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STOCHASTIC IDENTIFICATION OF TRANSMISSIVITY AND EFFECTIVE RECHARGE IN STEADY GROUNDWATER FLOW: 2. CASE STUDY

by

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ABSTRACT

A first-order analytical solution of the inverse problem for aquifer steady flow, presented in Part 1 [Rubin & Dagan, this issue], is applied to the Avra Valley aquifer [Clifton & Neuman, 1982]. First, the parameters characterizing the statistical structure of the logtransmissivity (Y) and water-head (H) fields are estimated by a maximum likelihood procedure. The results for Y are in good agreement with those of Clifton and Neuman [1982], in spite of the different methodologies. The incorporation of head measurements is shown to have definite advantages in reducing the estimation variances of Y-parameters. Next, the best estimates of Y at various points are obtained by simultaneous conditioning on the measurements of Y and H. It is shown that a substantial reduction in the variance of the conditioned Y is achieved by accounting for H measurements, justifying a-posteriori the solution of the inverse problem. Finally, the effective recharge, which is assumed to be uniform, but random, is estimated as part of the process. Although the latter is relatively small for Avra Valley, it might be a parameter of considerable interest in other cases. Further applications of the methodology are suggested.

1. INTRODUCTION

A solution of the identification (or inverse) problem in steady groundwater flow has been presented in two recent articles. The basic theoretical approach, in the spirit of the study of Kitanidis & Vomvoris [1984], is outlined in Dagan [1985]. The approach is further extended and generalized in the first part of this sequence [Rubin & Dagan, this issue] refered here as RD for briefness.

The main distinctive feature of the present approach is its analytical nature, as compared to previous stochastic models which rely heavily on numerical schemes (e.g. Neuman [1979], Clifton & Neuman, denoted CN for brevity [1982], Hoeksema & Kitanidis [1984]). Closed form, simple, analytic expressions are obtained in RD for Γ_H, the head variogram, and for C_{YH}, the logtransmissivity-head cross-covariance, for a given exponential C_Y. They are derived from a first order approximation of the equations of steady flow in an unbounded aquifer and are expressed in terms of a vector $\underline{\theta}$ of unknown parameters. Two additional features of RD are: the method is capable of identifying the effective natural recharge R, which is considered to be a random variable, and the estimation of $\underline{\theta}$ is associated with uncertainty, which is incorporated in the prediction of the logtransmissivity field. The detailed steps required for implementation of the method are presented in RD, Sect. 4, and are not repeated here.

The aim of this part is to subject the method to the acid test of a real-life case, illuminating various aspects and problems associated with its implementation. A proper setting for this purpose was found at the Avra Valley aquifer in Arizona, U.S.A, which has served as a case study in a few previous

articles. A detailed description of the aquifer hydrogeology may be found in Clifton [1981] and CN [1982], who applied to it the methodology first outlined in Neuman et. al. [1979]. Additional studies, relying in part on Clifton's [1982] data, were conducted by Hoeksema & Kitanidis [1984], Neuman & Jacobson [1984] and Nelson et al [1985]. Our choice of the same aquifer has been motivated by the abundance of measurements and their availability on one hand, and the possibility to compare the present results with the ones obtained previously by a different methodology.

The plan of the paper is as follows: in section 2, some aspects concerning the parameters' identification method are elucidated. Section 3 discusses a few differences between the current method and CN's, which will later serve as a reference for some of our results. In section 4 the specific model for Avra will be presented, followed by results in section 5, and a brief summarizing discussion in section 6.

2. Properties of the maximum-likelihood estimators.

Following the methodology developed by Kitanidis & Vomvoris [1983], the estimate of the vector of parameters $\underline{\theta}$, as well as the associated estimation error covariance matrix $\underline{\underline{\Sigma}}$, are obtained by applying a maximum-likelihood procedure (MLP). The MLP, unlike the least-squares or weighted least-squares methods, requires defining first the probability distribution function of the errors of estimation, i.e. the likelihood function. The latter may be quite general and apply to situation in which the error structure may be very complicated, non-linear or event-dependent (see Troutman, 1985). Another work which employs the maximum-likelihood principle in the context of the inverse problem is that of Carrera & Neuman [1986].

Although it seems that the question of selecting a best identification method does not have a clear-cut answer, MLP has some definite advantages. Thus, in order to compare it with the least-squares method, it is worthwhile to mention the four common assumptions underlying the latter [Troutman, 1985]: (i) the errors are statistically independent of the prediction and are of identical distribution; (ii) they are statistically independent of each other; (iii) the errors have zero mean and finite variance; (iv) the errors are normally distributed. In contrast, if any of these assumptions is violated, the MLP can account for it explicitely. Similarly, Bard [1974, p.57] states that in the case of a linear model, or if the number of observations is large and the errors are normally distributed, the weighted least-squares (WLS) method display some optimal properties. In the general case, however (non-normal distribution or non-linear model, or when the number of observations is relatively small), optimal properties of WLS cannot be proven and MLP may be more advantageous. Furthermore, at the stage of post-calibration checks, the MLP, being based on an outright statistical model, enables to employ some simple well-known procedures for that purpose. In the WLS case, however, strong consistency of the estimator and the presence of a large sample are required in order to establish the normality assumption.

MLP estimators have some features which are particularly important in our case: (i) they are asymptotically efficient and best asymptotically normal estimators; (ii) they are consistent estimators; (iii) a unique feature, which is particularly advantageous in light of the log-transformation commonly applied to transmissivity data, is invariance. That is, optimality of any MLP parameter applies also to any one-to-one function of that parameter.

3. A comparison of the present method with Clifton & Neuman [1982].

Since we are going to compare our results for the Avra Valley aquifer with those of CN, the last being representative of a different approach, it is worthwhile to recapitulate first a few differences of principle in methodology:

(i) The solution of CN is based on the minimization of an objective function which combines two criteria. The first criterion measures the closeness of the model numerically-derived heads to the measured heads. The second measures the fit of the predicted T-field to an a-priori given one, obtained by kriging. This is equivalent to a simultaneous identification of H_i, T_i $(i=1,\ldots\ldots,N)$, where N is the number of nodal points in a predetermined finite element grid, by WLS applied to measurements. The relative weight given to each criterion is chosen arbitrarily a-priori, with judicious corrections later. It was pointed out by Sun & Yeh [1985] that some instability may occur in such a procedure depending on the size of N and on the location of the nodal points (the last point is usually refered to as the problem of zonation). In contrast, the present method, following Kitanidis & Vomvoris [1983] and Hoeksema & Kitanidis [1984] aims to estimating and validating a stochastic model through MLP based on measurements. This model is later used to identify the areal distribution of H and T.

(ii) The equation of flow is incorporated into the scheme of CN by a numerical procedure, defining the model fit criterion. In the present approach, however, the flow equation serves to fully characterize the joint Y,H spatial random field through its statistical moments. CN thus offer the advantage of generality and flexibility, whereas the present analytical method has some

basic limitations. It should be realized, however, that the a-priori defined zonation in the numerical scheme, is not necessarily optimal, and may suffer from rank deficiency as N equals the number of measurements. Furthermore, as pointed out by Dagan [1985], zonation creates a numerical correlation scale of the order of the dimension of the zone. The alteration of the actual correlation structure may be avoided by increasing N, which might incur a heavy computational effort.

(iii) In the present model, well established statistical validation tests are employed by taking advantage of the a-priori postulated error structure in the MLP. CN, on the other hand rely on the trial-and-error method of cross validation of the model.

(iv) As mentioned before, the present method accounts for the impact of the variance of estimation of various parameters on the conditional variance of the logtransmissivity $\tilde{\sigma}^2_{Y|}^{C}$ (RD eq.32). In contrast, the parameters characterizing the statistical structure of Y are assumed to be deterministic in CN.

(v) Similarly, the present method accounts for the presence of an effective recharge R and its randomness, which are not considered by CN because of its presumed negligible magnitude. It has been shown, however (see RD figs. 1,2) that even a small recharge may have a large impact on the head correlation structure, especially at large separation distances. It is felt that neglect of R should be justified only a-posteriori, by analysing the head measurements.

