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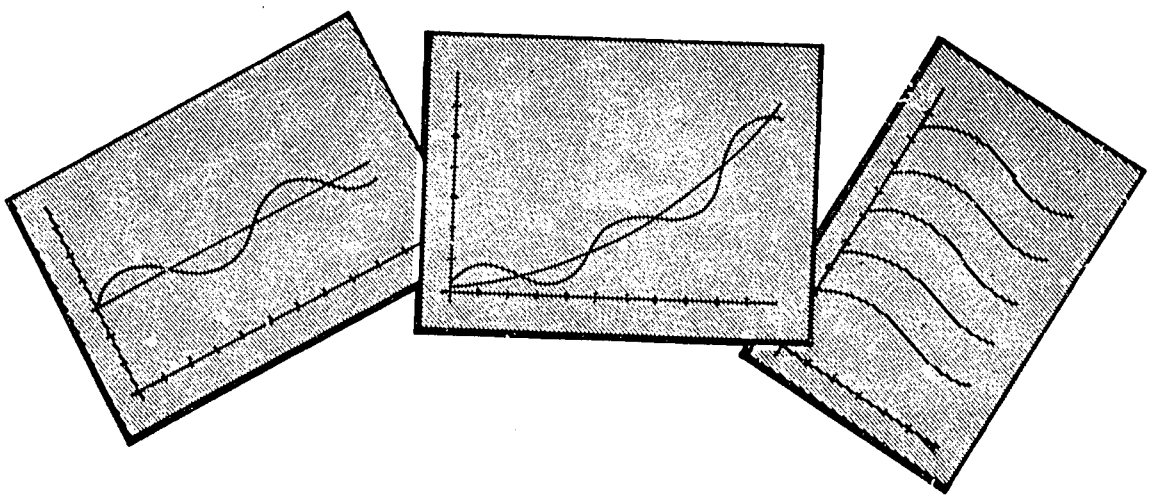
TIME SERIES ANALYSIS HANDBOOK

$$O = T + C + S + I$$



U. S. Agency for International Development
Manila, Philippines

October, 1975

TIME SERIES ANALYSIS

HANDBOOK

Kenneth F. Smith
Management Systems Advisor

RP Time Series Analysis Handbook.
001.422 USAID/Philippines.
S653a Time Series Analysis Handbook. Kenneth F. Smith. Oct. 1975.
86 p.

1. Statistical method - RP. 2. Time study - Evaluation methodology - RP, 3. Manuals - RP, I. Smith, Kenneth F. II. Title.

U.S. Agency for International Development
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PREFACE

Data Analysis is an important aspect of Management Information Systems. Publication of the Statistical Survey and Analysis Handbook in March 1975 for the Philippine National Food and Agriculture Council (NFAC) and the Bureau of Agricultural Economics (BAECON) generated several requests for similar treatment of certain other statistical concepts.

A particular need was expressed for Time Series Analysis by the USAID Provincial Development Staff to evaluate existing programs and situations and for forecasting purposes. Time Series concepts are also important to BAECON in analyzing agricultural activity and to NFAC in the analysis of present projects and formulation of future ones to improve the national wellbeing. This booklet has therefore been prepared to meet these various needs.

As before, no attempt has been made to provide an exhaustive treatment, nor to cover all the fine points of those topics selected. However, I have endeavored to familiarize the reader/user with the subject, and provide him/her with a working knowledge of the various concepts, together with "cook-book" examples to follow.

Generally, governmental organizations spend a large portion of their effort gathering data, but very little in analyzing it once it has been obtained beyond simple arithmetic tallying. The prime reasons for this are (1) a lack of awareness of the hidden value in available data, and (2) a lack of skilled manpower to perform the necessary statistical analysis. This booklet is a step towards filling that gap. I hope that it will result in the application of more quantitative analysis in Provincial Development work, as well as National Agricultural Program management; - first by familiarizing managers and their staffs with some of the available tools, and second, by providing encouragement to them that mastery is not beyond their grasp.

October 1975

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ON INDUCTIVE STATISTICS & COMMON SENSE

A CAUTION

In the space of one hundred and seventy-six years, the Lower Mississippi has shortened itself two hundred and forty-two miles. That is an average of a trifle over one mile and a third per year. Therefore, any calm person, who is not blind or idiotic, can see that in the Old Oolitic Silurian Period, just a million years ago next November, the Lower Mississippi River was upward of one million three hundred thousand miles long, and stuck out over the Gulf of Mexico like a fishing rod. And by the same token any person can see that seven hundred and forty-two years from now the Lower Mississippi will be only a mile and three-quarters long, and Cairo and New Orleans will have joined their streets together, and be plodding comfortably along under a single mayor and a mutual board of aldermen. There is something fascinating about science. One gets such wholesale returns of conjecture out of such a trifling investment of fact.

Mark Twain

INTRODUCTION

The Future Belongs to Those Who Plan for it Best

Managers, and planners of all types, have an insatiable appetite for facts (usually numbers) to assist them in making decisions. These include setting policies, recommending implementation, and/or changing the direction of ongoing projects. Some of the most critical facts they seek in this regard concern the future! Of course, since the future has not yet arrived, no one can say what the facts will be. However, "educated guesses" can be made by applying common sense to trends projected from the historical data which is available.

Such projections can be extremely important to public administrators. If changes of any magnitude are likely, and the existing socio/economic structure is inadequate to accomodate the change, adjustments (in the form of development projects) may be required to avert crisis situations; or at least to lessen the impact. Furthermore, investments of public funds should be made where they are likely to do the most good. Development and analysis of time series data can be helpful in this regard.

The projection of time series data in forecasting is very similar to correlation and regression analysis, discussed earlier,[1] with a slight difference:-

In Regression Analysis, we attempt to determine

the effect of one variable (independent)
upon another (dependent)

whereas in Time Series Analysis, we attempt to determine

the change in one variable (dependent)
over time (independent)

In other words, the independent variable is a unit of time.

This booklet then is a "how-to-do-it" guide for would-be managerial crystal-ball gazers.

---

[1] Since regression and time series are very closely related, I will cover some additional aspects of regression appropriate to time series analysis, beyond the simple linear form discussed in my Statistical Survey and Analysis Handbook, pp 58-62, USAID/Manila, March 1975.

FOUR FACTORS OF TIME SERIES

There are four (4) principal factors which affect the fluctuation of a variable over time:-

-- Long Term Trends
-- Cyclical Variation
-- Seasonal Variation
-- Erratic or Unpredictable Variation

In other words, data at any time represents the net effect of each of these factors. In analyzing time series data therefore, we attempt to isolate and estimate each factor separately, according to our particular need.

$$O = T + C + S + I$$

Where
O = Data Value
T = Long Term Trend
C = Cyclical Variation
S = Seasonal Variation
I = Erratic/Unpredictable Variation

The factors can be categorized briefly as follows:

Long Term Trends

Over long periods of time, things rarely stay the same. Although for the short run, there may be many "ups and downs", which seem to deny this, nevertheless over the long run, there is usually an increase (positive trend)or decrease (negative trend) in the value of a variable. "Trend" is thus the underlying tendency of a variable to grow or decline consistently over a period of time.

Cyclical Variation

Even when the long terms trend has been identified as positive or negative it is rarely perfectly smooth. The tendency of data to fluctuate around the trend line in a regular pattern is called the cyclical variation.

Seasonal Variation

Regardless of the long term trend, and the cyclical variation, for the shorter run, social customs and climate often play a major role. Most things seem to have their "seasons" and "off-seasons", rather than a constant level of activity all year round.

Erratic/Unpredictable Variation

Occasionally, sharp deviations from long term and seasonal trends occur which upset all carefully calculated projections. After-the-fact we may be able to review the historical data, single out major variations, and attribute a particular cause to it. This may be helpful in adjusting the data to produce a better trend line, however, it cannot help us to predict the unpredictable. Hence the usual caveat in forecasting -- "other things being equal". . . !

IMPORTANT CRITERIA

There are three (3) important considerations to observe in time series analysis:

1. Equal Time Intervals The appropriate time unit can be seconds, minutes, hours, days, weeks, months, seasons, years, etc., depending upon the subject and variable being studied. However, once the time unit has been selected, the data should be drawn from equal time intervals throughout the period.

2. Adequate Data Base Since analysis of time series data is a form of sampling, you must have a sufficiently large number of sample items in order for the data to be meaningful. Once you have determined the appropriate time unit for your study, it is desirable to have data from at least 10 sequential time intervals.

3. Comparable Data The data should also be comparable. Although a seemingly obvious point, this is not a trivial consideration. Due to the extensive historical time period requirement , unless a regular collection effort is established, data for analysis must be obtained from secondary sources.[1] Particular care is required in using such data, because in the "real world" people occasionally change their ways of doing and recording things to reflect improvements and/or felt needs. Unless these changes are noted and suitable adjustments made, erroneous conclusions will be drawn from the data by the unwary.

For example, the Cavan, as a reporting unit for Palay production has recently been changed from 44 kg to 50 kg at the national level. Any time series analyses which overlap these different size reports must be expressed in comparable units.

Similarly, in 1974, the official Philippine estimate for the per capita consumption of palay was adjusted from 93 kg per annum to 104 kg per annum. Thus, analyses over these time periods would have to take the adjustment into consideration.

---

1 Data that already exists (usually gathered by someone else for a different purpose) rather than organized and gathered particularly for this analysis.

Another example is the difference between a cavan of palay, and a cavan of Bigas (milled Rice). When palay is milled, the light outer husk is removed, along with a portion of the rice grain itself. The amount removed depends upon the efficiency of the milling equipment used. The milled rice is usually expressed as a "recovery rate" from palay. For example, a 66% recovery indicates that 100 kg of palay may end up as 66 kgs. of milled rice. Since the lighter portion of the palay has been removed, the remaining milled rice is heavier per unit volume. Furthermore, the milled rice is customarily sold in larger "cavans" of 56 kgs per cavan. Thus a cavan of palay and a cavan of milled rice are not interchangeable units. The net effect of combining the milling loss (real) with the volume change (apparent loss) results in a conversion factor from palay to milled rice as follows:

1 cavan (@ 44 kgs/ca) palay = .52 cavans (@ 56 kgs/ca) milled rice @ 66% recovery, or 29.04 kilograms milled rice

or 1 cavan (@56 kgs/ca) milled rice can be obtained from 1.92 cavans (@ 44 kgs) palay, or 84.48 kilograms palay.

With both palay and milled rice cavans standardized at 50 kgs, using the same recovery factor of 66%

1 cavan (@ 50 kgs/ca) palay = .66 cavans (50kgs/ca) milled rice or 33 kilograms milled rice

or 1 cavan (@ 50 kgs/ca) milled rice can be obtained from 1.52 cavans (@ 50 kgs) palay.

or = 75.76 kilograms palay

A final example (which creates a lot of difficulty) is in measuring incomes and expenditures over time. Even though the <u>unit</u> may be the same -- pesos -- the actual <u>value</u> or purchasing power of the peso may have changed over the time period under study. Thus expenditure of 5 million pesos in road construction in 1975 may not buy you the same number of kilometers as in 1973. In this situation, the value of the peso during each time period should be equated for a meaningful comparison.

In summary, then, be very careful in data gathering and preparation prior to analysis.

DATA BASE

After comparable data has been obtained for the time period to be studied (FIGURE 1) we can begin our study of trends.

FIGURE 1 HYPOTHETICAL TIME SERIES DATA

	1964	65	66	67	68	69	70	71	72	73	74	Total
January	3	3	3	4	4	4	4	5	5	5	5	45
February	3	2	2	3	3	4	5	5	4	4	5	40
March	4	3	4	4	3	4	4	5	5	5	4	45
April	4	4	4	3	4	3	5	3	4	3	4	41
May	3	4	3	3	3	3	4	4	4	3	4	38
June	5	5	5	5	4	5	5	6	4	5	6	55
July	4	4	4	3	3	4	4	5	5	4	5	45
August	3	3	3	4	4	4	3	4	5	4	4	41
September	3	3	3	3	4	3	4	3	4	3	4	37
October	4	4	4	4	5	5	5	5	4	4	5	49
November	6	6	6	7	6	5	6	7	6	6	7	68
December	5	6	6	5	5	6	7	7	8	8	7	70
Σ =	47	47	47	48	48	50	56	59	58	54	60	

This data table will be referred to in the analysis of both long Term and Seasonal Trends.

THE LONG TERM TREND[1]

To reduce or eliminate the effect of seasonal variation in the long term trend, the data can be "condensed". Monthly data can be summed so that only annual totals (or annual monthly averages) are utilized for the long term trend analysis.

FIGURE 2 EXTRACT OF ANNUAL DATA FROM FIGURE 1

X Time:	1964	65	66	67	68	69	70	71	72	73	74
Time Sequence:	1	2	3	4	5	6	7	8	9	10	11
Y Value:	47	47	47	48	48	50	56	59	58	54	60

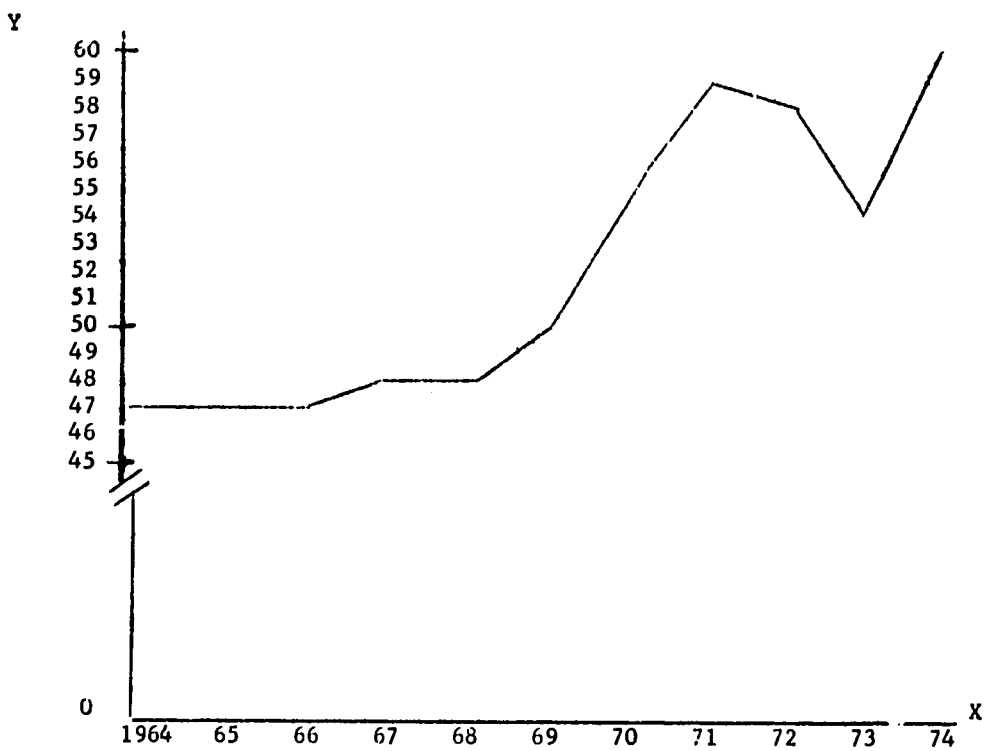
Before plunging into trend analysis too deeply you should first examine a general picture of the data to determine

(1) whether it is <u>linear</u> in nature, <u>curvilinear</u>, or whether perhaps there is <u>no</u> discernible trend at all; and

(2) the general direction of the trend, - increasing (positive slope) or decreasing (negative slope).

A simple line graph with the time on the "X" Axis and the measurement unit of the variable under study on the "Y" axis should reveal this trend.

FIGURE 3 SIMPLE LINE GRAPH OF ANNUAL DATA IN FIGURE 2



The trend can now be found by the Semi-Average-Method or Least Squares Method. Each of these will be discussed on the following pages.

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[1] Also known as the Secular Trend

CALCULATING THE LONG TERM TREND

THE SEMI-AVERAGE METHOD

Using the semi-average method, the time series is divided into two parts. The arithmetic average of each part is found, and then a trend line is calculated through these points, based on the formula for a straight line

$$Y = a + bX$$

Where

Y = Value of data at Time X
X = Selected Time period
a = value of Y where X = 0
b = amount or rate of change in Y per unit of time X.

In a situation where there is an odd number of time periods, either eliminate the middle time, or alternately include the middle three time periods in both averages, to facilitate computation.

Thus, the arithmetic average for the period 1964 through 1970 is

$$\text{arithmetic average} = \frac{47 + 47 + 47 + 48 + 48 + 50 + 56}{7} = \frac{343}{7} = 49$$

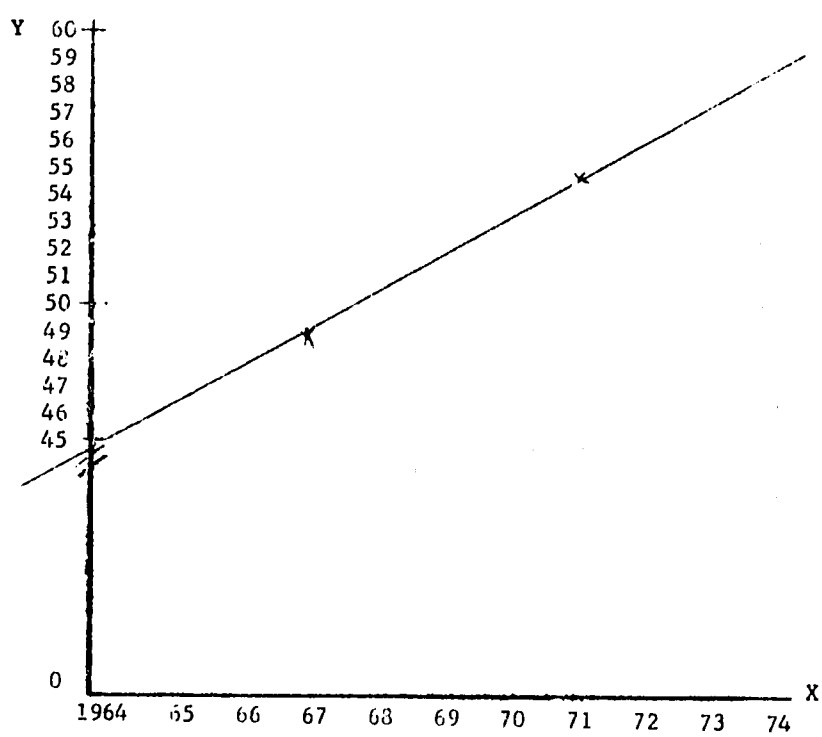
and the arithmetic average for 1968 through 1974 is

$$\frac{48 + 50 + 56 + 59 + 58 + 54 + 60}{7} = \frac{385}{7} = 55$$

This indicates that the long term trend for the midpoint of the first period (4) (1967) is 49, and for the midpoint of the second period (8) (1971) is 55.

Another line graph will give a quick picture of the overall trend.

FIGURE 4 ANNUAL TIME SERIES DATA - SEMI-AVERAGE TREND LINE



Mathematically, we substitute the calculated values of Y at time period X in the straight line formula for two simultaneous equations as follows:

$$Y = a + bX$$

$$(1)\ 49 = a + b4$$

$$(2)\ 55 = a + b8$$

By subtraction

$$\begin{array}{rrcrcr} (2) & 55 & = & a & + & 8b \\ (1) - & 49 & = & a & + & 4b \\ \hline & 6 & = & / & + & 4b \end{array}$$

Therefore if $4b = 6$ $\quad b = \frac{6}{4} = 1.5$

Given that b is 1.5, we can now calculate the value of a, by substituting in either (1) or (2) above, thus:-

$$(2)\ 55 = a + 8 \times 1.5$$

$$55 = a + 12$$

Therefore, if $a + 12 = 55$

$$a = 55 - 12$$

$$= 43$$

The estimated trend /value of (Y)/ for any year (X) can now be calculated by substituting in the formula

$$Y = 43 + 1.5\ X$$

and solving for Y

Thus where $X = 6$,

$$Y = 43 + 1.5\ (6)$$

$$= 43 + 9$$

$$= 52$$

THE METHOD OF LEAST SQUARES

Another method, and probably the most widely used for identifying trend lines is the Method of Least Squares. This is where the sum of the squares of the deviations from the trend line are calculated to be at a minimum.

For a linear trend the least squares is obtained by solving for the two equations.[1]

(1) $\Sigma Y = na + b\Sigma X$

(2) $\Sigma XY = a\Sigma X + b\Sigma X^2$

with the data developed in the following table:

FIGURE 5 WORKSHEET FOR DEVELOPING LEAST SQUARES DATA

Time Period	Independent Variable X (Time Sequence)	Dependent Variable Y	XY	X^2	Y^2
1964	1	47	47	1	2209
1965	2	47	94	4	2209
1966	3	47	141	9	2209
1967	4	48	192	16	2304
1968	5	48	240	25	2304
1969	6	50	300	36	2500
1970	7	56	392	49	3136
1971	8	59	472	64	3481
1972	9	58	522	81	3364
1973	10	54	540	100	2916
1974	11	60	660	121	3600
	66	574	3600	506	30232
Substituting	ΣX	ΣY	ΣXY	ΣX^2	ΣY^2

(1) $574 = 11a + 66b$

(2) $3600 = 66a + 506b$

First we must eliminate one of the unknowns ("a") by multiplying (1) by 6.

(3) $3444 = 66a + 396b$

and subtracting (3) from (2) thus

$$\begin{array}{lrcrcr} (2) & 3600 & = & 66a & + & 506b \\ (3) & -\ 3444 & = & 66a & + & 396b \\ \hline & 156 & = & / & & 110b \end{array}$$

$$b = \frac{156}{110} = 1.418$$

Therefore $b = 1.418$ or 1.42 rounded off.

Substituting this value for "b" in (1) we can now find "a" as follows

(1) $574 = 11a + 66\ (1.42)$

$574 = 11a + 93.72$

$11a = 574 - 93.72$

$$a = \frac{574 - 93.72}{11}$$

$$= \frac{480.28}{11}$$

$= 43.66$ or 43.7 rounded off.

[1] As outlined on page 61 and 62 of the Statistical Survey and Analysis Handbook.

LEAST SQUARES - ALTERNATE FORMULA

An alternate procedure can be used to calculate the linear trend line value for Y, directly at the mid-point of the time period under study, together with the values of a & b, as follows: -

(1) The trend line passes through the point X, Y

where $X = \frac{\Sigma X}{N}$ and $Y = \frac{\Sigma Y}{N}$

(2) The coefficient "a" is the point where X = 0 (Time period 0, or 1963)

$$a = \frac{\Sigma Y}{N} - \frac{\Sigma X}{N} \left(\frac{N \Sigma XY - \Sigma X \Sigma Y}{N \Sigma X^2 - (\Sigma X^2)} \right)$$

and

(3) The coefficient "b" is the change in Y per unit of time (X) (slope of line)

$$b = \frac{N \Sigma XY - \Sigma X \Sigma Y}{N \Sigma X^2 - (\Sigma X)^2}$$

Substituting the data from FIGURE 5, in these equations, we have

(1) $X = \frac{66}{11} = 6$ and $Y = \frac{574}{11} = 52.1818$

Thus, at time period 6 (1969), the value of Y was 52.2 rounded off

(2) $$a = \frac{574}{11} - \frac{66}{11} \left(\frac{11\ (3600) - 66\ (574)}{11\ (506) - (66)^2} \right)$$

$$= 52.1818 - 6 \left(\frac{39600 - 37884}{5566 - 4356} \right)$$

$$= 52.1818 - 6 \left(\frac{1716}{1210} \right)$$

$$= 52.1818 - 6\ (1.41818)$$

$$= 52.1818 - 8.5091$$

$$= 43.6727 \text{ or } 43.7 \text{ rounded off}$$

Thus, at time period 0 (1963) the trend line was 43.7 rounded off

(3) $$b = \frac{11\ (3600) - 66\ (574)}{11\ (506) - (66)^2}$$

which was already calculated above as 1.41818, or 1.42

Thus, the long range trend is an increase of 1.42 per year.

