PN-ABK-625
75986

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INTERNATIONAL FOOD POLICY RESEARCH INSTITUTE

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Washington, D.C. 20036 1998 U.S.A.

Reprint No. 229

15 July 1991

FALLING TRANSPORT COSTS AND AGRICULTURAL DEVELOPMENT: IMPLICATIONS OF VON THUENEN'S MODEL OF THE DUAL ECONOMY[1]

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Prepared for presentation at the Annual Meeting of the American Agricultural Economics Association, Manhattan, Kansas, 4-7 August 1991.

[1] We are indebted to Heinz Koenig, John Mellor, and Hans van Beeck for helpful comments, and to Viktoria Dalko for her able research assistance. Financial assistance from the Foerder Institute for Economic Research and the International Food Policy Research Institute is gratefully acknowledged. A version of the paper also appears as Working Paper No. 989 of the Foerder Institute and Discussion Paper No. A 269 of the University of Bonn.

ABSTRACT

Von Thuenen's general equilibrium model of the dual economy shows that falling transport costs alone are sufficient to induce a movement of labor from the rural/agricultural to the urban/industrial sector. More rapid rural population growth or higher income elasticities of demand for manufactures are not necessary.

Imagine a very large town, at the center of a fertile plain which is crossed by no navigable river or canal. Throughout the plain the soil is capable of cultivation and of the same fertility. Far from the town, the plain turns into uncultivated wilderness which cuts off all communication between this State and the outside world.

There are no other towns on the plain. The central town must therefore supply the rural areas with all manufactured products and in return it will obtain all its provisions from the surrounding countryside.

J.H. von Thuenen (1826; 1842, p.11).

I. Introduction

The concept of dualism typically involves some essential asymmetry between two economic sectors and is frequently dynamic, involving disequilibrium and a misallocation of resources between the two sectors. (See Dixit, 1973; Kanbur and McIntosh, 1987; and Ranis, 1988.) In our discussion of the von Thuenen model, however, we will develop duality implicit in his work in the context of a general equilibrium model in which resources are allocated efficiently. The asymmetry between the two sectors, agriculture and industry, arises because land is immobile and required only in agricultural production.

While preferences and demand, especially differing income elasticities of demand, are frequently given a central role (Jorgenson, 1961; and Kelley, Williamson and Cheetham, 1972), assumptions made in this connection are frequently ad hoc in order to arrive at the principal conclusions of dual economy models concerning the movement of resources from the "traditional" to the "modern" sector. In our general development we assume homotheticity of preferences so that all income elasticities are one. In a more specialized development of the model, we assume both unitary elasticities of substitution in preferences and in agricultural production; the former implies that all price elasticities of consumer demand and derived demand for factors of production used in agriculture are -1, i.e., that relative budget and factor shares are constant. We purposely make such drastically simplifying assumptions to see how far we can get without introducing assumptions about empirical magnitudes, which are made primarily to guarantee particular conclusions. These simplifications also, incidently, enable us to demonstrate the uniqueness of equilibrium in the von Thuenen model, a uniqueness which is necessary to give meaning to our comparative static analysis.

Finally, although changes in relative prices, the terms of trade, during the course of dual economic development do figure prominently in the work of Lewis (1954), Ranis and Fei (1961) and Jorgenson (1961), wages are determined in one or both sectors exogenously, so that the connection between relative prices and wages, as well as the spatial variation in prices, is overlooked. In our development wages are determined endogenously in a perfectly functioning labor market, and the von Thuenen model thus provides a link between prices and wages which has played so central a role in the fields of urban and regional economics.

Recently Samuelson (1983) has given an extended appreciation of von Thuenen's contributions to location theory and, above all, to the development of the neoclassical theory of marginal factor productivity and distributive shares.[1] In his appreciation, however, Samuelson also brings out very nicely the general equilibrium nature of von Thuenen's location theory. Except by geographers concerned with economic development, the spatial aspects of growth have been generally neglected in the economic literature. In particular, the role of transportation costs in determining the spatial distribution of commodity prices and, therefore, real wages and the spatial distribution of labor in developing economies is generally de-emphasized.[2] The general spatial equilibrium model of von Thuenen, as exposited by Samuelson, may be used to fill this gap and to provide a basis for both equilibrium and disequilibrium models of growth of the dual economy. Although von Thuenen is widely regarded as the father of location theory and now appreciated as the independent discoverer of the marginal productivity theory of distribution, he is also, in our view, the author of the first, and in some ways the best, model of the dual economy, a model which, with a little effort can be turned into a model of dual economic development having considerable relevance to the developing economies

of the world today in which high costs of transport are pervasive and significant determinants of relative commodity prices.[3] Section 2 provides a general discussion of von Thuenen's general equilibrium model of the location of economic activity following Samuelson's appreciation. Section 3 brings out some spatial features of von Thuenen's model. In Section 4, we deal with the nature of equilibrium in the von Thuenen model with logarithmically additive preferences. Section 5 provides some interesting comparative statics results, concerning, in particular, the possibilities of technological progress and growth and the effects of falling transport costs.

2. Von Thuenen's General Equilibrium Model

First, following von Thuenen, we assume constant returns to scale in agriculture, transportation and industry. The assumption of constant returns to scale is important in a comparative static analysis of a general equilibrium model for two reasons: It is necessary to demonstrate the uniqueness of equilibrium, without which comparative statics is meaningless. Second, without constant returns, at least in the vicinity of equilibrium, exhaustion of total product when each factor is paid its marginal product cannot be guaranteed. This then leaves open the question of what happens to the remainder, under diminishing returns, or from where the deficit is made up, under increasing returns. Assuming constant returns to scale does, of course, rule out an important rationale for the concentration of manufacturing activity in one place. However, for our purposes, it is not necessary to assume constant returns everywhere but only in the vicinity of equilibrium: An initial region of increasing returns at very low output levels would suffice to bring the town into existence, after which constant returns <u>in the town</u> could be a reasonable

assumption. Constant returns to scale in transport may also strike the reader as very restrictive in view of extensive discussions about indivisibilities in infrastructure and the consequent need for public development. Such matters would have to be discussed in an analysis of development policy focused on the role of transport in economic development. Our purpose here, however, is to show exactly how the von Thuenen model focuses attention on the role of transport and of relative prices in dual economic development. For this purpose, such realism is unnecessary and distracting.