4. Details of methodology applied to the Avra Valley aquifer.

The measured data available for Avra Valley are 106 transmissivity values from wells scattered irregularly over the aquifer (see Fig.2 in CN and Table C-2 in Clifton), as well as 98 water heads (Fig.1 in Neuman & Jacobson, Table

D-1 in Clifton). These measurements, i.e. Y_i ($i=1,....M$, $M=106$) and H_j ($j=M+1,....,N$, $N=204$), and their location, constitute the needed data base. It was proposed in CN to subdivide the aquifer into two regions, north and south (CN, Fig.6), because the transmissivity in the northern part is significantly higher on the average than that of the southern part. For the purpose of comparison, we adopted this subdivision in our study as well. It is pointed, however, that this subdivision is not essential as the logtransformation of Y-data diminishes significantly the relative variability of the sample. From a computational point of view however, large computers are needed to invert the matrices of the order of the above N.

Hence, our aim is to identify the logtransmissivity $Y(\underline{x})$ of the aquifer, as well as the effective recharge R and this task is performed separetely and simultaneously for the two subregions of the aquifer. First, the parameters which constitute the vector $\underline{\theta}$ are selected: (i) $\theta_1=m_Y$, the logtransmissivity constant expected value; (ii) the three parameters characterizing directly the autocovariance C_Y (RD, eq.10), and indirectly $\bar{\Gamma}_H$ (RD, eq.19) and C_{YH} (RD, eq.16), namely $\theta_2=w$, $\theta_3=\bar{\sigma}_Y^2$, and $\theta_4=I$, denoting the nugget, the correlated residuals variance and the integral scale respectively, and (iii) the five parameters characterizing the head quadratic trend, (RD, eq.12) namely $\theta_5=J_{o,x}$,$\theta_6=J_{o,y}$, the Cartesian components of the head gradient at the origin, and $\theta_7=c_{11}$, $\theta_8=2c_{12}$ and $\theta_9=c_{22}$ the coefficients of the quadratic trend in RD, eq.12.

For each part of the aquifer the observations vector is defined by

$$Z_i=\begin{cases} Y_i & i=1,.....,M \\ h_i-h_N & i=M+1,...,N-1 \end{cases} \qquad (1)$$

where $h_j=H_j-\langle H_j\rangle$, H_j being the measured head and $\langle H_j\rangle$ its expected value (RD, eq.12). Head residual increments are adopted for theoretical reasons discussed

in RD, namely that h has a well defined variogram.

The next step is the derivation of the optimal vector $\underline{\theta}$ which maximizes the Gaussian likelihood function

$$L=p(\underline{Z} \mid \underline{\theta})=(2\pi)^{\frac{-(N-1)}{2}} |\underline{\underline{Q}}|^{-(\frac{1}{2})} \exp\left[-(\tfrac{1}{2})\ (\underline{Z}-\langle\underline{Z}\rangle)^T\ \underline{\underline{Q}}^{-1}\ (\underline{Z}-\langle\underline{Z}\rangle)\right] \tag{2}$$

where $\underline{\underline{Q}}$ (a double underline denotes matrices) is the covariance matrix of the order (N-1)(N-1) defined as $Q_{ij}=\langle Z_i' Z_j' | \underline{\theta}\rangle$. In eq.(2):

$$\langle Z_i\rangle = \begin{cases} \theta_i & i=1,\ldots,M \\ 0 & j=M+1,\ldots,N \end{cases}$$

$$\langle Z_i' Z_j'\rangle = \begin{cases} C_Y(\underline{x}_i,\underline{x}_j) & i,j=1,\ldots,M \\ C_{YH}(\underline{x}_i,\underline{x}_j)-C_{YH}(\underline{x}_i,\underline{x}_N) & i=1,\ldots,M;\quad j=M+1,\ldots,N-1 \\ \bar{\Gamma}_H(\underline{x}_i,\underline{x}_N)+\bar{\Gamma}_H(\underline{x}_j,\underline{x}_N)-\bar{\Gamma}_H(\underline{x}_i,\underline{x}_j) & i,j=M+1,\ldots,N-1 \end{cases} \tag{3}$$

Hence, L (2) depends in a complex and strong nonlinear fashion on $\underline{\theta}$ directly through the functions C_Y, C_{YH} and Γ_H in (3) and indirectly through the averages $\langle H\rangle$ and $\langle Y\rangle$. The MLP in our case was carried out using a gradient-based iterative method. There are some well-documented computer programs available for that purpose, and we shall not dwell on this aspect of the solution.

The outcome of the MLP is the estimate $\hat{\underline{\theta}}$ of $\underline{\theta}$ with its associated estimation error covariance matrix $\underline{\underline{\Sigma}}$. $\underline{\underline{\Sigma}}$ is called the Cramer-Rao lower bound (Schweppe, 1973 pp.374) or the information inequality and it is computed simultaneously with the optimal estimator $\hat{\underline{\theta}}$. $\underline{\underline{\Sigma}}$ is the inverse of the matrix $\underline{\underline{B}}$, where $B_{ij}=E\left[\frac{\partial}{\partial\theta_i}(\frac{\partial}{\partial\theta_j}\ln L)\right]$, $\underline{\underline{B}}$ being the Fisher information matrix, and $\underline{\underline{B}}^{-1}$ is the asymptotic estimator of $\underline{\underline{\Sigma}}$.

At this stage the recharge R can be inferred (RD, eq.33) and also an analysis of the statistical significance of the results can be carried out (see, for example- Sorooshian & Gupta, 1985 or Bard, 1974 who suggested to employ response surface techniques. This can be carried out straightforwardly based on the Fisher information matrix).

The last step towards identifying the normal Y at a generic point $\underline{x}$ is to calculate the conditional estimator $\tilde{Y}^c(\underline{x})$ (RD, eq.31) and the best estimate of the variance $\tilde{\sigma}^2_{Y^c}$ (RD, eq.32). These require computing the coefficients λ_i, μ_j, Λ (RD, eq.30) and λ_{ik}, μ_{jk} (RD, Appendix B).

5. Results

(i) Estimation of $\underline{\theta}$

The derivation of $\hat{\underline{\theta}}$ and $\underline{\underline{\Sigma}}$ can be carried out in three stages of increasing complexity, as outlined in RD Sect.4. The simplest mode is the one in which only θ_i $(i=1,..,4)$ are determined, by applying MLP to Y_i only, no use being made of head measurements and flow equations. Accurate estimates may be obtained if a large number of measurements, evenly distributed in separation distances between points is available. However, in the typical field situation in which T

measurements are scarce, this mode is less effective than the solution of the inverse problem, discussed in the sequel.

The next mode is the one in which only the measurements H_j $(j=M+1,\ldots,N)$ are used in order to identify θ_i $(i=3,\ldots,9)$. This can be done in absence of T measurements, in order to verify the consistency of the model by comparing statistical mcments derived separetly from Y_i and H_j and in the case in which the Y statistical structure is given and only θ_i $(i=5,..,9)$ are to be identified. Furthermore, knowing the values of θ_i $(i=3,..,9)$ suffice in some cases to carry out some calculations leading to the prediction of the migration of solutes. As mentioned above, a good agreement between mode 1 and 2 is indicative of the suitability of the flow model with its underlying assumptions to the case at hand . It is emphasized that $\theta_1=m_Y$ cannot be inferred on this basis, as it is well known from previous studies of the inverse problem. Finally, mode 3, in which T and H measurements are used jointly in order to identify θ_i $(i=1,\ldots 9)$, is the most comprehensive, taking maximum advantage of available data. The structure of three modes is summarized in Table 1.

In Table 2 we present the results of the MLP for Avra Valley aquifer for the Y parameters, as well as those of CN. The latter estimated θ_1 by a geometric mean, while estimates for the other parameters were obtained by a curve-fitting between measurements and a semi-spherical Y variogram.