CALCULATING THE LONG TERM TREND

PROJECTIONS INTO THE FUTURE

Given that the trend for Y was 52.2 at the time period 6 (1969), and that the long term trend is an increase of 1.42 per year, the long term trend for any period can be projected into the future. For instance, to project Y for 1975

(1) Determine the number of years of the desired time from the measured point or (Base Period)

$$\begin{array}{r} 1975 \\ -\ \underline{1969} \\ 6 \end{array}$$ or 6 years beyond the measuring point.

(2) Increase at Projected Time Period = Time Period for projection x Rate of Increase per Time Period

= 6 x 1.42

= 8.52 increase over Value of Y for the base period

(3) Value of Y at Projected Time Period = Value of Y at Base Period + Increase at Projected Time Period

= 52.2 + 8.52

Thus the Value of Y for 1975 = 60.72 or 60.7 rounded off.

Because our data is based on a few sample points, any projections are likely to be tinged with uncertainty at least to the same degree as the sample base. Therefore we should also indicate the limits of confidence which we have in our projection.

STANDARD ERROR & CONFIDENCE LIMITS

In order to do this we must calculate the standard error of estimate, as follows:

Standard Error

First, the standard error of estimate for the linear trend is calculated from the formula

$$1\ Se = \sqrt{\frac{\Sigma Y^2 - a\Sigma Y - b\Sigma XY}{N - 2}}$$

assuming that all distributions for values of X have the same standard deviation.

Substituting in the above formula from FIGURE 5

$$1\ Se = \sqrt{\frac{30232 - (43.7)(574) - (1.42)(3600)}{11 - 2}}$$

$$= \sqrt{\frac{30232 - 25083.8 - 5112}{9}}$$

$$= \sqrt{\frac{36.2}{9}}$$

$$= \sqrt{4.02}$$

$$= 2$$

Limits of Prediction

Second, to calculate the "limits of prediction" for the mean value of Y for any specified time period (X_s) the following formula is used

$$L = t \times Se \sqrt{\frac{1}{N} + \frac{\left(X_s - \frac{\Sigma X}{N}\right)^2}{\Sigma X^2 - N\left(\frac{\Sigma X}{N}\right)^2}}$$

Where

L = limit of prediction
X_s = Specified time period for prediction
T = Value of t from the table* for n - 2 degrees of freedom for desired percentage confidence level

Alternately, to calculate the "limits of prediction" for a single specific value of 'Y' for any specified time period (X_s) the following formula is used:

$$L = t \times Se \sqrt{1 + \frac{1}{N} + \frac{\left(X_s - \frac{\Sigma X}{N}\right)^2}{\Sigma X^2 - N\left(\frac{\Sigma X}{N}\right)^2}}$$

* Page 77

LIMITS OF PREDICTION

Therefore, predicting the mean value of Y for 1975 (time period 12) with a 95% confidence, we have

$$L = 2.262 \times 2 \times \sqrt{\frac{1}{11} + \frac{\left(12 - \frac{66}{11}\right)^2}{506 - 11\left(\frac{66}{11}\right)^2}}$$

$$= 4.524 \times \sqrt{.091 + \frac{(12 - 6)^2}{506 - 11\ (6)^2}}$$

$$= 4.524 \times \sqrt{.091 + \frac{6^2}{506 - (11 \times 36)}}$$

$$= 4.524 \sqrt{.091 + \frac{36}{506 - 396}}$$

$$= 4.524 \sqrt{.091 + \frac{36}{110}}$$

$$= 4.524 \times \sqrt{.091 + .327}$$

$$= 4.524 \times \sqrt{.418}$$

$$= 4.524 \times .6465$$

$$= 2.925 \quad \text{or} \quad 2.93 \quad \text{rounded off}$$

And predicting the <u>specific</u> value of Y with a 95% confidence would give us

$$L = 2.262 \times 2 \times \sqrt{1 + \frac{1}{11} + \frac{\left(12 - \frac{66}{11}\right)^2}{506 - 11\left(\frac{66}{11}\right)^2}}$$

which is essentially the same as adding 1 to the previous example at

$$= 4.524 \times \sqrt{.418 + 1}$$

$$= 4.524 \times \sqrt{1.418}$$

$$= 4.524 \times 1.1908$$

$$= 5.387 \quad \text{or } 5.39 \text{ rounded off}$$

thus the mean projection for 1975 and limit of prediction should be stated as

$$60.7 \pm 2.93$$

or 57.77 thru 63.63

Alternately, the specific projection for 1975 and limits of prediction could be stated as

$$60.7 \pm 5.39$$

or 55.31 thru 66.09

Although these are quite wide ranges, it is the <u>best estimate</u> that can be made given the circumstances, and Management should be advised of this rather than merely give the mid-point without qualification.

CALCULATING THE LONG TERM TREND

NON LINEAR TRENDS

All the foregoing trend calculations were based on the assumption that the trend in the time series was linear. This is the simplest assumption to make, but unfortunately, not necessarily the most accurate. There are many curvilinear forms which the data could take. However, recognizing and developing solutions for most of them is beyond the scope of this text.

Four of the more common non-linear curves which may be encountered are as follows:

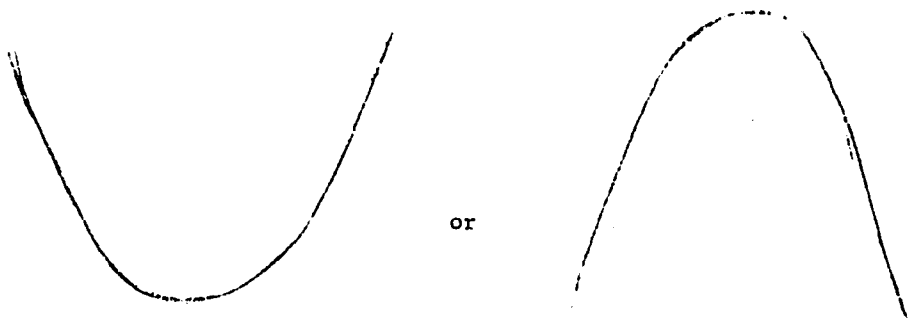
1. Second Degree Parabola
2. Logarithmic (Exponential) Curve
3. Modified Exponential Curve
4. Logistic Curve

I will discuss each of these and the method for their regression solution on the following pages.

SECOND DEGREE PARABOLA

A Second Degree Parabola trend line graphically takes the following general shape.

FIGURE 6 GENERAL SHAPE OF SECOND DEGREE PARABOLA



and the following equation $Y = a + bX + cX^2$

The coefficients a, b, and c are found by the method of least squares.

The normal equations are:

(1) $\Sigma Y = na + b\Sigma X + c\Sigma X^2$

(2) $\Sigma XY = a\Sigma X + b\Sigma X^2 + c\Sigma X^3$

(3) $\Sigma X^2Y = a\Sigma X^2 + b\Sigma X^3 + c\Sigma X^4$

However by shifting the origin of X to the center of the series, X becomes 0 which simplifies the above equations to:

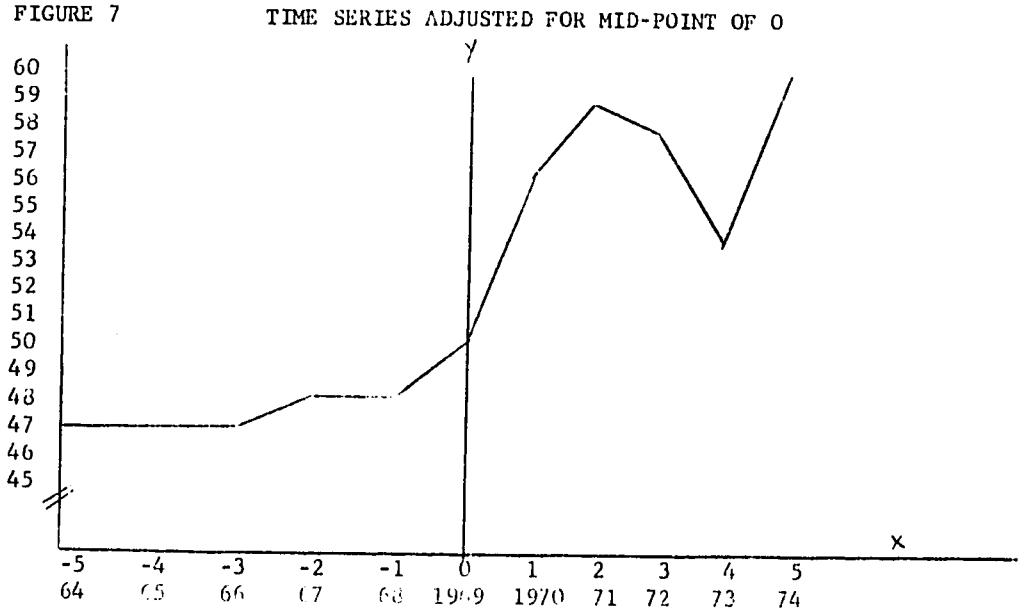
(1) $\Sigma Y = na + c\Sigma X^2$

(2) $\Sigma XY = b\Sigma X^2$

(3) $\Sigma X^2Y = a\Sigma X^2 + c\Sigma X^4$

Thus in our series (in FIGURE 3) 1969 would become the 0 time period where the X and Y axis meet, and time periods before 1969 would become negative, as indicated below

FIGURE 7 TIME SERIES ADJUSTED FOR MID-POINT OF 0



Our data in FIGURE 5 would also be adjusted as follows:

FIGURE 8 WORKSHEET OF DATA FOR ADJUSTED MID-POINT O AND CORRELATION

Time Period	Independent Variable X	Dependent Variable Y	Y^2	XY	X^2	X^4	X^2Y
	Time Sequence						
1964	- 5	47	2209	- 235	25	625	1175
1965	- 4	47	2209	- 188	16	256	752
1966	- 3	47	2209	- 141	9	81	423
1967	- 2	48	2304	- 96	4	16	192
1968	- 1	48	2304	- 48	1	1	48
1969	0	50	2500	0	0	0	0
1970	1	56	3136	56	1	1	56
1971	2	59	3481	118	4	16	236
1972	3	58	3364	174	9	81	522
1973	4	54	2916	216	16	256	864
1974	5	60	3600	300	25	625	1500
	0	574	30232	156	110	1958	5768
	ΣX	ΣY	ΣY^2	ΣXY	ΣX^2	ΣX^4	ΣX^2Y

From equation (2) we can find "b" directly by substitution since

$$(2) \quad \Sigma XY = b \Sigma X^2$$

Therefore

$$b = \frac{\Sigma XY}{\Sigma X^2} = \frac{156}{110} = 1.41818$$

By further substitution of the values that we have obtained from our worksheet (FIGURE 8) in equations (1) and (3) we have then reduced the problem to solving for two unknowns in a pair of simultaneous equations as follows:

$$(1) \quad \Sigma Y = na + c \Sigma X^2 \qquad (1) \quad 574 = 11a + 110c$$

$$(3) \quad \Sigma X^2Y = a \Sigma X^2 + c \Sigma X^4 \qquad (3) \quad 5768 = 110a + 1958c$$

Multiplying (1) by 10 gives us (4) $5740 = 110a + 1100c$ which can be subtracted from (3) as follows:

$$\begin{array}{rrcrcr} (3) & 5768 & = & 110a & + & 1958c \\ -\ (4) & 5740 & = & 110a & + & 1100c \\ \hline & 28 & = & & & 858c \end{array}$$

$$c = \frac{28}{858} = .0326$$

This value can then be substituted in (1) to find "a"

$$574 = 11a + 110\ (.0326)$$

Transposing, $11a = 574 - 3.586$

$$a = \frac{570.4}{11} = 51.85$$

Projecting, to estimate the value of Y for 1975 (ie where X = 6), substitute the values for a, b & c in the formula

$$Y = a + bX + cX^2$$

thus

$$Y = 51.85 + 1.41818\ (6) + .0326\ (6)^2$$

$$= 51.85 + 3.509 + 1.174$$

$$= 61.533$$

STANDARD ERROR & LIMITS OF PREDICTION

We must now estimate the limits of prediction around this projection.

Standard Error of Estimate

The Standard error of estimate for a second degree parabola is

$$Se = \sqrt{\frac{\Sigma Y^2 - a\Sigma Y - b\Sigma XY - c\Sigma X^2 Y}{N - 3}}$$

The "limits of prediction" can then be calculated as before, with

$$L = t \times Se \sqrt{\frac{1}{N} + \frac{\left(X_s - \frac{\Sigma X}{N}\right)^2}{\Sigma X^2 - N\left(\frac{\Sigma X}{N}\right)^2}}$$

thus, the standard error of estimate is

$$Se = \sqrt{\frac{30232 - (51.85)574 - 1.41818\ (156) - .0326\ (5768)}{11 - 3}}$$

$$= \sqrt{\frac{30232 - 29761.9 - 221.236 - 188.037}{8}}$$

$$= \sqrt{\frac{60.327}{8}}$$

$$= \sqrt{7.603}$$

$$= 2.757$$

and the limits of prediction for 95% confidence for 1975 (X = 6) are

$$L = 2.306 \times 2.757 \sqrt{\frac{1}{11} + \frac{\left(6 - \frac{0}{11}\right)^2}{100 - 11\left(\frac{0}{11}\right)^2}}$$

$$= 6.3576 \times \sqrt{.091 + \frac{6^2}{100}}$$

$$= 6.3576 \times \sqrt{.091 + \frac{36}{100}}$$

$$= 6.3576 \times \sqrt{.091 + 3273}$$

$$= 6.3576 \times \sqrt{.4183}$$

$$= 6.3576 \times .6468$$

$$= 4.112$$

So the answer for management is that Y is predicted with 95% confidence to be
Y = 61.53 ± 4.112 or between 57.42 and 65.64 for 1975

CALCULATING THE LONG TERM TREND - SECOND DEGREE PARABOLA

TO DETERMINE THE BEST FIT - LINEAR OR CURVILINEAR?

With a trend and projection calculated from a linear equation, and another calculated from the same data with a parabolic equation -- you may wonder which to use? The answer is -- the one that gives the best fit. This can be determined mathematically by calculating a correlation coefficient between the data and the trend for each equation. The line with the highest correlation will be the best fit.

A faster (but less rigorous) method is to plot it on a graph and "eyeball" it.

With the particular set of data already used, there isn't much to choose between the two. However, I will go through the motions so that you have a model to follow.

Linear Correlation Coefficient

Use the formula

$$(1) \qquad r = \frac{N\Sigma XY - \Sigma X \Sigma Y}{\sqrt{N\Sigma X^2 - (\Sigma X)^2}\ \sqrt{N\Sigma Y^2 - (\Sigma Y)^2}}$$

Substituting in (1)

$$r = \frac{11\ (156) - 0\ (574)}{\sqrt{11\ (110) - 0^2}\ \sqrt{11\ (30232) - 574^2}}$$

$$= \frac{1716 - 0}{\sqrt{1210 - 0}\ \sqrt{332552 - 329476}}$$

$$= \frac{1716}{34.785054 \times \sqrt{3076}}$$

$$= \frac{1716}{34.785054 \times 55.461698}$$

$$= \frac{1716}{1929.2381}$$

$$= .8894703$$

The Correlation Coefficient (for a 2nd Degree Parabola) is determined from the following formula:

(2)

$$r = \sqrt{\frac{a\sum Y + b\sum XY + c\sum X^2Y - \frac{(\sum Y)^2}{N}}{\sum Y^2 - \frac{(\sum Y)^2}{N}}}$$

Substituting in (2)

$$r = \sqrt{\frac{51.85(574) + 1.41818(156) + .0326(5768) - \frac{574^2}{11}}{30232 - \frac{574^2}{11}}}$$

$$= \sqrt{\frac{29761.9 + 221.23608 + 188.0368 - \frac{329476}{11}}{30232 - \frac{329476}{11}}}$$

$$= \sqrt{\frac{29761.9 + 221.23608 + 188.0368 - 29952.363}{30232 - 29952.363}}$$

$$= \sqrt{\frac{218.809}{279.637}}$$

$$= \sqrt{.7824751}$$

$$= .8845762$$

Summary	Comparing the linear	(1)	.8894703
	with the non-linear	(2)	.884576

the linear is a slightly superior fit.

LOGARITHMIC (EXPONENTIAL) CURVE

An exponential trend is one where the rate of change of Y for each time interval (X) is constant, rather than the amount of change as in a linear trend. Such trends are often seen in biological situations such as population growth, savings type situations with compound interest, and the like.

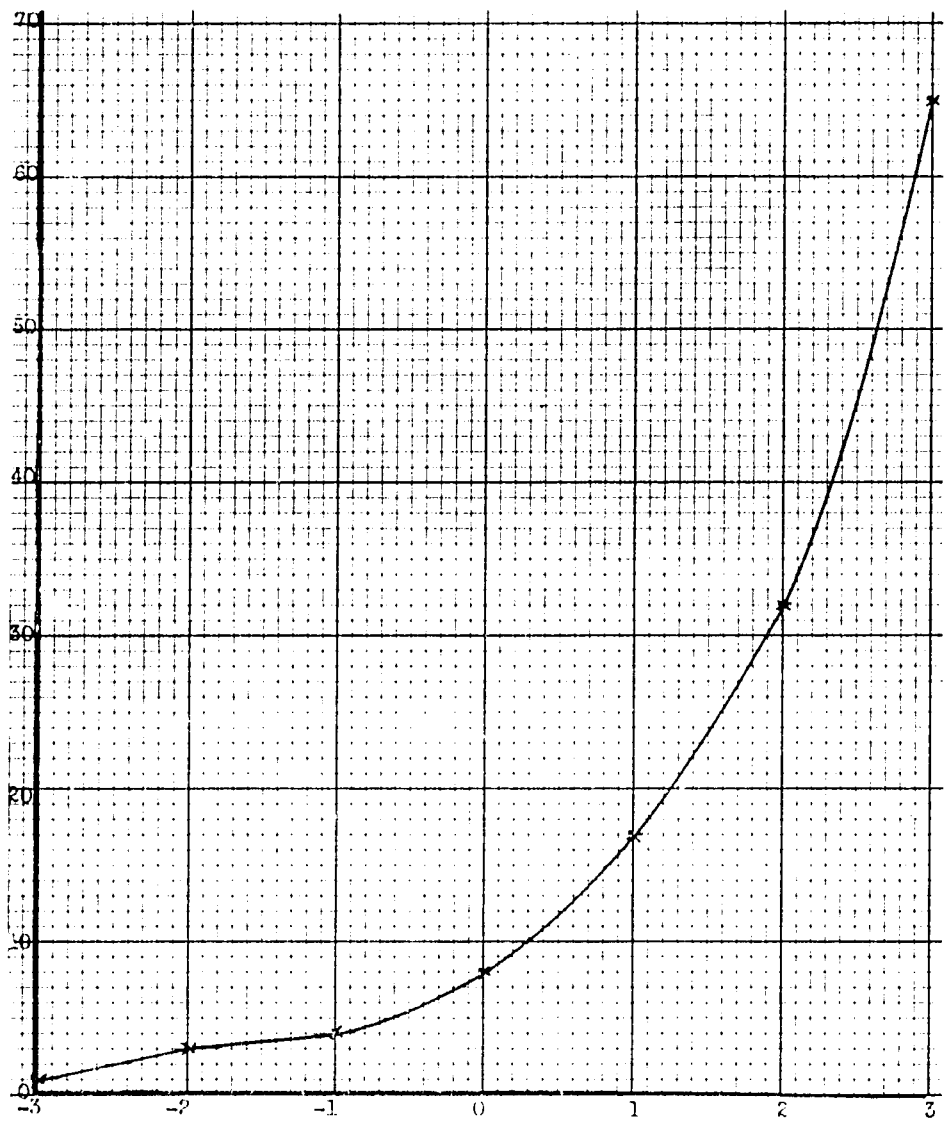
Since our preceding data is obviously not exponential, it is a waste of time trying to fit an exponential curve to it. I will therefore use the following hypothetical data as a model for solving a problem of this type.

FIGURE 9 DATA FOR AN EXPONENTIAL CURVE

Time Period X			Dependent Variable Y
Year	Sequence	Adjusted	
1969	1	-3	1
1970	2	-2	3
1971	3	-1	4
1972	4	0	8
1973	5	+1	17
1974	6	+2	32
1975	7	+3	65

A Plot of this data on arithmetic graph paper would appear thus:

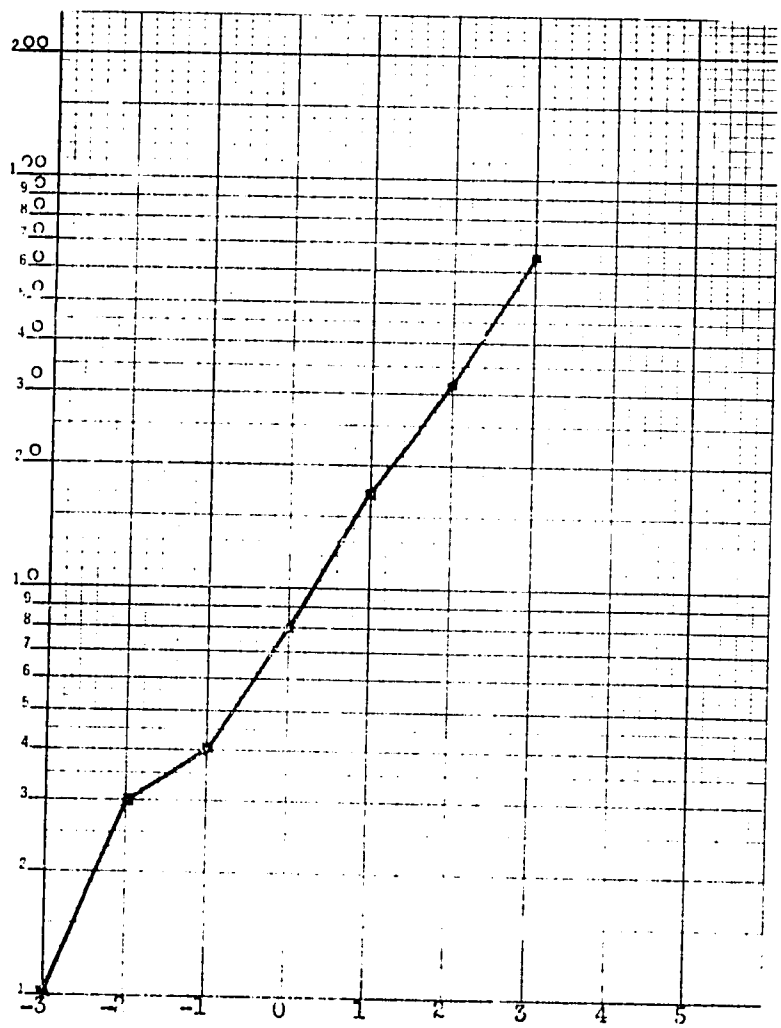
FIGURE 10 GENERAL SHAPE OF AN EXPONENTIAL CURVE



SEMI-LOG GRAPH

The exponential growth curve depicted in FIGURE 10 on arithmetic graph paper can be plotted on semi-log paper. A semi log scale shows the same relative change for an equal vertical distance thus the data appears as follows:-

FIGURE 11 DATA PLOTTED ON SEMI-LOG GRAPH



It becomes a straight line. The reason is because logarithms of exponential progressions form an arithmetic progression.