We find that we cannot solve the von Thuenen model explicitly or demonstrate the uniqueness of equilibrium in general, and so we make two additional specific assumptions not made by von Thuenen. We assume unitary elasticities of substitution in agricultural production and in consumers' preferences and homothetic preferences. The latter guarantees that income distribution doesn't matter, since, given relative prices, the consumption of each commodity by a consumer with twice the income of another is simply double, that is, all income elasticities of demand are one. The assumption of homotheticity allows a considerable simplification and allows us to focus on the analysis on matters of more central concern, namely, transport costs and relative prices. The assumption of unitary elasticities is not so benign (one could hardly say innocuous), for it implies constant factor shares in agriculture and unitary price elasticities of consumer demand. In that part of the analysis, therefore, in which we must perforce make these assumptions, the particular results may depend on them. We can ask in what direction it might be modified if the elasticity of substitution in question were a bit greater than one or a bit less than one, that is adopt a kind of variational technique in our analysis.

We do understand that different income elasticities of demand for manufacturers play a crucial role in many other discussions of dual economic development. Homothetic preferences not only rule out interesting distributional problems in economic development, but also move the focus of the analysis from income effects of growth to price effects. While not denying the irreality of the von Thuenen model in this respect, it is nonetheless remarkable that one can get so far with it in understanding an important reason for the movement of labor from countryside to city in the course of economic development.

To begin, we make the following assumptions:

(a) All land is homogeneous except for distance from the town. The town exists as a point.

(b) There are only two goods. A manufactured commodity, say cloth, produced in the town, and an agricultural commodity, say corn, produced in the countryside.

(c) Cloth is produced by labor alone (any raw materials used are available at the site of the town). We assume constant returns to scale, for the reasons outlined above.

(d) Corn is produced by labor and land according to a mainly constant returns production function. It is useful, however, to assume that a minimum application of labor per unit of land is necessary to grow anything at all. This translates into a minimal, nonzero land rent at the extensive margin of cultivation. This assumption is made largely for convenience, but see below. It allows us to discuss changes in the extensive margin of cultivation (an important concept in von Thuenen's work) without a relatively complicated analysis of the density of labor use per unit of land. With infinitely divisable factors of production and constant returns, there would otherwise be <u>no</u> point at

which, "Far from the town, the plain turns into uncultivated wilderness...."

(e) All commodity markets clear instantaneously at every point; the total demand for cloth equals its supply as does the total demand for and supply of corn; prices adjust accordingly at every point.

(f) Transport costs for both commodities, although different, are proportional to distance (logarithmically linear in distance). Again, note our caveat above.

(g) Labor is homogeneous and assumed perfectly mobile. All laborers have identical preferences. The distribution of a fixed labor force between town and country, and in the country at different distances from town, is determined so as to equalize the real wage (or utility) everywhere. Because the prices of cloth and corn may differ from location to location, however, money wages vary from point to point. Labor is assumed to live where it works.

(h) Landowners live in town and use all their rent (profit) income to consume corn and cloth. Their wealth consists entirely of the value of the land they own. All landowners have identical, homothetic preferences. This assumption simplifies matters since, as remarked, given relative prices, landowners' demand will be proportional to total rent income and will not depend on the number of landowners among whom it is divided.

Independent, exogenous variables of the analysis are the technology of production, preferences for corn and for cloth, transport costs (or technology), and population or labor force. Subsuming the necessary minimal labor per unit of land as a technological parameter, all other variables, such as relative prices at each point, the location of the extensive margin of cultivation, the real wage, numeraire wages at each point, the amount of labor applied per unit of land, land rent at each point, and the total consumption of cloth and corn, and the distribution of these in space, are all endogenous variables determined within the von Thuonen system by the equilibrium of factor and product markets.

To see how the system works, proceed as follows: All distances may be measured in terms of the distance from the town, r. Indeed, all behavior at every point on a circle of radius r, with the town at the center, is identical. Let the subscript 0 denote cloth (manufactures) and 1 denote corn (agricultural products). Production of cloth, Q_0, may be assumed to be proportional to the labor used in cloth production, L_0:

$$(1) \qquad Q_0 = f_0 L_0,$$

where $f_0 > 0$ is assumed constant. Production of corn at a distance r from the town, $Q_1(r)$, depends on the labor applied, $L_1(r)$, to the quantity of land planted to corn, $A_1(r) = 2\pi r$. We assume that agricultural production takes place according to a constant returns production function, so that the yield of corn per unit area, $q_1(r)$, is a function of the labor applied per unit area, $l_1(r) = L_1(r)/A_1(r)$, i.e.:

$$(2) \qquad q_1(r) = f_1[l_1(r)].$$

The function f_1 is strictly increasing ($f_1' > 0$) and strictly concave ($f_1'' < 0$).[4] If R is the location of the extensive margin of cultivation, the distance from the town at which "... the plain turns into uncultivated wilderness," then the total production of corn is

$$(3) \qquad Q_1 = 2\pi\int_0^R q_1(r)r\,dr.$$

The total labor force in agriculture is, similarly,

$$(4) \qquad L_1 = 2\pi\int_0^R l_1(r)r\,dr.$$

If L is the total labor force available, given exogenously, then

$$(5) \qquad L = L_0 + L_1.$$

Let $w(r)$, $p_0(r)$, and $p_1(r)$ be, respectively, the wage rate, the price of cloth, and the price of corn, all at a distance r from town. Thus, equating the wage to the value of the marginal product of labor yields:

(6) $$w(0) = p_0(0)f_0 = f_0$$

and

(7) $$w(r) = p_1(r)f_1'(l_1(r)), \text{ for all } r > 0,$$

where the price of cloth in town is set as the numeraire ($p_0(0) = 1$). Wage payments in the town exhaust the total product of cloth. However, wage payments do not exhaust the product of corn at r; there is a positive residual equal to rent per unit of land:

(8) $$y_1(r) = p_1(r)\{f_1(l_1(r)) - l_1(r)f_1'(l_1(r))\} > 0,$$

because of the assumed concavity of the production function f_1. Total income of landowners is equal to aggregate land rent:

(9) $$Y = 2\pi\int_0^R y_1(r)rdr.$$

We shall show in the next section (see (26) below) that the labor/land ratio (l_1) is diminishing as distance from the town increases. Thus, land is increasingly substituted for labor until some minimal labor/land ratio (say m) is reached. At this point, say R, all cultivation ceases:

(10) $$l_1(R) = m.$$

Denote the expenditure function by $E(\cdot)$, i.e., $E(p_0, p_1, u)$ is the (minimum) income required in order to attain the utility level u at prices p_0 and p_1 for cloth and corn, respectively. Since laborers are homogenous and perfectly mobile, they must all enjoy the same utility level, irrespective of where they choose to work and reside. This yields the fundamental locational equilibrium condition of the model:

(11) $$E(p_0(r), p_1(r), u) = w(r) \text{ for all } r \geq 0.$$

For every r, the price of corn must be such that producers are just indifferent between selling the grain locally for $p_1(r)$ or shipping it to the town where the price is higher but having to deduct the costs of transport. Von Thuenen makes a series of careful calculations in which he reckons the costs of

transport to consist largely of corn consumed on the way by the oxen pulling the load; as a first approximation, we can assume transport costs in grain to be proportional to the quantity transported per mile. A bushel of grain in the countryside r miles from the town becomes $e^{-a_1 r}$ bushels in town. It follows that

$$(12) \qquad p_1(r) = p_1(0)e^{-a_1 r},$$

where a_1 is a parameter reflecting the costs of transport; the higher a_1, the greater the costs of transport. Although there is no argument for doing so other than symmetry and simplicity, assume that the costs of transporting cloth to the countryside also have the same proportionate form

$$(13) \qquad p_0(r) = p_0(0)e^{a_0 r} = e^{a_0 r}. \quad 5$$

At every point r, the excess supply of corn consists of output minus consumption of laborers residing there, i.e.:

$$2\pi r\{f_1(l_1(r)) - H_1(p_0(r), p_1(r), u)l_1(r)\},$$

where $H_i(\cdot)$ is the Hicksian demand function for commodity i, i = 0,1. Note that everyone in the countryside has a surplus of corn since all their income is received in corn and they wish to trade some for cloth. Hence, there is an excess supply of corn which must be shipped to town. However, due to transport costs, a bushel of corn which is shipped from point r to town shrinks to $e^{-a_1 r}$ bushels in town. Thus, the corn market clearing equation (in town) requires that

$$(14) \qquad 2\pi\int_0^R r\{f_1(l_1(r)) - H_1(p_0(r), p_1(r), u)l_1(r)\}e^{-a_1 r}dr$$
$$= L_0H_1(p_0(0), p_1(0), u) + D_1(p_0(0), p_1(0), Y),$$

where L_0H_1 is the demand for corn by cloth laborers and $D_i(\cdot)$ is the Marshallian demand function of landowners for commodity i, i = 0,1.

Similarly, the cloth market clearing equation requires that:

$$(15) \qquad 2\pi\int_0^R r\, H_0(p_0(r),\ p_1(r),\ u)\ l_1(r)\ e^{a_0 r}\ dr$$
$$+\ L_0 H_0(p_0(0)\ \ p_1(0),\ u) + D_0(p_0(0),\ p_1(0),\ Y) = Q_0.$$

Due to Walras's Law, one of the two market clearing equations, (14) or (15), is redundant. We henceforth drop equation (14). The 14 equations remaining determine the following endogenous variables:

total cloth output, Q_0;

corn output per unit area at location r, $q_1(r)$;

total corn output, Q_1;

labor/land ratio in corn production at location r, $l_1(r)$;

total labor input in corn production, L_1;

total labor input in cloth production, L_0;

wage rate in town, $w(0)$;

wage rate at location r, $w(r)$;

cloth price at location r, $p_0(r)$;

corn price at location r, $p_1(r)$;

land rent at location r, $y_1(r)$;

aggregate land rent, Y;

the location of the extensive margin of cultivation, R;

the laborers' common utility level, u.

3. Spatial Features of the von Thuenen Model

In this section we investigate the spatial properties of the endogenous variables. Specifically, we examine the gradients (namely, the derivatives with respect to r) of the wage, cloth and corn prices, rent, cloth and corn

consumption and factor intensity (equivalently, population density).

Before proceeding further we will state some familiar results from demand theory that will be employed in our analysis. First, the envelope theorem implies the derivative property of the expenditure function:

$$(16) \qquad \frac{\partial E(p_0,p_1,u)}{\partial p_i} = H_i(p_0,p_1,u), \quad i=0,1.$$

Second, we have the following signs for the Hicks-Slutsky substitution terms ($\partial H_i/\partial p_j$, $i,j = 0,1$):

$$(17) \qquad \frac{\partial H_i(p_0,p_1,u)}{\partial p_j} \begin{cases} < 0, \text{ if } i=j, \\ > 0, \text{ if } i\neq j. \end{cases}$$

A dot ($\cdot$) over a variable will denote a derivative with respect to location r. The price gradients are trivially obtained from (12) and (13):

$$(18) \qquad \dot{p}_0 = a_0 p_0 > 0$$

and

$$(19) \qquad \dot{p}_1 = -a_1 p_1 < 0.$$

That is, the price of cloth rises and the price of corn declines as one moves further away from the town. The consumption gradients are derived by making use of (17):

$$(20) \qquad \dot{H}_0 = \frac{\partial H_0}{\partial p_0}\dot{p}_0 + \frac{\partial H_0}{\partial p_1}\dot{p}_1 < 0$$

and

$$(21) \qquad \dot{H}_1 = \frac{\partial H_1}{\partial p_0}\dot{p}_0 + \frac{\partial H_1}{\partial p_1}\dot{p}_1 > 0.$$

That is, as the price of cloth rises and the price of corn declines, the consumption of cloth declines and the consumption of corn rises further away from the town: laborers substitute the cheaper commodity (corn) for the more

expensive commodity (cloth).

To obtain the wage gradient, differentiate the locational equilibrium condition (11) with respect to r to obtain:

$$\dot{w} = \frac{\partial E}{\partial p_0}\dot{p}_0 + \frac{\partial E}{\partial p_1}\dot{p}_1 = \dot{p}_0 H_0 + \dot{p}_1 H_1,$$

where use is made of the derivative property (16). Substituting (18) and (19) for $\dot{p}_0$ and $\dot{p}_1$, respectively, one obtains:

$$(22) \qquad \dot{w} = a_0 p_0 H_0 - a_1 p_1 H_1.$$

The interpretation of (22) is straightforward: as the laborer moves further away from the town, he has to be compensated by $a_0 p_0 H_0$ for the increase in transport costs (and price) of cloth and can give up $a_1 p_1 H_1$ due to the decline in the price of corn. Thus, the sign of the wage gradient is not generally determined without making further assumptions. Denote by $\alpha = p_0 H_0/w$ and by $1 - \alpha = p_1 H_1/w$, the budget shares of cloth and corn, respectively. Then, (22) can be rewritten as

$$(23) \qquad \dot{w} = [a_0\alpha - a_1(1-\alpha)]w.$$

Thus, assuming symmetry in transport cost (namely, $a_0 = a_1$) and that, as is plausible in a low income economy, corn's budget share is higher than cloth's budget share, then $\dot{w} < 0$. That is, money wage declines further away from the town. Note that, in general, the shares of expenditures will not be constant, except when preferences are characterized by a unit elasticity of substitution. We make this assumption in the next section, but analyze the direction of change to be expected if the elasticity departs from one in either direction.

