its axes. The point at which the two axes intersect is the **centre** of the ellipse.

One fact is worth noting about the ellipse. For many years it was thought the sun moved about the earth in a circular path. Then people learned that the earth moves about the sun, but still they thought the path was a circle. Kepler, a scientist of the 17th century, first discovered and showed that the path of the earth as it goes around the sun is an ellipse, with the sun at one focus.

**EXERCISE 48-9A**

1. Here are some drawings. Determine which ones are paths; for those that are paths, say whether they are closed or not and whether they are simple or not.
2. Draw two points on a piece of paper. Using a straight-edge and compass, try to draw a circle that goes through these two points.

3. If you have done Question 2 successfully, try this one. Given any three points in a plane but not lying together on any straight line, there is exactly one circle going through all three points. Draw three such points and see if you can construct this circle.

4. Given a point and a circle with its centre, how would you construct a second circle that is congruent to the given circle and has the given point for its centre? How would you construct a third circle that is congruent to the given circle and goes through the given point?

5. Name as many objects from the real world as you can which have the shape of an ellipse. Tell how these could be used in teaching children about ellipses.

6. Draw an ellipse with a very long major axis and a very short minor axis, and compare it with an ellipse where the axes are of very nearly the same length.

7. If in drawing an ellipse we used just one point (focus) and one pin, instead of two points and two pins, but otherwise followed the instructions in the text, what sort of figure would we make? Perform this construction and see.

48-10 Regions in the plane

When we draw a circle on paper, it is clear to us that there is a part of the paper that is inside the circle and another part that is outside the circle, and that these two parts together with the circle itself make up all the paper.

The part that is inside any circle is called a **circular region**: the shaded portion in the next figure is such a region.
More generally, any simple closed path has an inside part and an outside part. The inside part is called a region. Here are three simple closed paths,

and here are the three regions that are inside them:

Whenever we have a special kind of simple closed path—say, a circle, a triangle or a rectangle—we give to the region inside the path a name like that of the path. The inside of a circle is a circular region, the inside of any triangle is a triangular region, the inside of any quadrilateral is a quadrilateral region and so on. If you invent a special path for yourself and call it by some special name, for example, "google", then the inside is a googley region.

For the special paths called "simple closed polygons"—for example, this "hexagon"—

we already have names for certain line segments and points in the figures. This figure is made of line segments $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$ and $\overline{FA}$. These are called the edges of the hexagonal region inside the hexagon. Note that $\overline{AB}$ and $\overline{BC}$ are taken to be edges, but $\overline{AC}$ is not. The end-
points of these edges are called the vertices of the region, just as they are called the vertices of the polygon. Similarly, this quadrilateral region lying inside the quadrilateral \( PQRS \) has for edges the line segments \( \overline{PQ}, \overline{QR}, \overline{RS}, \overline{SP} \) and for vertices the four points \( P, Q, R, S \).

Each simple closed path divides its plane into three parts: the path itself, the inside of the path and the outside. A simple path that is not closed, like this,

\[ \sim \]

divides its plane into only two parts: the path, and the rest of the plane. There is no inside region of the path. For closed paths that aren't simple, the questions "Does it have an inside? an outside? Into how many pieces does it divide its plane?" become difficult to answer.

A line segment in a plane is a simple path that is not closed, and it divides its plane into two parts: the line segment, and the rest of its plane.

But if we make the entire straight line determined by this line segment,

\[ \sim \]

then we have a different story. Into how many pieces is the plane divided? Are \( V \) and \( G \) points in the same piece? Are \( H \) and \( L \)? Are \( Z \) and \( M \)?

Any straight line in a plane divides the plane into three parts: one is the line itself, and the other two parts are the sides of the line. Thus \( H \) and \( M \) lie on one side of the line \( \overline{GV} \), and \( Z \) and \( T \) lie on one side (the side other than the one \( H \) and \( M \) lie on). The points \( L \) and \( Z \) lie on opposite sides of \( \overline{GV} \).

Just as any line divides any plane that contains it into three parts, any point \( P \) divides any straight line containing it into three parts.
In this case, there is the part to the left of $P$, the part to the right of $P$, and $P$ itself.

What sort of figure would divide all of space into three parts in the same sort of way that a point on a line divides the line, and a line in a plane divides the plane?

48-11 Figures in space

In Section 47-6 we separate physical objects into three groups: those whose surfaces have only flatness and straightness, those whose surfaces have no flatness or straightness and those whose surfaces have some flatness or straightness and some roundness.

As an example of the first group we mentioned what we called a brick. This book when closed is a brick. The surface, or "outside" or "skin", of a brick looks like this.

It has six faces, each consisting of a rectangular region. Each face meets four of the other five faces, one at each straight edge of the face. When two faces of a brick do not meet, they are called opposite. There are twelve straight edges, each being part of two faces; and there are eight corner tips or corner points, each being a common endpoint for three of the straight edges and each being part of three faces.

In summary: a brick has one surface or skin or outside. This surface is called a box. It consists of six parts called faces, each face being made of a rectangle and the rectangle region inside. The straight edges making up these rectangles are twelve in number, and their endpoints, which are the tips of the corners of the brick, total eight. These points are called the vertices of the brick and of the box. The straight edges of the brick are the edges of the brick and of the box. The faces of the brick are also the faces of the box.

The second solid figure whose surface has only flatness and straightness that we wish to mention is something we will not name here and now. In discussing the brick and its surface, the box, we began with the solid object (the brick), then talked about its surface (the box), and then discussed the parts of the box (faces, edges and vertices). For the next figure, we shall reverse this order.

We begin with an equilateral triangle $PQR$ and its inside region,
and find, as the drawing shows, the points $X$, $Y$, $Z$ that are the midpoints of the line segments $PQ$, $QR$ and $RP$. We then draw the line segments $XY$, $YZ$ and $ZX$ as shown in the figure. The triangular region $PQR$ has been subdivided into four smaller triangular regions. Take the region $XQY$ and fold it up along the line segment $XY$, so that region $XQY$ sticks up from the region $PXYR$. Do the same with $ZYR$, folding up along $ZY$, and with $PXZ$ along $XZ$. Bring the points $P$, $Q$, $R$ together. Then $PX$ fits exactly together with $XQ$, $QY$ exactly together with $YR$, and $PZ$ with $ZR$. The result is a shell with four faces, each a triangular region, that looks like this,

![Diagram of a shell formed by four triangular regions](image)

with the points $P$, $Q$, $R$ together forming a top tip that is above the region $XYZ$. Such a surface, one that has four faces each a triangular region, is called a tetrahedron shell. If we fill the shell with sand or water so that we have a solid, the solid is called a tetrahedron. Another name for this solid is triangular pyramid.

We remember that a circular region is a circle together with the inside, and that we can think of it as being made of all line segments reaching from the centre of the circle to the circle.

In the same way we can make tetrahedrons. We begin with any triangular region $ABC$ in space and any point $D$ not in the plane of the region.

![Diagram of a tetrahedron formed from a triangular region](image)

Now we start drawing line segments that have one end at $D$ and the other end in the triangular region $ABC$ and continue until we have drawn all such segments.
All such line segments taken together make a tetrahedron.

To make the tetrahedron above, we began with a triangular region. Could we start with other regions? What about a polygonal one?

Can you imagine this filled in with all possible line segments having one endpoint at $G$ and the other endpoint in the polygonal region shown? The resulting solid is called a pyramid. (This is why a tetrahedron can be called a "triangular pyramid"). Can you see what a quadrilateral pyramid would be? A square pyramid?

If we become more general and change from a polygonal region to any region enclosed by a simple closed path in a plane, we have a cone. If the path is a circle, we have a circular cone. Here are examples:

Thus, pyramids are special cases of cones.

For our next two figures, we vary the construction. In each we shall have a plane region enclosed by a simple closed curve, but instead of having also a point in position let us have a line segment in position, with one endpoint in the region, thus:

The construction is made by drawing all line segments that have one endpoint in the region,
are parallel to and congruent to the given line segment, and lie on the same side of the region as the given line segment. For the region and line segment just above, the construction gives this figure:

![Figure showing a solid with a base and a side face]

The result is a solid called a **cylinder**. The original plane region is the **base** of the cylinder and the original line segment the **directrix** of the cylinder. If the base is a polygonal region, the cylinder is called a **prism**:

- Quadrilateral prism
- Triangular prism

If the simple closed path is a circle, the result is a **circular cylinder**.

- Circular cylinder
- Circular cylinder

The surface or shell of a cylinder is called a **cylinder surface**; the surface of a prism is a **prism surface**. A cylinder always has a top face and a bottom face that are congruent and parallel. It always has a "side face" that in the case of a prism can be broken up into three or more parallelogram faces. Does a prism always have straight edges? How many? Does every cylinder have some straight edges?

The final figure we wish to mention starts with a point and a line segment having that point as one of its endpoints:

![Point and line segment diagram]
Now draw all line segments in space that have this point as an endpoint and are congruent to the original line segment. The result is a solid we call a ball. The surface or shell of this figure is made up of the other endpoints of all the line segments drawn: this surface is a sphere.

**EXERCISE 48-11A**

1. A circular cone has one vertex, no straight edges, one round edge, one flat face and one face that isn't flat. Identify these in the figure in the text.
2. For each of the following, count the number of vertices, straight edges, edges that are not straight, flat faces and faces that are not flat: a. triangular prism; b. ball; c. quadrilateral prism; d. circular cylinder.
3. A cube is a quadrilateral prism all of whose faces are square regions.
   a. Is every cube a brick? Why?
   b. Try to give an argument to show that any two faces of a cube must be congruent.
4. Is every brick a quadrilateral prism? Give your reasons.

**48-12 Points, lines and planes in space**

Let us return now to the points, lines and planes in space that we described in Chapter 47. We want to see how lines and planes may intersect in space and how they may be perpendicular or parallel. We recall some facts we learned in Chapter 47.

The first is that there is an important property shared by any straight line and any plane: if either happens to contain two points of some straight line, it contains all of that line. This is a useful fact to remember.

Given any two points A and B in space, there is exactly one line segment \(AB\) that has endpoints A and B, and exactly one straight line \(AB\) going through A and B. The straight line \(AB\) is the combination of two rays, \(\overrightarrow{AB}\) and \(\overrightarrow{BA}\). It is also the combination, or union, of all line segments in space that contain \(AB\); that is, it is the combination of all lines of sight through A and B.

Given any three points A, B and C in space, either all three lie on some line, like this:

\[
\begin{array}{c}
A \\
\hline
B \\
\hline
C
\end{array}
\]

or each is off the straight line that the other two points determine, like this:

\[
\begin{array}{c}
A \\
\hline
B \\
\hline
C
\end{array}
\]

In the first case, there will be many planes that contain the line on which A, B and C lie. You can see this by letting your left forefinger be the line and putting the little finger of your
flat right hand alongside it. This shows one plane through the line. If you move your right hand back and forth but keep the little finger alongside your left forefinger, you will see many planes through the line. In the second case, as we saw in Chapter 47, there is just one plane through these three points; you can see this by spreading three fingers of your left hand and fitting your right hand flat on the tips of these three fingers. The plane of your flat right hand is the plane through the three points of your finger tips.

There are other ways of making planes. Given just one line there are many planes that contain the line: we saw this in the paragraph above (when \(A, B\) and \(C\) were on a line). But suppose we are given a line and a point not on the line, is there a plane that contains both the line and the point? For example, is there a plane that goes through the bottom edge of this page and through the tip of your nose? Put your right hand flat on the page but with your thumb spread out. Keep your hand flattened this way but move it to the position where your little finger is along the edge of this page and your thumb is pointing towards your nose. If you move your hand straight towards your nose you will see it tracing part of a plane in the air. From this we can see that a straight line and a point not on the line are contained together by just one plane. You can think of the plane as consisting of all straight lines through the point that intersect the given line.

Still another way to make a plane is to start with two intersecting straight lines. These determine a plane, as you can see by looking at the floor of your room and choosing two straight edges of the floor that meet at some corner. The floor shows the plane that contains the lines of the two edges. Given two straight lines intersecting at a point \(C\), we can think of the plane they determine as being made of the point \(C\) together with all straight lines that intersect both lines at points different from \(C\).

It is easy to find, or make, a pair of perpendicular lines. There are some on this book; and tracing a square corner shows two perpendicular lines. More than this, we remember how, given a line and a point on it,

we can construct a second line to be perpendicular to the given line at the given point: we fit one edge of a square corner to the line so that the tip is at the point, and then trace the other edge,

and extend it below.)
We can do the same with a line in space,

except that now there are many positions the square corner can be put in with one edge on the line and the corner tip at the point. Three of these are shown in the figure above: there are many more. If we visualize how some of these lines would look after they were drawn we would see something like this:

If we filled in all of the lines of this sort, we would get a flat layer of lines forming what looks like (and actually is) a plane.

The next step is to see what could be meant by a line and a plane being perpendicular. Let us choose a table top for the plane and on it choose any point $P$. If you put a pencil point down at $P$ and move or rotate the other end of the pencil around in the air, the pencil will show many straight lines that intersect the plane at $P$. Is there one of these that we could think of as being "perpendicular to the plane at $P"? If there is one, it should be one that "stands up the straightest". And there is one that does this. To get a picture of it, take a paper straight-edge and fold it to make a square corner. Then fold it half-way. The paper will look like this,
with three edges that are straight. These are marked 1, 2 and 3 in the picture above. Edges 1 and 2 are the two parts of the original straight-edge of the paper; edge 3 is the edge made in the paper when the paper was folded to make the square comer. The three edges meet at the tip of the square comer.

Stand the paper on the table, with edges 1 and 2 resting on the table and with the tip of the square corner resting on $P$. You will see this picture:

Edge 3 will then be standing straight up from the table top at $P$. To see that this is really so, shift the paper around on the table but keep edges 1 and 2 on the table and the square corner tip at $P$. You will see that edge 3 occupies the same position it did before. It will, in fact, occupy this same position no matter how you stand the paper on the table, as long as you have edges 1 and 2 resting on the table and the square comer tip resting at $P$. This unique position shows the straight line that pierces the table at $P$ and "stands straight up". This line is called the line perpendicular to the table-top plane at $P$. There is a shorter name: the normal to the plane at $P$.

This straight line has an important property: it is perpendicular to every straight line that goes through $P$ and lies in the plane. This is easy to see, as follows. Let $QP$ be any line in the plane and going through $P$. Put the paper on the table so that edge 1 fits on $QP$, edge 2 rests somewhere on the table top, and the square comer tip is at $P$. Then edge 3 is on the normal line, and the half of the paper that has 1 and 3 for its edges is a square comer that fits the normal line and the line $QP$! So the two lines are perpendicular: there is a square comer fitting at their intersection.

It is this property that is the precise and real meaning of the words "standing up the straightest".

There is another fact about this particular line. All lines in space that are perpendicular to this line at $P$ form a fan of lines that taken together are the plane of the table top.

We now see that given any plane there are many lines perpendicular to it, but at each point of the plane there is only one such line, namely, the line normal to the plane at that point. We also see that given any straight line in space, there are many planes perpendicular to the line, but through each point of the line there goes only one such plane.

Two planes should be considered perpendicular if they intersect and each "stands up straight from the other". If we tried to see what this means, we would find this as an answer: each plane contains a line that is perpendicular to the other plane. To see an example of two perpendicular planes, we can go back to our table top and the paper resting on it. Open the halves of the paper so that edges 1 and 2 make a right angle. Look at the plane containing edges 1 and 3 and the plane containing the table top. They are perpendicular because the line of edge 3 is in the first plane and perpendicular to the second plane, and the line of edge 2
is in the second plane and perpendicular to the first plane. If we put this paper down on any
plane, each half of the paper will show a plane perpendicular to the one the paper is resting on.
Finally we get to parallelism, and briefly: two lines in space are parallel if there is a
plane that contains both and if the lines are parallel in this plane. It is easy to find on this
book two edges whose straight lines are parallel. A line and a plane are parallel if the plane
contains a line parallel to the given line. And two planes are parallel if there is a line that is
perpendicular to both.

**EXERCISES 48-12A**

1. There are many straight-edges around us: edges of books, edges of table tops, of doors,
pencils, papers, cardboards and so on. Find some of these, and from them choose a pair
whose straight lines do not intersect. Find two more such pairs.

2. Find a pair of straight-edges whose straight lines do intersect. (There are several such
pairs on this book.) Show the plane that this pair of straight-edges determine.

   Do the same with other pairs of edges. Do the two cutting edges of a pair of scissors
make two straight lines that intersect? Will opening and closing the scissors show
the plane these lines make?

3. A door is a "swinging plane": it swings on the line of its hinges and when it swings, it
shows many planes through that line. Find another swinging plane and point out the line
it swings around; put the swinging plane in several positions to show several different
planes through that line.

4. Choose two points on a table top, not too far apart. Fit an edge of a cardboard on these
two points. Swing the cardboard like a door or book cover to show many different planes
that go through these two points.

5. Open and close a stiff book cover to show another swinging plane. There is a line around
which the cover swings (like the hinge line of a door). Choose two points on this line.
Hold a pencil tip in the air and open the book cover so that it rests against the pencil tip.
This will show the plane determined by the three points that are the pencil tip and the
two points on the hinge line. Do this for different positions of the pencil tip.

   Find another swinging plane and repeat this procedure to show the plane deter-
mined by three points.

6. A cardboard is a flat object and so is a table top. If you put an edge of a cardboard
down on a table top, the edge shows the line at which the plane of the cardboard inter-
sects the plane of the table top.

   Find three other pairs of flat objects and show the lines of intersection of their
planes.

7. Make two paper square corners and unfold them halfway so that they will stand up on a
table top. They should be like the unfolded square corner shown in the text. Put them
on your table top and bring them together so that their stand-up edges (edge 3 in the
text illustration) are together. If the papers are stiff enough, the longer stand-up edge
should fit all along the shorter one. Put the two papers in new positions and repeat the
procedure; the stand-up straight edges should again fit together. (If they don’t, either
your table top isn’t flat or your square corners weren’t properly made!)

   What line does the stand-up edge show?

8. Swing a stiff book cover on its hinge. The bottom edge of the cover will trace part of a
plane that is perpendicular to the hinge line of the cover. Is the hinge line perpendicular
to the plane? Where? What does the top edge of the cover trace?

   Find another swinging plane and answer the same questions for yourself.
49-1 Introduction

If we have two straight-edges and fit them edge to edge one on top of the other, we remember that they are congruent if they fit like this:

```
Top

Bottom
```

In this case, we say that the top one is exactly as long as the bottom one; we can also say that the bottom one is exactly as long as the top one.

If, however, our straight-edges fitted like this,

```
Top

Bottom
```

we would say that the top one is shorter than the bottom one, and the bottom one is longer than the top one.

If we had only these two straight-edges, we would do no more. But if we had another straight-edge just like the top one (that is, congruent to the top one), we could place it end to end with the top one, getting a picture like this:

```
Top

Bottom
```

Now we could say that one of the top straight-edges is shorter than the bottom straight-edge but two of the top ones together are longer than the bottom one. If we decide to use the top edge as a way of measuring length, we could say that in the last picture the bottom straight-edge has length that is more than one top edge and is less than two top edges.

If we had many copies of the top edge, we could use them to measure across this page by placing an end of one of them at the edge of the page,
and then placing another end to end with the first, like this,

and then others straight across the page until they get close to the other edge, like this:

Five of these almost reach the edge, and six would go beyond, so we say, "It is more than five top edges across the page and less than six top edges", or "The width of the page is more than five and less than six top edges".

It is easy to lay copies of the top edge across the page of this book when the book is flat. But that is a hard way to measure the height of a man who is standing. So we make other things, like knotted string or a ruler. A knotted string has knots evenly spaced, perhaps one inch apart, a foot apart, or some other distance, depending on the unit of length we are using:

Here the part of the string between one knot and the next is congruent to any other piece between two next-to-each-other knots: piece $AB$ is congruent to piece $CD$ and also to piece $DE$.

How many units long is the string in the picture?

A foot ruler looks like this,

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
</table>

when divided into inches. It is like having 12 little blocks of wood, all congruent to each other, all 1 inch long, and all glued end to end to make the ruler. The little black marks above the numbers show where the little blocks would have been stuck together. A ruler is much easier to use than little blocks; and we don't have to count, just read numbers.

What we have been saying shows how we make a way of measuring lengths. We choose a particular straight-edge, call it our "unit", give a special name to it (some names used are inch, foot, metre, yard, mile, furlong), make many copies of the unit, and measure any length or distance by putting these copies end to end until they stretch from one end of what we are measuring to the other end. We may connect the copies end to end in the form of a ruler, knotted string or tape measure, in order to measure more easily and accurately.

We follow this same idea in every measuring, whether we are measuring time, length, area or the size of angles. We choose for unit a particular thing of the kind we are measuring (a certain interval of time, a certain straight-edge, a certain square region, a certain angle)
and make many copies of this unit. Then, to measure something of the same kind (an interval of time, the edge of a board, the area of a floor, the size of an angle in a triangle) we put together copies of the unit end to end or edge to edge, until they cover as closely as possible the thing we are measuring. We then count the number of copies we have used and call this number the measure, in our unit, of what we are measuring. Sometimes the copies are already put together, as in a ruler, knotted string, protractor (see the next section) or clock.

This way of making a system of measuring is the first principle of measuring. The second principle is that measuring is additive. This really comes from the first principle, and means that if we tie together two strings, A and B,

\[ A \quad \text{Knot} \quad B \]

and the length of A from one end to the knot is measured and found to be 5 feet, and the length of B is found to be 8 feet, then the length of the whole string when we measure it will be 13 feet. If five foot rulers end to end stretch exactly along A and eight foot rulers do the same along B, then these foot rulers together are all fitted end to end and stretch exactly along the whole length of the combined string. Since there are thirteen of them, 13 is the length of the string. We can state the second principle thus: if we have a system of measuring a certain kind of thing and two of these things are measured separately and then put together end to end or edge to edge to make a third thing of the same kind, then the measure of the third thing will turn out to be the sum of the measures of the first two things.

The third principle is that we make our units from material that won’t change with time or movement; a wooden ruler changes very little in length from day to day and from place to place. As a result, our measurements of the length of an object will give the same answers, or almost the same, tomorrow that they give today, and the same answers when we measure the object in Lagos that they would give if we carried the object to Nairobi and measured it there, provided the object being measured doesn’t change with movement or time.

The fourth principle is that any two objects that are congruent will have the same measure when measured with the same unit.

In this chapter we shall develop informally, and just as described above, some standard ways of measuring line segments, angles, certain plane regions (rectangular, triangular, and so on) and certain solids. Although you already know most and perhaps all of what will be said here, there might be some new facts or viewpoints for you.

49.2 Measuring angles in degrees

Angles can be measured using different units, just as distances can be measured using feet, miles, metres or furlongs (among others). Probably the most common way to measure angles is to use degrees. This form of measurement is based on the fact that “a straight angle measures $180^\circ$”. The symbol “$^\circ$” at the upper right-hand corner of the number denotes degrees. Following our measurement principle, a right angle must measure $90^\circ$; and if we divide a right angle into 90 smaller angles, all congruent to each other, then they should all have the same measure (since they are congruent) and the sum of their measures in degrees should be 90. So each such angle would measure $1^\circ$. Thus, if you want to see an angle measuring $1^\circ$, divide some right angles into 90 angles all congruent to each other! And if you want to measure any angle in degrees, get many of these angles that measure $1^\circ$ and put them together side by side until they fill your angle, then count the number you used; this is the measure, in degrees, of your angle.
But there is a much more convenient way of measuring angles, and that is to use a protractor. If you want to measure in degrees, you should have a degree protractor. In fact, you should have a degree protractor for each of the children you teach, as well as a large one for the blackboard. If they are not supplied, you can make them, as follows. Take a piece of cardboard with a straight-edge base. Choose a point on the base near its centre as the centre of a circle. Choose a radius which is about half the length of the base of the piece of cardboard, and draw an arc from one side of the base to the other. This will be a half-circle. Cut the cardboard along the half-circle. Your cutout should look like this:

![Cutout diagram]

At this point you can begin to mark angle measures on the protractor. Mark $0^\circ$ at the right-hand side and $180^\circ$ at the left. On a sheet of paper, construct a right angle as you learned to do above. Mark the correct point on the protractor for $90^\circ$, using this right angle. Then bisect the right angle on each side, to get $45^\circ$ and $135^\circ$. Now you can approximately find the points for $15^\circ$, $30^\circ$, $60^\circ$, $75^\circ$, $105^\circ$, $120^\circ$, $150^\circ$, and $165^\circ$. Your protractor should look like this:

![Protractor diagram]

Let us recall how we use a ruler to measure the length of a line segment. We put the ruler along the line segment with the "zero" end of the ruler fitting at one end of the segment. Then, where the other end of the segment touches the ruler, we read the number on the ruler as the length of the segment. (Do you see why this does give the measure of the length of the segment?) We follow the same idea when we measure angles with a protractor. We think of part of the protractor as being a base edge, or zero edge,
of the protractor. We fit this base or zero edge to one side of any angle we are measuring and see where the other side of the angle crosses the edge of the protractor:

Here side $\overrightarrow{BC}$ of angle $\overrightarrow{ABC}$ is fitted to the zero edge; the other side, $\overrightarrow{BA}$, of the angle crosses the protractor edge at about the 40 mark, so we say the angle measures 40°.

**EXERCISE 49-2A**

1. Construct an angle on a sheet of paper. Using the method described above, copy that angle on the same sheet of paper. Now put another piece of paper over the first angle, and cut out a copy of that angle. Use it to test the second angle to see if it is, in fact, a copy of the first.

2. Construct an angle on a sheet of paper. Using the method for bisecting angles, divide your angle into two congruent angles. Now bisect each of these two angles, so that you have cut the original angle into four congruent angles. Make a protractor as described above, and measure each of the angles you made in the previous exercise as carefully as you can. Is the measure of each of the four angles half of the measure of the two angles and one-fourth of the measure of the original angles?

3. The method we have described for making a protractor involves guessing the positions for 15°, 30°, 60°, 75°, 105°, 120°, 150° and 165°. With straight-edge and compass can you construct a 60° angle, using the method you know of constructing an equilateral triangle? Make a protractor using the construction you have found for all of the angles that you previously guessed.

4. Make a paper square corner and unfold it once. When you look at it, you can see a straight angle. You can also see some right angles; how many? If we used a right angle as a unit to measure angles, how many right angles would a straight angle measure? How many right angles fit together to make a straight angle?

5. This problem teaches a very important fact.

From a piece of paper cut any triangle you wish. Let us suppose you have named the vertices $A$, $B$ and $C$ and your triangle looks something like this:
Fold $A$ straight down onto side $BC$, like this:

\[
\begin{array}{c}
\text{P} \\
\text{B} \\
\text{A} \\
\text{C} \\
\text{Q}
\end{array}
\]

so that the line $PQ$ of the fold is as parallel to $BC$ as you can make it. Now fold $B$ over to touch $A$, and $C$ over to touch $A$. Your paper will look like this,

\[
\begin{array}{c}
\text{P} \\
\text{R} \\
\text{A, B, C} \\
\text{Q} \\
\text{S}
\end{array}
\]

Underneath is a rectangle, and on top a rectangle made of three triangles, $PRB$, $PAQ$, $QSC$. Sides $PA$ and $PB$ should fit along each other, and $QA$ and $QC$ should do likewise. Unfold your paper to show the original triangle; notice where the three angles of the triangle are. Now fold the paper back to make the two rectangles again. The three angles of the original triangle now fit together. What sort of angle do they make fitted together?

This shows that you can join the angles of a triangle to make an angle of what kind? Since a straight angle measures $180^\circ$, we can say that the sum of the measures in degrees of the angles of any triangle is $180$.

49-3. Measuring length

It is clearly possible to measure segments. You remember the number line, which looks as follows:

\[
\begin{array}{c}
-3 \\
-2 \\
-1 \\
0 \\
1 \\
2 \\
3 \\
4 \\
5
\end{array}
\]

One of the first uses of the number line was to compare two segments, matching them against the number line. Each segment is placed on the line, with its left-hand end against zero. The number at the right-hand end is given to the segment, and called its length. This process of getting this number is called "measuring the length" of the segment. If one of two segments is larger than the other, then its length is the larger number, as in this picture where $AC$ is longer than $AB$ since it contains $AB$:

\[
\begin{array}{c}
A \\
B \\
C
\end{array}
\]

Clearly, the measure of the length of a segment depends on what the unit segment is on.
the number line. It can be much smaller than that above, or much larger, depending on what you want. You know that you can measure things in miles or in inches or in any other unit you may choose. It is all a question of what you wish to do. If you measure in miles, your number line has very big unit pieces. If in inches, the pieces are very small. The important thing is to choose your unit of length, and then go ahead and measure.

The measures of two congruent line segments must be the same, if you use the same unit to measure their lengths. And if two line segments have the same length when measured on a certain number line, then they must be congruent. They fit the same segment on the number line, so they must fit each other. Thus, there are two ways in which you can tell whether two line segments are congruent. The first is to make the one segment fit on top of the other, so that at each end the two endpoints coincide. The second is to measure the lengths of the two segments with the same measuring stick or ruler and find that their lengths are the same.

One way of measuring the length of a given line segment is to take line segments of unit length and place them end to end along the given segment to make a string of segments that most closely fits the given segment. By counting the number of unit segments in the strings, we get an approximate measure of the length of the given segment.

**EXERCISE 49-3A**

1. Take a cardboard straight-edge and make it into a number line by choosing some point on the edge as the zero point, another as unit point (1), and marking unit lengths in each direction, numbering them as you go. Take another cardboard straight-edge, use the left-hand endpoint as the zero point, and mark to the right, using the positive numbers 1, 2, 3 and so on. Which straight-edge would you choose as a ruler? Do you need the negative numbers in measuring?

2. Using the second ruler you constructed in Question 1, measure the lengths of straight edges of a number of different objects in the room. Use it also to draw line segments congruent to some of these straight edges.

3. Make two rulers with different unit lengths. Draw some line segments, and for each segment compare the two lengths given by the two rulers. Is the ratio of the two lengths the same for all the segments?

**49-4 Measuring area**

In discussing area we begin with the figure we called the square, a quadrilateral whose four sides are all congruent to each other and whose four angles are all right angles. Each square encloses a plane region which we called a square region. Square regions are important because we will use them as basic regions in building other regions and measuring area. You remember that to measure length we constructed a ruler, using unit segments on a line, and you remember that the number line was the model for such a ruler. Now we will see how to measure plane regions by using certain square regions that we shall call "square units". For example, we sometimes measure length in inches. In this case we use "inch-square units" to measure area. An inch-square unit is a square region as shown in this picture: each edge of the region is one inch in length.
You can use these square units to measure plane regions bounded by squares, rectangles, triangles and parallelograms. To give a very simple case, look at the rectangular region below. You can see that three of our inch-square units will exactly cover it. We therefore say that its area is *three inch-square units* or, more briefly, *3 square inches*.

If you compare this rectangular region with the inch-square unit above, you will find that its horizontal edge is three times as long as that of the square unit, while its vertical edge has the same length as that of the vertical edge of the square unit. Putting these facts together, we can say that the number of inch-square units in the rectangular region is the same as the number we get by multiplying together the number of inch units in the horizontal edge of the region and the number of inch units in the vertical edge of the region.

You can think of the following, more complicated case in the same way. Take the following region enclosed by the outside rectangle:
As the dashed lines show, you can fit into this rectangular region exactly 12 inch-square units. Thus, you can say that the area of the rectangle is 12 square inches. On the other hand, the base, or horizontal edge, of the rectangle has length 3 inches and the altitude, or vertical edge, has length 4 inches. And \(3 \times 4 = 12\).

These two examples should lead you to make a guess about the area of a rectangle. But before putting that guess on paper, look at one more example, more complicated than the last one. Here is a rectangle which is \(3 \frac{1}{2}\) inches at the base and \(1 \frac{1}{2}\) inches at the altitude.

What can you say about its area? From the drawing you see that you can put in 3 inch-square units, 4 halves of inch-square units, and 1 quarter of an inch-square unit. Thus, its area must be 5 \(\frac{1}{4}\) inch-square units, or \(5 \frac{1}{4}\) square inches, which is the same as \(\frac{21}{4}\) square inches. But \(\frac{21}{4} = 3 \frac{1}{2} \times 1 \frac{1}{2}\).

Look at the results you have obtained. For the rectangle 1 inch by 3 inches, the area was 3 square inches. For the rectangle 3 inches by 4 inches, the area was 12 square inches. For the rectangle \(\frac{7}{2}\) by \(\frac{3}{2}\) inches, the area was \(\frac{21}{4}\) square inches. From this you can guess the following rule for the area of a rectangular region: the area of the region (measured in inch-square units) equals length times height (measured in inches). This rule would still be true if we used feet, or any other unit of length, instead of inches. These are the facts that are meant when we say, very briefly, that for a rectangular region, area equals length times height.

You should lead your students to discover this fact, rather than teaching them right away to memorize it. Eventually they must learn it by heart, but at the beginning it is better to find it out by discovery.

A very similar problem concerns the region inside a parallelogram. You can proceed in the same way to find the area, if you notice the following trick. Look at this parallelogram:

You notice that the right-angled triangle at the right-hand end is congruent to the triangle in dotted lines at the left-hand end. If you remove the triangle at the right from the parallelogram
and place it over the triangle at the left, you change the parallelogram region to a rectangular region that has the same area. And the two regions have the same length and the same height. But you know that the area of a rectangle is length times height. Thus, all you need to do in this case is to find the length and the height of the parallelogram. If you measure the parallelogram above, you will see that its height is 1 inch and its length is 4 inches, so that its area in square inches is $1 \times 4$, or 4.

Finally, you can consider a triangular region in the same way. Look at the triangle $ABC$ below. You can see that you can construct a second triangle $ABD$ congruent to the first and connected to it on the side $AB$. Since triangle $ABC$ and triangle $ABD$ are congruent, clearly their areas are equal. But the complete figure $ACBD$ is a parallelogram and it has the same length and the same height as triangle $ACB$. You know that for a parallelogram its area is its length times its height. Thus, the area of each triangle must be half that of the parallelogram; that is, it must be half the product of the length of the base of the triangle by the height of the triangle.

\[ \frac{1}{2} \times \left( \frac{5}{2} \times \frac{3}{2} \right) = \frac{15}{8} = \frac{7}{8}, \]

so that the area of the triangle is $\frac{7}{8}$ square inches.

**EXERCISE 49-4A**

1. Construct a rectangle and find its area by marking off the number of inch-square units within the region enclosed by the sides. Then measure the length of each side and find the area by taking the product of length and height. Are the results the same?

2. Construct inch-square units of cardboard by drawing squares one inch on each side and cutting them out. Measure the area of the cover of this book by placing the inch-square units on it until there is no space left. If you need to, you can cut up your inch-square units to fill the edge regions with smaller pieces. Keep count of the number you use. Then measure the lengths of the sides of the book and compute the area as described above. Compare the two answers.

3. Use the inch-square units you made for Question 2 to find the area of some irregular region. Fit as many as you can into the region and cut out pieces of others to fit the small edge regions. Keep count of the number you use. Would you have obtained a better
result if you had used smaller square units, for example, pieces $\frac{1}{2}$ or even $\frac{1}{10}$ of an inch on a side? Why?

4. Draw a parallelogram. Then construct a rectangle so that regions inside these two quadrilaterals have the same area. Draw a triangle and then construct a parallelogram whose inside region has twice the area of the region inside the triangle.

5. Construct a series of triangles and parallelograms and find the area of the region inside each.

The last region for which we wish to find a way to measure area is the region bounded by a circle. This region gives us a hard problem, since it turns out that there is no simple way to express the area of a circular region. We have to make the best approximation for it that we can. Let us attempt to find that approximation by using inch-square units. The circle drawn below has a radius of 2 inches and, thus, a diameter of 4 inches.

![Circle with inch-square units]

We have drawn a set of inch-square units, and have attempted to find the area of the circular region in this way. If you count up the total number of square units you find 9 wholes, 4 halves and 8 quarters, making a total area of 13. This figure is probably too high, since the parts of the quarters not covering any of the circular region seem to have more area than the uncovered parts of the circular region. However, it does not appear to be a bad approximation to the actual area.

We can try to find a better approximation to the area by using smaller square units. Let us take a congruent circle and use $\frac{1}{10}$-of-an-inch-square units. In this way we will get a very small grid, and we will find some difficulty, therefore, in counting the total. This approximation is shown in the following picture:
If you add up the total number of square units in the picture, not counting some of the edge pieces because they are too small, and counting other as full pieces to make up, you find a total of 1244 unit squares. Obviously 100 such square units make up an inch-square unit, and so this gives the area of the circle as 12.44 square inches, instead of the previous estimate of 13 square inches. The second estimate is very likely to be more accurate.

You will notice another circle, with the same centre as the circle we have been discussing. This second circle has a radius of one inch and a diameter of two inches. If you add up the total number of square units in this smaller circle, you find a total of 312 unit squares. As above, this represents an area of 3.12 square inches, which is approximately \( \frac{13}{4} \) of the area of the previous circle. You can see from these results that the area of the circular region seems to be multiplied by four when the radius doubles. From this you might guess that the area depends on the square of the radius; that is, the area might be the square of the radius multiplied by some constant. It can be shown that there is actually such a constant. It is so useful and famous in mathematics and science that it has a special name, the Greek letter "\( \pi \)", called "pi" in English. From the first calculation, when you used inch-square units, the number \( \pi \) works out to be approximately:

\[
\frac{\text{Area}}{\text{Square of radius}} = \frac{13}{4} = 3.25
\]

From the more accurate case, we get

\[12.44 \div 4 = 3.11\]

as a number that is closer to \( \pi \). \( \pi \) is an irrational number whose value is approximately 3.1416,
or more roughly \( \frac{22}{7} \). Since it is not a rational number, there can be no completely accurate way of writing the number \( \pi \) in decimal or fractional form, but we can use either of these two approximations in most cases.

It turns out that this same number \( \pi \) appears when you wish to find the length of the circle itself, and the resulting formula tells you to multiply twice the radius by \( \pi \) in order to find that length, which is also called the circumference.

**EXERCISE 49-4B**

1. Using a compass, draw as carefully as you can on a large sheet of paper a circle whose radius is 4 inches. Using a ruler marked in tenths of an inch, construct square units \( \frac{1}{10} \) of an inch on a side, in the same way as in the previous drawing. Count the number of square units which cover the circular region, and from that number try to find a more accurate approximation to \( \pi \).

2. Lay a string or a thread as carefully as you can along the circle which bounds the region considered in Question 1. Measure the length of the string to the nearest tenth of an inch. Remember from above how \( \pi \) and the circumference of a circle are related. Use this relation to find a value for \( \pi \). See if the value you have found is almost the same as that found from the area.

**49-5 Measuring volumes**

Just as a line segment could be used as a unit for measuring length and a square region could be used as a unit for measuring area, so a cube can be used as a unit for measuring volume. In our case we shall choose an inch-cube unit; that is, for our unit we choose a cube with all edges one inch long. We shall say that the volume of this cube is one cubic inch. The drawing below shows a cube with each edge 2 inches long, and as you can see it has a volume of 8 cubic inches.
Other cube units can be used to measure the volume of solids, just as we used squares of various sizes to measure an area enclosed by given lines.

Here is a picture showing the six faces of a box. All six faces are rectangular regions. If we filled up the box, we would have a brick. What is the volume of the brick?

Two of the faces of the box are 1 inch by 3 inches. Two of them are 1 inch by 2 inches. And two of them are 2 inches by 3 inches. If you attempt to fit inch-cube units into the box, as in the following picture, you find that 6 such cubes can fill the box, so that 6 of our inch-cube units make the brick, and the volume of the brick is 6 cubic inches.

In general, if you study a number of such figures carefully, you will find that the volume of a brick is given by the product of the lengths of any three edges which meet at one vertex.

"Brick" is a short name for a rectangular prism. Another kind of prism is a triangular prism, which has for its faces five plane regions, two of them triangular and three of them rectangular.
(You can construct a triangular prism surface by cutting out a copy of the next diagram, folding along the dotted lines and pasting or taping the tabs on the adjacent faces to close the figure.)

It is also possible for a prism to have some slanting sides and edges. However, we will not deal with such here.

The volume of a triangular prism is the area of the triangular base times the height of the prism.

Another flat-faced figure we shall recall is the tetrahedron. You will remember that it looks like this,

and can be constructed if you copy the following drawing on stiff paper, cut out on the solid lines, fold on the dotted lines and past or tape the tabs on adjacent surfaces.
Another solid is the square pyramid, pictured below.

A tetrahedron and a square pyramid are both pyramids. The volume of any pyramid is \( \frac{1}{3} \) the product of its height by the area of its base.

**Exercise 49-5A**

1. Construct out of stiff paper the surfaces of a cube, a rectangular prism, a triangular prism, a pyramid with a triangular base and a pyramid with a square base. The construction diagrams for the triangular prism and the triangular pyramid are given in the text.
Using these models, you should be able to make the diagrams for the cube, the rectangular prism and the square pyramid and construct the surfaces from your diagrams.

2. On a piece of paper draw a picture of a box 2 inches by 3 inches by 4 inches. Show the inch-cube units which fill the box. In this way, compute the volume of the brick that fills the box.

3. Using a trick similar to that used for the area of a parallelogram, attempt to find the volume of a prism two of whose faces are parallelogram regions and four of whose faces are rectangular regions. Construct the prism surface to help you do the problem.

4. In the same way as in the previous problem, attempt to find the volume of a prism, two of whose faces are triangles and four of whose faces are rectangles. Again, you should construct the prism surface or at least draw a picture of it.

The final figures which we must consider are those with curved faces. There are three of importance. The first we have already looked at, namely, the sphere. The only additional thing we need to say about the sphere is that it is possible to find its volume using the same constant, $\pi$. It turns out that this volume is given by $\frac{4}{3}$ times $\pi$ times the cube of the radius.

We must also think about the figure which we see so commonly as a container for tinned food. This is the cylinder. An example is the circular cylinder which, you will remember, looks like this.

It turns out that the volume of a cylinder is the area of its base multiplied by its height. In order to find the volume of a circular cylinder, multiply together the height of the cylinder, the square of the radius of the base and $\pi$. You should be able to work out a formula for the area of the surface of a circular cylinder, including the top and bottom, for yourself.

The final figure to be considered is the cone, which has the following appearance.

You are familiar with the cone from many objects in the natural world. You can construct a cone by taking a circular piece of paper, cutting along one radius from the edge to the centre and folding the two halves of that radius across each other. As you continue folding them across, the cone will roll into shape. Tape or paste the top side to the bottom when you reach the desired shape.
EXERCISE 49-5B

1. Find the volumes of spheres whose radii are, respectively, 3 inches, 7.5 inches, $\frac{13}{5}$ inches, 16 feet, 4000 miles (the last number is approximately the radius of the earth).

2. Construct a circular cylinder with height 6 inches and radius of the base 2 inches. Find the volume and the surface area of this cylinder.

3. Construct a cone as described in the text. Name instances of cones in the natural world and bring some to class. How can you use these to teach about cones?

4. Track down in some book the formula that gives the volume of a cone in terms of the height of the cone and the area of the base of the cone.

5. The area of an ellipse is given by the product $\pi ab$ where $a$ is half the length of the major axis of the ellipse and $b$ is half the length of the minor axis of the ellipse. A manufacturer of electric razors has designed a razor container that is an elliptical cylinder having height 5 inches and with 6 inches and 4 inches as the lengths of the two axes of the ellipse around the base. If the container were solid, what would be its volume?

6. A teacher has a box and 150 children’s wooden blocks to put in the box. Each block is a cube that measures 2 inches on each edge. The inside of the box measures 18 inches by 10 inches by 6 inches. Can he fit all the blocks into the box?

7. Abu and Ben argue as to who is better at guessing volumes. A school teacher says he will test them, and shows them a wooden triangular prism. Abu guesses the volume to be 100 cubic inches, and Ben guesses it to be 150 cubic inches. The teacher measures the prism with a ruler and obtains these measurements,

![Diagram of a triangular prism with dimensions 10 inches, 5 inches, and 4 inches.]

and says Ben’s guess is better. Was the teacher correct?

Abu and Ben think the test was not fair. They want to use a box instead of a triangular prism. So the teacher chooses a book called Basic Concepts of Mathematics and has them guess its volume in cubic inches. Ben looks at it and says, “60 cubic inches.” Abu thinks a moment, measures the book by eye and by his handspan, thinks another moment, and says “90 cubic inches.” The teacher then measures the book with a ruler and finds the dimensions to be $\frac{7}{8}$ of an inch, $8\frac{1}{2}$ inches and 11 inches. Did Ben or Abu make the better estimate? How do you think Abu arrived at his figure? Who is the better guesser of volumes?
Chapter 50

IDEALIZED GEOMETRIC FIGURES

50-1 Introduction

Until now we have talked about points as dots made with chalk or a pencil, or as holes made by pinpoints. We have thought of line segments as straight marks made on the blackboard or paper or as stretched strings, and of plane regions as sheets of paper on a flat desk and so on. We have thought of them as physical objects. But in geometry "points", "line segments" and "planes" are not physical objects, but rather are ideas that come from physical objects. We come to think of lines and planes as sets of points, and we think of points as having no size and therefore no shape. How do we do this, and why? It will be the purpose of this chapter to discuss our ideas of point, line and plane.

50-2 Thinking about points

Let us make a chalk mark $C$, an actual dot on the blackboard. We often call this a point. But is this really what we want to mean by a point? Suppose that we make a small dot $P$ with a pencil inside the chalk mark. Would we want to call $P$ a point? It comes closer to what we want than $C$ does. Why? For one thing, we could have made the pencil mark somewhere else inside the chalk mark, say at $P'$. We would then want to say that $P'$ was a different point from $P$. Then if $C$ is a point, it has other points inside it. This does not seem to be what we want.

Would we want to call $P$ a point? No, even a dot made with a very sharp pencil is not really a point, because with a very sharp needle you could locate "points" inside it. A prick made with a needle in a piece of paper is probably as close as we can get physically to the idea which we have in our minds when we think of a point. We use the word to express an absolutely precise location.
50-3 Thinking about lines

Let us begin again with two chalk marks (dots), $A$ and $B$, and draw the chalk along a straight-edge to join them. We often call $AB$ a line segment. But is this what we really mean by a line segment?

Let $P$ be a pencil point inside $A$ and $Q$ a pencil point inside $B$.

![Diagram of line segment AB with points P and Q]

Draw the pencil along a straight-edge to connect $P$ to $Q$. Then we have a straight line segment $PQ$ entirely inside the line segment $AB$. This does not seem right either, but $PQ$ comes closer to what we want to mean by a line segment. Can we come closer still? Yes. Let $P$ and $Q$ now be dots made with a sharp pencil on a piece of paper. Connect them with a pencil mark, using a straight-edge as a guide. Now use a needle to prick a hole $P'$ inside $P$ and a hole $Q'$ inside $Q$. Take a razor blade and with a straight-edge as a guide cut the paper from $P'$ to $Q'$. This will separate the previous line segment $PQ$ into two pieces lengthways. Does this fit your idea of what a line segment should be?

As a matter of fact, if we now push the two cut edges together as tightly as possible, the cut itself is about as good a realization of a line segment as we can hope for.

50-4 Thinking about planes

How can we think of a plane region? We can imagine a piece of paper laid flat on the top of a desk or table. Is this really a plane region? No, because it has thickness. We can imagine a still thinner piece of paper, say a piece of tissue paper. This we feel is more like what we have in our minds when we think of a plane region.

We can do still better. Take a pan of water. The surface of the water is close to a plane region. How thick is it? We can think of the surface as separating the water below it from the air above it. Does the surface have water in it? If so, we could make it thinner. In the same way, if the surface had air in it we could also make it thinner. In imagination, we think of the plane region as without any thickness.

You can help your imagination by taking a tiny drop of oil, such as is used to oil a sewing machine. if this drop is placed on the water, it will spread out over the top of the water and cover the whole of it. It must be very thin.

With a matchstick remove a little of the oil film and put it on the surface of a fresh pan of water. This oil will spread out to cover the water in the second pan. It must form a film which is much thinner than before. If you repeat this experiment in imagination, you will get a picture which comes close to the idea of a plane region that we use in geometry.
Chapter 51
GEOMETRIC FIGURES
AS SETS OF POINTS

51-1 Revision of sets

Do you remember when we talked about sets in Chapter 1? Sets have an important place in geometry, as you will soon see. Let us begin by revising the idea of set. A set is any collection of objects. They may be mathematical objects, such as numbers, or they may be non-mathematical objects, such as birds or people. The objects in a set are called the members of the set. For example,

\{1, 3, 5, 7, 9\}

is the set whose members are 1, 3, 5, 7 and 9; and

\{football, star, triangle, man\}

is the set whose members are a football, a star, a triangle and a man.

Here is a picture of three points. Remember that this is merely a picture of the three points. You have to imagine the ideal points without size.

Since a set can be a collection of any objects, we can imagine the set of those three ideal points. Our set will have these three points as members and it will have no other members. Certainly, this is a set of mathematical objects.

How shall we describe our set? We can say that it is "the set of the three points pictured above". Just to make sure that we know exactly which points we are talking about, it might be a good idea to give the points names, such as A, B and C.
Then we can list the names of the points and say that we are thinking of the set of points \( \{A, B, C\} \). You know that we are thinking of the set of three points which we named \( A, B \) and \( C \). Of course, we are NOT thinking of the set whose members are the letters \( A, B \) and \( C \).

**EXERCISE 51-1A**

1. Give a name to each point in the following pictures of sets of points and list the sets.

Let us look at the number line.

![Number line](image)

We can consider the set of points for the integers \(-3\) to \(4\). How many points are there in this set of points? We can describe our set as "the set of points for the integers \(-3\) to \(4\)". Can you think of a good way of listing the set? There are many different ways of naming the points, but one of the easiest ways is to name the points \(-3, -2, -1, 0, 1, 2, 3, 4\), because the points are already labelled with those numerals. Then you can write our set as the set of points \(\{-3, -2, -1, 0, 1, 2, 3, 4\}\). And since you know you are talking about a set of points, you will not confuse the set with the set of numbers \(\{-3, -2, -1, 0, 1, 2, 3, 4\}\).

All of the examples we have had of sets of points have been finite sets. We can also have infinite sets of points. The set of points for all counting numbers on a number line is an infinite set of points, and we can write it the same way we listed the set of counting numbers:

\[1, 2, 3, 4, 5, \ldots\]

The set of points for all integers on a number line can be written

\[1, -2, -1, 0, 1, 2, \ldots\]

There are dots at both ends because the list has no beginning and no end. As you know, the order in which you list the members of a set is unimportant. So another way of writing down the set of points for all integers is

\[\{0, 1, -1, 2, -2, 3, -3, \ldots\}\]
EXERCISE 51-1B

1. Think of some more examples of infinite sets of points. Which of the sets you thought of can you list? Can you find more than one way of listing any of your sets?

51-2 Straight lines as sets of points

Can you think of the set of points for all of the fractions between 0 and 1 on a number line? You know that this set must be an infinite set because the set of fractions between 0 and 1 is an infinite set. You cannot completely list all of the fractions, since you cannot even completely write down all the fractions with 1 as the numerator. Here is the set of fractions with 1 as the numerator:

\[
\left\{ \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots \right\}
\]

It is necessary to use dots at the end to indicate that the list goes on endlessly.

It may be difficult for you to imagine the set of points for all fractions between 0 and 1, because there are infinitely many points and because some of them are so close together. Before discussing this set of points, let us study the fractions themselves. It is surprising, if you have never seen it done, that you can arrange all fractions between 0 and 1 in a list with dots at the right-hand end. The way to do this is first to write down all fractions with denominator 2, then all fractions with denominator 3, then all fractions with denominator 4 and so on:

\[
\left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, \frac{8}{8} \right\}
\]

As you write them down, cross off every fraction which you have already written in a different way. For example, you should cross out \(\frac{2}{4}\) because it was written down already as \(\frac{1}{2}\). You should cross out \(\frac{3}{6}\) for the same reason. Why should you cross out \(\frac{2}{6}\) and \(\frac{4}{6}\)? The list you are left with will begin like this:

\[
\left\{ \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{8}, \frac{2}{8}, \frac{3}{8}, \frac{4}{8}, \frac{5}{8}, \frac{6}{8}, \frac{7}{8}, \frac{8}{8} \right\}
\]

Every fraction between 0 and 1 would be in the list (if you could write it out endlessly), and every fraction would appear only once because of all the cross-outs you made.

Now we can return to the set of points for these fractions. To imagine this set, first locate the unit piece between 0 and 1 on the number line.
Remember that we are thinking of the number line as an ideal line without any width. Then imagine that on to the line we put one of our ideal points at the location for the fraction \( \frac{1}{2} \), then a point for \( \frac{1}{3} \), then one for \( \frac{2}{3} \). By now you have a set of three points, and your number line can be pictured like this.

![Number line with points at \( \frac{1}{3}, \frac{1}{2}, \frac{2}{3} \)]

Keep putting on points one at a time in the same order in which you listed all the fractions. When you have arrived at \( \frac{7}{8} \), you have a set of twenty-one points and your number line will look like this.

![Number line with points from \( \frac{1}{8} \) to \( \frac{7}{8} \)]

Since the fractions get very close together, the points on the number line will get quite close together. If you thought of points as dots you draw with a pencil, the number line would become very crowded and soon the points would begin to overlap. But we do not have this trouble now because we are imagining the points as being without size, and they will not overlap at all.

**Exercise 51-2A**

1. If you make the unit piece of your number line 1 inch long, what would be the distance between the points for the numbers \( \frac{1}{2} \) and \( \frac{3}{4} \)?

2. On the same unit piece as in Question 1, what would be the distance between the points for the numbers \( \frac{1}{8} \) and \( \frac{7}{11} \)? for the numbers \( \frac{1}{7} \) and \( \frac{4}{9} \)?

3. On the same unit piece as in Question 1, what would be the distance between the points for the numbers \( \frac{1}{100} \) and \( \frac{1}{101} \)? Can you imagine a distance that small between two points?

Now suppose you could keep putting on points endlessly until there is a point for every fraction between 0 and 1. Next you could cover the piece from -1 to 0 with points for fractions. Then cover the pieces from 1 to 2 and from -2 to -1, then from 2 to 3 and from -3 to -2 and so on. If you go on you will cover every fraction on the entire number line with points.
Of course, you may not have put on a point for every number on the number line. Some such numbers were discussed in Chapter 37 on real numbers. For example, there is a number for the length of the diagonal of a square with side of length 1, and this number is not equal to any fraction. So we must put on a point for that number, too. Then after you have put on a point for every other number which is not equal to any fraction, you will have covered the entire number line with points.

It really doesn't matter whether we think of the number line as the original ideal line or whether we agree to think of it instead as simply the set of points which we just constructed to cover the line. You may usually think of it in either way that you prefer. But it is often convenient in geometry to use the idea that the number line is the set of points. Therefore, for the remainder of this chapter let us agree to regard the number line as the infinite set of points which lie in a straight line which we just constructed.

We have seen how number lines can be thought of as sets of points. This is not our final goal. We would like to regard all straight lines as sets of points. Do you see how we can do this? If we take any straight line, we can freely choose a zero point and a unit piece on it.

Then we regard our straight line as a number line and, thus, as a set of points, as we have just done. Therefore, we will agree to regard any straight line as a certain infinite set of points which lie in a straight line.

51-3 Operations on point sets

Do you recall that we talked about union and intersection of sets? Remember that the union of sets $A$ and $B$ is the set of all the things which are members of $A$ or are members of $B$. We write the union as $A \cup B$. The intersection of sets $C$ and $D$ is the set of all things that are members of $C$ and members of $D$. We write the intersection as $C \cap D$. If the sets that we are talking about are sets of points, we can consider their unions and intersections.

For example, let $P$ be the set of points $\{A, C, E\}$, let $Q$ be the set of points $\{B, D, F\}$ and let $R$ be the set of points $\{A, D, G\}$.

Then $P \cup Q = \{A, B, C, D, E, F\}$. We have drawn a shape around $P \cup Q$. Also $P \cap Q = \{\}$, Do you remember that $\{}$ is the way we write the empty set, which has no members?
**EXERCISE 51-3A**

1. List the sets of points $P \cup R$, $P \cap R$, $Q \cup R$, $Q \cap R$.
2. Draw shapes around each of the sets that you have found.

Did you find that $P \cup R = \{A, C, D, E, G\}$ and $P \cap R = \{A\}$? Point $A$ is the only point that is a member of both $P$ and $R$. Therefore $P \cap R = \{A\}$. But members that the sets have in common (such as the point $A$) appear only once in the union.

We can also find the unions and intersections of more than two sets. For example, $(P \cup Q) \cup R = \{A, B, C, D, E, F, G\}$ and $(P \cap Q) \cap R = \{A\}$.

Suppose now we let $A$ be the set of points on the number line for the whole numbers from 1 to 10.

\[0 1 2 3 4 5 6 7 8 9 10\]

Let $B = \{2, 4, 6, 8, 10\}$, the set of even whole numbers between 1 and 11. And let $C = \{1, 3, 5, 7, 9\}$ and $D = \{2, 4, 7, 8, 9\}$. Then $(B \cup D) \cup (A \cup C) = \{1, 2, 4, 5, 6, 7, 8, 9, 10\}$ and $(B \cap D) \cap A = \{2, 4, 8\}$.

**EXERCISE 51-3B**

Find these sets.

1. $A \cap (B \cap D)$
2. $(C \cap D) \cap A$
3. $B \cup C$
4. $(B \cup D) \cup C \cup A$
5. $(A \cup D) \cup C$
6. $[B \cap (D \cap A)] \cap (A \cap C)$

We can also find intersections and unions of sets with infinitely many members. There are many examples that we can find of sets of points on the number line. Of course, we will not be able to write down complete lists of the sets we are using. Let $I$ be the set of points for all integers: $I = \{..., -2, -1, 0, 1, 2, ...\}$. Let $C$ be the set of points for the counting numbers: $C = \{1, 2, 3, ..., \}$. Let $E$ be the set of points for even integers: $E = \{..., -4, -2, 0, 2, 4, ..., \}$. Then $E \cap C$ is the set of points for even counting numbers: $E \cap C = \{2, 4, 6, ..., \}$. $E \cup C$ is the set of points for all even integers and all positive odd integers.

**EXERCISE 51-3C**

Look at the following sets of points:

$I$ is the set of points for all integers.

$E$ is the set of points for all even integers.

$N$ is the set of points for negative integers.

$F$ is the set of points for all fractions between 0 and 1.

$H$ is the set of points for all fractions with denominator 2.

1. List these sets of points as well as you can and locate them on the number line.
2. Find the following sets, list them if you can and locate them on the number line:

\[
\begin{align*}
I \cap E & \quad H \cap F \\
E \cap N & \quad E \cup (N \cup H) \\
(I \cap N) \cap E & \quad (I \cup (N \cup E))
\end{align*}
\]
Let us look at some more difficult examples. If we have infinitely many sets, we may think of the union of all of these sets and the intersection of all of them. The union of infinitely many sets will be the set of all things which are members of any of the individual sets, and the intersection will be the set of all things which are members of every one of the sets. Suppose we let $A$ be the set of points for rational numbers from 0 to 1, inclusive. Let $B$ be the set of points for rational numbers from 1 to 2, $C$ the set of points for rational numbers from 2 to 3 and so on. Do you see what the union of the sets $A$, $B$, $C$ and so on will be? It will be the set of points for zero and all positive rational numbers on the number line. Can you find the intersection of the sets $A$, $B$, $C$ and so on? If a point is in the intersection, it must belong to all of the sets. In particular, it must belong to the set $A$. Therefore, the point will be on the unit piece. Furthermore, it must belong to the set $C$, for example. This means that the point must be one of the points from 2 to 3. Can a point on the unit piece be between 2 and 3? No, this is impossible. Therefore, the intersection is the empty set, $\emptyset$.

As another example, consider the three lines $k$, $m$ and $n$ shown here. We have agreed to view these lines as sets of points.

![Diagram of three lines](image)

The intersection $k \cap m$ is just the set consisting of the point $B$. What are $k \cap n$ and $m \cap n$? Do you see that $(k \cap m) \cap n = \emptyset$? The union $(k \cup m) \cup n$ is the set of all points making up the geometric figure consisting of the three lines together.

**EXERCISE 51-3D**

1. Imagine all of the lines passing through a single point $P$. Each of these lines is a set of points. What is the intersection of all of these sets?
2. What is the union of all the lines in Question 1?

51-4 Planes as sets of points

One more thing we would like to do is regard planes as sets of points. In an ideal plane there are infinitely many straight lines. What we would like to do is to consider enough lines in the plane so that the union of all the sets of points making up the lines will completely cover the entire plane. There are many ways of doing this. One way is as follows.

Choose one line in the plane and then choose one point which is not on the line. Let us call the line $k$ and the point $P$. 

![Diagram of line and point](image)
For each point on \( k \) we can construct a straight line joining that point with the point \( P \). Now let us take as our lines all of the lines which join points of \( k \) with \( P \). The points making up these lines will cover almost every point in the plane. Can you find any points which will not be covered?

Suppose we draw a line \( m \) through \( P \) which is parallel to the original line \( k \). No point on \( m \) (other than the point \( P \)) will be on a line joining \( P \) with a point of \( k \). Do you see why this is so? Suppose some point \( Q \) on \( m \) is also on a line joining \( P \) with \( k \). Then there is a line containing \( P \) and \( Q \) which intersects the line \( k \). But remember, only one straight line may contain two different given points. Therefore, the line \( m \) which contains points \( P \) and \( Q \) must be none other than the line containing \( P \) and \( Q \) which intersects the line \( k \). Can \( m \) be parallel to \( k \) and also intersect \( k \)?

We have just seen that the lines joining \( P \) with all the points on the line \( k \) do not quite cover the plane. In addition we will take the line \( m \) through \( P \) and parallel to \( k \). Are you convinced that the union of the sets of points making up all of these lines completely covers the plane? There is a point in the union for every location along the line \( m \) because line \( m \) was taken in the union. There is also a point in the union for every location along the line \( k \), because for each point on \( k \), we have a line in the union passing through it, namely, the line joining that point with \( P \). What points are left?

If any point \( R \) is left over, it will surely be on some line containing \( P \), namely, the line joining \( P \) and \( R \). It cannot be on any line joining \( P \) with a point of \( k \). Therefore, \( R \) would have to be on the line through \( P \) parallel to \( k \). This line is \( m \), which is already in the union. Therefore, \( R \) could not have been left over.

In the remainder of this chapter, we shall agree to regard a plane as the infinite set of points which covers the entire plane.
EXERCISE 51-4A

1. Find some other ways of choosing infinitely many lines in a plane, so that the union of all the sets of points making up the lines covers the entire plane.

51-5 Subsets

When discussing sets, we can speak of subsets. Recall that if \( A \) and \( B \) are two sets and every member of set \( A \) is also a member of set \( B \), then we say that \( A \) is a subset of \( B \). For example, the set \( \{1, 3, 5\} \) is a subset of the set \( \{1, 2, 3, 4, 5, 6\} \). The idea of subset is useful in geometry, because if we think of geometric figures as sets of points, we can see the parts of the figures as subsets.

EXERCISE 51-5A

1. Say which of the following sets of points are subsets of others of the sets.
   - a. The set of points for integers on the number line
   - b. The number line
   - c. The point for zero on the number line
   - d. The set of points for rational numbers on the number line

2. Make up six other subsets of the number line.

3. Make up six subsets of a plane.

   One of the subsets you could have given for the plane is a straight line. A straight line in the plane is the subset consisting of those points of the plane that lie on the straight line. Any two intersecting straight lines intersect in exactly one point, and of course the point is a subset in the plane that contains just one member.

   One of our basic geometric figures is the line segment. Can we think of it as a set of points?

   Certainly we can. If \( k \) is a line and \( A \) and \( B \) are two different points on \( k \), then the line segment \( \overline{AB} \) is the set whose members are all points on the line between \( A \) and \( B \) and, in addition, the points \( A \) and \( B \) themselves. Then any line segment is also a subset of the plane in which it lies. Note that the endpoints of the line segment are also included in the subset.

   Let us look at the following example consisting of two line segments, \( \overline{AB} \) and \( \overline{CD} \).
The union of the line segments, $\overline{AB} \cup \overline{CD}$, is the figure consisting of the two segments together. The intersection, $\overline{AB} \cap \overline{CD}$, is just the single point $E$. Let $p$ be the plane in which the two line segments lie. Then the following statements are true.

- $\{E\}$ is a subset of $\overline{AB}$.
- $\overline{AB} \cup \overline{CD}$ is a subset of $p$.
- $\overline{AB} \cap \overline{CD}$ is a subset of $\overline{CD}$.
- $\overline{CD}$ is a subset of $\overline{AB} \cup \overline{CD}$.

The empty set, $\emptyset$, is a subset of $\overline{AB}$.

Notice that we just said that the empty set is a subset of $\overline{AB}$. Do you see why this is so? $\overline{AB}$ would have to contain any member which is in the empty set. The empty set has no members, and so there are no members of it which are not members of $\overline{AB}$. Do you see why the same reasoning shows that the empty set is a subset of any set?

**EXERCISE 51-5B**

1. Name six other pairs consisting of a set and a subset of it that occur in the previous figure.
2. Draw a quadrilateral and give the vertices letter names. Express the quadrilateral as the union of its four edges. Express each vertex as the intersection of two edges.
3. Suppose $k$ and $m$ are two straight lines in the same plane.
   a. If $k$ and $m$ are different names for the same straight line, what can you say about their intersection? their union? Write your conclusions in symbols using $\cap$ and $\cup$.
   b. If $k$ and $m$ are different straight lines, what can you say about $k \cap m$?
4. Suppose $k$ is a straight line in a plane and $\overline{AB}$ a line segment in the same plane.
   a. If $\overline{AB}$ lies in the straight line $k$, what can you say about $k \cup \overline{AB}$ and $k \cap \overline{AB}$?
   b. If $\overline{AB}$ lies in a straight line different from $k$, what can you say about $k \cup \overline{AB}$ and $k \cap \overline{AB}$?

In Question 3b you should have seen that if $k$ and $m$ intersect, then their intersection is a single point $A$: $k \cap m = \{A\}$.

On the other hand, $k$ and $m$ might be parallel, so that you would write $k \cap m = \emptyset$.

There is another simple fact that we can observe about two straight lines. Their intersection is a subset of each of the original lines. If $k$ and $m$ intersect, the intersection point $A$ lies on both $k$ and $m$, so that $\{A\}$ is a subset of $k$ and a subset of $m$. If the lines are parallel, the intersection is the empty set, and $\emptyset$ is certainly a subset of every set in the plane, including $k$ and $m$.

Do you see that we can formulate a general property of intersections? If $A$ and $B$ are two sets, then their intersection, $A \cap B$, is a subset of the set $A$ and a subset of the set $B$. To see that this is always so, just recall what the intersection is. If an object is to be a member of $A \cap B$, it must be a member of $A$ and a member of $B$. But if every object in $A \cap B$ must be a
member of \( A \), \( A \cap B \) must be a subset of the set \( A \). Every object in \( A \cap B \) must also be a member of \( B \), and so \( A \cap B \) will be a subset of \( B \).

51-6 Paths and circles as sets of points

We have not yet thought of other geometric figures, such as circles and angles, as sets of points. Furthermore, the straight line is the only path we have considered as a set of points. We can now consider all paths, because we have the idea of a subset of a set of points. Here is a picture of a path.

Of course, we are thinking of the path as not having any thickness, just as a straight line has no thickness. Then we can think of the path as a subset of the plane consisting of all the points in the plane which lie on the path. In other words, the path is the set of all points which the path goes through. This applies, for example, to a circle.

A circle is the subset of the plane consisting of all points lying on the circle.

Since circles and straight lines are sets of points, let us see what we can say about their intersections. If you have a circle \( C \), you can draw a straight line \( k \) that does not intersect the circle at all.

\[
k \cap C = \{1\}.
\]

It is also easy to find a straight line \( m \) which intersects the circle in two points. You may ask if it is possible to find a line which intersects the circle in only one point. Any line which just touches the circle but which does not go through the inside of the circle will intersect the circle in only one point. Here is a picture of such a line.
A line which intersects a circle in only one point is called a \textit{tangent} to the circle.

\textbf{EXERCISE 51-6A}

1. Can you find more than one tangent to a circle?
2. Discuss the possible sets in which a line segment can intersect a circle.
3. Picture examples of paths that intersect a circle in the empty set, in a set with one member, two members, five members.
4. Picture examples of paths that intersect a straight line in the empty set, in a set with one member, four members, ten members.
5. Can you think of a path that intersects a straight line in an infinite set?

\textbf{51-7 Ray, half-plane and angle}

We would like to connect our idea of \textit{angle} with the idea of a set of points. Let us see how this can be done.

If we look at a straight line and a point \( A \) on it, then the set of points consisting of \( A \) and all points in one direction along the straight line from \( A \) will form a \textit{ray}, and the set of points consisting of \( A \) and all points in the other direction along the straight line from \( A \) will form the opposite ray. So we can view a ray as a subset of the straight line in which the ray lies.

Another subset of the plane that will be useful in thinking of angles will be a half-plane.
Here is a picture of a straight line in a plane \( p \). The straight line together with all of the points of the plane which lie on one side of it make up a subset of the plane called a half-plane. One of the half-planes formed by the straight line \( k \) in the picture is labelled \( H \). Of course, you see straightaway that \( k \) separates the plane into two half-planes, \( G \) and \( H \). What is the intersection \( G \cap H \)? What is the union \( G \cup H \)?

Now look at two straight lines \( k \) and \( m \) which intersect in a point \( A \). Suppose that \( H \) is a half-plane formed by \( k \) and \( J \) is a half-plane formed by \( m \). What is the intersection \( H \cap J \)?

In the picture, \( H \) and \( J \) are shaded. The intersection is the set whose members are elements of both \( H \) and \( J \). So in the picture, \( H \cap J \) is shaded twice. Do you see that \( H \cap J \) consists of the angle \( \angle BAC \) together with its inside? Earlier in geometry we saw that an angle was the union of two rays with the same endpoint. What are the two rays here? What is their common endpoint? One of the rays is \( \overrightarrow{AB} \), and using the union and intersection symbols, you can write \( \overrightarrow{AB} = \overline{(H \cap J) \cap k} \).

**EXERCISE 51-7A**

1. Using the union and intersection symbols, write each of the following sets:
   
   a. \( \overline{AC} \)
   
   b. \( \overline{BAC} \)
   
   c. \( \overline{A} \)

2. Draw a picture of a triangle \( ABC \). Explain how you can express the set consisting of the triangle together with its inside as the intersection of three half-planes.

3. Many \( n \)-gons and their insides can be expressed as the intersection of \( n \) half-planes. Explain how this can be done.

4. Find an example of a quadrilateral which cannot be expressed as the intersection of four half-planes.
Chapter 52
SYMMETRY

52-1 Symmetric geometric figures

Look at the four geometric figures pictured here. What do you notice about them?

Do you see that the left half of each figure is just the mirror image of the right half? Here are some drawings of figures whose left halves are not mirror images of their right halves.

EXERCISE 52-1A

1. Draw six geometric figures whose left halves are mirror images of their right halves.
2. Draw five geometric figures whose bottom halves are mirror images of their top halves.
3. Draw four geometric figures whose left halves are not mirror images of their right halves.
If you have a figure drawn on a piece of paper, there is a way that you can test it to see if its left half is the mirror image of its right half. Try to fold the figure in half along a straight line. This will fold the left half over on to the right half, and if the two halves fit exactly over each other, then they are mirror images of each other. Trace the first four figures in this section on pieces of paper and try the folding experiment. If you have carefully traced the figures, the halves will fit on each other when you fold the paper. Here are some figures with folds shown as dotted lines. If you trace them and fold each figure along the line, the halves will fit over each other.

If you fold a figure into two halves that will fit over each other, the straight line along which you make the fold is called a line of symmetry for the figure. We will say that the figure is symmetrical with respect to the line of symmetry. Notice that the circle shown above has more than one line of symmetry. Can you find other lines of symmetry for the circle? Where do all of its lines of symmetry intersect?

Of course, drawings on paper are not ideal geometric figures and you cannot test an ideal geometric figure by folding. You can only imagine ideal geometric figures and lines of symmetry for them. But this work with drawing on paper will be good to use with your pupils, because of the activity that it involves.

**EXERCISE 52-1B**

1. Find all of the lines of symmetry for a square. How many lines of symmetry are there?
2. Find all of the lines of symmetry for a rectangle that is not a square. How many lines of symmetry are there?
3. How many lines of symmetry are there for an equilateral triangle? a regular pentagon? a regular hexagon?

**52-2 A symmetry test**

One of the simplest of geometric figures is a line segment. You can see that the perpendicular bisector of a line segment is a line of symmetry for it.
Since the distances \( AP \) and \( BP \) are equal, the points \( A \) and \( B \) coincide when the fold is made.

Let us look at one of our earlier figures together with a line of symmetry for it.

Suppose a point \( A \) is on the figure. If we construct the line \( m \) through the point \( A \) and perpendicular to the line of symmetry, this line will intersect the line of symmetry in a point \( P \). Now let us fold the paper again. The two halves of the picture fit exactly on top of each other, so that some point \( B \) of the figure must be on top of the point \( A \). This will be a point in which the line \( AP \) intersects the other half of the figure. Trace the figure and try folding it. The points \( B \), \( P \) and \( A \) are all on the straight line \( m \), and since \( B \) and \( A \) coincide when the paper is folded, the distance \( BP \) must be equal to the distance \( AP \). The line of symmetry for the entire figure is also a line of symmetry for the line segment \( AB \).

If you are given a figure and a straight line, you can test to see if the line is a line of symmetry by folding the paper that the figure is drawn upon. But perhaps the figure is not drawn on a piece of paper. Perhaps it is scratched on a flat piece of wood. How would you test it?

Imagine that these figures are scratched on a flat board.

This is what you can do. Choose a point \( A \) on the figure, as shown below. Then construct the straight line segment \( AB \), which is perpendicular to the dotted line, intersects the dotted line in a point \( P \) and such that the distances \( AP \) and \( BP \) are equal. If the dotted line is a line of symmetry, then \( B \) must be a point of the figure.
And if $B$ is always on the figure, no matter which point of the figure we choose for $A$, then the line is a line of symmetry. You could not test every point $A$, because the set of points on the figure is endless. But you can get a good idea whether or not the line is a line of symmetry for the figure. Of course, if a single pair of points $A$ and $B$ fails the test, the line is not a line of symmetry. The figure shown on the left has the dotted line shown as a line of symmetry. The figure shown on the right does not.

52-3 Symmetry of a straight line

We can use this test for some ideal geometric figures that we cannot draw. For example, consider an ideal straight line. It extends endlessly in two directions, and we cannot put it on a piece of paper and fold it. But we would like to show that any line perpendicular to it is a line of symmetry for it. So let $k$ be an ideal line and $m$ be any line perpendicular to $k$. If $A$ is a point on the line $k$, we test the line $m$ as a line of symmetry by constructing a perpendicular $AP$ to the line $m$ and extending it to the point $B$ such that $BP = AP$.

Since the ideal line $k$ itself is perpendicular to the line $m$, $P$ and $B$ both lie on the line $k$. The important thing is that $B$ is on $k$. No matter which point $A$ we try, the corresponding point $B$ will be on $k$. So $m$ must be a line of symmetry for the straight line $k$.

**EXERCISE 52-3A**

1. Let $k$ and $m$ be two ideal parallel lines and $n$ be the line parallel to them which is exactly midway between them. Show that for the geometric figure consisting of the lines $k$ and $m$, the line $n$ is a line of symmetry.
2. Make up more exercises for your pupils involving lines of symmetry for ideal geometric figures.

52-4 Making symmetrical figures

There are some interesting ways that your pupils can make symmetrical geometric figures. Take a piece of paper and fold it in half.

Then cut a design in the paper, starting at some point along the fold and returning to the fold with the cut. You can even make several cuts. When you open up the paper, the piece will have the fold as a line symmetry. (Can you see why this is so by folding the paper together again?)

Your pupils can make more interesting figures by folding a paper twice or more before cutting. Each fold will then be a line of symmetry.

Another easy way that your pupils can make complicated geometrical figures is this. Fold a piece of paper in half. Then put a drop of ink between the halves and press them together. The ink will spread out between the paper and make the same blot on one half as on the
other. When you open the paper, the fold will be a line of symmetry for the ink blots.

EXERCISE 52-4A

1. By folding and cutting paper, make some symmetrical cutouts.
2. By folding pieces of paper once in half and using drops of ink, produce some symmetrical ink blots.
3.1 Introduction—bisection of a line segment

In teaching geometry, we must first teach the pupil to recognize and name certain shapes and objects. We also must teach certain properties of these shapes or figures. We must devise ways of drawing or making some of these figures.

We have seen that line segments and circles were used as "building blocks" in drawing many figures. We can draw more complicated figures by using line segments and circles according to certain rules. For example, let us recall how to draw a line segment perpendicular to a given line segment \( \overline{AB} \) at a point \( P \) on \( \overline{AB} \). We first locate points \( C \) and \( D \) on \( AB \) equally distant from \( P \) by drawing a circle with centre \( P \) [Fig. 1(a)]. Then with \( C \) and \( D \) as centres, we draw two more circles with equal radii larger than \( PC \). These intersect at \( E \) and \( EP \) is the desired segment [Fig. 1(b)]. In the figure we made, it seems reasonable that \( EP \) is perpendicular to \( \overline{AB} \). But how do we convince the pupil that this method will always give the perpendicular we want? Does the method depend upon the radii of the circles we used? Will it work if we use larger radii? It would be unfortunate if the method works on this one line but not on another line, or at one point \( P \) but not at another.

The pupil should also realize that the figures we draw are not really what we want to consider as line segments or circles, for they will be "too thick". Does this method work if we use thinner chalk or pencil marks?

**EXERCISE 53-IA**

Draw three different points \( P, Q \) and \( R \) on a line \( \overline{AB} \). By using the method described, draw lines perpendicular to \( \overline{AB} \) through \( P, Q \) and \( R \), in each case using a differ-
ent radius to make the drawing. Do all three line segments look perpendicular to $\overline{AB}$? Use your protractor to check.

In the example we just discussed, the construction of the perpendicular to a given line segment seems to work and we shall later show that it does work. Here is another construction to consider. Let us bisect an angle; that is, divide it into two congruent angles. We first recall the simple method for finding the midpoint of a segment $\overline{AB}$. We draw two circles of the same radius, one with centre at $A$ and the other with centre at $B$. If the radius is chosen large enough, these intersect at $C$ and $D$. The segment $\overline{CD}$ intersects $\overline{AB}$ at $M$ [Fig. 2(a)]. The two segments $\overline{AM}$ and $\overline{BM}$ seem of equal length, so we say that $M$ is the midpoint of $\overline{AB}$.

**53-2 A pitfall**

Now let us try to bisect an angle $\angle ABC$. We propose to try the following method. Connect $A$ and $C$ with the segment $\overline{AC}$ [see Fig. 2(b)]. Now use the method given above to find the midpoint $M$ of $\overline{AC}$. Since $M$ is half-way between $A$ and $C$, we may surely hope that the ray $\overline{BM}$ bisects the angle $\angle ABC$. If we measure the two angles in Figure 2(b), we shall see that they do turn out to be congruent, so perhaps the method is a good one.

But we did not say how to select the points $A$ and $C$ on the two sides of the angles. Suppose we had not taken them as in Figure 2(b), where they are both the same distance from $B$. Let us try the same method again using different points $A$ and $C$, as in Figure 2(c). It looks as though the method works in Figure 2(c), even when the two points $A$ and $C$ are not taken at the same distance from $C$, for $M$ is again half-way between $A$ and $C$ and $\overline{BM}$ should be "half-way" between $\overline{BA}$ and $\overline{BC}$. Measure the angles $\angle ABM$ and $\angle CBM$ with your protractor to see if they are congruent. It looks as though we have found a very good method for bisecting an angle.

Let us try it once again with still other points $A$ and $C$, as in Figure 2(d).
Here again we have selected \(A\) and \(C\) on the two sides and found the midpoint \(M\) of \(AC\). Thus, \(M\) is half-way between \(A\) and \(C\), as you can easily verify by measuring \(\overline{AM}\) and \(\overline{CM}\). But is \(\overline{ABM}\) congruent to \(\overline{CBM}\)? It is quite clear that \(\overline{BM}\) is not the bisector of \(\angle ABC\)! One does not even have to measure it; it is obvious from just looking at it. What went wrong with our "wonderful" method?

Perhaps the trouble is in the way we chose the two points \(A\) and \(C\). Maybe \(C\) is too far from \(B\), or \(A\) too close. Try a few more obvious choices of \(A\) and \(C\) yourself and see when the method seems to work. Does it seem as though \(A\) and \(C\) must be nearly the same distance from \(B\), as in Figures 2(b) and 2(c)? Perhaps, but then at what distances does the method begin to fail? It is clear that we have found a method which seems to work in some cases but not in others, and it does not seem clear exactly when it begins to go wrong. It shows us that we must always be careful not to say that because the method seemed to work in a few cases that we try, it will work in all cases. We want to have a way that will enable us to convince ourselves and others that the method we are going to use will give us the result we want. That is the object of this chapter. We are going to study such ways of convincing people that the rules we give for constructions give us what we want from them.

The argument that we use to convince someone that a statement is true is called a "proof". When we have given such a convincing argument, we say that the statement has been "proved".

Incidentally, the method given above for bisecting an angle works only when \(A\) and \(C\) are equally the same distance from \(B\). Even though it seemed to work in Figure 2(c), it did not, for \(\overline{BM}\) is very close to the true bisector and we could not easily see the difference. Even our measurements with a protractor lack the accuracy to be sure that \(\overline{ABM}\) and \(\overline{CBM}\) are not congruent, but it can be proved that they are definitely not congruent unless \(A\) and \(C\) are equally distant from \(B\). This again shows the importance of a method of proof, for sometimes our eyes deceive us and we must appeal to logical reasoning to help us out.

**EXERCISE 53-2A**

1. What does "to bisect an angle" mean? Answer in your own words.
2. With your protractor, draw an angle of \(75^\circ\), and then bisect it using the same instrument. Are you sure that you have done a perfect job? Explain.
3. On a large piece of paper, use a protractor and ruler to draw a right angle \(\angle A\hat{B}C\). On \(\overline{BA}\), mark a point \(P_4\) four inches from \(B\), \(P_5\) five inches from \(B\) and \(P_6\) six inches from \(B\).
   
   Locate, by sliding your ruler along \(\overline{BC}\), points \(Q_4\), \(Q_5\) and \(Q_6\) on \(\overline{BC}\) such that \(\overline{P_4Q_4}\)
and $P_5Q_5$ each has length 10 inches. Let $M_4$, $M_5$ and $M_6$ be the respective midpoints of $P_4Q_4$, $P_5Q_5$ and $P_6Q_6$. Draw $BM_4$, $BM_5$ and $BM_6$. Are these all different?

53-3 A strategy of proof

We have devised ways to copy triangles in Unit I. We showed the method to our pupils on a few triangles. The method seems to work well on the triangles we tried it on, but will it work for all triangles? We should hope so, but in order to be sure, we must devise some way to prove "once and for all" that the methods we have used do actually give us what we want in all cases. But how does one prove that something is true for all possible situations that can ever arise? We cannot draw all possible triangles and see if our method of copying them works for every one. We would never finish. It is certainly a triumph of man's reasoning power that he can devise methods of proving such far-reaching statements which apply to all possible situations, without having to try each one separately.

How is this done? Sometimes one tries to convince people that a statement is true by showing that it is true for some cases and then claiming that it is always true. Such an argument is called an inductive argument. It is often used in the physical sciences where natural phenomena are observed many times and then conclusions are drawn. But this kind of reasoning is not convincing in mathematics, for what works in some cases may not work in all cases. We gave one example of this in the incorrect method for bisecting an angle. Here is another example from arithmetic.

Suppose we want to "prove" the statement, "Every positive odd integer different from 1 is a prime". We begin with 3; it is a prime. The next one is 5; it is a prime. The next one is 7; it is a prime and so on. We have tried several cases and they all seem to work, so we may feel confident that the statement is true. But then we try the next odd integer, 9, and find that $9 = 3 \times 3$, and we find that our argument fails.

On the other hand, consider the statement, "Every prime number different from 2 is odd". We begin by looking at some cases: 3 is odd, 5 is odd, 7 is odd, 11 is odd and so on. Have we proved the statement? No! But we can give a proof which does show that every prime other than 2 is odd. For what are the possibilities for an integer? It can be even or odd. If it is even, it can be divided by 2. But if it has 2 as a factor, then it is not a prime, unless it is 2 itself. Thus, if a number is a prime different from 2, it must be an odd number. This proof does show that all primes except 2 are odd!