Table 2 reveals a few point of interest: (a) The estimation of $\underline{\theta}$ is indeed associated with uncertainty as demonstrated by the values of Σ_{ii}; (b) Mode 3 procedure, as might be anticipated, is generally favorable in most cases, since it leads to smaller values of Σ_{ii}. (c) It is evident that θ_1 is statistically the soundest parameter, as might be revealed by a t-ratio test. Other θ_i $(i=2,3,4)$, on the other hand, are more susceptible to error. The fact that the MLP estimates for θ_1 and θ_i $(i=5,..,9)$ converged to the same values at both

mode 1 and 3 and mode 2 and 3 respectively leads to the conclusion that θ_i (i=2,3,4) constitute a somewhat flat joint response-surface around the global optimum. That is to say, the optimal value of L (Eq.2) is less sensitive to θ_i (i=2,3,4) than to the other parameters. The flateness of the response surface is associated with small second derivatives, which lead to relatively large variances when inverted.

It was found, however, that at the stage of conditioning, the single most influential term in $\underline{\underline{\Sigma}}$ on $\tilde{\sigma}^{2,C}_Y(\underline{x})$ is Σ_{11}. Therefore, the estimation error associated with θ_i (i=2,3,4) constitutes only a minor problem.

(d) There are some discrepancies between the present results and those of CN, but the overall agreement is fair in view of the different computational schemes.

(e) Σ_{ii} of the various parameters change of course when estimated by different modes with a possible trade off in between. The comparison of the efficiencies of different modes is not a trivial matter, since it cannot be judged simply by inspection of the various Σ_{ii}. One possible approach is to map the generalized variance $\tilde{\sigma}^{2,C}_Y$ and to compare the overall performance of the modes. The result, however, might be inconclusive if θ_3 is underestimated by mode 1. This strategy will be elaborated in the sequel.

Another answer may be found in the realm of Information Theory. Measuring the uncertainty I^* associated with some p.d.f. $p(\underline{Z} \mid \underline{\theta})$ was suggested by Shannon [1948] and his formulae is considered to be a unique measure. Modifying Shannon's I^* for the case of a multi-variate-normal p.d.f. yields $I^*=\log||\underline{\underline{\Sigma}}||$ (see Bard, 1974) and information is gained upon minimizing I^*. Some figures obtained by employing Shannon's technique are summarized in Table 3. These figures show that mode 3 is invariably favorable as far as reducing the uncertainty associated with the estimation of $\underline{\theta}$, strengthening the claim

that incorporation of head measurements has a beneficial effect upon parameter identification.

(f) Last not least, the moderate values of $\theta_3=\bar{\sigma}_Y^2$ confirm a-posteriori the applicability of the first-order approximation to the equations of flow. This also suggests that a full numerical scheme, which can supposedly cope with larger $\bar{\sigma}_Y^2$, is not really needed in this case.

(ii) Computational aspects.

Some steps may be taken in order to expedite and ensure convergence of the MLP procedure: (a) initialization of $\underline{\theta}$. Every non-linear search technique is based on some initial set of values for $\underline{\theta}$. Our experience has showed that regressing C_Y and Γ_H on the raw Y and H variogram respectively supply a good starting point. However, although L (Eq.2) is generally a convex and smooth function, different initial values for $\underline{\theta}$ should be tried to ensure global convergence; (b) it is recommended that upper and lower bounds be placed on variables whenever sensible values are known (see for example Hoeksema & Kitanidis, 1984). This makes a problem easier to solve, since they confine the solution to a smaller region. However, when some constraint is found binding, it should be relaxed; (c) in case of large N, the inversion of $\underline{\underline{Q}}$ in (2) may require a degree of precision that is beyond the limit to some computers. In this case it was found that scaling of the terms in $\underline{\underline{Q}}$ may be beneficial.

(iii) The identification of the logtransmissivity field.

We present now the main results of the analysis, namely the logtransmissivity field $\bar{Y}^c$, which has been evaluated in two stages: (i) Stage 1

in which $\underline{\theta}$ is derived by mode 1 of Table 1 and $\tilde{Y}^c(\underline{x})$ is conditioned only on measurements of Y (RD, eq.31, with $\mu_j=0$) similar to kriging; (ii) Stage 2, in which $\underline{\theta}$ is derived by mode 3 of Table 1 and $\tilde{Y}^c$ is conditioned on both Y_i and H_j (RD, eq.31), which is similar to cokriging.

In order to draw maps of $\tilde{Y}^c$, the latter has been computed at the nodes of a grid repetitively. It is emphasized, however, that the l.h.s. of RD Eq.27-29 has to be computed and inverted only once for each subregion, thus reducing the computational effort considerably. The results of the two identification procedures are illustrated in Fig.1 and 2. (graphical results are presented for South Avra only, similar results and findings were obtained for North Avra).

The influence of head conditioning upon $\tilde{Y}^c$ may be assessed by comparing Figs.1 and 2. It can be also evaluated by representing the frequency of $\tilde{Y}^c$ values in a histogram (Fig.3). Although Fig.3 does not tell anything about the location at which $\tilde{Y}^c$ may be found, it is indicative of the impact of the full inverse problem upon identification of transmissivity.

The present results of Fig.1 and 2 have been compared with those of CN Figs.9 and 15, showing fair agreement in general.

The next major step is the computation of the variance $\tilde{\sigma}^{2,c}_Y$ (RD eq.32). This has been carried out in four stages: (i) by mode 1 of Table 1 with $\underline{\theta}$ deterministic (RD, eq.26,32 with $\mu_j=0$ and $\underline{\underline{\Sigma}}=\underline{\underline{0}}$), corresponding to the usual kriging procedure, see fig.4; (ii) the same with $\underline{\underline{\Sigma}}$ taken into account, by the procedure of RD Sect.3 (RD, eq.26, 32, with $\mu_j=0$), see fig.5; (iii) by mode 3 of Table 1 with $\underline{\theta}$ deterministic (RD, eq.26, 32 with $\underline{\underline{\Sigma}}=\underline{\underline{0}}$) see fig.6 and (iv) the same, with $\underline{\underline{\Sigma}}$ taken into account, see fig.7. Inspection of Figs. 4-7 reveals that the impact of head conditioning in reducing the logtransmissivity variance was evident whenever applied. Its influence is moderate along zones displaying heavy concentration of T-measurements, and is more evident in zones

lacking T-measurements. As might be expected, the sum of θ_2, θ_3 and Σ_{11} which is equal to 0.33 in south Avra creates an upper limit on $\tilde{\sigma}^{2,C}_Y$.

As already mentioned, the major benefit of solving the inverse problem for transmissivity, as compared to conditioning (i.e. kriging) on transmissivity measurements only, resides in the assumed reduction of the prediction variance (for a thorough discussion see Dagan, 1985). To assess this benefit in the present case we have represented histograms of $\tilde{\sigma}^{2,C}_Y$ by the four computational models mentioned before in Fig.8 . Examination of this figure reveals that: (i) the variance of estimation of various parameters has a definite impact on $\sigma_Y^{-2,C}$ which is otherwise underestimated and (ii) incorporation of head measurements definitely reduces the logtransmissivity variance. Similar results were obtained for North Avra. Thus, even in the case of a large amount of T measurements as is evidently the case in Avra Valley, incorporation of H measurements is advantageous in reducing uncertainty associated with the prediction of T. This demonstrates convincingly the usefullness of solving the inverse problem.

(iii) Identification of effective recharge and of quadratic trend

The MLP estimation of θ_i (i=5,...,9) are summarized in Table 4. These results serve to estimate the recharge through Eq.33 in RD, whose values are also given in Table 4.

Before discussing the outcome, it is worthwhile to remind that a possible use of the results for <H> is to draw the maps of head H by conditioning the latter on both measurements of Y and H. This is part of the direct problem (Dagan,1982) which is not discussed here.

CN assumed the absence of natural recharge though some indications may be found in Clifton [1981], about the existence of recharge of meager magnitude. The sources of recharge listed are mountain front recharge and stream-channel recharge amounting to 18000 acre-feet per year according to some estimates. According to other mentioned studies, the recharge is negligible. As to discahrge, during the relevant year a quantity of 10,000 acre-feet of water was pumped out.

