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BASIC CIRCUIT ANALYSIS for Electrical Engineering

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Juta & Co, Ltd

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Notes to the student

This book covers basic network analysis at an intermediate level. It is designed as a one-semester course for technikon students studying Electrical Engineering.

The language in the text is simple, conversational English. Difficult concepts and technical terminology are explained throughout the book.

The book consists of 8 units. Each unit starts with a list of outcomes or study objectives. These outcomes set out what you should be able to do at the end of each unit.

The text is set out in such a way that you should be able to work through the book by yourselves. New concepts are explained and reinforced by providing examples with solutions to work through. Try to work through these on your own first, before looking at the solution for guidance. The figures throughout the text help you to understand the material and clarify concepts. The mathematics in this course is thereby made clear and understandable.

Because this is a problem-solving course, there are also many activities for you to work through. These activities allow you to make sure that you have understood the work you have just covered.

The summary at the end of each unit enables you to see at a glance what you should have learnt in the unit. The summary is followed by a section with self-evaluation exercises for assessing your progress.

Answers to the activities and self-evaluation exercises appear at the end of each unit.

We have used four icons in this book:

Activity 2

This is an ACTIVITY icon. When you see this icon you will know that it is time to DO something! The activities are active and enjoyable, and they help you to understand the subject. Feel free to do them with a friend or group of friends. The solutions to the activities are given at the end of each unit.

This is a DEFINITION icon. Read the definitions carefully because the details are important.

The TAKE NOTE icon appears alongside all the extremely important information.

Self-evaluation

This is a self-evaluation icon. The self-evaluation exercises at the end of each unit enable you to assess your own progress.

Symbols used in this book:

Symbol	Physical parameter	Unit
V, v	voltage	V (Volt)
I, i	current	A (Ampere)
R	resistance	Ω (Ohm)
L	inductance	H (Henry)
C	capacitance	F (Farad)
Z	impedance	Ω (Ohm)
Y	admittance	S (Siemens)
t	time	s (second)
ω	angular velocity	rad/s $\left(\frac{\text{radian}}{\text{second}}\right)$
T	period	s (second)
f	frequency	Hz (Hertz)
α, β, ϕ, θ ...	angles	°, rad (degree or radian)
P	active power	W (watt)
Q	reactive power	VAr (volt-ampere reactive)
S	total (apparent) power	VA (volt-ampere)
PF $(=cos_{angle})$	power factor	—
W, w	energy	J, Wh (joule or watt-hour)
S	resistivity	Ωm (ohm meter)
ε	permittivity	$\frac{\text{F}}{\text{m}}$ $\left(\frac{\text{Farad}}{\text{meter}}\right)$
μ	permeability	$\frac{\text{H}}{\text{m}}$ $\left(\frac{\text{Henry}}{\text{meter}}\right)$
	complex number	—
$a + jl$ $r \angle\theta°$	in: rectangular form polar form	—

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unit 1

Basic concepts

OUTCOMES

After studying this unit, you should be able to:

- review the concepts of electric current and voltage
- apply Ohm's law
- explain Kirchhoff's current and voltage laws and use these to determine currents and/or voltages in simple resistive circuits
- determine when resistors are connected in series or in parallel, and calculate the equivalent resistance for any combination of these
- explain what voltage and current dividers are and how to use their rules
- calculate electrical power and energy for simple resistive circuits
- explain the meaning of 'short circuit' and 'open circuit' in circuit analysis
- practically analyse any common resistive DC circuit, and solve for its parameters using any of the above methods (in terms of current, voltage, power and energy)
- review your mathematical knowledge in the fields of trigonometry, complex numbers, calculus and matrices

1.1 Introduction

In today's world, most of us live with electricity around us. Electricity can be a very good friend if it is used properly, but a deadly enemy if it is misused. Because we cannot see, hear or smell electrical current, it is important that we have an understanding of basic electrical principles.

All basic electrical systems have the following main components:

- a source;
- a transmission system;
- a control apparatus; and
- a load.

An ideal electric circuit is a mathematical model of a physical circuit that allows us to transmit energy from a supply or **source** to a point of application or **load**.

The parameters of such an ideal circuit that we need to determine are the **voltage** and the **current** associated with any element of the circuit.

The ideal circuits we study allow us to theoretically predict the behaviour of any electric circuit and its components, without having to use real (physical) elements.

Electrical engineers and technicians should be able to analyse or design models for any circuit and then, based on their observations, construct the physical system that performs the desired task. When there is a fault in an existing circuit, they should be able to make observations and take measurements that enable them to pinpoint the fault and repair the circuit. These are some of the aspects we will consider in this subject.

In this course we use the international unit system known as the **SI unit system**. This system has the advantage of being consistent and, as you already know, is based on **seven basic quantities** (length, mass, time, electric current, temperature, luminous intensity and amount of substance) and their base units (**metre, kilogram, second, ampere, degree Kelvin or Celsius, candela and mole**). I hope you can remember some of the units commonly used in electrical engineering, like ampere for current, coulomb for charge, volt for voltage, and ohm for resistance.

In most technical fields, measurements range from very small to very large. So we need to refresh our memories about the SI unit prefixes, like 'tera-', 'giga-', 'mega- etc. They are very convenient and often used instead of writing long numbers.

The symbols and the Greek letters used in this book are the ones that are commonly used in our field and appear in all electrical engineering textbooks.

To study Electrical Engineering, you will need a basic mathematical background. You may remember studying some special functions called **trigonometric functions** in high school. For easy reference, there are basic trigonometric identities in Appendix I at the end of this unit.

When you do calculations, remember that all answers should have **the correct number of significant digits** and must adhere to the rules for rounding off (you probably have studied this in Maths). Where appropriate, try to use **engineering notation**. Find out how your calculator works in 'engineering mode' if you do not already know this.

In the first part of this unit, we will revise some of the basic concepts you studied in the previous level. After that, we will review some mathematical concepts, but since this is engineering and not a mathematics course, we will not go into great detail.

All of these concepts form the basic theoretical knowledge you will need to study circuit components and their interaction. Let's get started!

1.2 Revision of basic concepts

Electrical engineering involves the production, transmission, application and control of electrical energy.

In this book we will focus on direct current (DC) and alternating current (AC) electrical circuits, which are mathematical models of physical circuits. The ideal electrical circuit includes current, voltage, resistance, inductance and capacitance (I, V, R, L and C).

The **materials used in electrical circuits** are typically **conductors**, **insulators**, and **semiconductors**.

1.2.1 Current and voltage

DEFINITION

Current is the rate of movement of charge in an electrical circuit:

$$i(t) = \frac{dQ}{dt}$$

Current can be described as the movement of free electrons in a circuit. For current to flow continuously, we must have a **complete circuit** that includes an electromotive force (emf) provided by the voltage source.

When any electric current flows, it produces forces known as 'magnetic' forces.

DEFINITION

Voltage is the 'potential' difference between two points and characterises the ability of an electrical force field to do work.

$$V_{ba} = \frac{W_{ba}}{q}$$

TAKE NOTE

In the following equation, the + sign indicates the assumed high potential point and the − sign indicates the low potential point:

$$V_{ba} = -V_{ab}$$
$$V_{ab} = V_b - V_a$$
$$V_{ba} = V_b - V_a$$

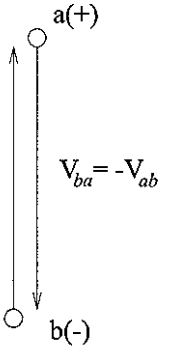


Fig. 1.0

1.2.2 Electrical resistance and Ohm's law

Resistance is a measure of the extent to which a material opposes the flow of current through it.

Conductors have a **small resistance**. **Insulators** have a **large resistance**.

For a conductor $R = \rho \frac{l}{A}$ and $\frac{1}{R} = G$ (conductance), where ρ = resistivity of material, l = length, A = cross-sectional area

When we want to **replace a resistor**, or if we want to build a circuit and need a resistor, it is very important to **know the resistor's power rating and the effect of temperature on the value of the resistor**.

For metals, resistance increases with the increase in temperature. We say that metals have a positive temperature coefficient ($\alpha_0 > 0$):

$$R_1 = R_0(1 + \alpha_0\theta_1)$$

where R_0 = resistance at temperature T_0
R_1 = resistance at temperature T_1
$\theta_1 = T_1 - T_0$, difference in temperature
α_0 = temperature coefficient

In this unit, we will refer to linear resistors only. For these, Ohm's law states that $v = Ri$, and is a linear relationship. (Do you remember that the graph $v = f(i)$ is a straight line that passes through the origin and has a slope equal to R?)

The current flowing through resistors found in resistive networks is always **directly proportional to the voltage** applied to the resistor. The resistor is then called a **linear resistor**.

Alternative mathematical forms for Ohm's law are:

$$I = \frac{V}{R} \text{ or } R = \frac{V}{I}$$

Activity 1.1

The field coil of a shunt motor has a resistance of 10.5 Ω at 0°C. Determine the current that will flow through the coil if the supply voltage is constant at 220 V and the temperature of the coil increases to 60°C. The resistance temperature coefficient of copper at 0°C is 0.00426/°C

1.2.3 Kirchhoff's Laws

Kirchoff's **current law** (KCL) states that the sum of the currents entering a junction (node) is equal to the sum of the currents leaving the junction:

$\sum_{1}^{n} i = 0$ (for the junction)

$\sum$ currents entering the node $= \sum$ currents coming out of the node

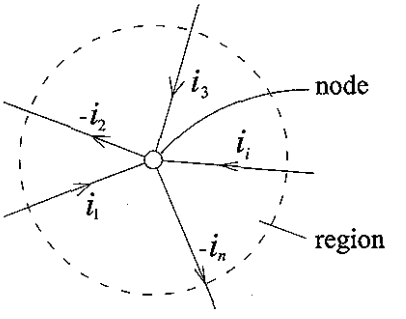


Fig. 1.1

Just as KCL applies to any node of a circuit (in other words, to satisfy the physical law of conservation of charge, the current going in must equal the current coming out), so KCL must hold for any closed region (see fig. 1.1).

DEFINITION

Kirchoff's **voltage law** (KVL) states that the sum of the voltage drops around any closed loop equals the sum of the voltage rises around the loop.

$\sum_{1}^{n} V = 0$ (for the closed loop)

$\sum$ voltage rises $= \sum$ voltage drops

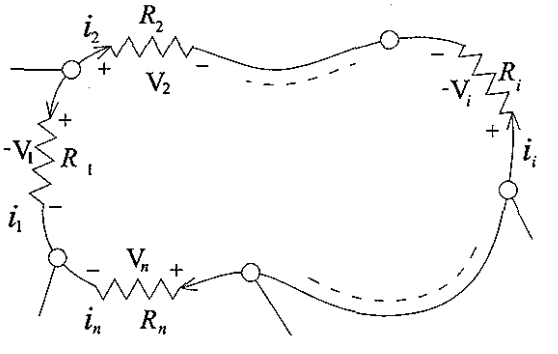


Fig. 1.2

KVL is, in fact, an expression of conservation of energy.

1.2.4 Series and parallel combinations of resistors

The **series connection** of resistors means that the **same current** flows through each resistor, since the output node of one is the input node for the next one, and so on.

TAKE NOTE

We can have two, three, four or more resistors in series. To write a general formula, we will call the last resistor the 'n'th resistor, where $n \geq 2$.

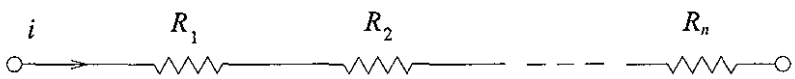


Fig. 1.3

The equivalent series resistance:

$$R_{eq(series)} = \sum_{1}^{n} R_i = R_1 + R_2 + \ldots + R_n$$

The **parallel connection** of resistors, as in fig. 1.4, means that the resistors have the **same voltages (potential difference) across them.**

$$\frac{1}{R_{eq||}} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n}$$

or,

$$G_{eq} = G_1 + G_2 + \ldots + G_n$$

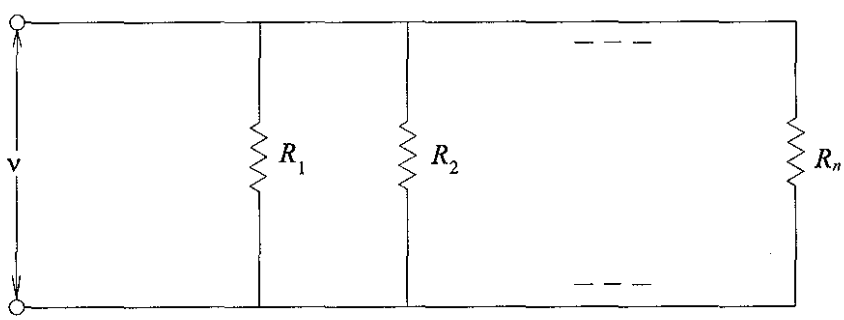


Fig. 1.4

For two parallel resistors, the equivalent resistance $R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$

For 'n' resistors with equal value, in parallel: $R_{eq} = \frac{R}{n}$

The 'n'th resistor is the number of resistors in a combination. For example, for three resistors $n = 3$, for 21 resistors $n = 21$.

Activity 1.2

1. Determine the combined resistance of four resistors in series of 3 Ω, 6 Ω, 7 Ω and 12 Ω respectively.
2. Determine the combined resistance of four resistors in parallel of 3 Ω, 6 Ω, 7 Ω and 12 Ω respectively.

1.2.5 Voltage and current dividers

A **voltage divider** is a circuit with 'n' **resistors in series**. The rule for the voltage divider is:

$$V_i = \frac{R_i V}{\sum_1^n R_i}$$

And for two resistors in series we have:

$$V_1 = \frac{R_1}{R_1 + R_2} V_s$$

$$V_2 = \frac{R_2}{R_1 + R_2} V_s$$

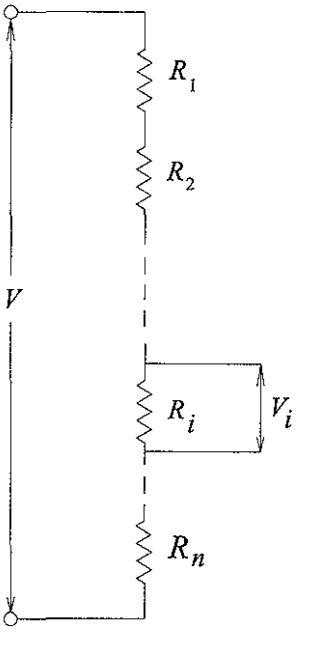


Fig. 1.5(a)

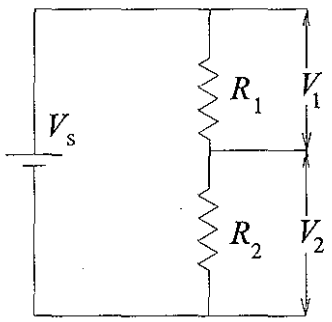


Fig. 1.5(b)

A **current divider** is a circuit with 'n' **resistors in parallel**. The rule for the current divider is:

$$I_i = \frac{V}{R_i} = \frac{G_i I}{\sum_1^n G_i}$$

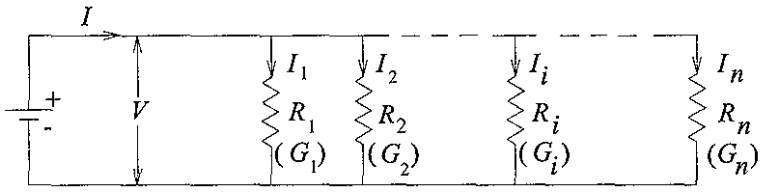


Fig. 1.6(a)

And for two resistors in parallel we have:

$$I_1 = \frac{G_1}{G_1 + G_2} I = \frac{R_2}{R_1 + R_2} I$$

and

$$I_2 = \frac{G_2}{G_1 + G_2} I = \frac{R_1}{R_1 + R_2} I$$

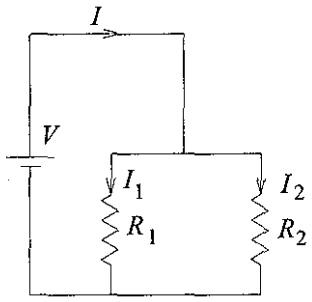


Fig. 1.6(b)

1.2.6 Short and open circuits

Short circuits are paths in which current flows through **zero resistance**. The **voltage** drop across the short circuit is always **zero** and in some cases this can lead to very high currents. These currents are known as **short-circuit currents**.

Open circuits are paths of **infinite resistance**. Since the resistance is infinite, the current practically is always zero and the voltage across it is known as open-circuit voltage.

1.2.7 Power and energy

Can you remember the concepts below? We are going to use them in our studies.

DEFINITION

Energy = ability to do work
Power = the rate of doing work

The relationship between electrical power (P) and electrical energy (W) is:

$$P = \frac{W}{t}$$

When there is an electrical resistance in the circuit:

$$P = VI = I^2R = \frac{V^2}{R}$$

EXAMPLE 1.1

Look at the circuit in fig. 1.7 and:
(a) calculate the current I in the resistance;
(b) find the current when the voltage is doubled; and
(c) calculate the current when the voltage is 24 V and R is doubled.

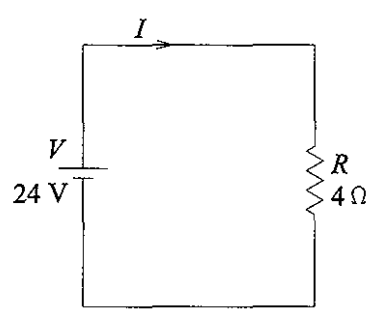


Fig. 1.7

Solution

(a) $I = \frac{V}{R} = \frac{24}{4} = 6$ A

(b) $I = \frac{2V}{R} = \frac{48}{4} = 12$ A

Note: Double voltage ⇒ double current

(c) $I = \frac{V}{2R} = \frac{24}{8} = 3$ A

Note: Double resistance ⇒ half of the current

EXAMPLE 1.2

Study the network in Figure 1.8(a) and calculate the current in each resistor.

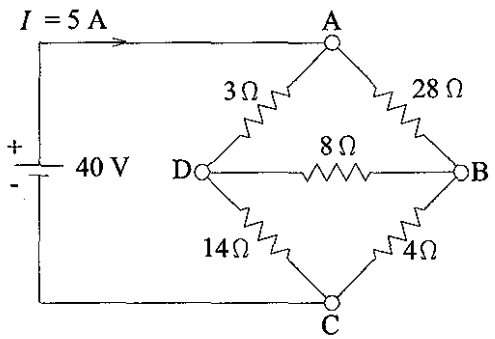


Fig. 1.8(a)

Solution

Did you notice that there are five unknowns? Let's apply KCL at the nodes to reduce the number of unknowns.

At first we must **assume the direction** of the **currents**.

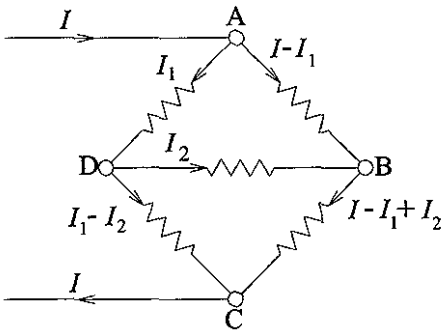


Fig. 1.8(b)

We can add in the direction we think the current will flow.

In node A we assume I_1 is the current flowing between A and D.

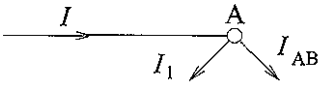


Fig. 1.8(c)

The current flowing between A and B will be $I - I_1$. So, we add this to fig. 1.8(b).

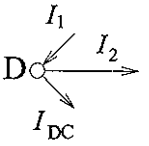


Fig. 1.8(d)

In node D we assume I_2 flows between D and B, so the current between D and C is $I_1 - I_2$. We add this to fig. 1.8(b).

In node B the current between B and C is $I_2 + I - I_1$. We add this to fig. 1.8(b).

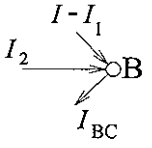


Fig. 1.8(e)

Let's check node C. We know the current that comes out must be I (equal to the current going into this section of the network).

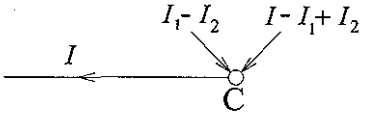


Fig. 1.8(f)

$$I_1 - I_2 + I + I_2 - I_1 = I$$

So, we now know that we have worked correctly up to here.

We still have two unknowns to solve, namely I_1 and I_2.

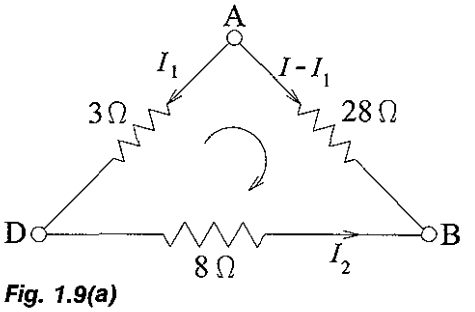


Fig. 1.9(a)

We will need to use KVL. Let's look at loop ABD.

$$28\,(I - I_1) - 8\,I_2 - 3\,I_1 = 0$$
$$28 \times 5 - 28\,I_1 - 8\,I_2 - 3\,I_1 = 0$$
$$140 = 31\,I_1 + 8\,I_2$$

Now let's consider the loop source ADC.

$$40 = 3\,I_1 + 14\,(I_1 - I_2)$$
$$40 = 17\,I_1 - 14\,I_2$$

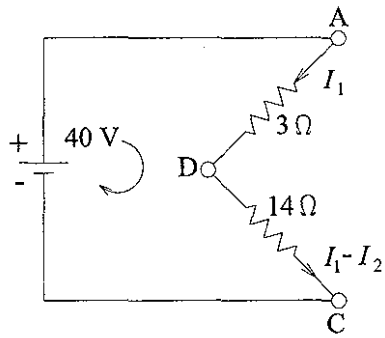


Fig. 1.9(b)

Now we have two equations for the two remaining unknowns:

$140 = 31\ I_1 + 8\ I_2$; multiply the equation by 7
$40 = 17\ I_1 - 14\ I_2$; multiply the equation by 4

Now we can solve the system:

$$980 = 217\ I_1 + 56\ I_2$$
$$160 = 68\ I_1 - 56\ I_2$$

$$(217 + 68)\ I_1 = 980 + 160 = 1\ 140$$

$$I_1 = 1\ 140/285 = 4 \text{ A and } I_2 = 2 \text{ A}$$

So the flow of current in the circuit will be:

$I_{AD} = 4$ A
$I_{DB} = 2$ A
$I_{AB} = 1$ A
$I_{BC} = 3$ A
$I_{DC} = 2$ A

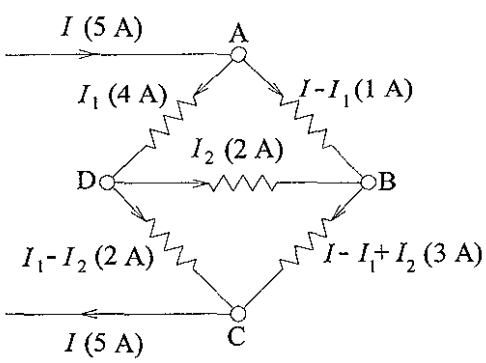


Fig. 1.10

If we check the BCD loop we get:

$$2 \times 14 = 8 \times 2 + 4 \times 3$$
$$28 = 28$$

Finally, we know the magnitude of all the currents and their directions. This is shown in fig. 1.10.

Analysing systems like these requires a lot of care. It is very easy to make mistakes in your calculations.

In the example, we had to apply KCL and KVL simultaneously to solve the unknowns. We needed to obtain five independent equations to find the five unknowns.

TAKE NOTE

If you come across a similar problem, try to follow these rules:

- First determine the number of unknowns.
- Then try to use Kirchhoff's first law (current law) as many times as possible. (KCL equations are simpler than KVL equations.) Apply KCL to the number of the nodes minus one. (In the example we applied KCL three times and checked the last node.)
- Choose a number of independent loops for KVL equal to the number of unknowns, minus the number of times you applied KCL.

Note that the fourth equation for current came as an identity ($I = I$) so it could not be used. However, it showed us that we worked correctly.

EXAMPLE 1.3

In example 1.2 we could have considered the loop DBC instead of ABD, as in fig. 1.11.

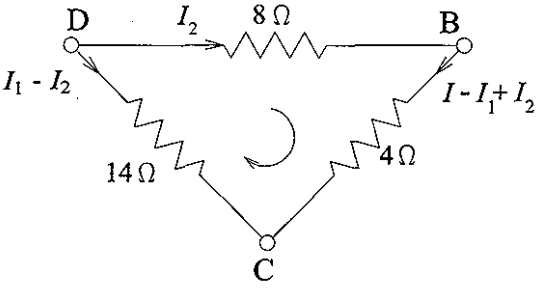


Fig. 1.11

Solution

$$8I_2 + 4(I - I_1 + I_2) = 14(I_1 - I_2) \Rightarrow 20 = 18I_1 - 26I_2$$

If we simultaneously solve this equation and the one for the loop source ADC, this gives us the same solution for the system $I_1 = 4$ A and $I_2 = 2$ A.

EXAMPLE 1.4

Now find the power dissipated by the resistor in fig. 1.12 if a 15 A current flows through the resistor. The voltage across the resistor is 30 V.

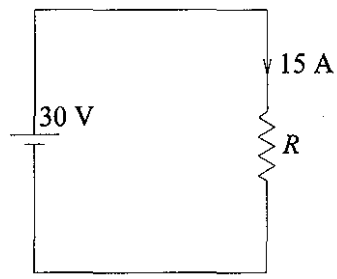


Fig. 1.12

Solution

$$P = V\,I = 30 \times 15 = 450 \text{ W}$$

or

$$R = \frac{V}{I} = \frac{30}{15} = 2\ \Omega$$

$$P = I^2\,R = 15^2 \times 2 = 225 \times 2 = 450 \text{ W}$$

or

$$P = \frac{V^2}{R} = \frac{30^2}{2} = \frac{900}{2} = 450 \text{ W}$$

TAKE NOTE

Did you notice that you only need to know two parameters of the circuit (voltage and current, or voltage and resistance, or current and resistance) to work out the power in a resistive circuit? And, if you know the value of **any two parameters** (R, I, V, P) of a circuit, you can work out the other two.

Also remember that, when you calculate the power dissipated by a resistor, you must use the voltage drop **across** that resistor and/or the current flowing **through** that resistor.

If you have several resistors in series or parallel in a circuit, you can work out the power of the circuit based on the equivalent resistances and the rules mentioned earlier in this unit.

1.3 Useful mathematical concepts

1.3.1 Complex numbers

DEFINITION

A **complex** number is a number written as $z = a + jb$, where a and b are real numbers and j, called the imaginary unit, has the property that $j^2 = -1$. The real numbers a and b are called, respectively, the real and imaginary parts of $a + jb$.

TAKE NOTE

Some books use i instead of j, but to avoid confusing it with the symbol for instantaneous value of current, we will only use j.

▶ **Properties of complex numbers in rectangular form**

The following five rules are **very important** and should always be used when working with complex numbers:

1. Equality of complex numbers
   $A + jB = C + jD$ if and only if $A = C$ and $B = D$
2. Addition of complex numbers
   $$(A + jB) + (C + jD) = (A + C) + j(B + D)$$
3. Subtraction of complex numbers
   $$(A + jB) - (C + jD) = (A - C) + j(B - D)$$

TAKE NOTE

Addition and subtraction of complex numbers are easier to do using the above rectangular form.

4. Multiplication of complex numbers
   $$(A + jB)(C + jD) = (AC - BD) + j(AD + BC)$$

TAKE NOTE

When two complex conjugate numbers $A + jB$ and $A - jB$ are multiplied we get:
$$(A + jB)(A - jB) = A^2 + B^2$$

5. Division of complex numbers
   $$\frac{A + jB}{C + jD} = \frac{A + jB}{C + jD} \times \frac{C - jD}{C - jD} = \frac{AC + BD}{C^2 + D^2} + j\frac{BC - AD}{C^2 + D^2}$$

You have just revised the properties of complex conjugates!

However, we suggest you use the **polar form** for multiplying and dividing complex numbers, as we will explain below.

▶ Polar form of a complex number

Look at fig. 1.13. Point P represents the complex number $x + jy$ with the coordinates (x, y).

Point P can also be represented by polar coordinates (r, θ).

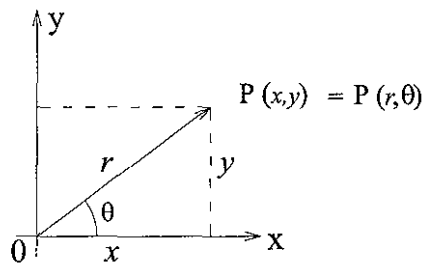


Fig. 1.13

Now we're going to use trigonometry functions. Have you revised them?

Since $x = r\cos\theta$ and $y = r\sin\theta$ we can write $x + jy = r(\cos\theta + j\sin\theta)$, which is the **polar form** of the complex number.

We often call 'r' the modulus and 'θ' the phase of a complex number.

So, we get $r = \sqrt{x^2 + y^2}$

The abbreviated notation for this is $r\underline{/\theta^\circ}$

Multiplication in polar form

$$[r_1(\cos\theta_1 + j\sin\theta_1)] \times [r_2(\cos\theta_2 + j\sin\theta_2)]$$
$$= r_1r_2[\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)]$$

or, in the abbreviated form: $r_1r_2\,\underline{/(\theta_1 + \theta_2)}$

Division in polar form

$$\frac{r_1(\cos\theta_1 + j\sin\theta_1)}{r_2(\cos\theta_2 + j\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + j\sin(\theta - \theta_2)]$$

or, abbreviated, $\frac{r_1}{r_2}\underline{/(\theta_1 - \theta_2)}$

EXAMPLE 1.5

Calculate $Z_{eq} = Z_1 + Z_2$ if $Z_1 = 2 + j5$ and $Z_2 = 7 + j9$.

Solution

$$Z_{eq} = (2 + 7) + j\,(5 + 9) = 9 + j14$$

Divide $V = 4.5\underline{/0°}$ by $Z = 1.5\underline{/30°}$.

Solution

$$\frac{4.5}{1.5}\,\underline{/(0 - 30)°} = 3\,\underline{/-30°}.$$

TAKE NOTE

You can check your result by recalculating using rectangular coordinates. Try it! This will be good practice for you.

Activity 1.3

Find the real and imaginary part of the complex number $z = \dfrac{2 + 3j}{4 + 5j}$

1.3.2 Calculus in Electrical Engineering

▶ **Derivatives**

DEFINITION

If $y = f(x)$, the derivative of $f(x)$ with respect to x is defined as:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Some other possible notations for derivatives are $\frac{df}{dx}$, $f'(x)$ or simply y'.

The **derivative** gives **the slope of the curve** associated with the initial function.

If we can calculate the **value of the derivative** at a specified point, we can see if the function is **increasing or decreasing** at that point. In other words, we can see the trend of the function.

To calculate the derivatives of any given function we need to know the rules for differentiation. These can be found in any calculus book.

EXAMPLE 1.6

Differentiate the following exponential function: $v = e^{5t}$

Solution

Let $u = 5t \Rightarrow \frac{d}{dt}(e^{5t}) = e^{5t}\frac{du}{dt} = e^{5t} \times 5 = 5e^{5t}$

Differentiate the following sinusoidal function: $i = \sin 5t$

Solution

$$\frac{d}{dt}\sin 5t = 5\cos 5t$$

Calculate $V = L\frac{di}{dt}$ *where* $L =$ *constant and* $i = I_m \cos\omega t$
(I_m *and* ω *are also constants)*

Solution

$$V = L\frac{di}{dt} = L\frac{d}{dt}(I_m \cos\omega t) = LI_m\frac{d}{dt}(\cos\omega t) = -LI_m\omega \sin\omega t =$$
$$-V_m \sin\omega t = V_m \cos(\omega t + \frac{\pi}{2}) \text{ where } V_m = LI_m\omega = \text{constant}$$

(Remember trigonometric identities!)

▶ Integrals

DEFINITION

If $\frac{dy}{dx} = f(x)$, then y is the function with derivative $f(x)$ and is called the anti-derivative of $f(x)$ or the indefinite integral of $f(x)$. The symbol used for the integral is $\int f(x)dx$. Since the derivative of a constant is zero, all indefinite integrals differ by an arbitrary constant, C.

To integrate, we need to know the rules for integration. Make sure you know and can apply these rules.

If we have an interval $[a, b]$ and $f(x)$ is piecewise continuous on this interval, the definite integral of $f(x)$ between $x = a$ and $x = b$ is defined as:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\{f(a)\Delta x + f(a + \Delta x)\Delta x + \ldots f[a + (n - 1)\Delta x]\Delta x\},$$

and the limit exists.

If $f(x) = \frac{d}{dx}g(x)$, the definite integral can be evaluated as:

$$\int_a^b f(x)dx = \int_a^b \frac{d}{dx}g(x)dx = g(x)\Big|_a^b = g(b) - g(a)$$

(a fundamental calculus theorem)

Another fundamental theorem is the **mean value theorem** that states:

$$\int_a^b f(x)dx = (b-a)f(m), \text{ where } m \in [a, b]$$

Application of integrals:

- The definite integral is the area under the curve f(x) between the limits a and b.
- The average value of a function on a given interval is defined:

$$f_{av(erage)} = \frac{1}{b-a}\int_a^b f(x)dx$$

TAKE NOTE This formula shows that the area under the curve is equal to the area of a rectangle with dimensions $(b-a) \times f_{av}$. The average of the function can be determined from this.

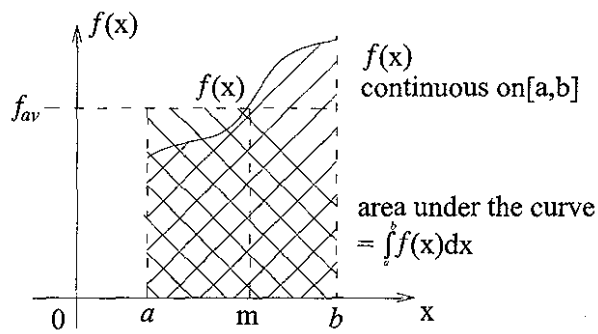


Fig. 1.14

EXAMPLE 1.7

Assume a typical cosine wave $i = i(t)$ where $i = I_m \cos\omega t$, with I_m = constant = maximum value for the function i, and ω is also a constant.

The square of i is then $i^2 = I_m^2 (\cos\omega t)^2$

Find the average of i^2 for a full period.

Solution

$$\omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega}$$

Then,

$$i_{av}^2 = \frac{1}{\frac{2\pi}{\omega}}\int_0^{2\pi/\omega} i^2\,dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} I_m^2(\cos\omega t)^2\,dt = \frac{I_m^2\omega}{2\pi}\int_0^{2\pi/\omega}(\cos^2\omega t)dt$$

$$= \frac{I_m^2\omega}{2\pi}\left[\frac{1}{2}t + \frac{1}{4\omega}\sin 2\omega t\right]_0^{2\pi/\omega} =$$

$$= \frac{I_m^2\omega}{2\pi}\left[\frac{1}{2}\times\frac{2\pi}{\omega} - \frac{1}{2}\times 0 + \frac{1}{4\omega}\times\sin\left(2\omega\times\frac{2\pi}{\omega}\right) - \frac{1}{4\omega}\times\sin(2\omega\times 0)\right] =$$

$$= \frac{I_m^2\omega}{2\pi}\left[\frac{\pi}{\omega} - 0 + \frac{1}{4\omega}\times 0 - \frac{1}{4\omega}\times 0\right] = \frac{I_m^2\omega}{2\pi}\times\frac{\pi}{\omega} = \frac{I_m^2}{2}$$

If we use the following notation: $\sqrt{i_{av}^2} = I_{rms\ (root\ mean\ square)} \Rightarrow$

$$i_{av}^2 = I_{rms}^2 = \frac{I_m^2}{2} \Rightarrow I_{rms} = \frac{I_m}{\sqrt{2}} = \frac{\sqrt{2}}{2}I_m$$

TAKE NOTE

Remember this result, because you'll see it again!

1.3.3 Matrices in Electrical Engineering

In Mathematics we learned to solve systems of equations with the help of their associated matrices. Let's see if we can use matrix calculations to find the unknown elements of our electrical circuits.

Do you remember the examples of solving a system of equations in the previous semester? You had to solve different given circuits by applying Kirchhoff's laws.

Let's solve the problem in Example 1.8.

EXAMPLE 1.8

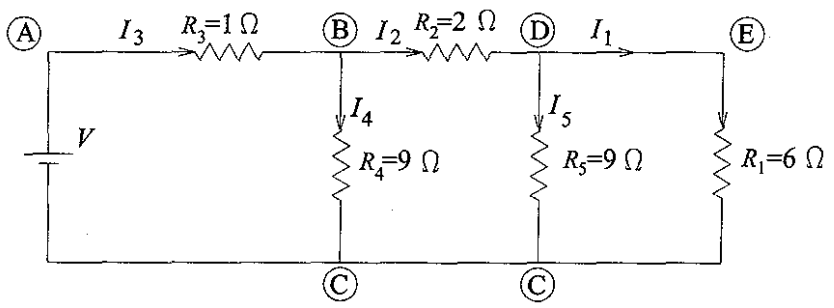


Fig. 1.15

Find the currents flowing in each branch of the circuit. The value of the supply voltage is 100 V, and the resistors have the values indicated in fig. 1.15.

Solution

If we use the equations for the nodes first, we notice the following:

The current through R_4 is equal to $I_4 = I_3 - I_2$ and the current through R_5 is equal to $I_5 = I_2 - I_1$.

TAKE NOTE Remember to use the current law before writing the voltage equations. This will make it easier for you to solve the problem!

After making the above observations, we only need to find the three remaining unknowns: I_1, I_2 and I_3.

The loop equations are:

For loop ABC: $100 = 1\ I_3 + 9\ (I_3 - I_2)$
For loop BCD: $9\ (I_3 - I_2) = 2\ I_2 + 9\ (I_2 - I_1)$
For loop CDE: $9\ (I_2 - I_1) = 6\ I_1$

This can be written as:

$$\begin{aligned} 100 &= 10\ I_3 - 9\ I_2 + 0\ I_1 \\ 0 &= -9\ I_3 + 20\ I_2 - 9\ I_1 \\ 0 &= 0\ I_3 - 9\ I_2 + 15\ I_1 \end{aligned}$$

Or in matrix format:

$$\begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 & -9 & 0 \\ -9 & 20 & -9 \\ 0 & -9 & 15 \end{bmatrix} \begin{bmatrix} I_3 \\ I_2 \\ I_1 \end{bmatrix}$$

Hint: If you worked correctly, you will have noticed that, for simple networks, the square matrices are symmetrical ($a_{ij} = a_{ji}$).

To find the unknown currents, we have to solve the following matrix equation. Rewrite it as:

$$\begin{bmatrix} I_3 \\ I_2 \\ I_1 \end{bmatrix} = \begin{bmatrix} 10 & -9 & 0 \\ -9 & 20 & -9 \\ 0 & -9 & 15 \end{bmatrix}^{-1} \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} 10 & -9 & 0 & 10 & -9 \\ -9 & 20 & -9 & -9 & 20 \\ 0 & -9 & 15 & 0 & -9 \end{matrix}$$

Do you remember how to calculate the determinant for a 3 x 3 matrix like the one above? Let's see how it's done ...

The determinant of the square matrix is:

$$\begin{aligned} &10 \times 20 \times 15 + (-9) \times (-9) \times 0 + 0 \times (-9) \times (-9) - 0 \times 20 \times 0 - \\ &\quad (-9) \times (-9) \times 10 - 15 \times (-9) \times (-9) \\ &\quad = 3\,000 + 0 + 0 - 0 - 810 - 1\,215 \\ &\quad = 975 \end{aligned}$$

So the inverse of the square matrix is:

$$\frac{1}{975}\begin{bmatrix} 20 \times 15 - (-9) \times (-9) & -[(-9) \times 15 - 0 \times (-9)] & (-9) \times (-9) - 0 \times 20 \\ -[(-9) \times 15 - (-9) \times 0] & 10 \times 15 - 0 \times 0 & -[10 \times (-9) - 0 \times (-9)] \\ (-9) \times (-9) - 20 \times 0 & -[10 \times (-9) - (-9) \times 0] & 10 \times 20 - (-9) \times (-9) \end{bmatrix}$$

Which gives, after elementary calculations:

$$\frac{1}{975}\begin{bmatrix} 219 & 135 & 81 \\ 135 & 150 & 90 \\ 81 & 90 & 119 \end{bmatrix}$$

Now we can solve the matrix equation:

$$\begin{bmatrix} I_3 \\ I_2 \\ I_1 \end{bmatrix} = \frac{1}{975}\begin{bmatrix} 219 & 135 & 81 \\ 135 & 150 & 90 \\ 81 & 90 & 119 \end{bmatrix}\begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{975}\begin{bmatrix} 21\,900 \\ 13\,500 \\ 8\,100 \end{bmatrix} = \begin{bmatrix} 22.46 \\ 13.85 \\ 8.31 \end{bmatrix}$$

$\Rightarrow I_1 = 8.31$ A, $I_2 = 13.85$ A and $I_3 = 22.46$ A

The flow of the currents in the circuit is shown in fig. 1.16:

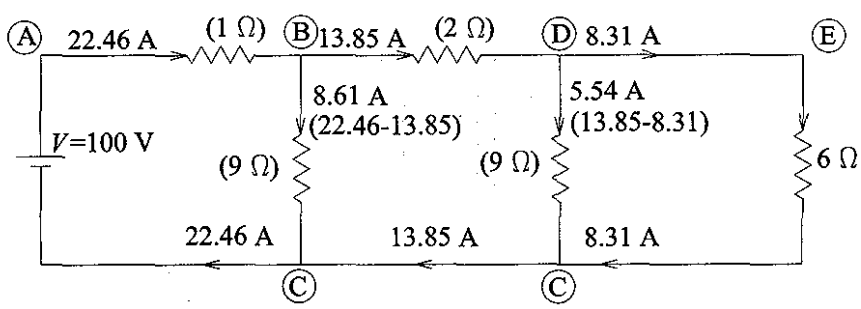


Fig. 1.16

We could have arrived at the same result in a much easier way, but we wanted to give you an example of matrix calculations in electrical engineering. This was a very simple example, since we only worked in DC and used pure resistors. In future, however, you can use these basic principles for any typical network problem.

The matrix method is also very useful when there is a large number of unknown values in circuits.

Let's use another method for solving the same problem.

The total resistance of the circuit is $R_{eq} = \dfrac{\left(\frac{6\times 9}{6+9} + 2\right)\times 9}{\frac{6\times 9}{6+9} + 2 + 9} + 1 = 4.45\ \Omega$

So, the current $I_3 = 100 / 4.45 = 22.46$ A, etc. (Apply Kirchhoff's laws. This also proves that you are working correctly.)

1.4 Summary

In the first part of this unit we revised some of the fundamental principles you encountered in Electrical Engineering I:

- Current and voltage
- Electrical resistance
- Ohm's Law
- Kirchhoff's Laws
- Series and parallel combinations of resistors
- Voltage and current dividers
- Short circuits and open circuits
- Electrical power and energy

These principles form the basis of all other (more advanced) theories in the field of electrical engineering.

The second part of this introductory unit, and appendices I and II, reviewed some of the basic concepts of trigonometry, complex numbers, calculus and matrix calculations.

In Unit 2 we will start applying these concepts and principles. So, if you are uncertain about anything, please page back to the relevant section and make sure that you understand it.

Self-evaluation

1 Calculate $I = I_1 - I_2$ if $I_1 = 6 - j\,3$ and $I_2 = 4 + j3$.

2 Write $V = 5{,}6\ \underline{/-110°}$ in rectangular form.

3 Divide $V = 4{,}5\ \underline{/0°}$ by $I = 2{,}25\,\underline{/-45°}$.

4 Calculate $i = C\frac{dv}{dt}$ if C is a constant and $v = V_m \cos \omega t$, where V_m and ω are given constants. [$v = f(t)$]

5 Assume a typical cosine wave $i = f(t)$ with the following expression: $i = I_m \cos \omega t$, where I_m = constant = the maximum value for the function i. Calculate $i_{av(erage)}$ for an interval of time equal to the first half-period of the wave.

Hint: Remember that $2\pi = \omega T \Rightarrow T/2 = \pi/\omega$.
So, you will have to integrate between 0 and π/ω.

6 Look at the following circuit and answer the questions below:

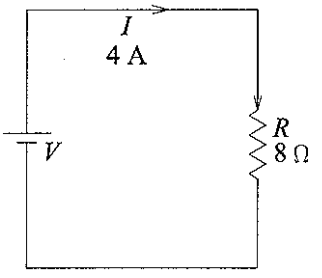


Fig. 1.17

(a) Find the voltage across the resistor.
(b) Find the voltage when the current passing through the resistor is doubled.
(c) Find the voltage if the current is again 4 A but the resistance is doubled.

7 Find the currents flowing in, and the voltages across each resistor, in the following circuit:

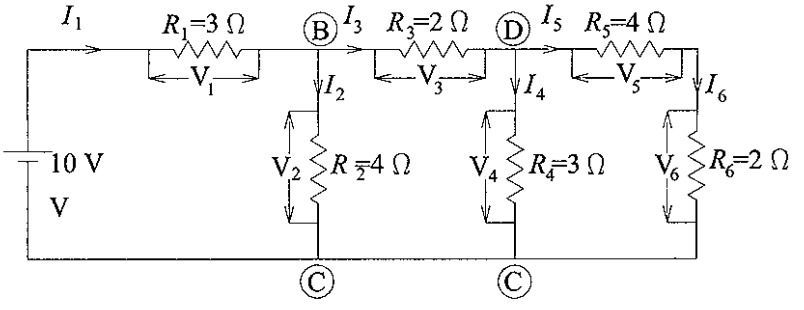


Fig. 1.18

8 Look at the following circuit:

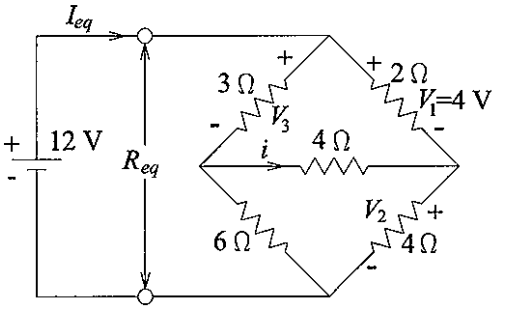


Fig. 1.19

Knowing that $V_1 = 4$ V, find V_2, V_3, i and R_{eq} for the bridge.

9 Find the total resistance R_T of the following circuit:

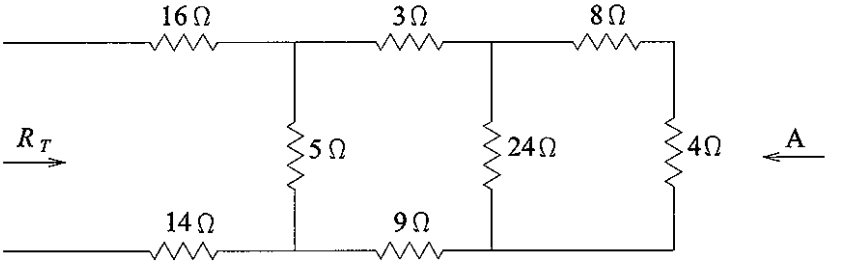


Fig. 1.20

Hint: Work from A towards the terminals!

10 A heater takes a current of 8 A from a 220 V source for 12 hours. Calculate the energy consumed in kWh.

Answers

Activity 1.1 (page 5)

16.67 A

Activity 1.2 (page 8)

1 28 Ω
2 1.38 Ω

Activity 1.3 (page 18)

Real part = 0.561
Imaginary part = 0.048

Self-evaluation (page 25)

1 $I = 2 - j6$ A
2 $V = -1{,}91 - j\,5{,}26$ V
3 $V / I = 2\underline{/45^\circ}$
4 $i = -\omega\, C\, V_m \sin \omega t = \omega C\, V_m \cos(\omega t + \pi/2)$
5 0
6 (a) $V = 32$ V
(b) $V = 64$ V
(c) $V = 64$ V
7 $I_1 = 2$ A, $I_2 = 1$ A, $I_3 = 1$ A, $I_4 = 0.67$ A, $I_5 = I_6 = 0.33$ A,
$v_1 = 6$ V, $v_2 = 4$ V, $v_3 = 2$ V, $v_4 = 2$ V, $v_5 = 1.32$ V, $v_6 = 0.67$ V
8 $v_2 = 8$ V, $v_3 = 4$ V, $i = 0$, $R_{eq} = 3.6\ \Omega$
9 $R_T = 34\ \Omega$
10 $W = 21.12$ kW

Appendix I

▶ Basic trigonometric identities

Let ϕ and θ represent two different angles. Then the following can be proved:

1 $\sin(\phi + \theta) = \sin\phi \cos\theta + \cos\phi \sin\theta$
2 $\sin(\phi - \theta) = \sin\phi \cos\theta - \cos\phi \sin\theta$
3 $\cos(\phi + \theta) = \cos\phi \cos\theta - \sin\phi \sin\theta$
4 $\cos(\phi - \theta) = \cos\phi \cos\theta + \sin\phi \sin\theta$
5 $\sin\phi + \sin\theta = 2 \sin \frac{1}{2}(\phi + \theta) \cos \frac{1}{2}(\phi - \theta)$
6 $\sin\phi - \sin\theta = 2 \cos \frac{1}{2}(\phi + \theta) \sin \frac{1}{2}(\phi - \theta)$
7 $\cos\phi + \cos\theta = 2 \cos \frac{1}{2}(\phi + \theta) \cos \frac{1}{2}(\phi - \theta)$
8 $\cos\phi - \cos\theta = 2 \sin \frac{1}{2}(\phi + \theta) \sin \frac{1}{2}(\theta - \phi)$
9 $\sin\phi - \sin\theta = \frac{1}{2}[\cos(\phi - \theta) - \cos(\phi + \theta)]$
10 $\cos\phi - \cos\theta = \frac{1}{2}[\cos(\phi - \theta) + \cos(\phi + \theta)]$
11 $\sin\phi \cos\theta = \frac{1}{2}[\sin(\phi - \theta) + \sin(\phi + \theta)]$
12 $\sin 2\theta = 2 \sin\theta \cos\theta$
13 $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$
14 $\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$
15 $\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$
16 $e^{j\theta} = \cos\theta + j \sin\theta$

Appendix II

▶ Indefinite integrals

Here are some important indefinite integrals that you need to understand

to complete your studies successfully.

1 $\int \sin \omega t dt = -\frac{1}{\omega} \cos \omega t$

2 $\int \cos \omega t dt = \frac{1}{\omega} \sin \omega t$

3 $\int \sin^2 \omega t dt = \frac{t}{2} - \frac{\sin 2\omega t}{4\omega}$

4 $\int \cos^2 \omega t dt = \frac{t}{2} + \frac{\sin 2\omega t}{4\omega}$

5 $\int \sin \omega t \cos \omega t dt = \frac{1}{2\omega} \sin^2 \omega t$

6 $\int t \sin \omega t dt = \frac{1}{\omega^2} (\cos \omega t - \omega t \cos \omega t)$

7 $\int t \cos \omega t dt = \frac{1}{\omega^2} (\cos \omega t + \omega t \sin \omega t)$

8 $\int t^2 \sin \omega t dt = \frac{1}{\omega^3} (2\omega t \sin \omega t + 2 \cos \omega t - \omega^2 t^2 \cos \omega t)$

9 $\int t^2 \cos \omega t dt = \frac{1}{\omega^3} (2\omega t \cos \omega t - 2 \sin \omega t + \omega^2 t^2 \sin \omega t)$

10 $\int e^{\omega t} dt = \frac{1}{\omega} e^{\omega t}$

11 $\int t e^{\omega t} dt = \frac{1}{\omega^2} e^{\omega t}(\omega t - 1)$

12 $\int t^2 e^{\omega t} dt = \frac{1}{\omega^3} e^{\omega t}(\omega^2 t^2 - 2\omega t + 2)$

13 $\int e^{\omega_1 t} \sin \omega_2 t dt = \frac{1}{\omega_1^2 + \omega_2^2} e^{\omega_1 t}(\omega_1 \sin \omega_2 t - \omega_2 \cos \omega_2 t)$

14 $\int e^{\omega_1 t} \cos \omega_2 t dt = \frac{1}{\omega_1^2 + \omega_2^2} e^{\omega_1 t}(\omega_1 \cos \omega_2 t + \omega_2 \sin \omega_2 t)$

15 $\int \frac{dt}{a^2 + b^2 t^2} = \frac{1}{ab} \tan^{-1}\left(\frac{bt}{a}\right)$

TAKE NOTE

1 $\int f(t)dt = y(+ C)$ means that:
(1) $f(t)$ is the derivative of y (dy/dt), and
(2) y is the anti-derivative (or indefinite integral, as it is usually called) of $f(t)$.

