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REAL TIME CONTROL
OF
RESERVOIS SYSTEMS

by

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PREFACE

This report is one of a series of publications which describe various studies undertaken under the sponsorship of the Technology and Development Program at the Massachusetts Institute of Technology.

The United States Department of State, through the Agency for International Development, awarded the Massachusetts Institute of Technology a contract to provide support at M.I.T. for the development, in conjunction with institutions in selected developing countries, of capabilities useful in the adaptation of technologies and problem-solving techniques to the needs of those countries. This particular study describes research conducted in conjunction with Cairo University, Cairo, Egypt. The analysis, conclusions, and recommendations are those of the author and do not necessarily reflect the views of the Agency for International Development.

In the process of making the TDP supported study, some insight has been gained into how appropriate technologies can be identified and adapted to the needs of developing countries per se, and it is expected that the recommendations developed will serve as a guide to other developing countries for the solution of similar problems which may be encountered there.

Fred Moavenzadeh
Program Director

ABSTRACT

This work presents a new method for the real time operation of reservoir systems.

The system is represented by a set of nonlinear differential equations describing the reservoir and river dynamics in state space form form. The formulated reservoir operation problem calls for finding policies which maximize the expected benefits of one system's objective while satisfying the remaining objectives at prespecified reliability levels. The solution is obtained by a new method named Extended Linear Quadratic Gaussian (ELQG) controller. ELQG draws on and extends stochastic control theory results and it is well suited for the optimization of constrained dynamical systems. It is a trajectory iteration algorithm theoretically expected to exhibit reliability and computational efficiency.

The new method is employed in case studies of two actual reservoir systems located in the River Nile Basin. The results indicate that ELQG's theoretical properties are indeed realized in practice and that the method has distinct advantages over state-of-the-art procedures.

ELQG control offers a powerful approach in reservoir system management and opens a host of further research directions.

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To my friends at the Parson's Lab. I extend my greetings. Their presence made my effort easier.

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Chapter 1

INTRODUCTION

1.1 Thesis' General Framework and Outline

Reservoir Systems are large-scale projects generating a variety of influential outputs. Water supply for municipal, industrial, or agricultural use, flood protection, energy generation, and navigation are some of the many benefits these systems offer to the societies living within their inflence range. To fulfil these objectives, reservoir systems must successfully manage the water volumes provided by the uncertain natural processes. This is where the element of risk originates and often becomes the cause of costly operational failures. The need of combining mathematical tools with the ever-growing computer technology in the management process has well been realized by both researchers and practitioners (Proceedings of the National Workshop on Reservoir Systems Operations, University of Colorado, Boulder, August 1979). Reservoir operation research has been supported for over 25 years and has suggested various mathematical optimization models. Although some of these models perform satisfactorily in the particular applications for which they are developed, a methodology able to handle the reservoir operation problem in its full complexity is still lacking. As a result, in practise the majority of reservoir systems are operated by heuristic rules derived by computer-aided simulation and engineering intuition. However, the reservoir operation problem can be particularly complex and its solution is not necessarily apparent.

Optimization of uncertain dynamical systems is the subject of optimal stochastic control theory. Originally motivated by aerospace

applications, this field provides a well integrated mathematical framework for the treatment of similar problems in other engineering disciplines as well. A stochastic control study is completed in two basic steps:

a. System model development. This is the phase where a model able to adequately reproduce the system's behavior is identified. The identification process requires intuitive understanding of the laws governing the system dynamics and can take advantage of input-output data records to properly adjust certain model parameters.

b. Stochastic controller design. Based on the identified model and after adopting performance criteria correctly reflecting the system's objectives, an optimization algorithm (controller) is designed to guide the system in successful operation.

This study's developments will also proceed along these lines with emphasis on the second of the previous steps. The identification of a reservoir system model will be discussed in Chapter 3. The objective is to arrive at a flexible model form to represent the reservoir systems' idiosyncracies in desirable detail. In Chapter 4 the reservoir system operation problem will first be formulated and then the development of a suitable control algorithm will follow. Chapters 5 and 6 include two case studies where the previous algorithm is used and tested in the operation of actual systems. Chapter 7 identifies further research directions and concludes the study. The immediately following Chapter 2 is a review of representative reservoir control models.

Chapter 2

STOCHASTIC MODELS IN RESERVOIR CONTROL

2.1 Introduction and Overview

Over the last 25 years the reservoir control literature has grown impressively (see Rosenthal, 1980, and Yeh, 1982, for survey discussions). This continuing research effort is sustained not only because of the many influential, social, and environmental effects each reservoir system generates, but also because a comprehensive methodology capable of handling the problem in its general form has not yet been developed. In essence, the reservoir control publications are tailored to and perhaps adequately handle a particular reservoir system, or a particular class of systems with common predominant characteristics. However, if these characteristics were to change, the methods would no longer be adequate. This chapter will review representative stochastic reservoir control studies to identify past deficiencies and to benefit from some successful ideas. The discussion will summarize some target properties used in the upcoming control design.

2.2 Reservoir Control Studies

Classified by the optimization philosophy adopted, the reservoir control models fall into one of the following general categories (Yeh, 1982):

a. Linear Programming models,

b. Dynamic Programming models,

c. Nonlinear Programming models,

d. Simulation models.

Linear Programming models which consider uncertainty are the Chance Constraint Programming (Revelle et al., 1969, Joeres et al., 1971, Eisel, 1972, Sobel, 1975, Sniedovich, 1980, etc.) and the Reliability Programming models (Colorni and Fronza, 1976, Simonovic and Marino, 1980, 1982, Marino and Mohamadi, 1983). Generally, L.P. models cannot adequately reproduce the system's stochasticity (temporal and spatial correlation of the uncertain inputs and the induced similar probabilistic structure on the system's state variables), and a global linear approximation of the system dynamics is not likely to give accurate results. It will also be seen (Chapter 4) that although the assumption of linear decision rules is optimal for unconstrained systems with linear dynamics and quadratic performance measures, it performs poorly in situations where the above characteristics are absent. For these reasons, the models of this category are considered appropriate for preliminary design studies rather than for operation purposes (Loucks, 1970, Loucks and Dorfman, 1975).

The second model category adopts an optimization procedure which is more suitable to the dynamic nature of a reservoir system. Young, 1967, introduced the Monte Carlo D.P. According to this implicitly stochastic approach, a model fitted to the input process generates many possible time series of input realizations. For each realization Deterministic Dynamic Programming is used to determine optimal releases at each time step of the control horizon. These optimal release trajectories are next related to various system variables via regression techniques. The result is a set of decision functions which dictate the "optimal" decisions as functions of a particular set of variables

statistically correlated with the optimal solutions of the deterministic optimization. Major potential disadvantages of the method are the following points: (1) the assumption of a certain functional relationship in the decision function regressions, (2) poor performance at the extreme values of the decision function independent variables, (3) excessive storage and computer time requirements. With respect to the first two, it is true (we shall extensively discuss this in Chapter 4) that the optimal decision action at a particular point in time is a function of the conditional (on any available observations relevant to the system's evolution) joint probability density of the system's storage variables. Since the shape of this multi-dimensional function is arbitrary, it follows that its complete characterization requires an infinite number of parameters. Thus, a decision function of a few variables may not suffice.

Similar to the previous approach is the Alternate Stochastic Optimization (ASO) presented by Croley, 1974. ASO is a sequential procedure repeated at each time step of the control horizon, and therefore it can incorporate any additional information regarding the system's present conditions. At each time step the procedure relies on deterministic optimization of many synthetic input process realizations. A probability density function of the first period's optimal decisions is derived and the mode or the mean is suggested as the best decision choice. ASO approximately yields the optimal action obtained by a deterministic optimization of the input realization with the maximum likelihood of occurrence. Hence, with substantially fewer computations, the maximum likelihood input trajectory can be generated, and the corresponding

optimal decision can be obtained through a deterministic procedure. This suggestion is somewhat attractive since it substitutes the stochastic control problem for the most likely to occur deterministic approximation. The advantage results from the fact that deterministic problems are comparatively easier to solve. However, this approach (known in control theory as Naive Feedback controller) is optimal only for systems characterized by the Certainty Equivalence (CE) property[1] and may yield suboptimal controls. Thus, unless a more reliable control scheme cannot be implemented, this approach is not suggested for use. Apart from the sequential scheme, Croley, 1974, suggests another interesting possibility; namely to consider only the time span having a bearing on the first period's decision action as the control horizon. That is, if the nature of the algorithm is sequential, and therefore its purpose is to identify the best decision for the first upcoming period, it is computationally efficient to consider the smaller control time horizon achieving this goal.

A class of D.P. models explicitly considering uncertainty in the optimization procedure includes the models using Markov Chain input process description and backward D.P. (Schweig and Cole, 1968, Butcher, 1971, Su and Deininger, 1972, 1974, Arunkumar and Yeh, 1973, Alarcon and Marks, 1979, Buchanan and Bras, 1981, etc.). In general the models of this class perform satisfactorily in small systems (of 2 - 3 reservoirs). However, their extension to multireservoir systems is seriously limited

[1] A stochastic problem possesses this simplifying property if it accepts the same solution with the deterministic approximation obtained as mentioned in the text. It is not possible to know a priori whether the CE property is present except in the case of Linear systems with Quadratic criteria and Gaussian statistics (LQG).

due to dimensionality problems. Consider, for example, a system of N_s reservoirs and N_w inputs. Then the state space of this formulation will have to include N_s reservoir storages as well as the previous period's N_w inputs. (It is assumed that the inputs follow a lag-1 Markov process). Also assume that each state variable is discretized in N_d values. Then, the Dynamic Programming backward recursions require the evaluation and storage of the "cost to go" —a D.P.'s functional of key importance— at $N_d^{(N_s + N_w)}$ discrete locations. This exponential (with respect to the number of state variables) growth of computer storage requirements limits this method's practicability to only small reservoir systems. Yeh, 1982, mentions an extension of the previous model to multireservoir systems (Arunkumar and Yeh, 1973). In a system of m reservoirs, the approach consists of optimizing one reservoir's decision functions at a time while keeping the others fixed until no further objective function improvement is achieved. The method is applied to a system of two parallel reservoirs for the optimization of the firm energy output. The approach cannot be applied to general reservoir systems. If the input process is correlated, and/or the reservoirs are in series, and/or the objective function terms are not simply additively separable with respect to each reservoir, then the state space affecting each control law cannot be simplified. In other words, each policy as well as the corresponding "costs to go" at each time step will be dependent on all the previous period's inputs and current period's reservoir storages creating the known problems in the discretized D.P. solution. Yet, if such an interconnected system is still within the D.P.'s solution feasibility range, this successive approximation idea does have an advantage. The

minimization operations performed at each time step need only be with respect to one control variable. This, however, is possible at the expense of losing the D.P.'s solution global optimality property; i.e., the policies found can only be locally optimal.

Concerning the lag-1 Markov Chain modeling of the input process, one should recognize that it becomes less satisfactory as the process' true correlation structure involves more lags. The extension to a multi-lag model would require that all the previous periods' inputs which improve the current period input's prediction be included in the state space. Again, dimensionality limitations will soon prohibit this attempt. Toward overcoming the multi-lag correlation problem in the context of the Markov Chain D.P. formulation, the procedure suggested by Buckanan and Bras, 1981, has proved efficient. According to their approach, a multi-lag stochastic model fitted to the input process uses real time observations to forecast the inputs of the upcoming periods. The forecasts update the Markovian transition probabilities as far into the future as there is a significant gain over the a priori estimates. Then the stochastic D.P. algorithm determines the optimal policies from which only the immediate period's decision is applied with the procedure repeated at the next decision time (sequential optimization). This model, making use of real time input observations, is found to perform better compared with the original Markov Chain D.P. formulation. In fact, the underlying idea can be used in a more general setting (see Bertsekas, 1976, Section 5.5) and will be exploited here.

A last comment regarding the Markov Chain D.P. formulation refers to its inability to explicitly produce policies satisfying probabilistic constraints. Such a constraint, for example, could be a reservoir

storage requirement not to exceed (or fall below) a certain level with probability γ ($0 \leq \gamma \leq 1$). Statements like the above are very useful in evaluating the performance of control policies for stochastic reservoir systems. If it is not possible for the optimization program to meet such specifications, the practice is to follow a trial-and-error approach. Namely, a certain policy obtained by the optimization model is evaluated through simulation, and in the event that its performance is not satisfactory, the optimization model is modified (certain constraints are tightened, penalty terms are introduced to or dropped from the objective function, etc.) with the hope that the new policies will give better results. Clearly, the approach is not set up in a well defined manner and can be costly.

Recognizing the importance of policies satisfying probabilistic constraints, Askew, 1974a, b, 1975, Sniedovich, 1979, 1980a, b, and others investigated the matter for a single reservoir with independent inflows. The attempts were successful in constructing algorithms which maximize the objective function while satisfying certain system constraints at prespecified probabilistic levels. Unfortunately, these methods are not extendible to a general multireservoir system, again, because of dimensionality problems.

Becker and Yeh, 1974, developed a monthly reservoir optimization model which was coupled to a daily (Yeh et al., 1976) and an hourly (Yeh, Becker, and Chu, 1979) model and constituted a multilevel (or hierarchical) structure currently used to operate the California Central Valley Project. The procedure is a deterministic sequential formulation utilizing deterministic forecasts and exhibits good computational efficiency. As mentioned earlier, the method falls in the category of the Naive

Feedback Controllers which assume validity of the Certainty Equivalence principle. However, as a general rule, the more nonlinear (with respect to its dynamics) and nonquadratic (with respect to its objective function) a system is, the less it is likely for the Certainty Equivalence property to hold, and the greater is the chance that the controls so obtained are far from optimal. Nevertheless, it is noted that the practical importance of the reduced deterministic formulation should be fully recognized in the event that a computationally efficient stochastic controller is not feasible. Another weak point of the previous model is the subjective judgement required to determine the terminal system state. It will be seen that substantial suboptimalities can be introduced by incorrect terminal state choices and that the proper selection law depends on many parameters (e.g., system objectives, control capability, length of optimization horizon, etc.) and is not always apparent.

Regarding the multilevel structures, it has been realized (Haimes, 1977, Unny et al., 1981, Bechard et al., 1981, etc.) that they are particularly suitable for reservoir systems, the reason being that these systems' objectives are usually associated with a variety of time bases. (For example, finding the operating policy minimizing the damages of a particular flooding event and operating the system for long-term maximum energy generation are two objectives of substantially different frequencies.) The multilevel approach represents the overall system by a hierarchy of models. The higher standing models involve less detailed system descriptions; they are meant to develop operating strategies which optimize the long-run system performance (strategic decisions). The more accurate lower level models are concerned with increasingly shorter interval optimization (tactical decisions) and finally dictate

the operation to be implemented. According to the objective priority ranking, each level properly constrains the adjacent lower level by its decisions, but feedback is possible, as well as upward information flow from real time observations. Functioning in the above interactive fashion, the hierarchy can assure that the system will perform well over all relevant time scales while keeping each model at a tractable complexity. Overall, hierarchical structures constitute a suitable approach in implementing a real time reservoir control scheme.

Thus far the basic problem of the Dynamic Programming models has been the dimensionality limitation stemming from the need to implement the D.P. solution in discretized form. The recent publications by Georgakakos, A., 1983, and Wasimi and Kitanidis, 1983, presented a methodology avoiding this numerical difficulty.

Wasimi and Kitanidis, 1983, modelled the system dynamics by a set of actual and conceptual (corresponding to the river reaches) linear reservoirs and employed a quadratic penalty cost functional to force the system's state trajectory on a prespecified track. The inputs were assumed Gaussian random variables and the formulation did not consider state or control constraints. Because of the LQG problem characteristics, the Dynamic Programming solution was obtained in analytical form which minimized the computational burden.

The model developed by Georgakakos, A., 1983, also employed the actual-conceptual reservoir system configuration, but it allowed for nonlinear dynamics and general performance functional. Here the objective was to identify the most rewarding state and control trajectories and the procedure was an iterative optimization scheme. At each iteration, a quadratic approximation of the objective function and a local

linearization of the dynamics around the current trajectories were performed to construct a local LQG approximation of the original problem. The solution obtained in a recursive analytical form gave rise to new state and control trajectories until convergence. The method exhibited fast convergence rate and had a provision for probabilistic state constraints which, nevertheless, did not perform efficiently. No control constraints were accounted for.

Both of the previous models allowed for updating of the current state estimates via a Kalman (or an Extended Kalman) Filter estimator.

Based on the many potential advantages of the previous approaches, this work will continue along these same lines. The intention is to perfect the control design so as to be efficient in handling most system idiosyncracies.

Regarding the nonlinear programming model category, Rosenthal, 1980, notes that the state-of-the-art needs a breakthrough in computational efficiency to become a practical solution technique for multireservoir system control.

Although predominantly used in reservoir operation, simulation models have no optimization structure and are suitable for performance evaluation rather than for optimal policy identification.

The preceding survey of stochastic reservoir control studies indicated that a comprehensive model efficiently accounting for all the peculiar reservoir system characteristics is still lacking. It also brought up a variety of properties which such a model should possess. These structural requirements of an efficient controller are summarized below to set the standards with which the subsequent control study will be expected to comply.

a. Explicitly account for the system's uncertainty.

b. Use local linear approximations of the system's dynamics, if any.

c. Optimize sequentially. (Repeat the procedure at each decision time.)

d. Take advantage of real time observations to update the estimates of the system model and make efficient use of the input process' forecasts.

e. Consider the shortest control horizon that has a bearing on the immediate period's decision action.

f. Account for control and probabilistic state constraints.

g. Overcome dimensionality problems and guarantee computational efficiency.

h. Design to facilitate a multilevel control structure's setup.

The following Chapter presents a model for a general reservoir system. This is similar to the models introduced by Georgakakos, A., 1983, and Wasimi and Kitanidis, 1983, but it is included here for completeness.

Chapter 3

MULTIRESERVOIR SYSTEM MODEL

3.1 Introduction and Overview

The design of a controller is the second basic step of a control study. Prior to this a credible system model must be developed. Based on physical considerations, this chapter will show how to represent a general reservoir system by a set of ordinary differential equations. This model's parameters can be optimally adjusted via estimation techniques to comply with available input-output data records. Some issues arising in parameter estimation will be discussed, but detailed presentation of the associated theory will not be offered; instead, the reader will be referred to selected relevant publications. The purpose of this chapter is to introduce a flexible model structure adjustable to the reservoir system idiosyncracies. Chapter 4 will consider the detailed and self-contained design of a control algorithm.

3.2 General Characteristics of a Reservoir System

A reservoir system is a number of reservoirs whose coordinated operation is suggested from hydrologic considerations and/or institutional requirements. Figure 3.2.1 is a schematic representation of a three reservoir system which will introduce the elements that when combined in series and/or in parallel synthesize more complicated configurations. Basic characteristics of a typical reservoir system are the following elements:

a. A set of hydrologic inputs (as, for example, the river flows $w_{a1}(t)$, $w_{b1}(t)$, $w_{d1}(t)$ in Figure 3.2.1) which enter the system at various boundary locations.

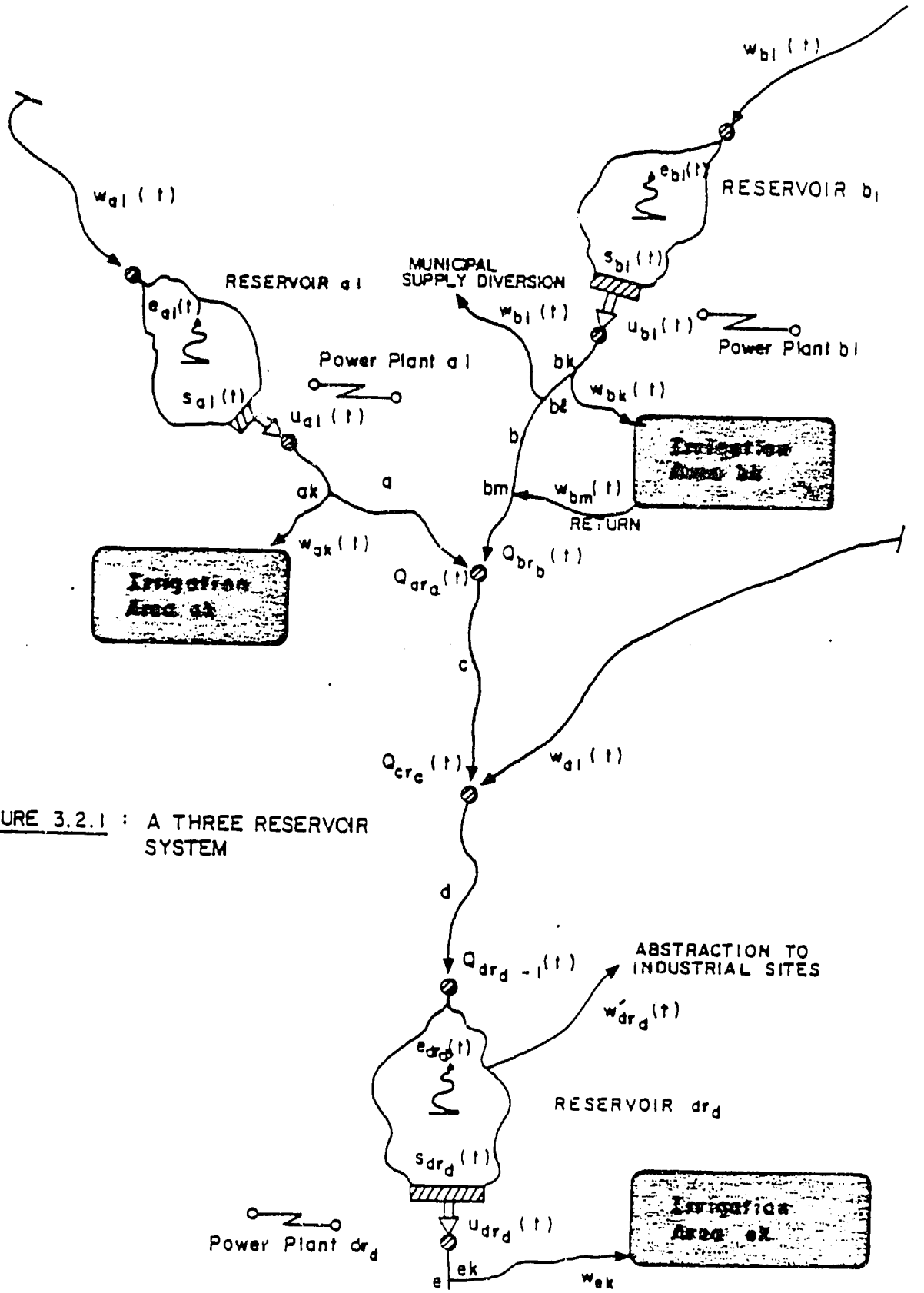

FIGURE 3.2.1 : A THREE RESERVOIR SYSTEM

b. A set of reservoirs located at various river branches. Each reservoir accepts an inflow from the upstream system, contains s(t) water volume, loses or gains additional water due to evaporation, seepage, or rainfall at a rate e(t), and releases at the controllable rate u(t). (In Figure 3.2.1 all quantities associated with a particular reservoir are indexed by two subscripts ij denoting the jth reservoir of the ith river branch.) The storage variables s(t) are restricted to vary within a positive bounded range, $s^{min}(t) \leq s(t) \leq s^{max}(t)$, from physical and operational considerations similar bounds govern the releases, $u^{min}(t) \leq u(t) \leq u^{max}(t)$.

c. A set of river segments providing the hydrologic linkage among the existing reservoirs. Linkage due to ground water movement will not be considered here; yet, if such a situation is suggested by the physical conditions, it too can be accounted for in the proposed framework. (The river segments of Figure 3.1 are denoted by a, b, c, and d.) Along the river segments and from the reservoirs, a number of water diversions (possibly generating return flows) may supply water to municipal, agricultural, or industrial sites. (The reason why two subscripts are used to signify the elements in Figure 3.2.1 will become clear in Section 3.7.)

d. A set of objectives that the system is expected to serve. A general set of typically established objectives includes flood protection, water supply for municipal, agricultural, and industrial use, energy generation, recreation, navigation, water quality improvement, and wildlife enhancement.

Loosely speaking, the reservoir control problem is to identify optimal schedules of the controllable variables which will guide the system

to successfully meet its objectives. This is a rather difficult task due to the following system idiosyncracies: (1) uncertainty, (2) nonlinearity, (3) high dimensionality, and (4) multiplicity of conflicting objectives.

In turn, reservoir systems' uncertainty may be due to the following sources:

1a. Hydrologic input or natural uncertainty. Once the system is defined, the inputs are used to represent the real world lying outside its boundaries. For example, the river flows $w_{a1}(t)$, $w_{b1}(t)$, $w_{d1}(t)$ should reflect the characteristics of the rainfall-runoff processes of their drainage basins. Consequently, the uncertainty of the omitted processes (e.g. atmospheric randomness) will carry over to the inputs, while the corresponding dynamics will induce temporal and spatial correlation structure in their behavior (e.g. seasonalities, etc.).

1b. Uncertainty due to imperfect knowledge concerning the response of the reservoirs and of the river segments (also known as model uncertainty).

1c. Uncertainty due to the objectives (e.g., random demand fluctuations, or economic uncertainty).

Nonlinearities in the system dynamics may be due to the reservoirs as well as the river segments. The reservoir nonlinearities are caused by the evaporation and seepage processes taking place through the reservoir's irregular surfaces and also by the various constraints. The river segments are nonlinear elements due to many reasons: e.g. bottom frictional effects, fluid viscosity, turbulence, channel nonuniformity, seepage, etc.

Examples of large reservoir systems in the U.S. are the California

Central Valley System, the Arkansas System, the Columbia System, the Lower Colorado System. Each consists of at least 30 major hydroprojects.

Reservoir systems are usually characterized by conflicting objectives; namely, a certain control sequence which successfully performs for one objective may be undesirable for another. (Conflicts arise, for example, between hydroenergy generation and flood protection, or due to incompatible water use demand patterns.) Furthermore, if the system is capable of long-term river control (i.e. if the existing storage capacity gives overyear management flexibility of the input process' water volumes), then there exist multiple time scales to which the objectives pertain. Namely, the system performs successfully, if it performs well from short to considerably longer time periods (e.g. hours to a few years).

After this preliminary introduction to the reservoir system characteristics, we next discuss the development of a representative system model.

3.3 System Model Development

Consider the following quantities characterizing a reservoir at any given time t:

$s_{ij}(t)$: water volume contained in the reservoir's lake,

$I_{ij}(t)$: inflow rate to the lake from the upstream system,

$u_{ij}(t)$: rate of controlled releases,

$L_{ij}(t)$: net water loss rate from evaporation, seepage, and rainfall.

Then, the conservation of mass law for the ij^{th} reservoir can be expressed by

$$\frac{d\, s_{ij}(t)}{dt} = I_{ij}(t) - u_{ij}(t) - L_{ij}(t). \qquad (3.3.1)$$

Consider the loss term $L_{ij}(t)$ and denote the following: (1) $e_{ij}(t)$ the net [(evaporation) - (rainfall)] rate per unit of the lake's surface area, (2) $A_{ij}(s_{ij}(t))$ the reservoir's surface area in terms of the stored volume, (3) g_{ij} a seepage coefficient per unit of water elevation, and (4) $E_{ij}(s_{ij}(t))$ the function giving water elevation in terms of the lake's volume. Then

$$L_{ij}(t) = e_{ij}(t)\, A_{ij}(s_{ij}(t)) + g_{ij}\, E_{ij}(s_{ij}(t)). \qquad (3.3.2)$$

In reality the relationship yielding the seepage losses is much more complicated. (The coefficient g_{ij} varies spatially and the form of the seepage term depends on the general groundwater hydrology of the lake's area.) However, because of the distributed parameter nature of the underlying phenomena (spatial nonuniformity), it is most likely that they will require a lumped parameter representation (as in Eq. (3.3.2)) to be hypothesized and estimated from available input-output data. In fact, even this approach can be impossible due to identifiability problems. However, issues such as this will be discussed in Section 3.4.

The inflow $I_{ij}(t)$ in Eq. (3.3.1) may be a hydrologic input, if the reservoir lies at the system's edges, or the output from the system lying upstream of reservoir ij. If apart from the release $u_{ij}(t)$ there also exists some other water abstraction (say, in the form of a water supply diversion), it can be incorporated in the inflow $I_{ij}(t)$ which should then be reduced by the corresponding abstraction rate.

Consider now the river segment j shown in Figure 3.3.1a. It can be thought of (Kitanidis and Bras, 1978, Georgakakos, K. and Bras 1980, 1982), as a number of cascaded conceptual (fictitious) reservoirs (Figure 3.1.3b) each of which stores and releases water according to the conservation of

mass law:

$$\frac{ds_{ji}(t)}{dt} = I_{ji}(t) - Q_{ji}(t) - L_{ji}(t) \tag{3.3.3}$$

where $s_{ji}(t)$ is the water volume contained in the ith conceptual reservoir of reach j at time t, $I_{ji}(t)$ is the inflow rate from the upstream system, $L_{ji}(t)$ represents the seepage loss rate (it might be that $L_{ij}(t)$ is negative, implying gains from groundwater), and $Q_{ji}(t)$ is the outflow rate. $I_{ji}(t)$ may be the outflow $Q_{ji-1}(t)$ of the previous conceptual reservoir, or the previous outflow minus the rate of any existing diversion (as at the conceptual reservoir $j\ell$ in Figures 3.3.1a, b), or the outflow plus any return flow. If $i = 1$, $I_{j1}(t)$ may be a hydrologic input, or an actual reservoir's release, or the final outflow of another branch, or some combination of the above (see Figure 3.2.1).

From kinematic wave type considerations (Fagleson, 1970) one can deduce a relationship of the following form for the outflow $Q_{ji}(t)$:

$$Q_{ji}(t) = \alpha_{ji}\, s_{ji}^{m_{ji}}(t). \tag{3.3.4}$$

Under certain uniformity assumptions for the channel geometry and flow condition, one can express the coefficients α_{ji} and m_{ji} in terms of physical parametrers such as bottom longtitudinal slope, channel length and width, roughness coefficient, and water depth (Georgakakos, K. and R. Bras, 1980). However, owing to the distributed parameter nature of the physical situation, it is suggested that these coefficients be estimated rather than obtained from the previous relationships. (In an estimation scheme the coefficients derived by physical considerations may serve as initial guesses.)

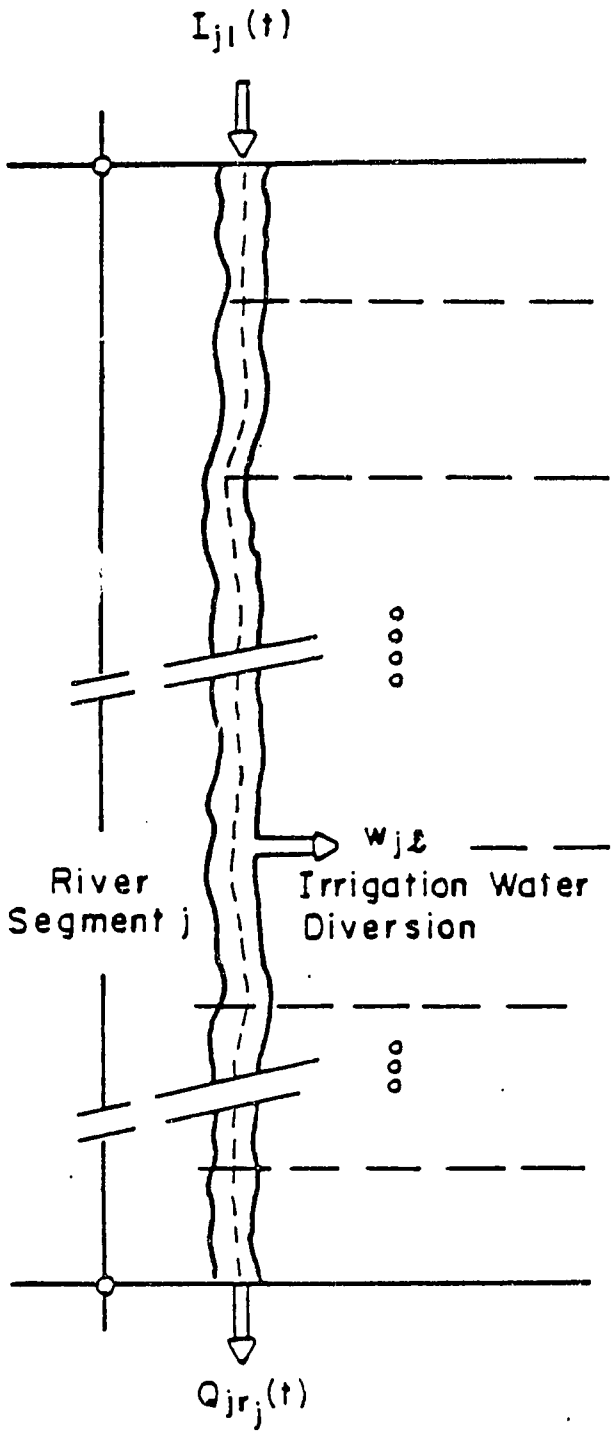

FIGURE 3.3.1a

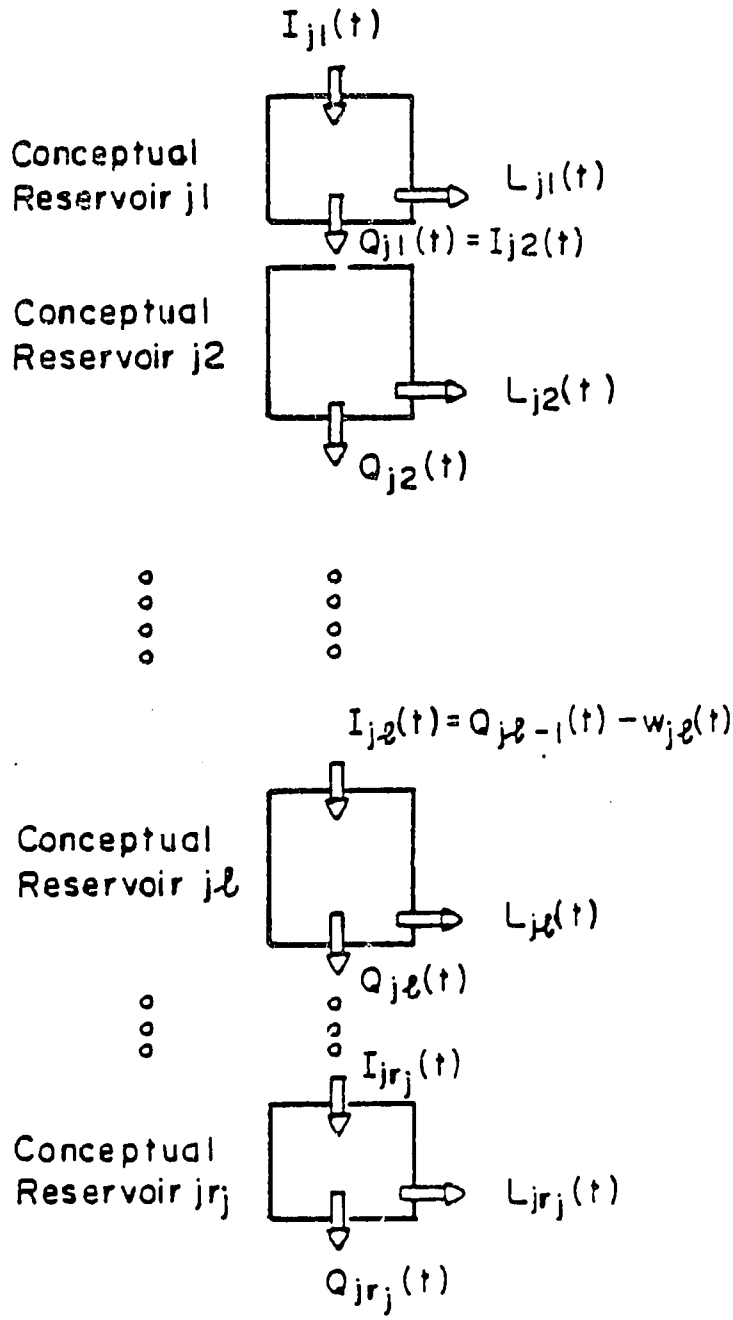

FIGURE 3.3.1b

MODELING OF A RIVER SEGMENT

The seepage (or infiltration) losses can be represented (Burkham, 1970a, b, Granados and Bras, 1983) by a similar expression of the following form

$$L_{ji}(t) = \mathcal{K}_{ji}\, s_{ji}^{\beta_{ji}}(t) \tag{3.3.5}$$

where β_{ji} lies approximately in the range [0.8, 1.2] and $\mathcal{K}_{ji}$ is the infiltration coefficient. Typically, these coefficients are assumed constant over the same river segment or constant over the whole basin. However, we shall continue considering them spatially varying to arrive at a general structure and later discuss possible (or mandatory) simplifications.

After these specifications the dynamics of the river segment j portrayed in Figure 3.3.1a can be represented by the following system of first-order ordinary differential equations:

$$\left.\begin{aligned}
\frac{ds_{j1}(t)}{dt} &= -\alpha_{j1}\, s_{j1}^{m_{j1}}(t) - \mathcal{K}_{j1}\, s_{j1}^{\beta_{j1}} + I_{j1}(t) \quad , \\
\frac{ds_{j2}(t)}{dt} &= \alpha_{j1}\, s_{j1}^{m_{j1}}(t) - \alpha_{j2}\, s_{j2}^{m_{j2}}(t) - \mathcal{K}_{j2}\, s_{j2}^{\beta_{j2}}(t), \\
&\vdots \\
\frac{ds_{j\ell}(t)}{dt} &= \alpha_{j\ell-1}\, s_{j\ell-1}^{m_{j\ell-1}}(t) - \alpha_{j\ell}\, s_{j\ell}^{m_{j\ell}}(t) - \mathcal{K}_{j\ell}\, s_{j\ell}^{\beta_{j\ell}}(t) - w_{j\ell}(t), \\
&\vdots \\
\frac{s_{jr_j}(t)}{dt} &= \alpha_{jr_j-1}\, s_{jr_j-1}^{m_{jr_j-1}}(t) - \alpha_{jr_j}\, s_{jr_j}^{m_{jr_j}}(t) - \mathcal{K}_{jr_j}\, s_{jr_j}^{\beta_{jr_j}}(t)
\end{aligned}\right\} \tag{3.3.6}$$

where it has been assumed that r_j conceptual reservoirs are necessary to model segment's j response.

A property which makes this conceptual-reservoir routing model favorable for the control purposes is its wide range of adaptability. Namely, by properly adjusting the parameters $\{\alpha_{ji}, m_{ji}, \kappa_{ji}, \beta_{ji}, r_j\}_{i=1}^{r_j}$ any river response can be adequately reproduced. Thus, this basic structure can be employed at any control hierarch level at commensurate accuracy. For example, a representation of one linear conceptual reservoir (if any) suffice for a monthly decision model, while more complex structures and more sophisticated parameter estimation procedures may be necessary for shorter decision time intervals. Another convenient property of the preceding model is its rather close compatibility with the model for the actual reservoir dynamics.

The dynamics of the entire system can then be modelled by combining the previous representations of the individual river segments with the actual reservoirs. (For example, it is straightforward to write the set of differential equations modeling the 3 reservoirs and the 4 river branches of the system in Figure 3.2.1). To facilitate the notation let the system dynamics be expressed by the following vector differential equation

$$\frac{d\underline{s}(t)}{dt} = \underline{F}\,(\underline{s}(t),\ t) + \underline{b}\ \underline{u}(t) + \underline{\underline{G}}\ \underline{w}(t) \qquad (3.3.7)$$

where $\underline{s}(t)$ is an n_s dimensional vector including all actual and conceptual reservoir storage variables,

$\underline{u}(t)$ is an n_u dimensional vector of the controllable releases,

$\underline{w}(t)$ is an n_w dimensional vector including the hydrologic inputs and any water diversions,

$\underline{F}(\cdot,\cdot)$ is an n_s dimensional time varying nonlinear function including all terms $e(t)\ A(s(t))$, $gE(s(t))\ \alpha s^m(t)$, $\kappa s^\beta(t)$

indexed after the corresponding reservoir ,

and $\underline{\underline{L}}$, $\underline{\underline{G}}$ are respectively $n_s \times n_u$, $n_s \times n_w$ dimensional matrices associating the control and input vector elements with the appropriate differential equation. (Obviously, these matrices are sparse.)

Notice if at any time t_k all actual and conceptual storage variables are known (i.e. the vector $\underline{s}(t_k)$ is given) and if a particular trajectory is specified for the inputs $\underline{w}(t)$ and the releases $\underline{u}(t)$ over an interval $[t_k , t_\ell]$, then the trajectory $\{\underline{s}(t);\ t\epsilon[t_k , t_\ell]\}$ of the storage variables can be computed by integrating Eq. (3.3.7). In other words, the vector $\underline{s}(t_k)$ is the only necessary information from the system's history prior to time t_k in order to predict its future response to certain input and release sequences. Then by definition the vector $\underline{s}(t_k)$ constitutes a state for the reservoir system, and the previous representation is said to be in state space form. The state is a pivotal quantity in the study of systems. For a reservoir system the previously defined state[1] at any time describes the water content distribution throughout the system. The control study's objective will be to specify reservoir release schedules which result in desirable time trajectories of this distribution.

As mentioned, under the availability of input-output data records, it is best to determine the various model parameters by a parameter estimation scheme. Some related discussions follow .

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[1] Often, the individual storage variables will also be referred to as states.

3.4 System Identification

System identification theory is concerned with identifying a model whose input-output characteristics correspond to those of the actual system. An identification study is an iterative process involving the four steps shown in Figure 3.4.1 (Schweppe, 1976). Section 3.3 is an example of the "HYPOTHESIZE STRUCTURE" step where physical model structure is hypothesized for further identification. The second step uses input-output observations and parameter estimation techniques (Schweppe, 1973, Sorenson, 1980) to optimally adjust the parameters of the hypothesized structure. Following are validity tests evaluating whether the model can adequately explain the system's observed behavior and the fourth step, "DIAGNOSTIC ANALYSIS," suggests a different model structure in the event that validity tests fail.

A prerequisite element for applying the above procedure is the availability of measurements (observations) of certain system variables over a time period. The measurements usually available in reservoir systems concern river flow discharges, reservoir water elevations and releases, and evaporation rates. The river flow discharge measurements (e.g. the inflow and outflow discharges of a river segment, or the inflow discharge to a reservoir) are generally of fairly good quality with an associated error whose standard deviation is typically equal to a small fraction (say, 0.01) of the observed value. The reservoir water elevation observations can be transformed to observations of the stored water volume via the elevation-storage relationship. This function is determined through detailed geographical mapping of the lake's area (from surveys, etc.) and is generally accurate at least within the

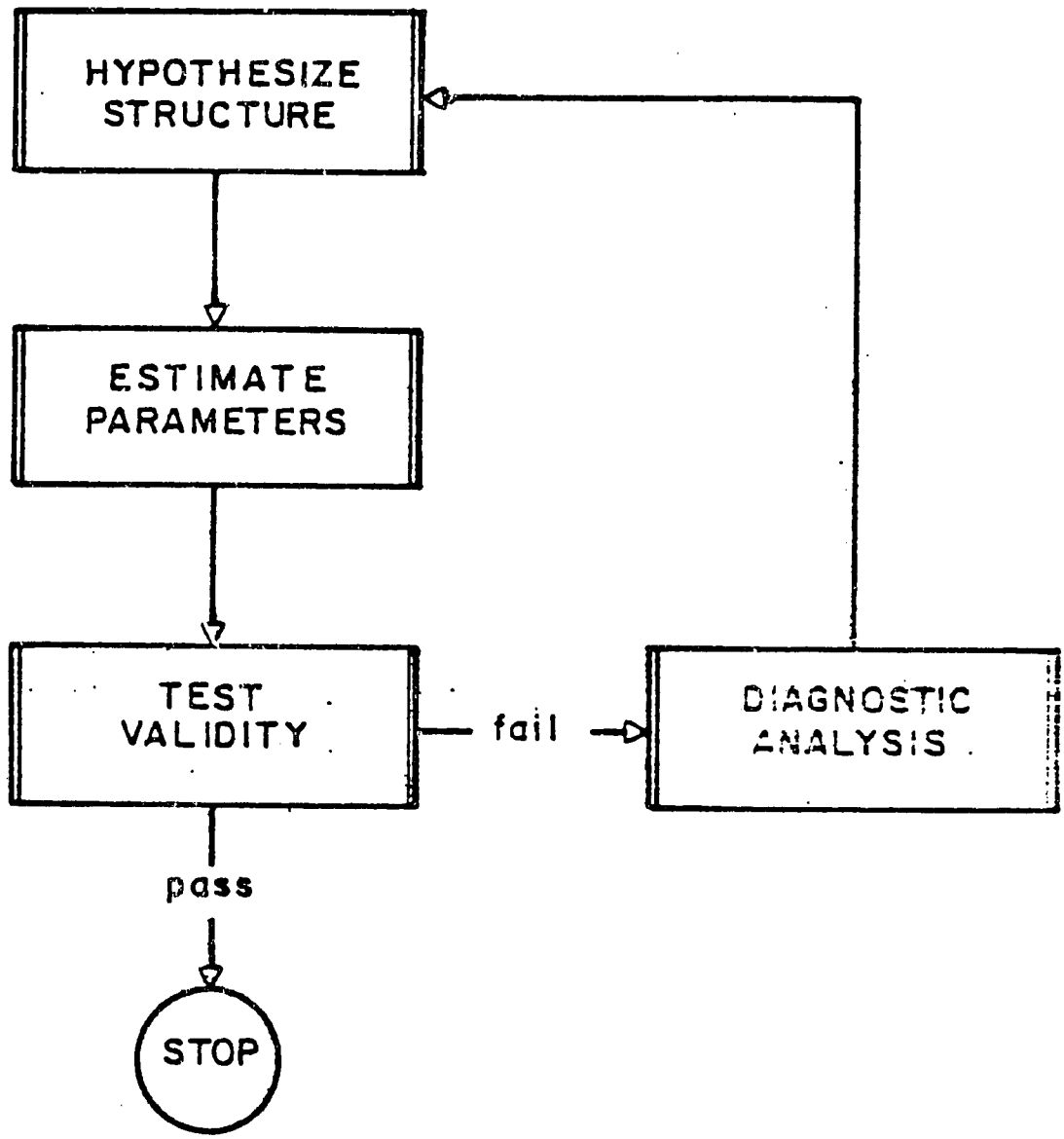

FIGURE 3.4.1: THE FOUR STEPS OF SYSTEM IDENTIFICATION

water storage normal fluctuation range. The reservoir releases are almost always passed through hydropower turbines or other accurately designed structures and can be safely considered free from measurement error. Good quality average evaporation rate measurements are rarely available owing to the high temporal and spatial process' variability.