This form takes the general equation $Y = ab^x$ By taking the logarithm, the formula becomes a linear function with the form

$$\text{Log } Y = \text{Log } a + \text{Log } b\ (X)$$

The normal equations to solve this are: -

(1) $\sum \log Y = N \log a + \log b \sum X$

(2) $\sum ((X) \log Y) = \log a \sum X + \log b \sum X^2$

Where $\sum X = 0$ (by setting the time period 0 in the middle of the series)

Then $\log a = \dfrac{\sum \log Y}{n}$ and $\log b = \dfrac{\sum (X(\log Y))}{\sum X^2}$

The data to solve this can be obtained from the following worksheet

FIGURE 12 DATA TABLE FOR AN EXPONENTIAL TIME SERIES

Time Period	Independent Variable X Time Sequence	Dependent Variable Y	Log Y	$(\text{Log } Y)^2$	x (Log Y)	Y^2	XY	X^2
1969	-3	1	.0000	.0000	0	1	-3	9
1970	-2	3	.4771	.2276	- .9542	9	-6	4
1971	-1	4	.6021	.3625	- .6021	16	-4	1
1972	0	8	.9031	.8156	0	64	0	0
1973	+1	17	1.2304	1.5139	1.2304	289	17	1
1974	+2	32	1.5051	2.2653	3.0102	1024	64	4
1975	+3	65	1.8129	3.2860	5.4387	4225	195	9
	0	130	6.5307	8.4715	8.123	5628	263	28
	$\sum X$	$\sum Y$	$\sum \text{Log } Y$	$\sum \text{Log } Y^2$	$\sum (X \log Y)$	$\sum Y^2$	$\sum XY$	$\sum X^2$

Substituting,

$\log a = \dfrac{6.5307}{7}$ and $\log b = \dfrac{8.123}{28}$

$\log a = .9329571$ and $\log b = .29$

Using the logarithm table on pages 78 & 79 you can determine the value of "a", and "b".

a = antilog of .9329571 = 8.57

b = antilog of .29 = 1.95

The predicted value of Y for 1976 (X = 4) is found by substituting in the formula $Y = ab^x$

$$Y = 8.57\ (1.95)^4$$

$$= 8.57\ (14.459)$$

$$= 123.914$$

or $\log Y = \log a + \log b\ (X)$

$$\log Y = .9329571 + .29\ (4)$$

$$= .9329571 + 1.16$$

$$= 2.0929571$$

STANDARD ERROR & LIMITS OF PREDICTION

The standard error of estimate for the exponential curve

$$\log Y = \log a + \log b(X)$$

is

$$Se = \sqrt{\frac{\Sigma(\log Y)^2 - \log a \Sigma \log Y - \log b \Sigma (X \log Y)}{N - 3}}$$

Thus substituting

$$Se = \sqrt{\frac{3.4715 - .9329571 \ (6.5307) - .29 \ (8.123)}{7 - 3}}$$

$$= \sqrt{\frac{8.4715 - 6.09286 - 2.35567}{4}}$$

$$= \sqrt{\frac{.02297}{4}}$$

$$= \sqrt{.0057425}$$

$$= .07578$$

and the limits of prediction for a mean value of Y with 95% confidence are

$$L = 2.306 \times .07578 \sqrt{\frac{1}{7} + \frac{\left(4 - \frac{0}{7}\right)^2}{28 - 7\left(\frac{0}{7}\right)^2}}$$

$$= .1747486 \sqrt{.14286 + \frac{4^2}{28}}$$

$$= .1747486 \sqrt{.14286 + \frac{16}{28}}$$

$$= .1747486 \sqrt{.14286 + .57143}$$

$$= .1747486 \sqrt{.71429}$$

$$= .1747486 \times .84516$$

$$= .1476905$$

Caution: The standard error of estimate is logarithmic, not arithmetic. Therefore the error range must be calculated in this form first before converting by antilogs to actual data values.

Thus the estimated mean value of Y for 1976 (X = 4) with limits of prediction for 95% confidence is

log 2.0929571 ± .1476905

= 1.9456 through 2.2407

or 123.9, with an upper limit of 174.1

and a lower limit of 88.17

#	log
123.9	2.0930
88.17	1.9453
174.1	2.2407

Other values for Y and limits can be calculated as follows from the formula

log Y = log a + log b (X)

Where X = 3 log Y = .933 + .29 (3)

= .933 + .87

= 1.803

Limits are then 1.803 ± .1476905

= 1.6553 through 1.9507

or 63.53 with upper limit 89.3

lower limit 45.2

#	log
63.53	1.803
45.2	1.6553
89.3	1.9507

Similarly, where X = 2

Log Y = .933 + .29 (2)

= .933 + .58

= 1.513

Limits are thus

1.513 ± .1476905

1.3653 through 1.6607

or 32.58 with upper limit 45.8

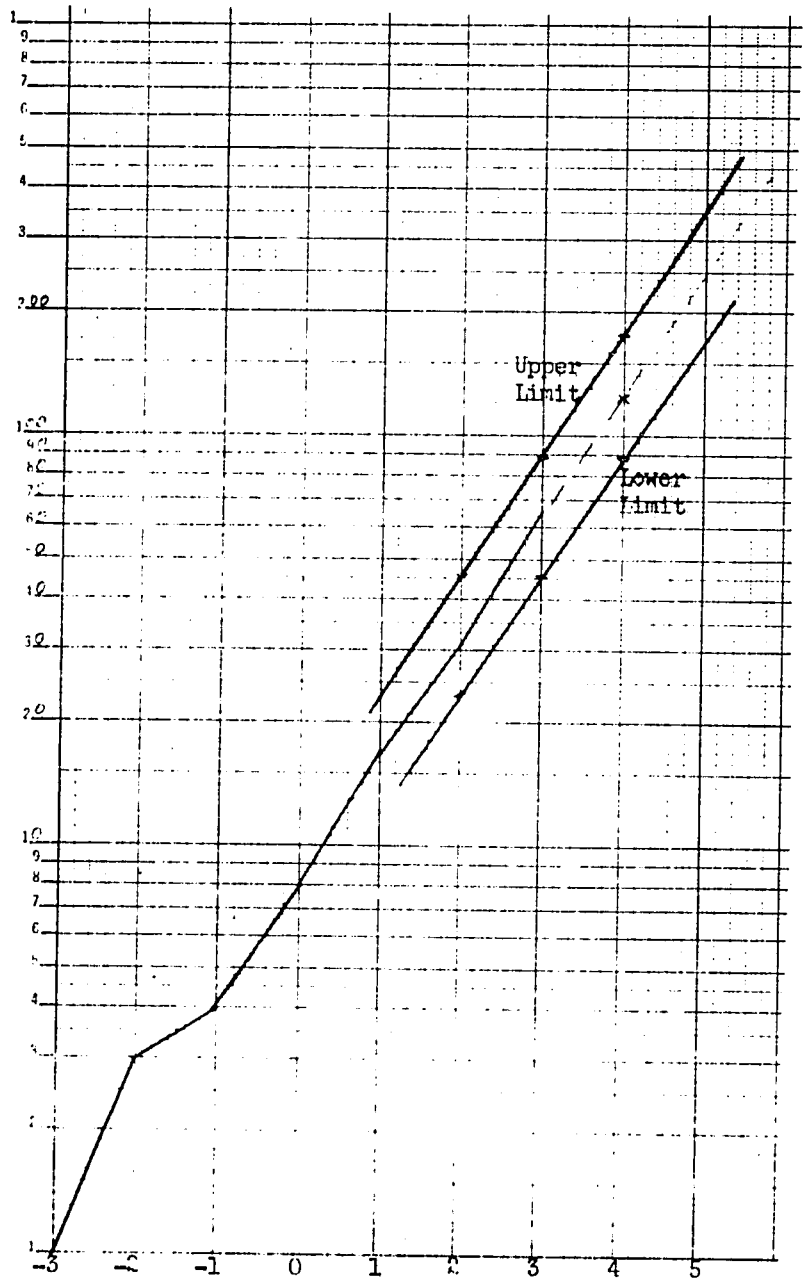
lower limit 23.2

#	log
32.58	1.513
23.2	1.3653
45.8	1.6607

This is shown graphically in Figure 13

FIGURE 13 LIMITS OF PREDICTION FOR

$$\log Y = \log a + \log b\,(X)$$



EXPONENTIAL GROWTH CURVE
CORRELATION COEFFICIENT

To determine the correlation coefficient of an exponential growth curve,

$$r = \frac{N \sum (X \log Y) - \sum X \; (\sum \log Y)}{\sqrt{N \sum X^2 - (\sum X)^2} \; \sqrt{N \sum (\log Y)^2 - (\sum \log Y)^2}}$$

Thus

$$r = \frac{7(3.123) - 0(6.5307)}{\sqrt{7(23) - 0} \; \sqrt{7(8.4715) - (6.5307)^2}}$$

$$= \frac{56.861}{\sqrt{196} \times \sqrt{59.3005 - 42.65}}$$

$$= \frac{56.861}{\sqrt{196} \times \sqrt{16.6505}}$$

$$= \frac{56.861}{14 \times 4.0805}$$

$$= \frac{56.861}{57.127} = .995$$

An alternate formula to achieve the same result can be used thus

$$r = \sqrt{\frac{\log a \sum \log Y + \log b \sum (X \log Y) - \frac{(\sum \log Y)^2}{N}}{\sum (\log Y)^2 - \frac{(\sum \log Y)^2}{N}}}$$

thus

$$r = \sqrt{\frac{.9329571(6.5307) + .29(8.123) - \frac{6.5307^2}{7}}{8.4715 - \frac{6.5307^2}{7}}}$$

$$= \sqrt{\frac{6.0928629 + 2.35567 - \frac{42.650042}{7}}{8.4715 - \frac{42.650042}{7}}}$$

$$= \sqrt{\frac{6.0928629 + 2.35567 - 6.0928631}{8.4715 - 6.0928631}}$$

$$= \sqrt{\frac{2.3556698}{2.3786369}}$$

$$= \sqrt{.9903444}$$

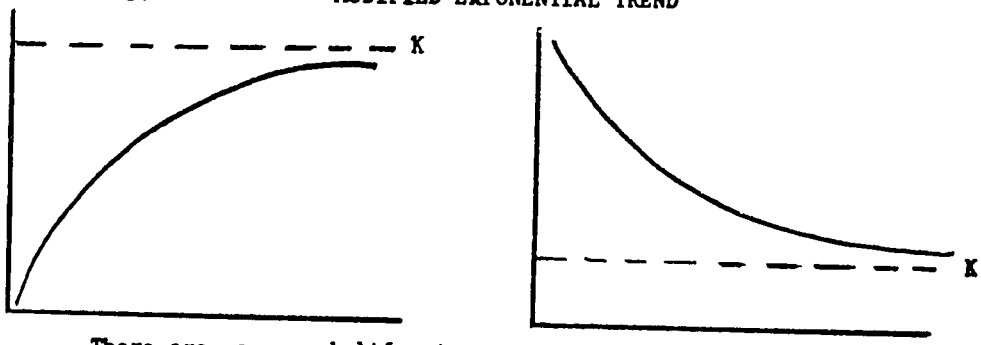
$$= .9951604$$

NOTE: Be very careful in substituting in the formulae: $\sum (\log Y)^2$ is not the same as $(\sum \log Y)^2$.

MODIFIED EXPONENTIAL TREND

A variation of the exponential trend is obtained by adding a constant "K" to the formula. K is the ceiling above which the curve is not expected to rise, or the base, below which it is not expected to fall.[1]

FIGURE 14 MODIFIED EXPONENTIAL TREND



There are many real life situations where this is appropriate - for instance an increase in solar energy will not give increasing yields indefinitely -- after a while the effect of "diminishing returns" appears, and for each incremental addition, a lesser return is received.

The formula for this curve is $Y = K + ab^x$

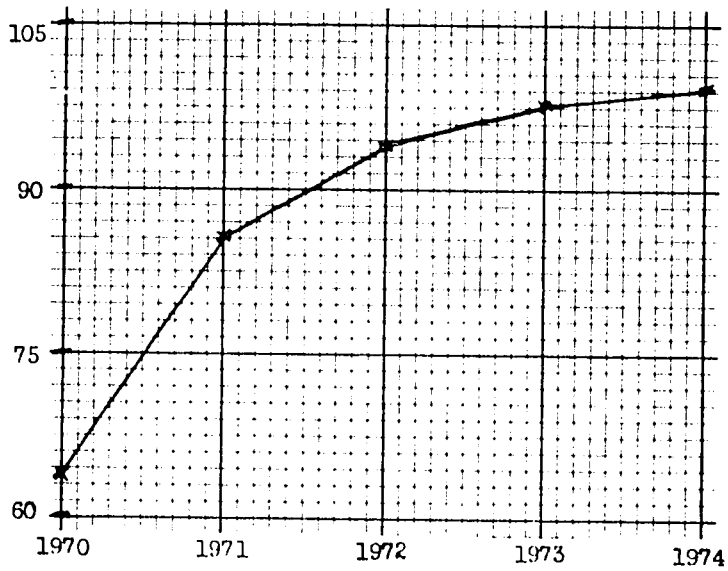
For example the following data is modified exponential

FIGURE 15 HYPOTHETICAL MODIFIED EXPONENTIAL DATA

X	Y
1970	64
1971	85.6
1972	94.24
1973	97.696
1974	99.078

and graphed would appear thus

FIGURE 16 GRAPH OF MODIFIED EXPONENTIAL DATA



The coefficients K, a and b can be found by the methods of selected points, or the method of semi-averages, as follows.

---

[1] "K" is known as the asymptotic limit

SELECTED POINT METHOD FOR CALCULATING MODIFIED EXPONENTIAL TREND

Three points at equal time intervals are selected and the values of Y at these points are substituted in the equations.

(1) $b^t = \frac{P_3 - P_2}{P_2 - P_1}$

(2) $a = \frac{P_2 - P_1}{b^t - 1}$

(3) $K = P_1 - a$

Where

t = time interval between selected points
P_1 = value of Y at point 1
P_2 = value of Y at point 2
P_3 = value of Y at point 3
k = asymptotic limit
a = value for Y where X is the first selected measuring point in the time series.
b = slope of the curve

Thus, we have three equations with three unknowns which can be solved sequentially. For example

FIGURE 17 MODIFIED EXPONENTIAL TREND DATA

X	=	X	Y
1970		X_1	64
1971		X_2	85.6
1972		X_3	94.24
1973		X_4	97.696
1974		X_5	99.078

and selecting every other point

$P_1 = X_1$
$P_2 = X_3$
$P_3 = X_5$

"t" is then 2.

Substituting in (1)

$$b^2 = \frac{99.073 - 94.24}{94.24 - 64}$$

$$= \frac{4.838}{30.24}$$

$$= .16$$

Therefore $b = \sqrt{.16}$

$= .4$

Substituting in (2) $a = \frac{30.24}{.16 - 1}$

$$= \frac{30.24}{.84}$$

$$= -36$$

Note: "a" is measured from "K". Thus where "K" is an upper asymptote, "a" is negative

Substituting in (3) $K = 64 - (-36)$

$= 64 + 36$

$= 100$

PROJECTIONS

To estimate the value of Y for X_6 substitute in the formula

$$Y = K + ab^X$$

$$Y = 100 + (-36)\ (.4)^6$$

$$= 100 + (-36)\ (.0041)$$

$$= 100 + (-.1476)$$

$$= 99.8524$$

Alternate Methods

To project later values of Y, an alternate method is to substitute in the formula

$$P_4 = P_3 + b\ (P_3 - P_2)$$

Where
P_4 = 1 time frame higher than P_3

For example to estimate the value of Y from the data in Figure 17 for X = 6

$$= 99.078 + .4(99.078 - 97.696)$$

$$= 99.078 + .4(1.382)$$

$$= 99.078 + .5528$$

$$= 99.6308$$

$t = 1$
$b = 4$
$X_6 = P_4 = ?$
$X_5 = P_3 = 99.078$
$X_4 = P_2 = 97.696$

To project earlier values of Y, substitute in formula

$$P_o = P_1 - \left(\frac{P_2 - P_1}{b^t}\right)$$

Where
P_o = 1 time frame lower than P_1

For example, for X = 0, the value of Y is

$$= 64 - \left(\frac{85.6 - 64}{.4}\right)$$

$$= 64 - \left(\frac{21.6}{.4}\right)$$

$$= 64 - 54$$

$$= 10$$

$t = 1$
$b = .4$
$X_o = P_o = ?$
$X_1 = P_1 = 64$
$X_2 = P_2 = 85.6$

METHOD OF SEMIAVERAGES FOR CALCULATING MODIFIED EXPONENTIAL TREND

1. Group the data into 3 even groups

$$G_1, \quad G_2, \quad G_3$$

with N = number of data items per group.

2. Sum each group.

Use the following three equations to solve b, k and a.

$$(1) \qquad b^n = \frac{G_3 - G_2}{G_2 - G_1}$$

$$(2) \qquad nk = G_1 - \frac{G_2 - G_1}{b^n - 1}$$

$$(3) \qquad a = \frac{G_2 - G_1}{(b^n - 1)(b + 1)}$$

An example of this will be shown on the following page.

For example using the data from FIGURE 17 in our previous problem for the selected point method,

Group the data into 3 even groups and sum each group.

$G_1 = 10 + 64 = 74$
$G_2 = 85.6 + 94.24 = 179.84$
$G_3 = 97.696 + 99.078 = 196.774$

	X	Y
G_1	0	10
	1	64
G_2	2	85.6
	3	94.24
G_3	4	97.696
	5	99.078

and n = number of data items per group
= 2

then substituting in (1)

$$b^2 = \frac{196.774 - 179.84}{179.84 - 74}$$

$$= \frac{16.934}{105.84} = .1599962$$

Thus

$$b = \sqrt{.1599962} = .399996$$

Substituting in (2)

$$2k = 74 - \frac{105.84}{.399996^2 - 1}$$

$$= 74 - \frac{105.84}{.1599962 - 1}$$

$$= 74 - \frac{105.84}{-.84}$$

$$= 74 - (-125.99952)$$

$$= 74 + 125.99952$$

Thus $K = \frac{199.9995}{2}$

$= 99.99976$

and substituting in (3)

$$a = \frac{105.84}{(.2572 - 1)\ (.5 + 1)}$$

$$= \frac{105.84}{(-.7428)\ (1.5)}$$

$$= \frac{105.84}{1.1142}$$

$$= 94.9919$$

Thus on the graph "a" is 99.99976 - 94.9919 = 5.0078

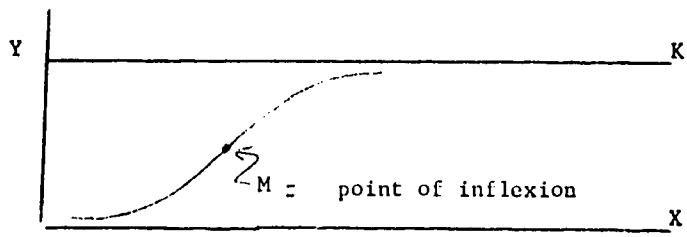
To estimate the value of Y where X = 6 substitute in the formula $Y = k + ab^X$ as before

$$Y = 99.99976 + (5.0078)\ (.39996)^6$$
$$= 99.99976 + (5.0078)\ (.0040935)$$
$$= 99.99976 + 0204994$$
$$= 100.0203$$

THE LOGISTIC CURVE

The logistic curve (also known as an "S" Curve) is one that starts out with an increasing rate of growth, then gradually declines as the saturation point is approached. The point at which the curve changes shape from growth to decline (M) is called the point of inflexion.

FIGURE 18 A LOGISTIC CURVE



The formula for this type of curve is

$$Y = \frac{K}{1 + 10^{a + bx}}$$

This curve cannot be plotted as a straight line but the variables b, a and k can be solved by the Selected Points Method.

Selected Points Method for Calculating a Logistic Curve

The Selected Points method requires the solution of three equations for three unknowns as follows:-

(1) $$tb = \log \frac{P_1 \quad (P_3 - P_2)}{P_3 \quad (P_2 - P_1)}$$

(2) $$a = \log \frac{P_1 - P_2}{P_2 \quad (10^{tb}) - P_1}$$

(3) $$K = P_1 \quad (1 + 10^a)$$

Where

P_1 = 1st Point

P_2 = 2nd Point

P_3 = 3rd Point

For Example

To find b, a and K,
Substituting in (1)

$$^2b = \log \frac{10 \ (90 - 50)}{90 \ (50 - 10)}$$

$$= \log \frac{10 \ (40)}{90 \ (40)}$$

$$= \log \frac{400}{3600}$$

$$= \log .1111$$

$$= -1.0453$$

Where

	X	Y
P_1 =	1	10
	2	30
P_2 =	3	50
	4	70
P_3 =	5	90

and

t = 2

From log tables on pages 78 & 79

#		log
1111	=	.0453
.1	=	-1.

$$\text{therefore } b = \frac{-1.0453}{2}$$

$$= -.5227$$

Substituting in (2)

$$a = \log \frac{10 - 50}{50\ (.1111) - 10}$$

$$\doteq \log \frac{-40}{5.555 - 10}$$

$$= \log \frac{-40}{-4.445}$$

$$= 8.99 \text{ or } 9. \text{ rounded up}$$

from the log tables

#	Log mantissa
.9542	9

$$= 0.9542$$

Substituting in (3)

$$K = 10\ (1 + 9)$$

$$= 10\ (10)$$

$$= 100$$

Then, to find Y for any value of X, substitute in the equation

$$Y = \frac{K}{1 + 10^{a + bx}}$$

<u>For example</u>

Where $X = 6$

$$Y = \frac{100}{1 + 10^{.9542 + (-.5227)\ (6)}}$$

$$= \frac{100}{1 + 10^{.9542 + (-3.1362)}}$$

$$= \frac{100}{1 + 10^{-2.1820}}$$

$$= \frac{100}{1.0152}$$

$$= 98.5$$

From the the log tables

$$10^{-2.1820} = .0152$$

MOVING AVERAGES

A moving average is simply an arithmetic mean of a specified time period (such as a three year moving average) which keeps renewing itself by adding the most recent data, and dropping the oldest. The net effect of using a moving average to study a trend is that the impact of unpredictable, cyclical and seasonal variation, although not completely eliminated, is greatly dampened. This is because the highs and lows tend to be smothered by the longer range trend effect which is what we are trying to study.

As a general rule, the time period for which the moving average is established (i.e. three year moving average, twelve month moving average, etc.) should correspond to or exceed the best estimate of the average cycle length of the data. If a period less than the cycle length is chosen, part of the cycle effect will remain in the average.

The easiest way to calculate a moving average is to construct a table, as follows:

FIGURE 18 B WORKSHEET FOR DEVELOPMENT OF THREE YEAR MOVING AVERAGE DATA

Year	X	Y Variable Studied		3Yr Moving Total	3 Yr Moving Average
1964	1	47			
1965	2	47		141	47
1966	3	47		142	47.3
1967	4	48		143	47.6
1968	5	48		146	48.6
1969	6	50	etc.	154	51.3
1970	7	56		165	55
1971	8	59		173	57.6
1972	9	58		171	57
1973	10	54		172	57.3
1974	11	60			

When calculated, the data represents the average for the middle time period of the series.

