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Economics, Debt Investments, Derivatives Abstracts

Study Session 17

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STUDY SESSION 17
Asset Valuation
Derivatives: Options

Study Session 17: *Futures, Options & Swaps*, 3rd. Edition, Robert W. Kolb (Blackwell, 1997)

1 - A. Option Payoffs and Option Strategies (Chapter 11, pp. 316—346)

1. This reading considers a number of option strategies.
   a) In each case the cost of the spread and the profit and loss at expiration are a simple combination of the costs of and payoffs to the individual option positions making up the strategy.
   b) Note that the options considered in this section are <u>European</u> options - this avoids the complications arising from the early exercise possible with American options.
2. Straddles and strangles are the purchase of a combination of a put and a call, both on the same underlying asset and with the same expiration date.
   a) In a straddle, the strike prices of the put and the call are equal, in a strangle the strike price of the call is higher than the price of the underlying asset, while the strike price of the put is below the price of the underlying asset.
   c) The straddle or strangle trader is predicting that the long straddle or strangle trader is expecting the price to change, while the short straddle or strangle trader is expecting the price to remain where it is.
      • The short and long positions are mirror images.
3. Bull spreads and bear spreads can be created using either call options or using put options.
   a) A bull spread consists of either:
      • Two calls on the same underlying asset with the same expiration date, but the call with the higher strike price is sold, while the call with the lower strike price is bought, <u>OR</u>
      • Two puts on the same underlying asset with the same expiration date, but the put with the higher strike price is sold, while the put with the lower strike price is bought.
   b) A bear spread is the mirror image of the bull spread and consists of either:
      • Two calls on the same underlying asset with the same expiration date, but the call with the lower strike price is sold, while the call with the higher strike price is bought, <u>OR</u>
      • Two puts on the same underlying asset with the same expiration date, but the put with the lower strike price is sold, while the put with the higher strike price is bought.
   c) The bull spread trader expects the value of the underlying asset to increase, but only by a limited amount; the bear spread trader expects the value of the underlying asset to decrease, but only by a limited amount.
      • The bear spread is the mirror image of the bull spread.
4. A box spread consists of a bull spread with calls plus a bear spread with puts.
   a) Regardless of the final price of the underlying asset, at expiration this results in buying the underlying asset at the lower strike price and selling the underlying asset at the higher strike price.
      • This is a riskless return - to prevent arbitrage, the cost of establishing a box spread must be the present value of the certain payoff.
5. Butterfly spreads are made up of four options on the same underlying asset and with the same expiration date, but with three different strike prices.

a) For a long butterfly spread using either call or put options, the long trader buys the low and high strike price options, and sells two of the intermediate strike price option.
b) For short butterfly spreads, the low and high strike price options are sold, while the intermediate strike price options are bought.
c) This strategy would appeal to the trader who expects the price of the stock to remain relatively constant, but wishes to limit the possibility of large losses.

6. Condor spreads are also made up of four options on the same underlying asset and with the same expiration date, but with four different strike prices.
   a) For a long condor spread using either call options or put options, the trader buys the option with the lowest strike price, sells the option with the next lowest strike price, buys the option with the third lowest strike price, and sells the option with the highest strike price.
   b) For a short condor spread, the trader sells the option with the lowest strike price, buys the option with the next lowest strike price, sells the option with the third lowest strike price, and buys the option with the highest strike price.
   c) This strategy would appeal to the trader who expects the stock to trade within some range, but wishes to limit the possibility of large losses due to large price changes.
7. A ratio spread is a spread transaction in which two or more related options are traded in a specified proportion or ratio.
   a) Since there are many possible proportions in which options can be traded, there are a wide variety of possible ratio spreads. The effect of the ratio is to change the slope of the relationship.
   b) There are a variety of possible motivations for ratio spreads.
8. Portfolio insurance is a purchase of a put on a stock portfolio held long (actually, the purchase of a put on a stock index).
   a) The result is a minimum value to the portfolio, although gains are reduced by the put premium.
   b) Note that portfolio insurance, by setting a maximum level of loss, alters the shape of the distribution.
9. A mimicking portfolio is a portfolio which has the same profits and losses as the underlying asset, but not the same value.
   a) A mimicking portfolio for stock can be created from <u>European</u> options by using a combination of a long call and a short put, both with the same strike price.
      - Above the strike price, the put expires worthless but the profit on the call equals the profit on the underlying stock.
      - Below the strike price, the call expires worthless but the loss on the put equals the loss on the underlying stock.
10. A synthetic instrument is a portfolio that has the same profits and losses, and the same value, as the instrument it replicates.
    a) A synthetic stock can be created from <u>European</u> options by using a combination of a long call and a short put with the same strike price, and bonds in amount equal to the present value of the strike price.
       - Above the strike price, the put expires worthless but the profit on the call equals the profit on (change in value of) the underlying stock, while the value of the bonds is equal to the beginning stock value.
       - Below the strike price, the call expires worthless but the loss on the put equals the loss on (change in value of) the underlying stock while the value of the bonds is equal to the beginning stock value..
    b) The cost of creating this synthetic instrument is $c_t - p_t + Xe^{-r(T-t)}$.

11. Since a synthetic stock has the same payoffs as the underlying stock, in order to avoid arbitrage the cost of the synthetic stock must be equal to the price of the underlying stock:

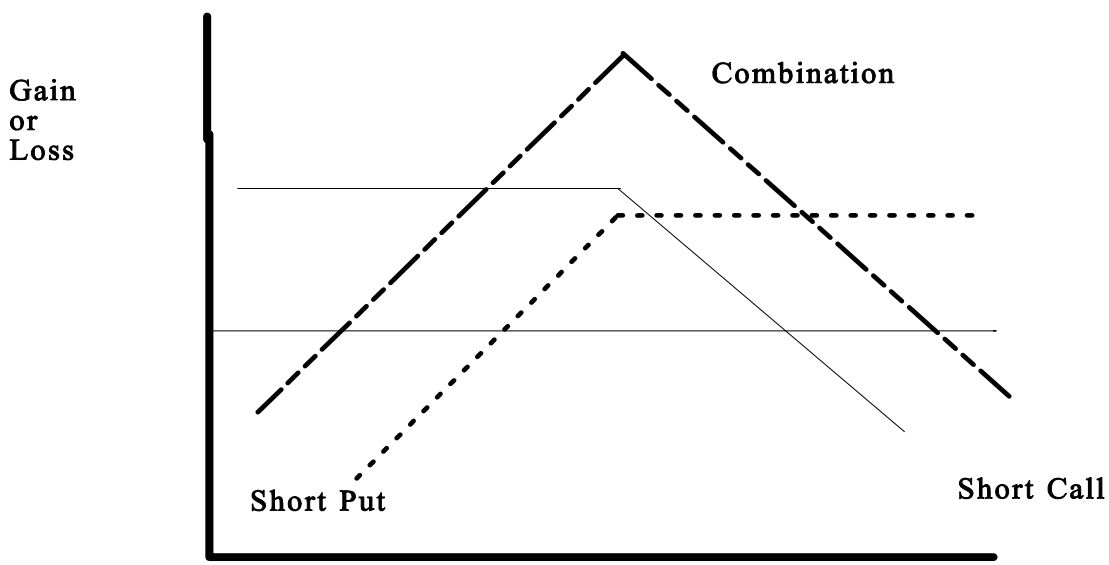
$$S_t = c_t - p_t + Xe^{-r(T-t)}$$

or, equivalently,

$$c_t - p_t = S_t - Xe^{-r(T-t)}.$$

This condition is known as put-call parity.0

SS 17. 1. A. a) Costs and payoffs of an option trading strategy. *There are many option trading strategies, but for all of the trading strategies the cost can be found by simply summing the underlying premiums (positive for a long position, negative for a short position) of the underlying positions, and the payoff can be found by simply summing the results of the underlying options. For example, a short straddle consists of a combination of short call and short put positions on the same underlying asset, having the same expiration. The cost of this position would be* $-C_t - P_t$, *a cash inflow*. Assuming *the same strike price, we can superimpose the payoff diagrams for the two positions and sum them to arrive at the payoff diagram for the combined position:*

