is establishes the validity of (22).

A.3. A trivial modification of the above argument covers the Gini coefficient, which $l'=(-2/K+1,\ldots,-2/K+1, K/K+1)$. The value of the objective ction in the dual linear program (A6) cannot exceed $\sum_{i=1}^{K}(2/K+1)d_i$. Choose $=-d_i$, $i=1,\ldots,K$, and $x_{K+1}=d_{K+1}=0$. This choice is feasible and indeed ves the dual problem so that when

$$l'=(-2/K+1,\ldots,-2/K+1, K/K+1),$$

$$H(l)=\sum_{i=1}^{K}(2/K+1)d_i=(2/K+1)\sum_{i=1}^{K}v_{ii}^{*\frac{1}{2}}$$

ablishing (28).

REFERENCES

CH, C. M. AND R. DAVIDSON, "Unrestricted Statistical Inference with Lorenz Curves and Income Shares," *The Review of Economic Studies*, 50 (October, 1983) 723–735.

LE, D., *The Theory of Linear Economic Models* (New York.: McGraw-Hill, 1960).

STWIRTH, J. L., "The Estimation of the Lorenz Curve and Gini Index," *Review of Economics and Statistics*, 54 (August, 1972), 306–316.

GARD, F. AND A. SANDSTROM, *Measuring Income Inequality*, Stockholm Studies in Statistics I (Stockholm: Almquist and Wicksell International 1981).

CHMOND, J., "A General Method for Constructing Simultaneous Confidence Intervals," *Journal of the American Statistical Association*, 77 (June, 1982), 455–460.

IN, N. E., "The Bonferroni and the Scheffé Multiple Comparison Procedures," *The Review of Economic Studies*, 47 (April, 1980), 255–273.

ER, G. A. F., *Linear Regression Analysis* (New York.: John Wiley & Sons, 1977).

DLINE, M. R. AND H. K. URY, "Tables of the Studentized Maximum Modulus Distribution and an Application to Multiple Comparisons among Means," *Technometrics*, 21 (February, 1979), 87–93.

INTERNATIONAL ECONOMIC REVIEW
Vol. 26, No. 2, June, 1985

64799

EFFICIENT TAXATION IN A STYLIZED MODEL OF INTERTEMPORAL GENERAL EQUILIBRIUM*

BY CHRISTOPHE CHAMLEY[1]

1. INTRODUCTION

A number of recent works have analyzed the efficiency and the incidence of second-best tax systems in models of general equilibrium. These models attempt to be realistic through a high level of disaggregation for the intratemporal allocation of resources.[2] Their extension to a dynamic framework raises difficult issues about the structure of intertemporal preferences, and the bequest motive. As a first step, it is possible to extend the intratemporal model to an intertemporal framework by considering utility functions which are separately additive between periods.[3]

The purpose of this study is to reexamine the problem of efficient taxation in such a dynamic framework. The analysis is conducted in a stylized model, and addresses three issues.

The first issue is to consider the dynamic problem where the tax rates are optimized at each instant, and to investigate whether its solution converges to a steady state which can be compared to the solution of a standard problem formulated in a static framework. The second issue is the measurement of the marginal excess burden of the tax distortions. The method presented here can easily be extended to arbitrary tax structures. For commodity taxes, it is different than the one presented in a previous study, Chamley [1981], where the tax rates are constant over time.

The third issue concerns the determination of the public debt in the long run. This debt is an important fiscal instrument because the government budget constraint should apply over the entire horizon and not at each instant. As for any individual, transitory differences between public revenues and expenditures imply borrowing or savings. Barro [1979] has recently shown the interesting result that the long-run level of the public debt may be indeterminate, i.e., that it depends on its value at the beginning of the policy horizon. An essential aspect of Barro's analysis is that the excess burden of taxation in each period is an *ad hoc* quadratic function of the tax revenues raised at the same time. Although the

* Manuscript received August, 1983; revised May, 1984.

[1] Discussions with Paul Champsaur, Ngo Van Long, Pierre Pestieau, Laurence Kotlikoff were very helpful. Financial support from the Sloan Foundation and C. O. R. E. is gratefully acknowledged. This is a revised version of CFDP 554.

[2] For a summary of the literature, see Fullerton and Gordon [1981].

[3] An axiomatic foundation of these functions is given by Koopmans [1972] (see also the other references cited there).

mulation may be justified by administrative costs, it is not consistent with the dard approach to the excess burden which considers the distortions in the ividual optimizing behavior. For example, individual decisions at any instant end not only on the contemporary set of tax rates but also on the entire future gram of the fiscal policy.

hese issues are analyzed here in a model which is simplified as much as possible. private sector is represented by a single family which lives forever.[4] owing an argument of Barro [1972], this family is representative of finitely d individuals with an operative bequest motive. This assumption introduces important difference between this study and previous work on intertemporal ation in a life-cycle framework with no bequest (see Pestieau [1974]; Atkinson Sandmo [1980]). The assumption on an infinite horizon is also useful in dressing the problems described previously.

n the next section, the static problem of efficient commodity taxation is reformud in a way which is slightly different from the standard method. This presen- on is more convenient for the computation of the excess burden of taxation, d for the comparison with the dynamic approach. To simplify the exposition, analysis of the dynamic problem uses the wage tax as an example of fiscal trument (together with the public debt). The extension to other commodity es would be straightforward. The case of a tax on the income of capital, wever, is somewhat different, and is treated in another paper Chamley [1984]. The main part of the paper is the presentation of the dynamic problem of cond best in sections 3.1 to 3.3, which can be read independently of section 2. important aspect of this study is that factor prices are endogenous to factor puts. This implies that the incidence of the wage tax is partially shifted to pital (the tax reduces the labor supply and, therefore, the rate of return of pital). Since the capital income tax is time inconsistent, the wage tax has the me property. This effect explains the difference between the steady state in the ynamic framework and the stationary state characterized in section 2. They e compared in section 3.4. The computation of the efficiency cost is illustrated y an example.

A novel aspect of this paper is a proof of the local stability of the steady state the dynamic problem of second best (when the marginal efficiency cost of xation is not too large). With this proof, one can also suggest a method for a umerical solution. A qualitative summary is presented in the conclusion.