Now we consider how we intend to convince people that the statements we made are true for all cases under consideration. We begin with certain facts that are so simple and obvious that no one would question them. These facts are called "postulates" or "axioms", and they tell us certain basic things about the building blocks of our geometry: points, lines, planes, triangles, circles and so on. These postulates are accepted as being true in all situations. We then proceed from these postulates to other facts which follow from them by logical reasoning. These facts, which are deduced from the axioms by logical reasoning, are called theorems. This process ultimately will lead us from the truth of the simple observations, which we called postulates, to the truth of much more complicated theorems, which are not at all obvious. This was our original goal!

Why will these postulates be necessary? Why can't we prove everything? Suppose we wish to prove some statement. We say, "This statement is true because ..." and then we give a second statement as the reason. But then someone asks, "Why is the second statement true?" We reply, "The second statement is true because ..." and give a third statement. But
why is this third statement true? This can go on endlessly unless we agree to the truth of some
more obvious statements, so that ultimately the sequence of questions "Why is that statement
true?" will be answered by, "It is one of our postulates which we have accepted as true", and
we can stop what would otherwise be an endless chain of questions.

**EXERCISE 53-3A**

1. Answer the following question. Why are you present in this class? Then ask a question
   about the given answer, and so on as far as you can. When you finally decide to stop at
   one answer, ask yourself why you stop. Could you not continue?
2. State any geometric "fact" you have learned. Can you prove this fact? Could you suc­
   cessfully answer the question "why" to each statement you make in the proof?

   Our postulates and theorems will all be about geometrical figures. Since we want to be
   sure that our argument is convincing, we want to be very careful that we all agree about the
   meaning of the things we are talking about. It would make no sense at all if one person thought
   that a triangle had four sides while another thought a triangle had three sides. They certainly
   then could not agree on theorems about triangles. In order to avoid possible confusion, we shall
   very carefully define each of the geometrical figures about which we speak. Once again, we de­
   fine things using words which themselves must be defined in terms of other words, which must
   be defined in terms of ... and so on. Where does this process stop? Here again we shall take
   certain words as being so common to our experience that we do not have to define them again.

   In the last few sections, we have been talking about "points", "lines", "planes" and
   "space". We shall now assume that we know what these words mean. We have also used the
   words "set of points", "union of two sets", "intersection of two sets", "subset of points",
   often enough to know what they mean. We shall as before use capital letters to denote points.
   If two distinct points $A$ and $B$ are on a line, we shall denote the line by $AB$. We also have seen
   what we mean when $A$, $B$ and $C$ are on the same line and $B$ is between $A$ and $C$. In Figure 3,
   $B$ is between $A$ and $C$, while in Figure 4 and Figure 5, $B$ is not between $A$ and $C$.

   ![Fig. 3](image1)
   ![Fig. 4](image2)
   ![Fig. 5](image3)

   Thus, we shall begin our more careful study of geometry by agreeing that we know what
   we mean by the following list of words which we refer to as "undefined te­ms".

   1. Point
   2. Line
   3. Plane
   4. Space
   5. Between; that is, "$B$ is between $A$ and $C$".
   6. Set of points, subset, union and intersection.
**EXERCISE 53-3B**

A pupil may ask the following questions:

a. What is a theorem?
b. What is a postulate?
c. What does "undefined term" mean?
d. When can a word be left "undefined" and when does it require a definition?
e. When does a statement require a proof and when can we accept it as obviously true?

How would you answer these questions?

53-4 Some simple definitions

We shall now set out to define some of the other geometrical words in terms of these undefined terms. We are going to begin our own little dictionary. The words will not be in alphabetical order, but in the order we shall need them. They will be preceded by the word "definition". For example:

**DEFINITION 1:** A geometrical figure is a set of points.

**DEFINITION 2:** Points are collinear if they are on the same line and coplanar if they are in the same plane.

**DEFINITION 3:** A line segment with endpoints $A$ and $B$ consists of $A$ and $B$ and all points on $AB$ which are between $A$ and $B$ (Fig. 6(a)]. We shall denote the line segment with endpoints $A$ and $B$ by $AB$.

How would you now define "ray"? See if your definition gives the same points as the one given below.

**DEFINITION 4:** A ray with endpoint $A$ passing through $B$ is the union of $AB$ and the set of all points $C$ on $AB$ such that $B$ is between $A$ and $C$. We shall write $\overrightarrow{AB}$ for the ray with endpoint $A$ through $B$ (Fig. 6(b)]. (The arrow indicates that $AB$ extends endlessly beyond $B$.)

Some of the constructions that we shall try to justify in this chapter have to do with copying figures. What do we want when we ask for a copy of some figure? In a sense we want a figure with the same size and shape. We want one to be an exact duplicate of the other. We have been using the word "congruent" to describe two such figures. We have then:

**DEFINITION 5:** Two figures are congruent if one is an exact duplicate (or copy) of the other. We use the symbol $\cong$ to mean "is congruent to".

For plane figures, one can sometimes place one of the figures on top of the other to see if they fit together exactly, that is, to see if they are congruent. Sometimes it is more convenient to trace a given plane figure and place the tracing over the other figure to see if they are congruent. We shall later discuss tests which will enable us to determine whether certain figures are congruent just by measuring certain parts of each and comparing these measures. This is very often easier than moving the whole figure or making a tracing.

**EXERCISE 53-4A**

1. Can you say what it means for two figures not to be congruent?
2. Can two solid figures be congruent? Can you explain what it would mean for a solid figure to "fit together exactly" with another solid figure? What would it mean for two chairs to be congruent?

53-5 The initial postulates

We can now state our first few postulates. Recall that these are to be facts or statements that we are willing to accept as being true. Do you find each of them reasonable?

POSTULATE 1: There is one and only one line segment having two given points as endpoints.

This certainly agrees with our intuitive concept of a line segment. Draw two points on a sheet of paper. Take a ruler and join them. Can you have two line segments having the same two endpoints?

We next consider the length of a line segment and observe that when we measure lengths, the following things are true.

POSTULATE 2: Lengths of line segments have the following properties.

(a) After a unit of length has been adopted, the length of any line segment $AB$ is a positive real number, which we will write as $|AB|$.

(b) Line segments are congruent if and only if they have the same length.

(c) If $C$ is between $A$ and $B$ on $AB$, then $AB = AC + CB$.

(d) If $A$, $B$ and $C$ are not collinear, then $AB < AC + CB$.

The last part says that in going from $A$ to $B$ the straight line path from $A$ directly to $B$ is shorter than the path which first goes to $C$ and then from $C$ to $B$ (Fig. 7).

Postulate 2(d) can be "tested" by drawing a triangle $ABC$. Take a string which just reaches directly from $A$ to $B$. Now try to make the same piece of string reach from $A$ to $C$ and go around $C$ to $B$. Will it reach from $A$ to $C$ and then on to $B$? Will it reach from $A$ to $C$ and then on to $B$?

It is convenient to define the sum of two segments which have a common endpoint.

DEFINITION 6: If $C$ is between $A$ and $B$ on $AB$, then we say that $AB$ is the sum of $AC$ and $CB$, and write $AB = AC + CB$. This enables us to rewrite Postulate 2(c) as follows: If $AB = AC + CB$, then $AB < AC + CB$.

DEFINITION 7: The distance from $A$ to $B$ is the length of the line segment $AB$.

Once we accept what distance means, we can say what we mean by a circle. Try writing your own definition of a circle and see if it agrees with the one below.

DEFINITION 8: A circle with centre at $A$ and radius $r$ is a figure consisting of all points which are in a plane containing $A$ and which have distance $r$ from $A$. A line segment $AB$, where $A$ is the centre and $B$ is on the circle is called a radial segment. The figure consisting of all radial segments of the given circle is called the circular region or disc with centre $A$ and radius $r$.

EXERCISE 53-5 A

1. Suppose you are asked to do the following: given a line segment $AB$ and a point $C$, draw a circle with centre $C$ and radius $AB$. Say in your own words what this means. Draw a
line segment \( \overline{AB} \) and label its endpoints \( A \) and \( B \). Locate a point \( C \). Now carry out the instructions given above.

2. Notice that all of the radial segments of a circle have the same length \( r \) and will all be congruent to one another. [Postulate 2(b) says segments of the same length are congruent.] Why was it necessary to say that the circle consists of points in a given plane? What figure would one get if one merely took all points in space that are at a distance \( r \) from \( A \)? Write your own definition for a "sphere" with centre \( A \) and radius \( r \).

53-6 Construction of congruent line segments

We are now ready to try to verify that one of our constructions actually gives us what we wanted. We begin with a very simple one to illustrate the method of writing the argument. We shall discuss how one copies a given line segment. That is, suppose we are given a line segment \( \overline{AB} \), and we want to make a segment congruent to \( \overline{AB} \) but having one endpoint at \( C \) and the other endpoint on a given ray \( \overrightarrow{CD} \). We shall first describe the construction and then verify that what we made actually is a segment congruent to \( \overline{AB} \).

**CONSTRUCTION 1.** Given a line segment \( \overline{AB} \) and a ray \( \overrightarrow{CD} \), construct a line segment \( \overline{CE} \) with \( C \) on \( \overrightarrow{CD} \) so that \( \overline{CE} \cong \overline{AB} \).

**Method:** With radius \( \overline{AB} \), draw a circle \( \overline{FGH} \) with centre \( C \) (Fig. 8). This circle intersects \( \overrightarrow{CD} \) in a point \( E \). Then \( \overline{CE} \) is the desired segment.

**THEOREM 1.** The segment \( \overline{CE} \) constructed above is congruent to \( \overline{AB} \).

**Proof:** \( \overline{CE} \) is a radial segment of a circle of radius \( \overline{AB} \), by our construction. Thus, \( \overline{CE} = \overline{AB} \).

But since segments of the same length are congruent [Postulate 2(b)], we conclude that \( \overline{CE} = \overline{AB} \), and the proof is completed.

We have written this proof in the form of a short paragraph. It consists of several assertions (statements) each supported by a reason. There is a very convenient form for writing such arguments which displays more clearly the assertions and the reasons. This is what is called the two-column, or formal, style of proof. We shall rewrite the above proof in two-column style and see if you do not agree that it sets things off more clearly.

**Proof** (in two-column style):

<table>
<thead>
<tr>
<th>Assertion</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \overline{CE} ) is a radial segment of the circle ( \overline{FGH} ).</td>
<td>1. The segment ( \overline{CE} ) was so constructed.</td>
</tr>
<tr>
<td>2. ( \overline{CE} = \overline{AB} )</td>
<td>2. All radial segments of a circle have length equal to the radius by Definition 8, and ( \overline{AB} ) is the radius of ( \overline{FGH} ).</td>
</tr>
<tr>
<td>3. ( \overline{CE} = \overline{AB} )</td>
<td>3. Segments of the same length are congruent by Postulate 2(b).</td>
</tr>
</tbody>
</table>

Now that we have finished the verification of this simple construction, let us take a little closer look into what was involved. We made use of Postulate 2(b) and Definition 8. But have we used any more? You might say that we used Definitions 3 and 4 when we spoke of line...
segments and rays. That is true. We have actually used much more than this when we made the actual construction. For in the very first step, we said "Draw a circle of radius $AB$ with centre $C$". If we want to be very careful about all small details, we should now ask, "How do we know that we can draw such a circle?" We next said, "The circle $FGH$ intersects the ray $CD$ in a point $E$". How do we know that the circle intersects a ray from its centre in one and only one point? Once we have the points $C$ and $E$, can we draw the segment $CE$? The answers to these questions are all intuitively yes, and one would hardly worry about them. But if one wanted to base everything on explicitly stated postulates, he would have to add some postulates to justify that the steps in the constructions can actually be carried out as described. In what follows, we shall not worry about this aspect of the problem. We shall assume that the steps of each construction can actually be performed as described and then only prove that the resulting construction gives us what we want.

You may, however, be interested in seeking some of the postulates you would have to add to those given already or to be given later to justify the constructions. We shall list a few of these here just to give an idea of what is involved.

(A) There is one and only one line through two given points.
(B) It is possible to draw a line segment with given endpoints $A$ and $B$.
(C) It is possible to draw a circle in a given plane with given centre and radius.
(D) Given a circle with centre $A$ and a coplanar ray $AB$, there is a single point where $AB$ and the circle intersect.
(E) If the sum of the radii of two coplanar circles is greater than the distance between their centres and the difference of the radii less than that distance, the circles intersect in two distinct points.
(F) If $C$ is any point of $AB$, then there is a positive integer $k$ such that $k \times AB > AC$.

These are some of the postulates one would use to justify that the steps described in each construction are possible. The last one assures us that we can extend a line segment as far as we wish in either direction. You may enjoy using these postulates yourself to give the reason why each step in the following constructions is possible. In the text itself, we shall only verify that the end result does what we want it to do.

53-7 Construction instruments

One further comment should be made about what we are doing in the next chapters. We are going to discuss geometrical constructions—that is, the drawing of certain geometrical figures. We shall try to draw these figures using only a pencil, a ruler to draw line segments and a compass to draw circles. There are many other devices available for drawing geometrical figures and your pupils will be familiar with some of them already and will learn how to use others later.

For many centuries people have been fascinated by the prospect of being able to draw a great many figures just using the ruler and compass. It had become somewhat of a game to see how far one can go using only these two simple tools. We shall continue to play this game in our study. We hope that when you teach this subject, you can help your students to enjoy seeing how many figures they can draw with these two tools.

Our methods will have an advantage over using measurements with the ruler and protractor, for with a ruler or protractor, one is limited in accuracy by the markings on the instruments. We shall learn to copy line segments and angles without having to rely upon the accuracy of a measurement.

Let us now continue on our way!
Chapter 54

ANGLES AND TRIANGLES

54-1 Angle and triangle definitions

The next figures that we shall construct will involve angles and triangles. Let us first define these figures carefully.

**DEFINITION 9:** An angle is a figure which consists of two distinct rays which have the same endpoint. The common endpoint is called the vertex of the angle, and the two rays are called sides or edges. If the two edges of an angle lie on the same line, the angle is called a straight angle. The symbol $\triangle ABC$ denotes the angle with vertex $B$ and edges $\overline{BA}$ and $\overline{BC}$. (Fig. 9) It is sometimes convenient to name an angle with a single letter. In such cases we shall write $\alpha$, and read this as "angle $\alpha". 

In Figure 9, the two rays separate the plane containing them into two parts, one of which we should like to call the inside or interior of the angle and the other part the outside or exterior. For example, we would like to say that $D$ is inside $\triangle ABC$ and $E$ is outside $\triangle ABC$. Can we give a rule to help us decide in general whether we should call a given point inside or outside an angle? First let us observe that the straight angle $\triangle ABC$ in Figure 10 also separates any plane containing it into two parts, $D$ lying in one part and $E$ in the other. But neither of these parts looks more "inside" than the other, so we shall not try to define inside and outside for a straight angle.

We shall merely say that a straight angle (or for that matter, a line) separates any plane containing it into two parts, each of which is called a half-plane.

**DEFINITION 10:** If $\triangle ABC$ is not a straight angle, then a point $D$ not on $\triangle ABC$ is inside (or interior to) $\triangle ABC$ if the ray $\overrightarrow{BD}$ intersects the line segment $\overline{AC}$. Likewise, when $\triangle ABC$ is not a straight angle, a point $E$ not on $\triangle ABC$ and in the same plane as $\triangle ABC$ is outside (or exterior to) $\triangle ABC$ if $\overrightarrow{BE}$ does not intersect $\overline{AC}$. 

Fig. 9

Fig. 10
Look back at Figure 9 to see if this definition agrees with the intuitive notion we had of inside and outside an angle. Notice that $\overline{BD}$ does intersect $\overline{AC}$ whereas $\overline{BE}$ does not.

**EXERCISE 54-1A**

1. Copy these angles and shade the inside of each of them.

![Figure 11](image)

![Figure 12(a)](image)

![Figure 12(b)](image)

![Figure 12(c)](image)

Just for completeness, it should be mentioned that there is another way of formulating the definition of the inside or interior of an angle which uses half-planes. For an angle $\angle ABC$ which is not a straight angle, $C$ lies in one half-plane lying on one side of the line $\overline{AB}$. Call this half-plane $H$. (Fig. 12(a).) Similarly $A$ lies in a half-plane $K$ on one side of the line $\overline{BC}$ (Fig. 12(b)). The interior of $\angle ABC$ could also have been defined as those points which are common to both $H$ and $K$; that is, $H \cap K$ (Fig. 12(c)).

Just as we describe the size of a line segment when we measure its length, we now wish to discuss the size of an angle. You recall that in Unit 1 you measured angles with a protractor, reading off a certain number of degrees. Let us consider what we did. We assigned a number to an angle to serve as its measure, just as we assigned a number to be the length of a segment. In the case of segments, we assigned a certain number of units, such as inches, feet or miles. In the case of angles, we assigned a certain number of degrees to each angle. We wrote $m(\angle ABC)$ for the measure of the angle $\angle ABC$. We used the symbol ° to denote degrees, so that $45°$ was read "forty-five degrees". Now what measures did we assign to a given angle? We can
agree to assign the measure $180^\circ$ to any straight angle. To talk further about measures of other angles, it is convenient to say what we mean by a ray being between two other rays.

**DEFINITION 11:** If $\overline{ABC}$ is not a straight angle, we say that the ray $\overline{BD}$ is between $\overline{BA}$ and $\overline{BC}$ if $\overline{BD}$ is inside the angle $\angle ABC$ (That is, the points of $\overline{BD}$, except for $B$ itself, are inside $\angle ABC$). In Figure 13, $\overline{BD}$ is between $\overline{BA}$ and $\overline{BC}$, while $\overline{BE}$ is not. If $\angle ABC$ is a straight angle, any ray $\overline{BD}$, with $D$ not on $\overline{ABC}$, is between $\overline{BA}$ and $\overline{BC}$. In Figure 14, $\overline{BD}$ is between $\overline{BA}$ and $\overline{BC}$, and so is $\overline{BE}$.

We can now talk about the sum of two angles. When $\overline{BD}$ is between $\overline{BA}$ and $\overline{BC}$ (as in Figures 13 and 14), we can consider the two angles $\angle ABD$ and $\angle DBC$. We then say:

**DEFINITION 12:** If $\overline{BD}$ is between $\overline{BA}$ and $\overline{BC}$, then $\angle ABC$ is the sum of $\angle ABD$ and $\angle DBC$ and we write $\overline{ABC} = \overline{ABD} + \overline{DBC}$.

When we measure angles $\angle ABD$ and $\angle DBC$ in either Figure 13 or Figure 14, we get two numbers, and we would certainly expect that the measure of $\angle ABC$ would be the sum of these two numbers. Try it and see if this is the case. We can write this fact in the following way. If $\overline{ABC} = \overline{ABD} + \overline{DBC}$, then $m(\angle ABC) = m(\overline{ABD}) + m(\overline{DBC})$.

**EXERCISE 54-1B**

1. In the figure below, state which rays are between which other rays (for example, $\overline{BE}$ is between $\overline{BF}$ and $\overline{BA}$). Express as many angles as possible as sums of other angles.
If two angles are congruent, one is an exact duplicate or copy of the other. We would certainly expect congruent angles to have the same measures. Let us now summarize the observations we have made about the measure of an angle to give us our next postulate.

**POSTULATE 3:** Measures of angles have the following properties:

(a) The measure of an angle is a positive real number.
(b) Angles are congruent if and only if their measures are the same.
(c) If \( \overline{BD} \) is between \( \overline{BA} \) and \( \overline{BC} \) (that is, if \( \angle ABC = \angle ABD + \angle DBC \)), then \( m(\angle ABC) = m(\angle ABD) + m(\angle DBC) \).
(d) The measure of any straight angle is 180°.

Let us look at straight angles again.

**DEFINITION 13:** If two rays \( \overline{AB} \) and \( \overline{AC} \) (with common endpoint \( A \)) form a straight angle, they are called **opposite rays**.

The rays \( \overline{AB} \) and \( \overline{AC} \) in Figure 15(a) form a pair of opposite rays.

![Fig. 15(a)](image)

Now let \( \overline{AB} \) and \( \overline{AC} \) be opposite rays and let \( \overline{AD} \) be between \( \overline{AB} \) and \( \overline{AC} \). Then \( \overline{AD} \) forms two angles with \( \overline{AB} \) and \( \overline{AC} \); namely, \( \angle CAD \) and \( \angle BAD \) [Fig. 15(b)].

**DEFINITION 14:** The two angles \( \angle CAD \) and \( \angle BAD \) formed by a ray \( \overline{AD} \) and each of two opposite rays \( \overline{AB} \) and \( \overline{AC} \) are called a **linear pair** of angles.

Since \( \overline{AD} \) is between \( \overline{AB} \) and \( \overline{AC} \), we may use Postulate 3(c) to conclude that \( m(\angle CAD) + m(\angle BAD) = m(\angle CAB) \). But \( \angle CAB \) is a straight angle, so \( m(\angle CAB) = 180° \), according to Postulate 3(d).

Thus we are led to the fact that \( m(\angle CAD) + m(\angle BAD) = 180° \); that is, **the sum of the measures of a linear pair of angles is 180°**. In general, any two angles whose measures add up to 180° are called **supplementary angles**.

**DEFINITION 15:** If the two angles of a linear pair of angles are congruent, they are called **right angles**.

Thus, when the linear pair of angles \( \angle CAD \) and \( \angle BAD \) are congruent (as in Figure 16), we say that each of them is a right angle. Since congruent angles have the same measures [Postulate 3(b)] we have \( m(\angle CAD) = m(\angle BAD) \). But their sum is 180°, so we see that \( m(\angle CAD) = 90° \) and \( m(\angle BAD) = 90° \). Thus, **the measure of a right angle is 90°**. Since angles with the same measure are congruent [again Postulate 3(b)], we have given a simple proof of the theorem that all right angles are congruent to each other.

**DEFINITION 16:** If \( \overline{AB} \) and \( \overline{AC} \) are opposite rays, they separate any plane in which they lie into two half-planes, one on each side of the line \( \overline{BC} \). Let \( \overline{AD} \) and \( \overline{AE} \) be two distinct rays which lie in the same half-plane. Then the three angles \( \angle BAD, \angle DAE \) and \( \angle CAE \) form a **linear triple** of angles (Fig. 17).
EXERCISE 54-1C

1. Can you add the angles which form a linear triple? Prove that the sum of the measures of the angles of a linear triple is 180°. Can you formulate the definition of a linear quadruple of angles? A linear n-tuple of angles? What is the sum of the measures of the angles in a linear n-tuple?

2. The two rectangles \( ABDC \) and \( DEGF \) are placed in such a way that \( BF \) and \( TE \) are two line segments intersecting at \( D \). Let \( JF \) be a line segment passing through \( F \) such that \( m(JFD) = 45° \). Find \( m(JFH) \).

Let us next turn to triangles.

DEFINITION 17: Let \( A, B \) and \( C \) be three points which are not collinear. Then the triangle with vertices \( A, B \) and \( C \) is the figure consisting of the three line segments \( AB, BC \) and \( CA \). The segments \( AB, BC \) and \( CA \) are called the sides or edges of the triangle (Fig. 18). We write \( \triangle ABC \) for this triangle.

In addition to speaking of the three
sides of a triangle, it is customary to speak of the three angles of a triangle. Since an angle consists of two rays having a common endpoint, and since a triangle does not contain any rays, one can only mean that the triangle determines the angles which are obtained when the sides are suitably extended to be rays. More precisely, we extend side \( \overline{AB} \) to get the ray \( \overrightarrow{AB} \), and we extend side \( \overline{AC} \) to get the ray \( \overrightarrow{AC} \). These rays form angle \( \angle BAC \) which we call the angle of \( \triangle ABC \) at the vertex \( A \), or the angle included by the sides \( \overline{BA} \) and \( \overline{AC} \). Similarly, we call \( \angle ABC \) the angle at vertex \( B \) or the angle included by the sides \( \overline{AB} \) and \( \overline{BC} \), while \( \angle BCA \) is the angle at vertex \( C \), or the angle included by the sides \( \overline{BC} \) and \( \overline{CA} \).

**DEFINITION 18:** A point is inside (or interior to) a triangle if it is inside all three angles of the triangle. If a point is in the plane of a triangle and is neither on the triangle nor inside it, then we say the point is outside (or exterior to) the triangle. A triangle together with those points which are interior to it form what is called a triangular region.

**EXERCISE 54-1D**

1. Draw a triangle \( \triangle ABC \). Name a. the vertices, b. the sides, c. the angles. State the location of each vertex in relation to each side, and each side in relation to each angle.

2. Extend the sides of \( \triangle ABC \) to form the three angles. Shade the interiors of two of these angles with different shadings. What is the intersection of these two interiors? Compare the result with Definition 18 and state any conclusion which seems appropriate.

54.2 Congruence—the SSS postulate

We shall be occupied soon in deciding whether two triangles are congruent, that is, whether one is a copy of the other. To decide this, we shall set up a one-to-one correspondence between the vertices of one and the vertices of the other. For example, if the two triangles are \( \triangle ABC \) and \( \triangle DEF \), we may set up the correspondence

\[
\begin{align*}
A & \rightarrow D \\
B & \rightarrow E \\
C & \rightarrow F.
\end{align*}
\]

There are other possibilities, for example,

\[
\begin{align*}
A & \rightarrow F \\
B & \rightarrow D \\
C & \rightarrow E.
\end{align*}
\]

The notation for the triangles gives us a convenient way of indicating the correspondence between the vertices which we have in mind. The vertices can be written down after the "\( \Delta \)" symbol so that corresponding vertices occur in the same order. Thus \( \Delta ABC \rightarrow \Delta DEF \) means \( A \rightarrow D, B \rightarrow E \) and \( C \rightarrow F \); while \( \Delta ABC \rightarrow \Delta FDE \) means \( A \rightarrow F, B \rightarrow D \) and \( C \rightarrow E \).

The correspondence between the vertices of two triangles also gives us a one-to-one correspondence between the sides of the two triangles. For if \( \Delta ABC \rightarrow \Delta DEF \), we shall take this to mean that \( \overline{AB} \rightarrow \overline{DE}, \overline{BC} \rightarrow \overline{EF} \) and \( \overline{CA} \rightarrow \overline{FD} \); that is, segments determined by corresponding vertices also correspond to each other. The same holds for the
angles of the two triangles. The correspondence $\triangle ABC \rightarrow \triangle DEF$ means that $\overrightarrow{AB} \rightarrow \overrightarrow{DE}, \overrightarrow{BC} \rightarrow \overrightarrow{EF}$ and $\overrightarrow{CA} \rightarrow \overrightarrow{FD}$.

**EXERCISE 54-2A**

1. Write the correspondence between vertices indicated by a. $\triangle PQR \rightarrow \triangle MLK$;  
b. $\triangle ABC \rightarrow \triangle BCA$. (Notice that the two triangles need not be distinct.)
2. List the correspondences defined between the sides and angles by a. $\triangle ABC \rightarrow \triangle PRQ$;  
b. $\triangle ABC \rightarrow \triangle ACD$.

It should seem reasonable from what was done in Unit I that two triangles $\triangle ABC$ and $\triangle DEF$ are congruent if there is a correspondence between their vertices, say $\triangle ABC \rightarrow \triangle DEF$, such that all pairs of corresponding sides are congruent (of the same length) and all pairs of corresponding angles are congruent (have the same measure).

We also saw in Unit I that in order to copy a triangle (that is, to produce a new triangle congruent to a given one), it was enough to consider only the sides. That is, to copy $\triangle ABC$, we only had to take three segments (for example, made of thin straight sticks) $\overrightarrow{DE}, \overrightarrow{EF}$ and $\overrightarrow{FD}$, whose lengths are given by $DE = AB, EF = BC$, and $FD = CA$. When we fitted these segments together, the resulting triangle $\triangle DEF$ was congruent to $\triangle ABC$ (Fig. 19). Since we are taking as our postulates some of the facts which seem the simplest, we are led to our next postulate.

**POSTULATE 4(a):** If there is a one-to-one correspondence between the vertices of two triangles such that corresponding sides are congruent, then the two triangles are congruent.

Since three sides of one triangle are made congruent to three sides of the other, this is often referred to as the SSS postulate. Writing $\triangle ABC = \triangle DEF$ means that under the correspondence $\triangle ABC \rightarrow \triangle DEF$, all corresponding parts are congruent.

Congruence plays a very important role in our study of geometry, so let us list some of the obvious things that are true of congruent figures. We shall often refer to these properties of congruence in later proofs.

**Properties of Congruence**

1. Any figure is congruent to itself.
2. If a figure is congruent to a second figure, then the second is congruent to the first.
3. If two figures are congruent to the same figure, they are congruent to each other.
4. Corresponding parts of congruent triangles are also congruent.

![Fig. 19](image)
EXERCISE 54-2B

1. Decide which of the following statements are true and give reasons.
   a. \( \triangle ABC \cong \triangle A'B'C' \).
   b. If \( \triangle ABC \cong \triangle DEF \) and \( \triangle DEF \cong \triangle GHK \), then \( \triangle ABC \cong \triangle GHK \).
   c. If \( \triangleABC \cong \triangle DEF \), then \( AB = EF \).
   d. If \( \triangle ABC \cong \triangle DEF \), then \( B'C' = D'E' \).

54.3 Copying angles and triangles

We can now proceed to the verification of the methods for copying an angle and a triangle. Let us begin with copying an angle.

EXERCISE 54-3A

1. Draw an angle \( \hat{A}BC \) so that \( m(\hat{A}BC) < 90^\circ \), and \( BA \) and \( BC \) are about 2 inches. Then using \( B \) as centre, draw a circle with radius \( 1 \frac{1}{4} \) inches. Mark \( A' \) and \( C' \), the points where the circle intersects \( BA \) and \( BC \). Answer the following questions.
   a. What is the length of \( BA' \) (give reasons)?
   b. Put a symbol into the boxes to make the statements true (give reasons).
      \[ BA' \square BC' \]
      \[ BA' \square BC' \]

2. Given the angle and the circle in Figure 20(a) with \( BC' = 1 \frac{1}{4} \) in. and a ray \( DE \) [Fig. 20(b)], draw a circle with centre \( D \) and radius \( BA' \) intersecting \( DE \) in point \( F \). Choose any point \( M \) on this circle and draw the line segments \( DM \) and \( MF \). Answer the following questions.
   a. What is the measure of \( \angle DFM \)?
   b. Is \( \angle DFM \) congruent to \( BC' \)? (Why?)
   c. Is the following statement true?
      \[ \angle A'C' = \angle MF \]
3.

Suppose we are given an angle $\angle ABC$ such that $m(\angle ABC) = 60^\circ$ and a circle with centre $B$ and radius $1\frac{1}{2}$ inches. Suppose we also have a ray $\overline{DE}$ and a circle with centre $D$ and radius $BC'$ which intersects $\overline{DE}$ in a point $F$. Draw a circle of radius $A'C'$ with centre $F$ intersecting the circle with centre $D$ at $G$ and $H$. Answer the following questions.

a. What is the measure of $\overline{GF}$?

b. Give two segments congruent to $\overline{HF}$.

c. $\triangle A'B'C'$ and $\triangle GDF$ are congruent. Why?

d. Find another triangle congruent to $\triangle GDF$.

e. Is $\triangle A'B'C' \rightarrow \triangle HDF$? Why?

f. List all segments shown which are congruent to $\overline{HI}$.

g. What is the measure of $\overline{GDF}$?

h. What is the measure of $\overline{HDF}$?

CONSTRUCTION 2. Given an angle $\angle ABC$ and a ray $\overline{DE}$ in a given plane, locate three points $F$, $G$ and $H$ in the given plane such that $F$ is on $\overline{DE}$ and $GDF = HDF = \angle ABC$.

Method: Draw a circle in the plane of $\angle ABC$ with centre at $B$ and with a convenient radius $r$ (Fig. 22). This circle intersects $\overline{BA}$ at $A'$ and $\overline{BC'}$ at $C'$. Now draw a circle of radius $r$. 

[Diagram of construction process]

Fig. 22
with centre at \( D \) in the given plane. This circle intersects \( \overline{DE} \) at a point \( F \). Now draw a circle of radius \( \overline{A'C'} \) with centre at \( F \) in the given plane. The circles with centres at \( D \) and \( F \) intersect at the two points \( G \) and \( H \). Then \( F, G \) and \( H \) are the desired points.

We next verify that the construction described above actually gives us two copies of \( \triangle ABC \). We must show that \( \triangle ABC = \triangle GDF = \triangle IDF \). It will probably be easier to follow the steps of the verification if we write them in two-column form listing the reasons on the right. We aim at showing first that \( \triangle A'BC' = \triangle GDF = \triangle IDF \), using the SSS postulate.

**THEOREM 2.** The three points \( F, G \) and \( H \) determined by Construction 2 satisfy \( \triangle GDF = \triangle IDF = \triangle ABC \).

**Proof:**

<table>
<thead>
<tr>
<th>Assertion</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \overline{BC'} = \overline{DF} )</td>
<td>1. Both are of length ( r ) since they are radial segments of circles of radius ( r ), and segments of equal length are congruent by Postulate 2(b).</td>
</tr>
<tr>
<td>2. ( \overline{A'B} = \overline{GD} = \overline{HD} )</td>
<td>2. Again all are of length ( r ) and are hence congruent by Postulate 2(b).</td>
</tr>
<tr>
<td>3. ( \overline{A'C'} = \overline{GF} = \overline{HF} )</td>
<td>3. ( \overline{GF} ) and ( \overline{HF} ) are both radial segments of the circle with centre ( F ) and radius ( \overline{A'C'} ).</td>
</tr>
<tr>
<td>4. ( \triangle A'BC' = \triangle GDF = \triangle IDF )</td>
<td>4. The SSS postulate, Postulate 4(a)</td>
</tr>
<tr>
<td>5. ( \overline{A'B} = \overline{GD} = \overline{HD} )</td>
<td>5. Corresponding parts of congruent triangles are congruent.</td>
</tr>
</tbody>
</table>

This completes the proof which we set out to give.

**EXERCISE 54-3B**

1. Given a triangle \( \triangle ABC \) and a ray \( \overline{DE} \) such that \( \overline{AB} \neq \overline{AC} \), draw circle of radius \( \overline{AB} \) with centre \( D \) and intersecting \( \overline{DE} \) at \( F \). Choose a point \( G \) along the circle you have just drawn such that \( \overline{GF} = \overline{CB} \) and draw the segment \( \overline{GD} \). Is \( \triangle DFG = \triangle ABC \)? Why?

2. Given a triangle \( \triangle ABC \) and a ray \( \overline{DE} \) such that \( \overline{AB} \neq \overline{AC} \), draw circle of radius \( \overline{AB} \) with centre \( D \) and intersecting \( \overline{DE} \) at \( F \). Choose a point \( G \) along the circle you have just drawn such that \( \overline{GF} = \overline{CB} \) and draw the segment \( \overline{GD} \). Is \( \triangle DFG = \triangle ABC \)? Why?
Suppose we are given a triangle $\triangle ABC$, a ray $\overrightarrow{DE}$ and a circle with radius $AB$ and centre $D$, intersecting $\overrightarrow{DE}$ at a point $F$. Draw a circle of radius $AC$ and centre $D$. Then choose a point $M$ on the circle just drawn and draw $MD$ and $MF$.

a. What property should $M$ have to make $\triangle DFM$ congruent to $\triangle ABC$?

b. Can you think of an easy way of locating this point $M$ at the proper position?

The next construction is the problem of copying a given triangle. We want to construct a triangle congruent to a given triangle. Stated more precisely, we want the following.

CONSTRUCTION 3. Given a triangle $\triangle ABC$, and a ray $\overrightarrow{DE}$ in a given plane, locate three points $F$, $G$ and $H$ in the given plane such that $F$ is on $\overrightarrow{DE}$ and the two triangles $\triangle DFG$ and $\triangle DFH$ are congruent to $\triangle ABC$.

Here, as in the previous construction, one must not only give the ray $\overrightarrow{DE}$ but also a plane in which the ray lies, for one can construct triangles in each different plane containing the ray.

Method:

Copy the segment $AB$ on $\overrightarrow{DE}$ as in Construction 1; that is, draw a circle (marked $j$ in Figure 23) with radius $AB$ and centre $D$. This intersects $\overrightarrow{DE}$ at $F$. Now draw a circle (marked $k$ in Figure 23) with radius $AC$ and centre at $D$. Finally draw a circle (marked $m$ in Figure 23) which has radius $BC$ and centre at $F$. The circles $k$ and $m$ intersect at the two points $G$ and $H$. Then $F$, $G$ and $H$ are the desired points, for if we draw the segments $DG$, $DH$, $FG$ and $FH$, we get two triangles $\triangle DFG$ and $\triangle DFH$, each congruent to $\triangle ABC$.

THEOREM 3. Construction 3 gives us two triangles $\triangle DFG$ and $\triangle DFH$ each congruent to $\triangle ABC$.

Proof:

To prove that the triangles are congruent, we shall show that the three sides of each are congruent to the three sides of $\triangle ABC$.

<table>
<thead>
<tr>
<th>Assertion</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{DF} = \overline{AB}$</td>
<td>1. $\overline{DF}$ is a radial segment of a circle with radius $AB$, so $DF = AB$. Segments of equal length are congruent by Postulate 2(b).</td>
</tr>
<tr>
<td>$\overline{DG} = \overline{DH} = \overline{AC}$</td>
<td>2. $\overline{DG}$ and $\overline{DH}$ are both radial segments of circle $k$ which has radius $AC$. Thus, $DG = DH = AC$, and Assertion 2 holds by Postulate 2(b).</td>
</tr>
</tbody>
</table>
Assertion

3. \( \overline{FG} = \overline{FH} = \overline{BC} \)

Reason

3. \( \overline{FG} \) and \( \overline{FH} \) are both radial segments of circle \( m \) which has radius \( BC \). Thus, \( \overline{FG} = \overline{FH} = \overline{BC} \) and Assertion 3 holds by Postulate 2(b).

4. \( \triangle DFG = \triangle DFH = \triangle ABC \)

This completes the proof!

**EXERCISE 54-3c**

1. Suppose we are given a circle with centre \( O \) and a diameter \( \overline{AB} \). Let \( C \) and \( D \) be points on the circle on opposite sides of \( \overline{AB} \) such that \( \angle BOC = 50^\circ \) and \( \angle AOD = 30^\circ \).

   Construct an angle \( \hat{A} \hat{C} \hat{D} \) congruent to \( \angle BOC \) such that \(\hat{F}\) and \(\hat{C}\) are on the same side of \(\overline{AB}\). Construct an angle \( \hat{D} \hat{G} \hat{C} \) congruent to \( \angle FOC \) such that \(\hat{G}\) and \(\hat{D}\) are on the same side of \(\overline{AB}\).

   What is \( \angle G\hat{O}B \)?

54-4 The SAS and ASA postulates

The need will arise very often in the next sections to prove that triangles are congruent. The SSS postulate is a very useful tool, but sometimes it is not so easy to show that the three sides of one are congruent to the three sides of the other. In such cases, we can try one of the other methods of copying a triangle used in Unit I. For example, you may recall that the copy can be made using two sides and their included angle (Fig. 24). Another method uses two angles

![Fig. 24](image1)

and their common side (Fig. 25). These two methods lead us

![Fig. 25](image2)

to take the following as postulates.
POSTULATE 4(b): If there is a one-to-one correspondence between the vertices of two triangles such that two sides and their included angle in one are congruent to the corresponding two sides and their included angle in the other, the two triangles are congruent. (We shall often refer to this as the SAS postulate.)

POSTULATE 4(c): If there is a one-to-one correspondence between the vertices of two triangles such that two angles and their common side in one are congruent to the corresponding two angles and their common side in the other, then the two triangles are congruent. (We shall refer to this as the ASA postulate.)

It should be pointed out here that some geometry books assume only one of the three parts of Postulate 4, say 4(b), as a postulate and prove the other two, using it and some of the other postulates which were assumed. In order to speed up our study, we are going to assume more postulates than are absolutely necessary. This reduces the number of theorems we shall have to prove before we get to the interesting ones for our constructions.

**EXERCISE 54-4A**

1. Determine in which of the following cases $\triangle ABC = \triangle DEF$ and state the postulate used to justify the conclusion.
   a. $AB = DE$, $BC = EF$ and $AC = DF$.
   b. $AB = DE$, $\triangle ABC = \triangle DEF$ and $BC = EF$.
   c. $AB = DE$, $\triangle ABC = \triangle DEF$ and $AC = DF$.
   d. $\triangle ABC = \triangle DEF$, $BCA = EFD$ and $CAB = FDE$.
   e. $\triangle ABC = \triangle DEF$, $BC = EF$ and $\triangle CAB = \triangle FDE$.

2. Suppose we are given a circle with radius $r$, centre $O$ and diameter $AB$. Let $\triangle ADO$ and $\triangle BOC$ be two congruent triangles such that $m(\angle COB) = 35^\circ$. Let $m(\angle BDE) = 70^\circ$.
   Say whether or not $\triangle AOE = \triangle DOC$ and why.
Chapter 55
PERPENDICULAR LINES

55-1 Construction of perpendicular lines

We shall study perpendicular lines in this section. Let us begin with the definitions of a few useful words.

If two lines intersect at a point \( Q \), they form four rays, each with endpoint \( Q \). We can label one pair of opposite rays \( QA \) and \( QB \), and the other pair of opposite rays \( QC \) and \( QD \) (Fig. 26). There are four angles formed by pairs of these rays which are not opposite each other. These are \( AQC \), \( AQD \), \( BQC \), and \( BQD \). We shall refer to these four angles formed by the two intersecting lines.

**DEFINITION 19:** Two intersecting lines are **perpendicular** if an angle formed by the two lines is a right angle. The symbol \( \perp \) will mean **is perpendicular to**.

Since a line segment or a ray is contained in a unique line, we can speak about perpendicularity of any combination of line segments, rays or lines as meaning that the lines containing them are perpendicular.

In Definition 19, we asked only that one of the four angles formed by the intersecting lines be a right angle. But it is not hard to see that if one such angle is a right angle, so are the other three. (Can you prove this?) For example, if in Figure 27, \( AQC \) is a right angle, then \( m(AQC) = 90^\circ \). But \( AQC \) and \( CQB \) form a linear pair and their measures add to \( 180^\circ \). Thus, since one has measure \( 90^\circ \), the other must also have measure \( 90^\circ \). Thus, \( CQB \) is also a right angle. But \( CQB \) and \( BQD \) form a linear pair, with \( m(CQB) = 90^\circ \), so \( m(BQD) = 90^\circ \). Applying the same argument to \( BQD \) and \( QAQ \), we find that all four angles are right angles.

Remember that when we began this unit, we said we wanted to see how one convinces his pupils that the constructions for some of...
the figures actually give what we say they do. One example given there was the construction of a perpendicular line. We can now look into the problem of raising a perpendicular from a given point on a given line.

**EXERCISE 55-1A**

Suppose we are given a line segment $AB$, a circle of radius $r$ with centre $C$ on $AB$ and intersecting $AB$ at $A'$ and $B'$, and two other circles of radius $s$ ($s > r$) having $A'$ and $B'$ as centres and intersecting each other at $D$ and $D'$.

1. What name would you give to the sum $A'D + B'D$?
2. Complete the following equation by filling in the box with a number of degrees.
   \[
   m(A'CD) + m(B'CD) = \Box
   \]
3. According to Definition 15, say what condition is required so that $m(A'CD) = m(B'CD) = 90^\circ$.
4. Prove that $\triangle A'CD = \triangle B'CD$.
5. How could you use the fact that $\triangle A'CD$ and $\triangle B'CD$ are congruent to verify the condition laid down in Question 3?

**CONSTRUCTION 4.** Given a line segment $AB$ and a point $C$ on $AB$, construct a segment $CD$ in a given plane through $AB$ such that $\overline{CD} \perp \overline{AB}$.

*Method:* Draw a circle of some convenient radius, say $r$, with centre $C$. This circle intersects $\overline{CA}$ at $A'$ and $\overline{CB}$ at $B'$ (Figure 28). Now take a radius $s$ larger than $r$ and draw two
circles of radius $s$, one with centre at $A'$ and the other with centre at $B'$. These circles intersect at two points $D$ and $D'$. Both $\overline{CD}$ and $\overline{CD'}$ are perpendicular to $\overline{AB}$.

We shall verify that $\overline{CD}$ is perpendicular to $\overline{AB}$. (The proof that $\overline{CD'}$ is perpendicular to $\overline{AB}$ is exactly the same. One merely replaces $D$ by $D'$ where it appears in the proof.)

**THEOREM 4.** The segment $\overline{CD}$ constructed above is perpendicular to $\overline{AB}$.

**Proof:**

<table>
<thead>
<tr>
<th>Assertion</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $\overline{AC} = \overline{BC}$</td>
<td>1. $\overline{AC}$ and $\overline{BC}$ are radial segments of a circle of radius $r$ so both segments have length $r$. By Postulate 2(b), segments of the same length are congruent.</td>
</tr>
<tr>
<td>2. $\overline{A'D} = \overline{B'D}$</td>
<td>2. $\overline{A'D}$ and $\overline{B'D}$ are radial segments of circles of radius $s$ and hence both have length $s$. By Postulate 2(b), segments of the same length are congruent.</td>
</tr>
<tr>
<td>3. $\overline{CD} = \overline{C'D}$</td>
<td>3. Any segment is congruent to itself.</td>
</tr>
<tr>
<td>4. $\triangle A'CD = \triangle B'C'D$</td>
<td>4. We have shown above that the corresponding sides are congruent, so the congruence of the triangles follows from the SSS postulate, Postulate 4(a).</td>
</tr>
<tr>
<td>5. $\angle A'CD = \angle B'C'D$</td>
<td>5. Corresponding parts of congruent triangles are also congruent.</td>
</tr>
<tr>
<td>6. $\angle A'CD$ and $\angle B'C'D$ are right angles.</td>
<td>6. $\angle A'CD$ and $\angle B'C'D$ are a linear pair of angles and they are congruent. But this is exactly how we defined right angle in Definition 15.</td>
</tr>
<tr>
<td>7. $\overline{CD} \perp \overline{AB}$</td>
<td>7. Two segments are perpendicular if an angle formed at the intersection of the lines containing them is a right angle. (See the remark following Definition 19.)</td>
</tr>
</tbody>
</table>

This completes the proof!

We have raised a perpendicular from a line segment at a point on the line. Now we can take a point not on a given line and drop a perpendicular from that point to the given line.

**EXERCISE 55-1B**
Suppose we are given two line segments $\overline{AB}$ and $\overline{CD}$ intersecting at $E$ such that by joining the endpoints of the line segments as shown, we obtain $\overline{AC} = \overline{AD}$ and $\overline{CB} = \overline{DB}$.

We shall see if we can prove that $\overline{CEA} = \overline{DEA}$.

There will be three steps in the solution:

(a) List the known data.
(b) Ask questions about the problem.
(c) Write a proof.

**Known data:**
- $\overline{CD} \cap \overline{AB} = E$,
- $\overline{AC} = \overline{AD}$,
- $\overline{CB} = \overline{DB}$.

Here are some reasonable questions to ask yourself about the problem. Answer them.

**Questions:**

1. In what triangles should angles $\overline{CEA}$ and $\overline{DEA}$ be considered so as to prove that they are congruent?
2. What property should these triangles have to give $\overline{CEA} = \overline{DEA}$?
3. How can we prove that $\triangle CEA = \triangle DEA$?
4. In what figures should we consider $\triangle C\hat{A}E$ and $\triangle D\hat{A}E$ so as to prove their congruence?
5. What property should these two triangles have so as to give $\triangle C\hat{A}E = \triangle D\hat{A}E$?
6. How can we prove that $\triangle ABC = \triangle BDA$?
7. Does the postulate SSS work?
8. What proofs need to be written down?

**Actual solution:**

Now do the proofs in order.

**NOTE:** When you are solving problems, the question and answers are often done mentally or on scrap paper, but listing the known data and the actual proofs must be clearly set down on your copy.

**CONSTRUCTION 5.** Given a line segment $\overline{AB}$ and a point $C$ not on $\overline{AB}$, construct a line segment $\overline{CD}$ such that $\overline{CD} \perp \overline{AB}$.

**Method:** In the plane containing $C$ and $\overline{AB}$, draw a circle with center $C$ and radius $r$ large enough so that the circle intersects $\overline{AB}$ in the two points $A'$ and $B'$ (Fig. 29). Now with
radius \( r \), draw two circles, one with centre at \( A' \) and the other with centre at \( B' \). These circles intersect at \( C \) and at another point \( D \), on the side of the line \( \overline{AB} \) not containing \( C \). \( \overline{CD} \) is then perpendicular to \( \overline{AB} \). (It was not necessary to take the radii of the last two circles drawn to be \( r \). We could have made both radii any number \( s \), chosen large enough so that the two circles intersect in two points. We then take as \( D \) the point of intersection on the side of \( \overline{AB} \) not containing \( C \). The verifications in both cases are essentially the same.)

**THEOREM 5.** The segment \( \overline{CD} \) constructed above is perpendicular to \( \overline{AB} \).

**Proof:**

We shall do this in two steps.

We first show that \( \triangle CA'D \equiv \triangle CB'D \).

We then will be able to show that \( \triangle A'EC \equiv \triangle B'EC \), from which we deduce that \( A'EC = B'EC \). This will tell us that \( \overline{CD} \perp \overline{AB} \).

**Assertion**

1. \( \overline{CA'} = \overline{CB'} \)

2. \( \overline{A'D} = \overline{B'D} \)

3. \( \overline{CD} = \overline{CD} \)

4. \( \triangle CA'D \equiv \triangle CB'D \)

5. \( A'\widehat{CD} = B'\widehat{CD} \)

6. \( \overline{CE} = \overline{CE} \)

7. \( \triangle A'EC \equiv \triangle B'EC \)

8. \( A'\hat{EC} = B'\hat{EC} \)

9. \( A'\hat{EC} \) is a right angle.

10. \( \overline{AB} \perp \overline{CD} \)

This completes the proof!

**EXERCISE 55-1C**

Explain why it was not necessary that \( DB' = CB' \).
55-2 Midpoints and angle bisectors

The methods used to construct perpendiculars given above are very similar to methods used to make some other figures. We shall look at some of these next. The first of these is finding the midpoint of a segment.

**DEFINITION 20:** A point C on AB is called the midpoint of AB if AC = BC. Thus the midpoint divides the segment into two segments of equal length.

**CONSTRUCTION 6.** Locate the midpoint C of a segment AB.

*Method:* Draw two circles with centres A and B with a radius r so chosen that the circles intersect in two points D and E. The segment DE intersects AB in the point C, which is the midpoint of AB (Fig. 30).

![Fig. 30](image)

**EXERCISE 55-2A**

Divide the page of paper into two columns, heading one "Assertion" and the other "Reason". Write the verification that the method described above gives a point C that is the midpoint of AB; that is, AC = BC. The argument is similar to the one given in the verification of Construction 5.

Construction 6 has actually given us more than we had asked of it. The segment DE which we constructed not only intersects AB at its midpoint but it is also perpendicular to AB. A segment that is perpendicular to a given segment AB and also passes through its midpoint is called a **perpendicular bisector** of AB.

**EXERCISE 55-2B**

Continue the argument which you wrote to verify Construction 6 to show that DE is perpendicular to AB.

**DEFINITION 21:** A triangle is called an **equilateral triangle** if its three sides are congruent to each other.

**CONSTRUCTION 7.** Given AB, construct an equilateral triangle ΔABC.

*Method:* If the radius r used in Construction 6 is taken to be equal to AB, the triangle ΔABD constructed there is equilateral. Verify this!
Just as we divided a segment into two congruent parts with its midpoint, we shall now divide an angle into two congruent parts. This is called bisecting the angle.

**DEFINITION 22:** A ray $\overrightarrow{BD}$ interior to an angle $\angle ABC$ is called the *bisector* of the angle if $\overrightarrow{ABD} = \overrightarrow{CBD}$.

**EXERCISE 55-2C**

**CONSTRUCTION 8.** Construct the bisector $\overrightarrow{BD}$ of an angle $\angle ABC$.

**Method:** Select a convenient radius $r$ and draw a circle with centre $B$. This circle intersects $\overrightarrow{BA}$ at $A'$ and $\overrightarrow{BC}$ at $C'$. Now with a convenient radius $s$, draw two circles, one with centre at $A'$ and the other with centre at $C'$. These two circles intersect in two points. At least one...
of these two points lies inside \( \triangle ABC \). Call this point \( D \) and draw the line segment \( BD \). Then \( BD \) is the bisector of \( \triangle ABC \). (Notice that if \( r = s \), the two circles of radius \( s \) with centres at \( A' \) and \( C' \) intersect at \( B \) and at another point interior to \( \triangle ABC \), which we then name \( D \).)

We must now verify that this construction does give us \( \angle AB\hat{D} = \angle CB\hat{D} \).

**THEOREM 8.** Construction 8, \( \angle AB\hat{D} = \angle CB\hat{D} \).

**Proof:**

<table>
<thead>
<tr>
<th>Assertion</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( BA' = BC' )</td>
<td>1. Both are radial segments of a circle of radius ( r ) and hence both have the same length ( r ). But by Postulate 2(b), segments of the same length are congruent.</td>
</tr>
<tr>
<td>2. ( A'D = C'D )</td>
<td>2. Both are radial segments of a circle of radius ( s ) and hence have the same length ( s ). Again by Postulate 2(b), segments of the same length are congruent.</td>
</tr>
<tr>
<td>3. ( BD = BD )</td>
<td>3. A line segment is congruent to itself.</td>
</tr>
<tr>
<td>4. ( \triangle A'BD = \triangle C'BD )</td>
<td>4. We have shown in steps 1, 2 and 3 that three sides of the one triangle are congruent to the corresponding three sides of the other triangle. The SSS postulate, (Postulate 4(a), tells us that the two triangles are congruent.</td>
</tr>
<tr>
<td>5. ( A'\hat{D} = C'\hat{D} )</td>
<td>5. Corresponding angles of congruent triangles are also congruent.</td>
</tr>
</tbody>
</table>

This completes the proof!

**EXERCISE 55-2D**

We have seen how to construct an angle whose measure is \( 90^\circ \). (We need only raise a perpendicular from a given segment.) If this angle of measure \( 90^\circ \) is bisected, what is the measure of each of the two parts? If you bisect one of the resulting angles, what is the measure of each of the angles obtained? What other angles can you construct? Can you construct an angle whose measure is \( 67 \frac{1}{2}^\circ \)?

**55-3 Isosceles triangles**

**DEFINITION 23:** If two sides of a triangle are congruent to each other, the triangle is called an *isosceles triangle*. The remaining side is called the *base* of the isosceles triangle, and the two angles of the triangle which have the base as a common side are called *base angles*.

In this definition of isosceles triangle, we have stated that two sides are congruent. We have made no statement about the angles. It is interesting to know that we can show that the base angles are also congruent. We shall now write a proof of this pretty fact, which we shall call a theorem. Since we may also consider each of the constructions which we have so far verified as theorems, we shall call this Theorem 9.
THEOREM 9. The base angles of an isosceles triangle arc congruent.

We are given an isosceles triangle, say \( \triangle ABC \) with \( AB = AC \). We are asked to prove that \( \hat{A}BC = \hat{A}CB \) (Fig. 32)

Proof:

**Assertion**
1. Let \( D \) be the midpoint of the base \( BC \).

**Reason**
1. The midpoint \( D \) divides \( BC \) into two congruent segments.

![Fig. 32](image)

2. \( \overline{AB} = \overline{AC} \)
3. \( \overline{AD} = \overline{AD} \)
4. \( \triangle ABD = \triangle ACD \)
5. \( \hat{A}BD = \hat{A}CD \)

This completes the proof!

There is another way to prove this theorem which does not make use of any additional line segments such as \( AD \). You may be interested in seeing such a proof, so it will be given below.

**Another proof of Theorem 9:**

**Assertion**
1. Consider \( \triangle ABC \) and \( \triangle ACB \); that is, take the same triangle but let the vertices correspond as follows: \( A \to A, B \to C \) and \( C \to B \). We then have \( \overline{AB} = \overline{AC} \).

2. \( \overline{AC} = \overline{AB} \)
3. \( \overline{BC} = \overline{CB} \)
4. \( \triangle ABC = \triangle ACB \)
5. \( \hat{A}BC = \hat{A}CB \)

This completes the proof!

DEFINITION 24: If one angle of a triangle is a right angle, the triangle is called a right-angled triangle.
**EXERCISE 55-3A**

1. Prove that the three angles of an equilateral triangle are congruent to each other.

2. Construct an isosceles right-angled triangle in which each of the two congruent sides has length 2 inches. Verify that the construction actually gives an isosceles right-angled triangle.

**55-4 Vertical angles**

We shall end this chapter with another interesting and useful fact concerning intersecting lines.

**DEFINITION 25:** Two angles are called vertical angles if the edges of one angle are opposite the edges of the other angle.

For example, in Figure 33, $\angle AQC$ and $\angle BQD$ are vertical angles, for $QA$ is opposite $QB$ and $QC$ is opposite $QD$. It is quite obvious from the figure that vertical angles are congruent. It is interesting to see how this fact follows immediately from our postulate about the measure of an angle, that is, from Postulate 3. We shall show how one writes a proof of this fact based on Postulate 3. We shall call this Theorem 10.

**THEOREM 10.** Vertical angles are congruent.

We shall assume that the angles are named $\angle AQC$ and $\angle BQD$ (as in Figure 33) and that $QA$ is opposite $QB$ and $QC$ is opposite $QD$. We shall prove that $\angle AQC = \angle BQD$. The proof is based upon the simple observation that $\angle AQB$ and $\angle CQD$ are straight angles.

**Proof:**

<table>
<thead>
<tr>
<th>Assertion</th>
<th>Reason</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m(\angle AQB) + m(\angle CQD) = 180^\circ$</td>
<td>1. $\angle AQB$ and $\angle CQD$ form a linear pair of angles. We have already seen how the assertion then follows from Postulates 3(c) and 3(d).</td>
</tr>
</tbody>
</table>
| $m(\angle AQC) = 180^\circ - m(\angle AQB)$ | 2. $\angle AQC$ and $\angle AQB$ also form a linear pair.
| $\angle BQD = \angle CQD$ | 3. These are equivalent ways of writing the statements in steps 1 and 2. |
| $m(\angle BQD) = m(\angle AQC)$ | 4. Both are equal to $180^\circ - m(\angle AQB)$ by step 3. |
| $\angle AQC = \angle BQD$ | 5. Angles having equal measures are congruent [Postulate 3(b)]. |

This completes the proof!

If two lines intersect, two pairs of vertical angles are formed. We may look at Figure 33 as being two lines $\overline{AB}$ and $\overline{CD}$, which intersect at $Q$. They form two pairs of vertical angles, namely, the pair $\angle AQC$ and $\angle BQD$ and the pair $\angle AQB$ and $\angle CQD$. The angles in each pair are thus congruent.
EXERCISE 55-4A

1. Draw a line segment $\overline{AB}$, 4 inches long. Locate points $C$ and $D$ on $\overline{AB}$ such that $AD = BC = 1$ inch. Construct a perpendicular $\overline{DE}$ to $\overline{AB}$ at the point $D$ such that $DE = 1$ inch. On the same side of $\overline{AB}$ as $E$, construct a perpendicular $\overline{CF}$ to $\overline{AB}$ at $C$ such that $CF = 1$ inch. Suppose line segments $\overline{EC}$ and $\overline{FD}$ intersect at $G$.

   a. Using triangles $\triangle IDC$ and $\triangle CFD$, prove that $\angle IDC = \angle CFD$.
   b. Using $\triangle EFC$ and $\triangle FED$, as well as the previous triangles, prove that $\angle EFD = \angle FCE$.
   c. Prove that $\triangle EGD = \triangle FGC$.

2. Two of the angles represented on the above figure are supplementary; say which ones.
56-1 Similar triangles

In making a scale drawing, one makes a figure which has the same shape but not necessarily the same size as the original. It may be necessary to increase or decrease the size of the figure when making the drawing. For example, one may make a scale drawing in which the figure in the drawing is one-tenth as large as the original figure. Each segment in the drawing will be $\frac{1}{10}$ as large as the corresponding segment in the original figure. We call $\frac{1}{10}$ the scale factor.

Let us look at triangles to fix our ideas somewhat better. When do two triangles have the same shape? It is clear that we would say that triangles (a) and (b) of Figure 34 have the same shape (are similar) but that (a) and (c) do not.

![Fig. 34](image)

Each of the angles in (a) is congruent to a corresponding angle of (b), while each side of (a) is $\frac{1}{2}$ as large as the corresponding side of (b). This leads us to the definition of similarity of triangles.

**DEFINITION 26:** Two triangles are similar if there is a one-to-one correspondence between their vertices such that both of the following conditions hold:

(a) corresponding angles are congruent, and

(b) there is a positive real number $k$ (the scale factor) such that the lengths of the sides of one triangle are $k$ times the lengths of the corresponding sides of the other triangle.

The symbol $\sim$ shall mean is similar to.

When condition (b) holds, we say that the lengths of the sides of one triangle are proportional to the lengths of the corresponding sides of the other triangle. The scale factor $k$ is often called the constant of proportionality.
It is certainly reasonable to think that we can enlarge or decrease the size of a triangle as much as we wish, preserving its shape; that is, that we can choose any scale factor \( k \) we wish and find a triangle similar to a given triangle but with scale factor \( k \). This fact will be our Similarity postulate, which we state below.

**POSTULATE 5**: Given \( \triangle ABC \) and a positive real number \( k \), there is a triangle \( \triangle A'B'C' \) similar to \( \triangle ABC \) with \( A'B' = k \times AB \).

**EXERCISE 56-1A**

1. Here are three triangles with the measures of some of their parts.

   ![Triangle Diagram]

   - **a.** Is triangle III similar to triangle II? Why?
   - **b.** According to Postulate 5, can you say immediately that triangle I is also similar to triangle II? Why?
   - **c.** If you compare I and III, what can you say about them? (Try to remember all that we have said already about triangles.)
   - **d.** Can you say that \( I \mid\mid II \)? Why?

In order to prove the similarity of two triangles, it would appear that we have to prove (a) the corresponding angles congruent and (b) the lengths of corresponding sides proportional. Actually, we can prove that two triangles are similar by doing much less. We shall now show that it is enough to prove either (a) or (b) and that the other one then follows. We shall first show that when (b) is true, then (a) must also be true, and the triangles are similar. This means that in order to prove that two triangles are similar, one has only to show that the three sides of one are proportional to the three sides of the other. This is our next theorem.

**THEOREM 11.** Given \( \triangle ABC \) and \( \triangle A'B'C' \), such that for some positive real number \( k \),

\[
A'B' = k \times AB, \quad B'C' = k \times BC \quad \text{and} \quad A'C' = k \times AC,
\]

then \( \triangle ABC \mid\mid \triangle A'B'C' \). In other words, if the lengths of corresponding sides are proportional, the two triangles are similar.

**Proof:**

(Make your own drawing as you read the proof!)

**Assertion**

1. There is a triangle \( \triangle A*B*C* \) such that \( \triangle A*B*C* \mid\mid \triangle ABC \) and \( A*B* = k \times AB \).
2. \( B*C* = k \times BC \) and \( C*A* = k \times CA \).

**Reason**

1. This is what the Similarity postulate (5) tells us.
2. Since \( \triangle A*B*C* \mid\mid \triangle ABC \) and \( A*B* = k \times AB \), Assertion 2 follows from the fact that corresponding sides of similar triangles have proportional lengths (Definition 26).
3. $A \hat{B} \hat{C} \hat{*} = A \hat{A} \hat{B} \hat{C}$, $B \hat{C} \hat{A} \hat{*} = B \hat{C} \hat{A}$ and $C \hat{A} \hat{B} \hat{*} = C \hat{A} \hat{B}$.


5. $\triangle A'B'C' = \triangle A'B*C$.


8. $\triangle A'B'C' \parallel \triangle ABC$

This completes the proof!

Thus, we can prove that the two triangles are similar by showing that their sides are proportional. We pointed out already that it is also enough to show that corresponding angles are congruent. In fact we can do even better than that. We shall next prove that it is enough to show that only two angles of one triangle are congruent to two angles of the other triangle in order to prove that the triangles are similar. This is our next theorem. Before proving it, consider the following exercises.

**EXERCISE 56-1B**

1.

Here are two triangles $\triangle ABC$ and $\triangle A'B'C'$ with some of their parts measured.

a. Construct a third triangle $\triangle A*B*C*$ similar to $\triangle A'B'C'$ with $k = \frac{AB}{A'B'}$ and such that $A*B* = k \times A'B'$.

b. Since you have constructed $\triangle A*B*C*$ similar to $\triangle A'B'C'$, what can you say about the angles of $\triangle A*B*C*$? Mark their measures on your figure.
c. What is the measure of $A*B*$?
d. If you compare $\triangle ABC$ to $\triangle A*B*C*$, what can you say?
e. Does the answer to Question d help you to conclude that $\triangle ABC \parallel \triangle A'B'C'$?

2.

a. Find the fraction $\frac{B'C'}{BC}$.
b. Find the fraction $\frac{A'B'}{AB}$.
c. Can you say from a and b why $\triangle A'B'C'$ is not similar to $\triangle ABC$?
d. Construct a triangle $A*B*C*$ similar to $\triangle ABC$ with $k = \frac{B'C'}{BC}$ such that $B*C* = k \times BC$. (Measure the figure in centimetres.)
e. Construct a triangle $\triangle A''B''C''$ similar to $\triangle ABC$ with $k = \frac{A'B'}{AB}$ such that $A''B'' = k \times AB$. (Make your measurements in centimetres.)

THEOREM 12: Given $\triangle ABC$ and $\triangle A'B'C'$, with $\triangle ABC \sim \triangle A'B'C'$ and $\triangle ABC \sim \triangle A'B'C'$, then $\triangle A'B'C' \parallel \triangle ABC$. In other words, if two angles of one triangle are congruent to two angles of another, the triangles are similar.

Proof: (Make a drawing as you read the proof.)

The idea of the proof is the following. We first take a new triangle $A*B*C*$ similar to $\triangle A'B'C'$ with scale factor $k$. We show that $\triangle A*B*C*$ is also similar to $\triangle ABC$ with scale factor $k$.

Assertion

1. $k = \frac{A'B'}{AB}$ is a positive real number. There is a triangle $\triangle A*B*C*$ similar to $\triangle ABC$ such that $A*B* = k \times AB$.
2. $B*C* = k \times BC$
   $C*A* = k \times CA$

Reason

1. This follows from the Similarity postulate (5).
2. Definition 26(b)
3. Definition 26(a)
4. We were given that $\triangle ABC \sim \triangle A'B'C'$ and we have shown in step 3 that $\triangle ABC \sim \triangle A'B'C'$. Angles congruent to the same angle are congruent. The other part of Assertion 4 follows in the same way.
5. \( A^*B^* = A'B' \)

6. \( A^*B^* = A'B' \)
7. \( \triangle A^*B^*C^* = \triangle A'B'C' \)

8. \( \frac{B^*C^*}{B'C'} = \frac{A^*A'}{C'A'} \) and \( \frac{B^*C^*}{B'C'} \) and \( \frac{C^*A^*}{C'A'} \).

9. \( \overrightarrow{B^*C^*} = \overrightarrow{BC} \) and \( \overrightarrow{C'A'} \).

10. \( A'B' = k \times AB, \) \( B'C' = k \times BC \) and \( C'A' = k \times CA. \)

11. \( \overrightarrow{ABC} \parallel \overrightarrow{A'B'C'}. \)

This completes the proof!

Let us now revise the method given in Unit IX for constructing a triangle similar to a given triangle.

CONSTRUCTION 13. Given \( \triangle ABC \) and a segment \( \overrightarrow{A'B'} \) on a given plane, locate a point \( C' \) in the given plane such that \( \overrightarrow{ABC} \parallel \overrightarrow{A'B'C'}. \) (How many such points are there?)

Method:

Draw ray \( \overrightarrow{A'D'} \), so that \( B'A'D' \) is a copy of \( BAC \). Then draw \( \overrightarrow{B'E'} \) so that \( A'B'E' \) is a copy of \( AB'C' \). The rays \( \overrightarrow{A'D'} \) and \( B'E' \) intersect at the desired point \( C' \) (Fig. 35).

Before verifying that this construction actually does what we claim, let us do an exercise.

**Exercise 56-1C**

1. Draw a triangle \( \triangle ABC \) and a line segment \( \overrightarrow{AB} \) with the measures indicated in the figure. Using Construction 13, locate a point \( C' \) in your drawing such that \( \overrightarrow{ABC} \parallel \overrightarrow{A'B'C'}. \)
a. Test the similarity of the two triangles by actual measuring.
b. List what conditions are sufficient to make a triangle similar to another.
c. Are some of these conditions realized here?

THEOREM 13. Construction 13 gives us \( \triangle A'B'C'\) which is similar to \( \triangle ABC \).

Proof: In \( \triangle ABC \) and \( \triangle A'B'C' \), we have constructed \( A'B'C' \cong ABC \) and \( B'A'C' \cong BAC \).

Thus, two angles of one triangle are congruent to two angles of the other triangle, and Theorem 12 then tells us that the triangles are similar. This completes the proof.

Notice that we have written this proof in paragraph style instead of the two-column style. Both ways of writing proofs are acceptable. When writing a proof in paragraph style, you must also state the reasons for your assertions and be sure that the steps follow one another logically.

**EXERCISE 56-ID**

1. One day a man called Olajide went to see one of his friends, Temida, and boasted that he could measure the height of a building without making any measurement on the building itself. He wanted to prove it, and they called on their common friend Akpa to be a witness.

Olajide said that he needed an instrument he had made himself out of two sticks, one 4 feet long and a smaller one, rotating at the end of the long one. More than this he needed only a peg and a piece of string. He would tie one end of the string to one end of the short stick. Then he would drive the peg into the ground in a position so that when the other end of the string is tied to the peg, he could sight the top of the building along the short stick.

Olajide showed his friend that he needed only the following three measurements:

1. The height of the instrument (4 feet)
2. The length CI in feet
3. The length CB in feet

And with this he would know the height \( AB \) of the building. Can you explain how he is able to do this?

For a certain building, Olajide found that \( CI = 40 \) feet and \( CB = 160 \) feet. What was the height of the building?
3. Garba says that he has his own way to find the answer to that problem. He would not compute with these three figures, but instead make a scale drawing of the situation. All he needs is the length $CB$ and the measure of $\hat{B}A$. He would proceed as follows (look at the figure).

(1) On a piece of paper he would draw a segment, $C'B'$ let us say, of a convenient length.
(2) At the endpoint $C'$ he would draw an angle $\hat{B}'\hat{A}'$ congruent to $\hat{B}A$.
(3) At endpoint $B'$ he would raise a perpendicular $B'D'$ intersecting $C'A'$ at $E'$.
(4) Then he would say that $\frac{B'E'}{C'B'} = \frac{AB}{CB}$. He could measure the length of $B'E'$ and compute the height $AB$ of the building.

a. Suppose that $C'B'$ is 4 inches, $CB$ is 160 feet, and $m(\hat{A}'\hat{C}'\hat{B}') = 52^\circ$. Make the scale drawing and compute $AB$.

b. Is the answer to part a the same as you obtained by using Olajide's method? Why?

4. Suppose you are on the side of a river and that you want to measure the width of the river between points $A$ and $B$. But you are on the side of point $B$ and you cannot cross the river.

Design an instrument somewhat similar to Oljide's instrument which would help you to measure the width of the river. Say how you could apply both Olajide's and Garba's method of computation to find the width $AB$.

5. Make up measurements that one might obtain in Question 4, and compute the width of the river.

56-2 Parallel lines

DEFINITION 27: Two lines in the same plane are parallel if they have no points in common. The symbol $\parallel$ means "is parallel to".

How do we decide whether two lines are parallel? Since two lines extend indefinitely in both directions, how can we be sure that they do not intersect very far out. It would be good to have a test which we could apply to the two lines, which does not involve inspecting the whole of the lines to see whether they intersect. Such a test is easy to find. One observes that parallel lines must have the same "direction", so to speak, so that if one of them is perpendicular to some line, the other must also be perpendicular to that line (Fig. 36). This leads us to our next postulate.
POSTULATE 6: Two coplanar lines are parallel if there is a line perpendicular to both of them.

Another question we shall ask is how many lines can one draw parallel to a given line \( k \) and passing through a given point \( P \) (Fig. 37). The important fact that there is only one such line is called the Parallel postulate. It is our seventh postulate.

Fig. 36

Fig. 37

POSTULATE 7 (Parallel postulate): There is one and only one line parallel to a given line and passing through a given point not on the given line.

We shall say that two line segments or rays are parallel if the lines containing them are parallel. Postulate 6 gives us a very good way of constructing a line parallel to a given line.

CONSTRUCTION 14. Given a line segment \( \overline{AB} \) and a point \( C \) not on \( \overline{AB} \), construct a line segment \( \overline{CD} \) through \( C \) and parallel to \( \overline{AB} \).

Method: Drop a perpendicular \( \overline{CE} \) from \( C \) to the line \( \overline{AB} \). Next raise a perpendicular \( \overline{CD} \) to \( \overline{CE} \) from \( C \). Then \( \overline{CD} \parallel \overline{AB} \) (Fig. 38).

THEOREM 14. The line segment \( \overline{CD} \) constructed above is parallel to \( \overline{AB} \).

Proof: The line segments \( \overline{AB} \) and \( \overline{CD} \) have both been constructed to be perpendicular to \( \overline{CE} \). Thus Postulate 6 tells us that \( \overline{CD} \parallel \overline{AB} \), and the proof is finished.
Postulate 6 says that two coplanar lines are parallel if there is a line perpendicular to both of them. Now suppose we take some other line perpendicular to one of the given two parallel lines. Will it necessarily be perpendicular to the other? Our intuition says that it will. We shall now see how this can be proved!

**Theorem 15.** If two lines are parallel, any coplanar line which is perpendicular to one is also perpendicular to the other. In other words, given $\overline{AB} \parallel \overline{CD}$ and $EF \perp \overline{AB}$, then $EF \perp \overline{CD}$.

**Proof:** First of all, how do we know the line $EF$ which is perpendicular to $\overline{AB}$ at $E$, actually intersects $\overline{CD}$? If it did not intersect $\overline{CD}$, then $EF$ is parallel to $\overline{CD}$, for two lines which do not intersect are parallel. Furthermore, $EF$ passes through $E$. But $\overline{AB}$ is also parallel to $\overline{CD}$ and passes through $E$. By Postulate 7, there is only one line parallel to $\overline{CD}$ passing through $E$, so $\overline{AB}$ and $EF$ would have to coincide. But this is impossible, since $\overline{AB}$ and $EF$ were given to be perpendicular. Thus, the assumption that $EF$ and $CD$ do not intersect leads to an impossible situation, so they must intersect and we call their point of intersection $F$.

![Diagram](image)

Now we show that $EF \perp \overline{CD}$. From Construction 4 we know that we can construct a line segment $FG$ which is perpendicular to $EF$ and passes through $F$. Then $EF$ is perpendicular to both $\overline{AB}$ and to $\overline{FG}$. Postulate 6 tells us that $\overline{FG}$ and $\overline{AB}$ are parallel. We now have two lines, $\overline{FG}$ and $\overline{CD}$, both passing through $F$ and parallel to $\overline{AB}$. Postulate 7 tells us that these two lines, $\overline{FG}$ and $\overline{CD}$, must be the same line, for there is only one line through $F$ parallel to $\overline{AB}$. Since $FG$ was constructed perpendicular to $EF$, the line $\overline{CD}$ must also be perpendicular to $EF$ and we have completed the proof that if $EF$ is perpendicular to $\overline{AB}$, it is also perpendicular to $\overline{CD}$.

The proof of Theorem 15 has a novel feature that has not appeared in earlier proofs. In the very first step, we wanted to show that $EF$ intersects $\overline{CD}$. There are two possibilities: either it does, or it does not, intersect $\overline{CD}$. We first supposed it did not intersect $\overline{CD}$ and arrived at an impossible situation. What possibility did that leave us? Only that $EF$ and $\overline{CD}$ do intersect. Thus, we conclude that they intersect. Such a proof which shows that the denial of what we want to prove leads to a contradiction is called an "indirect proof." It frequently is easier to give an indirect proof than to show directly that a statement is true.

**Definition 28:** A line which cuts two coplanar lines in exactly two distinct points is called a transversal of the two lines.

A transversal makes four angles with each of the two lines.
In Figure 40, $PQ$ is a transversal of the lines $AB$ and $CD$. Of the eight angles formed by the lines $PQ$, $AB$ and $CD$, only four of them have $PQ$ on one of their edges, namely $APQ$, $BPQ$, $CQP$ and $DQP$. These four are called the interior angles formed by the transversal and the two lines. Any two interior angles formed by a transversal and two lines are called alternate interior angles if they have different vertices and have no common interior points.

Figure 41 indicates the two pairs of alternate interior angles formed by a transversal and two lines. In Figure 41(a) the angles $APQ$ and $DQP$ are marked, while in Fig. 41(b) $BPQ$ and $CQP$ are marked.

**EXERCISE 56-2A**

1. Measure in degrees the sizes of the alternate interior angles $APQ$ and $DQP$ in Figure 41(a). Do the same for the alternate interior angles $BPQ$ and $CQP$ in Figure 41(b).

Notice that a pair of alternate interior angles do not always have the same measures! Can you draw a case in which the alternate interior angles do have the same measure? If you were able to draw such a case, what could you say about the two lines that were cut by the transversal?
Suppose you know that \( m(ABC) = 90^\circ \) and \( \triangle BAC \cong \triangle DCE \). Using the postulates, prove that \( \overline{DE} \parallel \overline{BA} \).

**THEOREM 16.** If two parallel lines are cut by a transversal, any pair of alternate interior angles formed are congruent.

**Proof:** Let the lines \( \overline{AB} \) and \( \overline{CD} \) be parallel and let \( \overline{PQ} \) be a transversal (as in Figure 42). Select a point \( R \) between \( P \) and \( Q \). Drop a perpendicular \( \overline{RS} \) from \( R \) to the line \( \overline{CD} \). The ray \( \overline{SR} \) intersects \( \overline{AB} \) at a point \( T \). Then \( \overline{TS} \) is perpendicular to both \( \overline{AB} \) and \( \overline{CD} \) by Theorem 15. Then \( \angle TRP = \angle RSQ \), since both are right angles and all right angles are congruent. Furthermore \( \angle TRP = \angle SRQ \), since vertical angles are congruent (see Theorem 10). We have now shown that two angles of \( \triangle TRP \) are congruent to two angles of \( \triangle SRQ \). Theorem 12 enables us to conclude that \( \triangle TRP \parallel \triangle SRQ \). But by Definition 26, corresponding angles of similar triangles are congruent, so the alternate interior angles \( \angle TRP \) and \( \angle SRQ \) are congruent.

In order to prove that the other pair of alternate interior angles \( \angle CQR \) and \( \angle BPR \) are also congruent, we observe that \( \angle TRP \) and \( \angle PBR \) form a linear pair, and \( \angle CQR \) and \( \angle QSR \) form a linear pair. Thus, \( m(\angle PBR) = 180^\circ - m(\angle TRP) \), and \( m(\angle CQR) = 180^\circ - m(\angle QSR) \). But \( \angle TRP = \angle QSR \), so \( m(\angle PBR) = m(\angle CQR) \), and we see that \( m(\angle PBR) = m(\angle CQR) \). But angles with the same measure are congruent, so \( \angle PBR \cong \angle CQR \) and the theorem is completely proved.
EXERCISE 56-2B

1. Consider two lines \( k \) and \( n \) cut by a transversal \( m \). If \( \hat{a} \) and \( \hat{b} \) are alternate interior angles and \( \hat{d} \) and \( \hat{c} \) are vertical angles formed by three lines, then \( \hat{a} \) and \( \hat{c} \) are called corresponding angles (see Figure 43). Prove the following theorem: If lines \( k \) and \( n \) are parallel and \( m \) is a transversal, corresponding angles formed are congruent (Fig. 44).

![Fig. 43 and Fig. 44]

2. Measure each of the angles in the triangle in Figure 45. Next add the three measures that you found. Draw three triangles of different shapes and find the sum of the measures of the angles of each triangle with a protractor. Are the sums of the measures of the angles of each triangle near 180°? We are now going to prove that the sum of the measures of the angles of any triangle is 180°. If your sums did not turn out to be exactly 180° each time, explain why in terms of the approximate nature of measurements.

**Theorem 17.** The sum of the measures of the angles of any triangle is 180°.

**Proof:** We may label the vertices of the triangle \( A \), \( B \) and \( C \) (Fig. 46). Through vertex \( A \) we can draw a line segment \( \overline{DE} \) which is parallel to \( \overline{BC} \) (see Construction 14). The angles

![Fig. 46]
$CBA$ and $EAB$ are alternate interior angles of the transversal $AB$ and the parallel lines $DC$ and $DE$. Thus, $CBA$ and $EAB$ are congruent. Likewise, $BCA = DCA$, for they are alternate interior angles of the transversal $AC$ and the parallel lines $DE$ and $BC$. These follow from Theorem 16. We next observe that $EAD$ is a straight angle and that the angles $EAB$, $BAC$ and $CAD$ form a linear triple. The sum of their measures is $180^\circ$, so we may write $m(EAB) + m(BAC) + m(CAD) = 180^\circ$. But since congruent angles have equal measures, we have $m(EAB) = m(BCA)$ and $m(CAD) = m(BCA)$. This leads us to $m(CBA) + m(BAC) + m(BCA) = 180^\circ$, and we have proved our theorem!

**EXERCISE 56-2C**

1. What are the measures of each of the angles of an equilateral triangle? Can you construct with straight-edge and compass an angle of $60^\circ$? Can you construct an angle of $30^\circ$? Write again a list of the angles that you can construct. Does your list include $105^\circ$? Explain.

2. What are the measures of each of the angles of an isosceles right-angled triangle?

3. In the following figure, lines $k$ and $n$ are parallel. Look at angle $\hat{a}$ and say which angle (or angles) is (or are):
   a. Alternate interior with it
   b. Supplementary to it
   c. Vertical to it
   d. Adjacent to it
   e. Corresponding to it

4. In the following figure, lines $k$, $n$ and $m$ are parallel and the measures of certain angles are as marked. What are the measures of $\hat{a}$ and $\hat{b}$?

5. In the figure at the right, lines $k$ and $n$ are parallel. What are the measures of $\hat{a}$, $\hat{b}$ and $\hat{c}$ if the angles marked have the given measures?
6. In the accompanying figure, \( \triangle CDB \) is isosceles with \( DB = DC \), \( m(\hat{D}C) + m(\hat{C}DB) = 75^\circ \),

\[ \overrightarrow{AE} \parallel \overrightarrow{CD}, \text{ what kind of triangles are } \triangle ABC, \triangle BDE \text{ and } \triangle CDF? \]

56-3 Extending and subdividing line segments

We began this chapter with a few remarks about scale drawings. We shall return to this subject to end this chapter.

First of all, if we are given a line segment \( \overrightarrow{AB} \), can we construct a segment 3 times as long? First we draw a segment which is at least 3 times as long as \( \overrightarrow{AB} \) with our straight-edge. We could then measure the original segment, multiply this length by 3 and measure this new length along the longer segment. We have already discussed the inaccuracies that come into measurements and would like to find a better way of doing this. Once we have drawn a segment \( \overrightarrow{CD} \) which is more than 3 times as long as \( \overrightarrow{AB} \), we can more accurately draw the segment which is 3 times as long as \( \overrightarrow{AB} \), as follows. We open our compass so that its two points are at \( A \) and \( B \). With this, we can accurately draw circles of radius \( \overrightarrow{AB} \). Now with centre \( C \), we draw a circle of radius \( \overrightarrow{AB} \). This intersects \( \overrightarrow{CD} \) in a point \( D_1 \). Now with center \( D_1 \), draw the second circle of radius \( \overrightarrow{AB} \). This intersects \( \overrightarrow{CD} \) at \( D_2 \). With \( D_2 \) as centre, we draw the third circle of radius \( \overrightarrow{AB} \), and this intersects \( \overrightarrow{CD} \) at \( D_3 \). Then \( \overrightarrow{CD} \) is 3 times as long as \( \overrightarrow{AB} \).

**EXERCISE 56-3A**

1. Can you modify the construction given above to get a segment 5 times as long as \( \overrightarrow{AB} \)? What about \( n \) times as long, where \( n \) is a positive integer?

You may also ask how we draw the segment \( \overrightarrow{CD} \) long enough in the first place. If our ruler is long enough, there is no problem. If it is not long enough, we can draw a segment as long as the ruler and then slide the ruler along the line so that part of it overlaps and continue the segment. This may be repeated as many times as necessary.

Once we have a segment \( \overrightarrow{AB} \) drawn, there is a nice way of locating a point \( C \) on \( \overrightarrow{AB} \) beyond \( B \) without sliding the ruler along \( \overrightarrow{AB} \). It can be done as follows. First select a convenient radius and draw two circles with this radius and centres at \( A \) and \( B \). These intersect at \( D \) and \( E \) (Fig. 47). Now we draw two circles of radius \( s \) and centres at \( E \) and \( D \).
If \( s \) is sufficiently large, these last two circles intersect in points which are not on \( \overline{AB} \). The one closer to \( B \) will be called \( C \). Then \( C \) is on \( \overline{AB} \), and \( B \) is between \( A \) and \( C \). By drawing \( \overline{BC} \), we get an extension of \( \overline{AB} \).