In order to derive the factor intensity gradient, differentiate the wage equation (7) with respect to r:

$$(24) \qquad \dot{w} = \dot{p}_1 f_1' + p_1 f_1'' \dot{l}_1 = -a_1 p_1 f_1' + p_1 f_1' f_1'' \dot{l}_1 / f_1' = (-a_1 + f_1'' \dot{l}_1 / f_1')w,$$

where use is made of (19). Compare (24) to (23) to conclude that

(25) $$f_1''\dot{l}_1/f_1' = (a_0 + a_1)\alpha > 0.$$

Since $f_1'' < 0$, by the concavity of f_1, it follows from (25) that

(26) $$\dot{l}_1 < 0.$$

That is, the labor/land ratio diminishes further away from the town. Hence, even though the marginal product of labor increases, the (money) wage does not necessarily increase since the price of corn diminishes.

Nevertheless, as the factor intensity (l_1) diminishes, the factor price ratio (w/y_1) must rise further away from the town:

(27) $$d(w/y_1)/dr > 0.$$

The rent gradient must be downward sloping because both the marginal product of land and the price of corn diminish further away from the town:

(28) $$\dot{y}_1 < 0.$$

4. Equilibrium of the von Thuenen Model for Logarithmically Additive Preferences and Agricultural Technology.

In this section we derive the equilibrium for von Thuenen's dual economy for a simplified version in which Cobb-Douglas preferences and technology are assumed. In the Mathematical Appendix, we show the equilibrium is unique. In this case equation (2) becomes

(2a) $$q_1(r) = f_1[l_1(r)] = \gamma\ (l_1(r))^{\beta},$$

where $0 < \beta < 1$ is the labor's share in output and $\gamma > 0$ is a Hicks-neutral coefficient of technological change. Correspondingly, we have the following factor price equations:

(7a) $$w(r) = \gamma\ \beta\ p_1(r)\ (l_1(r))^{\beta-1}$$

and

(8a) $$y_1(r) = \gamma\ (1-\beta)\ p_1(r)\ (l_1(r))^{\beta}.$$

The aggregate land rent is

$$(9a) \qquad Y = 2\pi\ \gamma\ (1-\beta) \int_0^R p_1(r)\ (l_1(r))^{\beta}\ r\ dr.$$

The expenditure function for the Cobb-Douglas preferences takes the following form:

$$(29) \qquad E(p_0,\ p_1,\ u) = p_0^{\alpha}\ p_1^{1-\alpha}\ u,$$

where $0 < \alpha < 1$ is the budget share of cloth. (Note that, not as in general as in equation (23), α is a constant which does not depend on prices and distance (r)).

The Hicksian demand functions are:

$$(30) \qquad H_0(p_0,\ p_1,\ u) = \alpha\ p_0^{\alpha-1}\ p_1^{1-\alpha}\ u,$$

and

$$(31) \qquad H_1(p_0,\ p_1,\ u) = (1-\alpha)\ p_0^{\alpha}\ p_1^{-\alpha}\ u.$$

The landowners' demands are:

$$(32) \qquad D_0(p_0(0),\ p_1(0),\ Y) = \alpha\ Y\ /\ p_0(0) = \alpha\ Y$$

and

$$(33) \qquad D_1(p_0(0),\ p_1(0),\ Y) = (1-\alpha)\ Y\ /\ p_1(0).$$

Equation (23) may be solved for w:

$$(34) \qquad w(r) = w(0)\ \exp((a_0\alpha - a_1(1-\alpha))r) = f_0\ \exp((a_0\alpha - a_1(1-\alpha))r),$$

where use is made of (6). On the other hand, equation (7a) implies that

$$(35) \qquad w(r) = \gamma\ \beta\ p_1(0)e^{-a_1 r}(l_1(r))^{\beta-1}.$$

Thus, equating (34) and (35) we conclude that

$$(36) \qquad l_1(r) = \left[\frac{f_0\ \exp((a_0\alpha + a_1\alpha)r)}{\gamma\ \beta\ p_1(0)}\right]^{\frac{1}{\beta-1}}$$

and, upon substituting (10):

$$(37) \qquad m = \left[\frac{f_0\ \exp((a_0\alpha + a_1\alpha)R)}{\gamma\ \beta\ p_1(0)}\right]^{\frac{1}{\beta-1}}$$

Equations (11) and (34) imply that

$$\left(e^{a_0 r}\right)^{\alpha}\left(p_1(0)e^{-a_1 r}\right)^{1-\alpha} u = f_0 \exp((a_0\alpha - a_1(1-\alpha))r),$$

so that,

$$(38) \qquad u = f_0 \ (p_1(0))^{\alpha-1}.$$

On substituting (30), (32), (36) and (38), equation (15) (the cloth market clearing equation) becomes:

$$(39) \qquad 2\pi\ \alpha\ f_0^{\frac{\beta}{\beta-1}}\ (\gamma\ \beta\ p_1(0))^{\frac{1}{1-\beta}} \int_0^R \exp(\delta_1 r)\ r\ dr$$

$$+\ 2\pi\ \alpha\ (f_0/\beta)^{\frac{\beta}{\beta-1}}\ (\gamma\ p_1(0))^{\frac{1}{1-\beta}}\ (1-\beta) \int_0^R \exp(\delta_1 r)\ r\ dr = (1-\alpha)\ f_0 L_0,$$

where $\delta_1 = \dfrac{a_0\alpha\ \beta + a_1(1-(1-\alpha)\beta)}{\beta-1} < 0$, because $\beta < 1$. See the Mathematical Appendix for the derivation.

Similarly, on substituting (4) and (36), equation (5) (the labor market clearing equation) becomes:

$$(40) \qquad L = L_0 + 2\pi\ f_0^{\frac{1}{\beta-1}}\ (\gamma\ \beta\ p_1(0))^{\frac{1}{1-\beta}} \int_0^R \exp(\delta_2 r)\ r\ dr,$$

where $\delta_2 = \dfrac{(a_0 + a_1)\alpha}{\beta-1} < 0$, because $\beta < 1$. Recalling (37), we can rewrite (39) and (40) as:

$$(41) \qquad 2\ \alpha\ \pi\ m\ \beta^{-1} \exp(-\delta_2 R) \int_0^R \exp(\delta_1 r)\ r\ dr = (1-\alpha)\ L_0$$

and

$$(42) \qquad L = L_0 + 2\pi\ m \exp(-\delta_2 R) \int_0^R \exp(\delta_2 r)\ r\ dr.$$

Equations (37), (41) and (42) contain only three endogenous variables: $p_1(0)$, L_0 and R. We can use this set of three equations in order to study the equilibrium of the model, and, in the next section, its comparative statics. In the Mathematical Appendix it is proved that an equilibrium exists and is unique in

the range $R > 0$. Specifically, we study the effects of changes in the exogenous variables on the endogenous variables $p_1(0)$, L_0 and R first and then we consider the effects on other endogenous variables (e.g., u, Y, Q_0, Q_1, etc.). Throughout, we emphasize the restrictive nature of the assumption of unit elasticities of substitution and analyze how our conclusion would be modified by small departures from unity.