After the associated data base is established, the model parameters are estimated according to, say, the Maximum Likelihood, the Extended Kalman Filter, or some other estimation methodology. However, a reservoir system has certain characteristics that can simplify the estimation process. As mentioned, the reservoir releases are virtually perfectly measureable. By the system's serial nature these releases are the only connecting link between the system part lying upstream with that extending downstream of a reservoir. Thus, all information of the upstream system measurements regarding the downstream system is carried through these links and is captured by their perfect measurements. This intuitive argument implies that the parameter estimation of the two system parts' models can be separately performed. As an example, reservoir a1, reservoir b1, and the rest of the system in Figure 3.2.1 may be separately identified. Theoretically each of the resulting individual parts should not be further decomposed. For instance, this means that the hypothesized system models for the river branches a, b, c, d, and for the reservoir e_{dr_d} in Figure 3.2.1 be put together and identified from all relevant measurements. However, there exist serious numerical difficulties in the identification of complex structures (see Morris and Schweppe, 1981) mainly from excessive computational requirements and irregularities in the likelihood function. For this reason and because

river flow discharge measurements are of fairly good quality, we suggest that further decomposition be employed first to determine smaller yet more easily identifiable substructures. In the event that the identified models fail the validity tests of the identification process, more complex configurations may be hypothesized and treated. Hence, if enough gage locations exist, then the identification process may be decomposed and reservoirs and river segments treated separately.

A frequently arising problem in system identification applications is related to the identifiability of the hypothesized structure parameters. Namely, it may happen that the unknown parameters cannot be estimated by the available observation record and some model structure changes are necessary before the process can go through. Although the "DIAGNOSTIC ANALYSIS" step will usually detect the problem, it is sometimes possible to a priori anticipate and correct such a deficiency. Appendix I discusses this issue for the reservoir model hypothesized in the previous section (Eq. (3.3.1), (3.3.2)). It is shown that it is not possible to estimate both of the parameters e(t) and g. It is suggested that either one be fixed at a physically reasonable value and the other be estimated by some parameter estimation methodology. Appendix I also briefly outlines how parameter estimation can be performed for a nonlinear system. The presentation requires linear system (Kailath, 1980) and estimation theory (Schweppe, 1973) background, and it may be skipped without affecting the control developments in Chapter 4. Appendix J takes up the same issues for the model of a river segment. It finally suggests that linear structures should be identified first. ($\beta_{ji} = 1 = m_{ji}$ for all i) and that both parameter sets $\{K_{ji}\}_{i=1}^{r_j}$,

$\{\alpha_{ji}\}_{i=1}^{r_j}$ cannot be assumed unknown (due to identifiability problems). Finally, the river segment model is converted to a form amenable to the basic estimation procedures such as the Maximum Likelihood or the extended Kalman Filter. For more detailed applications of system identification theory, the reader is referred to Kitanidis and Bras, 1978, Georgakakos, K. and Bras, 1980, 1982, and Restrepo Posada and Bras, 1982.

Hereon, we shall assume that the system identification phase has been performed and resulted in the following system model:

$$\frac{d\underline{s}(t)}{dt} = \underline{F}(\underline{s}(t),t) + \underline{\underline{L}}\,\underline{u}(t) + \underline{\underline{G}}\,\underline{w}(t) + \underline{q}(t) \tag{3.4.1}$$

with $\underline{F}(\cdot,\cdot)$, $\underline{\underline{L}}$, $\underline{\underline{G}}$ as in Eq. (3.3.7) and $\underline{q}(t)$ a continuous time n_s-vector white Gaussian process which is added and identified (see Appendices I and J) to account for model and other error sources (linearizations and other approximations of the parameter estimation procedure. etc.). This is a zero mean stochastic process with spectral density matrix $\underline{\underline{Q}}_q(t)$:

$$E\{\underline{q}(t)\,\underline{q}^T(s)\} = \underline{\underline{Q}}_q(t)\,\delta(t-s) \tag{3.4.2}$$

($\delta(t-s)$) is the dirac delta function which is zero everywhere except at the origin, $t=s$, where it goes to infinity.)

A separate identification procedure concerns the hydrologic inputs $\underline{w}(t)$. Depending on the available data, one can identify a black-box correlation model (e.g. Curry and Bras, 1980, Wasimi and Kitanidis, 1983), or a model based on the physics of the rainfall process (Georgakakos, K. and Bras, 1982). In any case, the hydrologic input model will be used to forecast the future inputs and allow for better decision making. Here it will be assumed that $\underline{w}(t)$ is a white Gaussian vector

with $\bar{\underline{w}}(t)$ mean vector and $\underline{\underline{Q}}_w(t)$ spectral density matrix. In a later chapter it will be shown how to derive these moments from the discrete time moments usually provided by a forecasting model and how to account for the process' correlation. Given that $\underline{w}(t)$ is a white Gaussian process and the same holds for $\underline{q}(t)$, we can lump the two and represent them by the following process $\underline{\varepsilon}(t)$:

$$\underline{\varepsilon}(t) = \underline{\underline{G}}\ \underline{w}(t) + \underline{q}(t) \quad . \tag{3.4.3}$$

As the sum of two Gaussian vectors, $\underline{\varepsilon}(t)$ is also Gaussian (see Jazwinski, 1970) with mean vector

$$\begin{aligned} E\{\underline{\varepsilon}(t)\} &= E\{\underline{\underline{G}}\ \underline{w}(t) + \underline{q}(t)\} = \underline{\underline{G}}\ E\{\underline{w}(t)\} + E\{\underline{q}(t)\} \\ &= \underline{\underline{G}}\ \bar{\underline{w}}(t) = \bar{\underline{\varepsilon}}(t) \end{aligned} \tag{3.4.4}$$

and spectral density matrix which can be easily obtained as follows:

$$\begin{aligned} & E\left\{(\underline{\varepsilon}(t) - \bar{\underline{\varepsilon}}(t))(\underline{\varepsilon}(s) - \bar{\underline{\varepsilon}}(s))^T\right\} \\ &= E\left\{(\underline{\underline{G}}\ \underline{w}(t) + \underline{q}(t) - \underline{\underline{G}}\ \bar{\underline{w}}(t)(\underline{\underline{G}}\ \underline{w}(s) + q(s) - \underline{\underline{G}}\ \bar{\underline{w}}(s))^T\right\} \\ &= E\ (\underline{\underline{G}}\ \underline{w}(t) - \underline{\underline{G}}\ \bar{\underline{w}}(t))(\underline{\underline{G}}\ \underline{w}(s) - \underline{\underline{G}}\ \bar{\underline{w}}(s))^T \quad + \\ &+ \underline{q}(t)\ \underline{q}^T(s) + (\underline{\underline{G}}\ \underline{w}(t) - \underline{\underline{G}}\ \bar{\underline{w}}(t))\ \underline{q}^T(s) \quad + \\ &+ \underline{q}(t)(\underline{\underline{G}}\ \underline{w}(s) - \underline{\underline{G}}\ \bar{\underline{w}}(s))^T\} \quad = \\ &= \underline{\underline{G}}\ \underline{\underline{Q}}_w(t)\ \delta(t-s)\ \underline{\underline{G}}^T + \underline{\underline{Q}}_q(t)\ \delta(t-s) \quad = \\ &= (\underline{\underline{G}}\ \underline{\underline{Q}}_w(t)\ \underline{\underline{G}}^T + \underline{\underline{Q}}_q(t))\ \delta(t-s) = \underline{\underline{Q}}_\varepsilon(t)\ \delta(t-s) \end{aligned} \tag{3.4.5}$$

Thus,

$$\underline{\underline{Q}}_\varepsilon(t) = \underline{\underline{G}}\ \underline{\underline{Q}}_w(t)\ \underline{\underline{G}}^T + \underline{\underline{Q}}_q(t) \tag{3.4.6}$$

(In the above, the superscript "T" denotes the transpose of a matrix and the derivation makes use of the fact that $\underline{w}(t)$, $\underline{q}(s)$ are uncorrelated for all times t, s.)

Lastly, the actual reservoir water elevation and river discharge measurements will be grouped in a vector $\underline{z}(t_\ell)$ related to the system's state vector through

$$\underline{z}(t_\ell) = \underline{H}\ (\underline{s}(t_\ell)) + \underline{v}(t_\ell) \tag{3.4.7}$$

where $\underline{v}(t_\ell)$ is a discrete time white Gaussian process added to account for observation errors (see Appendices I and J). $\underline{v}(t_\ell)$ has zero mean and covariance matrix $\underline{\underline{R}}_{v\ell}$ (assumed for generality time varying). Note, due to decomposition of the identification process, matrices $\underline{\underline{Q}}_{\varepsilon}(t)$ and $\underline{\underline{R}}_{v\ell}$ are block diagonal.

Equations (3.4.1) and (3.4.7) summarize the results of the system identification phase. Based on this system model, Chapter 4 will take up the design of an optimization algorithm (controller).

Chapter 4

EXTENDED LINEAR QUADRATIC GAUSSIAN CONTROL

4.1 Introduction and Overview

Thus far, the discussion has covered the first of the two major steps in a control study; namely, the identification of a representative system model. This chapter will be concerned with the second step which regards the design of an optimization procedure (controller) able to guide the model (and thereby the actual system) in successfully meeting its operational objectives.

We shall begin by addressing issues arising in the operation of multiobjective and stochastic systems. The various system objectives will be modelled and a general perspective of the control study will be given. Then we shall formulate the reservoir operation problem and discuss its idiosyncracies. A brief review of stochastic optimal control theory will follow to provide the framework for the subsequent developments. The approach taken here in solving the reservoir operation problem is first to simplify it and then work with increasingly more complicated versions until the complete problem is reconstructed and solved. Along this line the unconstrained problem, the problem with control (release) constraints, and the problem with both control and state constraints are successively considered. Apart from being insightful, this presentation also reflects the sequence of operations performed by the controller. It solves the unconstrained problem first and halts to check for any constraint violations. If there are none, it terminates because an optimal solution has been found.

Otherwise it takes care of the violated control and state bounds.

This is a long chapter including the major contributions of this thesis. To the best of our knowledge, this is the first published multireservoir control procedure capable of solving the general operation problem formulated here. Computational experience with the new method will be offered in subsequent chapters and separate publications. This work has been motivated by problems in reservoir operation; however, it is generally applicable for the control of any stochastic, dynamical, nonlinear, multiobjective system in state-space form.

4.2 Modeling of the System's Objectives

Typically, reservoir systems are expected to serve several objectives. Water supply for municipal, agricultrual, or industrial use, flood protection, hydropower generation, navigation, water quality control, recreation, and wildlife enhancement, constitute a set of generally accepted reservoir system objectives (Water Resources Council's Principles and Standards 1973, Revision 1979). Although often underestimated, appropriate modeling of the system objectives is a prerequisite of a successful control design.

What complicates the reservoir control problem even more than the objectives' multiplicity is their antagonistic natures. Take, for instance, hydropower generation and flood protection. For the purposes of the former, it is profitable to maintain reservoir storages close to capacity so that the power turbines are under the highest possible hydraulic head. At the same time for the fear of severe flooding events, one would prefer to operate the system at lower reservoir elevations to safely attenuate the flood wave hydrograph. Similar conflicts also exist between water supply and hydropower generation, navigation and water supply, as well as among other objectives.

In multiobjective optimization one is interested in solutions which belong on the problem's Tradeoff Surface otherwise known as Transformation or Pareto Optimal Surface. A Pareto Optimal solution is noninferior with respect to any other feasible solution in the sense that it performs strictly better toward at least one objective. In deterministic problems, the Pareto Optimal Surface

and methods for its reconstruction are well studied (Haimes, 1977, chapter 7). Stochastic multiobjective problems are not as well explored and the Pareto Optimal Surface needs yet to be defined.

Reservoir systems are usually provided by operating priorities mandated by institutional agreements (e.g., the Tennessee Valley Authority Act passed by Congress in 1933). The agreements establish a specific priority ranking on the system objectives and prescribe mandatory performance levels. A commonly encountered ranking (Yeh, 1982) appoints flood protection, water supply, water quality control and navigation as the primary objectives and states that the remaining objectives be met so far as they are consistent with the primary ones. From the analyst's point of view, these specifications restrict the investigation of the problem's Tradeoff Surface on that portion which meets the set requirements. Including recreation and wildlife enhancement in the set of primary objectives or assuming that they are implicitly satisfied when water quality standards are met, the following approach for multiple objective reservoir operations can be proposed. Constrain the operation to always meet the primary objectives and maximize the hydropower production as far as possible. Conceptually, this transforms the multiobjective problem into a single objective optimization, but at the same time it allows reconstruction of the Tradeoff Surface by varying the constraint levels or changing the optimized objective (Cohon and Marks, 1975). This section will discuss how this general approach for treating multiple objectives can be quantified in relation to the system model developed in the

previous chapter. The intention is to present a modeling framework which can be flexibly adjusted in specific applications.

Consider, for example, that a system's manager agreed to supply a user with $q(t)$ volume of water per unit of time (discharge) for a specified time period $t\varepsilon[t_0, t_T]$. Assume further that the diversion is located downstream of conceptual reservoir jk. If the system was not random, this agreement could be met by requiring the reservoir's outflow $Q_{jk}(t)$ to be greater than or equal to some flow level $q^{ws}(t)$ which makes the agreed diversion possible. Namely, it would suffice to maintain

$$Q_{jk}(t) \geq q^{ws}(t), \qquad (4.2.1)$$
$$t\varepsilon[t_0, t_T].$$

Replacing $Q_{jk}(t)$ by $\alpha_{jk} s_{jk}^{m_{jk}}(t)$ (c.f. Chapter 3), we find that the diversion can be made as long as reservoir storage $s_{jk}(t)$ satisfies

$$s_{jk}(t) \geq \left| \frac{q^{ws}(t)}{\alpha_{jk}} \right|^{1/m_{jk}}, \qquad (4.2.2)$$
$$t\varepsilon[t_0, t_T].$$

Consider next the same water supply requirement in a stochastic system where storage $s_{jk}(t)$ as well as outflow $Q_{jk}(t)$ are only known in a probabilistic sense. Statements such as "$Q_{jk}(t)$ must be greater than or equal to $q^{ws}(t)$" are now meaningless unless they are supplied with probabilistic specifications regarding their validity. In this case the reservoir manager can choose a probability level $\gamma_{jk}^{ws}(t)$

and see if it is feasible to operate the system so that

$$P_r[Q_{jk}(t) \leq q^{ws}(t)] \leq \gamma_{jk}^{ws}(t), \qquad (4.2.3)$$

$$t \varepsilon [t_0, t_T].$$

In words (4.2.3) specifies that the agreement should be met at reliability not less than $[1 - \gamma_{jk}^{ws}(t)]$. As before an equivalent probabilistic statement can be written with respect to the corresponding storage variable:

$$P_r\left[s_{jk}(t) \leq \left[\frac{q^{ws}(t)}{\alpha_{jk}}\right]^{1/m_{jk}}\right] \leq \gamma_{jk}^{ws}(t), \qquad (4.2.4)$$

$$t \varepsilon [t_0, t_T],$$

or alternatively

$$\int_{-\infty}^{s_{jk}^{ws}(t)} p(s_{jk}(t),t)\, ds_{jk}(t) \leq \gamma_{jk}^{ws}(t), \qquad (4.2.5)$$

$$t \varepsilon [t, t_T]$$

In (4.2.5) $p(\cdot,\cdot)$ is the probability density function (p.d.f.) of $s_{jk}(t)$ at time t,

$$s_{jk}^{ws}(t) = \left[\frac{q^{ws}(t)}{\alpha_{jk}}\right]^{1/m_{jk}}, \qquad (4.2.6)$$

and the lower limit of the integral represents the value below which $s_{jk}(t)$ cannot lie with probabilistic significance. Thus, the requirement to meet the water supply demand at $[1 - \gamma_{jk}^{ws}(t)]$ reliability level is mathematically equivalent to constraining the

p.d.f. of the storage variable $s_{jk}(t)$ in a region above a certain bound. This bound is characterized by two parameters: (1) $s_{jk}^{ws}(t)$ which is specified by the demand level through (4.2.6) and (2) $\gamma_{jk}^{ws}(t)$ which gives the probability of failing to satisfy this demand. It is noted that as the demanded level becomes greater and/or the probability of violation gets smaller, constraint (4.2.5) increasingly confines the p.d.f.'s feasible region.

The previous example was presented to illustrate the claim that the requirement to satisfy system objectives at prespecified reliability levels implies probabilistic constraints on the system state variables.

As a second case, consider flood protection. The associated probabilistic constraints can be easily derived after specifying two characteristic parameters per storage variable: That is, the flood level $s_{jk}^{f}(t)$ which the storage is not allowed to exceed and the probability of exceedance $\gamma_{jk}^{f}(t)$. For an actual reservoir, $s_{jk}^{f}(t)$ can be taken equal to the reservoir's capacity (above which water flows down the spillway), while for a conceptual reservoir $s_{jk}^{f}(t)$ can be defined by the flood discharge level $q^{f}(t)$ and the relationship

$$s_{jk}^{f}(t) = \left[\frac{q^{f}(t)}{\alpha_{jk}} \right]^{1/m_{jk}} . \qquad (4.2.7)$$

The flood protection reliability constraint can then be stated by

$$\int_{s_{jk}^{f}(t)}^{+\infty} p(s_{jk}(t),t)\, ds_{jk}(t) \leq \gamma_{jk}^{f}(t), \qquad (4.2.8)$$

$$t \in [t_0, t_T].$$

This constraint restricts the state's p.d.f. to lie below some upper bound and becomes more restrictive for smaller $\gamma_{jk}^{f}(t)$ and/or as less severe flood condition levels are established.

The navigation objective calls for water level fluctuations within a certain range determined by the adjustability of the port structures and the navigability of the river branches. These ranges together with the reliabilities assigned to navigation by the system manager imply two more probability constraints per storage variable.

Water quality control can similarly be treated by requiring river flows to be greater than a critical level above which adequate effluent diffusion takes place. In fact, the actual-conceptual reservoir system model allows for more detailed account of the water quality control objective. This can be seen in Beck, 1974, where the dynamics of the biochemical oxygen demand (BOD) and dissolved oxygen (DO) concentrations are modeled by differential equations very compatible with the system-model of the previous chapter. However, in this work it will suffice to translate meeting water quality standards into reliability constraints on the system's states; different treatment of the subject along the lines mentioned will be undertaken as a separate study.

Similar bounds on the system's storage variables can be imposed by the recreation objective after specifying the operating range of the recreational facilities along with the desirable reliability level.

Thus, it is generally valid that the system objectives can be quantified by a set of reliability constraints on each actual and

conceptual storage variable. (As explained, hydropower production will not be treated by a constraint, but rather it will be maximized given that all other system objectives are satisfied at the prespecified levels.) The constraints are categorized into upper and lower types and the most severe one can be determined from each group. (Note that both the storage thresholds as well as the probabilities of violation determine which are the most severe constraints. Figure 4.2.1 illustrates a case where flood protection is more binding than navigation.) It is evident that if the p.d.f. of a storage variable meets the most severe constraints from the upper and lower group, it also meets all others. In the sequel, these two constraints will be denoted by

$$\int_{-\infty}^{s_i^{min}(t)} p(s_i(t),t)ds_i(t) \leq \gamma_i^{min}(t), \tag{4.2.9}$$

$$\int_{s_i^{max}(t)}^{+\infty} p(s_i(t),t)ds_i(t) \leq \gamma_i^{max}(t), \tag{4.2.10}$$

$$t \varepsilon [t_0, t_T]$$

where i scans the storage variables: $i = 1, \ldots, n_s$. (For notational convenience, from here on the state variables will be referred to by one index (i) rather than by two (jk).)

A Pareto Optimal point corresponding to the previous reliability constraints can now be obtained by optimizing hydroelectric energy generation. Towards representing this objective, we can assume that

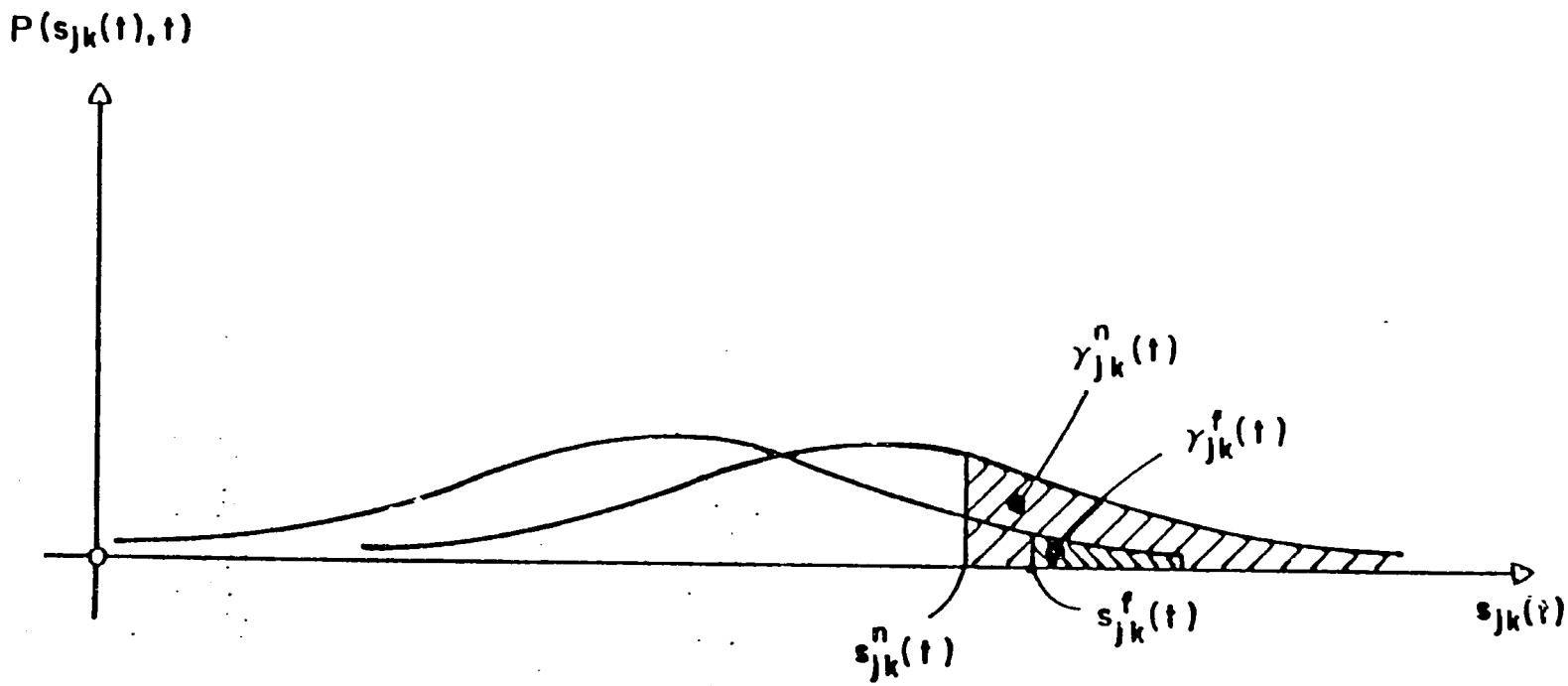

FIGURE 4.2.1: DETERMINING THE MOST SEVERE UPPER PROBABILITY CONSTRAINT

the power production function for reservoir j is given by $g_j(u_j(t), s_j(t))$ where $u_j(t)$ is the total downstream release and $s_j(t)$ is the reservoir storage at time t. (The assumption is made that $u_j(t)$ is passed through the turbines as long as this is feasible.) The state $s_j(t)$ enters the power production function by establishing the necessary difference in hydraulic head. Over a period $[t_0, t_T]$ reservoir j will generate

$$\int_{t_0}^{t_T} g_j(u\ (t),\ s_j(t))\ dt \tag{4.2.11}$$

units of energy where $u_j(t)$ and $s_j(t)$ are the actual releases and realized storage levels. Over the same period the energy generation of the entire system will be equal to

$$\sum_j \int_{t_0}^{t_T} g_j(u_j(t),\ s_j(t))dt \tag{4.2.12}$$

where the summation includes all reservoirs with power plants.

Note that it makes no sense to optimize (4.2.12) because a priori this is a random quantity owing to the stochasticity of the storage variables. In turn one should look for some deterministic scalar quantity characterizing the underlying p.d.f. A typical choice is to optimize the expected value:

$$\underset{\substack{\underline{u}(t) \\ t\varepsilon[t_0,t_T]}}{\text{Maximize}} \quad E\left\{ \sum_j \int_{t_0}^{t_T} g_j(u_j(t), s_j(t))\ dt \right\} \tag{4.2.13}$$

or alternatively

$$\underset{\substack{\underline{u}(t) \\ t\in[t_0,t_T]}}{\text{minimize}} \quad E\left\{\sum_j \int_{t_0}^{t_T} \left[T_j - g_j(u_j(t),s_j(t))\right]^2 dt\right\} \tag{4.2.14}$$

where $E\{\cdot\}$ denotes expectation with respect to all random quantities, $\underline{u}(t)$ is a vector including all $u_j(t)$, and T_j can be taken equal to reservoir's j installed power capacity. Performance index (4.2.14) maximizes energy generation by quadratically penalizing energy deficits from the maximal possible production. For optimization purposes this second index is more attractive than the first because $g_j(u_j(t),s_j(t))$ is generally mildly nonlinear and therefore $[T_j - g_i(u_j(t),s_j(t))]^2$ will usually be convex. (Problems of convex structure are generally easier to solve.)

With regard to hydropower, a reservoir system is operated either to generate the maximal possible energy or to cover a preassigned portion of the power demand. (The rest of the load is undertaken by other types of power plants.) Either (4.2.13) or (4.2.14) can be employed to represent the first mode of operation, while the following performance index can be used to represent the second:

$$\underset{\substack{\underline{u}(t) \\ t\in[t_0,t_T]}}{\text{minimize}} \quad E\left\{\int_{t_0}^{t_T} [T - \sum_j g_j(u_j(t),s_j(t))]^2 \, dt\right\} \tag{4.2.15}$$

where T is the portion of the power demand assigned to the hydroelectric units.

Note that we can represent both indices (4.2.14) and (4.2.15) by

$$\underset{\substack{\underline{u}(t) \\ t\varepsilon[t_0,\, t_T)}}{\text{minimize}} \quad \left[J_c = E \left\{ \int_{t_0}^{t_T} g(\underline{u}(t),\, \underline{s}(t))dt \right\} \right] \tag{4.2.16}$$

where $g(\underline{u}(t),\, \underline{s}(t)) = \sum_j [T_j - g_j(u_j(t),\, s_j(t))]^2$

in the first case, and

$$g(\underline{u}(t),\, \underline{s}(t)) = [T - \sum_j g_j(u_j(t),\, s_j(t))]^2$$

in the second. Additionally note that other possibilities such as run-on-the-river power plants can also be represented by this more general performance index. (In such,cases the power production function will solely depend on the corresponding conceptual storage variable.)

As discussed in the beginning of this section, our objective has been to identify release (control) trajectories corresponding to points on the problem's Pareto Optimal Surface. A Pareto Optimal release trajectory $\underline{u}^*(t)$, $t\varepsilon[t_0, t_T]$ is defined here as the one which achieves the minimum value of performance index (4.2.16) given that it produces a probabilistic state trajectory satisfying the reliability constraints imposed by all other system objectives. (At this stage, the reader should only try to follow the general concept. More precise understanding will be acquired as this chapter progresses.) Other

Pareto Optimal trajectories can be obtained by varying the probabilistic constraint levels and the entire Pareto Optimal surface can be generated and explored. Notice that the priority ranking of the system's objectives can be conveniently reflected by the allowable levels of probabilistic constraint violation. Between two objectives, the one of higher priority should have smaller $\gamma(t)$.

Having discussed the modeling of the system's dynamics as well as its objectives, we shall next proceed to formulate and consider the associated optimization problem.

4.3 The Reservoir Operation Problem

From the discussions of Chapter 3, the following mathematical system model was seen to result:

$$\frac{d\,\underline{s}(t)}{dt} = \underline{F}(\underline{s}(t),t) + \underline{\underline{L}}\,\underline{u}(t) + \underline{\varepsilon}(t), \qquad (4.3.1)$$

where $\underline{s}(t)$ is the n_s dimensional state vector including all actual and conceptual reservoir storage variables,

$\underline{s}(0)$ is assumed to be a Gaussian random vector with mean $\bar{\underline{s}}(0)$ and covariance $\underline{\underline{P}}_s(0)$,

$\underline{u}(t)$ is the n_u dimensional control vector,

$\underline{\varepsilon}(t)$ is the n_s dimensional random disturbance vector assumed to be Gaussian white noise (non autocorrelated in time) with mean $\bar{\underline{\varepsilon}}(t)$ and positive semidefinite spectral density matrix $\underline{\underline{Q}}_\varepsilon(t)$:

$$E\{\underline{\varepsilon}(t)\} = \bar{\underline{\varepsilon}}(t),$$

$$E\{[\underline{\varepsilon}(t) - \bar{\underline{\varepsilon}}(t)][\underline{\varepsilon}(s) - \bar{\underline{\varepsilon}}(s)]^T = \underline{\underline{Q}}_\varepsilon(t)\,\delta(t-s),$$

$$t,s \in [t_0, t_T]$$

($\delta(t-s)$ is the dirac delta function); $\underline{\varepsilon}(t)$ is also assumed uncorrelated with $\underline{s}(0)$,

$\underline{F}(\cdot,\cdot)$ is the n_s dimensional time varying nonlinear function of the state, and

$\underline{\underline{L}}$ is the $n_s \times n_u$ constant control coefficient matrix.

Associated with the system in Eq. (4.3.1) is the state observation model expressed by the following vector equation:

$$\underline{z}(t_k) = \underline{H}(\underline{s}(t_k)) + \underline{v}(t_k). \tag{4.3.2}$$

The output $\underline{z}(t_k)$ includes measurements of actual reservoir elevations (for the actual storage variables) as well as measurements of discharge at gauged channel sections (for the observation of the conceptual storages) taken at discrete time instants. $\{\underline{v}(t_k);\ k = 1, 2, \ldots, T-1\}$ is a discrete Gaussian white random process with zero mean and positive definite covariance matrix $\underline{\underline{R}}_{vk}$, $k = 1, 2, \ldots, T-1$. It is assumed uncorrelated with $\underline{s}(0)$ and $\underline{\varepsilon}(t)$ for all t. The observation equation may exclusively refer to discharge measurements if it is assumed that the actual reservoir elevation observations can specify (through the elevation-volume relationships) the corresponding actual storages with no significant error.

According to the discussion of the previous section, the objective of the reservoir operation problem is to identify the release trajectory $\{\underline{u}^*(t);\ t\ [t_0, t_T]\}$ which minimizes

$$J_c = E\left\{\int_{t_0}^{t_T} g[\underline{s}(t),\ \underline{u}(t)]\ d_t\right\} \tag{4.3.3}$$

subject to the following constraint:

$$
\left.
\begin{array}{ll}
\text{a.} & \text{Eq. (4.3.1) of the system dynamics.} \\
\text{b.} & u_j^{min}(t) \le u_j(t) \le u_j^{max}(t) \\
 & j = 1, 2, \ldots, n_u, \quad t\varepsilon[t_0, t_T] \\
\text{c.} & \displaystyle\int_{-\infty}^{s_i^{min}(t)} p(s_i(t),t)\, ds_i(t) \le \gamma_i^{min}(t), \\
 & \displaystyle\int_{s_i^{max}(t)}^{+\infty} p(s_i(t),t)\, ds_i(t) \le \gamma_i^{max}(t) \\
 & \text{where } i = 1, 2, \ldots, n_s, \qquad t\varepsilon[t_0, t_T]. \\
\text{d.} & \text{Continuing operations after } t_T
\end{array}
\right\} \qquad (4.3.4)
$$

Calling the problem above 4.1 we note the following comments:

1. The expectation in (4.3.3) is with respect to all random variables; namely $\{\underline{\varepsilon}(t);\ t\varepsilon[t_0,t_T]\}$, $\{\underline{v}(t_k);\ k = 1, 2, \ldots, T-1\}$, $\underline{s}(0)$.

2. The scalar function $g(\cdot,\cdot)$ concerns the power objective as discussed in Section 4.2.

3. Constraints b may reflect physical or operational considerations (e.g., low flow requirements, etc.).

4. The probability densities $p(s_i(t),t)$ are, for all i, conditioned on all available measurements and all previously applied controls.

5. It may be argued that the reservoir operation problem should seek to minimize

$$E\left\{\int_{t_0}^{+\infty} g(\underline{s}(t),\ \underline{u}(t))d_t\right\} \tag{4.3.5}$$

where $+\infty$ exhausts the system's lifetime. Although this suggestion reflects a pragmatic concern, it could mathematically lead to an ill-conditioned problem if the integral in (4.3.5) is not convergent. (For a discussion of infinite-horizon control problems, see Bertsekas, 1976, Chapter 6.) To avoid such pathologies, this work instead seeks to optimize functional (4.3.3), but the finite control horizon $[t_0, t_T]$ and the constraint d are specified so that the true optimal control policy is identified.

6. The solution of the stated problem will be a policy realizing a specific point on the objectives' tradeoff surface. This is the point corresponding to the specified reliability levels $\gamma_i^{max}(t)$, $\gamma_i^{min}(t)$ and respective thresholds $s_i^{max}(t)$, $s_i^{min}(t)$, $i = 1, 2, \ldots, n_s$. Varying these levels over their range of interest, portions of the tradeoff surface can be generated and the most desirable location can be selected.

7. Still, Problem P.1 is not meant to be formally complete; it is rather meant to set the form of the optimization problem we are about to tackle. Several important specifications, as for example, the admissible form of the control policy, remain to be discussed. Such supplements will be provided as more stochastic control theory background is communicated.

Like most interesting real world problems, the solution of the

reservoir operation problem will involve a digital computer. Consequently, we find it appropriate to convert the previous formulation to the discrete time representation and work with difference rather than differential equations. This conversion is also justified because in a real application solutions of the operation problem are desired in monthly, weekly, daily, etc., time intervals.

Consider a deterministic, nominal control trajectory which is constant within each interval $[t_k, t_{k+1})$, $k = 0, 1, \ldots, T-1$:

$$\{\underline{u}^{nom}(t) = \underline{u}^{nom}(t_k),\ t\varepsilon[t_k, t_{k+1}),\ k=0,1, \ldots, T-1\}. \quad (4.3.6)$$

Set the random distrubance $\underline{\varepsilon}(t)$ equal to its mean level $\bar{\underline{\varepsilon}}(t)$, and integrate Eq. (4.3.1) over $[t_0,t_T]$ taking the expected value $\bar{\underline{s}}(0)$ as initial state condition. Then by construction the resulting nominal state trajectory satisfies

$$\frac{d\ \underline{s}^{nom}(t)}{d\,t} = \underline{F}(\underline{s}^{nom}(t),t) + \underline{\underline{L}}\ \underline{u}^{nom}(t) + \bar{\underline{\varepsilon}}(t). \quad (4.3.7)$$

Using a Taylor series expansion, we linearize Eq. (4.3.1), (4.3.2) around the nominal state and control trajectories to obtain the following linear system:

$$\frac{d\ \underline{s}(t)}{d\,t} = \underline{F}(\underline{s}^{nom}(t),t) + \underline{\underline{F}}_1(\underline{s}^{nom}(t),t)(\underline{s}(t)-\underline{s}^{nom}(t)) +$$

$$\underline{\underline{L}}\ \underline{u}(t) + \underline{\varepsilon}(t), \quad (4.3.8)$$

$$\underline{z}(t_k) = \underline{H}(\underline{s}^{nom}(t_k)) + \underline{\underline{H}}_1(\underline{s}^{nom}(t_k))(\underline{s}(t_k)-\underline{s}^{nom}(t_k)) + \underline{v}(t_k) \quad (4.3.9)$$

where

$$\left.\begin{aligned} \underline{\underline{F}}_1(\underline{s}^{nom}(t),t) &= \left[\frac{\partial \underline{F}(\underline{s}^{nom}(t),t)}{\partial s_1} \cdots \frac{\partial \underline{F}(\underline{s}^{nom}(t),t)}{\partial s_{n_s}}\right], \\ \frac{\partial \underline{F}(\underline{s}^{nom}(t),t)}{\partial s_i} &= \begin{bmatrix} \dfrac{\partial F_1(\underline{s}^{nom}(t),t)}{\partial s_i} \\ \vdots \\ \dfrac{\partial F_1(\underline{s}^{nom}(t),t)}{\partial s_i} \end{bmatrix}, \end{aligned}\right\} \quad (4.3.10)$$

and similarly for $\underline{\underline{H}}_1(\underline{s}^{nom}(t_k))$.

Define next

$$\left.\begin{aligned} &\delta\underline{s}(t) = \underline{s}(t) - \underline{s}^{nom}(t), \quad \delta\underline{s}(0) = \underline{s}(0) - \bar{\underline{s}}(0). \\ &\delta\underline{u}(t) = \underline{u}(t) - \underline{u}^{nom}(t), \\ &\underline{\xi}(t) = \underline{\varepsilon}(t) - \bar{\underline{\varepsilon}}(t), \\ &t\varepsilon[t_0,t_T], \\ &\delta\underline{z}_k = \underline{z}(t_k) - \underline{H}(\underline{s}^{nom}(t_k)) \quad \underline{v}_k = \underline{v}(t_k), \\ &k = 1, 2, \ldots, T-1, \end{aligned}\right\} \quad (4.3.11)$$

and invoke Eq. (4.3.7) to get the following linear perturbation system model in continuous time:

$$\frac{d\delta\underline{s}(t)}{dt} = \underline{\underline{F}}_1(\underline{s}^{nom}(t),t)\,\delta\underline{s}(t) + \underline{\underline{L}}\,\delta\underline{u}(t) + \underline{\xi}(t), \quad (4.3.12)$$

$$\delta\underline{z}_k = \underline{\underline{H}}_1(\underline{s}^{nom}(t_k))\,\delta\underline{s}(t_k) + \underline{v}_k \quad (4.3.13)$$

Towards the discrete time formulation we need to solve the system of differential equations in (4.3.12). The solution, which is guaranteed to exist and is unique, consists of a homogeneous and a forced part (Brockett, 1970, Chen, 1970, Kailath, 1980). The homogeneous part solves

$$\frac{d\ \delta \underline{s}(\tau)}{d\tau} = \underline{\underline{F}}_1 (\underline{s}^{nom}(\tau), \tau)\ \delta \underline{s}(\tau), \tag{4.3.14}$$

and gives the state $\underline{s}(t)$ at time t as a function of the state at any other time t_k via

$$\delta \underline{s}(t) = \underline{\underline{\phi}}(t, t_k)\ \delta \underline{s}(t_k). \tag{4.3.15}$$

$\underline{\underline{\phi}}(\cdot,\cdot)$ is the state transition matrix which satisfies

$$\frac{d\ \underline{\underline{\phi}}(\tau,s)}{d\tau} = \underline{\underline{F}}_1 (\underline{s}^{nom}(\tau), \tau)\ \underline{\underline{\phi}}(\tau,s) \tag{4.3.16}$$

with initial condition

$$\underline{\underline{\phi}}(s,s) = \underline{\underline{I}}. \tag{4.3.17}$$

Among the properties of the state transition matrix are that

$$\underline{\underline{\phi}}(t,s) \text{ is invertible for all } t,s, \tag{4.3.18}$$

$$\underline{\underline{\phi}}(t,s)^{-1} = \underline{\underline{\phi}}(s,t), \text{ and} \tag{4.3.19}$$

$$\underline{\underline{\phi}}(t,s) = \underline{\underline{\phi}}(t,r)\ \underline{\underline{\phi}}(r,s) \text{ for any } t,s,r \text{ ordering.} \tag{4.3.20}$$

The forced part of the solution solves

$$\frac{d\delta \underline{s}(\tau)}{d\tau} = \underline{\underline{F}}_1(\underline{s}^{nom}(\tau),\tau)\ \delta\underline{s}(\tau) + \underline{\underline{L}}\ \delta\underline{u}(\tau) + \xi(\tau) \tag{4.3.21}$$

with initial condition $\delta\underline{s}(t_k) = \underline{0}$,

and is given by (variation of constants formula)

$$\delta\underline{s}(t) = \int_{t_k}^{t} \underline{\underline{\phi}}(t,\tau)\ \underline{\underline{L}}\ \delta\underline{u}(\tau)d\tau + \int_{t_k}^{t} \underline{\underline{\phi}}(t,\tau)\ \underline{\xi}\ (\tau)d\tau. \tag{4.3.22}$$

If the control $\delta\underline{u}(\tau)$ is assumed constant over the interval (t_k,t) and equal to $\delta\underline{u}_k$, then Eq. (4.3.22) becomes

$$\delta\underline{s}(t) = \left(\int_{t_k}^{t} \underline{\underline{\phi}}(t,\tau)d\tau\right)\ \underline{\underline{L}}\ \delta\underline{u}_k + \int_{t_k}^{t} \underline{\underline{\phi}}(t,\tau)\ \underline{\xi}(\tau)\ d\tau\ . \tag{4.3.23}$$

Combining Equations (4.3.23) and (4.3.15), the complete solution can be obtained:

$$\delta\underline{s}(t) = \underline{\underline{\phi}}(t,t_k)\ \delta\underline{s}(t_k) + \left(\int_{t_k}^{t} \underline{\underline{\phi}}(t,\tau)d\tau\right)\ \underline{\underline{L}}\ \delta\underline{u}_k + \int_{t_k}^{t} \underline{\underline{\phi}}(t,\tau)\ \underline{\xi}(\tau)\ d\tau. \tag{4.3.24}$$

Repetitive use of (4.3.24) within the intervals (where by the assumption (4.3.6) the controls are constant) results in the following discrete time system:

$$\delta \underline{s}_{k+1} = \underline{\underline{\phi}}_k \, \delta \underline{s}_k + \underline{\underline{B}}_k \, \delta \underline{u}_k + \underline{\xi}_k, \tag{4.3.25}$$

$$k = 0, 1, \ldots, T-1,$$

where

$$\left.\begin{aligned} \delta \underline{s}_k &= \delta \underline{s}(t_k), \\ \underline{\underline{\phi}}_k &= \underline{\underline{\phi}}(t_{k+1}, t_k), \\ \underline{\underline{B}} &= \left(\int_{t_k}^{t_{k+1}} \underline{\underline{\phi}}(t_{k+1}, \tau) d\tau \right) \underline{\underline{L}}, \\ \underline{\xi}_k &= \int_{t_k}^{t_{k+1}} \underline{\underline{\phi}}(t_{k+1}, \tau) \, \underline{\xi}(\tau) \, d\tau. \end{aligned}\right\} \tag{4.3.26}$$

The integrals and derivatives involving random processes are defined in the mean square sense (Jazwinski, 1970, Chapter 3). Apart from this change in interpretation, however, the operations go through, and the final results (mostly involving deterministic quantities such as statistical moments) are meaningful in the ordinary sense (e.g., Riemann integrals). With this clarification the statistics of the discrete time random process are now derived:

$$E\{\underline{\xi}_k\} = E\left\{ \int_{t_k}^{t_{k+1}} \underline{\underline{\phi}}(t_{k+1}, \tau) \, \underline{\xi}(\tau) d\tau \right\} =$$

$$= \int_{t_k}^{t_{k+1}} \underline{\underline{\phi}}(t_{k+1}, \tau) \, E\{\underline{\xi}(\tau)\} \, d\tau = \underline{0}. \tag{4.3.27}$$

The last equality holds because the expected value of $\xi(\tau) = \underline{\xi}(\tau) - \bar{\underline{\xi}}(\tau)$ is equal to the zero vector while the previous equation is true because expectation and integration can be interchanged.

$$E\{\underline{\xi}_k \underline{\xi}_\ell^T\} = E\left\{\left(\int_{t_k}^{t_{k+1}} \underline{\underline{\phi}}(t_{k+1},\tau)\underline{\xi}(\tau)d\tau\right)\left(\int_{t_\ell}^{t_{\ell+1}} \underline{\underline{\phi}}(t_{\ell+1}, p)\ \underline{\xi}(p)dp\right)^T\right\}$$

$$= \int_{t_k}^{t_{k+1}} \int_{t_\ell}^{t_{\ell+1}} \underline{\underline{\phi}}(t_{k+1},\tau)\ E\{\underline{\xi}(\tau)\underline{\xi}^T(p)\}\ \underline{\underline{\phi}}(t_{\ell+1},p)^T\ dp\ d\tau.$$

Substituting $E\{\underline{\xi}(\tau)\underline{\xi}^T(p)\}$ by $\underline{\underline{Q}}_\varepsilon(\tau)\delta(\tau-p)$ and using the shifting property the dirac delta function, we finally obtain:

$$E\{\underline{\xi}_k \xi_\ell^T\} = \begin{cases} \underline{\underline{0}}, & \text{if } k \neq \ell \\ \displaystyle\int_{t_k}^{t_{k+1}} \underline{\underline{\phi}}(t_{k+1},\tau)\ \underline{\underline{Q}}_\varepsilon(\tau)\ \underline{\underline{Q}}(t_{k+1},\tau)^T d\tau, & \text{if } k=\ell. \end{cases} \quad (4.3.28)$$

Equations (4.3.28), (4.3.27) completely characterize the probability density of the process $\underline{\xi}_k$ which is Gaussian by definition (4.3.26). $\underline{\xi}_k$ results from a linear operation on Gaussian white noise. Similar to (4.3.28), one can show that $E\{\underline{\xi}_k\ \delta\underline{s}_0\} = E\{\underline{\xi}_k \underline{v}_\ell\} = \underline{\underline{0}}$ for all k,ℓ.

On our way to the discrete time control problem, we now need to discretize the performance index (4.3.3) and the constraints (4.3.4).

Integral (4.3.3) can be rewritten as follows:

$$J_c = E\left\{ \sum_{k=0}^{T-1} \left[\int_{t_k}^{t_{k+1}} g[\underline{s}(t), \underline{u}(t)]\, dt \right] \right\} . \tag{4.3.29}$$

Approximating the terms of the above summation by

$$\int_{t_k}^{t_{k+1}} g(\underline{s}(t), \underline{u}(t))dt = g(\underline{s}(t_k), \underline{u}(t_k))(t_{k+1}-t_k), \tag{4.3.30}$$

$$k = 0, 1, \ldots, T-1,$$

one obtains

$$J_d = E\left\{ \sum_{k=0}^{T-1} \left(g(\underline{s}(t_k), \underline{u}(t_k))\ (t_{k+1}-t_k) \right) \right\} \tag{4.3.31}$$

Assuming that the intervals $[t_k, t_{k+1}]$ are of the same length for all k and substituting $\underline{s}(t_k)$, $\underline{u}(t_k)$ by

$$\left.\begin{aligned} \underline{s}(t_k) &= \underline{s}^{nom}(t_k) + \delta\underline{s}_k, \\ \underline{u}(t_k) &= \underline{u}^{nom}(t_k) + \delta\underline{u}_k \end{aligned}\right\} \tag{4.3.32}$$

in (4.3.31), one can consider the following functional for minimization:

$$J = E\left(\sum_{k=0}^{T-1} \ell_k(\delta\underline{s}_k, \delta\underline{u}_k) \right) , \tag{4.3.33}$$

where by definition $\ell_k(\delta\underline{s}_k, \delta\underline{u}_k) = g(\delta\underline{s}_k + \underline{s}^{nom}(t_k), \delta\underline{u}_k + \underline{u}^{nom}(t_k))$.