2. THE STATIC PROBLEM OF SECOND BEST

In this section, the standard optimal tax problem with one representative indiidual is presented in a way which is slightly different from the usual method of exposition; this will provide a useful point of reference for the dynamic problem

[4] Brock and Turnovsky [1981] analyze tax incidence in the model of a single infinitely lived individual, and adress mainly issues about capital income taxation.

in the next section. Assume that there are N consumption goods. The utility function of the representative individual is of the form $u(x, l, g)$, where x represents the vector of consumption, l the labor supply, and g the level of government expenditures. It is strictly concave.

Without loss of generality, the resource constraint is equal to

$$\sum_{i=1}^{n} x_i + g = wl + \pi,$$

where w is the (fixed) gross wage rate, and π represents a term of "pure profit," which accrues to the individual income. This term is exogenous. It is introduced mainly for the sake of exposition, and will facilitate later "thought experiments" and comparisons with the dynamic case. The fiscal instruments are linear *ad valorem* taxes of commodities $i=1,\dots,M$ $(M<N)$, and on labor. The components of vector of net consumer prices are denoted by q_i ($q_i=1$ for $M<i\leqslant N$), and $\bar{w}$ represents the net wage rate.[5]

The standard method for the problem of second-best taxation is to use the indirect utility function as the objective function of the central planner. This function embodies the optimizing behavior of the private sector, i.e. its first-order conditions and the private budget constraint. These two types of constraints are distinguished here: when the consumer faces the net prices q_i $(i=1,\dots,N)$, $\bar{w}$, the lump-sum income π, and the government expenditures g, he chooses a consumption vector x, and a labor supply l which satisfy the marginal conditions

$$\text{(1)} \qquad \frac{\partial u}{\partial x_i} = \alpha q_i, \qquad i = 1,\dots,N, \qquad \frac{\partial u}{\partial l} = -\alpha\bar{w},$$

and the budget constraint

$$\text{(2)} \qquad \sum_{i=1}^{N} q_i x_i = \bar{w}l + \pi.$$

The variable α is the marginal utility of income. Its value in the optimal consumer's program depends on q_i, $(i=1,\dots,N)$, $\bar{w}$, π and g. Although this variable is not a fiscal instrument of the government, it is an endogenous variable in the mathematical solution of the second-best problem. By inversion of the system of first-order conditions (1), the vector x and the level of l can be expressed as functions of the vector q, and of $\bar{w}$, α, g:

$$\text{(3)} \qquad x = x(q, \bar{w}, \alpha, g), \qquad l = l(q, \bar{w}, \alpha, g).$$

[5] It is assumed that not all the goods are taxable, in order to avoid a minor technical difficulty when pure profits exist. See Munk [1980].

..y substitution in the function $u(x, l, g)$, one obtains what will be called the ..ni-indirect utility function:

$$v(q, \bar{w}, \alpha, g) = u(x(q, \bar{w}, \alpha, g), l(q, \bar{w}, \alpha, g), g).$$

..te that the indirect utility function is obtained by elimination of α in Equation through the private budget constraint Equation (2). Since this constraint is embodied in the semi-indirect form given in Equation (4), it has to be taken ɔ account separately in the mathematical formulation of the problem. This ıstraint is equivalent to the resource constraint when the government budget ıstraint is satisfied. The second-best optimization problem is then formulated follows:

Problem (P_0). $\underset{q_i(1\leq i\leq M), \bar{w}, \alpha, g}{\text{Maximize}}$ $v(q, \bar{w}, \alpha, g)$,

..ubject to

$$\pi + wl - \sum_{i=1}^{N} x_i - g = 0,$$

$$\tau + (w-\bar{w})l + \sum_{i=1}^{M} (q_i - 1)x_i - g = 0,$$

..nere x and l are functions of $(q, \bar{w}, \alpha, g)$ in Equation (3). In the government ..dget constraint in Equation (6), the term τ represents a fixed lump-sum tax. : is usually equal to zero, but is introduced here for the sake of exposition (as ..re profits are included in Equation (5)).

One should emphasize at this point the difference between the mathematical ..lution of the problem of second-best (P_0), and the implementation of the ..ution through the fiscal instruments. The variables q_i $(1 \leqslant i \leqslant M)$, $\bar{w}$, α and g ..e endogenous in the programming problem (P_0). To implement the optimal olution, the planner needs to announce only the values of the fiscal instruments $_i$ $(1 \leqslant i \leqslant M)$, and $\bar{w}$. The private individual will then choose values for x, l and which are identical to those found in the solution of (P_0), because his optimizing ehavior is already taken into account in the functions of the system (3) (for the ..rst-order conditions), and in the relations (5) and (6) (for the budget constraint).

The Lagrangean associated to the problem (P_0) is equal to:

$$\begin{aligned} L(q, \bar{w}, \alpha, g, \lambda, \mu, \pi, \tau) = {} & v(q, \bar{w}, \alpha, g) \\ & + \lambda(\pi + wl - \sum_{i}^{N} x_i - g) \\ & + \mu(\tau + (w-\bar{w})l + \sum_{i=1}^{M} (q_i - 1)x_i - g). \end{aligned}$$

The interpretation of the multipliers is straightforward: the shadow price of the ..esources is equal to λ. Assume that the lump-sum tax increases by $d\tau$, keeping g ..onstant. Distortionary taxes are thus reduced by an equal amount. According ɔ (7), the level of the individual's welfare increases by $\mu d\tau$. Therefore the shadow ..rice μ represents the marginal efficiency cost of distortionary taxation (in utility ..erms).

One can deduce here a very simple rule for the optimal level of government expenditures. Differentiating the Lagrangean L with respect to g, the first-order condition is

$$\frac{\partial v}{\partial g} = \lambda + \mu. \tag{8}$$

The marginal utility of g in the semi-indirect utility function v at the optimum should be equal to the sum of the social values of the resources used, and of the marginal efficiency cost of distortionary taxation which are necessary to finance the government expenditures. The term $\partial v/\partial g$ is in general different from the marginal utility of government expenditures in the direct utility function, unless this function is additively separable between the vector (x, l) and g.[6]

The main purpose of the formulation (P_0) is to provide an easy comparison between the static and the dynamic results. The following lemma will be useful to highlight the differences and the similarities.