For example, the 3 year moving average for 1973 is calculated from the period 1972 through 1974 and is

$$\text{1973 moving average} = \frac{58 + 54 + 60}{3} = \frac{172}{3} = 57.3$$

Note: 1. 1965 is the first year for which we can obtain a moving average from our data.

2. It is not possible to develop a moving average for 1974, until the 1975 data is recorded. The latter is a more frustrating problem, especially when presenting the data to "uninitiated" managers. It is difficult to overcome their impression that in 1975 the latest trend information you can develop is based on a two year old (1973) average !!!

3. <u>The moving average is not really a calculated trend line</u>. It is merely a smoothing out of the time series. A trend line can be calculated from the moving average data, using the semi-average or least squares methods discussed earlier. Moving averages can also be used for seasonal data analysis.

CYCLICAL TRENDS AND ERRATIC DEVIATION

With the long term trend computed, it is now possible to isolate and examine the magnitude of cyclical and erratic factors.

Cyclical fluctuations generally cover long time periods. They can be explained in terms of the negative feedback loop and momentum.

Negative Feedback Loop

In general, negative feedback is the tendency for a situation to be self-correcting. Economists also refer to this phenomenon as the "cobweb theorem." For example, in agriculture, farmers make decisions about planting based on current prices. Current prices however are based on current harvests, and there is a lag time for growth and production which prevents the farmer from planting and harvesting immediately. Thus where the market price responds to changes in supply volume, fluctuations occur. In essence the phenomena operates as follows:-

At TIME 1, There are very few tomatoes available. Consequently the price of tomatoes is high. Farmers observe the current high price being paid for tomatoes. Many decide to benefit themselves and so they plant.

At TIME 2, Because so many farmers decided to plant in time 1, there is now an oversupply of tomatoes in the market. Consequently the price of tomatoes falls. Farmers receive low prices for their tomato crop, some decide not to plant again (or not to plant as much). Others who did not plant, observe the situation and also decide not to plant.

At TIME 3, Because so few farmers planted in time 2, there is now a shortage of tomatoes in the market again. Consequently the price of tomatoes increases; in turn tempting more farmers to plant -- at iba pa!

Momentum

This is the tendency for things to continue in the same direction once they got started, regardless of the fact that the original cause may no longer exist.

In the agricultural business, once a crop is planted, it is usually cared for through to harvest despite the ups and downs in the market. Thus, depending upon the life cycle of the crop, the initial momentum may continue to have an effect for several months (or even years) beyond the time that the need was satisfied.

Thus, once an upward economic spiral starts, there is a tendency for it to continue; also once a downward trend sets in, it is difficult to turn around. Despite the theory, we don't really understand too much about cyclical fluctuations, but if they do appear to follow a systematic pattern, once this pattern is established, it can be used predictively as the best guess available.

To identify the cyclical effect, the long term trend is compared with the actual data at previous points in time and the difference noted.

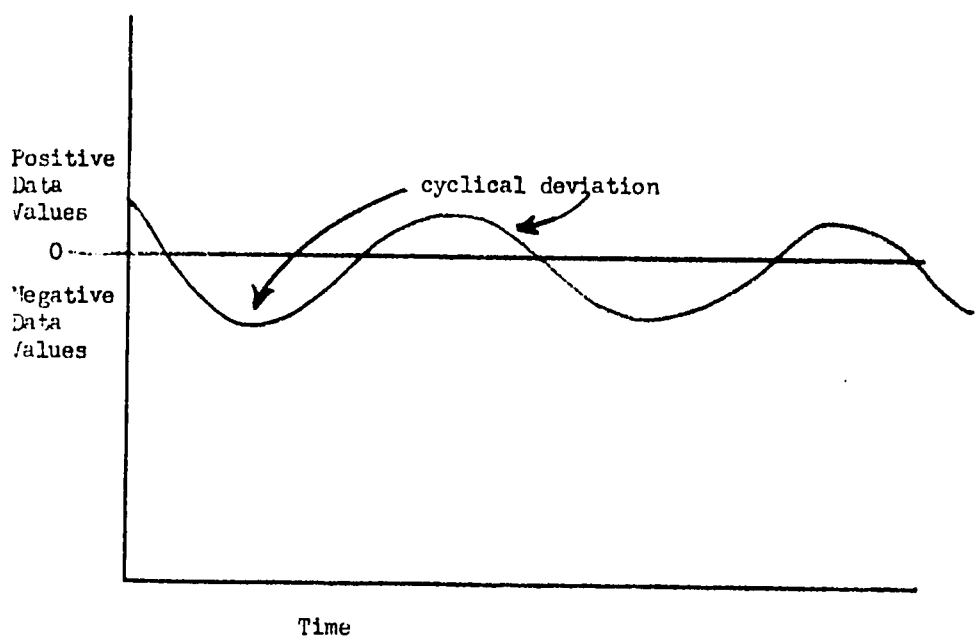
FIGURE 19 ANALYSIS OF CYCLICAL AND ERRATIC DEVIATION FROM TRENDS

Period	Year	Actual Value of Y	Trend Value of Y	Cyclical and Erratic Deviation of Actual from Trend (Actual - Trend)	Cyclical & Erratic Deviation as a Percentage of Trend $\left(\frac{\text{Actual - Trend}}{\text{Trend}} \times 100\right)$
1	1964	47	45.1	2	4%
2	1965	47	46.5	1	2%
3	1966	47	47.9	-1	-2%
4	1967	48	49.4	-1	-2%
5	1968	48	50.8	-3	-6%
6	1969	50	52.2	-2	-4%
7	1970	56	53.6	2	4%
8	1971	59	55.0	4	7%
9	1972	58	56.5	2	4%
10	1973	54	57.9	-4	-7%
11	1974	60	59.3	1	2%

STATIONARY TIME SERIES

When we have removed the trend from the original data, the remaining time series is known as "Stationary Time Series." This series can best be shown on a control type chart where the nature of any cyclical deviation is revealed, as follows:-

FIGURE 20 STATIONARY TIME SERIES WITH % CYCLICAL AND ERRATIC DEVIATIONS (HYPOTHETICAL)



Either the number or the percentage may be depicted in this manner.

The type of curve above is called a Periodic Curve.

THE PERIODIC CURVE

This is the type of curve most commonly found in cyclical trends and is known as the Periodic, or Fourier Curve.[1]

The general equation for a periodic curve is

$$Y = a_o + a_1 \text{Cos}CX + b_1 \text{Sin}CX$$

Where

a_o = the central value around which the wave fluctuates

a_1 = "a" for a given value of X

b_1 = "b" for a given value of X

Cos = Cosine

Sin = Sine

$C = \dfrac{360}{\text{\# of time units per cycle}}$

X = a given time unit X

and the normal equations are

(1) $\Sigma Y = a_o N + a_1 \Sigma \text{Cos}CX + b_1 \Sigma \text{Sin}CX$

(2) $\Sigma \text{Cos}CXY = a_o \Sigma \text{Cos}CX + a_1 \Sigma \text{Cos}CX^2 + b_1 \Sigma \text{Cos}CX \ \text{Sin}CX$

(3) $\Sigma \text{Sin}CXY = a_o \Sigma \text{Sin}CX + a_1 \Sigma \text{Cos}CX \ \text{Sin}CX + b_1 \Sigma \text{Sin}CX^2$

Where

N = Number of time periods.

Analysis

a. The Period is the number of time intervals between peaks.

b. The amount that the curve fluctuates above and below the central point is called the Semi-Amplitude, and is found from the formula

$$A = \sqrt{a_1^2 + b_1^2}$$

c. The range from the highest to the lowest point on the wave is called the Amplitude, and is equivalent to

2A

d. The point in the cycle where the wave reaches its maximum value is known as the Phase Angle. This is found as follows:-

1. Find the angle (Theta) whose tangent is $\dfrac{b_1}{a_1}$ 2/

	Phase Angle is
2. Then if b_1 is positive, and a_1 is positive	= Theta
b_1 is positive, and a_1 is negative	= 180° - theta
b_1 is negative and a_1 is negative	= 180° + theta
b_1 is negative and a_1 is positive	= 360° - theta

A specific example is developed on the following pages to illustrate the analysis of this type of curve, using the data in Figure 19.

---

1 If the series repeats, but does not seem to have a fixed period, it is known as an oscillatory series, and is beyond the scope of this text.

2 Also known as the Arc Tangent of $\dfrac{b_1}{}$

From FIGURE 19, the basic data can be calculated as follows:-

FIGURE 21

WORKSHEET FOR CALCULATING PERIODIC CURVE DATA

Time Period	X	Y	CX	CosCX 1/	SinCX 1/	$CosCX^2$	$SinCX^2$	CosCX Y	SinCX Y	CosCX Sin CX
1964	1	4	72	+.3090	+.9511	.0955	.9046	1.236	3.8044	.2939
1965	2	2	144	-.8090	+.5878	.6545	.3455	-1.618	1.1756	-.4755
1966	3	-2	216	-.8090	-.5878	.6545	.3455	1.618	1.1756	.4755
1967	4	-2	288	+.3090	-.9511	.0955	.0946	-. 618	1.9022	-.2939
1968	5	-6	360	+1.	0	1.	0	-6.	0	0
1969	6	-4	432	+.3090	+.9511	.0955	.9046	-1.236	-3.8044	.2939
1970	7	4	504	-.8090	+.5878	.6545	.3455	-3.236	2.3512	-.4755
1971	8	7	576	-.8090	-.5878	.6545	.3455	-5.663	-4.1146	.4755
1972	9	4	648	+.3090	-.9511	.0955	.9046	1.236	-3.8044	-.2939
1973	10	-7	720	+1.	0	1.	0	-7	0	0
1974	11	2	792	+.3090	+.9511	.0955	.9046	.618	1.9022	.2939
Σ		2	4752	+.3090	.9511	5.0955	5.9050	-20.663	.5878	.2939

Then $a_o = 0$ N = 11, Period = 5 (between 1968 (-6%) and 1973 (-7%) and $C = \frac{360^o}{5} = 72^o$

1/ If you are unsure of the procedure for calculating trigonometric ratios, see the summary review on pages 75 & 76

Then the calculated data is substituted in the normal equations

(1) $2 = 11a_o + .3090a_1 + .9511b_1$

(2) $-20.663 = .3090a_o + 5.0955a_1 + .2939b_1$

(3) $1.9022 = .9511a_o + .2939a_1 + 5.9050b_1$

To eliminate some of the unknowns, multiply (1) by .2939 and (2) by .9511

(4) $.5878 = 3.2329a_o + .0908a_1 + .2795b_1$

(5) $-19.6526 = .2939a_o + 4.8463a_1 + .2795b_1$

Subtract (5) from (4)

(6) $20.2404 = 3a_o - 4.7555a_1$

Multiply (2) by 5.9050 & (3) by .2939

(7) $-122.015 = 1.8247a_o + 30.0889a_1 + 1.7355b_1$

(8) $.5591 = .2795a_o + .0864a_1 + 1.7355b_1$

Subtract (8) from (7)

(9) $-122.5741 = 1.5452a_o + 30.0025a_1$

Multiply (6) by 1.5452 and (9) by 3

(10) $31.2755 = 4.6356a_o - 7.3482a_1$

(11) $-367.7223 = 4.6356a_o + 90.0075a_1$

Subtract (10) from (11)

(12) $-398.9978 = 97.3557a_1$

$$a_1 = -\frac{398.9978}{97.3557}$$

$$= -4.0984$$

Substituting this value of a_1 in (6)

(6) $20.2404 = 3a_o - 4.7555 \ (-4.0984)$

$20.2404 = 3a_o + 19.4897$

$3a_o = 20.2404 - 19.4897$

$$a_o = \frac{.7507}{3}$$

$a_o = .2502$

Substituting these values of a_o and a_1 in (1)

(1) $2 = 11\ (.2502) + .3090\ (-4.0984) + .9511\ b_1$

$2 = 2.7522 - 1.2664 + .9511b_1$

$2 = 1.4858 + .9511b_1$

$.9511b_1 = 2 - 1.4858$

$$b_1 = \frac{.5142}{.9511}$$

$= .5406$

The semi-amplitude is then

$$A = \sqrt{a_1^2 + b_1^2}$$

$$= \sqrt{-4.0984^2 + .5406^2}$$

$$= \sqrt{16.7969 + .2923}$$

$$= \sqrt{17.0892}$$

$$= 4.1339$$

and the amplitude is $2A =$

$$= 4.1339 \times 2$$

$$= 8.2678$$

The Phase Angle is found as follows

$$\text{Theta} = \text{Tan} \frac{b_1}{a_1}$$

$$= \frac{.5406}{-4.0984}$$

$$= -.1319$$

From the tangent table .1319 is $7^o 31'$

The sign is negative, however;

b_1 is positive and a_1 is negative.

The Phase Angle is 180° - theta

or $180^o - 7^o\ 31' = 172^o 29'$

Since there are 5 years in the period, for each year in the period, and $C = \frac{360}{5} = 72^o$ this means that on the average, the maximum point in the cycle occurs at

$$\frac{172^o 29'}{72} = \frac{172\ \frac{29}{60}}{72}$$

$$= \frac{172.483}{72}$$

$$= 2.396 \text{ years of the cycle}$$

$$1 \text{ month} = \frac{1}{12} \text{ year} = .0833 \text{ year}$$

$$\text{therefore } .396 = \frac{.396}{.0833}$$

$$= 4.75 \text{ months}$$

or 2 years 4 months and 3 weeks after the beginning of the cycle.

CALCULATING THE CYCLICAL TREND

FORECASTING

To project the cyclical value of Y for 1975 (X = 12) substitute in the formula

$$Y = a_o + a_1 \text{CosCX} + b_1 \text{SinCX}$$

Where X = 12 and C = 72°

CX = 12 x 72

= 864

CosCX

Cos 864° is equivalent to Cos 144°

Cos 144° = Quadrant II

Theta = 180° - 144

= 36°

= .8090, and the sign is negative

Sin CX

Sin 864° is equivalent to Sin 144°

Sin 144° = Quadrant II

Theta = 180° - 144

= 36°

= .5878, and the sign is positive

thus

Y = .2502 + (-4.0984) (-.8090) + (.5406) (.5878)

= .2502 + 3.3156 + .3178

= 3.8836

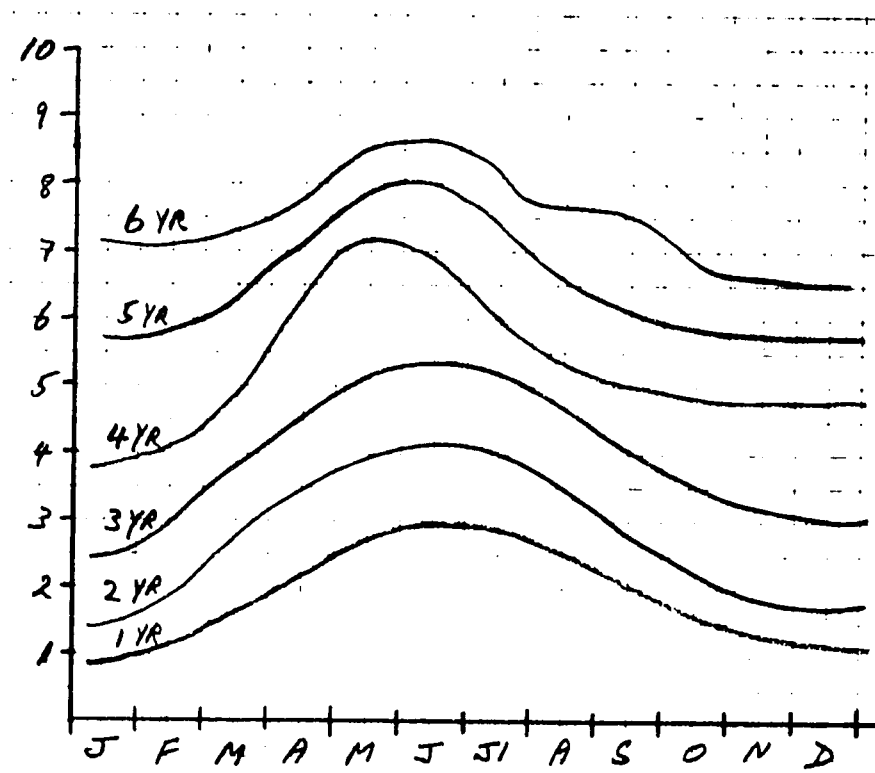
Since the Y data was in percentage units of a stationary time series, the projected cyclical effect is also a percentage of deviation from the long term trend line.

SEASONAL VARIATION

Previously, the data in FIGURE 1 was summarized by year and analyzed for the long term trend. That summarization was useful to suppress the effect of seasonal variation. Now however, we specifically wish to study the seasonal effect (in this case the season is monthly). Thus the data must be rearranged and the long term trend, cyclical and erratic effect removed so that one month can be effectively compared with another. Seasonal variation is itself a periodic or cyclical effect. However this time we will analyze it a different way.

An excellent graphic device to give a quick picture of the seasonal pattern is the "tier" graph. Here the months are laid out on the X axis without regard to year, and each year's history is charted over the same 12 months base, as follows

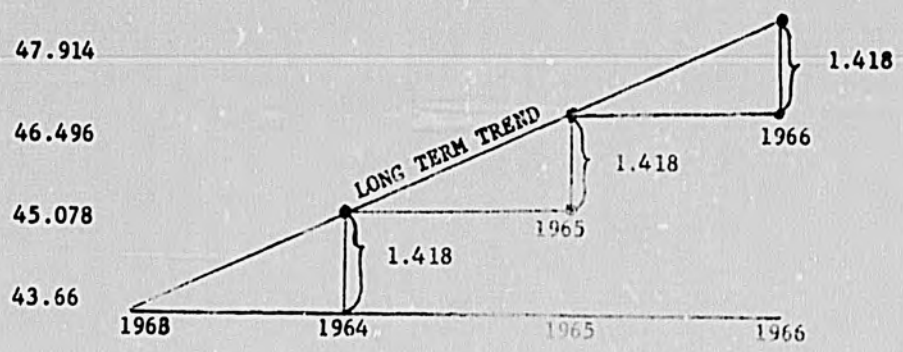
FIGURE 22 TIER GRAPH TO ILLUSTRATE SEASONAL VARIATION (HYPOTHETICAL)



Essentially, the difference between each year is the result of long term and cyclical trends while the basic general pattern reflects the seasonal trend.

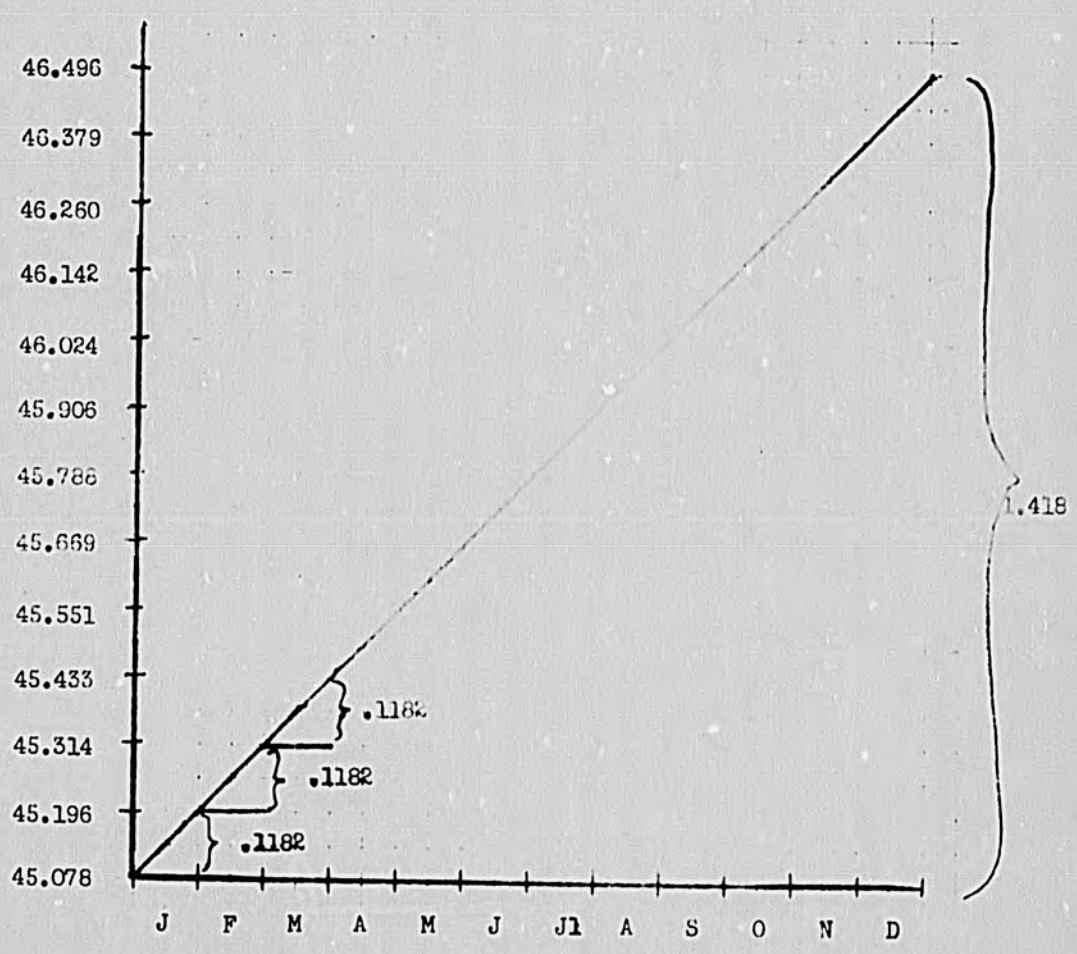
In analyzing the series annually, the base point of reference (a) was 43.66 and the annual long trend (b) was found to be 1.418

FIGURE 23 ANNUAL LONG TERM TREND



On a monthly basis this means an increment of $\frac{1.418}{12} = .1182$

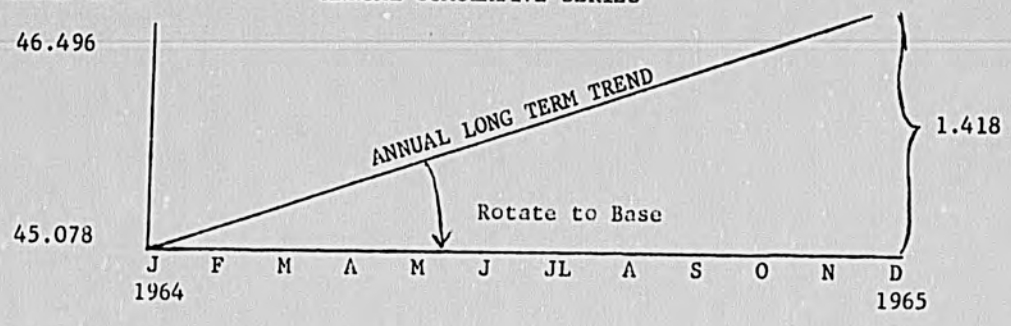
FIGURE 24 MONTHLY INCREMENT OF ANNUAL LONG TERM TREND



The Seasonal variation is the time series that is left after the Annual Long Term Trend effect has been removed. Thus, we are no longer interested in looking at the cumulative trend increment for the year (of 1.418) or even the monthly trend increment (of .1182), but rather how does one month differ from another month when all the long term effects are eliminated.