Regarding the constraints (4.3.4), we shall require their validity at the discrete time instants $t_0, t_1, \ldots, t_T$. For their representation in terms of the variables $\delta\underline{s}_k$, $\delta\underline{u}_k$ we need to know the probability

densities $p(\delta\underline{s}_k, k)$ given $p(\underline{s}(t_k), t_k)$, $k = 0, 1, \ldots, T-1$. Since $\underline{s}(t)$ is the output of a linear system of Gaussian initial state and input, we conclude (this can also be seen by (4.3.24)) that it too is Gaussian for all times. It then follows from $\delta\underline{s}_k = \underline{s}(t_k) - \underline{s}^{nom}(t_k)$ that $\delta\underline{s}_k$ is also Gaussian with mean vector $\bar{\underline{s}}(t_k) - \underline{s}^{nom}(t_k)$ and covariance $\underline{\underline{P}}_s(t_k)$ equal to the one of $\underline{s}(t_k)$. The thresholds $\underline{s}_i^{max}(t_k)$, $\underline{s}_i^{min}(t_k)$, $\underline{u}_j^{max}(t_k)$, $\underline{u}_j^{min}(t_k)$ will respectively translate to

$$\left.\begin{aligned}
&\delta s_{ik}^{max} = s_i^{max}(t_k) - s_i^{nom}(t_k),\\
&\delta s_{ik}^{min} = s_i^{min}(t_k) - s_i^{nom}(t_k),\\
&i = 1, 2, \ldots, n_s, \quad k=0, 1, 2, \ldots, T,\\
&\\
&\delta u_{jk}^{max} = u_j^{max}(t_k) - u_j^{nom}(t_k),\\
&\delta u_{jk}^{min} = u_j^{min}(t_k) - u_j^{nom}(t_k),\\
&j = 1, 2, \ldots, n_u, \quad k = 0, 1, 2, \ldots, T-1,
\end{aligned}\right\} \qquad (4.3.34)$$

and for compatibility we define $\gamma_{ik}^{max} = \gamma_i^{max}(t_k)$, $\gamma_{ik}^{min} = \gamma_i^{min}(t_k)$, $i = 1, 2, \ldots, n_s$, $k = 0, 1, \ldots, T$.

In summary, the discrete time reservoir operation problem which we shall refer to as Problem P.2, is of the following form:

Problem P.2: The Reservoir Operation Problem in Discrete Time

$$\underset{\{\delta \underline{u}_k\}_{k=0}^{T-1}}{\text{Minimize}} \left\{ J = \underset{\substack{\delta \underline{s}_0 \\ \{\underline{\xi}_k\}_{k=0}^{T-1} \\ \{\underline{v}_k\}_{k=1}^{T}}}{E} \left\{ \sum_{k=0}^{T-1} \ell_k(\delta \underline{s}_k, \delta \underline{u}_k) \right\} \right\}$$

subject to

a. $\delta \underline{s}_{k+1} = \underline{\underline{\phi}}_k \, \delta \underline{s}_k + \underline{\underline{B}}_k \, \delta \underline{u}_k + \underline{\xi}_k,$

$k = 0, 1, \ldots, T-1,$

$\delta \underline{z}_k = \underline{\underline{H}}_1 [\underline{s}^{nom}(t_k)] \, \delta \underline{s}_k + \underline{v}_k,$

$k = 1, 2, \ldots, T-1$

b. $\delta u_{jk}^{min} \leq \delta u_{jk} \leq \delta u_{jk}^{max},$

$j = 1, 2, \ldots, n_u, \quad k = 0, 1, \ldots, T-1.$

c. $$\int_{-\infty}^{\delta s_{ik}^{min}} p(\delta s_{ik}, k) \, d\delta s_{ik} \leq \gamma_{ik}^{min},$$

$$\int_{s_{ik}^{max}}^{+\infty} p(\delta s_{ik}, k) \, d\delta s_{ik} \leq \gamma_{ik}^{max},$$

$i = 1, 2, \ldots, n_s, \quad k = 0, 1, \ldots. T.$

d. Continuing operations after T.

Problem P.2 is not the exact equivalent of Problem P.1. This is mainly due to the linearization of the continuous time dynamics introduced at (4.3.8). As a consequence the more $\underline{s}(t)$ deviates from $\underline{s}^{nom}(t)$,

the less Problem P.2 is representative of P.1 with this effect becoming more significant as the degree of nonlinearity of $\underline{F}(\underline{s}(t),t)$ increases. The validity of the approximation is further jeopardized by the presence of the random noise for, if intense, the noise originates frequent departures of $\underline{s}(t)$ away from $\underline{s}^{nom}(t)$.

In order to mitigate this effect, one could consider using the Describing Function (Gelb and Vender Velde, 1968), otherwise known as Quasi- (Phaneuf, 1968) or Statistical (Gelb, 1973) linearization. In hydrology this technique -- first introduced by Kitanidis and Bras, 1978, and also employed by Georgakakos, K. and Bras, 1980, 1982 -- has been shown to possess distinct advantages over the Taylor Expansion lineraization at the expense of heavier computer effort. The advantages result from this method's explicit concern to minimize the average approximation error. The high computational requirements are a consequence of the additional probabilistic information utilized on the variables involved. More specifically, the linear approximation

$$\underline{F}_L(\underline{s}(t),t) = \underline{F}_0 + \underline{\underline{F}}_1(\underline{s}(t)-\underline{s}^{nom}(t)) \tag{4.3.35}$$

of the function $\underline{F}\ \underline{s}(t),t$ is sought which

$$\left.\begin{array}{l} \underset{\underline{F}_0,\underline{\underline{F}}_1}{\text{minimizes}} \quad J_e = E\{\underline{e}^T\ \underline{e}\} \\ \underline{e} = \underline{F}(\underline{s}(t),t) - \underline{F}_L(\underline{s}(t),t) \end{array}\right\} \tag{4.3.36}$$

The expectation is taken with respect to the probability density of $\underline{s}(t)$. This problem can be easily solved by directly differentiating

J_e with respect to the vector $\underline{F}_0$ and the matrix $\underline{\underline{F}}_1$ and setting the resulting linear expressions equal to the zero element of the corresponding dimension. Assuming that $\bar{\underline{s}}(t) = \underline{s}^{nom}(t)$, the optimal coefficients are given by the following expressions:

$$\left.\begin{aligned} \underline{F}_0 &= E\{\underline{F}(\underline{s}(t),t)\} \\ \underline{F}_1 &= E\{\underline{F}(\underline{s}(t),t)\ (\underline{s}(t)-\underline{s}^{nom}(t))^T\}\ \underline{\underline{P}}_s(t)^{-1}, \end{aligned}\right\} \quad (4.3.37)$$

where $\underline{\underline{P}}_s(t)$ is the covariance matrix of $\underline{s}(t)$. As can be seen from (4.3.35) and the first of the (4.3.37) relationships, the resulting approximation is unbiased which is an important gain over the Taylor Expansion approach. On the other hand, the evaluation of the expectations in (4.3.37) could be cumbersome and may prohibit its use in a real time application. Other properties of the Describing Function linearization are that it preserves statistical relationships up to the 2nd moment between $\underline{F}(\underline{s}(t),t)$ and $\underline{s}(t)$, and it can also be employed when $\underline{F}(\underline{s}(t),t)$ is nondifferentiable.

Apart from using the appropriate linearization type, the credibility of Problem P.2 can also be enforced by restricting the range where the controls are allowed to vary from their nominal path. On the average, this will result in state trajectories in the neighborhood of the nominal one and will justify the use of the perturbation model.

This section has formulated the optimization problem associated with the operation of a reservoir system. The formulation is based on the system model developed in Chapter 3 and the representation of the

system objectives discussed in Section 4.2. The resulting optimization problem (with dynamics in difference or differential equation form and time separable objective function and constraints) is of the optimal control variety. Working toward a solution, we shall next review some theoretical material developed for the treatment of similar problems.

4.4 A Review of Optimal Control Theory

Optimal control problems frequently arise in the study of physical and socio-economic systems. Although they can be treated according to ordinary nonlinear programming techniques, a solution process can significantly benefit by exploiting their special structure. Depending on the modeling approach, one can distinguish two types of optimal control problems: deterministic and stochastic. Deterministic problems are conceptually better behaved to solve. Stochastic formulations can be conceptually subtle and, in many cases, computationally intractable. Both types can be treated via the main optimal control solution methodologies; namely, the Minimum Principle of Pontryagin (Pontryagin, et al., 1962, Athans and Falb, 1966, Kushner and Schweppe, 1964) and Bellman's Dynamic Programming (Bellman, 1961, Bellman and Dreyfus, 1962, Larson, 1968, Bertsekas, 1976, Larson and Casti, 1978). Yet the Minimum Principle is more efficient in deterministic cases where Dynamic Programming is more appropriate in stochastic problems. This assertion is based (1) on the premise that the Minimum Principle is an open-loop methodology while Dynamic Programming is a closed-loop or feedback one (these terms will soon be defined), and (2) on the fact that in deterministic problems feedback and open-loop solutions are the same (with the latter more easily obtainable) while in stochastic problems feedback solutions are more desirable.

This is a review section intended to provide the unfamiliar reader with an understanding of basic time-domain optimal control theory background. Other design philosophies, such as frequency-

domain and root-locus techniques (Jamshidi, 1980), are also available but less intuitive at least in the water resources literature. Since our problem is stochastic where feedback solutions are preferable, this work will focus on the Dynamic Programming approach.

Consider a nonlinear dynamical system evolving according to the following deterministic difference equation:

$$\underline{s}_{k+1} = \underline{f}_k(\underline{s}_k, \underline{u}_k); \ \underline{s}_0 \text{ known}, \qquad (4.4.1)$$

$$k = 0, 1, \ldots, T-1,$$

where $\underline{s}_k \in S_k$ is the n_s dimensional state vector,

$\underline{u}_k \in U_k$ is the n_u dimensional control vector,

$\underline{f}_k$: $S_k \times U_k \rightarrow S_{k+1}$ is the n_s dimensional, real-valued state transition function,

S_k, U_k are sets of real numbers where the state and the control vectors are constrained to lie at any time k.

Additionally, consider a scalar, real-valued performance measure g_k: $S_k \times U_k \rightarrow \mathbb{R}$ (real numbers) assumed to represent the cost incurred at time k when the state takes a value $\underline{s}_k$ and the control $\underline{u}_k$ is applied. At the terminal finite time T, there will be a terminal cost $g_T: S_T \rightarrow \mathbb{R}$ associated with each terminal state of the process.

The objective is to determine the control sequence $(\underline{u}_0^*, \underline{u}_1^*, \ldots, \underline{u}_{T-1}^*)$ which minimizes

$$J = \sum_{k=0}^{T-1} g_k(\underline{s}_k, \underline{u}_k) + g_T(\underline{s}_T) \qquad (4.4.2)$$

subject to the dynamical Equation (4.4.1) and the accompanying constraints on the states and controls.

Such a formulation can be seen to result from Problem P.2 of the previous section, if one replaces all random quantities with their expected values. Although in reality there exist no deterministic systems, one may be satisfied with this approximation either because the underlying uncertainties are not as influential or simply because the stochastic problem is very difficult to solve.

Two methods of solution are possible: According to the first, one assumes a particular control sequence $\{\underline{u}_0', \underline{u}_1', \ldots, \underline{u}_{T-1}'\}$, and propagates (4.4.1) forward to get the corresponding state trajectory $\{\underline{s}_0', \underline{s}_1'. \ldots, \underline{s}_T'\}$ and the associated performance index value. Then, based on this well-defined mapping $\{\underline{u}_k\}_{k=0}^{T-1} \to J[\{\underline{u}_k\}_{k=0}^{T-1}]$, one searches over all feasible control sequences to find the one incurring the minimum cost. This is the open-loop approach -- directly along the ordinary nonlinear programming principles -- where the properties of the state are not given much attention. The second method relies on the definition of the state $\underline{s}_k$ which at any time k summarizes all currently available information. One then attempts to specify the optimal feedback or closed-loop control laws $\mu_k^*(\cdot)$: $S_k \to U_k$, $k = 0, 1, \ldots, T-1$ which give the optimal controls as functions of the states: $\underline{u}_k^* = \mu_k^*(\underline{s}_k)$, $k = 0, 1, \ldots, T-1$.

In a deterministic problem, the two approaches will result in the application of the same control sequence as can be seen from the following experiment: Suppose the optimal feedback laws have been

obtained. Starting from the initial state $\underline{s}_0$, the system (4.4.1) is run forward with the feedback controls $\underline{u}_k^F = \underline{\mu}_k^F(\underline{s}_k)$ evaluated, applied, and recorded. At the end the record will contain the optimal open-loop sequence $\{\underline{u}_0^{OL}, \underline{u}_1^{OL}, \ldots, \underline{u}_{T-1}^{OL}\}$. If this was not true, at some time j the feedback procedure failed to generate the corresponding open-loop element $(\underline{u}_j^{OL} \neq \underline{\mu}_j^F(\underline{s}_j))$, and the law $\underline{\mu}_j^F(\cdot)$ could not have been optimal. (The argument assumes all the feedback and open loop controls prior to time j coincided and that there is a unique control sequence achieving the minimum cost.) Instrumental in the above equivalence is the fact that the control input is the only influence affecting the state trajectory. In the stochastic case this is no longer true due to the presence of the random inputs which act in addition to the controls and invalidate their unique correspondence with the states.

In principle, the optimal feedback laws can be obtained via the following Dynamic Programming algorithm:

<u>Algorithm A4.4.1</u>: Optimal Feedback Controller for Deterministic Systems.

a. Initialize: $J_T(\underline{s}_T) = g_T(\underline{s}_T)$.

b. Recurse: For k = T-1, T-2, ..., 0, select

$$\underline{\mu}_k^*(\underline{s}_k) = \arg\left\{ \min_{\underline{\mu}_k(\underline{s}_k) \in M} \left\{ g_k(\underline{s}_k, \underline{\mu}_k(\underline{s}_k)) + J_{k+1}(\underline{s}_{k+1}) \right\} \right\},$$

where M is the set of functions satisfying $\underline{u}_k = \underline{\mu}_k(\underline{s}_k) \in U_k$, and evaluate

$$J_k(\underline{s}_k) = g_k(\underline{s}_k, \underline{\mu}_k^*(\underline{s}_k)) + J_{k+1}\left[\underline{f}_k(\underline{s}_k, \underline{\mu}_k^*(\underline{s}_k))\right]$$

c. Terminate: If $J_0(\underline{s}_1) = J^*$ exists[1], then it is the globally minimal cost and

$$\pi^* = \{\underline{\mu}_0^*(\cdot), \ldots, \underline{\mu}_{T-1}^*(\cdot)\} \text{ is the minimizing policy.}$$

Crucial in the derivation of Algorithm A4.4.1 is that the costs are additively separable in time and that each control $\underline{u}_k$ does not affect the costs $g_j(\cdot,\cdot)$ for $j < k$.

Apart from its impressive generality (no restrictive assumptions on the functions' or sets' type were made), in the majority of interesting cases, the above algorithm cannot be implemented due to excessive high-speed memory requirements mainly associated with the backward propagation of the "cost to go" $J_k(\underline{s}_k)$ in Step b. This quantity, except for very few special cases, will have to be recursed in a discretized manner -- each value corresponding to every quantized $\underline{s}_k \in S_k$. When the dimensionality of the state is higher than 4 or 5, a reasonable quantization saturates the capacity of the existing computers. (This is what is known as the "curse of dimensionality".)

In order to remedy this situation many efficient techniques have been developed (Larson, 1968 , Jacobson and Mayne, 1970, Heidari, et al., 1971, Yakowitz, 1982, Turgeon, 1982, etc.). The underlying idea has been to start from some initial nominal control trajectory and

[1] This specification referes to the cases where $J_0(\underline{s}_0)$ becomes increasingly negative for certain control sequences but this is not likely to occur in real world problems.

iteratively move from local considerations to trajectories which reduce the objective function's value. If the problem does not have a special structure, the identified trajectory is only locally optimal, but the procedure can be reinitiated from a different initial trajectory to reveal other local optima and eventually detect the global one.

Unfortunately, deterministic modeling is not an adequate approximation for the majority of real systems. Uncertainty - be it natural or an artifact of our inadequate state of knowledge - is alsmot always present impeding our means for effective systems' control. We, therefore, turn now to stochastic formulations.

Consider the following modification of the system equation:

$$\underline{s}_{k+1} = \underline{f}_k(\underline{s}_k, \underline{u}_k, \underline{w}_k); \underline{s}_0 \text{ known}, \qquad (4.4.3)$$
$$k = 0, 1, \ldots, T-1$$

where $\underline{s}_k$, $\underline{u}_k$, as in the previous formulation,

$\underline{w}_k \in W_k$ is the n_w dimensional random disturbance vector,

$\underline{f}_k: S_k \times U_k \times W_k \to S_{k+1}$ is the n_s dimensional, real-valued state transition function,

W_k is the set from which the random disturbance takes on values at time k.

Also consider a cost $g_k(\underline{s}_k, \underline{u}_k, \underline{w}_k)$ incurred at time k when the system is at state $\underline{s}_k$, the control $\underline{u}_k$ is applied, and the disturbance $\underline{w}_k$ occurs. ($g_k: S_k \times U_k \times W_k \to \mathbb{R}$).

Concerning the modeling of the disturbance, one usually follows

the Bayesian philosophy postulating some probabilistic model. Other descriptions are possible (e.g., the Set Theoretic approach, Schweppe, 1973, Usoro and Schweppe, 1980), but as yet they are not developed for general control processes. It will then be assumed that probability densities are given for the $\underline{w}_k$'s, k=0, 1, ..., T-1 (e.g. obtained by fitting available data), and furthermore (1) each disturbance $\underline{w}_k$ is independent of all previous disturbances, and (2) the p.d.f. of any $\underline{w}_k$ is conditioned only on the contemporary state and control values.

$$p(\underline{w}_k/\underline{s}_k, \ldots, \underline{s}_0, \underline{u}_k, \ldots, \underline{u}_0, \underline{w}_{k-1}, \ldots, \underline{w}_0) = p(\underline{w}_k/\underline{s}_k, \underline{u}_k). \tag{4.4.4}$$

A special case of the above is when the p.d.f. of $\underline{w}_k$ cannot be influenced by $\underline{s}_k$ or $\underline{u}_k$ as, for example, in Problem P.2. If the disturbances are not independent over time, the problem can still be reformulated in the above format by state augmentation (Bertsekas, 1976, Chapter 2).

The uncertainty introduced by the random disturbances creates two distinguishing differences over the deterministic formulation: The state of the system is now a random process and the cost functional $\{\sum_{k=0}^{T-1} g_k(\underline{s}_k, \underline{u}_k, \underline{w}_k) + g_T(\underline{s}_T)\}$ is a random variable. Since minimization of a random variable is meaningless, one usually specifies an operation which maps the space of random variables to the real numbers and continues with the minimization at a deterministic quantity. One such intuitively satisfying map is the expected value operation but other choices are also possible (Sain, 1969). Assume then that the

control objective is to

$$\underset{\substack{\underline{u}_k \\ k=0,1,\ldots,T-1}}{\text{minimize}} \left\{ J = \underset{\substack{\underline{w}_k \\ k=0,1,\ldots,T-1}}{E} \left\{ \sum_{k=0}^{T-1} g_k(\underline{s}_k, \underline{u}_k, \underline{w}_k) + g_T(\underline{s}_T) \right\} \right\} . \quad (4.4.5)$$

Still, the control problem is not totally defined. There are two ways that the minimization (4.4.5) can be performed: First we could attempt to specify the entire control sequence $\{\underline{u}_k\}_{k=0}^{T-1}$ independently of the system's evolution, or alternatively we could assume that at any time k the controller (or decision maker) will apply the control $\underline{u}_k$ after the state $\underline{s}_k$ has become known. The difference in these two choices is that the second assumes future information gathering and takes advantage of the fact by determining optimal feedback laws $\{\underline{\mu}_k^*(\cdot)\}_{k=0}^{T-1}$, while the first ignores this possiblity and determines controls which will be applied independently of the state values. As the optimal open-loop sequence is a subset of the optimal feedback solutions (where the only allowable $\underline{\mu}_k(\cdot)$ functions are the constant ones), it follows that utilizing information gathered during the process's evolution can only improve the system's performance. The purpose then of using feedback in stochastic problems is to reduce the adverse influences of the uncertain inputs.

The solution to the stochastic control problem with perfect state measurements can be obtained via Dynamic Programming by the following backward recursive algorithm:

Algorithm A.4.4.2: Optimal Controller for Uncertain Systems with Perfect State Information

a. Initialize: $J_T(\underline{s}_T) = g_T(\underline{s}_T)$.

b. Recurse: For $k = T-1, T-2, \ldots, 0$, select

$$\underline{\mu}_k^*(\underline{s}_k) = \arg\left\{ \min_{\mu_k(\underline{s}_k)\in M} \left\{ E_{\underline{w}_k} \{ g_k(\underline{s}_k, \underline{\mu}_k(\underline{s}_k), \underline{w}_k) + J_{k+1}(\underline{s}_{k+1}) \} \right\} \right\}$$

where M is the set of functions satisfying $\underline{u}_k = \mu_k(\underline{s}_k) \in U_k$, and evaluate

$$J_k(\underline{s}_k) = E_{\underline{w}_k} \left\{ g_k(\underline{s}_k, \underline{\mu}_k^*(\underline{s}_k), \underline{w}_k) + J_{k+1}\left[f_k(\underline{s}_k, \underline{\mu}_k^*(\underline{s}_k), \underline{w}_k) \right] \right\}$$

c. Terminate: If $J_0(\underline{s}_0) = J^*$ exists, then it is the global minimum and

$$\pi^* = \{\underline{\mu}_0^*(\cdot), \ldots, \underline{\mu}_{T-1}^*(\cdot)\} \quad \text{is the minimizing policy.}$$

Figure 4.4.1 schematically presents the sequence of events taking place in an actual system evolution and the feedback nature of the optimal controller.

Like the one before it, this D.P. algorithm is not practically useful for most real world problems (due to the "curse" of dimensionality). Nevertheless, there exists a particular class of problems - the class of linear systems and quadratic performance indices presented later in this section - where the above procedure leads to analytical solutions of significant practical value.

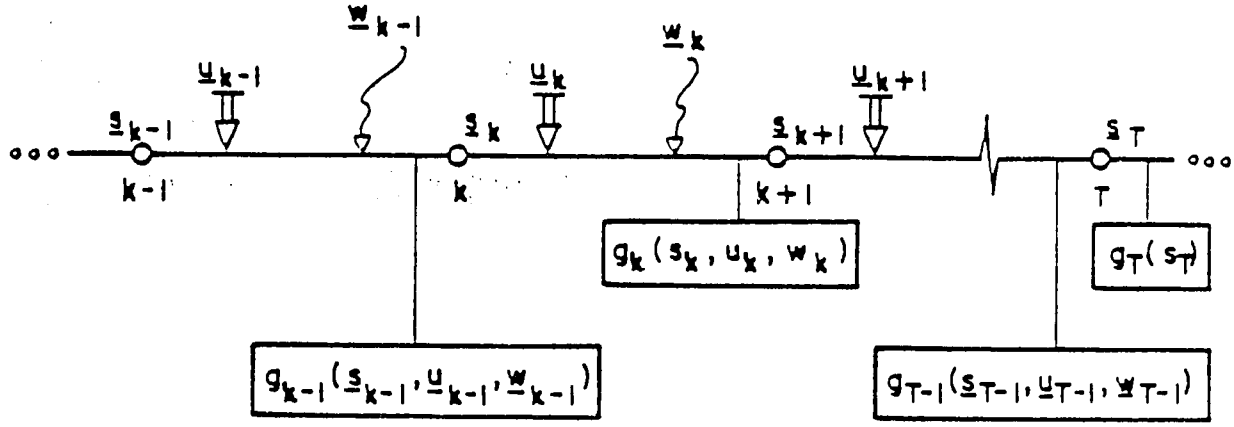

FIGURE 4.4.1a: THE SEQUENTIAL NATURE OF SYSTEMS WITH EXACTLY KNOWN STATE

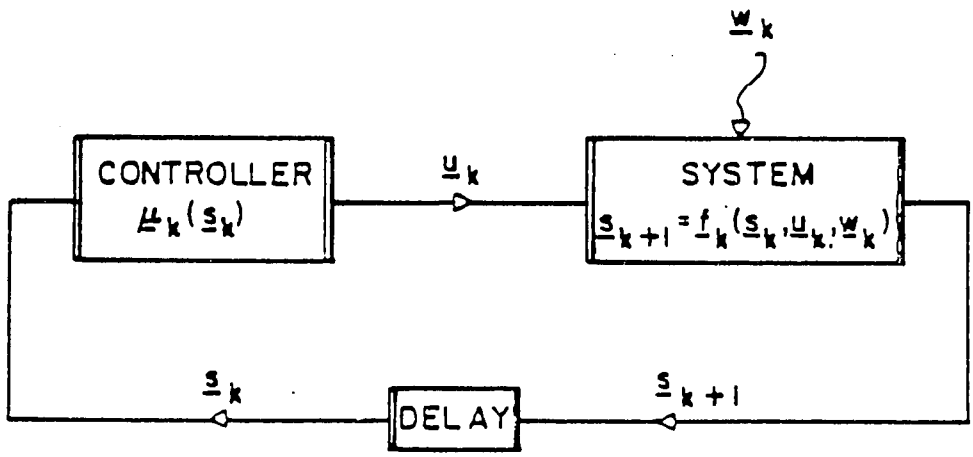

FIGURE 4.4.1b: THE STRUCTURE OF THE OPTIMAL FEEDBACK CONTROLLER

Referring back to Problem P.2, we note the following difference over the stochastic formulation previously treated. Namely, its state at each time k is not perfectly known; some other quantity functionally related to the state can imperfectly be measured instead. This difference is a very important one because, after all, it was the properties of the state that made feedback solutions possible. These problems with imperfect state information (or inexactly known state) are subjects of the forthcoming discussion.

The system dynamical Equation (4.4.3) previously given is also valid in this case with the exception of the initial state $\underline{s}_0$ being a random variable. s_0 is assumed to have an a priori p.d.f $p(\underline{s}_0)$ and to be independent from $\underline{w}_k$ for all k. In addition to that formulation, however, the state here is assumed observable only through the measurements $\underline{z}_k$:

$$\underline{z}_k = \underline{h}_k(\underline{s}_k, \underline{u}_{k-1}, \underline{v}_k), \tag{4.4.6}$$

$$k = 1, 2, \ldots, T-1,$$

where $\underline{z}_k$ is the n_z dimensional measurement (or observations) vector,

$\underline{v}_k$ is the n_v dimensional random error (noise) the p.d.f. of which is assumed to satisfy

$$p(\underline{v}_k/\underline{s}_k,\ldots,\underline{s}_0,\underline{u}_{k-1},\ldots,\underline{u}_0,\underline{v}_{k-1},\ldots,\underline{v}_0,\underline{z}_{k-1},\ldots,\underline{z}_0) =$$

$$= p(\underline{v}_k/\underline{s}_k,\underline{u}_{k-1}), \tag{4.4.7}$$

$\underline{h}_k$: $S_k \times U_{k-1} \times V_k \rightarrow Z_k$ is the real-valued measurement function, V_k, Z_k are the error and measurement sets.

(The case of Problem P.2, where $p(\underline{v}_k/\underline{s}_k, \underline{u}_{k-1}) = p(\underline{v}_k)$, is again a special case of the one postulated above.

As before, the control objective is to

$$\underset{\{u_k\}_{k=0}^{T-1}}{\text{minimize}} \left\{ J = \underset{\substack{\underline{s}_0 \\ \{\underline{w}_k\}_{k=0}^{T-1} \\ \{\underline{v}_k\}_{k=1}^{T-1}}}{E} \left\{ \sum_{k=0}^{T-1} g_k(\underline{s}_k, \underline{u}_k, \underline{w}_k) + g_T(\underline{s}_T) \right\} \right\} \tag{4.4.8}$$

with the expectation here additionally taken over the uncertainty introduced by the observations errors.

Feedback control laws are again desired, the idea of feedback being to utilize all available information in the control process. In the case of exactly known state, the feedback was on the state because this was all the controller needed to make the best possible decision. Since in this formulation the state is not accessible, the best the controller can do is to base its decisions on all pieces of information gathered so far:

$$\left.\begin{aligned} I_k &= \{\underline{z}_1, \underline{z}_2, \ldots, \underline{z}_k, \underline{u}_0, \underline{u}_1, \ldots, \underline{u}_{k-1}\}, \text{ or} \\ I_k &= I_{k-1} \cup \{\underline{z}_k\}, \; k=1,2,\ldots,T-1, \\ I_0 &\triangleq \{\emptyset\}. \end{aligned}\right\} \tag{4.4.9}$$

In that sense the information vector I_k can be viewed as another system's state. In fact, it can be shown that there is an equivalent to the previous problem with exactly known state I_k (Tenney, 1982)

and feedback controls laws $\underline{u}_k(I_k)$. However, although I_k theoretically qualifies for a system's state, in practice it is not so appealing because of its expanding nature.

In search of an alternative, recall that the vector I_k was considered in the first place because it contained all information of possible use to the controller. If there was a way to process this information in a more usable but adequate form for control purposes, the difficulty could be alleviated. This idea leads to the notion of sufficient statistics (Striebel 1973) one of which is the conditional density $p(\underline{s}_k/I_k)$. We shall next present a reformulation of the present problem into one with an exactly known state equal to the conditional probability density $p(\underline{s}_k/I_k)$, the solution of which can be obtained by an algorithm similar to A4.4.2.

First, we must construct the dynamics of the new state $p(\underline{s}_k/I_k)$. This will be accomplished in two steps: the propagation step and the update step schematically illustrated in Figure 4.4.2.

a. Propagation Step.

The problem is as follows:

Given $p(\underline{s}_k/I_k)$ from the previous update step,

$$\underline{s}_{k+1} = \underline{f}_k(\underline{s}_k,\underline{u}_k,\underline{w}_k),$$

$$p(\underline{w}_k/\underline{s}_k,\underline{u}_k),$$

find $p(\underline{s}_{k+1}/I_k,\underline{u}_k)$.

Notice first that from $\underline{s}_{k+1} = \underline{f}_k(\underline{s}_k,\underline{u}_k,\underline{w}_k)$ and $p(\underline{w}_k/\underline{s}_k,\underline{u}_k)$ one can obtain the density $p(\underline{s}_{k+1}/\underline{s}_k,\underline{u}_k)$ employing, for example, the method

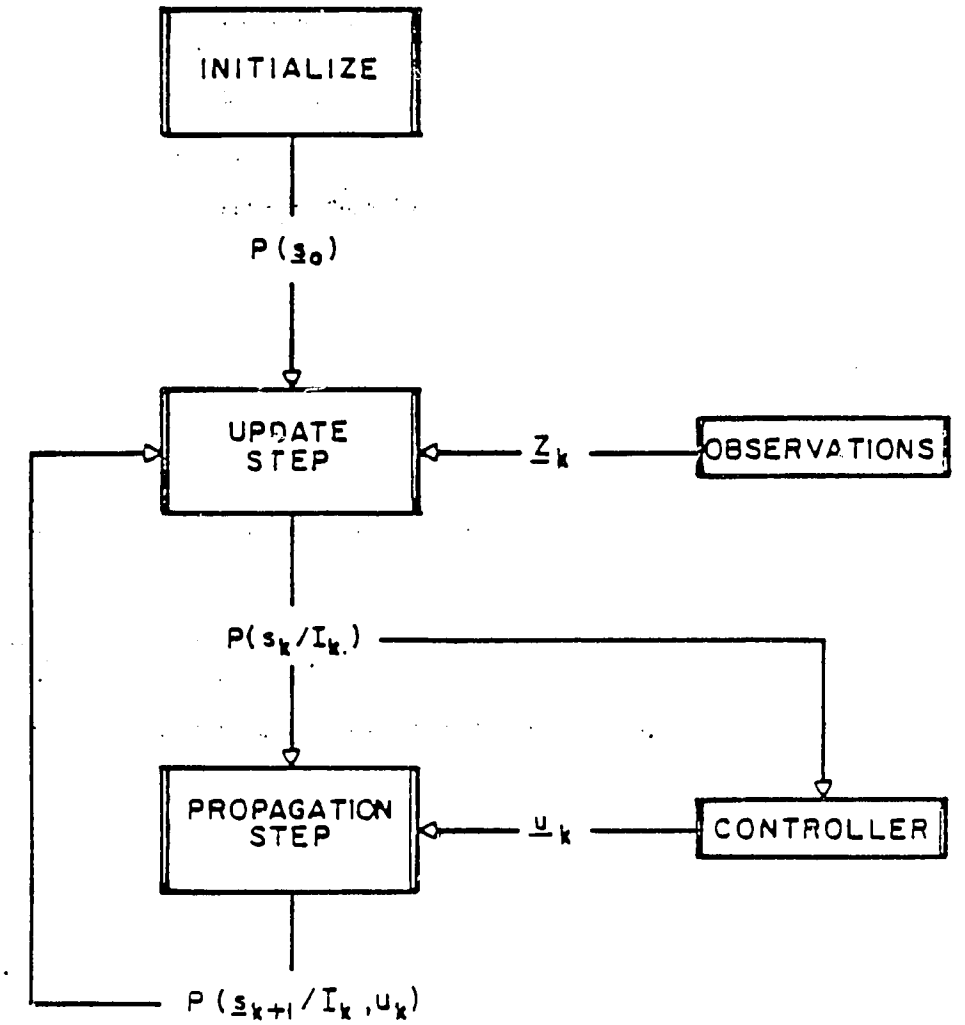

FIGURE 4.4.2: THE ESTIMATION SCHEME FOR THE RECURSIVE DERIVATION OF THE CONDITIONAL DENSITY $P(\underline{s}_k/I_k)$

of events (Sandell and Shapiro, 1976). Next, there holds

$$p(s_{k+1}/I_k,\underline{u}_k) = \int_{\underline{s}_k} p(\underline{s}_{k+1},\underline{s}_k/I_k,\underline{u}_k)d\underline{s}_k =$$

$$= \int_{\underline{s}_k} p(s_{k+1}/\underline{s}_k,I_k,\underline{u}_k)\ p(\underline{s}_k/I_k,\underline{u}_k)d\underline{s}_k. \qquad (4.4.10)$$

The notation implies integration over the range of $\underline{s}_k$'s variation and the equalities are true because for any two jointly distributed random vectors $\underline{x}$, $\underline{y}$ there holds

$$p(\underline{x}) = \int_{\underline{y}} p(\underline{x},\underline{y})d\underline{y} = \int_{y} p(\underline{x}/\underline{y})\ p(\underline{y})d\underline{y}. \qquad (4.4.11)$$

However, by the Markov state property, I_k does not add any information if $\underline{s}_k$ is known and therefore

$$p(\underline{s}_{k+1}/\underline{s}_k,\ I_k,\ \underline{u}_k) = p(\underline{s}_{k+1}/\underline{s}_k,\ \underline{u}_k). \qquad (4.4.12)$$

Also, since $\underline{s}_k$ does not depend on $\underline{u}_k$

$$p(\underline{s}_k/I_k,\underline{u}_k) = p(\underline{s}_k/I_k). \qquad (4.4.13)$$

Using (4.4.13) and (4.4.12) in (4.4.10) we obtain the following equation known as the Chapman-Kolmagorov equation (Jazwinski, 1970):

$$p(\underline{s}_{k+1}/I_k,\underline{u}_k) = \int_{\underline{s}_k} p(\underline{s}_{k+1}/\underline{s}_k,\underline{u}_k)\ p(\underline{s}_k/I_k)d\underline{s}_k \qquad (4.4.14)$$

b. Update Step

The problem is to evaluate the density $p(\underline{s}_{k+1}/I_{k+1})$ given $p(\underline{s}_{k+1}/I_k, \underline{u}_k)$ from the propagation step,

$$\underline{z}_{k+1} = \underline{h}_{k+1}(\underline{s}_{k+1}, \underline{u}_k, \underline{v}_{k+1})$$
$$p(\underline{v}_{k+1}/\underline{s}_{k+1}, \underline{u}_k).$$

Employing again the method of events, we can derive $p(\underline{z}_{k+1}/\underline{s}_{k+1}, \underline{u}_k)$ from $\underline{z}_{k+1} = \underline{h}_{k+1}(\underline{s}_{k+1}, \underline{u}_k, \underline{v}_{k+1})$ and $p(\underline{v}_{k+1}/\underline{s}_{k+1}, \underline{u}_k)$. Then using the Bayes rule we get:

$$p(\underline{s}_{k+1}/I_{k+1}) = p(\underline{s}_{k+1}/I_{k+1}, \underline{z}_{k+1}, \underline{u}_k) =$$

$$= \frac{p(\underline{z}_{k+1}/\underline{s}_{k+1}, I_k, \underline{u}_k)\ p(\underline{s}_{k+1}/I_k, \underline{u}_k)}{p(\underline{z}_{k+1}/I_k, \underline{u}_k)} =$$

$$= \frac{p(\underline{z}_{k+1}/\underline{s}_{k+1}, \underline{u}_k)\ \ p(\underline{s}_{k+1}/I_k, \underline{u}_k)}{p(\underline{z}_{k+1}/I_k, \underline{u}_k)} \qquad (4.4.15)$$

However, $p(\underline{z}_{k+1}/I_k, \underline{u}_k) = \int_{\underline{s}_{k+1}} p(\underline{z}_{k+1}, \underline{s}_{k+1}/I_k, \underline{u}_k) d\underline{s}_{k+1} =$

$$= \int_{\underline{s}_{k+1}} p(\underline{z}_{k+1}/\underline{s}_{k+1}, \underline{u}_k)\ \ p(\underline{s}_{k+1}/I_k, \underline{u}_k)\ d\underline{s}_{k+1} \qquad (4.4.16)$$

Substituting (4.4.16) in (4.4.15) we have an expression of the density $p(\underline{s}_{k+1}/I_{k+1})$ as a function of the propagation output $p(\underline{s}_{k+1}/I_k, \underline{u}_k)$ and the observation $\underline{z}_{k+1}$.

Thus, the dynamics of $p(\underline{s}_k, I_k)$ can be symbolically represented by

$$p(\underline{s}_{k+1}/I_{k+1}) = \text{Update}[\text{Propagate}[p(\underline{s}_k/I_k), \underline{u}_k], \underline{z}_{k+1}], \qquad (4.4.17)$$

$$k = 0, 1, \ldots, T-1.$$

$$p(\underline{s}_0/I_0) = p(\underline{s}_0),$$

where the update $[\cdot,\cdot]$ and Propagate $[\cdot,\cdot]$ operators were previously defined. Notice that Equation (4.4.17) represents the dynamics of an exactly known state with the role of the system's disturbance assigned to $\underline{z}_k$.

Furthermore, the cost functional (4.4.8) can be reexpressed (Tenney, 1982) in the following form:

$$J = \underset{\substack{\underline{s}_0 \\ \{\underline{z}_k\}_{k=1}^{T-1}}}{E} \left\{ \sum_{k=0}^{T-1} \tilde{g}_k(p(\underline{s}_k/I_k), \underline{u}_k) + \tilde{g}_T(p(\underline{s}_T/I_T)) \right\}, \qquad (4.4.18)$$

where $\tilde{g}_k(p(\underline{s}_k/I_k), \underline{u}_k) = \int_{\underline{s}_k} \int_{\underline{w}_k} g_k(\underline{s}_k, \underline{u}_k, \underline{w}_k)\ p(\underline{w}_k/\underline{s}_k, \underline{u}_k)\ p(\underline{s}_k/I_k) d\underline{w}_k d\underline{s}_k$

Lastly, and in principle only, the optimal feedback controller can be recursively obtained by the following algorithm (see Figure 4.4.3 for an illustration of the system dynamics and the structure of the optimal controller):

Algorithm A4.4.3: Optimal Controller for Uncertain Systems with Imperfect State Information

a. Initialize: $J_T[p(\underline{s}_T/I_T)] = g_T(p(\underline{s}_T/I_T))$

b. Recurse: For k = T-1, T-2, ,,,, 0, select

$$\underline{\mu}_k^*[p(\underline{s}_k/I_k)] = \arg\left\{\min_{\underline{\mu}_k(\cdot)\varepsilon M}\left\{\tilde{g}_k(p(\underline{s}_k/I_k)),\underline{\mu}_k(p(\underline{s}_k/I_k)) + E_{\underline{z}_{k+1}}\{J_{k+1}[p(\underline{s}_{k+1}/I_{k+1})]\}\right\}\right\}$$

where M is the set of functions satisfying $\underline{u}_k = \underline{\mu}_k(p(\underline{s}_k/I_k)) \varepsilon\ U_k$, and evaluate

$$J_k[p(\underline{s}_k/I_k)] = \tilde{g}_k(p(\underline{s}_k/I_k),\ \underline{\mu}_k^*(p(\underline{s}_k/I_k)) + E_{\underline{z}_{k+1}}\left\{J_{k+1}\ [\text{Update}[\text{Propagate}[p(\underline{s}_k/I_k),\underline{\mu}_k^*(p(\underline{s}_k/I_k)],\ \underline{z}_{k+1}]]\right\}$$

c. Terminate: If $J_0[p(\underline{s}_0/I_0)] = J_0[p(\underline{s}_0)] = J^*$ exists, then it is the global minimum and

$\pi^* = \{\mu_0^*(\cdot),\ \ldots,\ \mu_{T-1}^*(\cdot)\}$ is the minimizing policy.

This solution can be seen to obey the Nonlinear Separation Theorem which states that the structure of the optimal controller consists of (1) a Bayesian estimator followed by (2) a memoryless

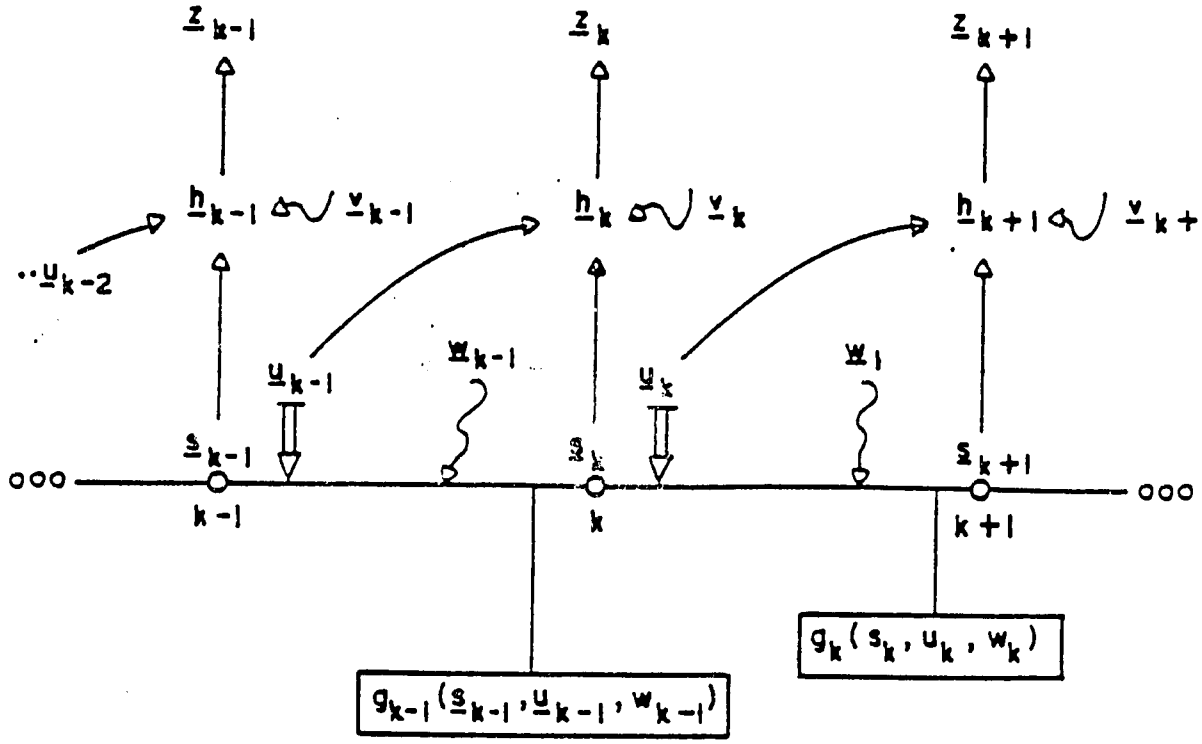

FIGURE 4.4.3a: THE SEQUENTIAL NATURE OF SYSTEMS WITH INEXACTLY KNOWN STATE

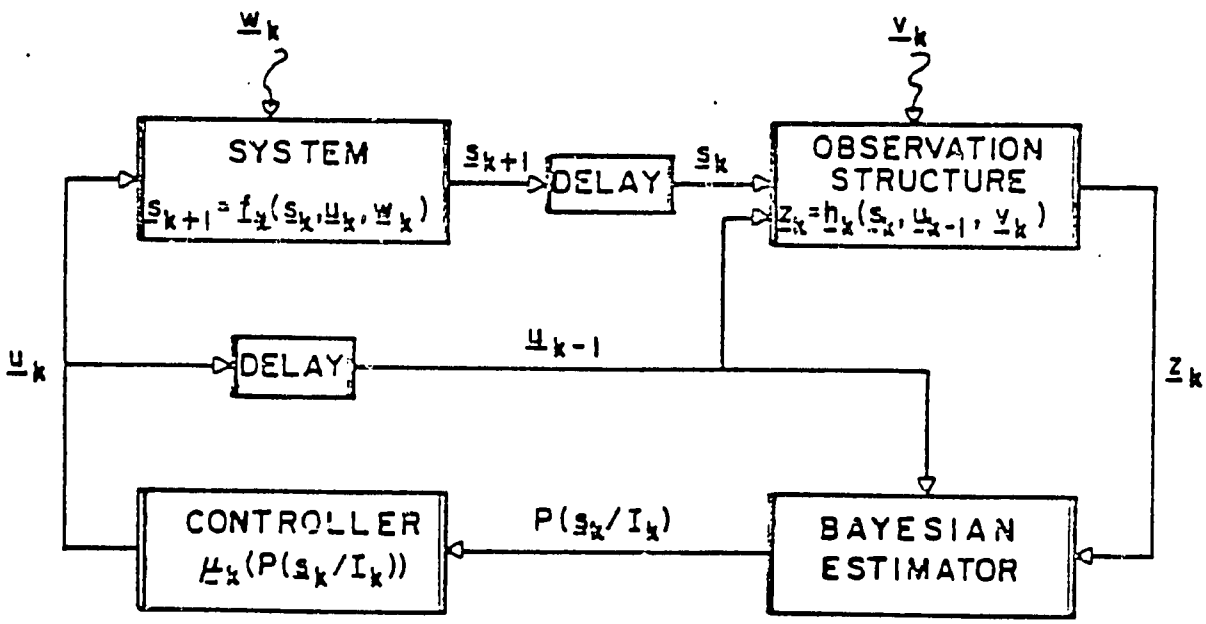

FIGURE 4.4.3b: THE STRUCTURE OF THE OPTIMAL FEEDBACK CONTROLLER

control law using $p(\underline{s}_k/I_k)$.