Our results for $\hat{R}$ indicate to the existence of discharge, although quite small. This may be explained by the possible existence of other sources of discharge beside pumpage, like evapotranspiration or underflow through the boundaries. The total discharge sums up to approximately 27000 acre-feet per year over the entire Avra Valley aquifer. The likelihood-ratio test may be employed to conduct a simple-hypothesis testing regarding CN assumption of $\hat{R}=0$ as null-hypothesis. The null hypothesis may be rejected with a level of significance $\alpha=1\%$ by $|R|>3085$ for North Avra and $|R|>820$ for South Avra (in acre-feet per year). Thus our results indicate that the null hypothesis may be rejected on quite safe ground. Although $\hat{R}$ is relatively small in the present case and compatible with CN's assumption, it is important to incorporate it in the process of detrending of head measurements. Our experience has shown that a head variogram that assumes a-priori the absence of recharge (see Eq.4e in Dagan [1985]) leads to an overestimation of θ_3 and θ_4, as well as to some bias in the head trend' parameters. The smaliness of $\hat{R}$ in the present case should not deter from pursuing its identification in the many cases in which $\hat{R}$ might be the parameter of major concern. A refinement of the areal distribution of $\hat{R}$ may be achieved by further subdivision of the aquifer, subjecting then each zone to either mode 2 or mode 3 identification procedures.

6. Summarizing discussion and conclusions.

A method to solve the inverse problem is suggested. Its aim is to provide a quick and inexpensive mean to estimate the transmissivity field and the effective recharge, which may also be used as a preliminary estimate in those cases in which the assumptions underlying the method are not met.

Herewith a few advantages of the solution:

(i) Conceptual simplicity. A stochastic model is suggested, based on three common basic assumptions: Y is a normal random space function, the aquifer is unbounded and a first-order approximation of the flow equation is adopted.

(ii) Easy to implement. The solution requires only a non-linear minimization algorithm, which is offered as a ready-made package by almost any mainframe computer library, like IMSL.

(iii) Computationally efficient. There is no need to carry out a costly numerical solution of the flow equation, which was solved analytically. The application of the method to Avra Valley led to consistent results, agreeing in general lines with those obtained previously by a numericall method. A solution can also be carried out for only a part of the aquifer, estimating $Y(\underline{x})$ at some location without solving for the whole aquifer.

(iv) Unbiased estimation of the error of prediction. The vector of parameters $\underline{\theta}$ is considered random, and its variance is acconted for in $\tilde{\sigma}^{2,c}_{Y}$, based on a first-order approximation of $Y^{c}(\underline{x})$ in $\underline{\theta}'$.

(v) Validity check. An a-priori postulated error structure enables to conduct post-calibration validity checks.

(vi) Estimation of the effective recharge. The recharge is considered to be a random variable, and its variance of estimation is also given.

The solution's main limitations are:

(i) The requirement $\bar{\sigma}^2_Y<1$ (see Dagan, 1985) is the most fundamental one,

preventing its application to cases of large variability. In the latter cases, the requirement may be satisfied sometimes by concentrating on small zones, risking insufficiency of data.

(ii) The assumption of unbounded aquifer. This assumption, adopted in order to simplify the analytic solution, is underlain by the requirement that the aquifer extent is much larger than the logtransmissivity integral scale. It can be circumvented though, by conditioning the head by a series of points for boundaries of given head, or by images of measurements points for a boundary of given discharge. It is emphasized that many times boundaries and boundary conditions are pourly defined, and regarding the aquifer as a part of a much larger one may be a sound approximation in any case.

There are additional possible applications of the method like: (i) Computation of block values. The algorithm for that purpose is given in Dagan [1985]; (ii) Estimation of the head field by unconditional probability (RD, eq.12) or by conditioning on Y, H measurements or cokriging (by formulae similar to RD, eq.31); (iii) Developing a scheme for variance reduction by additional measurements. Since the estimation of λ_i and μ_j is based on the inverted covariance matrix of observations, it is possible to use it in order to locate the points which are the most beneficial as far as reduction of variance is concerned.

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Figure captions

Figure 1- Contour map of transmissivity obtained by mode 1, South Avra (in ft^2/day)

Figure 2- Contour map of transmissivity obtained by mode 3, South Avra (in ft^2/day)

Figure 3- Hystogram of the conditional Y, South Avra.

Figure 4- Contour map of the conditional variance $\hat{\sigma}^{2,C}_{Y}$ obtained by mode 1, South Avra.

Figure 5- Contour map of the generalized conditional variance $\tilde{\sigma}^{2,C}_{Y}$ obtained by mode 1, South Avra.

Figure 6- Contour map of the conditional variance $\hat{\sigma}^{2,C}_{Y}$ obtained by mode 3, South Avra.

Figure 7- Contour map of the generalized conditional variance $\tilde{\sigma}^{2,C}_{Y}$ obtained by mode 3, South Avra.

Figure 8- Hystogram of variance of estimation of Y in South Avra: (a) the conditional variance $\hat{\sigma}^{2,C}_{Y}$ (b) the generalized conditional variance $\tilde{\sigma}^{2,C}_{Y}$.

Table 1

Summary of parameters and data base for modes 1 ÷ 3

Mode	Sample Data	Parameters in $\underline{\theta}$
1	Y_i $(i=1,\ldots M)$	$\langle Y\rangle;\bar{\sigma}_Y^2;w;I$
2	$H_j(j=M+1,\ldots,N)$	$\bar{\sigma}_Y^2,w,I;J_{o,x};J_{o,y};c_{11};c_{12};c_{22}$
3	$Y_i(i=1,\ldots,M);H_j(j=M+1,\ldots,N)$	$\langle Y\rangle;\bar{\sigma}_Y^2;w;I;J_{o,x};J_{o,y};c_{11};c_{12};c_{22}$

Zone	parameters / method	$\theta_1 = \langle y \rangle$		$\theta_2 = W$		$\theta_3 = \bar{\sigma}_y^2$		$\theta_4 = I$(miles)	
		$\hat{\theta}_1$	Σ_{11}	$\hat{\theta}_2$	Σ_{22}	$\hat{\theta}_3$	Σ_{33}	$\hat{\theta}_4$	Σ_{44}
South Avra	CN*	8.88**	NA	0.55	NA	0.21	NA	2.25***	NA
	Mode-1	8.61	0.04	0.07	0.008	0.29	0.022	1.29	0.99
	MODE-2	-	-	-	-	0.46	0.037	1.20	0.32
	MODE-3	8.73	0.039	0.05	0.004	0.29	0.010	1.21	0.18
North Avra	CN*	9.97**	NA	0.17	NA	0.21	NA	2.25***	NA
	MODE-1	9.91	0.026	0.03	0.007	0.33	0.017	0.99	0.31
	MODE-2	-	-	-	-	0.32	0.12	3.88	1.29
	MODE-3	9.80	0.018	0.0	0.007	0.31	0.008	2.42	0.28

* based on Y - data only

** by geometric mean

*** based on linear integral scale for semi-spherical variogram

NA - not available

Table 2 - Results for $\theta_i (i=1,\ldots,4)$

Parameters \ Mode		1	2	3
3,4	NORTH	-2.27	-0.81	-2.64
	SOUTH	-1.66	-1.92	-2.74
1,2	NORTH	-3.73	-	-3.89
	SOUTH	-3.49	-	-3.80
1,2,3,4	NORTH	-6.00	-	-6.54
	SOUTH	-5.15	-	-6.59
3-9	NORTH	-	-4.82	-8.01
	SOUTH	-	-5.03	-6.52

table 3 - Shanon's measurement of information I^* for several groups of parameters.