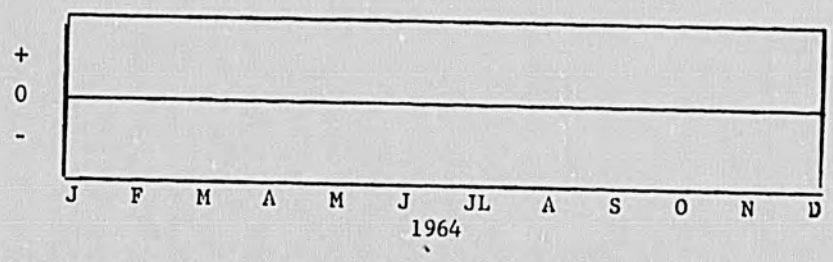
In order to do this, we must change the annual cumulative series to a monthly stationary series. Graphically,

FIGURE 25 ANNUAL CUMULATIVE SERIES



is converted to

FIGURE 26 MONTHLY AVERAGE STATIONARY SERIES



We then compare actual monthly fluctuations of data from this base line.

Mathematically, a simple way that we can convert an annual series to a monthly stationary series is as follows:

$$\frac{\text{annual "a"}}{12} = \text{monthly "a"}$$

and

$$\frac{\text{annual "b"}}{144} = \text{monthly "b"}$$

Thus Monthly $a = \frac{43.66}{12} = 3.6383$

and Monthly $b = \frac{1.418}{144} = .0099$

The monthly "b" is then cumulatively subtracted from the actual data for each month to establish the seasonal variation.

Although this is the simplest, it is the least accurate method.

The actual initial point and slope can be found by computing the entire table of data in monthly increments over the total time period and solving for"a" and"b"using the following equations:-

$$a = \frac{\Sigma Y}{N} - \frac{\Sigma X}{N} \left(\frac{N\Sigma XY - \Sigma X \Sigma Y}{N\Sigma X^2 - (\Sigma X)^2} \right)$$

and

$$b = \frac{N\Sigma XY - \Sigma X \Sigma Y}{N\Sigma X^2 - (\Sigma X)^2}$$

However, this is an extremely laborious, time consuming effort, and is not recommended unless computers are available to perform the calculations.

FIGURE 27 DATA FOR SEASONAL TREND

TIME	X	Y	XY	X^2	Y^2
1964 Jan.	1	3	3	1	9
Feb.	2	3	6	4	9
Mar.	3	4	12	9	16
Apr.	4	4	16	16	16
May	5	3	15	25	9
June	6	5	30	36	25
July	7	4	28	49	16
Aug.	8	3	24	64	9
Sept.	9	3	27	81	9
Oct.	10	4	40	100	16
Nov.	11	6	66	121	36
Dec.	12	5	60	144	25
1965 Jan.	13	3	39	169	9
Feb.	14	2	28	196	4
Mar.	15	3	45	225	9
Apr.	16	4	64	256	16
May	17	4	68	289	16
June	18	5	90	324	25
July	19	4	76	361	16
Aug.	20	3	60	400	9
Sept.	21	3	63	441	9
Oct.	22	4	88	484	16
Nov.	23	6	138	529	36
Dec.	24	6	144	576	36
1966 Jan.	25	3	75	625	9
Feb.	26	2	52	676	4
Mar.	27	4	108	729	16
Apr.	28	4	112	784	16
May	29	3	87	841	9
June	30	5	150	900	25
July	31	4	124	961	16
Aug.	32	3	96	1024	9
Sept.	33	3	99	1089	9
Oct.	34	4	136	1156	16
Nov.	35	6	210	1225	36
Dec.	36	6	216	1296	36
1967 Jan.	37	4	148	1369	16
Feb.	38	3	114	1444	9
Mar.	39	4	156	1521	16
Apr.	40	3	120	1600	9
May	41	3	123	1681	9
June	42	5	210	1764	25
July	43	3	129	1849	9
Aug.	44	4	176	1936	16
Sept.	45	3	135	2025	9
Oct.	46	4	184	2116	16
Nov.	47	7	329	2209	49
Dec.	48	5	240	2304	25
1968 Jan.	49	4	196	2401	16
Feb.	50	3	150	2500	9
Mar.	51	3	153	2601	9
Apr.	52	4	208	2704	16
May	53	3	159	2809	9
June	54	4	216	2916	16
July	55	3	165	3025	9
Aug.	56	4	224	3136	16
Sept.	57	4	228	3249	16
Oct.	58	5	290	3364	25
Nov.	59	6	354	3481	36
Dec.	60	5	300	3600	25
1969 Jan.	61	4	244	3721	16
Feb.	62	4	248	3844	16
Mar.	63	4	252	3969	16
Apr.	64	3	192	4096	9
May	65	3	195	4225	9

TIME		X	Y	XY	X^2	Y^2
	June	66	5	330	4356	25
	July	67	4	268	4489	16
	Aug.	68	4	272	4624	16
	Sept.	69	3	207	4761	9
	Oct.	70	5	350	4900	25
	Nov.	71	5	355	5041	25
	Dec.	72	6	432	5184	36
1970	Jan.	73	4	292	5329	16
	Feb.	74	5	370	5476	25
	Mar.	75	4	300	5625	16
	Apr.	76	5	380	5776	25
	May	77	4	308	5929	16
	June	78	5	390	6084	25
	July	79	4	316	6241	16
	Aug.	80	3	240	6400	9
	Sept.	81	4	324	6561	16
	Oct.	82	5	410	6724	25
	Nov.	83	6	498	6889	36
	Dec.	84	7	588	7056	49
1971	Jan.	85	5	425	7225	25
	Feb.	86	5	430	7396	25
	Mar.	87	5	435	7569	25
	Apr.	88	3	264	7744	9
	May	89	4	356	7921	16
	June	90	6	540	8100	36
	July	91	5	455	8281	25
	Aug.	92	4	368	8464	16
	Sept.	93	3	279	8649	9
	Oct.	94	5	470	8836	25
	Nov.	95	7	665	9025	49
	Dec.	96	7	672	9216	49
1972	Jan.	97	5	485	9409	25
	Feb.	98	4	392	9604	16
	Mar.	99	5	495	9801	25
	Apr.	100	4	400	10000	16
	May	101	4	404	10201	16
	June	102	4	408	10404	16
	July	103	5	515	10609	25
	Aug.	104	5	520	10816	25
	Sept.	105	4	420	11025	16
	Oct.	106	4	424	11236	16
	Nov.	107	6	642	11449	36
	Dec.	108	8	864	11664	64
1973	Jan.	109	5	545	11881	25
	Feb.	110	4	440	12100	16
	Mar.	111	5	555	12321	25
	Apr.	112	3	336	12544	9
	May	113	3	339	12769	9
	June	114	5	570	12996	25
	July	115	4	460	13225	16
	Aug.	116	4	464	13456	16
	Sept.	117	3	351	13689	9
	Oct.	118	4	47[illegible]	13924	16
	Nov.	119	6	714	14161	36
	Dec.	120	8	960	14400	64
1974	Jan.	121	5	605	14641	25
	Feb.	122	5	610	14884	25
	Mar.	123	4	492	15119	16
	Apr.	124	4	496	15376	16
	May	125	4	500	15625	16
	June	126	6	756	15876	36
	July	127	5	635	16129	25
	Aug.	128	4	512	16384	16
	Sept.	129	4	516	16641	16
	Oct.	130	5	650	16900	25
	Nov.	131	7	917	17161	49
	Dec.	132	7	924	17424	49
	Σ	8778	574	40310	774390	2674

Then, substituting,

$$a = \frac{574}{132} - \frac{3778}{132}\left(\frac{132\ (40310) - (3778)\ (574)}{132\ (744390) - (3778)^2}\right)$$

$$= 4.3485 - 66.5\left(\frac{5320920 - 5038572}{102219480 - 77053284}\right)$$

$$= 4.3485 - 66.5\left(\frac{282348}{25166196}\right)$$

$$= 4.3485 - 66.5\ (.01122)$$

$$= 4.3485 - .7461$$

$$= 3.6024$$

and $b = \dfrac{132\ (40310) - (3778)\ (574)}{132\ (774390) - (3778)^2}$

which from the above

$$= .01122$$

The Worksheet is developed as follows:

FIGURE 28

ANALYSIS OF SEASONAL VARIATION

Month A	1964-74 Total B	11 Yr. Monthly Average C	Unadjusted Index D	Cumulative Monthly Trend Increment .01122 E	Adjusted Monthly Average F	Adjusted Seasonal Index G
January	45	4.09	94.02	0	4.09	95.34
February	40	3.64	83.63	.01122	3.63	84.62
March	45	4.09	94.02	.02244	4.07	94.87
April	41	3.73	85.75	.03366	3.70	86.25
May	38	3.46	79.54	.04488	3.42	79.72
June	55	5.0	114.94	.05610	4.94	115.15
July	45	4.09	94.02	.06732	4.02	93.71
August	41	3.73	85.75	.07854	3.65	85.08
September	37	3.36	77.24	.08976	3.27	76.22
October	49	4.46	102.53	.10098	4.36	101.63
November	68	6.18	142.07	.11220	6.07	141.49
December	70	6.36	146.21	.12342	6.24	145.46
TOTAL		52.19			51.46	
MONTHLY AVERAGE FOR ALL MONTHS COMBINED		4.35			4.29	

An explanation of some of the columns and data elements is in order:-

B. Total This is the totals by month for the entire period.

C. Monthly Average = $\frac{\text{Total}}{\text{\# of times the monthly data was reported}}$

Monthly Average for all months combined = $\frac{\Sigma \text{ Monthly Averages}}{12}$ in this instance $\frac{52.19}{12} = 4.35$

Note: Since this is a long term average, the effect of erratic variation is smoothed out.

D. Unadjusted Index This represents the seasonal activity of the variable under study for each month, compared with the monthly average. Since the data upon which it has been computed was the basic raw data, the effect of long term cyclical trends, and erratic variation is still included. Hence the term "unadjusted." It is calculated as follows:

Unadjusted Index = $\frac{\text{Average for Month}}{\text{Average for all months}} \times 100$ for example $\frac{4.09}{4.35} \times 100 = 94.66$

E. Cumulative Monthly Trend Increment The long term trend for this data series was calculated earlier as .01122 per month.

F. Adjusted Monthly Average

By subtracting the cumulative long term trend from the monthly average, what remains in the series is then solely the seasonal effect. Note however, that no adjustment is made for the first month, as this is the base point of each annual increment.

Since the average monthly trend data is cumulative throughout the year, it is subtracted cumulatively throughout the year from the original monthly average data in order to arrive at an Adjusted Monthly Average. The Adjusted Monthly Average for each month is totalled for the year, and an average developed of these.

In this instance the total is 51.46, with a monthly average of

$$\frac{51.46}{12} = 4.29$$

G. Adjusted Seasonal Index

The final step in calculating the seasonal variation (adjusted for the long term and cyclical trend effects), is to develop the Adjusted Seasonal Index as follows:

$$\text{Adjusted Seasonal Index} = \frac{\text{Adjusted Monthly Average}}{\text{Average Adjusted Monthly Average for all months combined}} \times 100$$

For example, the Seasonal Adjusted Index for January is

$$\frac{4.09}{4.29} \times 100$$

$$= .95337 \times 100$$

$$= 95.34$$

An important effect of adjusting the seasonal index in this manner is that with a long term trend for increase, the relative weight of occurences earlier in the year is increased. For example, an original monthly average of 4.09 in January is now weighted higher than the same value in July.

Erratic Variation. As stated earlier, by its nature, the erratic variation is hard to deal with. If the contribution of a previous erratic variation can be estimated, the index can be adjusted for it. Otherwise, adjustment is only made for the long term and cyclical variation.

SEASONAL VARIATION RATIO TO MOVING AVERAGE METHOD

This method attempts to find the seasonal variation by the following formula

$$O = \frac{T \times C \times S \times I}{T \times C} = \frac{\text{Original Data}}{\text{Moving Average}} = S \times I$$

Where

O = Output Level for Y
T = Long Range Trend
C = Cyclical Trend
S = Seasonal Variation
I = Erratic Variation

Using the data in FIGURE I again, the seasonal (monthly) data can be analyzed using this method as follows:

FIGURE 29 WORKSHEET FOR COMPUTING MOVING AVERAGES

A	B	C	D	E	F = B/E x 10
Month	Original Data TCSI	12 Month Moving Total	12 Month Moving Average	Centered 12 Month Moving Average T x C	S (I)
1964 J	3				
F	3				
M	4				
A	4				
M	3				
J	5				
Jl	4	47	3.917		
A	3	47	3.917	3.917	102.119
S	3	46	3.833	3.875	77.419
O	4	45	3.75	3.792	79.114
N	6	45	3.75	3.75	106.667
D	5	46	3.833	3.792	158.228
		46	3.833	3.833	130.446

Column C The 12 month moving total is calculated by summing the 12 month period 6 months behind and six months ahead.

Column D The average is found by dividing "C" by 12. Note both these figures are placed between the months -- the first number is between June and July.

Column E The Average of the two "off-center" months is found and entered between them, thus placing the data back on line with a month.

Column F The monthly index is calculated by dividing column B by Column E and multiplying by 100. These monthly indexes are then reconsolidated in another worksheet.

A	B	C	D	E	F
1965 J	3	46	3.833	3.833	78.268
F	2	46	3.833	3.833	52.179
M	3	46	3.833	3.833	78.268
A	4	46	3.833	3.833	104.357
M	4	46	3.833	3.833	104.357
J	5	47	3.917	3.875	129.032
Jl	4	47	3.917	3.917	102.119
A	3	47	3.917	3.917	76.589
S	3	48	4.	3.959	75.777
O	4	48	4.	4.	100
N	6	47	3.917	3.959	151.553
D	6	47	3.917	3.917	153.179
1966 J	3	47	3.917	3.917	76.589
F	2	47	3.917	3.917	51.06
M	4	47	3.917	3.917	102.119
A	4	47	3.917	3.917	102.119
M	3	47	3.917	3.917	76.589
J	5	47	3.917	3.917	127.649
Jl	4	48	4	3.957	101.087
A	3	49	4.083	4.042	74.221
S	3	49	4.083	4.083	73.475
O	4	48	4	4.042	98.961
N	6	48	4	4	150
D	6	48	4	4	150
1967 J	4	47	3.917	3.959	101.036
F	3	48	4	3.959	75.777
M	4	48	4	4	100
A	3	48	4	4	75
M	3	49	4.083	4.042	74.221
J	5	48	4	4.042	123.701
Jl	3	48	4	4	75
A	4	48	4	4	100
S	3	47	3.917	3.957	75.815
O	4	48	4	3.957	101.087
N	7	48	4	4	175
D	5	47	3.917	3.957	126.358

A	B	C	D	E	F
1968 J	4			3.917	102.119
		47	3.917		
F	3			3.917	76.589
		47	3.917		
M	3			3.957	75.815
		48	4		
A	4			4.042	98.961
		49	4.083		
M	3			4.042	74.221
		48	4		
J	4			4	100
		48	4		
Jl	3			4	75
		48	4		
A	4			4.042	98.961
		49	4.083		
S	4			4.125	96.97
		50	4.167		
O	5			4.125	121.212
		49	4.083		
N	6			4.083	146.951
		49	4.083		
D	5			4.125	121.212
		50	4.167		
1969 J	4			4.209	95.034
		51	4.25		
F	4			4.25	94.118
		51	4.25		
M	4			4.209	95.034
		50	4.167		
A	3			4.167	71.994
		50	4.167		
M	3			4.125	72.727
		49	4.083		
J	5			4.125	121.212
		50	4.167		
Jl	4			4.167	95.992
		50	4.167		
A	4			4.209	95.034
		51	4.25		
S	3			4.25	70.588
		51	4.25		
O	5			4.335	115.34
		53	4.417		
N	5			4.459	112.133
		54	4.5		
D	6			4.5	133.333
		54	4.5		
1970 J	4			4.5	88.889
		54	4.5		
F	5			4.459	112.133
		53	4.417		
M	4			4.459	89.706
		54	4.5		
A	5			4.5	111.111
		54	4.5		
M	4			4.542	88.067
		55	4.583		
J	5			4.625	108.108
		56	4.667		
Jl	4			4.709	84.944
		57	4.75		
A	3			4.75	63.158
		57	4.75		
S	4			4.792	83.473
		58	4.833		
O	5			4.75	105.263
		56	4.667		
N	6			4.667	128.562
		56	4.667		
D	7			4.709	148.652
		57	4.75		

A	B	C	D	E	F
1971 J	5	58	4.833	4.792	104.341
F	5	59	4.917	4.875	102.564
M	5	58	4.833	4.875	102.564
A	3	58	4.833	4.833	62.073
M	4	59	4.917	4.875	82.051
J	6	59	4.917	4.917	122.026
Jl	5	59	4.917	4.917	101.683
A	4	58	4.833	4.875	82.051
S	3	58	4.833	4.833	62.073
O	5	59	4.917	4.875	102.564
N	7	59	4.917	4.917	142.363
D	7	57	4.75	4.834	144.808
1972 J	5	57	4.75	4.75	105.263
F	4	58	4.833	4.792	83.473
M	5	59	4.917	4.875	102.564
A	4	58	4.833	4.875	82.051
M	4	57	4.75	4.792	83.473
J	4	58	4.833	4.792	83.473
Jl	5	58	4.833	4.833	103.455
A	5	58	4.833	4.833	103.455
S	4	58	4.833	4.833	82.764
O	4	57	4.75	4.792	83.473
N	6	56	4.667	4.709	127.416
D	8	57	4.75	4.709	169.887
1973 J	5	56	4.667	4.709	106.18
F	4	55	4.583	4.625	86.487
M	5	54	4.5	4.542	110.084
A	3	54	4.5	4.5	66.667
M	3	54	4.5	4.5	66.667
J	5	54	4.5	4.5	111.111
Jl	4	54	4.5	4.5	88.889
A	4	55	4.583	4.542	88.067
S	3	54	4.5	4.542	66.050
O	4	55	4.583	4.542	88.067
N	6	56	4.667	4.625	129.73
D	8	57	4.75	4.709	169.887

A	B	C	D	E	F
1974 J	5			4.792	104.341
		58	4.833		
F	5			4.833	103.455
		58	4.833		
M	4			4.875	82.051
		59	4.917		
A	4			4.959	80.661
		60	5		
M	4			5.042	79.334
		61	5.083		
J	6			5.042	119
		60	5		
Jl	5				
A	4				
S	4				
O	5				
N	7				
D	7				

FIGURE 30

SUMMARY WORKSHEET FOR SEASONAL VARIATION

	J	F	M	A	M	J	JL	A	S	O	N	D
1964							102.119	77.419	79.114	106.667	158.223	130.446
65	78.268	52.179	78.268	104.357	104.357	129.032	102.119	76.589	75.777	100	151.553	153.179
66	76.589	51.06	102.119	102.119	76.589	127.649	101.087	74.221	73.475	98.961	150	150
67	101.036	75.777	100	75	74.221	123.701	75	100	75.315	101.087	175	126.358
68	102.119	76.589	75.315	98.961	74.221	100	75	98.961	96.97	121.212	146.951	121.212
69	95.034	94.118	95.034	71.994	72.727	121.212	95.992	95.034	70.588	115.34	112.133	133.333
70	88.339	112.133	89.706	111.111	88.067	108.108	84.944	68.158	88.473	105.263	128.562	148.652
71	104.341	102.564	102.564	62.073	82.051	122.026	101.688	82.051	62.073	102.564	147.368	144.808
72	105.263	83.473	102.564	82.051	83.473	83.473	103.455	103.455	82.764	83.473	127.416	169.837
73	106.13	86.687	110.034	66.667	66.667	111.111	88.339	88.067	66.050	88.067	129.73	169.837
74	104.341	103.455	82.051	80.661	79.334	119						
TOTAL	962.06	837.335	938.205	854.994	801.707	1145.312	930.293	858.955	766.099	1022.634	1421.936	1447.762
MEANS	96.206	83.734	93.821	85.499	80.171	114.531	93.029	85.896	76.61	102.263	142.194	144.776
LEVELED SEASONAL INDEX		83.869	93.917	85.536	80.253	114.648	93.124	85.984	76.688	102.367	142.339	144.923
INDEX(S)		83.9	93.9	85.6	80.3	114.7	93.1	86.0	76.7	102.4	142.3	144.9

In this table, by averaging the monthly indexes for the total period, the erratic variation is eliminated, thus leaving only the seasonal variation.

The sum of the seasonal indexes should add up to 1200. If it does not, due to rounding errors, each index is adjusted by a factor that uses 1200 as the base. For example, in this instance the sum of the indexes is 1198.78, thus the adjustment factor is

$$\frac{1200}{1198.78} = 1.0010177$$

Then each mean is multiplied by this adjustment factor.

Finally, it is rounded off to 1 decimal place.

FORECASTING

Now, equipped with data for the various components of our time series - Long Term Trend, Cyclical Trend and Seasonal Variation -- disregarding erratic variation, - we can put the parts back together again to build a forecast for 1975, as follows:-

Long Term Trend 1.42 per annum, or .129 per month. Since 60.74 is projected for 1975, the projection excluding the long term trend is

$$60.74 - 1.42 = 59.32$$

$$\text{or} \quad \frac{59.32}{12} = 4.943 \text{ per month}$$

Cyclical Trend 3.3836% of the long term trend. Monthly, this would be

$$\frac{3.3336}{100} \times .129$$

$$= .033336 \times .129$$

$$= .005\% \text{ per month}$$

Seasonal Level calculated from the seasonal variation as indicated in the table, (FIGURE 28) on page 50.

Thus, at last the 1975 forecast can be calculated as follows:

FIGURE 31 WORKSHEET FOR CONSTRUCTING 1975 FORECAST

A	B	C	D	E	F = C+D+E
Month	Adjusted Seasonal Index	Predicted 1975 Seasonal Level	Monthly Trend	Average Cyclical Increment	Projected Level 1975
January	95.34	4.713	.129	.005	4.847
February	34.62	4.183	.129	.005	4.317
March	94.37	4.689	.129	.005	4.823
April	36.25	4.263	.129	.005	4.397
May	79.72	3.941	.129	.005	4.075
June	115.15	5.692	.129	.005	5.826
July	93.71	4.632	.129	.005	4.766
August	35.08	4.206	.129	.005	4.34
September	76.22	3.768	.129	.005	3.902
October	101.63	5.024	.129	.005	5.158
November	141.49	6.994	.129	.005	7.128
December	145.46	7.190	.129	.005	7.324
	Total	59.32			60.903
	Average	4.943			

CONFIDENCE AND CONTROL LIMITS

From our earlier work in statistics, we know that a range of ± 2 standard deviations will provide a confidence of 95.44%. This range will also serve adequately as a control limit for monitoring the activity.

We should calculate the standard deviation from our original 11 year monthly average values for Y, and apply it to the 1975 projections.