In the general case both the estimator and the controller will have to be implemented by some numerical discretizing scheme raising severe limitations on the practicability of Algorithm A4.4.3. Notable exceptions where analytical solutions exist is the class of problems involving linear dynamics, gaussian uncertainties, and quadratic performance indices. These problems, largely unknown in the water resources area, will now be briefly examined as they can provide a useful framework in the study of systems. Athans and Falb, 1966, Kushner, 1971, Bertsekas, 1976, Sage and White, 1978, have written texts including extensive LQG theory treatment. Furthermore, in the prolific journal literature the papers by Athans, 1971, Meier et al., 1971, Tse, 1971, provide instructive related presentations.

In the LQG control problems the state difference equation is of the following form:

$$\underline{s}_{k+1} = \underline{\underline{A}}_k \underline{s}_k + \underline{\underline{B}}_k \underline{u}_k + \underline{w}_k \tag{4.4.19}$$

$$k = 0, 1, \ldots, T-1$$

where $\underline{\underline{A}}_k$, $\underline{\underline{B}}_k$ are real $n_s \times n_s$, $n_s \times n_u$ matrices respectively. The state at the initial time and the random disturbances at all times are assumed gaussian with first and second order moments given by

$$\left.\begin{aligned} E\{\underline{s}_0\} &= \bar{\underline{s}}_0, \\ E\{(\underline{s}_0 - \bar{\underline{s}}_0)(\underline{s}_0 - \bar{\underline{s}}_0)^T\} &= \underline{\underline{P}}_{s0}, \end{aligned}\right\} \tag{4.4.20}$$

$$E\{\underline{w}_k\} = \underline{0}$$

$$E\{\underline{w}_k \underline{w}_\ell^T\} = \begin{cases} \underline{\underline{0}}, & \text{if } k \neq \ell, \\ \underline{\underline{Q}}_{wk}, & \text{if } k = \ell \end{cases} \tag{4.4.21}$$

$$E\left\{(\underline{s}_0 - \bar{\underline{s}}_0)\underline{w}_k^T\right\} = \underline{\underline{0}} \text{ for all } k. \tag{4.4.22}$$

Additionally, $\underline{w}_k$ is assumed independent of $\underline{s}_k$ and $\underline{u}_k$:

$$p(\underline{w}_k / \underline{s}_k, \underline{u}_k) = p(\underline{w}_k). \tag{4.4.23}$$

The system is observable through the vector $\underline{z}_k$ which is related to the state via

$$\underline{z}_k = \underline{\underline{H}}_k \underline{s}_k + \underline{v}_k, \tag{4.4.24}$$
$$k = 1, 2, \ldots, T-1.$$

The random noise $\underline{v}_k$ is Gaussian for all k and it is characterized by

$$\left.\begin{aligned} & p(\underline{v}_k / \underline{s}_k, \underline{u}_{k-1}) = p(\underline{v}_k), \\ & E\{\underline{v}_k\} = \underline{0}, \\ & E\{\underline{v}_k \underline{v}_\ell^T\} = \begin{cases} \underline{\underline{0}}, & \text{if } k \neq \ell, \\ \underline{\underline{R}}_{vk}, & \text{if } k = \ell, \end{cases} \\ & E\{\underline{w}_k \underline{v}_\ell^T\} = \underline{\underline{0}} \text{ for all } k, \ell, \\ & E\{(\underline{s}_0 - \bar{\underline{s}}_0)\underline{v}_k^T\} = \underline{\underline{0}} \text{ for all } k. \end{aligned}\right\} \tag{4.4.25}$$

The state, control, and disturbance sets are the full real spaces of the respective dimension; namely the problem is unconstrained.

The objective is to construct the optimal feedback policy which minimizes

$$J = \underset{\substack{\underline{s}_0 \\ \{\underline{w}_k\}_{k=0}^{T-1} \\ \{\underline{v}_k\}_{k=1}^{T-1}}}{E} \left\{ \sum_{k=0}^{T-1} (\underline{s}_k^T \underline{\underline{N}}_{sk} \underline{s}_k + \underline{u}_k^T \underline{\underline{N}}_{uk} \underline{u}_k) + \underline{s}_T^T \underline{\underline{N}}_{sT} \underline{s}_T \right\}, \qquad (4.4.26)$$

where $\underline{\underline{N}}_{sk}$, k=0, 1, ..., T are positive, semidefinite real $n_s \times n_s$ matrices and $\underline{\underline{N}}_{uk}$, k=0, 1, ..., T-L are positive, definite[1] real $n_u \times n_u$ matrices.

To facilitate the presentation of the solution we make the following definitions:

$$\left.\begin{aligned}
\bar{\underline{s}}_{k/k-1} &= E\{\underline{s}_k/I_{k-1},\underline{u}_{k-1}\}, \\
\bar{\underline{s}}_{k/k} &= E\{\underline{s}_k/I_k\}, \\
\underline{\underline{p}}_{sk/k-1} &= E\{(\underline{s}_k - \bar{\underline{s}}_{k/k-1})(\underline{s}_k - \bar{\underline{s}}_{k/k-1})^T/I_{k-1},\underline{u}_{k-1}\}, \\
\underline{\underline{p}}_{sk/k} &= E\{(\underline{s}_k - \bar{\underline{s}}_{k/k})(\underline{s}_k - \bar{\underline{s}}_{k/k})^T/I_k\},
\end{aligned}\right\} \qquad (4.4.27)$$

where I_k is the information set defined in (4.4.9) and the expectations are taken with respect to the appropriate conditional densities.

If employed for the above system, the Bayesian Estimation scheme (Eq.(4.4.14), (4.4.15)) yields probability densities $p(\underline{s}_k/I_{k-1},\underline{u}_{k-1})$, $p(\underline{s}_k/I_k)$ which are Gaussian at all times k with mean vectors $\bar{\underline{s}}_{k/k-1}$, $\bar{\underline{s}}_{k/k}$ and covariance matrices $\underline{\underline{p}}_{sk/k-1}$, $\underline{\underline{p}}_{sk/k}$. These conditional statistics are provided by the following equations which constitute

[1] A symmetric matrix $\underline{\underline{A}}$ is positive (semi)definite if $\underline{x}^T\underline{\underline{A}}\underline{x}$ is greater (or equal to) zero for all $\underline{x} \neq \underline{0}$. These assumptions guarantee uniqueness of the optimal solution.

the well-known Kalman Filter:

Initialization:

$$\left.\begin{aligned} \bar{\underline{s}}_{0/0} &= \bar{\underline{s}}_0 \\ \underline{\underline{P}}_{s\,0/0} &= \underline{\underline{P}}_{s0} \end{aligned}\right\} \tag{4.4.28}$$

Propagation step: For $k = 0, 1, \ldots$

$$\left.\begin{aligned} \bar{\underline{s}}_{k+1/k} &= \underline{\underline{A}}_k \bar{\underline{s}}_{k/k} + \underline{\underline{B}}_k \underline{u}_k, \\ \underline{\underline{P}}_{sk+1/k} &= \underline{\underline{A}}_k \underline{\underline{P}}_{sk/k} \underline{\underline{A}}_k^T + \underline{\underline{Q}}_{wk}. \end{aligned}\right\} \tag{4.4.29}$$

Update step: For $k=0,1, \ldots$

$$\left.\begin{aligned} \bar{\underline{s}}_{k+1/k+1} &= \bar{\underline{s}}_{k+1/k} + \underline{\underline{\Pi}}_{k+1} [\underline{z}_{k+1} - \underline{\underline{H}}_{k+1} \bar{\underline{s}}_{k+1/k}], \\ \underline{\underline{\Pi}}_{k+1} &= \underline{\underline{P}}_{sk+1/k} \underline{\underline{H}}_{k+1}^T [\underline{\underline{H}}_{k+1} \underline{\underline{P}}_{sk+1/k} \underline{\underline{H}}_{k+1}^T + \underline{\underline{R}}_{vk+1}]^{-1}, \\ \underline{\underline{P}}_{sk+1/k+1} &= [\underline{\underline{I}} - \underline{\underline{\Pi}}_{k+1} \underline{\underline{H}}_{k+1}] \underline{\underline{P}}_{sk+1/k}. \end{aligned}\right\} \tag{4.4.30}$$

It is notable that the observations and the controls do not enter in the covariance computations which, therefore, can be performed before the process takes its actual course (off-line). This is an important property providing precomputable estimates of the procedure's accuracy.

Combining (4.4.30) with (4.4.29) we can reformulate the problem into one with the exactly known state $\bar{\underline{s}}_{k/k}$:

$$\bar{\underline{s}}_{k+1/k+1} = \underline{\underline{A}}_k \bar{\underline{s}}_{k/k} + \underline{\underline{B}}_k \underline{u}_k + \tilde{\underline{w}}_k, \tag{4.4.31}$$

$k = 0, 1, \ldots, T-1,$

where $\bar{\underline{s}}_o$ is given and

$$\tilde{\underline{w}}_k = \underline{\underline{\Pi}}_{k+1}[\underline{z}_{k+1} - \underline{\underline{H}}_{k+1}\bar{\underline{s}}_{k+1/k}]. \tag{4.4.32}$$

The quantities $\underline{r}_{k+1} = \underline{z}_{k+1} - \underline{\underline{H}}_{k+1}\bar{\underline{s}}_{k+1/k}$, k=0, 1, ... are known as filter residuals (or as innovations sequence) and they are Gaussian with zero mean and covariance

$$E\{\underline{r}_{k+1}\underline{r}_{\ell+1}\} = \left\{ \begin{array}{l} \underline{\underline{0}}, \text{ if } k \neq \ell \\ \\ \underline{\underline{H}}_{k+1}\underline{\underline{P}}_{sk+1/k}\,\underline{\underline{H}}^T_{k+1} + \underline{\underline{R}}_{vk+1}, \text{ if } k = \ell \end{array} \right\} \tag{4.4.33}$$

It follows from (4.4.32) that $\tilde{\underline{w}}_k$, k=0, 1, ..., T-1, are also Gaussian with

$$E\{\underline{w}_k\} = \underline{0},$$

$$E\{\tilde{\underline{w}}_k\tilde{\underline{w}}^T_\ell\} = \left\{ \begin{array}{l} \underline{\underline{0}}\text{ , if } k \neq \ell \\ \\ \underline{\underline{\Pi}}_{k+1}[\underline{\underline{H}}_{k+1}\,\underline{\underline{P}}_{sk+1/k}\,\underline{\underline{H}}^T_{k+1} + \underline{\underline{R}}_{vk+1}]\underline{\underline{\Pi}}^T_{k+1}, \text{ if } k=\ell \end{array} \right\} = \tag{4.4.34}$$

$$= \underline{\underline{Q}}_{\tilde{w}k}$$

The reformulation of the cost function can proceed according to the general procedure given by Equation (4.4.18). Using

$$E\{\underline{s}^T_k\,\underline{\underline{N}}_{sk}\,\underline{s}_k/I_k\} = \bar{\underline{s}}^T_k\,\underline{\underline{N}}_{sk}\bar{\underline{s}}_k + tr\{\underline{\underline{N}}_{sk}\,\underline{\underline{P}}_{sk/k}\} \tag{4.4.35}$$

we find that the objective is to derive the feedback laws which minimize

$$J = \underset{\substack{\tilde{\underline{w}}_k \\ k=0,1,\ldots,T-1}}{E} \left\{ \sum_{k=0}^{T-1} (\bar{\underline{s}}_{k/k}^T \underline{\underline{N}}_{sk} \bar{\underline{s}}_{k/k} + \underline{u}_k^T \underline{\underline{N}}_{uk} \underline{u}_k) + \bar{\underline{s}}_{T/T}^T \underline{\underline{N}}_{st} \bar{\underline{s}}_{T/T} \right.$$

$$\left. + \sum_{k=0}^{T} \left[\mathrm{tr}\{\underline{\underline{N}}_{sk} \underline{\underline{P}}_{sk/k}\} \right] \right\} \qquad (4.4.36)$$

As mentioned earlier, the covariance computations are independent of observations and controls and so the last term in (4.4.36) is a precomputable constant.

The solution of the reformulated problem can be obtained by directly applying Algorithm A4.4.2; it takes the following analytical form:

<u>Algorthm A4.4.4</u>: Optimal Controller for LQG Systems with Imperfrect State Information.

The optimal feedback control laws are given by

$$\underline{\mu}_k^*(p(\underline{s}_k/I_k)) = -\underline{\underline{L}}_k \bar{\underline{s}}_{k/k}, \qquad (4.4.37)$$

$k=0,1,\ldots,T-1,$

where control gain $n_u \times n_s$ matrices $\underline{\underline{L}}_k$ are obtained from

$$\underline{\underline{L}}_k = [\underline{\underline{N}}_{uk} + \underline{\underline{B}}_k^T \underline{\underline{K}}_{k+1} \underline{\underline{B}}_k]^{-1} \underline{\underline{B}}_k^T \underline{\underline{K}}_{k+1} \underline{\underline{A}}_k, \qquad (4.4.38)$$

$k=0,1,\ldots,T-1$

and the $n_s \times n_s$ matrices $\underline{\underline{K}}_k$ are the positive semidefinite solution of the following matrix Riccati equation:

$$\underline{\underline{K}}_T = \underline{\underline{N}}_{sT},$$

$$\underline{\underline{K}}_k = \underline{\underline{A}}_k^T[\underline{\underline{K}}_{k+1} - \underline{\underline{K}}_{k+1}\,\underline{\underline{B}}_k[\underline{\underline{B}}_k^T\,\underline{\underline{K}}_{k+1}\,\underline{\underline{B}}_k + \underline{\underline{N}}_{uk}]^{-1}. \tag{4.4.39}$$

$$\underline{\underline{B}}_k^T\,\underline{\underline{K}}_{k+1}]\,\underline{\underline{A}}_k + \underline{\underline{N}}_{sk}$$

$$k = T-1, T-2, \ldots, 0.$$

The optimal "costs to go"[1] are recursively given by:

$$J_k[p(\underline{s}_k/I_k)] = \bar{\underline{s}}^T_{k/k}\,\underline{\underline{K}}_k\,\bar{\underline{s}}_{k/k} \pm \tilde{c}_k, \tag{4.4.40}$$

$$k=T, T-1, \ldots, 0$$

where scalars $\tilde{c}_k$ are such that

$$\left.\begin{aligned} \tilde{c}_T &= \mathrm{tr}\{\underline{\underline{N}}_{sT}\,\underline{\underline{P}}_{st/t}\}, \\ \tilde{c}_k &= \tilde{c}_{k+1} + \mathrm{tr}\{\underline{\underline{K}}_{k+1}\,\underline{\underline{Q}}_{\tilde{w}k} + \underline{\underline{N}}_{sk}\,\underline{\underline{P}}_{sk/k}\}, \\ k &= T-1, T-2, \ldots, 0. \end{aligned}\right\} \tag{4.4.41}$$

Lastly, the minimum total expected cost is

$$J^* = J_0[p(\underline{s}_k/I_k)] = \bar{\underline{s}}_0^T\,\underline{\underline{K}}_0\,\bar{\underline{s}}_0 + \tilde{c}_0. \tag{4.4.42}$$

1

$$J_k[p(\underline{s}_k/I_k)] = \underset{\substack{\underline{\tilde{w}}_j \\ j=k,\ldots,T-1}}{E}\left\{\sum_{j=k}^{T-1}(\bar{\underline{s}}^T_{j/j}\,\underline{\underline{N}}_{sj}\,\bar{\underline{s}}_{j/j} + \underline{u}_j^T\underline{\underline{N}}_{uj}\underline{u}_j) + \bar{\underline{s}}^T_{T/T}\,\underline{\underline{N}}_{sT}\,\bar{\underline{s}}_{T/T} + \sum_{j=k}^{T}\{\mathrm{tr}\,\underline{\underline{N}}_{sj}\,\underline{\underline{P}}_{sj/j}\}\right\}$$

The computational requirements of this solution are mainly associated with the calculation of the matrices $\underline{\underline{K}}_k$ through Equation (4.4.39). It can be seen that only the matrices at two consecutive times are needed for the recursions. This and the fact that $\underline{\underline{K}}_k$, k=0, 1, ..., T, are symmetric guarantee high computational efficiency.

The solution is governed by the Separation Theorem where the estimates $\bar{\underline{s}}_{k/k}$ are obtained by a Bayesian Estimator (Kalman Filter), and the control laws are linear functions of $\bar{\underline{s}}_{k/k}$, k=0, 1, ..., T-1. The coefficient matrices $\underline{\underline{L}}_k$ are independent of all random quatitites' statistics.

If one assumes perfect state information (i.e., if it is possible to completely specify $\underline{s}_k$ from Eq. (4.4.24)), then the feedback laws minimizing

$$J = \underset{\substack{\underline{w}_k \\ k=0,1,\ldots,T-1}}{E} \left\{ \sum_{k=0}^{T-1} (\underline{s}_k^T \underline{\underline{N}}_{sk} \underline{s}_k + \underline{u}_k^T \underline{\underline{N}}_{uk} \underline{u}_k) + \underline{s}_T^T \underline{\underline{N}}_{st} \underline{s}_T \right\} \quad (4.4.43)$$

are still given by Eq. (4.4.37), (4.4.38), (4.4.39). ($\bar{\underline{s}}_{k/k}$ in Eq. (4.4.37) should now be replaced by $\underline{s}_k$.) The optimal costs to go in this case take the form

$$J_k(\underline{s}_k) = \underline{s}_k^T \underline{\underline{K}}_k \underline{s}_k + c_k \quad (4.4.44)$$

$$k=T, T-1, \ldots, 0,$$

where the scalars c_k are obtained recursively from

$$c_T = 0$$
$$c_k = c_{k+1} + \mathrm{tr}\,\{\underline{\underline{K}}_{k+1}\,\underline{\underline{Q}}_{wk}\}, \tag{4.4.45}$$
$$k = T-1,\ T-2,\ \ldots,\ 0$$

The minimum total expected cost is now

$$J^* = J_0(\underline{s}_0) = \underline{s}_0^T\,\underline{\underline{K}}_0\,\underline{s}_0 + c_0. \tag{4.4.46}$$

Additionally, if the disturbances $\underline{w}_k$ in Eq. (4.4.19) are replaced by their expected values ($\underline{0}$ in our formulation) and the feedback laws are sought which minimize the deterministic function

$$J = \sum_{k=0}^{T-1}(\underline{s}_k^T\,\underline{\underline{N}}_{sk}\,\underline{s}_k + \underline{u}_k^T\,\underline{\underline{N}}_{uk}\,\underline{u}_k) + \underline{s}_T^T\,\underline{\underline{N}}_{sT}\,\underline{s}_T, \tag{4.4.47}$$

then the solution is again provided by Eq. (4.4.37), (4.4.38), (4.4.39) with $\underline{s}_k$ replacing $\bar{\underline{s}}_{k/k}$. The optimal costs to go are now given by

$$J_k(\underline{s}_k) = \underline{s}_k^T\,\underline{\underline{K}}_k\,\underline{s}_k, \tag{4.4.47}$$
$$k=T,\ T-1,\ \ldots,\ 0,$$

and the total minimum cost is

$$J^* = J_0(\underline{s}_0) = \underline{s}_0^T\,\underline{\underline{K}}_0\,\underline{s}_0. \tag{4.4.48}$$

In summary, a control problem with imperfect state information takes on the same solution as the corresponding problem with perfect state information and the determin stic problem resulting when all random quantities are replaced by their expected values. When these are true, we say that the problem possesses the Certainty Equivalence

(C.E.) property. Thus, the C.E. property, if valid, reduces the stochastic control problem into an easier deterministic one. However, one must note that the more the characteristics of a certain problem diverge from LQG, the less likely it is that the C.E. property will hold.

The advantageous form of the solution is lost if any of the problem specifications change. For example, if the system is linear, the disturbances are Gaussian, but the costs are nonquadratic, and the Kalman Filter is still the optimal estimator; however, the control laws will in general be nonlinear functions of both $\bar{\underline{s}}_{k/k}$ and $\underline{\underline{P}}_{sk/k}$. For nonlinear systems, the conditional p.d.f. of the state $p(\underline{s}_k/I_k)$ is no longer Gaussian (even if all disturbances are so), and the optimal estimator (4.4.17) cannot be simplified. In these problems the control laws are nonlinear functions of the so-called infinite dimensional $p(\underline{s}_k/I_k)$. Hence, a significant advantage of linear systems is that the first two conditional moments are sufficient statistics for the control purposes, whereas an infinity of moments are needed when nonlinear systems are controlled.

Figure 4.4.4 schematically shows the structure of the LQG controller.

Unfortunately, many interesting actual problems do not possess LQG characteristics. For instance, the reservoir operation problem of the previous section is both nonquadratic and constrained (nonlinear). For such problems the optimal feedback laws are practically inaccessible and consequently certain suboptimal techniques are of

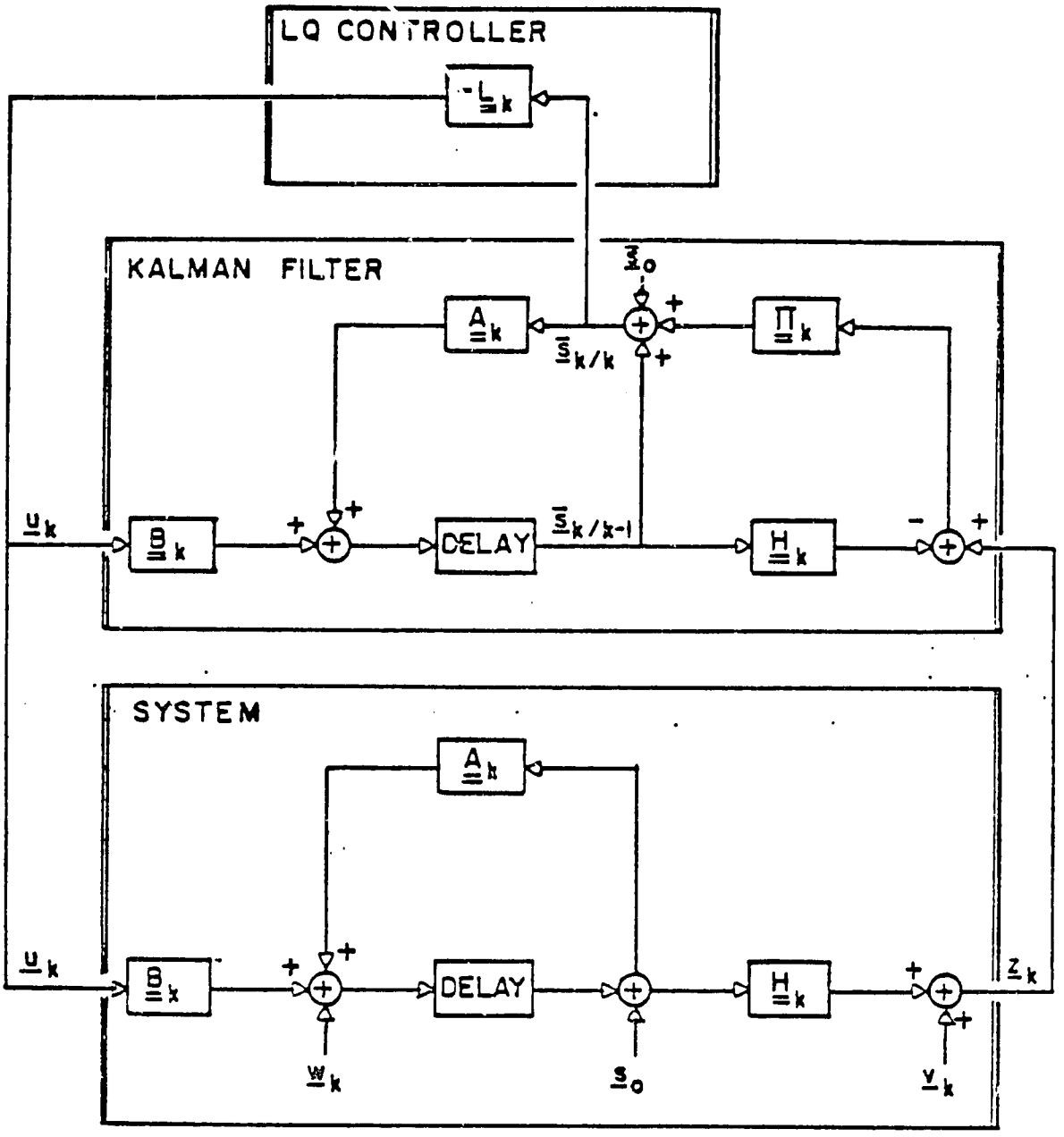

FIGURE 4.4.4: LQG IMPLEMENTATION

great practical interest. The Open Loop Feedback Control procedure is one such technique which can be employed to approximate the optimal controller.

Algorithm A4.4.5: Open Loop Feedback Controller

At any time k of the control horizon, the OLF Controller performs the following operations:

a. Estimates the conditional density $p(\underline{s}_k/I_k)$ using the information set $I_k=\{\underline{z}_1, \ldots, \underline{z}_k, \underline{u}_0, \ldots, \underline{u}_{T-1}\}$.

b. Assumes that no measurements will be made in the future.

c. Finds the open-loop trajectory $\{\underline{u}_k, \underline{u}_{k+1}, \ldots, \underline{u}_{T-1}\}^{OLFC}$ which minimizes

$$J_k = \underset{\substack{\underline{s}_k \\ \underline{w}_\ell \\ \ell = k,\ldots,T-1}}{E} \left\{ \sum_{\ell=k}^{T-1} g_j(\underline{s}_\ell, \underline{u}_\ell, \underline{w}_\ell) + g_T(\underline{s}_T)/I_k \right\}$$

d. Applies $\underline{\mu}_k^{OLFC}(p(\underline{s}_k/I_k)) = \underline{u}_k^{OLFC}$

e. Redefines the information set at time k+1

$$I_{k+1} = I_k \cup \{\underline{u}_k^{OLFC}, \underline{z}_{k+1}\}$$

and repeats the previous steps.

The above is an open-loop procedure because at each decision time the entire future control trajectory is determined assuming no

information gathering. It is also a feedback procedure because the applied controls $\underline{u}_k^{OLFC}$ are functions of all currently available information. The advantage of the OLFC idea is that the computations required to obtain $\underline{u}_k^{OLFC}$ are considerably simpler than those of the true optimal controls. This holds because the assumption of no future measurements at Step b allows one to use the deterministic Equation 4.4.14) rather than the stochastic (4.4.17) to represent the dynamics of the state's p.d.f. (Remember that the future measurements are random variables.) As a result, if the reformulation to perfect state information is employed, the reduced problem is essentially deterministic (in the p.d.f.'s space) and, hence, easier to solve. The OLFC procedure is suboptimal because in the specification of the feedback OLFC controls, it is not taken into account that measurement information will be gathered in the future and used.

Other suboptimal control techniques are the Open Loop Controller and the Naive Feedback or Certainty Equivalence Controller. The first is based on the assumption of no information gathering and has been discussed previously. The second arbitrarily assumes that the problem possesses the Certainty Equivalence property, solves the associated deterministic problem, and applies the resulting feedback laws on the stochastic system. In the general case, both have been found to be less reliable as compared to the OLFC procedure (Bertsekas, 1976).

After this section's introduction to optimal and suboptimal stochastic control theory, we next begin to design a suitable controller for the reservoir operation problem. The controller will be

of the Open Loop Feedback type and will be completed in three main steps. In the first, a procedure capable of efficiently solving the unconstrainted nonquadratic control problem will be developed; in the second and third, the basic procedure will be modified to account respectively for possible control or state constraint violations. Care is taken to maintain coherency in the presentation and for this reason lengthy mathematical derivations and background material are relegated to the appendices.

4.5 Solving the Unconstrained Reservoir Operation Problem

Neglecting all control and state constraints, the unconstrained reservoir operation problem can be stated as follows (cf. Problem P.2 in Section 4.3):

Problem P.3: The Unconstrained Reservoir Operation Problem

$$\underset{\{\delta \underline{u}_k\}_{k=0}^{T-1}}{\text{Minimize}} \left\{ J = \underset{\substack{\delta \underline{s}_0 \\ \{\underline{\xi}_k\}_{k=0}^{T-1} \\ \{\underline{v}_k\}_{k=1}^{T-1}}}{E} \left[\sum_{k=0}^{T-1} \ell_k(\delta \underline{s}_k, \delta \underline{u}_k) + \ell_T(\delta \underline{s}_T) \right] \right\} \tag{4.5.1}$$

subject to

$$\delta \underline{s}_{k+1} = \underline{\underline{\phi}}_k \, \delta \underline{s}_k + \underline{\underline{B}}_k \, \delta \underline{u}_k + \underline{\xi}_k, \tag{4.5.2}$$

$$k=0, 1, \ldots, T-1$$

$$\delta \underline{z}_k = \underline{\underline{H}}_1 (\underline{s}^{nom}(t_k)) \, \delta \underline{s}_k + \underline{v}_k, \tag{4.5.3}$$

$$k=1, 2, \ldots, T-1.$$

The term $\ell_T(\delta \underline{s}_T)$ which is added to the cost functional will be used later on to model the continuing operations requirement of the system. The various quantitites in Eq. (4.5.1), (4.5.2), (4.5.3) have been defined in Section 4.3.

This section will be concerned with developing a computationally efficient procedure to solve Problem P.3. According to the terminology established in the control theory review, Problem P.3 is one of

imperfect state information. It involves linear state dynamic and observation equations, Gaussian statistics, and a nonquadratic performance index. Due to the nature of the performance index, optimal feedback solutions cannot be obtained in analytical form. As computational efficiency for large systems is a major concern, we shall proceed by adopting the Open Loop Feedback control philosophy.

4.5.1 Open Loop Feedback Control

The OLF controller solves Problem P.3 in the manner presented by Algorithm A4.4.5; in place of the imperfect state information problem it solves a sequence of open loop control problems of the following type:

$$\underset{\{\delta \underline{u}_\ell\}_{\ell=k}^{T-1}}{\text{Minimize}} \quad \underset{\substack{\delta \underline{s}_k \\ \{\xi_\ell\}_{\ell=k}^{T-1}}}{E} \left\{ \sum_{\ell=k}^{T-1} \ell_\ell(\delta \underline{s}_\ell, \delta \underline{u}_\ell) + \ell_T(\delta \underline{s}_T)/I_k \right\} \tag{4.5.4}$$

subject to

$$\delta \underline{s}_{\ell+1} = \underline{\underline{\phi}}_\ell \, \delta \underline{s}_\ell + \underline{\underline{B}}_\ell \, \delta \underline{u}_\ell + \underline{\xi}_\ell , \tag{4.5.5}$$

$$\ell = k, \ldots, T-1.$$

The probabiality density of the state at time k has been estimated by using all available information summarized the $I_k = \underline{z}_1, \ldots, \underline{z}_k, \underline{u}_0, \ldots, \underline{u}_{k-1}$. It is Gaussian with

$$E\{\delta \underline{s}_k / I_k\} = \underline{0}$$

$$E\{\delta \underline{s}_k \, \delta \underline{s}_k^T / I_k\} = \underline{\underline{P}}_{sk/k}. \tag{4.5.6}$$

The random disturbances $\underline{\xi}_\ell$, $\ell = k, \ldots, T-1$ are also Gaussian independent of $\delta\underline{s}_k$ and have zero mean and covariance

$$E\{\underline{\xi}_\ell \, \underline{\xi}_m^T\} = \begin{cases} \underline{\underline{0}}, \text{ if } \ell \neq m, \\ \underline{\underline{Q}}_{\xi\ell}, \text{ if } \ell = m \end{cases} \tag{4.5.7}$$

$\underline{\underline{Q}}_{\xi\ell}$ has been defined by Equation (4.3.28). The terminal time T will be properly determined later. For now it will be assumed as given at each decision time k.

Equivalently, we can transform the above problem by following the reformulation procedure presented in Section 4.4. The advantage is that under the assumption of no future measurements, the transformed problem is deterministic in the space of the state's p.d.f.

The dynamics of the transformed problem obey Eq. (4.4.14). Because the system is linear and the uncertainties Gaussian, it can be shown that $p(\delta\underline{s}_\ell/I_k)$ will also be Gaussian with mean vector and covariance matrix given by

$$\delta\bar{\underline{s}}_{\ell+1/k} = \underline{\underline{\phi}}_\ell \; \delta\bar{\underline{s}}_{\ell/k} + \underline{\underline{B}}_\ell \; \delta\underline{u}_\ell, \delta\bar{\underline{s}}_{k/k} = \underline{0}, \tag{4.5.8}$$

$$\ell = k, \ldots, T-1,$$

$$\underline{\underline{P}}_{s\ell+1/k} = \underline{\underline{\phi}}_\ell \; \underline{\underline{P}}_{s\ell/k} \; \underline{\underline{\phi}}_\ell^T + \underline{\underline{Q}}_{\xi\ell} \; , \; \underline{\underline{P}}_{sk/k} \text{ given}, \tag{4.5.9}$$

$$\ell = k, \ldots, T-1.$$

(Compare also with Equation (4.4.29).)

It is easy to see how these equations can be derived. Take, for

instance $\ell=k$. Then by the dynamics

$$\delta \underline{s}_{k+1} = \underline{\underline{\phi}}_k \, \delta \underline{s}_k + \underline{\underline{B}}_k \, \delta \underline{u}_k + \underline{\xi}_k \tag{4.5.10}$$

and the Gaussianness of $\delta \underline{s}_k$, $\underline{\xi}_k$, we conclude (Jazwinski, 1970, Section 2.6) that $\delta \bar{\underline{s}}_{k+1}$ is also Gaussian. Taking expectations in (4.5.10) we find:

$$\delta \bar{\underline{s}}_{k+1/k} = E\{\delta \underline{s}_{k+1}/I_k\} = E\,\{\underline{\underline{\phi}}_k \, \delta \underline{s}_k + \underline{\underline{B}}_k \, \delta \underline{u}_k + \underline{\xi}_k/I_k\} =$$

$$= \underline{\underline{\phi}}_k E\{\delta \underline{s}_k/I_k\} + \underline{\underline{B}}_k \, \delta \underline{u}_k + E\,\{\underline{\xi}_k/I_k\} =$$

$$= \underline{\underline{\phi}}_k \, \delta \bar{\underline{s}}_{k/k} + \underline{\underline{B}}_k \delta \underline{u}_k, \tag{4.5.11}$$

where apart from the linearity of expectations we used the fact that $\delta \underline{u}_k$ is deterministic and that $\underline{\xi}_k$ is zero mean and independent of I_k. Similarly by definition

$$\underline{\underline{P}}_{sk+1/k} = E\{(\delta \underline{s}_{k+1} - \delta \bar{\underline{s}}_{k+1/k})(\delta \underline{s}_{k+1} - \delta \bar{\underline{s}}_{k+1/k})^T/I_k\} \; .$$

Using (4.5.10) and (4.5.11) and performing the indicated multiplications we obtain:

$$\underline{\underline{P}}_{sk+1/k} = E\Big\{ \underline{\underline{\phi}}_k(\delta \underline{s}_k - \delta \bar{\underline{s}}_{k/k})(\delta \underline{s}_k - \delta \bar{\underline{s}}_{k/k})^T \underline{\underline{\phi}}_k^T \; +$$

$$\underline{\xi}_k(\delta \underline{s}_k - \delta \bar{\underline{s}}_{k/k})^T \underline{\underline{\phi}}_k^T + \underline{\underline{\phi}}_k(\delta \underline{s}_k - \delta \bar{\underline{s}}_{k/k}) \; \underline{\xi}_k^T +$$

$$+ \underline{\xi}_k \underline{\xi}_k^T/I_k$$

Since $\underline{\xi}_k$ and $\delta\underline{s}_k$ are independent random vectors, the second and third term above vanish and we have

$$\underline{\underline{P}}_{s\ k+1/k} = \underline{\underline{\phi}}_k\ E\ \{(\delta\underline{s}_k - \delta\bar{\underline{s}}_{k/k})(\delta\underline{s}_k - \delta\bar{\underline{s}}_{k/k})^T/I_k\}\ \underline{\underline{\phi}}_k^T\ +$$

$$+\ E\ \{\underline{\xi}_k\underline{\xi}_k^T/I_k\} =$$

$$= \underline{\underline{\phi}}_k\ \underline{\underline{P}}_{sk/k}\ \underline{\underline{\phi}}_k^T + \underline{\underline{Q}}_{\xi k}. \qquad (4.5.12)$$

The operations and assumptions employed in the derivation of Equations (4.5.11) and (4.5.12) are valid for any time ℓ and thus Equations (4.5.8) and (4.5.9) can be derived by induction.

Consider now the reformulation of the cost functional:

$$J_{OLFC} = \underset{\substack{\delta\underline{s}_k \\ \{\xi_\ell\}_{\ell=k}^{T-1}}}{E} \Big\{\ell_k(\delta\underline{s}_k,\delta\underline{u}_k)+\ell_{k+1}(\delta\underline{s}_{k+1},\delta\underline{u}_{k+1}) \ \ldots + \ell_{T-1}(\delta\underline{s}_{T-1},\delta\underline{u}_{T-1})+$$

$$+\ \ell_T(\delta\underline{s}_T)\Big\},$$

where it is understood that the p.d.f. of $\delta\underline{s}_k$ is conditioned on I_k. Since $\underline{\xi}_\ell$ enters in the cost functional through the state $\delta\underline{s}_{\ell+1}$ and does not affect the cost terms at previous times, we can write:

$$J_{OLFC} = \underset{\delta\underline{s}\,k}{E}\ \{\ell_k(\delta\underline{s}_k,\delta\underline{u}_k) + \underset{\underline{\xi}_k}{E}\ \{\ell_{k+1}(\delta\underline{s}_{k+1},\delta\underline{u}_{k+1}) + \ldots +$$

$$+\ \underset{\underline{\xi}_{T-2}}{E}\ \{\ell_{T-1}(\delta\underline{s}_{T-1},\delta\underline{u}_{T-1}) + \underset{\underline{\xi}_{T-1}}{E}\ \{\ell_T(\delta\underline{s}_T)\}\}\ \ \ldots\ \}\}\ . \qquad (4.5.13)$$

or equivalently

$$J_{OLFC} = \underset{\delta \underline{s}_k}{E} \{\ell_k(\delta \underline{s}_k, \delta \underline{u}_k)\} + \underset{\delta \underline{s}_{k+1}}{E} \{\ell_{k+1}(\delta \underline{s}_{k+1}, \delta \underline{u}_{k+1})\} + \dots +$$

$$+ \underset{\delta \underline{s}_{T-1}}{E} \{\ell_{T-1}(\delta \underline{s}_{T-1}, \delta \underline{u}_{T-1})\} + \underset{\delta \underline{s}_T}{E} \{\ell_T(\delta \underline{s}_T)\} \quad . \tag{4.5.14}$$

The expectations in (4.5.13) are taken with respect to the Gaussian densities whose statistics are obtained through Equations (4.5.8) and (4.5.9). The equivalence of (4.5.13) and (4.5.14) can be seen by the fact that the randomness of each cost term $\ell_\ell(\delta \underline{s}_\ell, \delta \underline{u}_\ell)$ is solely due to the randomness of $\delta \underline{s}_\ell$. This together with the Markov property of state establish the validity of the following auxiliary lemma:

$$\underset{\substack{\delta \underline{s}_k \\ \{\underline{\xi}_j\}_{j=k}^{\ell-1}}}{E} \{\ell_\ell(\delta \underline{s}_\ell, \delta \underline{u}_\ell)\} = \underset{\substack{\delta \underline{s}_{\ell-1} \\ \underline{\xi}_{\ell-1}}}{E} \{\ell_\ell(\delta \underline{s}_\ell, \delta \underline{u}_\ell)\} = \underset{\delta \underline{s}_\ell}{E} \{\ell_\ell(\delta \underline{s}_\ell, \delta \underline{u}_\ell)\} \tag{4.5.15}$$

Furthermore, if we define

$$\hat{\ell}_\ell(\delta \bar{\underline{s}}_{\ell/k}, \underline{\underline{P}}_{s\ \ell/k}, \delta \underline{u}_\ell) \overset{\Delta}{=} \underset{\delta \underline{s}_\ell}{E} \{\ell_\ell(\delta \underline{s}_\ell, \delta \underline{u}_\ell)\}, \tag{4.5.16}$$

we arrive at the following reformulated problem:

$$\underset{\{\delta \underline{u}_\ell\}_{\ell=k}^{T-1}}{\text{Minimize}} \quad \hat{J} = \sum_{\ell=k}^{T-1} \hat{\ell}_\ell(\delta \bar{\underline{s}}_{\ell/k}, \underline{\underline{P}}_{s\ell/k}, \delta \underline{u}_\ell) + \hat{\ell}_T(\delta \bar{\underline{s}}_{T/k}, \underline{\underline{P}}_{sT/k}) \tag{4.5.17}$$

subject to

intention is to design a control methodology applicable to other systems as well. If analytical specification of $\hat{\ell}_\ell(\cdot,\cdot,\cdot)$ and $\tilde{\ell}_\ell(\cdot,\cdot)$ through (Equations (4.5.16) and (4.5.20))

$$\tilde{\ell}_\ell(\delta\bar{\underline{s}}_{\ell/k},\delta\underline{u}_\ell) \triangleq \hat{\ell}_\ell(\delta\bar{\underline{s}}_{\ell/k}, \underline{\underline{P}}_{s\ell/k}, \delta\underline{u}_\ell) \triangleq \underset{\delta\underline{s}_\ell}{E} \{\ell_\ell(\delta\underline{s}_\ell, \delta\underline{u}_\ell)\}$$

is not readily possible, one can expand $\ell_\ell(\cdot,\cdot)$ in power series and obtain as accurate analytical approximations of $\hat{\ell}_\ell(\cdot,\cdot,\cdot)$ and $\tilde{\ell}_\ell(\cdot,\cdot)$ as desired by using a property of Gaussian random variables known as Gaussian moment factoring. The details are explained in Appendix B. In the case where $\ell_\ell(\cdot,\cdot)$ are quadratic, the reformulated costs are also quadratic as can be seen by the result in Equation (4.4.35).

The question is how to efficiently solve the reformulated deterministic problem. Since this is a general nonlinear programming problem where analytical solutions are not likely to exist, one must employ some minimization procedure. The design of a suitable such procedure is taken up in the following section.

4.5.2 A Newton's Method for Optimal Control Problems

There are two major concerns about a minimization method: reliability and efficiency. A method is reliable if it is guaranteed to converge to optimal (in some sense) points, and it is efficient if it exhibits a fast convergence rate. As can be seen in Luendberger, 1970, Lasdon, 1970, Bertsekas, 1978, 1982, there exists an impressive plethora of minimization algorithms. Perhaps the most important class of algorithms is the class of the Generalized Gradient methods. When minimizing a scalar

real valued function $f(\underline{x})$ with respect to the n dimensional real vector $\underline{x}$, a method of this class at the ith iteration "moves" from a point $\underline{x}_i$ to another $\underline{x}_{i+1}$ according to (Bertsekas, 1978)

$$\underline{x}_{i+1} = \underline{x}_i + \alpha_i \underline{d}_i \tag{4.5.22}$$

where for all i we have $\alpha_i \geq 0$ and

$$\left.\begin{array}{ll} [\nabla_x f(\underline{x}_i)]^T \underline{d}_i < 0, & \text{if } \nabla_x f(\underline{x}_i) \neq \underline{0}, \\ \underline{d}_i = \underline{0}, & \text{if } \nabla_x f(\underline{x}_i) = \underline{0}. \end{array}\right\} \tag{4.5.23}$$

$\nabla_x f(\underline{x}_i)$ denotes the Gradient vector of $f(\cdot)$ at $\underline{x}_i$, $\underline{d}_i$ is the descent direction, and α_i is the stepsize of the iteration. The conditions in (4.5.23) guarantee that a reduction of the objective function value will be realized as a result of the iteration and that the method will stop at a stationary point of $f(\cdot)$ where the gradient is zero. If $f(\cdot)$ is a convex function, this point is the global minimum; otherwise, it may be a local minimum or some other stationary point (e.g., an inflection point).

A method of this class is completely defined once the rules for selecting the direction $\underline{d}_i$ and the stepsize α_i are established, these rules exclusively determining whether the method is reliable and efficient.

Concerning the $\underline{d}_i$ selection rule, the best, yet not always possible, choice is to use the Newton's direction:

$$\underline{d}_i = -[\nabla^2_{xx} f(\underline{x}_i)]^{-1} \nabla_x f(\underline{x}_i), \tag{4.5.24}$$

where $\nabla^2_{xx} f(\underline{x}_i)$ is the Hessian matrix of $f(\cdot)$ evaluated at x_i (see Appendix A). At each iteration, the Newton's direction is specified using first and second order derivative information concerning the shape of the objective function, this being both its strength and weakness. The additional information can realize a faster convergence rate, yet it also requires heavier computational load. Our objective is to design an implementation of Newton's method which is computationally efficient. Despite our problem's large dimensionality (the vector $\underline{x}_i$ includes now all control vectors at all times: $\underline{x}_i^T = [\delta\underline{u}_k^T \ldots \delta\underline{u}_{T-1}^T]$) this will be accomplished by taking advantage of its special dynamical structure.

In general, the Newton's direction (4.5.24) is obtained by minimizing the second order Taylor series expansion of $f(\cdot)$ around x_i:

$$\tilde{f}(\underline{x}) = f(\underline{x}_i) + [\nabla_x f(\underline{x}_i)]^T(\underline{x}-\underline{x}_i) + \frac{1}{2}(\underline{x}-\underline{x}_i)^T \nabla^2_{xx} f(\underline{x}_i)(\underline{x}-\underline{x}_i). \qquad (4.5.25)$$

This can be easily seen by differentiating (4.5.25) with respect to $\underline{x}$, setting the resulting expression equal to zero, and solving for the minimizing vector $\underline{x}_{i+1}$. (It is assumed that the Hessian is non-singular.) Then the Newton's direction is given by

$$\underline{d}_i = \underline{x}_{i+1} - \underline{x}_i = - [\nabla^2_{xx} f(\underline{x}_i)]^{-1} \nabla_x f(\underline{x}_i). \qquad (4.5.26)$$

Notice, furthermore, that the same direction would result if in place of $\tilde{f}(\underline{x})$ we had minimized some other quadratic function $q(\underline{x})$ such that

$$\nabla^2_{xx} q(\underline{x}_i) = \nabla^2_{xx} f(\underline{x}_i), \ \nabla_x q(\underline{x}_i) = \nabla_x f(\underline{x}_i); \tag{4.5.27}$$

namely,

$$q(\underline{x}) = q_0 + [\nabla_x q(\underline{x}_i)]^T(\underline{x}-\underline{x}_i) + \frac{1}{2}(\underline{x}-\underline{x}_i)^T \ \nabla^2_{xx} q(\underline{x}_i)(\underline{x}-\underline{x}_i). \tag{4.5.28}$$

This observation can be very useful in optimal control problems where the dimensionality of the vector $\underline{x}$ is very large and a direct evaluation of the Gradient and Hessian is impractical.