LEMMA. *The solution to the problem* (P_0) *satisfies the first-order conditions* $\frac{\partial L}{\partial \bar{w}} = \frac{\partial L}{\partial \lambda} = \frac{\partial L}{\partial \mu} = \frac{\partial L}{\partial g} = \frac{\partial L}{\partial q_i} = 0$ $(i = 1, \ldots, M)$, $\frac{\partial L}{\partial \alpha} = 0$, *where the Lagrangean* L *is defined in* (7).

The same method applies for the solution of a second-best problem where the set of fiscal instruments is restricted to one tax in a one-good economy (with variable labor supply and exogenous government expenditures). In this case, there is no degree of freedom in the policy determination (in the static framework). However, the method is still useful for the determination of the marginal efficiency cost of taxation. This assumption will be made in the following section to simplify the notation.

3. DYNAMIC OPTIMIZATION

3.1. *The framework.* In this section, the static second-best problem is extended to the intertemporal framework. To simplify the exposition, and without loss of generality, it is assumed that there is a single good produced with capital and labor. This good can be consumed or saved to augment the stock of capital. It will be the numéraire. The private sector is represented by a family with an infinite life, growing at the rate n, with a utility function given by:

[6] A different form of the formula (8) is given by Atkinson and Stern [1974], who discuss extensively the problem of the optimal supply of public goods with second-best taxation. Represent the vector (x, l) by z in (4). Since $v(z, g) = u(z(\alpha, q, g), g)$, $\frac{\partial u}{\partial g} = {}^t\frac{\partial u}{\partial z}\frac{\partial z}{\partial g} + \frac{\partial u}{\partial g}$. Using matricial notation, the first term of the right hand side is equal to, $-{}^t\frac{\partial u}{\partial z}\left(\frac{\partial^2 u}{\partial z^2}\right)^{-1}\frac{\partial^2 u}{\partial z \partial g}$. It is nil when u is additively separable in g, $(\partial^2 u/\partial z \partial g = 0)$, but it is otherwise generically different from zero.

$$U = \int_0^\infty e^{-(\rho-n)t} u(c_t, l_t, g_t)dt,$$

ere ρ, c_t and l_t represent the pure rate of time preference, the consumption level 3 the labor supply, respectively. All quantities are measured *per capita*. The ue of ρ is greater than n, for convergence of U.

The level of output y, net of capital depreciation, is given by a neoclassical duction function $f(k, l)$ which has the usual properties. For simplicity, function is homogenous of degree one.[7] The factor inputs are not perfect ostitutes, unless specified explicitly.

In an intertemporal framework, there is *a priori* no reason to assume that the ernment budget is balanced at each instant. Differences between receipts and enditures are met by issuing or redeeming bonds. Since there is no uncertainty, se government bonds are perfectly substitutable with capital and have the same e of return. The level of the debt will be represented by b (it is negative when government owns capital). For notational reasons, we also introduce the el of public assets $h = -b$. The level of private assets a, is therefore equal to $-b$, or $k-h$.

The tax instruments are the rates of the linear tax on labor, at different instants. s equivalent to consider the net wage rate $\bar{w}_t$ as the instrument. Also, in order concentrate on efficient taxation, the stream of government expenditures is nstant over time.[8] Therefore, one can omit g from the utility function u. e government chooses the most efficient fiscal program to finance its expendi- es under the constraints imposed by the optimizing behavior of the representa- family.

This family behaves competitively and is endowed with perfect foresight; it ximizes the intertemporal utility U given in (9), under its budget constraint. e intertemporal program is the solution of an optimal control problem, and the els of consumption and labor satisfy the relations $\frac{\partial u}{\partial c} = \alpha$, $\frac{\partial u}{\partial l} = -\alpha\bar{w}$, where the shadow price of the private accumulation of assets.[9] By inversion of se relations, c and l can be expressed as functions of $\bar{w}$ and α:

$$c = c(\bar{w}, \alpha)$$

$$l = l(\bar{w}, \alpha).$$

ese expressions correspond to the expressions in (3) for the static case.

The variable α defines the marginal utility of wealth at each instant. Its vari- on is given by the relation:[10]

This assumption simplifies the results in the steady state.

In a complete optimization, the value of g_t is determined by a cost-benefit rule as in (8).

The time subscript will be omitted when there is no ambiguity.

[10] This relation is equivalent to $\frac{\partial \mathscr{H}}{\partial a} = (\rho-n)\alpha - \dot{\alpha}$, where $\mathscr{H}$ is the Hamiltonian of the repre- tative family and is equal to:

$$\mathscr{H} = u(c, l) + \alpha((r-n)a + \bar{w}l - c).$$

$$\dot{\alpha} = \alpha(\rho - r),$$

where r is the rate of return. The private sector chooses the initial value α_0 such that its budget constraint is satisfied. (This constraint is equivalent here to the transversality condition.) The value of α_0 depends on the entire set of future net prices and on the initial private wealth. As in the static case, the variable is an endogenous variable in the mathematical formulation of the second-best problem which can now be written as follows:

Problem (P). $\underset{\bar{w},\alpha,k,h}{\text{Maximize}} \int_0^\infty e^{-(\rho-n)t} v(\bar{w}, \alpha)dt$ subject to the constraints

(11) $$k = f(k, l) - nk - c - g,$$

(12) $$h = (r-n)h + (w-\bar{w})l - g,$$

(13) $$\dot{\alpha} = (\rho - r)\alpha,$$

where c and l are functions of $\bar{w}$ and α (in (10)), and the gross factor prices r and w, are equal to $\frac{\partial f}{\partial k}$ and $\frac{\partial f}{\partial l}$ respectively; they depend on k and l.

The initial levels of the capital stock k_0 and of the public assets h_0 are exogenous. Indeed, if the government could manipulate h_0, lump-sum transfers of wealth between the private sector and the government sector would be possible (the private wealth a_0 is equal to $k_0 - h_0$). This is ruled out by the second-best assumption.