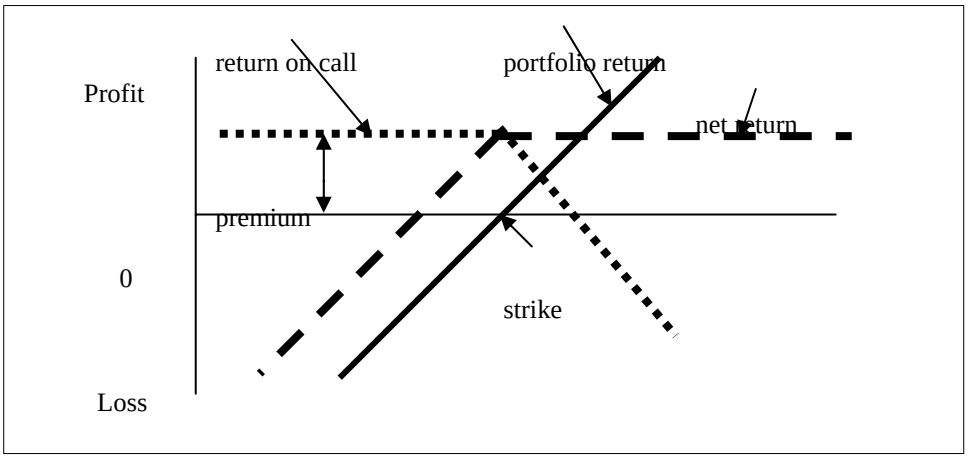


I.e.,a) If the expiration stock price is equal to the strike price neither option will be exercised, and the short straddle holder will gain the sum of the two premiums.
b) If the expiration stock price is below the strike price, the payoff to the put decreases but the value of the position is increased by the premium on the now worthless call.
c) If the expiration stock price is above the strike price, the payoff to the call decreases but the value of the position is increased by the premium on the now worthless put

The intercepts for the short straddle are found by summing the profit and loss for the underlying options when the stock price is equal to zero. I.e., if the expiration stock price is zero, the put payoff is a loss equal to the strike price, but the loss is reduced by the premium on the now worthless call.

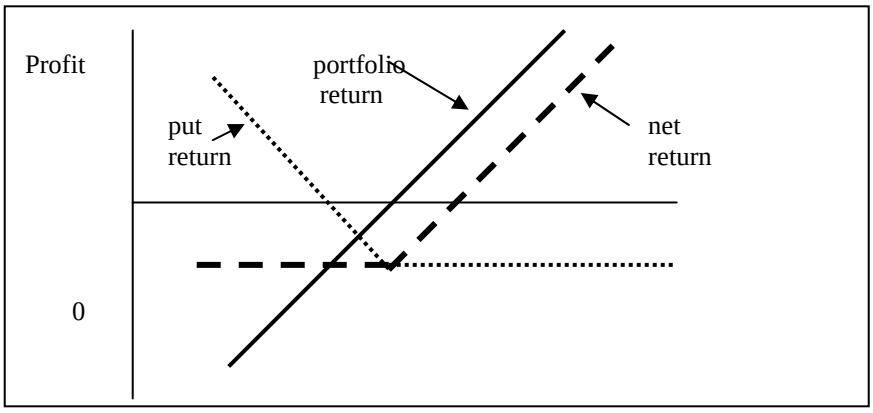
SS 17. 1. A. b) Potential profit or loss of an option trading strategy. *The profit or loss to an option trading strategy is simply the sum of the profit or loss on the underlying options. The payoff diagram simply indicates the resulting sum. The exact value may be difficult to read, but the payoff diagram can simply be used as a guideline indicating the relationship and the actual number inferred.*

SS 17. 1. A. c) Probability distribution for covered calls vs. probability distribution for a hedged portfolio. *A "covered" call is the sale of a call on a stock held long. The profit and loss diagram for this combination (assuming that the strike is equal to the portfolio value at the time the position is established) is:*

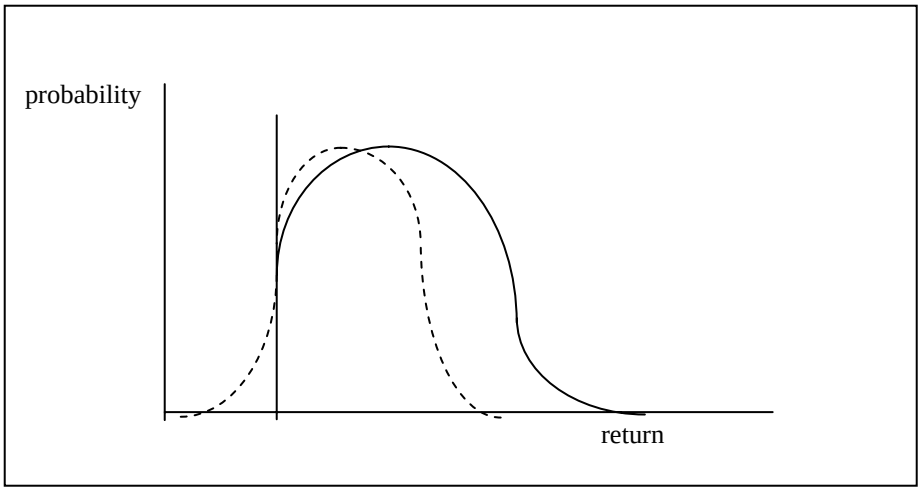


The net return is the sum of the return to the two individual positions. As shown, the net return is limited on the upside – it will not be larger than the premium. Below the strike price, the net return decreases, but is always greater than the return to the base portfolio.

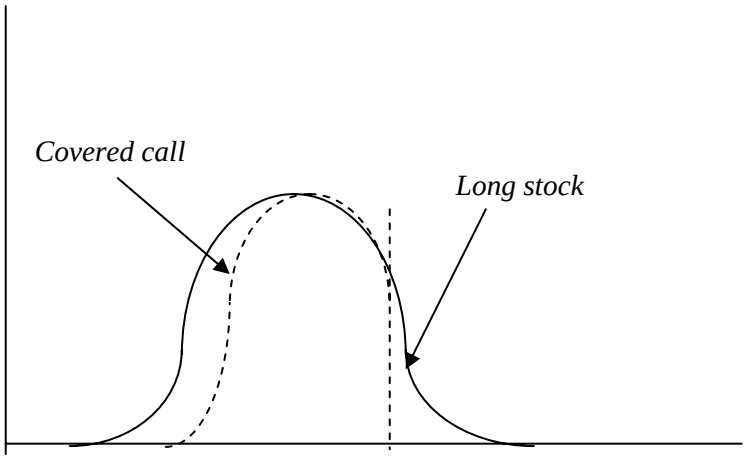
Portfolio insurance is the purchase of a put on a stock portfolio held long (actually, the purchase of a put on a stock index). Assuming that the portfolio was at the strike price when the put was purchases, the profit and loss diagram for this combination is:



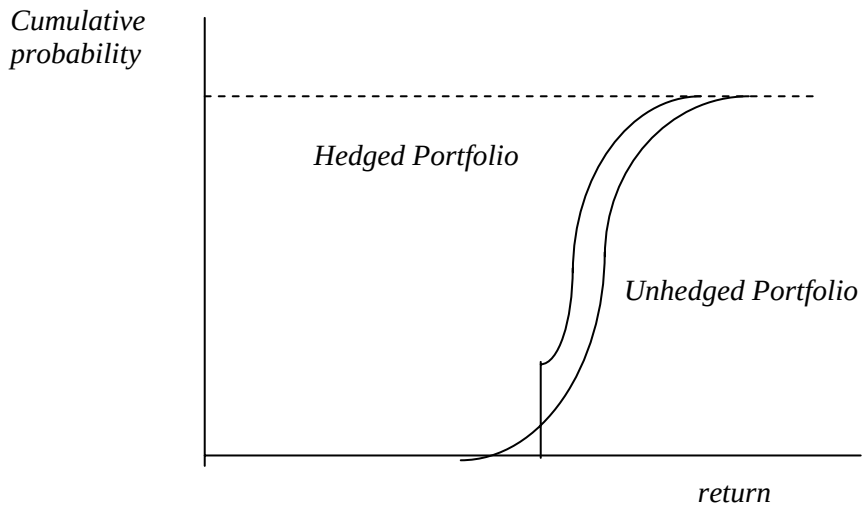
If we combine the profit and loss diagram with the probability distribution for the terminal value of the index, we obtain the probability distribution for the profit and loss on the position:



Note the impact portfolio insurance - by setting a maximum level of loss, the purchase of a put alters the shape of the distribution. Similarly, by combining the probability distribution for the price of the underlying asset with the profit and loss diagram for a covered call, we could see the impact of the sale of a covered call:

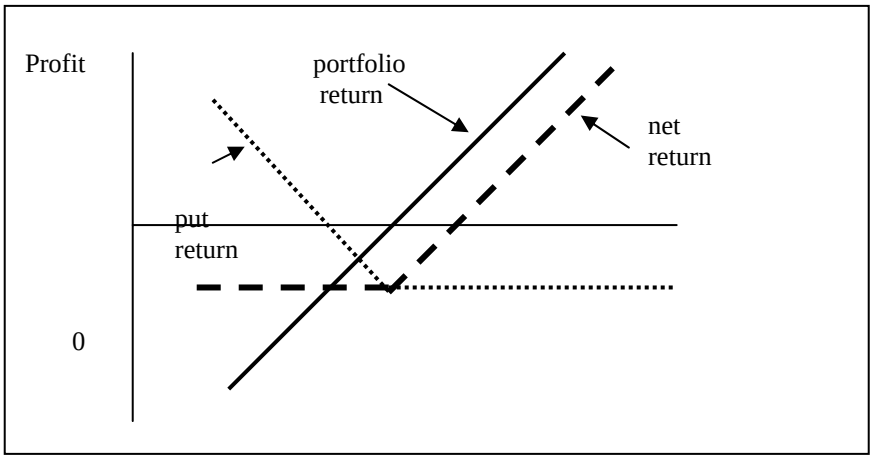


The effect of portfolio insurance and a covered call can also be shown using the cumulative probability distribution, where the cumulative probability is the probability that the outcome is equal to or less than the specified outcome:



Portfolio insurance establishes a minimum value for the position, effectively putting a floor under losses. A covered call strategy, on the other hand, establishes a maximum value for the position and effectively puts a ceiling on gains while providing some downside protection because of the premium received.

SS 17. 1. A. d) Expiration profits / losses of a hedged vs. unhedged portfolio,. *The resulting return for a hedged portfolio is the sum of the returns to the individual positions. In the case of a protective put, the result is as shown. The procedure of graphing the returns to the individual positions can then be used to analyze any combination of positions.*



SS 17. 1. A. e) Synthetic stock using a European put option, a European call option, and a risk-free discount bond. *A synthetic stock can be created from <u>European</u> options by using a combination of a long call and a short put with the same strike price, and bonds in amount equal to the present value of the strike price.*

a) Above the strike price, the put expires worthless but the profit on the call equals the profit on the underlying stock while the bonds equate the value: i.e., at expiration the value of the put =0, the value of the call = S_T - X, and the value of the bonds = X, so that the value of the position is:

$$S_T - X + X = S_T.$$

b) Below the strike price, the call expires worthless but the loss on the put equals the loss on the underlying stock while the bonds equate the value, i.e., at expiration the value of the put = $X - S_T$, the value of the call = 0, and the value of the bonds = X, so that the value of the position is:

$$X - S_T + X = S_T.$$

SS 17. 1. A. f) Value of a European call (put) given the European put (call) value. *Since a synthetic stock has the same payoffs as the underlying stock, in order to avoid arbitrage the cost of the synthetic stock must be equal to the price of the underlying stock:*

$$S_t = c_t - p_t + Xe^{-r(T-t)}$$

or, equivalently,

$$c_t - p_t = S_t - Xe^{-r(T-t)}.$$

This condition is known as put-call parity, and can be algebraically manipulated to give:

$$c_t = p_t + S_t - Xe^{-r(T-t)}$$

or

$$p_t = c_t - S_t + Xe^{-r(T-t)}.$$

Example: If a call on a stock with a current price of 42 has a strike price of 40 and expiration of 90 days has a premium of 8, and the risk-free rate is such that a 90-day riskless bond with a maturity value of 100 will cost 92, put-call parity requires that:

$$p_t = 8 - 42 + 40(0.92) = 2.8.$$

SS 17. 1. A. g) Put-call parity and European-style options. *If the put-call parity condition is violated, it is possible to construct a riskless, costless hedge that is immediately profitable. I.e., the synthetic stock has the same payoffs as the underlying stock. If the synthetic stock had a different price than the underlying stock, the payoffs would have two different prices. It would then be possible to arbitrage - i.e., to sell the payoffs at the higher price, while simultaneously buying them at the lower price.*
Example: As shown above, if a call on a stock with a current price of 42 has a strike price of 40 and expiration of 90 days has a premium of 8, and the risk-free rate is such that a 90-day riskless bond with a maturity value of 100 will cost 92, put-call parity requires that: $p_t = 8 - 42 + 40(0.92) = 2.8$. If the put were priced at 2, it would be underpriced. It would be possible for a trader to purchase the put at 2, sell the call at 8, buy the stock at 42, and borrow 0.4(92) = 36.8. This action would produce an immediate inflow of -2 +8 -42 +36.8 = 0.8. At maturity:

a) If the stock price is above the strike price the put would expire unexercised while the call would be exercised. The stock would be delivered against the call and the proceeds would used to pay the borrowing, with no cash flow.

b) If the stock price is below the strike price the call would expire unexercised. The put would be exercised by delivering the stock and the proceeds would be used to pay the borrowing, with no cash flow.

Self Test Question: Assuming a strike price of 40 and a premium of $2.00 per share, **indicate** the situation resulting in the maximum gain or loss and **calculate** the maximum gain and loss, for both the buyer and seller of a naked call and a naked put.
Suggested Answer: The maximum loss to the buyer of a call occurs when the option expires worthless (stock price < $40.00). The loss is the amount of the premium, $2.00 per share. The maximum gain of the buyer of a call is indeterminate, since it depends on the price of the underlying asset.
The maximum gain to the seller of a call occurs when the option expires worthless. The gain is the premium, $2.00 per share. The maximum loss to the seller of a call is indeterminate, since it depends on the price of the underlying asset.

The maximum loss to the buyer of the put occurs when the option expires worthless (stock price > \$40.00). The loss is the amount of the premium, \$2.00 per share. The maximum gain to the buyer of a put occurs when the stock price goes to zero. The gain is the strike price per share less the premium per share = \$40.00 - 2.00 = \$38.00 per share.
The maximum gain to the seller of a put occurs when the option expires worthless (stock price > \$40.00). The gain is the amount of the premium, \$2.00 per share. The maximum loss to the seller of a put occurs when the stock price goes to zero. The loss is the strike price per share less the premium per share = \$40.00 - 2.00 = \$38.00 per share.

1 - B. European Option Pricing (Chapter 13, pp. 381-386 and 396-405)

1. In a binomial model (model with two states of the world at the end of a single period) a European call option can be replicated by a combination of the underlying asset and a short position in a risk-free bond.
   a) If the synthetic call is composed of N shares of stock with price S_t and a short position in a bond of price B_t, the value of the portfolio at the end of the period, time T, can be one of two values:
      • In one state, the up state, the price of the stock has increased. The value of the synthetic call is then $NUS_t - RB_t$, where U is the holding period return on the stock (1 + the % return) and R is the risk-free rate.
      • In the other state, the down state, the price of the stock has decreased. The value of the synthetic call is then $NDS_t - RB_t$, where D is the holding period return on the stock (1 + the % return) and R is the risk-free rate.
   b) In order to replicate the call, the outcomes of the synthetic instrument must be the same as the outcomes for the call:
      • In the up state, $NUS_t - RB_t = C_U$.
      • In the down state, $NDS_t - RB_t = C_D$.

$$N^* = \frac{C_U - C_D}{(U - D)S_t}, \qquad B_t^* = \frac{C_{UD} - C_{DU}}{(U - D)\,R}$$

   c) Solving, we obtain:
   d) Thus, a position of N^* shares and B_t^* worth of the riskless bond will have the same outcomes as the call - i.e., the position will synthetically replicate the call.
   e) This binomial model can be extended to multiple final states by using subperiods - i.e., by splitting the original horizon into shorter intervals.
2. Since the synthetic instrument has the same outcomes as the call, the price of the call and the price of the synthetic instrument must be the same if arbitrage is not to exist:

$$\text{Price of Call} = N^* S_t - B_t^*$$

3. An alternative approach to call pricing is the Black-Scholes option pricing model. The model is built on seven assumptions:
   a) Prices adjust so as to prevent arbitrage.
   b) Stock prices change continuously.
   c) Stock returns are log-normally distributed.
   d) The option is a European call.
   e) The underlying stock does not pay a dividend.

f) The interest rate is constant over the life of the option.
g) The volatility of the stock remains constant over the life of the option.