Returning back to our problem, consider a second order Taylor expansion of each cost term $\tilde{\ell}_\ell$ $(\delta\bar{\underline{s}}_{\ell/k}, \delta\underline{u}_\ell)$ around the nominal state and control trajectories, i.e. around $\delta\bar{\underline{s}}_{\ell/k} = \underline{0}$, $\delta\underline{u}_\ell = \underline{0}$ (see Appendix A):

$$\begin{aligned}\tilde{\ell}_\ell(\delta\bar{\underline{s}}_{\ell/k},\delta\underline{u}_\ell) = \ & \tilde{\ell}_\ell(\underline{0},\underline{0}) + [\nabla_s \tilde{\ell}_\ell(\underline{0},\underline{0})]^T\delta\bar{\underline{s}}_{\ell/k} + [\nabla_u \tilde{\ell}_\ell(\underline{0},\underline{0})]^T \delta\underline{u}_\ell + \\ & + \frac{1}{2}\delta\bar{\underline{s}}^T_{\ell/k} \nabla^2_{ss} \tilde{\ell}_\ell(\underline{0},\underline{0}) \delta\bar{\underline{s}}_{\ell/k} + \frac{1}{2}\delta\underline{u}^T_\ell \nabla^2_{uu} \tilde{\ell}_\ell(\underline{0},\underline{0})\delta\underline{u}_\ell + \\ & + \delta\underline{u}^T_\ell[\nabla^2_{us} \tilde{\ell}_\ell(\underline{0},\underline{0})]^T\delta\bar{\underline{s}}_{\ell/k} + \text{(higher order terms)}\end{aligned} \tag{4.5.29}$$

In Equation (4.5.29) the subscript s implies differentiation with respect to $\delta\bar{\underline{s}}_{\ell/k}$ while differentiation with respect to $\delta\underline{u}_\ell$ is indicated by the subscript u. $\nabla_s \tilde{\ell}_\ell(\underline{0},\underline{0})$, $\nabla_u \tilde{\ell}_\ell(\underline{0},\underline{0})$ are n_s, n_u dimensional real vectors respectively, $\nabla^2_{ss} \tilde{\ell}_\ell(\underline{0},\underline{0})$, $\nabla^2_{uu} \tilde{\ell}_\ell(\underline{0},\underline{0})$ are $n_s \times n_s$, $n_u \times n_u$ real symmetric matrices respectively, and $\nabla^2_{us} \tilde{\ell}_\ell(\underline{0},\underline{0})$ is an $n_s \times n_u$ real matrix.

For notational convenience define

$$\left.\begin{aligned}
\underline{N}_{s\ell} &= \nabla_s \tilde{\ell}_\ell(\underline{0},\underline{0}),\\
\underline{N}_{u\ell} &= \nabla_u \tilde{\ell}_\ell(\underline{0},\underline{0}),\\
\underline{\underline{N}}_{ss\ell} &= \nabla^2_{ss} \tilde{\ell}_\ell(\underline{0},\underline{0}),\\
\underline{\underline{N}}_{uu\ell} &= \nabla^2_{uu} \tilde{\ell}_\ell(\underline{0},\underline{0}),\\
\underline{\underline{N}}_{us\ell} &= [\nabla^2_{us} \tilde{\ell}_\ell(\underline{0},\underline{0})]^T
\end{aligned}\right\} \qquad (4.5.30)$$

and consider the following problem

$$\left.\begin{aligned}
\underset{\{\delta\underline{u}_\ell\}_{\ell=k}^{T-1}}{\text{Minimize}} \quad \tilde{J} = \sum_{\ell=k}^{T-1} [&\underline{N}^T_{s\ell}\, \delta\bar{\underline{s}}_{\ell/k} + \underline{N}^T_{u\ell}\, \delta\underline{u}_\ell \;+\\
&+ \tfrac{1}{2}(\delta\bar{\underline{s}}^T_{\ell/k}\, \underline{\underline{N}}_{ss\ell}\, \delta\bar{\underline{s}}_{\ell/k} + \delta\underline{u}^T_\ell\, \underline{\underline{N}}_{uu}\, \delta\underline{u}_\ell\,) +\\
&+ \delta\underline{u}^T_\ell\, \underline{\underline{N}}_{us\ell}\, \delta\bar{\underline{s}}_{\ell/k}] \;+\\
&+ \underline{N}^T_{sT}\, \delta\bar{\underline{s}}_{T/k} + \tfrac{1}{2}\, \delta\bar{\underline{s}}^T_{T/k}\, \underline{\underline{N}}_{ssT}\, \delta\bar{\underline{s}}_{T/k}
\end{aligned}\right\} \qquad (4.5.31)$$

subject to the dynamical Equation (4.5.18).

Based on the comment concerning the Newton's method, if it were true that problems (4.5.31) and (4.5.21) had equal Gradients and Hessians along the nominal sequences $\{\delta\underline{u}_\ell = \underline{0}\}_{\ell=k}^{T-1}$, $\{\delta\bar{\underline{s}}_{\ell/k} = \underline{0}\}_{\ell=k}^{T}$, then we could obtain the Newton's direction for the latter by solving the former. If this direction is denoted by $\delta\underline{u}^{*(i)}$, a Newton's iteration could then be performed to identify a more rewarding nominal control trajectory $\underline{u}^{(i+1)}$:

$$\underline{u}^{(i+1)} = \begin{bmatrix} \underline{u}_k^{(i+1)} \\ \vdots \\ \underline{u}_{T-1}^{(i+1)} \end{bmatrix} = \begin{bmatrix} \underline{u}_k^{(i)} \\ \vdots \\ \underline{u}_{T-1}^{(i)} \end{bmatrix} + \alpha_i \begin{bmatrix} \delta \underline{u}_k^{*(i)} \\ \vdots \\ \delta u_{T-1}^{*(i)} \end{bmatrix} \tag{4.5.32}$$

The equality of the two problems' Gradients and Hessians is demonstrated in Appendix C by direct evaluation along the nominal control and state trajectories. The final expressions do not recommend straightforward implementation of the Newton's method. Comparatively, implementation through solution of problem (4.5.31) is much more advantageous. The computation of the Gradient $\nabla_u \tilde{J}(\delta\underline{u}_k=\underline{0}, \ldots, \delta\underline{u}_{T-1}=0)$ and especially of the Hessian $\nabla^2_{uu} \tilde{J}(\delta\underline{u}_k=\underline{0}, \ldots, \delta\underline{u}_{T-1}=\underline{0})$ and its inverse is bypassed and the Newton's direction

$$\delta\underline{u}^* = -[\nabla^2_{uu} \tilde{J}(\delta\underline{u}_k = \underline{0}, \ldots, \delta\underline{u}_{T-1}=\underline{0})]^{-1} \nabla_u \tilde{J}(\delta\underline{u}_k=\underline{0}, \ldots, \delta\underline{u}_{T-1}=\underline{0})$$

is obtained from the solution of an optimal control problem with linear dynamics and a general quadratic cost functional. This solution is derived in Appendix D via dynamic programming and is reproduced below as Algorithm A4.5.1. It has the advantage of being analytical and, therefore, computationally efficient.

<u>Algorithm A4.5.1</u>: Computation of the Newton's Direction for the Unconstrained Reservoir Operation Problem.

For $\ell=k, k+1, \ldots, T-1$,

$$\delta \underline{u}_\ell^* = \underline{\mu}^*(\delta \bar{\underline{s}}_{\ell/k}) = -\underline{\underline{D}}_\ell [\underline{\underline{L}}_\ell \, \delta \bar{\underline{s}}_{\ell/k} + \underline{\Lambda}_\ell], \tag{4.5.33}$$

$$\underline{\underline{D}}_\ell = [\underline{\underline{B}}_\ell^T \, \underline{\underline{K}}_{\ell+1} \, \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell}]^{-1}, \tag{4.5.34}$$

$$\underline{\underline{L}}_\ell = \underline{\underline{B}}_\ell^T \, \underline{\underline{K}}_{\ell+1} \, \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{us\ell}, \tag{4.5.35}$$

$$\underline{\Lambda}_\ell = \underline{\underline{B}}_\ell^T \, \underline{k}_{\ell+1} + \underline{N}_{u\ell}, \tag{4.5.36}$$

where the positive semidefinite matrices $\{\underline{\underline{K}}_\ell\}_{\ell=k}^T$ are obtained recursively by the following matrix Riccati equation:

$$\left.\begin{aligned} &\underline{\underline{K}}_T = \underline{\underline{N}}_{ss\,T} \\ &\underline{\underline{K}}_\ell = \underline{\underline{N}}_{ss\ell} + \underline{\underline{\phi}}_\ell^T \, \underline{\underline{K}}_{\ell+1} \, \underline{\underline{\phi}}_\ell - [\underline{\underline{B}}_\ell^T \, \underline{\underline{K}}_{\ell+1} \, \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{us\ell}]^T \cdot \\ &\quad \cdot [\underline{\underline{B}}_\ell^T \, \underline{\underline{K}}_{\ell+1} \, \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell}]^{-1} [\underline{\underline{B}}_\ell^T \, \underline{\underline{K}}_{\ell+1} \, \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{us\ell}] \\ &\ell = T-1, T-2, \ldots, k, \\ &\text{and the vectors } \{\underline{k}_\ell\}_{\ell=k}^T \text{ result from} \end{aligned}\right\} \tag{4.5.37}$$

$$\left.\begin{aligned} &\underline{k}_T = \underline{N}_{sT} \\ &\underline{k}_\ell = \underline{N}_{s\ell} + \underline{\underline{\phi}}_\ell^T \, \underline{k}_{\ell+1} - [\underline{\underline{B}}_\ell \, \underline{\underline{K}}_{\ell+1} \, \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{us\ell}]^T \\ &\quad \cdot [\underline{\underline{B}}_\ell^T \, \underline{\underline{K}}_{\ell+1} \, \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell}]^{-1} [\underline{\underline{B}}_\ell^T \, \underline{k}_{\ell+1} + \underline{N}_{u\ell}]^T \\ &\ell = T-1, T-2 \ldots, k. \end{aligned}\right\} \tag{4.5.38}$$

The optimal "costs to go" are given for $\ell = T, T-1, \ldots, k$, by

$$\left.\begin{aligned}
&\bar{J}_\ell(\delta\bar{\underline{s}}_{\ell/k}) = \frac{1}{2}\delta\bar{\underline{s}}^T_{\ell/k}\ \underline{\underline{K}}_\ell\ \delta\bar{\underline{s}}_{\ell/k} + \underline{k}^T_\ell\ \delta\bar{\underline{s}}_{\ell/k} + C_\ell \\
&\text{where } \underline{\underline{K}}_\ell,\ \underline{\underline{k}}_\ell \text{ as above and} \\
&C_T = 0, \\
&C_\ell = C_{\ell+1} - \frac{1}{2}\,[\underline{\underline{B}}_\ell\ \underline{k}_{\ell+1} + \underline{N}_{u\ell}]^T[\underline{\underline{B}}^T_\ell\ \underline{\underline{K}}_{\ell+1}\ \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell}]^{-1}\cdot \\
&\qquad\qquad \cdot\ [\underline{\underline{B}}_\ell\ \underline{k}_{\ell+1} + \underline{N}_{u\ell}], \\
&\ell = T-1,\ T-2,\ \ldots,\ k.
\end{aligned}\right\} \qquad (4.5.39)$$

Lastly, the optimal value of the problem is

$$\bar{J}_k(\delta\bar{\underline{s}}_{k/k}) = \frac{1}{2}\,\delta\bar{\underline{s}}^T_{k/k}\ \underline{\underline{K}}_k\ \delta\bar{\underline{s}}_{k/k} + \underline{k}^T_k\ \delta\bar{\underline{s}}_{k/k} + C_k. \qquad (4.5.40)$$

Essentially, implementation of this solution is completed in two steps: (1) the matrices $\{\underline{\underline{K}}_\ell\}^T_{\ell=k}$ and the vectors $\{\underline{k}_\ell\}^T_{\ell=k}$ are computed by a backward pass of the corresponding equations from time T to time k, and (2) the controls $\{\delta\underline{u}_\ell\}^{T-1}_{\ell=k}$ are computed by the previous quantities and the sequence $\{\delta\bar{\underline{s}}^*_{\ell/k}\}$ obtained from a forward pass of the dynamics. To illustrate how the second step is performed, assume that $\{\underline{\underline{K}}_\ell\}^T_{\ell=k}$, $\{\underline{k}_\ell\}^T_{\ell=k}$ have been computed. Then, the optimal control $\delta\underline{u}^*_k$ is obtained from

$$\delta\underline{u}^*_k = -\underline{\underline{D}}_k[\underline{\underline{L}}_k\ \delta\bar{\underline{s}}^*_{k/k} + \underline{\Lambda}_k] \qquad (4.5.41)$$

where $\delta\bar{\underline{s}}^*_{k/k} = \delta\bar{\underline{s}}_{k/k} = \underline{0}$. Thus,

$$\delta \underline{u}_k^* = -\underline{\underline{D}}_k \underline{\Lambda}_k = -[\underline{\underline{B}}_k^T \underline{\underline{K}}_{k+1} \underline{\underline{B}}_k + \underline{\underline{N}}_{uuk}]^{-1} [\underline{\underline{B}}_k \underline{k}_{k+1} + \underline{N}_{uk}]. \qquad (4.5.42)$$

Using this result in the dynamics, we find

$$\delta \bar{\underline{s}}_{k+1/k}^* = \underline{\underline{\Phi}}_k \, \delta \bar{\underline{s}}_{k/k} + \underline{\underline{B}}_k \, \delta \underline{u}_k^* = \underline{\underline{B}}_k \delta \underline{u}_k^* \qquad (4.5.43)$$

and the next period's control vector $\delta \underline{u}_{k+1}^*$ is computed from

$$\delta \underline{u}_{k+1}^* = -\underline{\underline{D}}_{k+1} [\underline{\underline{L}}_{k+1} \, \delta \bar{\underline{s}}_{k+1/k}^* + \underline{\Lambda}_k] =$$

$$= - [\underline{\underline{B}}_{k+1}^T \underline{\underline{K}}_{k+2} \underline{\underline{B}}_{k+1} + \underline{\underline{N}}_{uuk+1}]^{-1} \Big[[\underline{\underline{B}}_{k+1}^T \underline{\underline{K}}_{k+2} \underline{\underline{\Phi}}_{k+1} + \underline{\underline{N}}_{usk+1}] \cdot$$

$$\cdot \; \delta \bar{\underline{s}}_{k+1/k}^* + \underline{\underline{B}}_{k+1}^T \underline{k}_{k+2} + \underline{N}_{uk+1} \Big]. \qquad (4.5.44)$$

Continuing this process, all elements of the Newton's direction $\delta \underline{u}^{*T} = [\delta \underline{u}_k^{*T} \ldots \delta \underline{u}_{T-1}^{*T}]$ can be determined with minimal computational requirements. These are mainly associated with the computation of $\{\underline{\underline{K}}_\ell\}_{\ell=k}^T$, $\{\underline{k}_\ell\}_{\ell=k}^T$ and involve multiplications and inversions of $n_s \times n_s$, $n_u \times n_u$ $n_s \times n_u$ matrices. Compared to a straightforward implementation of Newton's method requiring the computation and inversion of a $(T-k)n_u \times (T-k)n_u$ Hessian, where $(T-k)$ could be of the order of several hundreds, substantial computational savings should be realized.

The second factor influencing the success of a minimization method is the stepsize selection rule. As can be seen in Bertsekas, 1982, there are many available good choices: the minimization rule, the limited minimization rule, the Goldstein rule, the Armijo rule, and many others. Here we shall present the Armijoy stepsize selection

rule because it is easily implemented and because it conveniently generalizes for a problem with control constraints. For the general problem of minimizing $f(\underline{x})$ with respect to $\underline{x}$, this rule can be stated as follows:

<u>The Armijo Stepsize Selection Rule</u>:

Let β and σ be scalars satisfying

$$0 < \beta < 1, 0 < \sigma < \frac{1}{2}. \tag{4.5.45}$$

Given a nonstationary point $\underline{x}_i$ and a direction $\underline{d}_i$, the stepsize α_i is obtained from

$$\alpha_i = \beta^{m_i} \tag{4.5.46}$$

where m_i is the first non-negative integer m for which

$$f(\underline{x}_i) - f(\underline{x}_i + \beta^m \underline{d}_i) \geq -\sigma \beta^m [\nabla_x f(\underline{x}_i)]^T \underline{d}_i. \tag{4.5.47}$$

If $\underline{x}_i$ is a stationary point of f(.), we set $\alpha_i = 0$.

The logic behind this rule is to guarantee that each iteration will reduce the objective function's value by an amount proportional to the Gradient and will thus prohibit convergence to a nonstationary point.

In the case of the control problem Equation (4.5.47) becomes

$$\tilde{J}(\underline{0},\ldots,\underline{0}) - \tilde{J}(\beta^m \delta \underline{u}_k^*,\ldots,\beta^m \delta \underline{u}_{T-1}^*) \geq -\sigma \beta^m [\nabla_u \tilde{J}(\underline{0},\ldots,\underline{0})]^T \delta \underline{u}^*, \tag{4.5.48}$$

where $(\delta \underline{u}^*)^T = [(\delta \underline{u}_k^*)^T \ldots (\delta \underline{u}_{T-1}^*)^T]$ is the Newton's direction obtained

by Algorithm A4.5.1. The Gradient $\nabla_u \tilde{J}(\underline{0},\ldots,\underline{0})$ can be efficiently computed by the procedure developed at the end of Appendix C as follows:

<u>Algorithm A4.5.2</u>: Computation of the Gradient

Consider the sequence of vectors $\{\underline{p}_i\}_{i=k}^{T}$ obtained recursively by

$$\left.\begin{aligned} \underline{p}_T &= \underline{N}_{sT} \\ \underline{p}_\ell &= \underline{\underline{\Phi}}_\ell^T \quad \underline{p}_{\ell+1} + \underline{N}_{s\ell} \\ \ell &= T-1, T-2, \ldots, k. \end{aligned}\right\} \tag{4.5.49}$$

Then

$$\nabla_u \tilde{J}(\underline{0},\ldots,\underline{0}) = \begin{bmatrix} \nabla\delta\underline{u}_k \quad \tilde{J}(\underline{0},\ldots,\underline{0}) \\ \vdots \\ \nabla\delta\underline{u}_{T-1} \tilde{J}(\underline{0},\ldots,\underline{0}) \end{bmatrix} = \begin{bmatrix} \underline{N}_{uk} + \underline{\underline{B}}_k^T \quad \underline{p}_{k+1} \\ \vdots \\ \underline{N}_{uT-1} + \underline{\underline{B}}_{T-1}^T \underline{p}_T \end{bmatrix} \tag{4.5.50}$$

Using the above results, the Armijo stepsize selection rule can be stated as follows: Select the stepsize

$$\alpha_i = \beta^{m_i}$$

where m_i is the first non-negative integer m for which

$$\tilde{J}(\underline{0},\ldots,\underline{0}) - \tilde{J}(\beta^m \delta\underline{u}_k^*,\ldots,\beta^m \delta\underline{u}_{T-1}^*) \geq -\sigma\, \beta^m \sum_{\ell=k}^{T-1} [\underline{N}_{u\ell}^T + \underline{p}_{\ell+1}^T \underline{\underline{B}}_\ell] \delta\underline{u}_\ell^*. \tag{4.5.51}$$

When the Newton's direction is combined with the Armijo stepsize selection rule, the following convergence properties are realized (Bertsekas, 1982):

1. Let $\{u\}_i$ be a sequence of control trajectories ($\underline{u}^T = [\underline{u}_k^T \cdots \underline{u}_{T-1}^T]$) generated by the previously defined minimization method. Then every limit point of $\{\underline{u}\}_i$ is a stationary point of $\tilde{J}(\underline{u})$.

2. In the vicinity of a local minimum $\underline{u}^*$ where $\tilde{J}(\cdot)$ is convex, the norm $|\underline{u}^{(i)} - \underline{u}^*| = [(\underline{u}^{(i)} - \underline{u}^*)^T(\underline{u}^{(i)} - \underline{u}^*)]^{1/2}$ converges to zero faster than superlinarly with order two. Namely, it convergest to zero faster than all sequences of the form $r_i = q\beta^{\bar{p}}$ where $q>0$, $\beta\varepsilon(0,1)$, $\bar{p}\varepsilon(1,2)$.

3. In the vicinity of a local minimum, the Armijo test is passed with $m_i=0$ and the stepsize α_i equals unity.

As will be seen in Chapter 5, these theoretical properties are indeed observed in practice making the previous design computationally appealing.

When the problem is characterized by a convex structure (as, for example, the case of linear dynamics and convex cost terms $\{\ell_\ell(\cdot,\cdot)\}_{\ell=k}^{T-1}$, $\tilde{\ell}_T(\cdot)$) the method will converge to the global minimum. Otherwise, the minimization process should be restarted from different initial trajectories and the global minimum should be determined by comparing the cost functional values at the indentified stationary points. In order to prohibit movements opposite to the direction of descent, one will have to ensure that the matrices $\{\underline{\underline{K}}_\ell\}_{\ell=k}^{T}$, $\{\underline{\underline{N}}_{uu\ell}\}_{\ell=k}^{T-1}$ are positive (semi)definite. A procedure that enforces these properties is included in Appendix E.

If the unconstrained global minimum violates no control or state constraints, it also solves the constrained problem. In the opposite event, a constrained minimization scheme should be adopted. The development of a control constrained minimization algorithm is the following section's topic. State constraints will be accounted for in Section 4.7.

4.6 Accounting for Control Constraints

Towards reconstructing and solving the reservoir operation Problem P.2, in this section we consider the following control constrained formulation:

Problems P.4: The Reservoir Operation Problem with Control Constraints.

$$\underset{\{\delta \underline{u}_\ell\}_{\ell=k}^{T-1}}{\text{Minimize}} \left\{ \tilde{J} = \sum_{\ell=k}^{T-1} \tilde{\ell}_\ell(\delta \bar{\underline{s}}_{\ell/k}, \delta \underline{u}_\ell) + \tilde{\ell}_T(\delta \bar{\underline{s}}_{T/k}) \right\} \tag{4.6.1}$$

subject to

$$\text{a.} \quad \delta \bar{\underline{s}}_{\ell+1/k} = \underline{\underline{\Phi}}_\ell \delta \bar{\underline{s}}_{\ell/k} + \underline{\underline{B}}_\ell \, \delta \underline{u}_\ell, \; \delta \bar{\underline{s}}_{k/k} = \underline{0} \tag{4.6.2}$$

$$\ell = k, \ldots, T-1,$$

$$\text{b.} \quad \delta u^{min} \leq \delta \underline{u}_\ell \leq \delta \underline{u}_\ell^{max}, \tag{4.6.3}$$

$$\ell = k, \ldots, T-1$$

Namely, we wish to develop a procedure accounting for control magnitude constraints within the framework of the Open Loop Feedback controller. Selected classes of constrained minimization methods are now briefly reviewed to find a suitable procedure.

4.6.1 Constrained Minimization Methods

Three broad classes of constrained minimization methods are (1) the Feasible Direction methods, (2) the Manifold Suboptimization methods, and (3) the Projection Newton methods.

The Feasible Direction methods (e.g., Frank-Wolfe, Goldstein-

Levitin-Polyak, Zoutendijk's, etc.)[1] are natural extensions of the unconstrained Generalized Gradient methods where the descent direction $\underline{d}_i$ is such that the iteration

$$\underline{x}_{i+1} = \underline{x}_i + \alpha_i \underline{d}_i \qquad (4.6.4)$$

yields feasible points $\underline{x}_i$, i=1, 2, ... (α_i being the stepsize parameter). At each iteration the feasible descent direction $\underline{d}_i$ is determined by solving a linear or a quadratic minimization problem. This requirement imposes considerable imputational overhead and makes these methods unattractive for problems with many variables (such as Problem P.4).

The Manifold Supoptimization methods (e.g. Gradient projection, Rosen's Gradient Projection, Reduced Gradient, etc.)[1] are also based on (4.6.4) type iterations where the descent direction $\underline{d}_i$ is obtained by minimizing the objective function over the subspace defined by the active constraints. If feasibility is violated, the minimization is repeated over a new subspace defined by the currently active constraint set, and the process is repeated until a point satisfying the necessary conditions for optimality is reached. Methods of this class perform quite well for problems of relatively small dimension, but are not efficient for problems with many binding constraints. The reason is that not more than one constraint can be added to the active set at each iteration.

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[1]For a complete discussion, see Luendberger, 1973, Bertsekas, 1978.

For the problem of minimizing $f(\underline{x})$ subject to $\underline{x} \geq \underline{0}$, a Projected Newton method will perform the following iteration (Bertsekas, 1982, Section 1.5):

$$\underline{x}_{i+1} = [\underline{x}_i + \alpha_i \underline{d}_i]^+ \tag{4.6.5}$$

where $[\cdot]^+$ denotes the projection operator:

$$[\underline{z}]^+ = \begin{bmatrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{bmatrix}^+ = \begin{bmatrix} \max\{0, z_1\} \\ \cdot \\ \cdot \\ \cdot \\ \max\{0, z_n\} \end{bmatrix}, \tag{4.6.6}$$

and $\underline{d}_i$ is the descent direction given by

$$\underline{d}_i = -\underline{\underline{D}}_i \nabla_x f(\underline{x}_i). \tag{4.6.7}$$

$\underline{\underline{D}}_i$ is a positive definite matrix of the following form.

$$\underline{\underline{D}}_i = \begin{bmatrix} \underline{\underline{H}}_i & & & \underline{\underline{0}} \\ & h_{r_i+1} & \cdots & 0 \\ \underline{\underline{0}} & \vdots & \ddots & \vdots \\ & 0 & \cdots & h_n \end{bmatrix}$$

where $\underline{\underline{H}}_i$ is a positive definite matrix, $\{h_\ell\}_{\ell=r_i+1}^{n}$ are positive numbers, and the set of indices

$$A^+(\underline{x}_i) = \{r_i+1, \ldots, n\} = \{\ell/0 \leq x_{\ell i} \leq \varepsilon_i, \ \frac{\partial f(\underline{x}_i)}{\partial x_{\ell i}} > 0\} \qquad (4.6.9)$$

correspond to the binding and nearly binding constraints. The specification $0 \leq x_{\ell i} \leq \varepsilon_i$ is employed rather than $x_{\ell i} = 0$, to avert possible "zigzagging" behavior of the method. ε_i can be chosen from

$$\varepsilon_i = \min\{\varepsilon, w_i\}, \quad w_i = |x_i - [\underline{x}_i - \nabla_x f(\underline{x}_i)]^+| \qquad (4.6.10)$$

with ε a small positive scalar.

The stepsize α_i can be obtained by an Armijo-like stepsize selection rule as follows:

$$\alpha_i = \beta^{m_i} \qquad (4.6.11)$$

where m_i is the first non-negative integer m for which

$$f(\underline{x}_i) - f([\underline{x}_i + \alpha_i d_i]^+) \geq -\sigma \Bigg[\beta^m \sum_{j \notin A^+(\underline{x}_i)} \frac{\partial f(\underline{x}_i)}{\partial x_{ji}} d_{ji} - \sum_{j \in A^+(\underline{x}_i)} \frac{\partial f(\underline{x}_i)}{\partial x_{ji}} (x_{ji} - [x_{ji} + \beta^m d_{ji}]^+) \Bigg] \qquad (4.6.12)$$

and $\sigma \in (0, \frac{1}{2})$, $\beta \in (0,1)$

(Notice that if none of the constraints are binding, the previous step-size selection rule becomes identical to the one given in Section 4.5.)

Under assumptions which are usually valid in actual problems, an algorithm of the above type is guaranteed to identify constrained critical points in a finite number of iterations, a constrained critical point being any point $\underline{x}^*$ such that

$$\left.\begin{aligned} &\frac{\partial f(\underline{x}^*)}{\partial x_j} = 0, \text{ if } x_j^* > 0, \\ &\frac{\partial f(\underline{x}^*)}{\partial x_j} \geq 0, \text{ if } x_j^* = 0, \; j=1, \ldots, n. \end{aligned}\right\} \tag{4.6.13}$$

If in addition $f(\cdot)$ is convex and $\underline{\underline{H}}_i$, $\{h_j\}_{j=r_i+1}^{n}$ are chosen equal to

$$\underline{\underline{H}}_i = \begin{bmatrix} \frac{\partial^2 f(\underline{x}_i)}{\partial x_{1i}^2} & \cdots & \frac{\partial^2 f(\underline{x}_i)}{\partial x_{1i} \partial x_{r_i i}} \\ \vdots & & \vdots \\ \frac{\partial^2 f(\underline{x}_i)}{\partial x_{r_i i} \partial x_{1i}} & \cdots & \frac{\partial^2 f(\underline{x}_i)}{\partial x_{r_i i}^2} \end{bmatrix}^{-1} \tag{4.6.14}$$

$$h_j = \left[\frac{\partial^2 f(\underline{x}_i)}{\partial x_{ji}^2}\right]^{-1}, \; j = r_i + 1, \ldots, n, \tag{4.6.15}$$

then the sequence $\{\underline{x}_i\}$ converges to the global minimum $\underline{x}^*$ faster than superlinearly of an order at least two (see Section 4.5).

These convergence properties make the

quite efficient for problems with simple constraints.

The previous algorithm easily generalizes to the case where $\underline{x}$ is constrained by upper and lower bounds:

$$\underline{b}_1 \leq \underline{x} \leq \underline{b}_2. \tag{4.6.16}$$

Now, the iteration takes the form

$$\underline{x}_{i+1} = [\underline{x}_i + \alpha_i \underline{d}_i]^{++} \tag{4.6.17}$$

where

$$([\underline{z}]^{++}) = \left\{ \begin{array}{l} b_{j2}, \text{ if } b_{j2} \leq z_j \\ z_j, \text{ if } b_{j1} \leq z_j \leq b_{j2} \\ b_{j1}, \text{ if } z_j \leq b_{j1} \\ j = 1, \ldots, n \end{array} \right\}, \tag{4.6.18}$$

and the set of active constraints is given by

$$A^{++}(\underline{x}_i) = \{\ell / b_{\ell 1} \leq x_{\ell i} \leq b_{\ell 1} + \varepsilon_i \text{ and } \frac{\partial f(\underline{x}_i)}{\partial x_{\ell i}} > 0$$

$$\text{or } b_{\ell 2} - \varepsilon_i \leq x_{\ell i} \leq b_{\ell 2} \text{ and } \frac{\partial f(\underline{x}_i)}{\partial x_{\ell i}} < 0\} \quad . \tag{4.6.19}$$

With these definitions replacing the ones given above, the Projected Newton method previously stated is also valid here.

It is a method of the third type that we shall employ in the solution of Problem P.4. The main difficulty is again related to the computation of the direction $\underline{d}_i$.

4.6.2 A Projected Newton Method in Optimal Control

The direction $\underline{d}_i$ of a Projected Newton method consists of two parts: The first corresponds to the nonbinding constraints $j=1, 2, \ldots, r_i$, $j \notin A^{++}(\underline{x}_i)$ and is obtained by a Newton iteration

$$\begin{bmatrix} d_{1i} \\ \cdot \\ \cdot \\ \cdot \\ d_{r_i i} \end{bmatrix} = - \begin{bmatrix} \dfrac{\partial^2 f(\underline{x}_i)}{\partial x_{1i}^2} & \cdots & \dfrac{\partial^2 f(\underline{x}_i)}{\partial x_{1i}\partial x_{r_i i}} \\ \vdots & & \vdots \\ \dfrac{\partial^2 f(\underline{x}_i)}{\partial x_{r_i i}\partial x_{1i}} & \cdots & \dfrac{\partial^2 f(\underline{x}_i)}{\partial x_{r_i i}^2} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial f(\underline{x}_i)}{\partial x_{1i}} \\ \cdot \\ \cdot \\ \cdot \\ \dfrac{\partial f(\underline{x}_i)}{\partial x_{r_i i}} \end{bmatrix} \tag{4.6.20}$$

assuming that the remaining variables are fixed at the respective bounds. The other is a scaled version of a Gradient method (Luendberger, 1973) where the scaling factor is equal to the inverse of the corresponding diagonal element of the Hessian at point $\underline{x}_i$:

$$d_{ij} = - \left[\frac{\partial^2 f(\underline{x}_i)}{\partial x_{ji}^2} \right]^{-1} \frac{\partial f(\underline{x}_i)}{\partial x_{ji}}, \quad j = r_i+1, \ldots, n. \tag{4.6.21}$$

The question is how to efficiently compute these two direction parts for the control Problem P.4. Regarding the computation of the first part, the approach employed for the uncontratined control problem in Section 4.5.2 is also applicable here. The difference is that now, due to the constraints, some control vectors cannot be adjusted towards the unconstrained minimum. Instead they must be fixed at their currently nominal values. Appendix F includes the derivation of an analytical procedure for computing the nonbinding direction part given the above requirement.

The scaled Gradient part of the direction can be conveniently obtained from the procedures derived at the end of Appendix C. These concern the evaluation of the Gradient and the diagonal submatrices of the Hessian at the nominal sequences.

Putting all these results together, we construct the following control constrained minimization algorithm:

<u>Algorithm A4.6.1:</u> A Projected Newton Method in Optimal Control

In a typical iteration the following sequence of operations is performed:

a. Compute the Gradient vectors $\{\nabla_{\delta \underline{u}\ell} \tilde{J}(\underline{0},\ldots,\underline{0})\}_{\ell=k}^{T-1}$

from

$$\nabla_{\delta \underline{u}\ell} \tilde{J}(\underline{0},\ldots,\underline{0}) = \underline{N}_{u\ell} + \underline{\underline{B}}_{\ell}^{T} \underline{P}_{\ell+1} \tag{4.6.22}$$

where

$$\left.\begin{aligned} \underline{P}_T &= \underline{N}_{sT}, \\ \underline{P}_\ell &= \underline{\underline{\Phi}}_{\ell}^{T} \underline{P}_{\ell+1} + \underline{N}_{s\ell}, \\ \ell &= T-1,\ldots,k \end{aligned}\right\} \tag{4.6.23}$$

b. Calculate

$$w_i = |\underline{u}^{(i)} - [\underline{u}^{(i)} - \nabla_u \tilde{J}(\underline{0},\ldots,\underline{0})]^{++}|. \tag{4.6.24}$$

If $w_i \simeq 0$, then stop; a stationary (or critical) control trajectory is reached. This is true because when $w_i=0$, any control element $u_{j\ell}$ for all $j\ell$ satisfies one of the following first order necessary conditions for optimality:

$$\left.\begin{array}{ll} u_{j\ell}^{min} \le u_{j\ell} \le u_{j\ell}^{max} & \text{and} \quad \dfrac{\partial \tilde{J}(\underline{0},\ldots,\underline{0})}{\partial \delta u_{j\ell}} = 0, \\ u_{j\ell} = u_{j\ell}^{max} & \text{and} \quad \dfrac{\partial \tilde{J}(\underline{0},\ldots,\underline{0})}{\partial \delta u_{j\ell}} < 0, \\ u_{j\ell} = u_{j\ell}^{min} & \text{and} \quad \dfrac{\partial \tilde{J}(\underline{0},\ldots\underline{0})}{\partial \delta u_{j\ell}} > 0. \\ j=1,\ldots,n_u, \quad \ell=k,\ldots,T-1 & \end{array}\right\} \tag{4.6.25}$$

(Under convexity assumptions $\underline{u}^{(i)}$ is then a global minimum.)

Otherwise, determine the set $A^{++}(\underline{u}^{(i)})$ of binding control constratins from

$$A^{++}(\underline{u}^{(i)}) = \left\{ j\ell / u_{j\ell}^{min} \le u_{j\ell}^{(i)} \le u_{j\ell}^{min} + \varepsilon_i \text{ and } \frac{\partial \tilde{J}(\underline{0},\ldots,\underline{0})}{\partial \delta u_{j\ell}} > 0 \right.$$

$$\left. \text{or } u_{j\ell}^{max} - \varepsilon_i \le u_{j\ell}^{(i)} \le u_{j\ell}^{max} \text{ and } \frac{\partial \tilde{J}(\underline{0},\ldots,\underline{0})}{\partial \delta u_{j\ell}} < 0 \right.$$

$$\left. j=1,\ldots,n_u, \quad \ell=k,\ldots,T-1 \right\} \tag{4.6.26}$$

where $\varepsilon_i = \min\{\varepsilon, w_i\}$, (4.6.27)

with ε being a small positive scalar.

c. Compute the Newton direction for the nonbinding control elements as follows:

Calculate the mattrices $\{\underline{\underline{K}}_\ell\}_{\ell=k}^{T}$ and the vectors $\{\underline{k}_\ell\}_{\ell=k}^{T}$ from

$$\left.\begin{array}{l} \underline{\underline{K}}_T = \underline{\underline{N}}_{ssT}, \\ \underline{\underline{K}}_\ell = \underline{\underline{N}}_{ss\ell} + \underline{\underline{\phi}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell - [(\underline{\underline{B}}^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{us\ell})^r]^T \cdot \\ \quad \cdot [(\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell})^{rc}]^{-1} [\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{us\ell}]^r, \\ \ell = T-1,\ldots,k, \end{array}\right\} \tag{4.6.28}$$

$$\left.\begin{aligned}
\underline{k}_T &= \underline{N}_{sT}, \\
\underline{k}_\ell &= \underline{N}_{s\ell} + \underline{\underline{\phi}}^T \underline{k}_{\ell+1} - [(\underline{\underline{B}}_\ell \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell + \underline{N}_{us\ell})^r]^T \cdot \\
&\quad \cdot [(\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell})^{rc}]^{-1} [\underline{\underline{B}}_\ell^T \underline{k}_{\ell+1} + \underline{N}_{u\ell}]^r,
\end{aligned}\right\} \tag{4.6.29}$$

where the notation $(\underline{\underline{X}})^r$ or $(\underline{\underline{X}})^{rc}$ implies that the rows or the rows and the columns of the matrix $\underline{\underline{X}}$ corresponding to all control elements $\delta u_{j\ell}$ with $j\ell \varepsilon A^{++}(\underline{u}^{(i)})$ have been deleted. (Note, $\underline{\underline{X}}$ may also be a vector.)

Then determine the Newton direction $\delta\bar{\underline{u}}^*$ from

$$\delta\bar{\underline{u}}_\ell^* = -\underline{\underline{D}}_\ell [\underline{\underline{L}}_\ell \, \delta\bar{\underline{s}}_{\ell/k}^* + \underline{\Lambda}_\ell] \tag{4.6.30}$$

$$\left.\begin{aligned}
\text{where} \quad \underline{\underline{D}}_\ell &= [(\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{us\ell})^{rc}]^{-1}, \\
\underline{\underline{L}}_\ell &= (\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{us\ell})^r, \\
\underline{\Lambda}_\ell &= (\underline{\underline{B}}_\ell^T \underline{k}_{\ell+1} + \underline{N}_{u\ell})^r, \\
\ell &= k, \ldots, T-1,
\end{aligned}\right\} \tag{4.6.31}$$

$$\text{and } \delta\bar{\underline{s}}_{\ell+1/k}^* = \underline{\underline{\phi}}_\ell \, \delta\bar{\underline{s}}_{\ell/k}^* + \underline{\underline{B}}_\ell \delta\underline{u}_\ell^*, \quad \delta\bar{\underline{s}}_{k/k}^* = \underline{0}, \tag{4.6.32}$$

$\ell = 1, \ldots, T-1.$

In (4.6.32) the binding controls $\delta u_{j\ell}$ are set equal to zero:

$$\delta u_{j\ell}^* = 0, \quad j\ell \varepsilon A^{++}(\underline{u}^{(i)}). \tag{4.6.33}$$

d. Compute the scaled Gradient direction for the binding constraints: Having computed the Gradient in Step a, we additionally need to calculate the diagonal elements of the Hessian which can be accomplished as follows

$$\nabla^2_{\delta \underline{u}_\ell \delta \underline{u}_\ell} \tilde{J}(\underline{0},\ldots,\underline{0}) = \underline{\underline{N}}_{uu\ell} + \underline{\underline{B}}_\ell^T \; \underline{\underline{G}}_{\ell+1} \; \underline{\underline{B}}_\ell \tag{4.6.34}$$

$\ell = k, \ldots, T-1$

where the matrices $\{\underline{\underline{G}}_\ell\}_{\ell=k}^T$ are given by

$$\left.\begin{aligned} \underline{\underline{G}}_T &= \underline{\underline{N}}_{ssT} \\ \underline{\underline{G}}_\ell &= \underline{\underline{\phi}}_\ell^T \; \underline{\underline{G}}_{\ell+1} \; \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{ss\ell} \\ \ell &= T-1, \ldots, 1. \end{aligned}\right\} \tag{4.6.35}$$

Then, the direction for the binding controls is found from:

$$\delta u_j^* = -\left[\frac{\partial^2 \tilde{J}(\underline{0},\ldots,\underline{0})}{\partial \delta u_{j\ell}^2}\right]^{-1} \frac{\partial \tilde{J}(\underline{0},\ldots,\underline{0})}{\partial \delta u_{j\ell}}, \tag{4.6.36}$$

$j\ell \in A^{++}(\underline{u}^{(i)})$.

e. Determine a stepsize α_i such that

$$\alpha_i = \beta^{m_i} \tag{4.6.37}$$

where m_i is the first non-negative integer m for which

$$\tilde{J}(\underline{0},\ldots,\underline{0}) - \tilde{J}([\beta^m \delta \underline{u}_k^*]^{++} \ldots [\beta^m \delta \underline{u}_{T-1}^*]^{++}) \geq$$

$$-\sigma\left[\beta^m \sum_{j\ell\varepsilon A^{++}(\underline{u}^{(i)})} \frac{\partial \tilde{J}(\underline{0},\ldots\underline{0})}{\partial \delta u_{j\ell}} \delta u_{j\ell}^* + \sum_{j\ell\varepsilon A^{++}(\underline{u}^{(i)})} \frac{\partial \tilde{J}(\underline{0},\ldots,\underline{0})}{\partial \delta u_{j\ell}} [\beta^m \delta u_{j\ell}^*]^{++}\right], \tag{4.6.38}$$

where $\beta\varepsilon(0,1)$, $\sigma\varepsilon(0,\frac{1}{2})$,

and

$$[\beta^m \delta u_{j\ell}^*]^{++} = \begin{cases} u_{j\ell}^{min} - u_{j\ell}^{(i)}, \text{ if } u_{j\ell}^{(i)} + \beta^m \delta u_{j\ell}^* \leq u_{j\ell}^{min} \\ \beta^m \delta u_{j\ell}^*, \text{ if } u_{j\ell}^{min} \leq u_{j\ell}^{(i)} + \beta^m \delta u_{j\ell}^* \leq u_{j\ell}^{max} \\ u_{j\ell}^{max} - u_{j\ell}^{(i)}, \text{ if } u_{j\ell}^{(i)} + \beta^m \delta u_{j\ell}^* \geq u_{j\ell}^{max} \end{cases} . \tag{4.6.39}$$

f. Perform the iteration

$$\underline{u}^{(i+1)} = \underline{u}^{(i)} + [\alpha_i \delta \underline{u}^*]^{++} \tag{4.6.40}$$

to obtain the new nominal control trajectory $\underline{u}^{(i+1)}$ and continue the iterations until a stationary control trajectory $\underline{u}^*$ is found with respect to Problem P.4.

Algorithm A4.6.1 is the core of the control design developed in this work. As will be seen in Section 4.7, the modification which is needed to account for the probabilistic state constraints only involves addition of some more quadratic terms in the present performance index.

A few comments are now noted:

1. In Step d, the computation of the diagonal Hessian submatrices can be avoided by using an ordinary Gradient direction where the scaling factor is unit or some other positive number. The resulting method is still reliable, yet it may take more itereations to converge.

2. In the absence of convexity assumptions, one must assure that a descending direction is obtained. This can be accomplished by enforcing positive definiteness properties on the matrices $\{\underline{\underline{K}}_\ell\}_{\ell=k}^{T}$, $\{\underline{\underline{N}}_{uu\ell}\}_{\ell=k}^{T-1}$ as in Appendix E.

3. If no control constraints are found binding (i.e., if the set $A^{++}(u^{(i)})$ is empty), then the algorithm conveniently becomes the Newton method for unconstrained minimization presented in Section 4.5.

4. Under convexity assumptions, every limit point of the sequence $\{\underline{u}^{(i)}\}$ generated by this algorithm is a stationary control trajectory. If the method converges to a local minimum $\underline{u}^*$, it identifies the set of active constraints at $\underline{u}^*$ in a finite number of iterations. Then it becomes the unconstrained Newton method and achieves a superlinear convergence rate (i.e. $|\underline{u}^{(i)}-\underline{u}^*|$ converges to zero faster than all sequences of the form $r_i = q\beta^{\bar{p}i}$, $q>0$, $\beta\varepsilon(0,1)$, $\bar{p}\varepsilon(1,2)$). Theoretical derivation of these properties for the nonlinear programming formulation presented in Section 4.6.2 can be found in Bertsekas, 1982.

5. After a new nominal control sequence $\underline{u}^{(i+1)}$ has been obtained, one can continue the iterations by reconstructing the corresponding Problem P.4 from P.1 as has been presented in Sections 4.3, 4.5, and 4.6. Concerning the continuous time nonlinear dynamics, this iteration is an ordinary Gradient method iteration. However, the milder the nonlinearity of the dynamics, the more Newtonlike the procedure becomes. With respect

to the reservoir system, if the releases cannot effect drastic storage changes, then linearization of the dynamics is a good approximation for a wide release range and the procedure will exhibit fast convergence rate.

Algorithm A4.6.1 is a well defined minimization procedure capable of efficiently handling large numbers of control constraints. Computational experience reported in the following chapters is suggestive of this conclusion.

Our last task is to account for probabilistic state constraints.