**EXERCISE 56-3B**

1. Observe that \( \overline{DE} \parallel \overline{AB} \). (Why?) Observe also that \( \overline{CF} \parallel \overline{DE} \). (Why?) Use these facts to give a proof that \( C \) is on \( \overline{AB} \).

   We have seen how we enlarge a given segment. We next ask how we draw a segment \( \frac{1}{2} \) or \( \frac{1}{4} \) as large as a given segment. This is the same as asking how we divide a segment into 3 or 4 equal parts. Consider first the following exercises.

**EXERCISE 56-3C**

1. In the given figure, \( \overline{FG} \) is parallel to \( \overline{BC} \) and intersects segments \( \overline{BA} \) and \( \overline{CA} \) of the triangle at \( D \) and \( E \), respectively. Furthermore, \( \overline{BD} = \overline{DA} \).
   a. What do we call angles \( \angle ADE \) and \( \angle DBC \)?
   b. Which of the following equations is true? Why?
m(\hat{A}D\hat{E}) > m(D\hat{B}C) \\
m(\hat{A}D\hat{E}) < m(D\hat{B}C) \\
m(\hat{A}D\hat{E}) = m(D\hat{B}C)

c. Consider triangle \(\triangle ADE\) and triangle \(\triangle ABC\). What can you say about their respective angles?

d. Is \(\triangle ADE || \triangle ABC\)? Why?

e. \(\frac{DA}{BA} = \frac{1}{2}\). Explain why.

f. Since \(\frac{DA}{BA} = \frac{1}{2}\), what is the numerical value of \(\frac{EA}{CA}\)? Why?

g. Which of the following equations is true and why?

\(EA = EC\) \\
\(EA > EC\) \\
\(EA < EC\)

2. Taking into account Question 1, how would you construct the midpoint of each of the segments \(\overline{CA}\) and \(\overline{BR}\)?

We can now easily prove the following general theorem.

**Theorem 18.** If two sides of a triangle \(\triangle ABC\) are cut by a transversal \(\overline{DE}\) (Fig. 48) which is parallel to the third side, then \(\triangle ADE || \triangle ABC\).

![Fig. 48](image)

**Proof:** First of all \(\hat{D}AE = \hat{B}AC\), for they are both names for the same angle. \(\overline{AB}\) is a transversal for the parallel lines \(\overline{DE}\) and \(\overline{BC}\), so the alternate interior angles \(\hat{ABC}\) and \(\hat{BDF}\) are congruent. But the vertical angles \(\hat{BDF}\) and \(\hat{ADE}\) are congruent, so we conclude that \(\hat{ADE} = \hat{ABC}\). We have now shown two angles of \(\triangle ADE\) congruent to two angles of \(\triangle ABC\), so the triangles are similar, and the proof is finished.

**Exercise 56-3D**

1. In the triangle in Figure 48, if the transversal \(\overline{DE}\) divides \(\overline{AB}\) so that \(AD = \frac{1}{3}AB\), what can you say about the way \(\overline{AC}\) is divided by the transversal?

Now let us return to the problem of subdividing a line segment \(\overline{AB}\) into \(n\) parts of equal length, \(n\) being some positive integer.

**Construction 15.** Given a line segment \(\overline{AB}\) and a positive integer \(n\), locate a point \(C\) between \(A\) and \(B\) such that \(AC = \frac{1}{n} AB\).
**Method:** Draw a line segment \( AD \) making some convenient angle with \( AB \) (Fig. 49). Select a convenient point \( D_1 \) between \( A \) and \( D \). Next use the method described above to locate the points \( D_1, D_2, \ldots, D_n \) such that \( AD_1 = D_1D_2 = D_2D_3 = \cdots = D_{n-1}D_n \). That is, draw a circle of radius \( AD_1 \) with centre at \( D_1 \). This intersects \( AD_1 \) at \( D_2 \) and so on. After locating \( D_n \), draw the line segment \( D_nB \). Now draw a line segment through \( D_1 \) and parallel to \( D_nB \). This intersects \( AB \) at the point \( C \).

**Theorem 19.** The construction described above locates a point \( C \) such that \( AC = \frac{1}{n}AB \).

**Proof:** Since \( D_1C \) is a transversal which cuts two sides of \( \triangle ABD_n \) and which is parallel to the third side \( D_nB \), we know that \( \triangle ACD_1 || \triangle ABD_n \). (What theorem applies?) But since corresponding sides of similar triangles have lengths that are proportional, we know there is a positive real number \( k \) such that \( AC = k \times AB \), \( AD_1 = k \times AD_n \) and \( CD_1 = k \times BD_n \). But we have constructed \( D_1 \) and \( D_n \) so that \( AD_1 = \frac{1}{n}AD_n \), so \( k = \frac{1}{n} \). Thus \( AC = \frac{1}{n}AB \), and we are finished with the proof.

**Exercise 56-3E**

1. Locate the point \( C \) on \( AB \) such that \( AC = \frac{1}{3}AB \); such that \( AC = \frac{2}{3}AB \). Draw a triangle \( \triangle ABC \). Now construct another triangle \( \triangle DEF \) which is similar to \( \triangle ABC \) and for which \( DE = 3AB \); for which \( DE = \frac{1}{3}AB \). Start with a line segment \( AB \) and locate a point \( C \) on \( AB \) such that \( AC = \frac{3}{3}AB \). Can you see how these methods are useful in making scale drawings?

2. In the accompanying figure, \( BC \parallel DE \) and \( AB = AC \). Prove that \( BD = CE \).

3. In the figure shown, \( XY \) and \( ZM \) are each parallel to \( BC \), \( AX = \frac{1}{4} AB \), \( AX = \frac{1}{3} AZ \). Find the value of the quotient \( \frac{ZM}{BC} \).
4. In the accompanying figure, $\overline{AE} \parallel \overline{DB}$, $\overline{AD} \parallel \overline{BC}$ and $\overline{BC} = \overline{CD}$. Show that:
   a. $\angle EAD = \angle CBD$.
   b. $\angle BAD = \frac{1}{2} \angle BDC$.
   c. If $\angle BCD = 40^\circ$, show that $\angle BAC + \angle ABD = 110^\circ$. 
Chapter 57
RIGHT-ANGLED TRIANGLES

A study of geometry is incomplete unless it includes some discussion of the remarkable properties of right-angled triangles and related topics. We shall close this study with some of this material.

57-1 Revision of polygons

We shall begin by discussing squares, rectangles and other polygons. We shall first revise the definitions of these words.

DEFINITION 29: A simple closed polygon of n sides (or an n-gon) is a figure consisting of n line segments \( P_1P_2, P_2P_3, P_3P_4, \ldots, PnP_1 \) (called sides) in which any two sides intersect only at their endpoints and each endpoint belongs to precisely two sides, which we shall assume do not lie on the same line. The endpoints \( P_1, P_2, \ldots, P_n \) are called vertices of the n-gon.

For example, a 3-gon is a triangle, a 4-gon is also called a quadrilateral, a 5-gon is also called a pentagon and a 6-gon is also called a hexagon.

Two sides of an n-gon are called consecutive sides if they meet in a common vertex. For a quadrilateral, two sides which do not meet are called opposite sides.

Two vertices of an n-gon are called consecutive if they are endpoints of the same side. In a quadrilateral, non-consecutive vertices are called opposite vertices. Similarly, in an n-gon two angles are called consecutive angles if their vertices are consecutive. In a quadrilateral, two angles are called opposite angles if their vertices are opposite. A diagonal of an n-gon is any line segment whose endpoints are non-consecutive vertices. For example, in Figure 50, \( A \) and \( C \) are opposite vertices and \( B \) and \( D \) are opposite vertices of the quadrilateral with vertices \( A, B, C \) and \( D \). We shall write this quadrilateral as \( ABCD \). The two diagonals of \( ABCD \) are \( AC \) and \( BD \).
**EXERCISE 57-IA**

1. How many diagonals does a pentagon have? How many diagonals does a hexagon have? Can you figure out how many diagonals an $n$-gon has?
2. Draw an equilateral triangle and mark its vertices with capital letters. List all pairs of consecutive sides for that triangle.
3. Draw a pentagon and say how many pairs of consecutive vertices it has.
4. Can you give a general statement about the number of pairs of consecutive vertices in an $n$-gon?

A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel. A rectangle is a parallelogram all of whose angles are right angles, and finally, a square is a rectangle all of whose sides are congruent to each other.

**EXERCISE 57-IB**

1. Construct a square each of whose sides has length 2 inches.
2. Construct a parallelogram $ABCD$ with $AB$ of length 2 inches, $BC$ of length 1 inch and $m(\angle DAB) = 45^\circ$. (Recall that we can construct a $45^\circ$ angle by bisecting a right angle.)
3. Draw a circle $k$ of radius 2 inches. Now select a point $A$ on this circle and with $A$ as centre, draw another circle of radius 2 inches which intersects $k$ at two new points which are to be labelled $B$ and $F$. Now with $B$ and $F$ as centres, draw two more circles of radius 2 inches, which intersect $k$ at new points which we label as $C$ and $E$. Now with $C$ and $E$ as centres, draw two circles of radius 2 inches. These should intersect $k$ in the same new point, which we can label $D$. Draw the hexagon $ABCDEF$. What is true about the lengths of each of the sides of this hexagon? What is true about each of the angles of this hexagon? By joining each vertex $A$, $B$, $C$, $D$, $E$, $F$ and $O$ to the centre $O$ of the circle $k$, draw the six triangles $\triangle OAB$, $\triangle OBC$, $\triangle OCD$, $\triangle ODE$, $\triangle OEF$ and $\triangle OFA$. What can you say about these triangles? Are they all equilateral triangles? Find the measure of each of the angles of the hexagon $ABCDEF$. A hexagon in which all sides and all angles are congruent is called a regular hexagon. In general a polygon all of whose sides are congruent and all of whose angles are congruent is called a regular polygon. By what other names do you know a regular triangle and a regular quadrilateral?

In the preceding chapters, we have assumed enough postulates and proved theorems so that you now have the tools to go on and prove some interesting properties of these new figures. For example, write your own proofs in the following exercises based upon the postulates and theorems of the preceding chapters.

**EXERCISE 57-IC**

Prove that a diagonal of a parallelogram separates it into two congruent triangles. Then prove that opposite sides of a parallelogram are congruent and that opposite angles are also congruent.

One of the reasons that triangles are studied so extensively in our lessons is that they form the "building blocks" of more complicated figures. In particular, we can define a polygonal region as follows.
DEFINITION 30: A *polygonal region* is the union of a finite number of coplanar triangular regions such that

a. If any two of these triangular regions intersect, their intersection consists of either a point or a line segment, and
b. each triangular region has at least one line segment in common with another triangular region.

For example, a square region and a rectangular region can be viewed as the union of two triangular regions obtained when a diagonal is drawn (as in Figure 51).

![Fig. 51](image)

Other polygonal regions are illustrated in Figure 52.

![Fig. 52](image)

Those edges of triangular regions in a polygonal region which belong only to one triangle from the *boundary* of the polygonal region. The boundary consists of a finite number of line segments which form a closed polygonal path. Those points of the polygonal region which are not on the boundary form the *interior* of the polygonal region.

EXERCISE 57-1D

Consider a quadrilateral $ABCD$ as in Figure 53. By drawing a diagonal such as $AC$ we obtain two triangles $\triangle ABC$ and $\triangle ACD$.

What can you conclude about the sum of the angles of a quadrilateral? Can you do something similar with a pentagon (5-gon)? with a hexagon? Can you figure out what the sum of the angles of an $n$-gon must be?
57-2 Area

We have already discussed how we measure line segments and angles. We now say a few words about measuring polygonal regions. The measure that we assign to a polygonal region is its area. The area of a polygonal region has properties very similar to the measures of line segments given in Postulate 2 and of angles given in Postulate 3. We give these properties in the following.

**POSTULATE 8:** The areas of polygonal regions satisfy the following properties.

After a unit of area has been agreed upon:

(a) The area of any polygonal region is a positive real number, denoting a number of square units.

(b) Congruent polygonal regions have the same area.

(c) If a polygonal region is the union of non-overlapping polygonal regions, its area is the sum of the areas of these polygonal regions.

(d) The area of a rectangular region with consecutive sides of length \(a\) units and \(b\) units is \(a \times b\) square units.

Using this postulate, we can derive the area of any polygonal region. We shall briefly sketch how this is done.

For the parallelogram region \(ABCD\) in Figure 54, the area can be found by dropping perpendiculars from \(A\) and \(B\) to \(CD\) and noticing that \(\triangle ADF \cong \triangle BCE\). (Prove this as an exercise.) Thus, the triangular regions \(ADF\) and \(BCF\) have equal areas. When we take the triangular region \(BCF\) away from the parallelogram region \(ABCD\) and add the triangular region \(ADF\), we obtain a figure \(ABEF\) with the same area as \(ABCD\). But \(ABEF\) is a rectangular region whose area is \(AB \times AF\). If we call \(AB\) the base of \(ABCD\) and \(AF\) the altitude, then we can say that **the area of a parallelogram region is the length of its base times the length of its altitude.**

Finally, if we start with a triangular region \(ABC\), we can draw a segment \(CD\) parallel to \(AB\), and \(AD\) parallel to \(BC\). We then obtain (as in Figure 55) a parallelogram region \(ABCD\) composed of two congruent triangular regions \(ABC\) and \(CDA\). This means that the triangular region \(ABC\) has just half the area of the parallelogram region \(ABCD\). But this means that the area of the triangular region is \(\frac{1}{2} AB \times CE\), where \(E\) is the end of the perpendicular dropped from \(C\) to \(AB\). Calling \(CE\) the altitude and \(AB\) the base of the triangle, we get the formula that **the area of a triangular region is one-half the length of its base times the length of its altitude.** Areas of polygonal regions are found by subdividing them into triangular regions.
EXERCISE 57-2A

1. Find the area of the rectangular region whose dimensions are indicated in the diagram below.

```
3 inches
5 inches
```

2. Find the area of the parallelogram region pictured below.

```
4 feet
6 feet
```

3. Find the area of the right-angled triangular region pictured below.

```
2 yards
4 yards
```

4. A farm is in the shape of a rectangular region and is 200 feet long and 150 feet wide. What is its area in square feet?

Although we have been very careful up to now to speak always of the area of a polygonal region and not of the area of a polygon, it is often more convenient to use the shorter expression. Since it is actually common practice to speak of the area of a polygon when one means the area of the polygonal region whose boundary is that polygon, we shall also do so from now on when there is no possibility for confusion. Thus, instead of saying "the area of the triangular region $ABC$", we shall merely say "the area of $\Delta ABC$".

57-3 Right-angled triangles

We conclude our study of geometry with two interesting properties of right-angled triangles. Draw a circle and a line segment through the centre of the circle, which intersects the circle in two points $A$ and $B$. Then $A$ and $B$ are called diametrically opposite points. (The segment $AB$ is called a diameter of the circle and the length $AB$ is also called the diameter of
the circle. Which of the two meanings of the word diameter is wanted can always be understood from the sentence in which it is used.)

Select any point \(C\) on the circle different from \(A\) and \(B\). Measure the angle \(\angle ACB\). You should find that it is a right angle. This remarkable result will hold whenever the vertices \(A\), \(B\) and \(C\) of a triangle lie on a circle with \(A\) and \(B\) diametrically opposite points. We shall prove this in the next theorem.

**THEOREM 20.** If the three vertices \(A\), \(B\) and \(C\) of a triangle lie on a circle and \(A\) and \(B\) are diametrically opposite points then \(\angle ACB\) is a right angle.

**Proof:** We wish to prove that \(\angle ACB\) is a right angle. The line segments \(\overline{OA}\), \(\overline{OB}\) and \(\overline{OC}\) are all radial segments of the same circle, and hence are all congruent (Fig. 56). Thus, \(\triangle OAC\) and \(\triangle OBC\) are isosceles triangles, and we conclude that \(\angle OAC = \angle OCA\) and \(\angle OBC = \angle OCB\).

This gives us \(m(\angle OAC) = m(\angle OCA)\) and \(m(\angle OBC) = m(\angle OCB)\).

Now we have \(m(\angle OAC) = m(\angle OCA)\) + \(m(\angle OCB) = m(\angle OAC) + m(\angle OBC)\). But \(m(\angle OAC) + m(\angle OBC) = 180^\circ\), and replacing \(m(\angle OCB) = m(\angle OAC)\) by \(m(\angle OAC)\), we get \(m(\angle OAC) + m(\angle OBC) = 180^\circ\), or simply \(m(\angle OBC) = 90^\circ\), so that \(\angle ACB\) is a right angle. This completes the proof.

Let \(\triangle ABC\) be a right-angled triangle with \(\angle ACB\) as its right angle; that is, \(C\) is the vertex of the right angle (Fig. 57). Then \(\overline{AB}\), the side opposite the vertex \(C\) of the right angle, is called the hypotenuse of the right-angled triangle. The other two sides which are contained in the right angle are called the legs of the right-angled triangle.

**EXERCISE 57-3A**

Construct very carefully a right-angled triangle \(\triangle ABC\) which has legs 3 inches and 4 inches long. Measure the length of the hypotenuse of the right-angled triangle. It should have length 5 inches. Now construct three squares which have the actual line segments \(\overline{AB}\), \(\overline{AC}\) and \(\overline{BC}\) as sides and lie outside \(\triangle ABC\). What are the areas of each of the squares which you drew? They are 9 square inches, 16 square inches and 25 square inches. Notice that the area of the square which has the hypotenuse as one of its sides is equal to the sum of the areas of the two squares which have the legs of the right-angled triangle as their sides. This fact is true for all right-angled triangles and is known as the Pythagorean theorem after the Greek mathematician in whose school it was discovered. We shall next prove this theorem.

**THEOREM 21 (The Pythagorean theorem).** The area of the square which has the hypotenuse of a right-angled triangle as its side is equal to the sum of the areas of the two squares which have the legs as their sides.
Proof: Let us name the vertices of the triangle $A, B, C$ with $C$ as the vertex of the right angle (Fig. 59). The area of the square with $AB$ as its side is $AB^2$, while the areas of the squares with $AC$ and $BC$ as their sides are $AC^2$ and $BC^2$, respectively. We want to prove that

$$AB^2 = AC^2 + BC^2.$$ 

One often simplifies the way of writing this by using the small letter $a$ to denote the length of the side opposite the vertex $A$, the small letter $b$ to denote the length of the side opposite the vertex $B$, and the small letter $c$ to denote the length of the side opposite $C$. Then $AB = c$, $BC = a$ and $AC = b$. We now want to prove that

$$c^2 = a^2 + b^2.$$

We begin by drawing two squares, $DEFG$ and $D'E'F'G'$ each having sides of length $a + b$ (Fig. 59). On the square $DEFG$, locate the points $H, I, J$ and $K$ such that $DI = EI = FJ = GK = a$. Then $II = IF = JG = KD = b$. The triangles $\triangle KDI, \triangle HEI, \triangle IFJ$ and $\triangle JGK$ are all right-angled triangles. They each have one leg of length $a$ and the other leg of length $b$. Thus, the SAS postulate enables us to conclude that all four of these triangles are congruent to $\triangle ABC$, which also is a right-angled triangle with legs of lengths $a$ and $b$. Thus, each of the four triangles $\triangle KDI, \triangle HEI, \triangle IFJ$ and $\triangle JGK$ has the same area as $\triangle ABC$.

Furthermore, since each of the four triangles is congruent to $\triangle ABC$, we can conclude that each has hypotenuse which is congruent to $AB$. Thus, $II = IH = JI = KJ = c$. The quadrilateral $HIJK$ has all of its sides of length $c$. We would like to show that $HIJK$ is a square, so
we must prove that its angles are all right angles. We know that \( m(\hat{DAK}) + m(\hat{DK}) + m(\hat{AK}) = 180^\circ \), since this is the sum of the measures of the angles of a triangle. But \( m(\hat{DAK}) = 90^\circ \) since it is a right-angle. Thus, \( m(\hat{DK}) + m(\hat{AK}) = 90^\circ \). Since \( \triangle KDK = \triangle KHE \), we have \( m(\hat{DK}) = m(\hat{KE}) \), and \( m(\hat{KD}) = m(\hat{HK}) \). We can then write \( m(\hat{HK}) + m(\hat{KD}) = m(\hat{KE}) \). We can then write \( m(\hat{HK}) + m(\hat{KD}) = m(\hat{KE}) \). Thus, \( m(\hat{HK}) = 90^\circ \). Since \( \triangle AIJKD \sim \triangle AIKJE \), we have \( m(\hat{AI}) = m(\hat{AI}J) \), and \( m(\hat{K}E) = m(\hat{K}t) \). We can then write \( m(\hat{K}E) + m(\hat{K}t) = m(\hat{K}H) \). Combining the last two equations gives us \( m(\hat{K}H) = 90^\circ \), so \( \hat{K}H \) is a right angle. The same argument shows that the other angles of the quadrilateral \( \triangle AIJK \) are all right angles and it is thus a square with sides of length \( e \).

Now locate the points \( II' \) and \( I' \) on the square \( D'E'F'G' \) such that \( D'I' = a \) and \( E'I' = a \) [Fig. 59(b)]. Then draw \( II' \parallel D'E' \) and \( I'K' \parallel E'F' \). It follows then that \( E'I' = b \), \( l'I' = b \), \( F'J' = b \), \( J'G' = a \), \( G'K' = b \) and \( K'D' = a \). Draw the diagonals \( \overline{II'} \) and \( \overline{K'K} \) for the rectangles \( E'I'L'I' \) and \( G'K'L'J' \). We then get four right-angled triangles \( \triangle E'I'I', \triangle I'L'I', \triangle J'G'K' \) and \( \triangle K'L'J' \), each having legs of lengths \( a \) and \( b \). Thus the triangles \( \triangle I'E'I', \triangle I'L'I', \triangle J'G'K' \) and \( \triangle K'L'J' \) each have the same areas. Thus the shaded regions in Figure 59 (a) and in Figure 59 (b) have the same areas, since each has area four times the area of \( \triangle ABC \). (The actual area of \( \triangle ABC \) is \( \frac{1}{2} a \times b \), but we shall not need this in the proof.)

Now we know that the squares \( DEFG \) and \( D'E'F'G' \) both have the same area (since both are squares with sides of length \( a + b \)). When we remove the shaded regions of each (which we have shown to have the same areas), we obtain unshaded figures whose areas are the same. The unshaded region in Figure 59(a) is a square region with side of length \( c \). Thus its area is \( c^2 \). The unshaded region in Figure 59(b) consists of two square regions, one with sides of length \( a \) and the other with sides of length \( b \). Thus its area is \( a^2 + b^2 \). We have thus shown that \( a^2 + b^2 \) is equal to \( c^2 \), and the theorem is proved.

This theorem has played a very important role in the further development of mathematics and the physical sciences. It is basic for the study of coordinate geometry and higher dimensional geometry. With this theorem, we shall have to end our brief introduction to geometry, but you should be aware of the fact that we have really only looked into a small corner of the field of geometry. It is a fascinating and rewarding study which shows us the power of man's reasoning, for from a few simple postulates one is able to produce a vast store of knowledge, which can be used to describe our universe and chart the way to new and unexplored regions of knowledge. We hope that this brief introduction will encourage you to go further in your study of geometry.

**EXERCISE 57-3B**

1. Which of the following triples of numbers can be the sides of a right-angled triangle?
   - a. 3, 4, 5.
   - b. 1, 1, 2.
   - c. 6, 8, 10.
   - d. 7, 11, 12.
   - e. 5, 12, 13.

2. If a man rides 6 miles north and then 8 miles east, how far is he from his starting point if we measure directly from his starting point to his terminal point?
Abacus

An arrangement of rods or grooves which illustrates the place value notation for numbers is called an abacus. Beads or stones are moved to show the number operations. It is of particular value in demonstrating grouping and regrouping.

Addition

Addition is an operation which we have defined on numbers. To any two such numbers, called addends, addition assigns a third number known as their sum.

Angle

Two different rays with a common endpoint form an angle. The common endpoint is called the vertex, and the two rays are known as edges. If the two rays are such that one is the opposite of the other, that is, part of the same line but on opposite sides of the vertex, the angle is said to be a straight angle. Two congruent angles which together form a straight angle are called right angles. The edges of a right angle are said to be perpendicular.

Approximations

When a count or measurement or rounding off gives a number which does not exactly match the original number or quantity, it is said to be given as an approximation.

Area

Area is the measure of a region. A square region bounded by line segments of unit length is said to have unit area.
**Associative property**

The associative properties of addition and multiplication state that if \(a, b, c\) are numbers, (i) \(a + (b + c) = (a + b) + c\) and (ii) \(a \times (b \times c) = (a \times b) \times c\).

**Average**

An average of a set of numbers is the number obtained by dividing the sum of all of the members in the set by the number of members in the set. It is also called the arithmetic mean of the set.

**Ball**

A ball is a solid figure which consists of a sphere and the set of points the sphere encloses.

**Base**

When a large collection of objects has to be counted, they can be grouped using any number for successive groupings. The decimal system uses groupings of ten, giving ones, tens, hundreds, ... for successive groupings. If the collection is grouped in threes, then ones, threes, nines, ... are used in the count. The number used in the grouping is called the base. The binary system uses base two and has successive groupings ones, twos, fours, eights, ... Fractions in these bases may be written in a form similar to that for decimal fractions. For example, \(132.45_{\text{six}}\) means \((1 \times 6^2) + (3 \times 6) + (2 \times 1) + \frac{4}{6} + \frac{5}{6}\).

**Binary**

Binary means relating to two. See Base and Operation.

**Circle**

The endpoints of the set of all congruent line segments in a plane which have a common endpoint form a circle. The common endpoint is called the centre and the length of the line segment is the radius. A circle and the points within it together form a circular region called a disc.

**Closure**

If a set of numbers is such that an operation on any two of its members produces a member of the set, that set is said to be closed under the operation.
**Commutative property**

The commutative properties of addition and multiplication state that if $a$ and $b$ are numbers, (i) $a + b = b + a$ and (ii) $a \times b = b \times a$.

**Cone**

A cone is a set of line segments with one endpoint in common and their other endpoints in a region inside a simple closed path in a plane which does not contain the common endpoint.

**Congruent**

Two figures which are in a plane (plane figures) are congruent if it is possible to place one of them (or a copy of it) over the other so that they fit exactly.

**Cube**

A solid figure formed from six congruent square regions joined together at their edges to form a closed bounded surface is called a cube.

**Decimal fractions**

See Fractions.

**Deviation**

The difference between any one number in a set and the average of the set is called the number's deviation from the average. The average deviation is the arithmetic mean of the deviations of all the numbers in a set from their average.

**Disc**

See Circle.

**Distributive property**

The distributive property states that if $a$, $b$, $c$ are numbers, $a \times (b + c) = (a \times b) + (a \times c)$. 
Division

Division is an operation which we have defined on numbers. To two such numbers, $a$ and $b$, known as dividend and divisor, division assigns a third number $a \div b$, called their quotient. If $b = 0$, no quotient can be assigned and $a \div 0$ is not any number.

Ellipse

An ellipse is a plane figure consisting of the set of all points, the sum of whose distances from two given points is constant. (For a given sum, the ellipse becomes more like a circle as the distance between the two points grows less.)

Figure

A geometrical figure is a set of points.

Fraction

The attempt to solve any equation $b \times \Box = a$, in which $b$ is not 0, leads to the idea of fractions. If we take a whole object, or a set of like objects, and break or separate it into $b$ equal parts, the number which denotes $a$ of such parts makes the given sentence true and is called a fraction.

A numeral for the fraction above is $\frac{a}{b}$. This is the common form; $a$ is called the numerator or and $b$ the denominator. If $a > b$, the numeral can also be written in mixed form, that is, as the sum of a whole number and a fractional part. For example, $\frac{15}{2}$ can also be written as $7 + \frac{1}{2}$, or briefly $7\frac{1}{2}$. Whole numbers can be written in fractional form with denominator 1.

For example, 3 can be written $\frac{3}{1}$. The fractions $\frac{a}{b}$ and $\frac{b}{a}$ are reciprocal fractions. The product of a pair of reciprocal fractions is always 1; for example, $\frac{3}{2} \times \frac{2}{3} = 1$.

A general definition of fraction can be stated as follows. Given two whole numbers $a$ and $b (b \neq 0)$, the quotient $\frac{a}{b}$ is said to be a fraction.

Any common fraction can be written as a decimal fraction, that is, as a fraction whose denominator is a power of ten or as an endless sum of such fractions. A fraction whose denominator cannot be a power of ten forms an endless decimal fraction, one or more of whose digits repeat. For example,

$$\frac{4}{11} = \frac{36}{10^1} + \frac{36}{10^2} + \frac{36}{10^3} + \ldots = 0.3636\ldots$$

An endless decimal fraction which neither ends nor repeats corresponds to an irrational number.
Identity

When 0 is added to any number, it leaves the number unchanged; 0 is said to be the **additive identity** element.

When any number is multiplied by 1, the number remains unchanged; 1 is said to be the **multiplicative identity** element.

Integer

The counting numbers have opposites which are shown to the left of 0 on the number line. The **integers** consist of the counting numbers, their opposites and zero.

Inverses

Two numbers which give 0 when added are opposites and are called **additive inverses**. Two numbers which give 1 when multiplied are **multiplicative inverses**; each is the **reciprocal** of the other.

If \( a \) and \( b \) are numbers, (i) \((a - b) + b = a\) and \((a + b) - c = a\); (ii) \((a \times b) \div b = a\) and \((a \div b) \times b = a\).

Subtraction and addition are called **inverse operations**; division and multiplication are also inverse operations.

Irrational number

An unending decimal which does not represent a rational number, is said to represent an **irrational** number.

Length

When a line segment is placed against the number line with its left endpoint at 0, the number at the right-hand endpoint is called the **length** of the segment. Clearly, this number depends on the size of the unit piece used on the number line.

Line segment

If any two distinct points \( A \) and \( B \) are joined by a straight-edge, a **line segment** is made which consists of \( A, B \), and all points located on the straight-edge. **Straightness** is tested by a tightly stretched string. The line segment is written \( \overline{AB} \); \( A \) and \( B \) are the **endpoints** of the segment. \( AB \) represents the length of the segment.

A line segment \( \overline{AB} \) may be extended indefinitely beyond an endpoint; this produces a **ray** with one endpoint. For example, \( \overline{AB} \) is the ray with endpoint \( A \). If a line segment \( \overline{AB} \) is extended indefinitely beyond both endpoints the result is a line, written \( \overline{AB} \).
Measure

A measure is the number on the number line which represents any quantity of which the unit is represented by the unit segment on the number line.

Multiplication

Multiplication is an operation which we have defined on numbers. To two such numbers \(a\) and \(b\), multiplication assigns a third number called their product written \(a \times b\) or \(ab\).

Operation

Two numbers can be linked in ways each of which would give a certain third number. The process which determines the third number is called a binary operation, because two numbers are needed to produce the third number. Addition, subtraction, multiplication and division are binary operations. For example, \(6 + 2 = 8\), \(6 - 2 = 4\), \(6 \times 2 = 12\), \(6 \div 2 = 3\).

Each operation has its own rules which it obeys. Some properties of operations include closure, the commutative, associative and distributive properties, and the properties of zero, of one, and of inverses.

Opposites

When the number line is extended to the left of 0, a point can be marked on it to correspond to each point to the right of 0. For example, 3 to the right of 0 can be written as pos 3 or 3; 3 to the left of 0 is then written as neg 3 or -3; 3 and -3 are opposites. The sum of two opposites is always 0; that is, for any number \(a\), \((a) + (-a) = (-a) + (a) = 0\).

Order

Any two different numbers have an order such that one is less than the other.

Any three different numbers can be placed in order so that one of them is between the other two; that is, \(a < b < c\). For example, on the number line:

(i) 4 is to the left of 7 and 7 is to left of 12: \(4 < 7 < 12\).

(ii) \(\frac{2}{3}\) is to the left of \(1\frac{1}{4}\); \(1\frac{1}{4}\) is to the left of \(1\frac{1}{2}\): \(\frac{2}{3} < 1\frac{1}{4} < 1\frac{1}{2}\).

(iii) -5 is to the left of -3; -3 is to the left of -2: \(-5 < -3 < -2\).

Order has the following properties:

1. **Comparison property**: for any numbers \(a\) and \(b\), one and one only of the following is true: \(a < b\), \(a = b\), \(b < a\).
2. **Transitive property**: for any numbers \(a\), \(b\), \(c\), if \(a < b\) and \(b < c\), then \(a < c\).
3. **Addition property**: for any numbers \(a\), \(b\), \(c\), if \(a < b\), then \(a + c < b + c\).
4. **Property of opposites**: for any rational numbers \(a\), \(b\), if \(a < b\), then \(-b < -a\).
5. **Multiplication properties**: for any rational numbers \(a\), \(b\), \(c\),
   (i) if \(a < b\) and \(c > 0\), then \(ac < bc\);
   (ii) if \(a < b\) and \(c < 0\), then \(bc < ac\).
**Parallel**

Two different straight lines in the same plane are said to be parallel if there can be a third line perpendicular to each of them. Parallel lines have no point in common.

**Parallelogram**

A parallelogram is a quadrilateral in which opposite sides are parallel.

**Path**

A path in a plane can be thought of as the trace made by a pencil point without its being lifted from the paper. When the pencil returns to its starting point without crossing its path, the path is said to be closed. A closed path has an inside and an outside region. A path may be curved or may consist of line segments successively connected at their endpoints.

**Percentage**

Percentage is a way of writing one number as hundredths of another. For example, 3 is \( \frac{1}{4} \) of 12; this can be written as \( \frac{25}{100} \) of 12, or 25 percent (25%) of 12.

Since the second decimal place shows hundredths, a decimal can easily be read as a percentage. For example, \(-375 = 37\frac{1}{2} \) hundredths = 37.5%.

**Perpendicular**

Two lines are said to be perpendicular if two adjacent angles at their intersection are congruent. See *Angle*.

**Plane**

An object in space has a surface which may be either flat or rounded. The test for flatness is that a straight-edge lies in the surface wherever it is placed. A flat surface indefinitely extended is a plane.

**Plane figure**

A plane figure is a set of points in one plane.
Point

The idea of a point arises from trying to represent the extreme tip of a corner. The word *point* is used to express an absolutely precise location. The intersection of two lines is a point. A point is ordinarily named by a capital letter.

Polygon

A *polygon* is a plane figure consisting of line segments placed end to end successively so that any consecutive pair have only an endpoint in common. The common endpoints are called *vertices* (singular: *vertex*) and the line segments are *sides*.

Quadrilateral

A *quadrilateral* is a polygon consisting of four line segments.

Rational numbers

The set of numbers which includes the positive fractions, zero and the negative fractions is the set of *rational numbers*. Since any integer can be expressed in fractional form, the integers are included in the set.

Ray

See *Line segment*.

Real numbers

The rational numbers and the irrational numbers together make up the *real numbers*. The real numbers correspond to all decimal fractions whether they end, repeat or neither end nor repeat. Any real number can be represented on the number line.

Rectangle

A parallelogram all of whose angles are right angles is a *rectangle*.

Region

A *plane region* is a set of points consisting of a simple closed path and all points inside it. A closed surface has an inside and an outside region.
**Significant digits**

By the significant digits of an approximate decimal fraction, we mean all its digits except (a) zeros written to the left of its first non-zero digit and (b) zeros written to the right of the non-zero digits if these zeros replace unknown (that is, non-reliable or rejected) digits. The number of significant digits in an approximate whole number depends on the degree of accuracy with which the number has been obtained.

**Similarity**

Two polygons are said to be similar if there is a one-to-one correspondence between their vertices such that:

(i) corresponding angles are congruent and

(ii) corresponding sides are proportional (that is, there is a positive real number—the scale factor—such that the lengths of the sides of one polygon are \( k \) times the corresponding sides of the other).

**Sphere**

A sphere consists of the endpoints of the set of all congruent line segments in space which have a common endpoint. The common endpoint is called the centre, and the length of the line segment is the radius.

**Square**

A square is a rectangle whose sides are all congruent.

**Subtraction**

Subtraction is an operation which we have defined on numbers. To two such numbers \( a, b \), subtraction assigns a third number called their difference and written \( a - b \).

**Symmetry**

Suppose we can divide a geometric figure into two parts by a line through the figure and that furthermore the parts are mirror images of each other. Then we say that the line is a line of symmetry for the figure and that the figure is symmetrical with respect to the line.

**Tetrahedron**

A tetrahedron is a solid figure consisting of the set of line segments which have one endpoint in common and their other endpoints in a triangular region, the common endpoint lying outside the plane of the triangle.
Transitivity

If \( a, b, c \) are numbers, an order relation which exists between \( a \) and \( b \) and also between \( b \) and \( c \) holds also for \( a \) and \( c \). Thus, the transitive property of order states that if \( a < b \) and \( b < c \), then \( a < c \).

Triangle

A closed polygon consisting of three line segments is a triangle.

Vertex

See Angle.

Volume

Volume is the measure of a space region bounded by a closed surface. A cubic region bounded by squares with sides of unit length is said to have unit volume.

Zero

The number which tells how many members are in the empty set is named zero. On the number line, 0 marks the empty set of unit lengths.

The property of zero in operations states that

(i) \( a + 0 = 0 + a = a \) and (ii) \( a \times 0 = 0 \times a = 0 \).
ANSWERS
FOUNDATIONS
OF GEOMETRY

The answers to the exercises in Chapters 53 to 57 consist mostly of figures to be drawn, and therefore no answers are given for these chapters.

Chapter 47

EXERCISE 47-2A

2. One. Eight. As many as you wish. As many as you wish.

EXERCISE 47-3A

5. The last two drawings on the right are not straight. The other drawings are straight.
9. The point $P$ should be on the line of sight from $J$ to $K$.

EXERCISE 47-4A

1. $A$ and $D$.
5. $Q$ is between $P$ and $R$. $T$ is between $Q$ and $V$.
7. Six; $PS$, $PR$, $PQ$, $SR$, $SQ$, $RQ$.
8. $XY$, $XZ$, $YZ$.
9. There should be six line segments showing in your figure.
10. Five.
11. $E$ should lie on $FG$.

EXERCISE 47-4B

1. The ray from $P$ through $O$, $PO$ or $OP$.
The ray from $Q$ through $X$, $QX$ or $XQ$.
The ray from $R$ through $N$, $RN$ or $NR$. 
2. a. $\overline{PQ}$ or $\overline{QP}$. b. $\overline{SK}$ or $\overline{KS}$. c. $\overline{OK}$ or $\overline{KO}$.

3. The ray $\overrightarrow{CD}$; the straight line $\overline{HG}$; the line segment $\overline{MN}$; the ray $\overrightarrow{FE}$. The points $D$, $H$, $M$ and $E$ on the figures have been chosen and named to provide names for the figures.

4. All line segments that end at $B$ and contain $A$.

5. The number line for whole numbers.

**EXERCISE 47-5A**

2. No. It does not contain three points that do not lie on any line.

3. No; a triangular region cannot contain straight lines.

4. a. Always. b. Always. c. Sometimes, but not always. Some four points do not lie in any plane. Some four points do lie in a plane. Take three points at three corners of the floor of a room, and the fourth point at one corner of the ceiling. If these four points were in some plane, the ceiling would have to touch the floor! d. Sometimes, but not always. e. Always. f. Never.

None of these is a plane region.

5. Any straight line.

6. Yes; papers are flat on flat table tops. The second figure is not full of straightness; in it, you can find two points whose line segment crosses the bitten-out part, and therefore does not lie in the figure.

7. (1) Yes, yes, no.
   (2) Yes, no, no.
   (3) No, yes, yes.
   (4) No, no, no.
   (5) Yes, no, no.
   (6) No, no, no.

**EXERCISE 47-6A**

1. 12

2. 4

3. 1

4. No.

5. 8. 8.

6. The top, bottom and side. 3.

   *Edges*: top circle, bottom circle: 2.

   *Vertices*: none; or else every point on an edge.

   "Vertex" has not really been defined here for such a figure.

7. One face; no edge; no vertices. Yes.

**REVIEW EXERCISES 47-7**

   e. Triangular region. f. Square. g. Circular region. h. Simple closed path. i. Path.

4. o, e, g, i, o.
6. Yes. Circle. Yes. Test with a straight-edge to see if it is a plane figure. If it is a plane figure, it must be a plane region since you can clearly draw a triangular region on the drum top.
7. a. Yes; it is a quadrilateral with all sides congruent. b. Yes; a square is a quadrilateral and has all four angles right angles. c. No; in section 47-6 there is an illustration labelled "Quadrilateral" that is not a rectangle.

Chapter 48

EXERCISE 48-3A

2. They are "crossing", "touching", or "connected end to end" according to whether the point of intersection is an endpoint of none, one, or both of them. Yes, they are.
6. $PR; XY; none.
9. $PJ and $DA; $MS and $XG; $AD and $QW.
12. As many as we wish.

EXERCISE 48-4A

2. a, c, d.
3. a. No; perpendicular lines intersect and so cannot be parallel. b. No, for the reason given in the answer to a.
4. They are perpendicular. The two lines are perpendicular.
5. a. Each of $HI and $LK with each of $CL, $FE and $HG. b. $IH and $LK; each of $CD, $EF and $HG with each other.

EXERCISE 48-5A

1. Cases e and i are impossible.
2. 3 angles. 2 right angles. $ZXY and $XZW; $XZW and $WZY; $WZY and $ZX.
3. Four and two.
4. Seven. Four.
11. $KL and $CD are perpendicular to $EF; $KL and $CD are parallel.

EXERCISE 48-6A

1. $b, d, e, g, j.
2. e, g and i are closed; e and g are simple.
3.  
   a. Nothing can be done to change this to a polygon.
   c. Erase any line segment.
   f. Erase arc \( AB \).
   h. Erase all but triangle \( ABC \).
   i. Draw \( \overline{AC} \) or \( \overline{CB} \) or \( \overline{BD} \) or \( \overline{DA} \).

**EXERCISE 48-7A**

1. Three.
5. b. As many as you wish.
   d. As many as you wish.
7. No.
8. No. Any two sides of a triangle intersect and so cannot be parallel.

**EXERCISE 48-8A**

1. b, d, g, i, j, l, m, o.
2. Squares - j; parallelograms - b, i, o; rectangles - b, j.
3. Yes.
5. Yes.
6. Yes.
8. Yes. Yes. No.
10. Yes. Yes. Yes, but this is sometimes physically difficult.
11. No, as explained and illustrated in Section 48-8.

**EXERCISE 48-8B**

2. Yes.
3. Yes.
4. The side included between the first pair of congruent angles is parallel to the side included between the second pair.
5. No. Opposite pairs of angles are congruent.
6. The angles are all right angles. Opposite sides are congruent and parallel.

**EXERCISE 48-9A**

1. The first and last are not paths; the second, third and fourth are simple.
3. Let \( A, B, C \) be the three points. Construct the perpendicular bisector of \( \overline{AB} \), and also that of \( \overline{BC} \). These two will intersect. The point of intersection is the centre of the circle.
7. A circle.
EXERCISE 48-11A

2.  a. 6 vertices, 9 straight edges, 5 flat faces; 
b. 1 face not flat;  c. 8 vertices, 12 straight edges, 6 flat faces;  d. 2 round edges, 
2 flat faces, 1 face not flat.
3.  a. Yes, a cube is a rectangular prism.
4.  Yes.

EXERCISE 48-12A

2.  Yes. Yes.
7.  The stand-up edge shows the line normal to the plane at the point at which the edge 
touches the table.
8.  Yes, at the bottom endpoint of the hinge line. The top edge of the cover traces a 
plane parallel to the one traced by the bottom edge.

Chapter 49

EXERCISE 49-2A

2.  Yes.
3.  Yes.

EXERCISE 49-3A

1.  The second. No.
3.  Yes.

EXERCISE 49-4A

1.  Yes.
3.  Yes; greater accuracy.

EXERCISE 49-5B

1.  The first two spheres have volumes $36\pi$ and $\frac{1125\pi}{2}$ cubic inches, respectively.
2.  $6 \times \pi \times (2)^2 = 24\pi$ cubic inches volume. 
Area: $32\pi$ square inches.
4.  $\frac{1}{3} \times \text{height} \times \text{area of base}.$
5.  $5 \times 3 \times 2 \times \pi = 30\pi$ cubic inches.
6. No; the volume of the box is 1080 cubic inches; the volume of the blocks is 1200 cubic inches.
7. The volume of the prism is 120 cubic inches, so Abu's guess was better. Abu made the better estimate for the volume of the book and is the better guesser. Abu probably guessed the dimensions of the book to be 1 inch, 9 inches and 10 inches.

Chapter 52

EXERCISE 52-1B

1. There are four lines of symmetry for a square.

2. There are two lines of symmetry for a rectangle that is not a square.

3. There are three lines of symmetry for an equilateral triangle, five for a regular pentagon, six for a regular hexagon.
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An Introductory Text for Teachers

Prepared at the 1965 Mombasa Mathematics Workshop

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This experimental teacher training text was prepared during the summer of 1965 at Mombasa, Kenya, as part of a program of curriculum study being conducted by the African Education Program of Educational Services Incorporated.

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The Mombasa Mathematics Workshop was directed by Professor W.T. Martin of Massachusetts Institute of Technology and Mr. John Oyelese of the University of Ibadan, Nigeria.

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This volume of Basic Concepts is concerned with different aspects of measurement. It is true that we have talked about measurement in previous volumes. We had something to say about measurement in Volume II when we discussed Approximations. We treated measurement again in Volume III when we discussed the measurement of angles, areas and volumes. It would be useful for the reader to revise these sections before reading this book. Now we want to go further. We want to talk about the measurement of other quantities like weight, time, cost or even the amount of information that a student has about a subject. There are many, many places where we measure things of one kind or another. Modern science and engineering are based on measurement. In ancient times some simple measurements were made, but the habit of measuring things really goes back to Galileo who around 1600 led the way to the age of science as we know it today. He wrote that we should "measure what is measurable and make measurable that which is not". We shall soon learn what this puzzling sentence means.

But measurement is important outside of science and technology. It is important in everyday life. It is also important in economics and the other social sciences. To plan for the future, the economist studies collections of numbers which measure population, exports and imports and other features of a country or region. The science of collecting and interpreting these numbers is called statistics.

In this volume of Basic Concepts we shall try to make clear how measurements can be used to help us to understand the world about us and to make plans to improve the conditions of our lives. We shall do this in Units XII, XIII and XIV. But first we must have clear ideas about how quantities can be measured. In Unit XI, then, we shall take a closer look at measurement itself. What kinds of things can be measured? How do we go about measuring a quantity? These are the kinds of questions which we must answer before we go on.
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ANSWERS

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Measurement, Functions and Probability
Chapter 58
MEASUREMENT

58-1 What is Measurement?

What is it that we do when we measure something, for example, the length of a table? We have something which we want to measure. In this example we call the something a length. To have a name for a thing to be measured, whatever it is, we use the word "quantity". Length is a quantity. Area is a quantity. Other quantities are the time something takes to happen (for example, the time it takes for the moon to rise), the cost of an article which we wish to buy, the speed of a bicycle, and the weight of a bunch of bananas.

After we have measured one of these quantities, we end up with a number. This number is called the measure of the quantity. But it would make no sense to say that the length of a table is 3. If someone told us this we would ask: "3 what?" or "Do you mean 3 feet or 3 yards?" So, to report a measurement it is not enough to give a number. We must give the unit of measure. We can measure a length in inches, in feet, in yards, or in miles. If we use the metric system we can choose such units as metres or centimetres. If we measure an area we report the result as a certain number of square inches, square feet, square yards, square miles or acres (in the case of land). We may measure time in days or hours or minutes or seconds. We measure cost in shillings and cents, in dollars and cents, in francs and centimes and so on.

So every measurement is reported in the form \[(\text{number}) \times (\text{unit})\] where \[\text{number}\] is to be filled by the name of a number and \[\text{unit}\] by the name of a unit, usually in the plural.

To measure any quantity we first choose a unit of measure and then determine the number of these units in the quantity to be measured. For example, we say that the length of a certain table is 3 feet.

58-2 Units

What exactly do we mean by a unit? First of all, a unit is a quantity of the same kind as the one that we wish to measure. We measure a length by a unit of length, an area by a unit of area, a time interval by a unit of time, and so on.

More precisely, we choose a particular example of the kind of quantity that we wish to know and agree to give the number 1 to it. That is, we agree to say that the measure of this example is 1. We call this example a unit for the quantity.

But it is also important to be able to use the unit any number of times. One way in which this can be done is to make as many copies of the unit as we please. If we measure length by using the length of a particular stick as a unit, we can make more sticks like it. You will remember how we constructed a number line by putting matches end to end. We had a supply of matches any one of which was a unit. We could test the equality of these matches so far as length is concerned by laying any two matches side by side to see if their ends correspond.
A more usual way to measure length is to use a ruler, say a ruler that people have agreed to call 1 foot long. We can of course make many copies of this ruler. We ordinarily do not do this. We use a single ruler and move it about. We assume that it does not change its length when we move it.

The point is that we can either copy our unit or use the same unit over and over again.

Let us turn to the measurement of area and choose for our unit a square 1 foot on a side. It is easy enough to make a copy of this unit or to move it about.

A certain angle is called a degree. Of course we can use this unit as many times as we like.

How about time? If we have a watch, we call an hour the time it takes the large hand to go around once and come back to the same place. This unit of time can be copied. For example, the time it takes to go around once more is equal to the hour just past.

So, too, when we measure money in shillings there is no difficulty in making shillings which are copies of each other.

Now we come to an interesting case. Can we measure how much a student knows about some subject which he has studied? Of course we try to do this when we give him an examination. Is this a real measurement? What is the unit of measurement? Let us say that a unit is a correct answer to a question so that we try to measure his knowledge by the number of questions that he answers correctly. But is a correct answer really a unit? Can we truthfully say that different questions are copies of each other? Obviously we cannot ask the student exactly the same question over and over again. That wouldn't do any good at all. So we use different questions. But how can we be sure that different questions are equal to each other as measures of knowledge? Of course we cannot be sure. So you see that measurement of knowledge is not entirely satisfactory, because we cannot find a unit that can be copied or moved about. It is usual to call it measurement, nevertheless. It fulfills the general purpose that measurement serves, even if it falls short of the more ideal measurements of length, area and weight.

Can we measure how much a mother loves her children? Can we measure how great a man is, or how important he is to his country? At least it would be difficult. In what units, however unprecise, could these things be measured?

Let us turn back to quantities like length which surely can be measured. How do we choose the unit? One part of the answer is that we use a unit that will make the numbers (the measures) come out to be neither too large nor too small. If we want to measure the distance from Nairobi to Mombasa we do not use inches or feet or yards. The measure would be too large. If we use a mile as a unit we do get a convenient number of units. In the other direction, if we measured a man's height in miles we would have a very small fraction of a unit for an answer.

Occasionally we use numbers of units which are very large or very small. For example, we say that the sun is about 93,000,000 miles away. Surely 93,000,000 is a large number. We use miles here to connect with earth measurements that are familiar. But the astronomer quickly chooses a larger and more convenient unit.

But before we discuss further the choice of units, we must look more closely at
the way in which a measurement is made. After we have decided on a unit, what do we do with it? We shall discuss this in the next section.

EXERCISE 58-2

Explain how you could measure each of the following quantities. In what units could they be measured?

a. The strength of a rope.
b. The annual rainfall at Lagos.
c. The strength of a salt solution (brine).
d. The speed of a bicycle.

58-3 The Process of Measurement

The measurement of length, say the length of a table, is perhaps the simplest example of a measurement. It is certainly one of the most important ones. So let us begin with it.

Suppose, then, that we have a stick 1 foot long, which is our chosen unit, and also that we can provide ourselves with as many copies of this unit as we wish.

We place a unit so that one end is exactly above the end of the table. We then put enough of our copies end to end to reach the other end of the table. If we are lucky the farther edge of the table will be even with the far end of one of our copies. Then we find the measure simply by counting the number of units we have used. If not, we must use fractions of a unit. When we talked about the number line we described how this could be done. We shall not repeat it here. For simplicity we assume that the measure is a counting number.

What we want to make clear now is that to measure a quantity we must be able to put units together in a way which it is reasonable to call addition. This will be true for the measurement of any quantity in terms of a given unit U. We shall want to write $U + U = 2U$, $U + U + U = 3U$, and so on.

Let us look at the measurement of quantities other than length with this in mind. We can surely put together units of area in a natural way. We can also fit together units of volume, for example, cubes 1 foot on a side. We can combine angle units, two or more pound weights and two or more shillings. Time units, say hours, can be fitted to each other in succession. In every case we have a way of putting units together that it seems natural to call addition.
This means that the way that we combine units must have the properties of addition with which we are familiar. We shall want it to be true, for example, that

\[3U + 5U = 5U + 3U\]

just as it is true that

\[3 + 5 = 5 + 3. \quad \text{(Commutative Property for Addition)}\]

Also we shall expect that

\[(3U + 5U) + 2U = 3U + (5U + 2U)\]

because

\[(3 + 5) + 2 = 3 + (5 + 2) \quad \text{(Associative Property for Addition)}\]

We can see very easily that these are true statements.

\[3U + 5U = 8U\]

and the 8 can be written as 5 + 3 as well as 3 + 5.

Generally

\[aU + bU = bU + aU\]

and

\[(aU + bU) + cU = aU + (bU + cU)\]

for any whole numbers \(a, b\) and \(c\). These results are just as valid when \(a, b\) and \(c\) are fractions as they are when \(a, b\) and \(c\) are whole numbers. We can describe these facts by saying that measurements must be additive.

**EXERCISE 58-3**

1. Give a reason for believing that if \(U\) is a chosen unit for any quantity

\[\frac{1}{3}U + \frac{4}{3}U = \frac{5}{3}U\]

and

\[\frac{4}{3}U + \frac{1}{3}U = \frac{5}{3}U\]

Hint: Take a new unit that is one third of \(U\).
2. As in Problem 1, show that
\[
\frac{1}{2} U + \frac{1}{3} U = \frac{5}{6} U = \frac{1}{3} U + \frac{1}{2} U
\]

58-4 The Need for Standard Units

Here is an experiment that you can try with your pupils. Let each of them bring a stick to class and use this stick as a unit with which to measure the length of the schoolroom. Will the pupils get the same number of units for an answer? Almost certainly not. It is very unlikely that all of the measuring sticks would have the same length.

Again you might ask different pupils to measure the length of the school room by pacing it off. Will they get the same answer? Probably not, because the unit of measure, the pace, for a tall child will be different from that for a short child.

It makes life simpler if people mean the same thing by the same words. If one man used the word "banana" to stand for what another man called a "pawpaw" it would be hard for them to talk to each other. This is the kind of trouble which people who speak different languages have in understanding each other. But within a group that speak the same language, people mean the same thing or nearly the same thing by the same words. This makes it possible for them to understand each other.

It is very much the same with measurement. If we have to buy and sell from each other, it helps to avoid misunderstandings if we can agree upon units of length, volume and weight and also agree upon units of money.

If a tradesman means something different by a yard of cloth than you do, you may easily find that you have bought more or less cloth than you need. If different tradesmen had different ideas of a yard or of a shilling, it would be very confusing. So one of the first things that a government does is to set standards, that is, agreed-upon units of measure.

If groups of people trade with each other, the standards of one group must be brought into some kind of agreement with those of the other. In the course of time it usually happens that one of these sets of standards is accepted. As the peoples of the world have come closer together, the number of different systems of standards has decreased. There are now only two principal sets of standards of measure for length, area, volume and weight. They are the English system and the Metric system. Of course there are many more than two systems of currency.

When two groups that deal with each other have different systems of measure or currency, there must be an agreement upon the way of changing from one to the other. For example, now that units have been standardized:

<table>
<thead>
<tr>
<th>Unit</th>
<th>Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 inch</td>
<td>2.54 cm</td>
</tr>
<tr>
<td>1 metre</td>
<td>39.37 inches</td>
</tr>
<tr>
<td>1 litre</td>
<td>1.057 liquid quarts</td>
</tr>
<tr>
<td>1 kilogram</td>
<td>2.2 pounds</td>
</tr>
</tbody>
</table>

For currency the "rate of exchange" changes somewhat with time. At present

1 U.S. dollar \(\approx\) 7 East African shillings

(\(\approx\) means "is approximately equal to").
The history of the English system is an interesting one. The distance from a man's nose to the tip of his middle finger when his arm and hand were outstretched was called a "yard". An "inch" long ago was the width of a man's thumb and a "foot" the length of a man's foot. Even with these crude units of measure, people soon found that a foot contained about 12 inches and that about 3 feet made a yard. An early king decreed that a yard was the distance from the tip of his nose to the tip of his finger.

The rod was a unit for measuring land, about 16 feet long. Another English king declared that a rod was 16 men's feet. To get this rod he ordered his officers to go to a certain church and on a particular Sunday take the first 16 men who came out of church, and measure a rod by having them stand so that the toe of one man just touched the heel of the man in front of him. This length was measured by a rope which was divided into 16 equal parts. One part became the official "foot" of England at that time. Later a yard was taken to be the length of a certain iron bar and a third of this length was taken to be a foot, and a thirty-sixth of it was called an inch.

The word "mile" comes from "mille" in the Latin expression "mille passum", which means "a thousand paces" ("mille" is the Latin word for 1,000). A pace was about 5 feet. Today we define a mile as 5,280 feet.

You will see that these English measures are more or less associated with the human body. The metric system broke away from these body units. The fundamental unit of length, the metre, was defined as one ten-millionth of the distance from the North Pole to the Equator. More recently, an international conference of scientists defined the metre in terms of the length of certain light waves. The metric system is used by scientists the world over.

In the metric system the connection between the different units for a given quantity is very simple. For example,

\[
\begin{align*}
1 \text{ micrometre (micron)} &= 0.000001 \text{ metre} \\
1 \text{ millimetre} &= 0.001 \text{ metre} \\
1 \text{ centimetre} &= 0.01 \text{ metre} \\
1 \text{ decimetre} &= 0.1 \text{ metre} \\
1 \text{ kilometre} &= 1,000 \text{ metres}.
\end{align*}
\]

It is therefore very easy to change units to get more convenient numbers. For example,

\[
1,371 \text{ metres} = 1.371 \text{ kilometres}.
\]

A statement like

\[
1 \text{ kilogram} = 2.2 \text{ pounds}
\]

really means that the two sides of the sign "=" name the same measurement. It does not mean that 1 equals 2.2! Some people read "=" as "equals", but it is better to read it as "is equivalent to".
Problem 1.

A bar measures 1.3 metres. How could this be expressed in feet?

Solution:

\[
1 \text{ metre} = 39.37 \text{ in.}
\]

\[
1.3 \text{ metres} = (1.3)(39.37) \text{ in.} = 51.181 \text{ in}
\]

\[
= \frac{51.181}{12} \text{ ft.}
\]

\[
= 4.265 \text{ ft.}
\]

Problem 2.

Change 10 lb. to kilograms (kg.)

Solution:

\[
2.2 \text{ lb.} = 1 \text{ kg.}
\]

\[
1 \text{ lb.} = \frac{1}{2.2} \text{ kg.}
\]

\[
10 \text{ lb.} = \frac{10}{2.2} \text{ kg.} = 4.55 \text{ kg.}
\]

EXERCISE 58-4

1. Have 10 pupils measure the length of a desk where each uses the width of his thumb as a unit. Record the results and find the average. Compare this average with the length of the desk in inches found by using a ruler.

2. Have 5 pupils measure the length of a room using the length of his right foot as a unit of measure. Average the results. Measure the room with a foot rule or a yard stick and compare the result with the average found.

3. Make up other experiments like these to try in your own school.

4. Fill in the boxes correctly. (Round off the results in a reasonable way.)
   a. 15 ft. = [ ] metres.
   b. 5 lb. = [ ] kg.
   c. 2 in. = [ ] cm.
   d. 100 sq. ft. = [ ] sq. metres.

5. Change 15 mi./hr. to ft./sec.
We have seen that measurement is a way of tying a number to a quantity by seeing how many units of the quantity can be put together to match the quantity. So far we have made things simple by assuming that a whole number of units would match the quantity that we wish to measure. We know that usually we must use fractions of units, but we shall keep on with whole number measures for a bit. What we have done should remind us of something. We have said that measurements are additive. Where have we met this idea before? In working with sets, of course. If we have two disjoint sets, say two sets of bananas, we can tie a number to each set. When we combine these sets to form their union we find that the number of members in the union is the sum of the numbers of members in each of the sets. We see that we can use the number of members in a set as a measure of the set. Measures are additive. We know that the sets must be disjoint, otherwise the number of the union would not be the sum of the numbers of the sets which are combined. It would be smaller. This is just like the case of adding lengths. If we had overlapping units (foot rules), we would not get the length of the table by addition. The length would be less than the sum of lengths we put together.

We begin to understand something else. We see why pictures are so useful in studying numbers. For example, we have used the number line to help us to understand fractions and how they are added. We also cut up circular and rectangular regions into equal pieces (of smaller units), again to help us to understand about fractions. This is because lengths and areas are additive. They act like numbers. For example, when we wish to test out the commutative law for fractions we may ask ourselves if it is true that

\[
\frac{1}{5} + \frac{3}{5} = \frac{3}{5} + \frac{1}{5}.
\]

We show \( \frac{1}{5} \) and \( \frac{3}{5} \) as lengths and add these lengths first one way and then the other.

We notice that the result is the same both times. What does this mean in the language of measurement? It really means that instead of using as a unit of length the distance \( OP \) from \( O \) to \( 1 \), we choose \( OQ \) as a new unit of length, say \( U \). In terms of this unit, \( OP = 5U \). We call this new unit a fifth. Then the equality that we wish to test can be written

\[
1 \text{ fifth} + 3 \text{ fifths} = 3 \text{ fifths} + 1 \text{ fifth}
\]

and this is true because they both amount to 4 fifths. This trick of choosing a new unit means that, if possible, we use whole number measures to make things simple again.

It is the same when we use rectangles. If the area of the figure \( ABCD \) is our
chosen unit of area we can choose a new unit to go with the shaded region. We call it a fifth (1 fifth). We verify by counting that

\[ 1 \text{ fifth} + 3 \text{ fifths} = 4 \text{ fifths} \]

\[ = 3 \text{ fifths} + 1 \text{ fifth}. \]

Of course, 5 fifths = 1 original unit. We write 1 fifth as \( \frac{1}{5} \). Then our statement reads:

\[ 1 \left( \frac{1}{5} \right) + 3 \left( \frac{1}{5} \right) = 4 \left( \frac{1}{5} \right) \]

\[ = 3 \left( \frac{1}{5} \right) + 1 \left( \frac{1}{5} \right) \]

which we shorten to

\[ \frac{1}{5} + \frac{3}{5} = \frac{4}{5} = \frac{3}{5} + \frac{1}{5}. \]

In this way everything becomes clear. It is hard to think of numbers but easy to think of pictures. Most people are more at home when they can picture what they are talking about.

In the Exercise you will be asked to use measurement language to talk about other properties of fractions. It all comes down to choosing a smaller unit that goes a whole number of times into the original one.

You will remember that this trick does not always work. There are other kinds of numbers that are not fractions — numbers that we call irrational numbers. We saw that we could not measure the length of the diagonal of a square whose sides have the length 1 by choosing any new unit which goes a whole number of times into the side. This was a very startling discovery. But this leads us to a new topic that we shall talk about in the next section.
EXERCISE 58-5

1. Show that
   a. \( \frac{1}{2} + \left( \frac{1}{4} + \frac{3}{4} \right) = \left( \frac{1}{2} + \frac{1}{4} \right) + \frac{3}{4} \)

   by choosing 1 fourth as a unit and using the associative property of whole numbers.

   b. What unit would you choose to show from the associative property of whole numbers that

   \( \left( \frac{1}{2} + \frac{1}{3} \right) + \frac{1}{4} = \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) \)?

2. We know that

   \( \frac{a}{b} + 0 = \frac{a}{b} \)

   for any fraction \( \frac{a}{b} \). Choose as a new unit one \( b \) th part of 1 to show that the result follows from the property of zero for whole numbers, that is, \( a + 0 = a \).

58-6 Approximations and Models

Let \( ABCD \) be a square 1 foot on a side and \( AD \) one of its diagonals. Now let us set out to measure the length \( AD \) of this diagonal. If we use a tenth of a foot as a unit of measure, we find that 14 units do not reach to \( D \), but 15 units extend beyond \( D \). We say that

\[ 14 \text{ tenths} < AD < 15 \text{ tenths}. \]
If we choose 1 hundredth of $AB$ as our unit and if we are very careful we find that $AD$ is more than 141 hundredths and less than 142 hundredths. We write this

$$141 \text{ hundredths} < AD < 142 \text{ hundredths}.$$ 

In terms of the original unit we would write

$$1.41 < AD < 1.42.$$ 

In practice we probably can not do any better than this because it is hard to put units together precisely. Even if they have been put together for us on a ruler with equally spaced marks, there is still a point beyond which your eyes are not sharp enough to tell two points apart. All measurement is therefore inexact or approximate. We have talked about this before and there is no need to go over the ground again.

The point that we should notice is that we assume that there is an exact value of the length of the diagonal that our actual measurements approximate. We imagine ideal lines with no thickness but exact lengths. We gave some reasons for doing so in the volume on geometry. The truth is that we replaced lines which have some thickness and lengths which we can only estimate with some uncertainty by imaginary lines which are sharp and clear.

This is an example of making a mathematical model to replace the somewhat hazy things that we observe. The real reason for doing this is to make our lives simpler. It is easier to think in terms of the model than the fuzzier reality.

The place of mathematics in the world of today can be understood only in terms of this idea of replacing the things and events that we observe by ideas. These ideas are the products of our imagination. They are clear and sharp. We think about them more easily than about the things themselves.

It is a little like using money instead of exchanging goods directly. Instead of finding someone who wants what we have in exchange for what he has, it is easier to find someone who wants what we have and will give us so many shillings for it. We can then use the shillings to buy what we want. It makes everything easier.

Mathematics is a kind of coinage which everyone can use for many purposes. Like coinage, it is something that had to be invented. Numbers can be used as measures of many, many sorts of things. Mathematics is universal. It applies in all countries and at all times. As we go on we shall see this more clearly. For now we add only one remark. It may seem impractical to talk about unending decimals. For who could carry out measurements even to a large number of decimal places? Suppose, therefore, that we limited ourselves to 4 decimal places. Wouldn’t that be sensible? There are two arguments against doing this. The improvement of measuring techniques has already made it possible to measure many quantities with much greater accuracy. It is sensible to be ready for these improvements and not be unprepared. It is also true that if we multiply two numbers with 4 decimal places we get answers with more places so that we have to be able to go farther, even if we later throw away some of the digits. Restrictions make things more complicated. In mathematics we seek both generality and simplicity. We want a coinage that can be used as widely as possible and with as much convenience as possible.
On the Use of Scales

Any quantity that can be measured by a unit which can be subdivided into smaller ones can be shown on a scale, that is, a line marked with units and subunits (a number line). The quantity does not need to be a length to use lengths to show it. For example, we can represent on a scale, intervals of time or weight or speeds or costs. When we do this we should write the name of the unit of measure near the line to make it easy to read off the measurement (numbers and units). For example, we might mark a scale in this way:

```
0 1 2 3
```
seconds

The dot shows $1 \frac{1}{2}$ seconds.

Of course a time is not a length. But intervals of time act like lengths. They add and subtract the way lengths do. So lengths can be used to show time intervals. This is a very important fact. Modern science owes much to Galileo who around 1600 used scales for times and speeds. Before his time people did not represent them as lengths. These quantities obviously are not lengths so it took a very original mind to treat them as if they were. When we study graphs we shall see that by choosing one scale to show one quantity and another scale to show another, we can draw pictures that help us to think about how one quantity depends on another. Graphs are one of the great discoveries of mankind. The first step in making graphs is to construct scales for quantities. So it is important to understand these scales.

Look at the following picture in which the scale is marked "ft." to show measures of length.

```
0 1 2 3 4
```
ft.

It is obvious that the interval from 0 to 1 is not really a foot long. It does not matter that it is not. We use the scale to picture things to ourselves. The picture does not need to be as big as what it is a picture of, any more than the picture of a house need be as big as the house. (If it were, we could not get it on the paper!) But, as we have said, our picture, that is, our scale, may not show lengths at all, but quantities like time intervals, speeds, amounts of money, and so on.

Direct and Indirect Measurements

We have seen that we can show on a line with a scale the measures of quantities which are not lengths at all. This leads us to notice that when we
measure some of these other quantities in practice we actually do find ourselves measuring lengths. Let us consider some examples.

Suppose that we wish to measure weights. There are two ways to do this. One is the direct way. We take a bar with pans hanging from its ends and balance this bar on a sharp edge in the middle. We have a set of standard weights (copies of a pound weight, for example). We put the object to be weighed in one pan and enough standard weights in the other so that the bar is level. This method is direct because it compares the quantity with a unit of the same kind.

But we can weigh the object in an entirely different way. We can hang it on a spring which stretches a certain amount. A pointer attached to the spring moves along a scale beside it. We find that twice as heavy a weight stretches the spring twice as much. More generally, the stretch is proportional to the weight. Of course this fact had to be discovered before we could use a spring to weigh with. That is, we should have to know the result of hanging 1 pound, 2 pounds, 3 pounds, etc., on the spring. When springs were first studied this had to be thoroughly investigated, using not only whole numbers of pounds but fractional units as well. But now that we know how a spring acts we can use the stretch to measure the weight. What we do is to mark the scale with 0, 1, 2, and so on, where the point marked 0 is at the end of the pointer when there is no weight attached and the point marked 1 at the end of the pointer when the weight is 1 pound, and so on. When we use the spring to measure the weights of objects whose weight is unknown we take advantage of what has been learnt with known weights.

The method of weighing with a spring scale is an example of an indirect measurement. It is easy to think of other examples of indirect measurements which show the result of the measurement by a reading on a scale. An ordinary watch or clock shows the time on a scale marked on a circle. The scale is marked in time units, say minutes, but these marks correspond of course to lengths along the circle. The fact that time can be measured in this way is the result of a long history going back to the early Egyptians. It is too long a story to tell here.

We find a simpler example of the two kinds of measurement (direct and indirect) if we think about volume. We might find the volume of a tank by
pouring water into it from a tin can filled to the brim. We simply count the number of cans of water that the tank holds. This is a direct measurement of volume in terms of a unit volume.

If the tank is of a simple shape we can also get the volume by a calculation. Suppose, for example, that the tank is box-shaped (a rectangular parallelepiped). By measuring the length, breadth and height of the tank in feet and multiplying the three measures together we obtain the volume in cubic feet. This is an indirect measurement which relies on geometry to obtain the result.

Our last example will be the measurement of speed. How fast is an automobile moving? We assume for simplicity that the motion is a steady one. We commonly say that the speed is so many miles per hour. It is very hard to think of how this might be measured directly. The "miles per hour" is usually written \( \frac{\text{mi.}}{\text{hr.}} \) which strongly suggests that we get the speed by dividing the number of miles by the number of hours. If, for example, it takes \( \frac{1}{2} \) hour to travel 25 miles, we would calculate the speed as

\[
\frac{25 \text{ mi.}}{\frac{1}{2} \text{ hr.}} = 50 \frac{\text{mi.}}{\text{hr.}}
\]

Of course, as you probably know, an automobile has an instrument which has a scale and a pointer. This instrument is called a speedometer. We can read the position of the pointer on the scale and get the speed without a calculation. This looks like a direct measurement but it really is not. It depends on knowing how the speedometer works in much the same way that weighing with a spring depends on knowing how a spring works. When we measure speed with a speedometer we do not directly compare the speed with a unit speed that goes into it a certain number of times.

**EXERCISE 58-8**

1. The ancient Egyptians measured time by a water clock. A large tank was kept filled with water. A small pipe at the bottom could be opened or closed. By letting water run into a cup they could measure the amount of water which collected in that time. How could they choose units for measuring the time? Why do you think that the tank was kept full of water?

2. Count your pulse or the beats of your heart. Explain how you could measure time in this way. Why isn't this a very good way? (What happens when you get excited?)

3. Can you think of a way to measure how clearly you can see at a distance?

4. Can you think of a way to measure how good your hearing is?

5. How could you measure how good a pupil's memory is?
6. Have you any idea how to measure the intelligence of a student? (We are not trying to test how well he has studied his lesson.) Are there perhaps different kinds of intelligence?

58-9 Some Indirect Measurements

It will be interesting to show how some other quantities can be measured indirectly. These examples will make it clear how mathematics is actually used in studying nature and studying ourselves.

Example 1

What angle is there between lines from the eye to opposite sides of the sun? How could this angle be measured?

Let us imagine that two sticks are nailed together at one end. By opening them so that we could sight one edge of the sun along one stick, and the other edge along the other stick, we could measure the angle between the two sticks.

But this experiment is not very practical. It is too hard to look at the sun without hurting our eyes. Also the angle between the two sticks would be so small that it would be very hard to measure.

What shall we do? The ancient Babylonians and Egyptians did the following. They started a water clock at the instant when the first light from the sun appeared on the horizon at sunrise. They let the water run until the sun had completely risen. That is, they timed the sunrise. The result was about 2 min. $\frac{1}{30}$ hr. $\frac{1}{720}$ of a day. But in one day the sun reappears at the horizon for a new sunrise. It has therefore moved 360°. It follows that the sun is $\frac{1}{720} \times 360^\circ = \frac{1}{2}^\circ$ wide. Of course it is assumed that the sun moves at a uniform rate across the sky.

Example 2

In the Introduction we quoted Galileo who said: "Measure what is measurable and make measurable that which is not." We are now able to understand the last part of this sentence. Of course Galileo meant that we should try to measure things which most people think cannot be measured. Galileo himself set us an example.

Since ancient times, people had used words like "hot" and "cold" to describe the properties of bodies. Bodies were said to have these qualities, but before Galileo there was no attempt to measure how hot or how cold a body was. He
thought of a *quantity* which could be measured to tell us how hot or cold a body was. We call this quantity the *temperature* of the body. And Galileo invented an instrument called a *thermometer* to measure this quantity.

How can temperature be measured? How does a thermometer work? The first thermometer was a bulb with a stem, partly filled with oil. If you put the bulb in your mouth, the oil will heat up and the oil will rise in the tube. If you put the bulb in cold water, the oil level will fall. By placing a scale alongside the stem, we can measure how much the oil rises. The reading on the scale is a measure of the temperature.

The modern thermometer is a development of this idea. The oil is replaced by mercury which expands more than oil. Two scales are commonly used.

On the Centigrade scale the position of the mercury level when the thermometer is in melting ice is marked $0^\circ$, and the position when it is in boiling water is marked $100^\circ$. The interval between these two points is divided into 100 equal parts. As we heat water from the freezing point to the boiling point the mercury level rises steadily.

On the Fahrenheit scale the position of the mercury at the freezing point is marked $32^\circ$ and the position of the boiling point is marked $212^\circ$. Therefore, $212 - 32 = 180$ degrees Fahrenheit are equivalent to 100 degrees Centigrade. A Fahrenheit degree is $\frac{180}{100} = \frac{9}{5}$ of a Centigrade degree.

Whichever scale we use we see that we have used length along the scale as a measure of temperature. Temperature has been made measurable when before Galileo it was not thought to be measurable.

**EXERCISE 58-9**

What reading on the Fahrenheit scale corresponds to a reading of 50 degrees on the Centigrade scale?

In the last section we asked some questions about measuring how intelligent a pupil is. It will be interesting to discuss these questions because they bring out some important points.

It is possible to set an examination to measure, at least roughly, how well a pupil has studied his lesson. We must of course ask good questions. Also we try to set questions that are as nearly equivalent as possible. It is hard to make a really good examination, but teachers learn to do it.

But sometimes we wish to know how *intelligent* a pupil is. We may think that
he is lazy so that, if he really wanted to, he could do much better school work than he does. How can we find out his inborn ability to learn? We imagine that he has something that we call intelligence. But this is not something that we can take out and look at. We cannot get inside his mind! We must judge in terms of things that he can do, things that we can see or hear.

Of course we can ask him questions. If we think that he does not study because he is lazy or lacks interest, it will not do to test how well he has learned his lesson. We must ask questions which he can answer if he is bright, whether he has studied or not. We must give him a test that he cannot prepare for. We must test his ability to reason correctly about things that he has never seen before. We may ask him to fill in missing words in a sentence as a test of understanding. We may ask him to put blocks of different shapes together to form a figure like a rectangle, to test his ability to see patterns. In brief, we give him some little puzzles to solve.

Now tests like this have been worked out after much study and experimentation. For children of a certain age from any given cultural background we know rather well what an "average child" can do on one of these tests. We know what per cent of children of this age group can get what scores. We can measure what is called their I.Q. (intelligence quotient).

Then we can study how well pupils who get such and such scores succeed on the average.

It is not easy to invent satisfactory intelligence tests. Testing has become a specialty which takes much time and thought. The point that we wish to make is that useful measurements can be made. One of the difficulties is to have a clear idea of what it is that we want to measure, that is, what we mean by "intelligence".

We have twice used the word "average". We spoke of the "average child" and "succeed on the average". In Unit XIII we shall have much to say about averages. For the present it is enough to notice that in cases like these where there are considerable differences in what children do, it is necessary to put together many scores (numbers) in some sensible way in order to know what to expect. When we try to use numbers in studying people we almost always use the methods of statistics.
Chapter 59
RATIO AND PROPORTION

59-1 Ratios of Numbers

Suppose that we are asked to compare the sizes of two given sets, that is to compare the numbers of elements of the two sets. For example, let us compare the number of banana trees on Hosea’s farm with the number of such trees on Ali’s farm. Counting we find that Hosea has 18 banana trees and that Ali has only 6. We may say that Hosea has 12 trees more than Ali, the number 12 being obtained by subtracting 6 from 18. We may also say that Hosea has three times as many trees as Ali, the number 3 being obtained by dividing 18 by 6.

Thus a comparison of two numbers is possible by one of the following two methods:

1. By subtracting the smaller number from the larger, we find by how much one number is larger than the other.
2. By dividing the larger number by the smaller, we find how many times greater the larger number is than the other. *)

In mathematics the method of comparing numbers by division is the more important and more accepted one. Instead of saying that we compare two numbers by division, we say that we find the ratio of one number to the other. In the example just considered, we look for the ratio of 18 to 6, denote it with the help of a colon “:” as 18:6 and read it “18 to 6”.

Since ratio indicates comparison by division,

\[ \frac{18}{6} \text{ means } 18 \div 6. \]

On the other hand, \( \frac{18}{6} \) can be written as \( \frac{18}{6} \), remembering that a fraction may be viewed as the quotient of the numerator by the denominator. Thus we have

\[ \frac{18}{6} = \frac{18}{6} = \frac{18}{6} \]

\[ \text{ratio quotient fraction} \]

Simplifying the last fraction, we obtain

\[ \frac{18}{6} = \frac{3}{1} = 3. \]

*) By dividing the smaller number by the larger, we find what part of the larger number the smaller is.
In order to emphasize that division is used here to compare numbers, that is, to find their ratio and not merely to obtain the result of division, the quotient, we keep two numbers in the final form, even if the quotient turns out to be a natural number. Thus it is customary to write

\[ 18:6 = \frac{18}{6} = \frac{3}{1} \text{ or } 18:6 = 3:1, \]

and we read it "The ratio 18 to 6 is the same as the ratio 3 to 1." But we do not read "The ratio 18 to 6 is 3." We may, however, say that 18 is three times 6. In comparing the banana trees on the two farms, we say that the ratio of the number of banana trees on Hosea's farm to the number of banana trees on Ali's farm equals 3 to 1.

Expressions like

- The ratio of the number 18 to the number 6,
- The ratio of the numbers 18 and 6, have the same meaning.

Let us take another example. In finding the ratio of the numbers 6 and 21, we have

\[ 6:21 = \frac{6}{21} = \frac{2}{7} = 2:7, \]

and read it "6 is to 21 as 2 is to 7", or "6 is related to 21 as 2 is related to 7", rather than "6 is \( \frac{2}{7} \) times 21".

In our previous examples we found ratios of natural numbers. The same procedure is applied to find a ratio of two fractions.

Examples

\[ \frac{9}{14} : \frac{3}{7} = \frac{14}{3} = \frac{9 \times 7}{3 \times 3} = \frac{9 \times 7}{13 \times 3} = \frac{3}{2} = 3:2. \]

\[ 5 : \frac{1}{20} = \frac{5}{20} = 5 \times \frac{20}{1} = \frac{100}{1} = 100:1. \]

We see that the ratio of any two numbers, the second of which is different from zero, is a number which we may write as a fraction in lowest terms.

**EXERCISE 59-1**

Express each of the following ratios in the simplest form:

(a) 56 to 7

(g) \( \frac{1}{3} \) to \( \frac{1}{2} \)
Ratios of like quantities

You have already learned about quantities. We will now discuss ratios of quantities. The lengths of two segments, whether expressed in the same unit or not in the same unit, are called like quantities.

Thus

17 inches and 5 inches are like quantities.
8 inches and 3 yards are like quantities.

Similarly each of the pairs
5 lb and 16 lb,
3 ounces and 4 lb,
2 minutes and 7 hours
represents like quantities. However, 4 lb and 3 yards are not like quantities.

In the same way that any two numbers are compared, we may compare any two like quantities as long as they are measured in the same unit. Suppose we want to compare the lengths of two given straight line segments. We measure the length of each and get, for example, 49 inches and 21 inches. To compare these two like quantities expressed in the same unit, or more precisely, to compare the measures of the two quantities expressed in the same unit, we find the ratio of their measures. That is, we divide the measure of one quantity by the measure of the other quantity.

In our case we write:

49 inches : 21 inches = \frac{49}{21} = \frac{7}{3} = 7:3,

and read it "The length of the larger segment to the length of the smaller segment is as 7 to 3."

We thus see how to compare the measures of two like quantities expressed in the same unit. If we are to compare like quantities such as 6 minutes and 3 hours, which are not expressed in the same unit, we cannot write

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<td>(b)</td>
<td>8 to 40</td>
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<td>(c)</td>
<td>.2 to 1</td>
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<td>(d)</td>
<td>1.3 to 6.5</td>
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<td>(e)</td>
<td>1 to 6</td>
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<td>(f)</td>
<td>4 \frac{1}{2} to 12</td>
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<td>(h)</td>
<td>\frac{3}{4} to \frac{5}{6}</td>
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<td>\frac{3}{4} to 3</td>
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<td>(k)</td>
<td>2 \frac{1}{3} to \frac{4}{9}</td>
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<tr>
<td>(l)</td>
<td>7 \frac{1}{2} to 1 \frac{1}{4}</td>
</tr>
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</table>
6 minutes : 3 hours = $\frac{6}{3}$ = $\frac{2}{1}$

and say that the ratio of 6 minutes to 3 hours is 2:1. We find this ratio as follows:

$$6 \text{ minutes} : 3 \text{ hours} = 6 \text{ minutes} : 180 \text{ minutes}$$

$$= \frac{6}{180} = \frac{1}{30} = 1:30$$

or 6 minutes : 3 hours = $\frac{1}{10}$ hour : 3 hours = $\frac{1}{3} = \frac{1}{30} = 1:30$.

In the same way,

1 yard : 18 inches = 36 inches : 18 inches

$$= \frac{36}{18} = \frac{2}{1} = 2:1,$$

or 1 yard : 18 inches = 1 yard : $\frac{1}{2}$ yard = $\frac{1}{2}$ = 2:1.

30 ounces : 5 lb = 30 ounces : 80 ounces

$$= \frac{30}{80} = \frac{3}{8} = 3:8,$$

or 30 ounces : 5 lb = $\frac{14}{16}$ lb : 5 lb = $\frac{14}{10}$ = $\frac{30}{16}$ = $\frac{6}{8}$ = 3:8.

We see that to find the ratio of two like quantities, we must express them in the same unit of measure. The ratio of two like quantities expressed in the same unit is the ratio of their measures, and is obtained by dividing the measure (number) of one quantity by the measure (number) of the other quantity. The ratio is a number (not a quantity), and no unit is attached to it.