5. Comparative Statics of the Simplified von Thuenen Model [6]

(a) Population Growth

In this subsection we examine the effect of a change in L. From (42) we conclude that

$$(43) \qquad 1 = \frac{\partial L_0}{\partial L} + 2\pi\, m \frac{\partial R}{\partial L} \left\{ R - \delta_2 \exp(-\delta_2 R) \int_0^R \exp(\delta_2 r)\, r\, dr \right\}.$$

Since $\delta_2 < 0$, it follows from (41) that $\partial L_0/\partial L$ and $\partial R/\partial L$ must have the same sign (because an increase in R increases the left hand side of (41) and an increase in L_0 increases the right hand side of (41)). Hence, (43) implies that

$$(44) \qquad \frac{\partial L_0}{\partial L} > 0$$

and

$$(45) \qquad \frac{\partial R}{\partial L} > 0.$$

That is, an increase in population increases the number of people living in the town and extends the extensive margin of cultivation. Because an increase in L increases R, (37) implies that $\partial p_1(0)/\partial L > 0$ and hence

$$(46) \qquad \frac{\partial p_1(r)}{\partial L} > 0.$$

So, an increase in population increases the price of grain everywhere.

Recalling that $\beta < 1$, it follows from (36) that

$$(47) \qquad \frac{\partial l_1(r)}{\partial L} > 0.$$

Thus, an increase in population increases the intensity of cultivation everywhere. Since $\alpha < 1$, it follows from (38) that

$$(48) \qquad \frac{\partial u}{\partial L} < 0.$$

That is, population increase leads to a fall in the welfare (real wage) of labor, but not to a fall in the nominal wage anywhere, since equation (34) implies that $w(r)$ does not change:

$$(49) \qquad \frac{\partial w(r)}{\partial L} = 0.$$

Since $\partial l_1(r)/\partial L > 0$ (and, consequently, $\partial q_1(r)/\partial L > 0$), $\partial R/\partial L > 0$ and $\partial p_1(r)/\partial L > 0$, it follows from (3) and (9a) that

$$(50) \qquad \frac{\partial Y}{\partial L} > 0$$

and

$$(51) \qquad \frac{\partial Q_1}{\partial L} > 0.$$

Rents rise and total grain production rises with an increase in population, and so does total cloth output, since $\partial L_0/\partial L > 0$ and $Q_0 = f_0 L_0$. It follows that

$$(52) \qquad \frac{\partial Q_0}{\partial L} > 0.$$

Summarizing: Population growth increases cloth and corn output and the labor/land ratio, extends the extensive margin of cultivation, increases the aggregate land rent and the (relativa) price of corn, but leaves the wage unchanged, and lowers the welfare (real wage) of labor.

(b) A Hicks-Neutral technological improvement in corn production.

In this subsection we examine the effect of a change in γ. It follows from (42) that

$$(53) \qquad 0 = \frac{\partial L_0}{\partial \gamma} + 2\pi\, m \frac{\partial R}{\partial \gamma} \left\{ R - \delta_2 \exp(-\delta_2 R) \int_0^R \exp(\delta_2 r)\, r\, dr \right\}.$$

Hence, $\partial L_0/\partial\gamma$ and $\partial R/\partial\gamma$ must have opposite signs. However, as in the preceding subsection, (41) implies that they must have the same sign. Therefore:

$$(54) \qquad \frac{\partial L_0}{\partial \gamma} = \frac{\partial R}{\partial \gamma} = 0.$$

Thus, a neutral technological change in agriculture leaves the labor force in town and in the country and the extensive margin of cultivation unchanged.

Since $\partial R/\partial\gamma = 0$, it follows from (37) that

$$(55) \qquad \frac{\partial p_1(0)\gamma}{\partial \gamma} = 0,$$

so that

$$(56) \qquad \frac{\partial p_1(r)}{\partial \gamma} < 0.$$

That is the price of grain must fall everywhere. The wage, $w(r)$, does not change (see (34)):

$$(57) \qquad \frac{\partial w(r)}{\partial \gamma} = 0,$$

and it follows from (36) and (55) that

$$(58) \qquad \frac{\partial l_1(r)}{\partial \gamma} = 0.$$

Therefore, corn output, which is equal to $2\pi\, \gamma \int_0^R (l_1(r))^\beta\, r\, dr$, increases in proportion to γ :

$$(59) \qquad \frac{\partial Q_1}{\partial \gamma} \frac{\gamma}{Q_1} = 1.$$

Since $\gamma p_1(0)$, R and $l_1(r)$ do not change, it follows that the aggregate land rent does not change either:

$$(60) \qquad \frac{\partial Y}{\partial \gamma} = 0.$$

This is a consequence of the assumption of constant factor shares in agricultural production. Changes in land rent, and therefore in the wage bill, would depart from constancy in a direction depending on whether our elasticity of substitution were greater or less than one.

Since $\alpha < 1$, it follows from (56) and (38) that

$$(61) \qquad \frac{\partial u}{\partial \gamma} > 0.$$

Technical change in agriculture increases the welfare (real wage) of labor.

Summarizing: A Hicks-neutral technological improvement in corn production increases corn output and the welfare (real wage) of labor; it does not change cloth output, the ratio of city to farm workers, the location of the extensive margin of cultivation, the nominal wage, factor intensity in agriculture or aggregate land rent; and it lowers the price of grain relative to cloth.

(c) A technological improvement in cloth production

In this subsection we examine the effect of a change in f_0. As in the preceding subsection, it follows from (42) and (41) that:

$$(62) \qquad \frac{\partial L_0}{\partial f_0} = \frac{\partial R}{\partial f_0} = 0.$$

Neither the distribution of the labor force nor the extensive margin of cultivation are changed by a technical change in manufacturing. Hence, it follows from (37) that $f_0/p_1(0)$ must remain constant and, consequently,

$$(63) \qquad \frac{\partial p_1(r)}{\partial f_0} \frac{f_0}{p_1(r)} = 1.$$

That is, the relative price of grain rises everywhere exactly in proportion to the improvement of manufacturing productivity, a consequence of unitary elasticity of substitution in preferences.

The wage rate is seen from (34) to rise proportionately with f_0 as well:

$$(64) \qquad \frac{\partial w(r)}{\partial f_0} \frac{f_0}{w(r)} = 1.$$

Since one can rewrite (38) as $u = (p_1(0))^{\alpha} f_0/p_1(0)$, and since, $f_0/p_1(0)$ remains constant, it follows that

$$(65) \qquad \frac{\partial u}{\partial f_0} > 0.$$

The welfare (real wage) of laborers is increased by a technical improvement in manufacturing. Again, since $f_0/p_1(0)$ does not change, it follows from (36) that

$$(66) \qquad \frac{\partial l_1(r)}{\partial f_0} = 0,$$

and consequently,

$$(67) \qquad \frac{\partial Q_1}{\partial f_0} = 0.$$

So, total grain output and labor intensity in agriculture do not change. Since $p_1(r)$ increases, while $l_1(r)$ and R stay constant, it follows from (9a) that

$$(68) \qquad \frac{\partial Y}{\partial f_0} > 0.$$

That is, total land rents are increased.