⇒ The indefinite integrals table can be used for finding integrals or derivatives!

2 These are only a few of the integrals you can use. If you need more, look them up in a mathematics textbook or try to calculate them yourself.

unit 2

AC Theory (Part I)

OUTCOMES

After studying this unit, you should be able to:

- **identify** the mathematical expression of sinusoidal alternating current and voltage, and **describe** what is meant by maximum value, cycle, period, frequency and root-mean-square value
- construct phasor diagrams and use them when studying simple AC circuits (see also complex number theory in Unit 1)
- describe the operating principles of all the passive components in AC (*R, L* and *C*)
- define and determine **impedances** (resistance, inductive and capacitive reactance) and **admittances** (conductance, susceptance)
- calculate equivalent impedance for simple combinations (series and/or parallel)

2.1 Introduction

In Unit 1 we showed you how to apply basic concepts to DC circuits. In this unit we will show you how to apply the same concepts in AC circuits.

Alternating current or AC is a current that **periodically reverses direction**. The **graph of the current versus time** is called the **waveform** of the current.

An **alternating voltage** is a voltage that **periodically reverses polarity**. The **positive and negative terminals** of an AC voltage source **interchange at regular intervals**. This means that current flows out of one terminal to the second terminal, then flows out of the second and returns to the first. This type of voltage is mainly produced by AC generators (or alternators, as they are often called).

The **AC waveform** studied most frequently is the **sinusoidal waveform** (the sine wave). Its **value** at any instant in time can be determined by using **trigonometric sine or cosine functions**. In sinusoidal waveform, the current (voltage) gradually increases in one direction, reaches a maximum, then decreases to zero, and finally reverses direction. After passing through zero it increases in the reverse direction until it again reaches a maximum, after which it returns to zero.

Sinusoidal currents and voltages are very important, mainly because **electric power is normally transmitted in this way** at a frequency of 50 Hz (or 60 Hz in some places).

2.2 Useful concepts

2.2.1 Mathematical aspects and rms values

Current can be expressed as a sine function: $i = I_m \sin \omega t$, but for our purposes the cosine function is more convenient. (This will be elaborated on in the following section on phasor and phasor diagrams.)

The mathematical expression used generally for sinusoidal alternating current is:

$$i = I_m \cos \omega t$$

where i is the value for current at the instant 't'
I_m is maximum current
ω is angular velocity ($\omega = 2\pi f$, where f = frequency)

In South Africa the supply frequency is $f = 50$ Hz

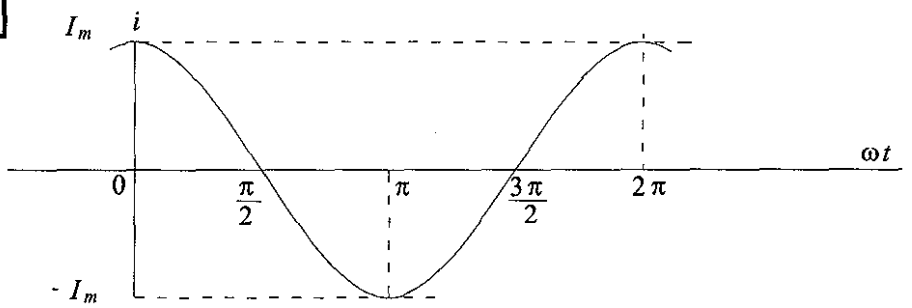


Fig. 2.1

A cycle is the time from one positive crest to the following positive crest:

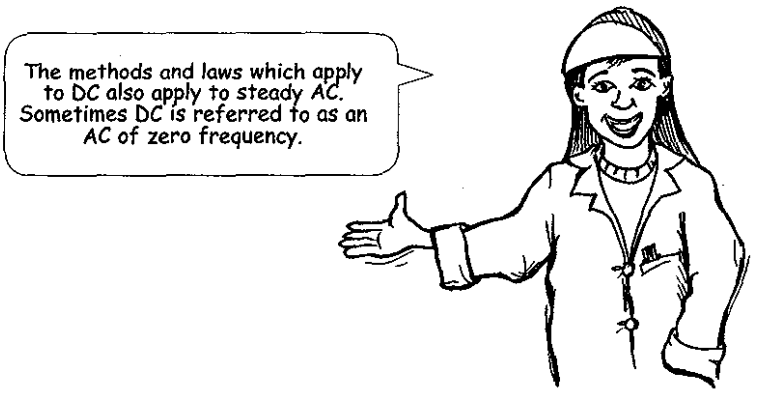
$$\omega T = 2\pi$$

where $T = \frac{2\pi}{\omega}$ = the time for one cycle = one period.

The number of cycles per second is called frequency:

$$f = \frac{\omega}{2\pi} = \frac{1}{T} \text{ (cycles per second), and}$$

$$\omega = 2\pi f = 314 \frac{rad}{s} \text{ (at 50 Hz).}$$



Did you know that the shape of ordinary household voltage is sinusoidal? If you want to see the **waveforms** when measuring voltage and current, you need to use an **oscilloscope**. Ask your lecturer to help you use an oscilloscope at any of the ordinary plug sockets in the lab. This will show you the waveform of the voltage.

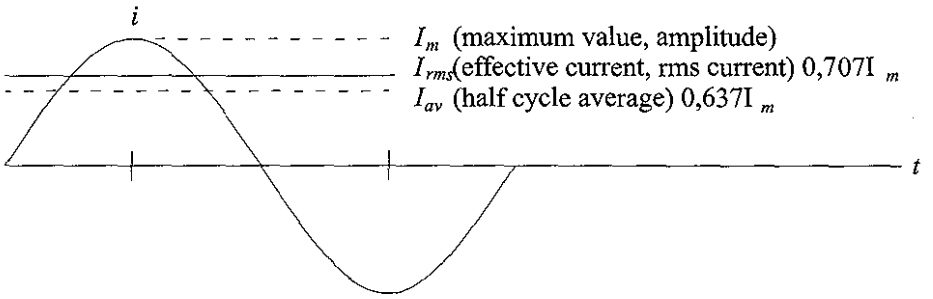


Fig. 2.2

In an AC circuit, the current varies from one instant to another. So the question is: how is the amount of current specified?

Several methods can be used to do this. Let's look at three of them.

Method 1. Specify the maximum current (the amplitude of the wave) I_m. This method is sometimes misleading because it gives the false impression that this would be the effective power-producing value of the current in the circuit.

Method 2. Specify an average value
It is obvious that the average value of a cycle is zero, so this is pointless. However, the **average current for half a cycle** can be worked out:

$$I_{av} = \frac{2}{\pi} I_m = 0.637 I_m$$

But even this method is not the most suitable for our purposes.

Method 3. Think: 'Why do we generally measure AC?'
Normally, we want **to be able to estimate what the current can do**. Whether it operates a bulb filament, a television set, a microwave oven or any other equipment, we want to know its **ability to do work** and, ultimately, what it will cost. We tend to think in terms of **power**, so we must find a value for AC that gives the same 'power' as a DC of equivalent value.

Do you remember that, for DC, power is IV or RI^2?

Let's assume that an AC flows through the same resistance. We should be able to write: $p = i^2R$ (remember that the basic principles are the same for AC and DC), so:

Average power $= p =$ average $(i^2\ R) = R$ (average i^2) (since R is constant)

So, what we are looking for is the (average i^2). This **average** is sometimes called the **effective value or root-mean-square value (rms)**:

$$I = I_{ef} = I_{rms} = \frac{I_m}{\sqrt{2}} = 0.707\ I_m$$

Page back to the section on integrals in Unit 1 and look at the example. Do you now understand why we did this calculation? What was the result? That's right: $\frac{I_m}{\sqrt{2}}$. This proves it is correct!

Another way to work out the average is:

Average power: $p = i^2R = I_m^2(\cos \omega t)^2R = I_m^2R\frac{1}{2}(\cos 2\omega t + 1)$

Since the average of cos $2\omega t$ for any integer number of cycles is zero, we can now write:

$$p = I_m^2\frac{R}{2} = \frac{I_m^2}{2}R = \left[\frac{I_m}{\sqrt{2}}\right]^2R$$

If you compare this result with the assumption $P = I^2_{rms}R$, you'll see that $I_{rms} = \frac{I_m}{\sqrt{2}}$!

The same method to determine average values applies to sinusoidal voltages.

$$V_{rms} = \frac{V_m}{\sqrt{2}}$$

TAKE NOTE

From now on, when a numerical value is given for an alternating voltage or current, this will always be **the rms value** unless otherwise stated.

Most **ordinary ac voltmeters and ammeters** are calibrated to **indicate rms values**. Some meters give the maximum value reading, but these are not commonly used (*see* Method 1).

2.2.2 Phasors and phasor diagrams

Phasors are rotating lines, and their projections on the X-axis are convenient representations for **alternating quantities**. They are very useful when studying AC circuits.

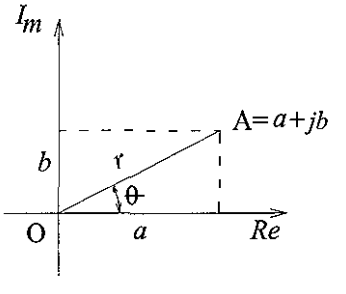


Fig. 2.3

The line OA in fig. 2.3 is at an angle θ with the real axis. If θ is constant, the line is fixed.

If the angle increases in time, the line rotates.

If the line rotates with a constant angular velocity ω, the angle θ constantly increases and can be given at any time (t).

We know from physics that:

$$\theta = \text{angle} = \omega t = \text{angular velocity} \times \text{time}$$

Each time the angle increases by $2\pi(\omega t = 2\pi)$, the line will be in its original position again.

The mathematical expression of the rotating line is: $re^{j\omega t}$, where:

- r is the (constant) **length of the line**, and
- ω is the **angular velocity** at which the line rotates (about the origin of the complex plane)

The real component of this line (the projection on the real axis) is:

$$\text{Re}\{re^{j\omega t}\} = \text{Re}\{r(\cos \omega t + j \sin \omega t)\} = r \cos \omega t$$

TAKE NOTE

You will see that 'sin' disappears. Because it has j in front of it, it is obviously the imaginary component!

This proves that the projection of the rotating line is a sinusoidal varying quantity, similar to the currents and voltages in AC.

Graphically, an **alternating quantity** can be conveniently represented by the **projection of a rotating line on a fixed axis**. This rotating line is called a **phasor**.

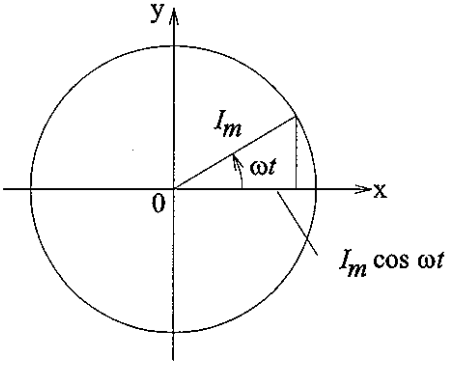


Fig. 2.4

If line I_m in Figure 2.4 rotates like a wheel-spoke at a constant velocity, the length of its projection on the X-axis will vary sinusoidally with time ($I_m \cos \omega t$). If I_m is the length of the line, then:

- the length of the projection on the X-axis is $I_m \cos \omega t$ and
- the length of the projection on the Y-axis is $I_m \sin \omega t$
  where:
  - ω is the constant angular velocity
  - we assume the direction of ω is counterclockwise

Now we can say that **a phasor can represent a sinusoidal current or a voltage.**

If we have an initial phase, we 'advance' the phasor by an angle θ, $I_m \cos(\omega t + \theta)$.

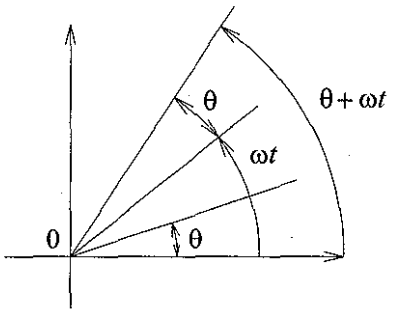


Fig. 2.5

This is a very easy way to **represent sinusoidally alternating currents and voltages**. The lines tell us all we need to know about these quantities.

The only problem is that we must **always assume a reference** (in time).

We can either choose the reference as the instant **the line for current is on the positive X-axis** (the angle of the phasor is then 0), or when the **voltage phasor has an angle zero**. The voltage reference is usually preferred.

The first step is always to **select one phasor as the reference**. Remember that all the other phasors must be drawn with the correct relative angles.

2.3 Passive elements in AC circuits

TAKE NOTE

An element that receives power from a network is called a passive element. An active element delivers power to a network.

All **passive components** have **three electrical properties**:

- Resistance
- Capacitance
- Inductance

2.3.1 Electrical resistance in AC

Here's a reminder of what we covered in Unit 1.

- **Resistance** is **the real component of any impedance**. Impedance generally implies sinusoidal excitation of the circuit (single frequency, sine wave). Although resistance is a simple electrical concept, it is probably the most difficult to describe accurately.
- Theoretically, **resistance is unaffected by the frequency** of the applied signal.
- Practically, real resistors are **frequency dependent** because they contain some inductance and capacitance.
- A typical wire-wound resistor will be purely resistive only at power frequencies below 1 kHz. At audio frequencies, it will show a strong inductive component and at much higher frequencies it will show a strong capacitive component.

Resistance is measured using:

- an **ohmmeter** (often one of the elements in the digital multimeter)
- the **Wheatstone bridge** (the most common of the resistance bridges)
- the **Kelvin (or double) bridge** (normally for small values of resistance)

Some of the basic practical resistor uses include:

- Room heating units
- Kitchen appliances, such as toaster and kettle heating coils, and the spiral grids of oven heating elements
- Soldering irons
- Incandescent light bulb filaments (remember the change in resistance with temperature)
- Limiting the current in other circuit components

TAKE NOTE

In general, the DC measurement is inadequate for evaluating resistances in power circuits because of the phenomenon known as 'skin effect' (*see Introduction to Electrical Engineering* (2nd ed.) by CR Paul, published in 1992 by McGraw-Hill). Resistances and losses tend to increase with the frequency due to the skin effects in the

wires. If possible, always try to measure resistance at the same level of excitation that will be used in the resistor's application.

In all the examples used in this book, we will consider only constant resistance, ignoring frequency effects.

Alternating current through a resistance can be represented using Ohm's law:

$$R = \frac{v}{i}$$

(or G = conductance, where $G = \frac{1}{R} = \frac{i}{v}$)

$$v = Ri = R(I_m \cos \omega t) = (RI_m)\cos \omega t = V_m \cos \omega t$$

in which $V_m = RI_m$ is the maximum value of the voltage wave.

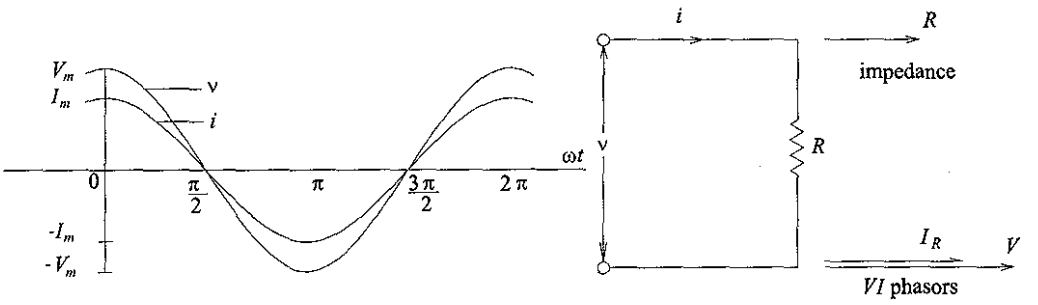


Fig. 2.6

As you can see, on the positive X-axis we can use ωt = angle, since ω = constant.

Using the conductance $G = \frac{1}{R}$

$$i = Gv = G(V_m \cos \omega t) = \left[\frac{V_m}{R}\right]\cos \omega t = I_m \cos \omega t$$

where $I_m = GV_m = \frac{V_m}{R}$

2.3.2 Inductance

▶ Self-inductance

At the beginning of the 19th century, J Henry discovered that electric circuits have a property similar to mechanical momentum: **once a current starts flowing, it tends to continue flowing.**

The mechanical momentum of an object is represented by Mv, where v is the velocity of the object and M its mass (a characteristic of the object). The 'electrokinetic' momentum, as it was originally called, is represented by Li, where i is the current flowing through the circuit and L a characteristic of the circuit known as 'inductance'.

Faraday explained this momentum in terms of the magnetic field around the conductor.

DEFINITION

The **symbol for inductance** in a circuit is L, and we **measure inductance in** Henry [H], in honour of J Henry who discovered the electromechanical analogy.

Newton's second law of mechanics states:

$$F = \frac{d}{dt}(Mv) = M\frac{dv}{dt} = Ma$$

where a represents acceleration. In words, 'the force necessary to change the velocity of an object is equal to the rate of change of mechanical momentum'.

The analogous electrical relation is:

$$\mathrm{v} = \frac{d}{dt}(Li) = L\frac{di}{dt}$$

or 'the voltage necessary to produce a change of current in an inductive circuit is equal to the rate of the change of electrokinetic momentum'.

As you can see, L is a constant and it is called the 'self-inductance' of the circuit.

The same relationship can be written:

$$di = \frac{1}{L}\mathrm{v}dt$$

or

$$i = \frac{1}{L}\int \mathrm{v}dt$$

▶ Mutual inductance

Faraday noticed that **a changing current in one circuit induces a voltage in another nearby circuit**. The circuits are said to be 'coupled' together. If we call the two circuits 'circuit 1' and 'circuit 2', the **voltage induced in circuit 2 by a changing current in circuit 1** can be written:

$$\mathrm{v}_2 = L_{21}\frac{di_1}{dt}$$

Let's assume that the current in circuit 2 is changing. The voltage induced in circuit 1 will be:

$$v_1 = L_{12}\frac{di_2}{dt}$$

L_{12} and L_{21} are called **mutual inductances.**

It can be proven from general energy relations that the two mutual inductances of a pair of coupled circuits are equal:

$$L_{12} = L_{21} = M$$

Faraday explained self-inductance and mutual inductance in terms of a magnetic field surrounding a conductor carrying current. This approach allows you to work out inductance based on the circuit parameters.

Faraday computed flux linkages, $N\phi$, which are equivalent to the electrokinetic momentum of the circuit:

$N\phi = Li$, where N = number of turns in the coil and ϕ = magnetic flux (Wb)

So, $L = \frac{N\phi}{i}$ (inductance is equal to the flux linkages produced by ampere of current).

$$v = \frac{d}{dt}(Li) = \frac{d}{dt}(N\phi)$$

DEFINITION

Farady's Law for induced voltages states: The magnitude of the voltage induced in a conductor moving through a magnetic field is directly proportional to the rate at which flux lines are intersected.

$$v = -\frac{d}{dt}(N\phi)$$

The minus sign shows that the induced voltage opposes the applied voltage that is responsible for the increase in current.

DEFINITION

According to Lenz's Law, an induced current will flow in such a direction that its own magnetic flux will be directed to oppose the change in flux that produced the induced current ⇒ The self-induced voltage opposes the increase in current.

If the current in the coil is decreasing, the induced and the applied voltages have the same direction. **In this case the induced voltage opposes the decrease in the current.**

It can be proved that the rules for series and parallel inductances are similar to those for resistors:

Series inductance: $L = L_1 + L_2 + \ldots + L_i + \ldots + L_n$
(without mutual inductance)

If you consider the mutual inductance, you must calculate accordingly.

Taking the mutual inductance (M) into consideration we have, for two inductances in series $L_{eq}(\pm) = L_1 + L_2 \pm 2M$.

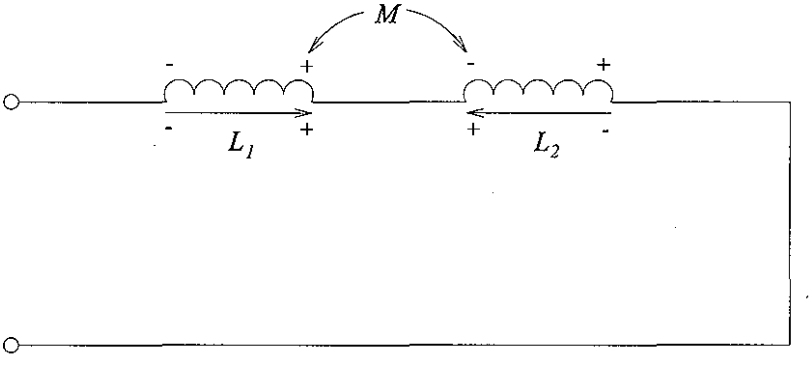


Fig. 2.7

Parallel inductance: $\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_i} + \ldots + \frac{1}{L_n}$;

For two inductances in parallel: $L_{eq} = \frac{L_1 L_2}{L_1 + L_2}$

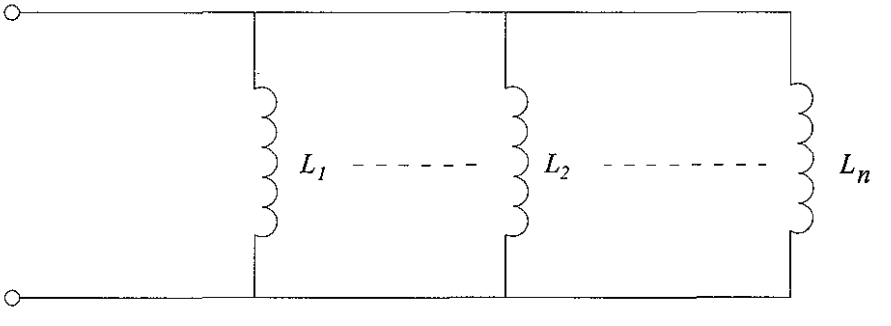


Fig. 2.8

Self-inductance of a coil depends on:

- the number of turns in the coil
- the cross-sectional area of the flux path
- the length of the flux path

$$L = \mu N^2 \frac{A}{l} = \mu_0 \mu_r N^2 \frac{A}{l}$$

where, μ = permeability = $\mu_0 \mu_r \left(\frac{H}{m}\right)$
N = number of turns in the coil
A = cross-sectional area of the flux path (m^2)
l = length of the flux path (m)

TAKE NOTE

We have worked with 'ideal' inductances, but remember that real inductors are a combination of resistance and inductance, since the wire of the coil has a resistance.

Assuming a sinusoidal alternating current is flowing through the conductor $I = I_m \cos \omega t$, we get:

$$v = L\frac{d}{dt}(I_m \cos \omega t) = LI_m \frac{d}{dt} \cos \omega t = -\omega L I_m \sin \omega t = \omega L I_m \cos[\omega t + \frac{\pi}{2}]$$

The current and voltage waveforms in this case are shown in fig. 2.9.

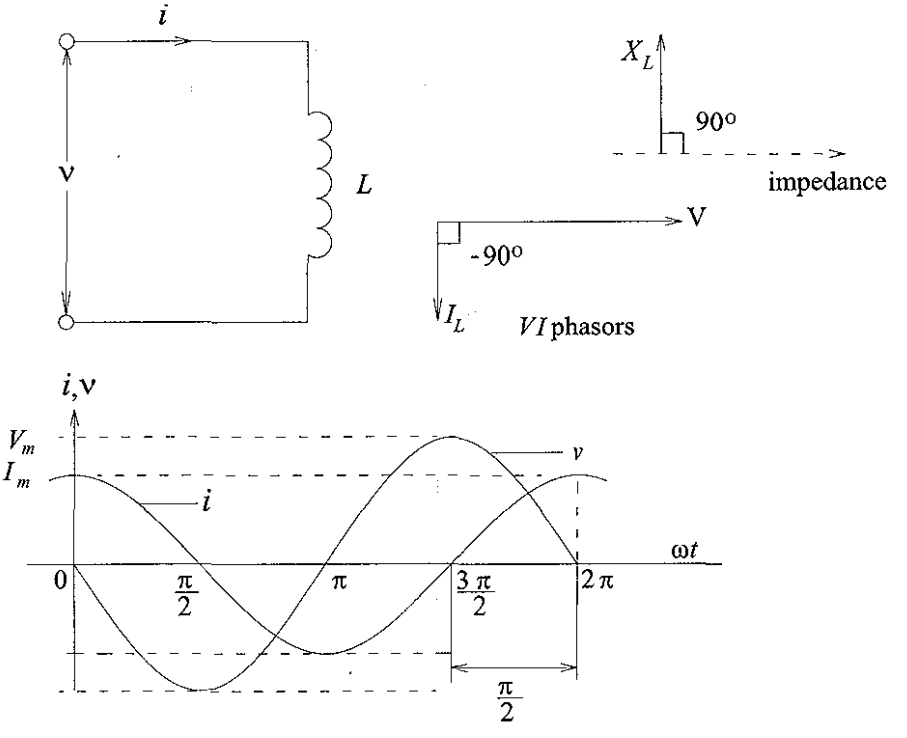


Fig. 2.9

Then $V_m = \omega L I_m$, the maximum voltage. We call ωL the 'inductive reactance' and normally use the symbol X for reactance:

$$V_m = XI_m$$

In fig. 2.9 **the maximum value of the current is reached later than the maximum value of the voltage**. We say that the current 'lags' behind the voltage. As you can see, the lag is ¼ cycle (or $\frac{\pi}{2}$, or 90°) on the axis.

The formula used to calculate the energy stored by an inductor is:

$$w_L(t) = \frac{1}{2}Li^2(t)$$

Let's move on to capacitance.

2.3.3 Capacitance

Capacitance has been recognised since the 18th century. For capacitance to exist, we need **two** electrical conductors at **different potentials, separated by an insulating material**.

In practice a **condenser** or **capacitor** (both terms are used) is a device in an electrical circuit that **stores electric charge**.

Most power capacitors we use are designed using **metal 'sheets'** as the two conductors, and **air, impregnated paper or plastic** is used as an **insulator** between the two conductors. There are other capacitor designs, but we will not discuss these in this unit.

The charge on a capacitor $q = Cv$, where:

- q is the **charge**;
- v is the **voltage** (or potential difference); and
- C is a factor of proportionality called **capacitance** (of the capacitor) measured in Farads (F).

▶ Capacitors in series

The following figure represents capacitors in series.

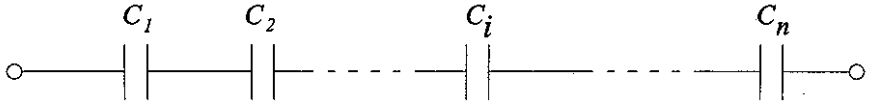


Fig. 2.10

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_n}$$

To remember this formula, make an association with resistors, but note that **for capacitors in series, the formula resembles that for resistors in parallel**, and **for capacitors in parallel, the formula resembles that for resistors in series!**

Capacitance depends on:

- the **size and form** of the plates;
- the **distance between** the plates; and
- the **insulating material** used.

$$C = \varepsilon \frac{A}{d} = \varepsilon_0 \, \varepsilon_r \frac{A}{d} \text{ [F]}$$

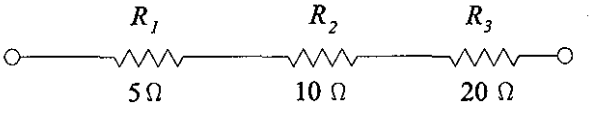
where $\varepsilon = \varepsilon_0 \, \varepsilon_r$ permittivity of the dielectric $[\frac{F}{m}]$
A = parallel area of the plates [m^2]
d = distance between the plates [m]

An electric field appears between the plates of a capacitor each time the capacitor is charged.

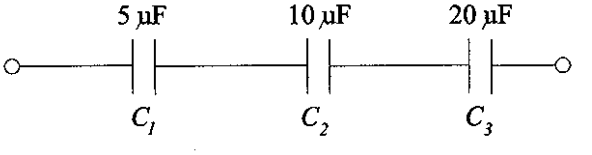
Remember that the symbol used for capacitors in a circuit is ⊣⊢, and that the value is measured in Farads (F).

Activity 2.1

a. What is the equivalent resistance of the circuit shown below?



b. Now what is the equivalent capacitance of the circuit shown below?



▶ Capacitors in parallel

The circuit below represents capacitors in parallel. To remember the formula, you must make an association with resistors in series.

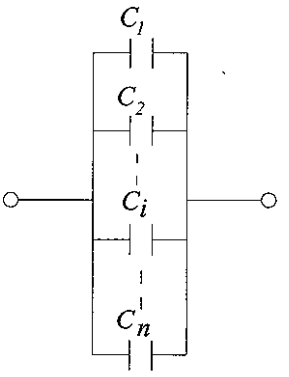
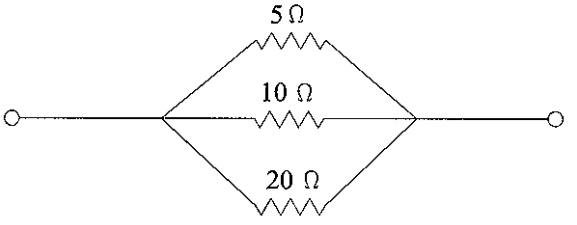


Fig. 2.11

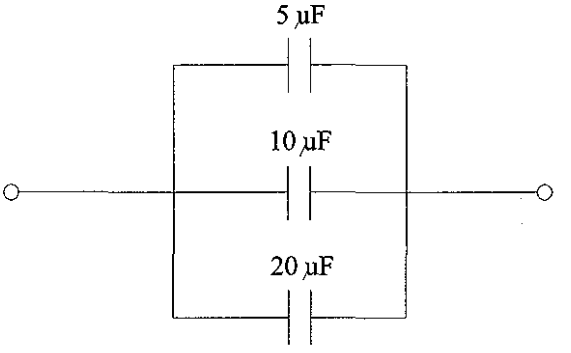
$$C_{eq} = C_1 + C_2 + \ldots + C_i + \ldots + C_n$$

Activity 2.2

a. What is the equivalent resistance of the circuit shown below?



b. What is the equivalent capacitance of the circuit shown below?



When voltage is applied to a capacitor's terminals, the current is given by the equation:

$$i = \frac{d}{dt}(Cv) = C\frac{dv}{dt}$$

If it is a sinusoidal alternating voltage:

$$v = V_m \cos \omega t$$

and the current is:

$$i = C\frac{d}{dt}(V_m \cos \omega t) = -\omega C V_m \sin \omega t = \omega C V_m \cos(\omega t + \frac{\pi}{2})$$

Current and voltage waves for sine excitation are shown in fig. 2.12.

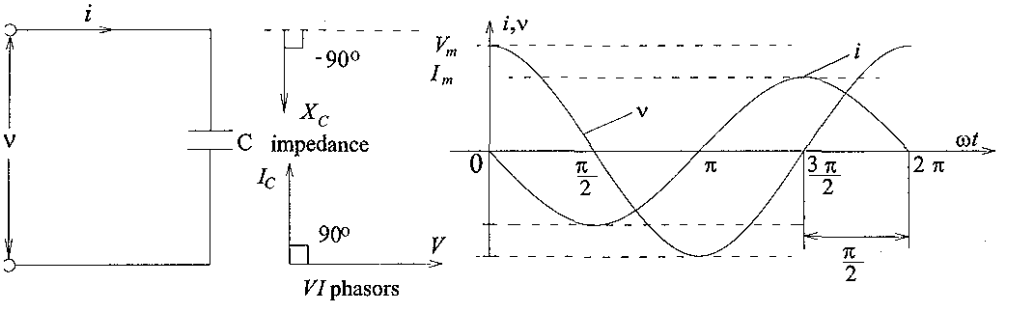


Fig. 2.12

Did you notice that the **current 'leads' the voltage** (by a quarter of a cycle, or 90°, or $\frac{\pi}{2}$ rad)? So, the maximum value of current is:

$$I_m = \omega C V_m$$

If we call $\omega C = B$ **susceptance** (in this case, 'capacitive susceptance'), we can write the equation as:

$$I_m = B_{(C)} V_m$$

Did you notice that all these equations are valid if we see the capacitor as a 'pure capacitor' – a capacitor that presents only capacitance.

We normally call $\frac{1}{\omega C}\left(=\frac{1}{B}\right)$ the capacitive reactance, by analogy with $\omega L = X$, the inductive reactance.

Mathematically, the inductor is the 'dual' of the capacitor.

The following formula represents energy stored in the electric field between the plates:

$$w_C(t) = \frac{1}{2}Cv^2(t)$$

Every practical component has a certain amount of **resistance, inductance** and **capacitance**, but in this course we will look mainly at the predominant characteristics.

Every inductor has resistance that must be taken into account. The overhead lines used to transmit electricity are studied as combinations of R, L and C, and for long lines you will have to consider distributed parameters and other factors.

2.4 Impedance (Z) and admittance (Y)

All **network theory** uses the concepts of **impedance** and **admittance**.

Impedance, Z, can be defined as the ratio of the applied voltage to the forced current:

$$Z = \frac{v}{i}$$

and is **not** a function of time.

Admittance, Y, is the reciprocal of impedance:

$$Y = \frac{1}{Z}$$

The real and the imaginary parts of the impedance Z are denoted by the letters R and X:

$$Z = R \pm jX$$

where R and X may be the resistance and the reactance of a physical element of the circuit. For example, if we have a coil of wire, R is its resistance ($\approx$ constant) and X its inductive reactance (proportional to frequency).

The admittance Y also has a real and an imaginary component. This can be written:

$$Y = G \mp jB$$

where G is conductance and B is susceptance.

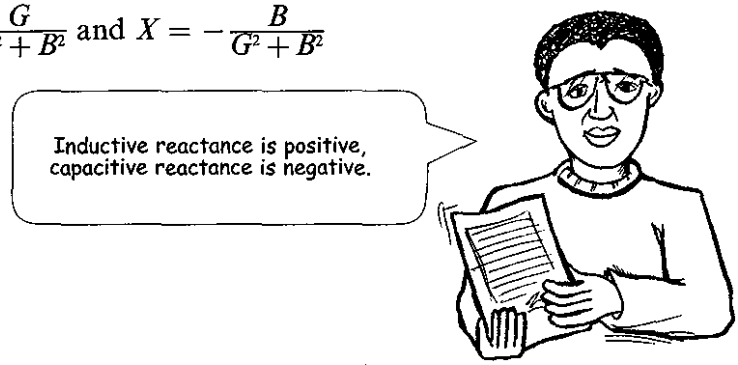
Since $Y = \frac{1}{Z}$, there are some relations between the components:

$$Y = G + jB = \frac{1}{Z} = \frac{1}{R+jX} = \frac{R-jX}{R^2+X^2} \Rightarrow$$

$$G = \frac{R}{R^2+X^2} \text{ and } B = -\frac{X}{R^2+X^2}$$

$$Z = R + jX = \frac{1}{Y} = \frac{1}{G+jB} = \frac{G-jB}{G^2+B^2} \Rightarrow$$

$$R = \frac{G}{G^2+B^2} \text{ and } X = -\frac{B}{G^2+B^2}$$



If you look back at the formulas for Z and Y you will notice that:

$$|Z| = \sqrt{R^2 + X^2} \text{ and } \theta_1 = \tan^{-1}\frac{X}{R}$$

$$|Y| = \sqrt{G^2 + B^2} \text{ and } \theta_1 = \tan^{-1}\frac{B}{G}$$

If you review the theory of capacitance and inductance, you will see that the **respective reactances** are:

$$X_L = \omega L = 2\pi f L$$

and

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

The **inductive reactance** is shifted 90° in front of the resistance and the **capacitive reactance** is shifted 90° behind the resistance.

In an RLC series, as seen in fig. 2.13, we normally add the two reactances (in fact, it is a subtraction!) and then calculate the impedance using Pythagoras in the right-angle triangle which remains:

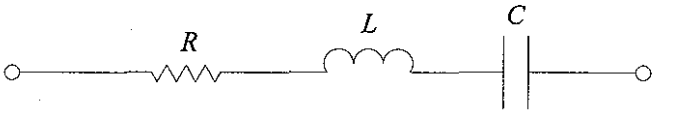


Fig. 2.13

$X = X_L - X_C$ in this case, and

$|Z| = \sqrt{R^2 + X^2}$

$\theta = \tan^{-1} \dfrac{X}{R}$

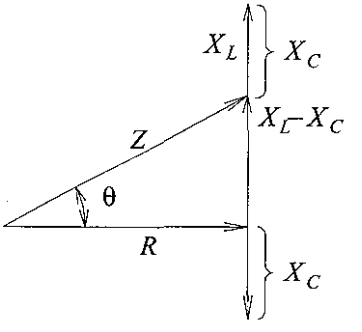


Fig. 2.14

2.5 Simple combinations of resistors (*R*), inductors (*L*) and capacitators (*C*)

2.5.1 *RL* in series

Look at fig. 2.15 below, where you have a resistance connected in series with an inductance.

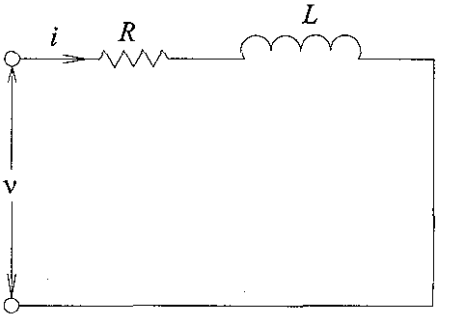


Fig. 2.15

$Z = R + j\omega L$

$Z = R + j X_L$

Phasor diagram

$$Z^2 = R^2 + X_L^2$$

$$|Z| = \sqrt{R^2 + X_L^2}\ \Omega$$

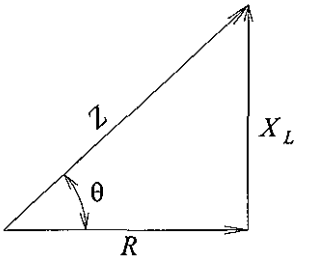


Fig. 2.16

$R\underline{/0°}$

$X_L\underline{/90°}$

$Z\underline{/\theta°}$

To show the direction $\tan\theta = \frac{X_L}{R}$, and $\theta = \tan^{-1}\frac{X_L}{R}$.

If we assume V as reference ($V\underline{/0°}$), it is obvious (*see* complex numbers theory) that:

$$\frac{V\underline{/0°}}{Z\underline{/\theta°}} = I\underline{/-\theta°}$$

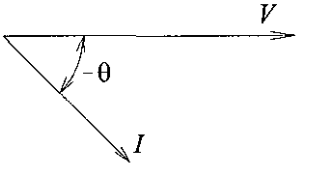


Fig. 2.17

The current lags behind the voltage with an angle $\underline{/-\theta°}$. Alternatively, we can say the voltage leads the current by the same angle.

So, the current will have a modulus $I = \frac{V}{Z}$ and a direction $\underline{/-\theta°}$ (don't forget that we assumed the voltage as the reference).

The waveforms in this case will be:

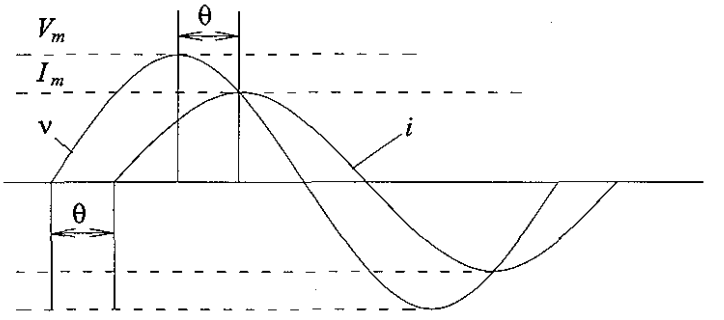


Fig. 2.18

You know that the angle for a **pure inductance** would have been 90° (or $\frac{\pi}{2}$). In this example, the new angle is θ.

2.5.2 *RC* in series

For fig. 2.19:

$$Z = R - j\frac{1}{\omega C}$$

$$Z = R - jX_C,\ Z^2 = R^2 + X_C{}^2$$

$$|Z| = \sqrt{R^2 + X_c^2}\ \Omega$$

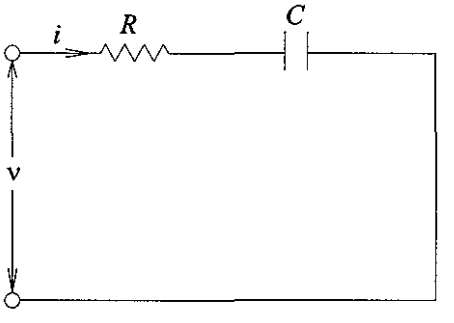


Fig. 2.19

To show the direction:

$$\tan\theta = \frac{X_C}{R}$$

$$\theta = \tan^{-1}\frac{X_C}{R}$$

but here you must choose the negative angle from the family of solutions!

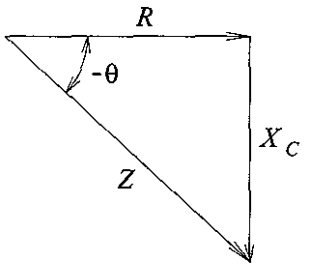


Fig. 2.20

$R\underline{/0^\circ}$
$X_C\underline{/-90^\circ}$
$Z\underline{/-\theta^\circ}$

Assuming V as the reference:

$$\frac{V\underline{/0^\circ}}{Z\underline{/-\theta^\circ}} = I\underline{/\theta^\circ}$$

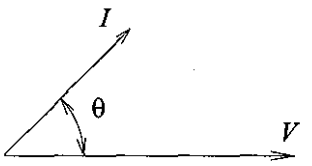


Fig. 2.21

and that the current 'leads' (in front of) the voltage with an angle θ:

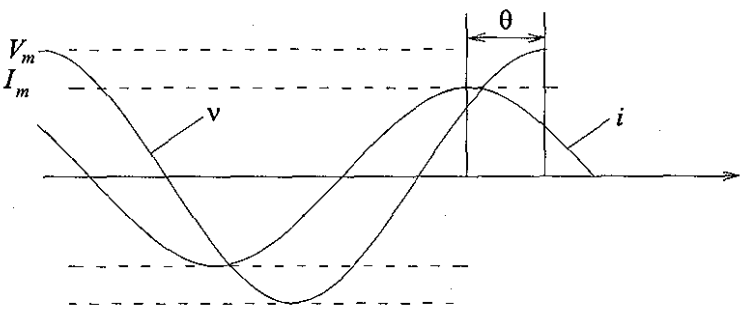


Fig. 2.22

(You already know that for pure capacitance $\theta = 90^\circ$.)

2.5.3 *RLC* in series

$$Z = R + j\omega L - j\frac{1}{\omega C}$$

$$Z = R + jX_L - jX_C, \text{ or}$$

$$Z = R + j(X_L - X_C)$$

$$|Z| = \sqrt{R^2 + (X_L - X_C)^2}, \text{ and}$$

$$\theta = \tan^{-1}\frac{X_L - X_C}{R}$$

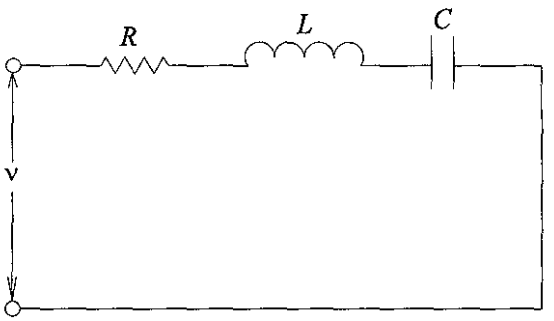
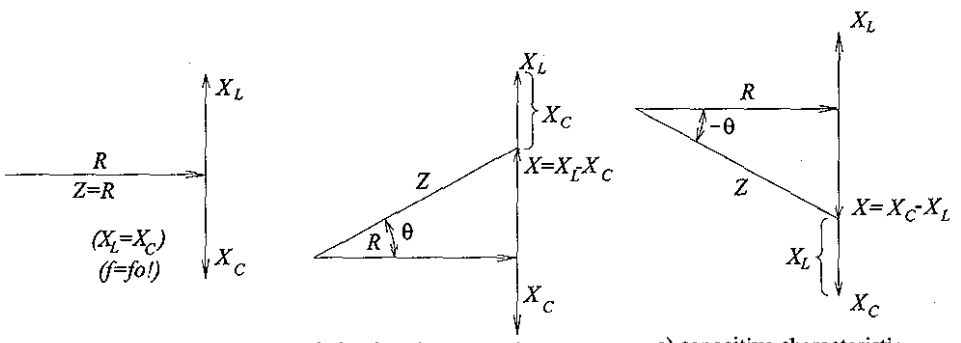


Fig. 2.23

Possible situations:



a) resonance: resistive characteristic for the circuit

b) inductive characteristic

c) capacitive characteristic

Fig. 2.24

In this example, if we assume the voltage as the reference point, the current can be leading, lagging or in phase with the voltage, depending on the overall character of the circuit.

EXAMPLE 2.1

Determine the current drawn by the following circuit:

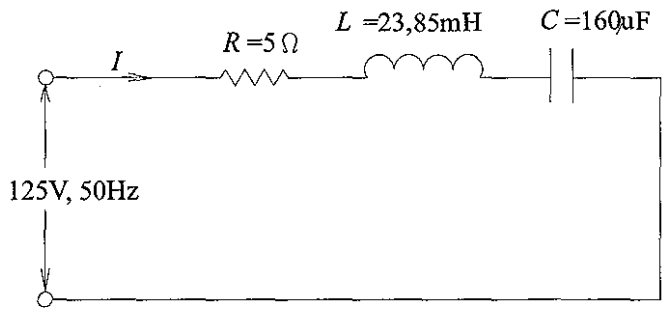


Fig. 2.25

Solution

$$X_L = 2\pi fL = 7.49\ \Omega$$

$$X_C = \frac{1}{2\pi fC} = 19.89\ \Omega$$

$$Z = 5 + j7.49 - j19.89 = 5 - j12.4 = 13.37\angle{-68^\circ}\ \Omega$$

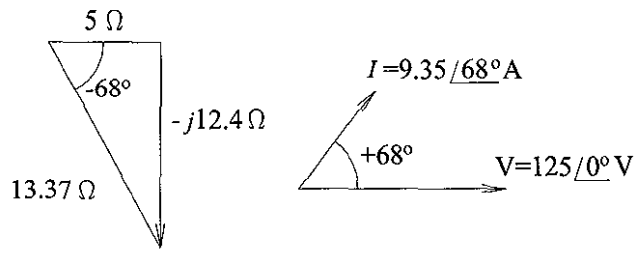


Fig. 2.26

Let's assume the supply voltage is the reference. Then:

$$I = \frac{V}{Z} = \frac{125}{13.37\angle{-68^\circ}} = 9.35\angle{-68^\circ}\ \text{A}$$

2.5.4 *RL* in parallel

Here one should think in terms of currents, since the currents flowing through the elements are no longer the same.

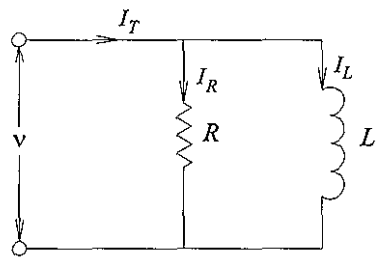


Fig. 2.27

$$I_R \underline{/0^\circ}$$

$$I_L \underline{/-90^\circ}$$

$$|I_T| = \sqrt{I_R^2 + I_L^2}$$

$$\theta = \tan^{-1}\frac{I_L}{I_R}$$

the angle for current is $\underline{/-\theta^\circ}$

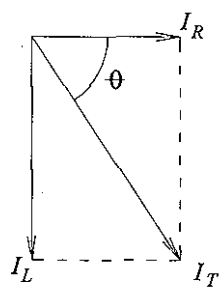


Fig. 2.28

The two impedances are now in parallel, so:

$$Z_1 = R_1 + j0$$

$$Z_2 = 0 + jX_2$$

$$Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(R_1 + j0)(0 + jX_2)}{R_1 + j0 + 0 + jX_2} = j\frac{R_1 X_2}{R_1 + jX_2} = Z\underline{/\theta^\circ}$$

You can check if this correct by doing the same calculation for current, and finding whether you get the same value.

We can now draw an equivalent series circuit for the *RL* parallel circuit. Since the impedance **must** be the same:

$R_{eq} = \text{Re}\{Z_{eq}\}$

$X_{eq} = \text{Im}\{Z_{eq}\}$

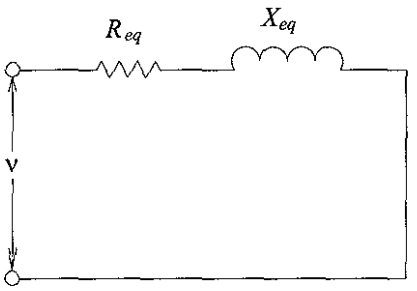


Fig. 2.29

2.5.5 *RC* in parallel

Applying KCL in node A for fig. 2.30:

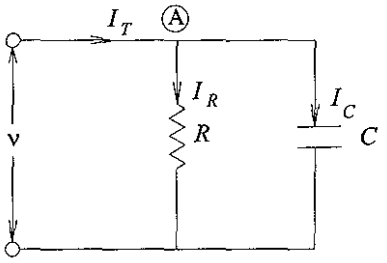


Fig. 2.30

$I_R \underline{/0°}$

$I_C \underline{/90°}$

$|I_T| = \sqrt{I_R^2 + I_C^2}$

$\theta = \tan^{-1} \frac{I_C}{I_R}$

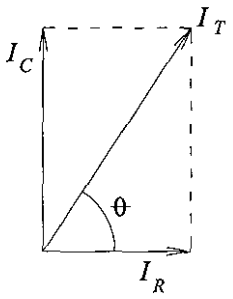


Fig. 2.31

Equivalent impedance:

$$Z_1 = R_1 + j0$$

$$Z_2 = 0 - jX_C$$

$$Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(R_1 + j0)(0 - jX_C)}{R_1 + j0 + 0 - jX_C} = \frac{-jR_1 X_C}{R_1 - jX_C} = Z\underline{/-\theta^\circ}$$

And in a series circuit with an equivalent impedance we will have:

$$R_{eq} = \mathrm{Re}\{Z_{eq}\}$$

$$X_{eq} = \mathrm{Im}\{Z_{eq}\}$$

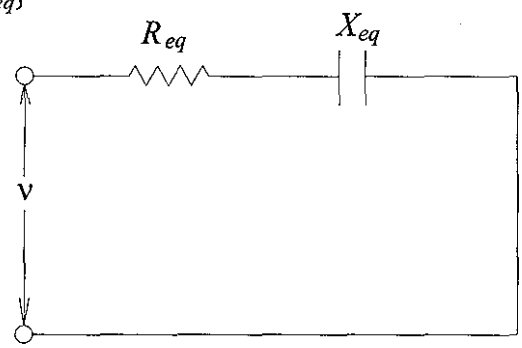


Fig. 2.32

2.5.6 *RLC* in parallel

Considering the three types of elements in parallel:

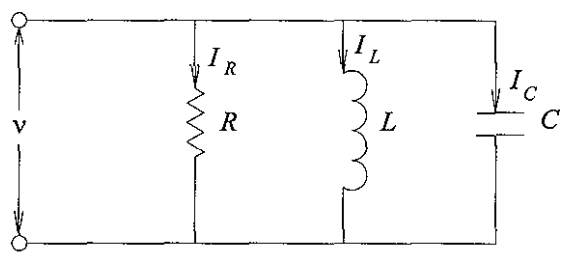


Fig. 2.33

$$I_R = \frac{V}{R}\underline{/0^\circ}$$

$$I_L = \frac{V}{X_L}\underline{/-90^\circ}$$

$$I_C = \frac{V}{X_C}\underline{/90^\circ}$$

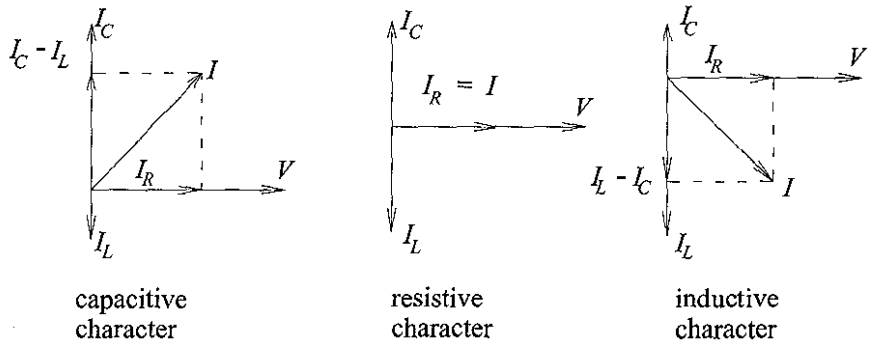


Fig. 2.34

If we combine I_L with I_C, and the result is combined with I_R using Pythagoras, we get:

$I_X = I_L - I_C$ (will have an angle of $+\frac{\pi}{2}$ or $-\frac{\pi}{2}$, depending which current is bigger)

$$I_T = \sqrt{I_R^2 + I_X^2}$$

$$\theta = \tan^{-1}\frac{I_X}{I_R}$$

Equivalent impedance:

$$\frac{1}{Z_e} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$$

$$Z_1 = R_1 + j0$$

$$Z_2 = 0 + jX_2 \text{ (or } jX_L)$$

$$Z_3 = 0 - jX_3 \text{ (or } jX_C)$$

$$\frac{1}{Z_e} = \frac{1}{R_1} + \frac{1}{jX_L} + \frac{1}{-jX_C} \Rightarrow$$

$$\frac{1}{Z_e} = \frac{(jX_L)(-jX_C) + R_1(-jX_C) + R_1(jX_L)}{R_1(jX_L)(-jX_C)} = \frac{X_L X_C + jR_1(X_L - X_C)}{R_1 X_L X_C} \Rightarrow$$

$$Z_e = \frac{R_1 X_L X_C}{X_L X_C + jR(X_L - X_C)}$$

Another way to solve parallel circuits is by using admittance (conductance and susceptance). Add the admittance for parallel branches, then the equivalent impedance of the parallel group is directly obtained as the reciprocal of its admittance.