The antique Greeks already used ratios in the 4th and 3rd centuries before our era. A ratio which has occupied mathematicians for over two thousand years is the ratio of the circumference (length) of a circle to the length of its diameter. This ratio is denoted by the letter $\pi$.

$$\pi = \text{Length of circumference of a circle} : \text{Length of diameter of the circle}.$$
EXERCISE 59-2

Express in the simplest form each of the following ratios of like quantities:

1. 4 hours to 12 hours
2. 18 inches to 4 inches
3. 15 lb. to 70 lb.
4. 1 ft. to 1 inch
5. 1 ft. to 1 yard
6. 1 lb. to 1 ounce
7. 1 minute to 1 hour
8. 1 second to 1 hour
9. 1 yard to 1 mile
10. 1 metre to 1 kilometre
11. 1 square ft. to 1 square inch
12. 1 square ft. to 1 square yard
13. 1 square yard to 1 square mile
14. 1 cubic metre to 1 cubic centimetre
15. 1 lb. to 4 ounces
16. 3 hours to 25 minutes
17. 3 hours to 25 seconds
18. 2 yards to 15 feet
19. 2 yards to 15 inches
20. 1 square ft. to 4 square inches
59-3 Properties of ratios

We have learned what a ratio is and how a ratio of two numbers or of like quantities is found by division. We will now use known facts about the operation of division to state some basic properties of ratios.

Since division by zero is impossible, we have

I. The members of a ratio may be any two numbers, except that the second member must be different from zero.

Recalling that a fraction may be viewed as the quotient of the numerator by the denominator, we have

II. Any fraction may be viewed as representing the ratio of its numerator to the denominator.

For example,

\[ \frac{2}{5} = 2 \div 5 = 2 : 5. \]

We will now use the following known property of fractions:

If \( c \) is any number different from zero, then

\[ \frac{a}{b} = \frac{ca}{cb} \]

(1)

and

\[ \frac{a}{b} = \frac{a}{c} \cdot \frac{c}{b} \]

(2)

Since ratios are expressed by fractions, we have, by (1),

for \( c \neq 0 \)

\[ ca : cb = \frac{ca}{cb} = \frac{a}{b} = a : b \]

hence

\[ ca : cb = a : b \]

(3)

Similarly, by (2),

\[ \frac{a}{c} : \frac{b}{c} = \frac{a}{c} \cdot \frac{c}{b} = \frac{a}{b} = a : b. \]
hence \[ \frac{a}{c} : \frac{b}{c} = a : b. \] (4)

Equalities (3) and (4) show the following property of ratios:

III. A ratio of two numbers does not change if each of its members is multiplied by the same nonzero number or if each of its members is divided by the same nonzero number.

Examples.

The ratio \( \frac{4}{7} : \frac{3}{5} \) is the same as the ratio \( 20 : 21 \), obtained by multiplying both members of the first ratio by \( 7 \times 5 = 35 \).

The ratio \( 46 : 115 \) is the same as the ratio \( 2 : 5 \), obtained by dividing both members of the first ratio by their common factor 23.

Let us consider the ratio of 10 to 4.

\[ 10 : 4 = \frac{10}{4} = \frac{5}{2}. \]

This equality may be viewed as indicated

First member of the ratio \[ \frac{10}{4} = \frac{5}{2} \] value of the given ratio

Second member of the ratio

Thus 10 may be viewed as the dividend, 4 as the divisor, and \( \frac{5}{2} \) as the quotient.

Since the product of the quotient by the divisor equals the dividend, we have

\[ \frac{5}{2} \times 4 = 10. \]

value of the ratio Second member of the ratio First member of the ratio

We obtain the following property of ratios:

IV. The product of the ratio of two numbers by the second member of the ratio equals the first member.
Examples.

\[ 2 : 7 = \frac{2}{7}, \text{ therefore } \frac{2}{7} \times 7 = 2. \]

\[ \frac{2}{3} : \frac{7}{15} = \frac{2}{7} \times \frac{7}{15}, \text{ therefore } \frac{10}{7} \times \frac{7}{15} = \frac{2}{3}. \]

Property IV is also valid for ratios of like quantities.

For example,

\[ 5 \text{ ft} : 60 \text{ ft} = \frac{5}{60} = \frac{1}{12}, \text{ thus } \frac{1}{12} \times 60 \text{ ft.} = \frac{5}{5} \text{ ft}. \]

As another example, the ratio

\[ 1 \text{ metre} : 1 \text{ kilometre} = \frac{1 \text{ metre}}{1,000 \text{ metre}} = \frac{1}{1,000}, \]

hence

\[ 1 \text{ metre} = \frac{1}{1,000} \times 1 \text{ kilometre}. \]

The relationship

\[ \frac{\text{First member}}{\text{Second member}} = \text{Value of the ratio} \]

gives us also the relationship

\[ \text{V.} \]

\[ \frac{\text{Second member}}{\text{First member}} = \frac{\text{value of the ratio}}{\text{of a ratio}} \]

Properties IV and V allow us to find one of the members of a ratio if the other member and the ratio are given.
Example 1.

The ratio of a number to 3 is as 5 is to 12.
Find the number.
We may write it as follows

\[ \frac{\square}{3} = \frac{5}{12}, \] hence by IV, \[ \square = \frac{5}{12} \times 3 = \frac{5}{4}. \]

The number is \( \frac{5}{4} \).

Example 2.

Ali is 35 years old. The ratio of Ali's age to his son's age is as 5 to 2. How old is Ali's son?

\[ \frac{35 \text{ years}}{\square} = \frac{5}{2}, \] therefore by V, \[ \square = \frac{35 \text{ years}}{\frac{5}{2}} = 35 \times \frac{2}{5} = 14 \text{ years}. \]

Ali's son is 14 years old.

EXERCISE 59-3

1. Express each of the following as a ratio of natural numbers in simplest form:

   (a) \( \frac{4}{5} : \frac{3}{15} \)

   (b) \( 2\frac{2}{3} : 3\frac{3}{4} \)

   (c) \( 1 : .08 \)

   (d) \( .2 : .125 \)

   (e) \( 1.2 : 4\frac{1}{5} \)

2. Express in the simplest form each of the following ratios:

   (a) \( 114 : 171 \)

   (b) \( 725 : 375 \)

   (c) \( 15,000 : 2,100 \)
3. Find the unknown member in each of the following ratios:

(a) □ : 3 = \frac{5}{12}

(b) □ : \frac{2}{3} = \frac{3}{8}

(c) 3 : □ = 4

(d) 1.25 □ = \frac{5}{2}

(e) 1 hour 40 minutes : □ = 5 : 3

(f) □ : 5 metres = \frac{16}{3}

(g) 12 ounces : □ = 3 : 7

59-4 Per cent expression of a ratio

In some cases it is more convenient and more instructive to represent a ratio as a per cent rather than as a fraction.

Suppose that a school has 288 pupils of whom 117 are boys. The ratio of the number of boys to the total student population is

\[ 117 : 288 = \frac{117}{288} = \frac{13}{32}. \]

Suppose now that another school has 225 pupils of whom 99 are boys. There the ratio of the number of boys to the total student body is

\[ 99 : 225 = \frac{11}{25} \]

If we want to compare the ratios in the two schools, if we want to know, for example, in which school this ratio is higher, to say that

The ratio in school #1 is \( \frac{13}{32} \)

The ratio in school #2 is \( \frac{11}{25} \)
does not answer the question clearly. In such cases it is convenient to express the
ratios as per cents. We shall see that comparison is then very easy.

Other instances when ratios are usually expressed in per cents:
The ratio of the number of literate people to the total population of a given
country.
The ratio of the amount spent on education in a year to the total budget of a
country.
The ratio of the number of children attending school to the total number of
children of school age.

In order to clarify what we mean by the per cent expression of a ratio let us
consider the following.

Example. A school ordered 200 mathematics textbooks and after a week
received 70 books. (The ratio of the number of books received to the number of books
ordered is 70 : 200 = \(\frac{7}{20}\)). What per cent of books were received?

We have to answer the question: What per cent of the number 200 does 70
represent? In other words, if 200 is taken to be 100% then what per cent does 70
represent?

We reason as follows:

If 200 is taken to be 100%, then to 1% there correspond \(\frac{200}{100}\) or 2 books. If 2
books correspond to 1%, then 70 books correspond to \(70 \div 2\) or 35%. This means that 35%
of the books were received. Thus the per cent expression of the ratio 70 : 200 is 35%.

In order to see the general procedure of expressing a ratio as a per cent, we will
review the operations we just performed.

\[
\begin{align*}
70 : \frac{200}{100} &= \frac{70 \times 100}{200} = \frac{70}{200} \\
&= \frac{35\%}{100} \\
&= 35\%.
\end{align*}
\]

This means that

From the last equality we see that

To find the per cent expression of a ratio we simply multiply the given ratio by
100.

Let us now return to our first example. The ratio of the number of boys to the
total population in school #1 is \(\frac{13}{32}\).

This ratio represents

\[
\frac{13}{32} \times 100 = \frac{325}{8} = 40.625\%.
\]
The same ratio in school #2 is $\frac{11}{25}$ which is

$$\frac{11}{25} \times 100 = 44\%.$$ 

We thus see that the ratio of boys to all students in school #2 is higher.

**EXERCISE 59-4**

1. Express each of the following ratios as a per cent.
   
   (a) $3 : 8$
   
   (b) $7 : 3$
   
   (c) $4.3 : 12.5$
   
   (d) $\frac{2}{3}$ yards : $\frac{3}{4}$ yards
   
   (e) 12 minutes : 3 hours

2. Express each of the ratios $6 : 11$ and $9 : 14$ as a per cent and find which ratio is higher.

3. Mohamed took an arithmetic test of 20 problems and gave 17 correct answers. He also took a language test of 25 questions and answered 21 correctly. Use per cent expressions of ratios to find on which test his performance was higher.

**59-5 Scale drawings**

On various occasions you have all seen geographical maps of your country, maybe a map of your capital city, the plan of a house, or the floor plan of your school. These are examples of drawings to scale or simply *scale drawings*. Each of them represents a diminished picture of the country, city, or school, respectively. Obviously none of the above could be reproduced on paper in its natural size.

A correct scale drawing or plan must give us the opportunity to estimate the actual measures of the object which is drawn. That is, the plan should indicate how many times a straight line segment on the drawing is smaller than the corresponding segment in reality.

If, for example, a window in a house whose plan is drawn is 1 yard wide and the window is represented on the plan by a segment of 1 inch, that is, by a segment 36 times smaller, then the plan is made in the ratio 1:36. In this case, we say that the *scale of*
the plan is 1 to 36, and indicate in a corner of the drawing one of the following:

Scale 1 : 36

Scale \(\frac{1}{36}\)

1 inch = 1 yard

The last statement simply means that to a segment of length 1 inch on the drawing there corresponds a segment of the object in reality of length 1 yard. Quite often this scale is indicated by a segment marked as follows:

\[
\begin{array}{cccc}
0 & 1 \text{ yard} & 2 \text{ yards} & 3 \text{ yards} & 4 \text{ yards} \\
\hline
\end{array}
\]

Actual length
1 inch

Having the plan of a house and knowing that it is drawn to the scale 1:36, we are able to find the actual size of any part of the house by measuring that part on the drawing. If, for example, on such a plan we measure the length of a room and find it to be 6 inches, then the actual length of the room is 36 times larger, that is 216 inches or 6 yards.

We thus see that the scale of a map or plan is the ratio of the length of any straight line segment on the drawing to the length of the corresponding line segment in reality.

Any scale may be chosen by us in drawing a plan. The choice is usually determined by the size of the object to be drawn and by the size of the paper on which we draw. For example, one geographic map of your country may have the scale of 1 inch to 100 miles. A larger map may be drawn to the scale of 1 inch to 10 miles.

If on the map of a country with the scale of 1 inch to 100 miles the straight line distance between two cities is 3.4 inches, then the actual air distance between the two cities is approximately 340 miles.

We discussed maps and plans representing diminished pictures. That means that the drawings are made to a scale which is less than 1. However in a zoology book we may often see a drawing or picture of an insect with an inscription like "Enlarged 30 times", which means that the scale is 30:1. If the length of the insect on paper is 2 inches, then the insect has the actual length \(\frac{1}{15}\) inch.

EXERCISE 59-5.

1. On a map with the scale 1 inch = 50 miles, two cities are \(3 \frac{1}{2}\) inches apart.
Approximately how far is it from one city to the other?

2. Find the scale of a map, if the distance between two signs equals 750 metres, and on the map it is $\frac{1}{2}$ cm.

3. A pupil is making a scale drawing of his schoolroom floor, which is 32 ft. by 24 ft. What dimensions will his drawing have if he uses the scale 1 inch $= 8$ ft.? If the top of the teacher's table is 3 ft. wide and 5 ft. long, what dimensions will it have in his drawing?

4. Modupe wants to make a scale drawing to represent the school's assembly hall 64 ft. long. His paper is $8\frac{1}{2}$ inches wide. Which of the following scales should he use?

   (a) 1 inch $= 1$ ft.
   (b) $\frac{1}{4}$ inch $= 1$ ft.
   (c) $\frac{1}{8}$ inch $= 1$ ft.

5. On a given map, 1 inch stands for 1 mile. Then each inch on the map stands for how many inches on the ground? What is the ratio of the length of any line segment on the map to the real distance that it represents?

6. The scale on a map is 1 inch to 20 miles.

   (a) Express this scale as a ratio.
   (b) On the map how many miles are represented by a segment of length $\frac{1}{4}$ inches?

7. A science book contains a picture of an insect marked "Enlarged 12 times". If the insect is $\frac{1}{2}$ inch long in the picture, what was its real length?

8. In a scale drawing of the floor plan of a house 1 inch represents 6 feet.

   (a) Express this scale as a ratio.
   (b) On the scale drawing how many feet are represented by a segment of length 20 inches?
   (c) In (b) what is the ratio of the length of the segment on the drawing to the number of feet represented by this segment?

9. Measure with a tape the length and width of your class room and draw its floor plan to the scale 1:96. Measure the doors, windows, etc., and their distances from the corners of the room. Indicate clearly the doors, windows, blackboard, the teacher's desk, etc. A line segment of length 1 ft. will on this plan be represented by a segment of what length?

10. The length and width of a rectangular piece of land are represented on a plan with the scale $\frac{1}{10,000}$ by 15 cm. and 12 cm., respectively. How many hectares does the piece of land measure? (1 hectare = 10,000 square metres)

11. On a map with the scale 5 cm. $= 1$ km, an island covers 42.25 square centimetres. What is the actual area of the island?
In 59-4 we considered an example where we needed to find which of two ratios is the larger one. These were the ratios of the number of boys to the total student population in two schools. Such situations arise quite often. A natural way, for example, of evaluating the economic, social or educational development of two or more countries consists precisely in comparing the values of certain ratios that reflect the country’s achievements in the relevant field. You will study the use of ratios for such purposes in Unit XIII.

We have already given one method of comparing ratios with different denominators, namely by expressing the ratios as per cents. In case the ratios have different denominators which are rather small numbers, we can easily compare their values as follows.

Example. Find which of the ratios \( \frac{3}{5} \) and \( \frac{4}{7} \) is the larger one.

Expressing the two fractions in a common denominator, we obtain, respectively,

\[
\frac{21}{35} \quad \text{and} \quad \frac{20}{35}.
\]

This shows that \( \frac{4}{7} \ < \ \frac{3}{5} \).

In many instances, however, especially in mathematics and in the physical sciences, we deal with equal ratios. Let us take a familiar example. Suppose that we measure on a plan of a house drawn to the scale 1:100 the width of a window and find it to be 0.7 inches. Then the width of the window in reality is approximately 70 inches. Again, if on the plan the length of a bedroom is 1.8 inches, then the length of the bedroom in reality is approximately 180 inches. Clearly the ratio of the width of the window on the plan to its actual width equals the ratio of the length of the bedroom on the plan to the length of the bedroom in reality, for each of these equals the ratio 1:100 representing the scale. Thus we have

\[
0.7 \text{ inches} : 70 \text{ inches} = 1.8 \text{ inches} : 180 \text{ inches}.
\]

A statement like this is called a proportion.

More precisely,

\[a \text{ proportion is a true statement of equality of two ratios.}\]

Thus

\[2 : 5 = 14 : 35\]

is a proportion, because the ratios are equal and the statement is true. On the other hand,
the statement

\[ 4 : 7 = 3 : 5 \]

is not a proportion, since the ratios are distinct and the statement of their equality is false. We have actually seen that the following statement is true:

\[ 4 : 7 \neq 3 : 5. \]

A proportion may be written with colons as above or in fractional form

\[ \frac{2}{5} = \frac{14}{35}. \]

In this form *a proportion simply means that two fractions are equivalent.*

In the general case, the proportion

\[ a : b = c : d \]

is read "a is to b as c is to d" or "a is related to b as c is related to d."

Other examples of proportions are

\[ 5 : 2 = 15 : 6, \]
\[ 4 \text{ inches} : 3 \text{ inches} = 28 : 21, \]
\[ 4 \text{ inches} : 3 \text{ inches} = 12 \text{ inches} : 9 \text{ inches}, \]
\[ 4 \text{ inches} : 3 \text{ inches} = 20 \text{ lb.} : 15 \text{ lb.} \]

**EXERCISE 59-6**

1. Determine which of the following statements is a proportion.

   (a) \( \frac{2}{5} = \frac{6}{15} \)

   (b) \( \frac{3}{4} = \frac{16}{20} \)

   (c) \( \frac{2}{7} = \frac{30}{105} \)

   (d) \( \frac{1}{3} = \frac{\frac{1}{2}}{7} \)

2. In each case form a proportion from the four given numbers.

   (a) 75 ; 45 ; 30 ; 18.
Basic properties of proportions

Let us take a simple proportion such as $3 : 7 = 9 : 21$, and write it in fractional form

$$\frac{3}{7} = \frac{9}{21}. \quad (1)$$

Multiplying both sides of this equality by $7 \times 21$, the product of the two denominators, we obtain

$$\frac{3}{7} \times 7 \times 21 = \frac{9}{21} \times 7 \times 21.$$

Simplification gives

$$3 \times 21 = 9 \times 7. \quad (2)$$

This means that for our equivalent fractions the products of the numerator of one fraction by the denominator of the other are equal. The products obtained in (2) are called cross products of (1). The diagram

explains the name.

Take some other proportions and see whether the cross products for each of them are equal. You will easily verify that this is true in each case. The statement that for any proportion the cross products are equal is always true, and represents a basic property of proportions. We state it in general form.

**Property 1.** If $b \neq 0$ and $d \neq 0$ and \( \frac{a}{b} = \frac{c}{d} \), then $a \times d = b \times c$.

We prove this statement by following the procedure of the previous example.
Multiplying both sides of the proportion by \( b \times d \), we obtain

\[
\frac{a}{b} \times b \times d = \frac{c}{d} \times b \times d,
\]

and after simplification we have

\[
a \times d = c \times b,
\]

which asserts the equality of the cross products.

Conversely, let us now start with two equal products

\[
3 \times 24 = 9 \times 8. \tag{3}
\]

Dividing both sides by \( 24 \times 8 \), we obtain

\[
\frac{3 \times 24}{24 \times 8} = \frac{9 \times 8}{24 \times 8}.
\]

Simplification gives

\[
\frac{3}{8} = \frac{9}{24}, \tag{4}
\]

which is a proportion. We have thus found that if a product of two numbers equals the product of two other numbers, then the four numbers form an appropriate proportion. This is an instance of the following general statement.

\textit{Property II.} If \( b \neq 0 \) and \( d \neq 0 \) and \( a \times d = b \times c \), \( \tag{5} \)

then

\[
\frac{a}{b} = \frac{c}{d}.
\]

The proof follows the same patterns as in the above numerical example.

If \( b \neq 0 \) and \( d \neq 0 \), then \( b \times d \neq 0 \). Dividing both sides of (5) by the nonzero number \( b \times d \), we have

\[
\frac{a \times d}{b \times d} = \frac{b \times c}{b \times d},
\]

and after simplification we obtain the proportion

\[
\frac{a}{b} = \frac{c}{d}.
\]
Combining Properties I and II, we have the following statement:

Property III. Let \( b \neq 0 \) and \( d \neq 0 \). Then

\[
\frac{a}{b} = \frac{c}{d} \text{ if and only if } a \times d = b \times c.
\]

We can use Property III to check whether a given statement of equality of two ratios is a proportion.

If the cross products are equal, then we have a proportion. (This follows from Property II.)

If the cross products are not equal, then we do not have a proportion. (This follows from Property I.)

Example. Determine whether the statement

\[
\frac{7}{0.791} = \frac{5}{0.565}
\]

is a proportion.

The cross products are

\[7 \times 0.565 = 3.955,
\]

and

\[5 \times 0.791 = 3.955.
\]

Thus the ratios are equal and (6) is a proportion.

You may note that verification of the truth of equality (6) either by expressing each ratio as a percent or by representing the fractions with a common denominator would be quite tedious.

On the other hand, the statement

\[
\frac{7}{5} = \frac{11}{8}
\]

is not a proportion. That is, the statement is false and the fractions are not equivalent, since their cross products are different:

\[7 \times 8 \neq 11 \times 5.
\]

**EXERCISE 59-7A**

1. Test by the method of cross products which of the following statements are proportions.

   (a) \[25 : 37 = 75 : 111\],
(b) \[ 16 : 35 = 48 : 95, \]
(c) \[ 0.15 : 0.06 = 0.2 : 0.08. \]

2. Can you form a proportion from the given four numbers?

(a) \[ 13 ; 16 ; 65 ; 80. \]
(b) \[ 1.5 ; 2.5 ; 7.5 ; 12.5. \]
(c) \[ \frac{1}{8} ; \frac{1}{16} ; \frac{1}{12} ; \frac{1}{4}. \]
(d) \[ \frac{2}{5} ; \frac{3}{10} ; \frac{4}{7} ; \frac{7}{16}. \]

3. Given the three numbers \[ 5, 25, 4. \] Find a fourth number which forms a proportion with the given three numbers. Show that there are three such fourth numbers.

* * *

Property II states that if the product of two numbers equals the product of two other numbers, then an appropriate proportion can be formed with these four numbers. In the particular example, by dividing both sides of the equality

\[ 3 \times 24 = 9 \times 8 \] (3)

by \( 24 \times 8 \), we obtained the proportion

\[ \frac{3}{8} = \frac{9}{24}, \] (4)

The question arises how many proportions can we actually form with these four numbers. Dividing both sides of equality (3) not by \( 24 \times 8 \) but by \( 24 \times 9 \), we obtain after simplification

\[ \frac{3}{9} = \frac{8}{24}, \] (7)

which is again a proportion, since its cross products are equal. Proportion (7) is obviously different from proportion (4).

If we divide both sides of equality (3) by \( 3 \times 9 \), we get the proportion

\[ \frac{24}{9} = \frac{8}{3}, \] (8)
and similarly division of both sides of (3) by $3 \times 8$ gives the proportion

$$\frac{24}{8} = \frac{9}{3}.$$  \hfill (9)

We thus see that from a true statement of equality of two products, each consisting of two nonzero numbers, we obtained four different proportions.

The same procedure applies to the general case, and leads us to the following statement:

**Property IV.** If $a$, $b$, $c$, $d$ are nonzero numbers and

$$a \times b = c \times d,$$ \hfill (10)

then the following four proportions are valid.

$$\frac{a}{d} = \frac{c}{b},$$ \hfill (11)

$$\frac{a}{c} = \frac{d}{b},$$ \hfill (12)

$$\frac{b}{c} = \frac{d}{a},$$ \hfill (13)

$$\frac{b}{d} = \frac{c}{a}.$$ \hfill (14)

That each of the statements (11) – (14) is indeed a proportion can be easily verified, since the cross products of each of them are equal by the given equality (10).

Note, for example, that the statement

$$\frac{a}{b} = \frac{c}{d}$$

does not follow from (10) and is not a proportion, except for the very special case when $a = c$ and $b = d$, or $a = -c$ and $b = -d$.

We thus see that with four given nonzero numbers such that the product of two of them equals the product of the remaining two, four proportions with different ratios in each case can be formed.

We emphasized that we are dealing with nonzero numbers for the following reason. If one of the given equal products consists of two zeros, for example,

$$3 \times 0 = 0 \times 0$$

then no proportion at all can be formed (Why?).
If each of the equal products has one zero factor, for example,

\[ 3 \times 0 = 0 \times 7, \]

then only one proportion can be formed, by dividing both sides by the nonzero product \(3 \times 7:\)

\[ \frac{0}{3} = \frac{0}{7}. \]

**EXERCISE 59-7B**

1. Write all possible proportions that can be formed from the equality

\[ 3 \times 21 = 7 \times 9. \]

### 59-8 Finding an unknown member of a proportion

We shall consider various problems involving four numbers or pairs of like quantities that form a proportion. If only three of them are known, then Property I allows us to find the fourth unknown member of the proportion.

Suppose, for example, that we measure on a plan of a house drawn to the scale 1 : 200 the radius of a circular lawn and find it to be 2.6 inches long. Then the lawn has in reality a radius of length \(r\) inches that appears in the following proportion:

\[ \frac{2.6 \text{ inches}}{r \text{ inches}} = \frac{1}{200}, \]

or in fractional form

\[ \frac{2.6 \text{ inches}}{r \text{ inches}} = \frac{1}{200}. \]

By Property I, we have the equality of the cross products

\[ r \text{ inches} \times 1 = 2.6 \text{ inches} \times 200, \]

therefore

\[ r = \frac{520 \text{ inches}}{43 \text{ ft.} 4 \text{ inches}}. \]

Note that in finding the unknown member of a proportion, it is convenient first to simplify the ratio if possible. For example, to find the unknown member of the proportion

\[ \square : 4 = 111 : 183, \]
the proportion on the right can be simplified by dividing both members by their common factor 37. Thus

\[ \frac{\Box}{4} = \frac{3}{5}, \]

\[ 5 \times \Box = 3 \times 4, \quad \Box = \frac{3 \times 4}{5} = 2.4. \]

The above is simpler than computing the unknown member from the original proportion as

\[ \Box = \frac{4 \times 111}{185}. \]

EXERCISE 59-8A

1. Find the unknown member of each of the proportions.

   (a) \( \frac{\Box}{6} = \frac{24}{16} \)

   (b) \( \frac{1.5}{\Box} = \frac{1}{4} \)

   (c) \( \frac{11}{3} = \frac{\Box}{36} \)

   (d) \( \frac{30}{42} = \frac{5}{\Box} \)

   (e) \( \frac{36}{432} = \frac{4}{\Box} \)

   (f) \( \frac{4.5}{\Box} = \frac{6\frac{1}{4}}{2} \)

2. Write the proportions and find the missing quantity in each case.

   (a) 3 inches are to 27 inches as 7 inches are to_______ inches.

   (b) 6 gallons are to_______ gallons as 38 gallons are to 19 gallons.

   (c) 5 lb. are to 45 lb. as 2 ft. are to_______ ft.

3. On a plan a piece of land is represented by a rectangle of length 7.5 cm. and width 4.5 cm. It is known that the real width of the piece of land is 900 metres. What is the length of the land? What is the scale of the plan?

4. Alma has a picture 4 inches wide and 5 inches long. She wants an enlargement that will be 12 inches wide. How long will the enlarged print be?

5. A road rises 6 feet for every 100 feet of road. How much does it rise in a mile? (Find the answer to the nearest foot.)
6. Complete the entries in the table.

<table>
<thead>
<tr>
<th>Real length of a straight line segment</th>
<th>Length of the corresponding segment on a plan</th>
<th>Scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>240 m.</td>
<td>24 cm.</td>
<td></td>
</tr>
<tr>
<td>15 cm.</td>
<td></td>
<td>1 : 500</td>
</tr>
<tr>
<td>300 m.</td>
<td></td>
<td>1 : 1,000</td>
</tr>
</tbody>
</table>

The first known use of finding an unknown member of a proportion dates back to the 6th century before our era. The Greek mathematician Thales then astonished his contemporaries by the fact that, while in Egypt, he computed the heights of several pyramids by measuring the lengths of their shadows. He did it on the basis of an equality of two ratios or a proportion.

The procedure used by Thales in finding an unattainable height is given in the following experiment, which you yourself can easily perform.

On a sunny day Modupo noticed that the shadow cast by his father was longer than the shadow cast by his younger brother. He measured the length of the shadow cast by his father, who is 6 ft. tall, and found it to be 48 inches long. At the same time his little brother, whom he measured to be 39 inches tall, cast a shadow 26 inches long. A straight pole 5 ft. in length put vertically cast a shadow 40 inches long, and a big tree on their farm cast a shadow 16 feet long. Modupe made the following picture of his experiment:
He wrote down all the data in a table as follows:

<table>
<thead>
<tr>
<th>Shadow length</th>
<th>Height</th>
<th>Ratio of shadow length to height</th>
</tr>
</thead>
<tbody>
<tr>
<td>Father</td>
<td>48 inches</td>
<td>72 inches</td>
</tr>
<tr>
<td>Brother</td>
<td>26 inches</td>
<td>39 inches</td>
</tr>
<tr>
<td>Pole</td>
<td>40 inches</td>
<td>60 inches</td>
</tr>
<tr>
<td>Tree</td>
<td>16 feet</td>
<td>?</td>
</tr>
</tbody>
</table>

Modupe computed first the ratio of the length of his father's shadow to his father's height.

\[
\frac{48 \text{ inches}}{72 \text{ inches}} = 2 : 3
\]

When he computed the ratio of the length of his brother's shadow to his brother's height

\[
\frac{26 \text{ inches}}{39 \text{ inches}}
\]
he again found it to be equal 2 : 3, and the same was true for the ratio of the length of the shadow of the pole to the length of the pole

\[
\frac{40 \text{ inches}}{60 \text{ inches}} = 2 : 3.
\]

He thus discovered that, for the given position of the sun, the ratio of the length of the shadow of an object to the length of the object equals in each case 2 : 3.

We know that the length of the shadow of an object changes during the day as the position of the sun changes. At various times of the day the ratio of the length of the shadow of an object to the height of the object assumes various values. Have you not noticed that soon after sunrise or before sunset the shadow cast by an object is quite long, and the ratio under consideration may then be 2 : 1 or 3 : 1. On the other hand, around noon, when the sun is high, the shadow cast by an object is very small, and the ratio under consideration may be 1 : 10 or 1 : 20.

At the time Modupe made his experiment, he found this ratio to be 2 : 3 for each of the measured objects. He assumed then that this principle holds for all objects and wrote therefore the following proportion.

\[
\frac{\text{Length of shadow of the tree}}{} : \frac{\text{unknown height of the tree}}{} = 2 : 3
\]

or

\[
\frac{\text{length of shadow of the tree}}{\text{unknown height of the tree}} = \frac{2}{3}.
\]
Cross products give:

\[ 2 \times \text{unknown height of the tree} = 3 \times \text{length of shadow of the tree} \]

\[ 2 \times \text{unknown height of the tree} = 3 \times 16 \text{ ft.} \]

hence \[ \frac{\text{unknown height of the tree}}{2} = 24 \text{ ft.} \]

A precise explanation of the equality of all the ratios of the length of the shadow cast to the length of the object at a given time, that is for a given position of the sun, and the same place is based on the similarity of the triangles in Modupe's picture (See Chapter 56 of Volume III.).

**EXERCISE 59-8B**

1. By the method used in our example, find the height of a tree, if the shadow of the tree is 15 feet long and the shadow of a 10-foot pole is 2 feet long.
2. If a water tower casts a shadow 75 ft. long and a 6-foot man casts a shadow 4 ft. long, how tall is the water tower?
3. A monument casts a 6-foot shadow. The shadow of a foot rule perpendicular to the ground is 4 inches. Find the height of the monument.

**59-9 Directly proportional quantities**

We shall now discuss a special relationship that may exist between two sets of numbers or between corresponding values of two quantities.

Let us start with a simple example:

Consider a square, and let us make a table of two rows as follows: In the first row we write various lengths of the side of a square and in the second row we insert the corresponding values of the perimeter of the square, that is, the sum of the lengths of all its four sides.

**Table I**

<table>
<thead>
<tr>
<th>Length of side of a square (in inches)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perimeter of the square (in inches)</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>16</td>
<td>20</td>
<td>24</td>
<td>28</td>
<td>32</td>
<td>36</td>
<td>40</td>
</tr>
</tbody>
</table>
We have here two quantities: the length of the side of a square and the perimeter of the square. These two quantities depend on each other, or as we may say, are related, in the following way: If a value of the first quantity (the length of the side) is "magnified" a given number of times, say from 1 to 8 inches, that is multiplied by 8, then the corresponding value of the second quantity (the perimeter) is also "magnified" the same number of times. In our case 4 inches is multiplied by 8 to give 32 inches. On the other hand, if we "minify" a value of the first quantity five times, for example, if we divide 1 inch by 5, then the perimeter will also be divided by 5 to give \( \frac{4}{5} \) inch.

Having such a relationship, we say that the length of the side and the perimeter of a square are directly proportional to each other.

More precisely: two quantities (or two sets of numbers) are directly proportional to each other, if by "magnifying" ("minifying") any value of one of them a given number of times, the corresponding value of the second quantity is "magnified" ("minified") the same number of times.

The three following statements have the same meaning.
1. Two quantities are directly proportional to each other,
2. Two quantities are directly proportional,
3. One quantity is directly proportional to another quantity.

We will now use our knowledge of ratio and proportion to analyze directly proportional quantities further.

If we take the ratio of any two values of the quantity in the first row (length of side of a square), for example,

5 inches : 2 inches, that is the ratio \( 5 : 2 \),
and then take the ratio of the corresponding values of the second quantity (perimeter of the square)

20 inches : 8 inches, that is, the ratio \( 20 : 8 \), we see that the two ratios are equal. Thus the four quantities form the proportion

\[
5 \text{ inches} : 2 \text{ inches} = 20 \text{ inches} : 8 \text{ inches}.
\]

The following statement is true in general:

**Property.** If two quantities are directly proportional, then the ratio of any two values of one quantity equals the ratio of the corresponding values of the second quantity.

In our example we had two directly proportional like quantities. Let us now discuss the following relationship between two unlike quantities.

**Example:** The cost of 1 yard of a fabric is 3 shillings. Find the cost of 2 yards, 3 yards, \ldots, 10 yards of the given fabric. We compose

Table II

<table>
<thead>
<tr>
<th>Length of fabric (in yards)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost of fabric (in shillings)</td>
<td>3</td>
<td>6</td>
<td>9</td>
<td>12</td>
<td>15</td>
<td>18</td>
<td>21</td>
<td>24</td>
<td>27</td>
<td>30</td>
</tr>
</tbody>
</table>
We have here again an example of directly proportional quantities, this time of unlike quantities. If a given length of the fabric is "magnified" an arbitrary number of times, say multiplied by 2, 3, ..., 10, then the cost of the fabric is "magnified" the same number of times, that is multiplied by 2, 3, ..., 10, respectively. Also, if we take the ratio of any two values of the first quantity, say 3 yards : 5 yards, then the ratio of the corresponding values of the second quantity is 9 shillings : 15 shillings, and these two ratios are equal. We can therefore form the proportion

\[ 3 \text{ yards} : 5 \text{ yards} = 9 \text{ shillings} : 15 \text{ shillings}. \]

We thus see that the length of the fabric and its cost are directly proportional quantities.

Other examples of pairs of directly proportional quantities are:

- The time of a uniform straight line motion and the distance covered.
- The volume and the weight of a uniform piece of metal (measured under the same conditions, like same temperature and same place).
- The number of work hours and the pay received by an employee (earning, for example, 1 shilling per hour).

In order to understand better the relationship of direct proportionality, let us consider the following Example.

We start again, as in the first example, with a square and compose a different table of two rows. In the first row we write various lengths of the side of a square, and in the second row we insert the corresponding values of the area of the square.

Table III

The relationship between the length of the side and the area of a square

<table>
<thead>
<tr>
<th>Length of side of a square (in inches)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area of the square (in square inches)</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td>25</td>
<td>36</td>
<td>49</td>
<td>64</td>
<td>81</td>
<td>100</td>
</tr>
</tbody>
</table>

As before, these two quantities depend on each other. Yet their relationship differs from those we have discussed hitherto. We easily notice that, as before, if the first quantity increases, then the second quantity increases. However, if the length of the side is "magnified" five times (for example, 2 inches are multiplied by 5 to give 10 inches), then the area of the square is "magnified" 25 times (from 4 to 100 square inches) and not the same number of times as required for directly proportional quantities. Also, if we take the ratio of any two values of the first quantity, for example

\[ 2 \text{ inches} : 8 \text{ inches}, \] that is, the ratio \( 1 : 4 \),
then the ratio of the corresponding values of the second quantity is 4 square inches : 64 square inches, that is, the ratio 1 : 16.

We thus see that the two ratios are different.

We have here an example of two quantities which are related, but they are not directly proportional quantities.

* * *

We shall now apply what we have learned about directly proportional quantities to solve problems of a kind which you may have encountered before.

Example.

Ahmed pays 20 shillings and 40 cents for 6 lb. of soap. How much does he have to pay for 15 lb. of the same soap?

In a previous example we saw that the cost of a fabric is directly proportional to the length of the fabric. Similarly here, the cost of the soap is directly proportional to its weight. Applying only the meaning of directly proportional quantities, we solve the problem as follows.

I. Method of reducing to a unit.

If 6 lb. of soap cost 20.4 shillings, then 1 lb. of soap costs one sixth as much, that is, 20.4 ÷ 6 = 3.4, or 1 lb. of soap costs 3.4 shillings. Hence 15 lb. of soap cost 15 times 3.4 shillings or

\[ 3.4 \text{ shillings} \times 15 = 51 \text{ shillings}. \]

We call this the method of reducing to a unit, since we computed first the cost of 1 lb. of soap. We actually solved the problem by performing successively two operations: 1. Division (to find the cost of 1 lb. of soap), and 2. Multiplication (to find the cost of 15 lb. of soap).

We may, however, also solve the same problem by recalling the statement that if two quantities are directly proportional, then the ratio of any two values of one quantity equals the ratio of the corresponding values of the second quantity.

II. Method of proportion.

In our problem we have two directly proportional quantities: the weight and the cost of soap. We can represent it as follows.

\[ 6 \text{ lb. of soap cost 20.4 shillings} \]

\[ 15 \text{ lb. of soap cost } \square \text{ shillings}. \]

From the above we immediately obtain the proportion

\[ \square \text{ shillings} : 20.4 \text{ shillings} = 15 \text{ lb.} : 6 \text{ lb.} \]

Simplification gives

\[ \frac{\square}{20.4} = \frac{5}{2}, \]
hence \[ \frac{5}{2} \times 20.4 = 51 \]

Thus 15 lb. of soap cost 51 shillings.

In this method we are looking for a quantity which forms with three given quantities an appropriate proportion. Problems of this kind are therefore called problems on finding the fourth proportional quantity (or number). We have already considered such problems in 59-8 in finding an unknown member of a proportion.

It is clear that Method II is more convenient than Method I.

**EXERCISE 59-9**

1. An airplane has an average velocity of 435 miles per hour. What distance (in miles) will the plane fly in 2 hours, 3 hours, 5 hours, 3 \( \frac{1}{2} \) hours? What is the relationship between the time of flight and the distance covered?

2. Five glass machines produce 1,900 bottles in 4 hours. How many bottles does one machine produce in an hour?

3. On a given plan drawn to the scale \( \frac{1}{40} \) the distance between two points equals 4.8 inches. Find the distance between the corresponding points on another plan drawn to the scale \( \frac{1}{75} \).

4. Solve the following problem by the two methods indicated at the end of the section.

5. A machine produces 240 toys in 8 hours. How many toys does the machine produce in 35 hours?

**59-10 Inversely proportional quantities**

To discuss another special relationship that may exist between two sets of numbers or between corresponding values of two quantities, we start with the following

*Example.*

Consider a rectangle whose area is 10 square inches. Let us compose a table of two rows as follows: In the first row write quantities representing various lengths of the rectangle, say 1, 2, \ldots, 10 inches, and in the second row insert the corresponding values of the width of the rectangle having the given fixed area. We obtain
The relationship between the lengths of the sides of a rectangle whose area is 10 square inches

<table>
<thead>
<tr>
<th>Length of rectangle (in inches)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Width of rectangle (in inches)</td>
<td>10</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

We may notice immediately that, if the quantity in the first row is being increased, the quantity in the second row is being decreased (and not increased as in the relationships represented by Tables I and II of the previous section). Further analysis shows that if the first quantity is "magnified" a given number of times (for example, 1 inch is multiplied by 4 to give 4 inches) then the corresponding value of the second quantity is "minified" the same number of times (10 inches are divided by 4 to give $2 \frac{1}{2}$ inches). If we "minify" the first quantity seven times (say, if we divide 1 inch by 7 to have $\frac{1}{7}$ inch), then the corresponding value of the second quantity is "magnified" seven times (from 10 to 70 inches).

Having such a relationship, we say that the length and the width of a rectangle with fixed area are inversely proportional quantities.

More precisely: two quantities (or two sets of numbers) are inversely proportional to each other, if by "magnifying" ("minifying") any value of one of them a given number of times, the corresponding value of the second quantity is "minified" ("magnified") the same number of times. The three following statements have the same meaning:

1. Two quantities are inversely proportional to each other,
2. Two quantities are inversely proportional,
3. One quantity is inversely proportional to another quantity.

Again, as in the previous section, let us use our knowledge of ratio and proportion to analyze inversely proportional quantities further. Take the ratio of any two values of the first quantity in Table IV, for example, 3 inches : 8 inches, that is, the ratio 3 : 8.

Then the ratio of the corresponding values of the second quantity is

$3 \frac{1}{3} : 1 \frac{1}{4}$ inches, that is, $\frac{10}{3} : \frac{5}{4}$ or $8 : 3$. 

50
We thus see that the two ratios are not equal. However, they reduce to the same members but taken in a reversed order. In the general case, if the ratio of two values of the first quantity is $a : b$, then the ratio of the corresponding values of the second quantity equals $b : a$. The ratio $b : a$ is called the inverted ratio of $a : b$.

To express in terms of ratios the relationship existing between the two quantities in Table IV, we denote

- Value No. 1 of the first quantity by $F_1$,
- Value No. 2 of the first quantity by $F_2$,
- Value of the second quantity corresponding to $F_1$ by $S_1$,
- Value of the second quantity corresponding to $F_2$ by $S_2$.

We saw by the example above that the following proportion holds

$$F_1 : F_2 = S_1 : S_2$$  \(1\)

Choosing any other two values of the first quantity, you may well verify that proportion (1) is correct. The following general statement is true.

**Property I.** If two quantities are inversely proportional, then the ratio of any two values of one quantity equals the inverted ratio of the corresponding values of the second quantity.

Let us now write proportion (1) in fractional form,

$$\frac{F_1}{F_2} = \frac{S_2}{S_1}.$$  

Since the cross products in any proportion are equal, we have

$$F_1 \times S_1 = F_2 \times S_2.$$  

This shows the following.

**Property II.** For a given pair of inversely proportional quantities the product of an arbitrary value of one quantity by the corresponding value of the other quantity is constant.

What is the product of an arbitrary value of one quantity by the corresponding value of the other quantity in Table IV?

In the example above we had two inversely proportional like quantities. We will now give an example of the same type of relationship between two unlike quantities.

**Example.**

Suppose that two cities are connected by a straight line highway of length 720 miles. A man walking with an average velocity of 3 miles per hour would need $720 \div 3$ or
240 hours to cover this distance. A young man riding on a bicycle with an average velocity of 15 miles per hour needs 720 ÷ 15 or 48 hours to go from one city to the other. A car traveling on the average 40 miles per hour will cover the distance in 720 ÷ 40 or 18 hours. Finally an airplane flying with an average velocity of 300 miles per hour will cover the same distance in 720 ÷ 300 or 2.4 hours or 2 hours and 24 minutes. The obvious fact is that the higher the velocity of the motion the smaller is the time needed to cover a fixed distance. To study the relationship further, we compose a table of the quantities discussed.

Table V

The relationship between the velocity of a motion and the time required to cover a distance of 720 miles.

<table>
<thead>
<tr>
<th>Velocity of the motion (in miles per hour)</th>
<th>3</th>
<th>15</th>
<th>40</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time needed to cover the above distance (in hours)</td>
<td>240</td>
<td>48</td>
<td>18</td>
<td>2.4</td>
</tr>
</tbody>
</table>

We could extend this table by considering various other velocities and the corresponding time required to complete the motion.

From the table we see that if the velocity is "magnified" five times (for example, from 3 to 15 miles per hour), then the corresponding time is "minified" five times (from 240 to 48 hours).

If we take the ratio of any two values of the first quantity, say 15 miles per hour : 300 miles per hour, that is, the ratio 1 : 20, and the inverted ratio of the corresponding values of the second quantity 2.4 hours : 48 hours, we also obtain the ratio 1 : 20.

We thus have the proportion

$$F_1 : F_2 = S_2 : S_1.$$

The average velocity of a moving body and the time required to cover a certain fixed distance are inversely proportional quantities.

In view of Property II, what is in this example the product of an arbitrary value of one quantity by the corresponding value of the other quantity?

Other examples of inversely proportional quantities:

A. I was asked to buy candy for the school party for 120 shillings. The amount of candy bought is inversely proportional to the price of the candy. The denominator and the value of a fraction whose numerator is constant are inversely proportional numbers.
One has to be rather careful in deciding whether two given quantities are inversely proportional to each other. We illustrate this statement by the following.

*Example.*

Ahmed has earned 40 shillings. If he spends 5 shillings, he has 35 shillings left. If he spends 16 shillings, he has 24 shillings left, and so on. We can compose the following table.

<table>
<thead>
<tr>
<th>Money spent by Ahmed (in shillings)</th>
<th>2</th>
<th>5</th>
<th>10</th>
<th>16</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
</tr>
</thead>
<tbody>
<tr>
<td>Money left with Ahmed (in shillings)</td>
<td>38</td>
<td>35</td>
<td>30</td>
<td>24</td>
<td>20</td>
<td>15</td>
<td>10</td>
<td>5</td>
</tr>
</tbody>
</table>

It is clear that if the first quantity increases then the second quantity decreases. Yet, as we can easily show, the two quantities are not inversely proportional to each other. If the first quantity is "magnified" four times, for example from 5 to 20 shillings, then the corresponding value of the second quantity is "minified", but not four times. The corresponding values of the second quantity are 35 and 20 shillings, respectively. Obviously,

\[ 35 \div 20 \neq 4. \]

**EXERCISE 59-10**

1. A toy factory accepted an order that can be completed by 120 workers in one week. How many workers are needed to complete the same order in 5 weeks, in 12 weeks, in \( \frac{1}{2} \) weeks?

   Compose a table of the given relationship.

2. A given manuscript can be printed on 144 pages if each page contains 32 lines. On how many pages will the manuscript be printed, if each page has 36 lines? What relationship are you using to solve the problem?

3. Indicate the pairs of directly proportional quantities (numbers) and of the pairs of inversely proportional quantities (numbers).
   a. The time of a uniform motion (constant velocity) and the distance covered.
   b. The velocity of a uniform motion and the time needed to cover a given distance.
   c. The volume of a uniform body and its weight.
   d. The length and the width of a rectangle whose area is constant.
   e. The length of the edge and the volume of a cube.
   f. The length and the area of a rectangle whose width is constant.
g. The weight of butter and its cost.
h. The number of workers and the amount of accomplished work during a given period of time.
i. The two factors of a product whose value is constant.
j. The product of two numbers and the value of one of its factors, if the second factor is constant.
k. The dividend and the quotient, if the divisor is constant.
l. The numerator and the value of a fraction whose denominator is constant.
m. The numerator and the denominator of a fraction whose value is constant.
n. The two terms of a sum that has a constant value.
UNIT XII
FUNCTIONS AND GRAPHS

Chapter 60
FUNCTIONS

60-1 Introduction

We shall now learn about one of the most important ideas in mathematics, the idea of a function. We shall lead up to this idea by examples and then make clear what we mean by the word.

Let us consider a square two inches on a side, and ask: What is the perimeter? The answer is obvious. It is $2\'' + 2\'' + 2\'' + 2\'' = 4 \times 2\'' = 8\''$. The point is that if the length of the side has been chosen, the perimeter is determined. It could not be anything else than it is. Similarly if the sides were 3\" the perimeter would necessarily be $4 \times 3\'' = 12\''$.

We can make a little table of values. Of course this table is incomplete. We could add to it the results for sides of 6, 7, 8 inches and so on. We could also find the perimeters of squares with a fractional number of inches on a side. What for example is the perimeter for a side of $2\frac{1}{2}$ inches?

<table>
<thead>
<tr>
<th>Side in inches</th>
<th>Perimeter in inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
</tr>
</tbody>
</table>

There are two ideas here. The first is that given the side there is a definite perimeter that goes with it. The second is that this definite perimeter can be found by a rule.

The first idea is an important one. We shall return to it, but the second is more obviously important. So let us talk about it now. We see that there is a rule which enables us to calculate the perimeter when we know the side. We do not need to measure the perimeter. The single measurement of the length of the side is enough.

What is this rule? It can be stated in a mathematical sentence:

The perimeter is 4 times the length of the side.

We can shorten this statement by letting $s$ stand for the number of inches in the side and $p$ the number of inches in the perimeter. Then the sentence becomes

$p = 4 \times s$

or still more briefly

$p = 4s$. 

55
Now we notice that when we have this rule it is no longer necessary to have the table. The rule takes the place of the table. If, for example, you want the perimeter of a square whose side is $3\frac{1}{2}$ in. you can replace $s$ by $3\frac{1}{2}$ and get $p = 4 \times 3\frac{1}{2} = 14$. The result is 14 in. So the rule is a great saver of time and space.

We usually call the mathematical statement an equation because it contains an equals sign. We also say that $p = 4s$ is a formula for the perimeter of the square. Does the formula work if $s$ is the number of feet, yards or miles in the side of a square?

**EXERCISE 60-1**

Find formulas to express each of the following relations. Choose appropriate units. Make a short table from each of the formulas.

1. The perimeter of an equilateral triangle with a given side.
2. The diameter of a circle with a given radius.
3. The circumference of a circle with a given diameter.
4. The circumference of a circle with a given radius.
5. The distance covered in a given time by an automobile that travels at the constant speed of 40 mi./hr.

**60-2 Other Examples**

Let us turn to another example. Instead of talking about the perimeter of a square, let us talk about its area. Once again it is true that if we are given the length of the side, the area is determined. For example if the side is 3", the area must be 9 sq. in. Here too the area is given by a rule that can be expressed in a mathematical sentence which is an equation. This rule is

$$\text{Area in sq. in.} = \text{length of the side in inches} \times \text{length of the side in inches}.$$

If $s$ is the number of inches in the side of the square and $A$ the number of square inches in its area we can write

$$A = s \times s$$

$$A = s^2$$

How much simpler this is than the rule in words! The formula for area replaces a table of values and makes this table unnecessary. You could make your own table from it!

Let us take another geometrical example. Consider different rectangles whose area is 36 sq. in.
If the base of the rectangle is 9" the height must be 4". No other height gives the correct area. We could go on and make a table. The easy values are those shown. But of course there are many other possibilities. For example, if the base is 10" the height must be 3.6". Actually we cannot make a complete table, that is, a table that lists all of the possibilities. Can we replace the incomplete table by a rule that includes every possibility? Of course we can. The height is related in a definite way to the length of the base. In fact if \( b \) is the length of the base in inches and \( h \) the height in inches it must be true that

\[
b \times h = 36
\]

and hence

\[
h = \frac{36}{b}.
\]

Do you see that we can get all of the results of our incomplete table by replacing the letter "\( b \)" by the numbers 36, 18, 9 and 6? What can you say about the units for \( b \) and \( h \)? Suppose that \( b \) is the measure in inches. Then \( h \) is also the measure in inches. Could \( b \) be the measure in feet or yards? For what units would \( h \) then be the measure? What then would 36 represent?

**EXERCISE 60-2**

Find formulas to express each of the following. Choose appropriate units. Make a short table from each of the formulas.

1. The volume of a cube with a given side.
2. The surface area of a cube with a given side.
3. The area of a circle with a given diameter.
4. The perimeter of a square with a semicircle on top, for a given side of the square.
5. The area of the figure in Problem 4.

**60-3 On Changing Units**

We know that 1 inch = 2.54 centimetres. From this fact we can make a rule for changing from centimetres to inches. Let \( c \) be the number of centimetres in a length measurement and \( i \) the corresponding number of inches. Then \( c = 2.54i \).

Check this formula by taking \( i = 1 \) and \( i = 2 \). Note that for each value of \( i \) there is a single definite value of \( c \) and that this value can be found from the formula.
From this formula we can obtain others. For example, suppose that we wish to change from area in sq. cm. to area in sq. in. The area of a square \( c \) cm. on a side has the measure

\[
A = c^2
\]

in sq. cm. Since \( c = 2.54 \text{ i} \), the area in sq. in. is \((2.54 \text{ i})^2 = (2.54)^2 \text{ i}^2 = 6.4516 \text{ i}^2\). [Can you show that \((2.54 \text{ i}) \times (2.54 \text{ i}) = (2.54)^2 \text{ i}^2\)?]

Another example of a rule for changing units is the change from Centigrade degrees to Fahrenheit degrees. The rule is given by the formula

\[
F = \frac{9}{5} C + 32
\]

where \( C \) is the number of Centigrade degrees and \( F \) the corresponding number of Fahrenheit degrees. Let us test this rule with a few numbers. When \( C = 0 \) (freezing point of water) the formula gives \( F = 32 \). Is this correct? When \( C = 100 \) (boiling point of water) what should \( F \) be? Does the result come out right?

What is the value of \( F \) when \( C = 5 \)? When \( C \) changes from 0 to 5 how much does \( F \) change? Did you get the answer 9? How much does \( F \) change when \( C \) changes 1°? Do you see that you can make a table for changing from Centigrade to Fahrenheit measure by using this formula?

**EXERCISE 60-3**

1. Fill in the missing numbers.
   
   a. The number of yards = \underline{________} \ times the number of feet.
   
   b. The number of feet = \underline{________} \ times the number of yards.
   
   c. The number of feet = \underline{________} \ times the number of miles.
   
   d. The number of quarts = \underline{________} \ times the number of gallons.
   
   e. The number of centimetres = \underline{________} \ times the number of metres.
   
   f. The number of metres = \underline{________} \ times the number of centimetres.

2. Translate each of the sentences in Problem 1 into a formula after choosing suitable letters.

3. For each of the following choose suitable letters and write a formula that relates the two numbers:
   
   a. The number of pounds and the number of tons.
b. The number of square inches and the number of square feet.
c. The number of cubic inches and the number of cubic feet.
d. The number of square centimetres and the number of square metres.

4. A car averages 30 miles on one gallon of petrol. Write an equation which relates the number of miles travelled to the number of gallons.
5. A man walks at the rate of 4 miles per hour. Write an equation that relates the number of hours to the number of miles he walked.
6. An aeroplane flies at an average rate of 200 miles per hour. Write a formula to show this relation.

60-4 Some Experiments

In the examples that we have met so far we have seen how a formula could replace a table of corresponding values. The formula was easy to write down because of our knowledge of geometry or because we knew how the units of measurement were related to each other. We turn now to some examples where experiment gives us a table of corresponding values and we must discover the rule for ourselves.

If a stone is dropped from rest it is found that the distance that it falls depends on the time since it was let go. Experiment gives us the following table. Can we replace this table by a rule?

<table>
<thead>
<tr>
<th>Time (sec)</th>
<th>Distance (ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16</td>
</tr>
<tr>
<td>2</td>
<td>64</td>
</tr>
<tr>
<td>3</td>
<td>144</td>
</tr>
<tr>
<td>4</td>
<td>256</td>
</tr>
</tbody>
</table>

If so, what is the rule? The problem is to see how the numbers in the second column are related to those in the first column. Do you notice that each of the numbers 16, 64, 144 and 256 is divisible by 16?

Let us rewrite these numbers as $16 \times 1$, $16 \times 4$, $16 \times 9$ and $16 \times 16$ so that the table now looks like this. How are the numbers 1, 4, 9 and 16 related to 1, 2, 3 and 4? Clearly $1 = 1^2$, $4 = 2^2$, $9 = 3^2$ and $16 = 4^2$.

It looks then as if we could now write the formula. Let us try.

Let $t$ be the number of seconds and $d$ the number of feet that the stone falls in $t$ seconds. Do you see that we seem to have

$$d = 16t^2$$

as our rule? At least this rule works for $t = 1, 2, 3$ and 4. Of course it might not work for other times. The truth of the formula must be tested for such values of $t$ as $\frac{1}{2}, \frac{1}{2}, 2\frac{1}{2}$ and so on, as well as for values of $t$ which are greater than 4. Actually it is found by experiment that the rule which we have guessed works very well. Since this is true we use the rule in place of the incomplete table and we can add to the table as many new entries as we please. But the rule is simpler and better than any table.
Something new has come in here. We have constructed a mathematical model for nature. We have invented a rule to describe how we think that things really happen. We use this rule as a way of thinking about the way falling bodies behave.

Let us turn to another example. Suppose that we measure the height of a maize stalk each day at noon and make a table of the results. We might get something like this (we have left out some of the entries). Can we replace this table by a formula from which the heights can be calculated? This is a much harder problem. If there is a rule it is not a very simple one. Unless it is fairly simple there is not much gain. We do not save much time by using it so that it is perhaps easier to work with the table of values.

Even if we could find a formula for the growth of a maize stalk we should probably find that it would not work for another maize stalk. All stones fall in the same way but maize stalks differ in the way they grow.

Some aspects of nature can be understood from simple mathematical models and some cannot. In the Exercise some experiments are suggested that you can perform. Sometimes you will be able to discover a simple model and sometimes you will not be able to do so.

**EXERCISE 60-4**

1. For each of the following tables invent a simple formula which gives the numbers in the B column for each of the numbers in the A column.

   | A | B |
   ---|---|
   a) | 1 | 2 |
   | 4 | 5 |
   | 2 | 3 |
   | 3 | 4 |

   | A | B |
   ---|---|
   b) | 1 | 5 |
   | 3 | 7 |
   | 16 | 20 |
   | 22 | 26 |

   | A | B |
   ---|---|
   c) | 1 | 1 |
   | 3 | 27 |
   | 4 | 64 |
   | 2 | 8 |

   | A | B |
   ---|---|
   d) | 1 | 2 |
   | 4 | 17 |
   | 3 | 10 |
   | 2 | 5 |

   | A | B |
   ---|---|
   e) | 1 | 2 |
   | 5 | 30 |
   | 2 | 6 |
   | 4 | 20 |

   | A | B |
   ---|---|
   | 3 | 12 |
Here are a few simple experiments which you can perform. In some cases, you can also find a formula to represent the results.

**Experiment 1**
Take and record your classroom temperature at noon each day for the five consecutive days in a school week. Record your findings thus.

<table>
<thead>
<tr>
<th>Set A: Day of Week</th>
<th>Set B: Temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Monday</td>
<td></td>
</tr>
<tr>
<td>Tuesday</td>
<td></td>
</tr>
<tr>
<td>Wednesday</td>
<td></td>
</tr>
<tr>
<td>Thursday</td>
<td></td>
</tr>
<tr>
<td>Friday</td>
<td></td>
</tr>
</tbody>
</table>

**Experiment 2**
Stretch a rubber band between two uprights. By means of a light string suspend from the midpoint of the rubber band weights of 1 oz, 4 oz, 8 oz, 12 oz, 14 oz, 16 oz, and 19 oz, respectively.

Measure the corresponding sags in the band in inches and record your findings:

<table>
<thead>
<tr>
<th>Set A: Weight in ounces</th>
<th>Set B: Sags in inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 oz</td>
<td></td>
</tr>
<tr>
<td>4 oz</td>
<td></td>
</tr>
<tr>
<td>8 oz</td>
<td></td>
</tr>
<tr>
<td>12 oz</td>
<td></td>
</tr>
<tr>
<td>14 oz</td>
<td></td>
</tr>
<tr>
<td>16 oz</td>
<td></td>
</tr>
<tr>
<td>19 oz</td>
<td></td>
</tr>
</tbody>
</table>

**Experiment 3**
Take a graduated yard rule and support it at the mid-point (18" mark) by a suitable upright which has a relatively thin top. By using a light thread hang a penny from one end of the ruler (0" mark). Then record how many pennies can be placed at points 36", 27", 24", 21", 20" to balance the rule.

<table>
<thead>
<tr>
<th>Set A: Point on rule</th>
<th>Set B: Number of pennies</th>
</tr>
</thead>
<tbody>
<tr>
<td>36&quot;</td>
<td></td>
</tr>
<tr>
<td>27&quot;</td>
<td></td>
</tr>
<tr>
<td>24&quot;</td>
<td></td>
</tr>
<tr>
<td>21&quot;</td>
<td></td>
</tr>
<tr>
<td>20&quot;</td>
<td></td>
</tr>
</tbody>
</table>

**Experiment 4**
Arrange the class into age groups and find the average weight of each age group. Make a record of your findings, using a table similar to this:

<table>
<thead>
<tr>
<th>Set A: Age in years</th>
<th>Set B: Weight in lbs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>15 - 16</td>
<td></td>
</tr>
<tr>
<td>17 - 18</td>
<td></td>
</tr>
<tr>
<td>19 - 20</td>
<td></td>
</tr>
<tr>
<td>21 - 22</td>
<td></td>
</tr>
<tr>
<td>23 - 24</td>
<td></td>
</tr>
</tbody>
</table>
Experiment 5
Get six lengths of thread of the same kind — 3”, 6”, 9”, 12”, 15”, and 18”. Take one at a time and tie one end to a horizontal bar and a penny at the other end.

Then pull the stringed penny 45° away from the vertical and let it go. Using a stop watch find the number of oscillations the penny makes in 30 seconds. Record your readings as follows:

<table>
<thead>
<tr>
<th>Set A: Length of string</th>
<th>18”</th>
<th>15”</th>
<th>12”</th>
<th>9”</th>
<th>6”</th>
<th>3”</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set B: Number of oscillations in 30 seconds</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

One oscillation may be taken as the complete movement from the start of the penny’s motion from left to right of the vertical and back again to a stop at the left.

Experiment 6
Fix firmly and erect on the ground a pole whose total length above the ground is about 10 feet. Take readings of the lengths of its shadow from 8:30 a.m. to 12 noon at 35 minute intervals or at the end of each lesson period.
Record the readings as in the following example:

| Set A: Time of day | 8:30a.m. | 9:05a.m. | 9:40a.m. | 10:15a.m. | 10:50a.m. | 11:25a.m. | 12noon |
|-------------------|----------|----------|----------|-----------|-----------|-----------|
| Set B: Length of shadow in ft. |

Experiment 7
Repeat experiment 6, but continue the measurements after 12 noon.
Record your findings:

<table>
<thead>
<tr>
<th>Set A: Time of day</th>
<th>8:30 9:05 9:40 10:15 10:50 11:25 12noon 12:35 1:10 1:45 2:20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set B: Length of shadow in ft.</td>
<td></td>
</tr>
</tbody>
</table>

**Experiment 8**
In experiment 6 and 7 use the following alternative ways of recording your findings:

(a) **Experiment 6**

<table>
<thead>
<tr>
<th>Set A: Time in minutes</th>
<th>0(8:30) 35 70 105 140 175 210</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set B: Length of shadow in ft.</td>
<td></td>
</tr>
</tbody>
</table>

(b) **Experiment 7**

<table>
<thead>
<tr>
<th>Set A: Time in minutes</th>
<th>0 at 8:30 35 70 105 140 175 210 245 280 315 350</th>
</tr>
</thead>
<tbody>
<tr>
<td>Set B: Length of shadow in ft.</td>
<td></td>
</tr>
</tbody>
</table>

**Experiment 9**
Ask the local meteorological station to supply you with average rainfall figures for the month of July for the years 1959 to the present year. Record the information in a table.

<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Set B: Rainfall in inches</td>
<td></td>
</tr>
</tbody>
</table>

60-5 The Idea of Function

It is time for us to go back over the road that we have travelled and consider what we have learned.

When we discussed the perimeter and the side of a square we made two statements:

(1) Given the side, there is a definite perimeter that goes with it.
(2) This definite perimeter can be found by a rule.
We have been emphasizing the rule or formula in most of the other examples. We have seen however that there are cases in which there seems to be no rule — at any rate no simple rule. The first statement therefore is the more fundamental one.

In all of our examples we have really been dealing with two sets of members that can be called A and B. Some of the members of the first set A may be listed in the first (left-hand) column of a table. Members of the second set B are then listed in the second (right-hand) column. Opposite each first number is a definite second number which belongs with it. We read this table from left to right. That is, we begin with a number in set A and pair with it a certain number in set B.

We can diagram this in a schematic way as follows.

<table>
<thead>
<tr>
<th>1st column from set A</th>
<th>2nd column from set B</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
</tbody>
</table>

The sets A and B are shown by the shaded regions. Let a be a member of set A. We show a by a point inside the region for A. We then draw an arrow from this point to the point b inside the B region, where b is the member of set B that corresponds to a. We write a → b which can be read "a goes into b". We imagine an arrow from every member of A to its corresponding member in B. No member a can be associated with more than one member of B. That is, the following situation: (a → b and a → b') is impossible.

When we can compare the members of two sets, A and B, in this way we can say that there is a function from A to B. In all of the examples that we have met there is indeed a function from a set A to a set B.

Commonly there are an infinite number of members of set A so that a complete table is impossible. In most of the cases that we are interested in there is a formula or rule that tells us what member of B is at the end of the arrow that begins at any member of A. If a → b we call b the value of the function at a.
Is it possible to have two different members of \( A \) lead to the same corresponding member of \( B \) as shown in this figure?

Consider the set \( A \) that consists of all integers, positive, negative or zero. Suppose that each member of \( A \) goes into its square so that \( B \) consists of the squares of the integers.

Then \( 1 \rightarrow 1, \ 2 \rightarrow 4, \ 3 \rightarrow 9, \ldots \). But it is also true that \( -1 \rightarrow 1, \ -2 \rightarrow 4, \ -3 \rightarrow 9, \ldots \). That is, two different members of \( A \) can have the same mate in \( B \). Do you think that we could have functions for which more than two members of set \( A \) go into the same \( B \)?

Suppose that \( A \) consists of 0 and all of the positive real numbers and \( B \) of all the whole numbers. The rule for finding the mate to a number \( a \) might be: take the whole number part of \( a \). Then \( 1.32 \rightarrow 1, \ 1.428 \rightarrow 1, \ 1.111\ldots \rightarrow 1 \) and so on.

In fact all real numbers between 1 and 2 (including 1 but not 2) go into 1.

The idea of function that we have explained is a very broad one, much broader in fact than might be guessed from these examples. But we have said enough for our present needs.

**EXERCISE 60-5**

In each of the following problems, two sets, \( A \) and \( B \), are described so as to relate their members to each other. In which of these problems is there a function from \( A \) to \( B \)?

1. \( A \): the set of all real numbers.  
   \( B \): the set of all doubles of real numbers.

2. \( A \): the set of all whole numbers  
   \( B \): the set of the remainders when the members of \( A \) are divided by 3.

3. \( A \): the set of whole numbers  
   \( B \): the set \( \{1\} \).

4. \( A \): the set \( \{2\} \)  
   \( B \): the set of all whole number multiples of 2.

5. \( A \): the set \( \{2\} \)  
   \( B \): the set of all real numbers that have 2 as a whole number part.

6. \( A \): the set of all real numbers  
   \( B \): the set of all the cubes of the real numbers of \( A \).

7. \( A \): the set of all whole numbers of the form \( 1^2 + 1 \) where 1 is an integer  
   \( B \): the set of all integers 1.
Chapter 61
GRAPHS

61-1 Introduction

We have seen that we can use a number line to help us to think about numbers. In Chapter 60 we studied functions from a set A to a set B. Numbers from set A can be shown on one number line and numbers from set B on another number line. We need two number lines. It is customary to use a horizontal line for set A and a vertical line for set B. We shall call the horizontal line the first axis and the vertical line the second axis. It is usual to make the 0 points on the two number lines agree and to choose the same unit of length on both of them. We shall do this. We show positive numbers on the first axis to the right of the point 0 and negative numbers to the left of 0. On the second axis we show positive numbers above 0 and negative numbers below 0.

The two axes divide the plane into four regions called quadrants which are numbered I, II, III and IV as shown in the figure. At first we shall be concerned with problems where there are no negative numbers in set A or in set B. We can then leave out the negative parts of the axes. The space on the paper can be used to best advantage by drawing the positive first axis near the bottom and the positive second axis near the left side of the available space as in the figure. Suppose that for a certain function, the first number 1 goes into the second number 3. How can we show this? We simply draw a vertical line through the point marked 1 on the first axis and a horizontal line through the point 3 on the second axis. These lines intersect at a point P. We mark this point (1, 3) to show that it corresponds to a first number 1 and a second number 3. P is 1 unit to the right of the second axis and 3 units above the first axis.
Again if the function takes the first number 2 into the second number 4, we can locate the point (2, 4) as the figure shows.

**EXERCISE 61-1**

Draw a pair of perpendicular lines that intersect near the lower left hand corner of a sheet of paper and mark scales on them whose 0 points agree. (Use 1” as a unit.)

1. Locate the points (1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4).

2. Locate the points (1/2, 1), (3/2, 2), (2, 1/3).

3. Locate the points (1, 0), (2, 0), (3, 0), (4, 0) and (0, 1), (0, 2), (0, 3), (0, 4). Where is (0, 0)?

4. Locate the points (−1, 1), (1, −1), (−1, −1) on a pair of axes.

5. Let us see if you understand how to find points named by pairs of numbers.

   - Find A(1, 5); B(2, 3); draw AB
   - Find C(3, 5); draw BC.
   - Find D(2, 1); draw DB.
   - Find E(6, 5); F(4, 5); draw FE.
   - Find G(4, 1); draw FG.
   - Find H(4, 3); I(5, 3); draw HI.
   - Find J(6, 1); draw GJ.
   - Find K(9, 5); L(7, 5); draw KL.
   - Find M(7, 3); draw LM.
   - Find N(9, 3); draw NM.
   - Find O(9, 1); draw NO.
Find $P(7, 1)$; draw $OP$.

If you understand, your answer will show it.

61-2 The Graph of a Function

By using the method of the last section we can make a picture of a function. This picture is called the graph of the function. Our first example described a function from the set of positive numbers $s$ (set A) to the set of positive numbers $p$ (set B) by the formula $p = 4s$. From this formula you can make a little table. Let us locate the points which picture these pairs of numbers. The points seem to lie on a straight line that passes through the common 0 which we call $(0, 0)$. Does $p = 0$ correspond to $s = 0$? This seems reasonable since a square with side 0 can be thought of as a point with perimeter 0. When we draw the straight line we are of course going beyond what is in the table. But you can extend the table, and see if the new points come where you expect that they should. For example when $s = \frac{1}{2}$,
you find that $p = 2$. The point $(\frac{1}{2}, 2)$ has been marked with a cross. Similarly you can mark the points $(\frac{3}{2}, 6)$ and $(\frac{5}{2}, 10)$. They come at the places that you could predict. Later we shall prove that we were right in thinking that all possible points do lie on the straight line. But even without this proof it seems very likely that this is true.

Let us turn to the second example where if $x$ is the length of one side of a rectangle of area 36, the length of the other side is given by the formula $y = \frac{36}{x}$.
The table shows a few corresponding values. Now let us locate the points in the way that you have learned. It is clear that you cannot draw a straight line through all of these points. You can sketch a smooth curve as the figure shows. Of course when you do this you are going beyond the evidence of the table. You
do not know for sure that the points on the curve that have been drawn are exactly the right points. But you can add as many sure points as you like. For example, you could try $x = \frac{3}{2}$, then the corresponding $y$ must be $\frac{36}{\frac{3}{2}} = 36 \times \frac{2}{3} = 24$. Let us locate the new point $\left(\frac{3}{2}, 24\right)$ and mark it with a cross.

Does it come about where you expected that it would? After a reasonable number of points have been marked we gain confidence that the curve is about right. We feel that we have a good picture of how the $y$ values (numbers) are related to the $x$ values (numbers). You can say that you have a graph of the function described by the formula $y = \frac{36}{x}$, or a graph of the equation $y = \frac{36}{x}$.

How does this graph help? First of all it allows you to see at a glance that as you take larger $x$ values, you get smaller $y$ values. As we usually say: when $x$ increases, $y$ decreases.

It also allows you to estimate the value of $y$ for any chosen value of $x$. The method is as follows. Begin with the point $a$ on the first axis that shows the chosen $x$ value. Then go up to the curve along a vertical line (see the arrow in the figure). From the point $P$ of the curve directly above $a$, follow the horizontal arrow to the second axis. The end of the arrow is the point that shows $b$. This is the value of the function at $a$.

Of course unless $P$ is a point on the graph which has been surely located from the table or by using the formula we are guessing the value of $b$. That is, we are assuming that the curve has been correctly drawn between the known points. In practice we are not likely to be far from the truth. The value $b$ is near enough to the correct value to serve our needs. For complete precision you must of course go back to the formula.

For example, if you wish the value of the function at $3.2$ you can read off the result $11.2$ or $11.3$ from the graph. You should show that the correct value is $11.25$.

**EXERCISE 61-2**

1. Draw a graph for $A = s^2$, choosing values of the first number $s$ from 0 to 4 inclusive at intervals of $\frac{1}{2}$. From the graph estimate the value of $s^2$ when
s = 1.2 and s = 2.9. Check your results by using the formula.

2. Draw a graph of \( F = \frac{9}{5}C + 32 \) using \( C = 0, 20, 40, 60, 80, 100 \). What kind of graph does this seem to be? Estimate the value of \( F \) at \( C = 5 \) and \( C = 50 \). Check your results from the formula.

3. Draw a graph of \( y = x + 2 \) from \( x = 0 \) to \( x = 5 \) marking at least eight points. Describe the graph.

4. Draw a graph of \( y = 3x \) from \( x = 0 \) to \( x = 5 \) marking at least eight points. Describe the graph.

5. Draw a graph of \( y = 3x + 2 \) from \( x = 0 \) to \( x = 5 \) using at least eight points. Describe the graph. How is it related to the graph in Problem 4?

### 61-3 Some Straight Line Graphs

When we drew the graph for \( p = 4s \) we seemed to obtain a straight line through the point \((0, 0)\). In Problems 2, 3, 4 and 5 of Exercise 61-2 you should have obtained graphs that look like straight lines. In each of these cases, the graph is really a straight line. We shall show that this is true. There is no guess work. You can have complete confidence about the position of the points between those that were actually found from the formula.

Also you will soon be able to recognize from the appearance of these formulas that when you draw the graphs you will certainly get straight lines. Some kinds of formulas always lead to straight line graphs. To be able to tell in advance that the picture will come out this way is a great step forward.

Let us turn to our examples. How do we know that the graph of \( p = 4s \) is a straight line through \((0, 0)\)? The last part is easy. When \( s = 0, p = 0 \) so that \((0, 0)\) is surely on the graph. The point \( P = (1, 4) \) is also certainly on the graph. Now we want to show that the straight line through \( O \) and \( P \) is really the graph of \( p = 4s \). How can we show this? (Actually the straight line extends into the third quadrant but we are interested only in the ray \( OP \) which is in the first quadrant.)
Let us draw the ray $\overrightarrow{OP}$ and let $s$ be any number different from 1. You can show $s$ by a point $S$ on the first axis. Now draw a vertical line through $S$ and let $Q$ be its intersection with $\overrightarrow{OP}$. Since $\overrightarrow{SQ}$ is parallel to $\overrightarrow{RP}$ (why?), the triangles $\triangle ORP$ and $\triangle OSQ$ are similar. Therefore you have the proportion

$$\frac{SQ}{OS} = \frac{RP}{OR},$$

that is

$$\frac{SQ}{s} = \frac{4}{1}.$$

From what you have learned about proportion you can write

$$SQ = 4s.$$

The point $Q$ is therefore the point $(s, 4s) = (s, p)$ and consequently $Q$ is a point on the graph of $p = 4s$. Therefore all points on $\overrightarrow{OP}$ lie on the graph of $p = 4s$.

Furthermore no point of the graph of $p = 4s$ could fail to be on $\overrightarrow{OP}$. For suppose that there were a point $Q'$ of the graph that is not on $\overrightarrow{OP}$. This point $Q'$ will be located by two numbers that we may call $s$ and $b$. So $Q' = (s, b)$. Draw a vertical line through $Q'$ and let $Q$ be the point where it intersects the ray $\overrightarrow{OP}$. The first number for $Q$ must be $s$, the same as the first number for $Q'$. Since $Q$ is on $\overrightarrow{OP}$, its second number must be $4s$. Then $Q = (s, 4s)$. If $Q'$ is not on $\overrightarrow{OP}$ it must be either below it or above it. This means that $b$ must be different from $4s$. But if $b$ is not equal to $4s$, the point is not on the graph. So we have a contradiction. There can be no point of the graph which is not on $\overrightarrow{OP}$.

Let us turn to the graph of $y = 3x$ (Problem 4 of Exercise 61-2). The point $0 = (0,0)$ is on the graph. Also $p = (1, 3)$ is a point on the graph. Draw the ray $\overrightarrow{OP}$. This is the graph of $y = 3x$. The proof follows exactly the same pattern as before. The equations $y = 3x$ and $p = 4s$ are much alike. If $s$ and $p$ are replaced by $x$ and $y$, the resemblance is more striking. You now have $y = 3x$ and $y = 4x$. These equations have a family resemblance.
Other equations of the same family are $y = 2x$, $y = x$ and $y = \frac{1}{2}x$. In each case, the graph is a straight line through $(0, 0)$. We draw all of them on the same pair of axes. The straight lines differ from each other in direction. The direction is determined by the second number that is paired with the first number 1. Thus $y = 3x$ goes through $(0, 0)$ and $(1, 3)$, $y = 2x$ through $(0, 0)$ and $(1, 2)$ and $y = \frac{1}{2}x$ through $(0, 0)$ and $(1, \frac{1}{2})$. This second number is called the slope because it measures the steepness of the line. Notice that the slope is also given by the number by which $x$ is multiplied on the right side of the equation.

All of these equations are of the form

$$y = cx$$

where $c$ is a number. The graph of such an equation goes through $(0, 0)$ and $(1, c)$ and therefore has the slope $c$. The larger the value of $c$, the steeper the line.

There is another way of looking at an equation of the form

$$y = cx.$$ If $x$ is "magnified" a given number of times (that is, multiplied by a given number), then $y$ is also "magnified" the same number of times (multiplied by the same number).

You can see this at once, because if $x$ is multiplied by $m$ then $cx$ becomes $c(mx) = m(cx) = my$ so that $y$ is also multiplied by $m$. We can then say that when $y = cx$, $y$ is directly proportional to $x$.

**EXERCISE 61-3**

1. Draw graphs of each of the following equations. In each case locate a few points for which $x$ has negative values as well as points with positive values of $x$.

   a. $y = 5x$

   b. $y = \frac{3}{2}x$
c. \( y = \frac{3}{4}x \)

d. \( y = \frac{1}{3}x \)

2. Follow the pattern of the proof in the text to show that the graph of \( y = cx \) (for a given number \( c \)) is a ray from the point \((0, 0)\) to the point \((1, c)\). Assume that the values of \( x \) are positive or zero. Show that for negative values of \( x \) we obtain points on the same straight line.

3. Look at your graph of \( y = 3x + 2 \) from Exercise 61-2. Can you prove that this graph is really a straight line? What is its slope? *Hint:* How is the graph related to the graph of \( y = 3x \)?

4. Look at your graph of \( y = x + 2 \). Can you prove that this graph is really a straight line? What is its slope?

5. Draw a graph of \( y = 2x + 1 \).

61-4 Some Other Straight Line Graphs

In Problem 3 of the previous Exercise you were asked to show that the graph of \( y = 3x + 2 \) is a straight line. It was suggested that you compare it with the graph of the equation \( y = 3x \). Of course you know that \( y = 3x \) graphs into a straight line through \( O = (0, 0) \) and \( P = (1, 3) \). The graph of \( y = 3x + 2 \) does not go through \((0, 0)\). To what does \( x = 0 \) correspond? To \( y = 2 \) of course. So you have the point \( R = (0, 2) \) on the required graph. And what number corresponds to the first number 1? The answer is certainly \( 3 + 2 = 5 \) so that you have another point \( S = (1, 5) \) on the graph of \( y = 3x + 2 \). If this graph is to be any straight line, this line must pass through \( R \) and \( S \). Can you prove that the shaded figure is a parallelogram so that \( RS \) is parallel to \( OP \)? Do you conclude that the graph of \( y = 3x + 2 \) is the line parallel to the graph of \( y = 3x \) through \((0, 2)\)?

In fact, to get the graph of \( y = 3x + 2 \) you can simply lift the graph of \( y = 3x \) by 2 units. The remaining problems in Exercise 61-3 are close relatives to this one. All of the equations have the form

\[ y = cx + b. \]
Their graphs can be found from the graph of \( y = cx \) by raising it \( b \) units. You obtain a straight line through \((0, b)\) and parallel to the graph of \( y = cx \). Does the slope or steepness of the line change when it is raised so as to remain parallel to its original position? Clearly not.

You should now be able to draw the graphs of equations in this family very quickly. For example, suppose that you wish to picture \( y = \frac{1}{2}x + 3 \). You begin with the points \( R = (0, 3) \) and \( S = (1, 3\frac{1}{2}) \) and draw \( RS \).

(We are now including points for which \( x \) is negative.)

You are now ready for the graph of \( F = \frac{9}{5}C + 32 \). You locate \((0, 32)\). You could also locate \((1, 3\frac{4}{5})\) but these points are a bit too close together to allow you to draw an accurate picture. Any second point will do. You could choose \((10, 50)\). Of course you know that \( 0^\circ C \) corresponds to \( 32^\circ F \) and \( 100^\circ C \) to \( 212^\circ F \) so that the points \((0, 32)\) and \((100, 212)\) can be marked at once. You know from the form of the equation that the graph must be a straight line. Therefore you can join these points by a ruler and draw the graph without difficulty.

So far we have always remained in the first quadrant. The time has come to consider other possibilities. There are Centigrade temperatures below \( 0^\circ \). To what Fahrenheit temperatures do they correspond?

Let us use the formula

\[
F = \frac{9}{5}C + 32
\]

for negative values of \( C \). We easily find that for \( C = -10 \), \( F = 14 \) and for \( C = -20 \), \( F = -4 \).

We have located these points on the graph. As you see, they lie on the same straight line as the other points. You can continue this line further. It has been discovered that it is impossible in nature to have a temperature below an "absolute zero" of about \(-273^\circ C\). So the straight line does not
have any physical meaning to the left of the point where \( C = -273 \).

**EXERCISE 61-4**

1. Find the Fahrenheit temperature when \( C = -273 \).

2. From the graph of \( F = \frac{9}{5} C + 32 \) estimate the value of \( C \) when \( F = 0 \). Can you think of a way to find its exact value?

3. Use the graph to find the value of \( C \) when \( F = 50 \). Check the accuracy of your result from the formula. Can you see how to work the formula backwards, that is, find the value of \( C \) for which

\[
50 = \frac{9}{5} C + 32 ?
\]

**61-5 On Going Backward**

In the previous Exercise you met the problem of working back from the value of \( F \) to the value of \( C \). This kind of problem comes up in other cases so we shall consider the general problem of which this is an example. But first we shall look at some simple cases. In mathematics it is always a good plan to begin with as simple problems as we can so that we do not get confused with details.

**Example 1**  Let us look at a straight-line graph with which you are now familiar: the graph of the equation

\[
y = x + 2.
\]

If you start with a value of \( x \), to get the corresponding value of \( y \) you follow the vertical arrow until you meet the line and then go across to the second axis. In terms of numbers what do you do? You add 2 to the given value of \( x \). For example for \( x = 3 \), you find \( y = 3 + 2 = 5 \).

Now suppose that you wish to go backwards, that is, find the value of \( x \) from which a given value of \( y \) was obtained. You *add* 2 to an \( x \) value to get a \( y \) value. To go backwards you start with the value of \( y \) and *subtract* 2 to get the value of \( x \). That is, \( x = y - 2 \). Thus \( y = 5 \) must have come from \( x = 5 - 2 = 3 \). (We remember that subtraction is the inverse of addition.)

The picture is this, where the arrows have been reversed from what we had before.
Example 2. Let us turn to another simple example, the straight-line graph of $y = 3x$. You follow the arrows to go from an $x$ value to its $y$ value. In terms of numbers you multiply the number $x$ by 3 to get the number $y$.

How can you reverse this process, that is, go from the number $y$ back to the number $x$ from which it came? On the picture, you simply reverse the arrows. In terms of numbers you undo the multiplication by 3 by dividing by 3. Then

$$x = \frac{y}{3}.$$ 

Example 3. Consider the equation $y = 3x + 2$ and its straight-line graph. What do you do to find the value of $y$ for a given value of $x$? You first multiply $x$ by 3 and then add 2. On the picture you first go up $3x$ to the dotted line and then go up 2 more. Finally you go across to get $y = 3x + 2$. 


To go from $y$ back to $x$ you must reverse the arrows. You first subtract 2 to get $3x$ and then divide by 3 to get $x$. That is $x = \frac{y - 2}{3}$.

For example, given $y = 5$ you subtract 2 and then divide by 3 to get

\[
\frac{5 - 2}{3} = \frac{3}{3} = 1.
\]

Let us return to our problem of the previous section where we had the equation

\[
F = \frac{9}{5}C + 32.
\]

You wish to go from $F$ back to the $C$ from which it came. You must reverse the steps which led from $C$ to $F$. To get from $C$ to $F$ you first multiplied by $\frac{9}{5}$ and then added 32. Starting with $F$ you must therefore first subtract 32 and then divide by $\frac{9}{5}$.