Parameter	$\theta_5 = j^{(1)}_{o,x}$		$\theta_6 = j^{(1)}_{o,x}$		$\theta_7^{(2)}$		$\theta_8^{(2)}$		$\theta_9^{(2)}$		Recharge		
	$\hat{\theta}_5$	Σ_{55}	$\hat{\theta}_6$	Σ_{66}	$\hat{\theta}_7$	Σ_{77}	$\hat{\theta}_8$	Σ_{88}	$\hat{\theta}_9$	Σ_{99}	$\hat{R}^{(3)}$	$\hat{R}^{(4)}$	$V(\hat{R})/\hat{R}^2$
NORTH	-4.3	0.19	8.09	0.32	-0.61	0.047	-0.77	0.024	-0.31	0.063	$-0.6 \cdot 10^{-3}$	-19,800	0.15
SOUTH	-0.5	0.88	13.83	0.51	-0.71	0.23	-0.77	0.04	-0.66	0.04	$-0.31 \cdot 10^{-3}$	-7,145	0.17

(1) in feet/mi

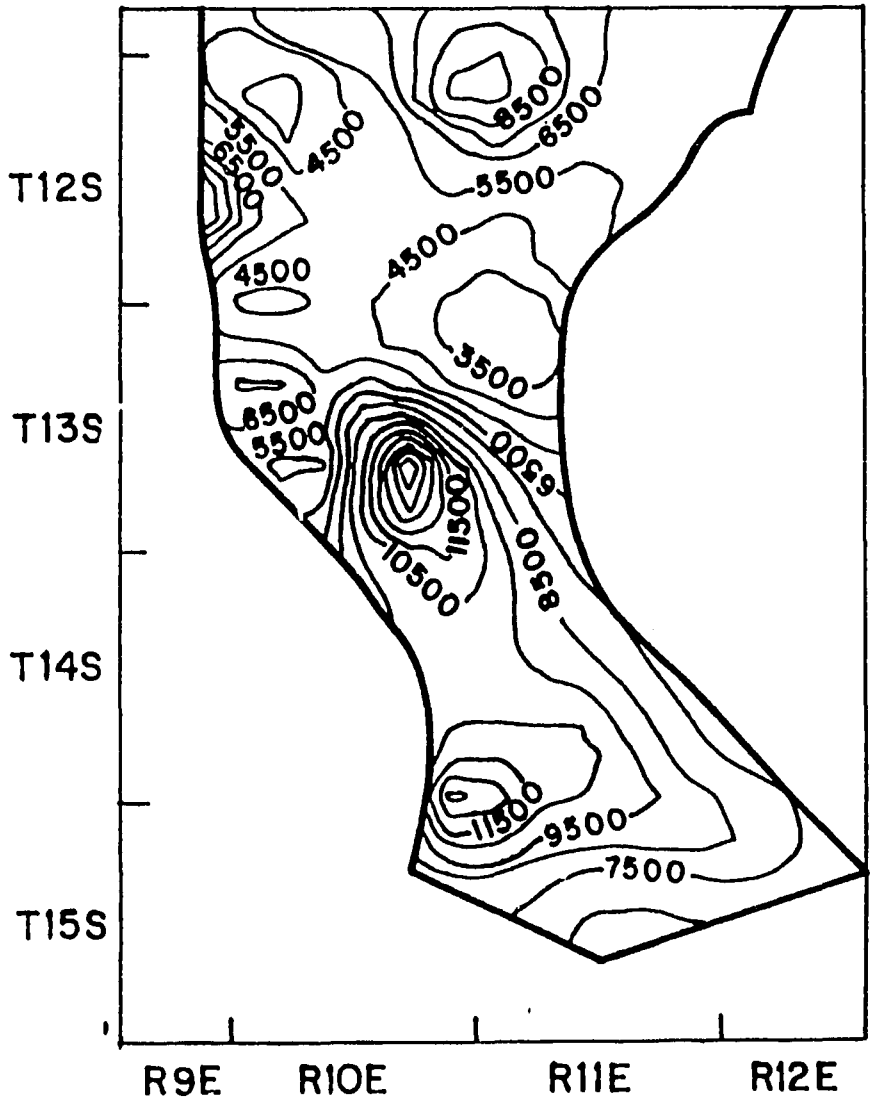
(2) in feet/mi^2

(3) in feet/day

(4) in acre-feet per year.

(5) Obtained by mode 3. Close results were obtained by mode 2.

Table 4 - The quadratic trend parameters and the corresponding recharge.


T12S
T13S
T14S
T15S
R9E
R10E
R11E
R12E
8500
8500
5500
4500
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4500
3500
6500
5500
5500
11500
10500
8500
11500
9500
7500