EXAMPLE 2.2

Determine the current drawn by the following circuit:

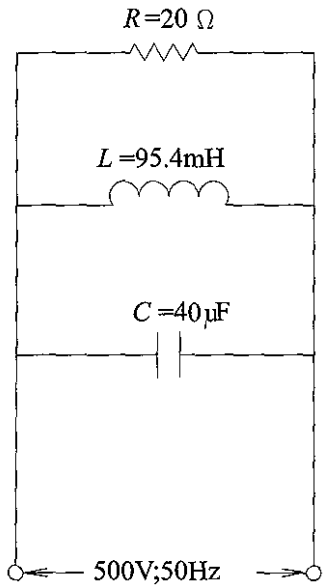


Fig. 2.35

Solution

$$I_R = \frac{V}{R} = \frac{500}{20} = 25 \text{ A}$$

$$Z_L = 0 + jX_L = 0 + j2\pi fL = 0 + j29.97\ \Omega$$

$$I_L = \frac{500}{29.97\underline{/90^\circ}} = 16.68\underline{/-90^\circ} = -j16.68 \text{ A}$$

$$Z_C = 0 - jX_C = -\frac{1}{j2\pi fC} = -j79.58\ \Omega$$

$$I_C = \frac{500}{79.58\underline{/-90^\circ}} = 6.28\underline{/90^\circ} = j6.28 \text{ A}$$

$$I_{tot} = 25 - j16.68 + j6.28 = 25 - j10.4 = 27\ \underline{/-22.58^\circ} \text{ A}$$

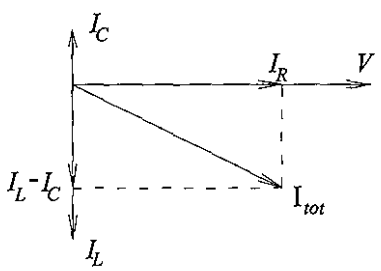


Fig. 2.36

EXAMPLE 2.3

A coil with a resistance of 5 Ω and an inductance of 0.03 H is connected in series with a resistor R = 10 Ω and a capacitor of 200 μF. The supply is giving 220 V at 50 Hz. Determine the current drawn in the circuit and the voltage drop across the coil.

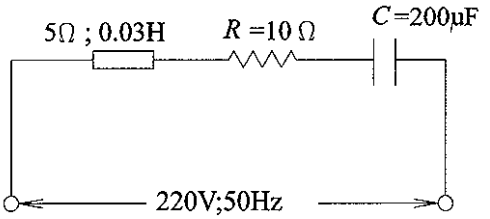


Fig. 2.37

Solution

$$Z_{coil} = 5 + j(2\pi fL) = 5 + j \times 2 \times \pi \times 50 \times 0.03 = 5 + j9.4 = 10.65\,\underline{/62^\circ}\ \Omega$$

$$Z_{tot} = 5 + j9.4 + 10 - j\frac{1}{2\pi fC} = 15 + j9.4 - j\frac{10^6}{2\pi \times 50 \times 200} = 15 + j9.4 - j15.9$$

$$= 15 - j6.5 = 16.35\,\underline{/-23.43^\circ}\ \Omega$$

Assume the voltage as reference:

$$I = \frac{V\angle 0^\circ}{Z_{tot}} = \frac{220\,\underline{/0^\circ}}{16.35\,\underline{/-23.43^\circ}} = 13.46\,\underline{/23.43^\circ}\ \text{A}$$

$$V_{coil} = I\,Z_{coil} = 13.46\,\underline{/23.43^\circ} \times 10.65\,\underline{/62^\circ} = 143.35\,\underline{/85.43^\circ}\ \text{V}$$

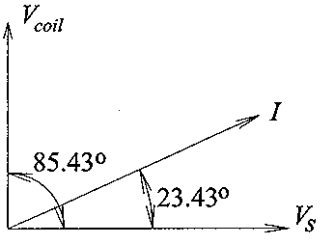


Fig. 2.38

EXAMPLE 2.4

Determine the total current drawn by the following circuit, assuming the supply voltage as reference:

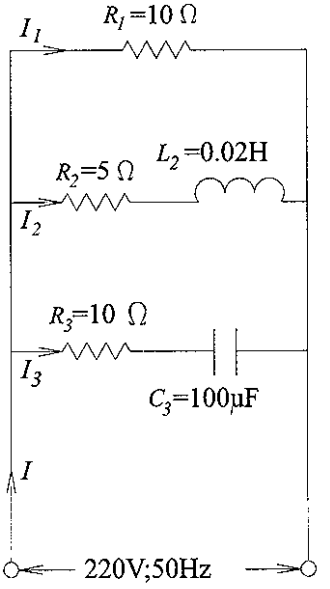


Fig. 2.39

Solution

$$Z_1 = 10 + j0 = 10\underline{/0^\circ}\ \Omega$$

$$Z_2 = 5 + j\omega L = 5 + j \times 2\pi f L = 5 + j \times 2 \times \pi \times 50 \times 0.02 = 5 + j6.28$$

$$= 8.03\underline{/51.47^\circ}\ \Omega$$

$$Z_3 = 10 - jX_C = 10 - j\frac{1}{2\pi f C} = 10 - j\frac{10^6}{2 \times \pi \times 50 \times 100} = 10 - j31.83$$

$$= 33.36\underline{/-72.56^\circ}\ \Omega$$

$$I_1 = \frac{220\underline{/0^\circ}}{10\underline{/0^\circ}} = 22\underline{/0^\circ} = 22 + j0\ \text{A}$$

$$I_2 = \frac{220\underline{/0^\circ}}{8.03\underline{/51.47^\circ}} = 27.4\underline{/-51.47^\circ} = 17.07 - j21.43\ \text{A}$$

$$I_3 = \frac{220\underline{/0^\circ}}{33.36\underline{/-72.56^\circ}} = 6.6\underline{/72.56^\circ} = 1.98 + j6.3\ \text{A}$$

$$I = I_1 + I_2 + I_3 = 22 + j0 + 17.07 - j21.43 + 1.98 + j6.3 = 41.05 - j15.13$$
$$= 43.75\underline{/-20.23^\circ}\ \text{A}$$

or, with admittances:

$$Z_1 = 10\,\underline{/0^\circ}\ \Omega$$
$$Z_2 = 8.03\,\underline{/51.47^\circ}\ \Omega$$
$$Z_3 = 33.36\,\underline{/-72.56^\circ}\ \Omega$$

$$\Rightarrow \quad Y_1 = \frac{1}{Z_1} = \frac{1}{10\underline{/0^\circ}} = 0.1\,\underline{/0^\circ} \text{ Siemens}$$

$$Y_2 = \frac{1}{8.03\underline{/51.47^\circ}} = 0.125\,\underline{/-51.47^\circ} \text{ S}$$

$$Y_3 = \frac{1}{33.36\underline{/-72.56^\circ}} = 0.03\,\underline{/72.56^\circ} \text{ S}$$

$$Y_1 + Y_2 + Y_3 = 0.1\,\underline{/0^\circ} + 0.125\,\underline{/-51.47^\circ} + 0.03\,\underline{/72.56^\circ} =$$

$$= 0.1869 - j0.0691 = 0.1992\,\underline{/-20.29^\circ} \text{ S}$$

$$\Rightarrow \quad I = 0.1992\,\underline{/-20.29^\circ} \times 220\,\underline{/0^\circ} = 43.82\,\underline{/-20.29^\circ} \text{ A}$$

(The difference between the results is due to the errors incurred by rounding the decimals.)

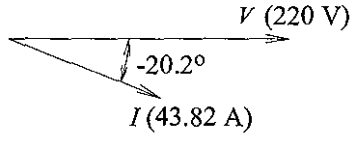


Fig. 2.40

EXAMPLE 2.5

Look at the following circuit:

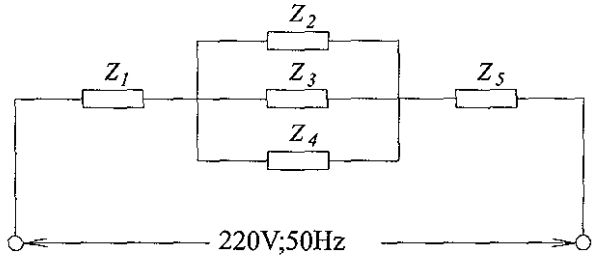


Fig. 2.41

Determine the total current drawn if you know that:

$Z_1 = 3 + j5\ \Omega$
$Z_2 = 10 + j0\ \Omega$
$Z_3 = 5 + j15\ \Omega$
$Z_4 = 10 - j30\ \Omega$
$Z_5 = 20 - j30\ \Omega$

Solution

$$Y_2 = \frac{1}{Z_2} = 0.1\,\underline{/0^\circ}\ \text{S}$$

$$Y_3 = \frac{1}{Z_3} = \frac{1}{15.81\underline{/71.57^\circ}} = 0.063\,\underline{/-71.57^\circ}\ \text{S}$$

$$Y_4 = \frac{1}{Z_4} = \frac{1}{31.62\underline{/-71.57^\circ}} = 0.0316\,\underline{/71.57^\circ}\ \text{S}$$

For the three parallel branches:

$$Y_{eq} = Y_2 + Y_3 + Y_4 = 0.13 - j0.03\ \text{S}$$

$$Z_{eq} = \frac{1}{0.13 - j0.03} = \frac{1}{0.1334\underline{/-12.99^\circ}} = 7.49\angle 12.99^\circ = 7.3 + j1.68\ \Omega$$

$$Z_{tot} = Z_1 + Z_{eq} + Z_5 = 3 + j5 + 20 - j30 + 7.3 + j1.68 = 30.3 - j23.32\ \Omega$$

$$I_{tot} = \frac{220\underline{/0^\circ}}{38.23\underline{/-37.58^\circ}} = 5.75\,\underline{/37.58^\circ}\ \text{A}$$

2.6 Summary

1 The typical mathematical **expressions for alternating currents/voltages** are:

$i = I_m \cos\omega t$
$v = V_m \cos\omega t$
where I_m, V_m – maximum values

2 The **frequency** is the **number of cycles per second**:
$f = \frac{1}{T}$, where T is the period

3 For a **sinusoidal alternating current**:
$I_{av} = 0.637 I_m$
$I_{rms} = \frac{\sqrt{2}}{2} I_m = 0.707 I_m$

4 The **passive components** in AC circuits have the following properties: **resistance**, **capacitance** and **inductance**.

For R: current and voltage are in phase

For L: current is lagging the voltage by 90° (or $\frac{\pi}{2}$ rad)

For C: current is leading the voltage by 90° (or $\frac{\pi}{2}$ rad)

For L: reactance is $X_L = 2\pi fL = \omega L$

For C: reactance is $X_C = \frac{1}{2\pi fC} = \frac{1}{\omega C}$

5 **Impedance**:

General: $Z = R \pm jX$ where R: resistance

$Z = R$: pure resistive circuit X: reactance

$Z = \pm jX$: pure reactive circuit

6 **Admittance**:

$Y = \frac{1}{Z}$

$Y = G \pm jB$ (+ for capacitive, − for inductive)

where G represents conductance and B represents susceptance

7 For **series** circuits:

– RL series: $Z = R + jX_L = R + j\omega L = R + j2\pi fL$

– RC series: $Z = R - jX_C = R - j\frac{1}{\omega C} = R - j\frac{1}{2\pi fC}$

– RLC series: $Z = R + j(X_L - X_C)$ for inductive circuit

$Z = R - j(X_C - X_L)$ for capacitive circuit

if $X_L = X_C$: resonance ($Z = R$, resistive circuit)

8 For **parallel** circuits:

- For two branches in parallel: $Z_{eq} = \frac{Z_1 Z_2}{Z_1 + Z_2}$
  Do not directly add resistance and reactance!
- For three branches in parallel: $\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}$
  or you can work with admittance.

Self-evaluation

1 A coil has an inductance of 200 mH. Calculate its inductive reactance at the following frequencies:

a. 1000 Hz

b. 100 kHz
c. 10 MHz

2 Assuming the supply voltage as reference, determine the current flowing in the coil in the circuit:
$R = 9.07\ \Omega$
$X_L = 4.21\ \Omega$

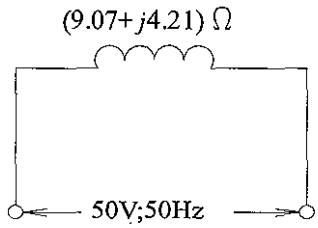


Fig. 2.42

Draw the impedance triangle for this coil.
Draw the phasor diagram for voltage and current.

3 A capacitor has a capacitance $C = 0.025\ \mu F$. Find the capacitive reactance for $\omega = 10^6\ \frac{rad}{s}$

4 A capacitor of 8 μF takes a current of 1 A when the alternating voltage applied across it is 250 V (rms). Calculate:
a. the frequency of the applied voltage
b. the resistance to be connected in series with the capacitor to reduce the current in the circuit to 0.5 A at the same frequency

5 A circuit consists of a 120 Ω resistor in parallel with a 40 μF capacitor and is connected to a 240 V, 50 Hz supply. Calculate the current drawn by the circuit from the supply.

6 A parallel network consists of three branches. If the currents in the three branches are $10\angle 30°$ A, $10\angle 0°$ A, and $10\angle 30°$ A, calculate the current flowing from the supply. Draw the phasor diagram for the currents.

7 Three branches with a resistance of 50 Ω, an inductance of 0.15 H, and a capacitance of 100 μF are connected in parallel across a 100 V, 50 Hz supply. Assume V_{supply} as reference and calculate the supply current. Draw the phasor diagram for the currents.

8 A series circuit consisting of a resistor of 20 Ω, an inductor of 0.3 H, and a capacitor of 33.8 μF is connected to a 230 V, 50 Hz supply. Calculate the value of the current in the circuit.

9 Analyse the following circuit:

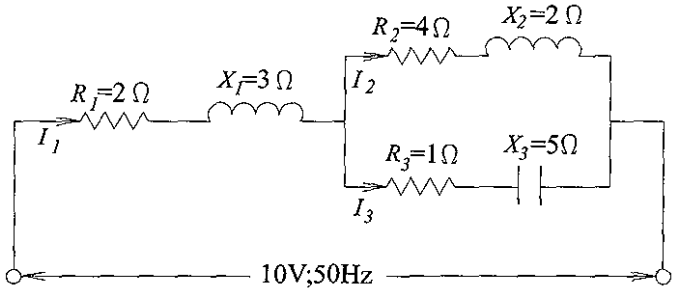


Fig. 2.43

Obtain the values of all currents and construct the phasor diagram. (Use the supply voltage as reference.)

Answers

Activity 2.1 (page 43)

a. $R_T = R_1 + R_2 + R_3$
$= 5 + 10 + 20$
$= 35\ \Omega$

b. $\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$
$\frac{1}{C_T} = \frac{1}{5} + \frac{1}{10} + \frac{1}{20}$
$\frac{1}{C_T} = \frac{7}{20}$
$C_T = \frac{20}{7} = 2.857\ \mu\text{F}$

Activity 2.2 (page 44)

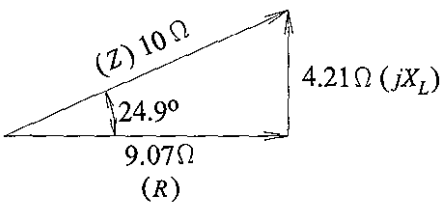
a. $\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$
$= \frac{1}{5} + \frac{1}{10} + \frac{1}{20}$
$= \frac{7}{20}$
$R_T = \frac{20}{7} = 2.857\ \Omega$

b. $C_T = C_1 + C_2 + C_3$
$= 5\ \mu\text{F} + 10\ \mu\text{F} + 20\ \mu\text{F}$
$= 35\ \mu\text{F}$

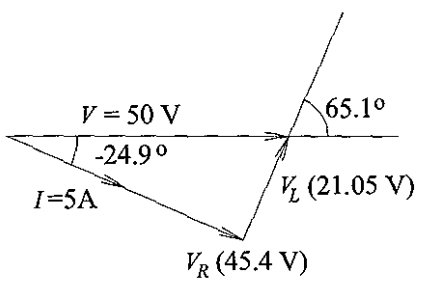
Self-evaluation (page 63)

1 1256 Ω; 1.256×10^5 Ω; 1.256×10^7 Ω

2 $I = 5\underline{/-24.9^\circ}$ A



3

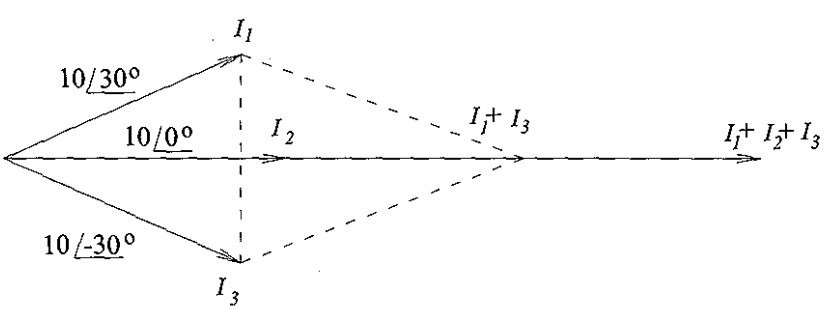


4 $X_C = 40$ Ω

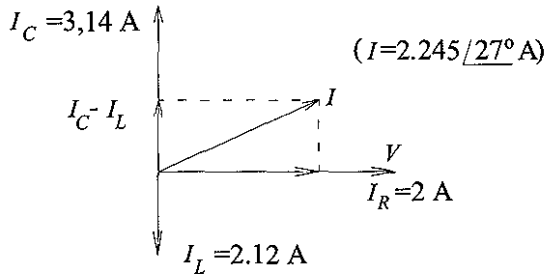
5 $f = 79.6$ Hz
$R = 433$ Ω

6 $I = 3.6\underline{/56.25^\circ}$ A

7 $I = 27.32\underline{/0^\circ}$ A

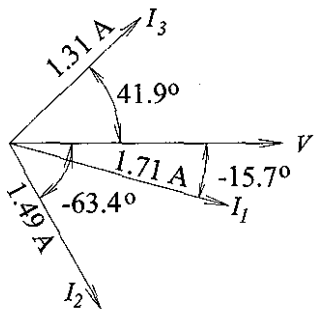


8 $I = 2.245\angle 27°$ A



9 $I = 11.5\angle 0°$ A
Observation: $X_L = X_C$; resonance.

10 $I_1 = 1.71\angle -15.7°$ A
$I_2 = 1.49\angle -63.41°$ A
$I_3 = 1.31\angle -41.9°$ A



Appendix I

▶ Dependent (controlled) sources

A dependent source is one with a value dependent on the voltage (or current) at some other point in a circuit.

These sources are sometimes called 'controlled' sources and can be either voltage or current sources.

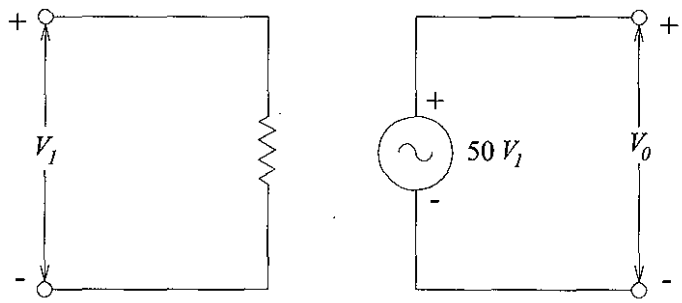
When we discuss problems, we usually refer to 'ideal', independent sources, but it helps to have some knowledge about these other sources.

The four types of dependent sources you may find in your circuits are:

- voltage-controlled voltage source
- voltage-controlled current source
- current-controlled voltage source
- current-controlled current source

These are quite easy to remember. There are current and voltage sources, both of which can be controlled by the two parameters I or V.

Let's have a look at the first type of dependant source: voltage-controlled voltage source:



Notice that the output voltage of the source (to the right of the drawing) depends on v_1 (the voltage applied on the left of the drawing).

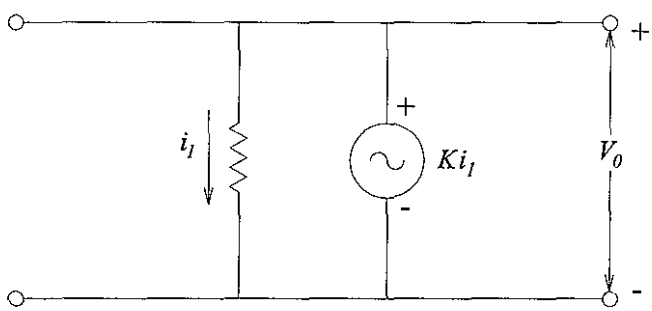
So, we can say that v_0 is 'controlled' by v_1, which, in this case, is external to the circuit containing your source.

If, for example, v_1 is 1 V then $v_0 = 50 \times 1 = 50$ V.

If v_1 is 1,5 V then $v_0 = 50 \times 1.5 = 75$ V, etc.

Note that the controlling voltage can sometimes belong to the same circuit as the controlled source.

Let's look at another example of controlled sources:



What do you see?

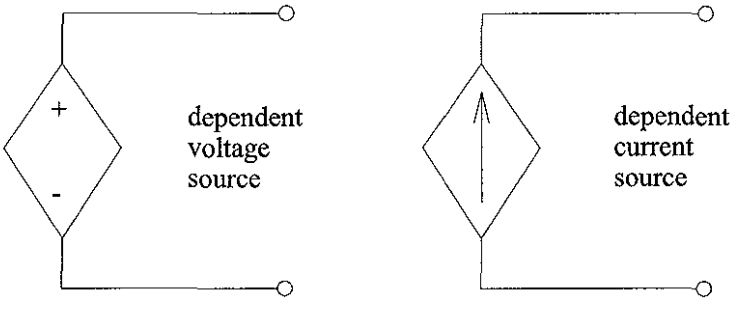
This time the controlling parameter belongs to the same circuit as the controlled source.

It should be obvious that this is a current-controlled source and that the source is a voltage source (the third type of dependent source).

K is the constant that multiplies the value (of the current) and is not dimensionless, like the 50 in the previous example.

In this example, the units are $\frac{V}{A} = \Omega$, so it is a resistance.

Other, more advanced books use a diamond symbol to represent a controlled source. This avoids confusing it with independent sources.



Appendix II

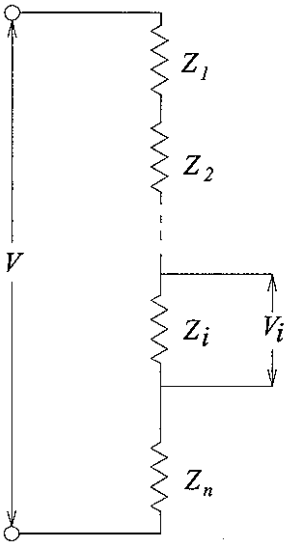
▶ Voltage and current dividers in AC

If you page back to Unit 1, you'll find the definitions for voltage and current dividers.

Previously, you studied DC circuits with resistors as loads. By now you should know that the basic principles hold for AC theory. However, for AC circuits you need to consider magnitudes and angles. This is quite easy if you use the complex numbers theory.

Since you know that the series and parallel combinations of impedances comply with the rules for resistances, it's obvious that:

In AC, the voltage divider is a circuit with '*n*' impedances in series:



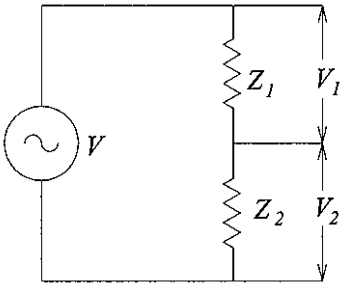
and the rule is:

$$V_i = \frac{Z_i}{\sum_1^n Z_i} V$$

For two impedances only:

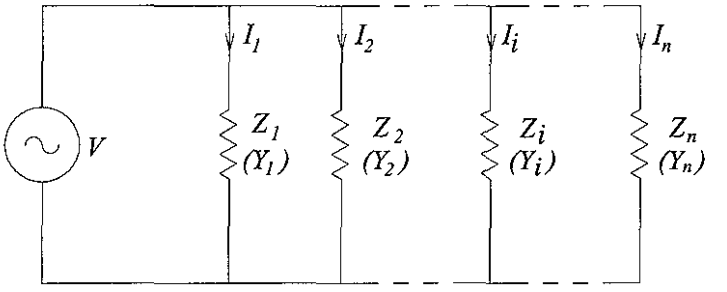
$$V_1 = \frac{Z_1}{Z_1 + Z_2} \cdot V$$

$$V_2 = \frac{Z_2}{Z_1 + Z_2} \cdot V$$



Remember that both impedances and voltages are complex numbers and you need to calculate accordingly!

In AC, the current divider is a circuit with 'n' impedances in parallel:



and the rule is:

$$I_i = \frac{V}{Z_i} = \frac{Y_i}{\sum_1^n Y_i} I = \frac{I}{\sum Y_i} \times Y_i$$

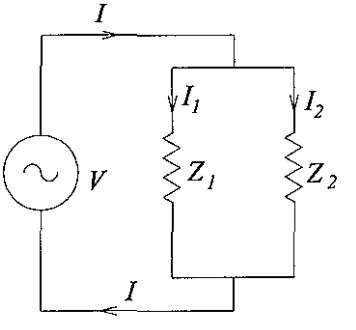
where Y_i (admittance) $= \frac{1}{Z_i}$ S

For two impedances in parallel:

$$I_1 = \frac{Y_1}{Y_1 + Y_2} I = \frac{Z_2}{Z_1 + Z_2} I$$

and

$$I_2 = \frac{Y_2}{Y_1 + Y_2} I = \frac{Z_1}{Z_1 + Z_2} I$$



These two rules are useful for solving problems with series and/or parallel combinations. You'll also come across them when you study electrical machines.

unit 3

AC Theory (Part II)

OUTCOMES

After studying this unit, you should be able to:

- understand the behaviour of any basic load (*R*, *L*, *C*) in terms of power and energy;
- calculate power – active, reactive and apparent – in AC networks;
- determine the power factor for any ac circuit;
- adjust the power factor of a given circuit to a desired value;
- understand the power triangle (P, S, Q) and its relationship to the PF for various types of load.

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3.1 Introduction

In Unit 1 we introduced power in a DC circuit. In this unit we will introduce you to power in an AC circuit.

Power is defined as the **rate** of doing work.

In circuits, power comes from the supply (battery, generator) and is transmitted to the load. The load, in turn, gives out the desired form of energy. Energy is defined as the **capacity** to do work.

For example, a resistor gives us heat (heat energy), a motor gives us the shaft mechanical motion (mechanical energy) and a pump can raise the water level (potential energy).

Up to now you only know that electrical power is measured in watts (W) and that $1\ \mathrm{W} = 1\frac{\mathrm{J}}{\mathrm{s}}$. While this is enough for the DC resistive circuits you have already studied, things can get a little more complicated for AC.

In this chapter, you'll learn more about power. So let's start!

3.2 Power in AC circuits

3.2.1. AC power in resistive circuits

Do you remember that, when we had a look at the rms concept for AC theory, we also discussed power? If you don't, page back to the section on 'Useful concepts' in Unit 2. We found the effective (or root-mean-square) value of an alternating current from the 'average' power dissipated in a resistor.

In Unit 2 we also found that, for resistance, the current was in phase with the voltage.

Keeping in mind that any combination of resistors can be equated with a single (equivalent) resistance, this means that the same applies for any resistive circuit.

We also know that instantaneous power is the product of instantaneous values of the voltage **across** and the current **through** a resistive network.

If we plot the curves for voltage, current and power of a resistor, we get:

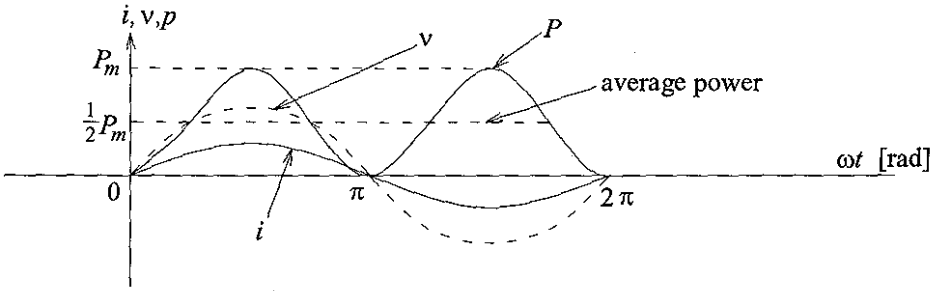

Fig. 3.1

The expression for instantaneous power can be written:

$p = vi = i^2R = \frac{v^2}{R}$ (where i, v are instantaneous values)

But this doesn't help much. A more practical and useful quantity is the average power of the resistor, which can be written:

$P_{av} = V_{rms}I_{rms}$ in watts (W)

$P_{av} = I^2_{rms}R$ W, or

$P_{av} = \frac{V^2_{rms}}{R}$ W

TAKE NOTE

Remember that $I_{rms} = 0.707\ I_{max}$ and $V_{rms} = 0.707\ V_{max}$.

The phasor diagram for the resistor was:

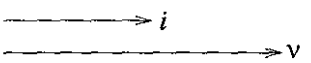

Fig. 3.2

No angle exists between current and voltage and, in this case, all the power will be **active** power.

So, $P = VI$

and the cosine of the angle between the current and the voltage is $\cos 0° = 1$. You'll see later that the cosine of the angle between current and voltage is called the 'power factor'.

Remember, for a pure resistive circuit:

- the angle between current and voltage is 0°;

- the cosine of this angle (power factor) is 1; and
- all the power in the circuit is active power.

There is an analogy between efficiency and power factor. Both have values from 0 to 1 (or 0–100%) and the efficiency shows how much input power can be used at the output of the element

$$\eta = \frac{P_{output}}{P_{input}} \times 100\%$$

Large power consumers are concerned with both the active and the total power. The ratio of these two powers gives the power factor of your network. In ideal conditions, we would like this ratio to be near one. This will become clearer as we work through this unit.

Remember that neither the PF (Power Factor) nor the efficiency of a network can ever be bigger than one (or 100%).

3.2.2 AC power in an inductance

Do you remember the phasor diagram for voltage and current through an inductance?

In Unit 2, we saw that, when a sinusoidal voltage is applied across a pure inductance, current lags voltage by $\frac{\pi}{2}$(90°).

In Figure 3.3 The current lags the voltage by 90°. So the angle between current and voltage (ϕ) is 90°, and the power factor ($\cos\phi$) is 0.

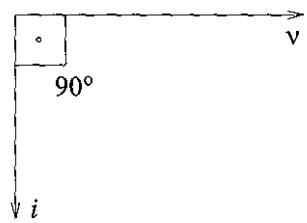

Fig. 3.3

You will never find 'pure' inductances in physical circuits – all the coils and windings present resistance. In this unit, we'll consider the case of a 'pure' inductor. After that, we can study the coils as a combination of the inductor and resistance.

There is a 'lagging' power factor because the current **lags** behind the voltage.

Let's look at the typical waveforms:

$$v_L = V_m \cos \omega t$$
$$i_L = I_m \cos\left(\omega t - \frac{\pi}{2}\right)$$

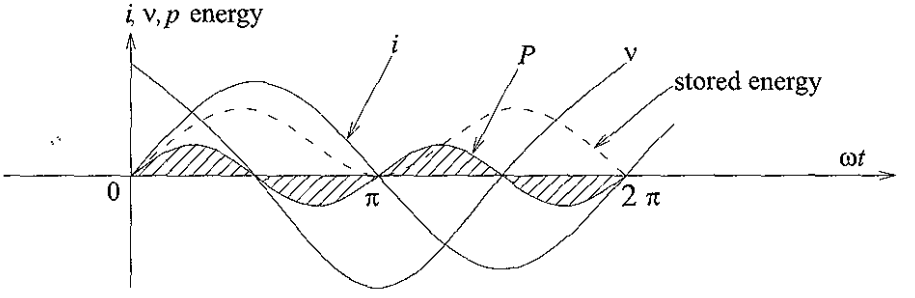

Fig. 3.4

The power curve crosses the zero axis every time the voltage or the current is zero. When both the voltage and the current are positive or negative, the power is positive. But when one is positive and the other is negative, the power is negative.

What is 'negative' power? We know that the energy stored in the magnetic field of an inductance is $\frac{1}{2}Li^2$. This means that, during the cycle, when power is positive, energy builds up in the field. When the power is negative the energy is returned to the source.

This has the immediate effect that the average power is zero because all the energy received from the source in a half cycle is returned to the source in the next half cycle.

Can you see that the average power in a pure inductance is always zero since the positive power values are cancelled out by the negative power values over one full cycle?

Instantaneous energy is:

$$w = \tfrac{1}{2}Li^2 = \tfrac{1}{2}L(I_m \sin \omega t)^2 = \tfrac{1}{2}LI_m^2 \sin^2 \omega t$$

From this formula it is clear that the energy has no negative values (constant > 0; I_m is squared; $\sin \omega t$ is squared), the zeros are at π, 2π ... etc., and the maximums are at $\frac{\pi}{2}$, $\frac{3\pi}{2}$, ..., etc.

I added the curve for energy in fig. 3.4.

The power curve is a double-frequency curve. One cycle = π.

Even if we cannot identify active power in an inductance, we still know a power associated with it: Q – the reactive power.

3.2.3 AC power in a capacitance

Have a look at the phasor diagram for capacitance which you studied in Unit 2. In this case, the current **leads** the voltage by 90°. So, the angle between current and voltage is 90° and $\cos\phi$, the power factor, will be zero.

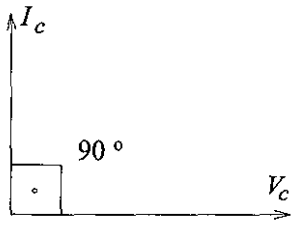

Fig. 3.5

We can now say that the power factor is a 'leading' power factor, since the current **leads** the voltage. Look again at the waveforms:

$$v_C = V_m \cos\omega t$$

$$i_C = I_m \cos\left(\omega t + \frac{\pi}{2}\right)$$

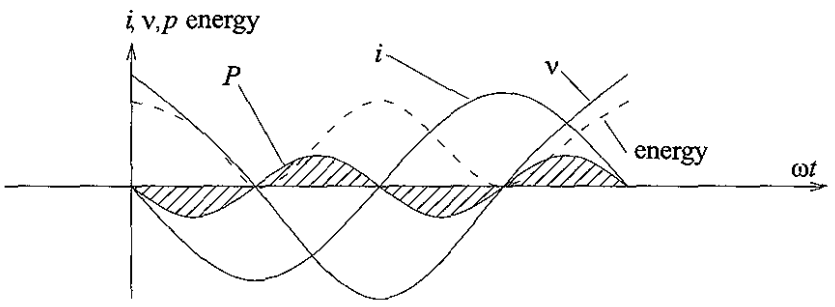

Fig. 3.6

When an alternating voltage is applied to a capacitance, the average power supplied is zero and the power represents a double-frequency wave.

When it comes to a capacitor, it is important that the energy is maximum when the voltage is maximum. This is different from inductance, where energy is maximum when the current is maximum.

Do you remember that energy in a capacitor is $\frac{1}{2}CV^2$? Energy is returned to the source in alternate half-cycles of the voltage. The capacitance consumes no energy.

From now on the product $v \times i$ will be measured in volt-amperes instead of watts.

Here's a simple method to help you remember when the power factor is leading or lagging: Think of the word 'CIVIL'. In the case of capacitance **C**, the current **I** is in front of (or leads) the voltage **V**, so the power will lead. Similarly, **V** leads the current **I** in inductance **L**, so the current and the power factor will be lagging.

EXAMPLE 3.1

The instantaneous current in a pure inductance of 8 H is given as:

$$i = 12 \cos\left(314t - \frac{\pi}{2}\right)$$

Determine:

a. the rms value of the current
b. the maximum energy produced in the magnetic field
c. the frequency

Solution

a. $I_{rms} = \frac{I_m}{\sqrt{2}} = \frac{12}{\sqrt{2}} = 8.49$ A

b. $W_{\max} = \frac{1}{2}LI^2_{\max} = \frac{1}{2} \times 8 \times 12^2 = 576$ J

c. $f = \frac{\omega}{2\pi}$

$\omega = 314\frac{\text{rad}}{\text{s}}$, from the expression above, and then

$f = \frac{314}{2\pi} \cong 50$ Hz

What is the maximum instantaneous voltage across the inductance in the example?

$$I_m = \frac{V_m}{\omega L} \Rightarrow V_m = \omega L I_m = 314 \times 8 \times 12 = 30\ 144 \text{ V}$$

EXAMPLE 3.2

If a capacitor needs to be connected across a voltage of 30 144 V, what should its capacitance be to receive a maximum energy of 576 J?

Solution

$$\frac{1}{2}CV^2_{\max} = 576 \Rightarrow C = \frac{576 \times 2}{30\,144^2} = 0.000001267 \text{ F} = 1.27\ \mu\text{F}$$

TAKE NOTE Get used to seeing small values for capacitors and try to use the prefixes for SI units when you write your results. Instead of using F here, we used μF (microfarad).

3.2.4 AC power in impedances

▶ **Power in a coil (RL series)**

We know that a coil consists of a resistance in series with an inductance.

For a combination of RL series, we know that the phasor diagram looked like this:

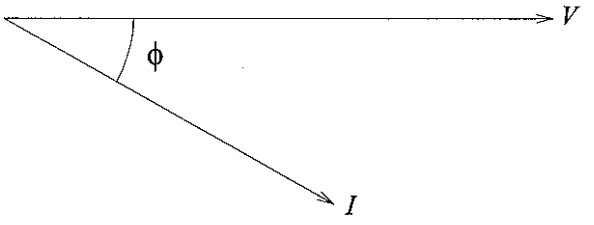

Fig. 3.7

or, more detailed now:

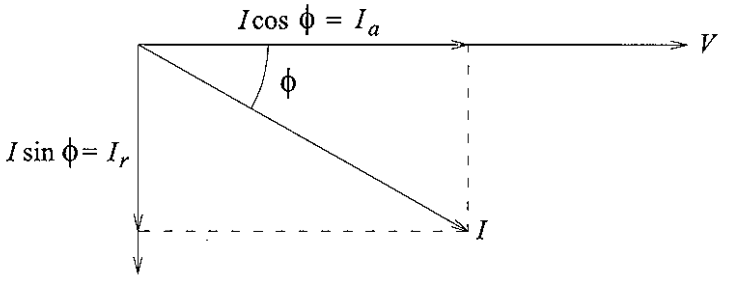

Fig. 3.8

I was projected on the direction of V and on the right-angle direction to V.

$\boldsymbol{I}\mathbf{cos}\boldsymbol{\phi}$ is the 'in-phase' component of the current (in phase with the voltage!) or the **active** current. This is the only power producing part of the total current.

We can calculate the power by multiplying this value by the voltage:

$$P = VI\cos\phi$$

What happens to the other part of the current? Remember that the power in the induction part, as we know it, is zero. $\boldsymbol{I}\mathbf{sin}\boldsymbol{\phi}$ is called the 'quadrature component' or the **reactive** current.

Although $\boldsymbol{I}\mathbf{sin}\boldsymbol{\phi}$ makes no active power contribution, when it is combined with voltage, it produces 'reactive' power:

$$Q = VI\sin\phi$$

Q is measured in VAr (volt-amperes reactive).

Since the product of V and I is another type of power, $|S| = VI$, measured in VA (volt-amperes):

$$S^2 = P^2 + Q^2$$

S is called the 'apparent power'.

The three powers can be represented in a right-angle triangle called the 'power triangle':

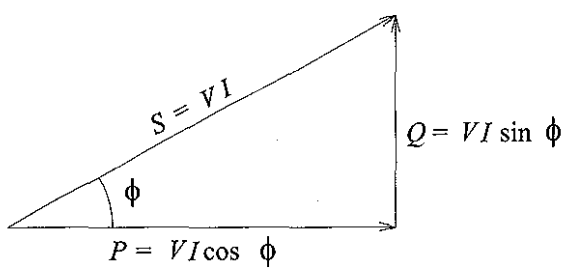

Fig. 3.9

The apparent power is the hypotenuse and the active and reactive powers are the two other sides of the right-angle power triangle.

The equation $P = VI\cos\phi$, which shows the product of the voltage by the in-phase current, is often used in power calculations. Since we are mainly interested in active power, this equation holds true for a single component, a network, and for a whole network.

Have you noticed anything interesting? If you go back to Unit 2, you'll see that your angle between current and voltage is the (–) angle from the impedance triangle:

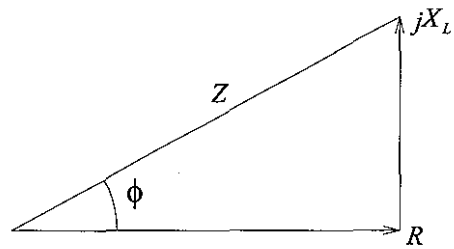

Fig. 3.10

The same angle appears in the power triangle, so we can say that the impedance and the power triangles are similar.

Later in this unit you'll find out that this angle is very important, since **cosϕ** is the power factor of this system.

EXAMPLE 3

A 100 V, 50 Hz voltage is applied across a coil with $R = 20\ \Omega$ and $X_L = 15\ \Omega$. Calculate:

a current
b active power
c apparent power
d reactive power
e power factor.

Solution

a. $Z = 20 + j15 = 25\underline{/36.87^\circ}\ \Omega$
$I = \frac{V}{Z} = \frac{100\underline{/0^\circ}}{25\underline{/36.87^\circ}} = 4\underline{/-36.87^\circ}$ A

b. $P = VI\cos\phi = 100 \cdot 4 \cdot \cos 36.87^\circ \cong 320$ W
(or, in our case, because we know R: $P = RI^2 = 20 \times 4^2 = 320$ W)

c. $|S| = VI = 100 \times 4 = 400$ VA

d. $Q = \sqrt{S^2 - P^2} = \sqrt{400^2 - 320^2} = 240$ VAr

e. $\cos\phi = \cos 36.87^\circ = 0.80$

▶ Power in a RLC series circuit

Let's have a look at a RLC series circuit.

You already know that any combination of loads can be simplified to a RL series circuit, a RC series circuit, or a R circuit. Since R, L and C cover (function of the values of the reactances) any of these possibilities, it means that after we discuss this case you'll be able to calculate practically the powers for any circuit.

For a RLC series circuit:

- $X_L > X_C \Rightarrow Z \angle\theta^\circ$
- $X_C > X_L \Rightarrow Z \angle -\theta^\circ$
- $X_L = X_C \Rightarrow Z \angle 0^\circ$; resonance (like a resistor)

For the above cases, the angles between currents and voltages are:

- $-\theta^\circ$
- θ°
- 0°

We can calculate further the real power (P), apparent (S) and reactive power (Q) using:

$$P = VI\cos\phi$$
$$Q = VI\sin\phi$$
$$|S| = VI = \sqrt{P^2 + Q^2}$$

EXAMPLE 3.4

An AC voltage of 100 V is applied to a series circuit of $R = 20\ \Omega$, $X_L = 15\ \Omega$, and $X_c = 30\ \Omega$. Calculate:

a. current
b. active power
c. reactive power
d. apparent power

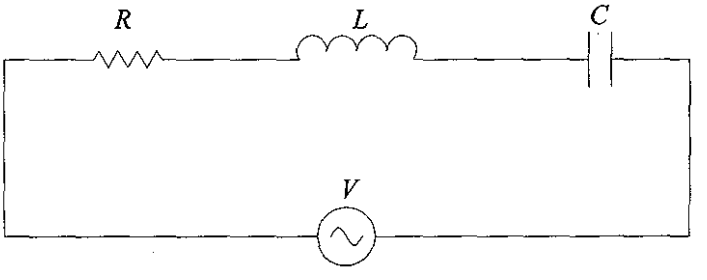

Fig. 3.11

Solution

Assuming V as the reference:

a. $Z = 20 + j15 - j30 = 20 - j15 = 25\,\underline{/-36.87^\circ}\,\Omega$
$I = \frac{V\angle 0^\circ}{Z} = \frac{100\underline{/0^\circ}}{25\underline{/-36.87^\circ}} = 4\angle 36.87^\circ$ A

b. $P = VI\cos\phi = 100 \times 4 \times \cos(-36.87^\circ) = 320$ W
(or $P = RI^2 = 20 \times 4^2 = 320$ W)

c. $Q = VI\sin\phi = 100 \times 4 \times \sin(-36.97^\circ) = -240$ VAr

TAKE NOTE

Keep the sign of Q in mind because, if you have to calculate sums of reactive powers (for example for finding out the total reactive power in a circuit), you'll have to know how to do the algebraic sum!

d. $S = P + jQ = 320 - j240 = 400\underline{/-36.87^\circ}$ VA

Let's work out the solution for $|S| = VI = 100 \times 4 = 400$ VA

OK! But where is the angle? Now try to calculate $S = VI = 100\underline{/0^\circ} \times 4\underline{/36.87^\circ}$ VA

What did you get?

The right modulus, but not the right angle?

Remember, when the power is written in polar form, the angle (sign included!) is the same as the one for impedance.

Where did you go wrong?

Let's try to recalculate:

$$S = VI^*$$

where I^* is the complex conjugate of I (for example if $I = \text{modulus}\,\underline{/\theta^\circ} \Rightarrow I^* = \text{modulus}\,\underline{/-\theta^\circ}$, or $I = a + jb \Rightarrow I^* = a - jb$) $\Rightarrow$
$S = 100\underline{/0^\circ} \times 4\underline{/-36.87^\circ} = 400\underline{/-36.87^\circ}$ VA

Have you got it right? But you still need to know why you have to use the complex conjugate of I.

Let's suppose we choose an arbitrary reference and we know the voltage for the circuit is $V = 220\underline{/45^\circ}$ V. If we calculate the current, we get a value of $I = 10\underline{/-15^\circ}$ A.

If we work correctly (and we haven't changed the reference!), the phasor diagram should look like this:

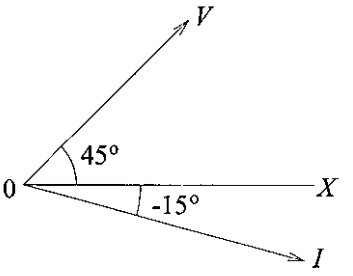

Fig. 3.12

It's clear that the angle between the voltage and the current is 60° (the phase angle). But $60° = 45° - (-15°)$, and not $45° + (-15°)$ which we would have got for the angle by multiplying $V \times I$.

This is why we have to consider the complex conjugate of I.

In the example, the power would be:

$$S = 220\underline{/45°} \times \underset{\substack{\downarrow \\ I^*}}{10\underline{/15°}} = 2\ 200\underline{/60°}\ \text{VA}$$

We have calculated the modulus of S and we can work out power factor, reactive power, etc. Why must we remember the formula?

Try to express S in rectangular form:

$$S = 2\ 200\underline{/60°} = 1\ 100 + j1\ 905\ \text{VA}$$

From this form we can identify the active (real) power and the reactive power:

$$P = 1\ 100\ \text{W};\ Q = 1\ 905\ \text{VAr}$$

We can also calculate the power factor (0.5).

As you can see, this a much faster method and it can be really useful when it comes to more complicated circuits.

Do you remember how to calculate the equivalent impedance for different series and parallel combinations of impedances? Every time you have a more complicated circuit, first try to find out the equivalent impedance and then calculate the power according to the formula.

EXAMPLE 3.5

Here is a circuit:

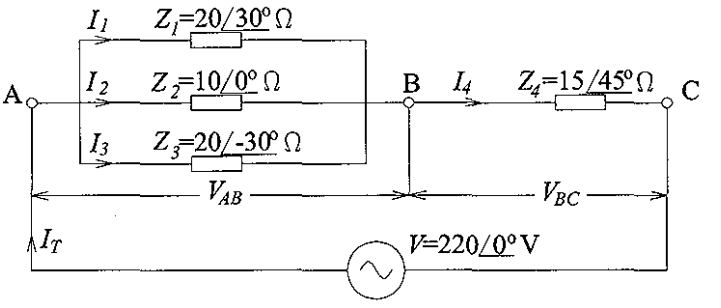

Fig. 3.13

From the information given in the figure, calculate:

1. Z_{eq} – the total equivalent impedance of the circuit
2. I_{total} – the current flowing through the circuit, I_T
3. The voltages V_{AB} and V_{BC}
4. The currents flowing through each impedance (I_1, I_2, I_3, I_4)
5. The active powers dissipated in each impedance, and the total active power for the circuit
6. The reactive powers dissipated in each impedance, and the total reactive power for the circuit
7. The apparent power for the circuit

Solution

TAKE NOTE

Z_1 has an inductive character (R in series with L), Z_2 is a resistor, and Z_3 has a capacitive character (R in series with C). These are all the types of loads we have studied (R, L, C). Look carefully at how each type of load influences the calculations.

1. $$Z_{eq} = \frac{Z_1 Z_2 Z_3}{Z_1 Z_2 + Z_1 Z_3 + Z_2 Z_3} + Z_4 =$$

$$= \frac{20\underline{/30^\circ} \times 10\underline{/0^\circ} \times 20\underline{/-30^\circ}}{20\underline{/30^\circ} \times 10\underline{/0^\circ} + 20\underline{/30^\circ} \times 20\underline{/-30^\circ} + 10\underline{/0^\circ} \times 20\underline{/-30^\circ}} + 15\underline{/45^\circ} =$$

$$= \frac{4\,000\underline{/0^\circ}}{200\underline{/30^\circ} + 400\underline{/0^\circ} + 200\underline{/-30^\circ}} + 15\underline{/45^\circ} = \frac{4\,000\underline{/0^\circ}}{746.4 + j0} + 15\underline{/45^\circ} =$$

$$= \frac{4\,000\underline{/0^\circ}}{746.4\underline{/0^\circ}} + 15\underline{/45^\circ} = 5.36\underline{/0^\circ} + 15\underline{/45^\circ} = 15.97 + j10.6 =$$

$$= 19.17\underline{/33.57^\circ}\ \Omega$$

or:

$$Y_1 = \frac{1}{Z_1} = \frac{1}{20\underline{/30^\circ}} = 0.05\underline{/-30^\circ}\ S$$

$$Y_2 = \frac{1}{Z_2} = \frac{1}{10\underline{/0^\circ}} = 0.1\underline{/0^\circ}\ S$$

$$Y_3 = \frac{1}{Z_3} = \frac{1}{20\underline{/-30^\circ}} = 0.05\underline{/30^\circ}\ S$$

$$Y_1 + Y_2 + Y_3 = 0{,}05\underline{/-30^\circ} + 0.1\underline{/0^\circ} + 0.05\underline{/30^\circ} = 0.186602 + j0 =$$
$$= 0.186602\ S = Y_{eq\text{A–B}}$$

$$Z_{eq\text{A–B}} = \frac{1}{Y_{eq\text{A–B}}} = 5.36\underline{/0^\circ}\ \Omega$$

$$Z_{eq} = 5.36\underline{/0^\circ} + 15\underline{/45^\circ} = 15.97 + j10.6 = 19.17\underline{/33.57^\circ}\ \Omega$$

Whatever method you use, you'll get the same result if you work correctly.