The formulation of the problem (P) is similar to that of the atemporal problem (P_0) and calls for the same remarks. Note that there is the constraint (13) on the evolution of α, but there is no constraint on the initial value α_0 (as there is no constraint on the variable α in the static case.) The equation of the private wealth accumulation $\dot{a} = k - h$ is omitted because it is redundant in view of (11) and (12). Since the dynamic path will converge to a steady state, the budget constraints of the economy and of the government will be satisfied, and the budget constraint of the private sector is redundant.

For the implementation of the second-best policy, the government first solves for the optimal values of $\bar{w}$, α, k, h in the mathematical problem (P). After this solution is computed, the government announces only the values of the wage tax rate (or the net wage rate). Then, the private sector, which is endowed with perfect foresight, computes its optimal solution. This program is, of course, identical to the solution of the problem (P) because the private optimization is taken into account in the latter.

3.2. *The Solution.* The problem (P) is solved by considering the Hamiltonian

(14) $$H(\bar{w}, \alpha, g, \lambda, \mu, \nu, k, h) = v(\bar{w}, \alpha) + \lambda(f(k, l) - nk - c - g) + \mu((r-n)h + (w-\bar{w})l - g) + \nu\alpha(\rho - r)$$

The interpretation of the shadow prices λ and μ has already been given in the

atic model. The shadow price v is associated to the intertemporal first-order condition of the private sector (equation (13)). Since there is no restriction *per se* on the initial value of α, the value of v at time zero, v_0, is equal to zero.

The solution of (P) is computed more easily by considering first the time variation of the marginal excess burden of taxation, μ,

$$\dot{\mu} = (\rho - n)\mu - \frac{\partial H}{\partial h}, \tag{15}$$

which is equivalent to

$$\dot{\mu} = (\rho - r)\mu. \tag{16}$$

Comparing this equation with (13), it follows that the ratio μ/α is constant over time, i.e., there exists a number φ such that:

$$\frac{\mu}{\alpha} = \varphi. \tag{17}$$

This ratio measures the marginal excess burden of taxation in terms of private consumption. When there is no restriction on the level of government deficit (or surplus), the current value of the marginal efficiency cost of taxation is constant over time.[11]

The solution to the problem (P) satisfies also the other first-order conditions given by the equations (11), (12), (13), and by

$$\dot{\lambda} = (\rho - n)\lambda - \frac{\partial H}{\partial k} \tag{18}$$

$$\dot{v} = (\rho - n)v - \frac{\partial H}{\partial \alpha} \tag{19}$$

$$\frac{\partial H}{\partial \bar{w}} = 0. \tag{20}$$

The computation of the solution proceeds as follows: assume first that a value of φ is given. The evolution of the quintuplet of variables $(k, h, \alpha, \lambda, v)$ is determined by the five dynamic equations (11), (12), (13), (18) and (19). The value of $\bar{w}$ is determined by (20).

For the initial conditions, the two values k_0 and v_0 are exogenous ($v_0 = 0$). There is a unique triplet of initial values $(\hat{h}_0, \hat{\alpha}_0, \hat{\lambda}_0)$ such that the dynamic system converges to a steady state (which is analyzed below). This stability is proven in the Appendix for a neighborhood of the steady state and when the elasticity of the labor supply is not too large.[12] The triplet $(\hat{h}_0, \hat{\alpha}_0, \hat{\lambda}_0)$ depends on the choice of φ in the first step. In particular, $\hat{h}_0 = \hat{h}(\varphi)$.

[11] This rule applies even if the tax on the wage rate is exogenous during some interval of time.

[12] These two assumptions are only sufficient. A formal analysis under more general assumptions is beyond the scope of this paper. Also, no solution may exist if the level of government expenditure is too high.

The computation of the solution is completed by choosing the value of φ such that $\hat{h}_0$ is equal to the initial level of the public assets h_0, which is predetermined[13]:

$$\hat{h}(\varphi) = h_0. \tag{21}$$

Finally, the transversality conditions of the problem (P) are satisfied because the dynamic solution converges to a steady state. The previous discussion is summarized by the following result.

PROPOSITION 1. *The solution to the second-best problem (P) has a dynamic path which converges to a steady state if the elasticity of the labor supply is not too large and the initial values of the capital stock and of the public debt are near their steady values. This steady state depends on the marginal value of the excess burden of taxation. When this efficiency cost is measured in units of private consumption, it is constant on the transition path. Its value depends on the levels of the capital stock and of the public dept at the beginning of the policy horizon.*

3.3. *The Steady State.* The steady state is characterized by the stationary forms of the dynamic equations. The gross rate of return r^* is equal to the discount rate ρ[14] (equation (13)). By the factor price function frontier, this rate determines also the gross wage rate w^*.

It is useful to consider first the steady state value of v. Assume that the factor inputs are not perfect substitutes (the case of fixed factor prices is entirely different and treated below). Because the production function has constant returns to scale, the slope of the factor price frontier is equal to $\partial w/\partial r = -k/l$. Using this relation in (18) (with $\dot{\lambda} = 0$ and $r = \rho$),

$$v = -\varphi a, \quad \text{with} \quad a = k - h, \quad \varphi = \mu/\alpha. \tag{22}$$

Note that in general, the value of v is different from zero.

To determine the other endogenous variables, it is convenient to introduce the function

$$\begin{aligned} H^*(\bar{w}, \alpha, g, \lambda, \mu, v, k, h) = {} & v(\bar{w}, \alpha) + \lambda((r^* - n)k + w^* l - c - g) \\ & + \mu((r^* - n)h + (w^* - \bar{w})l - g), \end{aligned} \tag{23}$$

where $r^* = \rho$, and w^* is determined by the factor price frontier. One can show with a straightforward manipulation that the stationary forms of the equations (11), (12), (20) and (19) are equivalent to $\frac{\partial H^*}{\partial \lambda} = 0$, $\frac{\partial H^*}{\partial \mu} = 0$, $\frac{\partial H^*}{\partial \bar{w}} = 0$, $\frac{\partial H^*}{\partial \alpha} = -\frac{\mu}{\alpha}$. $(\rho - n)(k - h)$, respectively.[15]

[13] The value of φ represents the marginal excess burden of taxation. It is positive if the level of public assets at time zero, h_0 is smaller than the present value of public expenditures. Economic intuition indicates that if positive, its value decreases with the level of initial government wealth h_0.

[14] Steady states values will be represented with an asterisk.