FIGURE 32 WORKSHEET FOR CALCULATING STANDARD DEVIATION OF MONTHLY AVERAGES

Y	Y^2
4.09	16.7281
3.64	13.2496
4.09	16.7281
3.73	13.9129
3.46	11.9716
5.0	25.0
4.09	16.7281
3.73	13.9129
3.36	11.2896
4.46	19.8916
6.18	38.1924
6.36	40.4496
$\Sigma Y = 52.19$	$\Sigma Y^2 = 238.0545$

$$\text{Mean} = \frac{52.19}{12} = 4.35$$

Substituting in the following formula for the Standard Deviation

$$SD = \sqrt{\frac{\Sigma X^2 - \frac{(\Sigma X)^2}{N}}{N}} \quad \text{or} \quad \sqrt{\frac{\Sigma X^2 - \frac{(\Sigma X)^2}{N}}{N - 1}}$$

Where N is less than 30

Substituting our ΣY and ΣY^2 data values for ΣX and ΣX^2

$$SD = \sqrt{\frac{238.0545 - \frac{(52.19)^2}{12}}{12 - 1}}$$

$$= \sqrt{\frac{238.0545 - \frac{2723.7961}{12}}{11}}$$

$$= \sqrt{\frac{238.0545 - 226.983}{11}}$$

$$= \sqrt{\frac{11.0715}{11}}$$

$$= \sqrt{1.0065}$$

$$= 1.0033$$

Thus 2 Standard Deviations for 95.44% Confidence

$= 1.0033 \times 2$

$= 2.0066$ or 2.01 rounded off.

This was based on a mean of 4.35.

Thus the percentage variation for 95.44% confidence is $\frac{2.01}{4.35}$

$= .4620689 \times 100$

$= 46.21\%$

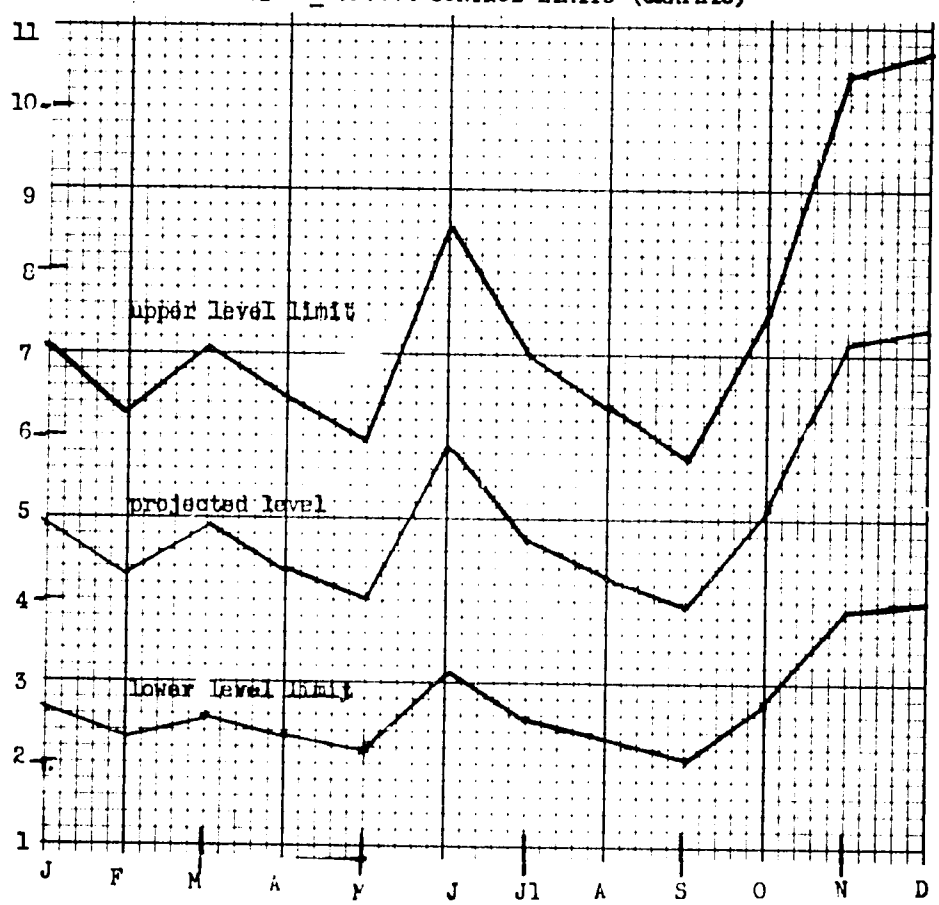
We can use this percentage to determine the lower and upper limits of our projected level of activity for 1975, as indicated on the next page.

FIGURE 33 PROJECTED SEASONAL ACTIVITY LEVELS FOR 1975 WITH ± 95.44% CONTROL LIMITS (TABULAR)

Month	Projected Level	46.21% of Projected Level	-2SD	+2SD
January	4.847	2.240	2.607	7.087
February	4.317	1.995	2.322	6.312
March	4.823	2.229	2.594	7.052
April	4.397	2.032	2.365	6.429
May	4.075	1.883	2.192	5.958
June	5.826	2.692	3.134	8.518
July	4.766	2.202	2.564	6.968
August	4.340	2.006	2.334	6.346
September	3.902	1.803	2.099	5.705
October	5.158	2.384	2.774	7.542
November	7.128	3.294	3.834	10.422
December	7.324	3.384	3.940	10.708

Graphically, these control limits can be shown as follows

FIGURE 34 PROJECTED SEASONAL ACTIVITY LEVELS FOR 1975 WITH ± 95.44% CONTROL LIMITS (GRAPHIC)



ALTERNATE METHOD FOR ESTIMATING STANDARD DEVIATION FROM MEAN RANGE

Given an average range from a small sample, it is possible to estimate the Standard Deviation of the population using the following formulae:

1. Range = Hi - Lo

2. Estimated Standard Deviation = Range x K

Where K varies according to Sample Size as indicated below

FIGURE 35 TABLE OF CONSTANTS (K) FOR ESTIMATING STANDARD DEVIATIONS OF SELECTED SAMPLE SIZES

Sample Size	K
2	.886
3	.591
4	.486
5	.430
6	.395
7	.370
8	.351
9	.337
10	.325
12	.307
14	.294
16	.283
18	.275
20	.268
30	.245
40	.231
50	.222

Application

From FIGURE 32, the data range for Y is from a low of 3.36 to a high of 6.36; a range of 3. The "sample size" is 12 so from FIGURE 35 K is .307. Thus the estimated standard deviation is

ESD = 3 x .307 = .921

And ± 2 standard deviations for control purposes would be .921 x 2 = 1.842 Proceeding as before, based on a mean of 4.35 the percentage of variation for 95.44% confidence is

$$\frac{1.842}{4.35} \times 100$$

$$= .4235 \times 100$$

$$= 42.35\%$$

which could be used as the lower and upper limits of the projected activity level for 1975. Note in this instance the short cut method is slightly lower than the percentage derived by the long method.

SUMMARY

In the preceding pages, we have dissected a series of data into its component parts to estimate how much was due to the long range trend, how much to the cyclical trend, and how much to the seasonal variation.

The final step was to put the pieces back together again with a projection for the future.

One caution should be observed in all of this type of analysis. Although it is possible to identify trends and mathematically project them into the future with "limits of prediction" to be used as guides for further management action, there is no guarantee that things will turn out as predicted. They are merely guesstimates albeit educated ones. In analyzing the data, a healthy dose of common sense must be applied to check that the findings are reasonable.

How can the manager use such information? In several ways:-

1. In the first place, he should recognize that Y is tending to increase over the long run. Given this fact, does anything need to be done about it? Is it a desirable trend or undesirable? Should we try to encourage it or curtail it? Can the situation accomodate it? What do we know that we can do to try to influence it?

2. Knowing that Y is not constant during the year but peaks at certain times, is this desirable? If we wish to curtail Y, priority areas for investigation are the December, November, June, January, and March time periods, in that order since these are the peak months.

3. For the coming year, we can analyze current operations to determine whether Y continues as expected, within the control limits, or whether major changes occur. Without such analysis, comparing raw data of one month with another, errors of judgement are often made because apparent increases may only be long term trends or seasonal variations rather than actual improvements in conditions. Similarly, apparent declines may not be cause for concern if they can be identified as seasonal effects.

4. Forecasts can be made for the coming year, as guides for management programming and support operations.

In short, time series analysis presents management with information which it needs for decision making. Although admittedly imperfect until someone comes up with a better crystal ball, it is probably the best basis upon which to formulate judgements about the future, for "those who ignore history are doomed to repeat its mistakes."

INDEX NUMBERS

Index Numbers are a special kind of average which express the state of a situation as a percentage compared to a previous situation. The time period with which it is compared is called a base. Thus each index has a base period.

INDEXES facilitate the comparison and analysis of different series of data over time. By establishing a base period at which all series of data to be compared are given the same base, and then comparing the fluctuations, significant differences can often be noted.

For example, for a given period you might wish to compare the production of palay, the price of palay, the price of fertilizer and the consumption of rice.

The value of the data at most bases is set at 100, however this is primarily for convenience and is not an absolute necessity. The actual data in its raw form could be used. Most indexes are carried to one decimal place, i.e .1

Finally, an index can also be developed to express the net change in a composite group of items.

The situation under observation is usually a complex one for which no real direct comparison exists. For instance - the Cost of Living Index, which is composed of many different elements.

The procedure in establishing an index is to define a "standard" situation. For example, we might want to identify the change in the net income of a typical small scale rice farm family since the introduction of the Masagana 99 program. To establish this base, we could run a sample survey to determine the characteristics of such a family, in any terms that we wish to measure from all the sources of income and all the outlets for expenditure; reduce them all to a relative weight and establish it as the base line of 100. Over time we would then check the changes that take place.

Many indexes are well established. Developing and maintaining time series data is an important prerequisite to the use of index numbers.

Important considerations in Constructing an Index

1. Selecting the base This should be established as a "normal" period. Unusual situations are not good times to choose as bases, as all the comparative data will be distorted. Bases may be changed from time to time as earlier established bases become obsolete with changes in patterns of living, inflation, and consumption habits.

2. Selection of Items for the Index Which items are going to be monitored as representative of the typical production or consumption pattern? It is impossible to get everything! As habits change, the weightings given some items may also have to be changed, or a new base and time series established.

The following pages illustrate how different indexes can be developed.

CALCULATING PERCENTAGE CHANGES

The percentage change from the base period to the period under study can be determined as follows

$$\text{Percent Change} = \left(\frac{\text{Study Index}}{\text{Base Index}} \times 100\right) - 100$$

Thus given a study index of 27
and a base index of 31
the percent change

$$= \left(\frac{27}{31} \times 100\right) - 100$$

$$= (.8709677 \times 100) - 100$$

$$= 87.09677 - 100$$

$$= -12.90323 \text{ or } -12.9\% \text{ rounded off}$$

Similarly, given a study index 42
and a base index 37
the percent change

$$= \left(\frac{42}{37} \times 100\right) - 100$$

$$= (1.1351351 \times 100) - 100$$

$$= 113.51351 - 100$$

$$= +13.51351 \text{ or } 13.5\% \text{ rounded off}$$

Where the base index is 100, the percentage change can be read almost directly as follows

$$\text{Percent Change} = \text{Study Index} - \text{Base Index}$$

for example

Given Study Index 123
and a Base Index 100
Percentage Change = 123 - 100

= 23%

This can be verified with the general formula

$$\text{Percent Change} = \left(\frac{123}{100} \times 100\right) - 100$$

$$= (1.23 \times 100) - 100$$

$$= 123 - 100$$

$$= 23\%$$

and Given a Study Index 97
and a Base Index 100
Percent Change = 97 - 100

= -3%

which can also be verified with the general formula

$$\text{Percent Change} = \left(\frac{97}{100} \times 100\right) - 100$$

$$= (.97 \times 100) - 100$$

$$= 97 - 100$$

$$= -3\%$$

CONSTRUCTION OF INDEXES

RELATIVE INDEX NUMBERS (also known as SIMPLE or UNWEIGHTED indexes)

Given the series of data in FIGURE 36 for time period 1970 through 1975 and arbitrarily selecting 1972 as the base,

FIGURE 36 TIME SERIES DATA

Year	Data Value
1970	46
1971	43
1972	49 ⇐ BASE
1973	50
1974	49
1975	51

The relative indexes can be computed as follows: -

(1972) Base Index $= \frac{49}{49} \times 100$

$= 1 \times 100$

$= 100$

(1970) Relative Index $= \frac{46}{49} \times 100$

$= .9387755 \times 100$

$= 93.87755$ or 93.9

(1975) Relative Index $= \frac{51}{49} \times 100$

$= 1.0408163 \times 100$

$= 104.08163$ or 104.1

and so on, as summarized in FIGURE 37

FIGURE 37 COMPUTING RELATIVE INDEXES FROM DATA VALUES

Time Period	Data Value	Relative Index
1970	46	93.9
1971	43	93.0
1972	49	100
1973	50	102.
1974	49	100
1975	51	104.1

Combining Several Items

Assume an individual's food consumption consisted typically of:

Rice
Fish
San Miguel Beer
Pork

and that the unweighted prices for 1975 (hypothetical) were compared against those of 1970 (hypothetical) as follows

FIGURE 37 DATA FOR DEVELOPING AN INDEX (UNWEIGHTED)

Item & Quantity	1970	1975
Rice (kilo)	2.30	1.95
Fish (kilo)	5.00	17.30
San Miguel Beer (liter)	2.00	2.00
Pork (kilo)	16.00	22.00
	1970 = 25.30	1975 = 43.25

The unweighted index for 1975 would be computed thus

$$\text{Unweighted Index} = \frac{(1975)\ \mathcal{E}}{(1970)\ \mathcal{E}} \times 100$$

$$= \frac{43.25}{25.30} \times 100$$

$$= 170.94361 \text{ or } 171 \text{ rounded off}$$

The unweighted method is not really a good measure in this instance because a major item - rice-decreased in price and beer stayed the same, while increases occured only in the items which are consumed less frequently and in less quantity.

To rectify this situation, more weight must be given to those items which are more heavily consumed.

To establish weighted index, we must analyze the total amount spent for the above items during the base time period as follows:

FIGURE 38 WORKSHEET FOR WEIGHTING AN INDEX

Item A	1970 Price B	Amount Consumed C	Total Amount Spent 1970 D = B x C	1975 Price E	Total Amount Spent (1975 F = E x C
Rice (kilo)	2.30	105	241.50	1.95	204.75
Fish	5.00	20	100.00	17.30	346.00
San Miguel (liter)	2.00	400	800.00	2.00	800.00
Pork (kilo)	16.00	10	160.00	22.00	220.00
		$\mathcal{E}$ 1970 =	1301.50	$\mathcal{E}$ 1975 =	1570.75

The weighted index for 1975 is then $\frac{\mathcal{E}\ 1975}{\mathcal{E}\ 1970} \times 100$

$$= \frac{1570.75}{1301.50} \times 100$$

$$= 120.68766 \text{ or } 120.7 \text{ rounded off}$$

Thus although there has been an increase due to the tremendous increases in the price of fish and pork, the overall effect is not quite as bad as it would have appeared in the unweighted form.

Of course, the above assumes that the consumption of each of the items remained the same during both periods. This is done to measure the effect of price change alone. (Known as the Laspeyres Method). Because the increase in fish and pork was so high, however, in actual practice, people would probably change their eating habits to match their income.

QUANTITY INDEX

A quantity index is an attempt to measure the change in volume of production in monetary terms of an item compared with a base period.

The Laspeyres Method is also used to compute quantity indexes; however in this situation it is the price in the base year which is held constant.

For example, consider the following hypothetical situation

FIGURE 39 WORKSHEET FOR QUANTITY INDEX OF PRODUCTION

Item	1970 Price	1970 Quantity Produced	1970 Production In Monetary Terms	1975 Production	1975 Production In Constant (1970) Monetary Terms
A	B	C	D = B x C	E	F = B x E
Palay (cav)	16 pesos	95 million	1.52 billion	136 M	2.176 billion

Then the Weighted Quantity index of palay production for 1975 compared with 1970 is

$$= \frac{F}{D} \times 100$$

$$= \frac{2.176}{1.52} \times 100$$

$$= 1.4315789 \times 100$$

$$= 143.15789 \text{ or } 143.2 \text{ rounded off}$$

COMPOSITE QUANTITY INDEXES OF PRODUCTION

Composite quantity indexes of production can also be developed in the same manner, thus

FIGURE 40 WORKSHEET FOR COMPOSITE QUANTITY INDEX OF PRODUCTION (HYPOTHETICAL)

Item	1969 Price	1969 Quantity Produced	1969 Production In Monetary Terms	1974 Production	1974 Production In Constant (1969) Monetary Terms
A	B	C	D = B x C	E	F = B x E
Palay (cav)	15	98	1470	103	1545
Mangoes (100)	100	62	6200	54	5400
Cocorut (100)	55	50	2750	51	2805
		Σ D 1969 =	10420	Σ F 1974 =	9750

Then the weighted quantity index of composite agricultural production for 1974 compared with 1969 is

$$= \frac{\Sigma F}{\Sigma D} \times 100$$

$$= \frac{9750}{10420} \times 100$$

$$= 93.57 \text{ or } 93.6 \text{ rounded off}$$

VALUE INDE:

A value index measures the change in the total value of a product as follows

FIGURE 41 WORKSHEET FOR COMPUTING A VALUE INDEX

Item	1969 Price	1969 Quantity	1969 Value	1975 Price	1975 Quantity Sold	1975 Value
A	B	C	D = B x C	E	F	G = E x F
Fertilizer (3 sacks)	65 pesos	400	26000	120	275	33000

$$\text{Value of Volume of sales} = \frac{G}{D} \times 100$$

$$= \frac{33000}{26000} \times 100$$

$$= 1.2692307 \times 100$$

$$= 126.92307 \text{ or } 126.9 \text{ rounded off}$$

The weighted change in value of a group of products can be computed in a similar manner

FIGURE 42 WORKSHEET FOR COMPUTING A WEIGHTED VALUE INDEX

Item	1970 Price	1970 Quantity Sold	1970 Value	1974 Price	1974 Quantity Sold	1974 Value
A	B	C	D = B x C	E	F	G = E x F
Jeep (ea.)	2500	5	12500	4000	3	12000
Tractor (ea.)	3000	1	3000	3500	2	17000
Truck (ea)	10000	2	20000	12000	1	12000
			$\Sigma D = 40500$			$\Sigma G = 41000$

$$\text{Value of Volume of sales} = \frac{\Sigma G}{\Sigma D} \times 100$$

$$= \frac{41000}{40500} \times 100$$

$$= 1.0123456 \times 100$$

$$= 101.23456 \text{ or } 101.2 \text{ rounded off}$$

Special purposes indexes are usually a combination of a number of price, quantity and value indexes. Various indexes are selected and weighted at the users discretion to develop a composite weighted index. For example, in the table below are three indexes (hypothetical) that I will combine into a single composite index.

FIGURE 43 THREE SEPARATE INDEXES WITH DIFFERENT BASE PERIODS

Time Period	Agricultural Production Index	Consumer Price Index	Value of Retail Sales Index
1969	100	95	89
1971	105	100	96
1973	120	103	100

Shifting the Base

Each of the above indexes are measured from a different base period. Thus, the first step is to establish them all on the same period. If I establish 1971 as the base year, then both the Agricultural Production, and Retail Sales Indexes will have to be adjusted, as follows:

$$\text{New Index} = \frac{\text{Old Index of Data Item to be Changed}}{\text{Old Index of the New Base period}} \times 100$$

Thus for Agricultural Production

The new index for 1969 $= \frac{100}{105} \times 100$

$= .9523809 \times 100$

$= 95.23809$ or 95.2 rounded off

and the new index for 1973 $= \frac{120}{105} \times 100$

$= 1.1428571 \times 100$

$= 114.28571$ or 114.3 rounded off

The new indexes for the Retail Sales can be calculated in a similar manner.

The new index for 1969 $= \frac{89}{96} \times 100$

$= .9270833 \times 100$

$= 92.70833$ or 92.7 rounded off

and the new index for 1973 $= \frac{100}{96} \times 100$

$= 1.0416666 \times 100$

$= 104.16666$ or 104.2 rounded off

These are summarized in FIGURE 44, below

FIGURE 44 THREE SEPARATE INDEXES ADJUSTED TO THE SAME BASE PERIOD

Time Period	Agricultural Production Index	Consumer Price Index	Value of Retail Sales Index
1969	95.2	95	92.7
1971	100	100	100
1973	114.3	103	104.2

The final stage is to assign relative weightings to each of the indexes in the composite, based on judgement and understanding of the purpose that the composite is supposed to serve. The next figure shows an assignment of weightings to these three indexes, and the calculation of the composite index.

FIGURE 45 CALCULATION OF COMPOSITE INDEX FROM THREE SEPARATE INDEXES

Time Period	Agricultural Production Index		Consumer Price Index		Value of Retail Sales Index		Composite Index
		Weight 45%		Weight 30%		Weight 25%	
A	B	C	D	E	F	G	H= C+E+G
1969	95.2	42.3	95	28.5	92.7	23.2	94.5
1971	100	45	100	30	100	25	100
1973	114.3	51.4	103	30.9	104.2	26.1	108.4

THE CONSUMER PRICE INDEX

The Consumer Price Index measures the change in prices of selected goods and services from one time period to another. Among other things, it is a useful indicator of an average individual's purchasing power or "Real Income."

For example

FIGURE 46 CONSUMER PRICE INDEX (HYPOTHETICAL) USED TO MEASURE "REAL" INCOME

Year A	Average Income B	Consumer Price Index C	Real Income $D = \frac{B}{C} \times 100$
1969	3000	100	3000
1971	4500	150	3000
1973	6000	220	2727.27

Real Income is calculated by

$$= \frac{\text{Average Income}}{\text{Consumer Price Index}} \times 100$$

Thus, from the 1969 base, although the average individual earned more cash in 1971, his cost of living also went up by the same amount so he broke even in terms of "real"income. In 1973, he received another pay increase, however his cost of living increased more than before, and the net result is that he has less purchasing power than he had in 1969! This is sometimes expressed as the current value of the peso compared to the base period as follows:

$$P1.00 \times \frac{\text{consumer price index for time period being compared}}{\text{consumer price index of time period under study}}$$

For example, from the above data (hypothetical) the 1973 peso could be compared with the 1969 peso as being worth only

$$P1.00 \times \frac{100}{200}$$

$$= P1.00 \times .4545$$

$$= P0.4545 \text{ or } P0.46 \text{ centavos, rounded off}$$

and compared with the 1971 peso as being worth only

$$P1.00 \times \frac{150}{220}$$

$$= P1.00 \times .6818$$

$$= P0.6818 \text{ or } P0.68 \text{ rounded off}$$

Thus, if you didn't get a pay raise <u>to equal the increase in the price index</u> you can see how much you are <u>losing</u> each year through inflation.

SUMMARY REVIEW FOR CALCULATING LOGARITHMS

From the laws of indices,

Multiplication

$$a^m \times a^n = a^{m+n}$$ The indices are added

Division

$$a^m \div a^n = a^{m-n}$$ The indices are subtracted

Powers

$$(a^m)^n = a^{m \times n}$$ The indices are multiplied

$$(ab)^n = a^n \times b^n$$

$$a^{\frac{x}{n}} = \sqrt[n]{a^x}$$

$$a^0 = 1$$ Whatever the value of a

Negative Indices

$$a^{-n} = \frac{1}{a^n}$$

Scientific Notation

Occasionally, you will see numbers written in Scientific Notation. In effect, this is using the Base 10 as a reference point to express other numbers. The table below is provided as guidance on converting these numbers.