That is, to get $C$ you first find $F - 32$ then $\frac{F - 32}{\frac{9}{5}}$.

But $\frac{F - 32}{\frac{9}{5}}$ is the same as $\frac{5}{9}(F - 32)$ (Why?).

Therefore finally

\[
C = \frac{5}{9}(F - 32).
\]

With this new equation you can easily answer the questions that we asked before. For what value of $C$ is $F = 0$? The answer of course is

\[
C = \frac{5}{9}(-32) = \frac{-160}{9} = -17\frac{7}{9}.
\]

Compare this exact result with your estimate from the graph. Did you succeed in getting the exact value for yourself?

For what value of $C$ is $F = 50$? This is now easy to answer. Does it correspond to your own result?
EXERCISE 61-5

1. Given \( y = \frac{1}{2} x - 3 \).
   
   For what value of \( x \) is \( y = 0 \)? \( y = 1 \)?
   
   Find a formula that gives the value of \( x \) for any given \( y \).

2. Suppose that \( y = 3(x - 1) \).
   
   For what value of \( x \) is \( y = 0 \)? \( y = 1 \)?
   
   Find a formula for the \( x \) which leads to any given \( y \).

3. Draw a graph of \( y = \frac{4}{x} \) showing at least the points for \( x = 1, 2, 3 \) and \( 4 \).
   
   Reverse this equation so that you can find \( x \) for a given \( y \).

4. Draw a graph of \( y = x^2 \) for values of \( x \) between \(-3 \) and \( 3 \).
   
   For what value of \( x \) is \( y = 1 \)? Is there more than one answer?
   
   For what value of \( x \) is \( y = 4 \)?
   
   How many answers are there?
   
   Can you write a formula for \( x \) in terms of a given \( y \)?

61-6 Can We Always Reverse Our Steps?

Here is part of the graph of \( y = x^2 \)
that we were asked to draw in the last problem
in Exercise 61-5. Notice that when \( x = 1 \),
\( y = 1 \). But also \( y = 1 \) when \( x = -1 \).
Therefore if we are asked to start with \( y = 1 \)
and find the \( x \) value from which it came we
are in doubt. Which \( x \) value is wanted, \(-1 \) or \( 1 \)?

Let us look at this situation in terms of a table. In the first column you will find
listed a few of the numbers from the set \( A \)
which contains all the real numbers (positive,
negative and \( 0 \)). In the second column you
will find the corresponding value of the
function described by the equation \( y = x^2 \).
There is a function from \( A \) to \( B \) because
there is only one member of \( B \) for each
member of \( A \). But we have trouble if we
try to reverse the order. Let us try to make
a table with members of \( B \) in the first
column and in the second column the

\[
\begin{array}{c|c}
A & B \\
\hline
-2 & 4 \\
-1 & 1 \\
\frac{1}{2} & \frac{1}{4} \\
0 & 0 \\
\frac{1}{2} & \frac{1}{4} \\
1 & 1 \\
2 & 4 \\
\end{array}
\]
members of $A$ that correspond to them. Except in the case of 0, there are two values of set $A$ that correspond to each member of set $B$.

So in this case there is no function from $B$ to $A$ which is the "inverse" of the original function.

If we consider only 0 and the positive members of $A$, there is no difficulty in going backwards. Each member of $B$ goes into a member of the restricted $A$ which we may call $A^+$. There is a function from $B$ to $A^+$ which is the inverse of the function from $A^+$ to $B$. We call it the square-root function.

Of course we could also restrict ourselves to the subset of $A$ which contains 0 and the negative numbers. If we call this subset $A^-$ we have a function from $B$ to $A^-$ that is the inverse of the function from $A^-$ to $B$. Here is part of this table.

<table>
<thead>
<tr>
<th>$B$</th>
<th>$A^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>-2</td>
</tr>
</tbody>
</table>

Can you tell by looking at the graph of a function from $A$ to $B$ whether there is also an "inverse function" from $B$ to $A$? Yes, very easily. What is the trouble with our graph of $y = x^2$? The trouble is that there are two points on the graph on a horizontal line. This means of course that two different values of $x$ lead to the same value of $y$. This will always happen if the graph has a decreasing piece and an increasing piece next to each other.

The graph of the temperature of a patient with a fever might look something like this. That is, the temperature goes up for a time and then comes down again. At each time there is a single temperature, but it is not true that for each temperature there is exactly one time at which this occurred. For example, from the knowledge that the temperature was 100° F at a certain time we cannot tell what that time was.
Another simple example is this. If a stone is thrown straight up into the air, the stone is at a definite height at any given time. (We are not concerned with the values of the time after the stone hits the ground.) But it is not true that for each height above the ground there is only one time at which this occurred. A graph of the heights at different times might look something like this. A horizontal line through the point 12 on the second axis cuts the graph in two points. There are two values of the time for which the height is 12 ft. This height was reached once on the way up and once on the way down.

**EXERCISE 61-6**

1. In the example of the stone let \( h \) feet be the height at \( t \) seconds after the stone is thrown. The equation that is pictured in our graph is

\[
h = 32t - 16t^2
\]

From this equation find the heights when \( t = 0, \ t = \frac{1}{2}, \ t = 1, \ t = \frac{3}{2} \) and \( t = 2. \)

Draw a graph showing these points.

2. In Problem 1, let \( A \) be the set of times from 0 to 2 and \( B \) the set of corresponding heights. How can \( A \) be restricted so that there is an inverse function from \( B \) to \( A \)?

3. A ship travels from Mombasa to Aden and returns. Let \( t \) be the number of days after it leaves Mombasa and \( d \) its distance from Mombasa in miles. Describe a function from the set of times to the set of distances. If you reverse the order of the sets is there an inverse function? Explain.

**61-7 Graphs of Experimental Results**

In each of the previous examples there has been a formula from which you could make a table and then a graph. When the scientist performs an experiment the situation is different. He measures two quantities and shows the results in a table. An example was given in Section 60-4 where there is a table of the distances in feet that a stone falls in certain times in seconds. In this case it was easy to guess the formula, \( d = 16t^2 \). You can of course draw a graph by using this formula to construct as complete a table as you need.

If you cannot find a formula or at any rate a simple formula, you can use the table of experimental results to draw a graph. It is true that there is some uncertainty.
You do not know for sure how things go between the points that are actually marked. It is usual to assume that the true picture is a fairly smooth curve through the known points.

There is, however, a complication. As you know, all actual measurements are approximate. You cannot therefore be certain that the points that you locate from the results of measurements are exactly where they ought to be. It usually happens that the graph that is drawn does not go exactly through all of the points that have been marked. It is always possible to draw a curve through all of the experimental points but the curve may not be a very smooth one. The art of fitting curves to experimental points takes some experience to learn. We cannot say much about it here.

It is often true that the points obtained from experiment come close to lying on a straight line. This should be true for some of the experiments that you performed in the last chapter. By marking the experimental points and laying a ruler on the paper you should be able to find a straight line that fits the points as well as possible. This is the sort of thing that is meant. The experiments that have been described are somewhat crude. Naturally with improved apparatus and great care more precise results can be obtained. This usually results in better-fitting curves or lines.

In the last section the graph of \( d = 32t - 16t^2 \) was drawn to represent the height of the stone thrown up in the air. You may have noticed that different scales were used on the two axes. Previously we used the same scale on both axes. The reason for the change here is a matter of convenience. The graph should not be too high to fit easily on the page.

In practice when you are drawing graphs of experimental results you should choose the scales on the axes very carefully so that you can use the space available on the page to best advantage.

**EXERCISE 61-7**

In the previous Chapter you were asked to perform a number of experiments and report the results in a table. For each of these experiments choose scales on the first and second axes so that most of a page will be filled. Mark all of the points from your table and draw as smooth a curve as you can through or near these points. In which cases do you appear to get a straight-line graph?
Chapter 62
MATHEMATICAL SENTENCES

62-1 Introduction

In this chapter we shall be concerned with mathematical sentences. You have met such sentences in the last two chapters, but we shall now consider them from a somewhat broader point of view. In particular you will learn how to change one mathematical sentence into another one which is equivalent to it. Even though you will meet some problems that you already know how to solve, it will give you more insight if you look at these problems in a new way.

You have used the statement

\[ F = \frac{9}{5} C + 32 \]

for converting degrees Centigrade to degrees Fahrenheit. Let us assume that the temperature is known to be 25 degrees Centigrade. What would this be in degrees Fahrenheit? Replacing \( C \) by 25 we have

\[ F = \frac{9}{5} (25) + 32 \]

\[ = 45 + 32 \]

\[ = 77 \]

and our answer is, of course, 77 degrees.

Now suppose that we were asked a somewhat different type question such as: If the temperature is 86 degrees Fahrenheit, what would this be in degrees Centigrade? In this case we cannot apply the formula directly.

We should like to develop a systematic approach for answering questions of this sort. Such an approach will also enable us to solve a variety of other problems.

Returning to the question of how many degrees Centigrade would correspond to 86 degrees Fahrenheit, we note that the formula

\[ F = \frac{9}{5} C + 32 \]

now becomes
\[ 86 = \frac{9}{5}C + 32 \]

or

\[ \frac{9}{5}C + 32 = 86. \]

From this we see that our question could be put in the following form: For what value of \( C \) will the statement, \( \frac{9}{5}C + 32 = 86 \), be true?

How would you work out a scheme for answering this type of question? The method we are about to explore involves a very important and very useful mathematical process known as

"Solving an equation"

or "Finding the truth set of an open sentence."

You have already encountered many examples of so-called mathematical sentences. These include such statements as

\[ 3 + 5 = 8, \quad 2 \times 7 = 14, \quad 18 \div 3 = 6 \]

and so forth. These are all examples of true sentences. We can say this because in every case the numerals on both sides of the "=" sign represent the same number.

On the other hand mathematical sentences such as

\[ 7 + 5 = 10, \quad 16 - 4 = 11, \quad \text{and} \quad 7 \times 8 = 45 \]

are clearly examples of false sentences.

What about the following?

\[ N + 7 = 12 \]

Is it a true sentence? Or is it false? You will probably agree that there is no way to answer this question as it stands. In other words, we cannot give a correct answer until we know what number \( N \) is supposed to represent. Since the question of whether or not the sentence is true remains "open" we call an expression like \( N + 7 = 12 \) an open sentence.

Is the sentence true if \( N = 5 \)? Is it true if \( N = 6 \), or 8, or 4? Are there any numbers other than 5 which when added to 7, give a sum of 12? Since the answer is "no", we say that 5 is the truth value of the sentence \( N + 7 = 12 \).

If the connecting mathematical symbol is an "=" sign, we call the sentence an equation. Thus \( N + 7 = 12 \) is an equation. The number 5 is also referred to as the solution of the equation.

An open sentence, like \( x < 5 \), is called an inequality. Here the sentence is true if \( x \) represents any number less than 5. Is it true if \( x = 5 \)? Is it true if \( x = 6 \)?
If in this problem we restrict the possible values of \( x \) to be the set of whole numbers, what numbers in this set make the sentence true? Under these conditions the set \( \{0, 1, 2, 3, 4\} \) is called the truth set of the open sentence (inequality), \( x < 5 \).

For the truth set of the sentence \( N + 7 = 12 \), we have the set \( \{5\} \) that consists of the single number 5.

In the sentences in the two examples shown, the letter \( N \) and the letter \( x \) are called variables. For open sentences a variable can be thought of as a symbol, usually a letter, used to represent any number in a given set. The given set is called the domain of the variable. In the previous example, \( x < 5 \), we restricted the possible values of \( x \) to be members of the set of whole numbers. Thus the domain of the variable in this case was the set

\[
\{0, 1, 2, 3, 4, 5, 6, 7, \ldots\}.
\]

Do you see that the truth set \( \{0, 1, 2, 3, 4\} \) is a subset of this domain?

In both of the sentences, \( N + 7 = 12 \) and \( x < 5 \), we could determine the respective truth sets without any difficulty.

There are many problems, however, which lead to more complicated open sentences, sentences whose truth sets cannot be easily determined by inspection.

One such is our Centigrade equation

\[
\frac{9}{5}C + 32 = 86.
\]

As another example suppose that we are asked the following question. Three more than six times a certain number is 45. What is the number? To begin with how can you translate this into a mathematical open sentence? It should be clear that such a sentence is

\[
6N + 3 = 45.
\]

For what value of \( N \) will this sentence be true? With a bit of trial and error the answer 7 will probably come to mind. To test this we note that \( 6(7) + 3 = 45 \) is a true sentence. If we replace \( N \) by any number other than 7, the result will be a false sentence. Can you give the reason?

### 62-2 Equivalent Sentences

The solution to the previous problem may have involved some experimentation, or guesswork. Let us see how such guesswork might be eliminated. To do this we need a basic concept which applies to open sentences in general. This is the concept of equivalence. To illustrate the idea, we return to the problem under discussion. For the equation \( 6N + 3 = 45 \) we have already obtained the solution 7. To state this, we write \( N = 7 \). Do you see that such a statement is itself an open sentence? The truth set of this sentence is also \( \{7\} \). We note then that the two open sentences
and

\[ 6N + 3 = 45 \]

and

\[ N = 7 \]

have the same truth sets. Under these conditions we say that the two sentences are equivalent. In other words, any two open sentences are equivalent if, and only if, they have the same truth set. To fix this idea, check each of the following pairs of open sentences to see if they are equivalent.

a. \[ 3x + 4 = 10. \quad x = 2 \]
b. \[ 5x - 2 = 18. \quad x = 4 \]
c. \[ 11N - 7 = 81. \quad N = 8 \]
d. \[ 12x + 14 = 26. \quad x = 1 \]
e. \[ 13x - 20 = 71. \quad x = 7 \]

In every case the above pairs of sentences are, in fact, equivalent. Comparing the sentences on the right, however, with those on the left, what basic difference suggests itself? It should be clear that for the sentences on the right, the truth sets are obvious. No guesswork or trial and error is needed. On the other hand for the sentences on the left this may not be the case. These sentences, in other words, do not show their truth sets in a self-evident manner.

Does this suggest how one might go about "solving" an equation? Suppose that we are given a complicated sentence. Can we make this into a simple sentence which is equivalent to the first (has the same truth set), but which is written in such a form as to enable us to determine immediately what its truth set is? For the problems that we shall be considering, the answer is "yes".

To see how this can be done, let us begin by working backwards. Suppose that we start with the simple form of sentence

\[ N = 7. \]

For a certain value of \( N \) (namely 7) this sentence is true. For other values of \( N \) the sentence is false. Now suppose that we add 3 to \( N \) and 3 to 7. This gives us \( N + 3 \) and \( 7 + 3 \). Here again we note that when \( N \) has the value 7, then the expression \( N + 3 \) represents the same number as \( 7 + 3 \). Thus \( N + 3 = 7 + 3 \) is a true sentence when \( N \) is 7. For other values of \( N \), the sentence is false. What can you say, then, about these two sentences?

\[ N = 7 \]

and

\[ N + 3 = 7 + 3 \]
Are they equivalent?

In this instance we may say that we have added 3 to both "sides" of the sentence, or equation. It should be clear, however, that the argument for equivalence does not depend on the choice of the particular number 3. What about the following sentences?

\[
N = 7 \\
N + 5 = 7 + 5 \\
N + 100 = 7 + 100 \\
N - 6 = 7 - 6
\]

Are they equivalent? In the last case we subtracted 6. Since we have seen that subtracting 6 is the same as adding -6, we can think of all of the sentences after the first as having been obtained by adding the same number to both sides of \(N = 7\).

A rule now suggests itself. If the same number is added to both sides of an open sentence (equation) the result will be an equivalent open sentence.

Does the same rule apply to multiplication? For the sentence \(N = 7\), multiply \(N\) by 6 and 7 by 6. Consider them, the expressions \(6N\) and \(6 \times 7\). If the value of \(N\) is 7, then \(6N\) names the same number as \(6 \times 7\) and for this value \(6N = 6 \times 7\) is a true sentence. For any other value of \(N\) the sentence \(N = 7\) is false, and the sentence \(6N = 6 \times 7\) is false also. Are the sentences \(N = 7\) and \(6N = 42\) equivalent?

Again the argument does not require that the number that we multiply by be specifically 6. Before we state the rule in general terms, however, we should look at one special case. Take the sentence \(N = 5\). Its truth set is clearly \{5\}. Now suppose that we multiply both sides by zero. This gives us \(0 \times N = 0 \times 5\) or \(0 \times N = 0\). For what values of \(N\) is this last sentence true? By the special property of zero, we can see that the sentence is true when \(N\) is any number at all. What, then, can we say about the two sentences \(N = 5\) and \(0 \times N = 0\)? Are they equivalent?

Can we now state a general rule? If a second equation is obtained from a given first equation by

1) Adding the same number to both sides

2) Multiplying both sides by any number other than zero

the resulting second equation will be equivalent to the first.

We are now ready to apply the rule to some of the equations that we have been discussing. For the first example we can write the following sequence of equivalent equations:
\[ N + 7 = 12 \]
\[ N + 7 - 7 = 12 - 7 \]
\[ N = 5 \]

The last one clearly proclaims its truth set. In the second step we have added \(-7\) or, what amounts to the same thing, subtracted 7 from both sides.

It is often necessary to apply the rule several times. As an example consider the following sequence of equivalent equations.

\[ 6N + 3 = 45 \]
\[ 6N = 42 \quad \text{(subtract 3)} \]
\[ \frac{1}{6} \times 6N = \frac{1}{6} \times 42 \quad \text{(multiply by \(\frac{1}{6}\))} \]
\[ N = 7 \]

The step involving multiplication by \(\frac{1}{6}\) could be more conveniently thought of as division by 6. Can you explain why?

Finally let us look at the Fahrenheit-Centigrade problem. We form equivalent equations as follows.

\[ \frac{9}{5}C + 32 = 86 \]
\[ \frac{9}{5}C = 54 \quad \text{(subtract 32)} \]
\[ 5 \times \frac{9}{5}C = 5 \times 54 \quad \text{(multiply by 5. Why?)} \]
\[ 9C = 270 \]
\[ C = 30 \quad \text{(divide by 9)} \]

All of these results are reassuringly consistent with our earlier findings!

**EXERCISE 62-2A**

1. Extend the rule for equivalent equations to include subtraction and division.
2. Solve the following equations.

a. \[3x + 5 = 26\]

b. \[17x + 1 = 52\]

c. \[14x + 23 = 219\]

d. \[42 + 25x = 292\]

e. \[\frac{3}{4}x - 8 = 1\]

f. \[\frac{5}{6}x = 25\]

g. \[\frac{2}{3}x - 5 = \frac{1}{2}\] (Hint: Multiply both sides by \(6\))

h. \[\frac{1}{4}x + \frac{1}{8} = \frac{1}{3}\] (Hint: Multiply both sides by \(24\). Why did we choose \(24\)?)

3. The temperature in degrees Fahrenheit is 41. Find the corresponding temperature in degrees Centigrade.

4. If 7 more than 5 times a certain number is 52, what is the number?

If a variable appears on both sides of an equation, the rule may be applied in the same manner using the distributive law. For example in the equation \[8x - 2 = 50 - 5x,\] we may consider \(5x\) as a number which can be added to both sides. Adding this and adding 2, we get \(8x + 5x = 52\). Since \(8x + 5x = (8 + 5)x = 13x\), we have \(13x = 52\) or \(x = 4\).

**EXERCISE 62-2B**

1. Solve the equations:

a. \[17x + 12 = 10x + 117\]

b. \[\frac{2}{3}x - 15 = 6 - \frac{1}{3}x\] (Hint: add \(\frac{1}{3}x\) first)

2. a. \[2 - 3x = 2x - 8\]

b. \[\frac{1}{2}x - 14 = 16 - \frac{3}{4}x\]

3. a. \[5m + 3m = 24\]

b. \[11t - 9 = 4t + 12\]

c. \[\frac{5}{8}y + 1 = \frac{3}{8}y + 2\]
4. If 4 times a number is decreased by 5, the result will be 70 more than the original number. Find the number.

5. A and B have 120/- between them. If A has x/- how many shillings has B? If it is also given that A's money is twice B's, what is x?

6. When A and B sit down to play, A has 2x/- and B x/-. A then wins 5/- from B and then has three times as much money as B. How much had each at first?

7. Two men starting from two points P and Q 60 miles apart ride toward one another. When they meet, A has done 3x miles to B's 2x miles. How far has the faster man travelled from his starting point?

8. If \( \frac{3x}{5} - 12 \) and \( 9 - \frac{1}{10} x \) stand for the same number what is that number?

9. John is x years old. Half John's age plus \( \frac{1}{4} \) of his age is 39 years. What is John's age?

10. The perimeter of a rectangular field is 24 ft. The longer side of the rectangle is three times the length of the shorter side. What are the dimensions of the field?

11. If you multiply a certain number by 3 and subtract 12 from the product you would get the same result as if you had divided the same number by 2 and added 58 to the quotient. What is that number?

12. A man keeps a number of weights for use in his grocery shop. The number of 1 oz. weights is twice the number of \( \frac{1}{2} \) oz. weights and half the number of 2 lb. weights, while the number of 1 lb. weights is three times the number of \( \frac{1}{2} \) oz. weights. The combined weights total 11 lb. 2 \( \frac{1}{2} \) oz. How many of each kind has he?

62-3 Changing the subject of the sentence

The sentence \( F = \frac{9}{5} C + 32 \) offers, as we have seen, a convenient device for determining the Fahrenheit temperature when the Centigrade temperature is given. We have also observed that when the situation is reversed, a difficulty arises. Given F, how do we find C?

The ideas in section 62-2 should now provide the answer. Since F is assumed to be given, we may regard it as a fixed number and the sentence

\[
\frac{9}{5} C + 32 = F
\]

can be thought of as an open sentence in the variable C. A procedure for developing a "chain" of equivalent sentences now suggests itself.
\[
\frac{9}{5}C + 32 = F \\
\frac{9}{5}C = F - 32 \quad \text{(subtract 32)} \\
9C = 5F - 160 \quad \text{(multiply by 5)} \\
C = \frac{5F - 160}{9} \quad \text{(divide by 9)}
\]

We now have a new sentence, which is equivalent to the original one in the following sense. For any specified F, say 50, the sentence

\[
C = \frac{5}{9} (50) - 160
\]

is equivalent to the sentence

\[
50 = \frac{9}{5}C + 32
\]

In both cases the truth set is \{10\}.

We now have the statement

\[
C = \frac{5i^2 - 160}{9}
\]

which provides a formula for determining the degrees Centigrade for any given number of degrees Fahrenheit.

Take another example. A man buys a radio on the hire-purchase plan. He makes a down payment of 150 shillings and must pay 24 shillings a month for 18 months. If \( P \) represents the total amount paid at the end of \( m \) months, we have the formula \( P = 18m + 150 \). At the end of 5 months he has paid \( P = 18(5) + 150 \) or 240 shillings.
How would you find out how many months it will take before the man has paid, say 456 shillings? We can change the subject of the sentence as follows.

\[ 18m + 150 = P \]
\[ 18m = P - 150 \]
\[ m = \frac{P - 150}{18} \]

Can you describe the steps used in forming the equivalent sentences?

For \( P = 456 \) we have

\[ m = \frac{456 - 150}{18} = 17 \]

and the answer is 17 months.

**EXERCISE 62-3**

1. A tank already holds 5 gallons of water and a tap puts in 2 gallons of water every minute. The number of gallons in the tank after any number of minutes can be found by \( G = 5 + 2m \) where \( G \) is the number of gallons and \( m \) stands for any number of minutes.

   Find the number of gallons in the tank after 1 hour; next rewrite the formula so that you could use it to find when the tank will hold a given number of gallons of water. Use this to find in how many minutes the tank will hold 185 gallons.

2. People say that to make good tea we must take a teaspoonful of tea for each person drinking the tea and one more for the pot in which the tea is to be made and this is the equation they use: \( T = 1 + n \) where \( T \) is the number of teaspoons of tea in the teapot and \( n \) the number of people for whom the tea is made.

   Use this equation to find the number of teaspoonsful required to make tea for 12 people.

   Now write the equation so that it could be used to find the number of people for whom tea could be made if the number of teaspoonfuls of tea is given. Use this last equation to find for how many people tea has been made if 14 teaspoonfuls of tea were put in the pot.
3. The volumes of several square prisms with congruent square bases each of side 4" are given by the equation \( V = 16h \) where \( h \) is the height in inches and \( V \) represents the volume in cubic inches.
Use the equation to find the volume of one of these square prisms whose height is 1 yd.
Next write the equation so that it could be used to find the height of one such square prism when its volume is known. Use it to find the height of one such square prism when its volume is 192 cu. inches.

4. Several right cylinders have congruent bases each of area 154 sq. inches. The total surface area of each can be determined by the equation \( S = 308 + 44h \) where \( S \) is the total surface area in square inches and \( h \) is the height in inches.
Determine the total surface area of such a right cylinder whose height is 8 inches.
Find an equation which can be used to find the height when the total surface area of such a right cylinder is given.
Use this equation to find the height of such a cylinder whose total surface area is 1892 square inches.
Assume that each of the following equations is derived from some actual physical situation. Then work as directed.

5. \[ E = \frac{1}{2}n + 8 \]
Find \( E \) when \( n = 15 \).
Then write the equation to find \( n \). Find \( n \) when \( E \) is 12.

6. \[ S = a + 3t \]
Find \( S \) when \( a = 3, \ t = 2 \).
Rewrite the equation to determine \( t \). Use it to find \( t \) when \( S = 24, \ a = 6 \).

7. \[ y = 5x - z \]
Find \( y \) when \( x = 2, \ z = 3 \).
Rewrite the equation to find \( z \) and use the result to find \( z \) when \( x = 12 \) and \( y = 5 \).
8. \[ T = \frac{R + r}{R} \]

Find \( T \) when \( R = 20 \), \( r = 8 \).
Find an equation which will give \( r \) and use it to find \( r \) when \( T = 2 \), \( R = 16 \).

9. \[ V = \frac{4}{3} \pi r^3 \]

Find \( V \) when \( r = 3 \).
Then write the equation to find \( r^3 \). Find \( r^3 \) when \( V = 38808 \).
Can you then find \( r \)?

62-4 Inequalities

We have already considered the open sentence \( x < 5 \), which as you remember, is called an inequality. When \( x \) is any number less than 5, the sentence is true. For any other value of \( x \) the sentence is false.

Given a slightly more complicated inequality, however, it might not be so easy to find the truth set. Take, for example, the sentence

\[ 3x - 5 < x + 5. \]

Can we see at a glance what its truth set is? The answer is probably "no". A second question might now be raised: Is this sentence equivalent to the first one, \( x < 5 \)? To answer this, you might be tempted to use the rules that we had for forming equivalent equations and proceed as follows:

\[
\begin{align*}
3x - 5 &< x + 5 \\
3x &< x + 10 & \text{(add 5)} \\
2x &< 10 & \text{(subtract } x) \\
x &< 5 & \text{(divide by 2)}
\end{align*}
\]

A plausible answer, then, might be "yes". It turns out that the answer is correct. There is a rule for equivalence of inequalities which follows the same pattern as the one for equations. This time we shall first state the rule, then see why it works. The rule follows:

If a second inequality is obtained from a given first inequality by
1) Adding the same number to both sides
2) Multiplying both sides by a positive number
the resulting second inequality is equivalent to the first.
On the basis of this rule we may be sure that the sentences

\[ x < 5 \quad \text{and} \quad 3x - 5 < x + 5 \]

are indeed equivalent.

We should perhaps note that the step "dividing by 2" can also be thought of as multiplying by \( \frac{1}{2} \). Thus the conditions of the rule are strictly fulfilled.

Note also the significant change. In place of multiplication by any number other than zero, we now have multiplication by any positive number. What, then, happens when we multiply by a negative number? We shall soon find out. But first let us see if the given rule is justified.

In an earlier study (Chapter 25) we considered the so-called addition and multiplication properties of order. These stated that if \( a, b, \) and \( c \) are any numbers and if \( a < b \) then \( a + c < b + c \). Furthermore if \( c > 0 \), then \( a \times c < b \times c \), or by the commutative property of multiplication

\[ c \times a < c \times b \]

You also learned in Chapter 25 that if \( a < b \) and \( c < 0 \) then

\[ c \times a > c \times b \]

It is not true that \( c \times a < c \times b \) ! The inequality is reversed! It will be worthwhile to convince ourselves of this important fact.

If \( a < b \)

we may write

\[ b = a + p \]

where \( p \) is some positive number. Now let us multiply each side of this equation by \( c \). We obtain the equivalent equality

\[ c \times b = c \times (a + p) = (c \times a) + (c \times p) \quad \text{(Why?)} \]

What is the sign of \( c \times p \)? Negative of course. Therefore \( c \times b = c \times a \) minus a certain positive number. In other words, \( c \times a \) is \( c \times b \) plus a positive number. Therefore

\[ c \times a > c \times b \]

as we wished to prove. Notice that this proof reduces the problem to one that concerns an equality.

When we apply these principles to equivalent inequalities we have the following results.

If we have a given inequality, we obtain an equivalent inequality if we
1. Add the same number to both sides.

2. Multiply both sides by the same positive number.

3. Multiply both sides by the same negative number and reverse the direction of the inequality sign.

Let us use this result to find the truth set of

$$4 - 3x < 7$$

An equivalent inequality is

$$-3x < 3$$

(Why?)

If we multiply by $$(-\frac{1}{3})$$ we get another equivalent inequality, namely

$$(-\frac{1}{3})(-3x) > (-\frac{1}{3})3$$

which means that

$$x > -1$$

is also an equivalent sentence. Its truth set is the set of all numbers greater than $$-1$$. We can obtain this result in a different way. We start again with

$$4 - 3x < 7.$$ 

Add $$3x$$ to both sides. The result is the equivalent inequality

$$4 < 7 + 3x.$$ 

This in turn is equivalent to

$$-3 < 3x$$

(Why?)

which is equivalent to

$$-1 < x$$

(Why?)

**EXERCISE 62-4**

Describe the truth set of each of the following inequalities.

1. $$3x + 1 > 4$$
2. \[ 1 - 3x < \frac{1}{4} \]

3. \[ 5x - 3 < 1 \]

4. \[ 2 - \frac{1}{3}c < 5 \]

5. \[ 8 - 3x < 11 \]

6. \[ -3 < 2x < 1 \]

7. \[ 3 > x > \frac{1}{2} \]

8. A rectangle is to be constructed such that its area shall be less than 400 square inches. If one side of the rectangle must be 25 inches, the length of the other side must be less than what number of square inches?

9. A train should cover a distance of 600 miles. Its speed must be less than 100 m.p.h. The number of hours it takes will be greater than what number?

10. A man wanting to build a pen cannot spend more than 100/- to get sticks for the job. If he must use 40 such sticks, he must pay less than how much per stick?

62-5 A New Kind of Inequality

We often encounter sentences of the form \( x \leq 5 \), where the symbol \( \leq \) stands for "is less than or equal to." The truth set of this sentence consists of all numbers which are less than 5, and also the number 5 itself. Actually the sentence \( x \leq 5 \) may be broken down into two sentences (a) \( x < 5 \) and (b) \( x = 5 \). In this sense the truth set of \( x \leq 5 \) may be regarded as the set of all numbers satisfying either (a) or (b).

Suppose that we are asked to solve the sentence \( 3x - 2 \leq 10 \). Again we may break this down into two sentences (a) \( 3x - 2 < 10 \) and (b) \( 3x - 2 = 10 \). From what has gone before we know that these are equivalent to \( x < 4 \) and \( x = 4 \) respectively. Putting the two together again we have \( x \leq 4 \). The truth set is the set of all numbers less than or equal to 4. In practice one may apply the multiplication and addition rules directly. To solve the sentence

\[ 5x - 3 \leq 27 \]
we may write the equivalent sentences

\[ 5x \leq 30 \]

and

\[ x \leq 6. \]

Care must be taken when multiplying by a negative. In this case it may be safer to separate the sentences as we did originally. For example, to solve the sentence

\[ 5 - 3x \leq 23, \]

we consider the two sentences

\[ 5 - 3x < 23 \text{ and } 5 - 3x = 23. \]

An equivalent sentence to the first is

\[ x > 6 \text{ and to the second } x = 6. \]

Thus we obtain \( x \geq 6 \), whose truth set consists of all numbers greater than or equal to 6.

**EXERCISE 62-5**

Repeat Problems 1 through 7, in Exercise 62-4, in each case replacing \( > \) by \( \geq \), or \( < \) by \( \leq \).
UNIT XIII

STATISTICS

Chapter 63

COLLECTION AND PRESENTATION OF DATA

63-1 Introduction

In this unit we shall learn about an important branch of mathematics called statistics. Sometimes we use the word statistics to mean number facts such as vital statistics, population statistics, rainfall statistics but often by statistics we mean the branch of knowledge which deals with the collection and interpretation of data.

Another important use of statistics is the prediction of future events. For example, a weather forecaster collects information about the weather over a long period of time, and from past experience he can use the analysis of his data to predict what the weather is going to be like. Of course his prediction may not always be correct since the conditions on which he bases the prediction are subject to sudden changes.

We shall not concern ourselves here with the use of statistics for prediction. Rather we shall learn about descriptive statistics. We shall study how best to present data in tabular form and how to represent data by a suitable graph. We shall learn the different meanings of the word "average" and in what way each kind of average may be used. Finally, we shall include a short discussion on measure of spread of a set of data.

Our examples to illustrate each of these ideas will be drawn from the classroom and from official records such as the Statistical Abstract and Annual Departmental Reports from different African countries. Whenever a table is used or a graph is drawn the source of the data will be indicated. This is good practice since it enables anyone who desires further details to go to the source of the information.

63-2 Collection and Presentation of Data

(a) Presenting Facts in Tabular Form

Often when we open our newspapers or books we find information presented to us in a form which is not easy to absorb at a glance. Here is an example. Below is given in each subject the number of candidates who took the University of London General Certificate of Education Examination at the Advanced Level in Sierra Leone, West Africa, in June 1963:
You will notice that the subjects are arranged in alphabetical order. If we wished to find out the most popular subject (in the sense that it was taken by the largest number of candidates) we would have to go carefully over the above data.

The data might be rearranged in order starting with the most popular subject and ending with the least popular. This rearrangement will help us to present the data in a form that is easier to absorb. Here it is.

<table>
<thead>
<tr>
<th>Subject</th>
<th>Number of Candidates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Economics</td>
<td>55</td>
</tr>
<tr>
<td>Chemistry</td>
<td>53</td>
</tr>
<tr>
<td>British Constitution</td>
<td>44</td>
</tr>
<tr>
<td>Physics</td>
<td>40</td>
</tr>
<tr>
<td>History</td>
<td>37</td>
</tr>
<tr>
<td>French</td>
<td>23</td>
</tr>
<tr>
<td>Biology</td>
<td>22</td>
</tr>
<tr>
<td>Zoology</td>
<td>22</td>
</tr>
<tr>
<td>Botany</td>
<td>17</td>
</tr>
<tr>
<td>English Literature (African Syllabus)</td>
<td>17</td>
</tr>
<tr>
<td>Latin</td>
<td>12</td>
</tr>
<tr>
<td>Pure Mathematics</td>
<td>12</td>
</tr>
<tr>
<td>History (English Economic)</td>
<td>10</td>
</tr>
<tr>
<td>English Literature (Home Syllabus)</td>
<td>7</td>
</tr>
<tr>
<td>Geography</td>
<td>7</td>
</tr>
<tr>
<td>Applied Mathematics</td>
<td>7</td>
</tr>
<tr>
<td>Mathematics (Pure &amp; Applied)</td>
<td>6</td>
</tr>
<tr>
<td>Physics</td>
<td>6</td>
</tr>
<tr>
<td>Religious Knowledge</td>
<td>6</td>
</tr>
<tr>
<td>Zoology</td>
<td>22</td>
</tr>
</tbody>
</table>

In the above form it is easier to see at a glance that the most popular subject (in the sense described above) was Economics and the two least popular ones were Mathematics (Pure and Applied) and Religious Knowledge. We can also answer much more quickly other questions on the data.

(b) *The Frequency Table*

Here is another example. Fifty pupils in a school were selected. Their
heights were measured to the nearest inch and recorded as follows:

56  50  62  54  59  60  62  55  56  59
60  59  56  63  58  59  58  62  59  54
54  60  58  56  59  63  56  55  60  59
62  55  59  60  56  58  59  59  55  60
60  58  55  58  59  55  58  56  59  60

As in the previous example we could arrange the names of the fifty pupils in order in one column and their heights in another column starting with the name of the tallest pupil. You will agree that this will indeed be a long column since there are fifty names altogether.

How best can we show this information without writing every name? Another look at the table shows that several pupils have the same height. In fact, it is in only one case that we find one pupil in a class all by himself.

This suggests that we arrange the heights in order from the tallest to the shortest and show in each case how often each height occurs. We may do this by going through the table and making a stroke (a tally mark) each time a height occurs; when a height occurs five times we draw the fifth stroke across as shown below. This makes it easier to count the tally marks in groups of five at a time. The number of times that each height occurs is the frequency of the height.

<table>
<thead>
<tr>
<th>Height in Inches</th>
<th>Tally Marks</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>/</td>
<td>1</td>
</tr>
<tr>
<td>54</td>
<td>///</td>
<td>3</td>
</tr>
<tr>
<td>55</td>
<td>//</td>
<td>6</td>
</tr>
<tr>
<td>56</td>
<td>///</td>
<td>7</td>
</tr>
<tr>
<td>58</td>
<td>///</td>
<td>7</td>
</tr>
<tr>
<td>59</td>
<td>/////</td>
<td>12</td>
</tr>
<tr>
<td>60</td>
<td>/////</td>
<td>8</td>
</tr>
<tr>
<td>62</td>
<td>/////</td>
<td>4</td>
</tr>
<tr>
<td>63</td>
<td>//</td>
<td>2</td>
</tr>
</tbody>
</table>

Total frequency = Total number of pupils = 50

From this table we see how a pupil who is 55 inches tall compares with other pupils in the class. We can say how many pupils are of the same height, or are taller or shorter. It would be more difficult to extract this information from the raw data which we had originally.

(c) Grouping of Data

It was stated above that the heights of the pupils were measured to the nearest inch. This means that if we had a height of 55.5 or 55.6 inches we would express either of them as a height of 56 inches while a height of 55.3 or 55.4 inches would each be expressed as 55 inches.

Marks (Scores) are also often expressed to the nearest whole number. For example a mark of 66½ may be expressed as 67 to the nearest whole number while a mark of 37½ out of 50 would be expressed as 75 out of 100.

The following is a record of scores out of 100 obtained by 100 candidates in
an Arithmetic examination; the scores are all expressed to the nearest whole number.

<table>
<thead>
<tr>
<th>Score</th>
<th>50</th>
<th>100</th>
<th>66</th>
<th>52</th>
<th>36</th>
<th>56</th>
<th>68</th>
<th>64</th>
<th>52</th>
<th>88</th>
</tr>
</thead>
<tbody>
<tr>
<td>18</td>
<td>96</td>
<td>56</td>
<td>44</td>
<td>30</td>
<td>50</td>
<td>50</td>
<td>40</td>
<td>10</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>54</td>
<td>64</td>
<td>50</td>
<td>58</td>
<td>84</td>
<td>12</td>
<td>60</td>
<td>46</td>
<td>56</td>
<td></td>
</tr>
<tr>
<td>74</td>
<td>34</td>
<td>50</td>
<td>18</td>
<td>50</td>
<td>96</td>
<td>56</td>
<td>44</td>
<td>30</td>
<td>48</td>
<td></td>
</tr>
<tr>
<td>52</td>
<td>60</td>
<td>18</td>
<td>50</td>
<td>46</td>
<td>20</td>
<td>42</td>
<td>16</td>
<td>68</td>
<td>48</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>58</td>
<td>73</td>
<td>32</td>
<td>66</td>
<td>56</td>
<td>52</td>
<td>58</td>
<td>88</td>
<td>66</td>
<td></td>
</tr>
<tr>
<td>82</td>
<td>62</td>
<td>52</td>
<td>28</td>
<td>38</td>
<td>76</td>
<td>86</td>
<td>30</td>
<td>72</td>
<td>54</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>52</td>
<td>21</td>
<td>33</td>
<td>54</td>
<td>58</td>
<td>58</td>
<td>62</td>
<td>52</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>50</td>
<td>58</td>
<td>27</td>
<td>24</td>
<td>27</td>
<td>66</td>
<td>48</td>
<td>32</td>
<td>29</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>23</td>
<td>19</td>
<td>64</td>
<td>29</td>
<td>48</td>
<td>36</td>
<td>70</td>
<td>42</td>
<td>89</td>
<td></td>
</tr>
</tbody>
</table>

In setting out the information contained in the above we could adopt the same approach as was used in dealing with the heights of the pupils. It would be found, however, that there are nearly fifty different scores represented in the data, and we shall thus need a column of about fifty rows to analyse the marks as was done with the heights. It would be convenient to group the scores so that it is easier to digest the data. We could group in tens: 0 - 9, 10 - 19, 20 - 29, etc. or in fives: 0 - 4, 5 - 9, 10 - 14, 15 - 19, etc.

A look at the scores shows that the lowest is 10 and that 10, 12, 16, 18, 19 fall in the interval 10 - 19. Thus we have seven strokes in the Tally Marks column.

We call 0 - 9, 10 - 19, 20 - 29 or 0 - 4, 5 - 9, 10 - 14 etc. the class intervals because they mark the groups or classes. As we stated above, the scores were expressed to the nearest whole number. This means that a score recorded as 20 could really have been anywhere between 19.5 and 20.5. Therefore the class that contains all the scores recorded as between 10 and 19 inclusive will contain all the scores which in fact lay between 9.5 and 19.5. That is, the class interval 10 - 19 has the class boundaries 9.5 - 19.5. Similarly the class interval 20 - 29 has the class boundaries 19.5 - 29.5 and so on. The mid-point of the class interval is equal to the average (arithmetic mean) of the class boundaries. Thus the mid-point of the interval 10 - 19 is $\frac{10 + 19}{2} = \frac{29}{2} = 14.5$.

Since the class boundaries are 9.5 and 19.5 the mid-point of the interval is also $\frac{9.5 + 19.5}{2} = 14.5$.

The above scores have been tabulated below:

<table>
<thead>
<tr>
<th>Class intervals</th>
<th>Class boundaries</th>
<th>Mid-point of interval</th>
<th>Tally marks</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 - 19</td>
<td>9.5 - 19.5</td>
<td>14.5</td>
<td>//</td>
<td>7</td>
</tr>
<tr>
<td>20 - 29</td>
<td>19.5 - 29.5</td>
<td>24.5</td>
<td>//</td>
<td>12</td>
</tr>
<tr>
<td>30 - 39</td>
<td>29.5 - 39.5</td>
<td>34.5</td>
<td>//</td>
<td>13</td>
</tr>
<tr>
<td>40 - 49</td>
<td>39.5 - 49.5</td>
<td>44.5</td>
<td>//</td>
<td>13</td>
</tr>
<tr>
<td>50 - 59</td>
<td>49.5 - 59.5</td>
<td>54.5</td>
<td>//</td>
<td>31</td>
</tr>
<tr>
<td>60 - 69</td>
<td>59.5 - 69.5</td>
<td>64.5</td>
<td>//</td>
<td>12</td>
</tr>
<tr>
<td>70 - 79</td>
<td>69.5 - 79.5</td>
<td>74.5</td>
<td>//</td>
<td>5</td>
</tr>
<tr>
<td>Class intervals</td>
<td>Class boundaries</td>
<td>Mid-point of interval</td>
<td>Tally marks</td>
<td>Frequency</td>
</tr>
<tr>
<td>-----------------</td>
<td>------------------</td>
<td>-----------------------</td>
<td>------------</td>
<td>-----------</td>
</tr>
<tr>
<td>80 – 89</td>
<td>79.5 – 89.5</td>
<td>84.5</td>
<td>//\</td>
<td>5</td>
</tr>
<tr>
<td>90 – 99</td>
<td>89.5 – 99.5</td>
<td>94.5</td>
<td>/</td>
<td>1</td>
</tr>
<tr>
<td>100 – 109</td>
<td>99.5 – 109.5</td>
<td>104.5</td>
<td>/</td>
<td>1</td>
</tr>
</tbody>
</table>

Total number of scores = Total frequency = 100

Look again at the table above. It presents information in a concise form. Can you think of any disadvantage of this particular form? Can you tell by looking at the table whether any of 16, 17, 18, 19 occurs in the interval 10 – 19 and with what frequency?

Although the table is concise we note that some information is lost. From the table we know that seven scores fall in the interval 9.5 – 19.5 but we do not know, without looking at the original data, what these scores are.

In general, in grouping data into classes, we should not have too many or too few classes; the usual number of classes is between 10 and 25 depending upon the data with which we are working. We should also avoid intervals which are not of the same length. For example we should not have intervals of length five mixed up with intervals of length ten; for example, 0 – 4, 5 – 9, 10 – 14, 15 – 19, and 20 – 29, 30 – 39, 40 – 49 etc.

**EXERCISE 63-2**

1. Below are the scores in an Arithmetic test. Arrange them in order from the highest to the lowest:
   90, 63, 66, 76, 83, 66, 66, 56,
   52, 25, 54, 11, 30, 51, 61, 51,
   54, 14, 54, 64, 30, 52, 58, 51,
   13, 2, 28, 6, 14, 68, 6, 28,
   24, 4, 6, 18

   (a) What is the sixth score from the lowest?
   (b) What percentage of the scores is above 25?
   (c) What is the difference between the highest and lowest scores?

2. Set out the scores below in a form which is easy to absorb, without grouping the data.
   87, 91, 82, 88, 87, 85, 92, 84, 81, 85, 86, 90
   83, 85, 82, 86, 88, 85, 88, 87, 90, 86, 83, 85

3. The table below, which is partly completed, gives the frequency distribution of the scores obtained by 100 students in a mathematics test. Complete the
(a) What is the mid-point of the interval with the most common score?

(b) How many students scored at most 70 marks?

(c) In which interval would a mark of 76.5 be recorded?

4. The average values in pounds per ton of Ghana cocoa in the London market for the twenty-four months of 1961 and 1962 are given as:

183, 171, 160, 180, 180, 175, 177, 170, 167, 181,
204, 208, 183, 166, 170, 171, 172, 171, 171, 167,
162, 164, 170, 171

Display this information by grouping with 160 – 164, 165 – 169 etc. as intervals. Show the tally marks.

5. The table below gives the number of pupils enrolled for each of the seven years in Primary Schools in Northern and Eastern Nigeria in 1963.

<table>
<thead>
<tr>
<th>Northern Nigeria</th>
<th>Eastern Nigeria</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st Year</td>
<td>91,567</td>
</tr>
<tr>
<td>2nd Year</td>
<td>83,082</td>
</tr>
<tr>
<td>3rd Year</td>
<td>73,409</td>
</tr>
<tr>
<td>4th Year</td>
<td>62,291</td>
</tr>
<tr>
<td>5th Year</td>
<td>42,010</td>
</tr>
<tr>
<td>6th Year</td>
<td>33,502</td>
</tr>
<tr>
<td>7th Year</td>
<td>24,845</td>
</tr>
</tbody>
</table>
(Source: Statistics of Education in Nigeria, 1963 Series, No. 1 Vol. III;
published by the Federal Ministry of Education, Lagos)

(a) What is the general trend that you can see in both sets of data?
(b) What proportion of the pupils in each Region is enrolled in the first year?

6. Ask 100 pupils in your school their ages. Write down the age of each child
   as he tells you. Arrange the ages in order starting with the youngest child.
   Group these ages at intervals of 6 months as follows:
   6 years 0 months — 6 years 6 months,
   6 years 7 months — 7 years 0 months,
   etc.
   Use tally marks and find the total frequency for each age group.

7. Collect from each class in your school the number of pupils present every day
   for a week. Arrange the data in a suitable manner.

8. Count the number of cars passing a particular point on a busy road at 5 minute
   intervals for one hour on a Monday morning. Do the same at the same hour on
   a Tuesday morning and compare your data.

9. List the most common cars and commercial vehicles in your town. Repeat the
   exercise in question 8 but state how many of each kind of vehicle passes the
   given point in one hour. (Prepare a frequency distribution.)

10. Find out from the police in your town the number of new vehicles registered
    in each category for the past three years
    (a) Motor - Cars;
    (b) Buses;
    (c) Lorries;
    (d) Motor - Cycles.

11. Play a game of ludo with your friends and make a table showing the number of
    sixes thrown in one hour by each player.

12. Find out the scores obtained in three different tests by a class at the end of
    the Term. Arrange the scores in order in each test; then find out by how many
    marks the first pupil in each test beat the last pupil.

13. Hold an election for a school captain of football in your school. Count the
    votes and show how each candidate stands.

14. Find out the weights of all the pupils in a Primary Three class. Group the
    weights.
15. Find out the heights of all the pupils in a Primary Four class. Group the heights.

63-3 Rounding off.

In Unit VIII of *Basic Concepts of Mathematics* you learned how to deal with approximations. It was pointed out in the unit that in some measurements exact results are required while in others it is sufficient to give an approximate result. For example, if we were to estimate the population of Kenya at any particular moment it would be sufficient to express the result to the nearest thousand persons.

In Chapter 41 you learned how to round off numbers to the nearest whole number, tens, hundreds or thousands. We use approximate numbers when we wish to get a general impression and if the error involved in the approximation is not great.

The table below shows the number of Out-Patients treated in Government Hospitals in the ten year period 1953 – 1962 in Kenya.

<table>
<thead>
<tr>
<th>Year</th>
<th>Out-Patients</th>
<th>Out-Patients Expressed to the Nearest thousand</th>
<th>Out-Patients Expressed to the nearest 100,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>1953</td>
<td>1,256,511</td>
<td>1,257,000</td>
<td>1,257,000</td>
</tr>
<tr>
<td>1954</td>
<td>1,246,330</td>
<td>1,246,000</td>
<td>1,246,000</td>
</tr>
<tr>
<td>1955</td>
<td>1,216,912</td>
<td>1,217,000</td>
<td>1,217,000</td>
</tr>
<tr>
<td>1956</td>
<td>1,313,041</td>
<td>1,313,000</td>
<td>1,313,000</td>
</tr>
<tr>
<td>1957</td>
<td>1,034,209</td>
<td>1,034,000</td>
<td>1,034,000</td>
</tr>
<tr>
<td>1958</td>
<td>931,924</td>
<td>932,000</td>
<td>932,000</td>
</tr>
<tr>
<td>1959</td>
<td>1,030,319</td>
<td>1,030,000</td>
<td>1,030,000</td>
</tr>
<tr>
<td>1960</td>
<td>1,166,765</td>
<td>1,167,000</td>
<td>1,167,000</td>
</tr>
<tr>
<td>1961</td>
<td>1,176,432</td>
<td>1,176,000</td>
<td>1,176,000</td>
</tr>
<tr>
<td>1962</td>
<td>1,316,631</td>
<td>1,317,000</td>
<td>1,317,000</td>
</tr>
</tbody>
</table>

(Source: Statistical Abstract 1963. Published by the Economics and Statistics Division, Ministry of Finance and Economic Planning, Kenya. The figures in the second column are those recorded in the Statistical Abstract.)

You will notice that the actual numbers are given. We may, however, be interested in getting a general impression rather than an exact picture. The third column gives the same figures expressed to the nearest thousand.

**EXERCISE 63-3**

1. As an exercise fill in the details in the fourth column of the above table, giving the number of out-patients correct to the nearest one hundred thousand patients in each case.

2. The total value of Postal Orders issued in East Africa for each year of the ten year period 1953–1962 is given in the table below.
(a) to the nearest thousand pounds
(b) to the nearest ten-thousand pounds

3. The value of the domestic exports of Nigeria for each of the years 1957 – 1963 is given in the table below.

<table>
<thead>
<tr>
<th>Year</th>
<th>£, thousand</th>
</tr>
</thead>
<tbody>
<tr>
<td>1957</td>
<td>10,348</td>
</tr>
<tr>
<td>1958</td>
<td>11,066</td>
</tr>
<tr>
<td>1959</td>
<td>13,375</td>
</tr>
<tr>
<td>1960</td>
<td>13,802</td>
</tr>
<tr>
<td>1961</td>
<td>14,172</td>
</tr>
<tr>
<td>1962</td>
<td>13,668</td>
</tr>
<tr>
<td>1963</td>
<td>15,405</td>
</tr>
</tbody>
</table>

The figures have been rounded off to the nearest thousand pounds. Round off further

(a) to the nearest one hundred thousand pounds;
(b) to the nearest million pounds.
64-1 Introduction

There are many ways of presenting data resulting from an experiment or a statistical investigation. One of these is the presentation of data in the form of a table. We have seen in Chapter 63, Section 63-2, that we can obtain a lot of useful information from a table if it is properly presented. However, some people find it difficult to get a mental picture of the impression that a table is supposed to convey. They find it easier to compare statistical information if it is presented in the form of a picture or illustration. Pictures, illustrations or charts which are used to help us to absorb statistical information readily are called graphs.

In this chapter, we shall learn about the most common types of graphs and how they can be used to convey information. There are good and bad graphs; we shall point out some of the characteristics of a good or a bad graph. Some graphs, especially in advertisements and in political propaganda leaflets, are often misleading, and often create the wrong impression. When we see such graphs we should be able to detect the error. Also when we are given a set of data we should be able to decide the most suitable type of graph to use.

Graphs, however, have a limitation. While they afford us a quick way of seeing the relationship between two quantities they often lack the detailed information which we can get from a table.

It is sometimes good practice to show the table from which the graph or chart is drawn so that anyone who wants the detailed information will have it ready at hand.

64-2 Choosing a Scale

The choice of a suitable scale is important in graphical work. One guiding principle is to choose a scale which will make the diagram large enough to be easily appreciated. The diagram should neither be too small nor too large. The data to be represented in the form of a graph will usually dictate the choice of scale. For example, in drawing a bar chart (this will be explained later), the scale must be chosen so that the longest bar fits conveniently on the paper on which it is drawn.
In most graphs we first draw a horizontal and a vertical line which meet at the lower left hand corner of the space in which the graph is to be drawn, like this:

These are the reference lines or axes. The scales are clearly marked along the axes. For example, if one inch on the graph represents five inches of rainfall, the interval between 5 and 10 as well as that between 12 and 17 on the graph should each be one inch. One error in drawing graphs is that people sometimes use one scale for one part of the graph and a different scale for another part. It is easy to draw the wrong conclusion from such a graph. It is not necessary, however, to use the same scale on both axes. The important point is that the scale on each axis should be included in every graph.

64-3 Characteristics of a Good Graph

The purpose of a graph is to give information in a quick and meaningful manner. If it does not do this it is not a good graph. In other words we should get from a graph the same impression that a table is intended to give but much more quickly than from the table. It is important to mark the scale clearly so that anyone reading the graph will experience no difficulty. Ordinarily the units should be equally spaced and there should be a clear indication of what is shown along each axis. The graph should have a title and the base line chosen suitably so that there is ample room left at the top. There should be no change of scale along any one axis and the scale used should be well chosen. As far as possible the origin, that is, the point where the two axes meet, should appear on the graph. Where this is not possible care should be exercised to avoid distortion. In graphical work simplicity is absolutely important. If the information is too complicated to interpret then the purpose of the graph will have been defeated.

64-4 Bar Graphs

One of the simplest kinds of graphs to draw and read is the bar graph. It is also sometimes called a column graph or a block graph and is used to compare data.

The bars may be vertical or horizontal depending on preference. The bars should be of the same width and should be evenly spaced; the spaces between the bars do not
need to be the same as the width of the bars themselves. The height (or length, when drawn across the page) of each bar is drawn in proportion to the size of the quantity it represents. Before drawing the graph it is important to work out a scale so that the height (or length) of the highest (longest) bar will conveniently fit on the page.

**Example 1**

Figure 1 shows a bar graph of the number of rainy days in each month of the year 1962 in Nairobi. Study the graph carefully and answer the following questions:

**EXERCISE 64-4A**

1. Which was the wettest month of the year in Nairobi in 1962? On how many days did the rain fall during that month?
2. Name the two driest months. On how many days did the rain fall in each of these months?
3. Which half of the year was the wetter: January to June or July to December?
4. How many dry days were there in Nairobi in 1962?
5. List the pairs of months in which the number of rainy days was exactly the same. State the number of rainy days in each case.
The following table gives the enrolment of boys and girls in the Nigerian Primary Schools for 1957, 1960 and 1963.


The figures in the original table have been rounded off to the nearest thousands of pupils.

<table>
<thead>
<tr>
<th>Year</th>
<th>Male pupils in thousands</th>
<th>Female pupils in thousands</th>
</tr>
</thead>
<tbody>
<tr>
<td>1957</td>
<td>1,594</td>
<td>854</td>
</tr>
<tr>
<td>1960</td>
<td>1,829</td>
<td>1,083</td>
</tr>
<tr>
<td>1963</td>
<td>1,772</td>
<td>1,125</td>
</tr>
</tbody>
</table>

The reason for rounding off to the nearest thousand is that we want to be able to choose a suitable scale for the graph. You will see that as the figures now stand they are still too large for a graph. It would therefore help to express the number of pupils enrolled in each category to the nearest million and the table will now look like this:

<table>
<thead>
<tr>
<th>Year</th>
<th>Male pupils in millions</th>
<th>Female pupils in millions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1957</td>
<td>1.6</td>
<td>0.8</td>
</tr>
<tr>
<td>1960</td>
<td>1.8</td>
<td>1.1</td>
</tr>
<tr>
<td>1963</td>
<td>1.8</td>
<td>1.1</td>
</tr>
</tbody>
</table>

You will notice that the final rounding off has obscured the differences between 1960 and 1963. This gives the impression that there was no growth between those two years whereas the original table shows a decrease of about 50,000 pupils for the males and an increase of about 40,000 pupils for the females. In order to correct this impression the final rounding off may be made to the nearest ten thousand pupils. This gives

<table>
<thead>
<tr>
<th>Year</th>
<th>Male pupils in tens of thousands</th>
<th>Female pupils in tens of thousands</th>
</tr>
</thead>
<tbody>
<tr>
<td>1957</td>
<td>15.9</td>
<td>8.5</td>
</tr>
<tr>
<td>1960</td>
<td>18.3</td>
<td>10.8</td>
</tr>
<tr>
<td>1963</td>
<td>17.7</td>
<td>11.3</td>
</tr>
</tbody>
</table>

Suppose that we wish to use a graph to compare the above data. There are two possible choices open to us depending on what we wish to compare. Figure 2 is a bar graph which compares the enrolment for boys with that of girls for the three years, whereas Figure 3 features the growth of the enrolment for boys separately and girls separately over the three year period. The general rule is that values which are to be compared should be, as far as possible, adjacent to each other.
ENROLMENT OF PUPILS BY SEX IN NIGERIAN PRIMARY SCHOOLS

Fig. 2

ENROLMENT OF PUPILS BY SEX IN NIGERIAN PRIMARY SCHOOLS

Fig. 3
From Figure 3 it is not as easy to compare the enrolment of boys and girls for 1957 or any of the other years whereas this is clearly brought out in Figure 2. Figure 3, however, compares the years very well. What is the scale used for this graph?

Example 2

The three longest rivers in Africa are the Nile 4,000 miles, the Congo 3,000 miles and the Niger 2,600 miles. If we wish to represent the lengths of these rivers, horizontal bars would be preferred to vertical bars. Figure 4 has been drawn to compare the lengths of these rivers. Vertical bars could tell the same story effectively.

THE THREE LONGEST RIVERS IN AFRICA

![Graph of the three longest rivers in Africa]

- The Nile
- The Congo
- The Niger

Length in miles
Another important use of horizontal bar graphs is comparison by subdivision of a graph into parts.

The 1960 Census in Ghana showed that the population of Ghana could be divided into the following categories:

<table>
<thead>
<tr>
<th>Age group</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 14 years</td>
<td>44.5</td>
</tr>
<tr>
<td>15 - 44</td>
<td>43.0</td>
</tr>
<tr>
<td>45 and over</td>
<td>12.5</td>
</tr>
</tbody>
</table>


In the original table the population of Ghana was divided into eight different categories, some of which were merged to produce the table above. The first group (0-14 years) shows the percentage of the population in the pre-working age.

The data can be represented on a single bar graph which we may divide into three parts in the ratio of the given percentages. This is done in Figure 5(a). For purposes of comparison a vertical bar graph has also been drawn below the horizontal one. Note that Figure 5(b) is not on the same scale as Figure 5(a). Which of the two graphs do you think better illustrates the data? Why?
EXERCISE 64-4B

1. The daily attendance for a week in a primary school was:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>18</td>
<td>17</td>
<td>24</td>
<td>26</td>
<td>29</td>
</tr>
</tbody>
</table>

Draw a bar graph to illustrate the above data. What scale have you used? Is your graph properly labelled?

2. The mean number of hours per day of sunshine for 1962 in Kisumu is given as:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>9.0</td>
<td>10.5</td>
<td>9.8</td>
<td>8.2</td>
<td>8.0</td>
<td>8.5</td>
<td>8.3</td>
<td>7.1</td>
<td>7.7</td>
<td>8.6</td>
<td>9.0</td>
<td></td>
</tr>
</tbody>
</table>

Draw a bar graph to show the data choosing one inch to represent two hours on the vertical axis.

Ensure that your bars are evenly spaced.


3. How many pupils are there in your class? How many are boys? How many are girls? Draw a horizontal bar graph to show the relation between the number of boys and girls in the class and shade appropriately.

4. How many pupils are there in your school? Find out the number of pupils in each class. Draw a bar graph using the information you have collected. For the whole school draw a bar graph to show how many of the pupils are boys and how many are girls.

5. Imports into Sierra Leone from various Sterling Area countries for the quarter October to December 1964 are given as:

<table>
<thead>
<tr>
<th>Country</th>
<th>Import (Le)</th>
</tr>
</thead>
<tbody>
<tr>
<td>United Kingdom</td>
<td>6,364,766</td>
</tr>
<tr>
<td>Eire</td>
<td>160,368</td>
</tr>
<tr>
<td>Hong Kong</td>
<td>246,509</td>
</tr>
<tr>
<td>India</td>
<td>399,174</td>
</tr>
<tr>
<td>Malawi</td>
<td>174,524</td>
</tr>
<tr>
<td>Other countries</td>
<td>349,857</td>
</tr>
</tbody>
</table>

Source: Sierra Leone Trade Journal, Vol. 5 No. 2, April/June 1965

Round off the figures in whatever way you think appropriate and then draw a bar graph to illustrate the data. (Le = Sierra Leone pounds)
6. Find out from the police in your area the number of fatal accidents for each month of the previous year. Draw a bar graph of this information.

7. Find out how many pupils are absent in your school for a week. Then draw a bar graph based on the data you have collected.

8. In 1964 the Nigerian Federal Government current revenue was derived from the three sources shown below:

<table>
<thead>
<tr>
<th>Source of Revenue</th>
<th>Value in £</th>
</tr>
</thead>
<tbody>
<tr>
<td>Customs and Excise</td>
<td>87,561,000</td>
</tr>
<tr>
<td>Direct Taxes</td>
<td>7,437,000</td>
</tr>
<tr>
<td>Other</td>
<td>29,578,000</td>
</tr>
</tbody>
</table>


The figures have already been rounded off to the nearest thousand pounds. Round off to the nearest million pounds and draw a horizontal bar graph divided into three parts as appropriate.

9. Draw a bar graph to show the number of pupils present in each class for one day in your school.

10. The table below gives the enrolment in Primary Schools in Ghana by sex in 1955 and 1960.

<table>
<thead>
<tr>
<th>Year</th>
<th>Boys</th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>1955</td>
<td>280,216</td>
<td>139,146</td>
</tr>
<tr>
<td>1960</td>
<td>311,857</td>
<td>166,285</td>
</tr>
</tbody>
</table>

Source: 1962 Statistical Year Book: Central Bureau of Statistics, Accra

Round off the figures. Then draw a bar graph to compare

(a) Enrolment of boys and girls
(b) Enrolment in the two years for boys and girls separately.

11. The graph in Figure 6 shows the number of schools in some African countries in 1959. Look at the graph carefully and answer the following questions:

(a) Which country had the largest number of schools?
(b) Which had the least number of schools?
(c) Read as accurately as you can from the graph the number of schools in Ghana and Congo (Leopoldville) in 1959.
12. The results of the West African School Certificate Examination in two schools in Ghana in June, 1964 were

<table>
<thead>
<tr>
<th></th>
<th>No. Presented</th>
<th>Grade I</th>
<th>Grade II</th>
<th>Grade III</th>
<th>Failures</th>
</tr>
</thead>
<tbody>
<tr>
<td>Achimota School</td>
<td>102</td>
<td>43</td>
<td>38</td>
<td>17</td>
<td>4</td>
</tr>
<tr>
<td>Mfantsipim School</td>
<td>93</td>
<td>26</td>
<td>22</td>
<td>29</td>
<td>17</td>
</tr>
</tbody>
</table>
Draw bar graphs to compare the data. Which is the better school?

13. Keep a record, during the rainy season, of the quantity of rain that falls on each school day for a week. Draw a bar graph of the information choosing an appropriate scale.

64-5 Histogram

Another method of representing data graphically is by the histogram in which the quantities to be represented are shown by the heights of columns next to each other as in a column graph. The histogram is often used when the data consist of grouped measurements each with a given frequency.

Example 4

Suppose that we have the following frequency distribution of marks in a Mathematical test.

<table>
<thead>
<tr>
<th>Score</th>
<th>Score boundaries</th>
<th>Mid-point of interval</th>
<th>Frequency</th>
<th>Cumulative frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>51-55</td>
<td>50.5 - 55.5</td>
<td>53</td>
<td>10</td>
<td>10</td>
</tr>
<tr>
<td>56-60</td>
<td>55.5 - 60.5</td>
<td>58</td>
<td>20</td>
<td>30</td>
</tr>
<tr>
<td>61-65</td>
<td>60.5 - 65.5</td>
<td>63</td>
<td>20</td>
<td>50</td>
</tr>
<tr>
<td>66-70</td>
<td>65.5 - 70.5</td>
<td>68</td>
<td>10</td>
<td>60</td>
</tr>
<tr>
<td>71-75</td>
<td>70.5 - 75.5</td>
<td>73</td>
<td>5</td>
<td>65</td>
</tr>
</tbody>
</table>

You will notice an additional column headed "cumulative frequency". In the second row of this column you will find 30, which is the sum of the frequencies 10 and 20 in the first and second rows of the previous column. In a similar way 50 in the third row is the sum of 10, 20, 20 in the previous column, and so on.

In order to represent the data by a histogram we first draw a horizontal scale and mark on it the boundaries of the intervals, making the width of all the intervals the same. We then show the frequencies on the vertical scale.
Here we note that the height of each rectangle represents the frequency. (This will always be true as long as the widths of the intervals are kept the same. However, if the widths of the intervals are different the height of each rectangle should no longer be made to represent the actual frequency. Instead of comparing the heights we should then compare the areas of the rectangles.)

64-6 Frequency Polygon

A frequency polygon is a slight modification of the histogram, and its construction is exactly the same as that of a histogram so far as the vertical and horizontal scales are concerned. We do not draw a rectangle over each interval. Instead, we join the mid-points of the tops of the rectangles that would have been drawn over the intervals. The result is the same as if we had joined the mid-points of the tops of the rectangles in a histogram and then erased the histogram itself.

The figure below shows the frequency polygon for the above data.
Two points are worthy of note in the above figure. First, the graph has been extended one interval beyond both lower and upper extremes and the polygon has been drawn to the middle of these intervals on the horizontal axis. Second, a jagged line is shown on the horizontal scale to indicate that the scale is not uniform between 0 and 45·5.

**64-7 Cumulative Frequency Histogram and Polygon**

Suppose that we wish to find the number of candidates who score not more than a given mark. How do we represent this on a graph? From the cumulative frequency table we know that 10 candidates scored not more than 55 marks, 30 candidates scored not more than 60 marks, 50 scored not more than 65 marks, and so on.

To represent this information on a graph we choose the horizontal axis to represent the scores as before but the vertical axis to represent the cumulative frequency.

![Cumulative Frequency Histogram and Polygon for the distribution of the Scores in a Mathematics test.](image)

For the cumulative frequency polygon we join as before the middle points of the cumulative frequency histogram.

**64-8 Dot Frequency Graph**

The histogram and the frequency polygon are useful for grouped data. If the frequency has not been grouped in intervals, the data may be represented by a dot frequency graph. This is very easy to draw.
Example 5

You will recall that we had a frequency distribution for the heights of 50 pupils in a school. This is reproduced below.

<table>
<thead>
<tr>
<th>Height in inches</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>1</td>
</tr>
<tr>
<td>54</td>
<td>3</td>
</tr>
<tr>
<td>55</td>
<td>6</td>
</tr>
<tr>
<td>56</td>
<td>7</td>
</tr>
<tr>
<td>58</td>
<td>7</td>
</tr>
<tr>
<td>59</td>
<td>12</td>
</tr>
<tr>
<td>60</td>
<td>8</td>
</tr>
<tr>
<td>62</td>
<td>4</td>
</tr>
<tr>
<td>63</td>
<td>2</td>
</tr>
</tbody>
</table>

The data in the above table are illustrated by a dot frequency graph below. The number of times each height occurred is represented by the same number of dots. For example, 56 inches occurred seven times and it is represented by seven dots.

---

Distribution of Heights in a Primary School.

Fig. 10

---
Suppose that we wish to classify test scores into groups. Then a dot frequency graph is very useful. We first write the scores in a row from the smallest to the largest. We then indicate by a dot every time a particular score comes up. From such a graph we see at a glance how the class has performed and we can group the scores as we wish.

**EXERCISE 64-8**

1. Draw a histogram to represent the grouped frequency distribution of the scores obtained by 100 candidates in the arithmetic test given in Section 63-2 of Chapter 63.
2. Use a dot frequency graph to represent the data in question 2 of Exercise 63-2 of Chapter 63.
3. Draw a histogram to show the data in question 4 of Exercise 63-2 of Chapter 63.
4. Draw a cumulative histogram and corresponding frequency polygon for the table in question 3 of Exercise 63-2 of Chapter 63.
5. The results in Additional Mathematics at the West African School Certificate Examination for Adisadel College, Ghana, in 1964 were:

<table>
<thead>
<tr>
<th>Grade</th>
<th>Count</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excellent</td>
<td>3</td>
</tr>
<tr>
<td>Very good</td>
<td>2</td>
</tr>
<tr>
<td>Good</td>
<td>6</td>
</tr>
<tr>
<td>Credit</td>
<td>19</td>
</tr>
<tr>
<td>Pass</td>
<td>2</td>
</tr>
<tr>
<td>Fail</td>
<td>-</td>
</tr>
</tbody>
</table>

Draw a dot frequency graph to illustrate the data.

**64-9 Circle Graphs (Pie Charts)**

You will recall that the percentage in each age group as given by the 1960 census of population in Ghana was shown on a horizontal bar graph divided into three parts in proportion to the percentage of persons in each age group. This information may also be conveniently represented by a circle graph (or pie chart).

*Example 6*

A circle graph is used when the number of things we wish to compare is not large and when our main interest is in comparing proportions and not actual numbers. In the distribution of the population of Ghana mentioned above the total population was divided into three age groups:

<table>
<thead>
<tr>
<th>Age Group</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 – 14 years</td>
<td>44.5%</td>
</tr>
<tr>
<td>15 – 44 years</td>
<td>43.0%</td>
</tr>
<tr>
<td>45 and over</td>
<td>12.5%</td>
</tr>
</tbody>
</table>
You will notice that these percentages add up to 100%. The total population may be represented by the area of a circle and our problem is to divide up the circular region in the same ratio as the percentages in the different age groups.

We know that we can divide the circle into 360 equal parts by drawing 360 angles of 1 degree, each with its vertex at the centre of the circle. The fraction of the circular region which represents the number of persons in each age group can be calculated as follows:

<table>
<thead>
<tr>
<th>Age group</th>
<th>Percentage</th>
<th>Fraction of total population</th>
<th>Number of degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 14</td>
<td>44.5%</td>
<td>$\frac{44.5}{100}$</td>
<td>$\frac{44.5}{100} \times 360^\circ = 160.2^\circ$</td>
</tr>
<tr>
<td>15 - 44</td>
<td>43.0%</td>
<td>$\frac{43}{100}$</td>
<td>$\frac{43}{100} \times 360^\circ = 154.8^\circ$</td>
</tr>
<tr>
<td>45 and over</td>
<td>12.5%</td>
<td>$\frac{12.5}{100}$</td>
<td>$\frac{12.5}{100} \times 360^\circ = 45^\circ$</td>
</tr>
</tbody>
</table>

With our ordinary protractor we cannot measure accurately fractions of a degree. We therefore express our answers to the nearest degree. Thus 160.2° will be expressed as 160° while 154.8° will be expressed as 155°. The data are shown below by a circle graph. We draw angles of 45°, 160° and 155° respectively at the centre of the circle. Since the area of a sector is proportional to the angle between its radii the sectors into which these angles divide the circle represent the percentage of persons in each age group.

Distribution of the population of Ghana. 1960 Census

![Circle Graph](image-url)
A circle graph is sometimes called a pie chart because the circle is divided into pie-shaped sections, each of which is proportional to the percentage of the sections into which the whole group has been divided. The graph is shaded or coloured and the percentage that each sector represents is often written in the appropriate portion.

**Example 7**

In the example above the percentages were given but often this has to be worked out. For example, suppose that we are given that the total number of Primary School pupils in Northern Nigeria in 1963 was 410,706 and that there were 295,644 boys and 115,062 girls, and suppose that we wish to represent the data by a circle graph. First we express 295,644 as a percentage of 410,706.

That is, \[ \frac{295,644}{410,706} \times 100\% = \frac{29,564,400}{410,706} \% \]

\[ = 71.98\% \]

The ratio of the percentage of boys to girls is 71.98% to 28.02%.

(Note: The percentage for girls should be worked out in the normal way in order to act as a check. A possible error would be missed if we merely subtract 71.98% from 100% to get the percentage of girls.)

<table>
<thead>
<tr>
<th>Sex</th>
<th>Number</th>
<th>Percentage</th>
<th>Degrees</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boys</td>
<td>295,644</td>
<td>72</td>
<td>(0.72 \times 360^\circ = 259.2^\circ) = 259° (to the nearest degree)</td>
</tr>
<tr>
<td>Girls</td>
<td>115,062</td>
<td>28</td>
<td>(0.28 \times 360^\circ = 100.8^\circ) = 101° (to the nearest degree)</td>
</tr>
</tbody>
</table>

Distribution of Primary School Pupils in Northern Nigeria by Sex in 1963

Fig. 12
In constructing circle graphs always make sure that the percentages add up to 100% and the degrees add up to 360°. Use an independent means of checking.

The examples given above showed circles divided into three and two parts respectively. It is of course possible to divide a circle into several more parts but when the number of parts becomes large a pie chart is no longer suitable for comparing the data.

**EXERCISE 64-9**

1. A father left £1,800 to his three sons and instructed that the money be shared in the ratio of their ages. If the boys were 16, 12, 8 years old respectively, find how much went to each son. Draw a circle graph to show how the money was divided. Draw also a bar graph to show the same information.

2. Draw a circle graph to show the ratio of boys to girls in your class.

3. Three pupils in your class are nominated for election as Class Prefect. Draw a circle graph to show how the votes were distributed. Draw also a bar graph.

4. In 1962 there were 2,430 Arab boys and 1,073 Arab girls in Primary Schools in Kenya. Illustrate with a circle graph.

5. According to the Report on “The Pattern of Expenditure and Consumption of Africans in Nairobi, 1957/58” published by the East African Statistical Department, May 1959, the average African in Nairobi spent 58% of his salary on food, 13% on rent and water charges, 7% on clothing and the rest on “Miscellaneous”. Draw a pie chart to illustrate the data.

6. Find out from the local police the number of accidents caused by motor vehicles in the previous twelve months. Divide into fatal and non-fatal. Draw both a circle graph and a bar graph to illustrate the data you have collected.

**64-10 Pictogram**

If, in the age distribution of the population of Ghana in 1960, we were interested in comparing actual numbers and not proportions we would not use a circle graph. We could use instead a pictogram (an ideogram). This is a graph in which we draw a row of pictures or symbols representing the data. Each little picture or symbol is used to stand for a fixed number of the given population and all the little pictures are of the same size. The little pictures should also be the same distance apart.

*Example 8*

The table gives the number of persons in each age group in Ghana according to the 1960 population census.
Distribution of the Population of Ghana: 1960 Census

<table>
<thead>
<tr>
<th>Age Group</th>
<th>Number of persons in thousands</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 14</td>
<td>2,996.5</td>
</tr>
<tr>
<td>15 - 44</td>
<td>2,308.2</td>
</tr>
<tr>
<td>45 and over</td>
<td>836.1</td>
</tr>
</tbody>
</table>


In order to represent the data in the above table by a pictogram we have to choose a scale which will enable us to fit the longest row of symbols conveniently on the page. We could use one stick-man to represent 200,000 persons. Thus those in the age group 0 - 14 will be represented by \( \frac{2,996.5}{200} = \frac{29,965}{2} = 14.9825 \) stick-men. Similarly 11.541 stick-men will represent those in the 15 - 44 age group and 4.1825 stick-men will represent the age group 45 and above. However, it is difficult for us to show accurately a fraction of a stick-man and so we may express our answers to the nearest half stick-man, or perhaps, to the nearest stick-man. Thus we have 15, 11.5, 4 stick-men respectively.

Age Distribution of the Population of Ghana
1960 Census

<table>
<thead>
<tr>
<th>Age Group</th>
<th>Key: represents 200,000 persons</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 - 14</td>
<td><img src="image" alt="Stick-Man 0-14" /></td>
</tr>
<tr>
<td>15 - 44</td>
<td><img src="image" alt="Stick-Man 15-44" /></td>
</tr>
<tr>
<td>45 &amp; over</td>
<td><img src="image" alt="Stick-Man 45+" /></td>
</tr>
</tbody>
</table>

Fig. 13
The graph is not very easy to interpret especially so far as a fraction of a symbol is concerned. To simplify matters we may decide to use only complete stick-men but in doing this some information is lost.

Example 9

The distribution of the pupils in Secondary Schools in Western Nigeria by sex in 1963 was:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Boys</td>
<td>104,411</td>
</tr>
<tr>
<td>Girls</td>
<td>46,277</td>
</tr>
</tbody>
</table>

This information can be illustrated by a pictogram. A suitable scale would be one symbol to represent 8,000 boys or girls.

**Distribution of pupils by sex in Secondary Schools in Western Nigeria in 1963**

<table>
<thead>
<tr>
<th>Boys</th>
<th>Girls</th>
<th>Fig. 14</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Boys" /></td>
<td><img src="image" alt="Girls" /></td>
<td></td>
</tr>
</tbody>
</table>

**Key:**
- ![Boy](image) represents 8,000 boys
- ![Girl](image) represents 8,000 girls

Pictograms, if drawn in a particular way, may be misleading. We have already seen that the heights of bar graphs and histograms are used to compare two quantities. Can we say the same about pictograms? Suppose that we wish to use a pictogram instead of a bar graph to represent the production of palm oil in two different years. Let 1,000 tons be the production in one year and 2,000 tons the production in another year. We may represent the production for the first year by an oil cask one inch high and that for
the second year by an oil cask two inches high.

In order to keep the same proportions, we have represented the production for the second year by an oil cask the radius of whose base is twice that of the first one. In doing this, however, the volume which represents the second year is eight times that for the first year, which of course is misleading.

In looking at the pictogram one would not know whether it is the area covered or the volume or the height that is being compared. For this reason, in using pictograms for illustrations we use little pictures all of the same size, using enough of these to represent the data. In the example above the correct thing to do would be to draw two casks of the same size to represent the production in the second year.

**EXERCISE 64-10**

1. How many pupils are there in your school? How many are boys? How many are girls? Draw a pictogram to illustrate your data, choosing an appropriate scale.
2. What are the principal exports of your country? Find out from the trade journal of your country the value of the principal ones for the previous year. (Name the four principal ones and put all the remaining ones together as 'others'.) Using the symbol to represent £1 million, draw a pictogram of your data.
3. How many members are there in the Central Legislature of your country? How many political parties are represented in the Legislature? How many members of the House belong to the different parties? Illustrate your data with a pictogram and a pie chart.
A broken line graph often shows a trend. This is the kind of graph we find above a patient's bed in a hospital showing how the patient's temperature rises and falls. It can also be used to show how exports rise and fall over the years or whether school population in a country increases or decreases.

Example 10

The table below shows the value of the domestic export trade in Kenya for the years 1959 - 1964.

<table>
<thead>
<tr>
<th>Year</th>
<th>Value (£)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1959</td>
<td>33,306,000</td>
</tr>
<tr>
<td>1960</td>
<td>35,191,000</td>
</tr>
<tr>
<td>1961</td>
<td>35,326,000</td>
</tr>
<tr>
<td>1962</td>
<td>37,913,000</td>
</tr>
<tr>
<td>1963</td>
<td>43,832,000</td>
</tr>
<tr>
<td>1964</td>
<td>47,115,000</td>
</tr>
</tbody>
</table>


Domestic Exports of Kenya 1959–1964

Fig. 16
As in the other kinds of graphs that we have drawn we choose a suitable scale to show the information contained in the table above. We first round off the figures to the nearest million. Then we show the year on the horizontal axis and the value of the exports on the vertical axis. (See Figure 16.)

Note that the years are spaced out evenly. Three-fourths inch on the vertical axis stands for ten million pounds. The points are placed at the appropriate positions and then joined by broken line segments. From the graph we see that the value of Kenya's exports has been steadily increasing over the years.

Can we, from the graph, find the value of the exports at the end of the six month period between 1959 and 1960, or between any other two years? The answer is No. This is because it is the crosses on the graph which really mark the value of the exports for each year and we join these crosses by broken lines only to help us get a better picture.

Example 11

Broken line graphs may also be used to compare not only the trend in the trade of one country but in that of two or more countries. If we wish to compare the trend in the domestic exports of Ghana and Kenya for the years 1959 - 1964, we may do this on the same broken line graph.

The graph below is such a graph. Look at it carefully and answer the questions which follow.

<table>
<thead>
<tr>
<th>YEAR</th>
<th>Ghana</th>
<th>Kenya</th>
</tr>
</thead>
<tbody>
<tr>
<td>1959</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1960</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1961</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1962</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1963</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1964</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

1. What is the scale on the vertical axis?
2. Describe the trend in the domestic exports for Ghana for the years 1958 - 1963.
3. Compare the graph for the trend in Kenya with the graph for Ghana. Do you notice any difference? Comment.
4. Read off as accurately as you can the value of the exports for 1960 in Ghana.
Chapter 65
AVERAGES OR MEASURES OF CENTRAL TENDENCY

65-1 Introduction

What do we mean when we say that the average attendance of a class for a week is 29? Do we mean that 29 pupils actually attended every day for five days? Again, what does it mean to say that a pupil has an average score of 84 in six tests? If a geography teacher says that the average monthly rainfall for Ibadan (Nigeria) is 5.5 inches, does he mean that this amount of rain falls every month?

Your experience with the school register tells you that an average attendance of 29 does not mean that 29 pupils came to school every day. In fact, it may happen that on none of the five days were 29 pupils present. If the attendances were: Monday 28, Tuesday 30, Wednesday 30, Thursday 30, Friday 27 we would have an average attendance of 29 even though the attendance was never 29 on any day. We could even have an average attendance of 28.4. Does this imply that we have a decimal fraction of a pupil attending?

The scores in the six tests mentioned above could have been: Arithmetic 90, English 98, History 66, Geography 95, Nature Study 75, Religious Knowledge 80. This would give an average score of 84.

It is known that the rainfall in Ibadan is distributed over the whole year in such a way that for seven months of the year there is less than five inches of rain in any month, yet enough rain falls in the remaining five months to bring the average monthly rainfall to 5.5 inches.

In a similar way when we say that a car is travelling at an average speed of 30 miles per hour we mean that it could do the journey from Nairobi to Mombasa, a distance of about 300 miles, in ten hours. Some of the time it may be travelling at a speed of 40 or 50 miles per hour; at other times the speed may be 20 or 25 miles per hour. When it is going up hill it will naturally travel at a slower speed than when it is going down hill or is on level ground.

65-2 The Arithmetic mean

The idea of an average is not strange to you; you are familiar with at least one kind of average. As a teacher you are using averages all the time.

In each of the above examples we have chosen a single number to represent the whole group. This representative of the group is an average and we call it the arithmetic mean (or simply the mean). We find the mean of a set of numbers by adding all the items in the set and dividing the total by the number of items in the set. For instance, the
The arithmetic mean is not the only kind of average in which we are interested even though it is perhaps more commonly used than other types of averages. Averages are often called measures of central tendency because they tell us numbers around which the group appears to cluster. Frequently an average is chosen as a typical value for some purpose.