Summarizing: A technological improvement in cloth production increases cloth output, wages, corn prices and aggregate land rent proportionately; such change also raises the real wage of labor, but does not change the location of the

extensive margin of cultivation, the distribution of the labor force between city and farms, factor intensity in agriculture or total corn output. These conclusions are sensitive to the unitary elasticity assumptions with obvious modifications, up or down, depending on the departures of the elasticities from one.

(d) Variations in Transport Costs, in Preferences, and in the Efficiency of Labor in Agriculture

Although the equilibrium value of the model is unique, it is a very complicated function of a_0, a_1, α and β, and the derivatives with respect to these variables cannot easily be signed. It is possible, however, to get some idea of the effects of changing these parameters numerically. Some numerical calculations have been made for fixed values of

- The marginal productivity of labor in cloth production, $f_0 = 1$.
- Hicks-neutral variations in efficiency in agriculture, $\gamma = 1$.
- Minimal labor/land ratio, $m = 10$.
- Population, $L = 10{,}000$.
- Transportation costs of cloth, $a_0 = 0.01$.

We examine variations in steps of 0.05 in the parameters:

- The share of cloth in total expenditure, $0.25 \leq \alpha \leq 0.75$.
- The share of labor in agriculture output, $0.50 \leq \beta \leq 0.95$.
- Transport costs of agricultural products, $0.05 \leq a_1 \leq 0.25$.

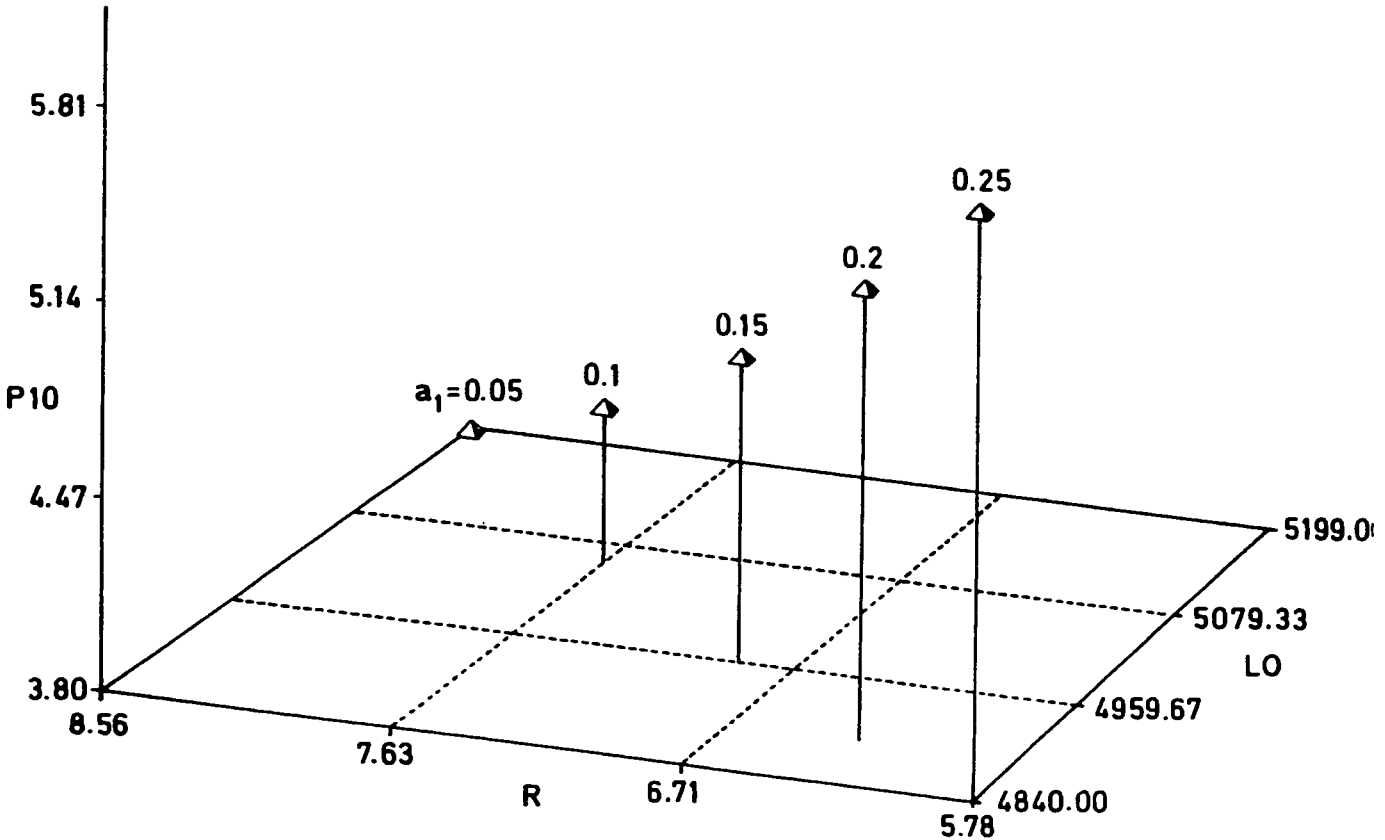
Transport costs of agricultural products vary from 2 to 25 times the costs of transporting manufacturers. The share of agricultural products in the total expenditure of each household, and therefore of society as a whole, $1-\alpha$, varies

from 25 to 75 percent. The elasticity of agricultural output with respect to labor input, or the share of labor in agricultural output, varies from 50 to 95 percent. These are ranges which span most of the plausible values of the parameters for both developed and developing countries.

The effects of changing transport costs are monotonic and relatively straightforward: For $\alpha=0.5$ and $\beta=0.75$, the extensive margin of cultivation shrinks (expands) monotonically as the relative cost of transporting agricultural products increases (declines). See Figure 1. This is also true for all other values of α and β, but the rate at which this occurs depends on α and β. More importantly, L_0, labor working in the town, increases monotonically with falling transport costs for agricultural products. Falling transport costs are sufficient to produce the "stylized fact" which most models of dual economic development seek to explain. Finally, falling transport costs result in a monotonic turning of the terms of trade against agriculture. The price of grain relative to cloth everywhere falls monotonically.

The following results may also be proved: Because falling transport costs for agricultural products extend the extensive margin of cultivation and reduce the proportion of population in rural areas, less labor per unit of land must be used everywhere. So yields must fall everywhere. The effect on total agricultural production is thus ambiguous without specific numerical calculation. Falling transport costs also affect the distribution of income in space and between rents and wages. Money wages decrease everywhere and the rate at which these wages decline with distance also decreases. Real wages are the same everywhere and increase as the result of falling transport costs. Aggregate land rent changes, but as in the case of total agricultural production cannot be signed unambiguously without numerical calculation.