4. The Black-Scholes option pricing model is expressed as:

$$c_t = S_t N(d_1) - Xe^{-r(T-t)}N(d_2),$$

where:

$$d_1 = \frac{\ln(\frac{S_t}{X}) + (r + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \qquad d_2 = d_1 - \sigma\sqrt{T - t}$$

a) The variables or inputs necessary to for computation of a call or put option value are:
   - The stock price S_t,
   - The risk-free rate r,
   - The exercise price X,
   - The time to expiration, T - t,
   - The standard deviation of the stock's returns, σ.

b) The Black-Scholes model applies to European calls. To apply the model to estimate the price of a European put, the Black-Scholes call price is used with the put-call parity relationship to derive a formula for the price of the put.

c) Of the necessary variables, S_t, T - t, and X are known, while r can be estimated from T-bill rates. Estimation of the standard deviation of the stock's returns, σ, is more difficult.
   - One approach to estimation of σ is to use historical data.
     · Care must be taken to adjust the σ to the time period being used in the model.
   - An alternate approach is to use observed prices on different options with the same underlying asset to arrive at an implied volatility i.e., instead of using the five inputs to solve for price, the observed price is used with S_t, T - t, X, and r to find σ.
     · The estimate is based on a weighted average of the implied volatilities from the different options. The weighting is usually highest on options near breakeven or "closest-to-the-money," because of infrequent trading or poor performance of the model for options which at "far-from-the-money."

5. The logic of the Black-Scholes model is quite similar to the logic of the binomial model, and indeed the binomial model will converge to the Black-Scholes model as the number of sub-periods increases.
   a) Both models are based on the assumption that prices adjust so as to make arbitrage impossible.
   b) Both models result in an expression of the general form:

$$c_t = S_t(\text{ Factor 1}) + Xe^{r(T-t)}(\text{Factor 2}),$$

   with the difference in the Factor expressions arising from the discrete vs. continuous nature of the models.

SS 17. 1. B. a) Synthetic European equity call option using stocks and bonds. *In a binomial model (model with two states of the world at the end of a single period) a European call option can be replicated by a combination of the underlying asset and a short position in a risk-free bond. If the synthetic call is composed of N shares of stock with price S_t and a short position in a bond of price B_t, the value of the portfolio at the end of the period, time T, can be one of two values:*

a) *In one state, the up state, the price of the stock has increased. The value of the synthetic call is then* $NUS_T - {}_{RBt}$, *where U is the holding period return on the stock (1 + the % return) and R is the risk-free rate.*
b) *In the other state, the down state, the price of the stock has decreased. The value of the synthetic call is then* $NDS_T - {}_{RBt}$, *where D is the holding period return on the stock (1 + the % return) and R is the risk-free rate.*

In order to replicate the call, we must choose an N^* *and a* B^*_t *such that the outcomes of the synthetic instrument will be the same as the outcomes for the call:*

- *In the up state,* $N^*UST - RB^*_t = C_U$.
- *In the down state,* $N^*DS_T - RB^*_t = C_D$.

Solving, we obtain:

$$N^* = \frac{C_U - C_D}{(U - D)S_t}, \qquad B_t^* = \frac{C_{UD} - C_{DU}}{(U - D)R}$$

SS 17. 1. B. b) Estimating volatility inputs for the Black-Scholes model. *One approach to estimation of σ is to use historical data. Care must be taken to adjust the σ to the time period being used in the model. An alternate approach is to use observed prices on different options with the same underlying asset to arrive at an implied volatility i.e., instead of using the five inputs to solve for price, the observed price is used with* S_t, $T - t$, X, *and r to find σ. The estimate is based on a weighted average of the implied volatilities from the different options. The weighting is usually highest on options near breakeven or "closest-to-the-money," because of infrequent trading or poor performance of the model for options which at "far-from-the-money."*

SS 17. 1. B. c) Value of a European call (put) option. *Given the binomial model, we can use the equation:*

$$\text{Price of Call} = N^* S_t - B_t^*,$$

where:

$$N^* = \frac{C_U - C_D}{(U - D)S_t}, \qquad B_t^* = \frac{C_{UD} - C_{DU}}{(U - D)R}$$

c_U = *the value of the call at the up state*
c_D = *the value of the call at the down state*
U = the holding period return on the stock
R = the risk free rate
S is the stock price in the up state
D is the price of the stock in the down state.

Using the Black-Scholes option pricing model the price of the call is:

$$c_t = S_t N(d_1) - Xe^{-r(T - t)}N(d_2),$$

where:

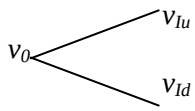
$$d_1 = \frac{\ln(\frac{S_t}{X}) + (r + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \qquad d_2 = d_1 - \sigma\sqrt{T - t}$$

and:

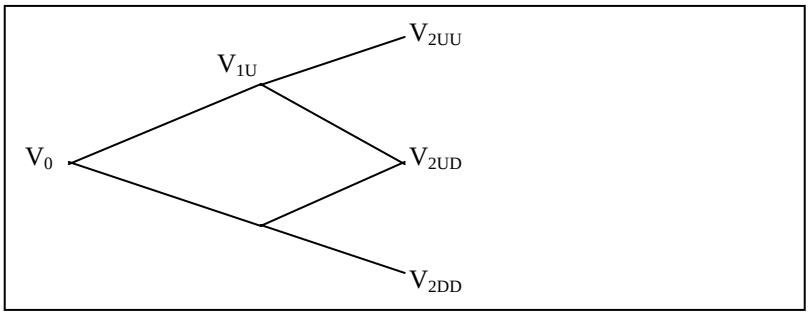
- *The stock price = S_t,*
- *The risk-free rate = r,*
- *The exercise price = X,*
- *The time to expiration is T - t,*
- *The standard deviation of the stock's returns = σ.*

Both the binomial model and the Black-Scholes model give the price of a call. In order to find the price of a put, the models are used to find the price of the call, and the put-call parity condition is used to compute the put price.

SS 17. 1. B. d) Single- and two-period binomial trees for both the underlying asset and the corresponding option. *The single period tree is:*



For the tree for the underlying asset, V_0 is the original value, V_{1U} the value after an increase in price, and V_{1D} the value after a decrease in price. For the option tree, V_0 is the value of the option at the original asset price, V_{1U} is the value of the option after the increase in the price of the asset, and V_{1D} is the value of the option after the increase in the price of the asset, The two-period binomial tree is:



Again, for the underlying asset tree the Vs are the value of the asset after successive increases in price (Us) and decreases in price, while for the option the Vs are the option values.

SS 17. 1. B. e) Annualizing standard deviation. *Standard deviation is annualized by multiplying the per-trading-period standard deviation by the square root of the number of trading periods in a year. I.e., the annualized standard deviation is:*

annualized standard deviation = (period standard deviation) X $(250/\text{trading days per period})^{½}$

Example: if the daily volatility of a stock is 0.02345, we would multiply by $(250)^1/2$ to find the annualized standard deviation, since there are 250 trading days in a year. If the standard deviation of a different stock over a 50 trading day period was 0.16543, we would multiply by $(250/50)^{½}$ to find the annualized standard deviation

SS 17. 1. B. f) Adjusting Black-Scholes for continuous dividends. *Basically, if dividends are paid at a continuous rate, they may be considered as negative interest rates.*

Self Test Question: The price of a stock is presently $55, may have one of two values at the end of the period: $60.00 or $40.00. The risk-free rate is 6%. Use this information to **compute** the value of a European call on the stock with a strike price of $50 and an expiration of one year.