Fig. 1

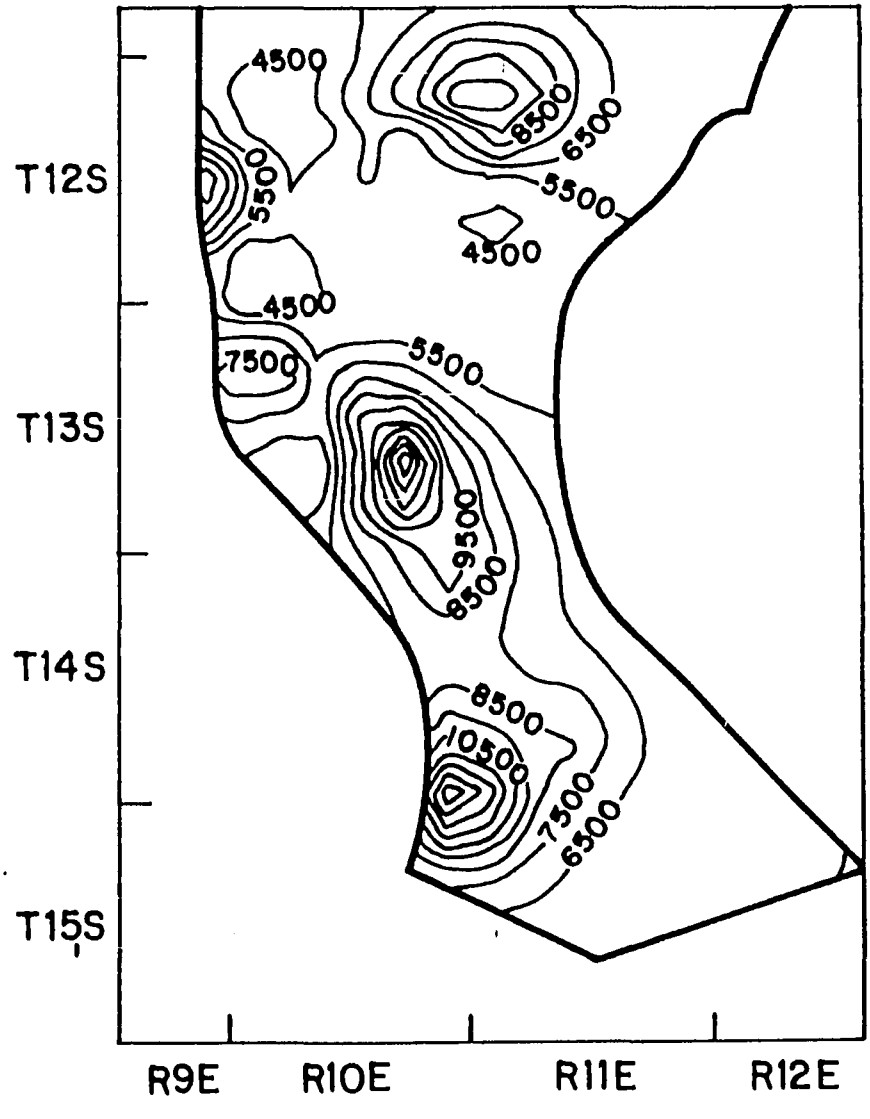
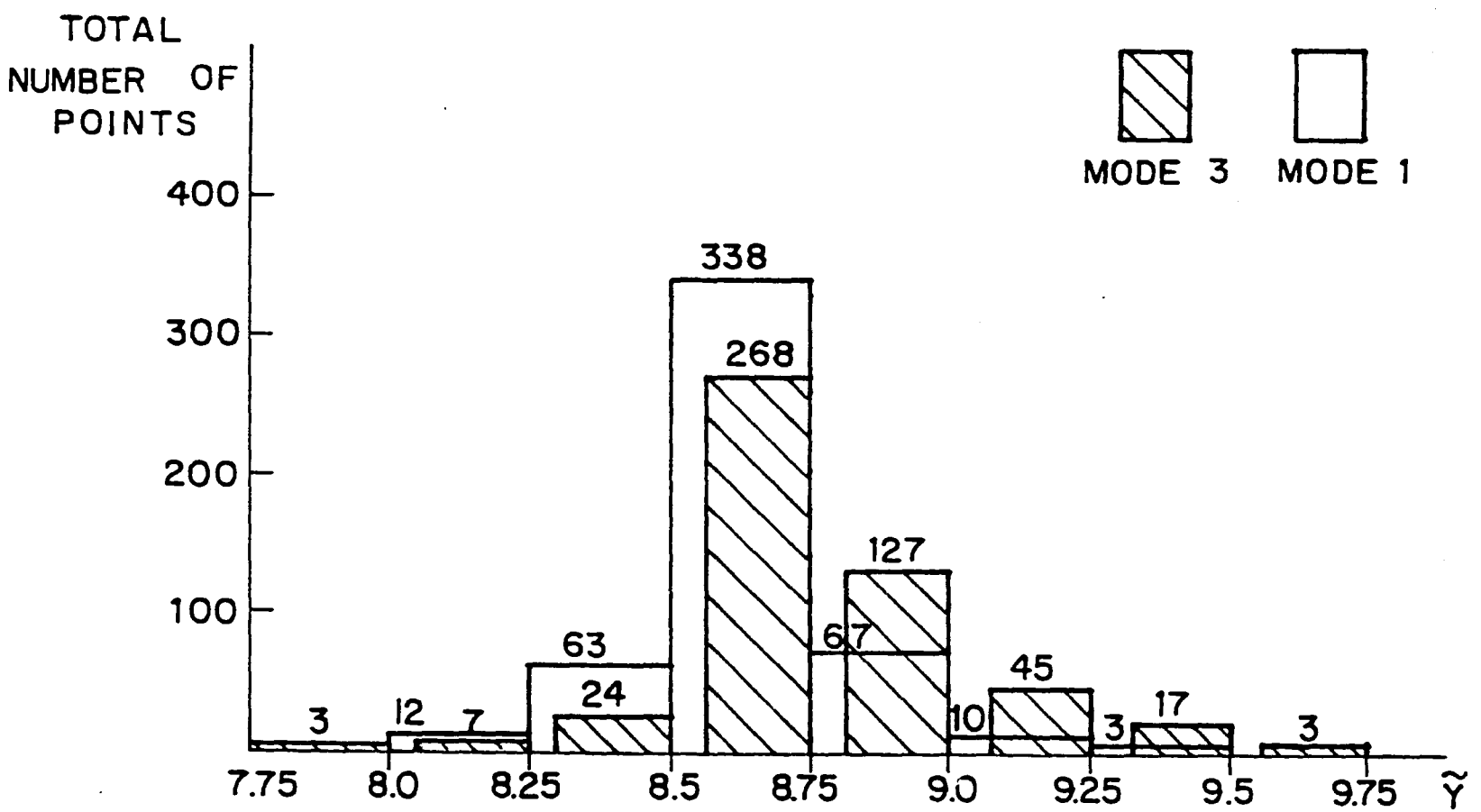
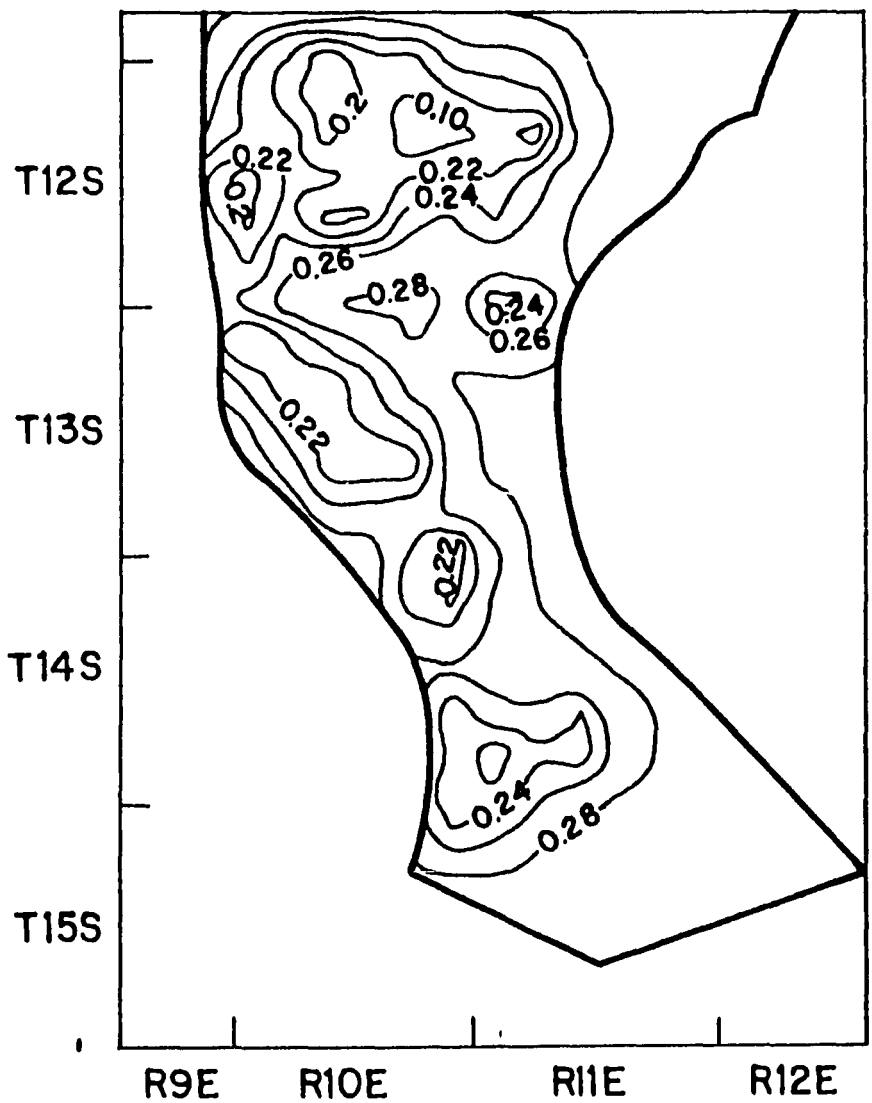