2. Knowing the supply voltage and the equivalent impedance of the circuit:

$$I_T = \frac{V}{Z_{eq}} = \frac{220\underline{/0^\circ}}{19.17\underline{/33.57^\circ}} = 11.48\underline{/-33.57^\circ}\ \text{A}$$

3. $V_{AB} = Z_{AB} \times I_T = 5.36\underline{/0^\circ} \times 11.48\underline{/-33.57^\circ} = 61.53\underline{/-33.56^\circ}$ V

$$V_{BC} = Z_{BC} \times I_T = 15\underline{/45^\circ} \times 11.48\underline{/-33.57^\circ} = 172.2\underline{/11.43^\circ}\ \text{V}$$

You can now check:

$$V_{AV} + V_{BC} = 61.53\underline{/-33.57^\circ} + 172.2\underline{/11.43^\circ}$$
$$= 220.05 + j0.1 \approx 220\underline{/0^\circ}\ \text{V}$$

4. $I_1 = \frac{V_{AB}}{Z_1} = \frac{61.53\underline{/-33.57^\circ}}{20\underline{/30^\circ}} = 3.08\underline{/-63.57^\circ}$ A

$$I_2 = \frac{V_{AB}}{Z_2} = \frac{61.53\underline{/-33.57^\circ}}{10\underline{/0^\circ}} = 6.15\underline{/-33.57^\circ}\ \text{A}$$

$$I_3 = \frac{V_{AB}}{Z_3} = \frac{61.53\underline{/-33.57^\circ}}{20\underline{/-30^\circ}} = 3.08\underline{/-3.57^\circ}\ \text{A}$$

Again we can check:

$$I_1 + I_2 + I_3 = 9.57 - j6.35 = 11.485\underline{/-33.565^\circ}\ \text{A}$$

And $I_4 = I_T = 11.48\underline{/-33.57^\circ}$ A

5. We know that the formula for active power is $P = VI\cos\phi$

   Replace in the formula the respective values for each impedance ⇒:

   $P_1 = 61.53 \times 3.08 \times \cos 30 = 164.12$ W

   $P_2 = 61.53 \times 6.15 \times \cos 0 = 378.41$ W

   $P_3 = 61.53 \times 3.08 \times \cos(-30) = 164.12$ W

   $P_4 = 172.2 \times 11.48 \times \cos 45 = 1\,397.85$ W

   The total active power will be:

   $P_1 + P_2 + P_3 + P_4 = 2\,104.5$ W

6. The formula for reactive power is $Q = VI\sin\phi$

   Again, by replacing the values we get:

   $Q_1 = 61.53 \times 3.08 \times \sin 30 = 94.76$ VAr

   $Q_2 = 61.53 \times 6.15 \times \sin 0 = 0$

   $Q_3 = 61.53 \times 3.08 \times \sin(-30) = -\ 94.76$ VAr

   Remember to keep the minus sign here!

   $Q_4 = 172.2 \times 11.48 \times \sin 45 = 1\,397.84$ VAr

   $Q_1 + Q_2 + Q_3 + Q_4 = 1\,397.84$ VAr

7. For the total power remember:

   $$S = VI^* = 220\underline{/0^\circ} \times 11.48\underline{/33.57^\circ} = 2\,525.6\underline{/33.57^\circ} =$$

   $$= 2\,104.36 \qquad\qquad + j1\,396.54 \text{ VA}$$

   $$\downarrow \qquad\qquad\qquad \downarrow$$

   $$\approx P_1 + P_2 + P_3 + P_4 \qquad\qquad \approx Q_1 + Q_2 + Q_3 + Q_4$$

   (checked!) (checked!)

   You could have written $S = P + jQ$, because we have already calculated P and Q for the circuit. But it's better to check that you have worked correctly.

Now, let's practice power calculations:

$$S_1 = V_1I_1^* = 61.53\underline{/-33.57^\circ} \times 3.08\underline{/63.57^\circ} = 189.51\underline{/30^\circ} \text{ VA}$$
$$S_2 = V_2I_2^* = 61.53\underline{/-33.57^\circ} \times 6.15\underline{/33.57^\circ} = 378.41\underline{/0^\circ} \text{ VA}$$
$$S_3 = V_3I_3^* = 61.53\underline{/-33.57^\circ} \times 3.08\underline{/3.57^\circ} = 189.51\underline{/-30^\circ} \text{ VA}$$
$$S_4 = V_4I_4^* = 172.2\underline{/11.43^\circ} \times 11.48\underline{/33.57^\circ} = 1\,976.86\underline{/45^\circ} \text{ VA}$$

If we write in rectangular coordinates:

$S_1 = 164.12 + j94.76$ VA $= P_1 + jQ_1$ (checked!)

$S_2 = 378.41 + j0$ VA $= P_2 + 0$ (checked!)

$S_3 = 164.12 - j94.76$ VA $= P_3 - jQ_3$ (checked!)

$S_4 = 1\,397.85 + j1\,397.85$ VA $= P_4 + jQ_4$ (checked!)

This means that we've also checked all the partial results for power!

And finally:

$$S_1 + S_2 + S_3 + S_4 = 2\,104.5 + j1\,397.84 \cong S_{total}$$

Obviously, this last part is an alternative solution for solving S_{total}

Activity 3.1

Calculate the **average** power of a network that has a supply voltage of 100 sin(ωt + 105°) V and draws a current of 4 sin(ωt − 15°) A
Is the network delivering or absorbing power?

3.3 Power factor

From here on we're going to use S as the **modulus**, rather than $|S|$.

3.3.1 Definition and generalities

The only difference between the equation for the amount of apparent power and the one for amount of real power is the factor **cosϕ**:

$$S = VI$$
$$P = VI\cos\phi$$

The cosine of the angle ϕ in the power triangle, or $\cos\phi$, has particular significance. It is called the **power factor** or **PF**.

We can write:

$$\text{PF} = \cos\phi$$
$$P = S \times \cos\phi = S \times \text{PF}$$

and

$$\cos\phi = \text{PF} = \frac{P}{S}$$

or

$$S = \frac{P}{\cos\phi}$$

The angle ϕ is also the phase angle between voltage and current in an element, branch or network.

The angle is the same as the one found in the impedance diagram.

TAKE NOTE

PF is **never** greater than one.

EXAMPLE 3.6

A single-phase, 200 V supply is connected to a single-phase motor delivering 2 kW at a power factor of 0.6. Also connected across the supply is an inductive load with impedance $Z = 15 + j20\ \Omega$. The efficiency of the motor is 91%. Calculate the power factor as seen from the supply.

Solution

First sketch the circuit using the given information:

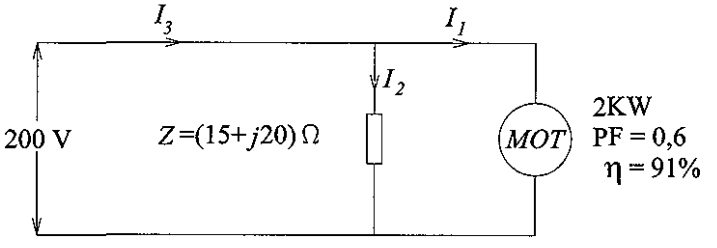

Fig. 3.14

Again, we must assume a reference. Let's say we choose V_{supply} as the reference.

Now, for the motor:

$$P_{in} = \frac{P_{out}}{\eta} = \frac{2\,000}{0.91} = 2\,198\,\text{W}$$

$$But\ P_{in} = VI_1 \cos\phi \Rightarrow$$

$$I_1 = \frac{P_{in}}{V\cos\phi} = \frac{2\,198}{200 \times 0.6} = 18.3\,\text{A}$$

$$\cos\phi = 0.6 \Rightarrow \phi = \pm 53.1°$$

The motor has an inductive characteristic, so we'll have to choose the negative angle from our values.

Remember that an inductive type of load means the power factor lags!

$$\Rightarrow I_1 = 18.3\underline{/-53.1^\circ} = 11 - j14.6 \text{ A}$$

For the inductive load:

$$Z = 15 + j20 = 25\underline{/53^\circ}\ \Omega$$

$$I_2 = \frac{200\underline{/0^\circ}}{25\underline{/53^\circ}} = 8\underline{/-53^\circ} = 4.8 - j6.4 \text{ A}$$

So the total current is:

$$I_3 = I_1 + I_2 = 11 - j14.6 + 4.8 - j6.4 = 15.8 - j21 = 26.3\underline{/-53^\circ} \text{ A}$$

There are a few things that need to be highlighted here...

Did you notice that both the loads had a similar PF? (angle $\approx 53^\circ \Rightarrow$ same PF ≈ 0.6)

It is logical that the overall power factor will be the same – i.e. equal PFs $\Rightarrow$ the total PF is equal to the PF of any load. That's what we got! The angle for the total current was -53°, so the PF for the whole circuit is again 0.6.

Did you notice that we didn't always stick to the 'rounding' rules? Sometimes you need to use your common sense to write down results and not follow the 'rules' word by word.

Let's look at how the current flowing through the **impedance**, I_2, can be calculated:

$$I_2 = \frac{200\underline{/0^\circ}}{25\underline{/53^\circ}} = 8\underline{/-53^\circ} \text{ A}$$

According to the rules of mathematics, when we use the rectangular form we do not have to write any decimals. Do you agree?

If we calculate:

$$8\underline{/-53^\circ} = 4.814\ldots - j6.389\ldots,$$

theoretically we should write:

$$8\underline{/-53^\circ} = 5 - j6!$$

Now, try this short calculation:

$$5^2 + 6^2 = 25 + 36 = 61$$
$$\sqrt{61} = 7.81 \text{ A}$$

If we choose to use only one decimal, we'll get:

$4.8 - j6.4$
$4.8^2 + 6.4^2 = 64$
$\sqrt{64} = 8,$

which is the current you really used in your calculations.

These issues are debatable and, unfortunately, nobody can give you a set of standard rules. These 'feelings' of significance usually come with practise.

If you find yourself in any doubt about, or when you don't trust yourself to choose the number of digits to work with (and be able to explain why!), **stick to the rules.**

In the example, if we had worked strictly according to the rules, we would have obtained an error of :

$$\varepsilon = \frac{\Delta x}{x} = 2.5\%$$

But for the active component:

$\varepsilon_a \approx 4\%$
$\varepsilon_r \approx 6\%$

If we accept the errors for current and for the active component, the error for the **reactive** part is already unacceptable. What we need to establish is how this will affect us. What was the purpose of solving the problem? Sometimes it matters, sometimes not. This brings us to the next section: the significance of power factor.

3.3.2 Significance of power factor

In the previous example, we worked out that the motor gets approximately 2 200 W from a supply of 200 V which delivers a current of 18.3 A, at a power factor of 0.6 (the phase angle being 53.1°).

Let's assume that the phase angle is only 20° and see what happens:

$$I_{new} = \frac{P}{V\cos\phi_{new}} = \frac{2\,200}{200 \times \cos 20} = 11.7 \text{ A}$$

What happened?

We still have the same real power supplied by our 200 V source, but the current decreased.

Why?

The initial current was more than 18 A and the power factor was 0.6 (which is a poor PF!). By changing only the PF, the current dropped to less than 12 A.

cos 20 = 0.94 (a very good power factor!)

So what?

Remember that the wires need to be larger to carry 18 A instead of 12 A, and that the resistive losses are proportional to the square of the current!

It's clearly uneconomical to have a poor power factor in this installation.

The high investment cost incurred (cables, transformers, etc.) means that the power company (for example, ESKOM) will charge you more on a monthly basis than they would if you had a good power factor.

So how can we improve the power factor of the consumer?

Activity 3.2

An electrical motor operates at 240 V. The motor absorbs an average power of 8 kW at a lagging power factor of 0.8. Calculate the complex power of the load and the impedance of the load.

3.3.3 Power factor correction

Let's look again at the example 3.6 on page 90.

The current was lagging (behind) the voltage by an angle of (–)53°. If we draw a phasor diagram for this situation, it will look like this:

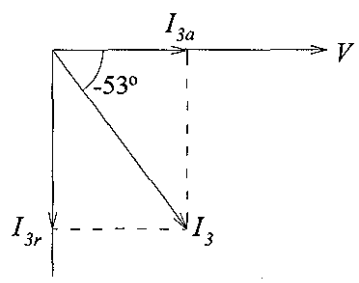

Fig. 3.15

Remember, we assumed the voltage as reference, so we've drawn it on the real axis!

The two projections of I_3 on the real and imaginary axes are I_{3a} (active component) and I_{3r} (reactive component).

The question now is: can we draw a leading current to reduce the reactive component of I_3?

The angle by which the current lags the voltage will decrease and the cosine of the angle will increase. In other words, we 'improve' or 'correct' the power factor. Ideally we would like a power factor of 1, which would mean that $S = P$, making all the power in the circuit active!

Since this is not practically possible, we aim for a power factor of approximately one. A PF of 0.90 – 0.95 is considered really good.

Let's say we would like a power factor of 0.92 for our circuit. What can we do?

Remember, if we can draw a leading current that reduces the reactive component of I_3, we will get a better PF.

The question is how?

Capacitors **draw** a leading current from the supply, so we'll connect a capacitor to our circuit:

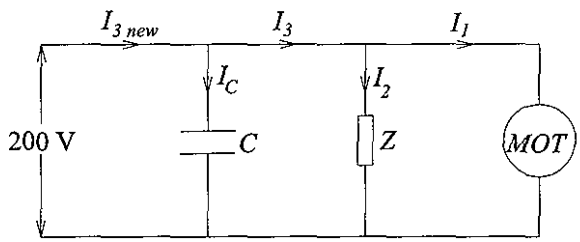

Fig. 3.16

Revise the section on current dividers!

EXAMPLE 3.7

What capacitor must I choose to obtain a power factor of 0.92?

Solution

To get a PF of 0.92, the angle between current and voltage must be:

$$\arccos 0.92 = \cos^{-1} 0.92 = 23.07°$$

I don't want to change the active current, so the phasor diagram will now look like this:

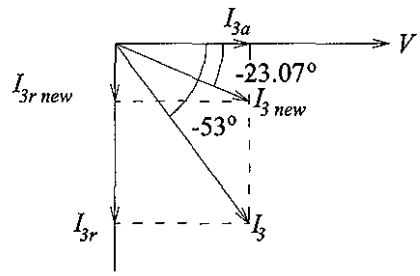

Fig. 3.17

From fig. 3.17 we can see that the capacitor must draw a current equal to $I_{3r} - I_{3r\ new}$.

But we dont know the value of $I_{3r\,new}$. Can you find it in the information you have?

I_{3a} is 15.8 A and I_{3r} is 21 A (both of which we already know).

We also know that the desired angle is 23.07° (to obtain the 0.92 power factor!).

If we calculate:

$$\tan 23.07° = 0.43 = \frac{I_{3r\,new}}{I_{3a}} = \frac{I_{3r\,new}}{15.8},$$

we get $I_{3r\,new} = 6.7$ A

(or, directly, $I_{3r\ new} = I_{3a} \times \tan 23.07$)

The problem is almost solved. The capacitor must draw a current of:

$$I_c = I_{3r} - I_{3r\ new} = 21 - 6.7 = 14.3\,\text{A},$$

and then:

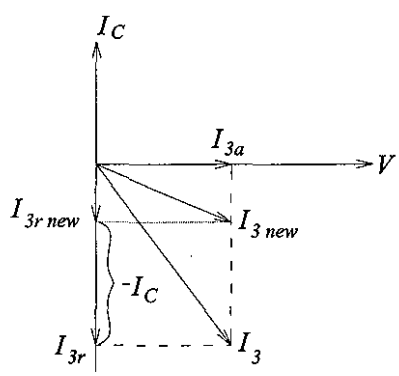

Fig. 3.18

and, for our capacitor:

$$X_c = \frac{V}{I_c} = \frac{200}{14.3} = 13.99\ \Omega$$

$$X_C = \frac{1}{2\pi fC} \Rightarrow C = \frac{1}{2\pi fX_C} = \frac{1}{2\pi 50 \times 13.99} = 0.000227 \text{ F} \approx 225\ \mu\text{F}$$

So, a capacitor of 225 μF would do the work.

There is another theoretical alternative. We didn't mention whether we wanted a leading or lagging power factor, so we could have chosen an angle of 23.07° with (+) (again trigonometry: cos 23.07° = cos (−23.07°) = 0.92!) Remember that $\cos\phi = \cos(-\phi)$!

The phasor diagram would have looked like this:

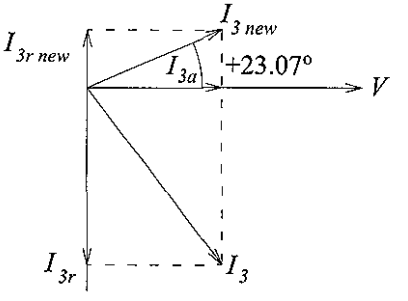

Fig. 3.19

The current drawn by the capacitor in this case must be:

$$I_c = I_{3r} - I_{3r\ new} = 21 + 6.7 = 27.7\,\text{A}$$

and

$$X_c = \frac{200}{27.7} = 7.22\ \Omega$$

$$C = \frac{1}{2\pi 50 \times 7.22} = 0.000440 \text{ F} \approx 440\ \mu\text{F}$$

A capacitor of 440 μF would have improved the PF to 0.92, this time leading.

If you study AC synchronous motors, you'll find that they also improve the PF of a factory. While doing useful work, they can draw a leading current with a quadrature component that has the same effect as current in a capacitor!

Activity 3.3

A 240 V 50 Hz induction motor of 500 W has a lagging power factor of 0.7. Calculate the value of the parallel connected capacitor required to raise the overall power factor to unity.

3.4 Power measurements in single-phase circuits

The instrument that measures active power in a circuit is called a **wattmeter** (from **watt**, the unit for power).

A wattmeter measures voltage, current, and power factor, since the expression for P is:

$$P = VI\cos\phi$$

TAKE NOTE

For DC power measurement (since we normally discuss resistive circuits), the power factor is 1 and we can use only an ammeter and a voltmeter to measure current and voltage respectively. The product of the two meausrements will give you the power.

A normal wattmeter measures average power and the scale calibration is normally not linear.

Basically, a wattmeter has a fixed coil for current measurement and a mobile coil for voltage measurement. Since the moving coil's torque is proportional to the product of current and voltage (in fact second current), it also considers the PF.

Due to the inertia of the meter, the readings are not reversed during negative power intervals.

Wattmeter connection in a circuit:

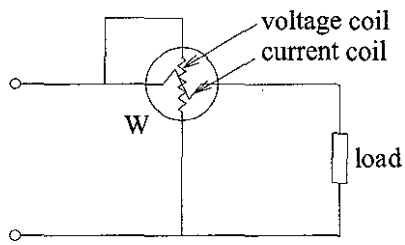

Fig. 3.20

In this example, the symbol for the wattmeter is:

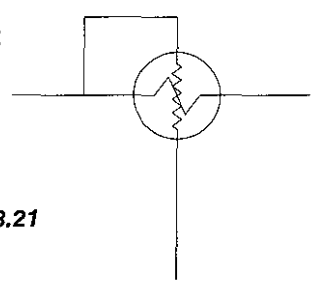

Fig. 3.21

and indicates both coils.

Remember to connect the current coil in series, like an ammeter, and the voltage coil in parallel with your load, like a voltmeter.

Other accepted symbols for wattmeters are:

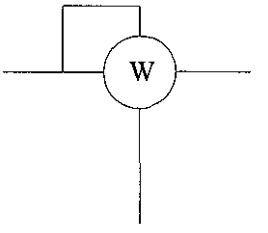

Fig. 3.22

where the coils are not represented (used for more schematic diagrams), or:

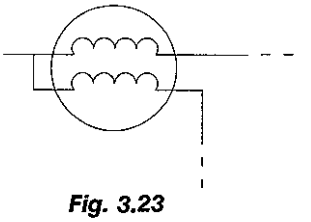

Fig. 3.23

for more detailed circuits or when we want to underline some connections.

There are two ways to connect a wattmeter. The first is illustrated in fig. 3.20.

The total line voltage (across the load and the current coil) is applied to the meter. This means that the loss in the current coil is included in the meter reading. To work out the loss for correction purposes would involve errors, because the resistance of the current coil is very small and not usually given or accurately measurable.

In this example, the real power P would be the indication of the wattmeter minus RI^2 (R of the series coil). Normally RI^2 is only a few watts and can be disregarded. This option is analog to the connection for measuring resistors:

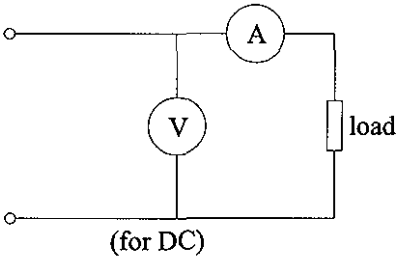

Fig. 3.24

The other possible connection is when through the current coil flows a current greater than the current through the load. (The current divides itself through the voltage coil and through the load, and the end of the current coil is connected to the voltage coil). This would be analog to the following (DC) measurement:

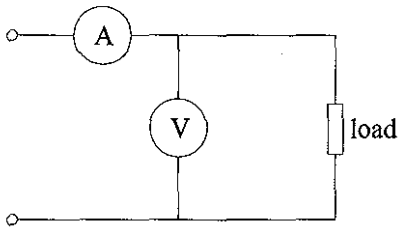

Fig. 3.25

The power in the voltage coil should be deducted from the reading. Since R_V is usually stated, we can calculate:

$$P = \text{(indication of the wattmeter)} - \frac{V^2}{R_V}$$

Then we must use a voltmeter to measure the load voltage V.

EXAMPLE 3.8

A wattmeter connected in a circuit reads 200 W. The line voltage is 200 V and the resistance of the potential circuit is 3800 Ω. What is the load power?

Solution

$$P = \text{wattmeter reading} - \frac{V^2}{R_v} =$$

$$= 200 - \frac{(200)^2}{3\,800} = 189.5\,\text{W}$$

We had to assume that the current coil supplies both the voltage coil and the load, although this wasn't stated.

The first connection (fig. 3.24) is preferred for small load currents and the second one (fig. 3.25) for high load currents and voltages.

TAKE NOTE

Sometimes wattmeters can be damaged below their full-scale readings. Can you work out why this happens? Suppose a wattmeter has a full-scale capacity of 1000 W, the rating for the current coil is 10 A and the rated voltage is 120 V. If we measure 720 W at 120 V for a load with a PF = 1 ⇒

$$P = VI \Rightarrow I = \frac{P}{V} = \frac{720}{120} = 6\text{ A!}$$

But we don't usually know the PF of the load beforehand. Let's see what would have happened if our load had a small power factor like 0.20!

$$720 = 120 \times I \times 0.20 \Rightarrow$$
$$I = 24 \text{ A!}$$

As you can see, this current will burn out the wattmeter's current coil ($\gg$10 A!).

Because a wattmeter reading gives no indication of current and voltage, always use an ammeter and a voltmeter in conjunction with a wattmeter (especially when a low PF is suspected):

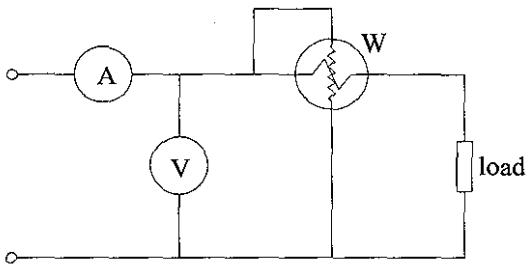

Fig. 3.26

This diagram is of immediate benefit. If we have all three measurements we can calculate:

$$S = VI$$
$$\cos\phi = \frac{P}{S} \text{ (the PF. of the load)}$$

and the reactive power:

$$Q = S\sin\phi = S\sqrt{1 - \cos^2\phi}$$

Another option is to connect the meters in the following order:

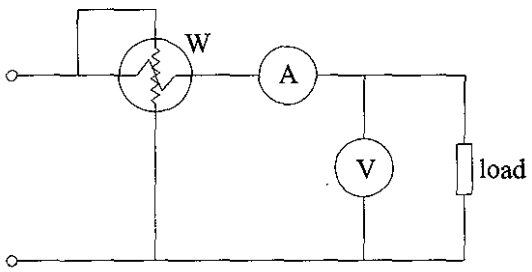

Fig. 3.27

Then you must deduct the losses in A and V from the wattmeters reading.

Have you noticed that both connections introduce errors to circuits? For example, the ammeter will read the load current and the current flowing through the voltmeter. Use your knowledge and common sense when you connect meters.

Activity 3.4

Using the circuit diagram in fig. 3.26, measure the active power for the following loads in the lab:

1 resistive load
2 (pure) inductive load
3 capacitive load

Now calculate the power factor for each one.

What did you find?

3.5 Summary

In this unit we studied mainly AC power (active, reactive and apparent)

For R, L and C loads we have:

1. For a resistor the power relations are:

   (Real) $P = S = VI$
   (Reactive) $Q = 0$
   (Power factor) PF = 1
   (Phase shift) $\phi = 0$

2. For inductances and capacitances:

   (Real) $P = 0$
   (Reactive) $Q = S = VI$
   (Power factor) PF = 0
   (Phase shift) $\phi = \pm 90°$

   (For C you'll have negative values for Q!)

3. For any impedance:

   $P = VI \cos\phi$
   $Q = VI \sin\phi$
   $|S| = VI = \sqrt{P^2 + Q^2}$
   $S = VI^*$
   $\mathrm{PF} = \cos\phi = \frac{P}{S}$
   $\mathrm{PF} \in [0, 1]$

4. The power factor of any element, branch or circuit is defined as:
   $PF = \cos\phi$
   where ϕ is the angle between the current flowing in and the voltage across the element, branch or circuit.
   ϕ is also the angle in the impedance triangle and power triangle. Power factor varies between 0 and 1.
5. A power factor as close to 1 as possible is preferred. Inductance and capacitors can be used to improve the power factor. Wattmeters are used to measure power.

Self-evaluation

1. A supply of 550 V is connected to a load with an impedance of $34.41\underline{/35.54°}$ Ω.
   Assuming the supply voltage as reference, calculate:
   a. the current flowing through the impedance
   b. active power
   c. reactive power
   d. apparent power
2. Two resistors of 8 000 Ω and 6 000 Ω are connected in parallel. A third resistor of 3 000 Ω is in series with the two. An AC supply applies 100 V to this network.
   a. What is the power in each resistor?
   b. Calculate the total power using the applied voltage and the current supplied by the source.
   c. What is the PF of the network?
3. A 120 V source supplies two loads: one carries 8 A at 0.85 PF lagging, and the other 5 A at 0.80 PF leading. Work out the total current, power, and overall power factor.
4. Determine the apparent power for each element in this circuit:

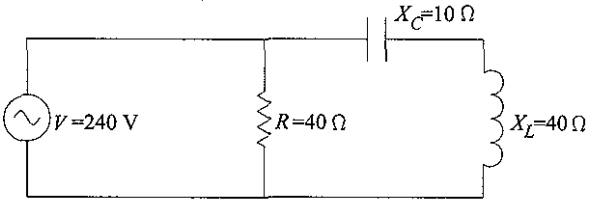

Fig. 3.28

Calculate the active, reactive, and apparent power for the circuit.

5. Study the following circuit:

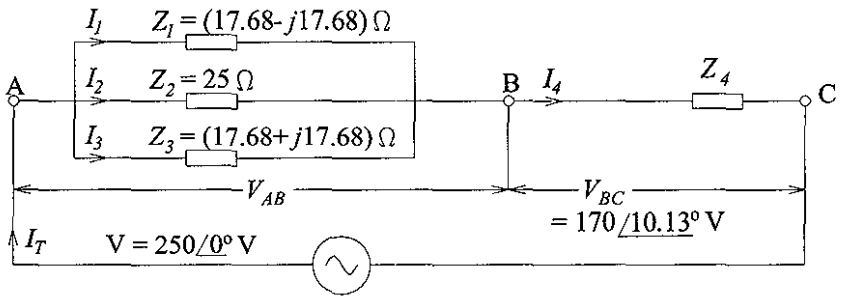

Fig. 3.29

From the information given in the circuit, calculate:

a. The voltage V_{AB}
b. The currents flowing through each of the impedances (I_1, I_2, I_3, I_4)
c. $I_{T\ (total)}$ – the current flowing through the circuit
d. Z_4 and Z_{eq} – the equivalent impedance of the circuit
e. The active power dissipated in Z_3
f. The reactive power dissipated in Z_4
g. The apparent power for the whole circuit

6. A single-phase 250 V supply is connected to a single-phase motor delivering 2.4 kW at a power factor of 0.62. Also connected to the supply is an inductive load with an impedance $Z = 25\angle 53^\circ\ \Omega$. The efficiency of the motor is 94%. Calculate:

a. The overall power factor (as seen from the supply)
b. The value of a parallel connected capacitor to improve the power factor to 0.94 lagging

Answers

Activity 3.1 (page 89)

$P = -100$ W
$Q = 173.2$ VAr
Delivering power.

Activity 3.2 (page 93)

$S = 8 + j6$ kV A
$Z = 4.61 + j3{,}46\ \Omega$

Activity 3.3 (page 96)

$C = 28.2\ \mu$F

Activity 3.4 (page 101)

The wattmeter will show you:

$P_{for\ resistor}$ = value (it depends on the values you have in the lab!)

$\left.\begin{array}{l} P_{for\ inductance} = 0 \\ P_{for\ capacitance} = 0 \end{array}\right\}$ (if you worked correctly!)

$PF_{\text{for resistor}} = 1\ (S = VI = P)$

$PF_{\text{for inductance}} = 0\ (S = VI;\ P = 0)$

$PF_{\text{for capacitance}} = 0\ (S = VI;\ P = 0)$

Alternatively, if you have a power factor meter, you could simply read off the power factors.

Your main observation is that, for the resistor, the apparent power is equal to the active power, while for the other two (L, C) the apparent power is equal to the reactive power. (Hence the denomination of 'active' and 'reactive' components.)

TAKE NOTE

Now you can try any combinations of loads (R, L, and C) and measure V, I, P. Then you can calculate: $S = VI$ VA; $\cos\phi = \text{PF} = \frac{P}{S}$; $Q = \sqrt{S^2 - P^2}$ VAr; etc

Self-evaluation (page 102)

1. a. $15.98\underline{/-35.54^\circ}$ A
   b. 7.15 kW
   c. 11 kVAr
   d. $8.79\underline{/35.54^\circ}$ kVA

2. a. $P_1 = 0.355$ W
   $P_2 = 0.474$ W
   $P_3 = 0.725$ W
   b. $P_{total} = 1.555$ W
   c. PF = 1

3. $I_T = 10.9\underline{/-6.4^\circ}$ A
   $S = 1\,300 + j146 = 1\,308\underline{/6.4^\circ}$ VA
   PF = 0.994

4. $S_R = P = 1\,440$ W
   $S_L = Q_L = j2\,560$ VAr
   $S_C = Q_C = -j640$ VAr
   $S_{total} = 2\,400\underline{/53.1^\circ}$ VA
   $P_{total} \cong 1\,440$ W
   $Q_{total} \cong 1\,920$ VAr

5. a. $V_{AB} = 87.98\underline{/-19.87^\circ}$ V
   b. $I_1 = 3.52\underline{/25.13^\circ}$ A
      $I_2 = 3.52\underline{/-19.87^\circ}$ A
      $I_3 = 3.52\underline{/-64.87^\circ}$ A
      $I_4 = 8.50\underline{/-19.87^\circ}$ A
   c. $I_T = 8.50\underline{/-19.87^\circ}$ A
   d. $Z_4 = 20\underline{/30^\circ}\ \Omega$
      $Z_{eq} = 29.42\underline{/19.87^\circ}\ \Omega$
   e. $P_3 = 218.98$ W
   f. $Q_4 = 722.5$ VAr
   g. $S = 2125\underline{/19.87^\circ} = 1\,998.5 + j722.3$ VA

6. a. $\mathrm{PF}_{\mathrm{total}} = 0.60$
   b. $C \cong 190\ \mu\mathrm{F}$ (191.52...)

unit 4

Resonant circuits

OUTCOMES

After studying this unit, you should be able to:

- identify the performances of networks subjected to varying frequencies
- analyse simple series and parallel resonant circuits
- design circuits that operate at pre-determined resonant conditions

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4.1 Introduction

In Unit 2 you were introduced to the analysis of *RLC* circuits. In this unit we will deal with the frequency response of such *RLC* circuits.

The frequency response of a circuit, instrument or electronic system is an important measure of its performance. Each one is designed with a specific operating range. For example, a stereo sound system has an operating range of 20 Hz to 20 KHz. Within this frequency range, the system components must detect and amplify signal frequencies that make up the music we like to hear.

In communication engineering we study the performance of a given network when subjected to varying frequencies. This is called the frequency response of the circuit. At certain frequency values, the current and/or voltage in a circuit reaches a maximum or minimum value. This frequency is called the resonant frequency of the circuit. While this is particularly significant in communications, resonance is also important in switching circuits, transmission lines and other electric power circuits. Resonance is fundamental in nature, finding its analog in mechanical systems in relation to such problems as shock and vibration, in fluid dynamics, and in many other fields. This unit focuses on electrical resonance.

4.2 Series circuit

4.2.1 Resonance in *RLC* series circuit

We have seen that the net reactance in an *RLC* circuit is:

$$X = X_L - X_C$$

$$Z = R + j\,(X_L - X_C)$$

and that

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$= \sqrt{R^2 + X^2}$$

If for some frequency of the applied voltage, $X_L = X_C$ (in magnitude) then

$$X = 0$$

and

$$Z = R \text{ (see fig. 4.1(b))}$$

In fig. 4.1(a):

$$V_L = I_0 jX_L \text{ and } V_C = I_0 jX_C$$

These are equal in magnitude but opposite in phase, so they cancel out each other. The two reactances together act as a short circuit since the combined voltage develops across the combination.

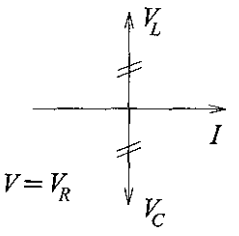


Fig. 4.1(a)

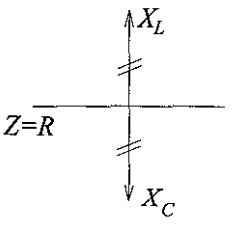


Fig. 4.1(b)

The applied voltage V drops across R so that $V = V_R$. The circuit impedance $Z = R$. The phasor diagram for series resonance is shown in fig. 4.1(c).

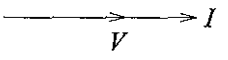


Fig. 4.1(c)

The frequency (f_r) at which net reactance is zero is given from the relation:

$$X_L - X_C = 0$$

$$\text{or } X_L = X_C$$

$$\text{or } \omega L = \frac{1}{\omega C}$$

$$\omega^2 = \frac{1}{LC}$$

$$(2\pi f_r)^2 = \frac{1}{LC}$$

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

If L is in henry and C in farad then f_r is given in hertz (H).

We have seen that, under these conditions, the impedance of the circuit equals the ohmic resistance R, so current is maximum, limited by the value of R alone. Its value is:

$$I_0 = \frac{V}{R}$$

and it is in phase with V.

This condition is called series resonance and the frequency at which it occurs is called resonant frequency (fr). So, a series RLC circuit is said to be in resonance when its net reactance is zero. Since $X = 0$, series resonance can be described as a condition of maximum admittance or minimum impedance (both terms mean the same thing).

As current is maximum it produces large voltage drops across L and C. These drops being equal and opposite to each other they cancel out.

Activity 4.1

A 1 H coil is connected in series with a 1 μF capacitor. What is the resonance frequency?

Activity 4.2

What capacitance is required to resonate with a 100 mH coil at 1 000 Hz?

Activity 4.3

What inductance is needed to resonate at 10 MHz with a capacitance of 100 pF?

4.2.2 Bandwidth of a resonant circuit

In an *RLC* circuit, the maximum current is solely determined by the circuit resistance *R*. However, at frequencies far from resonance, the amplitude of current becomes decreasingly dependant on *R*. For circuits with low value of *R*, the resonance curve sharply peaks and such a circuit is said to be sharply resonant or highly selective. Circuits with high value of *R* have a flat resonance curve and are said to have poor selectivity.

Selectivities of different resonant circuits are compared according to their bandwidths. Bandwidth of a circuit is given by the band of frequencies that lie between two points on either side of the resonant frequency where current falls to $\frac{1}{\sqrt{2}}$ of its maximum value (at resonance). The narrower the bandwidth, the higher the selectivity of the circuit.

In fig. 4.2 the bandwidth AB is given by

$$\Delta f = f_2 - f_1$$

or

$$\Delta\omega = \omega_2 - \omega_1 = \frac{R}{L}$$

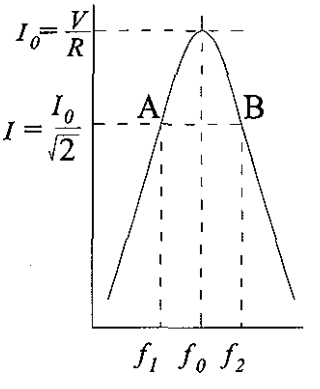


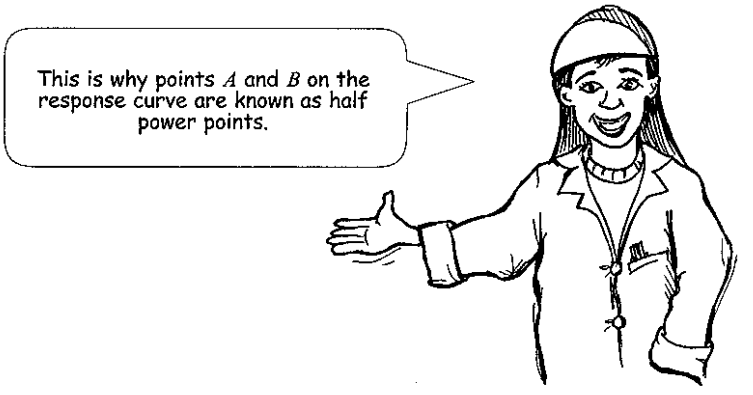
Fig. 4.2

Obviously, over this range of frequencies, current is equal to or greater than $\frac{I_0}{\sqrt{2}}$.

where $I_0 = \frac{V}{R}$ – the maximum current at resonance.

Power at points A and $B = I^2R$

$$= \left(\frac{I_0}{\sqrt{2}}\right)^2 \times R$$
$$= \frac{I_0{}^2R}{2}$$
$$= \frac{1}{2} I_0{}^2R$$
$$= \frac{1}{2} \times \text{power at resonance.}$$



It can be proven that the half power frequencies are:

$$f_1 = f_r - \frac{R}{4\pi L}$$

and $f_2 = f_r + \dfrac{R}{4\pi L}$

4.2.3 Q-factor of a series circuit

Where you have an RLC series circuit, the Q-factor is defined as being equal to the voltage magnification in the circuit at resonance.

We have seen that at resonance

$$I_0 = \frac{V}{R}$$

Voltage across either coil or capacitor $= I_0X_L$ or I_0X_C
Supply voltage $V = I_0R$

$$\therefore \text{ Voltage magnification} = \frac{V_L}{V} = \frac{I_0X_L}{I_0R} = \frac{X_L}{R} = \frac{\omega_0 L}{R}$$

$$\therefore Q\text{-factor} = \frac{\omega_0 L}{R} = \frac{2\pi f_0 L}{R} = \tan\phi$$

where ϕ is the power factor angle of the coil.

EXAMPLE 4.1

The parameters of a series R, L, C circuit are R = 1 KΩ, L = 8 mH, C = 20 PF. The applied voltage is 20 V rms of variable frequency.

Calculate:

1 the resonant frequency
2 the current at resonance
3 the Q-factor
4 the voltage across the inductor and capacitor at resonance
5 the lower and upper half power frequencies
6 the bandwidth

Solution

1 $$f_r = \frac{1}{2\pi\sqrt{LC}}$$
$$= \frac{1}{2\pi\sqrt{8 \times 10^{-3} \times 20 \times 10^{-12})}}$$
$$= 397.88 \text{ kHz}$$

2 $$I_0 = \frac{V}{R}$$
$$= \frac{20}{1\text{k}\Omega}$$
$$= 20 \text{ mA}$$

3 $$Q\text{-factor} = \frac{\omega L}{R}$$
$$= \frac{2\pi \times 397{,}88 \times 10^3 \times 8 \times 10^{-3}}{1 \times 10^3}$$
$$= 20$$

4 Voltage across inductor = Voltage across capacitor
i.e. $V_L = V_C$
$$V_L = V_C = (2\pi \times f_r L) \times I_0$$
$$= (2\pi \times 397{,}88 \times 10^3 \times 8 \times 10^{-3}) \times 20 \times 10^{-3}$$
$$= 400 \text{ V}$$

5 $$f_1 = f_r - \frac{R}{4\pi L}$$
$$= 397.88 \times 10^3 - \frac{1 \times 10^3}{4 \times \pi \times 8 \times 10^{-3}}$$
$$= 387.93 \text{ kHz}$$

$$f_2 = f_r + \frac{R}{4\pi L}$$

$$= 397.88 \times 10^3 + \frac{1 \times 10^3}{4\pi \times 8 \times 10^{-3}}$$

$$= 407.82 \text{ kHz}$$

6 $$\text{Bandwidth} = \frac{R}{L} = \frac{1 \times 10^3}{8 \times 10^{-3}}$$

$$= 1.25 \times 10^5 \text{ rad/s}$$

4.3 Parallel circuit

4.3.1 Resonance in a parallel circuit

In this section we will look at a practical case of a coil in parallel with a capacitor as shown in fig. 4.3.

Such a circuit is said to be in electrical resonance when the reactive component of line current is zero. The frequency at which this happens is known as resonant frequency.

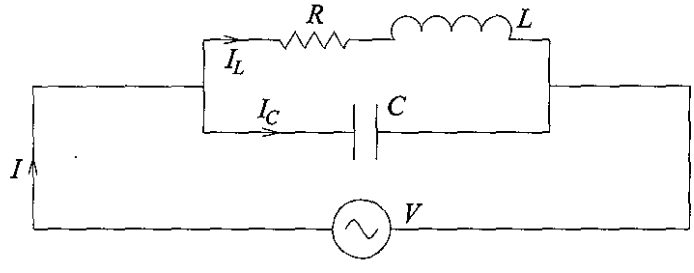


Fig. 4.3

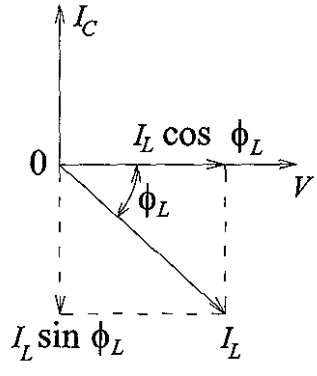


Fig. 4.4

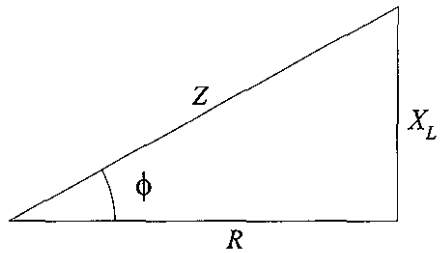


Fig. 4.5

The vector diagram for this circuit is shown in fig. 4.4.

At resonance, the reactance is zero, so

$$I_C - I_L \sin\phi_L = 0 \text{ or } I_L \sin\phi_L = I_C$$

The condition for resonance becomes:

$$\frac{V}{Z} \times \frac{X_L}{Z} = \frac{V}{X_C}$$

$$X_L \times X_C = Z^2$$

Now $X_L = \omega L$, $X_C = \frac{1}{\omega C}$

$$\therefore \frac{\omega L}{\omega C} = Z^2$$

$$\text{or } \frac{L}{C} = Z^2$$

$$\text{or } \frac{L}{C} = R^2 + X_L{}^2$$

$$= R^2 + (2\pi f L)^2$$

$$(2\pi f_r L)^2 = \frac{L}{C} - R^2$$

$$2\pi f_r = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

$$\text{or } f_r = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

This is the resonant frequency and is given in Hz if R is in ohm, L is in henry and C is in farad. If R is negligible (i.e. $R \ll X_L$) then:

$$f_r = \frac{1}{2\pi\sqrt{LC}} \text{ - - - - same as for series resonance}$$

▶ **Current at resonance**

As shown in fig. 4.4, since the wattless component of the current is zero, the circuit current is $I = I_L \cos\phi_L = \frac{V}{Z} \times \frac{R}{Z}$

$$I = \frac{VR}{Z^2}$$

Substituting $Z^2 = \frac{L}{C}$ we get:

$$I = \frac{VR}{\frac{L}{C}}$$

$$= \frac{V}{\frac{L}{CR}}$$

The denominator $\frac{L}{CR}$ is known as the equivalent or dynamic impedance of the parallel circuit at resonance. Since current is minimum at resonance, $\frac{L}{CR}$ must represent the maximum impedance of the circuit. In fact, parallel resonance is a condition of maximum impedance or minimum admittance.

Current at resonance is minimum, and such a circuit (when used in radio work) is sometimes known as a rejector circuit because it rejects (or takes minimum current of) the frequency to which it resonates.

4.3.2 Q-factor of a parallel circuit

The Q-factor of a parallel circuit is defined as the ratio of the current circulating between the two branches to the line current drawn from the supply – or, simply, the current magnification.

Obviously, the circulating current between capacitor and coil branches is I_C.

So Q-factor $= \frac{I_C}{I}$

Now $I_C = \frac{V}{X_C}$

$$= \frac{V}{1/\omega C} = \omega CV$$

and $I = \frac{V}{\left(\frac{L}{CR}\right)}$

$$\therefore Q\text{-factor} = \omega CV \div \frac{V}{\frac{L}{CR}}$$

$$= \omega\frac{L}{R}$$

$$= \frac{2\pi fL}{R} = \tan\phi \text{ (same as series)}$$

where ϕ is the power factor angle of the coil.

Resonant frequency when R is negligible can be written:

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

Substituting this value in $\frac{2\pi fL}{R}$ we get:

$$Q\text{-factor} = \frac{2\pi L}{R} \times \frac{1}{2\pi\sqrt{LC}}$$

$$= \frac{1}{R}\sqrt{\frac{L}{C}}$$

EXAMPLE 4.5

A choke (coil) connected across a 500 V, 50 Hz supply takes 1A at 0.8 power factor. What capacitance must be placed in parallel to make the power factor of the combination unity (PF = 1).

Solution

$$\text{Impedance of coil } Z = \frac{500}{1}$$

$$= 500\ \Omega$$

$$\cos\phi = 0.8 \therefore \sin\phi = 0.6$$

$$X_L = Z\sin\phi$$

$$= 500 \times 0.6$$

$$= 300\ \Omega$$

$$X_L = 2\pi f L$$

$$L = \frac{X_L}{2\pi f}$$

$$= \frac{300}{2 \times \pi \times 50}$$

$$= 0.954\ \text{H}$$

The power factor of the combined circuit will become unity when it is resonant at $f = 50$ Hz. The condition is:

$$C = \frac{L}{Z^2}$$

$$= \frac{0{,}954}{500^2}$$

$$= 3.81\ \mu\text{F}$$

EXAMPLE 4.6

A coil of resistance 20 Ω and inductance 200 μH is in parallel with a variable capacitor. The combination is in series with a resistor of 8 000 Ω. The voltage of the supply is 200 V at a frequency of 10^6 Hz.

Calculate:
1 the value of C to give resonance
2 the Q-factor of the coil
3 the current in each branch of the circuit at resonance.

Solution

First draw the circuit

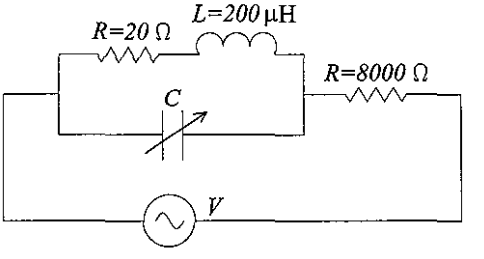


Fig. 4.6

$$X_L = 2\pi f L$$
$$= 2\pi \times 10^6 \times 200 \times 10^{-6}$$
$$= 1\ 256\ \Omega$$

Coil resistance is 20 Ω which is negligible compared to its reactance ($R \ll X_L$). Because coil resistance is negligible the resonant frequency is given by:

1 $$f_r = \frac{1}{2\pi\sqrt{LC}}$$
$$\therefore\ 10^6 = \frac{1}{2\pi\sqrt{200 \times 10^{-6} \times C}}$$
$$\therefore\quad C \simeq 125\ \text{pF}$$

2 $$Q = \frac{2\pi f L}{R}$$
$$= \frac{2\pi \times 10^6 \times 200 \times 10^{-6}}{20}$$
$$= 62.8$$

3 Dynamic impedance of the circuit is:
$$\frac{L}{CR} = \frac{200 \times 10^{-6}}{125 \times 10^{-12} \times 20}$$
$$= 80\ \text{k}\Omega$$

Total equivalent resistance of the tuned circuit is:

$$80 + 8$$

$$= 88\ \text{k}\Omega$$

$$\therefore \text{Current I} = \frac{200}{88\,000}$$

$$= 2.27\ \text{mA}$$

$$\therefore \text{p.d. across tuned circuit} = \text{current} \times \text{dynamic impedance}$$
$$= 2.27 \times 10^{-3} \times 80\,000$$
$$= 181.6\ \text{V}$$

$$\text{Current through inductive branch} = \frac{181{,}6}{\sqrt{20^2 + 1\,256^2}}$$
$$= 0.1445\ \text{A}$$
$$= 144.5\ \text{mA}$$

$$\text{Current through capacitive branch} = \frac{V}{1/\omega C}$$
$$= \omega V C$$
$$= 181.6 \times 2\pi \times 10^6 \times 125 \times 10^{-12}$$
$$= 142.7\ \text{mA}$$

4.4 Summary

1 Remember the following points about series resonance:

- Net reactance is zero, i.e. $X = 0$ or $X_L = X_C$
- Therefore $V_L = V_C$ or $V_L - V_C = 0$
- Impedance at resonance $= R$
- Line current at resonance is maximum and equal to $\frac{V}{R}$ and is in phase with voltage.
- Power factor is unity i.e. $\cos\phi = 1$

Remember the following points about parallel resonance:

- net susceptance is zero i.e. $\frac{1}{X_C} = \frac{X_L}{Z^2}$ or $X_L X_C = Z^2$ or $\frac{L}{C} = Z^2$
- reactive (or wattless) component of line current is zero.

- dynamic impedance $= \frac{L}{CR}\ \Omega$
- line current at resonance is minimum and equal to $\frac{V}{\frac{L}{CR}}$ but is in phase with the applied voltage.
- power factor of the circuit is unity.

Here is a table comparing series and parallel resonant circuits.

Item	Series Circuit	Parallel Circuit
Impedance at resonance	Minimum	Maximum
Current at resonance	Maximum $= \frac{V}{R}$	Minimum $= \frac{V}{\frac{L}{CR}}$
Effective impedance	R	$\frac{L}{CR}$
Power factor at resonance	Unity	Unity
Resonant frequency	$\frac{1}{2\pi\sqrt{LC}}$	$\frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$
It magnifies	Voltage	Current
Magnification (Q factor) is	$\omega\frac{L}{R}$	$\omega\frac{L}{R}$

Self-evaluation

1 A series circuit, consisting of a coil ($R = 30\ \Omega$, $L = 0.5$ H) and a capacitor resonates at a frequency of 48 Hz. Calculate the capacitance of a capacitor that, when connected in parallel with this circuit, will increase the resonant frequency to 60 Hz. Calculate in the latter case, the total current and the current in each branch of the circuit if the line voltage is 100 V.

2 A 20 Ω resistor is connected in series with an inductor, a capacitor and an ammeter across a 25 V variable frequency supply. When the frequency is 400 Hz, the current is at its maximum value of 0.5 A and the potential difference across the capacitor is 150 V. Calculate:

a. the capacitance of the capacitor.
b. the resistance and inductance of the inductor.

3 An *RLC* series circuit consists of a resistance of 1 000 Ω, an inductance of 100 mH and a capacitance of 10 PF. If a voltage of 100 V is applied across the combination, find:

a. the resonance frequency
b. *Q*-factor of the circuit
c. the half power points.

4 A constant voltage at 1 MHz is applied to an inductor in series with a variable capacitor. When the capacitor is set to 500 pF, the current has its maximum value but is reduced to one half when the capacitance is 600 pF. Find:

a. the resistance
b. the inductance
c. the Q-factor of the inductor

5. A coil of resistance 10 Ω and inductance 0.5 H is connected in series with a capacitor. On applying a sinusoidal voltage, the current is maximum when the frequency is 50 Hz. A second capacitor is connected in parallel with this circuit. What capacitance must this second capacitor have so that the combination acts like a non inductive resistor at 100 Hz?

Calculate the total current supplied in each case if the applied voltage is 220 V.