Suggested Answer: If the position is made up of N shares of the stock and $B of the bond, the values at expiration must be:
Up State: N(60) - B(1.06) = 10 Down State: N(40) - B(1.06) = 0

From the Down State condition, we find that N = B(1.06) / 40. Substituting this into the Up State condition, we obtain [(60/40)B(1.06)] - B(1.06) = 10
Solving, we obtain B = $18.68, and substituting into the Down State condition we obtain N = 0.5. The mimicking or replicating portfolio is then composed of $-18.68 of the bond and 0.5 shares of stock, having a cost (value) of (0.5)($55) - $18.68 = $8.63.

1 - C. Option Sensitivities and Option Hedging (Chapter 14, pp.422-437)

1. Payoff diagrams for European options are easily derived form the expiration values.
   a) At a given stock price, the value of the option will generally be above the expiration value
      • It is important to note that the value of a put can be below the intrinsic value.
2. The "delta" (δ) of an option is the first derivative of the value of the option with respect to the value of the underlying asset:
$$\delta = \partial c/\partial S.$$
   a) Expressed in finite or approximate terms, this is:
$$\delta = \Delta c/\Delta S,$$
   where Δ indicates a (small) change.
   b) I.e., delta is the change in the value of the option for a small change in the value of the underlying asset. Alternately, it is the sensitivity of the option value to changes in the value of the underlying asset.
   c) Geometrically (and intuitively) this is interpreted as the slope of the payoff diagram at a point, as indicated in the above diagram.
3. Delta is computed by simply applying the formula $\delta = \Delta c/\Delta S$ to observed price changes. Correspondingly, $\Delta c = \delta \text{ X } \Delta S$.
4. A delta-neutral portfolio is a portfolio made up of a short call plus δ units of the underlying asset.
   a) The portfolio is termed delta-neutral because (small) changes in the price of the underlying asset will have no effect on the value of the portfolio:
$$\Delta P = -\Delta c + \delta\Delta S = -\delta\Delta S + \delta\Delta S = 0$$
   b) This is a perfect hedge - no price change.
   c) Note that this applies exactly only to infinitesimally small changes in the price of the underlying asset:
      • The slope of this relationship is exactly zero only at the exact stock price at which the delta is computed, is approximately zero for small changes around that price, and is increasingly different from zero as the size of the change increases.
      • Because the delta-neutral hedge is exact only at one price, the portfolio must be continuously rebalanced as the asset price changes to maintain the delta-neutral hedge.
5. Option value is also related to a number of other variables. Each of the relationships can also be expressed as a first derivative or as the slope of the relationship or simply as the sensitivity of the option price to changes in the variable.
   a) Option value is related to the time remaining to expiration.
      • The measure of sensitivity is called theta (θ).

- The relationship between option value and time to expiration is positive.

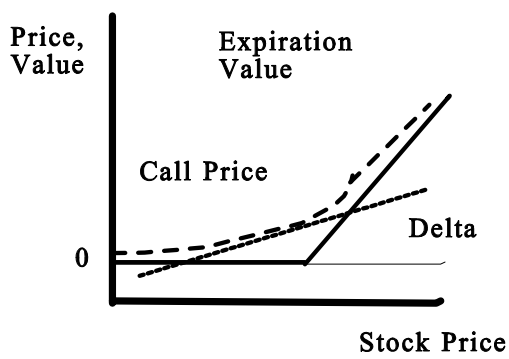
b) Option value is related to the volatility of the underlying asset.
- The measure of sensitivity is called vega (υ).
- The relationship between option value and volatility is positive.

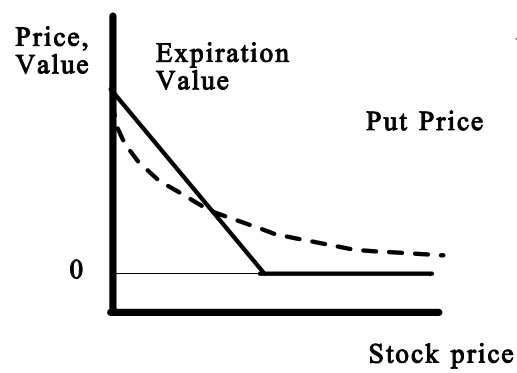
c) Option value is related to the interest rate:
- The measure of sensitivity is called rho (ρ).
- The value of a call is positively related to the interest rate, while the value of a put is negatively related to the interest rate.

6. The variable gamma (γ) is a second-order effect - it is the relationship between the delta of an option and the value of the underlying stock.

a) The relationship is exactly the same for puts and calls, but may be positive or negative, depending on the situation.

SS 17. 1. C. a) Payoffs to a European put or call option before and at expiration. *Payoff diagrams for European options at expiration are easily derived and are shown below. Note that the payoff is simply the cash flow at exercise and does not include the premium. Payoff diagrams (i.e., the value of the option vs. the value of the underlying asset) for a European option at times other than expiration, based on the Merton model, are also shown:*





a) The expiration value is simply the value of the shares to be received.

b) Prior to expiration, we would generally expect the price/value of the call to be above the corresponding expiration value, simply because of the positive time value of the option.

c) Not that for the put, however, the price/value before expiration may be below the expiration value. This occurs because if the price is low and the payoff to exercise high, the present value effect of the time to expiration may outweigh the time value of the put.

The "delta" (δ) of an option is the first derivative of the value of the option with respect to the value of the underlying asset:

$$\delta = \partial c / \partial S.$$

Geometrically this is interpreted as the slope of the payoff diagram at a point, as indicated in the above call diagram I.e., an option with a higher slope is more sensitive to changes in the underlying variable, in this case the value of the underlying security.

SS 17. 1. C. b) Change in option price using delta and the change in asset price; computing delta from the change in option price and the change in asset price. *The change in the option price is computed by simply applying the formula $\Delta c = \delta$ X ΔS. For example, if the δ is 0.75 and the ΔS is \$0.50, the* <u>*approximate*</u> *change in the option price is:*

$$\Delta c = 0.75 \text{ X } \$0.50 = \$0.375.$$

The delta is computed by simply applying the formula $\delta = \Delta c/\Delta S$. *For example, if the ΔS is \$0.50 and the change in the option price is \$0.0375, the* <u>*approximate*</u> *delta is:*

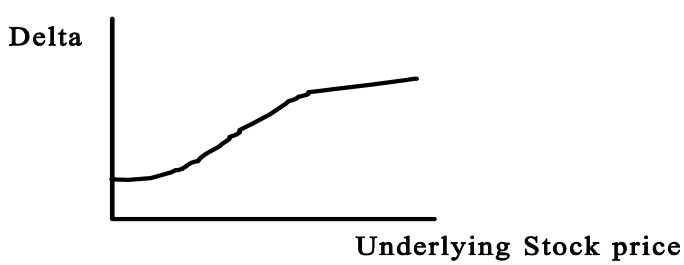
$$\delta = \$0.375 / \$0.50 = 0.75$$

SS 17. 1.C. c) Properties of a delta-neutral portfolio and their relationship to hedging. *A delta-neutral portfolio is a portfolio made up of a short call plus delta units of the underlying asset. The portfolio is termed delta-neutral because (small) changes in the price of the underlying asset will have no effect on the value of the portfolio:*

$$\Delta P = -\Delta c + \delta\Delta S = -\delta\Delta S + \delta\Delta S = 0$$

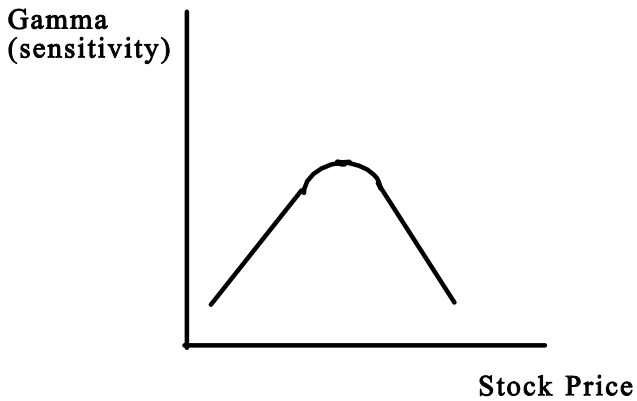
For infinitesimally small ΔS this is a perfect hedge - no price change.