4.7 Accounting for Probabilistic State Constraints

In this final effort, all state constraints are restored and the following control problem is considered:

Problem P.5: The Reservoir Operation Problem with Control and State Constraints.

$$\underset{\{\delta \underline{u}_\ell\}_{\ell=k}^{T-1}}{\text{Minimize}} \left\{ \tilde{J} = \sum_{\ell=k}^{T-1} \tilde{\ell}_\ell(\delta \bar{\underline{s}}_{\ell/k}, \delta \underline{u}_\ell) + \tilde{\ell}_T(\delta \bar{\underline{s}}_{T/k}) \right\} \tag{4.7.1}$$

subject to

a. $$\delta \bar{\underline{s}}_{\ell+1/k} = \underline{\underline{\phi}}_\ell \cdot \delta \bar{\underline{s}}_{\ell/k} + \underline{\underline{B}}_\ell \, \delta \underline{u}_\ell, \quad \delta \bar{\underline{s}}_{k/k} = \underline{0} \tag{4.7.2}$$

$\ell = k, \ldots, T-1,$

b. $$\delta \underline{u}_\ell^{min} \leq \delta \underline{u}_\ell \leq \delta \underline{u}_\ell^{max}, \tag{4.7.3}$$

$\ell = k, \ldots, T-1,$

c. $$\left.\begin{aligned} &\int_{-\infty}^{\delta s_{j\ell}^{min}} p(\delta s_{j\ell}/I_k) d\delta s_{j\ell} \leq \gamma_{j\ell}^{min}, \\ &\int_{\delta s_{j\ell}^{max}}^{+\infty} p(\delta s_{j\ell}/I_k) d\delta s_{j\ell} \leq \gamma_{j\ell}^{max}, \\ &i = 1, \ldots, n_s, \ \ell = k, \ldots, T, \end{aligned}\right\} \tag{4.7.4}$$

d. Continuing operations after T.

For consistency with the Open Loop Feedback control philosophy, the probability densities of the states at all times $\ell = k, \ldots, T$ are conditioned on all information available at time k. Namely, the possibility of future information gathering is ignored. Under this assumption,

in Section 4.5.2 it was shown that all probability densities are Gaussian with mean vectors obtained by Equation (4,7,2) and covariance matrices resulting from (c.f., Equation (4.5.9)))

$$\underline{\underline{P}}_{s\ell+1/k} = \underline{\underline{\phi}}_{\ell}\, \underline{\underline{P}}_{s\ell/k}\, \underline{\underline{\phi}}_{\ell}^{T} + \underline{\underline{Q}}_{\xi\ell}, \quad \underline{\underline{P}}_{sk/k} \text{ given}, \qquad (4.7.5)$$
$$\ell = k, \ldots, T-1.$$

It follows that $p(\delta s_{j\ell}/I_k)$ is given by

$$p(\delta s_{j\ell}/I_k) = \frac{1}{\sqrt{2\pi(\underline{\underline{P}}_{s\ell/k})_{jj}}} \exp\left[-\frac{1}{2}\,\frac{(\delta s_{j\ell} - \delta\bar{s}_{j\ell/k})^2}{(\underline{\underline{P}}_{s\ell/k})_{jj}}\right] \qquad (4.7.6)$$

where $(\underline{\underline{P}}_{s\ell/k})_{jj}$ denotes the jth diagonal element of $\underline{\underline{P}}_{s\ell/k}$. It is now possible to substitute the probabilistic constraints (4.7.4) by equivalent deterministic constraints on the mean value of the state. This is facilitated by the following two facts: First, the density (4.7.6) is completely characterized by its mean and variance. Second, the covariance Equation (4.7.5) does not depend on the controls $\delta\underline{u}_{\ell}$, $\ell=k,\ldots,$ T-1. As a result, we can change the position of $p(\delta s_{j\ell}/I_k)$ along the $\delta s_{j\ell}$ axis but we cannot alter its shape. The equivalent constraints on the mean value can be easily derived as follows: Consider the constraint

$$\int_{-\infty}^{\delta s_j^{min}} p(\delta s_{j\ell}/I_k)\, d\delta s_{j\ell} \leq \gamma_{j\ell}^{min} \qquad (4.7.7)$$

and the standard Gaussian variate $z \sim N(0,1)$. The level $z_{j\ell}^{min}$ can be found (from standard normal variate tables) for which

$$\int_{-\infty}^{z_{j\ell}^{min}} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}z^2} dz = \gamma_{j\ell}^{min}. \tag{4.7.8}$$

Then, as long as

$$z_{j\ell}^{min} \sqrt{(\underline{P}_{s\ell/k})_{jj}} + \delta\bar{s}_{j\ell/k} \geq \delta s_{j\ell}^{min} \tag{4.7.9}$$

or $$\delta\bar{s}_{j\ell/k} \geq \delta s_{j\ell}^{min} - z_{j\ell}^{min} \sqrt{(\underline{P}_{s\ell/k})_{jj}} \tag{4.7.10}$$

holds true, the probabilistic constraint (4.7.7) is also satisfied and vice versa. Now we can call

$$\delta\bar{s}_{j}^{min} = \delta s_{j\ell}^{min} - z_{j\ell}^{min} \sqrt{(\underline{P}_{s\ell/k})_{jj}} \tag{4.7.11}$$

and in place of (4.7.7) consider the inequality

$$\delta\bar{s}_{j\ell}^{min} \leq \delta\bar{s}_{j\ell/k}. \tag{4.7.12}$$

For example, if the reliability parameter $\gamma_{j\ell}^{min}$ is equal to 0.025 (i.e., if the state $\delta s_{j\ell}$ is allowed to violate its lower bound $\delta s_{j\ell}^{min}$ at most 2.5% of the time), the level $z_{j\ell}^{min}$ equals -1.96. Then the constraint (4.7.12) implies that the mean value $\delta\bar{s}_{j\ell/k}$ should be kept a distance of 1.96 standard deviations above $\delta s_{j\ell}^{min}$.

Similarly the upper probability constraints can be transformed to constraints of the following type

$$\delta\bar{s}_{j\ell/k} \leq \delta\bar{s}_{j\ell}^{max} \tag{4.7.13}$$

where $\delta\bar{s}_{j\ell}^{max} = \delta s_{j\ell}^{max} - z_{j\ell}^{max} \sqrt{(\underline{P}_{s\ell/k})_{jj}}$ (4.7.14)

with $z_{j\ell}^{max}$ such that

$$\int_{z_{j\ell}^{max}}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz = \gamma_{j\ell}^{max} , \tag{4.7.15}$$

$j=1, \ldots, n_s, \quad \ell= k, \ldots, T.$

Thus, we have shown how to convert the probabilistic constraints (4.7.4) into equivalent constraints on the state's mean:

$$\delta\bar{s}_j^{min} \leq \delta\bar{s}_{j\ell/k} \leq \delta\bar{s}_{j\ell}^{max} \tag{4.7.16}$$

$j=1, \ldots, n_s, \quad \ell=k, \ldots, T,$

or $$\delta\underline{\bar{s}}_{\ell}^{min} \leq \delta\underline{\bar{s}}_{\ell/k} \leq \delta\underline{\bar{s}}_{\ell}^{max}, \tag{4.7.17}$$

$\ell=k,\ldots,T$

in vector notation.

Although these constraints are similar to the ones imposed on the controls, they cannot be handled by the Projected Newton developed in the previous section. The reason is that the states are functionally related to the controls through the system dynamics, and it is either the controls or the states that can serve as an independent set of variables for the minimization of $\tilde{J}$. Optimal control problems with state constraints are best handled by Penalty Function of Multiplier methods, a discussion of which will be offered in the following section.

The continuing operations system requirements will be discussed later.

A last comment refers to the proposed treatment of the probabilistic constraints in relation to the Open Loop Feedback control procedure. According to algorithm A4.4.5. at each time k the OLF controller determines the entire control trajectory $\{\underline{u}_\ell^*\}_{\ell=k}^{T-1}$ assuming that no other decision specifications take place over the period [k, T-1]. In reality only the first member from each optimal trajectory is used and the process is repeated at the next decision time. Thus, owing to this sequential set-up, the applied controls are of a feedback nature $\underline{u}_k^* = \underline{\mu}_k^*(\bar{\underline{s}}_k)$, k=0,1, ..., although Open Loop controls are being determined. The OLF control structure relates to this section's developments through the covariance propagation Equation (4.7.5). The covariance is propagated under the assumption of Open Loop control sequences. For a certain class of systems[1] this may result in covariance matrices with diagonal elements growing unbounded over time. On the other hand, if the feedback laws were taken into account, the above elements would be either stabilized at a finite level or they would grow at a lower rate. It is evident that the Open Loop approach could result in suboptimal control policies in the sense that the applied controls would meet the probabilistic constraints at greater percentages of time than the one required. However, the feedback control functions cannot be obtained explicitly and therefore cannot be used in the covariance calculations. We suggest two ways to remedy the

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[1]This is the class of unstable systems for which some eigenvalues $\{\underline{\phi}_\ell\}_{\ell=k}^{T-1}$ matrices are in absolute magnitude greater than one. (For a short but informative treatment of stability, see Kailath, 1980, Section 2.6.)

situation. One strictly concerns the single reservoir case and will be presented in Chapter 5 while the other will be briefly discussed now. As explained in Section 4.5, at each time k, the optimal Open Loop trajectory $\{\underline{u}_\ell^*\}_{\ell=k}^{T-1}$ is derived through the Dynamic Programming Algorithm A4,5.1. The controls are obtained by

$$\delta\underline{u}_\ell^* = -\alpha_i \underline{\underline{D}}_\ell[\underline{\underline{L}}_\ell \, \delta\bar{\underline{s}}_{\ell/k} + \underline{\Lambda}_\ell], \qquad (4.7.18)$$
$$\ell=k,\ldots,T-1$$

where α_i is the stepsize of the ith iteration, and $\{\underline{\underline{D}}_\ell, \underline{\underline{L}}_\ell, \underline{\Lambda}_\ell\}_{\ell=k}^{T-1}$ are the control gains defined in (4.5.34), (4.5.35), and (4.5.36). Equation (4.7.18) is a linear feedback function of the state and can be adopted as a first order approximation of the actual feedback laws $\{\underline{\mu}_\ell^*(\delta\underline{s}_\ell)\}_{\ell=k}^{T-1}$. (Recall that (4.7.18) solves a locally linearized approximation of the nonlinear problem.) The idea is to substitute (4.7.18) in the system dynamics and construct the following "Closed Loop" system representation:

$$\delta\underline{s}_{\ell+1} = \underline{\underline{\phi}}_\ell \delta\underline{s}_\ell + \underline{\underline{B}}_\ell[-\alpha_i \underline{\underline{D}}_\ell[\underline{\underline{L}}_\ell \, \delta\underline{s}_{\ell/k} + \underline{\Lambda}_\ell]] + \underline{\xi}_\ell =$$

$$= [\underline{\underline{\phi}}_\ell - \alpha_i \underline{\underline{B}}_\ell\underline{\underline{D}}_\ell\underline{\underline{L}}_\ell] \, \delta\underline{s}_\ell - \alpha_i\underline{\underline{B}}_\ell\underline{\underline{D}}_\ell\underline{\Lambda}_\ell + \underline{\xi}_\ell, \qquad (4.7.19)$$
$$\ell=k,\ldots,T-1.$$

The covariance propagation equation for the system (4.7.19) is given by

$$\underline{\underline{P}}_{s\ell+1} = [\underline{\underline{\phi}}_\ell - \alpha_i \underline{\underline{B}}_\ell\underline{\underline{D}}_\ell\underline{\underline{L}}_\ell] \, \underline{\underline{P}}_{s\ell/k} \, [\underline{\underline{\phi}}_\ell - \alpha_i \underline{\underline{B}}_\ell\underline{\underline{D}}_\ell\underline{\underline{L}}_\ell]^T + \underline{\underline{Q}}_{\xi\ell} \qquad (4.7.20)$$
$$\ell = k, \ldots T-1$$

and results in tighter variances than the ones obtained by Equation (4.7.5) (Athans, 1984).

The approach can be extended to the control constrained case by using the results of Algorithm A4.6.1 for the matrices $\{\underline{\underline{D}}_\ell, \underline{\underline{L}}_\ell\}_{\ell=k}^{T-1}$ (Equation 4.6.31). For Equation (4.7.20) to continue being valid, the only modification needed is to restore the original dimensions of the matrices by including the deleted rows and columns but with zero elements. In other words, the suggestion is to propagate the covariance in an Open Loop manner when the controls are binding and use the Closed Loop approach for the unconstrained segments.

4.7.1 Penalty Function and Multiplier Methods

The idea of the Penalty Function and Multiplier methods is to obtain the solution of a constrained problem by solving a sequence of unconstrained ones. Their validity is based on the fact that for well-posed problems the sequence of the unconstrained solutions converges to the solution of the constrained problem.

For Problem P.5 a Penalty Function method can be derived as follows:

Each two-sided state inequality constraint (4.7.21) can be broken up into two one-sided constraints:

$$\delta \underline{\bar{s}}_{j\ell}^{min} - \delta \underline{\bar{s}}_{j\ell/k} \leq 0, \tag{4.7.22}$$

$$\delta \underline{\bar{s}}_{j\ell/k} - \delta \underline{\bar{s}}_{j\ell}^{max} \leq 0, \tag{4.7.23}$$

$$j=1,\ldots,n_s, \quad \ell=k,\ldots,T$$

and these can be converted into equality constraints by introducing the non-negative variables $y_{j\ell}^{min}$, $y_{j\ell}^{max}$:

$$\delta\bar{s}_{j\ell}^{min} - \delta\bar{s}_{j\ell/k} + y_{j\ell}^{min} = 0, \quad y_{j\ell}^{min} \geq 0, \tag{4.7.24}$$

$$\delta\bar{s}_{j\ell/k} - \delta\bar{s}_{j\ell}^{max} + y_{j\ell}^{max} = 0, \quad y_{j\ell}^{max} \geq 0, \tag{4.7.25}$$

$$j=1,\ldots,n_s, \quad \ell=k,\ldots,T.$$

Now define the Lagrangian function

$$L_{C_i}(\delta\underline{u}_k,\ldots,\delta\underline{u}_{T-1},\underline{y}_k^{min},\ldots,\underline{y}_T^{min}, \underline{y}_k^{max},\ldots\underline{y}_T^{max}) =$$

$$= \tilde{J}(\delta\underline{u}_k,\ldots,\delta\underline{u}_{T-1}) + \sum_{j\ell} \frac{1}{2} C_i(\delta\bar{s}_{j\ell}^{min} - \delta\bar{s}_{j\ell/k} + y_{j\ell}^{min})^2 +$$

$$+ \sum_{j\ell} \frac{1}{2} C_i(\delta\bar{s}_{j\ell/k} - \delta\bar{s}_{j\ell}^{max} + y_{j\ell}^{max})^2 \tag{4.7.26}$$

where C_i is a positive penalty parameter and consider the following sequence of problems:

$$\underset{\substack{\{\delta\underline{u}_\ell\}_{\ell=k}^{T-1} \\ \{\underline{y}_\ell^{min},\underline{y}_\ell^{max}\}_{\ell=k}^{T}}}{\text{Minimize}} \quad L_{C_i}(\delta\underline{u}_k,\ldots,\delta\underline{u}_{T-1},\underline{y}_k^{min},\ldots,\underline{y}_T^{min},\underline{y}_k^{max},\ldots,\underline{y}_T^{max}) \tag{4.7.27}$$

subject to

a. $$\delta\underline{\bar{s}}_{\ell+1/k} = \underline{\underline{\phi}}_\ell \; \delta\underline{\bar{s}}_{\ell/k} + \underline{\underline{B}}_\ell \; \delta\underline{u}_\ell, \; \delta\underline{\bar{s}}_{k/k} = 0 \tag{4.7.28}$$

$$\ell=k,\ldots,T-1,$$

b. $\delta \underline{u}_\ell^{min} \leq \delta \underline{u}_\ell \leq \delta \underline{u}_\ell^{max}$, (4.7.29)

$\ell = k, \ldots, T-1$,

c. $\underline{y}_\ell^{min} \geq \underline{0}$, (4.7.30)

$\underline{y}_\ell^{max} \geq \underline{0}$, (4.7.31)

$\ell = k, \ldots, T$

where $i = 0,1,\ldots,$ $C_{i+1} > C_i > 0$ and $C_i \to \infty$.

If $(\delta \underline{u}^*, \underline{y}^*)_i$ is a global minimum of the above problem, then it can be shown that every limit point of the sequence $\{\delta \underline{u}^*\}_i$ is a global minimum of problem P.5. (For a proof in a general nonlinear programming context, see Luendberger, 1973, Section 12.1 or Bertsekas, 1982, Sections 2.1, 3.1). Intuitively, this result is expected to hold because for large C_i values the penalty terms $C_i(\delta \bar{s}_{j\ell/k} - \delta \bar{s}_{j\ell}^{max} + y_{j\ell}^{max})^2$ are minimized when $(\delta \bar{s}_{j\ell/k} - \delta \bar{s}_{j\ell}^{max} + y_{j\ell}^{max})$ approaches zero or equivalently when the constraint $\delta \bar{s}_{j\ell/k} - \delta \bar{s}_{j\ell}^{max} \leq 0$ is satisfied. (Similar argument holds for the lower bound penalties as well.

It is still questionable why one should choose to solve the sequence of the above problems rather than the original one. Notice, however, that for a particular state and control trajectory the minimization with respect to $\underline{y}^{min}$ and $\underline{y}^{max}$ can be explicitly performed. Consider for instance, minimizing the Lagrangian in (4.7.26) with respect to $y_{j\ell}^{min}$ subject to $y_{j\ell}^{min} \geq 0$. This is equivalent to

$$\underset{y_{j\ell}^{min}}{\text{minimize}} \quad \{\tfrac{1}{2}C_i(\delta \bar{s}_{j\ell}^{min} - \delta \bar{s}_{j\ell/k} + y_{j\ell}^{min})^2\} \tag{4.7.32}$$

subject to $y_{j\ell}^{min} \geq 0$. Since (4.7.28) is a convex quadratic function with respect to $y_{j\ell}^{min}$, the solution is

$$(y_{j\ell}^{min})^* = \begin{cases} -(\delta\bar{s}_{j\ell}^{min} - \delta\bar{s}_{j\ell/k}), & \text{if } \delta\bar{s}_{j\ell}^{min} < \delta\bar{s}_{j\ell/k} \\ 0 & , \text{if } \delta\bar{s}_{j\ell}^{min} \geq \delta\underline{s}_{j\ell/k} \end{cases} \tag{4.7.33}$$

Similarly the optimal value of any $y_{j\ell}^{max}$ variable can be seen to be

$$(y_{j\ell}^{max})^* = \begin{cases} -(\delta\bar{s}_{j\ell/k} - \delta\bar{s}_{j\ell}^{max}), & \text{if } \delta\bar{s}_{j\ell}^{max} > \delta\bar{s}_{j\ell/k} \\ 0 & , \text{if } \delta\bar{s}_{j\ell}^{max} \leq \delta\bar{s}_{j\ell/k} \end{cases} \tag{4.7.34}$$

Substituting these results into the Lagrangian function we find

$$\begin{aligned} L_{C_i}(\delta\underline{u}_k,\ldots,\delta\underline{u}_{T-1}) &= \tilde{J}(\delta\underline{u}_k,\ldots,\delta\underline{u}_{T-1}) + \\ &+ \sum_{j\ell} \tfrac{1}{2}C_i[\max\{0,(\delta\bar{s}_{j\ell}^{min}-\delta s_{j\ell/k}^{-})\}]^2 + \\ &+ \sum_{j\ell} \tfrac{1}{2}C_i[\max\{0,\ (\delta\bar{s}_{j\ell/k}-\delta\bar{s}_{j\ell}^{max})\}]^2 \end{aligned} \tag{4.7.35}$$

and our problem becomes one of minimizing (4.7.35) with respect to $\{\delta\underline{u}_\ell\}_{\ell=k}^{T-1}$ subject to the dynamics (4.7.28) and the control magnitude constraints (4.7.29). The penalty terms in (4.7.35) are zero when the corresponding constraints are not violated while they prescribe quadratic costs whenever there is a violation. Notice, however, that this problem can be solved by the Projected Newton Algorithm A4.6.1. The only adjustment needed at each iteration is to determine

which of the state constraints (4.7.22) or (4.7.23) are currently violated and include in the objective function $\bar{J}(\delta\underline{u}_k,\ldots,\delta\underline{u}_{T-1})$ the corresponding quadratic and linear penalty terms. The procedure involves the following sequence of operations: For a level C_i of the penalty parameter use Algorithm A4.6.1 to find the minimizing control trajectory $\underline{u}^*(C_i)$; then increase C_i to $C_{i+1} > C_i$ and repeat the previous step until the sequence $\{\underline{u}^*(C_i)\}_i$ converges to the solution $\underline{u}^*$. For convex problems the method can also be operated by increasing C_i at each Projected Newton iteration, but in the general case this may cause failures.

The rate at which C_i should be increased can be determined by preliminary experimentation. Slow increment rate will result in slow convergence while an extremely fast rate will render the problem ill-conditioned. In the latter case a solution at some iteration may not exist or it may become increasingly difficult to obtain. However, our computational experience with the Penalty Function method in relation to reservoir operation problems shows that the method is quite reliable and that a penalty increment formula (see also Bertsekas, 1982, Chapter 2)

$$C_{i+1} = \beta C_i, \quad \beta\varepsilon[4,10] \tag{4.7.36}$$

performs well

Multiplier methods are theoretically characterized by better convergence properties than penalty function methods, yet in practise they are more difficult to implement.

For the control Problem P.5 the associated Lagrangian of a Multiplier method is defined by

$$L_{C_i}(\delta\underline{u}_k,\ldots,\delta\underline{u}_{T-1},\underline{y}_k^{min},\ldots,\underline{y}_T^{min},\underline{y}_k^{max},\ldots,\underline{y}_T^{max},\underline{\mu}_k^{min},\ldots,\underline{\mu}_T^{min},\underline{\mu}_k^{max},\ldots,\underline{\mu}_T^{max}) =$$

$$= \tilde{J}(\delta\underline{u}_k,\ldots,\delta\underline{u}_{T-1}) + \sum_{j\ell}\{\mu_{j\ell}^{min}(\delta\bar{s}_{j\ell}^{min}-\delta\bar{s}_{j\ell/k}+y_{j\ell}^{min})+\frac{1}{2}C_i(\delta\bar{s}_{j\ell}^{min}-\delta\bar{s}_{j\ell/k}+y_{j\ell}^{min})^2\}$$

$$+\sum_{j\ell}\{\mu_{j\ell}^{max}(\delta\bar{s}_{j\ell/k}-\delta\bar{s}_{j\ell}^{max}+y_{j\ell}^{max})+\frac{1}{2}C_i(\delta\bar{s}_{j\ell/k}-\delta\bar{s}_{j\ell}^{max}+y_{j\ell}^{max})^2\}$$

(4.7.37)

where $\{\underline{\mu}_\ell^{min},\underline{\mu}_\ell^{max}\}_{\ell=k}^T$ are the multiplier vectors and C_i is the penalty parameter as before. In the original method of multipliers (Hestenes, 1969), Problem P.5 can be solved as follows: First, for fixed values of $\underline{\mu}^{min,i}$, $\underline{\mu}^{max,i}$, C_i we

$$\underset{\substack{\{\delta\underline{u}_\ell\}_{\ell=k}^{T-1}\\ \{y_\ell^{min},y_\ell^{max}\}_{\ell=k}^T}}{\text{minimize}} \quad L_{C_i}(\delta\underline{u},\underline{y}^{min},\underline{y}^{max},\underline{\mu}^{min,i},\underline{\mu}^{max,i})$$

subject to

a. $\delta\bar{\underline{s}}_{\ell+1/k} = \underline{\underline{\phi}}_\ell\,\delta\bar{\underline{s}}_{\ell/k} + \underline{\underline{B}}_\ell\,\delta\underline{u}_\ell$, $\delta\bar{\underline{s}}_{k/k} = \underline{0}$,

$\ell=k,\ldots,T-1$,

b. $\delta\underline{u}_\ell^{min} \le \delta\underline{u}_\ell \le \delta\underline{u}_\ell^{max}$,

$\ell=k,\ldots,T-1$,

c. $y_\ell^{min} \geq \underline{0}$,

$y_\ell^{max} \geq \underline{0}$,

$\ell = k,\ldots,T-1$

As in the Penalty Function method, this problem can be explicitly solved with respect to $\{\underline{y}_\ell^{min}, \underline{y}_\ell^{max}\}_{\ell=k}^{T}$ since minimization of L_{C_i} with respect, say, to $y_{j\ell}^{min}$ is equivalent to

$$\text{minimizing}\{\mu_{j\ell}^{min,i}(\delta\bar{s}_{j\ell}^{min} - \delta\bar{s}_{j\ell/k} + y_{j\ell}^{min}) + \tfrac{1}{2}C_i(\delta\bar{s}_{j\ell}^{min} - \delta\bar{s}_{j\ell/k} + y_{j\ell}^{min})^2\} \tag{4.7.38}$$

subject to $y_{j\ell}^{min} \geq 0$.

The minimum of this subproblem is either the point

$$y_{j\ell}^{min,i} = -(\delta\bar{s}_{j\ell}^{min} - \delta\bar{s}_{j\ell/k}) - \frac{\mu_{j\ell}^{min,i}}{C_i} \tag{4.7.39}$$

minimizing the parabola (4.7.34), or

$$y_{j\ell}^{min,i} = 0, \tag{4.7.40}$$

if the previous point is a negative number. More compactly we can write

$$y_{j\ell}^{min,i} = \max\{0, -(\delta\bar{s}_{j\ell}^{min} - \delta\bar{s}_{j\ell/k}) - \frac{\mu_j^{min,i}}{C_i}\}. \tag{4.7.41}$$

Similarly for the case of upper state constraints, we find

$$y_{j\ell}^{max,i} = \max\left\{0, -[\delta\bar{s}_{j\ell/k} - \delta\bar{s}_{j\ell}^{max})] - \frac{\mu_{j\ell}^{max,i}}{C_i}\right\} \tag{4.7.42}$$

Substituting (4.7.41) and (4.7.42) in the Lagrangian, the previous problem reduces to the one that follows:

For fixed $\underline{\mu}^{min,i}$, $\underline{\mu}^{max,i}$, C_i

$$\underset{\{\delta\underline{u}_\ell\}_{\ell=k}^{T-1}}{\text{minimize}} \quad \tilde{J}(\delta\underline{u}_k, \ldots, \delta\underline{u}_{T-1}) +$$

$$+ \sum_{j\ell}\left[\mu_j^{min,i}\left(\max\left\{\delta\bar{s}_{j\ell}^{min} - \delta\bar{s}_{j\ell/k}, -\frac{\mu_{j\ell}^{min,i}}{C_i}\right\}\right) + \frac{1}{2}C_i\left(\max\left\{\delta\bar{s}_{j\ell}^{min} - \delta\bar{s}_{j\ell/k}, -\frac{\mu_{j\ell}^{min,i}}{C_i}\right\}\right)^2\right]$$

$$+ \sum_{j\ell}\left[\mu_{j\ell}^{max,i}\left(\max\left\{\delta\bar{s}_{j\ell/k} - \delta\bar{s}_{j\ell}^{max}, -\frac{\mu_{j\ell}^{max,i}}{C_i}\right\}\right) + \frac{1}{2}C_i\left(\max\left\{\delta\bar{s}_{j\ell/k} - \delta\bar{s}_{j\ell}^{max} - \frac{\mu_{j\ell}^{max,i}}{C_i}\right\}\right)^2\right]$$

subject to

a. $\delta\underline{\bar{s}}_{\ell+1/k} = \underline{\underline{\phi}}_\ell\, \delta\underline{\bar{s}}_{\ell/k} + \underline{\underline{B}}_\ell\, \delta\underline{u}_\ell$, $\delta\underline{\bar{s}}_{k/k} = \underline{0}$

$\ell = k, \ldots, T-1$

b. $\delta\underline{u}_\ell^{min} \leq \delta\underline{u}_\ell \leq \delta\underline{u}_\ell^{max}$

$\ell = k, \ldots, T-1.$

The penalty terms in the objective function of this problem become

$$\left.\begin{aligned} &\mu_{j\ell}^{min,i}(\delta\bar{s}_{j\ell}^{min} - \delta\bar{s}_{j\ell/k}) + \frac{1}{2}C_i(\delta\bar{s}_{j\ell}^{min} - \delta\bar{s}_{j\ell}^{min})^2, \\ &\mu_{j\ell}^{max,i}(\delta\bar{s}_{j\ell/k} - \delta\bar{s}_{j\ell}^{max}) + \frac{1}{2}C_i(\delta\bar{s}_{j\ell/k} - \delta\bar{s}_{j\ell}^{max})^2 \end{aligned}\right\} \tag{4.7.43}$$

when the corresponding constraints are violated and

$$\left.\begin{array}{l} -\frac{1}{2}(\mu_{j\ell}^{min,i})^2/C_i \\ -\frac{1}{2}(\mu_{j\ell}^{max,i})^2/C_i \end{array}\right\} \qquad (4.7.44)$$

when they are not. In any case this is a problem that can be efficiently solved by the Projected Newton Algorithm A4.6.1. Having obtained this solution, the procedure continues by updating the multipliers and the penalty parameter and repeating the previous step. The new multipliers can be specified by

$$\left.\begin{array}{l} \mu_{j\ell}^{min,i+1} = \mu_{j\ell}^{min,i} + C_i \max\{-(\mu_{j\ell}^{min,i}/C_i),(\delta\bar{s}_{j\ell}^{min} - \delta\bar{s}_{j\ell/k})\} \\ \mu_{j\ell}^{max,i+1} = \mu_{j\ell}^{max,i} + C_i \max\{-(\mu_{j\ell}^{max,i}/C_i),(\delta\bar{s}_{j\ell/k} - \delta\bar{s}_{j\ell}^{max})\} \\ j=1,\ldots,n_s,\ \ell=k,\ldots,T \end{array}\right\} \qquad (4.7.45)$$

and C_{i+1} can be obtained as in the penalty function method.

The main advantage of this over the Penalty Function method is related to the problem of "ill-conditioning" which sometimes occurs when very high penalty parameter values are needed to induce convergence. In the method of multipliers, owing to the multiplier iteration (4.7.45), convergence is brought about at moderate C_i values avoiding the previous problem and exhibiting a faster convergence rate. These benefits, however, are realized only if the initial multiplier values are close to the ones at the optimal point. In the event that the initial guesses are "bad", the method may diverge or take too many interations to converge. We have found that in optimal control problems, where there are many binding state constraints, guessing good initial multiplier values can be a formidable assignment. For this reason and

provided that there are no "ill-conditioning" problems, the simpler Penalty Function method is also safer to use.

This section discussed procedures for handling probabilistic state constraints and completed the third basic step in the design of the controller. A last issue concerning the appropriate specification of the control horizon T in relation to the Continuing Operations system requirement is the topic of this chapter's concluding section.

4.7.2. The Continuing Operations System Requirement

Reservoir systems are expected to satisfactorily operate throughout a long-time span. In fact the actual length of the operating horizon depends on a variety of physical and socioeconomic factors and cannot be a priori determined. At the same time mathematical formulations aiding month-to-month or day-to-day operation should be well defined and computationally efficient. Long-term optimal performance can be guaranteed by optimizing the system model over a control horizon T which is of the same order of magnitude as the system's expected lifetime. However, taking T longer will result in heavier computational requirements and this may render the control procedure inefficient. If the controller was one of the optimal feedback types presented in Section 4.4, there would be no other choice; the parameter T would have to represent an estimate of the system's operational life. In contrast, the OLF control procedure does not seek to a priori determine the best feedback laws at all decision times. Rather, it seeks to specify the best controls $\underline{u}_k^*$, $k=0,\ldots,T-1$ one at a time. Within this framework, the best T choice is evidently to consider the shortest control horizon

that reveals the optimal decision $\underline{u}_k^*$ for the immediately upcoming time period. In other words, if this optimal terminal time is denoted by T_k, then $[k, T_k]$ is the shortest interval $[k,T]$ for which the first member of the OLF optimal trajectories $\{\underline{u}_k^*, \ldots, \underline{u}_{T-1}^*\}$ has settled to a constant vector. The fact that $\underline{u}_k^*$ remains constant for T greater than some characteristic value can be formally proved by use of Contraction Mapping theory (Bertsekas, 1976, Chapter 6, and specifically, Section 6.7). This behavior also makes intuitive sense because system conditions in the remote future are not expected to influence present decision making.

Among other factors the length $[k,T_k]$ depends on the specification of the terminal cost term $\ell_{T_k}(\delta s_{T_k})$ or $\tilde{\ell}_{T_k}(\delta \bar{\underline{s}}_{T_k/k})$. (The other factors are related to the inputs' variability and periodicities, the state and control constraints, and the rest of the performance index, all of which are specified for a particular system.) The term $\tilde{\ell}_{T_k}(\cdot)$ allows one to drive the system's state to any desirable vector $\bar{\underline{s}}_{T_k/k}^*$ at the end of the control horizon. This is accomplished by penalizing the deviations of the terminal time state away from $\bar{\underline{s}}_{T_k/k}^*$. If the penalty is much higher than all the other costs $\sum_{\ell=k}^{T-1} \tilde{\ell}_\ell(\delta \bar{\underline{s}}_{\ell/k}, \delta \underline{u}_\ell)$, then the controller will first make sure that on the average the terminal state is close to $\bar{\underline{s}}_{T_k/k}^*$ and next will be concerned with minimizing $\sum_{\ell=k}^{T-1} \tilde{\ell}_\ell(\delta \bar{\underline{s}}_{\ell/k}, \delta \underline{u}_\ell)$. Yet, if the penalty is not significant, the generated trajectories will minimize $\sum_{\ell=k}^{T-1} \tilde{\ell}_\ell(\delta \bar{\underline{s}}_{\ell/k}, \delta \underline{u}_\ell)$ paying no attention to where the terminal state vector will end up. As the terminal system's state will also

determine the initial conditions for subsequent operations, correct specification of $\tilde{\ell}_{T_k}(.)$ is in general instrumental to the long-term system performance. However, owing to its structure, the Open Loop Feedback controller is not so critically dependent upon these specifications. In this procedure using some incorrect $\tilde{\ell}_{T_k}(\cdot)$ will only prolong the control horizon $[k,T_k]$ required to reveal the optimal $\underline{u}_k^*$. One can then determine T_k by assuming a reasonable $\tilde{\ell}_{T_k}(\cdot)$ and trying progressively longer control horizons $[k,T]$. T_k will be the time for which the first decision $\underline{u}_k^*$ remains constant in all optimal trajectories $\{u_k^*,\ldots,u_{T-1}^*\}$ for $T \geq T_k$.

For cases where the above procedure fails to identify relatively short control horizons, one might consider answering a more direct question: namely, how to determine $\tilde{\ell}_{T_k}(.)$ which corresponds to some prespecified T_k. Toward this end consider the problem of:

$$\underset{\{\delta\underline{u}_\ell\}_{\ell=k}^{T-1}}{\text{minimizing}} \quad \sum_{\ell=k}^{T-1} \tilde{\ell}_\ell(\delta\bar{\underline{s}}_{\ell/k},\delta\underline{u}_\ell) + \ell_T(\delta\bar{\underline{s}}_{T/k}) \tag{4.7.46}$$

subject to all system constraints where T is much greater than T_k and $\tilde{\ell}_T(\cdot)$ is dominated by the other running costs. According to the Dynamic Programming Algorithm A4.4.1, this problem can be solved in two steps where in the first we find the minimum:

$$J_{T_k}(\delta\bar{\underline{s}}_{T_k/k}) = \sum_{\ell=T_k}^{T-1} \tilde{\ell}_\ell(\delta\bar{\underline{s}}_{T_k/k},\delta\underline{u}_\ell) + \tilde{\ell}_T(\delta\bar{\underline{s}}_T) \tag{4.7.47}$$

solving the dynamic program from time T to time T_k and in the second we

$$\underset{\{\delta \underline{u}_\ell\}_{\ell=k}^{T_k-1}}{\text{minimize}} \quad \sum_{\ell=T_k}^{T_k-1} \tilde{\ell}_\ell(\delta \bar{\underline{s}}_{\ell/k}, \delta \underline{u}_\ell) + J_{T_k}(\delta \bar{\underline{s}}_{T_k/k}) \tag{4.7.47}$$

subject to all related system constaints. From Eq. (4.7.48) we conclude that the terminal cost term should approximate the "cost to go" $J_{T_k}(\cdot)$. In practice $J_{T_k}(\cdot)$ will have to be obtained in discretized form by solving (4.7.47) for different initial state vectors $\delta \bar{\underline{s}}_{T_k/k}$. However, these computations can be performed off-line (a priori) and the resulting $J_{T_k}(\cdot)$ can be approximated by some analytical function which will serve as the terminal cost term for all on-line (real time) operations.

The previous discussion completes the design of the Open Loop Feedback controller for the solution of the reservoir operation problem. Algorithm A4.6.1 modifed by the quadratic penalty terms from the probabilistic constraint violations is used at each decision time k=0,1,... to identify the optimal action $\underline{u}_k^*$ to be applied throughout the period $[t_k, t_{k+1}]$. (If it fails to do so, the conclusion to be drawn is that the system cannot control the input process so as to meet its objectives at the specified reliability levels. These levels should then be relaxed and the procedure repeated.) Between such computation cycles, measurements $\underline{z}_k$ of the observable system quantities become available and can help update the statistics of the state's Gaussian p.d.f. $p(\underline{s}_k/I_k)$. The equations for the update step (Extended Kalman Filter) are included in Appendix G.

The method developed here will be called Extended Linear Quadratic Gaussian control (ELQG) because it shares similar analytical

characteristics with the Linear Quadratic Gaussian control solution and in addition it can be employed to nonlinear, constrained, and non-quadratic problems. The following chapters report computational experience in reservoir operation case studies.

CHAPTER 5

A SINGLE RESERVOIR CASE STUDY

5.1 Introduction and Overview

The ELQG control method was theoretically designed to exhibit reliability and computational efficiency for addressing dimensionally large systems. The case study presented in this chapter will be a first verification and test of the new method's potential in real time decision making. It concerns the operation of Egypt's High Aswan Dam for more efficient use of the River Nile waters.

After discussing and formulating the control problem, ELQG will be employed in several experimental runs revealing various performance aspects. The runs will also identify certain system idiosyncracies which are subsequently exploited in setting up an appropriate sequential control scheme. A record of historical inflow data will then be used in a simulation analysis to compare ELQG's performance with other control methods. The chapter will conclude with reference to some system specific policy-making issues.

5.2 The High Aswan Dam: Hydrologic and Operational Characteristics

Calmly flowing from the Equatorial Lakes or turbulently from the Ethiopian highlands, the River Nile has historically been the only source of livelihood to Egypt. Being so vital, the Nile has long become the subject of studies seeking to modulate its markedly variable flows and create a source of dependable water supply. These efforts have begun nearly 6,000 years ago with the construction of a masonry dam about 20 km to the south of Cairo (El Assiouti et al., 1979) and culminated in 1968 with the completion of the High Aswan Dam (HAD) located by the Egyptian-Sudanese borders (see Figure 5.2.1).

Egypt experiences a semi-arid climate and except for a narrow coastal zone to the north of the Nile's Delta receives virtually no rain. A groundwater aquifer existing below the Delta region is solely recharged with seepage water from the river and, consequently, the Nile is Egypt's only source of fresh water supply. Figure 5.2.2 gives the average monthly Nile flows based on observed data over the period 1912 to 1965 at Wadi Halfa (see Figure 5.2.1 and Appendix H). The apparent seasonal variability is a consequence of the different hydrologic responses characterizing the major Main Nile tributaries, namely, the White and the Blue Nile. The first, contributing about one-third of the yearly total, flows evenly throughout the year, while the second is responsible for the violent floods arriving in August, September, and October. Apart from this pronounced variation within a year, the Nile also exhibits significant overyear variability. During high flood years, the water passing by Wadi Halfa may reach 140 milliard cubic metters (1 milliard m^3 = 1 billion m^3 = 10^9 m^3)

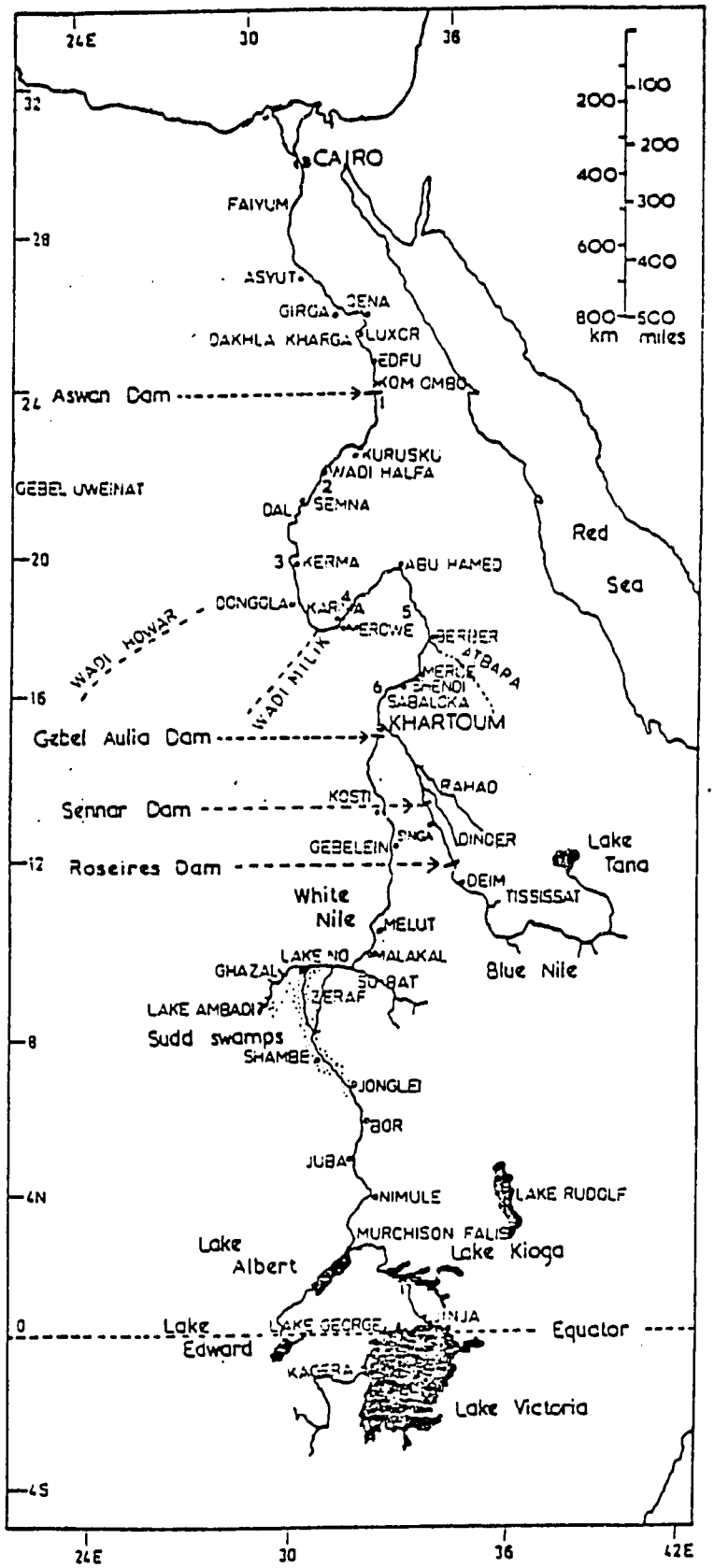

FIGURE 5.2.1: A MAP OF THE NILE BASIN

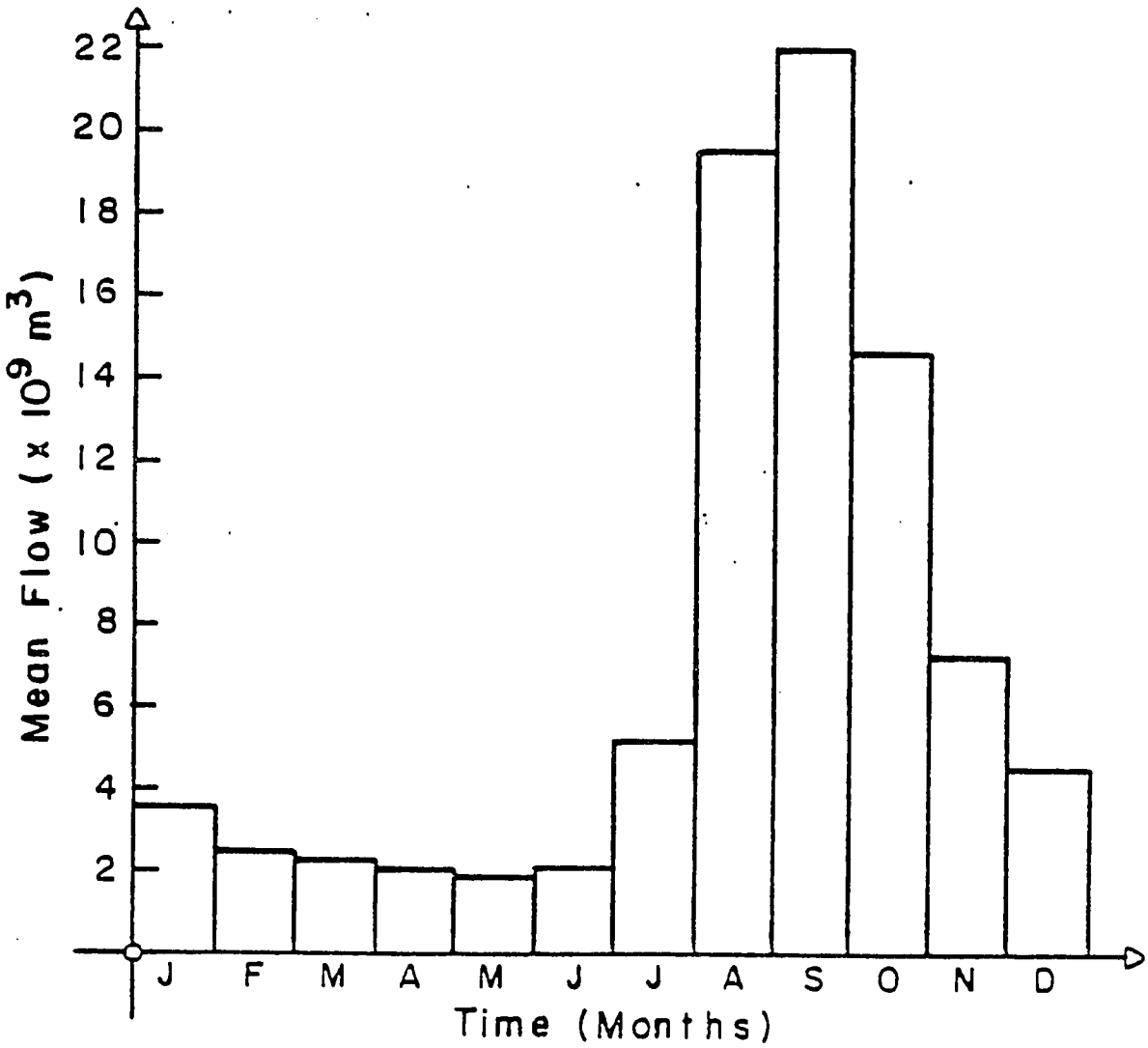

FIGURE 5.2.2: MONTHLY MEAN NILE FLOW (WADI HALFA)

while during drought periods, it may fall to 50 milliard cubic meters, the historical average being approximately 84 (Buchanan and Bras, 1981). Table 5.2.1 summarizes these characteristics by reporting the monthly means, standard deviations, correlation coefficients, minimum, and maximum observed flows calculated on the basis of the 1912-1965 data record included in Appendix H.