4500
8500
6500
5500
5500
4500
4500
7500
5500
9500
8500
8500
10500
7500
6500
T12S
T13S
T14S
T15S
R9E
R10E
R11E
R12E

Fig 2

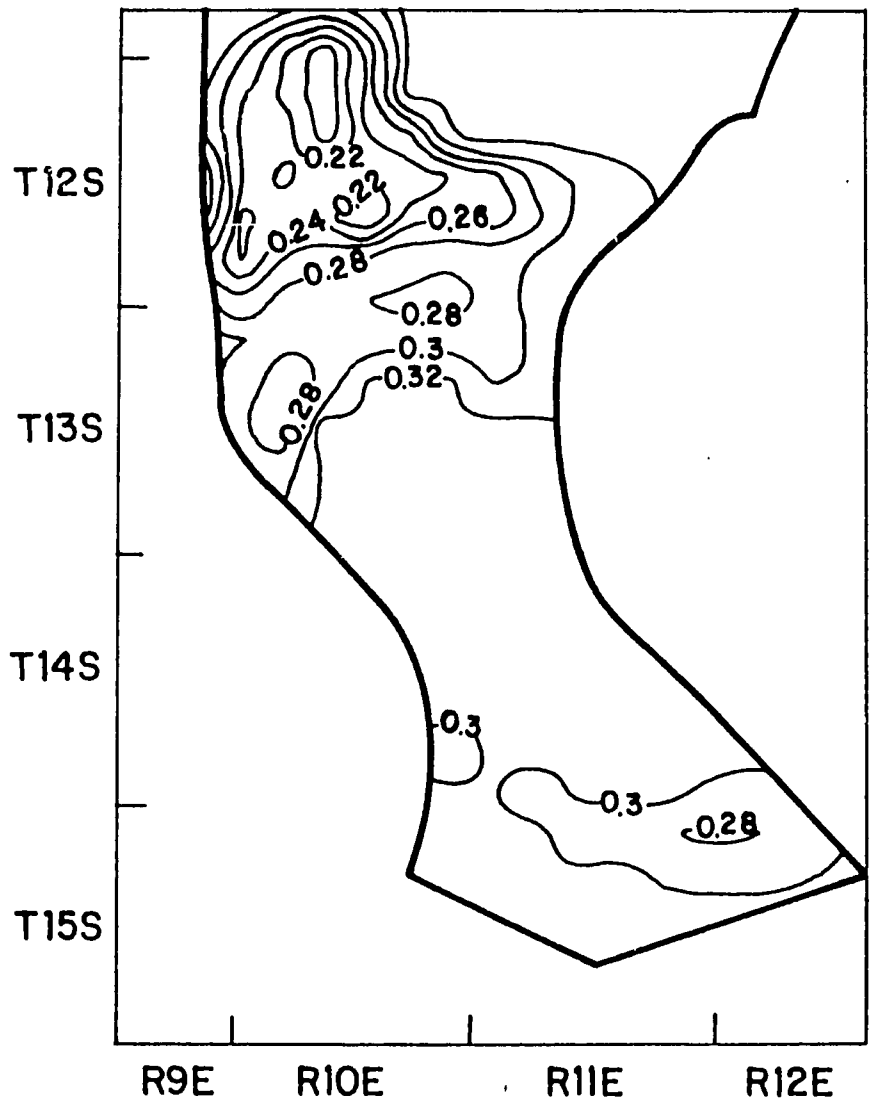
26


TOTAL NUMBER OF POINTS
400
300
200
100
MODE 3
MODE 1
338
268
127
67
63
24
45
10
3
12
7
3
17
3
7.75
8.0
8.25
8.5
8.75
9.0
9.25
9.5
9.75
$\tilde{Y}$

3


T12S
T13S
T14S
T15S
R9E
R10E
R11E
R12E
0.2
0.10
0.22
0.22
0.24
0.22
0.26
0.28
0.24
0.26
0.22
0.22
0.24
0.28

4


T12S
T13S
T14S
T15S
R9E
R10E
R11E
R12E
0.22
0.22
0.24
0.26
0.28
0.28
0.3
0.32
0.28
0.3
0.3
0.28

1 in 5

Fig. 6

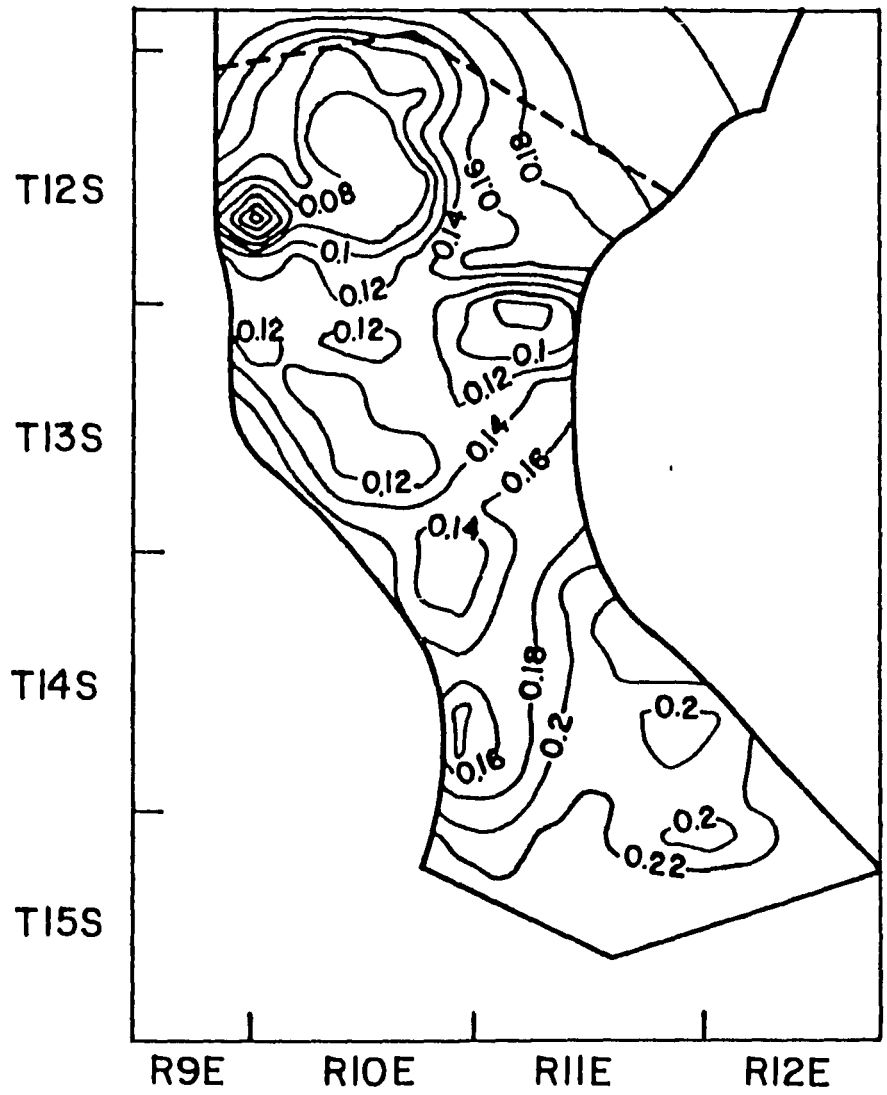
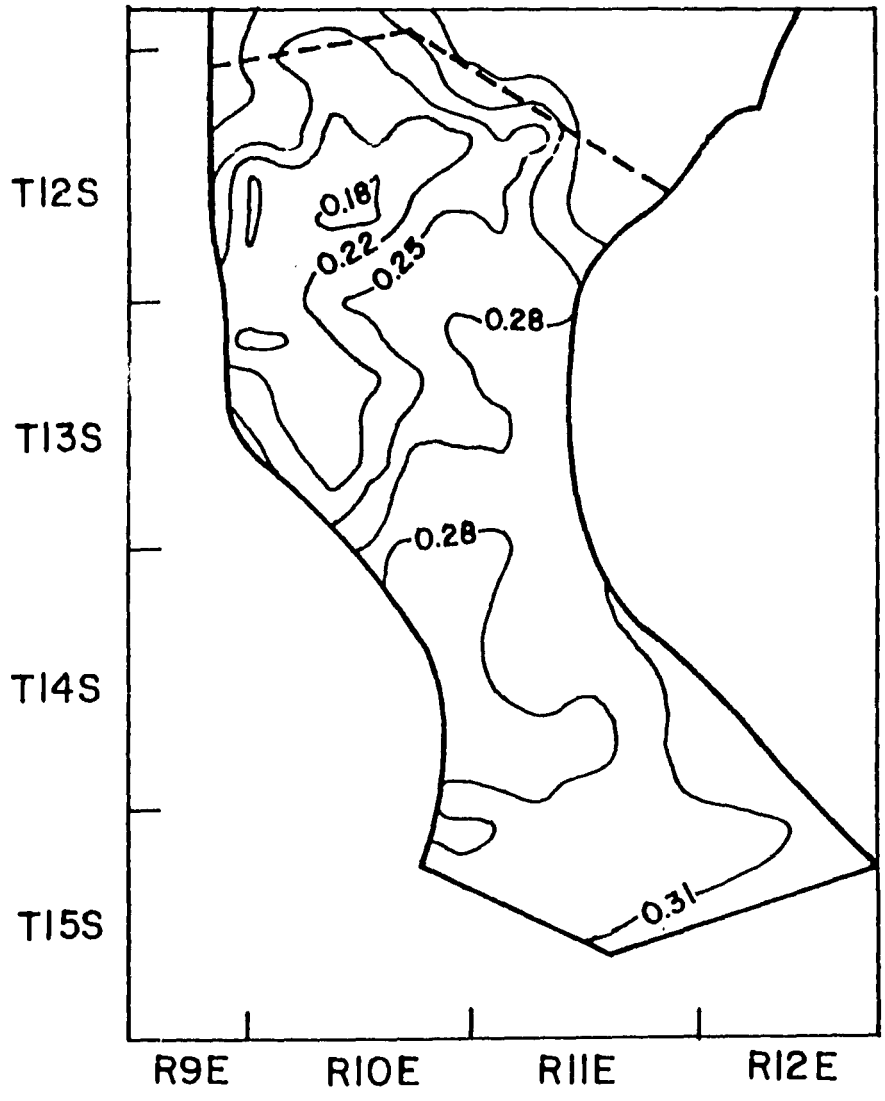
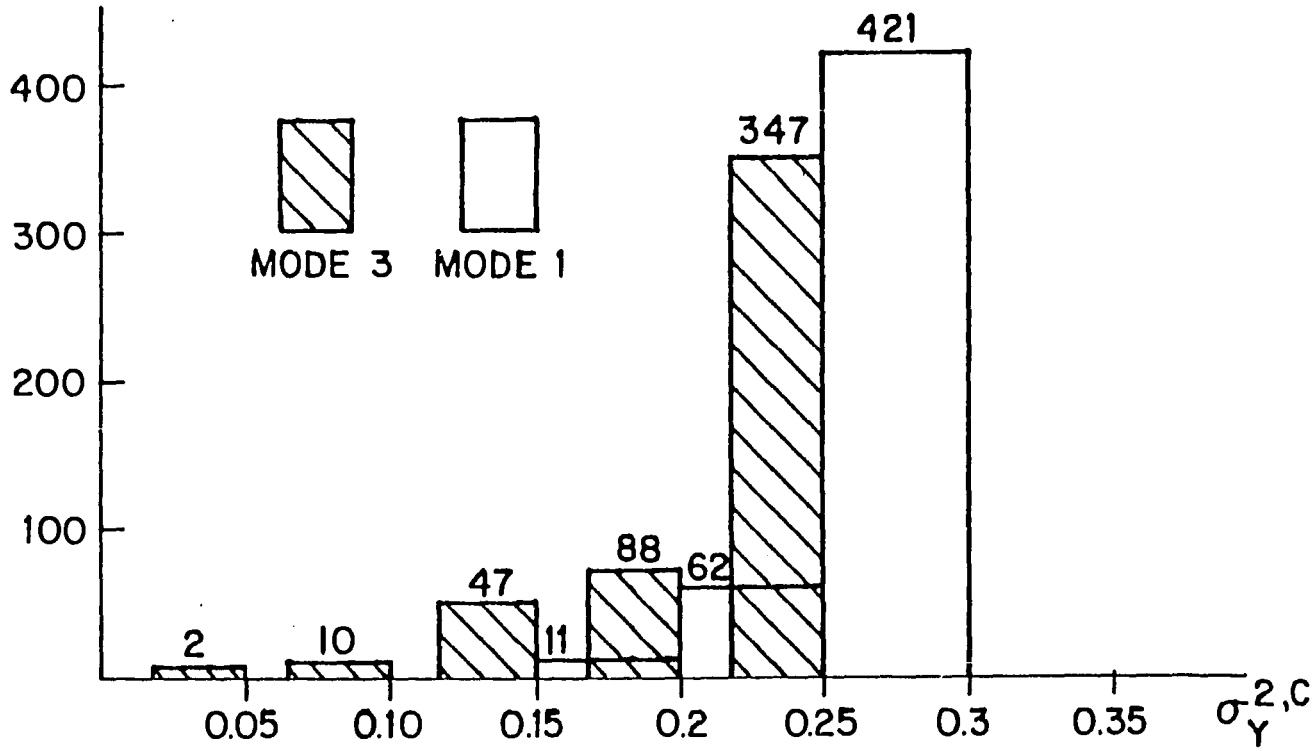