We shall now discuss two other kinds of averages which often prove useful.

65-3 The Mode

Suppose that you were a tailor and you were asked to make school caps for a secondary school class of 30 boys and suppose that the head measurements were given as:

<table>
<thead>
<tr>
<th>Measurement in inches</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>6(\frac{1}{4})</td>
<td>1</td>
</tr>
<tr>
<td>6(\frac{3}{8})</td>
<td>1</td>
</tr>
<tr>
<td>6(\frac{1}{2})</td>
<td>2</td>
</tr>
<tr>
<td>6(\frac{5}{8})</td>
<td>8</td>
</tr>
<tr>
<td>6(\frac{3}{4})</td>
<td>5</td>
</tr>
<tr>
<td>6(\frac{7}{8})</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
</tr>
<tr>
<td>7(\frac{1}{4})</td>
<td>5</td>
</tr>
</tbody>
</table>

Total ........ 30
It can easily be shown that the mean head measurement is \( \frac{6\frac{4}{5}}{} \) inches.

If, for one reason or other, you could make only one size of cap, what size would you choose as a representative of the group of the sizes of the boys' head measurements? You would probably not choose size \( \frac{6\frac{4}{5}}{} \), which is the average. Why would you not do this? Although size \( \frac{6\frac{4}{5}}{} \) is an average we note that the size will not fit very many of the 30 boys. The size which seems to represent the group better is \( \frac{6\frac{5}{8}}{} \) because more boys wear that size cap than any other. That is, \( \frac{6\frac{5}{8}}{} \) occurs the most number of times and since it is the most popular size cap we call it the mode.

You will agree that this is a useful kind of average or measure of central tendency in some situations even though it does not mark the middle point of the set of data in the same way as the mean. Of course there are situations in which the mode may be completely useless.

Look again at the table in Section 64-8 showing the distribution of the heights of 50 pupils in a primary school. Which height occurs more often than any other? Yes, a height of 59 inches is the modal height, that is, the height which occurs most often. We call 59 the mode of the distribution of the heights.

The scores out of 100 obtained by a teacher in the final Teachers' Certificate Examination were: English 95, History 90, Geography 75, Arithmetic 100, Rural Science 80, School Method 80, Elementary Mathematics 97, Practical Teaching 78, Health Education 80, Physical Education 85.

The scores could be re-arranged thus:

<table>
<thead>
<tr>
<th>Subject</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arithmetic</td>
<td>100</td>
</tr>
<tr>
<td>Elem. Mathematics</td>
<td>97</td>
</tr>
<tr>
<td>English</td>
<td>95</td>
</tr>
<tr>
<td>History</td>
<td>90</td>
</tr>
<tr>
<td>Physical Education</td>
<td>85</td>
</tr>
<tr>
<td>Health Education</td>
<td>80</td>
</tr>
<tr>
<td>Rural Science</td>
<td>80</td>
</tr>
<tr>
<td>School Method</td>
<td>80</td>
</tr>
<tr>
<td>Practical Teaching</td>
<td>78</td>
</tr>
<tr>
<td>Geography</td>
<td>75</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>860</td>
</tr>
</tbody>
</table>

The mean score = \( \frac{860}{10} = 86 \). The mode is 80 since this score occurs more times than any other score.
We may have more than one mode in a set of data. For instance, the scores of ten pupils in a mathematics test were:

\[
\begin{align*}
100 & \mid \text{Mode} \\
100 & \\
95 & \\
90 & \\
76 & \\
75 & \\
70 & \mid \text{Mode} \\
70 & \\
65 & \\
64 & \\
\hline
805
\end{align*}
\]

The mean score is \( \frac{805}{10} = 80.5 \).

For the mode, we note that two scores are equally "popular"; these are 100 and 70. Thus we have two modes; the scores are therefore bimodal (meaning 'having two modes'). It is possible to have three or more modes in a set of numbers.

65-4 The Median

If a pupil gets a score of 79 in an arithmetic test, would this be considered an excellent, good or average score? We cannot answer this question without more information. We may want to know how he compares with other pupils who took the same test. Does his score fall above or below the average mark? If we arrange the scores in order of magnitude, where would he fall? Is his score above or below the middle score?

Suppose the scores of all eleven pupils taking the test were:

\[
\begin{align*}
99 & \mid \quad \text{5 scores are greater than 70} \\
98 & \\
88 & \\
86 & \\
79 & \mid \quad \text{middle score} \\
70 & \\
65 & \mid \quad \text{5 scores are less than 70} \\
66 & \\
68 & \\
55 & \\
46 & \hline
820
\end{align*}
\]

The mean score is \( \frac{820}{11} = 74.5 \), whereas the middle score is 70. A score of 79 is not
only greater than the average or mean score of 74.5 but it is also greater than the middle score of 70 and hence may be considered as a good but not an excellent score. If on the other hand, the scores were 79, 64, 62, 61, 60, 58, 55, 48, 45, 43, 40 a score of 79 is the highest score and so would be regarded as excellent; it is 39 marks higher than the lowest score and 15 marks higher than the next highest.

Let us have another look at the middle score in the above example. We notice that five scores are greater than 70, while five scores are less than 70. This middle score represents the whole group in a special way and is thus a kind of average or a measure of central tendency. We call it the median. Thus, the median of a set of numbers is that number which is such that half the numbers in the set are greater and half are less than the number. When the data are not grouped both the median and the mode are easier to calculate than the mean. In fact, we have nothing to calculate; we merely arrange the numbers in order of size and then pick the middle one for the median or the most common one (or ones) for the mode (or modes.) The examples above illustrate this fact.

The mean of a set of numbers, we found, need not be a member of the set. This is also true of the median. For example, if there are an even number of items in the set, the median is defined as the mean of the two middle numbers. In the mathematics scores discussed above the two middle scores are 75 and 76 and their average, 75.5, is not a number in the set of scores. However, if the two middle scores are equal then the median is one of the two equal scores and is a member of the set.

**EXERCISE 65-4**

1. Find the arithmetic mean of the sets of counting numbers 1 to 3; 1 to 4; 1 to 5; 1 to 6; 1 to 7; 1 to 8; 1 to 9; 1 to 10. Can you find a simple rule for finding the mean of a set of consecutive counting numbers, beginning with 1?

2. What is the median of each set of numbers in problem 1?

3. The scores in an arithmetic test were: 17, 15, 16, 18, 13, 17, 13, 11, 14, 16, 15, 10, 11, 14, 5, 12, 14, 11, 12, 6. Find
   (a) the mean
   (b) the median
   (c) the modes
   of the scores.
   (First arrange the scores from the smallest to the largest.) How many scores are
   (i) greater than the mean (ii) less than the mean (iii) greater than the median
   (iv) less than the median?

4. The attendances for a four week period in a class were

<table>
<thead>
<tr>
<th></th>
<th>First Week</th>
<th>Second Week</th>
<th>Third Week</th>
<th>Fourth Week</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Boys</strong></td>
<td>31 31 30 32 32</td>
<td>32 32 32 32 32</td>
<td>31 32 31 31 31</td>
<td>31 32 32 32 32</td>
</tr>
<tr>
<td><strong>Girls</strong></td>
<td>3 3 3 3 3</td>
<td>3 3 3 3 3</td>
<td>3 3 3 3 3</td>
<td>3 3 3 3 3</td>
</tr>
</tbody>
</table>

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Find:
(a) the mean daily attendance for boys;
(b) the mean daily attendance for girls;
(c) the mean daily attendance for boys and girls together.

5. Find (a) the mode  (b) the median  (c) the mean of the following sets of data:
(i) 4, 6, 8, 14, 10, 14, 6, 6, 27, 8, 18
(ii) 30, 27, 26, 31, 26, 34
(iii) 13, 13, 11, 12, 12, 13, 12, 14, 15, 14, 13, 14, 12, 13, 13, 14.

6. The number of passengers who landed at Nairobi airport for each quarter of 1963 was:

<table>
<thead>
<tr>
<th>Quarter</th>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st Quarter</td>
<td></td>
<td></td>
<td></td>
<td>32,100</td>
</tr>
<tr>
<td>2nd Quarter</td>
<td></td>
<td></td>
<td></td>
<td>31,200</td>
</tr>
<tr>
<td>3rd Quarter</td>
<td></td>
<td></td>
<td></td>
<td>38,500</td>
</tr>
<tr>
<td>4th Quarter</td>
<td></td>
<td></td>
<td></td>
<td>34,500</td>
</tr>
</tbody>
</table>

Find the average (mean) number of passengers who landed per quarter.
(Source: Kenya Statistical Digest Vol. III – No. 1 – Published by the Statistics Division of the Ministry of Economic Planning and Development, Nairobi)

7. The number of reported cases of smallpox in Ghana for each of the years 1956 - 1962 was

<table>
<thead>
<tr>
<th>Year</th>
<th>Cases</th>
</tr>
</thead>
<tbody>
<tr>
<td>1956</td>
<td>259</td>
</tr>
<tr>
<td>1957</td>
<td>184</td>
</tr>
<tr>
<td>1958</td>
<td>160</td>
</tr>
<tr>
<td>1959</td>
<td>105</td>
</tr>
<tr>
<td>1960</td>
<td>139</td>
</tr>
<tr>
<td>1961</td>
<td>131</td>
</tr>
<tr>
<td>1962</td>
<td>231</td>
</tr>
</tbody>
</table>

Find (a) the average number of reported cases for the seven year period.
(b) the median number.
(Source: 1962 Statistical Year Book Published by the Central Bureau of Statistics, Accra.)

8. The mean hours per day of sunshine in Nairobi over a number of years and the
1962 mean hours per day of sunshine are given below:

<table>
<thead>
<tr>
<th>Month</th>
<th>1962 Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan</td>
<td>8.8</td>
</tr>
<tr>
<td>Feb</td>
<td>9.4</td>
</tr>
<tr>
<td>Mar</td>
<td>8.7</td>
</tr>
<tr>
<td>Apr</td>
<td>7.3</td>
</tr>
<tr>
<td>May</td>
<td>5.9</td>
</tr>
<tr>
<td>June</td>
<td>5.9</td>
</tr>
<tr>
<td>July</td>
<td>4.4</td>
</tr>
<tr>
<td>Aug</td>
<td>4.2</td>
</tr>
<tr>
<td>Sept</td>
<td>5.8</td>
</tr>
<tr>
<td>Oct</td>
<td>7.1</td>
</tr>
<tr>
<td>Nov</td>
<td>7.0</td>
</tr>
<tr>
<td>Dec</td>
<td>8.1</td>
</tr>
</tbody>
</table>

Find (a) the average number of hours of sunshine per day over many years.
(b) the mean number of hours of sunshine over many years and
for the year 1962. Was 1962 a year of above average monthly sunshine or below average?


9. The 1962 mean maximum temperature figures for Mombasa were:

<table>
<thead>
<tr>
<th>Month</th>
<th>Temp (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan</td>
<td>88.9</td>
</tr>
<tr>
<td>Feb</td>
<td>91.6</td>
</tr>
<tr>
<td>Mar</td>
<td>90.1</td>
</tr>
<tr>
<td>Apr</td>
<td>88.7</td>
</tr>
<tr>
<td>May</td>
<td>84.7</td>
</tr>
<tr>
<td>June</td>
<td>84.6</td>
</tr>
<tr>
<td>July</td>
<td>82.2</td>
</tr>
<tr>
<td>Aug</td>
<td>83.1</td>
</tr>
<tr>
<td>Sept</td>
<td>85.1</td>
</tr>
<tr>
<td>Oct</td>
<td>86.5</td>
</tr>
<tr>
<td>Nov</td>
<td>88.7</td>
</tr>
<tr>
<td>Dec</td>
<td>88.9</td>
</tr>
</tbody>
</table>

Find the mean of these mean maximum monthly temperatures for Mombasa in 1962.

10. The scores of twelve pupils in a Geography test were: 90, 60, 90, 50, 30, 90, 90, 45, 90, 85, 55, 65.

Find the mean, mode and median of this distribution. How many scores are (a) greater than (b) less than, the mean?

65-5 Averages for grouped Data

In question 5 (iii) above you were asked to find the mean of:
13, 13, 11, 12, 12, 13, 12, 14, 15, 14, 13, 14, 12, 13, 13, 14.

From what we already know the mean is given by:

Mean = \(\frac{13+13+11+12+12+13+12+14+15+14+13+14+12+13+13+14}{16}\)

Mean = \(\frac{208}{16}\) = 13.

We note, however, that 12, 13, 14 each occurs a number of times and so we could have written the mean as

Mean = \(\frac{11 + 12(4) + 13(6) + 14(4) + 15}{16}\) = \(\frac{208}{16}\) = 13.

The above calculation may be set out as shown below. Let \(X\) represent the values and \(f\) the corresponding frequencies. Let \(N\) be the sum of all frequencies, that is, the total number of measurements. Then \(fX\) is the product of each number and the frequency with which it occurs.

<table>
<thead>
<tr>
<th>Number (X)</th>
<th>Frequency (f)</th>
<th>fX</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>1</td>
<td>11</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>48</td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>78</td>
</tr>
</tbody>
</table>
You will recall that in the previous section the mean hat size was found to be $\frac{64}{3}$.

Did you wonder how this was obtained?

Try and see if you can compute the mean hat size using this new method.

When a set of data is grouped in intervals, as in the example of the 100 candidates in the arithmetic examination, the scores are considered to have the value at the mid-point of the interval. For example, all the seven scores which fell in the interval 10-19 are considered to have the value 14.5. The mean may be found as shown below.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Mid-point (X)</th>
<th>Frequency (f)</th>
<th>fX</th>
</tr>
</thead>
<tbody>
<tr>
<td>10-19</td>
<td>14.5</td>
<td>7</td>
<td>101.5</td>
</tr>
<tr>
<td>20-29</td>
<td>24.5</td>
<td>12</td>
<td>294.0</td>
</tr>
<tr>
<td>30-39</td>
<td>34.5</td>
<td>13</td>
<td>448.5</td>
</tr>
<tr>
<td>40-49</td>
<td>44.5</td>
<td>13</td>
<td>578.5</td>
</tr>
<tr>
<td>50-59</td>
<td>54.5</td>
<td>31</td>
<td>1689.5</td>
</tr>
<tr>
<td>60-69</td>
<td>64.5</td>
<td>12</td>
<td>774.0</td>
</tr>
<tr>
<td>70-79</td>
<td>74.5</td>
<td>5</td>
<td>372.5</td>
</tr>
<tr>
<td>80-89</td>
<td>84.5</td>
<td>5</td>
<td>422.5</td>
</tr>
<tr>
<td>90-99</td>
<td>94.5</td>
<td>1</td>
<td>94.5</td>
</tr>
<tr>
<td>100-109</td>
<td>104.5</td>
<td>1</td>
<td>104.5</td>
</tr>
</tbody>
</table>

Total freq. = $N = 100$  
Total fX = 4880.0

Mean $= \frac{\text{Total fX}}{\text{Total Frequency}} = \frac{4880.0}{100} = 48.8$

The sum of all the 100 scores is actually 4,835 and thus the mean is 48.35 which is quite close to the result obtained above. The problem above would of course be simpler if the end points of the interval were both even or both odd numbers, in which case the mid-point would be a whole number. The results obtained by the method of grouped data in finding averages described above will not always be as close as we have found since in grouped data we assume that all the scores are concentrated at the mid-point. We of course know that this is not really the case.

For grouped data the mode is defined to be the mid-point of the interval with the greatest frequency. In our example above the mode is 54.5 since it is the mid-point of the interval with the greatest frequency. The interval 50-59 is called the modal interval,
that is, the interval in which the mode occurs.

The median, on the other hand, is a little more complicated to calculate in the case of grouped data with intervals. It is sufficient, for the time being, to find the interval in which the median lies. This, you will recall, is the middle score. Since there are 100 candidates, the middle score will lie between the 50th and the 51st. Now counting from the top of the table, up to 45 (i.e. 7 + 12 + 13 + 13) candidates score marks below 49.5. We need five more candidates to make up the 50 and these five must be in the group of 31 candidates whose scores lie in the interval 50-59. This then is the interval in which the median lies.

**EXERCISE 65-5**

1. The scores out of 100 in a Science test in a school in Mombasa were:

   95, 81, 78, 67, 78, 84, 60, 72, 66, 60, 33, 36,
   60, 72, 72, 63, 42, 18, 42, 36, 27, 66, 54, 27,
   42, 63, 27, 18, 33, 30, 24, 39, 45, 24, 39, 30,
   33, 30, 30, 33, 45, 9

   First arrange the scores in order of magnitude starting with the smallest. Group the scores 0-9, 10-19, 20-29, etc. Find the mean score, the modal score and the interval which contains the median score by the method of the above example.

2. Find the mean for the scores in Problem 1 without grouping the data and compare your answer with the result obtained with the grouped data.

3. Find the mode and the mean of the following distribution

   20, 18, 20, 22, 16, 24, 20, 22, 16, 22, 20, 18, 24, 20, 18.

4. For the distribution given below find the mean and modal values.

<table>
<thead>
<tr>
<th>Score</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>48</td>
<td>3</td>
</tr>
<tr>
<td>46</td>
<td>4</td>
</tr>
<tr>
<td>44</td>
<td>6</td>
</tr>
<tr>
<td>42</td>
<td>9</td>
</tr>
<tr>
<td>40</td>
<td>5</td>
</tr>
<tr>
<td>38</td>
<td>4</td>
</tr>
<tr>
<td>36</td>
<td>2</td>
</tr>
</tbody>
</table>

5. The scores obtained by 42 pupils in a Health Education test were:

   25, 20, 18, 34, 26, 10, 17, 15, 15, 35, 30, 10, 16, 19, 35,
   20, 11, 10, 10, 5, 7, 25, 10, 23, 8, 5, 15, 5, 13, 26,
   20, 10, 5, 13, 7, 22, 6, 5, 10, 9, 12, 15.
Group the data 5-9, 10-14, etc. and complete the following table.

<table>
<thead>
<tr>
<th>Interval</th>
<th>Mid-point (X)</th>
<th>Frequency (f)</th>
<th>fX</th>
</tr>
</thead>
<tbody>
<tr>
<td>5-9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10-14</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Find
(a) the mean
(b) the mode
(c) the interval in which the median lies.

### 65-6 Some Important Properties of the Mean

In Exercise 65-4, question number 5, you were asked to find the mean of the following sets of data:

(i) 4, 6, 8, 14, 10, 14, 6, 6, 27, 8, 18
(ii) 30, 27, 26, 31, 26, 34
(iii) 13, 13, 11, 12, 12, 13, 12, 14, 15, 14, 13, 14, 12, 13, 13, 14

We shall now use these exercises to discuss some important properties of the mean. You will recall that for (i) you found the mean to be 11. If we now add 6 to each number we shall get 10, 12, 14, 20, 16, 20, 12, 12, 33, 14, 24. What is the mean of this new set of data?

New mean = \[
\frac{10 + 3(12) + 2(14) + 16 + 2(20) + 24 + 33}{11}
\]

= \[
\frac{10 + 36 + 28 + 16 + 40 + 24 + 33}{11} = \frac{187}{11} = 17.
\]
This may also be written in the form:

New mean = \[
\frac{(4+6) + 3(6+6) + 2(8+6) + (10+6) + 2(14+6) + (18+6) + (27+6)}{11}
\]

\[
= \frac{4 + 18 + 16 + 10 + 28 + 18 + 27 + (11 \times 6)}{11}
\]

\[
= \frac{121 + 66}{11} = 11 + 6 = 17.
\]

What do you notice about the mean of the new set of data? By how much did you increase each item in the original set of data? By how much is the mean increased?

If we now subtract 4 from each item we shall get: 0, 2, 4, 10, 6, 10, 2, 2, 23, 4, 14. The mean of this set of data is given by:

Mean = \[
\frac{0 + 3(2) + 2(4) + 6 + 2(10) + 14 + 23}{11}
\]

\[
= \frac{77}{11} = 7.
\]

By how much was each item decreased? By how much is the mean now less than the original mean of 11?

It would appear from these two examples that if we add or subtract the same amount from each observation in a set of data the mean of the new set of data is equal to the mean of the original set of data plus or minus the amount added or subtracted. In the first example we added 6 to each item and the new mean was 17 which is 6 more than the original mean. In the second example we decreased each set of data by 4 and the mean we got was the old mean decreased by 4.

Consider the two sets of data (i) and (ii) above. We know that the mean of (i) is 11 and of (ii) is 29. Suppose that we multiply each item of (i) by 2 and each item of (ii) by 3. Then the sets of data become:

(a) 8, 12, 16, 28, 20, 28, 12, 12, 54, 16, 36,

(b) 90, 81, 78, 93, 78, 102.

Mean of Set (a) = \[
\frac{8 + 3(12) + 2(16) + 20 + 2(28) + 36 + 54}{11}
\]

\[
= \frac{8 + 36 + 32 + 20 + 56 + 36 + 54}{11}
\]

\[
= \frac{242}{11} = 22.
\]
By what number did we multiply each item of (i)? The mean of set (a) = 22 = 2 \times 11 = 2 \times \text{Mean of (i)}. Is this an accident?

Let us find the mean of (b) in order to see whether we find a similar relation.

\[
\text{Mean of (b)} = \frac{2(78) + 81 + 90 + 93 + 102}{6} = \frac{522}{6} = 87 = 3 \times 29
\]

Here we see that the mean of (b) is 3 times the mean of (ii). From these examples it would appear that if each item of a set of data is multiplied by the same number, the mean of the new set is the mean of the original set of data multiplied by the given number. In the examples above we see that this is true. In order to convince yourself that it holds in general apply the principle to some of the examples you have worked in Exercise 64-4.

After showing that the principle holds for many examples we may guess that it will hold in general. But the general principle must still be rigorously proved.

Suppose that we are given two sets of data:

Set I: 34, 31, 30, 27, 26, 26
Set II: 98, 95, 90, 80, 75, 66

and suppose that we wish to find the mean of the sum of the two sets. Let us first find the mean of each set.

\[
\text{Mean of Set I} = \frac{2(26) + 27 + 30 + 31 + 34}{6} = \frac{174}{6} = 29.
\]

\[
\text{Mean of Set II} = \frac{66 + 75 + 80 + 90 + 95 + 98}{6} = \frac{504}{6} = 84.
\]

\[
\text{Sum of the two sets} = (26+66) + (26+75) + (27+80) + (30+90) + (31+95) + (34+98)
\]
\[
= 92 + 101 + 107 + 120 + 126 + 132 = 678
\]

\[
\text{Mean of the two sets} = \frac{678}{6} = 113.
\]

But \text{Mean of Set I} + \text{Mean of Set II} = 84 + 29 = 113.
Let us look back for a moment at the set of numbers 4, 6, 8, 14, 10, 14, 6, 6, 27, 8, 18.

You will recall that we added 6 to each member of this set. We may think of our result as adding the corresponding members of the two different sets below:

Set (c) 4, 6, 8, 14, 10, 14, 6, 6, 27, 8, 18
Set (d) 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6.

Notice that the mean of Set (c) = 11 and that of Set (d) = 6. Thus mean of (c) + mean of (d) = 11 + 6 = 17.

But we found that mean of (c + d) = 17. That is, the two means are equal.

These examples lead us to guess that the mean of the sum of two sets of numbers is the sum of the two separate means. As an exercise try this out with several examples so that you may convince yourself that the general rule holds.

Finally we shall look at another important principle of the mean by considering the sets of numbers (i) and (ii)

These sets are:

Set (i) 4, 6, 8, 14, 10, 14, 6, 6, 27, 8, 18
Set (ii) 30, 27, 26, 31, 26, 34.

You will remember that we found the mean of set (i) to be 11 and the mean of set (ii) to be 29. Let us now calculate the difference between each member of the set and the mean first for set (i) and then for set (ii).

Let $X$ represent each member of the set and let $M$ represent the mean. We may set out the calculation as follows:

$\begin{array}{c|c}
\text{Deviation from the Mean} \\
X & X - M \\
4 & 4 - 11 = -7 \\
6 & 6 - 11 = -5 \\
6 & 6 - 11 = -5 \\
6 & 6 - 11 = -5 \\
8 & 8 - 11 = -3 \\
8 & 8 - 11 = -3 \\
10 & 10 - 11 = -1 \\
14 & 14 - 11 = 3 \\
14 & 14 - 11 = 3 \\
18 & 18 - 11 = 7 \\
27 & 27 - 11 = 16 \\
\end{array}$

Notice that some of the deviations from the mean are positive while others are negative. The sum of these deviations $= -29 + 29 = 0$. 
Similarly for set (ii) we have

**Deviation from the Mean**

<table>
<thead>
<tr>
<th>X</th>
<th>X - M</th>
</tr>
</thead>
<tbody>
<tr>
<td>26</td>
<td>-3</td>
</tr>
<tr>
<td>26</td>
<td>-3</td>
</tr>
<tr>
<td>27</td>
<td>-2</td>
</tr>
<tr>
<td>30</td>
<td>+1</td>
</tr>
<tr>
<td>31</td>
<td>+2</td>
</tr>
<tr>
<td>34</td>
<td>+5</td>
</tr>
</tbody>
</table>

The sum of the deviations = \(-8 + 8 = 0\). These two examples suggest that if the mean is subtracted from each measurement, some of the differences will be positive and some will be negative but their sum will always be equal to zero. Since the sum of the deviations is zero it follows that their average is also equal to zero. In the next chapter when we talk about measure of spread we shall say a little more about these deviations.

Let us summarize these properties of the Mean:

1. If the same number is added to or subtracted from each observation in a set of data, the mean of the new set of data is equal to the mean of the original set of data plus the number added or minus the number subtracted.

2. If each number in a set is multiplied by a constant, then the mean of the new set of data is that constant times the mean of the original set of data. If \(X\) represents the set of data and if \(c\) is a constant, Mean \((cX) = c \times \text{Mean} X\).

3. Let \(X\) represent one set of data and \(Y\) another. Then Mean \((X + Y) = \text{Mean of} X + \text{Mean of} Y\).

4. If the mean of a set of data is subtracted from every number in the set and if we take note of the plus and minus signs, the sum of the deviations from the mean is always zero.

**EXERCISE 65-6**

1. (a) Add 10 to each number in question 3 of Exercise 65-4. Find the mean of the new set of data.

   (b) Subtract 5 from each of: 30, 27, 26, 31, 26, 34. Find the mean of the new set of numbers so formed and compare with the mean of the original set.

2. Multiply each of the following set of numbers by 3, and find the mean of the new set of numbers.

   \(13, 11, 12, 12, 13, 12, 14, 15, 14, 13, 14, 12, 13, 13, 14\)

3. After you have found the mean in 1(b), find the deviation of each number from the mean and hence show that sum of the deviations from the mean is zero.
4. Make up an example to illustrate the fact that \( \text{Mean} (X + Y) = \text{Mean} X + \text{Mean} Y. \)

65-7 A Short Method of Calculating the Mean

When we work with large numbers, especially with grouped data, it is sometimes tedious to calculate the mean. The work is often simplified if we work with an assumed mean. That is, we guess what the mean is likely to be and later we make the necessary correction after working with the mean that we have guessed.

The principle of working with an assumed mean is quite simple. Let \( M \) be the true mean, \( A \) the assumed mean and \( C \) the correction. Then we may express the relationship between them as

\[
M = A + C.
\]

If, by a stroke of luck, the assumed mean that we choose is the true mean then it is clear that \( C \) will be zero. If, on the other hand, \( A \) is greater than \( M \), then \( C \) will be negative, while if \( A \) is less than \( M \), \( C \) will be positive. From one of the properties of the mean we know that the average deviation from the true mean is zero. Our correction is thus seen to be the average deviation from the assumed mean.

We may now write

True mean value = Assumed mean + average deviation.

Suppose that we wish to find the average height of 8 men whose heights are given to be 5ft. 9in., 6ft. 0in., 6ft. 3in., 6ft. 2in., 5ft. 11in., 6ft. 1in., 5ft. 10in., and 5ft. 8in.

Let us assume that the mean height is 5ft. 11in. We may set out the calculation as follows:

<table>
<thead>
<tr>
<th>Height</th>
<th>Deviation from Assumed mean in inches</th>
</tr>
</thead>
<tbody>
<tr>
<td>5ft. 8in.</td>
<td>-3</td>
</tr>
<tr>
<td>5ft. 9in.</td>
<td>-2</td>
</tr>
<tr>
<td>5ft. 10in.</td>
<td>-1</td>
</tr>
<tr>
<td>5ft. 11in.</td>
<td>0</td>
</tr>
<tr>
<td>6ft. 0in.</td>
<td>1</td>
</tr>
<tr>
<td>6ft. 1in.</td>
<td>2</td>
</tr>
<tr>
<td>6ft. 2in.</td>
<td>3</td>
</tr>
<tr>
<td>6ft. 3in.</td>
<td>4</td>
</tr>
</tbody>
</table>

Sum of deviation = 4 in.

Mean deviation = \( \frac{4}{8} \text{ in.} = \frac{1}{2} \text{ in.} \)
Mean height = assumed mean + mean deviation = 5ft. 11in. + $\frac{1}{2}$ in.

mean height = 5ft. $1\frac{1}{2}$ in.

The mean height may also be found by the long method thus:

\[
\text{Mean height} = \frac{5\text{ft. 8in.} + 5\text{ft. 9in.} + 5\text{ft. 10in.} + \ldots + 6\text{ft. 3in.}}{8}
\]

\[
= \frac{47\text{ft. 8in}}{8}
\]

\[
= 5\text{ft. }1\frac{1}{2}\text{in.}
\]

We obtain the correct mean no matter which height we choose for the assumed mean. For example, we could just as well have chosen 6 ft. 1 in.

<table>
<thead>
<tr>
<th>Height</th>
<th>Deviation from assumed mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>5ft 8in</td>
<td>-5</td>
</tr>
<tr>
<td>5ft 9in</td>
<td>-4</td>
</tr>
<tr>
<td>5ft 10in</td>
<td>-3</td>
</tr>
<tr>
<td>5ft 11in</td>
<td>-2</td>
</tr>
<tr>
<td>5ft 0in</td>
<td>-1</td>
</tr>
<tr>
<td>6ft 1in</td>
<td>0</td>
</tr>
<tr>
<td>6ft 2in</td>
<td>+1</td>
</tr>
<tr>
<td>6ft 3in</td>
<td>+2</td>
</tr>
</tbody>
</table>

\[
\text{Sum of deviation} = -15 + 3 = -12
\]

\[
\text{Mean deviation} = -\frac{12}{8} = -1\frac{1}{2}\text{in.}
\]

True mean height = assumed mean + mean deviation

\[
= 6\text{ft. 1in.} -1\frac{1}{2}\text{in.}
\]

\[
= 5\text{ft. }1\frac{1}{2}\text{in.}, \text{as before.}
\]

The value of the method of an assumed mean becomes apparent when we work with
large numbers in grouped data. As an illustration of the method let us look once more at the problem of finding the mean score of the 100 candidates in an arithmetic test discussed earlier in this chapter. We have

<table>
<thead>
<tr>
<th>Mid-point of interval</th>
<th>deviation from assumed mean</th>
<th>frequency</th>
<th>Product fd</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>d</td>
<td>f</td>
<td>fd</td>
</tr>
<tr>
<td>14.5</td>
<td>-40</td>
<td>7</td>
<td>-280</td>
</tr>
<tr>
<td>24.5</td>
<td>-30</td>
<td>12</td>
<td>-360</td>
</tr>
<tr>
<td>34.5</td>
<td>-20</td>
<td>13</td>
<td>-260</td>
</tr>
<tr>
<td>44.5</td>
<td>-10</td>
<td>13</td>
<td>-130</td>
</tr>
<tr>
<td>54.5</td>
<td>0</td>
<td>31</td>
<td>0</td>
</tr>
<tr>
<td>64.5</td>
<td>10</td>
<td>12</td>
<td>120</td>
</tr>
<tr>
<td>74.5</td>
<td>20</td>
<td>5</td>
<td>100</td>
</tr>
<tr>
<td>84.5</td>
<td>30</td>
<td>5</td>
<td>150</td>
</tr>
<tr>
<td>94.5</td>
<td>40</td>
<td>1</td>
<td>40</td>
</tr>
<tr>
<td>104.5</td>
<td>50</td>
<td>1</td>
<td>50</td>
</tr>
</tbody>
</table>

Total = 100 Sum of fd = -570

We have assumed that the mean score is 54.5. Then in the second column we found the deviations from the assumed mean. The fourth column is the product of the deviations from the assumed mean and the frequencies. The sum of this column is -570.

\[
\text{Mean of fd} = \frac{-570}{100} = -5.7
\]

\[
\text{Mean Score} = 54.5 - 5.7 = 48.8
\]

You will notice that this is simpler than the method used when the problem was first considered. A further simplification may be made if the deviation of the mid-point of each class interval from the assumed mean is measured in units of the class interval.

In our grouped frequency table the class interval is 10. This means that deviations that read -40, -30, -20, -10 will now be expressed as -4, -3, -2, -1. The positive deviations will similarly be 1, 2, 3, 4, 5.

Let \( X_0 \) be the assumed mean and \( X \) the mid-point of the class interval. Then if \( t \) be the units in which we measure the deviations we will have \( X - X_0 = 10t \). In general, 10 will be replaced by \( c \), where \( c \) is the class interval. The correct value of the mean will now be given by:

\[
\text{true mean} = \text{assumed mean} + 10 \text{ average deviation}.
\]
The table above will now appear as

<table>
<thead>
<tr>
<th>Mid-point of interval</th>
<th>Deviation from assumed mean divided by 10</th>
<th>Frequency</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>t</td>
<td>f</td>
<td>ft</td>
</tr>
<tr>
<td>14.5</td>
<td>-4</td>
<td>7</td>
<td>-28</td>
</tr>
<tr>
<td>24.5</td>
<td>-3</td>
<td>12</td>
<td>-36</td>
</tr>
<tr>
<td>34.5</td>
<td>-2</td>
<td>13</td>
<td>-26</td>
</tr>
<tr>
<td>44.5</td>
<td>-1</td>
<td>13</td>
<td>-13</td>
</tr>
<tr>
<td>54.5</td>
<td>0</td>
<td>31</td>
<td>0</td>
</tr>
<tr>
<td>64.5</td>
<td>1</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>74.5</td>
<td>2</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>84.5</td>
<td>3</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>94.5</td>
<td>4</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>104.5</td>
<td>5</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

Total $ft = -103 + 46 = -57$

True Mean = $54.5 + 10 \left( \frac{-57}{100} \right) = 54.5 - 5.7 = 48.8$, as before.

**EXERCISE 65-7**

1. As an exercise choose 44.5 as the assumed mean. Measure the deviation from the assumed mean first in the original units and then in terms of the cell interval.
2. Do problems 3 and 5 of Exercise 65-4 of this chapter by the short method.

**65-8 Which Average?**

When we use the word average in its general sense, we have seen that it may refer to the mean, the mode or the median. It is therefore necessary, when we speak, to state quite clearly which average we are talking about. Some of the examples that we have considered and the exercises that you have done will have shown you that, in general, the three averages are quite distinct. In one or two of the exercises you found that the three averages coincided.

Let us look again at the averages we have discussed.

The Managing Director of a firm earns a salary of £11,000 per annum. His assistant earns £5,000, while two branch office managers each receive £3,000 per annum. Other posts are: two on a salary of £2,000 per annum, three on £1,500, three on £1,000, one on £600 and twelve on £400 per annum. The Managing Director, in
order to impress a visitor to the firm, says that the average salary paid by his firm is £1,716 per annum. This figure is in fact the arithmetic mean of the set of salaries. On the other hand the Secretary of the Workers’ Union who earns only £400 per annum puts the average salary at £400 per annum since this figure represents the most common salary earned by any one group; it is the mode of the earnings. Also, £600 may be regarded as the average earning since there are twelve persons who earn more than this sum and twelve who earn less. £600 is the median salary.

It is clear that only six persons earn more than £1,716 and twice that number earn only £400 per annum. In spite of this, the arithmetic mean is a good representative of the data on salaries; it tells us that if all the money available for salaries were distributed so that each person received the same amount, each employee of the firm would get £1,716. In calculating the arithmetic mean we take account of all the salaries. However, our average of £1,716 is not typical of the earnings of the employees of the firm because of the high salaries earned by a few persons and the low salaries earned by many.

If you score very high marks in one or two subjects in an examination your poor performance in one or two other subjects is sometimes obscured when the arithmetic mean of the scores is taken.

The mean is found useful in many situations. We use the mean for finding the average attendance, the average monthly rainfall or temperature, the average mark in an examination, the average population density per square mile, the average speed of a car, etc.

The Secretary of the Workers’ Union of the firm chose £400 as the average earning because it occurs more frequently than any other salary. Also he chooses this low figure because it is better for his propaganda purposes. He ignores the fact that there are other salaries on the payroll which are quite high. We have already found in the case of the school cap that the mode is sometimes the most appropriate average to use.

The median salary of £600 represents the middle salary. In considering the median salary, our main interest is whether or not a salary is higher or lower than this salary; we do not appear to care whether a salary is very much higher or very much lower than the salary of £600. If the salaries are evenly spread out on either side of the median salary of £600 then the median would certainly be the best measure of central tendency and in this case would coincide with the mean. We can use the median score to determine whether or not a student is above or below the average; in most cases the scores in a test are evenly spread out and so the median is a good measure of the way in which the scores are distributed. The median is not influenced as much as the mean by a few extreme scores.

Thus we see that some averages are useful in some situations and others in other situations. Our choice of an average will be dictated by the problem we have in hand. On the whole the mean is the most important measure of central tendency; as you will find out later on if you study more statistics the mean plays an important part in the theory of sampling and in tests of statistical hypotheses.
Chapter 66
MEASURE OF SCATTER

66-1 Introduction

In Section 63-2 of Chapter 63, we considered the distribution of scores in an arithmetic examination. We grouped the scores in tens: 10 - 19, 20 - 29, 30 - 39, ..., 100 - 109 and found the frequencies of scores in these intervals. Later, in chapter 65 we found the mean of the scores to be 48.8. In exercise 63-2, question 3, you were asked to complete a table showing the scores of candidates in a mathematics examination. You will recall that the scores were grouped in intervals of five: 36 - 40, 41 - 45, ..., 96 - 100. The mean of this distribution of scores can be shown to be 62.8 (to one decimal place). Look carefully at the two distributions once more.

These two distributions differ not only in the position of the mean but also in the way in which the scores are grouped about the mean. It would appear that the scores in the arithmetic test are more spread out than those in the mathematics test. It would be good for us to have some quantitative way of measuring the spread or scatter of these scores. This is what we shall discuss in this chapter. There are several measures of scatter, each of which is good for a particular purpose, and none best in all situations.

66-2 The Range

One of the simplest ways of measuring the spread of a distribution is the use of the range. We shall illustrate this with some examples.

The scores of two candidates in ten tests at the end of term were:

Candidate I: 100, 100, 99, 97, 85, 84, 75, 75, 75, 60
Candidate II: 90, 89, 88, 87, 85, 85, 83, 82, 81, 80

We may illustrate these scores by a dot frequency diagram as shown below.
This diagram makes the scores stand out more clearly than the table did. We need to know all the ten scores for each candidate to be able to say everything about them. Suppose that you wanted to say as much as possible about each set of these ten scores with just two numbers. As you have already seen we could choose for one number any of the three measures of central tendency which we discussed in Chapter 65. Apart from these measures could you choose some measure of the scatter of the scores? Let us consider some possible measures of scatter and see what they tell us about the distribution.

Look again at the scores and the diagram. What is the mean score for each candidate? Did you find that both candidates had the same mean score? Which of the two candidates would you regard as being more consistent in his performance? Why have you chosen one candidate rather than the other? Notice that candidate I had full marks in two tests but in one test his score was only 60; the difference between his highest and lowest scores is 40. What is the difference between the highest and lowest scores for candidate II?

When we have a set of data, the difference between the largest and smallest numbers is known as the range. Thus the range for candidate I is 40 while that for candidate II is 10. These two numbers 10 and 40 give us some idea of the way in which the scores are spread. The example also tells us that it is not sufficient for us to know the mean score; we would get a much better picture if we knew also how the scores were spread out. The range does give us some idea of how the scores are spread, but it is determined only by the highest and the lowest scores, and is not changed by changes in the other scores.

The range is not a completely satisfactory measure of spread because it is possible for us to have two distributions with the same range but which are scattered quite differently. Suppose that we have four candidates who took ten subjects each at both the Ordinary and the Advanced Levels in the General Certificate of Education Examination and suppose that their scores in corresponding subjects were distributed as follows:

<table>
<thead>
<tr>
<th>Candidate A</th>
<th>Candidate B</th>
<th>Candidate C</th>
<th>Candidate D</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>100</td>
<td>60</td>
<td>100</td>
</tr>
<tr>
<td>90</td>
<td>100</td>
<td>60</td>
<td>100</td>
</tr>
<tr>
<td>80</td>
<td>100</td>
<td>60</td>
<td>100</td>
</tr>
<tr>
<td>70</td>
<td>100</td>
<td>60</td>
<td>90</td>
</tr>
<tr>
<td>60</td>
<td>100</td>
<td>60</td>
<td>80</td>
</tr>
<tr>
<td>50</td>
<td>10</td>
<td>50</td>
<td>30</td>
</tr>
<tr>
<td>40</td>
<td>10</td>
<td>50</td>
<td>20</td>
</tr>
<tr>
<td>30</td>
<td>10</td>
<td>50</td>
<td>10</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
<td>50</td>
<td>10</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>50</td>
<td>10</td>
</tr>
</tbody>
</table>
The diagram below illustrates how these scores are distributed.

What is the mean score for each candidate? What is the range in each case? Did you find that all the candidates had a mean score of 55 and that candidates A, B and D each had a range of 90? Would you say then that the range measures the spread adequately? If your answer is No, then we shall have to seek some other measure of spread.

66-3 Mean Deviation from the Mean

Let us consider the distributions of the scores of the four candidates A, B, C, D at the General Certificate of Education Examination from another stand-point. We may get some useful information by looking at the deviation of each score from the mean.
<table>
<thead>
<tr>
<th>Score (X)</th>
<th>Candidate A deviation from mean (X-M)</th>
<th>Score (X)</th>
<th>Candidate B deviation from mean (X-M)</th>
<th>Score (X)</th>
<th>Candidate C deviation from mean (X-M)</th>
<th>Score (X)</th>
<th>Candidate D deviation from mean (X-M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>45</td>
<td>100</td>
<td>45</td>
<td>60</td>
<td>5</td>
<td>100</td>
<td>45</td>
</tr>
<tr>
<td>90</td>
<td>35</td>
<td>100</td>
<td>45</td>
<td>60</td>
<td>5</td>
<td>100</td>
<td>45</td>
</tr>
<tr>
<td>80</td>
<td>25</td>
<td>100</td>
<td>45</td>
<td>60</td>
<td>5</td>
<td>100</td>
<td>45</td>
</tr>
<tr>
<td>70</td>
<td>15</td>
<td>100</td>
<td>45</td>
<td>60</td>
<td>5</td>
<td>90</td>
<td>35</td>
</tr>
<tr>
<td>60</td>
<td>5</td>
<td>100</td>
<td>45</td>
<td>60</td>
<td>5</td>
<td>80</td>
<td>25</td>
</tr>
<tr>
<td>50</td>
<td>-5</td>
<td>10</td>
<td>-45</td>
<td>50</td>
<td>-5</td>
<td>30</td>
<td>-25</td>
</tr>
<tr>
<td>40</td>
<td>-15</td>
<td>10</td>
<td>-45</td>
<td>50</td>
<td>-5</td>
<td>20</td>
<td>-35</td>
</tr>
<tr>
<td>30</td>
<td>-25</td>
<td>10</td>
<td>-45</td>
<td>50</td>
<td>-5</td>
<td>10</td>
<td>-45</td>
</tr>
<tr>
<td>20</td>
<td>-35</td>
<td>10</td>
<td>-45</td>
<td>50</td>
<td>-5</td>
<td>10</td>
<td>-45</td>
</tr>
<tr>
<td>10</td>
<td>-45</td>
<td>10</td>
<td>-45</td>
<td>50</td>
<td>-5</td>
<td>10</td>
<td>-45</td>
</tr>
</tbody>
</table>

We shall illustrate these deviations from the mean for candidates A and C graphically. (You may do the same for the other two candidates as an exercise.)

*Spread of Candidate A's Scores*
The uprights in the diagrams above represent the deviations of individual scores from the mean of 55. The uprights above the mean line represent positive deviations while those below the mean represent negative deviations.

We cannot use the average of these deviations to measure the spread since, as we found in Chapter 65, the sum of the deviations from the mean is zero. In order to overcome this difficulty we may agree to ignore all minus signs and treat every deviation as though it were positive. The sum of the deviations of A's scores from the mean, ignoring negative signs, will thus be 250 and so the average or mean deviation $= \frac{250}{10} = 25$.

The corresponding value for candidate C is $\frac{50}{5} = 5$. This result is not unexpected since, from the diagrams above, we see that the scores for A are more spread out than those for C. If all C's scores were equal to 55 then the spread would be zero.

The average of the absolute values of the deviations from the mean is thus another measure of spread and is usually called *mean absolute deviation from the mean*. For the two sets of scores for A and C we find that both the range and the mean absolute deviation are greater for A than for C; the range for A was found to be 90 and that for C was 10.

Let us find the mean absolute deviations for candidate I and candidate II that we discussed in section 66.2 above. First the mean score of 85 is the same for both
candidates. Hence we have

<table>
<thead>
<tr>
<th>Score (X)</th>
<th>Deviation from mean (X-M)</th>
<th>Score (X)</th>
<th>Deviation from mean (X-M)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>15</td>
<td>90</td>
<td>5</td>
</tr>
<tr>
<td>100</td>
<td>15</td>
<td>89</td>
<td>4</td>
</tr>
<tr>
<td>99</td>
<td>14</td>
<td>88</td>
<td>3</td>
</tr>
<tr>
<td>97</td>
<td>12</td>
<td>87</td>
<td>2</td>
</tr>
<tr>
<td>85</td>
<td>0</td>
<td>85</td>
<td>0</td>
</tr>
<tr>
<td>84</td>
<td>-1</td>
<td>85</td>
<td>0</td>
</tr>
<tr>
<td>75</td>
<td>-10</td>
<td>83</td>
<td>-2</td>
</tr>
<tr>
<td>75</td>
<td>-10</td>
<td>82</td>
<td>-3</td>
</tr>
<tr>
<td>75</td>
<td>-10</td>
<td>81</td>
<td>-4</td>
</tr>
<tr>
<td>60</td>
<td>-25</td>
<td>80</td>
<td>-5</td>
</tr>
</tbody>
</table>

Sum of absolute values of deviations = 112 for candidate I
Sum of absolute values of deviations = 28 for candidate II

Mean absolute deviation for candidate I = \( \frac{112}{10} = 11.2 \)

Range for candidate I = 40

Mean absolute deviation for candidate II = \( \frac{28}{10} = 2.8 \)

Range for candidate II = 10

Here again both measures of spread are greater for candidate I than for candidate II as was to be expected from a look at the diagram.

The mean absolute deviation has some uses in statistics; it is easy to understand and each reading affects it about as much as any other and so it is not strongly influenced by some values and ignored by others as in the case of the range. However, it is not a very satisfactory measure of spread especially for later work in statistics.

66-4 Standard Deviation

In our search for a suitable measure of spread our guess is that we could do something with the deviations from the mean other than ignoring the minus signs. We could square all the deviations from the mean, thus getting rid of all the minus signs. We then add up all these squares and take their mean. In order to express such a measure in linear units, and not square units, we take the square root of the mean square deviation. This gives us a measure of spread which we call the standard deviation.

The standard deviation provides a quantitative basis for comparing how spread out a distribution is in relation to another. It has many important uses in more advanced
work in statistics. If the standard deviation is relatively small then we may expect a close clustering of the scores about the mean. On the other hand, a relatively large value of the standard deviation suggests wide scattering about the mean.

We shall now calculate the standard deviations for the two distributions of scores for candidates A and C.

Calculation of the Standard Deviation for Candidate A's Scores

<table>
<thead>
<tr>
<th>Scores</th>
<th>Deviations from the mean X - M</th>
<th>Squared deviations (X - M)^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>45</td>
<td>2025</td>
</tr>
<tr>
<td>90</td>
<td>35</td>
<td>1225</td>
</tr>
<tr>
<td>80</td>
<td>25</td>
<td>625</td>
</tr>
<tr>
<td>70</td>
<td>15</td>
<td>225</td>
</tr>
<tr>
<td>60</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>50</td>
<td>-5</td>
<td>25</td>
</tr>
<tr>
<td>40</td>
<td>-15</td>
<td>225</td>
</tr>
<tr>
<td>30</td>
<td>-25</td>
<td>625</td>
</tr>
<tr>
<td>20</td>
<td>-35</td>
<td>1225</td>
</tr>
<tr>
<td>10</td>
<td>-45</td>
<td>2025</td>
</tr>
<tr>
<td></td>
<td>Total</td>
<td>8250</td>
</tr>
</tbody>
</table>

Mean Squared deviations = $\frac{8250}{10} = 825$

Standard deviation = $\sqrt{825} = 28.8$

Calculation of Standard Deviation for Candidate C's Scores

<table>
<thead>
<tr>
<th>Scores</th>
<th>Deviations from mean</th>
<th>Squared deviations (X - M)^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>60</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>60</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>60</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>60</td>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>50</td>
<td>-5</td>
<td>25</td>
</tr>
<tr>
<td>50</td>
<td>-5</td>
<td>25</td>
</tr>
<tr>
<td>50</td>
<td>-5</td>
<td>25</td>
</tr>
<tr>
<td>50</td>
<td>-5</td>
<td>25</td>
</tr>
<tr>
<td>50</td>
<td>-5</td>
<td>25</td>
</tr>
</tbody>
</table>

Total = 250
Mean squared deviation \[= \frac{250}{10} = 25\]

Standard deviation \[= \sqrt{25} = 5\]

We see that the standard deviation for the scores of candidate C is very much smaller than that for the scores of candidate A. This confirms the general impression gained from the diagrams.

Let us summarise the values of these three measures of spread that we have found for the two sets of scores.

<table>
<thead>
<tr>
<th></th>
<th>Candidate A</th>
<th>Candidate C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Range</td>
<td>90</td>
<td>10</td>
</tr>
<tr>
<td>Mean absolute deviation from mean</td>
<td>25</td>
<td>5</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>28.8</td>
<td>5</td>
</tr>
</tbody>
</table>

We see from this analysis that all these measures of spread or scatter are useful; each is preferred to the others for some situations. They tend to be small for closely bunched data and large for widely spread data. None of them tell anything about individual values. One is often preferred to the other depending on how easy it is to calculate, how well it does the job on hand, or how useful it is in more advanced work in statistics.

66-5 Standard Deviation for Grouped Data

So far we have talked about finding the standard deviation for ungrouped data. We shall now look first at ungrouped data, in which each number has a frequency and later we shall consider distributions grouped into intervals.

Let us consider the distribution: 13, 13, 11, 12, 12, 13, 12, 14, 15, 14, 13, 14, 12, 13, 13, 14 which we have met before. The distribution may be tabulated as shown below

<table>
<thead>
<tr>
<th>Number (X)</th>
<th>Frequency</th>
<th>fX</th>
<th>Deviation from mean (X - M)</th>
<th>Squared deviation from mean (X - M)^2</th>
<th>f(X-M^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>1</td>
<td>11</td>
<td>-2</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>12</td>
<td>4</td>
<td>48</td>
<td>-1</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>13</td>
<td>6</td>
<td>78</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>4</td>
<td>56</td>
<td>1</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>15</td>
<td>1</td>
<td>15</td>
<td>2</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>Total</td>
<td>16</td>
<td>208</td>
<td></td>
<td>16</td>
<td></td>
</tr>
</tbody>
</table>
Average of f(X - M)^2 = \frac{16}{16} = 1

Standard deviation = \sqrt{1} = 1

If we have a distribution grouped into intervals, the only difference in the above set up is that the column headed "Number (X)" will be replaced by "Mid-point of Interval" and the procedure is then exactly the same as above.

**EXERCISE 66-5**

1. Calculate the range and the standard deviation of
   (a) 2, 4, 6, 8, 10, 12, 14.
   (b) 9, 10, 11, 12, 13, 14, 15, 16, 17.

2. Find the range and the standard deviation of these scores.
   (a) 100, 100, 99, 97, 85, 84, 75, 75, 75, 60
   (b) 90, 89, 88, 87, 85, 83, 82, 81, 80
   Which of these two sets of scores is more scattered than the other?

3. (a) Draw diagrams to show the deviation from the mean for each of the scores for B and D, in Section 66-3.
   (b) Find the mean absolute deviation from the mean for each distribution.
   (c) Calculate the standard deviation for each set of scores.

4. Multiply each number in 1(a) above by 2. Find the standard deviation of the new set of numbers and compare your answer with the answer for 1(a).

5. (a) Add 3 to each number in 1(b) above. Find the standard deviation of the new set of numbers. Compare with your previous answer for 1(b).
   (b) Subtract 4 from each number in 1(b) above. Find the standard deviation of the new set of numbers so formed. Compare your answer with the previous result in 1(b) and 5(a).
67-1 Introduction

We cannot be sure of many things we do each day. When we leave home for school, we cannot be sure we will arrive on time. When we give advice to pupils, we cannot be sure it will have a good result even if our advice is followed. When we play a game of snakes and ladders or ludo, we cannot be sure of the number that will show when we roll the die. When we choose sides at the start of a football game by tossing a coin we cannot know with certainty whether the coin will show heads or tails. When we plant our seed, we cannot be sure of the size of the crop we will harvest.

The things we do often have uncertain outcomes. Whether we arrive at school on time depends on the weather, on our state of health, perhaps on the mechanical condition of the bus or bicycle that transports us, and on many other things. When we roll our die we know it will fall with its face showing one of the numbers 1, 2, 3, 4, 5, 6, but we do not know in advance which particular number will show up. Similarly for the toss of a coin: we cannot know before we toss the coin whether it will fall heads or tails. Our harvest depends on the quality of our seed, the amount of sunshine and rainfall, the fertility of our soil, and on our care in warding off weeds and insect pests. Our world is full of uncertainty and risk.

The theory of probability is a branch of mathematics that deals with and shows how to measure risk and uncertainty. We will learn how to compute or measure the chance or probability of an uncertain event. We will perform simple experiments to become familiar with uncertain outcomes and we will study the way probability theory helps us understand how to analyze such experiments. If we know something about probability, we will understand our world better.

To develop the theory of probability we will need to recall some things we learned about sets and functions in our earlier work. We will revise old topics as we learn new ones.

Let us start by doing some experiments. First, we shall toss a coin and keep a record of which side of the coin shows as it lands. For each experiment, toss the coin at least 20 times. Perform at least five experiments and enter your results in a table like this:
Experiment Coin-Tossing Experiments

<table>
<thead>
<tr>
<th>Number of Coin Tosses</th>
<th>Number of Heads Obtained</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
</tr>
</tbody>
</table>

**EXERCISE 67-1**

1. Do you feel that the two possibilities “heads” and “tails” are equally likely outcomes? Do your experimental results seem to support this feeling? What ratio of number of heads to number of tosses did you get in each experiment? What ratio do you get when you total all your tosses?

2. If other students have done the coin-tossing experiments, combine all of the totals and determine the ratio of number of heads to total number of tosses.

3. Find a drawing pin and toss it. When it lands it will either have its head flat ( ) or it will fall on its side ( ). Throw the drawing pin on a flat table and keep a record of whether it falls flat or on its side. For each experiment, toss the pin at least 20 times. Perform at least five experiments and record your results in a table like this:

**Drawing-Pin Tossing-Experiments**

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Number of Drawing Pin Tosses</th>
<th>Number of Times the Drawing Pin Fell Flat</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Do you feel that the two possibilities "falling flat" and "falling on a side" are equally likely outcomes? Do your experimental results seem to support this feeling?

67-2 Experiments

We want now to describe some simple experiments with uncertain outcomes. We can imagine these experiments repeated again and again under identical conditions. We begin our mathematical theory of probability by using the idea of a set to say precisely what we mean by such an experiment. At this time, you may wish to re-read Unit 1 - "Basic Concepts and Language of Sets" - in the first volume of Basic Concepts of Mathematics.

A set, you will recall, is just a collection of things. The things we now are interested in are all the different possible outcomes when we actually do (or only imagine doing) an experiment. Let us consider some examples.

Example 1. We toss a coin. The set of possible outcomes has two members: "heads" and "tails." (Let us agree that the coin will never land on its edge.) If we let \( H \) stand for "heads" and \( T \) for "tails," we can write this set as \( \{ H, T \} \). You will recall that one way to describe a set is to list all the members of the set and to enclose this list in curly brackets, using commas to separate the members. So the set \( \{ H, T \} \) is used to describe the coin-tossing experiment. This set is called the universe set of the experiment and is denoted by the capital letter \( U \):

\[ U = \{ H, T \}. \]

Example 2. We throw a die to determine our first move in a game of ludo. The set of possible outcomes has six members: 1, 2, 3, 4, 5, 6. Each numeral stands for the outcome in which the die falls with that numeral showing on its uppermost face. So the universe set for this die-throwing experiment is the set

\[ U = \{ 1, 2, 3, 4, 5, 6 \}. \]

Example 3. We buy one lottery ticket. Let us assume that 1000 tickets are sold and that these tickets are numbered 1, 2, 3, \ldots, 1000. Then our ticket can have any one of these numbers on it. Therefore we take the set

\[ U = \{ 1, 2, 3, \ldots, 1000 \} \]
as the universe set for the experiment in which we buy a lottery ticket. This set has 1000 members and we read "\( U \) is the set consisting of tickets numbered 1, 2, 3, and on up to 1000." Recall that we use three dots when the members of a set are clearly indicated by some pattern, but when we cannot or do not want to list all the members of the set.
Example 4. We toss a shilling coin and then toss it again. The possible outcomes are given in the table:

<table>
<thead>
<tr>
<th>First Toss</th>
<th>Second Toss</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>H</td>
</tr>
<tr>
<td>H</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>H</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

So we can take the set

\[ U = \{ \text{HH, HT, TH, TT} \} \]

as the universe set for this experiment. Here we write HH to stand for the first toss resulting in heads and the second toss resulting in heads, we write HT for the first toss resulting in heads and the second in tails, and so on.

Example 5. We ask a friend to tell us his month of birth. His answer may be any one of the 12 months of the year. So we have 12 possible outcomes of this experiment. Therefore

\[ U = \{ \text{Jan., Feb., March, \ldots, Dec.} \} \]

can serve as universe set for this experiment.

Example 6. Suppose we are with two friends and ask each of them to tell us his month of birth. We can see the possible outcomes of this experiment by studying the chart.

<table>
<thead>
<tr>
<th>First Friend's Birth Month</th>
<th>Jan</th>
<th>Feb</th>
<th>March</th>
<th>April</th>
<th>May</th>
<th>June</th>
<th>July</th>
<th>Aug</th>
<th>Sept</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
</tr>
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Each box in the chart stands for one of the answers we can hear when both our friends tell us their birth months. For example, the box marked with an asterisk (*) stands for the first friend having March and the second friend having February as birth month. The box marked with a check (✓) stands for the first friend born in May and the second friend in July. What do the boxes marked with a cross (✗) have in common?

How many boxes are there in this chart? There are 12 rows and 12 columns so we see there are $12 \times 12 = 144$ boxes. We have an experiment with 144 possible outcomes. Instead of listing all 144 members of this set, it is more convenient to give a verbal description. We say that the universe set for this experiment is

$$U = \text{the set of all possible pairs of birth months of our two friends}.$$ 

**Example 7.** We play a game of draughts. We can win the game, lose the game, or the game can end in a draw (tie). So the universe set for this game can be the set

$$U = \{ \text{win, lose, draw} \}.$$ 

**Example 8.** An insurance company that writes a policy for a man aged 20 is interested in whether the man will survive one year or will die within the year. We can think of an experiment lasting one year and having the two possible outcomes given in the universe set:

$$U = \{ \text{man survives one year, man dies within the year} \}.$$ 

To summarize, each time we talk of a real or imaginary experiment, we must be prepared to describe the universe set of this experiment. This universe set has as its members all the possible outcomes that can occur when we perform the experiment.

You will notice that all of the experiments in our examples have finite universe sets. Can you think of an experiment which has an infinite (unending) number of possible outcomes? We shall restrict our study to experiments whose universe sets are finite.

**EXERCISE 67-2**

1. For each of the following experiments, define a suitable universe set $U$:
   (a) Play a game of snakes and ladders with your friend.
   (b) From a hat containing 10 slips of paper numbered 1, 2, 3, ..., 10, we select one slip of paper.
   (c) From the hat as in part (b), we select one slip of paper, return it to the hat, then again select one slip of paper.
   (d) From the hat as in part (b), we select one slip of paper, then select
another slip of paper without returning the first to the hat.

(e) We ask our friend to choose a prime number between ten and thirty.
(f) We choose a day of the week.
(g) We throw a green die and then throw a red die.
(h) We toss a drawing pin on a flat table.
(i) We toss a drawing pin on a flat table, then toss it again.

[ Note: Compare with Example 4 of the text. ]

(j) A survey of families with two children is made and the sexes of the children (in order of age, older child first) are recorded.

[ Note: Compare with Example 4 of the text. ]

(k) A survey of families with three children is made and the sexes of the children (in order of age, oldest child first) are recorded.

(l) We toss a shilling, then toss it again, then toss it a third time.
[ Note: Compare with part (k). ]

(m) We are with three friends and ask each to tell us his month of birth.

2. For each universe set in Exercise 1, state how many members U has.

3. Describe two experiments that have an infinite (unending) number of possible outcomes.

67-3 Events

When we do (or just think of doing) an experiment, we have learned that a universe set U must be defined for this experiment. This set U has as members all the possible outcomes of the experiment. But we may be interested in only some of the outcomes. For example, when we throw a die in a game of ludo, we may be interested in whether we get a five or a six. When we buy one lottery ticket, we are keenly interested in whether it has a winning number. When we ask two friends to tell us their birth months, we may want to know if they were born in the same month. When we survey families with two children, we can ask whether the parents have both a son and a daughter.

In each of these examples, we can write the universe set U for the entire experiment and then pick out of U those outcomes that are of special interest to us. These will form a set B. Each member of set B is also a member of the corresponding universe set U. In such a case, you recall that we say set B is a subset of set U. In the theory of probability, any subset of the universe set is called an event.

Let us look at some examples of events (or subsets of the universe sets U) for some of the experiments we have already studied.
Experiment | Universe Set U | Event B (described in words and then as a subset of U)
--- | --- | ---
a. Throw a die. | U = \{ 1, 2, 3, 4, 5, 6 \} | die shows five or six: B = \{ 5, 6 \}.
b. Buy a lottery ticket. (Assume 1,000 sold of which 10 are selected as winning numbers.) | U = \{ 1, 2, 3, \ldots, 1000 \} | you win: B = \{ w_1, w_2, w_3, \ldots, w_{10} \}, where w_1 denotes the first winning number, w_2 denotes the second winning number, and so on. (w_1 is read "w sub one", w_2 is read "w sub two" and so on.)
c. Ask two friends to tell you their birth month. 144 members, as given in Chart 1, page 161. | U = the set containing both friends have same birth months: B = the set containing those 12 members of U that correspond to the main diagonal entries (marked by crosses, x) in Chart 1 | d. Survey families with two children. | U = \{ BB, BG, GB, GG \} | both friends have same birth months: B = the set containing those 12 members of U that correspond to the main diagonal entries (marked by crosses, x) in Chart 1 parents have son and daughter: B = \{ BG, GB \}.

Do you see that each member of the events B on the right is also a member of the corresponding universe set U? This must be so since an event is defined as a subset of the universe set.

Now try to write down all the events that can be found if the universe set U = \{ win, lose \}, as in a game of snakes and ladders. Did you find all four events? They are:

- The subset with no members: the empty set \{ \}.
- The subsets with one member: \{ win \}, \{ lose \}.
- The subset with two members: \{ win, lose \} = U itself.

Did you recall from your previous study that every set has the empty set and the whole set itself as subsets? In probability theory, any event represented by the empty set is called an impossible event. And any event represented by the whole universe set U is
called a *sure event*. For example, suppose we roll a die and have \( U = \{ 1, 2, 3, 4, 5, 6 \} \) as universe set. Then the event "the number showing is greater than 9" is an impossible event since it is represented by the empty set: there are no members of \( U \) that are greater than 9. At the other extreme, we have for example the event "the number showing is less than 9." This is a sure event since it is represented by the entire set \( U \): every member of \( U \) is less than 9.

Can you find all eight events if the universe set \( U = \{ \text{win}, \text{lose}, \text{draw} \} \) as in a game of draughts? Here is a list of these events:

The impossible event: \( \emptyset \) = the empty set

The events with one member: \( \{ \text{win} \}, \{ \text{lose} \}, \{ \text{draw} \} \)

The events with two members: \( \{ \text{win}, \text{lose} \}, \{ \text{win}, \text{draw} \}, \{ \text{lose}, \text{draw} \} \)

The sure event: \( \{ \text{win}, \text{lose}, \text{draw} \} = \text{the universe set } U \)

How many events are there if the universe set \( U \) has just one member? Then the impossible event \( \emptyset \) and the sure event \( U \) are the only subsets. So there are two events if \( U \) has one member. And we have already seen that there are four events if \( U \) has two members, and eight events if \( U \) has three members. Let us make a little table summarizing these results:

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<th>Number of members in ( U )</th>
<th>Total Number of Events</th>
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<td>1</td>
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<td>2</td>
<td>4</td>
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Can you see the pattern here? Guess how many events there are if the universe set \( U \) has four members. Did you guess that there are \( 2^4 = 16 \) events? That is the correct answer. In fact, it can be proved that if \( U \) has \( n \) members where \( n \) is any counting number, then there are a total of \( 2^n \) different events.

**EXERCISE 67-3**

1. For each of the following experiments, first write the universe set \( U \) and then write the event (subset of \( U \)) for which the verbal description is given. [Note: The universe sets have been described in Section 67-2 and in the Answers to Exercise 67-2.]

(a) **Experiment:** I throw a die in a ludo game.  
**Events:** (i) die shows an even number.
(b) **Experiment:** Toss a shilling coin and then toss it again.

**Events:**
(i) obtain exactly one head in the two tosses.
(ii) obtain at least one head in the two tosses.
(iii) obtain at most one head in the two tosses.

(c) **Experiment:** Ask friend to tell us his month of birth.

**Events:**
(i) friend was born in first six months.
(ii) friend was born in month whose name begins with the letter "J".
(iii) friend was born in month whose name begins with the letter "B".

(d) **Experiment:** Ask two friends to tell us their birth months.

**Events:**
(i) first friend born in January
(ii) second friend born in January
(iii) first friend born in January and second friend born in August or September.
(iv) first friend born in January and second friend born in month whose name begins with the letter "B".

(e) **Experiment:** Select one slip of paper from a hat containing 10 slips numbered 1, 2, 3, . . . , 10.

**Events:**
(i) slip has an even number
(ii) slip has an odd number
(iii) slip does not have an even number
(iv) slip has an even or an odd number
(v) slip has a number bigger than 10
(vi) slip has a number less than 5
(vii) slip has a number less than or equal to 5.

(f) **Experiment:** From same hat as in (e), select one slip, then select another after replacing first slip.

**Events:**
(i) first slip has the number 1
(ii) second slip has the number 1
(iii) both slips have number 1
(iv) first slip has even number and second has number 2

(g) **Experiment:** From same hat as in (e), select one slip, then select another without replacing first slip.

**Events:**
(i) first slip has number 1
(ii) second slip has number 1
(iii) both slips have number 1
(iv) first slip has even number and second has number 2

(h) **Experiment:** Survey family with three children and record the sexes of the children in order of age, oldest child first.
2. For each part of the preceding problem, state how many members are in the universe set $U$ and how many are in each of the events described in the problem.

3. Consider the experiment in Problem 1, part (b). How many events (subsets of $U$) are there all together? List these events.

4. Consider the experiment in Problem 1, part (e). How many events are there all together?

5. Consider the experiment in Problem 1, part (h). How many events are there all together? How many events are there that contain no members? Exactly one member? Can you think of a way to count the number of events that contain exactly two members?

67-4 Combination of Events

In this section, we wish to revise what you have learned about sets, about picturing sets and subsets, and about operations on sets. But we shall now introduce examples and language that are part of the theory of probability.

First let us illustrate by means of a picture that the event $B$: "family has exactly one son" is a subset of the set

$$U = \{ \text{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG} \}.$$  

This set $U$ is the universe set for the experiment in which we survey families with three children and write the sex of each child, oldest child first.

You will recall that we can picture the universe set by a rectangle and think of all points inside the rectangle as members of the set $U$. Then we can picture the members of the event $B$ by the set of points within a circle labeled $E$, as in Diagram (a).
Event $E$ is the subset $\{\text{BGG, GBG, GGB}\}$ containing three of the eight members of the universe set $U$. The circle representing event $E$ therefore lies entirely within the rectangle representing $U$.

Someone may be interested in the event "not-$E$," that is, the event that the family does not have exactly one son. This new event contains all of the members of the universe set $U$ that are not in event $E$. So we find

$$\text{not-}E = \{\text{BBB, BBG, BGB, GBB, GGG}\}.$$  

We have pictured this new event in Diagram (b). The points within the rectangle but outside the circle represent the members of not-$E$. Do you see that each member of the universe set $U$ is in either $E$ or in not-$E$ and that no member can be in both $E$ and not-$E$? We have called two sets that have no members in common disjoint sets. So event $E$ and event not-$E$ are disjoint events.

Now suppose we think of a new event $F$ that the family's oldest child is a girl. Then $F = \{\text{GBB, GBG, GGB, GGG}\}$ and we can picture the universe set $U$ with event $E$ and with event $F$ as in Diagram (c).
Do you see why the circles for events E and F overlap? Event E and event F are not disjoint events for they have common members. In fact, E and F both have GBG, GGB as members. This is indicated by the overlapping circles in Diagram (c).

Do you recall that the set of members common to two sets is called their intersection? The event \{ GBG, GGB \} is the set of members common to both event E and event F. Hence \{ GBG, GGB \} is the intersection of event E and event F. We shall write this new event as $E \text{ and } F$. So \{ GBG, GGB \} = $E \text{ and } F$ and is pictured in Diagram (c) as the little section with both vertical lines (belonging to event E) and horizontal lines (belonging to event F).

We now have introduced two ways of combining events to get new events. One way is to take an event E and form the new event $\overline{E}$. Another way is to take the event $E$ and the event $F$ and form the new event $E \text{ and } F$. Let us now consider one other way of combining two events to get another event.

What are the members that belong to event $E \text{ or } F$? We find that BGG, GBG, GGB, GGG are the members in either event $E$ or event $F$ or in both. Do you recall that this new set is called the union of set $E$ and set $F$? We shall write this new event as $E \text{ or } F$. So

$$E \text{ or } F = \{ \text{BGG, GBG, GGB, GBB, GGG} \}.$$ 

It may help you understand this use of the word "or" to remember that a member in $E \text{ or } F$ is in at least one of the two sets $E$, $F$. We can picture the union as in Diagram (d):

![Diagram (d)](image)

We could also refer to Diagram (c) and note that the event $E \text{ or } F$ is pictured by the region that has either horizontal lines or vertical lines or both.

Let us consider another example to see how we can combine events to obtain new events of interest. We select a slip from a hat containing ten slips of paper with the numbers 1, 2, 3, …, 10 on them. The universe set is

$$U = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}.$$ 

Now consider these two events (subsets of $U$):

$E = \{ 1, 3, 5, 7, 9 \}$ = the event "number selected is odd"

$F = \{ 7, 8, 9, 10 \}$ = the event "number selected is greater than six".
Here are some new events we have learned how to form from the given events E, F and the universe set U:

\[ \text{not-E} = \{ 2, 4, 6, 8, 10 \} \text{ the event "number selected is not odd" or equivalently, "number selected is even."} \]

\[ \text{not-F} = \{ 1, 2, 3, 4, 5, 6 \} \text{ the event "number selected is not greater than six" or equivalently, "number selected is at most six."} \]

\[ E \text{ and } F = \{ 7, 9 \} \text{ the event "number selected is odd and is greater than six".} \]

\[ E \text{ or } F = \{ 1, 3, 5, 7, 8, 9, 10 \} \text{ the event "number selected is odd or is greater than six".} \]

There are still other events that can be formed. For example,

\[ \text{not-E and not-F} = \{ 2, 4, 6, 8, 10 \} \text{ and} \]

\[ \{ 1, 2, 3, 4, 5, 6 \} = \{ 2, 4, 6 \} \text{ event "number selected is neither odd nor greater than six".} \]

\[ \text{not-(E or F)} = \text{not-} \{ 1, 3, 5, 7, 8, 9, 10 \} = \{ 2, 4, 6 \}. \]

We see that \text{not-E and not-F} is the same set as \text{not-(E or F)}. This is true for all events and expresses the fact that in English the following sentences are equivalent:

"Neither E nor F occurs".

"It is not true that at least one of the events E, F occurs".

In the next section we finally arrive at the meaning of the phrase "the probability of an event".

**EXERCISE 67-4**

1. We throw a die in a game of ludo.
   (a) Write the universe set U for this experiment.
   (b) Suppose E is the event "die shows number greater than four" and F is the
event "die shows odd number". Identify the members of event E and of event F and picture these events in a diagram.

(c) Are E and F disjoint events?
(d) Identify the members of \( \text{not-E} \), \( \text{not-F} \), \( E \) and \( F \), and picture these events in diagrams.
(e) Show that the event \( \text{not-} (\text{not-E}) \) is the same as event E.
(f) Show that the event \( \text{not-E} \) and \( \text{not-F} \) is the same event as \( \text{not-} (E \text{ or } F) \).
(g) Show that \( \text{not-E} \text{ or } \text{not-F} \) is the same event as \( \text{not-} (E \text{ and } F) \).

2. We ask a friend to tell us his month of birth.
(a) Write the universe set \( U \) for this experiment.
(b) Suppose \( E \) is the event "friend is born in first six months" and \( F \) is the event "friend is born in month whose name begins with the letter J". Identify the members of event E and of event F and picture these events in a diagram.
(c) Are event E and event F disjoint sets?
(d) Identify the members of \( \text{not-E} \), \( \text{not-F} \), \( E \) and \( F \), \( E \) or \( F \), and picture these events in diagrams.

67-5 First Definition of Probability

If 1,000 lottery tickets are sold and ten have winning numbers, then the ticket we buy has ten chances in 1,000 of being a winning ticket. We can put this differently by recalling that the universe set for the experiment in which we buy one lottery ticket is \( U = \{ 1, 2, 3, \ldots, 1000 \} \) and the event \( E: "\text{our ticket is a winner}" \) is the subset \( \{ W_1, W_2, \ldots, W_{10} \} \) where \( W_1 \) is the first winning number, \( W_2 \) is the second winning number, and so on. Let us introduce "probability of event E" as the technical phrase for the "chance" or "likelihood" of our ticket being a winner. The universe set \( U \) has 1,000 members and the event \( E \) has ten members. So our intuitive answer of ten chances in 1,000 is the ratio of the number of members in \( E \) to the number of members in \( U \). That is,

\[
\text{Probability of Event } E = \frac{\text{Number of members in Event } E}{\text{Number of members in universe set } U} = \frac{10}{1,000} = \frac{1}{100}.
\]

Let us see if equation (\(*)\) gives reasonable results when applied to some other experiments. Suppose we select one slip from ten slips numbered 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. What is the probability of the event \( E: "\text{the number selected is odd}" \)? The universe set \( U \) for this experiment is \( U = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \} \) and the event \( E \) is the subset \( \{ 1, 3, 5, 7, 9 \} \). Applying formula (\*)}, we find

\[
\text{Probability of Event } E = \frac{5}{10} = \frac{1}{2}.
\]
a reasonable answer since there are an equal number of odd and even integers among the
ten slips. And any one slip is as likely to be selected as any other.

As another example, suppose we toss a die in a game of ludo. What is the
probability that we will get a six? Since there are six possibilities, no one of which is
more likely to occur than any other, we feel intuitively that the chance or probability of
getting a six is the ratio 1:6. To use equation (1) we need to recall that the universe
set \( U = \{ 1, 2, 3, 4, 5, 6 \} \) and the event "die falls six" is the subset \( E = \{ 6 \} \). Then

\[
\text{Probability of "six"} = \frac{\text{Number of members in \{6\}}}{\text{Number of members in \{1, 2, 3, 4, 5, 6\}}} = \frac{1}{6} \text{ as expected.}
\]

As a final example, consider the experiment in which we ask two friends their month
of birth. We wish to calculate the probability of Event \( E \) that both friends were born in the
same month. We have seen (refer to Chart 1, page 161) that there are 144 members in the
universe set \( U \) for this experiment and that event \( E \) has 12 members (corresponding to
those boxes in Chart 1 marked with an \( x \)). Intuitively we would guess that no one of the
144 possible outcomes is preferred or more likely than any other. (This would, of course,
not be true if our two friends were twins!) So it seems reasonable to say that there are 12
chances in 144 of both friends having the same birth month. From equation (1), we get

\[
\text{Probability of event} \ E = \frac{12}{144} = \frac{1}{12} \text{ as expected.}
\]

It appears from these examples that equation (1) would be a reasonable definition
for the probability of an event \( E \) provided we feel that no outcome of the experiment
(that is, no member of the universe set \( U \)) is preferred over any other. Under these
circumstances we do indeed adopt equation (1) as our definition. This is the classic
definition of probability given by the French philosopher-mathematician Laplace (1749-
1827), one of the first and most important contributors to the mathematical theory of
probability. In a later section we shall suggest a modification of formula (1) to make it
apply to even those experiments whose outcomes are not equally likely.

**EXERCISE 67-5**

1. Refer to Exercise 67-3, Problem 1, and now find the probability of each event
described in parts (a) - (h). Assume that Definition (1) applies.
Some Properties of Probability

For the time being we assume that Definition (**) applies to our experiments. It is convenient to introduce the following symbols in order to write this definition in more concise form:

(i) For any event $E$, we let $n(E)$ denote the number of members in $E$.

(ii) For any event $E$, we let $P(E)$ denote the probability of event $E$.

Now Definition (**) becomes

$$P(E) = \frac{n(E)}{n(U)}$$

The quantity on the left and the numerator and denominator of the fraction on the right were previously written using English words; all we have done is write the same defining equation using symbols.

Let us now turn to some properties of probability.

Property 1: If $E$ is an impossible event, then $P(E) = 0$.

Proof: If $E$ is impossible, then it is the empty subset of the universe set $U$ of the experiment. Since the empty set has no members, Definition (**) yields

$$P(E) = \frac{n(E)}{n(U)} = 0$$

Property 2: If $E$ is a sure event, then $P(E) = 1$.

Proof: If $E$ is sure, then it is the entire universe set $U$. Therefore

$$P(E) = \frac{n(E)}{n(U)} = \frac{n(U)}{n(U)} = 1$$

Property 3: If $E$ is any event, then $0 \leq P(E) \leq 1$.

Proof: (First, let us be sure that we understand the conclusion. You know that $P(E) < 1$ means that the number $P(E)$ is less than the number 1. We write $P(E) \leq 1$ to mean that the number $P(E)$ is less than or equal to 1. Similarly $0 \leq P(E)$ means that $P(E)$ is greater than or equal to zero. So $0 \leq P(E) \leq 1$ means that $P(E)$ is both greater than or equal to zero and less than or equal to one. In other words, the probability $P(E)$ is represented on the number line by a point that lies within the interval from 0 to 1 or at one of its end points.)

To prove Property 3, we have only to note that the number of members in Event $E$ cannot be negative [in symbols, $0 \leq n(E)$], Also, since $E$ is a subset of the universe set $U$, $n(E)$ cannot be larger than the number of members in $U$ [in symbols, $n(E) \leq n(U)$]. Dividing the first inequality by $n(U)$, a positive number, we get

$$0 = \frac{0}{n(U)} \leq \frac{n(E)}{n(U)} = P(E)$$
or $0 \leq P(E)$. Dividing the second inequality by $n(U)$, we get

$$P(E) = \frac{n(E)}{n(U)} \leq \frac{n(U)}{n(U)} = 1$$

or $P(E) \leq 1$. Putting our results together, we have $0 \leq F(E) \leq 1$ and the proof of Property 3 is complete.

We can think of Property 3 as establishing a probability scale. At one extreme are impossible events with probability zero. At the other extreme are sure events with probability one. All other events have probabilities between zero and one, with the likelihood or chance of an event increasing as its probability moves to the right on the number line from 0 to 1.

Property 4: If $E$ is any event, then $P(not-E) = 1 - P(E)$.

Proof: All members in the universe set $U$ are in either the subset $E$ or the subset $not-E$. Also, you recall that $E$ and $not-E$ are disjoint events (that is, have no common members). It follows that we count all the members in the Universe set $U$ by first counting the members of $E$ and then counting the members of $not-E$. We get

$$n(U) = n(E) + n(not-E).$$

Do you see how this equation is related to Diagram (b) in Section 67-4? Dividing by the nonzero number $n(U)$, we get

$$\frac{n(U)}{n(U)} = \frac{n(E) + n(not-E)}{n(U)} = \frac{n(E)}{n(U)} + \frac{n(not-E)}{n(U)}.\quad (*)$$

Using Definition ($*$), we have

$$1 = P(E) + P(not-E),$$

which is equivalent to

$$P(not-E) = 1 - P(E).$$

This is not a surprising result. We know that if the probability of an event $E$ is $\frac{1}{3}$, let us say, then the probability that $E$ does not occur is given by

$$P(not-E) = 1 - \frac{1}{3} = \frac{2}{3}.\quad (*)$$

Property 5: Let $E$ and $F$ be any events (subsets of the same universe set $U$). Then

$$P(E or F) = P(E) + P(F) - P(E and F).$$

In words, the probability that at least one of the events $E$, $F$ occurs is found by adding
the probability that \( E \) occurs and the probability that \( F \) occurs, and then subtracting the probability that both \( E \) and \( F \) occur.

**Proof:** It is convenient to refer to Diagram (c) in section 67-4. We count all the members of the event \( E \) or \( F \) (the striped region in the Diagram) as follows: First count all the members of set \( E \) (the region with vertical stripes) and obtain \( n(E) \). Next count all the members of set \( F \) (the region with horizontal stripes) and obtain \( n(F) \). But the members belonging to the region with both vertical and horizontal stripes have now been counted twice: once in \( n(E) \) and again in \( n(F) \). So we count all these duplicates and obtain \( n(E \ and \ F) \) since the region with both horizontal and vertical stripes is the event \( E \ and \ F \). From the number \( n(E) + n(F) \) we must subtract this number of duplicates. Therefore all members of the event \( E \) or \( F \) are counted once and only once when we compute \( n(E) + n(F) - n(E \ and \ F) \). Since the number of members in \( E \) or \( F \) is \( n(E \ or \ F) \), we have proved \( n(E \ or \ F) = n(E) + n(F) - n(E \ and \ F) \) and the conclusion of Property 5 results by dividing this equation by \( n(U) \).

Before illustrating how we use these properties of probability, we should stop to mention a corollary of Property 5:

**Property 6:** If \( E \) and \( F \) are disjoint events, then \( P(E \ or \ F) = P(E) + P(F) \).

**Proof:** Since \( E \) and \( F \) are disjoint events by hypothesis, we know that \( E \ and \ F \) is the empty set. Therefore, by Property 1 we get \( P(E \ and \ F) = 0 \). Now we have only to substitute in Property 5 to obtain our result.

Do you see that this result is a special case of Property 5? The probability of the occurrence of at least one of two disjoint events is just the sum of their individual probabilities. You must be sure the events \( E, F \) are disjoint before using Property 6, but Property 5 holds for any two events.

Let us consider two examples that illustrate how some of these properties of probability are used to solve problems.

**Example 1.** If the probability of winning a game is 0.7 and no ties are possible, what is the probability of losing this game?

We let \( E \) be the event that we win the game. We are given \( P(E) = 0.7 \). Do you see that the problem asks us to compute \( P(\text{not-}E) \)? This is so because the event "not-win" is the same as "lose" since the game cannot end in a tie.

From Property 4, we have

\[
P(\text{not-}E) = 1 - P(E)
\]

\[
= 1 - 0.7
\]

\[
= 0.3
\]

We find that our chance of losing the game is 0.3.

**Example 2.** We throw a die in a game of ludo. What is the probability of getting an even number or a number bigger than four?