Number \ Scientific Notation at head of column × Number × number in column	10^{-3}	10^{-2}	10^{-1}	10^{0}	10^{1}	10^{2}	10^{3}
1000.	1	10	100	1,000	10,000	100,000	1,000,000
100.	.1	1	10	100	1,000	10,000	100,000
10.	.01	.1	1	10	100	1,000	10,000
1.	.001	.01	.1	1	10	100	1,000
.1	.0001	.001	.01	.1	1	10	100
.01	.00001	.0001	.001	.01	.1	1	10
.001	.000001	.00001	.0001	.001	.01	.1	1

For example,

$$1 \times 10^{-2} = .01$$

and

$$100 \times 10^{-2} = 1$$

It is possible to express any number as a power of any other number.

The logarithm of a number to base 10 is the index of the power to which 10 must be raised to produce the number.

This series of numbers is known as the Common logarithm, and is often used for simplifying involved problems of multiplication and division.

TO USE THE LOGARITHM TABLE

The logarithm of a number is composed of two parts, a Characteristic, and a Mantissa.

Characteristic - This is the whole number of the logarithm and is determined by inspection, from the following general guidelines

Numbers between:			Characteristic
1	and	9	0
10	and	99	1
100	and	999	2
1000	and	9999	3

In other words, the characteristic is one less than the number of digits in the whole number part of the number.

Mantissa

This is the decimal part of the logarithm, and is shown in the table calculated to four places of decimals.

Example: (1) Find the logarithm of 353.

Look up 35 under N, and then go across the row to column 3. .5478 will be found in the table. 2 is added by inspection

	Number	Logarithm
	353	2.5478
(2) Find Logarithm of 3.53	3.53	0.5478

The same numbers are searched; and the Mantissa is the same. However the characteristic is now 0.

Check the following numbers and logarithms

Number	Logarithm
3	0.4771
3.07	0.4871
3070	3.4871
3075	3.4879 approx

Note in the latter case, the tables are not sufficiently accurate to provide an answer for the "5". Thus, we must interpolate between 3070 and 3080.

TO MULTIPLY

	Number	Logarithm
two numbers together 47 x 400	47	1.6721
Find logarithms, then add:	400	+2.6021
		4.2742

Now, look up .2742 in the table

It is located at N = 18, column 8. Thus the number you are seeking (called the antilog) is 188. To this you must add the digits for the Characteristic. A Characteristic of 4 means 5 digits before the decimal point. Thus the antilogarithm of 4.2742 is 18800.

TO DIVIDE

	Number	Logarithm
$\frac{600}{15}$		
	600	2.7782
Find logarithms and subtract	15	-1.1761
		1.6021

Thus the antilog of 1.6021 is 40.0 The antilog of .6021 is 400. The Characteristic 1 means 2 digits before the decimal point.

NEGATIVE LOGARITHMS

Logarithms are positive numbers and can only be calculated for positive numbers.

Negative numbers are sometimes encountered when dealing with small numbers.

For example the characteristic of .06 has to be expressed as -2.

The mantissa of 6 in the table is .7782.

The complete logarithm is thus -2.7782, however it should be noted that only the characteristic is negative.

Often this is written as $\bar{2}.7782$ called "bar 2" point seven seven eight two.

In multiplying (adding) or dividing (subtracting), care must be taken when dealing with a combined negative and positive log.

For example, in the following problem

$$\frac{.204}{6} = 0.034$$

Number	Logarithm
.204	$\bar{1}.3096$
6	$-\ .7782$
	$\bar{2}.5314$

To determine the characteristic you must go through the process

"change the sign of the bottom line and add"

thus $-1 - (+1) = -1 - 1 = -2$

The antilog of .5314 is 340, and the characteristic $\bar{2}$ indicates the number is two places after the decimal point, or .034

This is a very cumbersome method for dealing with manipulations involving addition and subtraction of logarithms.

An alternate method to covert the entire logarithm to a negative form is as follows:

1. Express the mantissa in terms of a wholely positive log and -10.

   Thus $\bar{1}.3096 = 9.3096 - 10$ (i.e. $+9 - 10 = -1$)

2. Add the wholely positive log to -10

   Thus $9.3096 - 10 = -.6904$

   Use this totally negative logarithm in computations.

   Thus in the problem

$$\frac{.204}{6}$$

Number	Logarithm
.204	-0.6904
6	$-\ .7782$
	-1.4686

3. Find the antilog, which is approximately 2942

   The characteristic 1 means 2 places before the decimal, thus 29.42

4. The negative sign means the reciprocal, or $-\frac{1}{29.42} = .034$

SUMMARY REVIEW FOR CALCULATING TRIGONOMETRIC RATIOS

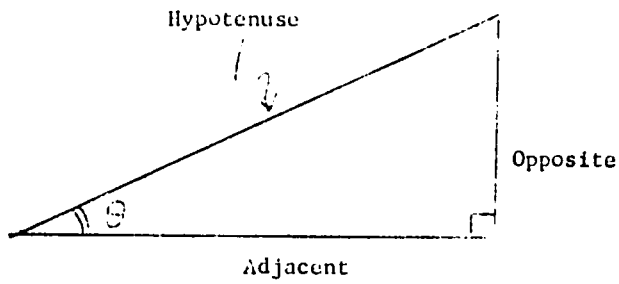
Definitions

The greek symbol θ (Theta) is commonly used to express the angle variable.

The Sine, Cosine and Tangent (Sin, Cos, Tan) are mathematical ratios which are used to express the angle in terms of a right angled triangle, as follows:

THE SINE, COSINE, AND TANGENT RATIOS

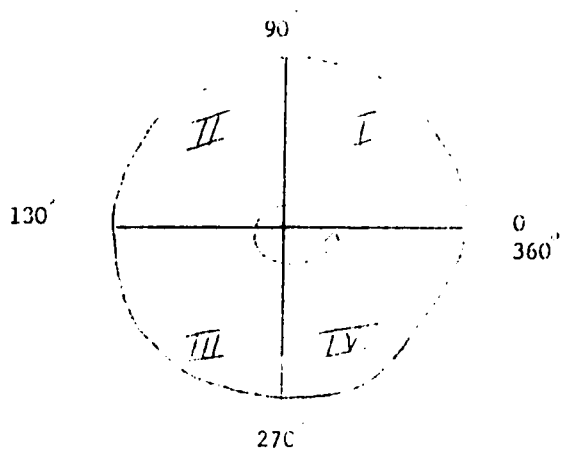
$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$



Mnemonically, they can be remembered as SOH, CAH and TOA

Angles can occur in any of four quadrants of a circle, starting at 3 o' clock and rotating in an anti-clockwise direction as indicated below

THE FOUR QUADRANTS OF A CIRCLE



For angles occuring in the first quadrant (i.e. between 0° and 90°) the Sine, Cosine, Tangent of Theta can be found directly from the tables on the following pages.

For angles in the other quadrants, a two stage adjustment must be made.

(1) The angle must be reduced to an angle with an equivalent trigonometric ratio in the first quadrant. In other words, Theta (θ) is an unknown which must be derived from other available information.

(2) The signs of some of the ratios change.

The net effect is indicated in the Reference Table on the following page.

REFERENCE TABLE TO FIND THE EQUIVALENT ANGLE AND TRIGONOMETRIC RATIO OF ANGLES IN THE FOUR QUADRANTS OF A CIRCLE

QUADRANT:	I	II	III	IV
Size of Angle:	0+ to 90°	90+ to 180°	180+ to 270°	270+ to 360°
(1) Equivalent Angle = (θ) in first quadrant	Same	180° - given angle	Given - 180° angle	360° - given angle
(2) Signs of the ratios				
Sin θ	+	+	-	-
Cos θ	+	-	-	+
Tan θ	+	-	+	-

For angles greater than 360°, reduce to an angle between 0° to 360° by subtracting 360° successively until a number in the appropriate range is reached.

Some specific examples will illustrate and further clarify the procedure.

<u>Find</u>

1. Sin of 32° = Quadrant I,
   Therefore read directly from table
   Sin 32° = .5299
   Sign + = .5299

2. Cos 103° = Quadrant II
   Therefore θ = 180 - 103
   = 77°
   Cos 77° = .2250
   Sign - = -.2250

3. Tan 190° = Quadrant III
   Therefore θ = 190 - 180
   = 10°
   Tan 10° = .1763
   Sign + = +.1763

4. Sin 295° = Quadrant IV
   Therefore θ = 360 - 295
   = 65°
   Sin 65° = .9063
   Sign + = +.9063

5. Cos 1432° Find Quadrant

   1432 - 360 = 1072°
   1072 - 360 = 712°
   712 - 360 = 352°

   Therefore = Quadrant IV
   Therefore θ = 360 - 352
   = 8°
   Cos 8° = .9903
   Sign + = .9903

STUDENT "T" DISTRIBUTION

Value of "T" for the following Percentage Confidence Levels

Degrees of Freedom*	80%	90%	95%	98%	99%
1	3.078	6.314	12.706	31.821	63.657
2	1.886	2.920	4.303	6.965	9.925
3	1.638	2.353	3.182	4.541	5.841
4	1.533	2.132	2.776	3.747	4.604
5	1.476	2.015	2.571	3.365	4.032
6	1.440	1.943	2.447	3.143	3.707
7	1.415	1.895	2.365	2.998	3.499
8	1.397	1.860	2.306	2.896	3.355
9	1.383	1.833	2.262	2.821	3.250
10	1.372	1.812	2.228	2.764	3.169
11	1.363	1.796	2.201	2.718	3.106
12	1.356	1.782	2.179	2.681	3.055
13	1.350	1.771	2.160	2.650	3.012
14	1.345	1.761	2.145	2.624	2.977
15	1.341	1.753	2.131	2.602	2.947
16	1.337	1.746	2.120	2.583	2.921
17	1.333	1.740	2.110	2.567	2.898
18	1.330	1.734	2.101	2.552	2.878
19	1.328	1.729	2.093	2.539	2.861
20	1.325	1.725	2.086	2.528	2.845
21	1.323	1.721	2.080	2.518	2.831
22	1.321	1.717	2.074	2.508	2.819
23	1.319	1.714	2.069	2.500	2.807
24	1.318	1.711	2.064	2.492	2.797
25	1.316	1.708	2.060	2.485	2.787
26	1.315	1.706	2.056	2.479	2.779
27	1.314	1.703	2.052	2.473	2.771
28	1.313	1.701	2.048	2.467	2.763
29	1.311	1.699	2.045	2.462	2.756
30	1.310	1.697	2.042	2.457	2.750
**	20%	10%	5%	2%	1%

* "Degrees of Freedom" is a statistical term which represents the number of independent pieces of information available about the variability of a population. There is no variability in a sample of one, one degree of freedom in a sample of two, and so forth. Each additional observation adds one additional independent piece of information about the population variance. In general, in a sample size of "n" there are "n-1" degrees of freedom. For determining correlations between two variables, in a sample size of "n" pairs, there are "n-2" degrees of freedom.

** When the table is read from the foot, the tabled values are to be prefixed with a negative sign.

Source: Derived from Fisher and Yates Statistical Tables for Biological, Agricultural and Medical Research, Oliver and Boyd, Ltd., Edinburgh.

LOGARITHMS

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

LOGARITIMS (Continued)

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

NATURAL SINES.

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	·0000	·0017	·0035	·0052	·0070	·0087	·0105	·0122	·0140	·0157	3	6	9	12	15
1°	·0175	·0192	·0209	·0227	·0244	·0262	·0279	·0297	·0314	·0332	3	6	9	12	15
2°	·0349	·0366	·0384	·0401	·0419	·0436	·0454	·0471	·0488	·0506	3	6	9	12	15
3°	·0523	·0541	·0558	·0576	·0593	·0610	·0628	·0645	·0663	·0680	3	6	9	12	15
4°	·0698	·0715	·0732	·0750	·0767	·0785	·0802	·0819	·0837	·0854	3	6	9	12	14
5°	·0872	·0889	·0906	·0924	·0941	·0958	·0976	·0993	·1011	·1028	3	6	9	12	14
6°	·1045	·1063	·1080	·1097	·1115	·1132	·1149	·1167	·1184	·1201	3	6	9	12	14
7°	·1219	·1236	·1253	·1271	·1288	·1305	·1323	·1340	·1357	·1374	3	6	9	12	14
8°	·1392	·1409	·1426	·1444	·1461	·1478	·1495	·1513	·1530	·1547	3	6	9	12	14
9°	·1564	·1582	·1599	·1616	·1633	·1650	·1668	·1685	·1702	·1719	3	6	9	12	14
10°	·1736	·1754	·1771	·1788	·1805	·1822	·1840	·1857	·1874	·1891	3	6	9	11	14
11°	·1908	·1925	·1942	·1959	·1977	·1994	·2011	·2028	·2045	·2062	3	6	9	11	14
12°	·2079	·2096	·2113	·2130	·2147	·2164	·2181	·2198	·2215	·2233	3	6	9	11	14
13°	·2250	·2267	·2284	·2300	·2317	·2334	·2351	·2368	·2385	·2402	3	6	8	11	14
14°	·2419	·2436	·2453	·2470	·2487	·2504	·2521	·2538	·2554	·2571	3	6	8	11	14
15°	·2588	·2605	·2622	·2639	·2656	·2672	·2689	·2706	·2723	·2740	3	6	8	11	14
16°	·2756	·2773	·2790	·2807	·2823	·2840	·2857	·2874	·2890	·2907	3	6	8	11	14
17°	·2924	·2940	·2957	·2974	·2990	·3007	·3024	·3040	·3057	·3074	3	6	8	11	14
18°	·3090	·3107	·3123	·3140	·3156	·3173	·3190	·3206	·3223	·3239	3	6	8	11	14
19°	·3256	·3272	·3289	·3305	·3322	·3338	·3355	·3371	·3387	·3404	3	5	8	11	14
20°	·3420	·3437	·3453	·3469	·3486	·3502	·3518	·3535	·3551	·3567	3	5	8	11	14
21°	·3584	·3600	·3616	·3633	·3649	·3665	·3681	·3697	·3714	·3730	3	5	8	11	14
22°	·3746	·3762	·3778	·3795	·3811	·3827	·3843	·3859	·3875	·3891	3	5	8	11	14
23°	·3907	·3923	·3939	·3955	·3971	·3987	·4003	·4019	·4035	·4051	3	5	8	11	14
24°	·4067	·4083	·4099	·4115	·4131	·4147	·4163	·4179	·4195	·4210	3	5	8	11	13
25°	·4226	·4242	·4258	·4274	·4289	·4305	·4321	·4337	·4352	·4368	3	5	8	11	13
26°	·4384	·4399	·4415	·4431	·4446	·4462	·4478	·4493	·4509	·4524	3	5	8	10	13
27°	·4540	·4555	·4571	·4586	·4602	·4617	·4633	·4648	·4664	·4679	3	5	8	10	13
28°	·4695	·4710	·4726	·4741	·4756	·4772	·4787	·4802	·4818	·4833	3	5	8	10	13
29°	·4848	·4863	·4879	·4894	·4909	·4924	·4939	·4955	·4970	·4985	3	5	8	10	13
30°	·5000	·5015	·5030	·5045	·5060	·5075	·5090	·5105	·5120	·5135	3	5	8	10	13
31°	·5150	·5165	·5180	·5195	·5210	·5225	·5240	·5255	·5270	·5284	2	5	7	10	12
32°	·5299	·5314	·5329	·5344	·5358	·5373	·5388	·5402	·5417	·5432	2	5	7	10	12
33°	·5446	·5461	·5476	·5490	·5505	·5519	·5534	·5548	·5563	·5577	2	5	7	10	12
34°	·5592	·5606	·5621	·5635	·5650	·5664	·5678	·5693	·5707	·5721	2	5	7	10	12
35°	·5736	·5750	·5764	·5779	·5793	·5807	·5821	·5835	·5850	·5864	2	5	7	9	12
36°	·5878	·5892	·5906	·5920	·5934	·5948	·5962	·5976	·5990	·6004	2	5	7	9	12
37°	·6018	·6032	·6046	·6060	·6074	·6088	·6101	·6115	·6129	·6143	2	5	7	9	12
38°	·6157	·6170	·6184	·6198	·6211	·6225	·6239	·6252	·6266	·6280	2	5	7	9	11
39°	·6293	·6307	·6320	·6334	·6347	·6361	·6374	·6388	·6401	·6414	2	4	7	9	11
40°	·6428	·6441	·6455	·6468	·6481	·6494	·6508	·6521	·6534	·6547	2	4	7	9	11
41°	·6561	·6574	·6587	·6600	·6613	·6626	·6639	·6652	·6665	·6678	2	4	7	9	11
42°	·6691	·6704	·6717	·6730	·6743	·6756	·6769	·6782	·6794	·6807	2	4	6	9	11
43°	·6820	·6833	·6845	·6858	·6871	·6884	·6896	·6909	·6921	·6934	2	4	6	8	11
44°	·6947	·6959	·6972	·6984	·6997	·7009	·7022	·7034	·7046	·7059	2	4	6	8	10

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	·7071	·7083	·7096	·7108	·7120	·7133	·7145	·7157	·7169	·7181	2	4	6	8	10
46°	·7193	·7206	·7218	·7230	·7242	·7254	·7266	·7278	·7290	·7302	2	4	6	8	10
47°	·7314	·7325	·7337	·7349	·7361	·7373	·7385	·7396	·7408	·7420	2	4	6	8	10
48°	·7431	·7443	·7455	·7466	·7478	·7490	·7501	·7513	·7524	·7536	2	4	6	8	10
49°	·7547	·7559	·7570	·7581	·7593	·7604	·7615	·7627	·7638	·7649	2	4	6	8	9
50°	·7660	·7672	·7683	·7694	·7705	·7716	·7727	·7738	·7749	·7760	2	4	6	7	9
51°	·7771	·7782	·7793	·7804	·7815	·7826	·7837	·7848	·7859	·7869	2	4	5	7	9
52°	·7880	·7891	·7902	·7912	·7923	·7934	·7944	·7955	·7965	·7976	2	4	5	7	9
53°	·7986	·7997	·8007	·8018	·8028	·8039	·8049	·8059	·8070	·8080	2	3	5	7	9
54°	·8090	·8100	·8111	·8121	·8131	·8141	·8151	·8161	·8171	·8181	2	3	5	7	8
55°	·8192	·8202	·8211	·8221	·8231	·8241	·8251	·8261	·8271	·8281	2	3	5	7	8
56°	·8290	·8300	·8310	·8320	·8329	·8339	·8348	·8358	·8368	·8377	2	3	5	6	8
57°	·8387	·8396	·8406	·8415	·8425	·8434	·8443	·8453	·8462	·8471	2	3	5	6	8
58°	·8480	·8490	·8499	·8508	·8517	·8526	·8536	·8545	·8554	·8563	2	3	5	6	8
59°	·8572	·8581	·8590	·8599	·8607	·8616	·8625	·8634	·8643	·8652	1	3	4	6	7
60°	·8660	·8669	·8678	·8686	·8695	·8704	·8712	·8721	·8729	·8738	1	3	4	6	7
61°	·8746	·8755	·8763	·8771	·8780	·8788	·8796	·8805	·8813	·8821	1	3	4	6	7
62°	·8829	·8838	·8846	·8854	·8862	·8870	·8878	·8886	·8894	·8902	1	3	4	5	7
63°	·8910	·8918	·8926	·8934	·8942	·8949	·8957	·8965	·8973	·8980	1	3	4	5	6
64°	·8988	·8996	·9003	·9011	·9018	·9026	·9033	·9041	·9048	·9056	1	3	4	5	6
65°	·9063	·9070	·9078	·9085	·9092	·9100	·9107	·9114	·9121	·9128	1	2	4	5	6
66°	·9135	·9143	·9150	·9157	·9164	·9171	·9178	·9184	·9191	·9198	1	2	3	5	6
67°	·9205	·9212	·9219	·9225	·9232	·9239	·9245	·9252	·9259	·9265	1	2	3	4	6
68°	·9272	·9278	·9285	·9291	·9298	·9304	·9311	·9317	·9323	·9330	1	2	3	4	5
69°	·9336	·9342	·9348	·9354	·9361	·9367	·9373	·9379	·9385	·9391	1	2	3	4	5
70°	·9397	·9403	·9409	·9415	·9421	·9426	·9432	·9438	·9444	·9449	1	2	3	4	5
71°	·9455	·9461	·9466	·9472	·9478	·9483	·9489	·9494	·9500	·9505	1	2	3	4	5
72°	·9511	·9516	·9521	·9527	·9532	·9537	·9542	·9548	·9553	·9558	1	2	3	3	4
73°	·9563	·9568	·9573	·9578	·9583	·9588	·9593	·9598	·9603	·9608	1	2	2	3	4
74°	·9613	·9617	·9622	·9627	·9632	·9636	·9641	·9646	·9650	·9655	1	2	2	3	4
75°	·9659	·9664	·9668	·9673	·9677	·9681	·9686	·9690	·9694	·9699	1	1	2	3	4
76°	·9703	·9707	·9711	·9715	·9720	·9724	·9728	·9732	·9736	·9740	1	1	2	3	3
77°	·9744	·9748	·9751	·9755	·9759	·9763	·9767	·9770	·9774	·9778	1	1	2	3	3
78°	·9781	·9785	·9789	·9792	·9796	·9799	·9803	·9806	·9810	·9813	1	1	2	2	3
79°	·9816	·9820	·9823	·9826	·9829	·9833	·9836	·9839	·9842	·9845	1	1	2	2	3
80°	·9848	·9851	·9854	·9857	·9860	·9863	·9866	·9869	·9871	·9874	0	1	1	2	2
81°	·9877	·9880	·9882	·9885	·9888	·9890	·9893	·9895	·9898	·9900	0	1	1	2	2
82°	·9903	·9905	·9907	·9910	·9912	·9914	·9917	·9919	·9921	·9923	0	1	1	2	2
83°	·9925	·9928	·9930	·9932	·9934	·9936	·9938	·9940	·9942	·9943	0	1	1	1	2
84°	·9945	·9947	·9949	·9951	·9952	·9954	·9956	·9957	·9959	·9960	0	1	1	1	1
85°	·9962	·9963	·9965	·9966	·9968	·9969	·9971	·9972	·9973	·9974	0	0	1	1	1
86°	·9976	·9977	·9978	·9979	·9980	·9981	·9982	·9983	·9984	·9985	0	0	1	1	1
87°	·9986	·9987	·9988	·9989	·9990	·9990	·9991	·9992	·9993	·9993	0	0	0	1	1
88°	·9994	·9995	·9995	·9996	·9996	·9997	·9997	·9997	·9998	·9998	0	0	0	0	0
89°	·9998	·9999	·9999	·9999	·9999	1·000	1·000	1·000	1·000	1·000	0	0	0	0	0

NATURAL COSINES.