[15] The last of these relations is proven by substituting for v in (19), the expression found in (22). Note that this derivation is possible only when the inputs are not perfect substitutes.

PROPOSITION 2. *When the elasticity of substitution between capital and labor is ite, the steady state of the solution to the problem (P) is characterized by the vations:*

$$.4) \qquad \frac{\partial H^*}{\partial \bar{w}} = \frac{\partial H^*}{\partial \lambda} = \frac{\partial H^*}{\partial \mu} = 0, \qquad \rho = f_1\left(\frac{k}{l}, 1\right)$$

$$.5) \qquad \frac{\partial H^*}{\partial \alpha} = -\frac{\mu}{\alpha}(\rho - n)(k - h).$$

nese equations determine the values of $\bar{w}$, λ, μ, k and α, for a given level of public sets h^*. The values of h^* depend on the initial values k_0 and h_0.

n equation (24), the labor supply l should be interpreted as a function of α $\bar{w}$, as given in (10). The term $k-h^*$ in (25) represents the level of private sets in the steady state. Also, the ratio μ/α represents the marginal efficiency st of taxation in consumption equivalent, and has the same non-zero value as the transition path (Proposition 1).

The values of the endogenous variables in the steady state depend on the rameters of production and demand such as the elasticity of substitution, the asticity of the labor supply, ρ, n, etc. $\cdots$, in the equations of Proposition 2. The aracterization of the steady state has been formulated in this way to facilitate the mparison with the static method. The interpretation of the excess burden, example, in terms of the fundamental parameters of the model is well known the latter framework, which is equivalent to the standard partial equilibrium. similar interpretation in the dynamic framework will be done after comparing e two methods. However, the two previous results imply first an interesting operty of the second-best dynamic path.

In the steady state, the level of the public is in general, not equal to zero. As or other endogenous variables, it depends on the marginal efficiency cost of cond-best taxation. This excess burden depends on the amount of revenues hich are raised through distortionary taxation. This amount depends itself on level of the public debt (positive or negative), at the beginning of time. In neral, one can expect that a relatively high level of the debt at time zero generates o a relatively high level of the debt in the steady state.[16]

The indeterminacy of the level of the debt in the long-run was first described by arro [1979], in a model with exogenous income and *ad hoc* quadratic ss function for the excess-burden. The above results show that this property lds in a general equilibrium model with endogenous capital accumulation and ctor prices, and a "standard" efficiency cost function which is derived explicitly om the choice distortions created by the entire path of tax rates.

3.4. *The Comparison Between the Static and the Dynamic Approaches.*

[16] This relation is monotonic at least when the marginal efficiency cost of taxation is suffi- ntly small.

The comparison between the steady states in the two methods follows from the characterizations in the Lemma and Proposition 2. The functions L and H^* are identical when the terms of "pure profits," π, and of "lump-sum revenues," τ, are replaced by the incomes of capital and public assets, net of growth, $(r^*-n)k$ and $(r^*-n)h$, respectively. The first-order conditions are identical in the two cases, except for the condition on $\partial H^*/\partial\alpha$. Therefore, the steady state solution of the dynamic problem is not equivalent in general, to the solution of a static problem (with exogenous pure profits and revenues from public wealth, positive or negative).

Other remarks can be made here.

1. The two solutions are identical if $\rho = n$. This result is obvious; when the discount rate and the growth rates are identical, the relative weight of any transition period is nil with respect to the long run, and the steady state analysis is appropriate.

2. The characterization of the steady state in Proposition 2 depends on the utility function and the excess-burden but not on the elasticity of substitution in production which is not an argument of H^* (this elasticity is obviously not an argument in the partial equilibrium result of the Lemma). However, we will see below that the excess-burden does depend on the elasticity of substitution.

3. The difference between the results in the Lemma and Proposition 2 arises in equation (25). This relation is valid only when the production factors are not perfect substitutes (in the derivation of (22)). The downward slope of the factor price frontier raises the issue of time-inconsistency which should be addressed now.

3.4.*a*. *The Time-Inconsistency of the Optimal Wage Tax.* In the dynamic solution of the problem (P), the value of the shadow price of the marginal utility of consumption v, varies over time from zero to a strictly negative limit which is determined by the steady state relation (22). Therefore, there exists an instant $t_0 > 0$ such that if $t \geq t_0$, $v_t < 0$. If the problem (P) were reformulated at this instant t_0 for the subsequent time span, the solution would imply a "new" value $\bar{v}_t$ equal to zero. This implies that the new solution would be different from the program chosen at the origin of time, for the same interval $(t_0, +\infty)$. The optimal wage tax is time-inconsistent. There are two (equivalent) interpretations of this situation.

The mathematical explanation is simple. In the problem (P_0), there is no restriction on the value of α_0. The values of α_t $(t>0)$, are then determined by integration of the dynamic equation (13) which corresponds to the intertemporal optimization of the private sector. This constraint prevents discontinuous jumps of the variable α. If the problem (P) is reformulated at a time t $(t>0)$, a jump of the variable α becomes admissible at this instant, and if this occurs, the new solution will be different from the initial program.

More important, there is an intuitive explanation: the time inconsistency of the wage tax occurs because factor prices are endogenous and the factor price

ntier is downwards sloping. When the wage tax reduces the labor supply, e rate of return on capital decreases: the tax incidence is partially shifted to pital. This effect is particularly clear in the special case where the elasticity of bstitution between capital and labor is equal to zero; the shift is then complete, least in the short run. It is well known that the taxation of capital is time-onsistent because it is a fixed factor in the short run and its supply is elastic the long run, Fisher [1980]. The shift of the incidence implies that the wage e is also time-inconsistent.

The smaller the elasticity of substitution, the greater the incidence shift. The vernment by increasing the wage tax at the beginning of the policy horizon, n raise a greater fraction of revenues with an indirect capital levy, and diminish e efficiency cost of indirect taxation. Therefore, a lower elasticity of substitution ween capital and labor implies, *ceteris paribus*, a smaller efficiency cost of ation. When the elasticity of substitution between capital and labor tends to finity, the benefits which can be reaped by a relatively higher initial wage tax ninish. At the limit, when the elasticity is infinite, factor prices are exogenous ith $r=\rho$ for the existence of a steady state), and no benefit can be gained by a gher initial wage tax.