SS 17. 1. C. d) Delta as a function of asset price. *The diagram of the payoffs to a European call indicates that at stock prices well below the strike price, the call price remains relatively constant. This indicates that the slope of the relationship, the delta, is small. As the price of the underlying asset approaches the strike price, however, the slope increases. At the other extreme, when the price of the stock is well above the strike price, the slope of the relationship is almost one, and again has only small changes. This indicates that the slope will be low if the price of the stock is well below the strike price, but the slope will increase rapidly when the price of the underlying asset is around the strike price. Once the price of the asset is well above the strike price, however, the slope will change less rapidly toward the maximum of one. The change in the slope of the relationship of a put to the price of the underlying asset is similar, but starts from a slope close to negative one and goes to a value close to zero. Graphically:*



In the general area of the strike price, however, the slope is changing rapidly, going from low sensitivity to changes in stock price to higher sensitivity to changes in stock price.

Graphically, this would be:



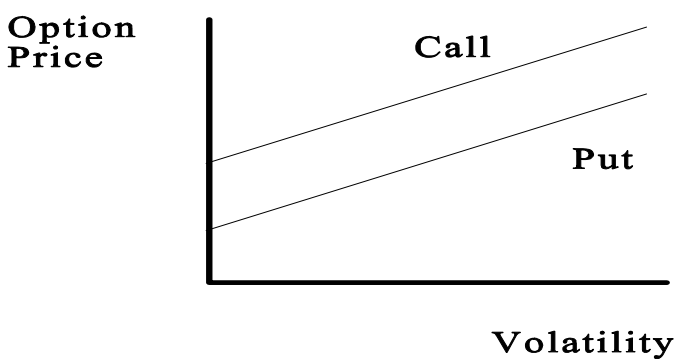
SS 17. 1. C. e) Option price and volatility. *Option value is related to the volatility of the underlying asset. Vega (v) is the first derivative of the value of the option with respect to the volatility of the underlying asset:*

$$v = \partial c / \partial \sigma_S,$$

or, expressed in finite or approximate terms, this is:

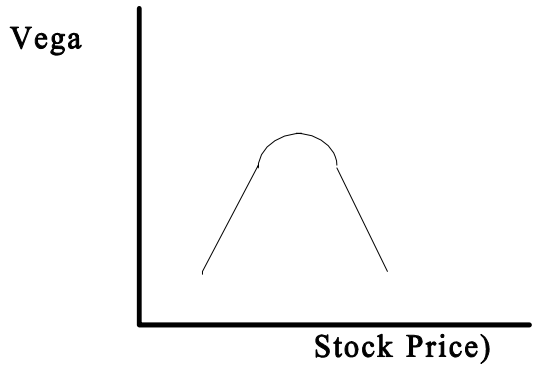
$$v = \Delta c / \Delta \sigma_S.$$

The relationship between option value and vega is positive and linear:



The simplest way to remember this relationship is to note that an option on a stock that has an unchanging price would be worthless.

SS 17. 1. C. f) Option sensitivity to volatility as asset price varies. *Option sensitivity to volatility (vega) will vary with the value of the underlying asset. Vega tends to be greatest for options that are "near-the-money" (i.e., the stock price is close to the strike price). As the stock price moves away from the strike price in either direction (i.e., further "out-of-the-money" or further "in-the-money"), the option price will be less sensitive to the stock price (vega will be smaller):*

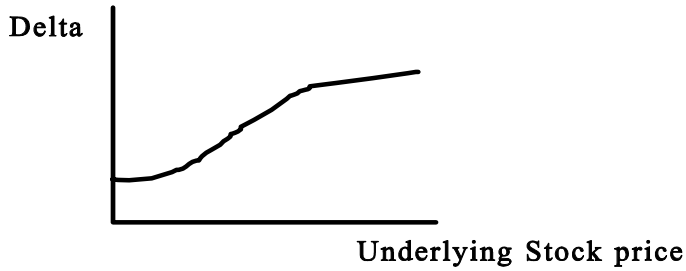


SS 17. 1. C. g) Rho and time to expiration and stock price. *(pp. 432-433) Rho is the first derivative of option price with respect to the interest rate. It is always positive for a call, and always negative for a put.*

a) Rho will vary with stock price, tending to be low for an out-of-the-money option and high for options that are near-the-money (far out-of-the-money option price are insensitive to changes in the price of the underlying asset).

b) Rho will also vary with time to expiration (interest rates have less effect when time to maturity is short).

SS 17. 1. C. h) Gamma and changes in delta and stock price. *Gamma is a measurement of how the delta of a stock changes as the stock price change - essentially, the slope of the relationship between delta and stock price. Examining the relationship between delta and stock price:*



we can see that the slope is low at when the option is deep out-of-the-money, increases as the stock price approaches the strike price, and then decreases again as the option becomes deep in-the-money.

Self test Question: Define the "delta" of an option, and **discuss** why it is difficult to apply.
Suggested Answer: The "delta" (δ) of an option is the first derivative of the value of the option with respect to the value of the underlying asset: $\delta = \partial c/\partial S$. Expressed in finite or approximate terms, this is: $\delta = \Delta c/\Delta S$ where Δ indicates a (small) change. Difficulties arise because delta, as a derivative, is only exact at one point. As stock price moves, delta will change, and for all but very small changes in price the portfolio must be rebalanced to maintain the delta-neutral property.

1 - D. Options on Stock Indexes, Foreign Currency, and Futures (Chapter 16, pp. 486-493)

1) Under the assumption of no dividends, the Black-Scholes option pricing model provide formulas for the price of a European call:

$$c_t = S_t N(d_1) - Xe^{-r(T-t)}N(d_2),$$

where:

$$d_1 = \frac{\ln(\frac{S_t}{X}) + (r + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \qquad d_2 = d_1 - \sigma\sqrt{T - t}$$

a) The variables or inputs necessary to for computation of a call or put option value are:
- The stock price S_t,
- The risk-free rate r,
- The exercise price X,
- The time to expiration, T - t,
- The standard deviation of the stock's returns, σ.

Merton adjusts this formulation for the case of continuous dividends at a rate δ:

$$c_t^M = e^{-\delta(T-t)} S_t N(d_1^M) - Xe^{-r(T-t)}N(d_2^M),$$

where:

$$d_1^M = \frac{\ln(\frac{S_t}{X}) + (r - \delta + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \qquad d_2^M = d_1^M - \sigma\sqrt{T - t}$$

When δ = 0, the Merton model reduces to the Black-Scholes model.

2. The adjustment is simply to replace S_t with $e^{-\delta(T-t)}S_t$ to reflect the (continuous) decrease in stock price over time due to the dividends. If a similar term can be determined for stock price indexes, foreign currency, and futures, the Black-Scholes model can be adjusted and applied.
   a) Application of the Merton model to options on price indexes is straightforward. The index is simply treated as a portfolio of stocks, and the model is applied to the portfolio as if it were an individual stock with continuous dividends.
   b) Application of the Merton model to options on foreign currency by substituting the dollar value of the foreign currency for the stock price, and the foreign interest rate for the continuous dividend δ.
   c) The Merton model may be applied to options on futures if the futures fit the relationship $F_t = S_t e^{r(T-t)}$, i.e., if the futures price and the spot price converge smoothly over time.
      - Futures will fit this condition if there is a large stock relative to consumption, the good is storable, and there is no seasonal consumption pattern.
      - In this case, the futures price F_t is substituted for the stock price, and the rate of convergence, r, is substituted for the continuous dividend rate δ.

$$N^* = \frac{C_U - C_D}{(U - D)S_t}, \qquad B_t^* = \frac{C_{UD} - C_{DU}}{(U - D)R}$$

SS 17. 1. D. a) Value of a European call (put) option using either the Black-Scholes model or the Binomial model with no continuous dividend paid by the underlying security. *Given the binomial model, we can use the equation:*

$$\textit{Price of Call} = N^* S_t - B_t^*,$$

where:
and :

c_U *= the value of the call at the up state*
c_D *= the value of the call at the down state*
U = the holding period return on the stock
R = the risk free rate
S is the stock price in the up state
D is the price of the stock in the down state.