It is convenient to call \( E \) the event that the die shows an even number and to call \( F \) the event that the die shows a number bigger than four. The problem asks us to compute
P (E or F) and so Property 5 comes to mind. There are 6 members in the universe set U for this experiment. (You should list the members of U.) Also we recall that E = \{ 2, 4, 6 \} and F = \{ 5, 6 \}. Hence E and F are not disjoint: in fact E and F = \{ 6 \}. By counting the members of these sets we find

\[
P(E) = \frac{3}{6}, \quad P(F) = \frac{2}{6}, \quad P(E \text{ and } F) = \frac{1}{6}.
\]

Now it is a simple matter to substitute these values in Property 5 and obtain our answer

\[
P(E \text{ or } F) = \frac{3}{6} + \frac{2}{6} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}.
\]

**EXERCISE 67-6**

1. The probability that your team wins a football game is 0.7. There is only a 0.1 probability of a tie. What is the probability that your team loses?

2. The probability that your friend was born in January, June or July is \(\frac{1}{4}\). What is the probability that he was not born in a month whose name begins with the letter "J"?

3. (a) Suppose event E is an impossible event. What can you say about event \(\text{not-E}\)?
   (b) Suppose event E is a sure event. What can you say about the event \(\text{not-E}\)?

4. We choose one number from among the first 20 counting numbers 1, 2, \ldots, 20. What is the probability that the chosen number is divisible by 6 or by 8? [Hint. Let E be the event that the number is divisible by 6 and let F be the event that the number is divisible by 8. Do you see that we must calculate \(P(E \text{ or } F)\)?]

5. We choose one number from among the first 30 counting numbers 1, 2, \ldots, 30. What is the probability that the chosen number is divisible by 6 or by 8? [Compare with Problem 4]

6. A club with five male and five female foundation members elects two other men and three other women to membership. From the 15 members, we select one person. What is the probability that the person selected is a foundation member or a man?

7. A student takes two examinations, one in Mathematics and one in Physics. He estimates that his probability of passing Mathematics is 0.7, that he will fail Physics with probability 0.4, and that the probability of failing at least one of the examinations is 0.6. What is the probability that he will pass at least one of the examinations?

8. Suppose E and F are two events (subsets of U) and that E is a subset of F.
   (a) Draw a diagram to illustrate this situation.
   (b) Do you see that event F occurs whenever event E occurs?
   (c) Use Definition (\(*\)) to show that \(P(E) \leq P(F)\). In words, we say that if F
occurs whenever \( E \) occurs, then the probability of \( F \) is at least as large as the probability of \( E \).

67-7 A Counting Principle

We have seen that the probability of event \( E \) is the ratio of the number of members in \( E \) to the number of members in the universe set \( U \). That is, in symbols,

\[
P(E) = \frac{n(E)}{n(U)}.
\]

To find \( n(E) \) and \( n(U) \) we count the number of members in each of the sets \( E \) and \( U \). When this number is small, we can easily count by listing all the members. But what if the number of members is very large? Then it is no longer practical to list all members: we need another method. In this section we present a fundamental method that is used very often in many counting and probability problems.

Consider first an example. Suppose we can go from city A to city B in three ways (by car, by train, by airplane) and from city B to our home in two ways (by car, by bicycle). In how many ways can we go from city A to our home? We can count the ways with the help of a picture known as a tree-diagram. Starting from a point that represents city A we draw 3 lines, one for each way we can go from A to B. From each of these lines, we can continue in two ways so we draw two lines. The total number of ways of going from A to our home is found to be \( 3 \times 2 = 6 \). This is the total number of "branches" in the tree-diagram. Now suppose you could go from your home to your school in three ways (by walking, by bicycle, by bus). Do you see how to continue the tree-diagram? Do you also see that there are \( 3 \times 2 \times 3 = 18 \) ways of going from A to your school with stops at B and your home? The Tree-diagram now looks like this:
This example illustrates the first part of the following fundamental principle of counting:

(a) If one task can be done in \( A \) different ways and, following this, a second task can be done in \( B \) different ways, then both tasks can be done in the given order in \( A \times B \) different ways.

(b) If one task can be done in \( A \) different ways and following this, a second task can be done in \( B \) different ways and following this, a third task can be done in \( C \) different ways, then all three tasks can be done in the given order in \( A \times B \times C \) different ways.

Can you see the pattern here? What if you had four different tasks to do in order? If the first could be done in \( A \) ways, then the second in \( B \) ways, then the third in \( C \) ways, and finally the fourth in \( D \) ways, do you see that there are \( A \times B \times C \times D \) ways of doing all four tasks in the given order? And this pattern continues for any number of tasks.

We will now use this principle to solve three illustrative problems.

Problem 1. A telephone number consists of four digits, the first of which cannot be zero. How many different telephone numbers are possible? We think of writing a telephone number by writing the first digit, then the second digit, then the third digit, and finally the fourth digit. We can write the first digit (task 1) in 9 different ways since it can be any of the numbers 1, 2, \ldots, 9. Then we can write the second digit (task 2) in 10 different ways, the third digit (task 3) in 10 different ways and the fourth digit (task 4) in 10 different ways. By the fundamental principle of counting, we can do all four tasks in

\[
9 \times 10 \times 10 \times 10 = 9,000
\]

different ways. There are therefore 9,000 possible four-digit telephone numbers.
**Problem 2.** "The World of Mathematics" is a 4-volume set of books. If one places these books on a shelf in a random fashion, what is the probability that they will be in exactly the correct order?

Each outcome of this experiment consists of an arrangement of the books on the shelf. How many such arrangements are there? We can choose any volume for the first book we put on the shelf. Therefore this task can be done in four ways. Having done this, we have only three choices for the second book we put on the shelf. Then there remain two volumes from which we choose the third book on the shelf. Finally, we put the last available volume on the shelf. By the fundamental principle of counting, there are \(4 \times 3 \times 2 \times 1 = 24\) such arrangements. So the universe set \(U\) for this experiment has 24 members and no arrangement is preferred over any other. (This is actually what we mean when we say the books are placed on the shelf in a *random* way.)

If we let \(E\) denote the event that the volumes are in exactly the correct order, then \(n(E) = 1\) since there is only one correct order. Therefore, by our definition of probability,

\[
P(E) = \frac{n(E)}{n(U)} = \frac{1}{24}.
\]

We have only one chance in 24 of placing the four volumes on the shelf in the correct order.

**Problem 3.** You ask three friends to tell you their month of birth. Find the probability that all three have different birth months.

When your friends give you their answers, how many possible answers are there? The first friend can be born in any month so there are 12 ways that he may answer (task 1). There are also 12 ways for your second friend to answer (task 2) and 12 ways for your third friend to answer (task 3). Therefore, the universe set contains \(12 \times 12 \times 12 = 1,728\) members, one for each possible outcome of our experiment. That is, \(n(U) = 1,728\).

Now let \(E\) be the event that all three friends have *different* birth months. How many members of the 1,728 in the universe set are in the subset \(E\)? We shall use the fundamental principle of counting again to arrive at this number. To write an outcome of our experiment that belongs to \(E\) we have 12 choices for the first friend’s birth month (task 1), 11 choices (Why?) for the second friend’s birth month, and 10 choices (Why?) for the third friend’s birth month. Therefore, there are \(12 \times 11 \times 10 = 1,320\) members in event \(E\). That is, \(n(E) = 1,320\).

By our definition of probability, we find

\[
P(E) = \frac{n(E)}{n(U)} = \frac{1,320}{1,728} = \frac{55 \times 24}{72 \times 24} = \frac{55}{72}.
\]

Suppose we had asked for the probability that *at least* two of your three friends had the *same* birth month. Do you see that we then want the probability of the event
not-E? For to say that all three of your friends do not have different birth months is equivalent to saying that at least two have the same birth month. Therefore, the probability that at least two of your three friends have the same birth month is

\[
P(\text{not-E}) = 1 - P(E)
\]

\[
= 1 - \frac{55}{72} = \frac{17}{72} = 0.24, \text{ approximately.}
\]

The same method could be used to find the probability that at least two of your friends have the same birth month when you ask four friends, five friends, and so on. The table below gives the results of such calculations.

<table>
<thead>
<tr>
<th>Number of Friends</th>
<th>Probability that at Least Two Have Same Birth Month</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(\frac{1}{12} = 0.08 ) (to two decimal places)</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{17}{72} = 0.24)</td>
</tr>
<tr>
<td>4</td>
<td>(\frac{123}{288} = 0.42)</td>
</tr>
<tr>
<td>5</td>
<td>(\frac{89}{144} = 0.62)</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>1</td>
</tr>
</tbody>
</table>

It appears that we have about a 62% chance of finding at least two people with the same birth month whenever there are five people in the room. Of course, in using our definition of probability you will recall that we need to assume that all outcomes are equally likely. In our example, this assumption means that births are equally distributed among the 12 calendar months. Although statistics of births indicate that this assumption is not completely true, it is close enough to make our calculations reasonably accurate.

**EXERCISE 67-7**

1. A school club with 15 members has to select a President, a Secretary, and a Treasurer. In how many different ways can this selection be made?
2. A restaurant offers two choices of soup, two different fish dishes, four different meat dishes, and three desserts. How many different meals consisting of soup, fish, meat, and dessert are there? Draw a tree-diagram to illustrate your answer.

3. How many three-digit numbers can be formed using the digits 1, 2, 3, 4, 5, if
   (a) no digit can be repeated?
   (b) digits can be repeated?
   (c) the number must begin with an even number?
   (d) the number must be an even number?

4. In how many different ways can a coach assign Alli, Kizza, Kofi and Sozi to positions on a football team so they can all play at the same time?

5. A three-digit number is selected at random. (That is, each of the numbers 100, 101, ..., 999 is equally likely.) Find the probability that the number selected
   (a) begins with an even number.
   (b) is an even number.
   (c) begins and ends with an even number.

6. A shilling coin is tossed five times.
   (a) How many different outcomes are there for this experiment?
   (b) Assuming all outcomes are equally likely, find the probability that you get exactly one head among the five tosses.

7. In the table, the probability that at least two people have the same birth month when there are five people in a room is listed as 0.89. Verify that this value is correct.

---

67-8 Another Look at the Definition of \( P(E) \)

Let us briefly revise what we have learned so far. An experiment is described mathematically by a universe set \( U \). The members of \( U \) represent all the possible outcomes of the experiment. We have seen that events are subsets of the universe set \( U \). If \( E \) is any event and if we assume that the outcomes of the experiment are all equally likely, then the probability of event \( E \), denoted by \( P(E) \), is defined by the equation

\[
P(E) = \frac{n(E)}{n(U)},
\]

where \( n(E) \) is the number of members in event \( E \) and \( n(U) \) is the number of members in the universe set \( U \). We have proved some properties of probability and we have also studied the fundamental principle of counting to help us determine the numbers \( n(E) \) and \( n(U) \) in some problems where it is not easy or not possible simply to list the members of \( E \) and \( U \).

Let us look more closely at what we mean by \( n(E) \) and \( n(U) \). When we count the members of a set, we give each member a weight of 1 unit and then add the weights of all the members of the set. For example, if \( n(U) = 4 \), we give each member of \( U \) the weight of 1 and then add the weights. Since there are four ones, we get the expected
result:

\[ n(U) = 1 + 1 + 1 + 1 = 4. \]

So our definition of probability can be rewritten in the following equivalent form: If \( E \) is any event and if each member of the universe set is assigned the same weight of 1, then

\[
P(E) = \frac{\text{sum of weights of members in } E}{\text{sum of weights of members in } U}.
\]

Do you see that this is merely another way of writing our definition? This is so because the numerator equals \( n(E) \) and the denominator equals \( n(U) \).

Now suppose that we think of an illustrative example in which \( n(U) = 4 \) and \( n(E) = 3 \). Then \( P(E) = \frac{3}{4} \) and we know that we can arrive at this answer by giving each member of \( U \) the same weight of 1 unit. (Of course, each member of \( E \) also gets weight 1 since it is necessarily a member of \( U \).) Then we add weights to get

\[
P(E) = \frac{1 + 1 + 1}{1 + 1 + 1 + 1} = \frac{3}{4}.
\]

What would happen if we gave each member of \( U \) the same weight, but changed it from 1 unit to 2 units? Then

\[
P(E) = \frac{2 + 2 + 2}{2 + 2 + 2 + 2} = \frac{6}{8} = \frac{3}{4},
\]

the same answer as before. Suppose we gave each member of \( U \) the weight \( \frac{1}{3} \). Then

\[
P(E) = \frac{1 + 1 + 1}{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = \frac{3}{\frac{4}{3}} = \frac{3}{4},
\]

again the same answer. Before reading further try assigning some other weight to each member of \( U \) and check to see whether you get \( P(E) = \frac{3}{4} \), as before.

That we will always get the same answer, no matter what weight we assign to each member of \( U \), follows from what you know about fractions. Do you recall that

\[
\frac{n(E)}{n(U)} = \frac{w \times n(E)}{w \times n(U)}
\]

for any nonzero number \( w \)? This means that assigning weight 1 or any other weight, say
w, to each member of \( U \) keeps the ratio or fraction

\[
\frac{\text{sum of weights of members in } E}{\text{sum of weights of members in } U}
\]

constant. In our example, where \( P(E) = \frac{3}{4} \), we could get this answer using weight 1 as before, or using any nonzero weight \( w \) for each member of \( U \). This is so because

\[
\frac{1 + 1 + 1}{1 + 1 + 1 + 1} = \frac{3}{4},
\]

and also

\[
\frac{w + w + w}{w + w + w + w} = \frac{3 \times w}{4 \times w} = \frac{3}{4}.
\]

Since our answer for \( P(E) \) is the same no matter what weight we assign to each member of \( E \), we may as well use a weight that makes everything as simple as possible. First of all, we shall avoid negative weights. If we choose our weight so that it is nonnegative and so that the sum of the weights of the members in \( U \) is 1, then the fraction

\[
P(E) = \frac{\text{sum of weights of members in } E}{\text{sum of weights of members in } U} = \frac{\text{sum of weights of members in } E}{1}
\]

simplifies since its denominator is 1. We then get

\[
P(E) = \text{sum of weights of members in } E.
\]

In our example, where \( n(U) = 4 \), we would assign weight \( \frac{1}{4} \) to each of the four members of \( U \). This assignment meets the above conditions since \( \frac{1}{4} \) is a nonnegative weight and the sum of the weights of all four members in \( U \) is indeed 1. If \( n(E) = 3 \), we would have

\[
P(E) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4},
\]

as before.

Let us summarize what we have learned in this section. An experiment is described in terms of a universe set \( U \) with \( n(U) \) members. Suppose an event \( E \) (subset of \( U \)) has \( n(E) \) members and we wish to determine \( P(E) \), the probability of event \( E \). We are still assuming that the outcomes of the experiment (members of \( U \)) are all equally likely. We then assign a nonnegative weight to each member of \( U \) so that the sum of all the weights is 1. Then our new definition is

\[
P(E) = \text{sum of weights of members in } E.
\]
Of course, if there are \( n(U) \) members in \( U \) and each must get the \textit{same} weight then the requirement that the weights sum to 1 forces us to choose the fraction \( \frac{1}{n(U)} \) as the weight assigned to each member of \( U \).

You should understand that this is just another way of writing and thinking about our previous definition. Both definitions will give the same answer for \( P(E) \) in any problem. We can see this by realizing that the first definition (in Section 67-5) asked us to compute

\[
P(E) = \frac{n(E)}{n(U)}.\]

We know from the rules for multiplication of fractions that this is equivalent to

\[
P(E) = n(E) \times \frac{1}{n(U)}
\]

But now we apply our understanding of the meaning of multiplication. Multiplying a number (in this case \( \frac{1}{n(U)} \)) by the positive integer \( n(E) \) is equivalent to adding that number \( n(E) \) times. So we can write

\[
P(E) = \frac{1}{n(U)} + \frac{1}{n(U)} + \cdots + \frac{1}{n(U)}.
\]

\( n(E) \text{ terms in sum} \)

But this last expression is what the new definition asks us to compute since \( \frac{1}{n(U)} \) is the weight assigned to each member of \( U \) and since there are \( n(E) \) terms when one sums the weights of members in event \( E \). So we see that the two definitions are equivalent. The reason for preferring the new definition is that it can easily be modified to apply to experiments whose outcomes are \textit{not} equally likely. We shall make this modification in the next section. But now let us see how a problem is done using our new definition.

\textit{Example:} We throw a die in a game of ludo. What is the probability of event \( E \): "even number shows on die"?

The universe set \( U = \{1, 2, 3, 4, 5, 6\} \) has \( n(U) = 6 \) members. Since we are assuming that all six are equally likely outcomes, we assign weight \( \frac{1}{6} \) to each. This weight is nonnegative and the sum of all the weights in \( U \) is indeed 1. Now event \( E = \{2, 4, 6\} \) so adding the weights of members in \( E \), we arrive at our answer:
EXERCISE 67-8

1. Suppose that the outcomes of an experiment are equally likely. What weight would you assign to each member of the universe set $U$ if (a) $n(U) = 2$ ? (b) $n(U) = 36$ ? (c) $n(U) = 981$ ?

2. Use the new definition of probability to find $P(E)$. (First write the universe set $U$, then assign weight to each member of $U$, and then determine $P(E)$.)
   (a) We toss a coin and then toss it again. Let $E$ be the event that we get one head and one tail.
   (b) We buy one lottery ticket. Assume 1000 tickets are sold and that there are ten winning tickets. Let $E$ be the event that we win.
   (c) We ask two friends to tell us their birth month. Let $E$ be the event that they are born in the same month.

3. Return to the experiments described in Examples 1-8 in the text of Section 67-2. For which of these does the assumption that the outcomes of the experiment are equally likely seem reasonable and for which does the assumption seem wrong?

4. Same question as in preceding problem but for the experiments described in Exercise 67-2, Problem 1 (a)-(m).

67-9 A General Definition of $P(E)$

As you have seen in the last two problems, there are experiments in which our assumption that all outcomes are equally likely is not reasonable. We must generalize our definition of the probability of an event to such experiments. The way of defining $P(E)$ that was discussed in the preceding section will be most helpful.

Suppose that you throw a drawing-pin on a flat surface and watch to see if it falls flat or on its side. You will recall that we did such an experiment and found (for our drawing-pin) that about $\frac{1}{4}$ of the tosses resulted in the pin falling flat. So the two outcomes of the experiment are not equally likely. It would therefore be unreasonable to assign the same weight to the two members of the universe set $U = \{ \text{falls flat, falls on side} \}$ for this experiment. If we wish to keep our requirement that the weights be nonnegative and that the sum of the weights of all members in $U$ be 1, but are willing to drop our previous requirement that the weights all be equal, then what weights should we choose?

We are helped to answer this question by our experimental results. It seems from

$$P(E) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$$

$$= \frac{3}{6} = \frac{1}{2}$$
these that the weight assigned to the outcome “falls flat” should be $\frac{1}{4}$ and the weight assigned to the outcome “falls on side” should be $\frac{3}{4}$. (Look back at the results of your drawing-pin tossing experiment (Exercise 67-1, Problem 3) and determine what weights you would assign to these two outcomes.)

Consider another example. Suppose you ask a person to name a prime number between 10 and 30. Let us also assume that among the people in your area, it is twice as likely for a person to choose the smallest prime 13 as to choose any one of the other available primes 17, 19, 23, 29. What weights would you assign to the members of the universe set $U = \{13, 17, 19, 23, 29\}$? Remember that we want nonnegative weights whose sum is 1. So we can proceed as follows: let $w$ be the (unknown) weight assigned to the outcome 29. Then $w$ is also the weight assigned to each of the outcomes 17, 19, and 23. And since the prime 13 is twice as likely to be chosen as each of the others, the weight assigned to outcome 13 is $2w$. The sum of all weights is then

$$2w + w + w + w + w$$

which equals $6w$. But this sum must be 1. Therefore

$$6w = 1$$

and $w = \frac{1}{6}$. So the weights of the outcomes 13, 17, 19, 23, 29 are $\frac{2}{6}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{1}{6}$, $\frac{1}{6}$ respectively.

Now that we have learned that the outcomes of an experiment do not have to be assigned equal weights we can return to our definition of probability and formulate it for any experiment.

An experiment is described in terms of a universe set $U$. We wish to determine $P(E)$, the probability of event $E$. We assign to each member of $U$ a nonnegative number (called the weight) so that the sum of all weights is 1. Then we define

$$P(E) = \text{sum of weights of members in event } E.$$ 

Notice that we have dropped the requirement that the weights must all be equal. That is now only one of many acceptable sets of weights for the members of $U$.

Do you know why we kept the requirements that each weight must be nonnegative
and the sum of all weights must be 1? The first requirement is necessary because we want \( P(E) \) to be nonnegative for every event \( E \). The second requirement guarantees that our probability scale will be restricted to the number line from 0 to 1, inclusive, since it makes the probability of a sure event equal to 1. In this way we automatically carry over to the general situation properties 1-3 of probability which were proved in Section 67-6 only for experiments with equally likely outcomes.

In fact, all of the properties of probability proved in Section 67-6 still hold true with our new definition. To see this we have only to re-do the proofs given in that section, being careful to replace "number" of members in event \( E \) by "sum of weights" of members in Event \( E \) and to use our new general definition. We illustrate by paraphrasing the proof of Property 4. You should carefully compare this proof with that given in Section 67-6.

**Property 4:** If \( E \) is any event, then \( P(\text{not-}E) = 1 - P(E) \),

**Proof:** All members in the universe set \( U \) are either in the subset \( E \) or the subset \( \text{not-}E \). Also, you recall that \( E \) and \( \text{not-}E \) are disjoint events (that is, have no common members.) It follows that we get the sum of the weights of all members in \( U \) (which is \( P(U) \) by our general definition) by first taking the sum of the weights of members in \( E \) (which is \( P(E) \) by our definition) and then the sum of the weights of members in \( \text{not-}E \) (which is \( P(\text{not-}E) \) by our definition.) We have in this way shown that

\[
P(U) = P(E) + P(\text{not-}E).
\]

But \( P(U) = 1 \), as we have already learned. Hence

\[
1 = P(E) + P(\text{not-}E)
\]

which is equivalent to the equation to be proved.

From now on we shall use our new definition of probability since it applies to all experiments with a finite number of possible outcomes, whether these outcomes are equally likely or not.

**EXERCISE 67-9**

1. An experiment has universe set \( \{ a, b, c, d \} \). Weights are assigned to the members of \( U \) to satisfy the following equations:

   - weight of \( a \) = weight of \( b \)
   - weight of \( c \) = weight of \( d \)
   - weight of \( d \) = 3 \times \) weight of \( a \)

   Find the weights and then determine the probability of event \( E = \{ a, c \} \).

2. A die is thrown so that we have universe set \( U = \{ 1, 2, 3, 4, 5, 6 \} \).
Suppose we assign weights in such a way that the weight of any member of $U$ is proportional to its value. That is, the weight of 1 is $k \times 1$, the weight of 2 is $k \times 2$, the weight of 3 is $k \times 3$, and so on, where $k$ is the constant of proportionality. Find the weights of each member in $U$ and then determine the probability that an odd number turns up when this loaded die is tossed.

3. Discuss how you might go about assigning weights to the members of the universe set $U$ for each of the following experiments.

(a) You play a game of draughts with your friend.

$$U = \{\text{you win, you lose, tie}\}$$

(b) An insurance company writes a 1-yr. policy for a man aged 20.

$$U = \{\text{man survives 1 yr., man dies within year}\}$$

(c) You ask twins for their month of birth. Take as universe set the set $U$ with 144 members as given in Chart 1, page 161.

(d) A person is selected from the population and asked the question, "Do you think that there will be another world war?" Let the universe set $U$ consist of three members corresponding to the possible answers "yes", "no", and "don't know". Suppose that you are told that 20% of the population expect another world war, 70% do not expect another world war and 10% are uncertain.

We give two solutions to the following problem: two coins are tossed. Let $U = \{HH, HT, TH, TT\}$ be the universe set for this experiment. The problem is to find the probability for the event $E$ that at least one head occurs.

**Solution 1**

Assign weight $\frac{1}{4}$ to each member of $U$. The event $E = \{HH, HT, TH\}$. Applying the definition of $P(E)$, we add the weights of the members in $E$ to get

$$P(E) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}.$$  

**Solution 2**

Assign weight $\frac{1}{2}$ to HH and to HT. Assign weight zero to TH and to TT. (Do you see that this is an assignment of weights that is permitted by our definition? This is so because each weight is nonnegative and the sum of all four weights is 1.) The event $E$ is still the subset $\{HH, HT, TH\}$. Applying the definition of $P(E)$, we add the weights of the members in $E$ to get

$$P(E) = \frac{1}{2} + \frac{1}{2} + 0 = 1.$$  

Explain the different answers obtained for $P(E)$ in these two solutions. Can both be correct solutions?
Suppose that an experiment is performed and we have a universe set $U$ with weights assigned to its members. You recall that the probability of event $E$ is then given by

\[(*) \quad P(E) = \text{sum of the weights of members in } E.\]

Now suppose that we are given the additional information that another event, say $F$, has occurred. How does this news affect the likelihood of event $E$?

Let us first consider an example. Our experiment is selecting one slip from a hat containing ten slips numbered 1, 2, \ldots, 10. Then the universe set is $U = \{1, 2, \ldots, 10\}$ and let us suppose that each member of $U$ is assigned the weight $\frac{1}{10}$, that is, each slip is as likely to be chosen as any other. Let $E$ be the event that the number selected is greater than 5. Then $E = \{6, 7, 8, 9, 10\}$ and we easily find

\[P(E) = 5 \times \frac{1}{10} = \frac{1}{2}.\]

Now suppose that we are told that event $F$: "slip selected has an even number" has occurred. Then we know that the slip selected has one of the numbers 2, 4, 6, 8, 10 on it. No longer is $U$ our universe set: the slips numbered 1, 3, 5, 7, 9 could not have been chosen once we know $F$ has happened. The set of possible outcomes is smaller than $U$ now. The probability of $E$ when we take into consideration the occurrence of event $F$ is called the conditional probability of $E$ given $F$ and is represented by the symbol $P(E|F)$. The vertical bar separates the event $E$ whose probability is to be found from the event $F$ which is given, that is, from the event known to have occurred.

Stop for a moment and try to calculate $P(E|F)$ in our example. How likely is it for the number to be greater than 5 if you know that an even number was selected? It seems reasonable to find $P(E|F)$ by noting there are the five possibilities 2, 4, 6, 8, 10 after the occurrence of event $F$ is taken into account. Of these five, there are three that are greater than 5. Since all slips were assumed to be equally likely, our intuition would have us say that

\[P(E|F) = \frac{3}{5}.\]

Notice that $P(E) = \frac{1}{2}$, but $P(E|F) = \frac{3}{5}$: the information that $F$ has occurred increases the likelihood of event $E$.

Our answer is better written in the equivalent form

\[P(E|F) = \frac{3}{5}.\]

Do you recognize the numerator as $P(E \text{ and } F)$ and the denominator as $P(F)$? It appears that we can get our answer of $\frac{3}{5}$ by using the equation
This equation is easy to picture in a diagram and can be interpreted as follows:
The conditional probability of event $E$ given event $F$ is the ratio of the total weight of those members of $E$ that are also in $F$ to the total weight of $F$. In the diagram below, we have pictured events $E$ (vertical stripes) and $F$ (horizontal stripes), with the event $E$ and $F$ having both vertical and horizontal stripes.

\[
( \star \star ) \quad P(E|F) = \frac{P(E \text{ and } F)}{P(F)}
\]

Before we know that $F$ has occurred, the probability of $E$ is the ratio of the total weight of members in $E$ to the total weight of members in the universe set $U$. But the weights have been chosen to make the total weight of $U$ equal to 1. So this ratio is just another way of thinking about our definition (1). After we are told that event $F$ occurred, then we restrict our attention to the new set of possible outcomes (those in $F$) and take the ratio of the weight of members in $E$ and $F$ to the weight of the new universe set $F$.

We shall take equation $\star \star$ as our definition of the conditional probability of any event $E$, given an event $F$. Of course, since division by zero is not allowed, this conditional probability is not defined if $P(F) = 0$.

Let us consider two more examples.

Example 1. You play a game of draughts with a friend. Suppose the members of the universe set $U =$ \{ you win, you lose, tie \} are assigned weights 0.3, 0.6, 0.1 respectively. Let $E$ be the event that you do not lose. Then $E =$ \{ you win, tie \} and, according to definition $\star$,

\[
P(E) = 0.3 + 0.1 = 0.4.
\]
Now suppose at some point in the game it becomes clear that a tie is no longer possible. What is your revised estimate of the probability of E given event F: "no tie"? The event F = { you win, you lose } so, again using (**),

\[ P(F) = 0.3 + 0.6 = 0.9. \]

To use definition (***) to compute the conditional probability of E given F, we need P(E and F). But E and F = { you win } so P(E and F) = 0.3. Now definition (***) gives

\[ P(E|F) = \frac{0.3}{0.9} = \frac{1}{3}. \]

So here we have an example where information that F has occurred decreases the likelihood of event E from \( \frac{2}{5} \) to \( \frac{1}{3} \).

**Example 2.** We toss a shilling coin and then toss it again. What is the conditional probability of a head on the second toss given that the first toss resulted in heads? We have the universe set

\[ U = \{ HH, HT, TH, TT \} \]

and let us assume that these outcomes are equally likely. Then each gets weight \( \frac{1}{4} \).

Let E be the event "head on second toss" and F the event "head on first toss". We are asked to find P(E|F). Now E = { HH, TH }, F = { HH, HT }, E and F = { HH } so, using definition (**), we get

\[ P(E) = \frac{2}{4}, \quad P(F) = \frac{2}{4}, \quad P(E \text{ and } F) = \frac{1}{4}. \]

Applying our definition of conditional probability, we have the answer

\[ P(E|F) = \frac{\frac{1}{4}}{\frac{2}{4}} = \frac{1}{2}. \]

Notice that in this example, the information that F occurred did not change the likelihood of event E.

This last example illustrates a very important possibility. If P(E|F) = P(E) then the information that F occurred has no effect on the probability of E. In this case, we have

\[ P(E|F) = \frac{P(E \text{ and } F)}{P(F)} = P(E). \]
so that cross-multiplying produces the equation

\[ (*** \text{ } P(E \text{ and } F) = P(E) \times P(F). \]

Two events E and F are said to be independent (sometimes called independent in the probability sense) if and only if equation (*** ) holds, that is, if and only if the probability of the joint occurrence of both E and F is the product of the probability of E by the probability of F.

If we assume that E and F are independent events, then we can show that \( P(E|F) = P(E) \) and \( P(F|E) = P(F) \). Can you carry out a demonstration of these facts? We see that for two independent events, the knowledge that either one has occurred does not change the probability of the other.

The idea of independent events is one of the most important ideas in the mathematical theory of probability. We cannot develop it any further in this brief introduction to the subject. But we shall be using the multiplication rule (*** ) for independent events when we study a decision problem in Section 12.

**EXERCISE 67-10**

1. You toss a die in a game of ludo. Find the conditional probability that you get a six if you are given the information that your number is greater than three.
2. We buy a lottery ticket. Assume that there are 1000 tickets numbered 1, 2, \ldots, 1000.
   (a) What is the probability that our ticket has a number whose first digit is 1?
   (b) What is the conditional probability that our ticket has a number whose first digit is 1 given that the number is greater than 250?
3. We ask our friend to tell us his birth month. Compare the probability that the name of his birth month begins with a "J" before and after you receive the information that he was born in the first six months of the year.
4. We throw a green die and then a red die.
   (a) Find the conditional probability that the red die results in 5, given that the green die resulted in 3.
   (b) The conditional probability that the green die results in 3, given that the red die resulted in 5.
   (c) The conditional probability of obtaining sum 7 given that the green die resulted in a number less than 4.
5. In the preceding problem, are the events "5 on red die" and "3 on green die" independent? Are the events "sum 7" and "number less than 4 on green die" independent?
6. A shilling coin is tossed and then tossed again. Let E be the event "not more than 1 head" and F the event "at least one of each face". Are E and F independent events?
7. Person A has probability 0.9 of surviving one year and person B has probability 0.8 of surviving that year. Assume that the events "A survives year" and "B survives year" are independent. Find the probability that
(a) A and B both survive the year.
(b) A survives the year, but B does not. Do you know what assumption you had to make to solve this part (b)?

8. An election with three candidates A, B, C is about to be held. A’s probability of winning is 0.6, B’s probability is 0.3, and C’s probability is 0.1. Candidate C suddenly withdraws. Find the conditional probability for each of A and B to win the election, given this new information.

67-11 Mean Value

When we perform an experiment, we often are interested not in the particular outcome that occurs, but rather in some number associated with that outcome. For example, if we toss a coin 50 times we may be interested in the number of heads obtained and not in the particular sequence of heads and tails recorded as they occurred; in a survey of families, we may be more interested in the total family income than in the particular makeup of the family; in selecting a sample of students in a college, we may want to study the proportion of students studying mathematics; and so on.

As an example, suppose that we toss a fair die and are offered one shilling if we get a 1, 2, or 3, two shillings if we get a 4 or 5, and three shillings if we get a 6. If we let X stand for the number of shillings we get, then we cannot say in advance what value X will have. All we can say is that the value of X is 1, 2, or 3 and we can determine the probability of each of these values. For example, we get one shilling with probability 1/2 since 1/2 is the probability that the die shows a 1, 2, or 3. In this way, we complete the following table:

<table>
<thead>
<tr>
<th>Value of X</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability of this value</td>
<td>1/2</td>
<td>1/3</td>
<td>1/6</td>
</tr>
</tbody>
</table>

As another example, consider a survey of families with two children so that \( U = \{BB, BG, GB, GG\} \) is the universe set. Let us give each member of U the same weight of \( \frac{1}{4} \). Let X stand for the number of boys in the family. Again we cannot be sure of the value of X before taking the survey. What we do know is summarized in the following table:
You will recall from your study of statistical data that when we are given a set of numbers it is often useful to have a measure of their central tendency, their average. The most important number used for this purpose is the arithmetic mean of the numbers.

We can also define and make use of the arithmetic mean of the possible values of X in the above examples. This mean value is also called the mathematical expectation of X or just the mean of X and the symbol E(X) is used for this value.

The mean of X is defined in the following way: one first forms the product of each possible value of X by the probability of this value. Then one adds all these products.

In our first example, E(X) would be the mean number of shillings we get. Following the definition, we find

\[
E(X) = 1 \times \frac{1}{2} + 2 \times \frac{1}{3} + 3 \times \frac{1}{6}
\]

\[
= \frac{1}{2} + \frac{2}{3} + \frac{3}{6}
\]

\[
= 1 \frac{2}{3} \text{ shillings.}
\]

In the second example, \(E(X) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4}\)

\[
= 0 + \frac{1}{2} + \frac{1}{2}
\]

\[
= 1 \text{ boy.}
\]

There is much more that we can learn about E(X) and its interpretation. This too, like the topic of independent events in the preceding section, forms an important chapter in probability theory. We have merely introduced you to these ideas, discussing only those fundamentals needed in the next section.

**EXERCISE 67-11**

1. A thousand tickets are sold in a lottery in which there is one top winner of £50, four winners of £10 each and five winners of £5 each. A ticket costs £1. Let X stand for your net gain when you buy one ticket. (It is understood that a negative gain means a loss.) Find E(X), the mean net gain (in £'s).
2. Let $X$ stand for the number showing when you throw a die in a ludo game. Find $E(X)$.

3. One slip is selected from a hat containing ten slips numbered 1, 2, ..., 10. Let $X$ stand for the number obtained. Find $E(X)$.

4. A student must guess on two questions in a multiple choice examination. Each question has three possible answers (a), (b), (c), only one of which is correct. The student answers both questions with (a). Let $X$ stand for the number of correct answers the student gets. Find $E(X)$.

67-12 A Decision Problem

You wish to buy a camera in a shop. The shop owner has the camera you want available in two models: A and B. Model A costs 200 shillings, model B 250 shillings. Model A cameras are not quite as carefully inspected at the factory and so there is a chance that a camera of model A will be defective (perhaps with a faulty shutter or a scratched lens). Let $p$ be this probability that a model A camera is defective. Model B cameras differ from model A only in their careful testing at the factory. They cost more, but when you buy a model B camera you are certain that it will not be defective.

If you buy the less expensive camera, you cannot return it to the shop if it turns out to be defective. You must spend 80 shillings to have it repaired and it is then guaranteed to be perfect.

A camera can be tested before you decide to purchase it. The test is foolproof and you will definitely learn whether the camera is defective or not. But this test costs 20 shillings to perform.

What would you do in this situation? Would you buy a camera of model A and take a chance on it not being a defective? Would you spend the 20 shillings to test a camera of model A before deciding whether to buy it? Let us consider a number of possible plans that you might follow.

Plan 1 Buy the more expensive model B camera.

Plan 2 Buy the less expensive model A camera without testing it. (You then take the chance of having to spend an additional 80 shillings to repair it if it turns out to be defective.)

Plan 3 Select a model A camera and test it. If it proves to be good, buy it. But if it proves to be defective, then buy a model B camera.

Plan 4 Select a model A camera and test it. If it proves to be good, buy it. But if it proves to be defective, then select another model A camera and test it. If this second camera is good, then buy it. But if this second one is also defective, then buy a model B camera.

How shall we determine which of these plans is better? First we have to understand what "better" means. Because of our uncertainty about the quality of model A camera, we cannot be sure of our cost unless we adopt plan 1. For plan 1 we know for sure that our total cost is 250 shillings. Let us compute for each of the plans the possible values of our cost and the probability with which each value occurs. Then we can calculate the
average or mean cost for each of these plans. We shall be interested in making the mean cost as small as possible. Let us carry out these computations and then we shall be in a better position to understand what is meant by one plan being “better” than another. And perhaps we can then determine the “best” plan.

Plan 1 We have already remarked that the only possible value of the cost is 250 shillings. So this value occurs with probability 1 and our definition of mean value gives

\[ \text{mean cost} = 250 \times 1 = 250 \]

Plan 2 Our cost will be 200 shillings if the camera is not defective, but will be \((200 + 80) = 280\) shillings if we have to repair a defective camera. The probability of a defective camera is \(p\) and the probability that the camera is not defective is therefore \(1-p\). So we have the following table:

<table>
<thead>
<tr>
<th>possible value of cost</th>
<th>200</th>
<th>280</th>
</tr>
</thead>
<tbody>
<tr>
<td>probability of this value</td>
<td>1-p</td>
<td>(p)</td>
</tr>
</tbody>
</table>

Our definition of mean value now gives

\[ \text{mean cost} = 200 \times (1-p) + 280 \times p \]

which is equivalent to

\[ \text{mean cost} = 80p + 200 \]

(Can you justify this last step by using the distributive and associative properties?)

Plan 3 Our cost will be \((20 + 200)\) shillings if the model A camera passes the test, but will be \((20 + 250)\) shillings if it proves to be defective. So we have the following table:

<table>
<thead>
<tr>
<th>possible value of cost</th>
<th>220</th>
<th>270</th>
</tr>
</thead>
<tbody>
<tr>
<td>probability of this value</td>
<td>1-p</td>
<td>(p)</td>
</tr>
</tbody>
</table>

By our definition of mean value:

\[ \text{mean cost} = 220 \times (1-p) + 270 \times p \]

which is equivalent to

\[ \text{mean cost} = 50p + 220 \]

Plan 4 Our cost will be \((20 + 200)\) shillings if the model A camera passes the
test. It will be \((20 + 20 + 200)\) shillings if the first camera tested is defective \textit{and} the second one tested is good. And the cost will be \((20 + 20 + 250)\) shillings if \textit{both} model A cameras fail the test. Let us assume that the test results for two different cameras are independent. Then the events "first camera defective" and "second camera defective" are independent and so the multiplication rule (*** ) of Section 67-10 applies. The probability that both cameras tested are defective is therefore \(p \times p\) or \(p^2\).

Similarly, the events "first camera defective" and "second camera good" are independent and so the probability of the first camera being defective \textit{and} the second camera being good is \(p \times (1 - p)\) or \((p - p^2)\). So we have the entries of our table for plan 4:

<table>
<thead>
<tr>
<th>possible value of cost</th>
<th>220</th>
<th>240</th>
<th>290</th>
</tr>
</thead>
<tbody>
<tr>
<td>probability of this value</td>
<td>1-(p)</td>
<td>(p-p^2)</td>
<td>(p^2)</td>
</tr>
</tbody>
</table>

(As we check on our work, we see that these probabilities sum to 1, as they should.) By our definition of mean value:

\[
\text{mean cost} = 220 \times (1 - p) + 240 \times (p - p^2) + 290 \times p^2
\]

which is equivalent to

\[
\text{mean cost} = 50p^2 + 20p + 220
\]

or

\[
\text{mean cost} = 10(5p^2 + 2p + 22)
\]

(Can you justify the steps in this simplification of the expression for the mean cost?)

Now we notice that the mean cost for each of plans 2, 3, and 4 is a function whose value depends on the unknown probability \(p\) that a camera of model A is defective. \textit{If we knew} the value of \(p\), then we could simply calculate the mean cost for each plan and then choose the plan which produces the lowest mean cost. For example, if \(p = 0.1\), then the mean cost is:

- for Plan 1: 250 shillings
- for Plan 2: 208 shillings
- for Plan 3: 225 shillings
- for Plan 4: 222.5 shillings,

so we would adopt plan 2 if we knew that \(p = 0.1\). On the other hand, if \(p = 0.9\) (an unlikely event since a manufacturer who produces 90% defectives would not survive in business very long!) then the mean cost is

- for Plan 1: 250 shillings
- for Plan 2: 272 shillings
- for Plan 3: 265 shillings
- for Plan 4: 278.5 shillings,
so we would now choose plan 1.

Our choice of plan becomes clearer if we draw on the same piece of graph paper the graphs showing the dependence of the mean cost for each plan as a function of the unknown probability $p$. (See Figure.) From the figure it is easy to see that Plan 4 always (that is, for every value of $p$) leads to a higher mean cost than Plan 2. We would therefore never choose to adopt Plan 4. But among plans 1, 2, and 3 there is no one graph that is always below the other. For values of $p$ from $p = 0$ to $p = \frac{5}{8}$, we see that plan 2 is best since its mean cost curve is lowest. But for $p$ from $p = \frac{5}{8}$ to $p = 1$, plan 1 is best. (You should verify that the graphs for plans 1 and 2 intersect when $p = \frac{5}{8}$.)

We could go on to discuss this problem further. But this would lead us into statistical decision theory. Instead we close this unit on probability with the hope that your interest has been aroused and that you will study other books and learn even more about probability and its applications.
Chapter 68
PRIME NUMBERS, PRIME FACTORIZATION
AND ELEMENTARY NUMBER THEORY

68-1 Introduction

This unit is supplementary to Volumes I and II. Its purpose is to give an introduction to the Theory of Numbers. By "numbers" we shall here mean "whole numbers". No other kinds of numbers will be considered. This work should generate an interest in primes, divisibility, least common multiples and greatest common factors. It gives an opportunity to explore and experiment in a fascinating field of study.

68-2 Prime Numbers

As you are aware, our system of writing numerals is based on place value. But numerals can also be written in many different ways. Thus 15, which is 1 ten and 5 units, represents the same number as 30 ÷ 2, 19 − 4, 7 + 8, 3 × 5, etc., etc. In this chapter we shall be concerned chiefly with the representation of numbers in factored form, that is, numbers written as products. When 15, for example, is written as 3 × 5, we say that

3 and 5 are factors of 15.
15 is the product of 3 and 5.
15 is a multiple of 3.
15 is a multiple of 5.

Note also that 15 = 5 × 3. However, the factored form 5 × 3 will be regarded as essentially the same as 3 × 5 because we are concerned with the factors 3 and 5 and not with the order in which they occur.

It should be clear that every whole number can be written as the product of itself and 1. What property makes this possible? On the other hand, if we examine the numbers 14, 26, 35, 56, 63, 17, we see that each of these except 17 can be written as the product of two other numbers. The number 17, however, can be written in product form only as 1 × 17. Thus 17 seems to be in a different class from the others.

EXERCISE 68-2A

Which of the following numbers can be written in factored form only as the product of itself and 1?
A whole number like 5, 13, or 17 that can be written in factored form only as the product of itself and 1 is called a prime number.

More precisely: A whole number is called a prime number or simply a prime, if it is greater than 1 and if it can be written in factored form only as the product of itself and 1.

Note that we do not consider 0 and 1 to be prime numbers. Although \(0 = 1 \times 0\), it is equally true that \(0 = 3 \times 0\), \(0 = 15 \times 0\), and so forth. That is, zero multiplied by any number results in zero. On the other hand, 1 can be multiplied only by itself to give 1. These two numbers, as we have seen in previous work, have special properties. In respect to factoring also they appear to be special, and we put them in a class by themselves.

A whole number that is greater than 1 and is not a prime is called a composite number.

Examples of composite numbers: 4, 9, 12, 45.

We thus classify whole numbers in terms of the ways in which they can be presented in factored form as follows:

The set \(S\) of special whole numbers, \(S = \{0, 1\}\).

The set \(P\) of prime numbers.

The set \(C\) of composite numbers.

It is clear that \(S\), \(P\), and \(C\) are mutually disjoint sets (that is, no two of them have any numbers in common) and their union is the set \(W\) of all whole numbers:

\[W = S \cup P \cup C.\]

Diagrammatically,

```
  S  P  C
```

How can you tell whether or not a given number is a prime?

One way, as we shall see, is to attempt to divide the given number by successive primes starting with the least, that is, 2, 3, etc. We shall also examine a device known as the Sieve of Eratosthenes, named for the Greek mathematician.

The sieve furnishes an interesting method of finding all primes less than a given number, say 100. We construct the sieve as follows:

a. Set out the numbers from 2 to 100 in an array.

b. Draw a circle around 2, which is the first prime. Then draw a slant line through every alternate number obtained by counting by two's, that is, 4, 6, 8, and so forth. This will eliminate all the multiples of 2 greater than 2.
c. Draw a circle around 3, the next prime. Then draw a line slanting in an opposite direction through every third number: 6, 9, 12, 15, and so forth. We thus line out the multiples of 3. The first two steps are shown below:

```
2  3  5  7  11  13  17  19
22  23  25  26  27  28  29
31  32  33  35  36  37  38  39  40
41  42  43  44  45  46  47  48  49  50
51  52  53  54  55  56  57  58  59  60
61  62  63  64  65  66  67  68  69  70
71  72  73  74  75  76  77  78  79  80
81  82  83  84  85  86  87  88  89  90
91  92  93  94  95  96  97  98  99  100
```

d. Circle the next prime, which is 5. Then, counting by 5's, draw a line through all other multiples of 5 not previously lined out.

e. Circle the next prime, which is 7. This time, counting by 7's, draw a line through all other multiples of 7 not previously lined out.

f. Finally circle all the remaining numbers which do not have lines through them. These circled numbers are all the primes less than 100. The end result is shown below:

```
2  3  5  7  11  13  17  19
22  23  25  26  27  28  29
31  32  33  35  36  37  38  39  40
41  42  43  44  45  46  47  48  49  50
51  52  53  54  55  56  57  58  59  60
61  62  63  64  65  66  67  68  69  70
71  72  73  74  75  76  77  78  79  80
81  82  83  84  85  86  87  88  89  90
91  92  93  94  95  96  97  98  99  100
```
In constructing the sieve, we lined out multiples of prime numbers only. Why were we not concerned with multiples of numbers other than primes?

To answer this, consider one of the numbers which is not a prime, say 6. Since 6 is composite, it must be a multiple of some prime number smaller than itself. Hence, by the time we reach 6 in the sieving process it will already have been lined out. The same argument holds for all composite numbers.

**EXERCISE 68-2B**

1. The pairs of primes (3, 5), (11, 13), (17, 19), (29, 31), (41, 43) are sometimes referred to as twin primes. What special characteristic earns them this name? Pick out the other pairs of twin primes less than 100. Find 3 pairs of twin primes between 100 and 200.

2. You noticed that the sieve process stopped after lining out multiples of 7. Do you think we should have gone on to eliminate multiples of 11? Give reasons for your answer. (Note: 11 \times 11 = 121. Therefore, if any of the numbers left are multiples of 11, then any other factor of such a number must be less than 11.)

3. Copy the following array and cross out all multiples of 2, 3, and 5.

   | 301 | 302 | 303 | 304 | 305 | 306 | 307 | 308 | 309 | 310 |
---|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
   | 311 | 312 | 313 | 314 | 315 | 316 | 317 | 318 | 319 | 320 |
   | 321 | 322 | 323 | 324 | 325 | 326 | 327 | 328 | 329 | 330 |
   | 331 | 332 | 333 | 334 | 335 | 336 | 337 | 338 | 339 | 340 |
   | 341 | 342 | 343 | 344 | 345 | 346 | 347 | 348 | 349 | 350 |

If the crossing-out process is correctly carried out, 14 numbers should remain uncrossed. We shall now illustrate a method of testing one of these, 349, to see if it is a prime.

We already know that 349 is not a multiple of 2, 3, or 5. Our next step is to determine whether or not 349 is a multiple of the next prime, namely 7. We can, of course, do this by ordinary division. In this case, however, we shall examine an
alternative approach to illustrate certain number patterns.

If 7 is a factor of 349, how large must the other factor be? Since \(7 \times 40 = 280\), we need look only at factors greater than 40. What units digit must this other factor have? The multiplication table for 7's shows \(7 \times 7\) as the only entry ending in 9. Thus we are limited to factors over 40 with units digit 7.

\[
\text{Try } 47, 57: \quad 7 \times 47 = 329 \text{ (too small)} \\
7 \times 57 = 399 \text{ (too large)}
\]

Thus 7 is not a factor of 349.

The next prime, 11, must have as its other factor a number less than 40 with units digit 9. (Why?)

\[
\text{Try } 39, 29: \quad 11 \times 39 = 429 \text{ (too large)} \\
11 \times 29 = 319 \text{ (too small)}
\]

Thus 11 is not a factor of 349.

The Case for 13: Again the other factor must be less than 40 and its units digit must be 3. (Why?)

\[
\text{Try } 33, 23: \quad 13 \times 33 = 429 \text{ (too large)} \\
12 \times 23 = 299 \text{ (too small)}
\]

The Case for 17: This time we can estimate that the other factor must be less than 30, and have 7 as its units digit.

\[
\text{Try } 27, 17: \quad 17 \times 27 = 459 \text{ (too large)} \\
17 \times 17 = 289 \text{ (too small)}
\]

It is worthwhile to stop at this point and ask a question. Do we need to test the case for 19, the next prime? Note that \(19 \times 19 = 361\). If 19 divides 349, the other factor must be less than 19. We may conclude, then, that 349 is a prime.

On the basis of what has gone before, we can now make the following general statement. Given the number \(N\); if \(p\) is a prime such that \(p^2 > N\) and if no prime number less than \(p\) is a divisor of \(N\), then \(N\) is a prime number.

**EXERCISE 68-2C**

1. Find two prime numbers other than 349 in the previous array.

2. Find the prime numbers in the set of numbers listed below. If they are not primes, list their factors.

\[
163, \quad 251, \quad 287, \quad 203, \quad 401, \quad 529.
\]
Let us examine the following numbers:

\[
2^3 - 1 = 7; \quad 2^4 - 1 = 15; \quad 2^5 - 1 = 31.
\]

Note that

\[
2^3 - 1 \text{ is a prime} \\
2^4 - 1 \text{ is not a prime} \\
2^5 - 1 \text{ is a prime.}
\]

Do you think that the result of subtracting 1 from an odd power of 2 will always be a prime number from here on?

**EXERCISE 68-2D**

Which of the following are primes?

\[
2^6 - 1, \quad 2^7 - 1, \quad 2^8 - 1, \quad 2^9 - 1, \quad 2^{10} - 1, \quad 2^{11} - 1.
\]

(Hint: \(2^9 - 1 = 511, \quad 2^{10} - 1 = 1023, \quad 2^{11} - 1 = 2047\).)

Do you still think that the result of subtracting 1 from an odd power of 2 will always be prime?

68-3 Composite Numbers and Prime Factorizations

Suppose that we now turn our attention to the set \(C\) of composite numbers. Let us take, for example, the composite number 42.

We can write 42 in factored form in several ways. For example, 42 may be written as \(2 \times 21\), or \(3 \times 14\), as well as \(7 \times 6\). If we examine these three forms, we note that in each case one of the factors is a prime, but the other is composite.

Since the numbers 21, 14, and 6 are composite, they too can be written in factored form. That is,

\[
21 = 3 \times 7, \quad 14 = 2 \times 7, \quad \text{and} \quad 6 = 2 \times 3.
\]

It follows that we may write 42 as the product of three factors. That is,

\[
42 = 2 \times 21 = 2 \times (3 \times 7), \\
42 = 3 \times 14 = 3 \times (2 \times 7), \\
42 = 7 \times 6 = 7 \times (2 \times 3).
\]

If we remove the brackets, the three "factorizations" are
(What properties of multiplication have we used?)
There are two important things to notice about these factorizations: (1) All factors are primes; (2) Except for the order in which the factors appear, all three factorizations are alike.

Because we are concerned with the set of prime factors and not their order, we shall regard these factorizations as being essentially the same. For convenience we usually write the factors in ascending order as in $2 \times 3 \times 7$.

A significant point has come out here. We started with three different factorizations of 42. We then found that if we continued to factor the remaining composite numbers until all factors were primes, the final results were no longer different.

Let us examine carefully the factorizations of 72. Some possibilities are:

$$72 = 6 \times 12 = (2 \times 3) \times (2 \times 6) = 2 \times 3 \times 2 \times (2 \times 3)$$
$$72 = 8 \times 9 = (2 \times 4) \times (3 \times 3) = 2 \times (2 \times 2) \times 3 \times 3$$
$$72 = 4 \times 18 = (2 \times 2) \times (3 \times 6) = 2 \times 2 \times 3 \times (2 \times 3)$$

Note that the final factorizations on the right are the same, except for order. A factored form containing only prime factors is called a prime factorization. Thus

$$2 \times 2 \times 2 \times 3 \times 3$$

is a prime factorization of 72. Since all such prime factorizations are alike (except for order), we say that prime factorization is unique.

**EXERCISE 68-3**

Give the prime factorizations of the following numbers:

$$70, \ 108, \ 180, \ 196, \ 231$$

**68-4 The Greatest Common Factor**

What is the greatest common factor of 9 and 12? Your answer is probably 3. This is correct, since 3 is the largest number that divides both 9 and 12.

Let us ask the same question about 30 and 42. With a little thought you can see that the correct number in this case is 6. For reasonably small numbers we could probably find the greatest common factor (GCF) by experiment, first listing all of the common divisors, then selecting the largest in the list. We should like, however, to develop a systematic method for calculating the GCF using the concept which we have just been discussing, namely, the concept of prime factorization.

Recall that $30 = 2 \times 3 \times 5$ and $42 = 2 \times 3 \times 7$, where the factors are all primes. Though the number 1 is not a prime, we shall write

$$30 = 1 \times 2 \times 3 \times 5 \quad \text{and} \quad 42 = 1 \times 2 \times 3 \times 7.$$
The reason for including 1 will be seen shortly. Now construct sets $P$ and $Q$ consisting of the prime factors and 1 for each of the given numbers. Thus

$$P = \{1, 2, 3, 5\}, \quad Q = \{1, 2, 3, 7\}.$$ 

We next determine the intersection of these two sets. It should be clear that in this case the intersection $P \cap Q = \{1, 2, 3\}$. The product of the elements in $P \cap Q$ is $1 \times 2 \times 3 = 6$. This, as you recall, is the GCF of 30 and 42.

What is the GCF of 70 and 105?

- $70 = 1 \times 2 \times 5 \times 7$
- $105 = 1 \times 3 \times 5 \times 7$
- $P = \{1, 2, 5, 7\}$
- $Q = \{1, 3, 5, 7\}$
- $P \cap Q = \{1, 5, 7\}$
- GCF = $1 \times 5 \times 7 = 35$

For the numbers 35 and 66 we have:

- $35 = 1 \times 5 \times 7$
- $66 = 1 \times 2 \times 3 \times 11$
- $P = \{1, 5, 7\}$
- $Q = \{1, 2, 3, 11\}$
- $P \cap Q = \{1\}$
- GCF = 1

Since $P \cap Q$ contains only the element 1, this is the GCF. Under these conditions we say that 35 and 66 are relatively prime. In general, two whole numbers whose GCF is 1 are called relatively prime. We now see the reason for including 1 in the factorization.

In order that this method may apply in all cases, we need a special device for those numbers whose prime factorizations contain repeated factors. For example, to find the GCF of 72 and 108 we note that

- $72 = 1 \times 2 \times 2 \times 2 \times 3 \times 3$
- $108 = 1 \times 2 \times 2 \times 3 \times 3 \times 3$

Using letters to distinguish like factors from each other, we write

- $P = \{1, 2a, 2b, 2c, 3a, 3b\}$
- $Q = \{1, 2a, 2b, 3a, 3b, 3c\}$

The intersection is $P \cap Q = \{1, 2a, 2b, 3a, 3b\}$ and the GCF = $1 \times 2 \times 2 \times 3 \times 3 = 36$. Do you see the reason for including exactly two 2's and two 3's?

Though we can verify by experiment that in all the given cases the result is, in fact, the GCF, we have not yet shown why this is so.

To discover this, we need to re-examine the relation between prime factors and divisibility. If we write 60, for example, in the form $60 = 1 \times 2 \times 2 \times 3 \times 5$, it
is clear that each factor is a divisor of 60. Furthermore, we know that the product of any combination of these factors is also a divisor of 60. For instance, 4, 6, 10, 15, 20, 30 are all products of certain of the given factors. They are therefore divisors of 60.

We can also see that products of these given factors are the only divisors of 60. How do we know this? Remember that factorization is unique! In a sense, this means that there are no "hidden" factors, factors which do not appear in the prime factorization. Clearly, then, 7 is not a factor, nor is $3 \times 3 = 9$.

Now, go back to the question of GCF. Let us look at 60 and 72 where $60 = 1 \times 2 \times 2 \times 3 \times 5$ and $72 = 1 \times 2 \times 2 \times 2 \times 3 \times 3$. Using letters to distinguish like factors from each other and omitting a few steps, we have

$P \setminus Q = \{1, 2a, 2b, 3\}$, whence the GCF $= 1 \times 2 \times 2 \times 3 = 12$. Since both 60 and 72 contain the product $1 \times 2 \times 2 \times 3$, we can be sure that 12 is a common factor.

Is it the greatest? Suppose that we include an additional factor, say a third 2, or a second 3, or a 5. Will the resulting product still divide 60 and 72? For example, does $1 \times 2 \times 2 \times 3 \times 3 = 36$ divide 60?

**EXERCISE 68-4**

Use prime factors to find the GCF of each of the following pairs of numbers.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>12, 18</td>
<td>b. 15, 36</td>
</tr>
<tr>
<td>d.</td>
<td>108, 117</td>
<td>e. 35, 44</td>
</tr>
<tr>
<td>g.</td>
<td>38, 140</td>
<td>h. 164, 200</td>
</tr>
<tr>
<td>j.</td>
<td>39, 91</td>
<td>k. 64, 81</td>
</tr>
</tbody>
</table>

68-5 The Least Common Multiple

The notion of the least common multiple of two given numbers has already been used in the process of adding fractions. It is interesting to see how closely this concept can be related to the GCF of the previous section by using sets and prime factorizations.

Given the numbers 60 and 72, you can verify by experiment that the LCM is 360. This, in other words, is the smallest number which is both a multiple of 60 and a multiple of 72.

It is possible, and more direct, to get this information from prime factorizations. We proceed, to start with, in the same way as we did in finding the GCF. From the fact that $60 = 1 \times 2 \times 2 \times 3 \times 5$ and $72 = 1 \times 2 \times 2 \times 2 \times 3 \times 3$ we obtain the sets $P = \{1, 2a, 2b, 3, 5\}$ and $Q = \{1, 2a, 2b, 2c, 3a, 3b\}$.

This time, however, we form the UNION, $P \cup Q$ of the two sets, instead of their intersection. Thus $P \cup Q = \{1, 2a, 2b, 2c, 3a, 3b, 5\}$. The product of these elements is

$1 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 360$. 
Do we know that this process will always yield the LCM of a given pair of numbers? Let us re-examine the three factorizations

\[
\begin{align*}
60 &= 1 \times 2 \times 2 \times 3 \times 5 \\
72 &= 1 \times 2 \times 2 \times 2 \times 3 \times 3 \\
360 &= 1 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5
\end{align*}
\]

The factored form of 360 contains all the factors of 60. It also contains the factors of 72. Hence, 360 is a multiple of both 60 and 72. It is certainly, then, a common multiple of the two numbers.

Is it the least? If one of the factors of 360, other than 1, were dropped, would the result still be a common multiple? Suppose, for example, that we omitted one of the 2's. Does \(1 \times 2 \times 2 \times 3 \times 3 \times 5\) contain all of the factors of 72?

**EXERCISE 68-5**

Using prime factorizations and set union, find the LCM of each of the following pairs.

- a. 12, 20
- b. 30, 45
- c. 27, 36
- d. 28, 49
- e. 17, 51
- f. 32, 48
- g. 18, 63
- h. 35, 56
- i. 66, 99
- j. 144, 160

**68-6 Divisibility Tests**

The problem of determining primes and prime factors involves a considerable amount of division. We have already seen how often we need to know whether a given number is divisible by 2, or 3, 5, 7, or 11, and so forth.

It is always possible, of course, to find this out by performing the actual division. There are certain aspects of our "place value" system of notation, however, which enable us to tell at a glance whether or not a number is a multiple of 2 or 5. With a very small amount of quick manipulation, we can also detect whether or not a number is divisible by 3, 7 or 11.

These tests of divisibility are useful in actual practice. They also provide some valuable insights into the nature of numbers and the special properties of a place value notation system.

In the next few sections we shall examine some of these tests in detail. Once the general principle is understood, it is hoped that the student may be able to discover new tests for himself.

The fundamental idea which underlies all the tests for divisibility is based on the following statement:

Let \(A\) and \(B\) be any two numbers and let \(N = A + B\).
If a given number \( d \) is a divisor of \( A \), then \( d \) will divide the number \( N \) if and only if \( d \) is also a divisor of \( B \).

In other words, suppose that \( N = A + B \) and we know that \( A \) is divisible by \( d \). Then only two situations are possible.

1. Both \( B \) and \( N \) are divisible by \( d \).
2. Neither \( B \) nor \( N \) is divisible by \( d \).

Let us first show that if \( d \) divides \( B \), then it must divide \( N \). Since we know that \( d \) is a divisor of \( A \), we can write \( A = d \times a \), for some whole number \( a \). If, then, we assume that \( d \) divides \( B \), we can write \( B = d \times b \), for some whole number \( b \). Thus we have

\[
N = A + B = (d \times a) + (d \times b) = d \times (a + b) \quad \text{(Distributive property)}.
\]

This shows us that \( d \) is a factor (divisor) of \( N \).

We now wish to show that if \( d \) divides \( A \), then it is not possible for \( d \) to divide \( N \) and not divide \( B \). To do this we first note that \( N = A + B \). We can write this as a subtraction statement

\[
N - A = B.
\]

Since we know that \( d \) divides \( A \), we can again write \( A = d \times a \). Furthermore, if \( d \) divides \( N \), we can write \( N = d \times n \) and our equation becomes

\[
(d \times n) - (d \times a) = B \quad \text{or} \quad d \times (n - a) = B.
\]

We see, then, that it is not possible to have \( d \) divide both \( A \) and \( N \) without also dividing \( B \).

We have now shown that in a situation where \( N = A + B \) and \( d \) divides \( A \), (1) If \( d \) divides \( B \) it must also divide \( N \); (2) If \( d \) does not divide \( B \), then it cannot divide \( N \).

We use this property in the following way: We are given a number \( N \) and wish to know whether or not \( N \) is divisible by a certain number \( d \). The procedure is to break \( N \) into two parts \( A \) and \( B \) where \( N = A + B \). Using decimal place value concepts we arrange it so that part \( A \) is always known to be a multiple of \( d \).

We next examine \( B \) to see if it is also a multiple of \( d \). Since \( B \) is a small number (by design), we can usually tell this by inspection. If \( B \) is divisible by \( d \), then the property tells us that the original large number \( N \) will also be divisible by \( d \). Similarly, if \( B \) is not a multiple of \( d \), then \( N \) is also not a multiple of \( d \).

To illustrate the point let us first consider whether or not a given number is
divisible by 2 or 5. You no doubt know the answer to this one from past experience. But it is good to understand the reason for the rules we use.

Let us consider the number 98275. Because we are using decimal notation, the number can be written as

\[(9827 \times 10) + 5\]

Since the quantity on the left is a multiple of 10, it is certainly divisible by 2, and also by 5. The key to the situation is clearly the units digit. If this is divisible by 2, then the original number is also divisible by 2. Otherwise not. Since the units digit is, in this case, 5, it follows that 98275 is not divisible by 2.

On the other hand, the units digit is divisible by 5. This means that the original number must be divisible by 5 also.

If a number ends in zero, it must be a multiple of 10. Such a number is therefore clearly divisible by both 2 and 5.

We may conclude from all this that a number is divisible by 2 if, and only if, the units digit is a member of the set \{0, 2, 4, 6, 8\}. Similarly, a number is divisible by 5 if, and only if, its units digit is in the set \{0, 5\}.

**EXERCISE 68-6**

1. Determine by inspection which of the following numbers are divisible by 2, and which are divisible by 5.

   a. 367  b. 295  c. 72460
   d. 12986  e. 34295  f. 2591
   g. 37264  h. 12865  i. 149278
   j. 34713

68-7 **A Divisibility Test for 3**

We have just seen how the decimal notation system can provide a quick and effective way to test divisibility by 2 or 5. The key lies in the fact that the base 10 itself is a multiple of both 2 and 5.

The question we now face is: How can this same system furnish a convenient means for testing divisibility by 3? Since 10 is not a multiple of 3 — nor is any other power of 10, such as 100, 1000, etc., — we must look about for a somewhat different procedure.

Can we use the fact that \(10 = 9 + 1\), \(100 = 99 + 1\), \(1000 = 999 + 1\), etc., where the quantities 9, 99, 999, and so forth, are clearly multiples of 3?

Let us look at an actual case. Take the number 453. We may write this as \((4 \times 100) + (5 \times 10) + 3\), which in turn may be written as
\[4 \times (99 + 1) + 5 \times (9 + 1) + 3\]

Applying the distributive property we see that this is equal to
\[\{(4 \times 99) + (4 \times 1)\} + \{(5 \times 9) + (5 \times 1)\} + 3\]

Some regrouping gives us
\[\{(4 \times 99) + (5 \times 9)\} + \{(4 \times 1) + (5 \times 9)\} + 3\]
\[= \{(4 \times 11 \times 9) + (5 \times 9)\} + (4 + 5 + 3)\]
\[= \{(4 \times 11) \times 9 + (5) \times 9\} + (4 + 5 + 3)\]
\[= \{(4 \times 11) + 5\} \times 9 + (4 + 5 + 3)\]

We now have the familiar set-up \(N = A + B\), where

\[A = \{(4 \times 11) + 5\} \times 9\]

is clearly a multiple of 3. The question, then, of whether or not \(N\) is divisible by 3 depends on \(B = \{4 + 5 + 3\}\). If \(B\) is divisible by 3, then so is the original number \(N\); otherwise not. Now what about the sum \(4 + 5 + 3\)? Since this is the sum of the digits of 453, we have discovered a "test" for divisibility by 3. Can you summarize the above results in the form of a rule?

To make sure that you understand the test for divisibility by 3, we strongly advise you to apply exactly the above procedure to the number 9875 to decide whether or not it is divisible by 3.

We are now in a position to assert, with confidence, that a whole number is divisible by 3 if, and only if, the sum of its digits is divisible by 3.

**EXERCISE 68-7**

1. Use the "test" to determine which of the following numbers are divisible by three.
   
   a. 285  
   b. 3415  
   c. 2718  
   d. 4860  
   e. 1902  
   f. 16241  
   g. 80415  
   h. 21178  
   i. 35124  
   j. 16125

2. On the basis of the discussion in this section, formulate a rule for divisibility by 9. Use the rule to find which numbers in Question 1 are divisible by 9.

3. Tests can be made for certain numbers by combining two other tests. Can you suggest a rule for divisibility by 6? Divisibility by 15? Which numbers in Question 1 satisfy either of these tests?

4. A test for divisibility by 4 might be based on the following:

   \[98232 = (982 \times 100) + 32\]
68-8 A Test for Divisibility by 11

In the previous section we noted that the numbers 9, 99, 999, etc., were all divisible by 3. We also used the fact that \(10 = 9 + 1\), \(100 = 99 + 1\), \(1000 = 999 + 1\), and so on, to set up our test for divisibility by 3. Can we apply a similar approach in the case of 11? First we note that 99 and 999 are both divisible by 11. So far so good! But what about 9 and 999? Since these are not multiples of 11, we need to explore an alternative. The number 999 will not be of any use. What about \(1000 = 999 + 1\)? This is a multiple of 11. So is \(10 + 1 = 11\). If we write 1000 as 1001 - 1 and 10 as 11 - 1, we should be able to construct a test similar to the test for 3.

Let us try the number 3927.

First, \(3927 = (3 \times 1000) + (9 \times 100) + (2 \times 10) + 7\) which we now write as 

\[
\left\{3 \times (1001 - 1)\right\} + \left\{9 \times (99 +1)\right\} + \left\{2 \times (11 - 1)\right\} + 7
\]

\[
= \left\{3 \times 1001\right\} - \left\{3 \times 1\right\} + \left\{9 \times 99\right\} + \left\{9 \times 1\right\} + \left\{2 \times 11\right\} - \left\{2 \times 1\right\} + 7
\]

\[
= \left\{3 \times 1001\right\} - 3\left\{ \right\} + \left\{9 \times 99\right\} + 9\left\{ \right\} + \left\{2 \times 11\right\} - 2\left\{ \right\} + 7
\]

\[
= \left\{3 \times 1001\right\} + \left\{9 \times 99\right\} + \left\{2 \times 11\right\} + (9 - 3 + 7 - 2)
\]

\[
= [(3 \times 91) \times 11 + (9 \times 9) \times 11 + (2 \times 11) + (9 - 3 + 7 - 2)]
\]

The quantity on the left in brackets is divisible by 11. Why? By the property that we have been using, we can again say that divisibility by 11 depends on the remaining part \(9 - 3 + 7 - 2\). If this quantity is divisible by 11, then so is 3927: otherwise not. In this case \(9 - 3 + 7 - 2\) is not the sum of the digits. It is surely related to the digits, however. From the way in which the form is set up, we see that the signs alternate. The quantity \(9 - 3 + 7 - 2\) can therefore be obtained by adding 9 and 7, then subtracting 3 and 2. Another way of describing this is to call it the difference of the sums of the alternate digits.

Let us try another example.

Is the number 3927 divisible by 11? We add the alternate digits \(3 + 2 = 5\), \(9 + 7 = 16\). Then find the difference \(16 - 5 = 11\). If the result is a multiple of 11 or zero, then the answer is "yes", otherwise no! Try 91718. Add alternate digits: \(9 + 7 + 8 = 24; 1 + 1 = 2\); \(24 - 2 = 22\). Is 22 a multiple of 11? Check by division to see if 11 divides 91718.

Let us look at 2384. Note that

\[
2 + 8 = 10 \\
3 + 4 = 7 \\
10 - 7 = 3
\]

*Note that the number zero is divisible by any nonzero whole number. For example, \(0 \div 1 = 0\), \(0 \div 3 = 0\), \(0 \div 15 = 0\), since \(0 = 1 \times 0\), \(0 = 3 \times 0\), \(0 = 15 \times 0\), respectively.
Is 3 a multiple of 11? What do we conclude about 2384?

**EXERCISE 68-8**

1. Determine which of the following numbers are divisible by 11.
   a. 825   b. 7832   c. 1546   d. 9471
   e. 68324  f. 94655  g. 918082  h. 416234
   i. 545655  j. 213824

2. Using the idea of combining two tests, formulate a rule for divisibility by 22, by 33, by 55.

3. Apply the rules from Problem 2 to the list in Problem 1.

4. If you were setting up a demonstration for a six-digit number, say 416234, the first step would be to write

   \[ 416234 = (4 \times 100000) + (1 \times 10000) + (6 \times 1000) + (2 \times 100) + (3 \times 10) + 4. \]

   For the next step, how would you represent 100000? How would you represent 100000?

**68-9 A Test for Divisibility by 7**

We turn now to a more challenging problem. Can we find a convenient method for determining whether or not a given whole number is divisible by 7? For most two-digit numbers, we can deal with the question simply by recalling the multiplication table. Numbers like 14, 21, 28, 35, 42, 49, 56, 63, ..., are easily spotted.

What about a three-digit number, say 483? Try the following! Remove the 3. Double it (2 \times 3 = 6). Subtract 6 from 48. The scheme looks like this:

\[
\begin{array}{c}
483 \\
-6 \\
42
\end{array}
\]

3 \times 2 = 6

Is 42 divisible by 7? The answer is "yes". Then so is 483. Check this by division! Now try 672. Is this a multiple of 7? Apply the test!

\[
\begin{array}{c}
672 \\
-4 \\
63
\end{array}
\]

2 \times 2 = 4

Success again.

For a larger number the test may be used a second time, or a third. Take 4564.
As before, we have

\[
\begin{array}{c}
4564 \\
-8 \\
\hline
448
\end{array}
\]

4 \times 2 = 8

We can apply the same test to 448.

\[
\begin{array}{c}
448 \\
-16 \\
\hline
28
\end{array}
\]

8 \times 2 = 16

What does this say about 4564?

To illustrate the negative, let us look at one more number. Take 594. For the test we have

\[
\begin{array}{c}
594 \\
-8 \\
\hline
51
\end{array}
\]

4 \times 2 = 8

Is 51 a multiple of 7? What about 594?

Now that we know how to use the test, we face the usual probing question. Why does it work?

To give a satisfactory mathematical answer, we shall have to examine our testing device in a more formal way.

Let us look again at the first example, 483. 3 \times 2 = 6. 48 - 6 = 42. In terms of mathematical operations, we could describe this as follows:

1) Subtract the units digit: 483 - 3 = 480
2) Multiply the units digit by 20: 3 \times 20 = 60
3) Subtract this result from 480: 480 - 60 = 420

The question here is: Does 7 divide 420? If the answer is "yes", then 7 also divides 483. We shall soon see why this is true.

First, however, we shall establish the connection between the formal mathematical approach and the original short cut. In the short version we, in a sense, ignored a zero and ended up with 42 rather than 420. Normally one cannot casually "drop" a zero. But in this case we are not concerned with the value of a number, only the question of whether or not it is a multiple of 7.

It should be clear that if 42 is a multiple of 7, then 420, which 42 \times 10, is also a multiple of 7, and vice versa. Furthermore, it can be shown generally that if a number, say d, is not a multiple of 7, then 10 times d will not be either. In short, the property of divisibility by 7 is the same for 42 as for 420. Similarly with other numbers.

Finally, then, we must show why the original number 483 is divisible by 7 if
and only if the derived number 420 has the same property. In applying the "formal" test we had

\[483 - 3 = 480, \quad 3 \times 20 = 60, \quad 480 - 60 = 420.\]

Can we describe this process in one step? First we subtract 3, then subtract 20 \times 3. Is this the same as subtracting 21 \times 3? It should be clear that we obtain the derived number by subtracting 21 times the units digit. That is,

\[483 - (21 \times 3) = 420.\]

Now suppose, as before, that we call the original number \(N\). Call \(B\) the number obtained after testing. Let \(D\) stand for the units digit of \(N\). In our example \(N = 483, D = 3, B = 420\). We see that \(N - (21 \times D) = B\), and \(N = (21 \times D) + B\). Now if \(B\) is a multiple of 7, then \(B = 7 \times b\) for some whole number \(b\). Under these conditions

\[N = (21 \times D) + (7 \times b)\]

Thus we can say, finally, that \(N\) is a multiple of 7, if \(B\) is. Otherwise not.

As a special case, we should realize that if our short-cut process results in zero, the original number is also divisible by 7. Examples: 147, 126, 168, etc.

We have seen why our test works. The reasoning is complicated. You should read this part very carefully, to be sure of the ideas involved.

**EXERCISE 68-9**

1. Apply the test for this section to determine which of the following numbers are divisible by 7.

   a. 483   b. 301   c. 594   d. 945   e. 854
   f. 1365  g. 1938  h. 2541  i. 21854  j. 25305

2. Devise tests for determining what numbers are divisible by 14, by 21, by 35.

3. Apply these tests to the list of Question 1.

4. Use the divisibility tests to cross out all multiples of 2, 3, 5, 7, and 11 in the following array. This would help in the determination of primes.

\[
\begin{array}{ccccccccccc}
761 & 762 & 763 & 764 & 765 & 766 & 767 & 768 & 769 & 770 \\
771 & 772 & 773 & 774 & 775 & 776 & 777 & 778 & 779 & 780 \\
781 & 782 & 783 & 784 & 785 & 786 & 787 & 788 & 789 & 790 \\
791 & 792 & 793 & 794 & 795 & 796 & 797 & 798 & 799 & 800 \\
801 & 802 & 803 & 804 & 805 & 806 & 807 & 808 & 809 & 810 \\
\end{array}
\]
68-10 Exploring and Experimenting with Numbers

Often one thinks of numbers merely as an aid in counting, measuring, making change, and the like. Actually, aside from their usefulness in everyday affairs, numbers themselves have many surprising properties (one might almost say "mysteries") which have never ceased to provide fascination and challenge.

In this section we shall look at some of the ways in which numbers can be experimented with. We shall see how theories about numbers may be tested, how new ideas can be created.

As a start, let us look at the LCM and GCF of given pairs of numbers.

Construct a table as follows:

<table>
<thead>
<tr>
<th>Pairs of Numbers</th>
<th>GCF</th>
<th>LCM</th>
</tr>
</thead>
<tbody>
<tr>
<td>8, 12</td>
<td>4</td>
<td>24</td>
</tr>
<tr>
<td>9, 15</td>
<td>3</td>
<td>45</td>
</tr>
<tr>
<td>6, 9</td>
<td>3</td>
<td>18</td>
</tr>
<tr>
<td>10, 15</td>
<td>5</td>
<td>30</td>
</tr>
<tr>
<td>5, 7</td>
<td>1</td>
<td>35</td>
</tr>
<tr>
<td>12, 18</td>
<td>6</td>
<td>36</td>
</tr>
<tr>
<td>30, 45</td>
<td>15</td>
<td>90</td>
</tr>
<tr>
<td>5, 8</td>
<td>1</td>
<td>40</td>
</tr>
</tbody>
</table>

For a first experiment see if you can find a relationship between the numbers in the two columns on the left and the numbers in the two columns on the right. Does $8 + 12 = 4 + 24$? Does $9 + 15 = 3 + 45$, or $6 + 9 = 3 + 18$? Though the answer is "no" in all cases, perhaps this suggests a similar experiment. Look at some of the smaller pairs.

Stop! Do not continue reading until you have come up with an idea!

* * * * *

By now you have probably observed that the product of the two numbers in each pair is equal to the product of the GCF and LCM.

Do you think that this will always be true? Try some other numbers!

As a way of exploring the reason why, let us look again at 60 and 72. In factored form we have $60 = 1 \times 2 \times 2 \times 3 \times 5$ and $72 = 1 \times 2 \times 2 \times 2 \times 3 \times 3$. The product $60 \times 72 = (1 \times 2 \times 2 \times 3 \times 5) \times (1 \times 2 \times 2 \times 2 \times 3 \times 3)$. Rearranging the factors and using 1 only once, we have $60 \times 72 = 1 \times 2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 5 = 4320$.

Recall, now, that the GCF is formed from the intersection set $A \cap B = \{ 1, 2a, 2b, 3 \}$ giving us $1 \times 2 \times 2 \times 3 = 12$.

The LCM is formed from the union set $A \cup B = \{ 1, 2a, 2b, 2c, 3a, 3b, 5 \}$ giving us $1 \times 2 \times 2 \times 2 \times 3 \times 3 \times 5 = 360$.

We have already seen that the product of the two numbers, in primes, contained five 2's,
three 3's, and one 5. It should be clear that the product of the GCF and the LCM will also contain exactly five 2's, three 3's, and one 5.

**EXERCISE 68-10**

1. Test the above relationship using the pairs of numbers in the Problems in Sections 5 and 6.

2. From the table, can you find any pairs \(a, b\) which have the relation \(a + b = \text{LCM} - \text{GCF}\)? List all such pairs. Can you find a common pattern?

3. Note that for the pair 3, 5 we have \(2(3 + 5) = 1 + 15\), that is, \(2(a + b) = \text{GCF} + \text{LCM}\). Is this true for 6, 10? What about 12, 20? List other pairs for which this same rule will apply.

**68-11 Divisibility in Other Bases**

You have seen that the divisibility tests and the special property of 9 all depend on the fact that 10 is the base of our decimal notation system.

Suppose that we now look at some other systems of notation. Take, for example, the base 2. The numeral \(110_\text{two}\) represents the number six. This numeral certainly ends in zero. But is six divisible by five?

Now recall the test for divisibility by 3 in the decimal system.

A number in base 10 is divisible by 3 if, and only if, the sum of its digits is divisible by 3.

Take the binary numeral \(110_\text{two}\). The number six, which it represents, is certainly divisible by three. But what about the sum of the digits?

These examples clearly illustrate the fact that the tests for divisibility which we have been using do not generally apply for bases other than ten. It should be interesting to see what ideas on divisibility tests we can come up with in connection with notation in different bases.

Before doing this, however, we should re-examine the notion of divisibility in general and the related concept of prime numbers.

One often hears the question: "What about prime numbers in other bases?" To answer this question we should emphasize the fact that a number greater than 1 is prime if it cannot be represented as the product of two numbers other than itself and one. We can see from this that the condition of being prime is a property of number that does not depend on the particular notation in which a number is written. For example, we have said that the number 13 is a prime number, since it can be factored only in the form \(1 \times 13\). Now let us consider the number thirteen as it would be written in other bases. We have \(13_{\text{ten}} = 1101_{\text{two}} = 23_{\text{five}} = 16_{\text{seven}} = 11_{\text{twelve}}\).

Is it possible to represent this number as the product of two numbers other than itself and one in any of the other bases? Try it!

Now take the number twenty. In our earlier discussion we classified this as a
composite number, since 20 could be written as \(4 \times 5\). Now let us look at the various representations in other bases. \(20_{\text{ten}} = 10100_{\text{two}} = 40_{\text{five}} = 26_{\text{seven}} = 18_{\text{twelve}}\). We can see that

\[
\begin{align*}
10100_{\text{two}} &= 100_{\text{two}} \times 101_{\text{two}} \\
40_{\text{five}} &= 4_{\text{five}} \times 10_{\text{five}} \\
26_{\text{seven}} &= 4_{\text{seven}} \times 5_{\text{seven}} \\
18_{\text{twelve}} &= 4_{\text{twelve}} \times 5_{\text{twelve}}
\end{align*}
\]

Thus, twenty remains a composite number regardless of the base in which it is represented.

The fundamental notion of divisibility is also a property of number that does not depend on a particular base.

In the previous example the various factorizations in the different bases all show that both four and five are divisors of twenty regardless of the bases in which these numbers are written.

The point to bring out here is that divisibility itself does not depend on a particular base notation. On the other hand, the various tests for divisibility do, indeed, depend on the choice of base.

Let us look again at base two. A sequence of counting numbers can be written as

\[1, 10, 11, 100, 101, 110, 111, 1000, 1001, \ldots\]

Here we have omitted the subscript "two".

In this notation, can you suggest a very simple rule for recognizing all even numbers? It is easy to see that a number written in base two is a multiple of four if, and only if, its last two digits are zeros. What can you say about divisibility by eight? by sixteen?

Let us now consider a notation system in base seven. First, we should remember that in decimal notation the number nine had the following special property.

A number in decimal notation is divisible by nine if, and only if, the sum of its digits is also divisible by nine.

We note that nine is one less than the base ten. What does this suggest in connection with base seven?

Let us examine some numbers written in this notation.
15 \text{seven} = 12 \text{ten} \\
33 \text{seven} = 24 \text{ten} \\
51 \text{seven} = 36 \text{ten} \\
105 \text{seven} = 54 \text{ten}

These numbers, as can be seen on the right, are all multiples of six. What can be said about the sum of the digits in the base seven notation on the left? To consider a larger quantity, the number 522 \text{ten} is a multiple of six. (Check this by the test in Section 7, Problem 3.) Its representation in base seven is 1344 \text{seven}. What about the sum of the digits? We note that the sum is equal to the number twelve, which is certainly a multiple of six.

It is interesting to see what the results of adding these digits would be if we use base seven addition. Here we have \( \text{se}_7 + \text{se}_7 + 4 \text{se}_7 \times 4 \text{se}_7 = 15 \text{seven}\) and \( \text{se}_7 + 5 \text{se}_7 \) is equal to \( 6 \text{seven}\).

From this discussion we can make a general statement about divisibility tests with respect to numbers in all bases. This will be developed in the exercises.

**EXERCISE 68-11**

1. Fill in the missing parts in the following rule.
   If numbers are written in a place value system using base \( n \), where \( n \) is a counting number greater than 1, then a number in this system is divisible

   by __________ if, and only if, the __________ of the __________ is divisible by __________.