6 The resonant frequency of series RLC circuit is 1.2 KHz, if $R = 5\ \Omega$ and $L = 5$ mH. Find C and the impedance and power factor:

a. at resonance
b. at twice resonance frequency
c. at half resonant frequency

Answers

Activity 4.1 (page 110)

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$= \frac{1}{2\pi\sqrt{1 \times 10^{-6}}}$$

$$= 0.159 \times 10^3$$

$$= 159 \text{ Hz}$$

Activity 4.2 (page 110)

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$

$$2\pi\sqrt{LC} = \frac{1}{f_r}$$

$$4\pi^2 LC = \frac{1}{f_r^2}$$

$$C = \frac{1}{4\pi^2 \times L \times f_r^{\,2}}$$
$$= \frac{1}{4\pi^2 \times 100 \times 10^{-3} \times (1\,000)^2}$$
$$= 0.253\ \mu\text{F}$$

Activity 4.3 (page 111)

$$f_r = \frac{1}{2\pi\sqrt{LC}}$$
$$2\pi\sqrt{LC} = \frac{1}{f_r}$$
$$4\pi^2 LC = \frac{1}{f_r^{\,2}}$$
$$L = \frac{1}{4\pi^2 \times c \times f_r^{\,2}}$$
$$= \frac{1}{4 \times \pi^2 \times 100 \times 10^{-12} \times (10 \times 10^6)^2}$$
$$= 2.53\ \mu\text{H}$$

Self-evaluation (page 120)

1 $C_1 = 32.7\ \mu\text{F}$
$I = 0.54$ A
$I_1 = j1.23$ A
$I_2 = 0.54 - j1.24$

2 $C = 1.325\ \mu\text{F}$
$L = 0.119$ H
$R = 30\ \Omega$

3 a. $f_r = 159$ KHz
b. $Q = 100$
c. $f_1 = 158.2$ KHz
$f_2 = 159.8$ KHz

4 a. $R = 30.6\ \Omega$
b. $L = 0.0507$ mH
c. $Q_f = 10.4$

5 $C = 6.9\ \mu\text{F}$
$I_{(50Hz)} = 22$ A
$I_T = 0.04$ A

6 a. $C = 3.5\ \mu\text{F}$
$Z = R = 5\ \Omega$
PF = 1
b. $Z = 56.8\ \Omega$
PF = 0.09 (lag)
c. $Z = 56.8\ \Omega$
PF = 0.09 (lead)

unit 5

Networks Theory (Part I)

OUTCOMES

After studying this unit, you should be able to:

- Analyse various DC and AC circuits using Kirchhoff's Laws.
- Solve different network problems using Superposition Theorem.
- Apply Thevenin's Theorem and solve problems using it (Thevenising).
- Explain Norton's Theorem and solve problems using it (Nortonising).
- Develop circuit problem-solving techniques and methods to check your answers.

5.1 Introduction

In Unit 1 you were introduced to Kirchoff's Laws. In this unit we will discuss these laws as applicable to networks.

We will study some principles and special theorems that can be used to analyse complex circuits. These techniques can be used to simplify both DC and AC circuits and reduce the amount of work needed to study the networks.

We will only refer to linear networks. A linear network responds directly proportional to the magnitude of any stimulus applied to it.

A linear circuit consists of linear elements and ideal sources of voltage and current. An ideal voltage source should supply an unlimited amount of energy without a decrease in its terminal voltage and an ideal current source delivers current that is independent of voltage. These are simplified models of the reality.

A passive network is one that contains no energy source and an active network is one that contains a minimum of one source of energy, for example a battery; an **active** element.

Various techniques can be used to solve a circuit for its voltages and currents. Remember, for any problem, that you can solve it at least two ways. Try and understand all the methods you are introduced to in this unit, because each one can be used to analyse circuits. Once you've mastered these 'tools' you should be able to economise on the amount of work and time you spend on solving problems. You will also be able to check your work. After all, the final solution should be the same regardless of which method you use!

Let's get started!

5.2 Network theorems

5.2.1 Kirchhoff's laws

Let's revise Kirchhoff's two laws:

▶ **Kirchhoff's current law (KCL)**

DEFINITION

At any node of a circuit, at every instant of time, the sum of the currents that enter the node is equal to the sum of the currents coming out of that node.

Another version of this law is: 'The currents algebraically sum to zero in any node of a circuit.'

Do you remember that, if we consider the currents directed **into** a node to be positive and the currents directed **out of** a node to be negative their algebraic sum will be zero?

This law can be applied to both DC and AC circuits.

This law is essentially the law of conservation of charge, which is why the first expression 'at every instant of time' was mentioned. The law can be generalised for any closed region of a network (the current going in the region must equal the current coming out of the region).

▶ **Kirchhoff's voltage law (KVL)**

DEFINITION

In any loop of a circuit, at every instant of time, the sum of the voltage rises equals the sum of the voltage drops.

or

Around any loop in a circuit, the voltages algebraically sum to zero (if we see the voltages across elements that are transversed from plus to minus as being positive and the voltages across elements that are transversed from minus to plus as being negative, or vice-versa).

Activity 5.1

Look back in your notes and make sure you know what we mean by 'node' and 'loop'. Try and solve again the example for Kirchhoff's laws in Unit 1 to remind you of this theory.

EXAMPLE 5.1

Let's start with a DC example.

Look at the circuit in Fig. 5.1. Find the currents in each branch using Kirchhoff's laws.

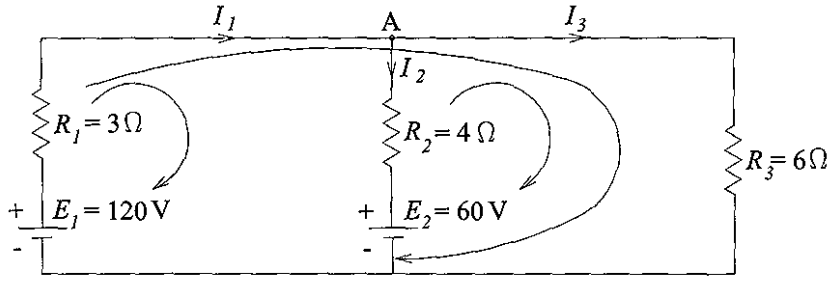


Fig. 5.1

Solution

In node A:

$$I_1 = I_2 + I_3 \Rightarrow I_2 = I_1 - I_3 \ldots ①$$

Remember, KCL can only be used once in this example, because we have only two nodes. However, KVL can be used twice because there are three unknowns and, so far, only one equation.

For loop:

$$E_2 \rightarrow R_2 \rightarrow R_3(\rightarrow E_2)$$

$$E_2 = R_3I_3 - R_2I_2 \ldots ②$$

TAKE NOTE

Remember, we conventionally assume the current flows from + to − in DC circuits.

For loop:

$$E_1 \rightarrow R_1 \rightarrow R_3(\rightarrow E_1)$$

$$E_1 = R_1I_1 + R_3I_3 \ldots ③$$

If we substitute $I_2 = I_1 - I_3$ in equation ② we get:

$$E_2 = R_3I_3 - R_2(I_1 - I_3)$$

$$= (R_3 + R_2)I_3 - R_2I_1$$

If we look at the system:

$$E_2 = (R_3 + R_2)I_3 - R_2I_1$$

$$E_1 = R_3I_3 + R_1I_1$$

There are two equations with two unknowns (I_1 and I_3)

$$R_1E_2 = (R_1R_2 + R_1R_3)I_3 - R_1R_2I_1$$
$$R_2E_1 = R_2R_3I_3 + R_1R_2I_1$$

Adding the two equations:

$$R_1E_2 + R_2E_1 = (R_1R_2 + R_2R_3 + R_1R_3)I_3 \Rightarrow$$
$$I_3 = \frac{R_2E_1 + R_1E_2}{R_1R_2 + R_2R_3 + R_1R_3} \ldots ④$$

Now substitute this value in equation ③ and:

$$E_1 = R_1I_1 + R_3I_3 = R_1I_1 + R_3\frac{R_2E_1 + R_1E_2}{R_1R_2 + R_2R_3 + R_1R_3}$$
$$\Rightarrow$$
$$R_1I_1 = \frac{R_1R_2E_1 + R_2R_3E_1 + R_3R_1E_1 - R_2R_3E_1 - R_3R_1E_2}{R_1R_2 + R_2R_3 + R_3R_1}$$
$$\Rightarrow$$
$$I_1 = \frac{(R_2 + R_3)E_1 - R_3E_2}{R_1R_2 + R_2R_3 + R_3R_1} \ldots ⑤$$

Now we can calculate I_2 as $I_1 - I_3$ but we should rather keep this equation for checking if we have worked correctly. So let's use the equation for loop:

$$E_1 \rightarrow R_1 \rightarrow R_2 \rightarrow E_2(\rightarrow E_1)$$
$$E_1 - E_2 = R_1I_1 + R_2I_2$$
$$\Rightarrow$$
$$R_2I_2 = (E_1 - E_2) - R_1\frac{(R_2 + R_3)E_1 - R_3E_2}{R_1R_2 + R_2R_3 + R_1R_3}$$
$$\Rightarrow$$
$$I_2 = \frac{R_3E_1 - E_2(R_1 + R_3)}{R_1R_2 + R_2R_3 + R_1R_3} \ldots ⑥$$

Using the data of the problem in equations ④, ⑤, ⑥ we get:

$$I_1 = \frac{(6 + 4) \times 120 - 6 \times 60}{3 \times 4 + 4 \times 6 + 6 \times 3} = \frac{840}{54} = 15.56 \text{ A}$$
$$I_2 = \frac{6 \times 120 - (3 + 6) \times 60}{54} = \frac{180}{54} = 3.33 \text{ A}$$

and

$$I_3 = \frac{4 \times 120 + 3 \times 60}{54} = 12.22 \text{ A}$$

Let's check these results:

In node A: $I_1 = I_2 + I_3$

Is $15.56 = 3.33 + 12.22$? Not exactly ($15.56 \neq 15.55$), but the slight error is due to the decimals rounding. So we worked correctly! Did you notice that all the values (I_1, I_2, I_3) are positive? This means that we incidentally chose the correct directions for the currents from the beginning.

Let's think about this problem. Why did we do such a simple problem? Basically, because the example could be solved from the beginning to the end literally (using only letters), in formulas. How does this help us?

The possiblity of you making a mistake in your calculations is reduced and, later on, you'll see that the results are general and can be extended to other problems.

More than that, you have the opportunity to check your results. Did you notice how we checked the currents? If you get the same answer (and you worked correctly) the problem is solved and you can rely on your result; if not (you've made a mistake) it is very easy to recalculate the values (only three calculations in this example!). We also tend to work more carefully and logically when we work in the formulas.

Try to use this way of solving problems as many times as you can, unless you need some partial results or special interpretations and observations. But if you feel more comfortable working with numbers, do so. If you work correctly you'll get the same result!

Let's take it a step further...

EXAMPLE 5.2

Study the circuit in fig. 5.2. Use Kirchhoff's laws to find the currents flowing in the circuit:

$E_1 = 60\underline{/0°}$ V
$E_2 = 90\underline{/30°}$ V
$Z_1 = j8\Omega$
$Z_2 = 12\Omega$
$Z_3 = 8 - j6\Omega$

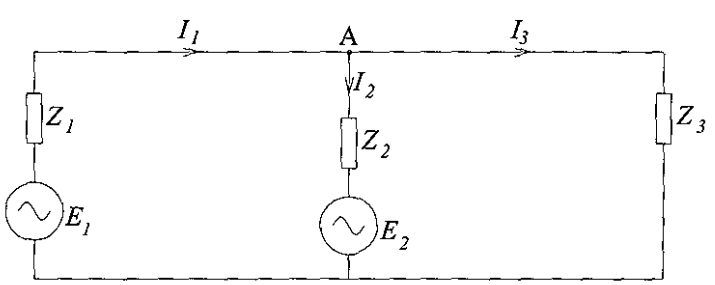


Fig. 5.2

Solution

TAKE NOTE

Unlike the previous example, here we have an AC circuit. So let's write our equations accordingly:

KCL

Node A:

$$I_1 = I_2 + I_3 \Rightarrow I_2 = I_1 - I_3 \ldots ①$$

Did you notice that we're now working in complex numbers?

KVL

Loop

$$E_2 \rightarrow Z_2 \rightarrow Z_3(\rightarrow E_2)$$
$$E_2 = Z_3 I_3 - Z_2 I_2 \ldots ②$$

$$E_1 \rightarrow Z_1 \rightarrow Z_3(\rightarrow E_1)$$
$$E_1 = Z_1 I_1 + Z_3 I_3 \ldots ③$$

$$E_2 = Z_3 I_3 - Z_2(I_1 - I_3) = (Z_3 + Z_2)I_3 - Z_2 I_1 \text{ (multiply equation by } Z_1)$$
$$E_1 = Z_3 I_3 + Z_1 I_1 \text{ (multiply equation by } Z_2)$$

$$Z_1 E_2 = (Z_1 Z_2 + Z_3 Z_1)I_3 - Z_1 Z_2 I_1$$
$$Z_2 E_1 = Z_2 Z_3 I_3 + Z_1 Z_2 I_1$$

$$Z_2 E_2 + Z_2 E_1 = (Z_1 Z_2 + Z_2 Z_3 + Z_1 Z_3)I_3$$
$$\Rightarrow$$
$$I_3 = \frac{Z_2 E_1 + Z_1 E_2}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1} \ldots ④$$

Substitute I_3 in equation ③ and:

$$E_1 = Z_1 I_1 + Z_3 \frac{Z_2 E_1 + Z_1 E_2}{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}$$

Working out I_1 from the last equation (where all E's and Z's are given data in the problem) we get:

$$I_1 = \frac{E_1(Z_2 + Z_3) - Z_3E_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} \quad \ldots ⑤$$

Again, write the equation for loop:

$$E_1 \rightarrow Z_1 \rightarrow Z_2 \rightarrow E_2(\rightarrow E_1)$$

and

$$E_1 - E_2 = Z_1I_1 + Z_2I_2$$

$$\Rightarrow$$

$$Z_2I_2 = \frac{(Z_1Z_2 + Z_2Z_3 + Z_3Z_1)(E_1 - E_2) - Z_1E_1(Z_2 + Z_3) + Z_3Z_1E_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1}$$

$$\Rightarrow$$

$$I_2 = \frac{Z_3E_1 - (Z_1 + Z_3)E_2}{Z_1Z_2 + Z_2Z_3 + Z_3Z_1} \quad \ldots ⑥$$

$$Z_1 = 8\underline{/90^\circ}\ \Omega$$
$$Z_2 = 12\underline{/0^\circ}\ \Omega$$
$$Z_3 = 10\underline{/-36.87^\circ}\ \Omega$$
$$Z_1Z_2 = j96\ \Omega^2$$
$$Z_2Z_3 = 96 - j72\ \Omega^2$$
$$Z_3Z_1 = 48 + j64\ \Omega^2$$
$$Z_1Z_2 + Z_2Z_3 + Z_3Z_1 = 144 + j88 = 168.76\underline{/31.43^\circ}\ \Omega^2$$
$$Z_2 + Z_3 = 20 - j6 = 20.88\underline{/-16.7^\circ}\ \Omega$$
$$Z_1 + Z_3 = 8 + j2 = 8.25\underline{/14.04^\circ}\ \Omega$$

Thus

$$I_1 = \frac{60\underline{/0^\circ} \times 20.88\underline{/-16.7^\circ} - 10\underline{/-36.87^\circ} \times 90\underline{/30^\circ}}{168.76\underline{/31.43^\circ}}$$

$$= \frac{1\,252.8\underline{/-16.7^\circ} - 900\underline{/-6.87^\circ}}{168.76\underline{/31.43^\circ}} = \frac{396.96\underline{/-39.47^\circ}}{168.76\underline{/31.43^\circ}}$$

$$= 2.35\underline{/-70.9^\circ}\ \text{A}$$

$$I_2 = \frac{10\underline{/-36.87^\circ} \times 60\underline{/0^\circ} - 8.25\underline{/14.04^\circ} \times 90\underline{/30^\circ}}{168.76\underline{/31.43^\circ}}$$

$$= \frac{600\underline{/-36.87^\circ} - 742.5\underline{/44.04^\circ}}{168.76\underline{/31.43^\circ}} = -\frac{877.52\underline{/86.49^\circ}}{168.76\underline{/31.43^\circ}}$$

$$= -5.2\underline{/55.06^\circ}\ \text{A}$$

$$I_3 = \frac{12\underline{/0^\circ} \times 60\underline{/0^\circ} + 8\underline{/90^\circ} \times 90\underline{/30^\circ}}{168.76\underline{/31.43^\circ}} = \frac{720\underline{/60^\circ}}{168.76\underline{/31.43^\circ}}$$

$$= 4.27\underline{/28.57^\circ}\ \text{A}$$

Let's check the results:

$$I_1 = 2.35\underline{/-70.9^\circ} = 0.77 - j2.22 \text{ A}$$
$$I_2 = -5.2\underline{/55.06^\circ} = -2.98 - j4.26 \text{ A}$$
$$I_3 = 4.27\underline{/28.57^\circ} = 3.75 + j2.04 \text{ A}$$
$$I_2 + I_3 = 3.75 - 2.98 + j2.04 - j4.26 = 0.77 - j2.22$$
$$= I_1$$

so we worked correctly!

TAKE NOTE

The general convention is to display current as positive with the modulus of the phase angle less that 180°. So,

$$I_2 = 5.2\underline{/-180^\circ + 55.06^\circ}$$
$$= 5.2\underline{/-124.94^\circ} \text{ A}$$

Before we go further let's do a quick recap of the two examples. Have you noticed any 'coincidences'?

Look at the final formulas obtained for the currents. Are they similar? The configurations of the two circuits are similar, the only difference being that one is a DC circuit, containing only resistances (except the two supplies) and the second is an AC circuit that, apart from the 2 (now) AC supplies consists of impedances.

Did you expect that, if we replaced the resistances in the initial current formulas with impedances, we would obtain the formulas for the currents flowing in the AC circuit? You probably did! And as you will see, it's true.

Look at the following diagram:

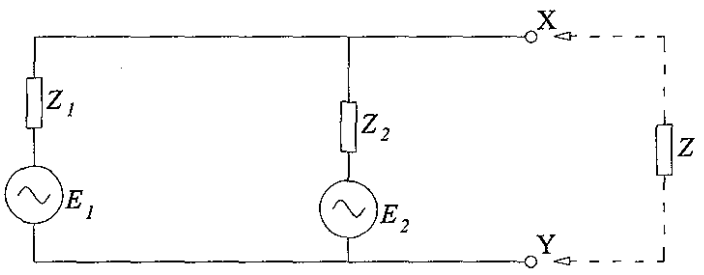


Fig. 5.3

Let's assume that the above circuit is part of a laboratory testing stand. Between the terminals X and Y, we want to study different types of loads (impedances), so we change Z continuously.

If we have the equations 4, 5 and 6 calculated in the previous example, each time we study a new load only Z (in the problem marked Z_3) will change. With three calculations only you can work out all currents.

Now you can see why it helps to delay substitution of values to the end of the problem (working literally).

Activity 5.2

A voltage of 2 V is applied between diagonally opposite corners of a framework in the form of a cube, each member comprising a resistor of 1 Ω. Use Kirchoff's laws to determine the combined resistance of the cube.

5.2.2 Superposition theorem

You may have already learnt that methods other than Kirchhoff's laws can be used to study electric circuits.

One of the most frequently used methods is based on the 'superposition theorem':

> 'In a linear network containing more than one emf source the resultant current in any branch is the algebraic sum of the currents that would be produced by each emf acting alone (all the other sources being set to zero or eventually replaced by their internal resistances).'

Simply put: when we have two or more active sources in a network, the current in any element of the network is the (phasor) sum of the separate currents that are created by each source as if it were acting alone in that circuit. The removed sources (all the others) must be replaced either by a short-circuit, or by their internal resistances (when they cannot be neglected).

You will see that the superposition principle has much broader applications. For example, you can use it when you'll study 'complex' waves as a superposition of two, three or more sine waves of different frequencies.

Steps for solving a problem using the superposition theorem:

1. Select one of the sources and replace all the others either by their internal impedance or by a zero resistance (short-circuit).
2. Determine the desired current(s) in the (different) branch(es) resulting from this single source (as if it is acting alone in the circuit).

3 Repeat steps 1 and 2 for each source (until the currents flowing in the branches have all been calculated).
4 Sum algebraically all the component currents to obtain the actual branch current (Remember, in AC, working with complex numbers will give us a phasor sum!)

EXAMPLE 5.3

(Superposition theorem applied in a DC circuit)

Calculate the currents in each branch of the following circuit:

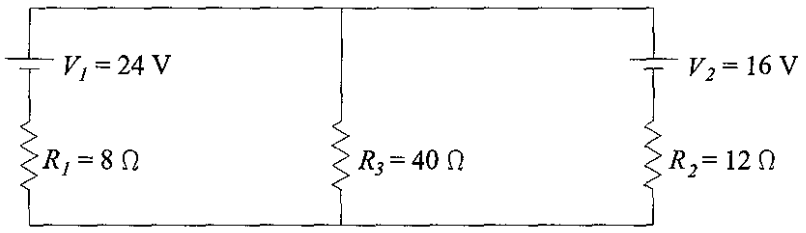


Fig. 5.4

Solution

Looking at only the first source the circuit will be:

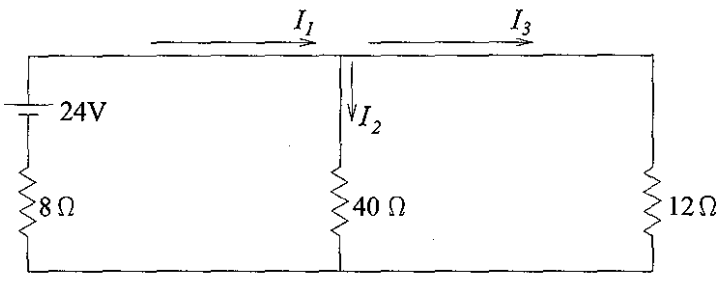


Fig. 5.5

The internal resistances of the two sources are not mentioned, so it is assumed they can be neglected.

$$R_{eq} = \frac{40 \times 12}{40 + 12} + 8 = 17.23\ \Omega$$

$$I_1 = \frac{24}{17.23} = 1.39\ \text{A}$$

With the rule of the current divider:

$$I_2 = 1.39 \times \frac{12}{52} = 0.32 \text{ A}$$

and

$$I_3 = 1.39 \times \frac{40}{52} = 1.07 \text{ A}$$

Now look at the second source:

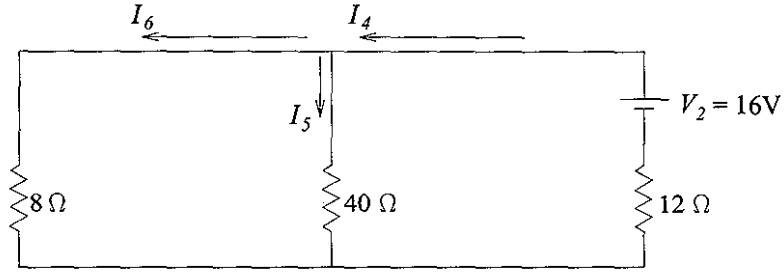


Fig. 5.6

$$R_{eq} = \frac{8 \times 40}{40 + 8} + 12 = 18.67\ \Omega$$

$$I_4 = \frac{V_2}{R_{eq}} = \frac{16}{18.67} = 0.86 \text{ A}$$

and, respectively

$$I_5 = I_4 \times \frac{8}{40 + 8} = 0.14 \text{ A}$$

$$I_6 = 0.86 \times \frac{40}{40 + 8} = 0.72 \text{ A}$$

Now do the algebraic sum in each branch:

For the branch containing the 8 Ω resistor:

$$I_1 - I_6 = 1.39 - 0.72 = 0.67 \text{ A}$$

For the branch containing the 40 Ω resistor:

$$I_2 + I_5 = 0.32 + 0.14 = 0.46 \text{ A}$$

For the branch containing the 12 Ω resistor:

$$I_3 - I_4 = 1.07 - 0.86 = 0.21 \text{ A}$$

So the currents flowing in the circuit will be:

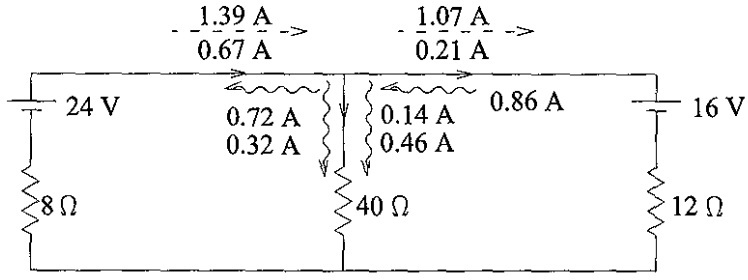


Fig. 5.7

with the directions indicated in this figure.

Check for node A: $1.39 = 1.07 + 0.32$ A
$0.86 = 0.14 + 0.72$ A
$0.67 = 0.46 + 0.21$ A, so we worked correctly.

Let's recalculate these currents using what we did in the first example with Kirchhoff's laws. You will notice that the configurations of the circuits are similar, so we'll use these formulas and replace the data we have for this problem.

In the 8 Ω resistor the current is:

$$= \frac{(40 + 12) \times 24 - 40 \times 16}{8 \times 12 + 12 \times 40 + 40 \times 8} = \frac{608}{896} = 0.678 \text{ A}$$

In the 40 Ω resistor the current is:

$$= \frac{12 \times 24 + 8 \times 16}{896} = \frac{416}{896} = 0.464 \text{ A}$$

In the 12 Ω resistor the current is:

$$= \frac{40 \times 24 - (40 + 8)16}{896} = \frac{192}{896} = 0.214 \text{ A}$$

$(0.464 + 0.214 = 0.678)$

There are small differences in the results due to the rounding of decimals and the accuracy used in other calculations. But it is obvious the results are correct. Remember, no matter what method you use, if you work correctly you will get the same result.

It is always a good idea to try and work out one problem using two different methods and compare the results. If you calculate correctly the results should coincide.

EXAMPLE 5.4

(Superposition theorem applied in an AC circuit)

Calculate the currents flowing in each branch of the following circuit:

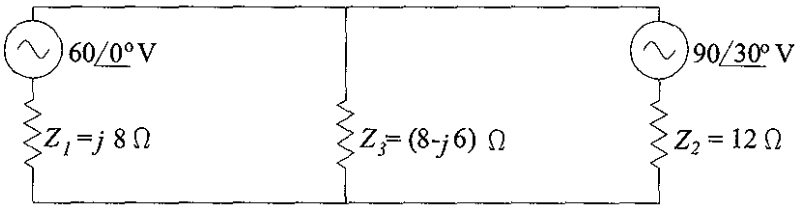


Fig. 5.8

Solution

Let's say that we have only the first source in the circuit. Then our circuit is:

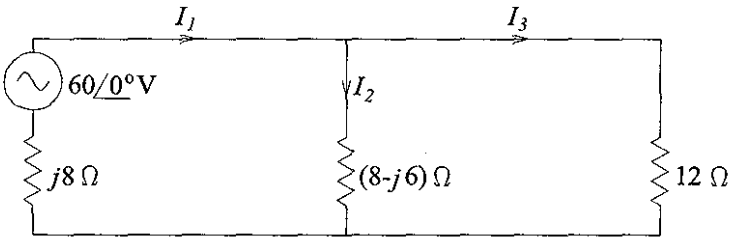


Fig. 5.9

Activity 5.3

Try to explain what character your three impedances have.

TAKE NOTE

The voltage given by this supply has an angle of 0°, so it is on the reference axis.

$$Z_{eq2-3} = \frac{12(8-j6)}{12+8-j6} = \frac{96-j72}{20-j6} = \frac{120\underline{/-36.87^\circ}}{20.88\underline{/-16.70^\circ}}$$

$$= 5.75\underline{/-20.17^\circ} = 5.4 - j1.98\ \Omega$$

$$Z_{eq1-2-3} = 5.4 - j1.98 + j8 = 5.4 + j6.02 = 8.09\underline{/48.11^\circ}\ \Omega$$

$$I_1 = \frac{60\underline{/0^\circ}}{8.09\underline{/48.11^\circ}} = 7.42\underline{/-48.11^\circ}\ \text{A}$$

$$I_2 = 7.42\underline{/-48.11^\circ} \times \frac{12\underline{/0^\circ}}{12 + 8 - j6} = 7.42\underline{/-48.11^\circ} \times \frac{12\underline{/0^\circ}}{20.88\underline{/-16.70^\circ}}$$

$$= 4.26\underline{/-31.41^\circ}\ \text{A}$$

and

$$I_3 = 7.42\underline{/-48.11^\circ} \times \frac{8 - j6}{20.88\underline{/-16.70^\circ}} = 7.42\underline{/-48.11^\circ} \times \frac{10\underline{/-36.87^\circ}}{20.88\underline{/-16.7^\circ}}$$

$$= 3.55\underline{/-68.28^\circ}\ \text{A}$$

We can now partially check:

$$I_2 + I_3 = 4.26\underline{/-31.41^\circ} + 3.55\underline{/-68.28^\circ} = 4.95 - j5.52$$

$$= 7.414\underline{/-48.116^\circ} \approx I_1$$

Up to here it seems right.

TAKE NOTE

Did you notice how we applied the rules for series and parallel impedances (they are similar to the ones you know for resistance) and those for current dividers in AC (again, similar to a DC current divider, only that we work with impedances instead of resistors and the currents are now complex numbers)?

Now use only the second source:

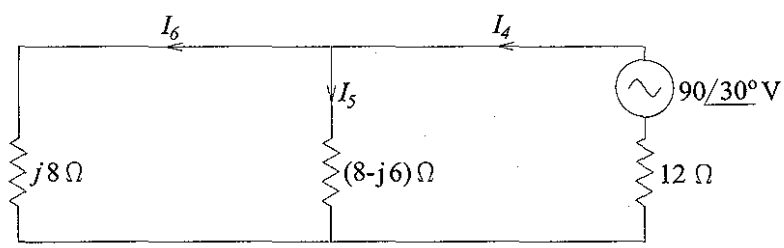


Fig. 5.10

$$Z_{eq13} = \frac{j8(8 - j6)}{j8 + 8 - j6} = \frac{48 + j64}{8 + j2} = \frac{80\underline{/53.13^\circ}}{8.25\underline{/14.04^\circ}}$$

$$= 9.70\underline{/39.09^\circ} = 7.53 + j6.12\ \Omega$$

$$Z_{eq123} = 7.53 + j6.12 + 12 = 19.53 + j6.12 = 20.47\underline{/17.40^\circ}\ \Omega$$

$$I_4 = \frac{90\underline{/30^\circ}}{20.47\underline{/17.4^\circ}} = 4.40\underline{/12.6^\circ}\ \text{A}$$

$$I_5 = 4.4\underline{/12.6^\circ} \times \frac{j8}{8 - j6 + j8} = 4.4\underline{/12.6^\circ} \times \frac{8\underline{/90^\circ}}{8 + j2}$$

$$= 4.4\underline{/12.6^\circ} \times \frac{8\underline{/90^\circ}}{8.25\underline{/14.04^\circ}} = 4.27\underline{/88.56^\circ}\ \text{A}$$

$$I_6 = 4.4\underline{/12.6^\circ} \times \frac{8 - j6}{8.25\underline{/14.04^\circ}} = 4.4\underline{/12.6^\circ} \times \frac{10\underline{/-36.87^\circ}}{8.25\underline{/14.04^\circ}}$$

$$= 5.33\underline{/-38.31^\circ}\ \text{A}$$

Partial check:

$$I_5 + I_6 = 4.27\underline{/88.56^\circ} + 5.33\underline{/-38.31^\circ} = 4.29 + j0.96$$

$$= 4.4\underline{/12.61^\circ}\ \text{A} \approx I_4$$

Up to here, everything seems to make sense. Let's calculate the real currents flowing in the three branches:

$$I_1 - I_6 = 7.42\underline{/-48.11^\circ} - 5.33\underline{/-38.31^\circ} = 0.77 - j2.22$$

$$= 2.35\underline{/-70.87^\circ}\ \text{A}$$

(for the branch containing Z_1)

$$I_2 + I_5 = 4.26\underline{/-31.41^\circ} + 4.27\underline{/88.56^\circ} = 3.74 + j2.05$$

$$= 4.26\underline{/28.73^\circ}\ \text{A}$$

(for the branch containing Z_3)

and

$$I_3 - I_4 = 3.55\underline{/-68.28^\circ} - 4.4\underline{/12.6^\circ} = -2.98 - j4.26\ \text{A}$$

(for the branch containing Z_2)

If we change the direction of the last current (it flows in the direction of I_4) we'll get:

$$I_4 - I_3 = 2.98 + j4.26 = 5.2\underline{/55.03^\circ}\ \text{A}$$

Now we can check:

$$2.35\underline{/-70.87^\circ} + 5.2\underline{/55.03^\circ} = 3.75 + j2.04$$

$$= 4.268\underline{/28.55^\circ}\ \text{A},$$

equal to the current flowing in Z_3.

Could we have checked using another method?

Page back to example 5.2 on page 132. Is this the same problem? Did you get the same results? Yes, you're right! It is the same problem and you got similar results (taking into account the small rounding errors).

This proves that no matter what method you use if you work correctly you'll always get the right answer, and that there are different ways of solving the same problem.

5.2.3 Thevenin's theorem

Thevenin's theorem states:

> 'Any linear active network with output terminals A and B can be replaced by a single voltage V_{th} in series with a single impedance Z_{th}.'

This means that the current in the load is the same if the network (besides load) is replaced with a source voltage with an equivalent series impedance.

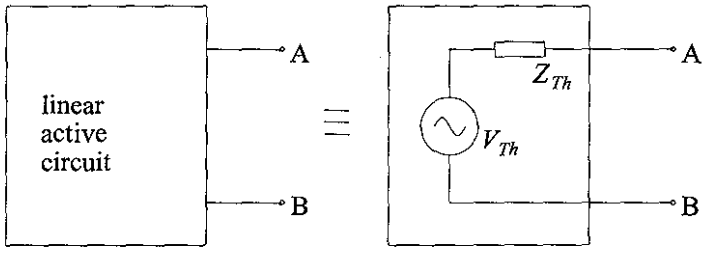


Fig. 5.11

Where:

V_{th} = Thevenin equivalent voltage (the open circuit voltage at the terminals from which the load has been removed)

Z_{th} = Thevenin equivalent impedance (impedance between A and B after all sources are removed and each has been replaced by its internal resistance)

You may be asking, 'What's the use of this theorem?' Let's assume we have a testing stand of different given elements and the only element we change according to the test procedure is the load that we are testing. It's only logical that, in this case, we wish to determine current, voltage or power in only one element of the circuit – the load – **without** obtaining other currents and voltages in the rest of the network. In this specific case, Thevenin's theorem becomes a powerful tool.

You'll also find the theorem useful for converting circuits containing transistors into equivalent circuits for analysis and design. For example, when you study amplifiers, you'll realise that the output resistance is very important (R_{th}).

Here are the steps for 'thevenising' a circuit:

- Calculate the open circuit voltage at the network terminals, A and B.
- Redraw the circuit with each voltage source replaced by its internal resistance or by a short circuit, and each current source replaced by an open circuit in parallel with its internal resistance.

- Calculate the equivalent impedance Z_{th} of this network as seen from the output terminals.

EXAMPLE 5.5

Determine the Thevenin equivalent circuit between terminals A and B:

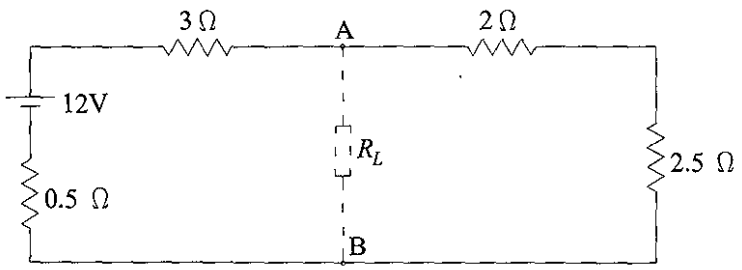


Fig. 5.12

Solution

First redraw the circuit, grouping all the known elements.

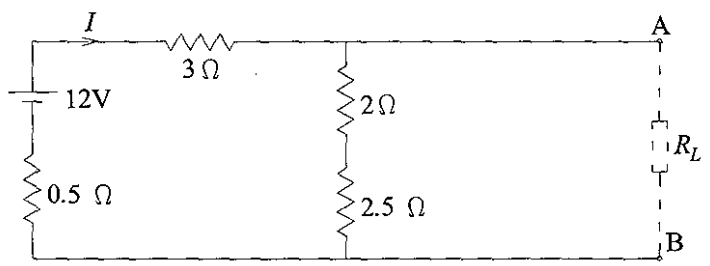


Fig. 5.13

It follows that:

$$I = \frac{V}{R_{eq}} = \frac{12}{3 + 2 + 2.5 + 0.5} = \frac{12}{8} = 1.5 \text{ A}$$

And then

$$V_{th} = V_{AB} = 1.5(2 + 2.5) = 1.5 \times 4.5 = 6.75 \text{ V}$$

To determine R_{th} (the equivalent resistance in our case) the circuit is:

$V = 0$ (short-circuited)

$$R_{th} = \frac{(0.5 + 3)(2 + 2.5)}{8} = \frac{3.5 \times 4.5}{8} = 1.97\ \Omega$$

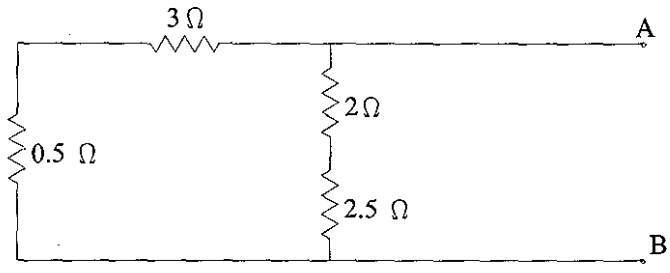


Fig. 5.14

So, the Thevenin equivalent for the simple circuit is:

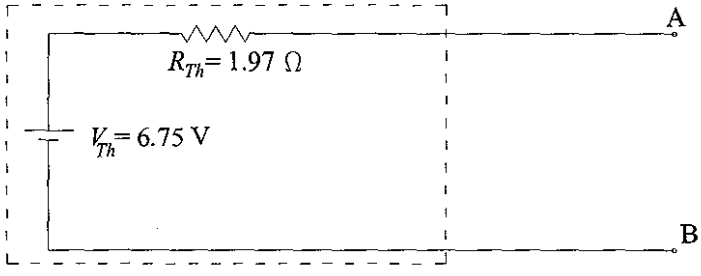


Fig. 5.15

EXAMPLE 5.6

Now work out the Thevenin equivalent for this circuit:

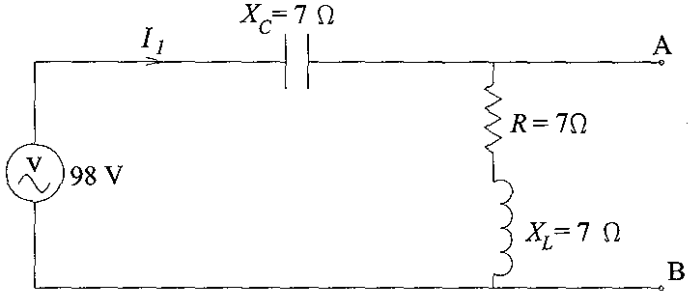


Fig. 5.16

Solution

Remember always to consider the phase when you calculate reactive impedances. For capacitive reactance it is $-jX_C$ and for inductive reactance it is $+jX_L$. If you don't remember, refer back to Unit 2.

First choose a reference. In this example, assume $V = 98\underline{/0^\circ}$ V is the reference.

The current I flowing in the circuit is:

$$I = \frac{V\underline{/0^\circ}}{Z_{eq}} = \frac{98\underline{/0^\circ}}{-j7 + 7 + j7} = \frac{98\underline{/0^\circ}}{7\underline{/0^\circ}} = 14\underline{/0^\circ}\text{ A}$$

TAKE NOTE

Did you notice that you obtained the current in phase with the voltage, since the equivalent impedance of the circuit is $Z = 7\ \Omega$? Keeping in mind that this is a RLC circuit, what do you have here? Does this have anything to do with resonance?

Now:

V_{th} will be the voltage across the $R + jX_L$ impedance

$$R + jX_L = 7 + j7 = 9.9\underline{/45^\circ}\ \Omega$$

So:

$$V_{th} = I \times Z = 14\underline{/0^\circ} \times 9.9\underline{/45^\circ} = 138.6\underline{/45^\circ} = 98 + j98\text{ V}$$

But we still have to calculate Z_{th}.

If we redraw the circuit with the voltage supply short-circuited, it looks like this:

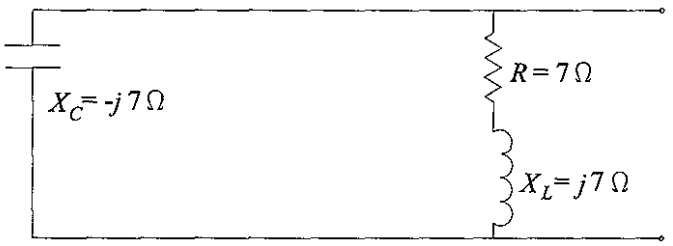


Fig. 5.17

And

$$Z_{th} = \frac{-j7(7 + j7)}{7 - j7 + j7} = \frac{-j7(7 + j7)}{7} = -j7 + 7 = 9.9\underline{/-45^\circ}\ \Omega$$

So, the Thevenin equivalent circuit is:

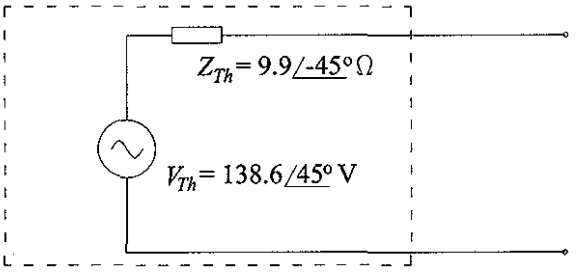


Fig. 5.18

Let's take this problem a step further ...

EXAMPLE 5.7

Using the information from the previous example, determine the current flowing through two resistors of 1 Ω connected in series across the terminals A and B:

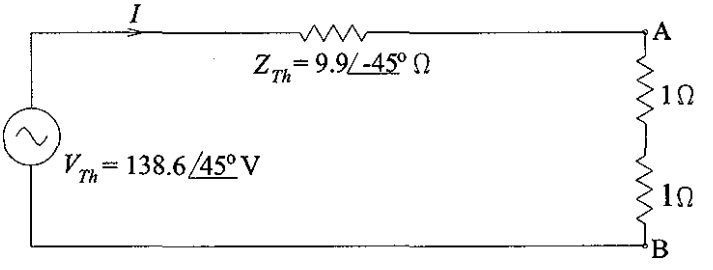


Fig. 5.19

Solution

$$I = \frac{V_{th}}{Z_{th} + 1 + 1} = \frac{138.6\angle 45°}{7 - j7 + 1 + 1} = \frac{138.6\angle 45°}{9 - j7} = \frac{138.6\angle 45°}{11.4\angle{-37.87°}}$$

$$= 12.16\angle 82.87° \text{ A}$$

Can you calculate the power dissipated by one of the resistors? And by both resistors? Yes!

$$P = I^2R = 12.16^2 \times 1 = 147.87 \text{ W}$$

$$2P = 295.74 \text{ W}$$

What is the voltage drop across one of the resistors? And across both?

$$V = 12.16\angle 82.87° \times 1\angle 0° = 12.16\angle 82.87° \text{ V}$$

$$V = 12.16\angle 82.87° \times 2\angle 0° = 24.32\angle 82.87° \text{ V}$$

Check if you worked correctly using another method.

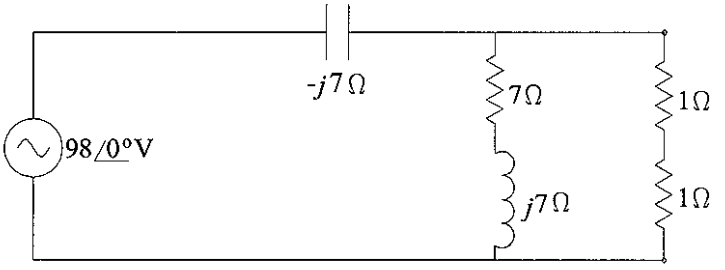


Fig. 5.20

$$Z_{eq||} = \frac{(7+j7)\times 2}{7+2+j7} = \frac{9.9\underline{/45^\circ}\times 2}{9+j7} = \frac{19.8\underline{/45^\circ}}{11.4\underline{/37.87^\circ}} = 1.74\underline{/7.13^\circ}\ \Omega$$

$$Z_{eq} = -j7 + 1.73 + j0.22 = 1.73 - j6.78 = 7\underline{/-75.68^\circ}\ \Omega$$

$$I = \frac{98\underline{/0^\circ}}{7\underline{/-75.68^\circ}} = 14\underline{/75.68^\circ}\,\text{A}$$

The current flowing through the resistors is:

$$I_{res} = 14\underline{/75.68^\circ}\times\frac{7+j7}{9+j7} = 14\underline{/75.68^\circ}\times\frac{9.9\underline{/45^\circ}}{11.4\underline{/37.87^\circ}}$$

$$= 12.16\underline{/82.81^\circ}\,\text{A}$$

This shows we worked correctly.

5.2.4 Norton's theorem

Norton's theorem states that:

> 'Any linear active network with output terminals A and B can be replaced by a single current source I_N in parallel with a single impedance Z_N.'

In other words:

> 'The current in the load is the same if the network (besides load) was replaced with a constant current source (I_N) in parallel with an impedance Z_N.'

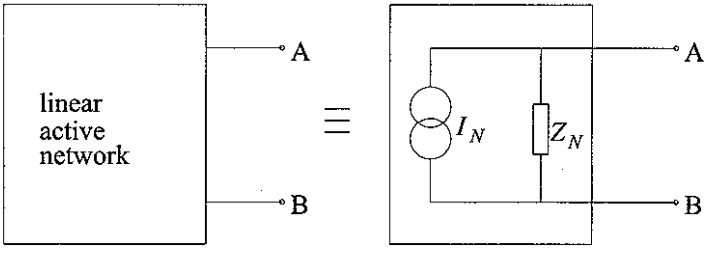


Fig. 5.21

Where:

Z_N = Norton equivalent impedance, seen 'looking back' into the circuit after the load has been disconnected and all the sources have been replaced by their internal resistances.

I_N = Norton equivalent current, that would flow in a short-circuit at the terminals AB in the original circuit.

This theorem is sometimes called the short-circuit theorem (since we calculate short-circuit current between A and B).

It can be proven that Thevenin resistance and Norton resistance are identical, and that:

$$V_{\text{th}} = I_N R_N$$

$$I_N = \frac{V_{th}}{R_{th}}$$

Here are the steps for 'nortonising' a circuit:

- Calculate the short-circuit current I_N at the network terminals.
- Redraw the circuit with all the sources replaced as explained in the theorem.
- Calculate the impedance Z_N of this network as seen from the output terminals.

EXAMPLE 5.8

Determine the Norton equivalent circuit between terminals A and B:

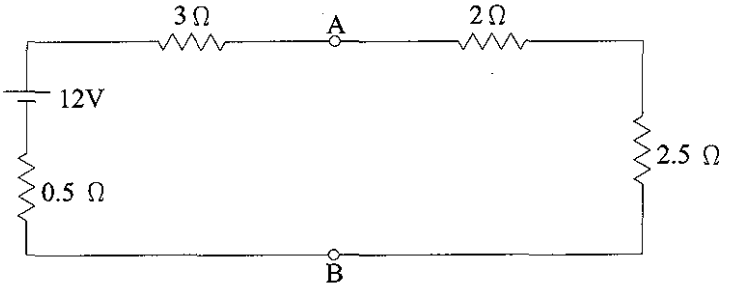


Fig. 5.22

Solution

The circuit can be redrawn:

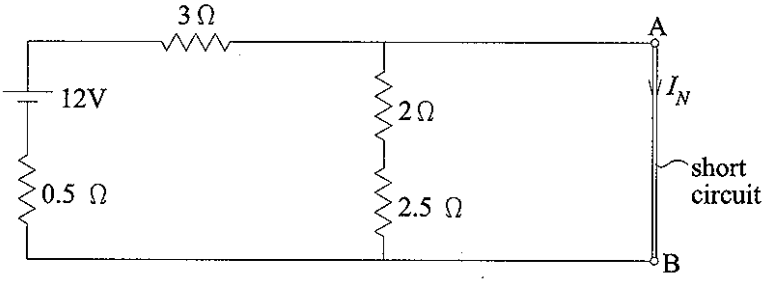


Fig. 5.23

and the short-circuit current calculated:

$$I_N = \frac{12}{3 + 0.5} = 3.43 \text{ A}$$

To determine R_N, we must set the voltage source to zero:

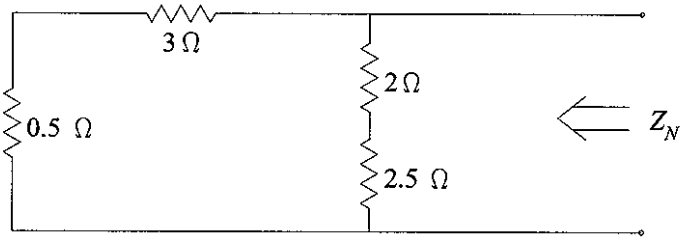


Fig. 5.24

$$R_N = \frac{(3 + 0.5)(2 + 2.5)}{3 + 0.5 + 2 + 2.5} = \frac{3.5 \times 4.5}{8} = 1.97\ \Omega$$

Then the Norton equivalent circuit is:

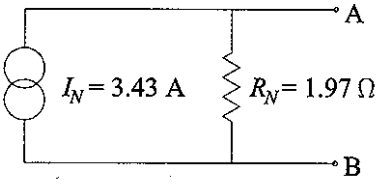


Fig. 5.25

If you look back at Example 5 you'll see that $R_N = R_{th}$, as stated when we introduced the theorem. But this time, R_N is in parallel with the current source, not in series.

Let's calculate $I_N R_N = 3.43 \times 1.97 = 6.75$ V

If you look at the Thevenin voltage in Example 5 you'll see that this is the value for V_{th}! What does this tell you? You can use the following alternative formulas to find the Norton equivalent of a circuit if you have already worked out the Thevenin equivalent:

$$R_N = R_{th}$$

$$I_N = \frac{V_{th}}{R_{th}}$$

Now draw the circuit, paying attention to how you connect the elements!

If you have a circuit that contains no (independent) sources the Thevenin and Norton equivalents are identical and comprise only output resistance ($R_N = R_{th}$)

The output resistance of a circuit can be worked out by finding V_{th} and I_N, the open-circuit voltage and short-circuit current respectively.

$$R_{th} = R_N = R_{output} = \frac{V_{th}}{I_N} = \frac{V_{oc}}{I_{sc}}$$

This is a very important observation. Keep it in mind, because you'll come across it again.

EXAMPLE 5.9

Determine the Norton equivalent circuit for the following circuit, between terminals A and B.

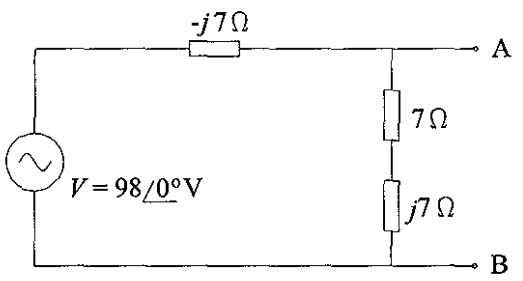


Fig. 5.26

Solution

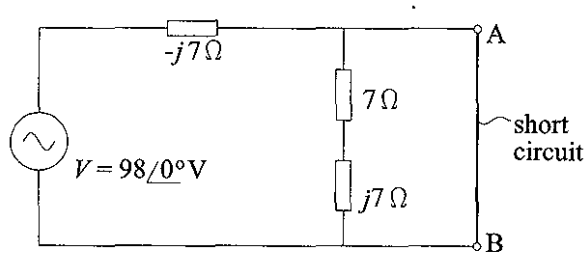


Fig. 5.27

$$I_N = \frac{V}{-j7} = \frac{98/0^\circ}{7/-90^\circ} = 14/90^\circ \text{ A}$$

Now redraw the circuit:

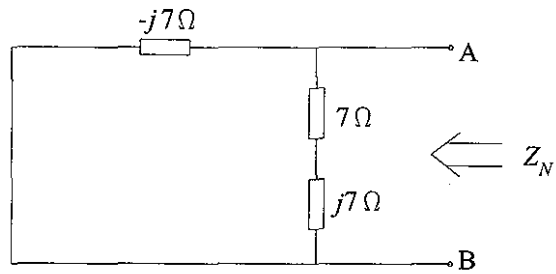


Fig. 5.28

$$Z_N = \frac{-j7(7+j7)}{7-j7+j7} = 7 - j7 = 9.9/-45^\circ \ \Omega$$

The Norton equivalent circuit is:

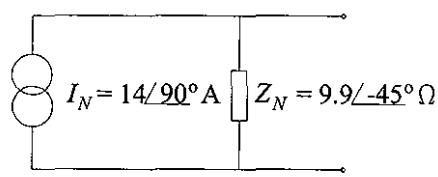


Fig. 5.29

Activity 5.4

Compare the results from Example 9 with those worked out for the Thevenin equivalent in Example 6.

Describe the equivalence of the two circuits.