Using the Black-Scholes option pricing model the price of the call is:

$$c_t = S_t N(d_1) - Xe^{-r(T-t)} N(d_2),$$

where:

$$d_1 = \frac{\ln(\frac{S_t}{X}) + (r + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \qquad d_2 = d_1 - \sigma\sqrt{T - t}$$

and:

- *The stock price =* S_t,
- *The risk-free rate = r,*
- *The exercise price = X,*
- *The time to expiration is T - t,*
- *The standard deviation of the stock's returns =* σ.

Both the binomial model and the Black-Scholes model give the price of a call. In order to find the price of a put, the models are used to find the price of the call, and the put-call parity condition is used to compute the put price.

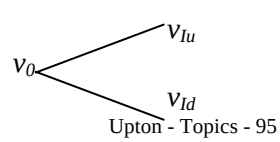
SS 17. 1. D. b) Value of a European call (put) using the Merton model with a continuous dividend paid by the underlying security. *When a continuous dividend is paid by the underlying security, the Merton model is given by:*

$$c_t^M = e^{-\delta(T-t)} S_t N(d_1^M) - Xe^{-r(T-t)} N(d_2^M),$$

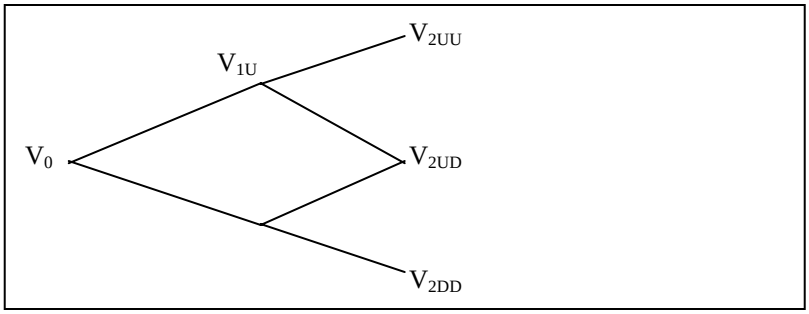
where:

$$d_1^M = \frac{\ln(\frac{S_t}{X}) + (r - \delta + 0.5\sigma^2)(T - t)}{\sigma\sqrt{T - t}}, \qquad d_2^M = d_1^M - \sigma\sqrt{T - t}$$

SS 17. 1. D. c) Single- and two-period binomial tree for both the underlying asset and the corresponding option, assuming no continuous dividends. *Note that this is discussed in Ch. 13, p.406, and the reading is not included in the page assignments for that chapter. The single period tree is:*



For the tree for the underlying asset, V_0 is the original value, V_{1U} the value after an increase in price, and V_{1D} the value after a decrease in price. For the option tree, V_0 is the value of the option at the original asset price, V_{1U} is the value of the option after the increase in the price of the asset, and V_{1D} is the value of the option after the increase in the price of the asset. For the case of continuous dividends, the V_1 values will reflect the decrease in price due to the cumulative continuous dividend over the first period. The two-period binomial tree is:



Again, for the underlying asset tree the Vs are the value of the asset after successive increases in price (Us) and decreases in price, while for the option the Vs are the option values. For continuous dividends, the values are adjusted accordingly.

SS 17. 1. D. d) Conditions necessary for the convergence of the Merton model and the Black-Scholes model. Simply put, the Merton model will be the same as the Black-Scholes model if the underlying stock does not pay any continuous dividend - i.e., if $\delta = 0$.

2. *Fixed income Analysis for the Chartered Financial Analyst Program: Level I and Level II Readings*, Frank J. Fabozzi (Frank J. Fabozzi Associates, 2000).

Interest Rate Derivative Instruments" (Ch. 7)

This reading covers the basic characteristics and uses of interest rate derivatives: forwards and futures, options, and swaps. The characteristics considered are the institutional trading arrangements and terms of the instruments.

1. Forwards and futures are highly similar instruments, the basic difference being that forwards are private contracts (over-the-counter or "OTC") with individualized provisions, while futures are exchange-traded instruments with highly specified provisions and trading arrangements. Features of futures which are different than forwards include:
   a) Easy liquidation through "reversing trades."
   b) Trading through a "Clearinghouse," removing counterparty risk.
   c) margin requirements and daily settlement ("mark to the market").
2. Futures contracts are traded on Treasury bonds. There are several "quirks" (oddities) in the delivery:
   a) The exact bond to be delivered is not specified, there are many issues to be delivered.
   b) Because of the defined conversion schedule among issues, however, one issue will be the "cheapest-to-deliver."
   c) The time of delivery can be any time in the delivery month, and the decision to deliver can be implemented after the close of the exchange.

3. Interest rate options are similar to other types of options. The underlying asset may be either a specific security or an interest rate futures contract. Options may be private contracts (otc) or exchange-traded, with exchange-traded options predominantly written on futures.
4. Interest rate swaps are an agreement to exchange interest payments (actually netted) on a notional principal. Both rates may be fixed or floating, but the usual arrangement is fixed for floating. Swaps are individual instruments, and counterparty risk can be significant. A swap has the same effect as a series of interest rate futures (with appropriate terms).
5. "Caps" and "floors" are instruments which have a payoff which is determined by the amount that a reference interest rate index is above (for caps) or below (for floors) some predetermined level. They have the same effect as options (of the appropriate terms).

SS 17. 2. a) Counterparty risk. *(pp. 574, 580). This is the risk that the counterparty will fail to perform as agreed. In the case of exchange-traded futures and options, this risk is virtually zero. In exchange trades, the "clearinghouse" steps between the buyer and seller, becoming a buyer to the seller and a seller to the buyer. The clearinghouse accepts the counterparty risk, for which is charges a small fee on every transaction.*

SS 17. 2. b) Basic features of an interest rate option contract. *(p. 586). These are over-the-counter (i.e., no traded on an exchange) instruments. The strike "price" is not actually a price, it is an interest rate. An interest rate call will pay if the reference rate is above the strike rate, a put will pay when the reference rate is below the strike rate.*

SS 17. 2. c) Trading mechanics and reasons for the popularity of futures options. *(pp.576-578). Futures options trade both on exchanges and over-the-counter. A futures call option buyer has the right to purchase a designated futures contract at the strike price, i.e., to acquire a long position in the future. Futures put option buyer has the right to sell a designated futures contract at the strike price – i.e., acquire a short position in the future.*

Assuming that an option is exercised, the futures price will be set at the strike price, and the position will be immediately marked-to-market, reflecting the difference between the strike price and the futures price at that moment.

There are three reasons for their popularity (as compared to options on bonds themselves):

a) options do not require compensation for accrued interest.
b) There is less chance of a delivery squeeze, because the supply of futures is more than adequate.
c) Bonds trade OTC, and there is not a single reporting system so that bond prices are harder to fix than exchange-traded futures prices.

SS 17. 2. d) Interest rate swaps and forward contracts. *(p578-589). An interest-rate swap is an agreement between two parties to exchange interest payments on some notional principal. The notional principal is for reference only and is not exchanged. In the most common swap, one party (the fixed-rate payer) pays interest at a fixed rate, while the other party (the fixed-rate receiver) pays interest at a floating rate referenced to some index. Typically, the payments are netted. In effect, this creates the same series of future payments that could be created by a series of forward futures contracts.*

SS 17. 2. e) Caps and floors and interest rate options. *(pp.584 – 586). A "cap" pays the buyer if interest rate rise above some set level, while a "floor" pays the buyer if interest rates fall below some set level. The cap can be used to set a maximum interest rate on a floating rate obligation, while the floor can be used to set a minimum rate of return on an floating rate investment. This same protection could be arranged using a series of interest rate options. For a call option on interest rates, there is a payoff if the reference rate is greater than the strike rate, so that the value of the option is positively related to interest rates. A cap, them is comparable to purchasing a series of interest rate call options. Conversely*

a put on fixed income securities pays when interest rates decrease, so that the value of an interest rate put is inversely related to interest rates. A floor, then, is comparable to purchasing a series of interest rate puts.