FIGURE 1: Effects of Falling Transport Costs on Relative Prices, the Extensive Margin, and Population in the Town



The effects of changing preferences, α, and changing efficiency of labor in agriculture, β, are complicated by the fact that for certain values of the parameters the relationships are nonmonotonic. For example, as the elasticity of output with respect to labor input, β, increases, when $a_1=0.05$ and $\alpha=0.5$, the extensive margin of cultivation first moves outwards, then begins to contract, although the labor force in the town falls monotonically. The relative price of grain decreases monotonically. (This is true for any $a_1<0.15$.) When a_1 is large, the extensive margin contracts uniformly, but the labor force in the town first falls and then begins to increase. For plausible values of the parameters, however, the extensive margin moves inwards, the relative price of grain falls, and the labor force in the town decreases, all as β increases. In this case it can be shown that the yield of corn and the labor intensity of cultivation fall everywhere as β increases. The real income of labor rises, but the change in money (in cloth units) wages is ambiguous. Rents fall everywhere as a proportion of total output. Because yields fall and the extensive margin of cultivation moves inwards from (3), the total output of grain decreases.

The behavior of R and L_0 with respect to variations in α is monotonic. Increasing the share of cloth in total expenditures increases the labor force in the town and shrinks the area under cultivation, but the behavior of relative prices are nonmonotonic: First, the relative price of grain rises, then it falls, as α exceeds 60%. This nonmonotonic behavior produces nonmonotonic behavior in yields and labor intensity and in the real income of labor. However, for a relatively large share of food in the total budget of consumers, the relative price of grain is rising so that, as that share decreases, <u>ceteris paribus</u>, real incomes fall.

Summarizing: Falling transport costs unambiguously increase the proportion of a fixed labor force working in the town. Agriculture is made more efficient, labor intensity per unit of land decreases and yields fall; money wages fall everywhere and real wages increase. The effects of changing preferences and a changing elasticity of output with respect to labor input in agriculture are, however, more problematic. For plausible ranges of the parameters, the labor force in town is decreasing as the elasticity of output increases and as the share of consumer expenditure devoted to food decreases. The real income of labor varies in the opposite direction; increasing for increasing elasticity and rising as the share of food in the budget increases. This suggests a more interesting role for the elasticity of substitution in agricultural production and in preferences. Such nonunitary elasticities could lead to multiple equilibria if the departures from unity were large.

6. Concluding Remarks

A model of economic development derived from von Thuenen's model of spatial price equilibrium is analysed in detail above only for the case of logarithmically additive preferences and technology in agriculture, i.e., for elasticities of substitution equal to one for grain vs. cloth in preferences and for labor vs. land in agricultural production. This is a very restrictive assumption, but small derivations of the elasticities from unity would not produce vastly different conclusions. In the simplified case, the von Thuenen model of the dual economy delivers results which contrast sharply with more traditional models of a stagnant agriculture and vibrant industrial sector. In this model, if we are to account for migration from the countryside to urban areas, we must have either a vibrant agriculture (irrespective of the state of

the industrial sector) or falling transportation costs for agricultural products. Casual observation suggest that the latter is characteristic of developing countries. This conclusion underscores the importance of the transportation sector in the process of development. We believe that it is robust to variations in the elasticities of substitution in preferences and in agriculture which we assumed to be constant in deriving the specific results presented in this paper.

Finally, we can ask how the von Thuenen model can be made dynamic. Economic development in most parts of the world has resulted in extensive rural to urban migration. Migration to the town does not obviously result from changes in the exogenous variables or parameters of the model, except for falling transport costs, unless it is assumed that population increase occurs only, or mostly, in the countryside; and that there is no reason why this should be so. We have not introduced any capital formation in the model in either sector. Substitution of capital for labor in agriculture or falling transport costs make agriculture relatively more labor-efficient. Given homothetic preferences, such increases in efficiency will generally result in an outflow of labor from the sector.

Such an outflow need not occur instantaneously in response to differences in real income (utility) between countryside and city. A natural point at which to introduce disequilibrium in the model is in the labor market; labor moves in response to differences in real income but not instantaneously. One problem which one encounters immediately in this formulation is that space is continuous and labor is distributed continuously over it. Labor does not merely move from point A, at which real income is low, to point B, at which it is higher, but movement occurs everywhere in between since real income must vary continuously from A to B if it differs at all. Methods for dealing with equilibrium models in continuous space (due to Palander, Loesch, and others) are explained and

extensively applied in a recent book by Beckmann and Puu (1985). Because all land is homogeneous in the von Thuenen model and because there is only one town, the problem is simplified because we need only look at movement along a radius of a circle with center at the town. But because the von Thuenen model involves intersections between demands and supplies, and between relative prices and wages, disequilibration of the labor market results in partial differential equations of a high order of complexity.

Von Thuenen was a pioneer in economic theory in three respects: (1) He laid the foundations for modern location theory. (2) He understood the marginal productivity theory of distribution many decades before John Bates Clark articulated the theory. (3) And, finally, von Thuenen clearly understood the nature of two-sector general equilibrium. While he did not, himself, derive the comparative static properties of his model, these properties can be derived as we have done in this paper. The results reemphasize the importance of the spatial dimension in economic equilibrium and growth and the pivotal role of transport costs, an importance and a role which von Thuenen was among the first to recognize, to understand, and to analyze.

ENDNOTES

1. The discussion concerning the marginal productivity theory of wages is continued in Dorfman (1986) and Samuelson (1986).

2. Not so by T.W. Schultz (1953, especially Ch.9, 146-151). See also Katzman (1977), who gives an extended discussion of the spatial aspects of economic development in Brazil. Of course, many economists have dealt with the problems of urbanization, urban labor markets, and rural-urban migration in the course of economic development. See Kelley, et al. (1972, Ch.7, 234-255). But space itself seems largely incidental to the models developed. There is, of course, an important literature on this topic in urban economics, much of which stems from Mills (1972).

3. Most development economists would probably disagree with our attribution to von Thuenen. Ranis (1988), for example, traces the origin of the concept of dualism back to the physiocrats. In terms of influence on the course of economic thought, they are, of course, correct.

4. It does not matter whether we assume that the laborers rent the land from the landowners or the landowners hire the laborers. We could also assume the existence of an m such that for labor use per unit of land greater than m, $l_1 \geq m$, $f_1''(m) = 0$; then there are no rents on marginal and on extra-marginal land. This, however, would be inconsistant with our later assumption of a Cobb-Douglas technology.