The High Aswan Dam is by far the most effective control project in the Nile basin. It has a storage capacity of 168 milliard cubic meters, 32.7 of which comprise its dead storage (see Figure 5.2.3) allocated for silt deposits and estimated to last 460-500 years. Water can be released at a controllable rate through a diversion channel which eventually divides into 24 branches. Twelve of these branches are feeding the 12 turbine power plants while the rest are designed to bypass the turbines and, if necessary, discharge up to 1 milliard cubic meters of water per day. However, the downstream river channel and water distribution network cannot transport releases higher than 0.25 to 0.275 milliard m^3/d for fear of severe damages due to bank erosion. To alleviate the possibility of excessive releases, the Toshka spillway was constructed on the western bank of the reservoir's lake (Lake Nasser). The spillway is a free-flow channel operating when the water elevation exceeds 178 m (above sea level) or equivalently when the storage exceeds the volume of 137.7 milliard m^3. It is designed to dispose of 0.25 milliard m^3 per day when the reservoir elevation reaches 182.6 m. The water is directed into the Toshka depression where it evaporates. An emergency spillway is also situated on the reservoir's western bank and it begins to operate at 182.6 meters water

TABLE 5.2.1: MONTHLY NILE FLOW STATISTICS (WADI HALFA)

Month	Expected Value ($x10^9$ m^3)	St. Deviation ($x10^9$ m^3)	Correlation Coefficient	Minimum ($x10^9$ m^3)	Maximum ($x10^9$ m^3)
Jan.	3.507	0.734	0.9080	2.040	5.750
Feb.	2.452	0.681	0.7503	1.420	5.080
Mar.	2.275	0.661	0.8649	1.260	4.810
Apr.	2.042	0.685	0.8714	1.050	4.540
May	1.924	0.750	0.7953	0.880	4.340
June	2.073	0.772	0.4955	1.000	4.520
July	5.170	1.516	0.6377	2.230	10.000
Aug.	19.448	3.821	0.7115	7.680	27.100
Sept.	21.991	3.745	0.7851	13,400	31.700
Oct.	14.605	3.125	0.8070	7.860	24.200
Nov.	7.166	1.752	0.8948	4.140	12.200
Dec.	4.538	0.857	0.9206	2.990	7,060

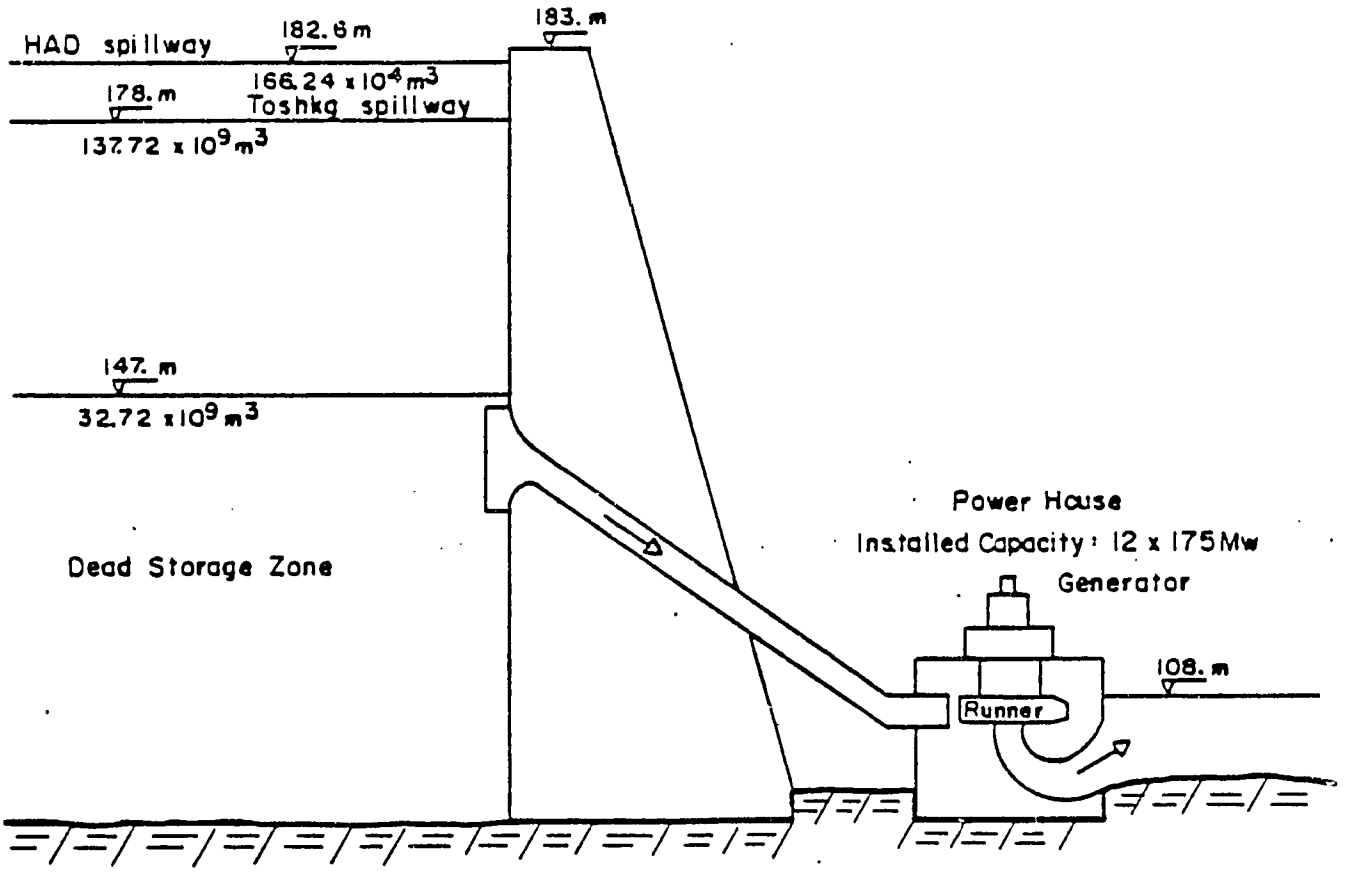

FIGURE 5.2.3: THE HIGH ASWAN DAM

level. Its purpose is to discharge in the downstream river channel all flood water that would raise the water level above 183.0 m.

Due to the semi-arid climate, HAD suffers heavy evaporation losses. The distribution of the average monthly evaportation rates presented in Figure 5.2.4 (Buchanan and Bras, 1981) results in a mean annual value of 2700 millimeters (mm). Multiplied by the yearly average surface area, this rate causes the loss of 10 to 15 milliard m^3 of water per year. Other losses of smaller magnitude are due to seepage and rock saturation (El Assiouti et al. 1979).

The primary operational objective of the HAD is to satisfy the Egyptian agricultural, municipal, and industrial water needs. On a monthly basis these downstream water supply requirements follow the distribution shown in Figure 5.2.5. They amount to 55.5 milliard m^3 per year. Comparing the previous distribution with that of the inflows (Figure 5.2.1), the HAD important role as a regulation project becomes apparent.

The other primary objective of the operation is to provide flood protection which, as mentioned, translates to preventing channel degradation.

Lastly but very importantly, HAD is the major energy supplier to the Egyptian Power system and currently satisfies about half of its total demand. Each of the twelve Francis turbines has 175 Megawatt power capacity and can operate under net hydraulic head ranging between 35 and 77 meters. Assuming a tailwater elevation of 108 m (see Figure 5.2.3) and neglecting hydraulic friction losees, it follows that the

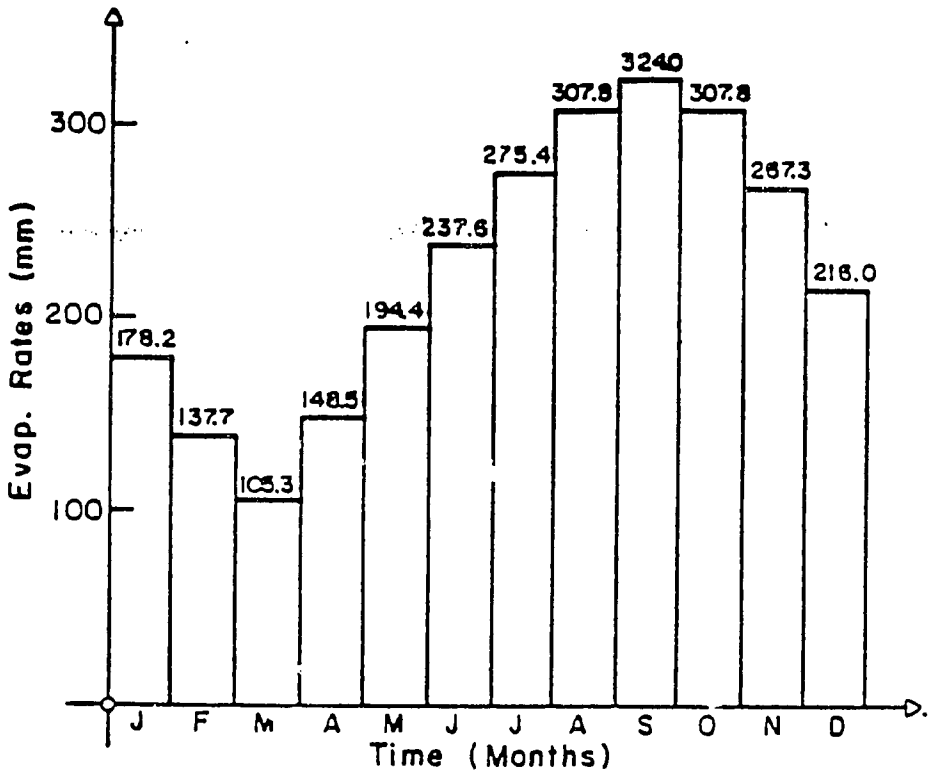

FIGURE 5.2.4: HAD MONTHLY AVERAGE EVAPORATION RATES

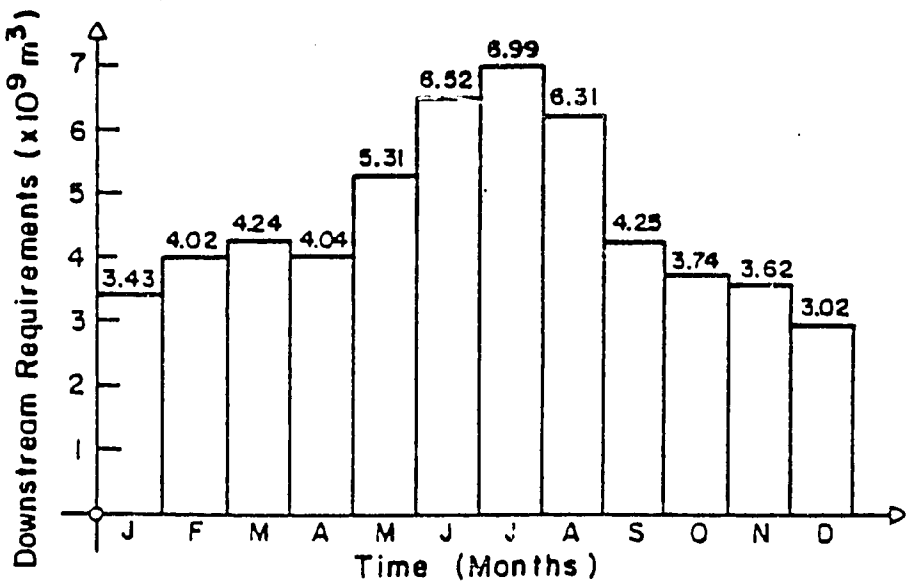

FIGURE 5.2.5: MONTHLY DOWNSTREAM WATER SUPPLY REQUIREMENTS

the turbines can operate between water levels 143 and 185 meters; that is, at all levels above the dead storage zone (147 m). Figure 5.2.6 (adopted from El Assiouti et al., 1979) shows the nominal relationship of the power with the discharge and net hydrualic head. The actual energy production is also a function of the load factor an average value which is 0.70. For maintenance purposes two turbines are always inoperative, and this reduces the available power capacity to 1750 Mw. Considering energy generation only, HAD should be operated at high reservoir elevations with uniform releases throughout the year (base power station). However, this mode of operation is in conflict with the water supply and flood control objectives for uses such as agriculture and flood control which suggests more detailed analysis.

The River Nile is also of great importance for the country of Sudan. By the Nile Water Agreement (1959) Egypt and Sudan established the following rights on the Nile waters: From the 84 milliard m^3 yearly average at Wadi Halfa, Egypt is entitled to use 55.5 milliard m^3 while Sudan's share is 18.5 milliard m^3. The remaining 10 milliard m^3 correspond to the yearly losses from the reservoir and cannot be used. Figure 5.2.7 gives the estimated monthly Sudan abstractions in percentages of the yearly total. This information is necessary for the 1912-1965 historical data to simulate present or future conditions because the record assumes very little or no Sudan abstractions. The agreement also determines that each country accrues equal shares from the water benefits of proposed upstream conservation projects. The projects (Jonglei I, Machar Marshes, Jonglei II, Bahr el Ghazal) are expected to increase the

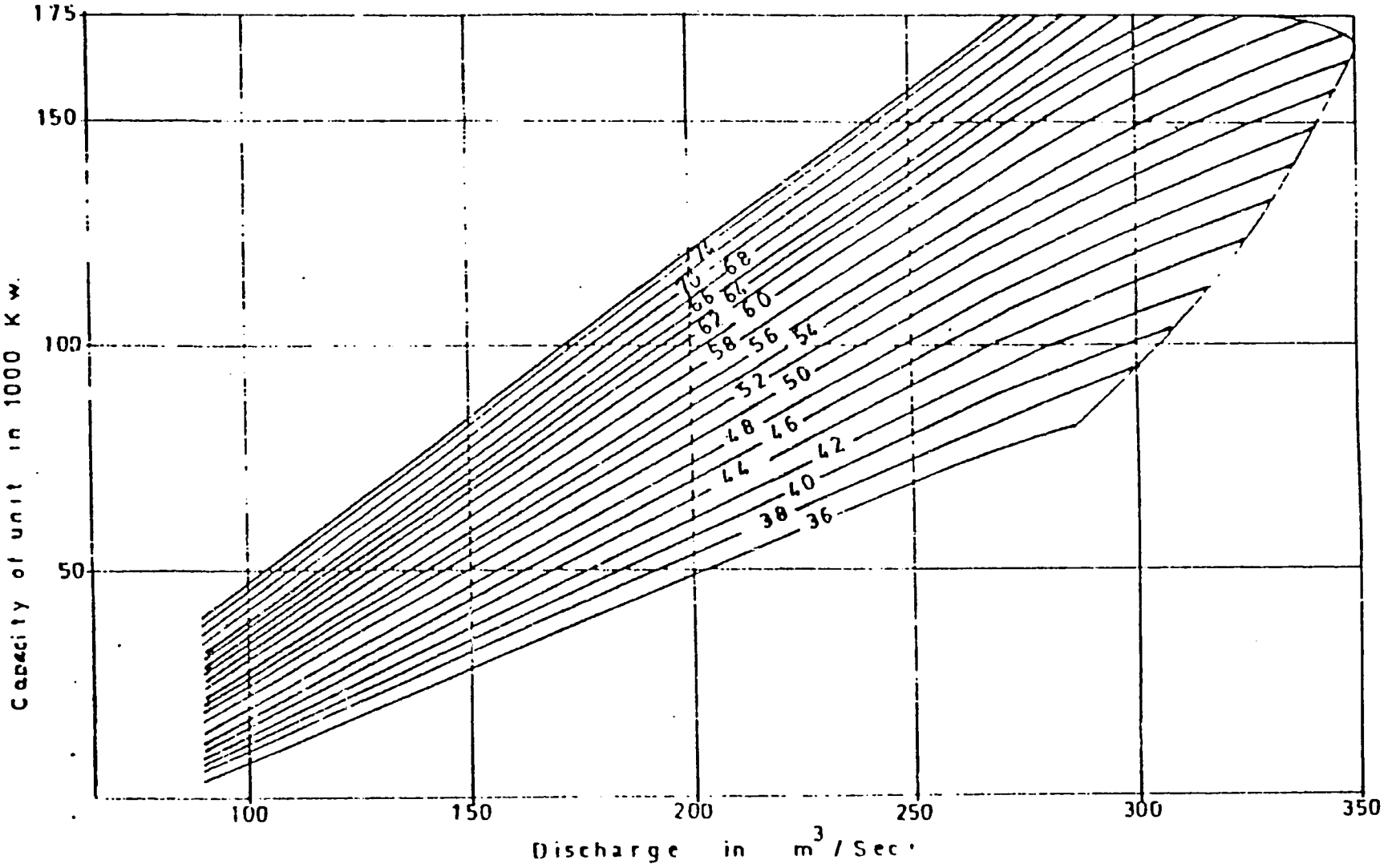

FIGURE 5.2.6: CAPACITY, HEAD AND DISCHARGE RELATIONSHIPS OF THE TURBINES

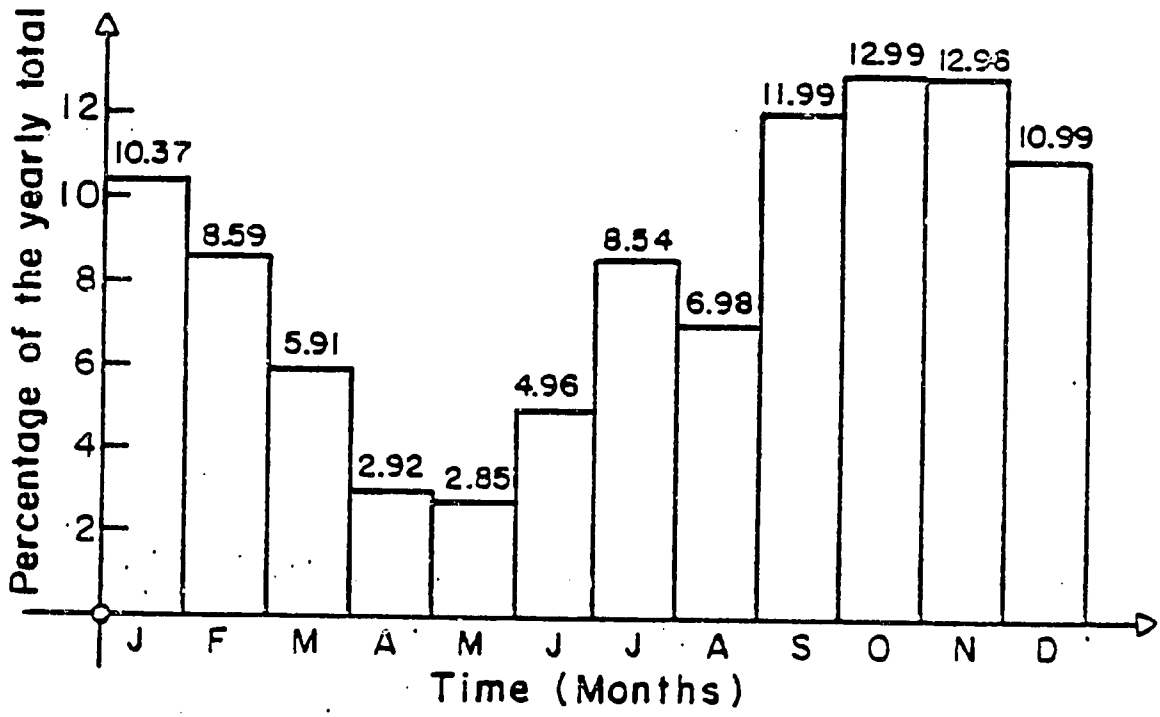

FIGURE 5.2.7: MONTHLY SUDAN ABSTRACTIONS AS PERCENTAGES OF THE YEARLY TOTAL

Nile's yearly average yield at Wadi Halfa by about 19 milliard m^3 (UNDP TR 14) adding another 9.5 milliard m^3 to each country's share. These additional quantities are necessary to counter balance the expected water shortages from intensive development scenarios in Egypt and Sudan.

In summary, the HAD operation calls for satisfying the growing downstream water supply requirements, providing reliable flood protection, and producing as much energy as possible under changing hydrologic conditions. With regard to flood control, it could be argued that the construction of the Toshka spillway has essentially alleviated any associated threat. This is indeed correct; however, it also has created another serious concern - that is, to avoid the Toshka spills as much as possible so that the otherwise wasted water can be used for energy generation and water supply purposes within the constraints imposed by channel degradation. In an era when the water availability constrains the country's development, Egypt can use all additional benefits resulting from a more effective HAD management.

5.3 Previous Studies on the HAD

Classified by the adopted optimization procedure, the studies on the HAD can be distinguished into Linear and Dynamic Programming models.

The models of the first category (Thomas and Revelle, 1966; Guariso, et al. 1979, 1980) assume a deterministic inflow sequence (average or historical monthly flows) and a linear water balance reservoir-model (fixed evaporation losses), and they attempt to determine the tradeoff curve between agriculture and hydropower. For a system of highly varying, nonstationary flows with multi-year control capability, this approach can produce questionable results and should not be used in real time operation planning. The reason for this is that the Pareto Optimal solutions identified on the basis of suboptimal modeling assumptions most likely belong to the inferior solution set of the true stochastic problem.

The Dynamic Programming models (El Assiouti, et al., 1979; Alarcon and Marks, 1979; Buchanan and Bras, 1981; Thompson, 1981) incorporate the river flow uncertainty through a Markov chain representation. For the control of a single reservoir in monthly time increments, the basic procedure is as follows:

Denote

m: the number of complete years remaining until the end of the control horizon

n: the number of months remaining until the end of the control horizon,

t: the number of months remaining in the current year (notice that if n is the present month, $n+1$ is the preceding one and that $n = 12m + t$)

s_n^i: the ith discrete value of the reservoir storage at the beginning of month n, $i = 1, 2, \ldots, I$,

w_n^j: The jth discrete value of the river flow during month n, $j = 1, 2, \ldots, J$,

p_n^{jk}: The probability that inflow w_n^k will occur given that w_{n+1}^j has just been realized.

The river flows are assumed to follow a lag one periodically stationary Markov chain where

$$p_{12m+t}^{jk} = p_t^{jk}, \quad \text{all } j, k, m, t. \tag{5.3.1}$$

The system incurs a cost $g_n(s_n^i, s_n^j, u_n)$ when the beginning of the month storage was s_n^i, the inflow w_n^j is realized and the amount u_n is released. The cost functions are also periodically stationary in the following sense:

$$g_{12m+t}(s_{12m+t}^i, w_{12m+t}^i, u_{12m+t}) = g_t(s_t^i, w_t^i, u_t), \text{ all } i,j, m,t. \tag{5.3.2}$$

It is assumed that both variables s_n and w_{n+1} are perfectly known at the end of month $n+1$, and the objective is to determine the sequence of feedback control laws

$$u_n^* = \mu_n^*(s_n^i, w_{n+1}^i), \text{ all } n, i, j \tag{5.3.3}$$

which minimize the expected cost

$$E\{ \sum_{\ell=0}^{T} g_\ell(s_\ell, w_\ell, u_\ell)\} \tag{5.3.4}$$

over a control horizon T subject to the system dynamics

$$s_{n-1}^k = s_n^i + w_n^j - u_n - L_n(s_n^i, w_n^i, u_n) \tag{5.3.5}$$

and the other system constraints on the reservoir release

$$u_n^{min} \leq u_n \leq u_n^{max}, \text{ all } n$$

The term $L_n(.,.,.)$ in Equation (5.3.5) represents the various reservoir losses and it is also assumed to be periodically stationary. The storage capacity constraints $s_n^{min} \leq s_n \leq s_n^{max}$ are accounted for when quantizing the range $[s_n^{min}, s_n^{max}]$ in I discrete levels.

The solution is obtained through the D.P. recursive equation which specifies that for each month the optimal policy is determined by

$$\mu_n^*(s_n^i, w_{n+1}^j) = \arg\{\min_{u_n} \{ \sum_k p_n^{jk}[g_n(s_n^i, w_n^k, u_n) + J_{n-1}(s_{n-1}^\ell, w_n)]\} \tag{5.3.7}$$

where s_{n-1}^ℓ is given by Equation (5.3.5). The "cost to go" $J_{n-1}(s_{n-1}^\ell, w_n^k)$ is obtained recursively via

$$J_n(s_n^i, w_{n+1}^j) = \sum_k p_n^{jk}[g_n(s_n^i, w_n^k, \mu_n^*(s_n^i, w_{n+1}^j)) + J_{n-1}(s_{n-1}^\ell, w_n^k)] \tag{5.3.8}$$

starting from some terminal condition $J_0(\cdot,\cdot)$.

This is the stationary or Steady State Dynamic Programming model (SSDP), which for an adequately long control horizon yields optimal policies also exhibiting periodic stationarity ·

$$\mu^*_{12m+t}(s^i_{12m+t}, w^j_{12m+t+1}) = \mu^*_t(s^i_t, w^j_{t+1}), \text{ all } i,j,m,t. \qquad (5.3.9)$$

Apart from the straightforward D.P. recursion, the stationary policies can be obtained more efficiently via the methods of Successive Approximations (Su and Deininger, 1972), or Policy Iteration (Howard, 1970; Bertsekas, 1976).

Due to its stationarity assumptions, SSDP can beneficially characterize the average system behavior. Toward extending its applicability to nonstationary systems, the modification suggested by Buchanan and Bras, 1981, has proved effective. According to their procedure, a multi-lag stochastic model fitted to the inflow process (Curry and Bras, 1980) uses real time observations to issue forecasts at each time step of the system's operation. The forecasts update the Markovian transition probabilities T_F months into the future as long as there is a significant gain over the a priori estimates. An ordinary D.P. procedure then determines the optimal policies from time T_F up to the present using SSDP's "cost to go" $J_{T_F}(\cdot,\cdot)$. Next, the current month's control action is taken and the procedure is repeated at the following decision time. This model, structured according to the Open Loop Feedback control philosophy, will be referred to as Adaptive Markov Dynamic Programming (AMDP). Having the ability to more accurately anticipate future

conditions, AMDP was found to exhibit improved performance with respect to SSDP, especially at times when the reservoir's storage approaches critical flood or drought levels. The idea behind utilizing forecasted information is the basis of an efficient suboptimal control procedure known as Partial Open Loop Feedback Control (Bertsekas, 1976, Section 5.5) and will also be exploited in this work in a different setting.

Inherent disadvantages of the previous D.P. procedures are (1) their requirement to discretize the reservoir storage and inflow variables, and (2) their inability to explicitly satisfy reliability constraints. As a result of the first, the identified policies are approximations of the true optimal ones with the "goodness" of the approximation directly dependent upon the "fineness" of the quantization. As a result of the second, the above procedures cannot reconstruct the Pareto Optimal Surface discussed in Section 4.2. Lastly, their application to multireservoir cases is prohibited by "dimensionality" limitations.

5.4 Extended Linear Quadratic Gaussian Control

The purpose of this section is to outline the steps of tuning the basic ELQC procedure to the idiosyncracies of a particular control problem. There is no doubt that the availability of powerful analytical tools potentially offers an advantage; yet, their "blind" usage could easily lead to costly implementation failures. Thus, these steps should be carried out with an intuitive understanding of the system's characteristics.

The emphasis of the case study is placed on the control aspects of the HAD management problem. The system identification part will not be addressed; instead, we shall employ modeling assumptions and parameter values which are consistent with those of previous studies. In this way, fair grounds can be established for comparisons and safe conclusions can be drawn.

5.4.1 Reservoir Dynamics

Let s(t) denote the HAD storage at time t, let w(t), u(t), q(s(t)), e(t) respectively denote the inflow, release, Toshka outflow, and evaporation rates, and let A(s(t)) be the area-storage reservoir relationship. Then, the following water balance differential equation can be written to model the reservoir dynamics:

$$\frac{ds(t)}{dt} = -e(t)\ A(s(t)) - q(s(t)) - u(t) + w(t) \qquad (5.4.1)$$

Equation (5.4.1) assumes that reservoir losses due to causes other than evaporation are negligible or that they can adequately be

accounted for by properly adjusting e(t). Regarding the form of A(s(t)), the following analytic function has been fitted to and well approximates available survey data (Alarcon and Marks, 1979):

$$A(s(t)) = -3164.28 + 25.4914\ s(t) + 1092.92\ \ln(s(t)). \qquad (5.4.2)$$

The above gives the reservoir's surface area in square kilometers (km^2) when the storage volume is expressed in milliard cubic meters ($10^9\ m^3$). This function is plotted in Figure 5.4.1 (solid line) together with the similarly obtained elevation-storage relationship (dashed line):

$$H(s(t)) = 79.9734 + 0.03698\ s(t) + 18.8705\ \ln(s(t)). \qquad (5.4.3)$$

Again, the reservoir elevation H(s(t)) is obtained in meters when the storage is expressed in milliard m^3. The Toshka outflow rate is related to the reservoir storage through (Buchanan and Bras, 1981):

$$q(s(t)) = \begin{cases} 0, \text{ if } H(s(t)) \leq 178.0 \text{ m} \\ \\ 0.019\ [H(s(t)) - 178.0]^{5/3}, \text{ if } H(s(t)) > 178.0 \text{ m} \end{cases} \qquad (5.4.4)$$

where q(s(t)) is obtained in milliard m^3 per day when the elevation is in meters.

Next, we consider a mean inflow and release time trajectories $\{\bar{u}(t)\}_{t=t_k}^{t_T}$, $\{\bar{w}(t)\}_{t=t_k}^{t_T}$ and ask how the system is expected to respond over the interval $[t_k, t_T]$ under these inputs. In other words, we are interested in tracing the corresponding mean storage trajectory

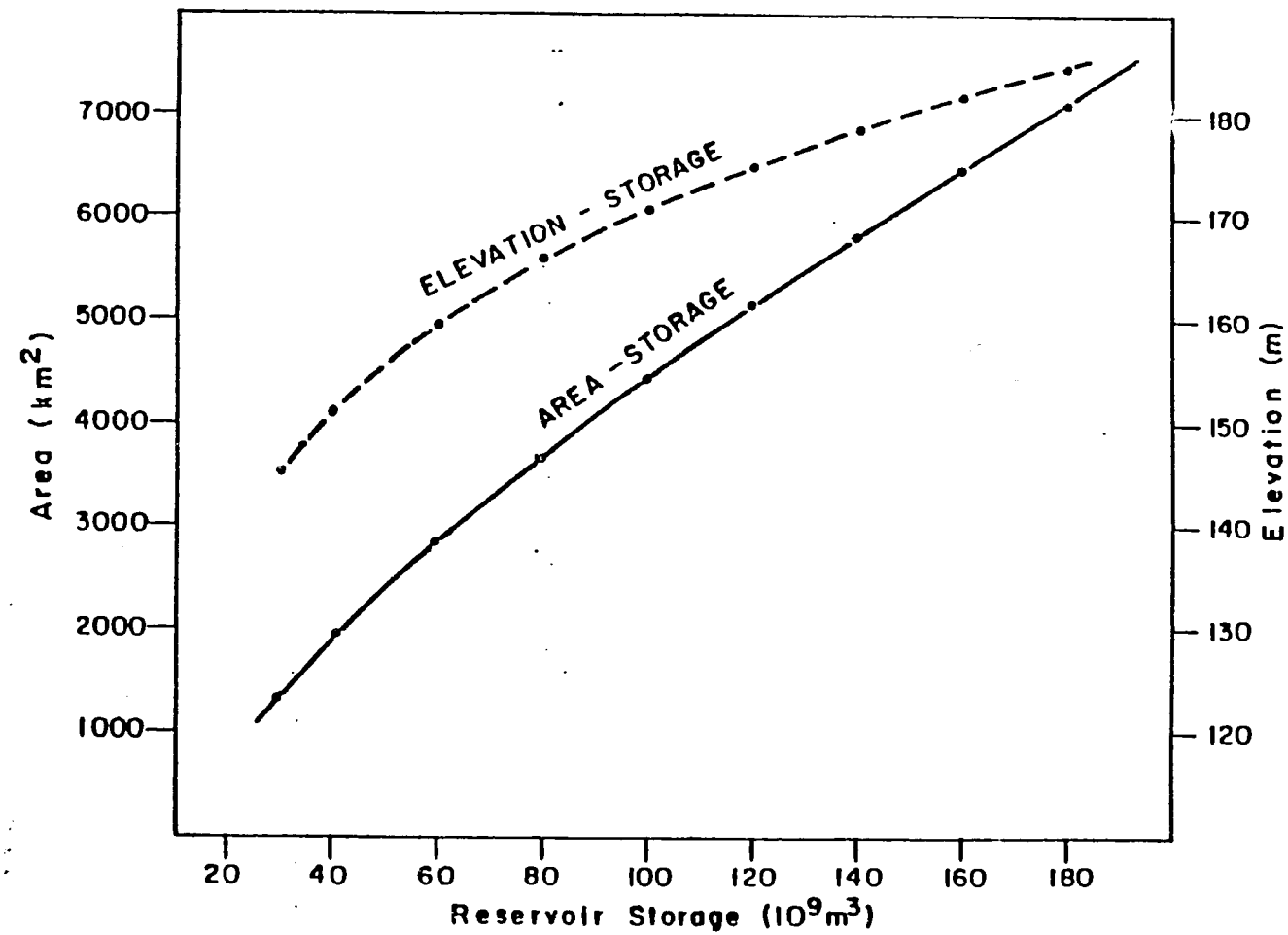

FIGURE 5.4.1.: HAD AREA - STORAGE AND ELEVATION-STORAGE RELATIONSHIPS

$\{\bar{s}(t)\}_{t=t_k}^{t_T}$ from some initial condition $\bar{s}(t_k)$. Mathematically this can be accomplished by integrating Equation (5.4.1) with inputs and initial condition as stated above:

$$\frac{d\bar{s}(t)}{dt} = f(\bar{s}(t),t) - \bar{u}(t) + \bar{w}(t), \tag{5.4.5}$$

where $f(\bar{s}(t) = -e(t)\ A(\bar{s}(t)) - q(\bar{s}(t))$ and $\bar{s}(t_k)$, $\{\bar{u}(t), \bar{w}(t)\}_{t=t_k}^{t_T}$ are given.

This is a nonlinear differential equation whose solution can be accurately obtained by some numerical integration routine (a 5th and 6th Runge Kutta numerical scheme was employed in this work). After obtaining $\{\bar{s}(t)\}_{t=t_k}^{t_T}$ we can construct the continuous time perturbation system model corresponding to the previous nominal trajectories. Define:

$$\left.\begin{aligned} \delta s(t) &= s(t) - \bar{s}(t), \\ \delta u(t) &= u(t) - \bar{u}(t), \\ \zeta(t) &= w(t) - \bar{w}(t), \\ t &\in [t_k, t_T], \end{aligned}\right\} \tag{5.4.6}$$

and expand $f(s(t),t) = -e(t)A(s(t)) - q(s(t))$ around $\bar{s}(t)$:

$$f(s(t),t) \simeq f(\bar{s}(t),t) + \left[\frac{\partial f(s(t),t)}{\partial s(t)}\right]_{s(t)=\bar{s}(t)} \delta s(t) =$$

$$= f(\bar{s}(t) + f_1(\bar{s}(t),t)\ \delta s(t). \tag{5.4.7}$$

Subtracting (5.4.5) from (5.4.1) and taking into account (5.4.6) and (5.4.7) we find

$$\frac{d\delta s(t)}{dt} = f_1(\bar{s}(t),t)\delta s(t) - \delta u(t) + \xi(t) \tag{5.4.8}$$

where $$f_1(\bar{s}(t),t) = -e(t)\left[\frac{\partial A(s(t))}{\partial s(t)}\right]_{s(t)=\bar{s}(t)} - \left[\frac{\partial \bar{q}(s(t))}{\partial s(t)}\right]_{s(t)=\bar{s}(t)}. \tag{5.4.9}$$

This is the continuous time linear perturbation model which we shall use in place of Equation (5.4.1) in the ELQG computations corresponding to the hypothesized nominal trajectories. Its credibility depends on how well the approximation $[f(\bar{s}(t),t) + f_1(\bar{s}(t),t)\delta s(t)]$ represents the function $f(s(t),t)$ over the possible variation range of $s(t)$. Figure 5.4.2 shows a plot of the nonlinear term $f(s(t),t) = -e(t)\ A(s(t))-q(s(t))$ for $e(t) = 0.0065\ m^3$ per day - an average evaporation rate from Figure 5.2.4 up to the storage where the Toshka spillway starts functioning (about $137.7 \times 10^9\ m^3$), this term is very mildly nonlinear. Beyond this value the Toshka outflow dominates and establishes a new trend drastically different from the previous one. Although the transition from segment ab to bc is highly nonlinear, it will not jeopardize the approximation (5.4.8). This is true because one of the system's operational constraints will be to maintain the reservoir's storage in the segment ab (to avoid wasting water). The possible variation range of $s(t)$ is specified by the corresponding probability density function. It will be seen that the interval $[\bar{s}(t) - 2\ P_s(t),\ \bar{s}(t) = 2P_s(t)]$, where $P_s(t)$ is the associated standard deviation, will not exceed $20 \times 10^9\ m^3$. Over this range and within the segment ab, the nonlinear term $f(s(t),t)$ and its linear approximation are practically

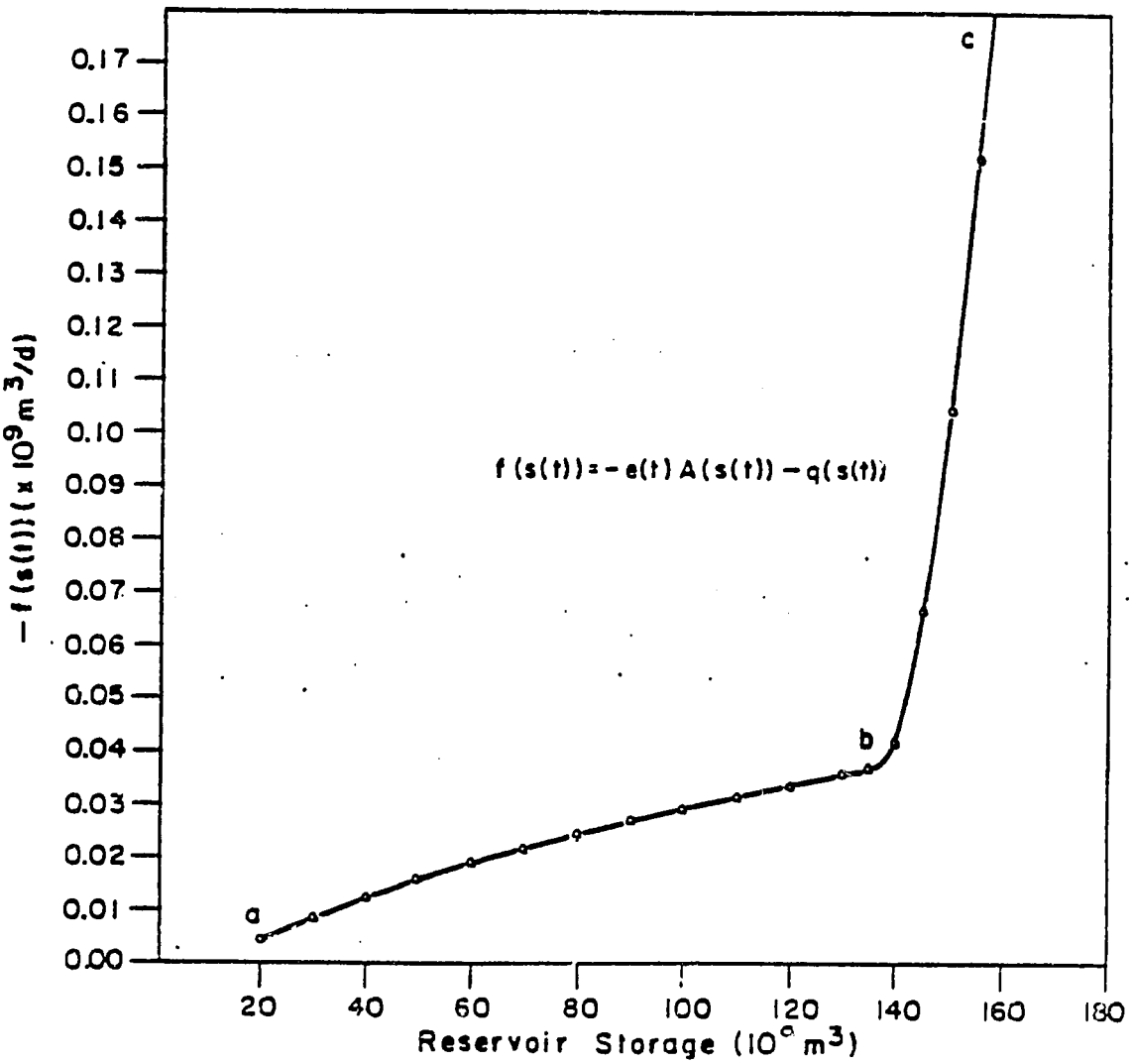

FIGURE 5.4.2: NONLINEARITY OF THE SYSTEM DYNAMICS

identical. As noted in Section 4.6, due to this attribute, the ELQC control method is expected to exhibit fast (at least quadratic) convergence rate. In cases where substantial nonlinearities exist in the system dynamics, one should prefer using statistical rather than Taylor series linearization (see end of Section 4.3).

Returning to the perturbation model (5.4.8), we have the following expression for the term $f_1(\bar{s}(t), t)$ over the segment ab on Figure 5.4.2:

$$f_1(\bar{s}(t),t) = 10^{-6} e(t) \cdot [25.4914 + 1092.92 \frac{1}{\bar{s}(t)}] \qquad (5.4.10)$$

where $f_1(\bar{s}(t),t)$ is obtained in milliard m^3 per x time units when $\bar{s}(t)$ is in milliard m^3 and $e(t)$ is expressed in millimeters per x time units (x being any time interval).

Being interested in monthly policies, we can convert Equation (5.4.8) into its discrete time counterpart by integrating it over the monthly periods $[t_\ell, t_{\ell+1}]$, ℓ=k, k+1, ..., T-1. As has been derived for the general case in Section 4.3, the result is the following discrete time system model:

$$\delta s_{\ell+1} = \phi_\ell \, \delta s_\ell + B_\ell \, \delta u_\ell + \xi_\ell \qquad (5.4.11)$$

ℓ=k, k+1, ..., T-1

where $\delta s_\ell = \delta s(t_\ell) = s(t_\ell) - \bar{s}(t_\ell)$, (5.4.12)

$\delta u_\ell = \delta u(t_\ell) = u(t_\ell) - \bar{u}(t_\ell)$, (5.4.13)

$\phi_\ell = \phi(t_{\ell+1}, t_\ell)$, (5.4.14)

$$\frac{d\phi(\tau,t_\ell)}{d\tau} = f_1(\bar{s}(\tau),\tau)\ \phi(\tau,t_\ell),\ \phi(t_\ell,t_\ell) = 1, \tag{5.4.15}$$

$$B_\ell = (-1)\int_{t_\ell}^{t_{\ell+1}} \phi(t_{\ell+1},\tau)d\tau, \tag{5.4.16}$$

$$E\{\xi_\ell\} = 0, \tag{5.4.17}$$

$$E\{\xi_\ell\xi_\mu\} = \left\{\begin{array}{ll} 0, \text{ if } \ell\neq\mu & \\ Q_w(t_\ell)\displaystyle\int_{t_\ell}^{t_{\ell+1}} \phi^2(t_{\ell+1},\tau)d\tau\ , & \text{if } \ell=\mu \end{array}\right\} \tag{5.4.18}$$

$\ell=k, \ldots, T-1$

The derivation assumes that the releases are constant within each monthly interval:

$$\{\delta u(\tau) - \delta u(t_\ell),\ \ \tau\epsilon[t_\ell,\ t_{\ell+1}),\ \ell=k,\ \ldots,\ T-1\} \tag{5.4.19}$$

and that the same is true for the spectral density of the random inflow:

$$\{Q_w(\tau) = Q_w(t_\ell),\ \tau\epsilon[t_\ell,\ t_{\ell+1}),\ \ell=k,\ \ldots,\ T-1\}\ . \tag{5.4.20}$$

Detailed discussion concerning the specification of the random inflow statistics will be given in the following section.

The basic computations in deriving the discrete time system model (5.4.11) are associated with the integrations in (5.4.15), (5.4.16), and (5.4.18). These can be carried out in a systematic way by taking advantage of the following state transition coefficient property (c.f. Equation (4.3.20):

$$\phi(t_{\ell+1},t_\ell) = \phi(t_{\ell+1},\tau)\ \phi(\tau,t_\ell) \tag{5.4.21}$$

from which

$$\phi(t_{\ell+1},\tau) = \frac{\phi(t_{\ell+1},t_\ell)}{\phi(\tau,t_\ell)} \quad . \tag{5.4.22}$$

Substituting (5.4.12) in (5.4.16) and (5.4.18) we find that the previous integraions can be performed as follows:

Simultaneously integrate

$$\frac{d\phi(\tau,t_\ell)}{d\tau} = f_1(\bar{s}(\tau),\tau)\ \phi(\tau,t_\ell),\ \phi(t_\ell,t_\ell) = 1, \tag{5.4.23}$$

$$\frac{d\ B(\tau,t_\ell)}{d\tau} = -\ \frac{1}{\phi(\tau,t_\ell)}\ , \qquad B(t_\ell,t_\ell) = 0, \tag{5.4.24}$$

$$\frac{d\ Q_\xi(\tau,t_\ell)}{d\tau} = \frac{1}{\phi^2(\tau,t_\ell)}\ , \qquad Q_\xi(t_\ell,t_\ell) = 0, \tag{5.4.25}$$

over the interval $[t_\ell,t_{\ell+1}]$, ℓ=k, ..., T-1. Then, compute ϕ_ℓ, B_ℓ, $E\{\xi^2\} = Q_{\xi\ell}$ from

$$\phi_\ell = (t_{\ell+1},t_\ell), \tag{5.4.26}$$

$$B_\ell = \phi_\ell\ B(t_{\ell+1},t_\ell), \tag{5.4.27}$$

$$Q_{\xi\ell} = Q_w(t_\ell)\ \phi_\ell^2\ \ Q_\xi(t_{\ell+1},t_\ell). \tag{5.4.28}$$

Actually, the integration of (5.4.23), (5.4.24), and (5.4.25) can also be combined with the one of Equation (5.4.5) to yield all the sequences $\{\bar{s}(t)\}_{t=t_k}^{t_T}$, $\{\phi_\ell,B_\ell,\ E\{\xi_\ell^2\}\}_{\ell=k}^{T-1}$ in one forward pass from time t_k to t_T.