2. The following numbers are written in base five. Apply a rule to determine which of these numbers is divisible by four. Rewrite the numbers in base ten and check the results by division.
   
   a. 44 \text{five}  
   b. 103 \text{five}  
   c. 123 \text{five}  
   d. 202 \text{five}  
   e. 341 \text{five}  
   f. 312 \text{five}  
   g. 141 \text{five}  
   h. 1102 \text{five}  
   i. 2011 \text{five}  
   j. 3011 \text{five}

3. The following numbers are written in base twelve. Apply a rule to determine which of these is divisible by eleven. Check your result as in Problem 2.
In the previous eleven sections we have opened for you a small door to an old and important branch of mathematics called the Theory of Numbers. Already Greek mathematicians proved the validity of the following two statements:

I. The set $C$ of all composite numbers is infinite,
II. The set $P$ of all prime numbers is infinite.

In other words, there is no largest composite number and there is no largest prime number.

We shall show the truth of both statements by a method called "Proof by contradiction."

I. The set $C$ of composite numbers is infinite.

Suppose on the contrary that there are a finite number of composite numbers. Then they can all be arranged in ascending order, and we obtain a largest composite number, say $N$. However, $2 \times N$ is greater than $N$ and is also composite. Thus there is no largest composite number, and $C$ is infinite.

II. The set $P$ of prime numbers is infinite.

The proof of this statement was given by Euclid in the 3rd century before our era. Suppose on the contrary that the set $P$ of all prime numbers is finite, that is

$P = \{2, 3, 5, 7, 11, 13, 17, \ldots, p\}$

has a largest prime number, say $p$.

To show that $P$ is an infinite set, it is enough to show that there is always a prime greater than the assumed largest prime $p$. To this end we form a new number, obtained by multiplying all members of $P$ and adding 1. That is, we consider the number $n$ such that

$$n = (2 \times 3 \times 5 \times 7 \times 11 \times \ldots \times p) + 1.$$
greater than the supposed largest prime $p$. In this case we have a contradiction with the assumption that $p$ is the largest prime.

2. If $n$ is a composite number, the structure of $n$ indicates that $n$ leaves the remainder 1 after division by any prime number not greater than $p$. This means that no prime number less than or equal to $p$ is a divisor of $n$. But $n$ is a composite number. Therefore, it must have a prime divisor that is greater than $p$. We have thus proved that there is a prime greater than the supposed largest prime number $p$, namely the prime divisor of $n$.

This completes the proof of II.

Since antiquity prime numbers have been an object of fascination for mathematicians. Some of these mathematicians have tried to discover a rule for constructing primes. For example, the famous French mathematician Fermat (17th century), who has contributed much to the theory of numbers, noticed that the formula

$$2^n + 1$$

gives prime numbers for some values of $n$ that are powers of 2. He indicated that

for $n = 2^0 = 1$, $2^n + 1 = 2^1 + 1 = 3$ is a prime,

for $n = 2^1 = 2$, $2^n + 1 = 2^2 + 1 = 5$ is a prime,

for $n = 2^2 = 4$, $2^n + 1 = 2^4 + 1 = 17$ is a prime,

for $n = 2^3 = 8$, $2^n + 1 = 2^8 + 1 = 257$ is a prime.

Fermat believed that this formula would always produce primes. In the 18th century, however, Euler disproved this conjecture by showing that for $n = 2^5 = 32$, the formula $2^n + 1$ gives a number divisible by 641.

Thanks to the invention of high-speed electronic computers, larger and larger prime numbers have been discovered. For example, the number $2^{3217} - 1$, which in its full form contains 1,000 digits (!), was proved to be a prime number.

Another conjecture has occupied mathematicians from the time it was suggested (1742) by Goldbach. He stated that

*Every even number greater than 2 can be represented as a sum of two prime numbers.*

For example,

$$4 = 2 + 2, \quad 6 = 3 + 3, \quad 8 = 3 + 5$$

$$10 = 3 + 7, \quad 48 = 7 + 41,$$

and so forth.

Take several even numbers and try to write each of them as a sum of two primes.

Nobody has ever found an even number for which this could not be done. This means that Goldbach's conjecture was never disproved. However, no mathematical proof has yet been invented to show that this is a generally true statement.
Some Last Words — Mathematics, the Great Adventure.

We are coming to the point where we must bid the reader good-bye. Before doing so we shall look back over the road that we have traveled together from the beginning. We stand on a mountain from which we can take a long view.

This volume has been largely concerned with ways in which numbers are used to help us to understand the world around us and to handle some of the problems of living together. We have looked at measurements which come out of experiment and we have seen how these measurements are often related to each other in simple ways. We have also seen how the results of many measurements can be treated so that useful information can be obtained from them. But now we shall take a look back to the very beginning of Volume 1.

As we do so we shall try to see how mathematics helps us to understand our world. We shall try to be clear about what mathematics is. Mathematics is one of the really great inventions of mankind. But it is not an invention which is completed. It is always being improved and added to. The story of its growth from small beginnings is an adventure story. Unfortunately we must bring the story to an end long before the present heights of development. We have had to confine ourselves to the simpler parts. Mathematics is such a vast subject that people spend their whole lives studying it. There seems to be no end to what can be learned.

We have said that mathematics is an invention. It is really a succession of inventions. Each of these inventions was first of all an idea in someone’s mind. Somebody was the first one to have this idea.

Arithmetic.

Probably the first mathematical ideas were the ideas of sets of things and the counting numbers which go with them. This is where we began our story. The ideas are so old that no one knows who invented them. They go back to prehistoric times. But they are ideas, not things like bananas or palm trees or houses. These ideas help us to think about things like bananas, palm trees and houses, but they are not things. “Three”, for example, is an idea. It cannot be seen or heard or touched. But numbers like three do help us to work with things that can be seen or heard or touched. They help us to handle things more easily.

Let us imagine ourselves present on that very important day many thousands of years ago when someone first had the idea of making a tally by notches in a stick, one
for each animal in his herd. What a discovery! For now it was easy to know whether any of the animals were lost. It must have been a long time before anyone thought of making tallies of days to keep a calendar. But think of the difference this idea made. The days had gone by but the marks were still there. There was a record.

Or think of the genius who invented numerals to replace tallies so that instead of \( /// \) one could write 9, and the greater geniuses who thought of the abacus, the place system and zero. How much easier it is to add 328 and 154, for example, than to combine two sets with these numbers and count the number of members in their union. A further step was to simplify repeated addition as a multiplication.

A big advance came when new symbols were invented to represent parts of objects or sets. Later, rules were devised to handle fractions, so that habits could be set up to work with them. Mankind learned to treat marks on paper as things.

It is so much easier to work with the marks than the ideas for which they stand. In time these symbols or marks took on a life of their own. We are now as familiar with them as with the objects of everyday life.

We come to notice how numbers behave, and discover, for example, the commutative and associative properties. We make a new world out of numbers, a world of our own creation which helps us to deal with the world around us.

When we work in this world of numbers that we have created we keep noticing things. For example we notice that we can keep on adding to get new numbers.

A small child who had just learned to count asked his father: "Is a thousand the largest number there is?" His father could have said "No! A million is larger". But he was a good teacher. He answered simply "A thousand and one is larger". The child came back after half an hour and said "There is no largest number, is there?" He was right. He had made the discovery that the counting numbers go on without end.

**Geometry.**

Let us turn from arithmetic to another invention of mankind—to geometry. From dots, stretched strings and stretched cloth or leather man created points, lines and planes. These are ideas, that is creations of the human imagination that exist only in the human mind. No one has ever seen a point without length, breadth or thickness or a line with length but no width or thickness but these products of our imagination make life simpler. They make it possible to think more clearly and precisely. And we think of a line segment as extended without end in both directions and a plane as extended without end in all directions. How does this make things simpler? We give an example. If you wish to find the altitude of a triangle it may happen that the perpendicular from the vertex P to the base fails to intersect the base. However, when the base is extended endlessly in both directions, the perpendicular always intersects the base line. In geometry we build up a world in which the rules are very simple and clear. This world too leads a life of its own which grows as people work with it. This creation, geometry, helps us to think about real things in a remarkable way. This
fact comes out most clearly after we combine arithmetic with geometry, that is, use numbers to measure geometric figures.

**Numbers and Geometry.**

Geometry and arithmetic were brought together by the invention of the number line. That is, the idea of marking a line with a scale was brought into mathematics. This was done by the ancient Greeks. In this way they could measure lengths, angles, areas and volumes. To see how this was useful a few examples will be helpful.

We have seen how Thales measured the height of a pyramid by using similar triangles. By proportion, the height of the pyramid: length of its shadow = height of a pole: length of the shadow of the pole. Three of these measurements were known so that the fourth could be found. This was an indirect measurement. It had to be since the line segment whose length was to be measured was inside the pyramid.

It should be noticed how geometry helps us here. We imagine that the real pyramid is replaced by an ideal one. We draw a picture which leaves out everything that is not essential for the purpose of measuring the height. We forget the fact that the actual pyramid is made of blocks of stone with a certain color and the fact that these stones are very heavy. We neglect the fact that there are certain passages into the tombs in the interior. We replace the actual object by a simplified idea of the object.

Let us turn to another problem. For various reasons the Greeks were led to the idea that the earth is a sphere. The question arose "How large is this sphere?". The question was answered by making some very simple measurements and using a picture. On a certain day the sun was directly overhead at noon at a place on the Nile called Syene. That is, at this place a vertical stick cast no noon shadow. At noon of the same day at Alexandria, 5000 stadia north of Syene, a vertical stick cast a shadow so that the sun was 7° 12' south of the point directly overhead. Here is the picture. The vertical stick at S (Syene) points away from O, the centre of the earth, and toward the sun. The vertical stick at A (Alexandria) points away from O but the line from A to the sun makes an angle of 7° 12' with the stick. (The arrows which show the direction to the sun are practically parallel because the sun is very far away). The angle at the centre of the earth between the rays \( \overrightarrow{OA} \) and \( \overrightarrow{OS} \) must be 7° 12'.

that is, \( \frac{1}{50} \) of the 360° around O. Therefore

5000 stadia is \( \frac{1}{50} \) of the circumference of the earth, and this circumference is

50 (5000) = 250,000 stadia. The details are interesting but the important thing for us to notice is that Eratosthenes, who first made this calculation, thought in terms of a picture which took the place of the real thing. That is, he used a mathematical model to
think with. Indeed when we say that the earth is a sphere we already use a math­ematical model – a geometrical one – to replace the earth in our thinking. This example is typical of how mathematics is put to work. It is an important example because without the idea that the earth is a sphere, Columbus would not have thought that he could reach the Indies by sailing west from Spain, and the new world, America, would not have been discovered.

We have seen that the arithmetic of numbers is a sort of world of its own that can be explored. The same is true of geometry. When the two were joined some surprising things happened. It was natural to assume that any two line segments could be measured in terms of some unit. We saw in Volume 2 that the attempt to measure the diagonal of a square led to the conclusion that there could be no common unit of measure for the side and diagonal. This means that a new kind of number had to be invented. Finally mankind invented a set of numbers, the real numbers, that satisfy all the needs of measurement. These numbers obey a very simple set of rules.

**Measurement in General.**

In the present volume we have seen how numbers can be attached to other quantities than those that occur in geometry. An example is the quantity weight. This too is an idea. That is, we imagine that bodies have a property called weight to which we can give a number (after the choice of a unit). Similarly, temperature is another quantity that we imagine and set out to measure. The history of modern science and technology is a history of the invention of mathematical models that we use to understand nature. When a successful model has been discovered we can understand and master a new aspect of nature. We think in the language of these models which are clear and easy to work with.

**Certainty and Uncertainty.**

There is one feature of mathematics that we must emphasize. This is the fact that mathematical statements seem so certain, so sure. When people are asked for an example of a statement that is absolutely certain they usually give “Two plus two equals four”. No one who understands the meaning of the words can possibly doubt it. Mathematical ideas are so clear-cut and the arguments so convincing that everyone who understands what is being said agrees with them. This feeling of certainty is carried over to science as successful mathematical models of nature are invented.

The fact is that it is possible to invent successful mathematical models of nature, mathematical pictures of nature that really work. By using these pictures and ideas we come to understand nature. As we come to understand we gain increasing control. In the present volume we have been able only to introduce you to the elements of scientific measurement and the mathematics that goes with it. Like Newton we stand on the seashore picking up pebbles while the boundless ocean of truth lies before us. Since Newton’s day man has sailed far out into this ocean but the adventure has only begun.
In talking of functions and formulas we have given only a hint of the mathematics of the last 300 years.

These ideas have proved to be very powerful but they are only a part of modern mathematics. We can only invite the reader to look elsewhere for further chapters of the great adventure. A bibliography has been added to suggest where he may turn for additional information. We hope that like all good teachers you will constantly strive to learn more. If you do go on you will share in the great adventure and be able to pass on to your pupils some of your knowledge and your enthusiasm. For it is in this way that each generation stands on the shoulders of the one that came before.

All of our scientific progress has been made on the basis of a certain faith, a certain belief. Modern man has believed that nature must make mathematical sense. That is, he has believed that with sufficient imagination we can invent mathematical models which represent nature. Nature must be mathematically reasonable. This is our faith.

All experience seems to bear this out, up to and including the exciting work with satellites and outer space exploration now going on. The world being opened by scientific investigation seems to have no limits. As the certainty of mathematics is carried into our knowledge of nature by the invention of successful models we gain increasing mastery and control of the circumstances of our lives.

What of human nature and the problems of living together? What can mathematics do to help? Certainly a great deal. Of course, the problems are different from those in the physical sciences. One stone falls in almost the same way as another but human beings are individuals that act differently. Variety is the rule more than regularity. For this reason the mathematics we use is often that of probability and statistics. Mathematical models are more difficult to invent and only the first steps have been taken. But attempts to make them are quite recent. It is hard to predict how successful they will be.

The hope of the world is that more understanding can be introduced into human affairs. There are, of course, other ways of understanding than the mathematical way, but we should not reject any method that leads to greater understanding. It seems to us that the greatest contribution that mathematics can make is the spirit in which it is conducted. Mathematics is universal. It appeals to the good sense of all mankind. Its truths are equally open to everyone. It is a common coinage of understanding. The world has something to learn from the fellowship of mathematicians who are united in a common cause of understanding.

And now finally we wish you all success in your chosen profession. Yours is a great calling. You can shape the attitudes of new generations and open to them the tremendous opportunities and responsibilities of a rapidly changing world. The future is in your hands and theirs.
A list of books for supplementary reading on mathematics, beyond the content of Basic Concepts. These books should broaden the reader’s idea of mathematics. Three titles that are somewhat more advanced have been marked with an asterisk.

Adler, Irving  
*The New Mathematics*, New York, New American Library, 1960, paper $ .60

Allendoerfer, C.B. and Oakley, C.O.  

Bell, E. T.  
*Men of Mathematics*, New York, Simon and Schuster, 1937, paper $ 2.95

Boehm, G.A.W.  
*New World of Mathematics*, New York, Wm. Morrow and Co., paper $1.50

*Courant R. and Robbins H.*  
*What is Mathematics?*, New York, Oxford University Press, 1941, text ed. $7.00

Davis, P.J.  
*The Lore of Large Numbers*, New York, Random House, 1961, paper $1.95

*Goldberg, S.*  

Haag, V.H.  

Kline, M.  
*Mathematics and the Physical World*, Garden City, N.Y. Doubleday Anchor Books, 1959, $1.95

*Kutuzov, B.V.*  
NCTM  

NCTM  
*Topics in Mathematics for Elementary Teachers*, (29th Yearbook), 1965, paper $2.15

Peter, R.  

Polya, G.  
*How to Solve It*, (2nd ed.), Garden City, N.Y. Doubleday Anchor Books, 1957, paper $ .95
**ANSWERS to EXERCISES**

Chapter 58

**EXERCISE 58-2**

a. By tying one end of the rope, and hanging weights on the other end until the rope breaks. Pounds or kilogrammes.

b. By collecting water in a vessel, measuring the depth after each rain, and adding the measures of depth for a year. Usually measured in inches.

c. By the weight of salt dissolved in a unit of volume. Pounds per cubic foot, or grammes per litre.

d. By dividing the distance covered in a given time (assuming that the speed is uniform) by the time. Miles per hour, or feet per second.

**EXERCISE 58-3**

1. 1 third + 4 thirds = 5 thirds
   4 thirds + 1 third = 5 thirds

2. Choose a new unit which is one sixth of U.

   \[
   \frac{1}{2} U + \frac{1}{3} U = 3 \text{ sixths} + 2 \text{ sixths} = 5 \text{ sixths}
   \]

   \[
   \frac{1}{3} U + \frac{1}{2} U = 2 \text{ sixths} + 3 \text{ sixths} = 5 \text{ sixths}
   \]

**EXERCISE 58-4**

4. a. 15 ft. = 4.57 metres
   b. 5 lb. = 2.27 Kg.
   c. 2 in. = 5.08 cm.
   d. 100 sq. ft. = 93 sq. metres

5. 15 mi./hr. = 22 ft./sec.

**EXERCISE 58-5**

1. a. 2 fourths + (1 fourth + 3 fourths) = (2 fourths + 1 fourth) + 3 fourths
   since \(2 + (1 + 3) = (2 + 1) + 3\) by the Associative Property of Addition of Whole Numbers.

   b. One twelfth. The result follows since \((6 + 4) + 3 = 6 + (4 + 3)\).

2. In terms of the new unit
   \[a + 0 = a\]
**EXERCISE 58-8**

1. We could measure the time by the volume or weight of water which collects in that time. The rate of flow from the tanks must remain constant. This will be true if the tank remains full, but if not, the water will flow more slowly as the water level falls.

2. You could measure an interval of time by the number of beats of the heart or pulse. (Galileo timed a pendulum in church in this way.) The pulse rate may vary. If one becomes excited, for example, the pulse will speed up.

3. You might have someone hold up one, two or three fingers and determine how far away he would have to walk in order for you to be certain to know how many fingers he held up.

4. Take a watch and find out how far away it can be held so that you can still hear it tick.

5. You might give him a list of unfamiliar words and see how many he could remember the next day or week or month. Again, you could have him look at a large number of pictures of people in a short time and see how many he recognized the next week or month.

   There are, however, differences in people in the kinds of things they remember, so that these two tests would not be equivalent.

6. You should devise a test with questions which require ability to reason on unfamiliar material. See the discussion in Section 58-9.

---

**EXERCISE 58-9**

122°F corresponds to 50°C. Reason: 50°C is half-way from 0°C to 100°C.

Half-way from 32°F to 212°F is \(32 + \frac{90}{2} = 122°F\).

---

**Chapter 59**

**EXERCISE 59-1**

a. 8:1,  

b. 1:5,  

c. 1:5,  

d. 1:5,  

e. 13:6,  

f. 1:15,  

g. 2:3,  

h. 9:10,  

i. 9:4,  

j. 7:12,  

k. 21:4,  

l. 6:1.
EXERCISE 59-3

1. a. 4:1, b. 32:45, c. 25:2, d. 8:5, e. 2:7.
2. a. 38:57, b. 29:15, c. 50:7.
3. a. \( \frac{5}{4} \), b. \( \frac{1}{4} \), c. \( \frac{3}{4} \), d. \( \frac{1}{2} \), e. 1 hour,
   f. \( 26\frac{2}{3} \) metres,
   g. 28 ounces.

EXERCISE 59-4

1. a. 37.5\%, b. 233.33\%, c. 34.4\%, d. 88.89\%, e. 6.67\%.
2. Second.
3. First.

EXERCISE 59-5

1. 175 miles.
2. 1:50,000.
3. 4 in. by 3 in.; \( \frac{3}{8} \) in. by \( \frac{5}{8} \) in.
4. c.
5. 63,360 inches; 1:63,360.
6. a. 1:1,267,200, b. 85 miles.
7. \( \frac{1}{24} \) inch.
8. a. 1:72, b. 120 ft., c. 1:72.
9. \( \frac{1}{8} \) inch.
10. 180 hectares.
11. 169 hectares.

EXERCISE 59-6

1. b. No, d. No.
2. a. 75:45 = 30:18, b. \( \frac{22}{3} : 16 = \frac{3}{4} \),
   c. 5:1 : 1:7 = 0.9 : 0.3, d. \( \frac{1}{4} : \frac{1}{2} = \frac{1}{10} : \frac{1}{5} \).
EXERCISE 59-7A

1. b. No.
2. d. No.
3. \( \frac{125}{4}; \frac{4}{5}; \) 20.

EXERCISE 59-7B

1. \( 3:9 = 7:21; \) \( 3:7 = 9:21; \) \( 21:9 = 7:3; \) \( 21:7 = 9:3. \)

EXERCISE 59-8A

1. a. 9, b. 6, c. 132, d. 7, e. 48, f. 1.44.
2. a. 63, b. 3, c. 18.
3. 1,500 metres; 1:20,000.
4. 15 in. hes.
5. 317 ft.
6. 1:1,000; 75 m.; 30 cm.

EXERCISE 59-8B

1. 75 ft.
2. \( 112\frac{1}{2} \) ft.
3. 18 ft.

EXERCISE 59-9

1. 870 miles; 1,305 miles; 2,175 miles; 1,522.5 miles. Direct proportionality.
2. 95
3. 2.56 inch.
4. 1,050

EXERCISE 59-10

1. 24; 10; 48.
2. 128
3. Direct proportionality: a., c., f., g., h., i., k., l., m.
   Inverse proportionality: b., d., i.
**Chapter 60**

**EXERCISE 60-2**

1. Let $s$ inches be the length of the side of the cube and $V$ cubic inches the volume. Then $V = s^3$. The same formula applies for $s$ feet and $V$ cubic feet, or $s$ cm. and $V$ cu. cm.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$V$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
</tr>
</tbody>
</table>

2. Let $s$ inches be the length of the side of the cube and $S$ square inches the surface area. Then $S = 6s^2$.

<table>
<thead>
<tr>
<th>$s$</th>
<th>$S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>27</td>
</tr>
</tbody>
</table>

3. If $d$ cm. is the diameter of the circle and $A$ sq. cm. its area

\[ A = \frac{\pi d^2}{4} \quad (\pi = \frac{22}{7}) \]

<table>
<thead>
<tr>
<th>$d$</th>
<th>$A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{\pi}{4}$</td>
</tr>
<tr>
<td>2</td>
<td>$\pi$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{9\pi}{4}$</td>
</tr>
</tbody>
</table>

4. If $s$ units is the length of the side of the square and $p$ units the perimeter of the figure

\[ p = 3s + \frac{\pi s}{2} = (3 + \frac{\pi}{2})s \]

<table>
<thead>
<tr>
<th>$s$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$3 + \frac{\pi}{2}$</td>
</tr>
<tr>
<td>2</td>
<td>$6 + \pi$</td>
</tr>
<tr>
<td>3</td>
<td>$9 + \frac{3\pi}{2}$</td>
</tr>
</tbody>
</table>
5. If \( s \) units is the length of the side of the square and \( A \) square units the area of the figure

\[
A = s^2 + \frac{\pi s^2}{8} = (1 + \frac{\pi}{8})s^2
\]

<table>
<thead>
<tr>
<th>( s )</th>
<th>( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 1 + \frac{\pi}{8} )</td>
</tr>
<tr>
<td>2</td>
<td>( 4 + \frac{\pi}{2} )</td>
</tr>
<tr>
<td>3</td>
<td>( 9 + \frac{9\pi}{8} )</td>
</tr>
</tbody>
</table>

**EXERCISE 60-3**

1. a. \( \frac{1}{3} \)
   b. 3
   c. 5280
   d. 4
   e. 100
   f. \( \frac{1}{100} \)

2. a. \( y = \frac{1}{3}f \)
   b. \( f = 3y \)
   c. \( f = 5280 \text{ m} \)
   d. \( q = 4g \)
   e. \( c = 100 \text{ m} \)
   f. \( m = \frac{c}{100} \)

3. a. \( p = 2000 \text{ t} \)
   b. \( i^2 = 144f^2 \)
   c. \( i^3 = 1728f^3 \)
   d. \( c^2 = 10,000 \text{ m}^2 \)

4. \( m = 30 \text{ g} \)

5. \( h = \frac{m}{4} \)

6. \( m = 200 \text{ h} \) (The number of miles is 200 times the number of hours.)
EXERCISE 60-4

1. Let \( n \) be the number in column A. Then the numbers in column B are given by the formulas

   a. \( n + 1 \)
   b. \( n + 4 \)
   c. \( n^3 \)
   d. \( n^2 + 1 \)
   e. \( n(n + 1) \) or \( n^2 + n \).

Experiment 3

<table>
<thead>
<tr>
<th>Experiment 3</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Set A: Point on rule:</td>
<td>36''</td>
<td>27''</td>
<td>24''</td>
<td>21''</td>
<td>20''</td>
</tr>
<tr>
<td>Set B: Number of pennies:</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>6</td>
<td>9</td>
</tr>
</tbody>
</table>

Experiment 5

<table>
<thead>
<tr>
<th>Experiment 5</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Set A: Length of string:</td>
<td>18''</td>
<td>15''</td>
<td>12''</td>
<td>9''</td>
<td>6''</td>
<td>3''</td>
</tr>
<tr>
<td>Set B: Number of oscillations in 30 seconds:</td>
<td>6.4</td>
<td>7^-</td>
<td>7.75</td>
<td>9^-</td>
<td>11</td>
<td>15.6</td>
</tr>
</tbody>
</table>

The results of the remaining experiments will depend upon circumstances—the rubber band (amount stretched) — the location and the time of year.

EXERCISE 60-5

1. A function
2. A function
3. A function
4. Not a function
5. Not a function
6. A function
7. Not a function. (For example, 2 in set A corresponds to \( 1 = 1 \) and \( 1 = -1 \) in set B.)
Chapter 61

EXERCISE 61-1

1.  

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(1,1)</td>
<td>(2,1)</td>
<td>(3,1)</td>
<td>(4,1)</td>
</tr>
<tr>
<td>1</td>
<td>(1,2)</td>
<td>(2,2)</td>
<td>(3,2)</td>
<td>(4,2)</td>
</tr>
<tr>
<td>2</td>
<td>(1,3)</td>
<td>(2,3)</td>
<td>(3,3)</td>
<td>(4,3)</td>
</tr>
<tr>
<td>3</td>
<td>(1,4)</td>
<td>(2,4)</td>
<td>(3,4)</td>
<td>(4,4)</td>
</tr>
</tbody>
</table>

2.  

- (1/2, 1)
- (3/2, 3/2)
- (2, 1/3)

3.  

(0,0), (0,1), (0,2), (0,3), (0,4)

4.  

(-1, -1), (-1, 1), (1, -1), (1, 1)
EXERCISE 61-2

1.

\[(1.2)^2 = 1.4\]
\[(1.2)^2 = 1.44\]
\[(2.9)^2 = 8.5\]
\[(2.9)^2 = 8.41\]
2.

Graph of \( y = 3x + 2 \) is parallel to the graph of \( y = 3x \) and 2 units above it.

3.

\[ y = x + 2 \]

4. and 5.

\[ y = 3x + 2 \]

Straight line through (0,0).

Graph of \( y = 3x + 2 \) is parallel to the graph of \( y = 3x \) and 2 units above it.
EXERCISE 61-3

1.

2. The point P = (1, c) is on the graph. So is the point O(0, 0). Draw ray OP. Let x be any positive number ≠ 1. Draw a vertical line through the point (x, 0) = S. The triangles ORP and OSQ are similar. Hence, \( \frac{SQ}{OS} = \frac{RP}{OR} \), that is, \( \frac{SQ}{x} = \frac{c}{1} \) and \( SQ = cx \). The point Q is therefore the point (x, cx) = (x, y). Hence Q is on the graph of \( y = cx \).

If a point \( Q' \) off this ray were part of the graph, let \( Q' = (x, b) \). Now \( Q = (x, cx) \) is on the graph. If \( Q' \neq Q \), we have a contradiction.
3. See the discussion at the beginning of Section 61-4.

4. \( y = x + 2 \) is parallel to \( y = x \) and therefore has the slope 1. The shaded figure is a parallelogram.

5. \( y = 2x + 1 \)

**EXERCISE 61-4**

1. Since \( F = \frac{9}{5} C + 32 \) when 
   
   \[ C = -273, \quad F = -\frac{9(273)}{5} + 32 \]
   
   \[ = -\frac{2457}{5} + 32 \]
   
   \[ = -491\frac{2}{5} + 32 = -459\frac{2}{5} \]
2. \[ 0^\circ F = -18^\circ C \]. If \[ \frac{9}{5} C + 32 = 0 \]

\[ \frac{9}{5} C = -32 \]

\[ C = \frac{5}{9} (-32) = -\frac{160}{9} = -17\frac{7}{9} \]

3. From the graph \[ C \approx 10 \]

Check!

\[ \frac{9}{5} \times 10 + 32 = \frac{90}{5} + 32 = 18 + 32 = 50 \]

\[ 50 = \frac{9}{5} C + 32 \]

\[ 18 = \frac{9}{5} C \]

\[ 2 = \frac{C}{5} \]

\[ C = 10 \]

---

**EXERCISE 61-5**

1. \[ y = 0 \] if \( x = 6 \)

\[ y = 1 \] if \( x = 8 \)

\[ x = 2(y + 3) \]

2. \[ y = 0 \] if \( x = 1 \)

\[ y = 1 \] if \( x = \frac{1}{3} \)

\[ x = \frac{y}{3} + 1 \]

3. \[ y = \frac{4}{x} \]

\[ x = \frac{4}{y} \]

4. 

\[ y = x^2 \]

\[ y = 1 \] if \( x = 1 \) or \( x = -1 \) (Two answers)

\[ y = 4 \] if \( x = 2 \) or \( x = -2 \) (Two answers)

\[ x = \sqrt{y} \] or \( x = -\sqrt{y} \) (Two formulas)
2. We can restrict the time $t$ to values from 0 to 1 inclusive, or from 1 to 2 inclusive.

3. The distance $d$ increases from 0 to about 1300 (miles) then decreases to 0. There is not an inverse function whole time of the trip.

$h = 32t - 16t^2$

$t = 0$, $h = 0$
$t = \frac{1}{2}$, $h = 12$
$t = 1$, $h = 16$
$t = \frac{3}{2}$, $h = 12$
$t = 2$, $h = 0$
EXERCISE 62-2A

2. a. 7  b. 3  c. 14  d. 10  
   e. 12  f. 30  g. $8\frac{1}{2}$  h. $\frac{5}{6}$

3. $C = 5$ degrees

4. 9

EXERCISE 62-2B

1. a. 15  
   b. 21

2. (a) 2  (b) 24

3. (a) 3  (b) 3  (c) 4

4. 25

EXERCISE 62-3

1. 125 gallons

   \[ m = \frac{G - 5}{2} \]

   \[ m = 90 \text{ minutes} \]

2. 13 teaspoons of tea

   \[ n = T - 1 \]

   13 people

3. 576 cu. ins.

   \[ h = \frac{V}{16} \]

   12 inches

4. 660 sq. inches

   \[ h = \frac{S - 308}{44} \]

   \[ n = 36 \text{ inches} \]
5. \[ E = 15 \frac{1}{2} \]
\[ n = 2E - 16 \]
\[ n = 8 \text{ when } E \text{ is } 12 \]

6. \[ S = 9 \text{ when } a = 3, \ t = 2 \]
\[ t = \frac{S - a}{3} \]
\[ t = 6 \text{ when } S = 24, \ a = 6 \]

7. \[ y = 7 \text{ when } x = 2, \ z = 3 \]
\[ z = 5x - y \]
\[ z = 55 \text{ when } x = 12, \ y = 5 \]

**EXERCISE 62-4**

1. \[ x > 1 \]

2. \[ x > \frac{1}{4} \]

3. \[ x < \frac{4}{5} \]

4. \[ x < -9 \]

5. \[ x > -1 \]

6. \[ x \text{ lies between } -1 \frac{1}{2} \text{ and } 1 \]

8. \[ T = \frac{1}{2} \text{ when } R = 20, \ r = 8 \]
\[ r = RT - R \]
\[ r = 16 \text{ when } T = 2, \ R = 16 \]

9. \[ V = 36\pi, \text{ when } r = 3 \]
\[ r^3 = \frac{3V}{4\pi} \]
\[ r^3 = 9,261 \text{ when } V = 38,808 \]
\[ r = 41 \]
EXERCISE 62-5

1. \[3x + 1 \geq 4\] \[3x \geq 3\] \[x \geq 1\]

2. \[1 - 3x \leq \frac{1}{4}\] \[\frac{3}{4} \leq 3x\] \[x \geq \frac{1}{4}\]

3. \[5x - 3 \leq 1\] \[x \leq \frac{4}{5}\]

4. \[2 - \frac{1}{3}c \leq 5\] \[\frac{1}{3}c \geq -3\] \[c \geq -9\]

5. \[8 - 3x \leq 11\] 
   \[-3 \leq 3x\]
   \[-1 \leq x\]
   \[x \geq -1\]

6. \[-3 \leq 2x \leq 1\] \[-\frac{3}{2} \leq x \leq \frac{1}{2}\]

7. \[3 \geq x \geq \frac{1}{2}\]

Chapter 63

EXERCISE 63-2

1. 90, 83, 76, 68, 66, 66, 66, 64, 63, 61, 58, 56, 54, 54, 54, 52, 52, 51, 51, 51, 51, 30, 30, 28, 28, 25, 24, 16, 14, 14, 13, 11, 6, 6, 6, 4, 2.
   (a) 11  (b) There are 24 scores above 25.

Hence the percentage \[= \frac{24}{36} \times 100\% = 66 \frac{2}{3}\%\].

(c) 90 - 2 = 88.
2. Score | Tally Marks | Frequency
--- | --- | ---
92 | / | 1
91 | / | 1
90 | / | 2
88 | /// | 3
87 | /// | 3
86 | /// | 3
85 | /// | 5
84 | / | 1
83 | / | 2
82 | / | 2
81 | / | 1

Total frequency = 24

3. Score interval | Tally marks | Score boundaries | Mid-point | Frequency
--- | --- | --- | --- | ---
96 - 100 | / | 95.5 - 100.5 | 98 | 1
91 - 95 | // | 90.5 - 95.5 | 93 | 2
86 - 90 | /// | 85.5 - 90.5 | 88 | 4
81 - 85 | /// | 80.5 - 85.5 | 83 | 4
76 - 80 | // | 75.5 - 80.5 | 78 | 5
71 - 75 | /////// | 70.5 - 75.5 | 73 | 8
66 - 70 | //////// /// | 65.5 - 70.5 | 68 | 10
61 - 65 | /////////// /// | 60.5 - 65.5 | 63 | 18
56 - 60 | /////////// /// | 55.5 - 60.5 | 58 | 20
51 - 55 | /////////// /// | 50.5 - 55.5 | 53 | 12
46 - 50 | //////// /// | 45.5 - 50.5 | 48 | 8
41 - 45 | /////// | 40.5 - 45.5 | 43 | 5
36 - 40 | /////// | 35.5 - 40.5 | 38 | 3

Total Frequency = 100

(a) 58 (b) 10 + 18 + 20 + 12 + 8 + 5 + 3 = 76
(c) In the interval 76 - 80.

4. Price Interval | Tally Marks | Frequency
--- | --- | ---
160 - 164 | /// | 3
165 - 169 | /// | 3
170 - 174 | /////////// | 9
175 - 179 | // | 2
180 - 184 | /////// | 5
185 - 189 | // | 0
190 - 194 | / | 0
195 - 199 | / | 0
200 - 204 | / | 1
205 - 209 | / | 1

Total Frequency = 24

246
5. (a) In both Northern Nigeria and Eastern Nigeria the number of pupils is largest in the first year of the Primary School course and this decreases gradually until the seventh year.

(b) Total number of Primary School pupils in Northern Nigeria in 1963 = 410,706
Number of pupils in the first year = 91,567
Ratio of those enrolled in the first year to the total school population = \[
\frac{91,567}{410,706}
\]

Total number of Primary School pupils in Eastern Nigeria in 1963 = 1,278,706
Number of pupils in the first year = 346,126
Ratio of those enrolled in the first year to the total school population = \[
\frac{346,126}{1,278,706} = 27\% \text{ approx.}
\]

6. Answers will vary. The question assumes that there will be no pupils under six years of age. If the assumption is wrong a necessary adjustment should be made.

It may be advisable to draw up a table showing:

<table>
<thead>
<tr>
<th>Age group</th>
<th>Tally marks</th>
<th>Frequency</th>
</tr>
</thead>
</table>

The tally marks should be put in the appropriate place as each pupil is asked to state his age. Alternatively the ages may be first recorded and sorted out later.

7. A suggested table is the following:

<table>
<thead>
<tr>
<th>Monday</th>
<th>Tuesday</th>
<th>Wednesday</th>
<th>Thursday</th>
<th>Friday</th>
</tr>
</thead>
<tbody>
<tr>
<td>Primary 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Primary 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Primary 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Primary 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Primary 5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Primary 6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The number of pupils present in each class should be entered in the appropriate column.

8. This exercise could be done by two persons. One person could count the number of cars and commercial vehicles going in a certain direction while the other counts the number travelling in the opposite direction. Each person should make a stroke each time a car passes the point. The two records should be
Jaguar
Hillman Minx
Chevrolet
Ford Consul

10. Your table will probably look like this:

<table>
<thead>
<tr>
<th>Year</th>
<th>Motor-cars</th>
<th>Buses</th>
<th>Motor-cycles</th>
<th>Lorries</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1964</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1965</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1966</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

11. Make a frequency distribution of the number of sixes thrown.
12. Tabulate appropriately.
13. Tabulate showing (a) name of candidate (b) number of votes received.
14. and 15. Tabulate as suggested in the answer to question 6.

**EXERCISE 63-3**

1. Year | Number of Out-Patients (Expressed to the nearest 100,000.)
--------|----------------------------------------------------
1953    | 1,300,000                                           
1954    | 1,300,000                                           
1955    | 1,200,000                                           
1956    | 1,300,000                                           
1957    | 1,000,000                                           
1958    | 900,000                                             
1959    | 1,000,000                                           
1960    | 1,200,000                                           
1961    | 1,200,000                                           
1962    | 1,300,000                                           

2. Year | Value of Postal Orders £ | Value of Postal Orders £ (to nearest thousand) | Value of Postal Orders £ (to nearest ten thousand)
--------|--------------------------|-----------------------------------------------|-----------------------------------------------
1953    | 1,975,813                | 1,976,000                                     | 1,980,000                                     
1954    | 2,055,243                | 2,055,000                                     | 2,060,000                                     
1955    | 2,344,964                | 2,345,000                                     | 2,340,000                                     
1956    | 2,402,413                | 2,402,000                                     | 2,400,000                                     
1957    | 2,446,669                | 2,447,000                                     | 2,450,000                                     
1958    | 2,327,144                | 2,327,000                                     | 2,330,000                                     
1959    | 2,154,417                | 2,154,000                                     | 2,150,000                                     
1960    | 2,139,895                | 2,140,000                                     | 2,140,000                                     
1961    | 2,253,302                | 2,253,000                                     | 2,250,000                                     
1962    | 2,285,532                | 2,286,000                                     | 2,290,000                                     

(Expressed to the nearest 100,000.)
3. Value of Domestic Exports

<table>
<thead>
<tr>
<th>Year</th>
<th>£</th>
<th>Expressed to Nearest 100,000 Pounds</th>
<th>Expressed to Nearest Million Pounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>1957</td>
<td>10,348,000</td>
<td>10,300,000</td>
<td>10,000,000</td>
</tr>
<tr>
<td>1958</td>
<td>11,066,000</td>
<td>11,100,000</td>
<td>11,000,000</td>
</tr>
<tr>
<td>1959</td>
<td>13,375,000</td>
<td>13,400,000</td>
<td>13,000,000</td>
</tr>
<tr>
<td>1960</td>
<td>13,802,000</td>
<td>13,800,000</td>
<td>14,000,000</td>
</tr>
<tr>
<td>1961</td>
<td>14,172,000</td>
<td>14,200,000</td>
<td>14,000,000</td>
</tr>
<tr>
<td>1962</td>
<td>13,668,000</td>
<td>13,700,000</td>
<td>14,000,000</td>
</tr>
<tr>
<td>1963</td>
<td>15,405,000</td>
<td>15,400,000</td>
<td>15,000,000</td>
</tr>
</tbody>
</table>

Chapter 64

**EXERCISE 64-4A**

1. May was the wettest month in Nairobi in 1962. Rain fell on 17 days.
2. February and July were the two driest months. Rain fell on two days in each month.
3. The number of rainy days from January to June was 56 and from July to December was 49. January to June was therefore wetter than July to December.
4. There were 105 rainy days and hence 260 dry days.
5. (a) January and October; 11 days.
(b) February and July; 2 days.
(c) March and June; 6 days.
(d) April and December; 14 days.

**EXERCISE 64-4B**

1. See Graph for question 1 of Exercise 64-4B.
2. See Graph for question 2 of Exercise 64-4B.
3. Answers will vary from school to school. This question applies to a co-educational school.
4. Answers will vary.
5. See Graph for question 5 of Exercise 64-4B. Note that we have rounded to nearest hundred thousand Le. Note also that highest bar is broken to indicate that it would go off the page if all of it appeared. For added clarity, the
The total of £125 million is divided into parts of 70.4% (Customs and Excise), 5.6% (Direct Taxes), and 24.0% (Others). These percentages are indicated as parts of the entire bar which represents the total 100%. The £-value of each source of income is also added to the graph.

(See Graph for question 8 of Exercise 74-4B.)

9. Space out the classes along the horizontal axis and the number of pupils present along the vertical axis.

10. If the figures are rounded off to the nearest hundred thousand pupils we should get

<table>
<thead>
<tr>
<th>Year</th>
<th>Boys</th>
<th>Girls</th>
</tr>
</thead>
<tbody>
<tr>
<td>1955</td>
<td>2.8</td>
<td>1.4</td>
</tr>
<tr>
<td>1960</td>
<td>3.1</td>
<td>1.7</td>
</tr>
</tbody>
</table>

(See Graph for question 10 of Exercise 64-4B.)

11. (a) Of the four countries shown on the graph, Congo (Leopoldville) had the largest number of schools in 1959.

(b) Ivory Coast had the least number of schools.

(c) Ghana: 3,700 schools.

Congo (Leopoldville): 16,000 schools.

12. See Graph for question 12 of Exercise 64-4B. Find the percentage of passes in each school. Achimota school will be found to be the better school if the school having the higher percentage of passes is regarded as "better".

13. As the quantity of rain on any one day is likely to be small you will have to choose a scale which will make the graph stand out.
Graph for Question 2 of

*EXERCISE 64-4B*

Mean Number of Hours Per Day of Sunshine in Kisumu
Graph for Question 5 of **EXERCISE 64-4B**

Imports into Sierra Leone, Oct.-Dec. 1964
(to nearest hundred thousand £)
Source of Nigerian Federal Government Revenue – 1964

Percentage due to different sources

- £88 million
- £7 million
- £30 million

- Customs and Excise
- Direct Taxes
- Other
**Graph for Question 10 of**

**EXERCISE 64-4B**

Enrollment of Pupils by Sex in Ghana Primary Schools

(a) Boys

(b) Girls

Enrolment of Pupils by Sex in Ghana Primary Schools

- Number of Pupils in hundred thousands
- 1955: 3.0
- 1960: 4.0
Results of West African School Certificate Examination in Two Schools in Ghana in June 1964

Number of Pupils

- Achimota School
- Mfantsipim School
Graph for Question 1 of

**EXERCISE 64-8**

Scores on Test

Frequency of Scores

![Histogram showing scores on a test](image-url)
Graph for Question 2 of

EXERCISE 64-8

Score

Frequency
Graph for Question 3 of
EXERCISE 64-8

<table>
<thead>
<tr>
<th>Price (pounds per ton)</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>159.5</td>
<td>3</td>
</tr>
<tr>
<td>164.5</td>
<td></td>
</tr>
<tr>
<td>169.5</td>
<td></td>
</tr>
<tr>
<td>174.5</td>
<td></td>
</tr>
<tr>
<td>179.5</td>
<td>9</td>
</tr>
<tr>
<td>184.5</td>
<td></td>
</tr>
<tr>
<td>189.5</td>
<td>5</td>
</tr>
<tr>
<td>194.5</td>
<td></td>
</tr>
<tr>
<td>199.5</td>
<td></td>
</tr>
<tr>
<td>204.5</td>
<td></td>
</tr>
<tr>
<td>209.5</td>
<td></td>
</tr>
</tbody>
</table>
Graph for Question 4 of

EXERCISE 64-8

Scores

Cumulative Frequency

30.5 35.5 40.5 45.5 50.5 55.5 60.5 65.5 70.5 75.5 80.5 85.5 90.5 95.5 100.5
Graph for Question 5 of

EXERCISE 64-8

Examination Results

Excellent Very Good Good Credit Pass Fail
1. The boys received £800, £600, and £400 respectively. The circle graph is divided into three sectors, with central angles 160°, 120°, and 80°.

4. The total number of boys and girls is 3,503. Therefore the percentage of boys is

\[
\frac{2,430}{3,503} \times 100\% = \frac{243,000}{3,503} = 69.5\%
\]

Therefore, the boys are represented by a sector having central angle equal to 69.5° \times 360° = 250.2° (≈ 250° to the nearest degree).
<table>
<thead>
<tr>
<th>Item</th>
<th>Percentage</th>
<th>Degrees in Pie Chart (to nearest degree)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Food</td>
<td>58%</td>
<td>(0.58 \times 360^\circ = 209^\circ)</td>
</tr>
<tr>
<td>Rent and Water</td>
<td>13%</td>
<td>(0.13 \times 360^\circ = 47^\circ)</td>
</tr>
<tr>
<td>Clothing</td>
<td>7%</td>
<td>(0.07 \times 360^\circ = 25^\circ)</td>
</tr>
<tr>
<td>Misc.</td>
<td>22%</td>
<td>(0.22 \times 360^\circ = 79^\circ)</td>
</tr>
</tbody>
</table>
Chapter 65

**EXERCISE 65-4**

1. 2; 2.5; 3; 3.5; 4; 4.5; 5; 5.5. The sum of the first n counting numbers is

\[ 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2} \]

Therefore the mean of the first n counting numbers is \( \frac{n+1}{2} \).

2. The median of the first n counting numbers is equal to their mean.

3. a. mean = 13
   b. median = 13.5
   c. modes are 11 and 14
      i. 10 scores are greater than the mean
      ii. 8 scores are less than the mean
      iii. 10 scores are greater than the median
      iv. 10 scores are less than the median

4. a. 31.55  b. 3  c. 34.55

5. i. mode = 6 median = 8 mean = 11
    ii. mode = 26 median = 28.5 mean = 29
    iii. mode = median = mean = 13

6. Mean number of passengers landing per quarter was 34,075.

7. a. mean = \( \frac{1209}{7} \) = 172.6, approximately.
    b. median = 160

8. a. i. Feb., March, April, June, Aug.
    b. Mean monthly number of hours of sunshine over many years = 82.6
        \( \frac{12}{} \) = 6.88, approx.

    Mean monthly number of hours of sunshine in 1962 = \( \frac{83.3}{12} \) = 6.94, approx.

    Therefore, 1962 was a year of above average monthly sunshine.

9. Mean = \( \frac{1043.1}{12} \) = 86.9, approx.

10. mean = 70
    mode = 90
    median = 75
    There are six scores greater than and six scores less than the mean.
**EXERCISE 65-5**

1. Interval | Mid-point (x) | Frequency (f) | fx  
|------------|---------------|--------------|----
| 0-9        | 4.5           | 1            | 4.5|  
| 10-19      | 14.5          | 2            | 29.0|  
| 20-29      | 24.5          | 5            | 122.5|  
| 30-39      | 34.5          | 12           | 414.0|  
| 40-49      | 44.5          | 5            | 222.5|  
| 50-59      | 54.5          | 1            | 54.5|  
| 60-69      | 64.5          | 8            | 516.0|  
| 70-79      | 74.5          | 5            | 372.5|  
| 80-89      | 84.5          | 2            | 169.0|  
| 90-99      | 94.5          | 1            | 94.5|  

Total frequency N = 42  
Total fx = 1999.0  

Mean score = \(\frac{1999}{42} = 47.6\), approx.  
Modal score = 34.5  
Median interval = 40-49  

2. Mean = \(\frac{1983}{42} = 47.2\), approx.  

3. Mode = Mean = 20  

4. Modal score = 42  

Mean score = \(\frac{1394}{33} = 42.4\), approx.  

**EXERCISE 65-6**

1. a. New mean = 23  
b. Original mean = 29  
   New mean = original mean \(-\) 5 = 24  

2. New mean = 39
3. number | number minus mean
---|---
25 | 1
22 | -2
21 | -3
26 | 2
21 | -3
29 | 5

sum of deviations from mean = (1+2+5) + (-2-3-3)

= 8 + (-8)

= 0

**EXERCISE 65-7**

1. **Mid-point of Interval** | **Deviation from Assumed Mean** | **Deviation from Assumed Mean Divided by 10** | **Frequency** | **Product** | **Product**
---|---|---|---|---|---
x | d | t | f | fd | ft
14.5 | -30 | -3 | 7 | -210 | -21
24.5 | -20 | -2 | 12 | -240 | -24
34.5 | -10 | -1 | 13 | -130 | -13
44.5 | 0 | 0 | 13 | 0 | 0
54.5 | 10 | 1 | 31 | 310 | 31
64.5 | 20 | 2 | 12 | 240 | 24
74.5 | 30 | 3 | 5 | 150 | 15
84.5 | 40 | 4 | 5 | 200 | 20
94.5 | 50 | 5 | 1 | 50 | 5
104.5 | 60 | 6 | 1 | 60 | 6

100 | 430 | 43

True mean = 44.5 + \frac{430}{100} = 48.8

True mean = 44.5 + 10(\frac{43}{10}) = 48.8

Chapter 66

**EXERCISE 66-5**

1. a. range = 14 - 2 = 12

mean = 8

sum of squared deviations from mean = (2 - 8)^2 + (4 - 8)^2 + (6 - 8)^2 + (8 - 8)^2 + (10 - 8)^2 + (12 - 8)^2 + (14 - 8)^2 = 112
Average of squared deviations from mean = $112/7 = 16$

Standard deviation = $\sqrt{16} = 4$

b. range = $17 - 9 = 8$
mean = $13$
Average of squared deviations from mean = $60/9 = 20/3$

Standard deviation = $\sqrt{20/3} = 2.6$, approx.

2. a. range = $100 - 60 = 40$
mean = $850/10 = 85$
Average of squared deviations from mean = $1716/10 = 171.6$

Standard deviation = $\sqrt{171.6} = 13.1$, approx.

b. range = $90 - 80 = 10$
mean = $850/10 = 85$
Average of squared deviations from mean = $108/10 = 10.8$

Standard deviation = $\sqrt{10.8} = 3.3$, approx.
The data in (a) are more scattered about the mean than the data in (b).

Spread of B’s scores

<table>
<thead>
<tr>
<th>Score</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>2</td>
</tr>
<tr>
<td>60</td>
<td>3</td>
</tr>
<tr>
<td>70</td>
<td>1</td>
</tr>
<tr>
<td>80</td>
<td>1</td>
</tr>
<tr>
<td>90</td>
<td>1</td>
</tr>
</tbody>
</table>

The data in (a) are more scattered about the mean than the data in (b).
b. For Candidate B: deviations from mean are 45, 45, 45, 45, 45, -45, -45, -45, -45, -45
   sum of absolute deviations is $10 \times 45 = 450$
   Mean absolute deviation $= \frac{450}{10} = 45$

For Candidate D: deviations from mean are 45, 45, 45, 35, 25, -25, -35, -45, -45, -45
   sum of absolute deviations is $6(45) + 2(35) + 2(25) = 390$
   Mean absolute deviation $= \frac{390}{10} = 39$

c. For Candidate B:
   Average of squared deviations from mean $= \frac{20250}{10} = 2025$
   Standard deviation $= \sqrt{2025} = 45$

For Candidate D:
   Average of squared deviations from mean $= \frac{15850}{10} = 1585$
   Standard deviation $= \sqrt{1585} = 39.8$, approx.

4. New standard deviation is twice the original standard deviation.

5. a. New standard deviation is same as old.
   b. New standard deviation is same as old. The general principle is the following: If the same number is added to (or subtracted from) each number in a set of measurements, then the standard deviation of the new set is the same as the original standard deviation.
**EXERCISE 67-1**

1. Most people feel that "heads" and "tails" are equally likely outcomes. Here is the result of one series of coin tossing experiments with the ratio of number of heads to number of tosses computed for each experiment and for the total of all five experiments:

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Number of Coin Tosses</th>
<th>Number of Heads Obtained</th>
<th>Ratio of Heads to Tosses</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>20</td>
<td>5</td>
<td>( \frac{9}{20} = 0.45 )</td>
</tr>
<tr>
<td>2</td>
<td>25</td>
<td>10</td>
<td>( \frac{10}{25} = 0.40 )</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>12</td>
<td>( \frac{12}{20} = 0.60 )</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>18</td>
<td>( \frac{18}{30} = 0.60 )</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
<td>14</td>
<td>( \frac{14}{25} = 0.56 )</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>120</strong></td>
<td><strong>63</strong></td>
<td>( \frac{63}{120} = 0.542 )</td>
</tr>
</tbody>
</table>

Notice the different ratio of heads to tosses obtained in each experiment. Did your results show this variation due to chance? We used an East African 50 cent coin. Perhaps different coins have different chances of landing heads. It is difficult to decide whether our experimental results support or oppose our feelings that heads and tails are equally likely outcomes. The method of making such a decision is part of the theory of mathematical statistics.

2. When you combine the experiments of all students you will be able to compute the overall ratio of heads to total tosses. For example, perhaps there are 2,000 tosses of which 1,084 are heads. Then the over-all ratio of heads to total tosses of the coin would be

\[
\frac{1,084}{2,000} = 0.542
\]

3. For the drawing pin tossing experiment, we obtained the following results:

<table>
<thead>
<tr>
<th>Pin Falls Flat to Total</th>
<th>Number of Drawing</th>
<th>Number of Times</th>
</tr>
</thead>
</table>
These results do not support the hypothesis that the two possibilities "falling flat" and "falling on a side" are equally likely outcomes. It appears that our drawing pin has only about one chance in four of falling flat. Experimental results will vary with the kind of drawing pin used and the table upon which it is tossed. How do your experimental results compare with ours?

**EXERCISE 67-2**

1. a. $U = \{\text{you win, you lose}\}$
   b. $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
   c.

<table>
<thead>
<tr>
<th>First Slip</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>*</td>
<td></td>
<td></td>
<td></td>
<td></td>
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We can list $10 \times 10 = 100$ possible outcomes of this experiment. These correspond to the 100 boxes of the above figure. For example, the box marked with a check (✓) indicates that the first slip had number 6 on it and the second slip had number 3 on it. The boxes marked with asterisks (*) correspond to those outcomes in which a slip is selected, returned to the hat, and then this same slip is again selected.

d. If it is the same as in part (c) except that the ten outcomes indicated by
Each of the $6 \times 6 = 36$ boxes represents one possible outcome of this experiment.

Each of the $2 \times 2 = 4$ boxes represents one possible outcome of this experiment.

So we can take $U = \{BB, BG, GB, GG\}$ where $B$ stands for boy, $G$ for girl. We could also see this from the following table:
So we can take \( U = \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\} \) as universe set for this experiment.

This universe set is the same as the one for part (k) except that H replaces B and T replaces G.

This universe set is the same as the one for part (k) except that H replaces B and T replaces G.
3. Toss a coin until it falls heads for the first time. This first head can occur at the first, second, third, . . . toss, or at any toss. For example we could conceivably have 953 straight tails and the first head at the 954th toss. So there are as many possible outcomes for this experiment as there are counting numbers. Since there is an unending sequence of counting numbers, we have an infinite (unending) number of outcomes in the universe set of this experiment.

As another example, think of throwing a very pointed dart at the part of the number line between 1 and 10. We could imagine the dart falling at any point of this line segment and the set of all these points is not finite. So the universe set for this experiment would have to have as many members as there are points on the line segment. U is therefore an infinite (unending) set.

EXERCISE 6.1-3

1. a. \(U = \{1, 2, 3, 4, 5, 6\}\)
   i. \(\{2, 4, 6\}\)
   ii. \(\{1, 3, 5\}\)
   iii. \(\{1, 3, 5\}\)
   iv. \(\{1, 2, 3, 4, 5, 6\} = U\)
   v. \(\emptyset = \text{the empty set}\)

b. \(U = \{HH, HT, TH, TT\}\)
   i. \(\{HT, TH\}\)
   ii. \(\{HH, HT, TH\}\)
   iii. \(\{HT, TH, TT\}\)

c. \(U = \{\text{Jan.}, \text{Feb.}, \text{March}, \text{April}, \text{May}, \text{June}, \text{July}, \text{Aug.}, \text{Sept.}, \text{Oct.}, \text{Nov.}, \text{Dec.}\}\)
   i. \(\{\text{Jan.}, \text{Feb.}, \text{March}, \text{April}, \text{May}, \text{June}\}\)
   ii. \(\{\text{Jan.}, \text{June}, \text{July}\}\)
   iii. \(\emptyset = \text{empty set since there is no month whose name begins with the letter "B"}\)

d. The universe set U is described in the text, page 161. We shall refer to Chart 1 on that page.
   i. This event consists of those outcomes corresponding to the boxes in the first row of Chart 1, i.e. \(\{\text{Jan.}-\text{Jan.}, \text{Jan.}-\text{Feb.}, \text{Jan.}-\text{March}, \text{Jan.}-\text{April}, \text{Jan.}-\text{May}, \text{Jan.}-\text{June}, \text{Jan.}-\text{July}, \text{Jan.}-\text{Aug.}, \text{Jan.}-\text{Sept.}, \text{Jan.}-\text{Oct.}, \text{Jan.}-\text{Nov.}, \text{Jan.}-\text{Dec.}\}\)
ii. \{1, 3, 5, 7, 9\}

iii. \{1, 3, 5, 7, 9\}

iv. \(U\), since every member of \(U\) is either even or odd.

v. \(\emptyset\) = empty set, since no member of \(U\) is larger than 10.

vi. \{1, 2, 3, 4\}

vii. \{1, 2, 3, 4, 5\}

f. \(U\) is described in the answer to Exercise 67-2 Problem 1, part (c). We shall refer to the Chart given in that answer.

i. This event contains as members those outcomes corresponding to the boxes in the first row of the Chart, i.e., the event is equal to \{1-1, 1-2, 1-3, 1-4, 1-5, 1-6, 1-7, 1-8, 1-9, 1-10\}

ii. \{1-1, 2-1, 3-1, 4-1, 5-1, 6-1, 7-1, 8-1, 9-1, 1-10\}

iii. \{1-1\}

iv. \{2-2, 4-2, 6-2, 8-2, 10-2\}

g. \(U\) is described in answer to Exercise 67-2 Problem 1, part (d).

i. \{1-2, 1-3, 1-4, 1-5, 1-6, 1-7, 1-8, 1-9, 1-10\}

ii. \{2-1, 3-1, 4-1, 5-1, 6-1, 7-1, 8-1, 9-1, 1-10\}

iii. \{\emptyset\} = Empty set since we cannot have the same slip withdrawn twice.

iv. \{4-2, 6-2, 8-2, 10-2\}

h. \(U = \{BBB, BBG, BGB, BGG, GBB, GGB, GGG\}\)

i. \{BBB\}

ii. \{BBB, BBG, BGB, GBB\}

iii. \{BBG, BGB, GBB\}

iv. \{BBG, BGB, GBB, BGG, GBB, GGB, GGG\}

v. \{BGG, GBB, GGB\}

vi. \{BBB, BBG, BGB, BGG, GBB, GGB, GGB\}

vii. \{BGG, GBB, GGB, GGG\}

viii. \{BGG, GGB, GBG, GGG\}

2. a. \(U\) has 6 members. i. 3, ii. 3, iii. 3, iv. 6, v. 0.

b. \(U\) has 4 members. i. 2, ii. 3, iii. 3.

c. \(U\) has 12 members. i. 6, ii. 3, iii. 0.

d. \(U\) has 144 members. i. 12, ii. 12, iii. 2, iv. 0.

e. \(U\) has 10 members. i. 5, ii. 5, iii. 5, iv. 10, v. 0, vi. 4, vii. 5.

f. \(U\) has 100 members. i. 10, ii. 10, iii. 1, iv. 5.

g. \(U\) has 90 members. i. 9, ii. 9, iii. 0, iv. 4.

h. \(U\) has 8 members. i. 1, ii. 4, iii. 3, iv. 7, v. 3, vi. 7, vii. 4, viii. 4.

3. If \(U = \{HH, HT, TH, TT\}\), then there are \(2^4 = 16\) events in all. These events are

The subset with no members: \{\} = the empty set.
The subset with four members: \{HH, HT, TH, TT\} = U itself

4. Since U has 10 members, there are \(2^{10} = 1,024\) events in all.

5. Since U has 8 members, there are \(2^8 = 256\) events in all. There is just the empty set which has no members. There are eight events containing exactly one member. There are 28 events containing exactly two members.

We can count these by noting that there are

- 7 of these events containing BBB
- 6 of these events containing BBG and not already counted in a.
- 5 of these events containing BGB and not already counted in a. or b.
- 4 of these events containing BGG and not already counted in a. or b. or c.
- 3 of these events containing GBB and not already counted in a. or b. or c. or d.
- 2 of these events containing GBG and not already counted in a. or b. or c. or d. or e.
- 1 of these events containing GGB and not already counted in a. or b. or c. or d. or e. or f.

Total: 28 events containing exactly two members. It may take a little time, but there is much to be learned by writing down all these events in the systematic way indicated in a. - g.

**EXERCISE 67-4**

1. a. \(U = \{1, 2, 3, 4, 5, 6\}\)
   b. \(E = \{5, 6\}, \ F = \{1, 3, 5\}\)

   ![Venn Diagram](image)

   or equivalently

   ![Venn Diagram](image)

   c. E and F are not disjoint events since they have the common member 5.

   d. not - E = \{1, 2, 3, 4\}
e. \( \text{not - (not-E)} = \text{not}\{1, 2, 3, 4\} = \{5, 6\} = E \)

f. \( \text{not - E and not - F} = \{1, 2, 3, 4\} \text{ and } \{2, 4, 6\} = \{2, 4\} \)

\( \text{not - (E or F)} = \text{not}\{1, 3, 5, 6\} = \{2, 4\} \)

g. \( \text{not - E or not - F} = \{1, 2, 3, 4\} \text{ or } \{2, 4, 6\} = \{1, 2, 3, 4, 6\} \)

\( \text{not - (E and F)} = \text{not}\{5\} = \{1, 2, 3, 4, 6\} \)

2. a. \( U = \{\text{Jan., Feb., March, April, May, June, July, Aug., Sept., Oct., Nov., Dec.}\} \)

b. \( E = \{\text{Jan., Feb., March, April, May, June}\} \)
\( F = \{\text{Jan., June, July}\} \)

c. E and F are not disjoint since they have the common members Jan., June.

d. \( \text{not - E} = \{\text{July, Aug., Sept., Oct., Nov., Dec.}\} \)
\( \text{not - F} = \{\text{Feb., March, April, May, Aug.}\} \)
EXERCISE 67-5

1. In Problem 2 of Exercise 67-3 we have counted the number of members in each universe set and in each event (subset). We have only to use the answers obtained there to obtain the required probabilities by means of Definition (*). Do you agree that in each experiment all possible outcomes (members of U) are equally likely so that Definition (*) does indeed apply? The required probabilities are:

a. i. \( \frac{3}{6} = \frac{1}{2} \)  
   ii. \( \frac{3}{6} = \frac{1}{2} \)  
   iii. \( \frac{3}{6} = \frac{1}{2} \)  
   iv. \( \frac{6}{6} = 1 \)  
   v. \( \frac{0}{6} = 0 \)  

b. i. \( \frac{2}{4} = \frac{1}{2} \)  
   ii. \( \frac{3}{4} \)  
   iii. \( \frac{3}{4} \)  

c. i. \( \frac{6}{12} = \frac{1}{2} \)  
   ii. \( \frac{3}{12} = \frac{1}{4} \)  
   iii. \( \frac{0}{12} = 0 \)  

d. i. \( \frac{12}{144} = \frac{1}{12} \)  
   ii. \( \frac{12}{144} = \frac{1}{12} \)  
   iii. \( \frac{2}{144} = \frac{1}{72} \)  
   iv. \( \frac{0}{144} = 0 \)  

e. i. \( \frac{5}{10} = \frac{1}{2} \)  
   ii. \( \frac{5}{10} = \frac{1}{2} \)  
   iii. \( \frac{5}{10} = \frac{1}{2} \)  
   iv. \( \frac{10}{10} = 1 \)  
   v. \( \frac{0}{10} = 0 \)  
   vi. \( \frac{4}{10} = \frac{2}{5} \)  
   vii. \( \frac{5}{10} = \frac{1}{2} \)  

f. i. \( \frac{10}{100} = \frac{1}{10} \)  
   ii. \( \frac{10}{100} = \frac{1}{10} \)  
   iii. \( \frac{1}{100} \)  
   iv. \( \frac{5}{100} = \frac{1}{20} \)  

g. i. \( \frac{9}{90} = \frac{1}{10} \)  
   ii. \( \frac{9}{90} = \frac{1}{10} \)  
   iii. \( \frac{0}{100} = 0 \)  
   iv. \( \frac{4}{90} = \frac{2}{45} \)  

h. i. \( \frac{1}{8} \)  
   ii. \( \frac{4}{8} = \frac{1}{2} \)  
   iii. \( \frac{3}{8} \)  
   iv. \( \frac{7}{8} \)  
   v. \( \frac{3}{8} \)  
   vi. \( \frac{7}{8} \)  
   vii. \( \frac{4}{8} = \frac{1}{2} \)  
   viii. \( \frac{4}{8} = \frac{1}{2} \)
EXERCISE 67-6

1. By Property 4, \( P(\text{lose}) = 1 - P(\text{not-lose}) \).
   By Property 6, \( P(\text{not-lose}) = P(\text{win}) + P(\text{tie}) = 0.7 + 0.1 = 0.8 \)
   Therefore \( P(\text{lose}) = 0.2 \)

2. Let \( E \) be the event that friend was born in January, June, or July. We are given \( P(E) = \frac{1}{4} \) and asked to find \( P(\text{not}-E) \).
   By Property 4, \( P(\text{not}-E) = 1 - \frac{1}{4} = \frac{3}{4} \).

3. a. If \( E \) is impossible, then \( \text{not}-E \) is a sure event.
   b. If \( E \) is sure, then \( \text{not}-E \) is an impossible event.

4. \( P(E) = \frac{3}{20} \) since \( E = \{6, 12, 18\} \)
   \( P(F) = \frac{2}{20} \) since \( F = \{8, 16\} \)
   We see that \( E, F \) are disjoint sets. Therefore, by Property 6,
   \( P(E \text{ or } F) = \frac{3}{20} + \frac{2}{20} = \frac{5}{20} \).

5. \( P(E) = \frac{5}{30} \) since \( E = \{6, 12, 18, 24, 30\} \)
   \( P(F) = \frac{3}{30} \) since \( F = \{8, 16, 24\} \)
   \( P(E \text{ and } F) = \frac{1}{30} \) since \( E \text{ and } F = \{24\} \)
   We use Property 5 to find \( P(E \text{ or } F) = \frac{7}{30} \).

6. Let \( E \) be the event that person selected is a foundation member. Let \( F \) be the event that person selected is male. Then we want to find \( P(E \text{ or } F) \).
   We have
   \( P(E) = \frac{10}{15} \quad P(F) = \frac{7}{15} \quad P(E \text{ and } F) = \frac{5}{15} \).
   Therefore, using Property 3, \( P(E \text{ or } F) = \frac{12}{15} = \frac{4}{5} \).

7. Let \( E \) be the event that he passes Mathematics and \( F \) the event that he passes Physics. We are asked to compute \( P(E \text{ or } F) \). We are given
   \( P(E) = 0.7, \ P(\text{not}-F) = 0.4, \ P(\text{not}-E \text{ or } \text{not}-F) = 0.6 \).
   By Property 4,
   \( P(F) = 1 - P(\text{not}-F) = 0.6 \)
   and \( P(E \text{ and } F) = 1 - P(\text{not}-E \text{ or } \text{not}-F) = 0.4 \)
   since the event \( \text{not}- (E \text{ and } F) \) is the same event as \( \text{not}-E \text{ or } \text{not}-F \).
   Therefore, by Property 5,
   \( P(E \text{ or } F) = 0.7 + 0.6 - 0.4 = 0.9 \).
   The student has probability 0.9 to pass at least one of the two examinations.
8. a. \[ U = \text{universe set} \]

\[ \begin{array}{c}
\text{F} \\
\text{E}
\end{array} \]

b. The circle for event E is entirely contained in the circle for event F. This pictures the situation when event E implies event F, that is, if E occurs, then F also occurs.

c. We have by hypothesis that \( n(E) \leq n(F) \). Therefore, by Definition (*) we get \( P(E) \leq P(F) \).

**EXERCISE 67-7**

1. \( 15 \times 14 \times 13 = 2,730 \) ways
2. \( 2 \times 2 \times 4 \times 3 = 48 \) meals

3. a. \( 5 \times 4 \times 3 = 60 \)
   
b. \( 5 \times 5 \times 5 = 125 \)
   
c. \( 2 \times 5 \times 5 = 50 \)
   
d. \( 5 \times 5 \times 2 = 50 \)

4. There are 11 positions on a football team. Hence there are \( 11 \times 10 \times 9 \times 8 = 7,920 \) ways for the coach to assign the four boys to the team.

5. There are 900 three-digit numbers that can be selected. (Since there are 9 choices for the first digit, and 10 choices for each of the second and third digits.)
   
a. There are \( 4 \times 10 \times 10 = 400 \) of these numbers that begin with an even number. Therefore this probability is \[ \frac{400}{900} = \frac{4}{9} \].
b. There are $9 \times 10 \times 5 = 450$ of these numbers that are even, that is end with an even number. Therefore this probability is $\frac{450}{900} = \frac{45}{90} = \frac{1}{2}$.

c. There are $4 \times 10 \times 5 = 200$ of these numbers that begin and end with an even number. Therefore this probability is $\frac{200}{900} = \frac{20}{90} = \frac{2}{9}$.

6. a. There are 2 outcomes for each toss. Head and Tail. Therefore, by the Fundamental Principle, there are $2 \times 2 \times 2 \times 2 \times 2 = 32$ different outcomes for the entire experiment. Can you write all the members of the universe set $U$?

b. The event $E$: "exactly one head in the five tosses" has $n(E)$ members. If we can count $n(E)$, then the required probability is the ratio of $n(E)$ to $n(U) = 32$. We see that $E = \{HTTTT, THTTT, TTHTT, TTTHT, TTTTH\}$ since every member in $E$ must represent the outcome in which you get one head and four tails.

Hence $n(E) = 5$ and $P(E) = \frac{5}{32}$.

7. There are now $12 \times 12 \times 12 \times 12 \times 12$ different answers when you ask the five people to tell you their birthmonths. So $n(U) = 12^5$.

If $E$ stands for the event that all five people have different birthmonths, than $n(E) = 12 \times 11 \times 10 \times 9 \times 8$.

Therefore

$$P(E) = \frac{12 \times 11 \times 10 \times 9 \times 8}{12^5}$$

$$= \frac{12 \times 9 \times 8 \times 2 \times 55}{12 \times 9 \times 8 \times 2 \times 144} = \frac{55}{144}$$

The probability that at least two people among the five have the same birthmonth is $P(not-E) = 1 - P(E)$

$$= 1 - \frac{55}{144}$$

$$= \frac{89}{144} \text{ as claimed.}$$
**EXERCISE 67-8**

1. a. weight is $\frac{1}{2}$  
   b. weight is $\frac{1}{36}$  
   c. weight is $\frac{1}{981}$

2. a. $U = \{HH, HT, TH, TT\}$. Assign weight $\frac{1}{4}$ to each member of $U$. The event $E = \{HT, TH\}$. Therefore $P(E) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$  
   b. $U = \{1, 2, 3, \ldots, 1000\}$. Assign weight $\frac{1}{1000}$ to each member of $U$. The event $E = \{w_1, w_2, \ldots, w_{10}\}$. Therefore  
      
      \[
      P(E) = \frac{1}{1000} + \frac{1}{1000} + \cdots + \frac{1}{1000} 
      
      10 \text{ terms in sum}
      \]
      or  
      
      $P(E) = 10 \times \frac{1}{1000} = \frac{1}{100}$
   c. There are $12 \times 12 = 144$ possible outcomes of this experiment, as we observed in Chart 1 of Section 67-2. We assign weight $\frac{1}{144}$ to each member in $U$. Since $E$ has 12 members, we get  
      
      \[
      P(E) = \frac{1}{144} + \frac{1}{144} + \cdots + \frac{1}{144} 
      
      12 \text{ terms in sum}
      \]
      or  
      
      $P(E) = 12 \times \frac{1}{144} = \frac{1}{12}$

3. It seems reasonable to assume equally likely outcomes in Examples 1-6, but not in Examples 7-8.

4. Parts (a) - (d), (g), (j) - (m) describe experiments where the assumption of equally likely outcomes seems reasonable. In parts (j), (k), (m) this assumes that each baby has an equal chance of being boy or a girl and that births are spread evenly over the 12 months. Vital statistics when carefully collected show these assumptions to be only approximate. But they are reasonable as a first approximation to the more complicated true state of affairs.
In part (e) we cannot be sure that our friend is equally likely to choose any prime between ten and thirty or whether he is more likely to choose some prime numbers than others. We would have to conduct an experiment among our friends to see which kind of assumption is reasonable. Similarly, our choice of a day of the week in part (f) is not likely to be easy to analyze. People probably have some preferences when they think of the names of the days and these are likely to show up in more choices of one day than another. Here too one would have to ask people to choose a day of the week and see how they respond.

In parts (h) and (i) it seems clearly unreasonable to assume that the outcomes of the experiment are equally likely.

**EXERCISE 67-9**

1. Let weight assigned to a be w. Then the weight of b is also w from the first equation. From the third equation the weight of d is 3w and from the second equation the weight assigned to c must also be 3w. So the sum of all the weights is
   \[ w + w + 3w + 3w \]
   or 8w. And this sum must equal 1. Hence \( w = \frac{1}{8} \).

   The \( P(E) \) is, by definition, the sum of the weights of the members in \( E \). Therefore
   \[
   P(E) = \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2}
   \]

2. The sum of all weights must be 1, so
   \[
   k + 2k + 3k + 4k + 5k + 6k = 1
   \]
   or \( 21k = 1 \). Therefore \( k = \frac{1}{21} \). The probability that an odd number turns up is the probability of event \( E = \{1, 3, 5\} \). From the definition.
   \[
   P(E) = \frac{1}{21} + \frac{3}{21} + \frac{5}{21} = \frac{9}{21} = \frac{3}{7}
   \]

3. a. If you have played draughts with your friend many times and have won half the time, tied 20% of the games, and lost the remaining 30%, then you might assign weights \( \frac{1}{2} \) to "won", \( \frac{3}{7} \) to "tied", and \( \frac{1}{21} \) to "lost".
b. Statistics of mortality are collected by insurance companies. Suppose 100,000 people were followed from birth and each year the number dying was recorded. Suppose there were 70,000 alive at age 20 and 1,000 of these died within the next year. Then we would assign weight $\frac{69}{70}$ to “man survives one year” and weight $\frac{1}{70}$ to “man dies within year”. Of course, one has to be sure that the mortality statistics apply to the population being insured since conditions of health, medical facilities, food supply, and so on, vary from one group to another.

c. Twins are born in the same month, but it seems reasonable to suppose that no month is more likely than any other month for the birth of twins. So we assign weight $\frac{1}{12}$ to each of the twelve outcomes marked with an X in Chart 1, and assign weight zero to each of the other 132 outcomes.

d. Assign weight 0.2 to “yes,” weight 0.7 to “no,” and weight 0.1 to “don’t know.”

4. The probability of an event, according to our definition, depends on the assignment of weights to the members of the universe set $U$. With different assignments of weights, as in Solutions 1 and 2, we should not be surprised if an event $E$ turns out to have different probabilities. Both are correct mathematical solutions. The situation here is like that in plane geometry. The conclusions “sum of angles of a triangle is 180 degrees” and “sum of angles of a triangle is less than 180 degrees” are different, but both can be correct mathematical conclusions from different hypotheses. The first conclusion is correct in Euclidean geometry, the second if we accept the postulate of Lobachewskian geometry. The different assignments of weights to the members of $U$ are like the different postulates in...
2. a. The numbers beginning with the digit 1 are 1, 10, 11, . . . , 19, 100, 101, . . . , 199, 1000, a total of 112 numbers in all. Therefore the probability is
\[
\frac{112}{1000}
\]

b. \[P(E \mid F) = \frac{1}{1000} = \frac{1}{750} \]

3. Let E be "birthmonth begins with J" and F be "birthmonth in first six months."
Then
\[P(E) = \frac{3}{12} = \frac{1}{4}\]
and
\[P(E \mid F) = \frac{2}{12} = \frac{2}{6} = \frac{1}{3}\]

4. Refer to the universe set of 36 members described in answer to Problem 1, part (g) of Exercise 67-2. (a) \(\frac{1}{6}\) (b) \(\frac{1}{6}\) (c) \(\frac{1}{6}\)

5. Events "5 on red die" and "3 on green die" are independent. Events "sum 7" and "number less than 4 on green die" are also independent.

6. \(P(E) = \frac{3}{4}\), \(P(F) = \frac{2}{4}\), \(P(E \text{ and } F) = \frac{2}{4}\). Therefore the product rule (***) is not satisfied and so E and F are not independent events.

7. a. \(0.9 \times 0.8 = 0.72\)

b. \(0.9 \times 0.2 = 0.18\). Here we used the fact that the events E and not-F are independent, that is, we used the equation
\[P(E \text{ and not-F}) = P(E) \times P(\text{not-F})\]

It can be proved (try it!) that if E and F are independent events, then E and not-F are also independent events.

8. Conditional probability that A wins is \(\frac{2}{3}\) and the conditional probability that B
EXERCISE 67-11

1. Value of $X$ (in £’s) | 49 | 9 | 4 | -1
--- | --- | --- | --- | ---
Probability of this value | $\frac{1}{1000}$ | $\frac{4}{1000}$ | $\frac{5}{1000}$ | $\frac{990}{1000}$

$E(X) = 49 \times \frac{1}{1000} + 9 \times \frac{4}{1000} + 4 \times \frac{5}{1000} + (-1) \times \frac{990}{1000}$

$= £(-0.885)$

2. Value of $X$ | 1 | 2 | 3 | 4 | 5 | 6
--- | --- | --- | --- | --- | --- | ---
Probability of this value | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$

$E(X) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$

$= 3.5$

3. Value of $X$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10
--- | --- | --- | --- | --- | --- | --- | --- | --- | --- | ---
Probability of this value | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$ | $\frac{1}{10}$

$E(X) = 5.5$

4. Value of $X$ | 0 | 1 | 2
--- | --- | ---
Probability of this value | $\frac{4}{9}$ | $\frac{4}{9}$ | $\frac{1}{9}$

$E(X) = 0 \times \frac{4}{9} + 1 \times \frac{4}{9} + 2 \times \frac{1}{9}$

$= \frac{2}{3}$ of a question

Note that here, in Problems 2 and 3, we get a value for $E(X)$ that is not one of the values of $X$. This often happens when you average test scores, too. The average of scores 60, 70, 95 is 75 and so the average test score is not one of the scores actually obtained.
Answers to CHAPTER 68

EXERCISE 68-2A

61, 7, 53, 31, 83, 43, 1, 79

EXERCISE 68-2B

1. (59, 61), (71, 73); (101, 103) or (107, 109) or (137, 139) or (149, 151) or (179, 181) or (191, 193)
   Twin primes are those which have only one number between them.
3. Numbers not crossed out are
   301, 307, 311, 313, 317, 323, 329, 331, 337, 341, 343, 347, 349

EXERCISE 68-2C

1. The primes are: 307, 311, 313, 317, 331, 337, 347, 349
2. The primes are: 163, 251, 401
   203 = 7 x 29
   529 = 23 x 23
   287 = 7 x 41

EXERCISE 68-2D

Primes: \(2^7 - 1 = 127\), \(2^9 - 1 = 511\),
\(2^{11} - 1 = 2047\) is not a prime since \(2047 = 23 \times 89\).

EXERCISE 68-3

\[70 = 2 \times 5 \times 7; \quad 108 = 2 \times 2 \times 3 \times 3 \times 3; \quad 180 = 2 \times 2 \times 3 \times 3 \times 3 \times 3; \quad 196 = 2 \times 2 \times 7 \times 7; \quad 231 = 3 \times 7 \times 11\]

EXERCISE 68-4

a. 6      b. 3      c. 12      d. 9
   e. 1      f. 1      g. 2      h. 4
   i. 17     j. 13     k. 1      l. 1

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EXERCISE 68-5

a. 60  b. 90  c. 108  d. 196  
  e. 51  f. 96  g. 126  h. 280  
  i. 198  j. 1440

EXERCISE 68-6

Divisible by 2: c, d, g, i; divisible by 5: b, c, e, h

EXERCISE 68-7

1. The following numbers are divisible by 3:
   a. 285  c. 2718  d. 4860  e. 1902
   g. 80415  h. 35124  i. 16125
2. A whole number is divisible by 9 if and only if the sum of its digits is a multiple of 9. Numbers divisible by 9:  c. 2718, d. 4860, g. 80415
3. A number is divisible by 6 if it satisfies both the test for 2 and the test for 3. That is, it must end in 0, 2, 4, 6, 8, and the sum of its digits must be a multiple of 3. We may thus select the even numbers from the answers to Exercise 1. These are c. 2718, d. 4860, e. 1902, i. 35124.
   Numbers divisible by 15 must satisfy the tests for both 3 and 5. These would be the numbers ending with 0 or 5 in the answers to Exercise 1, i.e., 285, 4860, 80415, 16125.
4. A number is divisible by 4 if and only if the number represented by the last 2 digits is a multiple of 4.

EXERCISE 68-8

1. The numbers divisible by 11 are:
   a. 825  b. 7832  d. 9471
   f. 94655  g. 918082  i. 545655
2. Divisible by 22:  b. 7832  g. 918082
   Divisible by 33:  a. 825  d. 9471  i. 545655
   Divisible by 55:  a. 825  f. 94655  i. 545655
3. 100000 = 100001 - 1
   10000 = 9999 + 1
**EXERCISE 68-9**

1. Numbers divisible by 7:  
   a. 483  
   b. 301  
   c. 854  
   d. 945  
   e. 21854  
   f. 1365  
   g. 2541  
   h. 25305  
   i. 25305  
   j. 21854

3. Divisible by 14:  
   a. 483  
   b. 301  
   c. 854  
   d. 945  
   e. 1365  
   f. 2541  
   g. 25305  
   h. 25305  
   i. 21854

Divisible by 21:  
   a. 483  
   b. 945  
   c. 21854  
   d. 945  
   e. 1365  
   f. 2541  
   g. 25305  
   h. 25305  
   i. 25305

Divisible by 35:  
   a. 483  
   b. 945  
   c. 1365  
   d. 2541  
   e. 25305


**EXERCISE 68-10**

2.  
   (8, 12), (6, 9), (10, 15), (12, 18), (30, 45)  
   These numbers are all in the ratio of 2 to 3.

3.  
   The formula $2(a + b) = \text{GCF} + \text{LCM}$ is good for all pairs in the ratio of 3 to 5.  
   Other pairs are (15, 25), (18, 30), (21, 35), (24, 40) and so forth.

**EXERCISE 68-11**

1. $n - 1 \text{ sum digits } n - 1$

2.  
   a. $4 + 4 = 8.$  
   yes.  
   b. $1 + 0 + 3 = 4.$  
   yes.  
   c. $1 + 2 + 3 = 6.$  
   no.  
   d. $2 + 0 + 2 = 4.$  
   yes.  
   e. $3 + 4 + 1 = 8.$  
   yes.  
   f. $3 + 1 + 2 = 6.$  
   no.  
   g. $1 + 4 + 1 = 6.$  
   no.  
   h. $1 + 1 + 0 + 2 = 4.$  
   yes.  
   i. $2 + 0 + 1 + 1 = 4.$  
   yes.  
   j. $3 + 0 + 1 + 1 = 5.$  
   no.  

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3. a. $4 + 4 = 8$.  
   no. $44_{twelve} = 52_{ten}$
   b. $6 + 5 = 11$.  
   yes. $65_{twelve} = 77_{ten}$
   c. $5 + 8 = 13$.  
   no. $58_{twelve} = 68_{ten}$
   d. $1 + 2 + 8 = 11$.  
   yes. $128_{twelve} = 176_{ten}$
   e. $2 + 4 + 5 = 11$.  
   yes. $245_{twelve} = 341_{ten}$
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