5. Any form of transportation costs, monotonically increasing with distance, which caused corn prices to fall the further from the town, and cloth prices to rise, would suffice. One possibility suggested by one of our referees is that, when cloth is hauled out to the countryside, the oxen do not consume cloth, as implied by (13), but rather corn, as they do on the

way in, necessary supplies of which are purchased by the teamsters at the prevailing local prices. Then the gradient of cloth price is given by

$$(13^*) \qquad p_0(r) = p_0(0) + \int_0^r \alpha\, p_1(r)\, dr$$

$$= p_0(0) + p_1(0)\,[1-e^{-a_1 r}],$$

where the physical units in which corn and cloth are measured, say kilos, are assumed to be the same.

Changing (13) of the text to (13*) has important ramifications for the market-clearing equations (14) and (15) since, if the oxen consume corn, regardless of whether they haul corn or cloth, the excess supply of corn at points distant from the town will be partly used up by the oxen hauling in both directions and there will be no physical loss of cloth on the way out. Because of the need to purchase feed for the oxen on the way, supplies of cloth will be greater nearer the town and less farther away; thus there is, in a sense, an "economic" loss of cloth on the way out.

The resulting wage gradient is not (22), as given below, but rather

$$(22^*) \qquad \dot{w} = \dot{p}_0 H_0 + \dot{p}_1 H_1$$

$$= a_1 p_1(0) e^{-a_1 r}\,[H_0(r) - H_1(r)].$$

If $H_0 < H_1$ for all r, i.e., laborers everywhere eat more kilos of corn than they wear kilos of cloth, then it follows straightforwardly from (22*) that $\dot{w} < 0$, which is an easier derivation of the result than we have given.

6. A similar comparative statics analysis for a purely urban model can be found in Pines and Sadka (1986).

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MATHEMATICAL APPENDIX

I. We first show (39):

Substituting (12), (13), (30) and (32) into the cloth market clearing equation (15) yields:

$$\text{(M1)} \qquad 2\pi\alpha(p_1(0))^{1-\alpha}u \int_0^R \exp\{a_0 r(\alpha-1)\} \exp\{-a_1 r(1-\alpha)\}\ r\ l_1(r)\ \exp(a_0 r)\ dr$$
$$+\ \alpha\ L_0(p_1(0))^{1-\alpha}u + \alpha\ Y = f_0 L_0.$$

Substituting (38) into (M1):

$$\text{(M2)} \qquad 2\pi\alpha f_0 \int_0^R \exp\{(a_0\alpha - a_1(1-\alpha))r\}\ r\ l_1(r)\ dr + \alpha\ L_0 f_0 + \alpha\ Y = f_0 L_0.$$

Substituting (9a) and (36) into (M2) yields:

$$\text{(M3)} \qquad 2\pi\alpha\ f_0^{\frac{\beta}{\beta-1}}\ (p_1(0)\ \gamma\ \beta)^{\frac{1}{1-\beta}} \int_0^R \exp\{(a_0\alpha - a_1(1-\alpha))r\}\ r\ \exp\left(\frac{(a_0+a_1)\alpha r}{\beta-1}\right) dr$$
$$+\ \alpha\ L_0 f_0$$
$$\because\ 2\pi\alpha\ f_0^{\frac{\beta}{\beta-1}}\ (p_1(0)\ \gamma\ \beta^{\beta})^{\frac{1}{1-\beta}}\ (1-\beta) \int_0^R r\ \exp(-a_1 r)\ \exp\left(\frac{(a_0+a_1)\alpha r\beta}{\beta-1}\right) dr$$
$$= f_0 L_0.$$

Rearranging terms yields:

$$\text{(M4)} \qquad 2\pi\alpha\ f_0^{\frac{\beta}{\beta-1}}\ (p_1(0)\ \gamma\ \beta)^{\frac{1}{1-\beta}} \int_0^R \exp(\delta_1 r)\ r\ dr$$
$$+\ 2\pi\alpha\ (f_0/\beta)^{\frac{\beta}{\beta-1}}\ (p_1(0)\ \gamma)^{\frac{1}{1-\beta}}\ (1-\beta) \int_0^R \exp(\delta_1 r)\ r\ dr$$
$$= (1-\alpha)\ f_0 L_0,$$

where:

$$\text{(M5)} \qquad \delta_1 = \frac{a_0\alpha\ \beta + a_1\{1-(1-\alpha)\beta\}}{\beta-1} < 0,$$

because $\beta < 1$. This proves (39).

II. We next show (40):

Substituting (36) into (4) yields:

$$\text{(M6)} \qquad L_1 = 2\pi\, f_0^{\frac{1}{\beta-1}}\, p_1(0)^{\frac{1}{1-\beta}}\, \gamma^{\frac{1}{1-\beta}}\, \beta^{\frac{1}{1-\beta}} \int_0^R r \exp\left(\frac{(a_0+a_1)\alpha r}{\beta-1}\right) dr$$

$$= 2\pi\, f_0^{\frac{1}{\beta-1}}\, (\gamma\, \beta\, p_1(0))^{\frac{1}{1-\beta}} \int_0^R \exp(\delta_2 r)\, r\, dr,$$

where

$$\text{(M7)} \qquad \delta_2 = \frac{(a_0+a_1)\alpha}{\beta-1} < 0,$$

because $\beta < 0$. Substituting (M6) into (5) yields:

$$\text{(M8)} \qquad L = L_0 + 2\pi\, f_0^{\frac{1}{\beta-1}}\, (\gamma\, \beta\, p_1(0))^{\frac{1}{1-\beta}} \int_0^R \exp(\delta_2 r)\, r\, dr.$$

This proves (40).

III. To show an equilibrium exists and is unique in the case of logarithmically additive preferences and technology, we proceed as follows :

Substitute L_0 from (42) into (41) to obtain a single equation with only one endogenous variable, R,

$$\text{(M9)} \qquad \frac{2\alpha\pi m}{\beta} \exp(-\delta_2 R) \int_0^R \exp(\delta_1 r)\, r\, dr$$

$$+ \; 2(1-\alpha)\, \pi m \exp(-\delta_2 R) \int_0^R \exp(\delta_2 r)\, r\, dr$$

$$= (1-\alpha)\, L.$$

Denote the left-hand side of (M9) by g(R). By differentiation of (M9) with respect to R, it can be verified that $g'(R) > 0$ for $R \geq 0$. Hence, a solution to (M9), if it exists, must be unique. To show a solution to (M9) exists, observe that $g(R) \to 0$, when $R \to 0$, because the integrals on the left-hand side of (M9) approach zero, when R does so. Moreover, $g(R) \to \infty$, when $R \to \infty$ because $\exp(-\delta_2 R) \to \infty$ as $R \to \infty$, since $\delta_2 < 0$, as shown above. Thus, a unique solution to (M9) exists by the continuity of g(R).