Now let's determine again the current flowing in and the voltage drop across two resistors of 1 Ω connected in series between the terminals of this circuit:

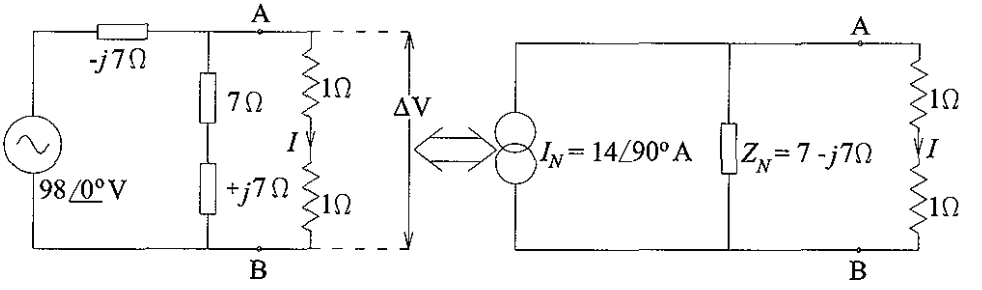


Fig. 5.30

$$V_{AB} = I_N \times \frac{(7-j7)2}{9-j7} = 14\underline{/90^\circ} \times 2\,\frac{9.9\underline{/-45^\circ}}{11.4\underline{/-37.87^\circ}} = 24.3\underline{/82.87^\circ}\,\text{V}$$

If you look at Example 7 you'll see that the same voltage drop occurs across the resistors. This shows that we worked correctly through both methods.

And

$$I_{AB} = \frac{V_{AB}}{2R} = \frac{24.3\underline{/82.87^\circ}}{2} = 12.15\underline{/82.87^\circ}\,\text{A}$$

– again, a value that we expected.

5.3 Summary

There are (almost) always alternative methods for studying the circuits, be it AC or DC, simple or complex. In this unit we studied four methods of analysis.

The most common way of solving network problems is using **Kirchhoff's laws**:

Kirchhoff's current law: The algebraic sum of the currents in any node of a circuit is zero.

Kirchhoff's voltage law: The algebraic sum of the voltages around any loop (mesh) in a circuit is zero.

$$\sum_{node} i = 0$$

$$\sum_{loop} v = 0$$

The **superposition theorem** states:

In a linear network containing more than one source of energy, the current in each branch can be calculated as the algebraic sum of the currents that would be produced by each source acting alone in the circuit.

We often use this to analyse circuits with two or more active elements. The results can be checked using another method.

Thevenin's theorem is:

Any linear active network with output terminals A and B can be replaced by a single voltage V_{th} in series with a single impedance Z_{th}.

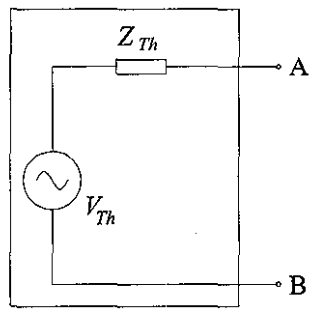


Fig. 5.31

Nortons theorem states:

Any linear active network with output terminals A and B can be replaced by a current source I_N in parallel with a single impedance Z_N.

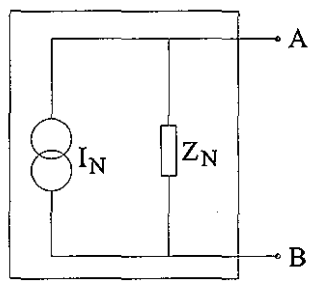


Fig. 5.32

The relation between Thevenin and Norton equivalent circuits is:

$$Z_{th} = Z_N$$
$$V_{th} = I_N Z_N$$
$$I_N = \frac{V_{th}}{Z_{th}}$$

Both methods can save you lots of time and energy, if they are used appropriately and correctly.

Remember that all these theories can be used to analyse DC and AC networks, and that it is always better to attempt to solve the problems using at least two methods and comparing the results.

Self-evaluation

1. a. Look at the circuit in fig. 5.33 and use Kirchhoff's laws to find the currents in each branch.

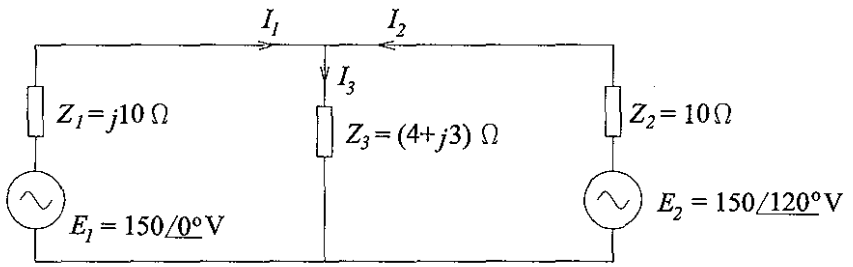


Fig. 5.33

b. Now attempt to solve the problem using the superposition theorem and check if you get the same result.

2. a. Study the following circuit.

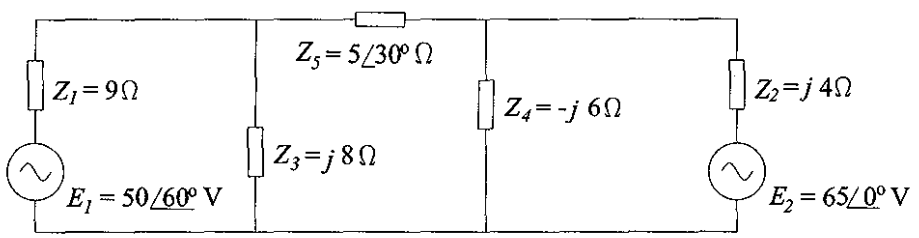


Fig. 5.34

b. Calculate the current flowing through Z_2 using the superposition theorem.
Check the result by applying Kirchhoff's laws.

3. Work out Thevenin's equivalent circuit for fig. 5.35, between the terminals A and B:

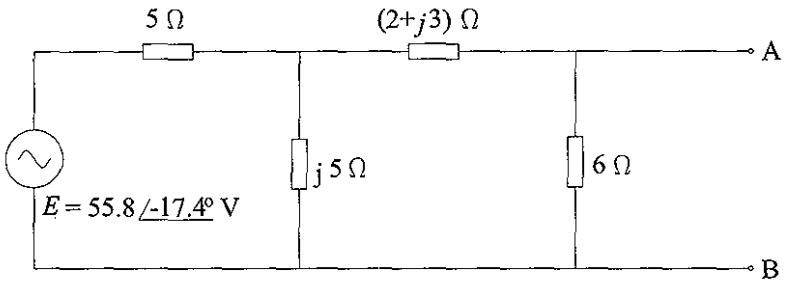


Fig. 5.35

4. For the circuit in Question 3, determine Norton's equivalent circuit between terminals A and B, without using the previous result. Then check the results.

Answers

Activity 5.1 (page 125)

Starting at a node in a circuit we form **a loop** traversing through elements and returning to the starting node, never encountering any other node more than once.

Activity 5.2 (page 132)

$R = 0.833\ \Omega$

Activity 5.3 (page 136)

$(j8)\ \Omega \rightarrow$ is a pure inductive reactance.
$(8 - j6)\ \Omega \rightarrow$ is a series combination of resistance and capacitance.
$12\ \Omega \rightarrow$ is a resistance.

We have all the possible elements in the circuit (R, L, C)

Activity 5.4 (page 148)

$Z_N = Z_{th}$, since we mentioned the same circuit and
$I_N Z_N = 14\underline{/90°} \times 9.9\underline{/-45°} = 138.6\underline{/45°} = V_{th}$ V from Example 6

So, the equivalence between Norton and Thevenin circuits holds for AC circuits. It can be expressed as:

$$Z_{th} = Z_N$$
$$V_{th} = I_N Z_N$$

or

$$I_N = \frac{V_{th}}{Z_{th}}$$

* *Fundamentals of Electrical Engineering*, 2nd edition, by LS Bobrow, published by Oxford University Press in 1996.

▶ Self-evaluation (page 151)

1. a. $I_1 = 16.25\underline{/-83.47^\circ}$ A
   $I_2 = 16.14\underline{/113.2^\circ}$ A
   $I_3 = 4.69\underline{/-163.85^\circ}$ A

   (With the directions of flow marked in the figure)

   b. The results should be the same (if you assume the same directions for the currents).

2. a. $I_{Z_2} = 8.46\underline{/23.7^\circ}$ A, (with direction from E_2 to Z_2).

   b. Same

3. $V_{\text{th}} = 19.86\underline{/0^\circ}$ V
   $Z_{\text{th}} = 3.6\underline{/23.06^\circ}\ \Omega$

4. $I_N = 5.52\underline{/-23.06^\circ}$ A
   $Z_N = 3.6\underline{/23.06^\circ}\ \Omega$

unit 6

Networks Theory (Part II)

OUTCOMES

After studying this unit, you should be able to:

- analyse different circuits using the Loop-Current Theory
- solve problems applying the Node-Voltages Analysis
- perform star-delta and delta-star transformations to simplify circuits when necessary
- apply the Maximum Power Transfer Theorem to networks
- generally solve any DC or AC circuit problem using an appropriate method and be able to check your solution by applying at least one other theory.

With the knowledge you accumulate from units 5 and 6, you should be able to solve almost any DC or AC circuit problem.

PREVIOUS PAGE BLANK

6.1 Introduction

In this unit we will introduce you to further techniques in analysing network using Kirchoff's laws.

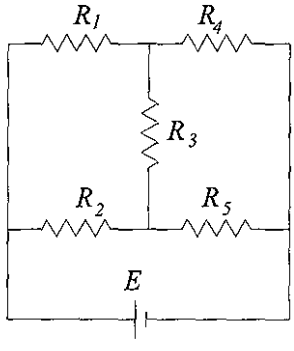


Fig. 6.1

Can you work out the current flowing in each of the five resistors? You should be able to do this using the knowledge you gained in Unit 5. But is there another method that can simplify the calculations?

Do you remember star-delta and delta-star transformations from Electrical Engineering I? Let's try to solve the problem with a delta-star transformation:

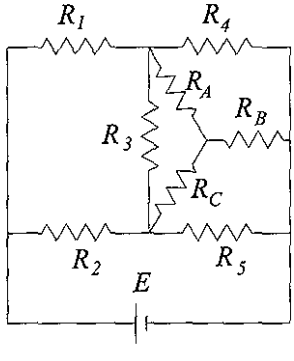


Fig. 6.2

We replaced the 'delta' configuration of R_3, R_4, R_5 with an **equivalent** 'star' configuration of R_A, R_B and R_C.

So, we now have R_1 in series with R_A, R_2 in series with R_C; the two branches in parallel, etc.

In this unit, we are going to look at alternative methods of circuit analysis that will help us to work much faster and easier.

6.2 Other methods used in circuit analysis

6.2.1 Loop-current analysis

If circuits contain multiple sources that are not connected in series (voltage) or in parallel (current), we need to use special techniques in our analysis.

One of these techniques is 'loop' or 'mesh' analysis, which is a variation of Kirchhoff's laws (KVL and KCL).

This technique requires that we write KVL around two or more loops and then solve the system of equations for the desired unknowns.

In this case, the unknowns are the **loop currents**. Once we know what these are, we can work out the current flowing in any component and the voltage across it.

For loop analysis, all the sources in the circuit are voltage sources. So, if we have current sources we must transform them to equivalent voltage sources before we go any further.

Then we need to select the closed loops in the circuit for which we will write KVL.

After that, we can choose a reference direction that we think is positive for that loop, be it clockwise or anti-clockwise.

For each loop we need to designate a current that flows around it: I_1, I_2 ... etc. These will be our unknowns in equations.

TAKE NOTE

Some books use J_1, J_2 ... etc. for these currents, to differentiate them from normal currents flowing in the **branches** of the circuit, which they mark I_1, I_2 ... etc.

We will need to write an equation for each loop to reach a solution.

Then, superimposing the currents we can calculate the currents flowing in each element (branch) of the circuit and the corresponding voltage drops if required (or powers), depending on the problem.

EXAMPLE 6.1

Study the following circuit:

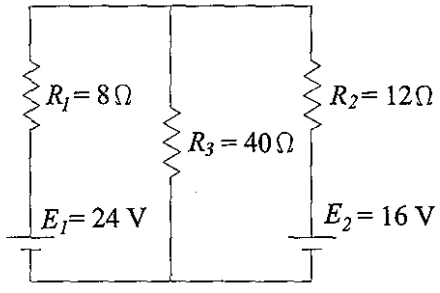


Fig. 6.3

Use loop-current analysis to find the current flowing in R_3.

Solution

First choose the directions of loop currents that will be analysed:

In this example, we'll make the clockwise direction positive. So, we'll use the following loops:

(1) $E_1 \rightarrow R_1 \rightarrow R_3 \rightarrow E_1$ and

(2) $R_3 \rightarrow R_2 \rightarrow E_2 \rightarrow R_3$

Now put the directions in the figure:

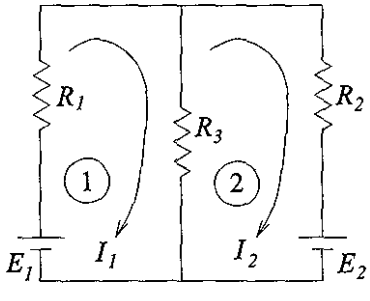


Fig. 6.4

Currents through the resistors R_1, R_2, R_3:
Through R_1 flows I_1
Through R_2 flows I_2
Through R_3 flows $I_1 - I_2$

The next step is to write the two equations for the two loops:

Loop (1)

$E_1 = R_1 I_1 + R_3 I_1 - R_3 I_2$ (because in R_3 flows $I_1 - I_2$)

Loop (2)

$-E_2 = R_2 I_2 + R_3 I_2 - R_3 I_1$

If we refer to the known data, we get:

$24 = 8 I_1 + 40 I_1 - 40 I_2 \quad \ldots (1)$

$-16 = 12 I_2 + 40 I_2 - 40 I_1 \ldots (2)$ (multiply the equation by -1)

$\Rightarrow$

$24 = 48 I_1 - 40 I_2$ (multiply the equation by 5)

$16 = 40 I_1 - 52 I_2$ (multiply the equation by 6)

$120 = 240 I_1 - 200 I_2$

$96 = 240 I_1 - 312 I_2$

Subtract equation (2) from equation (1):

$24 = 0 + 112 I_2 \Rightarrow I_2 = \frac{24}{112} = 0.214$ A

Replacing I_2 in the first equation $\Rightarrow$

$120 = 240 I_1 - 200 \times 0.214 = 240 I_1 - 42.8$

$\Rightarrow 162.8 = 240 \mathrm{I}_1 \Rightarrow$

$I_1 = \frac{162.8}{240} = 0.678$ A

In: R_1 flows $I_1 = 0.678$ A
R_2 flows $I_2 = 0.214$ A
R_3 flows $I_1 - \mathrm{I}_2 = 0.678 - 0.214 = 0.464$ A

The direction is shown in fig. 6.5:

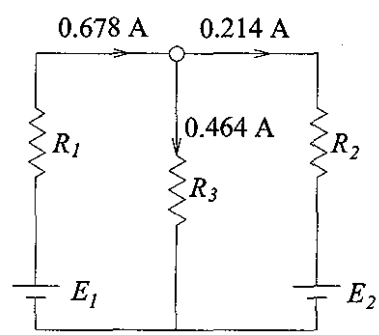


Fig. 6.5

If you page back to Example 3 in Unit 5, you'll see that we've just solved the same problem using loop (mesh) analysis instead of superposition. As expected, we get exactly the same results for the currents flowing in the resistors (only now we are working more accurately).

EXAMPLE 6.2

Calculate the current flowing through Z_3 in the following circuit using the loop-current analysis:

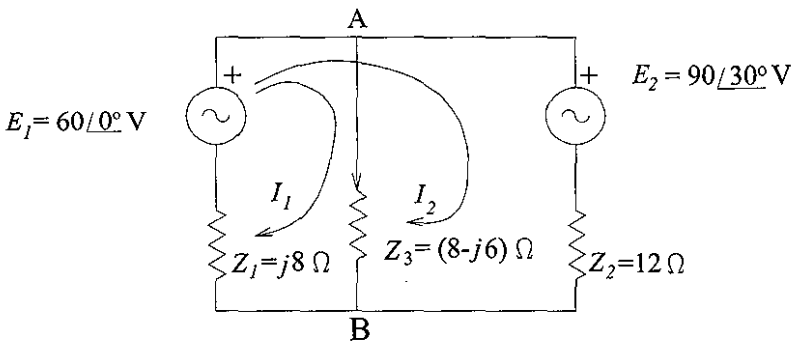


Fig. 6.6

Solution

Again, we had to choose two loops and the positive directions for the loops. This time, we tried loops (1) $E_1 \rightarrow Z_3 \rightarrow Z_1 \rightarrow E_1$ and (2) $E_1 \rightarrow E_2 \rightarrow Z_2 \rightarrow Z_1 \rightarrow E_1$. The positive direction is clockwise for both I_1 and I_2.

Now write the two equations:

$$E_1 = Z_3 I_1 + Z_1 (I_1 + I_2)$$
$$E_1 - E_2 = Z_2 I_2 + Z_1 (I_1 + I_2)$$
$$\Rightarrow \quad E_1 = (Z_1 + Z_3) I_1 + Z_1 I_2 \text{ (multiply the equation by } -1)$$
$$E_1 - E_2 = (Z_2 + Z_1) I_2 + Z_1 I_1$$
$$\Rightarrow \quad -E_1 = -(Z_1 + Z_3) I_1 - Z_1 I_2$$
$$-E_2 + E_1 = Z_1 I_1 + (Z_1 + Z_2) I_2$$

Add the two equations and you get:

$$-E_2 = -Z_3 I_1 + Z_2 I_2 \Rightarrow I_2 = \frac{Z_3 I_1 - E_2}{Z_2}$$

Now substitute this value in the first equation:

$$E_1 = (Z_1 + Z_3)I_1 + Z_1 \frac{Z_3 I_1 - E_2}{Z_2} \Rightarrow$$

$$E_1 + \frac{Z_1}{Z_2} E_2 = (Z_1 + Z_3 + \frac{Z_1 Z_3}{Z_2})I_1 \Rightarrow$$

$$I_1 = \frac{E_1 + \frac{Z_1}{Z_2}E_2}{Z_3 + Z_1 + \frac{Z_1 Z_3}{Z_2}}$$

In the example, this is the only current flowing through Z_3, so this is the value the problem asks for.

Did you notice that we no longer have to calculate I_2?

Substituting the known values we have:

$$I_1 = \frac{60\underline{/0^\circ} + \frac{8\underline{/90^\circ}}{12\underline{/0^\circ}} \times 90\underline{/30^\circ}}{8 - j6 + j8 + \frac{(8-j6)j8}{12}} = \frac{60\underline{/0^\circ} + 60\underline{/120^\circ}}{8 + j2 + j\frac{16}{3} + \frac{12}{3}}$$

$$= \frac{60\underline{/0^\circ} + 60\underline{/120^\circ}}{12 + j\frac{22}{3}} = \frac{60\underline{/60^\circ}}{14.06\underline{/31.42^\circ}} = 4.27\underline{/28.58^\circ}$$

$$= 3.75 + j2.04 \text{ A}$$

So, the current flowing in Z_3 is $4.27\underline{/28.58^\circ}$ and has the direction of I_1 (from node A to node B).

Let's check if we have worked correctly. Page back to Example 4 in Unit 5 and you'll find that we analysed exactly the same circuit using the superposition theorem and that the current flowing through Z_3 was $4.26\underline{/28.73^\circ}$ A.

Ignoring the small errors you can see that we obtained the same current, so we don't need to continue with the calculation.

We have solved literally up to the end (to the final result) and have worked correctly.

TAKE NOTE

The sources here are sinusoidal and both have the same frequency.

6.2.2 Node-voltage analysis

Node-voltage analysis or nodal analysis is another method of analysing multisource circuits.

This method requires that we solve a system of equations, so we could use matrices and determinants. The equations are obtained when KCL is applied for different nodes of the circuit.

First we must choose a 'reference' node, to which all the other nodes' voltages can be compared, measured or calculated. This node is better known as the 'ground' or 'common' node for the network.

The next step is to convert all voltage sources to their equivalent current sources.

Now number all the nodes (except the ground node) 1, 2, ... and, assuming the reference node has a 0(V) potential, designate all the corresponding voltages V_1, V_2, ...

Our task is to find the values of V_1, V_2, ... and, since we have 'n' equations with 'n' unknowns, the equations system should give us an unique solution.

To apply KCL we must assume the directions of our currents – in other words, which currents enter the node and which leave the node.

Here are the steps involved in using this method:

1 Assume a direction for the current flowing in each branch of the circuit (from the higher to the lower voltage).

2 Write on each arrow the value of the currents, expressed as a function of the node voltages

In fig. 6.7 we assumed $V_1 > V_2$, so the current flows from node 1 to node 2

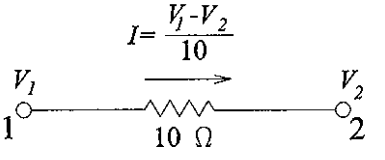


Fig. 6.7

3 Write KCL for each node and solve the resulting system, working out V_1, V_2, ... V_n.

EXAMPLE 6.3

Use node-voltages analysis to work out the current flowing in the following circuit and the value of V_1.

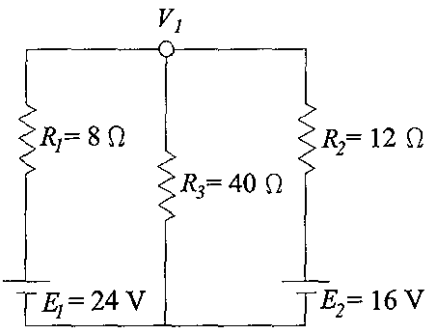


Fig. 6.8

Solution

First convert the two voltage sources to current sources:

$$I_1 = \frac{24}{8} = 3 \text{ A}$$

and

$$I_2 = \frac{16}{12} = 1.333 \text{ A}$$

The circuit is now:

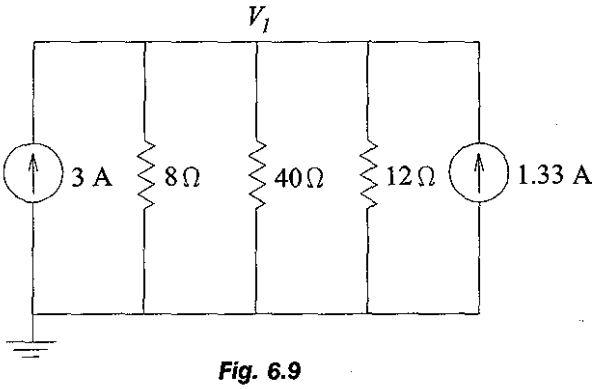


Fig. 6.9

Now we need to choose a reference. Since the problem asks us to determine the node voltage V_1 for the upper node of the circuit, it's obvious that we should choose the ground (reference) as the lower node. This is marked on the circuit in fig. 6.9.

Assuming that $V_1 > 0$, the currents in the three resistances will have the direction $\downarrow$.

Mark these on the circuit:

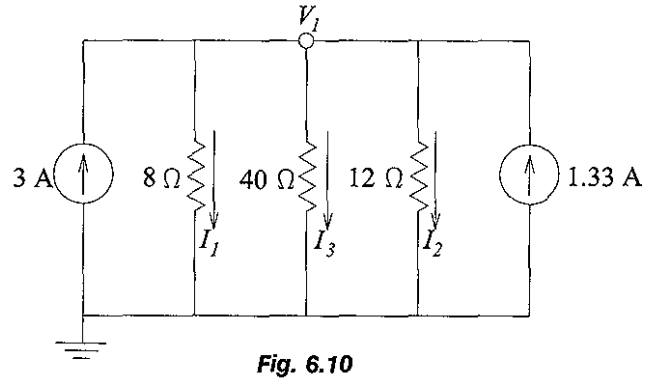


Fig. 6.10

According to Ohm's law, the current in each resistor will be:

$$I_1 = \frac{V_1}{8}\text{ A};\ I_2 = \frac{V_1}{12}\text{ A};\ I_3 = \frac{V_1}{40}\text{ A}$$

Now write KCL for node 1

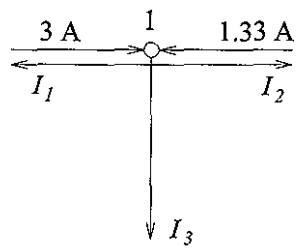


Fig. 6.11

Did you notice that the hypothetical currents from the two sources enter the nodes, whereas all the others go out of the node?

$$3 + 1.333 = \frac{V_1}{8} + \frac{V_1}{40} + \frac{V_1}{12}$$

$$\Rightarrow\ 4.333 = V_1\left(\frac{1}{8} + \frac{1}{40} + \frac{1}{12}\right) = 0.233V_1$$

$$\Rightarrow V_1 = \frac{4.333}{0.233} = 18.597\text{ V}$$

And the currents will be:

$$I_1 = \frac{18.597}{8} = 2.325\text{ A}$$

$$I_2 = \frac{18.597}{12} = 1.550\text{ A}$$

$$I_3 = \frac{18.597}{40} = 0.465\text{ A}$$

Adding algebraically the currents for the 8 Ω resistor:

$$I_{real\ 1} = 3 - 2.325 = 0.675 \text{ A}$$

Remember that I_{real} is the true current flowing in R_1.

Do the same for the 12 Ω resistor:

$$I_{real\ 2} = 1.333 - 1.550 = -0.217 \text{ A}$$

This means that this current is flowing from node 1 to E_2 , and not vice versa as we assumed.

Compare these results with those obtained in example 1 of this unit and you'll see that they are the same. We analysed the same circuit! This shows that we worked correctly.

EXAMPLE 6.4

Calculate the current flowing through Z_3 in the following circuit using nodal analysis:

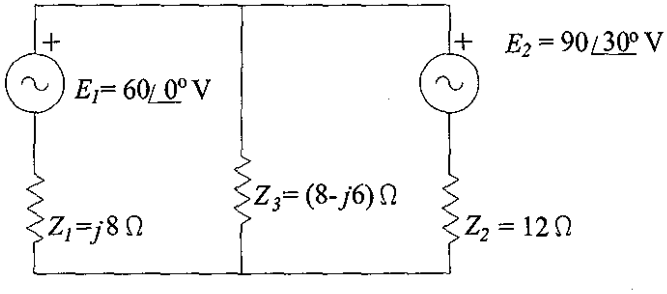


Fig. 6.12

Solution

First transform the two voltage sources to equivalent current sources:

$$I_1 = \frac{E_1}{Z_1} = \frac{60\angle 0°}{j8} = \frac{60\angle 0°}{8\angle 90°} = 7.5\angle -90° \text{ A}$$

$$I_2 = \frac{E_2}{Z_2} = \frac{90\angle 30°}{12} = \frac{90\angle 30°}{12\angle 0°} = 7.5\angle 30° \text{ A}$$

Then the circuit becomes:

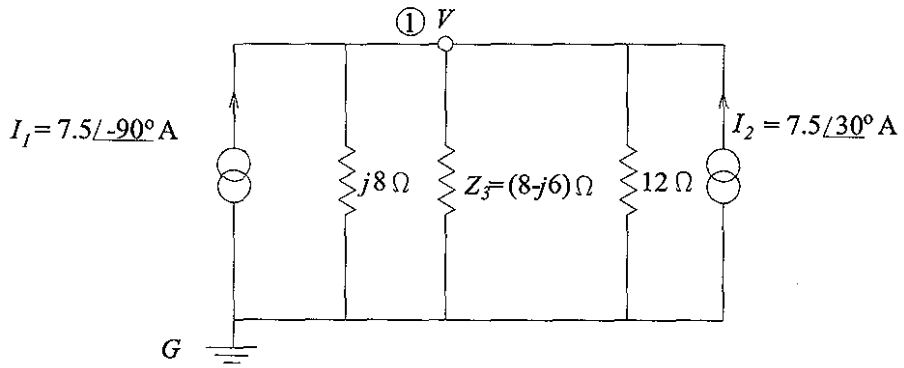


Fig. 6.13

Choose a reference node (ground) and assume that node 1 is at a higher potential than ground ($V > 0$). Then the three currents will have the direction from node 1 towards the ground (from up to down).

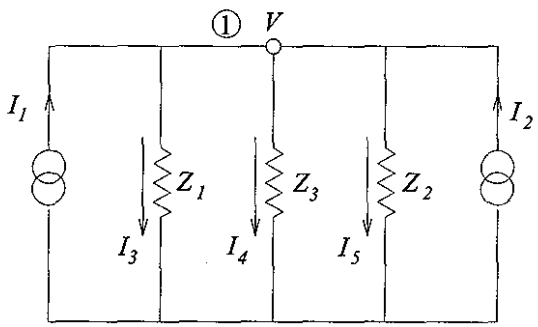


Fig. 6.14

The current in each impedance will be:

$$I_3 = \frac{V}{Z_1} = \frac{V}{8\underline{/90°}} \text{ A}$$

$$I_4 = \frac{V}{Z_3} = \frac{V}{8-j6} = \frac{V}{10\underline{/-36.87°}} \text{ A}$$

$$I_5 = \frac{V}{Z_2} = \frac{V}{12\underline{/0°}} \text{ A}$$

Now we can write KCL for node 1:

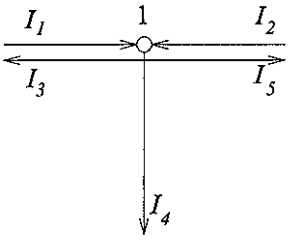


Fig. 6.15

$$7.5\underline{/-90^\circ} + 7.5\underline{/30^\circ} = V\left(\frac{1}{8\underline{/90^\circ}} + \frac{1}{10\underline{/-36.87^\circ}} + \frac{1}{12\underline{/0^\circ}}\right) \Rightarrow$$

$$6.5 - j3.75 = V(0.125\underline{/-90^\circ} + 0.1\underline{/36.87^\circ} + 0.083\underline{/0^\circ}) \Rightarrow$$

$$7.5\underline{/-30^\circ} = V(0.163 - j0.065) = V \times 0.175\underline{/-21.74^\circ} \Rightarrow$$

$$V = \frac{7.5\underline{/-30^\circ}}{0.175\underline{/-21.74^\circ}} = 42.857\underline{/-8.26^\circ}\ \text{V}$$

And the current in Z_3 will be:

$$I_4 = \frac{V}{Z_3} = \frac{42.857\underline{/-8.26^\circ}}{10\underline{/-36.87^\circ}} = 4.2857\underline{/28.61^\circ}\ \text{A}$$

$$= 3.76 + j2.05\ \text{A}$$

If we compare this result with the one obtained using a different method – loop-current analysis in example 2 of this unit – we can see that we worked correctly. Both nodal analysis and loop current analysis are useful when studying multisource circuits.

6.2.3 Delta-star and star-delta transformations

Let's analyse the following circuit:

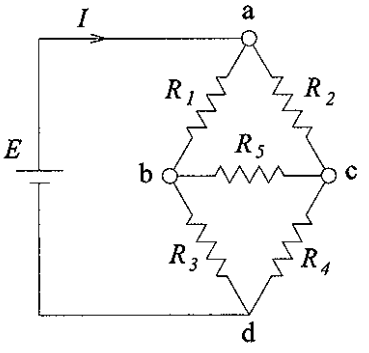


Fig. 6.16

In fig. 6.16, b, c and d form a triangle called 'delta'. This delta can be converted to star (or wye) configuration with the resistances connected to a common point with the ends b, c and d as shown in fig. 6.17 below.

There is a way to transform the configuration between the points b, c and d in the circuit (where R_3, R_4 and R_5 are connected in a triangle – the so-called 'delta' connection) into a new configuration containing three resistances joined in a common point and with the other ends connected to b, c and d (where these new resistances are connected in a 'star' or 'wye' configuration).

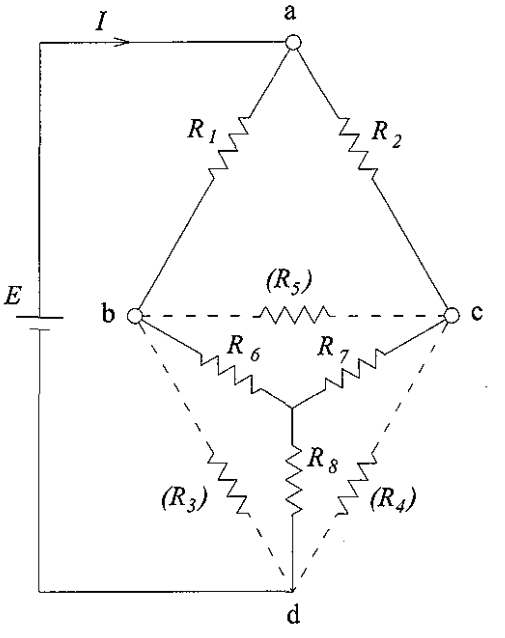


Fig. 6.17

where R_6, R_7, R_8 are the new resistors connected in a common point S (the star point) and between the star point and our initial nodes b, c, d.

You may be asking: 'Why do we do this?'

Look again at the circuit. Can you find the current I without using simultaneous equations?

R_1 is in series with R_6; R_2 is in series with R_7. These two branches are in parallel and everything is then in series with R_8.

The circuit can be re-drawn:

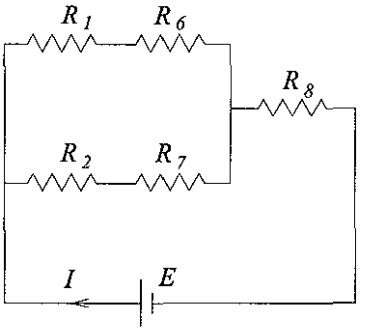


Fig. 6.18

The equivalent resistance of the circuit is:

$$R_{eq} = \frac{(R_1 + R_6)(R_2 + R_7)}{R_1 + R_2 + R_6 + R_7} + R_8$$

Then we can calculate the current flowing in the circuit as:

$$I = \frac{E}{R_{eq}}$$

without writing the system of equations that would be necessary if we tried to solve the problem using, for example, Kirchhoff's laws.

So what are the transformation formulas we can use to calculate the three replacement resistors R_6, R_7 and R_8?

Based on the fact that the resistance between two terminals must be the same before and after the change, we can deduct the formulas for the replacement resistors.

Consider these equivalent circuits:

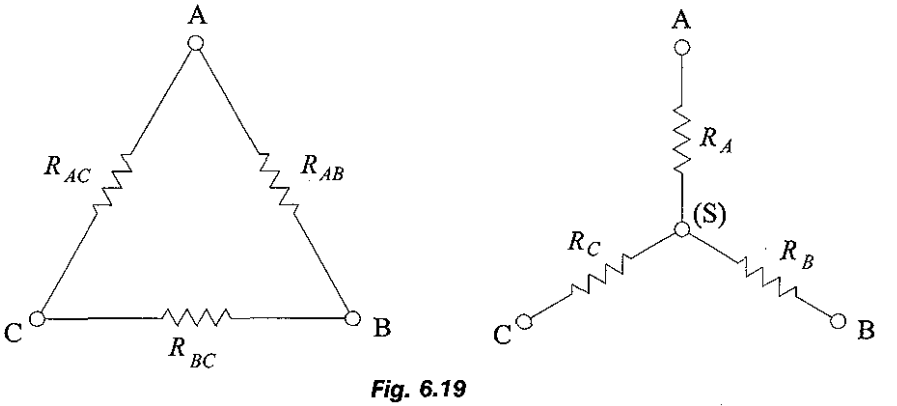


Fig. 6.19

The relations between the Δ and Y resistors can be written as:

$$R_A = \frac{R_{AB}R_{AC}}{R_{AB} + R_{AC} + R_{BC}}$$

$$R_B = \frac{R_{AB}R_{BC}}{R_{AB} + R_{AC} + R_{BC}}$$

$$R_C = \frac{R_{AC}R_{BC}}{R_{AB} + R_{AC} + R_{BC}}$$

These are the equations for delta-star transformation.

This is how we calculate the equivalence if we know the delta-connected resistors and want to transform the delta to a star.

Take note of the symmetry of these equations – it helps when you have to remember them in a hurry. More than that, the denominators are all the same. So, for each numerator we have the product of the 'adjacent' resistors (i.e. R_{AB} and R_{AC} for R_A) divided by the sum of the three.

Let's now assume that this time we know the star-connected resistors and we want to find the equivalent delta-connection.

With this set of equations, it can be mathematically deduced that:

$$R_{AB} = \frac{R_AR_B + R_BR_C + R_CR_A}{R_C}$$

$$R_{BC} = \frac{R_AR_B + R_BR_C + R_CR_A}{R_A}$$

$$R_{AC} = \frac{R_AR_B + R_BR_C + R_CR_A}{R_B}$$

These are the equations for star-delta transformation.

In some books, these equations are written:

$$R_{AB} = R_A + R_B + \frac{R_AR_B}{R_C}$$

$$R_{BC} = R_B + R_C + \frac{R_BR_C}{R_A}$$

$$R_{AC} = R_A + R_C + \frac{R_AR_C}{R_B}$$

They are exactly the same as the first set of equations we looked at earlier. Again the formulas are symmetrical, only this time the numerators are the same.

You may prefer to use the first set of equations, because once you have calculated $R_AR_B + R_BR_C + R_CR_A$, you can use it in all three.

The star-delta and delta-star sets of transformation can be generalised for AC circuits. Then the formulas for the impedances become:

For delta-star transformation:

$$Z_A = \frac{Z_{AB}Z_{AC}}{Z_{AB}+Z_{AC}+Z_{BC}}$$

$$Z_B = \frac{Z_{AB}Z_{BC}}{Z_{AB}+Z_{AC}+Z_{BC}}$$

$$Z_C = \frac{Z_{AC}Z_{BC}}{Z_{AB}+Z_{AC}+Z_{BC}}$$

where Z_{AB}, Z_{BC} and Z_{CA} are the impedances connected in delta and Z_A, Z_B and Z_C are the equivalent star impedances.

And for star-delta transformation:

$$Z_{AB} = \frac{Z_AZ_B+Z_BZ_C+Z_CZ_A}{Z_C}$$

$$Z_{BC} = \frac{Z_AZ_B+Z_BZ_C+Z_CZ_A}{Z_A}$$

$$Z_{AC} = \frac{Z_AZ_B+Z_BZ_C+Z_CZ_A}{Z_B}$$

A specific case of stars and deltas is when all the three impedances are equal. These are called 'balanced' stars (or deltas) and the relations between the elements become:

$Z_{star} = Z_{delta} / 3$ or, respectively

$Z_{delta} = 3 \times Z_{star}$

These transformations are very useful when studying special configurations of circuits, bridges etc. Later you'll see how important they are in three-phase circuits and electrical machines.

Activity 6.1

Three impedances each of $10\underline{/60°}$ Ω are connected in star. What will their values be if they are transformed into an equivalent delta configuration?

Activity 6.2

Three impedances each of $5\underline{/30°}$ Ω are connected in delta. What will their value be if they are transformed into an equivalent star configuration?

EXAMPLE 6.5

A network of six resistances is connected like this:

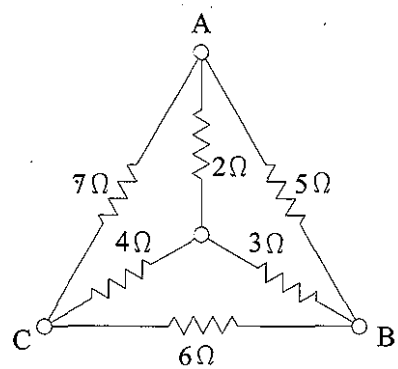


Fig. 6.20

Calculate the equivalent resistance of the circuit between points A and B.

Solution

Transform the star-connection to a delta connection:

For the new connection:

$$R_{AB} = \frac{2\times3+3\times4+4\times2}{4} = \frac{26}{4} = 6.5\ \Omega$$

$$R_{BC} = \frac{26}{2} = 13\ \Omega$$

$$R_{CA} = \frac{26}{3} = 8.67\ \Omega$$

Now we have two deltas in parallel. So the equivalent resistances will be:

$$R_{eqAB} = \frac{6.5\times5}{6.5+5} = 2.83\ \Omega$$

$$R_{eqBC} = \frac{13\times6}{13+6} = 4.11\ \Omega$$

$$R_{eqAC} = \frac{8.67\times7}{8.67+7} = 3.87\ \Omega$$

And the total resistance between A and B is :

$$R_{total} = (R_{eqBC} + R_{eqCA})//R_{eqAB}$$

$$= \frac{(4.11+3.87)\times2.83}{4.11+3.87+2.83} = \frac{7.98\times2.83}{10.81} \cong 2.1\ \ \Omega$$

EXAMPLE 6.6

Let's assume we have a star-connected load that consists of the following three impedances:

$$Z_A = 20\underline{/0^\circ}\ \Omega$$
$$Z_B = 20\underline{/-30^\circ}\ \Omega$$
$$Z_C = 20\underline{/30^\circ}\ \Omega$$

Work out the equivalent delta-connected impedance.

Solution:

TAKE NOTE Don't think, even though the modulus of the three impedances are equal, that this is a balanced star. As you can see, the phase angles differ (there is a resistive load Z_A, a resistive-capacitive combination Z_B, and a resistive-inductive combination Z_C).

So, we'll have to calculate the three delta impedances (Z_{AB}, Z_{BC}, Z_{CA}) individually as they are unequal:

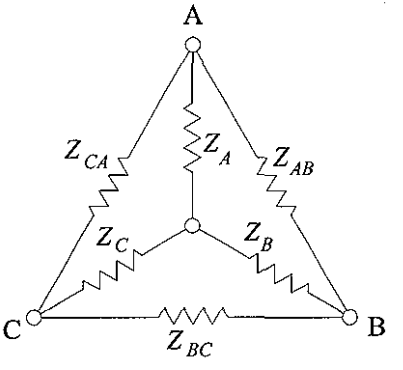


Fig. 6.21

Let's write the sum $Z_AZ_B + Z_BZ_C + Z_CZ_A = \Sigma$

Then:

$$\begin{aligned}\Sigma &= 20\underline{/0^\circ} \times 20\underline{/-30^\circ} + 20\underline{/-30^\circ} \times 20\underline{/30^\circ} + 20\underline{/30^\circ} \times 20\underline{/0^\circ}\\ &= 400\underline{/-30^\circ} + 400\underline{/0^\circ} + 400\underline{/30^\circ}\\ &= 1\,092.82 + j0\\ &= 1\,092.82\underline{/0^\circ}\ \Omega^2\end{aligned}$$

Then:

$$Z_{AB} = \frac{\Sigma}{Z_C} = \frac{1092.82\underline{/0^\circ}}{20\underline{/30^\circ}} = 54.64\underline{/-30^\circ}\ \Omega$$

$$Z_{BC} = \frac{\Sigma}{Z_A} = \frac{1\,092.82\underline{/0^\circ}}{20\underline{/0^\circ}} = 54.64\underline{/0^\circ}\ \Omega$$

$$Z_{CA} = \frac{\Sigma}{Z_B} = \frac{1\,092.82\underline{/0^\circ}}{20\underline{/-30^\circ}} = 54.64\underline{/30^\circ}\ \Omega$$

Notes

1 The three values are unequal. They have the same modulus, but different angles.

2 Now you can see why using this set of formulas, especially when we work with impedances (i.e. complex numbers) saves time. The only risk is that, if you make a mistake when you calculate Σ, all the values will be wrong.

EXAMPLE 6.7

Assume we have a delta-connected load that consists of the following three impedances:

$$Z_{AB} = 27.3\underline{/0^\circ}\ \Omega$$
$$Z_{BC} = 27.3\underline{/-30^\circ}\ \Omega$$
$$Z_{AC} = 27.3\underline{/-30^\circ}\ \Omega$$

Work out the equivalent star-connected impedances.

Solution

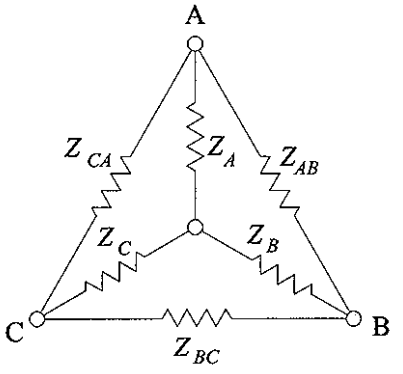


Fig. 6.22

$$\begin{aligned} & Z_{AB} + Z_{BC} + Z_{AC} \\ &= 27.3\underline{/0^\circ} + 27.3\underline{/-30^\circ} + 27.3\underline{/30^\circ} \\ &= 74.58 + j0 \\ &= 74.58\underline{/0^\circ}\ \Omega \end{aligned}$$

$$Z_A = \frac{Z_{AB}Z_{CA}}{Z_{AB}+Z_{BC}+Z_{CA}} = \frac{27.3\underline{/0^\circ} \times 27.3\underline{/30^\circ}}{74.5\underline{/0^\circ}} = 10\underline{/30^\circ}\ \Omega$$

$$Z_B = \frac{Z_{AB}Z_{BC}}{\Sigma} = \frac{27.3\underline{/0^\circ} \times 27.3\underline{/-30^\circ}}{74.58\underline{/0^\circ}} = 10\underline{/-30^\circ}\ \Omega$$

$$Z_C = \frac{Z_{CA}Z_{BC}}{\Sigma} = \frac{27.3\underline{/30^\circ} \times 27.3\underline{/-30^\circ}}{74.58\underline{/0^\circ}} = 10\underline{/0^\circ}\ \Omega$$

Did you expect this result? If you look at the previous example you'll see that the delta values were all double in modulus, but had the same angles. We expected the star impedances to be half those in the other example (although they are in a different sequence).

6.2.4 Maximum power transfer theorem

The theorem states that 'maximum output power is obtained from a network when the load impedance is the complex conjugate of the impedance of the network as seen from the terminals of the load'.

In DC (since we deal only with resistors $Z = R = Z^*$) the theorem can be expressed:

> 'Maximum output power is obtained from a network when the load resistance is equal to the output resistance of the network as seen from the load terminals.'

In AC we may have different situations depending of the types of loads available:

- if there is a variable (pure) resistive load (no reactance!), maximum power transfer takes place when:

$$R_{load} = |Z_{network}| = \sqrt{R^2_{network} + X^2_{network}}$$

- if there is a variable resistance and a fixed reactance, maximum power transfer takes place when:

$$R_{load} = \sqrt{R^2_{network} + (X_{network} + X_{load})^2}$$

- if only the reactance of the load varies, maximum power transfer takes place when:

$$X_{load} = -X_{network}$$

- when there is both variable resistance and reactance of load, maximum power transfer takes place when:

$$Z_{load} = Z^*_{network}$$

Remember, * is the symbol for complex conjugate.

EXAMPLE 6.8

Let's consider the following situation:

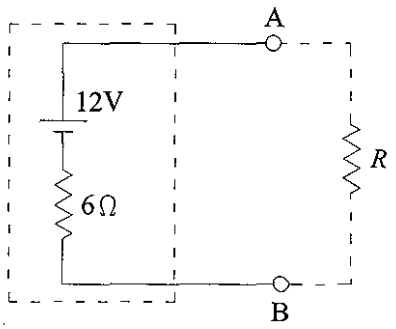


Fig. 6.23

The square block contains the Thevenin equivalent of a network. Between terminals A and B we can connect a resistive load R.

a. What will the value of R be when there is maximum power transfer between network and load?

b. Calculate the power for different loads:

$R_1 = 2\ \Omega$
$R_2 = 4\ \Omega$
$R_3 = 6\ \Omega$
$R_4 = 8\ \Omega$
$R_5 = 10\ \Omega$

and discuss the results.

Solutions

a. $R = 6\ \Omega$, equal to the resistance of the network, as seen from the output terminals A and B.

b. $\rightarrow R_1 = 2\ \Omega \Rightarrow R_{eq1} = 8\ \Omega \Rightarrow I_{T1} = \frac{12}{8} = 1.5$ A

and the power in the resistor $P_1 = 2 \times 1.5^2 = 4.5$ W

$\rightarrow R_2 = 4\ \Omega \Rightarrow R_{eq2} = 10\ \Omega \Rightarrow I_{T2} = \frac{12}{10} = 1.2\ \text{A}$

and the power in the resistor $P_2 = 4 \times 1.2^2 = 5.76$ W

$\rightarrow R_3 = 6\ \Omega \Rightarrow R_{eq3} = 12\ \Omega \Rightarrow I_{T3} = \frac{12}{12} = 1\ \text{A}$

and the power in the resistor $P_3 = 6 \times 1^2 = 6$ W

$\rightarrow R_4 = 8\ \Omega \Rightarrow R_{eq4} = 14\ \Omega \Rightarrow I_{T4} = \frac{12}{14} = 0.86\ \text{A}$

and the power in the resistor $P_4 = 8 \times 0.86^2 = 5.92$ W

$\rightarrow R_5 = 10\ \Omega \Rightarrow R_{eq5} = 16\ \Omega \Rightarrow I_{T5} = \frac{12}{16} = 0.75\ \text{A}$

and the power in the resistor $P_5 = 10 \times 0.75^2 = 5.625$ W

If we look at the power values we can see that the maximum power transfer is obtained for $R = R_{network} = 6\ \Omega$.

This is generally proven by calculating

$$\frac{dP_{load}}{dR_{load}}$$

and working out the value of R_{load} from the equation:

$$\frac{dP_{load}}{dR_{load}} = 0$$

Mathematical proof:
In this example, make E_g and R_g the equivalent Thevenin voltage and resistance for the active (generating) network, and R_L the load.

Then:

$$P_L = R_L \times I_T^2 = R_L \times \frac{E_g^2}{(R_L + R_g)^2}$$

$$\frac{dP_L}{dR_L} = \frac{E_g^2(R_g + R_L)^2 - 2R_L E_g^2(R_g + R_L)}{(R_L + R_g)^4} \equiv 0$$

$$E_g^2(R_g + R_L)^2 - 2R_L E_g^2(R_g + R_L) \equiv 0$$

for maximum power transfer.

$\Rightarrow$

$$R_g + R_L = 2R_L \Rightarrow R_g = R_L$$

EXAMPLE 6.9

Having a Thevenin equivalent circuit of a network:

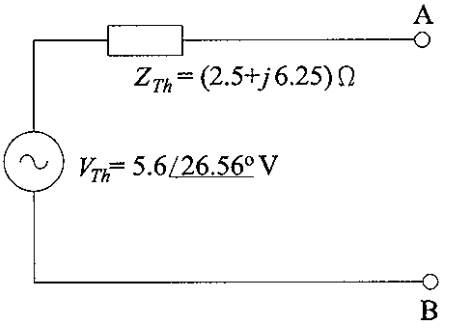


Fig. 6.24

how would you realise a load that, connected to the circuit, should absorb maximum power?

Solution

From the maximum power transfer theorem, we know that we must look for an impedance

$$Z_{load} = Z^*_{network} \Rightarrow$$

$$Z_{load} = 2.5 - j6.25\ \Omega$$

We can realise this load by connecting in series a resistor of 2.5 Ω with a capacitor.

$$X_C = \frac{1}{\omega c} = \frac{1}{2\pi fc} \Rightarrow C = \frac{1}{2\pi f X_C} = \frac{1}{2\pi \times 50 \times 6.25} = 0.000509\,\text{F}$$

So, what we are looking for is a capacitor $C \approx 500\ \mu$F

TAKE NOTE

We assumed $f = 50$ Hz as this is the supply frequency in South Africa.

So, the load would be:

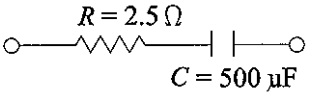


Fig. 6.25

Remember this is only one possible solution for this problem, but it is the simplest one!

Activity 6.3

Show how you can use different combinations of R, L and C to obtain the same load as in example 9.

Can you work out another combination for this load? Focus on the elements you would use and how you would connect them, not necessarily in calculated values!

6.3 Summary

- In this unit we studied two methods that can be used to analyse multisource circuits: loop (current) analysis and nodal (voltage) analysis.
- These are special applications of Kirchhoff's laws and, for both, you must solve a system of 'n' equations with 'n' unknowns.
- In loop analysis the unknowns are the loop currents. In nodal analysis the unknowns are the nodes voltages.
- In loop (mesh) analysis the circuits have only voltage sources. In nodal analysis the circuit must have only current sources.
- To determine the equations for loop analysis, you must write KVL for the loops you choose. To determine the equations for nodal analysis, you must write KCL for the nodes.
- Both methods need an assumed reference. For loop analysis choose a reference direction. For nodal analysis choose a reference node (point of zero potential).
- To transform a star to a delta use:

$$Z_{AB} = \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_C} \quad \text{etc.}$$

And for a balanced star:

$$Z_\Delta = 3Z_Y$$

To transform a delta to a star use:

$$Z_A = \frac{Z_{AB} Z_{CA}}{Z_{AB} + Z_{BC} + Z_{CA}} \quad \text{etc.}$$

And for a balanced delta:

$$Z_Y = \frac{Z_\Delta}{3}$$

Remember that in the first case the numerator is common ($\Sigma = Z_A Z_B + Z_B Z_C + Z_C Z_A$) and in the second case the denominator is common ($\Sigma = Z_{AB} + Z_{BC} + Z_{CA}$). This observation can save time when you are solving problems.