SS 17. 2. f) Payoff for a cap or a floor. *(p. 584). The payoff to the cap would be the greater of :*

a) the interest rate at the time minus the reference rate, times the notional principal,

b) zero (if the reference rate is above the interest rate at the time).

The payoff to a floor would be the greater of:

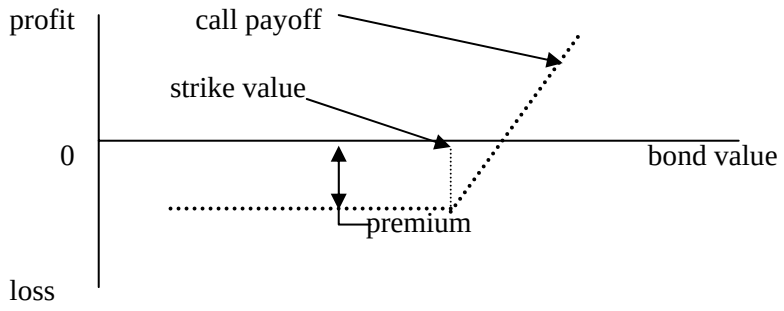
a) the reference rate minus the interest rate at the time, times the notional principal,

b) zero (if the interest rate at t time is above the reference rate).

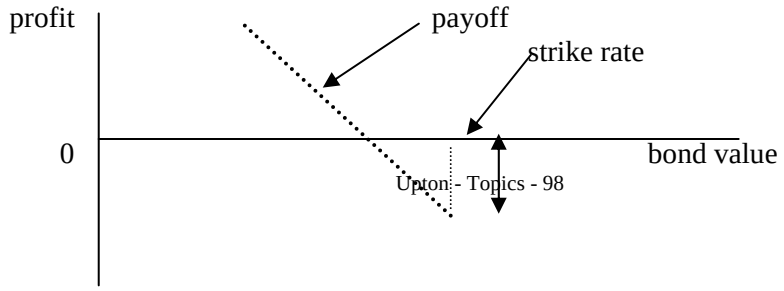
SS 17. 2. g) Options on interest rates vs. options on fixed income securities? *(p. 858). As explained above, interest rate calls are positively related to interest rates, while interest rate puts are inversely related to interest rates. Calls on fixed-income securities, however, pay off in reference to the price of the security, an inverse relationship. Puts on fixed-income securities, by similar reasoning, have a positive relationship. The result is that payoffs to positions in interest rate options are inverse to the payoffs to options on fixed income securities.*

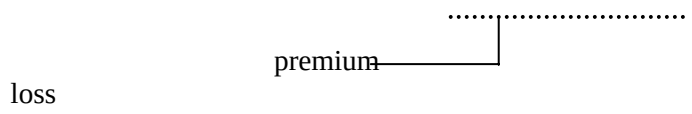
SS 17. 2. h) Caps and floors and their use with other positions. *(p. 585). Cap and floor positions, by themselves, can be interpreted as a series of "caplets" and "floorlets," In conjunction with an existing position, they would be used to set up either a minimum rate of return (floored position), a maximum liability (capped position), or in conjunction to set up a collar. Note that the collar may be an incidental result – i.e., a cap is often sold to reduce the cost of a floor, while a floor is often sold to reduce the cost of a cap.*

SelfTest Question: Discuss and diagram the payoff of a long call on a bond as a function of interest rates.
Suggested Answer: The payoff of the option as a function of the value of the bonds is as shown:



Since the value of the bond is inversely related to interest rates, however, the payoff as a function of interest rates is diagrammed as:





3. "What does an Option pricing Model Tell Us about Option Prices," Stephen Figlewski, *Financial Analysts Journal* (AIMR, September/October 1989).

1. The Black-Scholes model, and others like it, are theories that try to derive the value of an option so that it is consistent with the price of the underlying stock.
   a) The models assume a market in which a dynamic riskless arbitrage strategy with the stock and the option is possible. In this ideal market:
      • Continuous rebalancing is possible and keeps the hedge riskless.
      • The hedge will earn the riskless rate plus the amount of mispricing.
      • Unconstrained hedging will force the option price to the model values.
2. In the real world, the assumptions of the models do not hold:
   a) The arbitrage is neither riskless nor costless and not unlimited:
      • Risk exists because the markets are not open continuously.
      • There are transactions costs for less-than-continuous rebalancing.
      • The theoretical value itself is uncertain because the volatility of the underlying asset is uncertain.
3. In addition, option supply and demand are influenced by variables omitted from the models, including:
   a) taxes
   b) transaction costs
   c) margin costs
   d) delivery features
   e) constraints on purchases and short sales
   f) interaction between options and futures
4. This indicates that the prices of the option pricing models are uncertain estimates.
   a) Arbitrage is risky, and will take place only when the estimated mispricing is large enough to compensate for risk.
   b) The result is "arbitrage bounds," within which the perceived mispricing is not large enough to cause arbitrage.
   c) Within these bounds no arbitrage occurs, and the price can float freely.
   d) Simulation indicates that the arbitrage bounds can be "disturbingly wide."

5. There are two possible situations:
   a) The model is wrong or incorrect, giving an incorrect true value.
      • This is the case when the model assumptions are violated, as in long-maturity options.
   b) The model gives the true value, but the market prices options differently. This will most easily occur in situations in which arbitrage is more difficult:
      • Out-of-the-money options offer a large potential payoff with limited risk - a pattern that has historically resulted in a premium.
      • American options may be exercised early, although early exercise is contrary to theory.

- Assets with embedded options are valued in theory as a linear sum of the option and "straight" values, but this seems to conflict with market pricing.
- In times of crisis, arbitrage may be difficult or impossible.

SS 17. 3. a) Difficulties of using dynamic riskless arbitrage in real markets. *Note that relevant material also appears on pp. 428-430 in Kolb. The models assume a market in which a dynamic riskless arbitrage strategy with the stock and the option is possible. In this ideal market:*

a) Continuous rebalancing is possible and keeps the hedge riskless.
b) The hedge will earn the riskless rate plus the amount of mispricing.
c) Unconstrained hedging will force the option price to the model values.

In the real world, the assumptions of the models do not hold. The arbitrage is neither riskless nor costless and not unlimited:

a) Risk exists because the markets are not open continuously.
b) There are transactions costs.
c) The theoretical value itself is uncertain because the volatility of the underlying asset is uncertain.

SS 17. 3. b) Factors that may affect option prices but are not included in the Black-Scholes model. *Option supply and demand are influenced by variables omitted from the models, including:*

a) taxes
b) transaction costs
c) margin costs
d) delivery features
e) constraints on purchases and short sales
f) interaction between options and futures

SS 17. 3. c) Sources of error in the Black-Scholes model. *A source of error in the models is that the prices of the option pricing models are uncertain estimates.*

a) Arbitrage is risky, and will take place only when the estimated mispricing is large enough to compensate for risk.
b) The result is "arbitrage bounds," within which the perceived mispricing is not large enough to cause arbitrage.
c) Within these bounds no arbitrage occurs, and the price can float freely.

SS 17. 3. d) Why the market value of an option differ from the Black-Scholes model. *The model may give the true value, but the market may price options differently. This will most easily occur in situations in which arbitrage is more difficult, such as:*

a) Out-of-the-money options offer a large potential payoff with limited risk - a pattern that has historically resulted in a premium.
b) American options may be exercised early, although early exercise is contrary to theory.
c) Assets with embedded options are valued in theory as a linear sum of the option and "straight" values, but this seems to conflict with market pricing.
d) In times of crisis, arbitrage may be difficult or impossible.