T12S
T13S
T14S
T15S
R9E
R10E
R11E
R12E
0.08
0.1
0.12
0.12
0.12
0.12
0.1
0.12
0.14
0.16
0.14
0.14
0.16
0.18
0.16
0.18
0.2
0.2
0.2
0.22

Fig 7



(a) HYSTOGRAM OF THE CONDITIONAL VARIANCE $\hat{\sigma}_Y^{2,C}(\underline{x})$ $\underline{\underline{\Sigma}}=0$



(b) HYSTOGRAM OF THE GENERALIZED VARIANCE $\tilde{\sigma}_Y^{2C}(\underline{x})$ $\Sigma \neq 0$

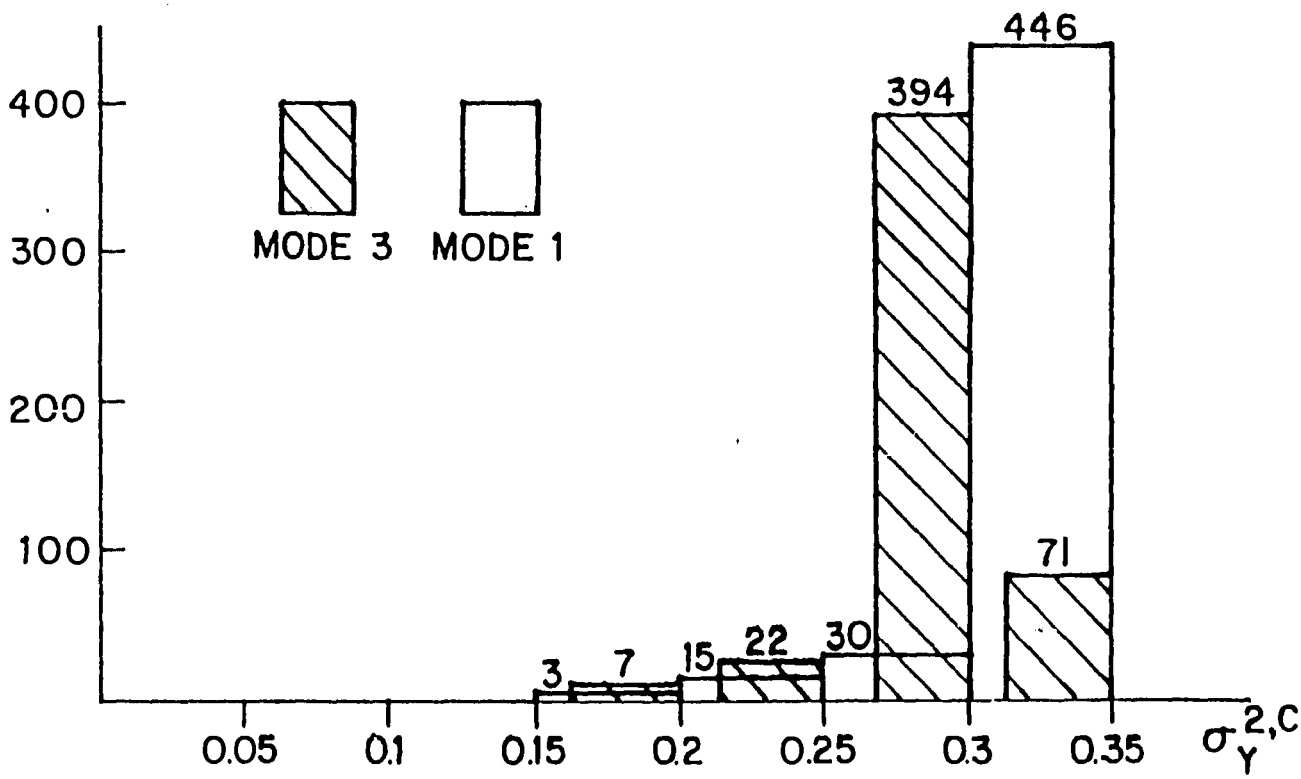


Fig 8