- Transformations are normally used when studying bridges, three-phase circuits, etc.
- The maximum power transfer theorem states that maximum power transfer takes place when:

  $R_{load} = R_{network\ (Thevenin)}$ for DC circuits

  and

  $Z_{load} = Z^*_{network\ (Thevenin)}$ for AC circuits
- When maximum power transfer is desired, 'match' the load and the source. If this is not required, avoid these values.

Self-evaluation

1 Find the total impedance of the following circuit and the total current flowing through it, i_T, where:

$Z_1 = j4\ \Omega$
$Z_2 = -j4\ \Omega$
$Z_3 = 8\ \Omega$
$Z_4 = j11\ \Omega$
$Z_5 = -j2\ \Omega$

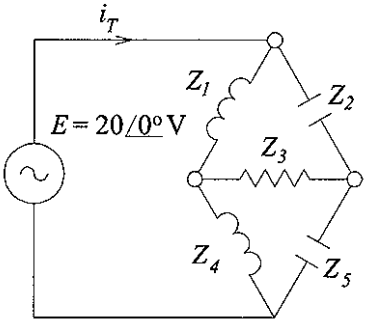


Fig. 6.26

2 Convert the following star system into an equivalent delta:

$Z_A = 100\angle -25°\ \Omega$
$Z_B = 175\angle -75°\ \Omega$
$Z_C = 215\angle 0°\ \Omega$

Where Z_A, Z_B and Z_C are the three star impedance's.

3 Study the following circuit:

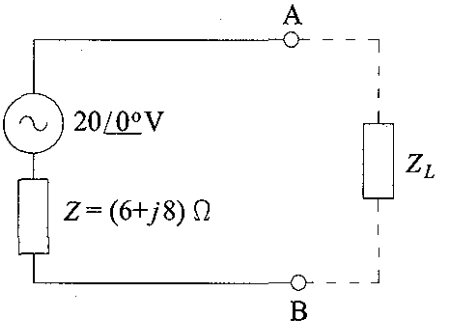


Fig. 6.27

Assume the following two loads:

1) $Z_L = (6 + j8)\ \Omega$
2) $Z_L = (6 - j8)\ \Omega$

In which situation will the power be transmitted to the load greater?

4 Use loop analysis to work out the current in the capacitor in the following circuit:

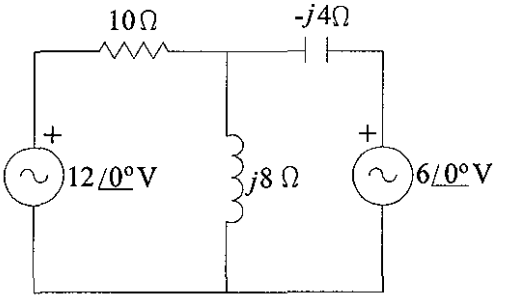


Fig. 6.28

Check the result using another method (for example, Kirchhoff's laws or superposition theorem).

Also work out the current flowing through the inductor in the circuit.

5 Use nodal analysis to find the current flowing in each element of the following circuit:

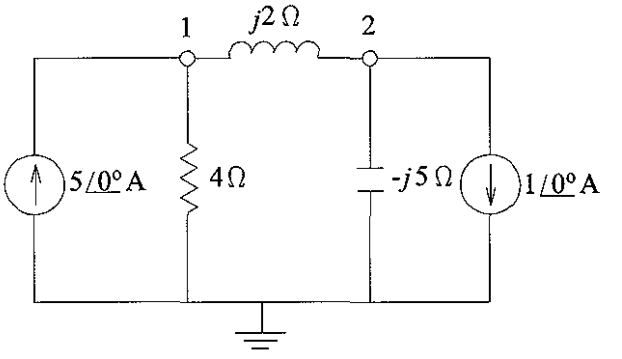


Fig. 6.29

Check Kirchhoff's current law at each node.

Hint: Choose the ground as in the figure (it's the only node to which four elements are connected), then you are left with two nodes (1 and 2).

Answers

Activity 6.1 (page 171)

$$Z_{star} = 10\underline{/60^\circ}\ \Omega$$
$$Z_{delta} = 3 \times Z_{star} \quad \text{(equal impedances)}$$
$$= 3 \times 10\underline{/60^\circ}$$
$$= 30\underline{/60^\circ}\ \Omega$$

Activity 6.2 (page 171)

$$Z_{delta} = 5\underline{/30^\circ}\ \Omega$$
$$Z_{star} = \frac{Z_{delta}}{3}$$
$$= \frac{5\underline{/30^\circ}}{3}$$
$$= 1.67\underline{/30^\circ}\ \Omega$$

Activity 6.3 (page 179)

Yes, a resistance (of 5 Ω) in parallel with another resistance (of 5 Ω), and the two resistors in series with a capacitance (in this case the same value $C = 509\ \mu\text{F}$) etc. (Theoretically there are infinite solutions to the problem.)

Self-evaluation (page 180)

1 $Z_{eq} = 10.2\underline{/-78.69^\circ}\ \Omega$
$i_T = 1.96\underline{/78.69^\circ}$ A

2 $Z_{AB} = 315.88\underline{/-67.3^\circ}\ \Omega$
$Z_{BC} = 679.14\underline{/-42.3^\circ}\ \Omega$
$Z_{CA} = 388.08\underline{/7.68^\circ}\ \Omega$

3 Obviously in the second case (see maximum power transfer theorem), so we don't need to calculate anything!

If we do calculate: $P_1 = 6$ W and $P_2 = 16.7$ W; so $P_2 > P_1$.

4 $I_C = 1.5\underline{/90^\circ}$ A $= j1.5$ A (flowing from left to right in the circuit)

$I_L = 1.5\underline{/-90^\circ}$ A (flowing from the upper part of the circuit towards the lower part)

5 $I_R = 2\underline{/-53.13^\circ}$ A (flowing up to down)
$I_L = 4.12\underline{/-22.83^\circ}$ A (flowing from left to right)
$I_C = 3.22\underline{/29.74^\circ}$ A (flowing from top to bottom)

You can check for each node that Σ currents entering the node $\equiv \Sigma$ currents leaving the node.

For example, in node 1:

$2\underline{/-53.13^\circ} + 4.12\underline{/22.83^\circ} = 5 - j0 = 5\underline{/0^\circ}$ A

unit 7

Harmonics

OUTCOMES

After studying this unit, you should be able to:

- Identify a non-sinusoidal waveform
- Analyse a non-sinusoidal waveform
- Determine the effective value, the rms value, power and the power factor in a circuit with a non-sinusoidal source.

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7.1 Introduction

In the previous unit, when dealing with alternating currents and voltages, we assumed they have a sinusoidal waveform or shape. Such a waveform is ideal and much sought after by alternator designers and manufacturers.

However, it is nearly impossible to realise such a waveform in practice. All the alternating waveforms deviate, to a greater or lesser degree, from the ideal sinusoidal shape. These waveforms are referred to as non-sinusoidal, distorted or complex waveforms.

7.2 Complex waveforms

Complex waveforms are produced by the superposition of sinusoidal waves of different frequencies. On analysis, a complex wave essentially consists of:

- a fundamental wave – it has the lowest frequency, say 'f'
- a number of other sinusoidal waves whose frequencies are an integer multiple of the fundamental or basic frequency, like $2f$, $3f$ and $4f$, etc.

The fundamental wave and its higher multiples form a harmonic series.

As shown in fig. 7.1, the fundamental wave is called the first harmonic. The second harmonic has a frequency twice that of the fundamental, the third harmonic has a frequency three times that of the fundamental, and so on.

Waves with frequencies of $2f$, $4f$ and $6f$ etc. are called **even** harmonics. Those with frequencies of $3f$, $5f$ and $7f$ etc. are called **odd** harmonics. Expressing these frequencies in angular form, we can say that successive odd harmonics have frequencies of 3ω, 5ω and 7ω etc., and that even harmonics have frequencies of 2ω, 4ω and 6ω etc.

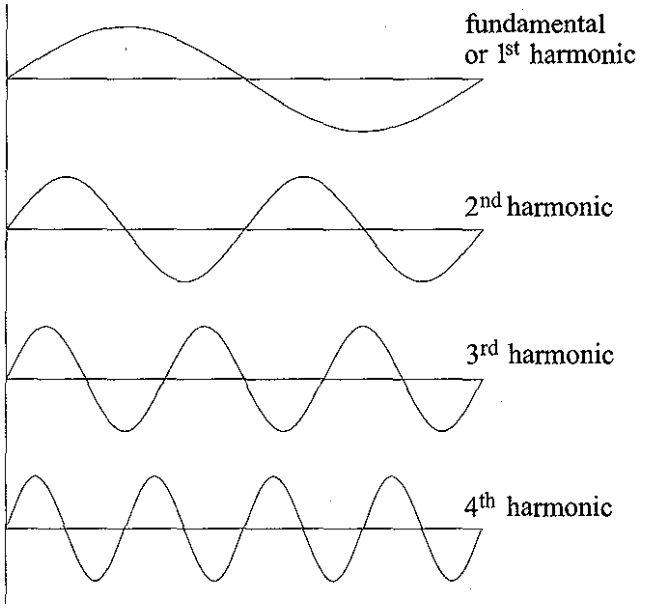


Fig. 7.1

Let's find out graphically the shape of a complex wave when we combine the fundamental with one of its harmonics.

In this example we only look at scenarios where the **fundamental and harmonic are in phase with each other and have equal amplitudes**.

The fundamental and second harmonic, both of **the same amplitude**, are shown in fig. 7.2(a) The complex waveform is plotted out by algebraically adding the individual ordinates as shown in fig. 7.2(a).

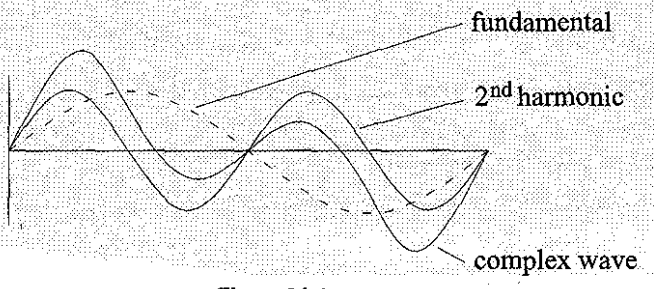


Fig. 7.2(a)

Fig. 7.2(b) indicates the fundamental and 4th harmonic, and fig. 7.2(c) consists of the fundamental and 3rd harmonic. Obviously, in all these cases, there is no phase difference between the fundamental and the harmonic.

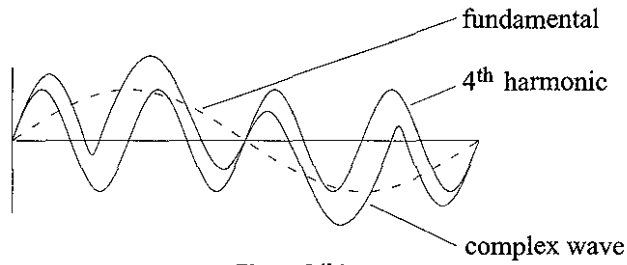


Fig. 7.2(b)

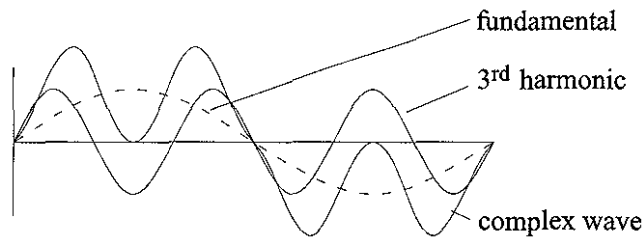


Fig. 7.2(c)

A careful examination of figs. 7.2(a), (b) and (c) leads us to the following conclusions:

1 With odd harmonics, the positive and negative halves of the complex waves are symmetrical, irrespective of the phase difference between the fundamental and the harmonic.

2 a. When even harmonics are present and their phase difference with the fundamental is 0 or π, the first and fourth quarters of the complex wave are of the same phase but inverted. The same holds true for the second and third quarters.

   b. When even harmonics are present and their phase difference with the fundamental is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, there is no symmetry.

3 The displacement of the complex wave (whether it comprises odd or even harmonics) is zero at $\omega t = 0$ only when the phase difference between the fundamental and the harmonics is 0 or π.

These conclusions are a great help when it comes to analysing a complex waveform into its harmonic constituents. Looking at the complex wave enables us to rule out the presence of certain harmonics. For example, if the positive and negative half cycles of a complex wave are symmetrical

(i.e. the wave is symmetrical about $\omega t = 0$), then we don't need to look for even harmonics. In some cases, we may be able to anticipate the types of harmonics to be expected from their mode of production. For example, in symmetricall designed alternators, we should expect odd harmonics only.

Activity 7.1

Determine the fundamental and harmonic frequencies of the following complex wave:

$$e = 60 \sin 2512t + 20 \cos 7536t$$

7.3 General equation of a complex wave

Think about a complex wave that is built up of the fundamental and a few harmonics, each having its own peak value and phase angle. The fundamental may be represented:

$$e_1 = E_{1m} \sin (\omega t + \psi_1)$$

the 2nd harmonic:

$$e_2 = E_{2m} \sin (2\omega t + \psi_2)$$

and the 3rd harmonic:

$$e_3 = E_{3m} \sin (3\omega t + \psi_3)$$

The equation for the instantaneous value of the complex wave is given by:

$$e = e_1 + e_2 + \ldots e_n$$
$$e = E_{1m} \sin (\omega t + \psi_1) + E_{2m} \sin (2\omega t + \psi_2) + \ldots + E_{nm} \sin (n\omega t + \psi_n)$$

where E_{1m}, E_{2m} and E_{nm} denote the maximum values (or the amplitude) of the fundamental, 2nd harmonic and n^{th} harmonic, etc., and ψ_1, ψ_2 and ψ_n represent the phase differences of the complex wave.

The number of terms in the series depends on the shape of the complex wave. In relatively simple waves, the number of terms in the series would be less; in others more.

Similarly, the instantaneous value of the complex current is given by:

$$i = I_{1m} \sin(\omega t + \phi_1) + I_{2m} \sin(2\omega t + \phi_2) + \ldots + I_{nm} \sin(n\omega t + \phi_n)$$

Obviously $(\psi_1 - \phi_1)$ is the phase difference between the harmonic voltage and current for the fundamental, $(\psi_2 - \phi_2)$ for the 2nd harmonic and $(\psi_n - \phi_n)$ for the nth harmonic.

7.4 RMS value of a complex wave

Let the equation of a given complex current wave be:

$$i = I_{1m}\sin(\omega t + \phi_1) + I_{2m}\sin(2\omega t + \phi_2) + \ldots + I_{nm}\sin(n\omega t + \phi_n)$$

Its rms. value is given by:

$$I = \sqrt{\text{average values of } i^2 \text{ over whole cycle}}$$

It can be mathematically proven that:

Average value of $i^2 = \frac{I_{1m}^2}{2} + \frac{I_{2m}^2}{2} + \ldots + \frac{I_{nm}^2}{2}$

$\therefore$ r.m.s. value of $I = \sqrt{\text{average value of } i^2}$

$$= \sqrt{\frac{I_{1m}^2}{2} + \frac{I_{2m}^2}{2} + \ldots \frac{I_{nm}^2}{2}}$$

This equation can also be written:

$$I = \sqrt{\left(\frac{I_{1m}}{\sqrt{2}}\right)^2 + \left(\frac{I_{2m}}{\sqrt{2}}\right)^2 + \ldots \left(\frac{I_{nm}}{\sqrt{2}}\right)^2}$$

which can be written as:

$$I = \sqrt{I_1^2 + I_2^2 + \ldots I_n^2}$$

where $I_1 = \frac{I_{1m}}{\sqrt{2}}$ is the rms value of fundamental,

$I_2 = \frac{I_{2m}}{\sqrt{2}}$ is the rms value of the 2nd harmonic, and

$I_n = \frac{I_{nm}}{\sqrt{2}}$ is the rms value of the nth harmonic.

7.5 Harmonics in single-phase circuits

If an alternating voltage, containing various harmonics, is applied to a single-phase circuit containing linear circuit elements, the current produced also contains harmonics. Each harmonic voltage will produce its own current, independent of others. By the principle of superposition, the combined current can be identified.

We will now look at some of the well known elements like pure resistance, pure inductance and pure capacitance and their various combinations. In each case, we will assume that the applied complex voltage is represented by:

$$e = E_{1m} \sin \omega t + E_{2m} \sin 2\omega t + \ldots + E_{nm} \sin n\omega t$$

7.5.1 Pure resistance

Let the circuit have a resistance of R that is independent of frequency. The instantaneous current i_1 due to fundamental voltage, is:

$$i_1 = \frac{E_{1m} \sin\omega t}{R}$$

Similarly, $i_2 = \frac{E_{2m} \sin 2\omega t}{R}$ ... for 2nd harmonic

and $i_n = \frac{E_{nm} \sin n\omega t}{R}$ ... for nth harmoic

Total current $i = i_1 + i_2 + \ldots + i_n$

$$= \frac{E_{1m} \sin \omega t}{R} + \frac{E_{2m} \sin 2\omega t}{R} + \ldots + \frac{E_{nm} \sin n\omega t}{R}$$

$$= I_{1m} \sin \omega t + I_{2m} \sin 2\omega t + \ldots + I_{nm} \sin n\omega t$$

This shows that the waveform of the current is similar to that of the applied voltage. In other words, the two waves are identical in form.

7.5.2 Pure inductance

Let the inductance of the circuit be L (henry) whose reactance varies directly with the frequency of the applied voltage.

Its reactance would be:

- for the fundamental $X_1 = \omega L$
- for the 2nd harmonic $X_2 = 2\omega L$
- for the nth harmonic $X_n = n\omega L$

However, for every harmonic term, the current will lag behind the voltage by 90°.

∴ Current due to fundamental

$$i_1 = \frac{E_{1m}}{\omega L} \sin(\omega t - \frac{\pi}{2})$$

Current due to 2nd harmonic

$$i_2 = \frac{E_{2m}}{2\omega L} \sin(2\omega t - \frac{\pi}{2})$$

Current due to 3rd harmonic

$$i_3 = \frac{E_{3m}}{3\omega L}\sin(3\omega t - \frac{\pi}{2})$$

Current due to n^{th} harmonic

$$i_n = \frac{E_{nm}}{n\omega L}\sin(n\omega t - \frac{\pi}{2})$$

$\therefore$ Total current $i = i_1 + i_2 + \ldots + i_n$

$$= \frac{E_{1m}}{\omega L}\sin(\omega t - \frac{\pi}{2}) + \frac{E_{2m}}{2\omega L}\sin(2\omega t - \frac{\pi}{2}) + \ldots + \frac{E_{nm}}{n\omega L}\sin(n\omega t - \frac{\pi}{2})$$

You can see from this equation that the current waveform differs from that of the applied voltage.

7.5.3 Pure capacitance

Let the capacitance of the circuit be C (farads) whose reactance varies indirectly with the frequency of the applied voltage. Its reactance is:

$X_1 = \frac{1}{\omega C}$ – for fundamental

$X_2 = \frac{1}{2\omega C}$ – for 2nd harmonic

$X_3 = \frac{1}{3\omega C}$ – for 3rd harmonic

$X_n = \frac{1}{n\omega C}$ – for n^{th} harmonic

$$\therefore \quad i_1 = \frac{E_{1m}}{\frac{1}{\omega C}}\sin(\omega t + \frac{\pi}{2})$$

$$= \omega C E_{1m}\sin(\omega t + \frac{\pi}{2})$$

and $i_2 = \frac{E_{2m}}{\frac{1}{2\omega C}}\sin(2\omega t + \frac{\pi}{2})$

$$= 2\omega C E_{2m}\sin(2\omega t + \frac{\pi}{2})$$

$$i_n = \frac{E_{nm}}{\frac{1}{n\omega C}}\sin(n\omega t + \frac{\pi}{2})$$

$$= n\omega C E_{nm}\sin(n\omega t + \frac{\pi}{2})$$

For every harmonic term the current will lead the voltage by 90°.

Now $i = i_1 + i_2 + \ldots i_n$

$$= \omega C E_{1m}\sin(\omega t + \frac{\pi}{2}) + 2\omega C E_{2m}\sin(2\omega t + \frac{\pi}{2}) + \ldots + n\omega C E_{nm}\sin(n\omega t + \frac{\pi}{2})$$

This equation shows that the current and voltage waveforms are dissimilar and that the percentage harmonic content of the current is larger than that of the applied voltage wave. This means that the current wave is more distorted that the voltage wave.

EXAMPLE 7.1

Determine the rms value of the current represented by:

$$i = 10 \sin\phi + \sin(3\phi + 20°) + 0.5 \sin(5\phi + 120°)$$

Solution

Remember that:

$$I_{rms} = \sqrt{\frac{I_1^2 + I_3^2 + I_5^2}{2}}$$

$$= \sqrt{\frac{10^2 + 1^2 + 0.5^2}{2}}$$

$$= 7.115 \text{ A}$$

EXAMPLE 7.2

Determine the rms value of the voltage represented by:

$$v = 50 \sin \omega t + 5 \sin(3\omega t + 15°) + 2 \sin(5\omega t + 120°)$$

Solution

Remember that:

$$V_{rms} = \sqrt{\frac{V_1^2 + V_3^2 + V_5^3}{2}}$$

$$= \sqrt{\frac{50^2 + 5^2 + 2^2}{2}}$$

$$= 35.56 \text{ V}$$

EXAMPLE 7.3

An emf is given by

$$e = 100 \sin \omega t + 30 \sin(3\omega t + 20°) + 10 \sin(5\omega t + 10°) \ V$$

Knowing that the fundamental frequency is 60 Hz work out:

1. the rms value of the current for a series circuit consisting of a resistor of 5 Ω and an inductance of 16.65 mH.

2. the current expression for the circuit.

Solution

1. $X_{L1} = 2\pi fL = 2\pi \times 60 \times 0.01665$
$= 6.28\ \Omega$
$X_{L3} = 3(2\pi fL) = 3 \times 6.28$
$= 18.84\Omega$
$X_{L5} = 5(2\pi fL) = 5 \times 6.28$
$= 31.4\Omega$

$\therefore\ Z_1 = 5 + j6.28\ \Omega = 8.03\underline{/51.47^\circ}\ \Omega$
$Z_3 = 5 + j18.84\ \Omega = 19.5\underline{/75.14^\circ}\ \Omega$
$Z_5 = 5 + j31.4\ \Omega = 31.8\underline{/80.85^\circ}\ \Omega$

$$\therefore I_1 = \frac{E_1}{Z_1} = \frac{100\underline{/0^\circ}}{8.03\underline{/51.47^\circ}}$$
$$= 12.45\underline{/-51.47^\circ}\ \text{A}$$
$$I_3 = \frac{E_3}{Z_3} = \frac{30\underline{/0^\circ}}{19.5\underline{/75.14^\circ}}$$
$$= 1.54\underline{/-75.14^\circ}\ \text{A}$$
$$I_5 = \frac{E_5}{Z_5} = \frac{10\underline{/0^\circ}}{31.8\underline{/80.95^\circ}}$$
$$= 0.314\underline{/-80.95^\circ}\ \text{A}$$
$$\therefore I_{rms} = \sqrt{\frac{12.45^2 + 1.54^2 + 0.314^2}{2}}$$
$$= 8.87\ \text{A}$$

2. $i = 12.45\sin(\omega t - 51.47^\circ) + 1.54\sin(3\omega t + 20^\circ - 75.14^\circ) + 0.314\sin(5\omega t + 10^\circ - 80.95^\circ)$
$\therefore\ i = 12.45\sin(\omega t - 51.47)^\circ + 1.54\sin(3\omega t - 55.14^\circ) + 0.314\sin(5\omega t - 70.95^\circ)$

EXAMPLE 7.4

The emf from Example 7.3 is applied to a circuit consisting of a 5 Ω resistor in series with a capacitor C = 423 μF. Work out the rms value of the current.

Solution

$$X_{c1} = \frac{1}{2\pi \times 60 \times 423 \times 10^{-6}}$$

$$= 6.27\ \Omega$$

$$X_{c3} = \frac{1}{3(2\pi \times 60 \times 423 \times 10^{-6})} = \frac{1}{3}X_{c1}$$

$$= 2.09\ \Omega$$

$$X_{c5} = \frac{1}{5}X_{c1}$$

$$= 1.254\ \Omega$$

and the impedances

$$Z_1 = 5 - j6.27 = 8.02\underline{/-51.4^\circ}\ \Omega$$
$$Z_3 = 5 - j2.09 = 5.42\underline{/-22.68^\circ}\ \Omega$$
$$Z_5 = 5 - j1.254 = 5.15\underline{/-14.08^\circ}\ \Omega$$

$$\therefore I_1 = \frac{100\underline{/0^\circ}}{8.02\underline{/-51.4^\circ}}$$

$$= 12.47\underline{/51.4^\circ}\ \text{A}$$

$$I_3 = \frac{30\underline{/0^\circ}}{5.42\underline{/-22.68^\circ}}$$

$$= 5.54\underline{/22.68^\circ}\text{A}$$

$$I_5 = \frac{10\underline{/0^\circ}}{5.15\underline{/-14.08^\circ}}$$

$$= 1.94\underline{/14.08^\circ}\text{A}$$

$$\therefore I_{rms} = \sqrt{\frac{12.47^2 + 5.54^2 + 1.94^2}{2}}$$

$$= 9.75\ \text{A}$$

Activity 7.2

The voltage given below is applied to a 15 Ω resistor.
$e = 27.01\cos(400\pi t - 45^\circ) + 19.1\cos(800\pi t - 90^\circ) + 9\cos(1\,200\pi t - 135^\circ)$

Calculate the rms value of the voltage and current in the 15 Ω resistor.

7.6 Power supplied by a complex wave

Let the complex voltage represented by the equation

$$e = E_{1m} \sin \omega t + E_{2m} \sin 2\omega t + \ldots E_{nm} \sin n\omega t$$

be applied to a circuit.

Let the equation of the resulting current waveform be:

$$i = I_{1m} \sin(\omega t + \phi_1) + I_{2m} \sin(2\omega t + \phi_2) + \ldots + I_{nm} \sin(n\omega t + \phi_n)$$

The instantaneous value of the power in the circuit is $p = ei$ W.

To obtain the value of this product, we will have to multiply every term of the voltage wave, in turn, by every term in the current wave. The average power supplied during a cycle is equal to the sum of the average values over one cycle of each individual product term.

However, the average value of all product terms involving harmonics of different frequencies will be zero over one cycle. We only need to consider the products of current and voltage harmonics of the same frequency.

Let's look at a general term of this nature $E_{nm} \sin n\omega t \times I_{nm} \sin(n\omega t - \phi_n)$ – and find its average value over one cycle of the fundamental.

$$\begin{aligned}
\text{Average value of power} &= \frac{1}{2\pi}\int_0^{2\pi} E_{nm}I_{nm} \sin n\omega t \sin(n\omega t - \phi_n) d(\omega t) \\
&= \frac{E_{nm}I_{nm}}{2\pi} \int_0^{2\pi} \sin n\theta \sin(n\theta - \phi_n) d\theta \\
&= \frac{E_{nm}I_{nm}}{2\pi} \int_0^{2\pi} \frac{\cos\phi_n - \cos(2n\theta - \phi_n)}{2} d\theta \\
&= \frac{E_{nm}I_{nm}\cos\phi_n}{2} \\
&= \frac{E_{nm}}{\sqrt{2}} \times \frac{I_{nm}}{\sqrt{2}} \times \cos\phi_n \\
&= E_n I_n \cos\phi_n
\end{aligned}$$

where E_n and I_n are the rms values of the voltage and current respectively. So, the total average power of a complex wave is the sum of the average powers of each harmonic component acting independently.

$\therefore$ total power $= E_1I_1 \cos\phi_1 + E_2I_2 \cos\phi_2 + \ldots + E_nI_n \cos\phi_n$

The overall power factor is given by:

$$\text{PF} = \frac{\text{total watts}}{\text{total voltamperes}} = \frac{W}{VA}$$

$$= \frac{E_1I_1\cos\phi_1 + E_2I_2\cos\phi_2 + \ldots}{E \times I}$$

where E = rms value of the complex voltage wave, and I = rms value of the complex current wave.

EXAMPLE 7.5

A single-phase voltage source 'e' is given by:

$$e = 141 \sin \omega t + 42.3 \sin 3\omega t + 28.8 \sin 5\omega t$$

The corresponding current in the load circuit is given by:

$$i = 16.5 \sin(\omega t - 54.5°) + 8.43 \sin(3\omega t - 38°) + 4.65 \sin(5\omega t - 34.3°)$$

Find the power supplied by the source.

Solution

Power at fundamental $= E_1 I_1 \cos\phi_1 = \frac{E_{1m}}{\sqrt{2}} \times \frac{I_{1m}}{\sqrt{2}} \cos\phi_1$

$$= \frac{E_{1m} I_{1m}}{2} \cos\phi_1$$

$$= \frac{141 \times 16.5}{2} \cos 54.5°$$

$$= 675.50 \text{ W}$$

Power at 3rd harmonic $= \frac{E_{3m} I_{3m}}{2} \cos\phi_3$

$$= \frac{42.3 \times 8.43}{2} \cos 38°$$

$$= 140.49 \text{ W}$$

Power at 5th harmonic $= \frac{E_{5m} I_{5m}}{2} \cos\phi_5$

$$= \frac{28.8 \times 4.65}{2} \cos 34.3°$$

$$= 55.31 \text{ W}$$

$\therefore$ Total power supplied $= 675.50 + 140.49 + 55.31$
$= 871.30$ W

EXAMPLE 7.6

A complex voltage given by:

$$e = 60\sin\omega t + 24\sin(3\omega t + \tfrac{\pi}{6}) + 12\sin(5\omega t + \tfrac{\pi}{3})$$

is applied across a circuit and the resulting current is given by:

$$i = 0.6\sin(\omega t - \tfrac{2\pi}{10}) + 0.12\sin(3\omega t - \tfrac{2\pi}{24}) + 0.1\sin(5\omega t - \tfrac{\pi}{12})$$

Calculate:

a. the total power supplied
b. the rms value of current and voltage
c. the overall power factor

Solution

Power at fundamental $= E_1 I_1 \cos\phi = \frac{E_{1m} I_{1m}}{2}\cos\phi_1$

$= \frac{60 \times 0.6}{2} \times \cos 36°$

$= 14.56$ W

Power at 3rd harmonic $= \frac{E_{3m} I_{3m}}{2}\cos 45°$

$= \frac{24 \times 0.12}{2} \times 0.707$

$= 1.02$ W

Power at 5th harmonic $= \frac{E_{5m} I_{5m}}{2}\cos 75°$

$= \frac{12 \times 0.1}{2} \times 0.2588$

$= 0.16$ W

a. Total power $= 14.56 + 1.02 + 0.16$
$= 15.74$ W

b. RMS current:

$$I = \sqrt{I_1^2 + I_3^2 + I_5^2}$$

$$= \sqrt{\left(\frac{I_{1m}}{\sqrt{2}}\right)^2 + \left(\frac{I_{3m}}{\sqrt{2}}\right)^2 + \left(\frac{I_{5m}}{\sqrt{2}}\right)^2}$$

$$= \sqrt{\frac{0.6^2}{2} + \frac{0.12^2}{2} + \frac{0.1^2}{2}}$$

$$= 0.438 \text{ A}$$

RMS voltage:

$$E = \sqrt{\frac{60^2}{2} + \frac{24^2}{2} + \frac{12^2}{2}}$$

$$= 46.475 \text{ V}$$

c. Overall PF $= \dfrac{\text{watts}}{\text{volt amperes}}$

$$= \frac{15.74}{46.475 \times 0.438}$$

$$= 0.773$$

EXAMPLE 7.7

An emf is represented by:

$$e = 200 \sin 314t + 40 \sin(942t + \tfrac{\pi}{3}) + 20 \sin(1\,570t + \tfrac{2\pi}{3})$$

The emf is applied to a series circuit consisting of a resistor of 10 Ω in series with an inductance of 0.02 H.

Determine the total power and the power factor when the frequency is 50 Hz.

Solution

$$X_{L1} = 2\pi f \times 0.02 = 6.28\ \Omega$$
$$X_{L3} = 3(2\pi f \times 0.02) = 18.84\ \Omega$$
$$X_{L5} = 5(2\pi f \times 0.02) = 31.41\ \Omega$$

$$\therefore\quad Z_1 = 10 + j6.28 = 11.8\underline{/32^\circ}\ \Omega$$
$$Z_3 = 10 + j18.84 = 21.3\underline{/62^\circ}\ \Omega$$
$$Z_5 = 10 + j31.4 = 32.9\underline{/72^\circ}\ \Omega$$

$$I_1 = \frac{E_1}{Z_1} = \frac{200\underline{/0^\circ}}{11.8\underline{/32^\circ}} = 16.94\underline{/-32^\circ};\ \cos\phi_1 = 0.85$$

$$I_3 = \frac{E_3}{Z_3} = \frac{40/0^\circ}{21.3/62^\circ} = 1.88/-62^\circ;\ \cos\phi_2 = 0.47$$

$$I_5 = \frac{E_5}{Z_5} = \frac{20/0^\circ}{32.9/72^\circ} = 0.6/-72^\circ;\ \cos\phi_3 = 0.31$$

$$E_{rms} = \sqrt{\frac{200^2}{2} + \frac{40^2}{2} + \frac{20^2}{2}}$$

$$= 44.9\ V$$

$$I_{rms} = \sqrt{\frac{16.94^2}{2} + \frac{1.88^2}{2} + \frac{0.6^2}{2}} = 12.05\ \text{A}$$

$$P_1 = \frac{E_{1m}I_{1m}}{2}\cos\phi = \frac{200 \times 16.94}{2} \times 0.85 =$$

$$= 440\ \text{W}$$

$$P_3 = \frac{40 \times 1.88}{2} \times 0.47$$

$$= 17.6\ \text{W}$$

$$P_5 = \frac{20 \times 0.6}{2} \times 0.31 =$$

$$= 1.86\ \text{W}$$

$$P_{TOTAL} = P_1 + P_3 + P_5$$
$$= 1440 + 17.6 + 1.86$$
$$= 1459.46\ \text{W}$$

$$\text{Power factor} = \frac{W}{VA}$$

$$= \frac{459.46}{44.9 \times 12.05}$$

$$= 0.83$$

EXAMPLE 7.8

A 6.30 μF capacitor is connected in parallel with a resistance of 500 Ω, and the combination is connected in series with a 500 Ω resistor. The whole circuit is connected across an AC voltage given by $e = 300 \sin\omega t + 100 \sin(3\omega t + \frac{\pi}{6})$. (See fig. 7.3.)

If $\omega = 314$ rad/s find:

1. power dissipated in the circuit; and
2. an expression for the voltage across the series resistor.

Solution

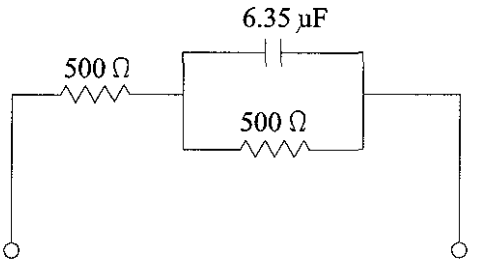


Fig. 7.3

For fundamental:

$$X_{c1} = \frac{1}{\omega C} = \frac{10^6}{314 \times 6.36}$$

$$= 500\ \Omega$$

The impedance of the whole series-parallel circuit is given by:

$$Z_1 = 500 + \frac{500 \times (-j500)}{500 - j500}$$

$$= 750 - j250$$

$$= 791\underline{/-18.5^\circ}\ \Omega$$

For third harmonic:

$$X_{c3} = \frac{1}{3\omega C} = \frac{500}{3} = 167\ \Omega$$

$$Z_3 = 500 + \frac{500 \times (-j167)}{500 - j167}$$

$$= 550 - j150$$

$$= 570\underline{/-15.3^\circ}\ \Omega$$

$$\therefore\ i = \frac{300}{791}\sin(\omega t + 18.5^\circ) + \frac{100}{570}\sin(3\omega t + 45.3^\circ)$$

$$= 0.379\sin(\omega t + 18.5^\circ) + 0.175\sin(3\omega t + 45.3^\circ)\ \text{A}$$

1. Power dissipated $= \frac{E_{1m}I_{1m}}{2}\cos\phi_1 + \frac{E_{3m}I_{3m}}{2}\cos\phi_3$

$$= \frac{300 \times 0.379}{2} \times \cos 18.5^\circ + \frac{100 \times 0.175}{2} \times \cos 15.3^\circ$$

$$= 62.35\ \text{W}$$

2. The volt drop across the series resistor is:
$$E_R = iR$$
$$= 500[0.379 \sin(\omega t + 18.5°) + 0.175 \sin(3\omega t + 45.3°)]$$
$$= 189.5 \sin(\omega t + 18.5°) + 87.5 \sin(3\omega t + 45.3°)$$

7.7 Summary

1. Complex waveforms are superimposed waveforms of different frequencies.
2. Waveforms with frequencies $2f$, $4f$ and $6f$ etc. are called even harmonics.
3. Waveforms with frequencies $3f$, $5f$ and 7f etc. are called odd harmonics.
4. $$I_{rms} = \sqrt{\left(\frac{I_{1m}}{\sqrt{2}}\right)^2 + \left(\frac{I_{2m}}{\sqrt{2}}\right)^2 + \ldots \left(\frac{I_{nm}}{\sqrt{2}}\right)^2}$$
$$V_{rms} = \sqrt{\left(\frac{V_{1m}}{\sqrt{2}}\right)^2 + \left(\frac{V_{2m}}{\sqrt{2}}\right)^2 + \ldots \left(\frac{V_{nm}}{\sqrt{2}}\right)^2}$$
5. In a pure resistive circuit, the current wave and voltage wave are similar (in phase).
6. In a pure inductive circuit, the current wave is behind (lags) the voltage wave by 90°.
7. In a pure capacitive circuit, the current wave is ahead of (leads) the voltage wave by 90°.
8. The overall power factor of a complex circuit is given by:
$$\text{PF} = \frac{\text{total watts}}{\text{total voltamperes}} = \frac{W}{VA}$$
$$= \frac{E_1 I_1 \cos\phi_1 + E_2 I_2 \cos\phi_2 + \ldots}{E \times I}$$
where E and I are the rms values of voltage (emf) and current respectively.

Self-evaluation

1. A voltage given by $V = 200 \sin 314t + 50 \sin(943t + 45°)$ is applied to a circuit consisting of a resistance of 20 Ω, an inductance of

20 mH and a capacitance of 56.3 μF, all connected in series. Calculate the rms values of the applied voltage and current, and find the total power absorbed by the circuit.

2. The current in a series circuit consisting of a 159 μF capacitor, a resistance of 10 Ω and an inductance of 0.025 H is given by:

$$i = \sqrt{2}(8 \sin \omega t + 2 \sin 3\omega t) \text{ amperes.}$$

Calculate the power input and the power factor if $\omega = 100\pi$ radian/sec.

3. A series circuit of a 60 Ω resistor, a 2 H inductor and a 3 μF capacitor is connected across a non-sinusoidal source. The emf of the source is $e = 20 \sin(377t) + 10 \sin(754t)$. Find:
   a. the sinusoidal expression for the current.
   b. the sinusoidal expression for the voltage across each component.
   c. the effective value of the source voltage and current.
   d. the power dissipated in the circuit.

4. The parallel circuit shown in fig. 7.4 is connected to a source of $e = 40 \sin(377t) + 20 \sin 754t$ V

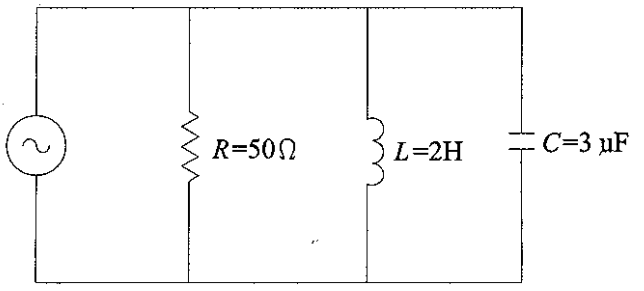


Fig. 7.4

Find:
   a. the sinusoidal expression for the current in each branch.
   b. the effective value of the source voltage and current.
   c. the power dissipated in the circuit.

Answers

Activity 7.1 (page 189)

Fundamental = 400 Hz
Third harmonic = 1 200 Hz

Activity 7.2 (page 195)

$V = 24.2$ V
$I = 1.62$ A

Self-evaluation (page 203)

1. 146 V; 3.16 A; 200 W

2. 680 W; 0.63

3. a. $i = 0.14 \sin(377t + 65.26°) + 0.0094 \sin(754t - 86.78°)$ A

   b. $V_R = 8.4 \sin(377t + 65.26°) + 0.564 \sin(754t - 86.78°)$ V
   $V_L = 105.56 \sin(377t + 155.26°) + 14.18 \sin(754t + 3.22°)$ V
   $V_C = 123.8 \sin(377t - 24.74°) + 4.16 \sin(754t - 176.78°)$ V

   c. $E_{eff} = 15.8$ V
   $I_{eff} = 0.099$ A

   d. 0.588 W

4 a. $i_R = 0.8 \sin(377t) + 0.4 \sin(754t)$ A
   $i_L = 0.053 \sin(377t - 90°) + 0.013 \sin(754t - 90°)$ A
   $i_C = 0.045 \sin(377t + 90°) + 0.045 \sin(754t + 90°)$ A

   b. $E_{eff} = 31.62$ V
   $I_{eff} = 0.63$ A

   c. 20 W

unit 8

Introduction to three-phase systems

OUTCOMES

After studying this unit, you should be able to:

- understand the concept of three-phase systems
- explain how three-phase voltages are generated
- represent three-phase voltages and currents using phasor diagrams
- determine the phase sequence rotation of phasors
- explain a balanced system
- understand star and delta three-phase connections and their applicable relationships between line and phase parameters (for balanced star or delta)
- convert star connections to equivalent delta connections and vice-versa;
- calculate power in three-phase balanced systems (for star and delta connections), as a function of various phase or line voltages and currents
- understand how power in three-phase circuits can be measured using the three- or two-wattmeter method

8.1 Introduction

In all the previous units, we have only considered single-phase systems. In this unit we will deal with three-phase systems. Three-phase systems are generally more economical than other systems.

From an electrical machines viewpoint, three-phase machines use copper and iron more efficiently.

Three-phase generators are also more balanced in behaviour than their single-phase counterparts – for example, in constant total power. Single-phase generators vibrate badly and are very noisy, especially in large machines, due to the pulsating power.

Three-phase transformers are more economical than three single-phase transformers (which would do exactly the same job), but using special types of three-phase transformers can even increase the number of phases – up to 6 phases – supplying rectifiers. This is possible without any significant extra costs.

Poly-phase systems are much more economical, especially for transmission lines and sometimes for distribution. After all, it is much cheaper to build one three-phase overhead line than to build three single-phase lines. Think how much material you save by using one tower instead of three!

Some loads have a more convenient behaviour in three-phase systems – for example in starting motors!

With these things in mind, it is important for us to elaborate on three-phase systems and compare them to the single-phase circuits we studied up to now.

Why don't we use other poly-phase systems? The answer to this is that the costs for additional phases are not compensated for by the slight increase in operating efficiency.

A three-phase network has three or four wires entering or leaving it. These networks are more complicated than single-phase networks, but most of the three-phase systems are approximately balanced and the analysis can be simplified.

DEFINITION

A balanced three-phase system is a system in which three voltages equal in magnitude and having the same frequency but differing by 120° in phase angle are generated. These balanced voltages are applied to equal impedances and balanced currents result.

Three-phase circuits offer several advantages over single-phase circuits. These include:

- more balance in power generation;
- less losses in transmitting power; and
- easier feeding of certain types of loads

For any balanced system, it is enough to analyse just one of the three phases, no matter how complicated the transmission networks or the generating facilities.

So, when we start studying the theory of three-phase circuits, we always look at balanced systems first.

In more advanced courses you will find out more about the three-phase system (unbalanced systems, power factor correction, etc.) and you will have to remember and use all of the basic concepts in this unit. You will also be able to use this knowledge in other courses like Electrical Machines, Transmission, Transport and Distribution.

Let's start!

8.2 Three-phase generation

8.2.1 Generation principles

In a large AC electric generator, the armature is the stationary part and the magnetic field structure rotates within the armature ring:

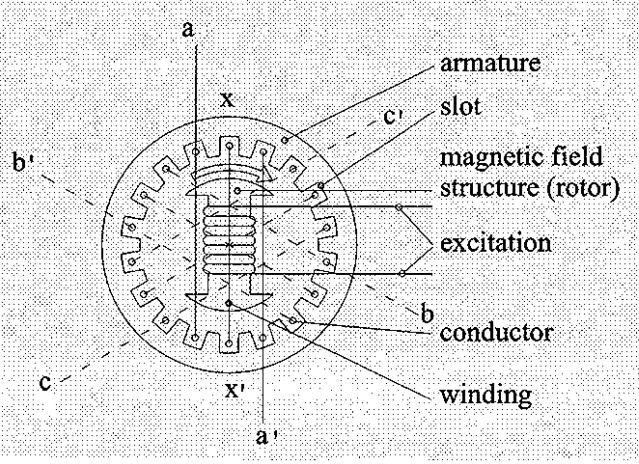
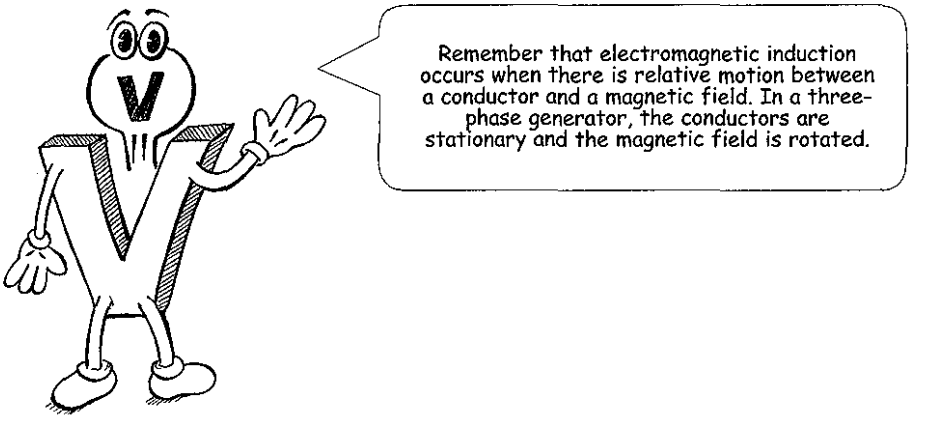


Fig. 8.1



As you can see in fig. 8.1, the **conductors are placed in slots** in the inner surface of the armature.

The active part of the conductors, where the voltage is generated, is the part lying in the armature slots. Fig. 8.1 is a schematic cross-section through the centre of a machine where the axis of the machine is perpendicular to the paper. The lines in the diagram only indicate the end connections of the conductors.

For the machine to be cost effective, we try to use all the slots available in the machine. So, how can we arrange the conductors to achieve this?

The first step would be to place a coil with its two active sides in opposite slots, for example in x-x′. The moment the machine starts operating, that is when the magnetic field structure starts turning within the armature, a relative motion appears between the magnetic field and conductors. This generates a voltage in the coil, similar to the single-phase principle of generation.

Since we have more slots available, lets add two more coils in series – one on each side of our initial x-x′ coil. The terminals of this coil are brought out of the machine at a and a′.

A coil formed by a conductor is also known as a winding when it is wound around a core.

Let's place another winding, also of three coils connected in series, **shifted 120°** compared to the first one. Its terminals will be b and b′. This second coil is not connected to a-a′.

The remaining slots will be filled with the third winding, which, according to our diagram, is shifted another 120° clockwise compared to the b-b′ winding and 240° compared to the initial a-a′ winding. The terminals for the third winding are marked c-c′.

At this point all the slots in the machine are filled.

Since the three windings a-a′, b-b′, and c-c′ are not yet connected in any way, we can study them as three separate single-phase generators. These three windings are the so-called 'three-phase' windings of the machine.

Remember that we have simplified the example and that when you study Electrical Machines, the theory of the three-phase generator is explained in more detail. In fact, you'll find that the three-phase windings are interconnected. However, for our purposes, the above description is adequate.

Let's assume the clockwise direction is the normal direction of the rotation of the rotor (the magnetic field structure) in the machine. In this example, the direction is a-b-c. The three voltages will reach their corresponding maximum values in the order $v_{aa'}$, $v_{bb'}$, and $v_{cc'}$, as in fig. 8.2:

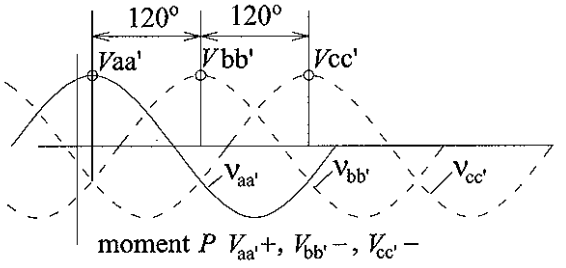


Fig. 8.2

A three-phase voltage is essentially three single-phase voltages, each separated from the next by 120° of phase angle.

A machine designer must always try to obtain undistorted sinusoidal voltages and identical waveforms for the three phases.

The corresponding phasor diagram for the three voltages is:

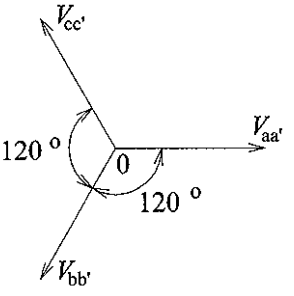


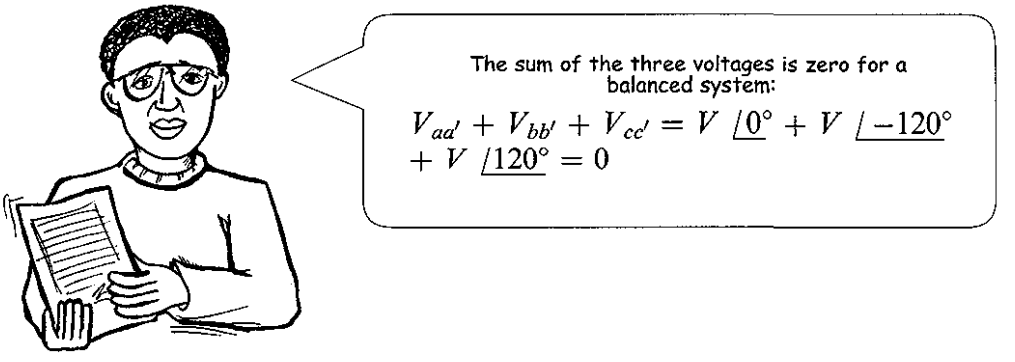
Fig. 8.3

You can see that $V_{aa'}$ (assumed as reference) is represented by $\underline{/0°}$. $V_{bb'}$ is shifted 120 clockwise from $V_{aa'}$ and $V_{cc'}$ is shifted 240° clockwise from $V_{aa'}$.

The concept of 'reference' phasor, as explained in unit 2, section 2.2, will also be used for three-phase systems (see example 1 in unit 2).

We can write these as:

$$\begin{cases} V_{aa'} = V\underline{/0°} \\ V_{bb'} = V\underline{/-120°} \text{ (or } V\underline{/+240°} \\ V_{cc'} = V\underline{/120°} \text{ (or } V\underline{/-240°}) \end{cases}$$



8.2.2 Phase sequence rotation

Let's redraw fig. 8.1 on a piece of paper and re-label the windings c-c′ instead of b-b′, and b-b′ instead of c-c′. Alternatively, assume the rotor rotates in the opposite direction.

It is easy to observe that the order in which the voltages will reach their corresponding maximums will now be $v_{aa'}$, $v_{cc'}$, and $v_{bb'}$. If we assume $V_{aa'}$ is our reference, the phasor diagram looks like this:

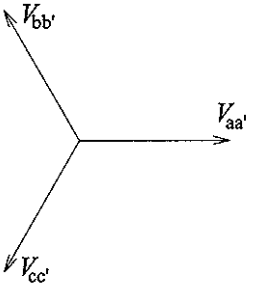


Fig. 8.4

and the corresponding waveforms for the voltages:

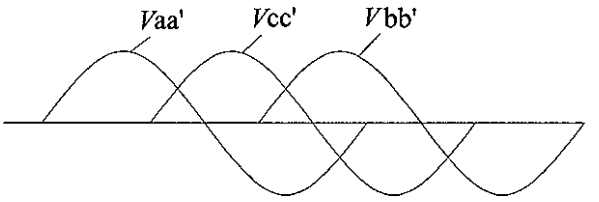


Fig. 8.5

Then we can write:

$V_{aa'} = V\underline{/0°}$
$V_{cc'} = V\underline{/-120°}$
$V_{bb'} = V\underline{/+120°}$

In the first case, $v_{aa'}$ leads $v_{bb'}$ by 120°, and $v_{bb'}$ leads $v_{cc'}$ by 120°. This is the a-b-c phase sequence called **'positive phase sequence'**.