Subtract Differences.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	1·0000	1·000	1·000	1·000	1·000	1·000	·9999	·9999	·9999	·9999	0	0	0	0	0
1°	·9998	·9998	·9998	·9997	·9997	·9997	·9996	·9996	·9995	·9995	0	0	0	0	0
2°	·9994	·9993	·9993	·9992	·9991	·9990	·9990	·9989	·9988	·9987	0	0	0	0	0
3°	·9986	·9985	·9984	·9983	·9982	·9981	·9980	·9979	·9978	·9977	0	0	1	1	1
4°	·9976	·9974	·9973	·9972	·9971	·9969	·9968	·9966	·9965	·9963	0	0	1	1	1
5°	·9962	·9960	·9959	·9957	·9956	·9954	·9952	·9951	·9949	·9947	0	1	1	1	1
6°	·9945	·9943	·9942	·9940	·9938	·9936	·9934	·9932	·9930	·9928	0	1	1	1	2
7°	·9925	·9923	·9921	·9919	·9917	·9914	·9912	·9910	·9907	·9905	0	1	1	2	2
8°	·9903	·9900	·9898	·9895	·9893	·9890	·9888	·9885	·9882	·9880	0	1	1	2	2
9°	·9877	·9874	·9871	·9869	·9866	·9863	·9860	·9857	·9854	·9851	0	1	1	2	2
10°	·9848	·9845	·9842	·9839	·9836	·9833	·9829	·9826	·9823	·9820	1	1	2	2	3
11°	·9816	·9813	·9810	·9806	·9803	·9799	·9796	·9792	·9789	·9785	1	1	2	2	3
12°	·9781	·9778	·9774	·9770	·9767	·9763	·9759	·9755	·9751	·9748	1	1	2	3	3
13°	·9744	·9740	·9736	·9732	·9728	·9724	·9720	·9715	·9711	·9707	1	1	2	3	3
14°	·9703	·9699	·9694	·9690	·9686	·9681	·9677	·9673	·9668	·9664	1	1	2	3	4
15°	·9659	·9655	·9650	·9646	·9641	·9636	·9632	·9627	·9622	·9617	1	2	2	3	4
16°	·9613	·9608	·9603	·9598	·9593	·9588	·9583	·9578	·9573	·9568	1	2	2	3	4
17°	·9563	·9558	·9553	·9548	·9542	·9537	·9532	·9527	·9521	·9516	1	2	3	4	4
18°	·9511	·9505	·9500	·9494	·9489	·9483	·9478	·9472	·9466	·9461	1	2	3	4	5
19°	·9455	·9449	·9444	·9438	·9432	·9426	·9421	·9415	·9409	·9403	1	2	3	4	5
20°	·9397	·9391	·9385	·9379	·9373	·9367	·9361	·9354	·9348	·9342	1	2	3	4	5
21°	·9336	·9330	·9323	·9317	·9311	·9304	·9298	·9291	·9285	·9278	1	2	3	4	5
22°	·9272	·9265	·9259	·9252	·9245	·9239	·9232	·9225	·9219	·9212	1	2	3	4	6
23°	·9205	·9198	·9191	·9184	·9178	·9171	·9164	·9157	·9150	·9143	1	2	3	5	6
24°	·9135	·9128	·9121	·9114	·9107	·9100	·9092	·9085	·9078	·9070	1	2	4	5	6
25°	·9063	·9056	·9048	·9041	·9033	·9026	·9018	·9011	·9003	·8996	1	3	4	5	6
26°	·8988	·8980	·8973	·8965	·8957	·8949	·8942	·8934	·8926	·8918	1	3	4	5	6
27°	·8910	·8902	·8894	·8886	·8878	·8870	·8862	·8854	·8846	·8838	1	3	4	5	7
28°	·8829	·8821	·8813	·8805	·8796	·8788	·8780	·8771	·8763	·8755	1	3	4	6	7
29°	·8746	·8738	·8729	·8721	·8712	·8704	·8695	·8686	·8678	·8669	1	3	4	6	7
30°	·8660	·8652	·8643	·8634	·8625	·8616	·8607	·8599	·8590	·8581	1	3	4	6	7
31°	·8572	·8563	·8554	·8545	·8536	·8526	·8517	·8508	·8499	·8490	2	3	5	6	8
32°	·8480	·8471	·8462	·8453	·8443	·8434	·8425	·8415	·8406	·8396	2	3	5	6	8
33°	·8387	·8377	·8368	·8358	·8348	·8339	·8329	·8320	·8310	·8300	2	3	5	6	8
34°	·8290	·8281	·8271	·8261	·8251	·8241	·8231	·8221	·8211	·8202	2	3	5	7	8
35°	·8192	·8181	·8171	·8161	·8151	·8141	·8131	·8121	·8111	·8100	2	3	5	7	8
36°	·8090	·8080	·8070	·8059	·8049	·8039	·8028	·8018	·8007	·7997	2	3	5	7	9
37°	·7986	·7976	·7965	·7955	·7944	·7934	·7923	·7912	·7902	·7891	2	4	5	7	9
38°	·7880	·7869	·7859	·7848	·7837	·7826	·7815	·7804	·7793	·7782	2	4	5	7	9
39°	·7771	·7760	·7749	·7738	·7727	·7716	·7705	·7694	·7683	·7672	2	4	6	7	9
40°	·7660	·7649	·7638	·7627	·7615	·7604	·7593	·7581	·7570	·7559	2	4	6	8	9
41°	·7547	·7536	·7524	·7513	·7501	·7490	·7478	·7466	·7455	·7443	2	4	6	8	10
42°	·7431	·7420	·7408	·7396	·7385	·7373	·7361	·7349	·7337	·7325	2	4	6	8	10
43°	·7314	·7302	·7290	·7278	·7266	·7254	·7242	·7230	·7218	·7206	2	4	6	8	10
44°	·7193	·7181	·7169	·7157	·7145	·7133	·7120	·7108	·7096	·7083	2	4	6	8	10

Subtract Differences.

	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	·7071	·7059	·7046	·7034	·7022	·7009	·6997	·6984	·6972	·6959	2	4	6	8	10
46°	·6947	·6934	·6921	·6909	·6896	·6884	·6871	·6858	·6845	·6833	2	4	6	8	11
47°	·6820	·6807	·6794	·6782	·6769	·6756	·6743	·6730	·6717	·6704	2	4	6	9	11
48°	·6691	·6678	·6665	·6652	·6639	·6626	·6613	·6600	·6587	·6574	2	4	7	9	11
49°	·6561	·6547	·6534	·6521	·6508	·6494	·6481	·6468	·6455	·6441	2	4	7	9	11
50°	·6428	·6414	·6401	·6388	·6374	·6361	·6347	·6334	·6320	·6307	2	4	7	9	11
51°	·6293	·6280	·6266	·6252	·6239	·6225	·6211	·6198	·6184	·6170	2	5	7	9	11
52°	·6157	·6143	·6129	·6115	·6101	·6088	·6074	·6060	·6046	·6032	2	5	7	9	12
53°	·6018	·6004	·5990	·5976	·5962	·5948	·5934	·5920	·5906	·5892	2	5	7	9	12
54°	·5878	·5864	·5850	·5835	·5821	·5807	·5793	·5779	·5764	·5750	2	5	7	9	12
55°	·5736	·5721	·5707	·5693	·5678	·5664	·5650	·5635	·5621	·5606	2	5	7	10	12
56°	·5592	·5577	·5563	·5548	·5534	·5519	·5505	·5490	·5476	·5461	2	5	7	10	12
57°	·5446	·5432	·5417	·5402	·5388	·5373	·5358	·5344	·5329	·5314	2	5	7	10	12
58°	·5299	·5284	·5270	·5255	·5240	·5225	·5210	·5195	·5180	·5165	2	5	7	10	12
59°	·5150	·5135	·5120	·5105	·5090	·5075	·5060	·5045	·5030	·5015	3	5	8	10	13
60°	·5000	·4985	·4970	·4955	·4939	·4924	·4909	·4894	·4879	·4863	3	5	8	10	13
61°	·4848	·4833	·4818	·4802	·4787	·4772	·4756	·4741	·4726	·4710	3	5	8	10	13
62°	·4695	·4679	·4664	·4648	·4633	·4617	·4602	·4586	·4571	·4555	3	5	8	10	13
63°	·4540	·4524	·4509	·4493	·4478	·4462	·4446	·4431	·4415	·4399	3	5	8	10	13
64°	·4384	·4368	·4352	·4337	·4321	·4305	·4289	·4274	·4258	·4242	3	5	8	11	13
65°	·4226	·4210	·4195	·4179	·4163	·4147	·4131	·4115	·4099	·4083	3	5	8	11	13
66°	·4067	·4051	·4035	·4019	·4003	·3987	·3971	·3955	·3939	·3923	3	5	8	11	13
67°	·3907	·3891	·3875	·3859	·3843	·3827	·3811	·3795	·3778	·3762	3	5	8	11	13
68°	·3746	·3730	·3714	·3697	·3681	·3665	·3649	·3633	·3616	·3600	3	5	8	11	14
69°	·3584	·3567	·3551	·3535	·3518	·3502	·3486	·3469	·3453	·3437	3	5	8	11	14
70°	·3420	·3404	·3387	·3371	·3355	·3338	·3322	·3305	·3289	·3272	3	5	8	11	14
71°	·3256	·3239	·3223	·3206	·3190	·3173	·3156	·3140	·3123	·3107	3	6	8	11	14
72°	·3090	·3074	·3057	·3040	·3024	·3007	·2990	·2974	·2957	·2940	3	6	8	11	14
73°	·2924	·2907	·2890	·2874	·2857	·2840	·2823	·2807	·2790	·2773	3	6	8	11	14
74°	·2756	·2740	·2723	·2706	·2689	·2672	·2656	·2639	·2622	·2605	3	6	8	11	14
75°	·2588	·2571	·2554	·2538	·2521	·2504	·2487	·2470	·2453	·2436	3	6	8	11	14
76°	·2419	·2402	·2385	·2368	·2351	·2334	·2317	·2300	·2284	·2267	3	6	8	11	14
77°	·2250	·2233	·2215	·2198	·2181	·2164	·2147	·2130	·2113	·2096	3	6	9	11	14
78°	·2079	·2062	·2045	·2028	·2011	·1994	·1977	·1959	·1942	·1925	3	6	9	11	14
79°	·1908	·1891	·1874	·1857	·1840	·1822	·1805	·1788	·1771	·1754	3	6	9	11	14
80°	·1736	·1719	·1702	·1685	·1668	·1650	·1633	·1616	·1599	·1582	3	6	9	11	14
81°	·1564	·1547	·1530	·1513	·1495	·1478	·1461	·1444	·1426	·1409	3	6	9	12	14
82°	·1392	·1374	·1357	·1340	·1323	·1305	·1288	·1271	·1253	·1236	3	6	9	12	14
83°	·1219	·1201	·1184	·1167	·1149	·1132	·1115	·1097	·1080	·1063	3	6	9	12	14
84°	·1045	·1028	·1011	·0993	·0976	·0958	·0941	·0924	·0906	·0889	3	6	9	12	14
85°	·0872	·0854	·0837	·0819	·0802	·0785	·0767	·0750	·0732	·0715	3	6	9	12	14
86°	·0698	·0680	·0663	·0645	·0628	·0610	·0593	·0576	·0558	·0541	3	6	9	12	15
87°	·0523	·0506	·0488	·0471	·0454	·0436	·0419	·0401	·0384	·0366	3	6	9	12	15
88°	·0349	·0332	·0314	·0297	·0279	·0262	·0244	·0227	·0209	·0192	3	6	9	12	15
89°	·0175	·0157	·0140	·0122	·0105	·0087	·0070	·0052	·0035	·0017	3	6	9	12	15

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
0°	0·0000	·0017	·0035	·0052	·0070	·0087	·0105	·0122	·0140	·0157	3	6	9	12	15
1°	0·0175	·0192	·0209	·0227	·0244	·0262	·0279	·0297	·0314	·0332	3	6	9	12	15
2°	0·0349	·0367	·0384	·0402	·0419	·0437	·0454	·0472	·0489	·0507	3	6	9	12	15
3°	0·0524	·0542	·0559	·0577	·0594	·0612	·0629	·0647	·0664	·0682	3	6	9	12	15
4°	0·0699	·0717	·0734	·0752	·0769	·0787	·0805	·0822	·0840	·0857	3	6	9	12	15
5°	0·0875	·0892	·0910	·0928	·0945	·0963	·0981	·0998	·1016	·1033	3	6	9	12	15
6°	0·1051	·1069	·1086	·1104	·1122	·1139	·1157	·1175	·1192	·1210	3	6	9	12	15
7°	0·1228	·1246	·1263	·1281	·1299	·1317	·1334	·1352	·1370	·1388	3	6	9	12	15
8°	0·1405	·1423	·1441	·1459	·1477	·1495	·1512	·1530	·1548	·1566	3	6	9	12	15
9°	0·1584	·1602	·1620	·1638	·1655	·1673	·1691	·1709	·1727	·1745	3	6	9	12	15
10°	0·1763	·1781	·1799	·1817	·1835	·1853	·1871	·1890	·1908	·1926	3	6	9	12	15
11°	0·1944	·1962	·1980	·1998	·2016	·2035	·2053	·2071	·2089	·2107	3	6	9	12	15
12°	0·2126	·2144	·2162	·2180	·2199	·2217	·2235	·2254	·2272	·2290	3	6	9	12	15
13°	0·2309	·2327	·2345	·2364	·2382	·2401	·2419	·2438	·2456	·2475	3	6	9	12	15
14°	0·2493	·2512	·2530	·2549	·2568	·2586	·2605	·2623	·2642	·2661	3	6	9	12	16
15°	0·2679	·2698	·2717	·2736	·2754	·2773	·2792	·2811	·2830	·2849	3	6	9	13	16
16°	0·2867	·2886	·2905	·2924	·2943	·2962	·2981	·3000	·3019	·3038	3	6	9	13	16
17°	0·3057	·3076	·3096	·3115	·3134	·3153	·3172	·3191	·3211	·3230	3	6	10	13	16
18°	0·3249	·3269	·3288	·3307	·3327	·3346	·3365	·3385	·3404	·3424	3	6	10	13	16
19°	0·3443	·3463	·3482	·3502	·3522	·3541	·3561	·3581	·3600	·3620	3	7	10	13	16
20°	0·3640	·3659	·3679	·3699	·3719	·3739	·3759	·3779	·3799	·3819	3	7	10	13	17
21°	0·3839	·3859	·3879	·3899	·3919	·3939	·3959	·3979	·4000	·4020	3	7	10	13	17
22°	0·4040	·4061	·4081	·4101	·4122	·4142	·4163	·4183	·4204	·4224	3	7	10	14	17
23°	0·4245	·4265	·4286	·4307	·4327	·4348	·4369	·4390	·4411	·4431	3	7	10	14	17
24°	0·4452	·4473	·4494	·4515	·4536	·4557	·4578	·4599	·4621	·4642	4	7	11	14	18
25°	0·4663	·4684	·4706	·4727	·4748	·4770	·4791	·4813	·4834	·4856	4	7	11	14	18
26°	0·4877	·4899	·4921	·4942	·4964	·4986	·5008	·5029	·5051	·5073	4	7	11	15	18
27°	0·5095	·5117	·5139	·5161	·5184	·5206	·5228	·5250	·5272	·5295	4	7	11	15	18
28°	0·5317	·5340	·5362	·5384	·5407	·5430	·5452	·5475	·5498	·5520	4	8	11	15	19
29°	0·5543	·5566	·5589	·5612	·5635	·5658	·5681	·5704	·5727	·5750	4	8	12	15	19
30°	0·5774	·5797	·5820	·5844	·5867	·5890	·5914	·5938	·5961	·5985	4	8	12	16	20
31°	0·6009	·6032	·6056	·6080	·6104	·6128	·6152	·6176	·6200	·6224	4	8	12	16	20
32°	0·6249	·6273	·6297	·6322	·6346	·6371	·6395	·6420	·6445	·6469	4	8	12	16	20
33°	0·6494	·6519	·6544	·6569	·6594	·6619	·6644	·6669	·6694	·6720	4	8	13	17	21
34°	0·6745	·6771	·6796	·6822	·6847	·6873	·6899	·6924	·6950	·6976	4	9	13	17	21
35°	0·7002	·7028	·7054	·7080	·7107	·7133	·7159	·7186	·7212	·7239	4	9	13	18	22
36°	0·7265	·7292	·7319	·7346	·7373	·7400	·7427	·7454	·7481	·7508	5	9	14	18	23
37°	0·7536	·7563	·7590	·7618	·7646	·7673	·7701	·7729	·7757	·7785	5	9	14	18	23
38°	0·7813	·7841	·7869	·7898	·7926	·7954	·7983	·8012	·8040	·8069	5	9	14	19	24
39°	0·8098	·8127	·8156	·8185	·8214	·8243	·8273	·8302	·8332	·8361	5	10	15	20	24
40°	0·8391	·8421	·8451	·8481	·8511	·8541	·8571	·8601	·8632	·8662	5	10	15	20	25
41°	0·8693	·8724	·8754	·8785	·8816	·8847	·8878	·8910	·8941	·8972	5	10	16	21	26
42°	0·9004	·9036	·9067	·9099	·9131	·9163	·9195	·9228	·9260	·9293	5	11	16	21	27
43°	0·9325	·9358	·9391	·9424	·9457	·9490	·9523	·9556	·9590	·9623	6	11	17	22	28
44°	0·9657	·9691	·9725	·9759	·9793	·9827	·9861	·9896	·9930	·9965	6	11	17	23	29

NATURAL TANGENTS.

Angle.	0′	6′	12′	18′	24′	30′	36′	42′	48′	54′	1′	2′	3′	4′	5′
45°	1·0000	1·0035	1·0070	1·0105	1·0141	1·0176	1·0212	1·0247	1·0283	1·0319	6	12	18	24	30
46°	1·0355	1·0392	1·0428	1·0464	1·0501	1·0538	1·0575	1·0612	1·0649	1·0686	6	12	18	25	31
47°	1·0724	1·0761	1·0799	1·0837	1·0875	1·0913	1·0951	1·0990	1·1028	1·1067	6	13	19	25	32
48°	1·1106	1·1145	1·1184	1·1224	1·1263	1·1303	1·1343	1·1383	1·1423	1·1463	7	13	20	26	33
49°	1·1504	1·1544	1·1585	1·1626	1·1667	1·1708	1·1750	1·1792	1·1833	1·1875	7	14	21	28	34
50°	1·1918	1·1960	1·2002	1·2045	1·2088	1·2131	1·2174	1·2218	1·2261	1·2305	7	14	22	29	36
51°	1·2349	1·2393	1·2437	1·2482	1·2527	1·2572	1·2617	1·2662	1·2708	1·2753	8	15	23	30	38
52°	1·2799	1·2846	1·2892	1·2938	1·2985	1·3032	1·3079	1·3127	1·3175	1·3222	8	16	24	31	39
53°	1·3270	1·3319	1·3367	1·3416	1·3465	1·3514	1·3564	1·3613	1·3663	1·3713	8	16	25	33	41
54°	1·3764	1·3814	1·3865	1·3916	1·3968	1·4019	1·4071	1·4124	1·4176	1·4229	9	17	26	34	43
55°	1·4281	1·4335	1·4388	1·4442	1·4496	1·4550	1·4605	1·4659	1·4715	1·4770	9	18	27	36	45
56°	1·4826	1·4882	1·4938	1·4994	1·5051	1·5108	1·5166	1·5224	1·5282	1·5340	10	19	29	38	48
57°	1·5399	1·5458	1·5517	1·5577	1·5637	1·5697	1·5757	1·5818	1·5880	1·5941	10	20	30	40	50
58°	1·6003	1·6066	1·6128	1·6191	1·6255	1·6319	1·6383	1·6447	1·6512	1·6577	11	21	32	43	53
59°	1·6643	1·6709	1·6775	1·6842	1·6909	1·6977	1·7045	1·7113	1·7182	1·7251	11	23	34	45	56
60°	1·7321	1·7391	1·7461	1·7532	1·7603	1·7675	1·7747	1·7820	1·7893	1·7966	12	24	36	48	60
61°	1·8040	1·8115	1·8190	1·8265	1·8341	1·8418	1·8495	1·8572	1·8650	1·8728	13	26	38	51	64
62°	1·8807	1·8887	1·8967	1·9047	1·9128	1·9210	1·9292	1·9375	1·9458	1·9542	14	27	41	55	68
63°	1·9626	1·9711	1·9797	1·9883	1·9970	2·0057	2·0145	2·0233	2·0323	2·0413	15	29	44	58	73
64°	2·0503	2·0594	2·0686	2·0778	2·0872	2·0965	2·1060	2·1155	2·1251	2·1348	16	31	47	63	78
65°	2·1445	2·1543	2·1642	2·1742	2·1842	2·1943	2·2045	2·2148	2·2251	2·2355	17	34	51	68	85
66°	2·2460	2·2566	2·2673	2·2781	2·2889	2·2998	2·3109	2·3220	2·3332	2·3445	18	37	55	73	92
67°	2·3559	2·3673	2·3789	2·3906	2·4023	2·4142	2·4262	2·4383	2·4504	2·4627	20	40	60	79	99
68°	2·4751	2·4876	2·5002	2·5129	2·5257	2·5386	2·5517	2·5649	2·5782	2·5916	22	43	65	87	108
69°	2·6051	2·6187	2·6325	2·6464	2·6605	2·6746	2·6889	2·7034	2·7179	2·7326	24	47	71	95	119
70°	2·7475	2·7625	2·7776	2·7929	2·8083	2·8239	2·8397	1·8556	2·8716	2·8878	26	52	78	104	130
71°	2·9042	2·9208	2·9375	2·9544	2·9714	2·9887	3·0061	3·0237	3·0415	3·0595	29	58	87	116	144
72°	3·0777	3·0961	3·1146	3·1334	3·1524	3·1716	3·1910	3·2106	3·2305	3·2506	32	64	96	129	161
73°	3·2709	3·2914	3·3122	3·3332	3·3544	3·3759	3·3977	3·4197	3·4420	3·4646	36	72	108	144	180
74°	3·4874	3·5105	3·5339	3·5576	3·5816	3·6059	3·6305	3·6554	3·6806	3·7062	41	81	122	163	204
75°	3·7321	3·7583	3·7848	3·8118	3·8391	3·8667	3·8947	3·9232	3·9520	3·9812					
76°	4·0108	4·0408	4·0713	4·1022	4·1335	4·1653	4·1976	4·2303	4·2635	4·2972					
77°	4·3315	4·3662	4·4015	4·4374	4·4737	4·5107	4·5483	4·5864	4·6252	4·6646					
78°	4·7046	4·7453	4·7867	4·8288	4·8716	4·9152	4·9594	5·0045	5·0504	5·0970					
79°	5·1446	5·1929	5·2422	5·2924	5·3435	5·3955	5·4486	5·5026	5·5578	5·6140					
80°	5·6713	5·7297	5·7894	5·8502	5·9124	5·9758	6·0405	6·1066	6·1742	6·2432	Mean differences not sufficiently accurate.				
81°	6·3138	6·3859	6·4596	6·5350	6·6122	6·6912	6·7720	6·8548	6·9395	7·0264					
82°	7·1154	7·2066	7·3002	7·3962	7·4947	7·5958	7·6996	7·8062	7·9158	8·0285					
83°	8·1443	8·2636	8·3863	8·5126	8·6427	8·7769	8·9152	9·0579	9·2052	9·3572					
84°	9·5144	9·6768	9·8448	10·019	10·199	10·385	10·579	10·780	10·988	11·205					
85°	11·430	11·664	11·909	12·163	12·429	12·706	12·996	13·300	13·617	13·951					
86°	14·301	14·669	15·056	15·464	15·895	16·350	16·832	17·343	17·886	18·464					
87°	19·081	19·740	20·446	21·205	22·022	22·904	23·859	24·898	26·031	27·271					
88°	28·636	30·145	31·821	33·694	35·801	38·188	40·917	44·066	47·740	52·081					
89°	57·290	63·657	71·615	81·847	95·489	114·59	143·24	190·98	286·48	572·96					

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The following texts were helpful in preparing this handbook. They contain much additional detail for further study if desired.

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