A simple exercise shows that in this case, the second best path is reduced to the eady state characterized in the Lemma. In particular, the convergence result Proposition 2 is not valid anymore. This apparent discontinuity of the conver-ence result is resolved by considering the stable eigenvalues of the dynamic ystem which defines the dynamic path. These eigenvalues provide an index of e speed of convergence of the economy towards the steady state of Proposition 2. is shown in the Appendix that when the elasticity of substitution tends to infinity, ese eigenvalues tend to zero: the steady state of Proposition 2 is more and more stant in the future when viewed from the initial instant. At the limit, it is infi-tely distant. The initial position (immediately after the time of tax reform)[17] ecomes the steady state. Also, if there is later another opportunity for a change f the tax rates, no further change is made since the policy is time-consistent.[18]

The problem of time-inconsistency is unavoidable in dynamic second-best xation with endogenous prices because the incidence of tax changes falls at least artially on the fixed factors. A benevolent social planner can always reduce the fficiency cost of taxation by reforms if they are unexpected. A solution of this problem is obviously beyond the scope of this paper. A possible answer has already been given by Auernheimer [1974] for the inflation tax,[19] and can also e applied here: if the social planner takes at each instant the private marginal

[17] There may be a difference between the tax rates before and after time zero if the first tax structure is not efficient.

[18] When the elasticity of substitution is large the dynamic path is relatively "flat" in a diagram with time or the horizontal axis. Its curvature is inversely related to the value of the elasticity.

[19] The constraint proposed by Auernheimer is to exclude a jump of the price of money from the value inherited from the past. The extension of this principle to a second-best situation is analyzed in Chamley [1983].

utility of wealth α (which is determined by policies announced before time t) as given, the optimal wage tax is time-consistent (jumps of α are excluded). Under this restriction, tax reforms are admitted only if they do not generate an indirect levy on existing assets. The shadow price of this constraint is equal to ν.

The steady state characterized by Proposition 2 can then be placed in two different contexts. In the first, this state is the point of convergence of the dynamic path as time goes to infinity. In the second context it can be a current state in which the policy maker foregoes indirect capital levies.

Both interpretations provide a simple explanation for the fact that the characterization of the steady state depends on the amount of tax revenues and on the excess-burden of taxation, but does not depend directly on the elasticity of substitution between capital and labor. In the second-best situation with commodity taxation, this elasticity matters only when the taxation of capital is desirable. the incidence of commodity taxes on capital income achieves at least partially this goal when the elasticity is finite. But in both the above interpretations a social planner would not use a capital income tax if it were an available instrument. In the first, the asymptotic result of zero capital income taxation is shown for a similar model in Chamley [1984]. In the second, the capital levies are excluded by assumption.[20]

In the two previous interpretations, the steady state of the dynamic policy of second best depends on the utility function and on other parameters such as the discount and the growth rates. An example will be useful for the comparison with the static method which was described in the Lemma.

3.4.b. *The Efficiency Cost of Taxation — An Example.* Assume that the current utility function u, is additively separable in c and l. This implies that in (10), the level of consumption c is only a function of α. Also, the level of labor depends only on the product of α and $\bar{w}$, and $\alpha l'_\alpha = \bar{w} l'_{\bar{w}}$.

Since $v'_{\bar{w}} = u'_l l'_{\bar{w}} = -\alpha \bar{w} l'_{\bar{w}}$, the equation $\partial H/\partial \bar{w} = 0$ in (24) is equivalent to

$$(-\alpha\bar{w} + \lambda w + \mu(w-\bar{w}))\bar{w} l'_{\bar{w}} = \mu \bar{w} l. \tag{26}$$

In the same way, the equation (25) takes the form, after multiplication by α,

$$(\alpha-\lambda)\alpha c'_\alpha + (-\alpha\bar{w} + \lambda w + \mu(w-\bar{w}))\alpha l'_\alpha = (\delta-1)\mu(\rho-n)a, \tag{27}$$

where $\delta=0$ in the dynamic approach (Proposition 2), and $\delta=1$ in the steady state method (Lemma). Using $\alpha l'_\alpha = \bar{w} l'_{\bar{w}}$, $\bar{w} l = c-(\rho-n)a$, and an obvious substi-

[20] The value of the shadow price of this constraint, ν, in the steady state has an intuitive interpretation. Assume that a small change occurs in the tax program, which decreases the value of q by $\Delta q<0$. This implies a capital loss of $\frac{\Delta q}{q}a$ on the private assets, which can be regarded as a lump-sum levy by the government. Distortionary taxes are reduced by the same amount, and there is an efficiency gain of $-\mu\frac{\Delta q}{q}a$. Therefore the shadow price of maintaining the value of q unchanged is equal to $-\frac{\mu}{q}a$, which is the value of ν in (22).

n in (26), one finds

$$(\lambda-\alpha)\left(-\alpha\frac{c'_\alpha}{c}\right)+\mu\left(1-\delta(\rho-n)\frac{a}{c}\right)=0.$$

represent the elasticity of marginal utility of consumption at the steady $1/\sigma=-\alpha c'_\alpha/c$:

$$\lambda=\alpha-\sigma\mu\left(1-\delta(\rho-n)\frac{a}{c}\right).$$

the wage tax rate: $\bar{w}=(1-\theta)w$. Substituting for λ in (26), a straightforward pulation gives the marginal efficiency cost of taxation:

$$\frac{\mu}{\alpha}=\frac{\theta}{\frac{(1-\theta)}{\varepsilon}+\theta+\sigma\left(1-\delta(\rho-n)\frac{a}{c}\right)}$$

$\varepsilon=\frac{\bar{w}}{l}l'_{\bar{w}}$ is the elasticity of labor with respect to the wage rate, keeping ant the marginal utility of income α.[21]

e welfare cost of taxation increases with the wage elasticity of labor as in the ard framework. It is also inversely related to the elasticity of the marginal of consumption σ. (This parameter is inversely related to the intertemporal city of substitution of consumption).[22]

e measurement of the excess-burden in the dynamic and the static methods is ssed by the same generic formula (30), with δ equal to zero and to one, ctively. For plausible stylized models, the ratio between asset income net rivate saving on the balanced growth path, and consumption, is relatively with respect to one. In this case, the quantitative difference between the ts of the two approaches is relatively small.[23] The same remark would to other endogenous variables.