In the second case, the a-c-b phase sequence is called the **'negative phase sequence'**.

It should be obvious that any negative phase sequence can be converted into a positive phase sequence simply by re-labelling the terminals. So we normally consider only positive phase sequences in our studies.

8.3 Three-phase connections

8.3.1 General features

If we consider that a three-phase generator supplies a load, then the equivalent circuit in which power is carried from the generator to the load can be drawn as follows:

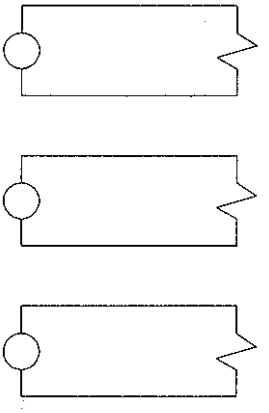


Fig. 8.6

Phase R, (A)
Phase Y, (B)
Phase B, (C)

The three phases are usually marked R,Y,B (red, yellow, blue) or a,b,c (A,B,C) in current textbooks.

If we use a single wire for the return path for all three phases, we'll get:

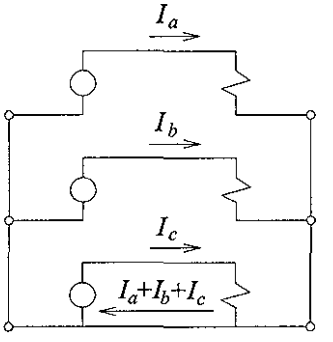


Fig. 8.7

What current does the common return wire carry?

If we look at a balanced system (balanced voltages, as normally given by any three-phase generator, and equal loads), the three currents are also balanced:

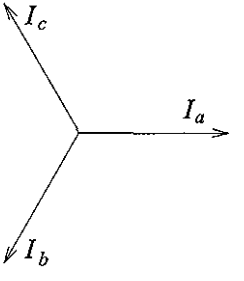


Fig. 8.8

Since the return wire will carry the sum of the three currents ($I_a + I_b + I_c$) and their sum is zero, there will be no current flowing in the return wire. But this only happens in ideal conditions. If the system is slightly unbalanced, the sum of the currents is no longer zero, but it will be a small value, I_n:

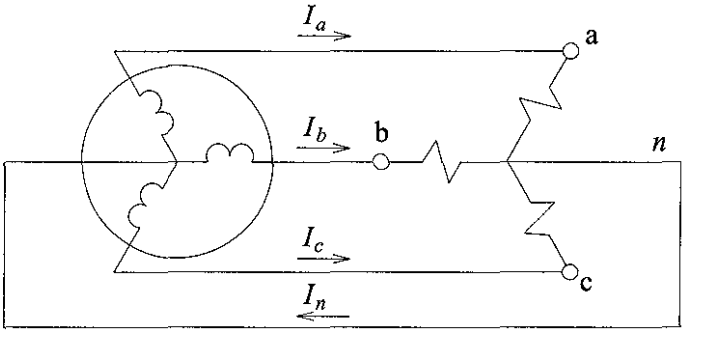


Fig. 8.9

Since I_n is very small, or ideally non-existent, the fourth wire can be left out completely for balanced loads:

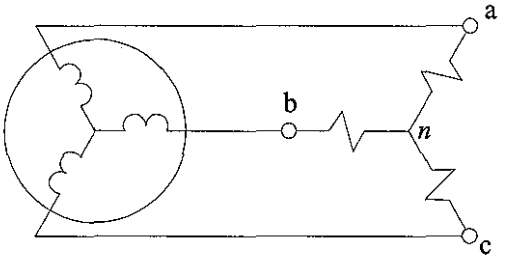


Fig. 8.10

This is a typical 'star' or 'wye' connection, where the loads are connected between one of the three lines *a*, *b*, or *c* and a neutral point n.

The star point is the point in which the 3(4) conductors come together. If the system is balanced, it is also called a 'neutral' point.

Looking back at fig 8.9 and fig. 8.10, you can see that we have two possible star connections – one with four wires and the other with three wires. Remember, for a balanced load we know that the current flowing through the fourth wire will always be zero, and that's the reason why there are only three wires in fig. 8.10.

Another way of connecting the load in a three-wire system is as follows:

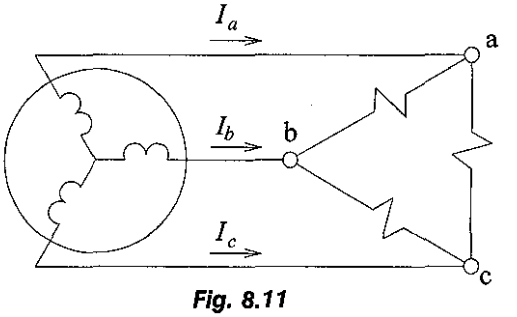


Fig. 8.11

Here the loads are connected between **lines** in a 'delta' or 'triangle' configuration.

For safety reasons, it is desirable to have a connection from the electrical system to the ground/earth. In a star system, the logical point to connect to ground/earth is the star point.

The star connection (3- or 4-wire) and the delta connection are the most popular for three-phase systems. There are other possible connections in three-phase systems, but for the time being we will only study these two. For our purposes, it is adequate to understand the delta and the star connections, and their relationships between voltages and currents, to tackle any of the following possible connections.

Possible combinations for star and delta connections

If the supply side connection is:	The load side connection can be:
star	delta (i.e. fig. 8.11)
star	star (i.e. fig. 8.9, fig. 8.10)
delta	star
delta	delta

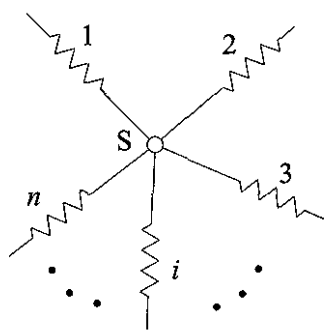
8.3.2 Three-phase star connection

▶ Circuit diagrams and properties

This three-phase network is called *Y* (or star) connection, the symbol suggested by the form of the circuit diagram.

This connection is a particular case of the general case of the '*n*' terminal star.

A '*n*' terminal star has '*n*' elements connected in a common point called the star point.



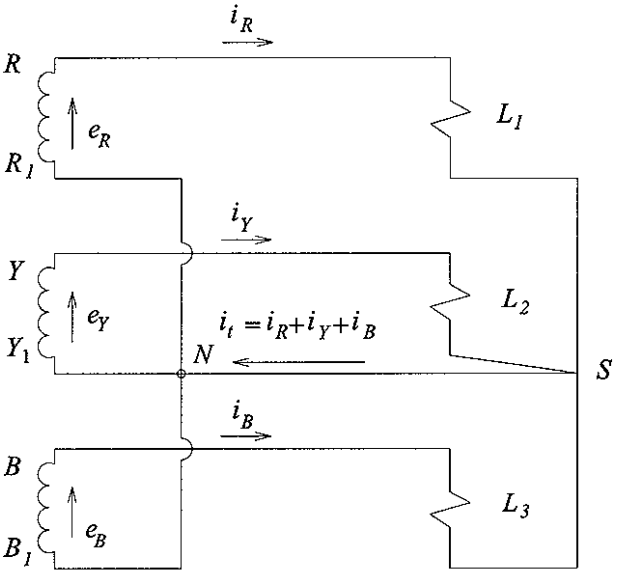


Fig. 8.12

If the system is balanced, the star point is also the neutral point of the system.

Assuming i_R as reference:

$$i_R = I_M \sin \theta$$
$$i_Y = I_M \sin (\theta - 120°)$$
$$i_B = I_M \sin (\theta + 120°)$$

and

$$i_t = i_R + i_Y + i_B =$$
$$= I_M[\sin \theta + \sin(\theta - 120°) + \sin(\theta + 120°)] = 0$$

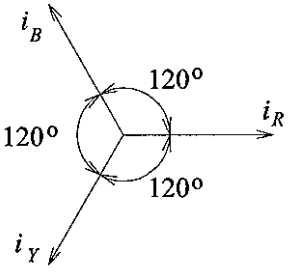


Fig. 8.13

In the phasor diagram for balanced systems given in fig. 8.13 you can see that if you add the three phasors, you always get zero.

So, a 4-wire star-connected system, as in fig. 8.14, is equivalent to fig. 8.9 although it has a different appearance.

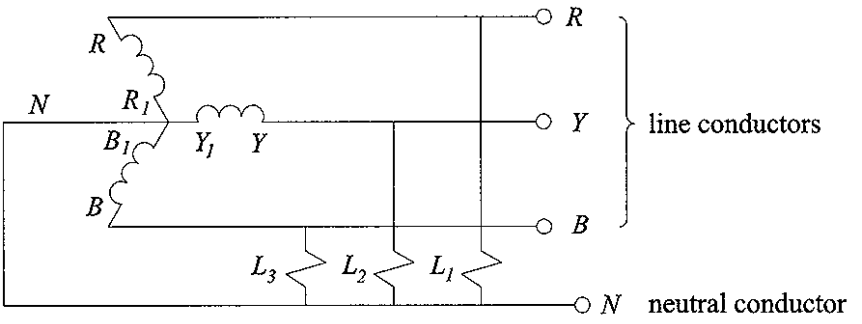


Fig. 8.14

Since this is a balanced load, we know that there is no current flowing in the neutral wire and that it can be discarded. The star-connected system now looks like this:

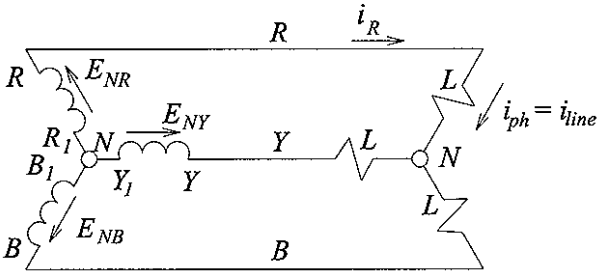


Fig. 8.15

▶ Relationships between line and phase voltages and currents in a star-connected system

It is very important to be able to explain the following two terms when using three-phase systems:

- line → between R, Y, B (between two lines)
- phase → between R and N; Y and N; B and N (between line and neutral)

We normally assume that the emf in each phase is positive when acting from the neutral point outwards. If this is true, the emf$_s$ generated in the three phases can be represented as E_{NR}, E_{NY} and E_{NB}.

If we now assume E_{YNR} positive, the value of the emf from Y to R is the phasor difference of E_{NR} and E_{NY}.

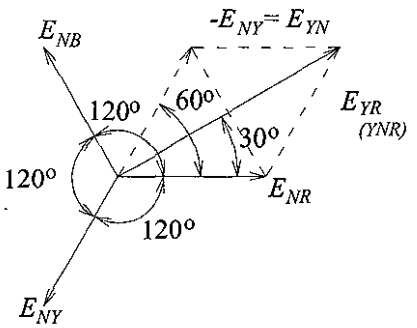


Fig. 8.16

The angle between E_{NR} and $-E_{NY}$ is $180° - 120° = 60°$ and the angle between E_{YR} and E_{NR} is thirty degrees $\Rightarrow$

$$E_{YR} = 2E_{NR} \cos 30° = 2 \times \frac{\sqrt{3}}{2} \times E_{NR} = \sqrt{3}E_{NR}$$

This means that the line voltage is $\sqrt{3} \times$ phase voltage. (The same can be proved for all three phases and lines.)

The line currents are equal to the phase currents. You can see this in the circuit diagram in fig. 8.15. The current flowing in the R line is the same as the current flowing in the load between that line and the star point.

DEFINITION

For a balanced star system:

- line voltage = $\sqrt{3}$ phase voltage
- line current = phase current

TAKE NOTE

Remember, **for a balanced load** the star connection can always be studied as a three-wire connection as there will be no current flowing through the neutral and this conductor can be discarded. Sometimes it is better to have the neutral in place, because then you can analyse only one phase of the circuit (between line and neutral). However, when you study unbalanced loads, you'll see that a current flows through the neutral and there will be a potential difference between the star and neutral points.

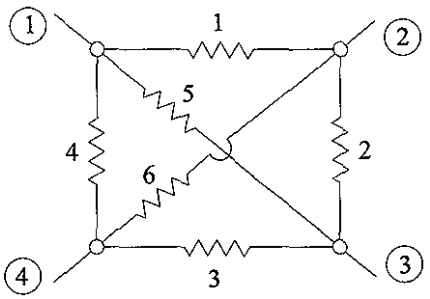
8.3.3 Three-phase delta connection

▶ Circuit diagrams and properties

This three-phase network is called Δ (delta) or mesh connection. Again the symbol is suggested by the form of the circuit diagram.

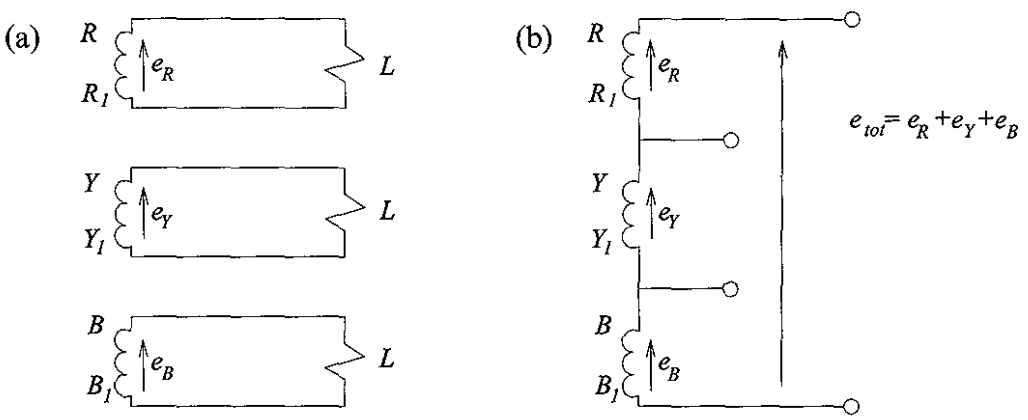
This connection is a particular case of the general case of the 'n' terminal mesh.

A four-terminal delta connection will look like this:



A 'n' terminal mesh has $\frac{n(n-1)}{2}$ elements.

The case of three terminals is a special situation. We will have three elements for star and three elements for delta. This is a very important observation, because it allows us to operate star-delta and delta-star transformations.



3-phase windings with 6 line conductors

resultant emf in a connected winding

Fig. 8.17

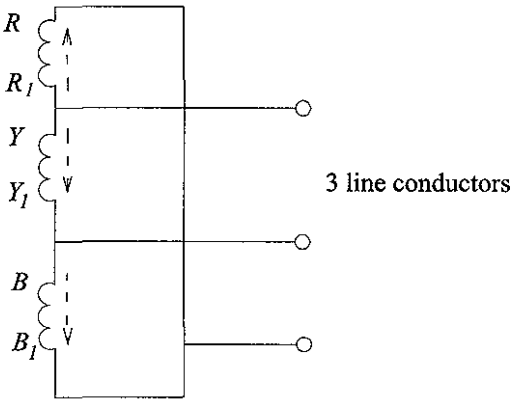


Fig. 8.18

The total emf between B_1 and R is:

$$e_t = e_R + e_Y + e_B =$$
$$= E_m[\sin\theta + \sin(\theta - 120°) + \sin(\theta + 120°)] = 0$$

As you can see, the algebraic sum of the emf$_s$ around the closed circuit is zero and there is no circulating current set up when we connect B_1 and R, So, we can connect the coils as in Figure 8.18.

▶ Relationships between line and phase voltages and currents in a delta-connected system

An equivalent representation for the delta is:

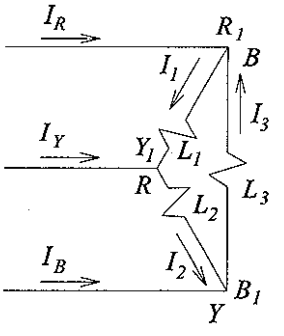


Fig. 8.19

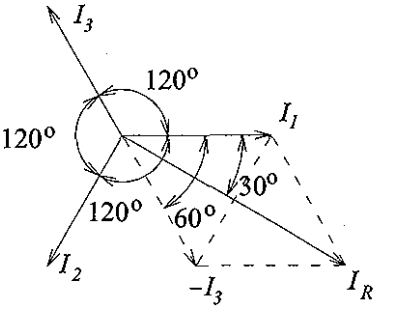


Fig. 8.20

Let I_1, I_2, and I_3 be the rms values of the phase currents having positive directions as in fig. 8.19. You will see that:

$$I_R = I_1 - I_3$$

$$I_R = 2 \times I_1 \times \cos 30° = 2 \times \frac{\sqrt{3}}{2} \times I_1 = \sqrt{3}\, I_1$$

$$\Rightarrow \text{line current} = \sqrt{3} \text{ phase current}$$

Look now at the circuit diagram in fig. 8.19 and you'll notice that line voltage = phase voltage. (The voltage between R and Y is the same as the voltage across L_1.)

DEFINITION

For a delta connection:

- line voltage = phase voltage
- line current = $\sqrt{3}$ phase current

Remember:

- In a three-phase star connection the loads are connected between the lines (R, Y, and B) and the neutral (N), and the point where the three loads join at the neutral is called the star point.
- In a three-phase delta connection the loads are connected between two lines (between R and Y, Y and B, and B and R).

EXAMPLE 8.1

In a balanced star-connected load, assume that the line voltages are balanced and that they are 100 V (rms) and each of the phase impedances Z_1, Z_2, and Z_3 is equal to $Z = 8 + j6\ \Omega$. Determine the currents flowing in the loads. Determine the current flowing through the neutral wire (consider normal phase sequence RYB).

Remember, to solve a problem, always start by sketching the circuit using the given information.

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{100}{\sqrt{3}} = 57.7 \text{ V}$$

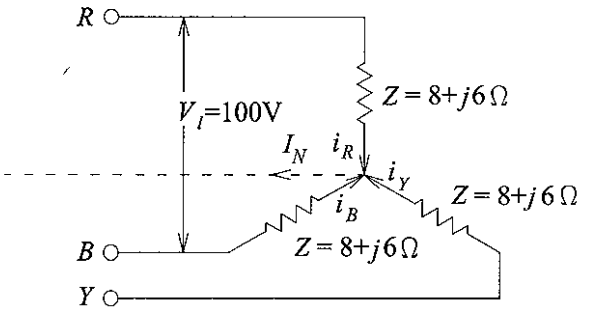


Fig. 8.21

Now its time to assume a reference.

Assuming V_{RN} as reference, we find:

$V_{RN} = 57.7\,\underline{/0°}$ V
$V_{YN} = 57.7\,\underline{/-120°}$ V
$V_{BN} = 57.7\,\underline{/+120°}$ V

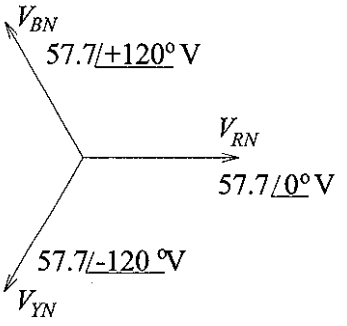


Fig. 8.22

Then $Z_L = 8 + j6 = 10\,\underline{/36.87°}\ \Omega$; the same for R, Y, B.

We can now calculate the three currents:

$$I_{R(N)} = \frac{V_{RN}}{Z} = \frac{57.7\underline{/0°}}{10\underline{/36.87°}} = 5.77\,\underline{/-36.87°} = 4.616 - j3.462 \text{ A}$$

$$I_{Y(N)} = \frac{V_{YN}}{Z} = \frac{57.7\underline{/-120°}}{10\underline{/36.87°}} = 5.77\,\underline{/-156.87°} = -5.306 - j2.267 \text{ A}$$

$$I_{B(N)} = \frac{V_{BN}}{Z} = \frac{57.7\underline{/120°}}{10\underline{/36.87°}} = 5.77\,\underline{/83.13°} = 0.690 - j5.729 \text{ A}$$

(phase current = line current in star)

$I_N = I_R + I_Y + I_B = 4.616 + 0.690 - 5.306 - j(3.462 + 2.267 - 5.729)$
$= 0 - j0 = 0!$

Do you remember from the theory that we were expecting this result? (In a balanced star load you can expect that the current flowing through the neutral is zero).

Now let's represent the current values in a phasor diagram:

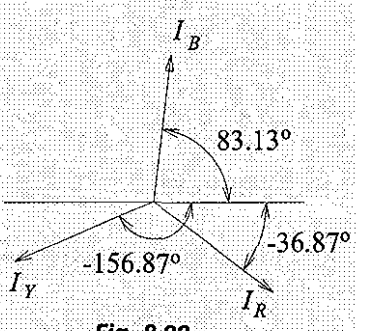


Fig. 8.23

If you encounter such problems, always assume a reference. Choose one of the phasors as reference (V_{RN} in the example). Depending on how you choose this reference, all the angles of voltages and currents must be calculated. In the example, we assumed the phase voltage for R as reference (angle 0°), since it is convenient for solving the problem. Once you choose your reference, you must stick to it in your calculations. If we assume any other reference, the angles of all the phasors will change accordingly. For example, if we choose V_{RY} as reference, $V_{RN} = 57.7\angle{-30°}$, $V_{YN} = 57.5\angle{150°}$, etc.).

EXAMPLE 8.2

A 380 V, 3-phase supply is connected to a ***delta****-connected load. The phase rotation sequence is A-B-C and the load consists of three equal impedances* $Z = 38\angle{-30°}\ \Omega$. *Assuming* V_{AB} *as reference, determine the values of the phase currents and the values of the line currents.*

Solution

Always choose the currents in delta so that they form a closed loop.

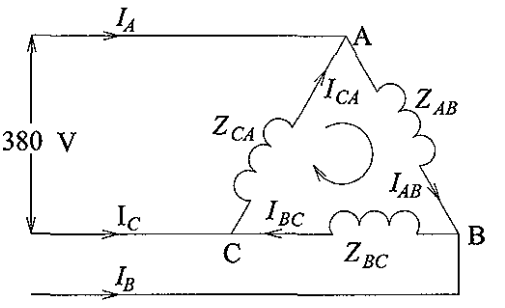


Fig. 8.24

If we have a 380 V supply, we normally assume this is the **line voltage**. For this delta connection we'll use $V_{ph} = V_l$

$V_{AB} = 380\angle 0°$ V
$V_{BC} = 380\angle -120$ V
$V_{CA} = 380\angle 120°$ V

$$I_{AB} = \frac{V_{AB}}{Z_{AB}} = \frac{380\underline{/0°}}{38\underline{/-30°}} = 10\underline{/30°} \text{ A}$$

$$I_{BC} = \frac{V_{BC}}{Z_{BC}} = \frac{380\underline{/-120°}}{38\underline{/-30°}} = 10\underline{/-90°} \text{ A}$$

$$I_{CA} = \frac{V_{CA}}{Z_{CA}} = \frac{380\underline{/120°}}{38\underline{/-30°}} = 10\underline{/150°} \text{ A}$$

Being a balanced load, the sum of the currents is zero.

Now let's calculate the line currents:

$I_A = I_{AB} - I_{CA} = 10\underline{/30°} - 10\underline{/150°} = 17.32 + j0 = 17.32\underline{/0°}$ A
$I_B = I_{BC} - I_{AB} = 10\underline{/-90°} - 10\underline{/30°} = -8.66 - j15 = 17.32\underline{/-120°}$ A
$I_C = I_{CA} - I_{BC} = 10\underline{/150°} - 10\underline{/-90°} = -8.66 + j15 = 17.32\underline{/120°}$ A

What do you notice?
$I_A + I_B + I_C = 0$!

In this example, I_A is in phase with V_{AB}, I_B is in phase with V_{BC}, and I_c is in phase with V_{CA}.

If we look at the relationship between phase and line currents, we can see that:

$I_{ph} = \frac{I_l}{\sqrt{3}}$

$(10 = \frac{17.32}{\sqrt{3}})$

as proved for the balanced delta. We can also see that the phase currents are shifted by $+30°$ compared to the line currents.

The phasor diagram for the problem is:

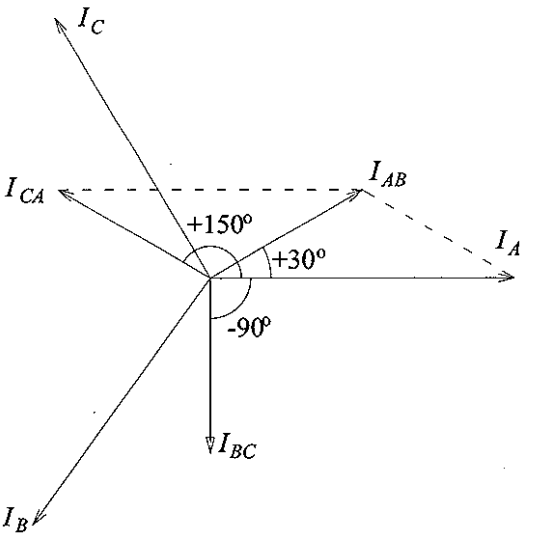


Fig. 8.25

8.4 Three-phase power

8.4.1 Three-phase power for a star connection

The power calculated in a single-phase representation of a three-phase system is the power produced, transmitted or consumed in one phase.

If the system is **balanced**, the **total power** for the whole system – not only for one phase – is then three times as much as the power for one phase.

Consider a star-connected load. For one phase the power will be:

$P_{ph} = V_{ph} I_{ph} \cos\phi$

where $\cos\phi$ is **the load power factor** (cosine of the phase angle between a sinusoidal voltage across the load and the current through that load).

The total power will be:

$P_{tot} = P_1 + P_2 + P_3$

(if the system is balanced, $P_1 = P_2 = P_3 = P$)

and

$P_{tot} = 3P$

$P_{tot} = 3V_{ph}I_{ph} \cos\phi$

Can you remember the relationship between phase and line voltages and currents for a star load?

$V_{ph} = \frac{V_l}{\sqrt{3}}$ and $I_{ph} = I_l$, where V_{ph}, I_{ph} are phase voltage and current, and V_l, I_l are line voltage and current.

If we substitute these in the expression of total power, we get:

$P_{tot} = 3\frac{V_l}{\sqrt{3}}I_l \cos\phi = \sqrt{3}V_lI_l \cos\phi$

We have just proved that, for a star connected balanced load, we can calculate the total power as:

$P_{tot} = 3V_{ph}I_{ph} \cos\phi$

or

$P_{tot} = \sqrt{3}V_lI_l \cos\phi$

TAKE NOTE

If you are unsure which expression has 3 as the multiplying factor and which has $\sqrt{3}$, remember that the power you're calculating is one and the same. If you keep in mind that the line parameters (in this case voltage) are $\sqrt{3}$ times larger than the phase parameters, it is only logical that you have to multiply by 3 where phase voltage and current are known, and by $\sqrt{3}$ if you worked with line values.

Do you remember from single-phase systems that in AC there is not only active (real) power, but also reactive power.

For single-phase AC circuits we calculated:

$S = VI$ VA
$P = VI \cos\phi$ W, and
$Q = VI \sin\phi$ VAr

If you have forgotten, page back to Unit 3 and refresh your memory.

For three-phase systems:

$S = 3V_{ph}I_{ph} = \sqrt{3}V_lI_l$,

$P = 3V_{ph}I_{ph}\cos\phi = \sqrt{3}V_lI_l\cos\phi$, and

$Q = 3V_{ph}I_{ph}\sin\phi = \sqrt{3}V_lI_l\sin\phi$

Activity 8.1

Did you notice that we used the same ϕ (angle) all the time? Are we allowed to do this? Explain your answer in ± 50 words.

8.4.2 Three-phase power for a delta connection

The formulae for a star connection also apply to a balanced load connected in delta. For one phase of the delta:

$P_{ph} = V_{ph}I_{ph}\cos\phi$

where $\cos\phi$ is once again the load power factor, and

$P_{tot} = 3V_{ph}I_{ph}\cos\phi$

Keeping in mind that, in this case, $V_{ph} = V_l$, but $I_{ph} = \frac{I_l}{\sqrt{3}}$, we can write:

$P_{tot} = 3V_l\frac{I_l}{\sqrt{3}}\cos\phi = \sqrt{3}V_lI_l\cos\phi$

Often, the line currents and the line-to-line voltages of a three-phase system are known because these quantities can be measured in any three-wire system. So, we can directly apply the formula. To measure the phase voltage of a star connected load, you need access to the star point to connect the wattmeter voltage coils.

A great advantage of a three-phase system is the uniformity of total power.

Remember, for a single-phase system, the power is pulsating, but the total power for a balanced three-phase load is constant. This is a very important observation and one of its immediate consequences is that three-phase motors present constant torque.

Activity 8.2

Try to prove the statement: 'the total power for a balanced three-phase load is constant in the case of a resistive load'.

Hint: Use trigonometry to prove that the sum of 3 cosine terms is zero. The variation with time disappears leaving instantaneous total power constant.

8.4.3 Measuring three-phase power

▶ Three-wattmeter method

When we use wattmeters to measure power in a three-phase system, the obvious method is to use three wattmeters, connecting each meter to measure the power consumed in one phase.

The simplest example is that of a 4-wire, star-connected load. For each wattmeter, we have I_{ph} in the current coil and V_{ph} across the voltage (potential) coil. The current coil for wattmeter 1 is connected in phase (line) 1, and the voltage coil is connected between line 1 and neutral. So, wattmeter 1 measures the power in phase one, as shown in fig. 8.26:

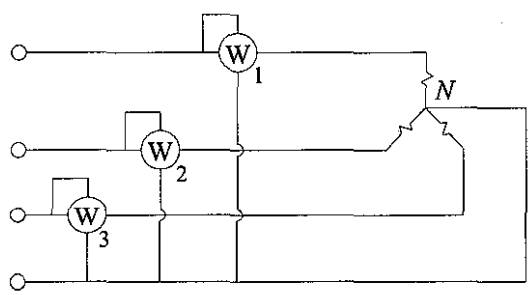


Fig. 8.26

where W_1 measures power for phase 1
W_2 measures power for phase 2
W_3 measures power for phase 3
$P_{tot} = P_1 + P_2 + P_3$

Since we are only dealing with balanced systems, this means that the powers for the three phases are equal. Instead of using three wattmeters, we can use one and multiply its reading by three:

$P_{tot} = 3P_{wattmeter}$

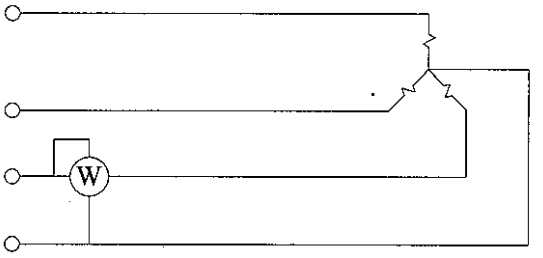


Fig. 8.27

Unfortunately, a delta load or a 3-wire star load doesn't have an available neutral. In fact, in most three-phase networks, the loads normally have only three terminals, like the induction motors. So, we have to find other methods to measure power using different wattmeter connections.

Using three wattmeters:

$P = P_1 + P_2 + P_3$

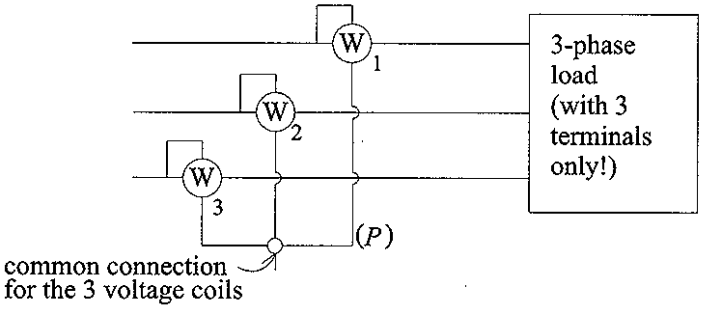


Fig. 8.28

This is a general case, whether the three-phase load is balanced or unbalanced, and the current and voltage can be complex waves. The only observation that must be made is that, in a 3-wire system, the following must be true: $i_a + i_b + i_c = 0$.

▶ Two-wattmeter method

As you can see in fig. 8.28, point P can have any potential. We can choose P so that it has the potential of one of the lines – for example, the potential of line b – which makes W_2 inoperative. There is no voltage across its voltage coil and it will always indicate zero, so it can be omitted.

We are left with the following connection:

$P_{tot} = P_1 + P_3$

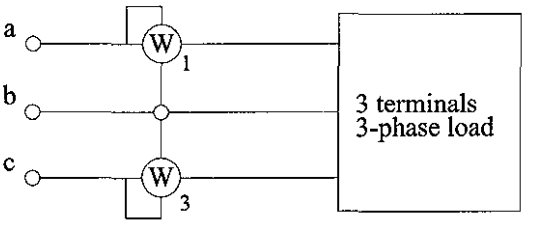


Fig. 8.29

This method is called the two-wattmeter method, and is the standard connection for measuring power in a 3-wire system.

It is a very popular method, as it applies to star and delta, to balanced and unbalanced systems, and for any waveform – even non-linear impedances! The only restriction is that the sum of the three currents $i_a + i_b + i_c = 0$!

Let's see if we can prove that two wattmeters measure the total power for balanced loads.

We assume the phase angle for the load is ϕ. Currents lag the correspondent phase voltages by ϕ and is the same for all three phases as it is a balanced system.

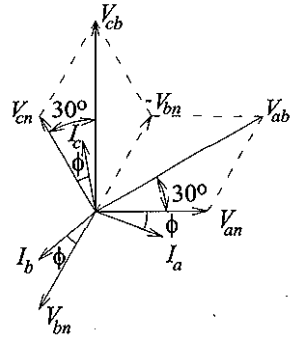


Fig. 8.30

Current through the coil of W_1:

$I_a = I_L$

Voltage through the coil of W_1:

$V_{ab} = V_L$

Current through the coil of W_3:

$I_c = I_L$

Voltage through the coil of W_3:

$V_{cb} = V_L$

Reading on W_1: $P_1 = I_a V_{ab} \cos(30° + \phi)$
Reading on W_3: $P_3 = I_c V_{cb} \cos(30° - \phi)$

$$\Rightarrow \quad P_1 = I_L V_L \cos(30° + \phi)$$
$$P_3 = I_L V_L \cos(30° - \phi)$$
$$P_1 + P_3 = I_L V_L [\cos(30° + \phi) + \cos(30° - \phi)] =$$
$$= V_L I_L \times 2 \times \cos\frac{30° + \phi + 30° - \phi}{2} \times \cos\frac{30° + \phi - 30° + \phi}{2} =$$
$$= V_L I_L \times 2 \times \cos 30° \times \cos\phi = V_L I_L \times 2 \times \frac{\sqrt{3}}{2} \times \cos\phi =$$
$$= \sqrt{3} V_L I_L \cos\phi$$

This is the same value as the total power for the balanced load indicated at the beginning of this section.

We can also prove that:

$P_3 - P_1 = I_L V_L \sin\phi$, and then

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \sqrt{3}\left(\frac{P_3 - P_1}{P_3 + P_1}\right)!$$

By using this formula we can calculate ϕ and work out cos ϕ, which is the power factor of the load.

TAKE NOTE

Sometimes you will find that one of the two wattmeters in the two-wattmeter connection indicates a negative power. Some wattmeters allow readings of negative power. If you come across a negative reading, you will need to reverse the connection either for the current coil or for the voltage coil. (Don't reverse both or the reading will be negative!)

Obviously, negative power has no physical support. What does this mean?

Let's look at our balanced three-phase load. The negative reading indicates that the power factor of the load is less than $\frac{1}{2}$. (The phase angle between load current and voltage is greater than 60° and the current in the current coil of one of the wattmeters is greater than 90° out of phase with the voltage across its voltage coil.)

Activity 8.3

Can you calculate the reactive power (Q) for a three-phase balanced system using the values measured by the two wattmeters in the two-wattmeter method?

Hint: Remember that $P_3 - P_1 = I_L V_L \sin\phi$!

Can you work out how to connect only one wattmeter in the same system and compute the reactive power?

Let's have a look at another method to measure total active power in a balanced delta system if the circuit allows it:

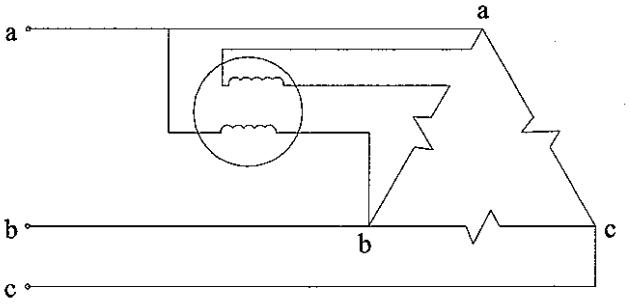


Fig. 8.31

This figure is a more detailed sketch of the wattmeter to enable us to see the connections of the two coils properly. The voltage coil is connected between a and b, and the current coil is connected in series in the a-b branch of the delta connection.

EXAMPLE 8.3

A delta-connected balanced three-phase load is supplied from a three-phase, 380 V supply. The line current is 20 A and the power absorbed by the load is 10 kW. Calculate:

a. the impedance in each branch
b. the line current, power factor and absorbed power if the same load is star-connected.

Solution

a. Delta:

$$V_{phase} = V_{line} = 380 \text{ V}$$

$$I_{phase} = \frac{I_{line}}{\sqrt{3}} = \frac{20}{\sqrt{3}} = 11.55 \text{ A, and}$$

$$|Z| = \frac{V_{ph}}{I_{ph}} = 32.9\ \Omega,$$

$$P = 3V_{ph}I_{ph}\cos\phi \Rightarrow \cos\phi = \frac{10\,000}{3 \times 380 \times 11.55} = 0.76, \text{ and}$$

$$\phi = \cos^{-1} 0.76 = \pm 40.53^\circ$$

$$Z = 32.9 \angle \pm 40.53^\circ\ \Omega$$

b. Since the load is the same, the PF of the load is the same: 0.76 ($\cos\phi = 0.76$)

Star:

$$V_{ph} = \frac{V_L}{\sqrt{3}} = \frac{380}{\sqrt{3}} = 219.39 \text{ V}$$

$$I_{ph} = \frac{V_{ph}}{Z_{ph}} = \frac{219.39}{32.9} = 6.67 \text{ A}$$

$I_{ph} = I_l$ for star $\Rightarrow I_l = 6.67$ A, and, since the PF is the same $\Rightarrow$

$$P = \sqrt{3}V_lI_l\cos\phi = \sqrt{3} \times 380 \times 6.67 \times 0.76 = 3336 \text{ W}$$

TAKE NOTE

The power is one-third of the power absorbed in the delta connection (the small error appeared due to the decimal rounding).

EXAMPLE 8.4

The input power to a 1 200 V, 50 Hz, 3-phase motor, having at full load an efficiency of 90%, is measured by two wattmeters. The readings of the two wattmeters are 300 kW and 100 kW. Determine:

a. input power
b. power factor
c. line current
d. output power

Solution

TAKE NOTE

Since it is not mentioned, we'll assume that 1 200 V is the **line** voltage for this motor.

a. Input power: $P_{tot} = P_1 + P_2 = 300 + 100 = 400$ kW

b. Power factor:

$$\tan\phi = \sqrt{3}\frac{P_2 - P_1}{P_1 + P_2} = \sqrt{3} \times \frac{100 - 300}{400} = -0.866$$

$$\phi = -40.893°$$

$$\cos\phi = 0.756$$

Since motors are inductive types of loads, we obtained a negative angle (current lagging the voltage) as expected.

c. Line current:

$$P = \sqrt{3}V_l I_l \cos\phi \Rightarrow$$

$$I_l = \frac{P}{\sqrt{3}V_l \cos\phi} = \frac{400 \cdot 10^3}{\sqrt{3} \times 1\,200 \times 0.756} = 254.564 \text{ A}$$

d. Output power:

$$\eta = \frac{P_{output}}{P_{input}} \Rightarrow$$

$$P_{output} = P_{input} \times \eta = 400 \times 0.9 = 360 \text{ kW}$$

8.5 Summary

- **Three-phase systems** are more **economical** than single-phase systems. Since most of them are relatively balanced, their analysis is not much more complicated than that for single-phase circuits.
- **Three-phase generators produce a three-phase voltage**, which is composed of three single-phase AC voltages, each **separated by 120°**.
- The positive (clockwise) phase sequence is the *a-b-c* phase sequence. The negative sequence (*a-c-b*) is not normally used in our studies at this stage.

- **The most popular three-phase connections are the star (Y) and the delta connections.** For a **balanced** system, we can prove that the 3- or 4-wire stars are equivalent. The following relationships can be proved:
  - for a balanced star connection:
    $I_{phase} = I_{line}$
    $V_{phase} = \frac{V_{line}}{\sqrt{3}}$
  - for a balanced delta connection:
    $V_{phase} = V_{line}$
    $I_{phase} = \frac{I_{line}}{\sqrt{3}}$
  - For equivalent balanced star and delta:
    $Z_{star} = \frac{Z_{delta}}{3}$
  - For both balanced star and delta connections we proved that:
    $P_{total} = 3V_{ph}I_{ph}\cos\phi$, or
    $P_{total} = \sqrt{3}V_lI_l\cos\phi$
    where $\cos\phi$ = the power factor of the load.

    Similarly:
    $S_{total} = 3V_{ph}I_{ph} = \sqrt{3}V_lI_l$, and
    $Q_{total} = 3V_{ph}I_{ph}\sin\phi = \sqrt{3}V_lI_l\sin\phi$

- For a three-phase balanced system, the **power in the system is constant**.
- The method normally used to measure power in a three-phase system is the **two-wattmeter method**.

If we make the following notations:

W_1 – the reading on one of the wattmeters, and
W_2 – the reading on the other wattmeter

we can calculate:

$$P_{total} = W_1 + W_2$$

$$\tan\phi = \sqrt{3}\frac{W_1 - W_2}{W_1 + W_2} \Rightarrow \phi \Rightarrow \cos\phi \text{ (the PF of the load)}$$

$$Q_{total} = \sqrt{3}(W_1 - W_2)$$

TAKE NOTE

$W_1 = W_2$ for power factor = 1 (resistive); $\phi = 0$.
W_1 = max for $\phi = 30°$

$W_2 = 0$ for $\phi = 60°$
$W_2 < 0$ for $\phi > 60°$

- Remember, for the purpose of this course, we focused on the study of three-phase balanced system only. (Three voltages equal in amplitude and having the same frequency, but shifted by 120° in phase angle, applied to equal impedances with resulting balanced currents.)

Self-evaluation

1. In the balanced star-connected load of the following figure, assume that the phase voltages are 240 V (rms) and each of the impedances are $Z = 25 + j40\,\Omega$. Calculate the currents flowing through each of the impedances and draw the phasor diagram for the currents. What do you notice? (Assume V_{RN} as reference and a normal phase sequence rotation.)

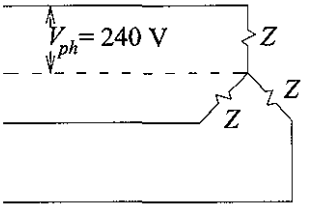


Fig. 8.32

2. A three-phase balanced delta-connected load is fed from a star that has a **phase** voltage of 240 V. Assuming a normal phase rotation sequence, and knowing that the impedances in each branch of the delta are equal to $40 + j25\ \Omega$, calculate the line and the phase currents.

3. Three similar coils, each having a resistance of 20 Ω and an inductance of 0.05 H are connected:
   a. in star
   b. in delta

   to a three-phase, 50 Hz supply with 400 V between the lines. Calculate the total power absorbed and the line current in each case. Draw the phasor diagrams of the voltages across the loads and of the currents flowing through the load in each case.

4. A 500 V, 3-phase motor has an output power of 37.8 kW and operates at a power factor of 0.85. The efficiency of the motor is 90%. Determine the reading on each of the two wattmeters used to measure the input power.

5. Let's consider a large three-phase induction motor. The two ends of each phase of the stator windings are brought out to a starter that, when moved to the starting position, connects the windings in a star. After the motor has accelerated, the starter is quickly moved to the running position, changing the connections to a delta.
   a. Find out the ratio between the starting voltage and the rated voltage (per phase).
   b. Knowing that the output torque of the motor is directly proportional to the square of the applied voltage, calculate the ratio between the starting torque and the torque that would be obtained if the motor is switched directly across the supply (with the stator windings connected in delta).
   c. What would the ratio be of the phase currents in this case (star connection/ delta connection)?
   d. What would the ratio of the line currents be?

Answers

Activity 8.1 (page 226)

Yes. If the load is balanced, the power factor for all three phases will be the same. The PF refers to the load and has nothing to do with the angle between V_l and I_l.

Activity 8.2 (page 226)

$$\begin{cases} v_a = V_m \cos\omega t \\ i_a = I_m \cos\omega t \end{cases}$$

$$p_a = v_a i_a = V_m I_m \cos^2 \omega t = V_m I_m \frac{\cos 2\omega t + 1}{2} = \frac{1}{2} V_m I_m (\cos 2\omega t + 1)$$

$$\begin{cases} v_b = V_m \cos(\omega t - 120°) \\ i_b = I_m \cos(\omega t - 120°) \end{cases}$$

$$p_b = V_m I_m \cos^2(\omega t - 120°) = V_m I_m \frac{\cos(2\omega t - 240°) + 1}{2}$$

$$= \frac{1}{2} V_m I_m [\cos(2\omega t - 240°) + 1]$$

$$\begin{cases} v_c = V_m \cos(\omega t + 120°) \\ i_c = I_m \cos(\omega t + 120°) \end{cases}$$

$$p_c = V_m I_m \cos^2(\omega t + 120°) = \frac{1}{2} V_m I_m [\cos(2\omega t + 240°) + 1]$$

$$p_a + p_b + p_c = \tfrac{1}{2}V_m I_m[\cos 2\omega t + 1 + \cos(2\omega t - 240°) + 1 + \cos(2\omega t + 240°) + 1] =$$
$$= \tfrac{1}{2}V_m I_m[3 + \cos 2\omega t + \cos(2\omega t - 240°) + \cos(2\omega t + 240°)]$$

$$\cos(2\omega t - 240°) + \cos(2\omega t + 240°) =$$
$$= 2\cos\frac{2\omega t - 240° + 2\omega t + 240°}{2}\cos\frac{2\omega t - 240° - 2\omega t - 240°}{2}$$
$$= 2\cos\frac{4\omega t}{2}\cos\frac{(-480°)}{2} = 2 \times \cos 2\omega t \times \cos(-240°) =$$
$$= 2 \times \cos 2\omega t \times \cos(120°) = 2 \times \cos 2\omega t \times (-\tfrac{1}{2}) = -\cos 2\omega t$$

$$\cos 2\omega t - \cos 2\omega t = 0$$
$$\Rightarrow p = 3\frac{V_m I_m}{2}$$

Do you notice anything?

For one phase:

$$\begin{cases} v_{rms} = \dfrac{\sqrt{2}}{2}V_m \\ i_{rms} = \dfrac{\sqrt{2}}{2}I_m \end{cases}$$

$$p_{av} = \frac{\sqrt{2}}{2}V_m \times \frac{\sqrt{2}}{2}I_m = \frac{2}{2\times 2}V_m I_m = \frac{V_m I_m}{2}$$

If you don't remember this, revise Unit 2.

We can say that the instantaneous total power is constant and equal to three times the average phase power.

The proof can be extended to include any load (PF $\neq$ 1) by inserting a phase angle ϕ in each expression for the instantaneous current. The result will be: $p = 3VI\cos\phi$ – again a constant, three times the average phase power. (Here $\cos\phi$ is the power factor of the load.)

However, remember that this proof is valid only for sinusoidal voltages and currents.

Activity 8.3 (page 231)

Multiply $P_3 - P_1$ by $\sqrt{3}$ and then you'll get exactly $\sqrt{3}I_L V_L \sin\phi$, which is the reactive power (Q) for the three-phase system.

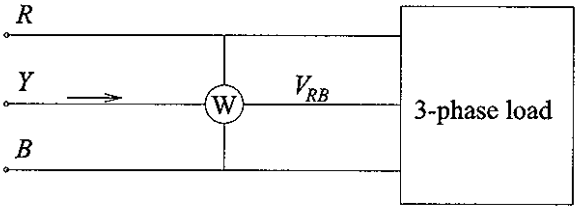


Fig. 8.33

The reading must also be multiplied by $\sqrt{3}$.

Self-evaluation (page 235)

1. $5.09\,\underline{/-58^\circ}$ A; $5.09\,\underline{/-178^\circ}$ A; $5.09\,\underline{/62^\circ}$ A

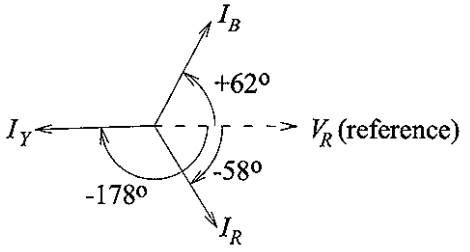


Fig. 8.34

If you look at the phasor diagram, you notice that $I_R + I_Y + I_B = 0$:

2. If we take V_{AB} as assumed reference:
   $I_{AB} = 8.81\,\underline{/-32^\circ}$ A
   $I_{BC} = 8.81\,\underline{/-152^\circ}$ A
   $I_{CA} = 8.81\,\underline{/88^\circ}$ A
   $I_A = 15.26\,\underline{/-62^\circ}$ A
   $I_B = 15.26\,\underline{/-182^\circ}$ A $= 15.26\,\underline{/178^\circ}$ A
   $I_C = 15.26\,\underline{/58^\circ}$ A

3. a. 4 948 W; 9.08 A

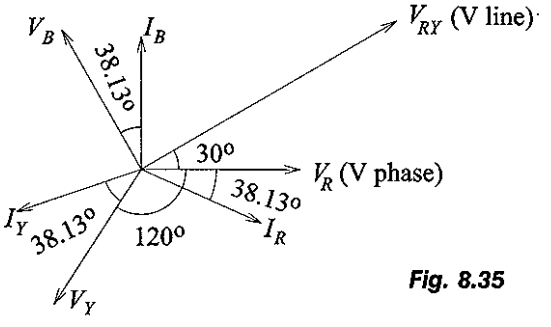


Fig. 8.35

b. 14 850 W; 27.25 A (≈ 3 times the star values!)

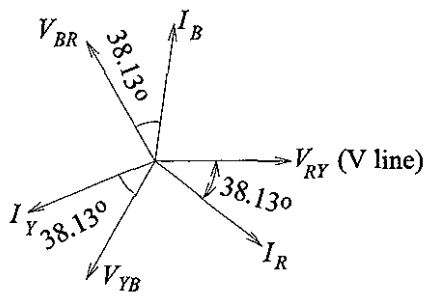


Fig. 8.36

4. 13.5 kW; 28.5 kW

5 a. $\frac{1}{\sqrt{3}}$; b. $\frac{1}{3}$; c. $\frac{1}{\sqrt{3}}$; d. $\frac{1}{3}$

Appendix I

▶ Transformations

The following transformations will help you change the star connection to an equivalent delta connection or vice-versa. Remember these when you are working with star or delta systems:

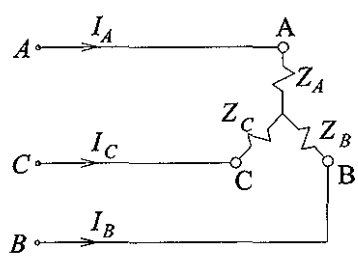


Fig. 8.37

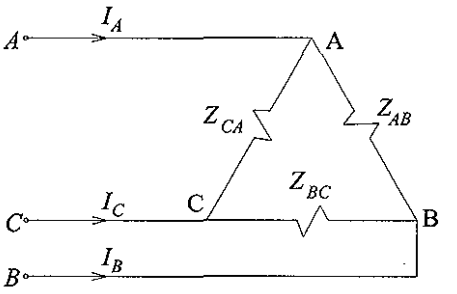


Fig. 8.38

Star to delta transformation:

$$Z_{AB} = \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_C} = Z_A + Z_B + \frac{Z_A Z_B}{Z_C}$$

$$Z_{BC} = \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_A} = Z_B + Z_C + \frac{Z_B Z_C}{Z_A}$$

$$Z_{CA} = \frac{Z_A Z_B + Z_B Z_C + Z_C Z_A}{Z_B} = Z_A + Z_C + \frac{Z_A Z_C}{Z_B}$$

Delta to star transformation:

$$Z_A = \frac{Z_{AB} Z_{CA}}{Z_{AB} + Z_{BC} + Z_{CA}}$$

$$Z_B = \frac{Z_{AB} Z_{BC}}{Z_{AB} + Z_{BC} + Z_{CA}}$$

$$Z_C = \frac{Z_{CA} Z_{BC}}{Z_{AB} + Z_{BC} + Z_{CA}}$$

For the special case of a balanced system, make the following notations:

Z_Y – impedance for the star connection (the same on all three phases)
Z_Δ – impedance for the delta connection (again the same for the three phases)

It can be proved that:

$Z_Y = \frac{Z_\Delta}{3}$ and respectively, $Z_\Delta = 3Z_Y$