4. CONCLUSION

e purpose of this paper is to analyze some aspects of efficient taxation in a mic framework of second-best. It is well known that the determination of ent intertemporal taxation has *a priori* no special feature which cannot be led by the standard approach presented by Diamond and Mirrlees [1971]. ever, the dynamic context introduces specific problems such as the charac-

This is also the elasticity of labor with respect to a temporary increase of $\bar{w}$ (during an tesimal interval of time).

The dynamic efficient cost of the wage tax has been determined in a previous study under striction of constant tax rates on labor income (Chamley, 1981). In the present framework, ormula (30) measures the efficiency cost of taxation when these tax rates are optimized over except for some finite intervals of time where they may be exogenously fixed.

Note that the static method overestimates the exact value of the welfare cost.

terization of a balanced growth path, the determination of the public debt, and the time inconsistency of policy.

The results of this paper show that when the structure of commodity taxes is efficient, the dynamic path converges to a steady state (or balanced growth path), with a simple characterization which is different from the standard solution of a static model which "repeats itself" through time. The variation of the public debt is equal to the deficit. In the second best, it is determined such that the current value of the marginal efficiency cost of taxation is constant over time. This value depends on the total amount of revenues raised through distortionary taxation, and therefore, on the level of public debt at the beginning of the policy implementation.

The policy which equalizes the marginal efficiency cost of commodity taxation through time does not generate a redemption of the debt, a result found also in Barro [1979], and in contradiction with the recommendation of Meade [1958]. This also implies that the steady state depends on the initial conditions of the dynamic path (contrary to the first-best situation, for the present model). A relation between the long-run and the initial values of the public debt occurs because of the operative bequest motive which is implicit in the framework of infinitely lived households. In the life-cycle model with no bequest, Pestieau [1974] has shown that the debt should adjust such that in the long-run the rate of return is equal to the discount rate.

A difference between the results in the dynamic and the static methods arises because the tax incidence is partially shifted to capital and commodity taxation is time inconsistent. The steady state can also be considered as a programme of second-best for policies which excludes such indirect capital levies and is time consistent. The characterization of this policy depends then only on the utility function and on the levels of the various aggregates.

Harvard University, U. S. A.

APPENDIX

1. The Stability of the Dynamic Solution

The Hamiltonian H is equal to:

$$H(w,\alpha,g,\lambda,\mu,\nu,k,h)=v(\bar{w},\alpha)+\lambda(f(k,l)-nk-c-g)$$
$$+\mu((r-n)h+(w-\bar{w})l-g)+\nu\alpha(\rho-r),$$

where c and l are functions of $\bar{w}$ and α (in (10)); r and w are functions of k and l. The ratio $\mu/\alpha=\varphi$ is constant over time (Section 3.2). Assume that a value of φ is given. The time variations of the five remaining dynamic variables, k, α, ν, λ and h are given by the equations:

$$(S)\quad \begin{cases} \dot{k} = f(k, l) - nk - c - g \\ \dot{\alpha} = \alpha(\rho - r) \\ \dot{\nu} = (\rho - n)\nu - \dfrac{\partial H}{\partial \alpha} \\ \dot{\lambda} = (\rho - n)\lambda - \dfrac{\partial H}{\partial k} \\ \dot{h} = (r - n)h + (w - \bar{w})l - g. \end{cases}$$

Consider the limit case where the elasticities of the labor supply l, with respect to $\bar{w}$ and α, become infinitesimal. This implies that the marginal excess burden of the wage tax, μ, is also infinitesimal. The linearization of the dynamic system (S), near the steady state, can then be written as follows:

$$\begin{bmatrix} \dot{k} \\ \dot{\alpha} \\ \dot{\nu} \\ \dot{\lambda} \\ \dot{h} \end{bmatrix} = \left[\begin{array}{c:cc:c} & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ \hdashline & & & 0 \\ & B & & 0 \\ \hdashline & & & \rho - n \end{array}\right] \begin{bmatrix} k - k^* \\ \alpha - \alpha^* \\ \nu - \nu^* \\ \lambda - \lambda^* \\ h - h^* \end{bmatrix}.$$

Values in the steady state are represented with an asterisk.

The matrices A and B are equal to:

$$A = \begin{bmatrix} \rho - n & -c'_\alpha \\ -\alpha r'_k & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \rho - n & c'_\alpha \\ \alpha r'_k & 0 \end{bmatrix}.$$

They have the same eigenvalues which are of opposite signs. It follows immediately that the linearized system has two negative eigenvalues and three positive ones. There are two predetermined initial values, k_0 and $\nu_0 = 0$. However, the equality between the numbers of negative eigenvalues and predetermined variables is not sufficient for local stability.

Consider first the couple of variables (k, α). The initial value of k_0 is exogenous. Since A has one positive and one negative eigenvalue, there is a unique value of α_0, such that the dynamic system (S) is locally stable. More precisely, when k_0 tends to k^*, the ratio $\dfrac{\alpha_0 - \alpha^*}{k_0 - k^*}$ tends to the ratio between the components of the eigenvectors of A, associated to the negative eigenvalue. The same argument applies to the couple of variables (ν, λ), where $\nu_0 = 0$ is predetermined, and it implies the unicity of λ_0.

The last eigenvalue of the linearized system is equal to $\rho - n$ and is positive. There exists a unique value for h_0 ($h_0 = h^*$) such that the dynamic system is stable.

unique choice of $(\alpha_0, \lambda_0, h_0)$ with the predetermined values k_0 and $\nu_0 = 0$, such that the dynamic system is stable near the steady state.[1] By continuity the above argument is valid, at least when the values of the labor elasticities are small. The dynamic system is completely determined by finding φ such that the value $\bar{w}$ (uniquely associated to φ) is equal to its predetermined value.

2. The Limit Case of Exogenous Factor Price

When the response of factor prices to the input levels becomes infinitesimal, we can consider r and w to be constant in the expression of H. The expression $\partial H/\partial \bar{w} = 0$ is then equivalent to:

$$v'_{\bar{w}} + \lambda(wl'_{\bar{w}} - c'_{\bar{w}}) + \mu((w - \bar{w})l'_{\bar{w}} - l = 0.$$

Therefore $\bar{w}$ is a function of α, λ, μ

$$\bar{w} = G(\alpha, \lambda, \mu).$$

In this case, the matrix of the linearized system near the steady state tends to

$$\begin{bmatrix} \rho - n & \cdot & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \rho - n & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \rho - n \end{bmatrix}.$$

A dot represents a term which may be different from zero.

This matrix has zero as an eigenvalue of order 2. This value is the limit of the two negative eigenvalues when the elasticity of substitution between capital and labor tends to infinity. Also, the other (positive) eigenvalues tend to $\rho - n$

REFERENCES

ARROW, K. AND M. KURZ, *Public Investment, the Rate of Return and Optimal Fiscal Policy* (Baltimore: John Hopkins Press, 1970).

ATKINSON, A. B. AND N. B. STERN, "Pigou, Taxation and Public Goods," *The Review of Economic Studies*, 41 (1974), 119–128.

——— AND A. SANDMO, "Welfare Implications of the Taxation of Savings," *Economic Journal* 90 (1980), 529–549.

AUERBACH, A., L. KOTLIKOFF AND J. SKINNER, "The Efficiency Gains from Dynamic Tax Reform," *International Economic Review*, 24 (1983), 81–100.

AUERNHEIMER, L., "The Honest Government's Guide to the Revenue from the Creation of Money," *Journal of Political Economy*, 82 (1974), 598–606.

[1] The uniqueness of a stable solution would not have been verified if the predetermined variables had been k and α, for example. This shows that the condition of two negative eigenvalues is only necessary. The sufficiency must be established by inspection of the matrix and

.O, R. J., "Are Government Bonds Net Wealth?" *Journal of Political Economy*, 82 (1972), 1095–1117.

—, "On the Determination of the Public Debt," *Journal of Political Economy*, 87 (1979), 940–971.

K, WILLIAM A. AND S. J. TURNOVSKY, "The Analysis of Macroeconomic Policies in Perfect Foresight Equilibrium," *International Economic Review*, 22 (1981), 179–209.

ILEY, C., "Efficient Tax Reform in a Dynamic Economy," mimeo, Yale University, *Quarterly Journal of Economics* (forthcoming).

—, "On a Simple Rule for the Optimal Inflation Rate in Second-Best Taxation," *Journal of Public Economics* (forthcoming).

—, *Optimal Capital Income Taxation in Economics with Identical Private and Social Discount Rates*, (mimeographed), CFDP 699, Yale University (1984).

OND, P. A., "Taxation and Public Production in a Growing Setting," in, J. A. Mirrlees and N. H. Stern eds., *Models of Economic Growth* (London: MacMillan, 1973).

— AND J. A. MIRRLEES, "Optimal Taxation and Public Production: II — Tax Rules," *American Economic Review*, 61 (1971) 261–278.

R, S., "Dynamic Inconsistency, Co-Operation and the Benevolent Dissembling Government," *Journal of Economic Dynamics and Capital*, (1980), 93–107.

ERTON, D. AND R. H. GORDON, *A Reexamination of Tax Distortions in General Equilibrium Models*, (mimeographed), National Bureau of Economic Research, No. 673, (1981).

MANS, T. C., "Representation of Preference Orderings over Time," in, C. B. McGuire and R. Radner eds., *Decision and Organization: A Volume in Honor of Jacob Marschak* (Amsterdam: North Holland, 1972).

OF, J. E., "Is the National Debt a Burden?" *Oxford Economic Papers*, 10 (1958), 163–183.

K, K. J., "Optimal Taxation and Pure Profit," *The Scandinavian Journal of Economics*, 80 (1980), 1–19.

EAU, P. M., "Optimal Taxation and Discount Rate for Public Investment in a Growth Setting," *Journal of Public Economics*, 3 (1974), 217–235.

TZ, J. E. AND P. DASGUPTA., "Differential Taxation, Public Goods, and Economic Efficiency," *Review of Economic Studies*, 38 (1971), 151–174.

INTERNATIONAL ECONOMIC REVIEW
Vol. 26, No. 2, June, 1985

ESTIMATION IN A LINEAR MODEL WITH SERIALLY CORRELATED ERRORS WHEN OBSERVATIONS ARE MISSING*

BY TOM WANSBEEK AND ARIE KAPTEYN[1]

1. INTRODUCTION

There exists an extensive literature on estimation and testing in linear regression models with first-order serially correlated errors. For the case where a string of consecutive observations is missing there have appeared a number of recent articles dealing with various tests of autocorrelation (cf. Savin and White [1978], Richardson and White [1979], Honohan and McCarthy [1982]). Obviously, many time series suffer from missing observations, like long annual series from which observations on war years are missing, or daily series that are not observed during weekends.

The purpose of this paper is to develop the ML estimator for a linear regression model with serially correlated errors when observations are missing. The results derived are generalizations of those by Beach and MacKinnon [1978]. Using both actual and simulated data we compare computational and statistical aspects of the ML estimator to those of some 'intuitive' estimators based on adaptations of suggestions by Cochrane and Orcutt [1949], Prais and Winsten [1954] and Maeshiro [1976, 1979].

In section 2, we present the model. In section 3, we present some results on the structure of the error covariance matrix and develop a convenient matrix notation which facilitates the algebraic derivations. Section 4 presents the ML estimator and the information matrix. In section 5, some alternative two-stage estimators are defined. In section 6, we present results of experiments designed to compare the computational and statistical properties of the ML and two-stage estimators. Section 7 concludes.

2. THE MODEL

Consider the single-equation regression model

$$y = X\beta + \varepsilon, \tag{1}$$

* Manuscript received March, 1981; revised August, 1984.

[1] We are grateful to the editor and the referees for their constructive comments, to H. van Terheijden, D. Bijlsma and C. Mijderwijk for extensive programming support and to A. ten Cate for comments and discussion. A first version was written while Kapteyn was with the University of Southern California, Los Angeles. Throughout the writing of the various versions, Wansbeek was with the Netherlands Central Bureau of Statistics (CBS). The views expressed in this paper