As has been derived in Section 4.5 (Equation (4.5.9)), the need to compute $Q_{\xi\ell}$, ℓ=k. ..., T-1, is related to the propagation of the state's variance:

$$P_{s\ell+1/k} = \phi_{\tau}^{2} P_{s\ell/k} + Q_{\xi\ell};\ P_{sk/k} = 0, \qquad (5.4.29)$$
$$\ell = k, \ldots, T-1.$$

The zero initial condition reflects the fact that at time k, i.e., the present time, reservoir storage is accurately known. Equivalent to (5.4.29), the state variance can be obtained from continuous time considerations. It can be shown (Gelb, 1974, Section 3.7) that

$$\frac{df_s(t)}{dt} = 2f_1(\bar{s}(t),t)\ P_s(t) + Q_w(t) \qquad (5.4.30)$$

where $P_s(t) = E\{[s(t)-\bar{s}(t)]^2\} = E\{\delta s^2(t)\}$. (5.4.31)

Starting from $P_s(t_k) = P_{sk/k} = 0$, Equation (5.4.30) can be integrated and provide the variance $P_{s\ell/k} = P_s(t_\ell)$ at any time t_ℓ, ℓ=k, ..., T-1. Hence, in place of (5.4.25) we can integrate (5.4.30) together with (5.4.5), (5.4.23), (5.4.24) and obtain in one forward pass the mean and variance trajectories along with the state and control coefficients of the discrete time model. This is how these computations were organized here with a 5th and 6th order Runge Kutta routine performing the integrations.

Throughout this section is has been assumed that the statistics of the random inflow process $\{\bar{w}(t),\ Q_w(t);\ t\varepsilon[t_k,t_T]\}$ are known for the computations performed at time t_k. However in reality, inflow data

are provided in discrete time form as in Appendix H. A procedure for retrieving the necessary information from such records and a procedure for incoporating forecasted information are discussed in the next section.

5.4.2 Inflow Statistics

In reality, inflow $w(t)$ is a continuous time stochastic process. Therefore, its complete characterization would require knowledge of all joint probability density functions $P(w(t_1), w(t_2), \ldots, w(t_n), t_1, t_2, \ldots, t_n)$ where $\{t_1, t_2, \ldots, t_n\}$ is any set of time instants. In general, however, such characterization is not only very difficult to establish but it also includes much more information than is actually needed. For example, a monthly control model is not so concerned with hourly or daily variations as it is with monthly characteristics. On the other hand, since water is being delivered and affects storage, elevation, power production, evaporation, etc., continuously over the month, we find it appropriate to derive the discrete time system model from the continuous time formulation rather than assume discrete system dynamics. The question is how to define a continuous time random process reproducing monthly inflow characteristics.

Assume for now that the amount of water delivered by the inflow process during a particular month is independent of any other month's water yield. Let w_i, $i=1, \ldots, 12$ denote the corresponding random variables. From a monthly data record (as, for instance, in Appendix H) one can determine the first two sample statistical moments $\{\bar{w}_i, Q_{w_i};$

$i=1, \ldots, 12\}$ (as in Table 5.2.1). Consider now a continuous time white Gaussian process $w_i(t)$ such that

$$w_i = \int_{t_i}^{t_{i+1}} w_i(s)ds \tag{5.4.32}$$

where time instants t_i, t_{i+1} designate the beginning and end of month i. Our intention is to determine the properties of the process $w_i(t)$ which preserves up to 2nd statistical moment the correspondence expressed by Equation (5.4.32). The arbitrary Gaussian assumption is a mathematical translation that we are only concerned with the first two statistical moments. Additionally, based on this section's introductory remarks, it will be assumed that over month i, $w_i(s)$ is characterized by consant mean $(E\{w_i(s)\} = \bar{w}_i(s),\ s\varepsilon[t_i,t_{i+1}])$ and constant spectral density $(E\{(w_i(r)-\bar{w}_i(r))(w_i(s) - \bar{w}_i(s))\} = Q_{w_i}(r)\delta(r-s),\ (r,s)\varepsilon[t_i,t_{i+1}])$. We can now specify the parameters $\bar{w}_i(s)$, $Q_{w_i}(s)$ by means of Equation (5.4.32) as follows: Taking expectations of both sides we find

$$E\{w_i\} = \bar{w}_i = E\left\{\int_{t_i}^{t_{i+1}} w_i(s)ds\right\} \tag{5.4.33}$$

or, since expectation and integration can be interchanged as linear operators,

$$\bar{w}_i = \int_{t_i}^{t_{i+1}} \bar{w}_i(s)ds. \tag{5.4.34}$$

However, by assumption $\bar{w}_i(s)$ is constant over $[t_i,t_{i+1})$ and

$$\bar{w}_i(s) = \frac{\bar{w}_i}{t_{i+1} - t_i}, s\varepsilon [t_i, t_{i+1}) \tag{5.4.35}$$

Regarding the variance of w_i we have

$$Q_{w_i} = E\left\{\left[\int_{t_i}^{t_{i+1}} w_i(s)ds - \bar{w}_i\right]^2\right\} =$$

$$= E\left\{\left[\int_{t_i}^{t_{i+1}} w_i(s)ds - \int_{t_i}^{t_{i+1}} \bar{w}_i(s)ds\right]^2\right\} =$$

$$= E\left\{\left[\int_{t_i}^{t_{i+1}} [w_i(s) - \bar{w}_i(s)]ds\right]^2\right\} =$$

$$= E\left\{\int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} [w_i(r) - \bar{w}_i(r)][w_i(s) - \bar{w}_i(s)]\, drds\right.$$

$$= \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} E\left\{[w_i(r) - \bar{w}_i(r)][w_i(s) - \bar{w}_i(s)]\quad drds\right\} =$$

$$= \int_{t_i}^{t_{i+1}} \int_{t_i}^{t_{i+1}} Q_{w_i}(r)\delta(r-s)\, drds =$$

$$= \int_{t_i}^{t_{i+1}} Q_{w_i}(r)dr. \tag{5.4.36}$$

And because $Q_{w_i}(r)$ was assumed constant throughout $[t_i, t_{i+1})$. we finally find

$$Q_{w_i}(s) = \frac{Q_{w_i}}{t_{i+1} - t_i}, \quad s\varepsilon[t_i, t_{i+1}). \tag{5.4.37}$$

Equations (5,4,37) and (5,4,35) completely define the continuous time Gaussian random process $w_i(t)$, i=1, ..., 12, which is needed to produce first and second moment effects consistent with the monthly lumped inflows.

Consider now the case where the independence assumption concerning the variables w_i, i=1, ..., 12, is no longer valid. Namely, assume that one could look at the previous months' inflows and more accurately predict the inflows of forthcoming months. To efficiently incorporate this possibility, the idea of the Partial Open loop Feedback Control (Bertsekas, 1976) can be very effective. First, a valid forecasting model should be identified based on the available inflow data. Then at each decision time t_k this model uses past observations to forecast the probability densities of future inflows over the control horizon $[t_k, t_T]$. Adopting the updated inflow p.d.f.s. and assuming statistical independence, the controller obtains the optimal policies and the scheme is repeated at the next decision time. In other words, at each decision time the forecasting model updates the monthly means and variances $\{\bar{w}_\ell, Q_{w_\ell}\}_{\ell=k}^{T-1}$. Then, the corresponding continuous time statistics $\{\bar{w}_\ell(s), Q_{w_\ell}(s);\ s\epsilon[t_\ell, t_{\ell+1})\}_{\ell=k}^{T-1}$ are also updated via Equation (5,4,37), (5,4,35) and used in the ELQG control method computations. According to the POLFC idea, although the existing correlation structure is called for to produce better inflow predictions, it is subsequently ignored in the control operations. However, the inconsistency is not so influential owing to the OLF procedure seeking not to determine all feedback laws at once but rather sequentially. On the other hand, if correlation was to be optimally accounted for, it

would require additional state variables (see Bertsekas, 1976, Section 2.3) perhaps reducing computational efficiency.

Incorporating forecasted information cannot result in worse policies than those obtained without it, but it will not necessarily lead to great improvements. In order to assess the worth of forecasting we shall consider here three models of varying predictive power. The first is based on the a priori monthly statistics and simply forecasts the monthly means with the corresponding variance. The second is known as Thomas-Fiering model and consists of twelve coupled equations of the following form:

$$w_{i+1} = \bar{w}_{i+1} + \rho_{i+1,i} \frac{\sigma_{i+1}}{\sigma_i}(w_i - \bar{w}_i) + \sigma_{i+1} \sqrt{1-\rho^2_{i+1,i}}\ \varepsilon_{i+1} \qquad (5.4.38)$$

where $\bar{w}_i$ and $\sigma_i = \sqrt{Q_{w_i}}$ represent the a priori mean and standard deviation of the month i inflows and $\rho_{i+1,i}$ denotes the correlation coefficient between the inflows of months i+1 and i (see Table 5.2.1). The random term ε_{i+1} is a zero mean Gaussian variable of unit variance. Given that inflow w_i has been realized, the previous equation can be used to forecast the conditional p.d.f. of w_{i+1} (which is Gaussian under the assumption for ε_{i+1} and the linear form of (5.4.35)) by specifying the conditional mean

$$\bar{w}_{i+1/i} = E\{\bar{w}_{i+1} + \rho_{i+1,i} \frac{\sigma_{i+1}}{\sigma_i}(w_i - \bar{w}_i) + \sigma_{i+1} \sqrt{1-\rho^2_{i+1,i}}\ \varepsilon_{i+1}/w_i\} =$$

$$= \bar{w}_{i+1} + \rho_{i+1,i} \frac{\sigma_{i+1}}{\sigma_i}(w_i - \bar{w}_i) \qquad (5.4.39)$$

and the variance

$$Q_{w_{i+1/i}} = E\{(w_{i+1} - \bar{w}_{i+1/i})^2/w_i\} = (1-\rho^2_{i+1,i})\, Q_{w_{i+1}}. \qquad (5.4.40)$$

Similarly, lead-2 forecasts can be obtained by invoking

$$w_{i+2} = \bar{w}_{i+2} + \rho_{i+2,i+1} \frac{\sigma_{i+2}}{\sigma_{i+1}} (w_{i+1} - \bar{w}_{i+1}) + \sigma_{i+2}\sqrt{1-\rho^2_{i+2,i+1}}\,\varepsilon_{i+2} \qquad (5.4.41)$$

and substituting w_{i+1} by the previously forecasted value:

$$w_{i+2/i} = E\{w_{i+2}/w_i\} = \bar{w}_{i+2} + \rho_{i+2,i+1}\rho_{i+1,i} \frac{\sigma_{i+2}}{\sigma_i} (w_i - \bar{w}_i) \qquad (5.3.42)$$

$$Q_{w_{i+2/i}} = E\{(w_{i+2} - \bar{w}_{i+2/i})^2/w_i\} = (1-\rho_{i+2,i+1}\rho^2_{i+1,i})Q_{w_{i+2}}. \qquad (5.4.43)$$

In general it can be shown (by induction) that the forecasted mean and variance at lead lime ℓ are given by

$$\bar{w}_{i+\ell/i} = \bar{w}_{i+\ell} + \rho_{i+\ell\, i+\ell-1}\rho_{i+\ell-1\, i+\ell-2} \cdots \rho_{i+1,i} \frac{\sigma_{i+\ell}}{\sigma_i} (w_i - \bar{w}_i), \qquad (5.4.44)$$

$$Q_{w_{i+\ell/i}} = (1-\rho^2_{i+\ell,i+\ell-1}\, \rho^2_{i+\ell-1,i+\ell-2} \cdots \rho^2_{i+1,i})Q_{w_{i+\ell}}. \qquad (5.4.45)$$

Since the correlation coefficients are always included in interval [-1,1] (with -1,1 limiting cases), the previous formulas show that as the forecast lead time increases, $\bar{w}_{i+\ell/i}$ and $Q_{w_{i+\ell/i}}$ tend more towards the corresponding a priori statistics. Equation (5.4.45) also shows that the forecasted variance $Q_{w_{i+\ell/i}}$ is always less than the a priori $Q_{w_{i+\ell}}$ with this reduction governed by the factor

$$R^2_{i+\ell/i} = \rho^2_{i+\ell,i+\ell-1} \cdots \rho^2_{i+1,i} \qquad (5.4.46)$$

also known as R-square statistic

Table 5.3.1 gives the coefficient $R^2_{i+\ell/i}$ of the Thomas-Fiering model for the Wadi Halfa Nile flow data included in Appendix H. They have been computed for lead times up to 12 months for forecasts issued from every month of the year.

The third forecasting model is the one developed by Curry and Bras, (1980, and also employed in Buchanan and Bras, 1981. It is a multivariate regression model using Nile flow obseervations at 8 sites (including Wadi Halfa). Its predictive capability is better than that of the Thomas-Fiering model as can be verified by comparing Table 5.3.1 to 5.3.2 where the R-square statistics for the Curry-Bras forecasting model are consistently higher (resulting in tighter forecast variances). However, as mentioned, the need of a forecasting model for control purposes should be evaluated according to the relative improvements of the system performance. Measures of the High Aswan Dam operation performance are now discussed.

LEAD	JAN.	FEB.	MAR.	APR.	MAY	JUNE	JULY	AUG.	SEPT.	OCT.	NOV.	DEC.
1	0.8245	0.5629	0.7481	0.7594	0.6325	0.2455	0.4066	0.5062	0.6165	0.6512	0.8007	0.8475
2	0.4641	0.4211	0.5681	0.4803	0.1553	0.0998	0.2058	0.3121	0.4014	0.5215	0.6786	0.6987
3	0.3472	0.3198	0.3593	0.1179	0.0631	0.0505	0.1269	0.2032	0.3214	0.4419	0.5595	0.3933
4	0.2636	0.2023	0.0882	0.0479	0.0320	0.0311	0.0826	0.1627	0.2724	0.3644	0.3149	0.2942
5	0.1668	0.0497	0.0359	0.0243	0.0197	0.0203	0.0662	0.1379	0.2246	0.2051	0.2356	0.2234
6	0.0409	0.0202	0.0182	0.0150	0.0128	0.0162	0.0561	0.1137	0.1264	0.1534	0.1789	0.1413
7	0.0166	0.0102	0.0112	0.0097	0.0103	0.0138	0.0462	0.0640	0.0946	0.1165	0.1132	0.0347
8	0.0084	0.0063	0.0073	0.0078	0.0087	0.0113	0.0260	0.0479	0.0718	0.0737	0.0278	0.0141
9	0.0052	0.0041	0.0058	0.0066	0.0072	0.0064	0.0195	0.0364	0.0454	0.0181	0.0113	0.0071
10	0.0034	0.0033	0.0049	0.0055	0.0040	0.0048	0.0148	0.0230	0.0112	0.0074	0.0057	0.0044
11	0.0027	0.0028	0.0041	0.0031	0.0030	0.0036	0.0093	0.0056	0.0045	0.0037	0.0035	0.0029
12	0.0023	0.0023	0.0023	0.0023	0.0023	0.0023	0.0023	0.0023	0.0023	0.0023	0.0023	0.0023

TABLE 5.3.1: $R^2_{i+\ell/i}$ COEFFICIENTS FOR THE THOMAS-FIERING MODEL (i DENOTES THE MONTH FROM WHICH AN ℓ-LEAD FORECAST IS ISSUED)

LEAD	JAN.	FEB.	MAR.	APR.	MAY	JUNE	JULY	AUG.	SEPT.	OCT.	NOV.	DEC.
1	0. 569	0.9281	0.9589	0.9822	0.9255	0.8358	0.7502	0.7180	0.7724	0.8918	0.9282	0.9864
2	0.7298	0.8299	0.9298	0.6928	0.4002	0.1037	0.4571	0.4262	0.4285	0.8707	0.9507	0.9089
3	0.8166	0.9013	0.5959	0.2291	0.0279	0.0533	0.3944	0.2111	0.6297	0.8992	0.8891	0.6503
4	0.8649	0.6018	0.2045	0.0132	0.0451	0.1818	0.1881	0.4598	0.8174	0.8637	0.6679	0.8289
5	0.5423	0.2132	0.0140	0.0466	0.1345	0.0826	0.3796	0.6505	0.7822	0.6191	0.1862	0.8284
6	0.2202	0.0146	0.0466	0.1343	0.0426	0.2725	0.5565	0.6277	0.6113	0.7985	0.7989	0.5154
7	0.0214	0.0466	0.0541	0.0465	0.2191	0.4489	0.5273	0.4940	0.7609	0.7962	0.4942	0.1986
8	0.0326	0.0475	0.0353	0.2049	0.3617	0.4194	0.4218	0.6992	0.7261	0.4791	0.1872	0.0107
9	0.0471	0.0298	0.2139	0.3355	0.3379	0.3595	0.7034	0.6512	0.4440	0.1678	0.1827	0.0800
10	0.0301	0.1997	0.3118	0.3137	0.2344	0.2788	0.6207	0.4066	0.1822	0.1759	0.0916	0.0349
11	0.1913	0.3035	0.2628	0.1861	0.6281	0.5835	0.3774	0.1734	0.1677	0.0897	0.1027	0.0130
12	0.2779	0.2684	0.1481	0.5849	0.5394	0.3631	0.1792	0.1651	0.0875	0.0989	0.0915	0.1700

TABLE 5.3.2: $R^2_{j+\ell/i}$ COEFFICIENTS FOR THE CURRY-BRAS MODEL

(i DENOTES THE MONTH FROM WHICH AN ℓ-LEAD

FORECAST IS ISSUED)

5.4.3 Modeling of Objectives

The High Aswan Dam is primarily a water supply and flood protection project and secondly a hydroelectric energy supplier.

The downstream water supply requirements presently amount to the monthly quantities reported in Figure 5.2.5 and are satisfied as long as HAD releases are sufficiently high:

$$u_\ell^{w.s.} \leq u_\ell, \ \ell=k, \ldots, T-1. \tag{5.4.47}$$

Implicitly, they also induce lower bounds on the reservoir storage to guarantee that there will be enough water to make the previous releases possible. These constraints can only be of a probabilistic nature specifying that

$$P_r[s_\ell < s_\ell^{min}] \leq \gamma_\ell^{min}, \ \ell=k, \ldots, T, \tag{5.4.48}$$

where s_ℓ^{min} can be taken equal to the ceiling of the dead storage zone (32.72×10^9 m^3) and γ_ℓ^{min} is the permissible probability of violation (i.e., the probability of not meeting the downstream water supply demand). Under the assumption of Gaussian state probability densities with means and variances obtained as discussed in Section 5.4.1 (Equation (5.4.5), (5.4.30)), constraint (5.4.48) was seen in Section 4.7 to be equivalent to

$$\bar{s}_{\ell/k} \geq s_\ell^{min} - z_\ell^{min} \sqrt{P_{s\ell/k}}, \ \ell=k,\ldots,T \tag{5.4.49}$$

where z_ℓ^{min} is such that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z_\ell^{min}} e^{-\frac{1}{2}z^2} dz = \gamma_\ell^{min}. \tag{5.4.50}$$

Some representative γ_ℓ^{min}, z_ℓ^{min} pairs follow:

γ_ℓ^{min}	z_ℓ^{min}
0.500	0.000
0.100	-1.281
0.050	-1.644
0.025	-1.960
0.005	-2.575
0.001	-3.090

The flood protection objective translates into channel degradation concerns which specify that monthly releases should not exceed the threshold at 7.6 milliard m^3 per month (or 0.253×10^9 m^3 per day):

$$u_\ell \leq u_\ell^{max}, \ \ell=k, \ldots, T-1. \tag{5.4.51}$$

The presence of the Toshka spillway dismisses other flood protection measures but it brings up the issue of minimizing spills so that the resulting water savings can be used in energy generation or for water supply purposes. Hence on upper probabilistic constraint is imposed:

$$P_r[s_\ell > s_\ell^{max}] \leq \gamma_\ell^{max}, \ \ell=k,\ldots T, \tag{5.4.52}$$

or

$$\bar{s}_{\ell/k} \leq s_\ell^{max} = z_\ell^{max} \sqrt{r_{s\ell/k}} \tag{5.4.53}$$

where s_ℓ^{max} denotes the storage of 137.72 milliard m^3, Toshka's operating

threshold. the previously given representative $(\gamma_\ell^{min}, z_\ell^{min})$ values are also valid for the $(\gamma_\ell^{max}, z_\ell^{max})$ pair with the difference that $z_\ell^{max} = -z_\ell^{min}$.

Based on historical data, the HAD energy production function is given by (Thompson, 1981):

$$E_\ell = 0.517 \quad u_\ell^{1.194} \; [\bar{H}_\ell - 108.]]^{1.268} \tag{5.4.54}$$

where E_ℓ is obtained in GWH per month when the monthly reservoir release is expressed in milliard m^3 and the monthly average water elevation is in meters (see Equation (5.4.3)). This function accounts for the system's load factor and is in close agreement with Figure (5.2.6)'s turbine characteristic curves. As the 10 operational HAD turbines can produce at most 1280 GWH per month (Buchanan and Bras, 1981; Thompson, 1981), according to the discussion in Sections 4.2, 4.3 we shall consider the following penalty cost function for minimization:

$$J = \underset{\{s_\ell\}_{\ell=k+1}^{T}}{E} \left\{ \sum_{\ell=k}^{T-1} g(s_\ell, u_\ell) + g_T(s_T) \right\} \tag{5.4.55}$$

where $g(s_\ell, u_\ell) = [1280. - 0.517 \; u^{1.194} [H(s_\ell) - 108.]^{1.268}]^2$, (5.4.56)

$$H(s_\ell) = 79.9734 + 0.03698 \; s_\ell + 18.8705 \; \ln(s_\ell), \tag{5.4.57}$$

$$s_\ell = s(t_\ell), \; s_k \text{ given.} \tag{5.4.58}$$

(The terminal cost term will be defined later.)

Following the procedure in Section 4.5.1 we should next proceed to specify the cost functions

$$\ell(\delta s_\ell, \delta u_\ell) = g(\bar{s}_\ell + \delta s_\ell, \bar{u}_\ell + \delta u_\ell), \tag{5.4.59}$$

$$\hat{\ell}(\delta\bar{s}_{\ell/k}, Ps_{\ell/k}, \delta u_\ell) = \underset{\delta s_\ell}{E} \{\ell(\delta s_\ell, \delta u_\ell)\}, \tag{5.4.60}$$

$$\tilde{\ell}(\delta\bar{s}_{\ell/k}, \delta u_\ell) = \hat{\ell}(\delta\bar{s}_{\ell/k}, Ps_{\ell/k}, \delta u_\ell), \tag{5.4.61}$$

where $$\delta\bar{s}_{\ell+1/k} = \phi_\ell \delta\bar{s}_{\ell/k} + B_\ell\, \delta u_\ell, \quad \delta\bar{s}_{k/k} = 0 \tag{5.4.62}$$

$$P_{s\ell+1/k} = \phi_\ell^2 P_{s\ell/k} + Q_{\xi\ell}, \quad P_{sk/k} = 0 \tag{5.4.63}$$

$\ell = k, \ldots, T-1.$

Now (5.4.56) is of a general nonlinear form and this a priori calls for the analytical approximation procedure presented in Appendix B. However, plotting this function (Figures 5.4.3, 5.4.4.) shows that it is well approximated by a quadratic functional. The particular approximation shown on these figures is a second order Taylor expansion of the function $g(s_\ell, u_\ell)$ around $s^0=125$, $u^0=6.5$ and is given by

$$\begin{aligned} g'(s_\ell, u_\ell) = {} & g(s^0, u^0) + N_s(s_\ell - s^0) + N_u(u_\ell - u^0) + \\ & + \tfrac{1}{2}N_{ss}(s_\ell - s^0)^2 + \tfrac{1}{2}N_{uu}(u_\ell - u^0)^2 + \\ & + N_{us}(u_\ell - u^0)(s_\ell - s^0) \end{aligned} \tag{5.4.64}$$

where $g(s^0, u^0) = 71593.05$,

$$\left.\begin{aligned} N_s &= -1906.931 = \frac{\partial g(125, 6.5)}{\partial s}, \\ N_u &= -99522.65 = \frac{\partial g(125, 6.5)}{\partial u}, \\ N_{ss} &= 36.23141 = \frac{\partial^2 g(125, 6.5)}{\partial s^2}, \\ N_{uu} &= 66203.65 = \frac{\partial^2 g(125, 6.5)}{\partial u^2}, \\ N_{us} &= 975.1393 = \frac{\partial^2 g(125, 6.5)}{\partial u \partial s}. \end{aligned}\right\} \tag{5.4.65}$$

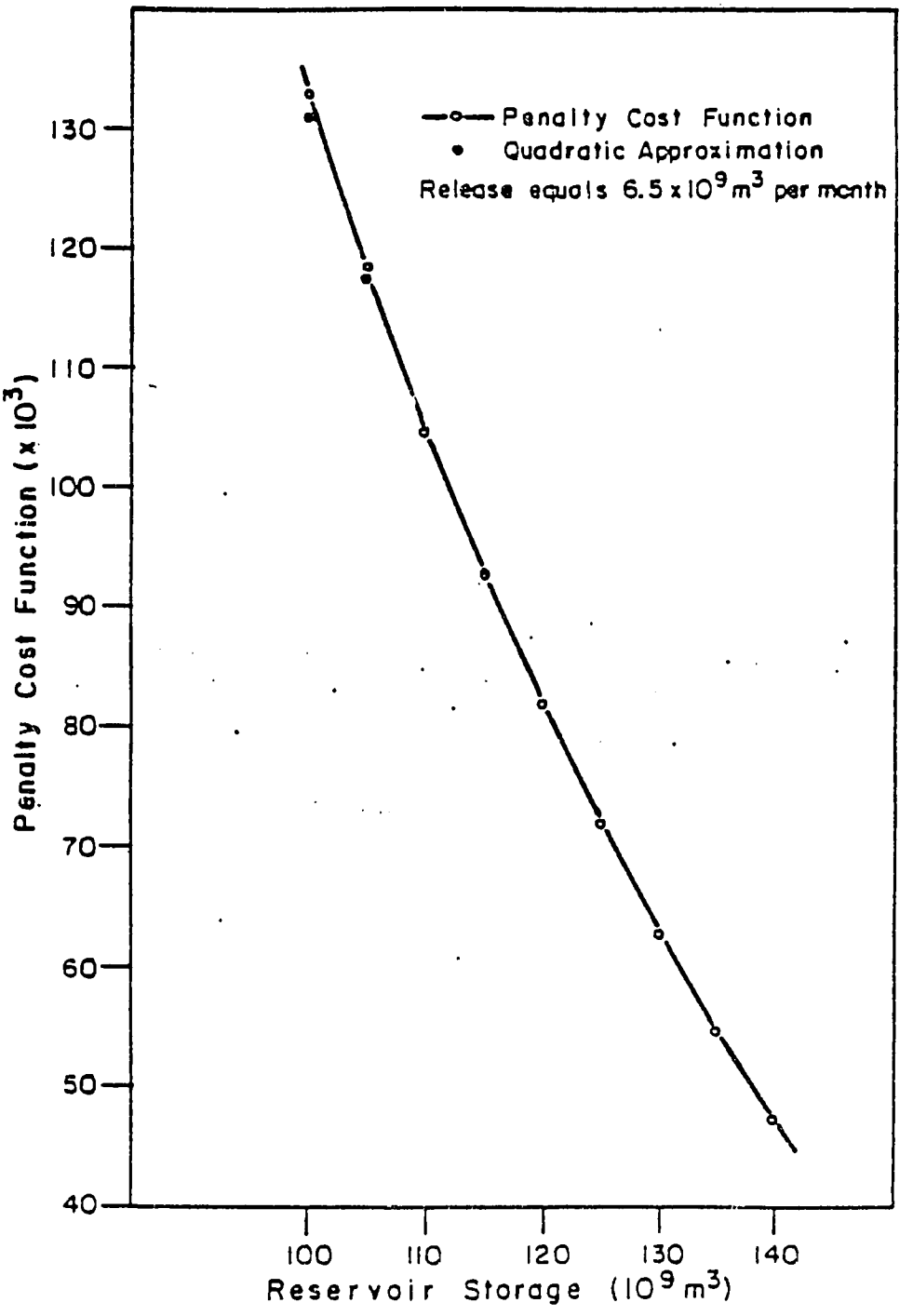

FIGURE 5.4.3: PENALTY COST FUNCTION VS RESERVOIR STORAGE

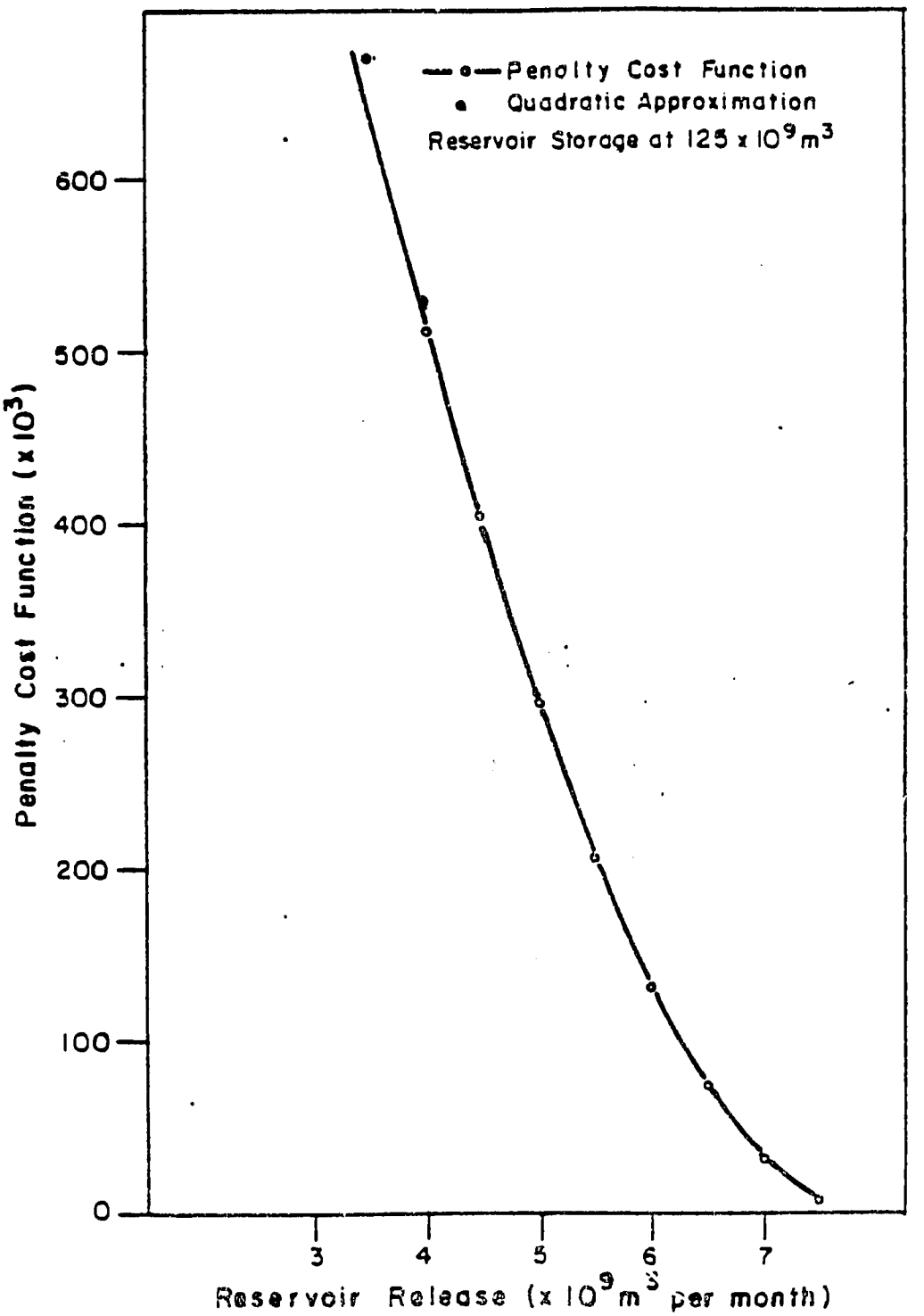

FIGURE 5.4.4: PENALTY COST FUNCTION VS RESERVOIR RELEASE

Since the approximation is satisfactory within the range of the possible storage variation, we shall assume that the cost function $g(s_\ell, u_\ell)$ (Equation (5.4.56)) is adequately represented by a quadratic approximation around the nominal state and control trajectories. Namely, in specifying the functions $\ell(\cdot,\cdot)$, $\hat{\ell}(\cdot,\cdot,\cdot)$, $\tilde{\ell}(\cdot,\cdot)$ (according to the Appendix B procedure) we shall not use Taylor series terms of order higher than two. Thus these functions are obtained as follows:

$$\ell(\delta s_\ell, \delta u_\ell) = g(\bar{s}_{\ell/k}, \bar{u}_\ell) + N_{s\ell}\delta s_\ell + N_{u\ell}\,\delta u_\ell + \\ + \frac{1}{2} N_{ss\ell}(\delta s_\ell)^2 + \frac{1}{2}N_{uu\ell}(\delta u_\ell)^2 + N_{us\ell}(\delta u_\ell)(\delta s_\ell), \tag{5.4.66}$$

$$\hat{\ell}(\delta\bar{s}_{\ell/k}, P_{s\ell/k}, \delta u_\ell) = g(\bar{s}_{\ell/k}, \bar{u}_\ell) + N_{s\ell}\,\delta\bar{s}_{\ell/k} + N_{u\ell}\delta u_\ell + \\ + \frac{1}{2}N_{ss\ell}(\delta\bar{s}^2_{\ell/k} + P_{s\ell/k}) + \frac{1}{2}N_{uu\ell}(\delta u_\ell)^2 + \\ + N_{us\ell}(\delta u_\ell)(\delta\bar{s}_{\ell/k}), \tag{5.4.67}$$

or neglecting the constant terms (which do not affect the control operations)

$$\tilde{\ell}(\delta\bar{s}_{\ell/k}, \delta u_\ell) = N_{s\ell}\delta\bar{s}_{\ell/k} + N_{u\ell}\,\delta u_\ell + \frac{1}{2}N_{ss\ell}(\delta\bar{s}_{\ell/k})^2 + \\ + \frac{1}{2}N_{uu\ell}(\delta u_\ell)^2 + N_{us\ell}(\delta\bar{s}_{\ell/k})(\delta u_\ell). \tag{5.4.68}$$

Thus, the reformulated cost function for minimization becomes:

$$\tilde{J} = \sum_{\ell=k}^{T-1} \tilde{\ell}(\delta\bar{s}_{\ell/k}, \delta u_\ell) + \tilde{\ell}_T(\delta\bar{s}_{T/k}) \tag{5.4.69}$$

where $\tilde{\ell}(\cdot,\cdot)$ as above and $\{N_{s\ell}, N_{u\ell}, N_{ss\ell}, N_{uu\ell}, N_{us\ell}\}_{\ell=k}^{T-1}$ given by the derivative (5.4.65) at the nominal state and control values.

With these specifications the HAD operation problem has taken the form required by the ELQG control procedure. The following section will discuss the mechanics and the performance of the method in some experimental runs.

5.4.4 Extended Linear Quadratic Gaussian Control

The ELQG method for the control of a single reservoir is summarized by the following algorithm. Some possible variations are subsequently mentioned.

Algorithm A5.4.1: ELQG Control of a Single Reservoir

a. Forecast the continuous time inflow statistics $\bar{w}(t)$, $Q_w(t)$ (mean and spectral density) over the control horizon $t \in [t_k, t_T]$ as explained in Section 5.4.2.

Assume that as a result of i previous iteration's nominal trajectory $\underline{u}^i = u_k^i, \ldots, u_{T-1}^i$ has been obtained. Then the operations that follow are performed during iteration i+1.

b. Compute the state's mean and variance trajectories as well as the coefficients of the discrete time perturbation model by integrating the following system of differential equations over the intervals $[t_\ell, t_{\ell+1}]$, $\ell=k,\ldots, T-1$:

$$\frac{d\bar{s}^i(\tau)}{d\tau} = f(\bar{s}^i(\tau),\tau) - u^i(\tau) + \bar{w}(\tau) \qquad (5.4.70)$$

$$\frac{dP_s^i(\tau)}{d\tau} = 2f_1(\bar{s}^i(\tau),\tau)\ P_s^i(\tau) + Q_w(\tau) \tag{5.4.71}$$

$$\frac{d\phi^i(\tau,t_\ell)}{d\tau} = f_1(\bar{s}^i(\tau),\tau)\ \phi(\tau,t_\ell) \tag{5.4.72}$$

$$\frac{dB^i(\tau,t_\ell)}{d\tau} = -\frac{1}{\phi^i(\tau,t_\ell)} \tag{5.4.73}$$

where $f(\cdot,\cdot)$ is the state transition function (Section 5.4.1) and

$$f_1(\bar{s}^i(\tau),\tau) = \left[\frac{\partial f(s(\tau),\tau)}{s(\tau)}\right]_{s(\tau)=\bar{s}^i(\tau)} . \tag{5.4.74}$$

The initial conditions for the mean state and variance equations are the results of the previous integration step: $\bar{s}^i(t_\ell)$, $P_s^i(t_\ell)$, while for (5.4.72) and (5.4.73) there holds

$$\phi^i(t_\ell,\ t_\ell) = 1,\ B^i(t_\ell,\ t_\ell) = 0 \tag{5.4.75}$$

for any ℓ, i. At time t_k, $P_s^i(t_k) = 0$ (perfect state information) and $\bar{s}^i(t_k)$ equals the currently observed $s(t_k)$.

Then the discrete time quantities are given by

$$\bar{s}^i_{\ell+1/k} = \bar{s}^i(t_{\ell+1}), \tag{5.4.76}$$

$$P^i_{s\ell+1/k} = P_s^i(t_{\ell+1}), \tag{5.4.77}$$

$$\phi^i_\ell = \phi^i(t_{\ell+1},t_\ell), \tag{5.4.78}$$

$$B^i_\ell = \phi^i_\ell\ B(t_{\ell+1},\ t_\ell), \tag{5.4.79}$$

$$\ell = k,\ \ldots,\ T-1$$

and the associated perturbation system model is as follows:

$$\delta \bar{s}^{i}_{\ell+1/k} = \bar{s}_{\ell+1/k} - \bar{s}^{i}_{\ell+1/k} = \phi^{i}_{\ell}\, \delta \bar{s}^{i}_{\ell/k} + \Xi^{i}_{\ell}\, \delta u^{i}_{\ell}\ , \tag{5.4.80}$$

$\ell = k, \ldots, T-1$

where $u^{i}_{\ell} = u_{\ell} - u^{i}_{\ell}$. (5.4.81)

c. Compute the coefficients of the quadratic costs

$$\tilde{\ell}(\delta \bar{s}^{i}_{\ell/k}, \delta u^{i}_{\ell}) = N^{i}_{s\ell}\, \delta \bar{s}^{i}_{\ell/k} + N^{i}_{u\ell} \delta u^{i}_{\ell} + $$

$$+ \frac{1}{2} N^{i}_{ss\ell} (\delta \bar{s}^{i}_{\ell/k})^{2} + \frac{1}{2} N^{i}_{uu\ell} (\delta u^{i}_{\ell})^{2} + $$

$$+ N^{i}_{us\ell} (\delta \bar{s}^{i}_{\ell/k})(\delta u^{i}_{\ell}), \tag{5.4.82}$$

for $\ell = k, \ldots, T-1$ by

$$\left.\begin{aligned}
N^{i}_{s\ell} &= \frac{\partial g(\bar{s}^{i}_{\ell/k}, u^{i}_{\ell})}{\partial s}, \\
N^{i}_{u\ell} &= \frac{\partial g(\bar{s}^{i}_{\ell/k}, u^{i}_{\ell})}{\partial u}, \\
N^{i}_{ss\ell} &= \frac{\partial^{2} g(\bar{s}^{i}_{\ell/k}, u^{i}_{\ell})}{\partial s^{2}} \\
N^{i}_{uu\ell} &= \frac{\partial^{2} g(\bar{s}^{i}_{\ell/k}, u^{i}_{\ell})}{\partial u^{2}}, \\
N^{i}_{us\ell} &= \frac{\partial^{2} g(\bar{s}^{i}_{\ell/k}, u^{i}_{\ell}}{\partial u \partial s}
\end{aligned}\right\} \tag{5.4.83}$$

($g(\cdot,\cdot)$ is given in Section 5.4.3) and similarly for the terminal cost term

$$\tilde{\ell}(\delta \bar{s}^{i}_{T/k}) = N^{i}_{sT}\, \delta \bar{s}^{i}_{T/k} + \frac{1}{2} N^{i}_{ssT} (\delta \bar{s}^{i}_{T/k})^{2}. \tag{5.4.84}$$

d) Check the validity of the probabilistic state constraints:

$$\left.\begin{aligned} &\bar{s}^{i}_{\ell/k} \geq s^{min}_{\ell} - z^{min}_{\ell}\sqrt{P^{i}_{s\ell/k}}, \\ &\bar{s}^{i}_{\ell/k} \leq s^{max}_{\ell} - z^{max}_{\ell}\sqrt{P^{i}_{s\ell/k}}, \\ &\ell = k+1, \ldots, T. \end{aligned}\right\} \tag{5.4.85}$$

where $\{z^{min}_{\ell}, z^{max}_{\ell}\}^{T}_{\ell=K+1}$ are specified as explained in Section 5.4.3 from the probabilistic allowances $\{\gamma^{min}_{\ell}, \gamma^{max}_{\ell}\}^{T}_{\ell=k+1}$. If some constraint at time m is violated, modify the corresponding quadratic costs in Step c by

$$N^{i'}_{ssm} = N^{i}_{ssm} + C_i\, N^{i}_{ssm} \tag{5.4.86}$$

$$N^{i'}_{sm} = N^{i}_{sm} + C_i\, N^{i}_{ssm}\, \Delta s_m \tag{5.4.87}$$

where

$$\Delta s_m = \left\{\begin{aligned} &\bar{s}^{i}_{m/k} - (s^{min}_{m} - z^{min}_{m}\sqrt{P^{i}_{sm/k}}), \text{ for lower constr. violoation} \\ &\bar{s}^{i}_{m/k} - (s^{max}_{m} - z^{max}_{m}\sqrt{P^{i}_{sm/k}}), \text{ for upper constr. violation} \end{aligned}\right\} \tag{5.4.88}$$

and $C_i = C^{i}$ (here i is the exponent, $C \in [4,10]$. (5.4.89)

e) Compute the Gradient vector of the performance index

$$\tilde{J} = \sum_{\ell=k}^{T-1} \tilde{\ell}(\delta\bar{s}^{i}_{\ell/k}, \delta u^{i}_{\ell}) + \tilde{\ell}(\delta\bar{s}^{i}_{T/k}) \tag{5.4.90}$$

at the current nominal trajectories - $\{\delta\bar{s}^{i}_{\ell/k}=0,\ \delta u^{i}_{\ell}=0\}$, $\delta\bar{s}^{i}_{T/k} = 0$ by

$$\left.\begin{aligned} &\left[\frac{\partial\tilde{J}}{\partial u_{\ell}}\right]_{i} = N^{i}_{u\ell} + B^{i}_{\ell}\, p^{i}_{\ell+1}, \\ &\ell=k, \ldots, T-1 \end{aligned}\right\} \tag{5.4.91}$$

$$\left.\begin{aligned} \text{where}\quad & p^{i}_{T} = N^{i}_{sT} \\ & p^{i}_{j} = N^{i}_{sj} + \phi^{i}_{j}\, p^{i}_{j+1}, \\ & j = T-1, \ldots, k+1 \end{aligned}\right\} \tag{5.4.92}$$

f) Compute the diagonal elements of the Hessian matrix at the ith nominal trajectories from

$$\left.\begin{aligned} &\left[\frac{\partial^{2}\tilde{J}}{\partial u^{2}_{\ell}}\right]_{i} = N^{i}_{uu\ell} + (B^{i}_{\ell})^{2}\, G^{i}_{\ell+1} \\ &\ell=k, \ldots, T-1 \end{aligned}\right\} \tag{5.4.93}$$

$$\left.\begin{aligned} \text{where}\ G^{i}_{T} &= N^{i}_{ssT} \\ G^{i}_{j} &= N^{i}_{ssj} + (\phi^{i}_{j})^{2} \\ j &= T-1, \ldots, k+1. \end{aligned}\right\} \tag{5.4.94}$$

g. Test the optimality of the ith nominal trajectory by the following criterion:

Compute

$$w_i = \left[\sum_{\ell=k}^{T-1} \left[u_\ell^i - \left[u_\ell^i - \left(\frac{\partial \tilde{J}}{\partial u_\ell} \Big/ \frac{\partial^2 \tilde{J}}{\partial u_\ell^2} \right)_i \right]^{++} \right]^2 \right]^{1/2} \tag{5.4.95}$$

where

$$\left[u_\ell^i - \left(\frac{\partial \tilde{J}}{\partial u_\ell} \Big/ \frac{\partial^2 \tilde{J}}{\partial u_\ell^2} \right)_i \right]^{++} = \begin{cases} u_\ell^{min}, \text{ if } u_\ell^i - \left(\frac{\partial \tilde{J}}{\partial u_\ell} \Big/ \frac{\partial^2 \tilde{J}}{\partial u_\ell^2} \right)_i < u_\ell^{min} \\ u_\ell^{max}, \text{ if } u_\ell^i - \left(\frac{\partial \tilde{J}}{\partial u_\ell} \Big/ \frac{\partial^2 \tilde{J}}{\partial u_\ell^2} \right)_i > u_\ell^{max} \\ u_\ell^i - \left(\frac{\partial \tilde{J}}{\partial u_\ell} \Big/ \frac{\partial^2 \tilde{J}}{\partial u_\ell^2} \right)_i, \text{ otherwise} \end{cases} \tag{5.4.96}$$

$$\ell = k, \ldots, T-1$$

Assuming that the problem is convex, if w_i is negligibly small, then the ith nominal trajectory is a global minimum. This holds because $w_i \simeq 0$ implies either that $\frac{\partial \tilde{J}}{\partial u_\ell} \simeq 0$ or that u equals one of their bounds with the descent direction pointing toward infeasible regions Given that $w_i \simeq 0$, then if C_i is sufficiently high, the ith nominal trajectory is the optimal solution to the control and state constrained problem. If C_i is not as high, some state constraints may be violated. In that case set

$$u_\ell^{i+1} = u_\ell^i \tag{5.4.97}$$

$$\ell = k, \ldots, T-1$$

and repeat steps d, e, f, g.

If the problem is not convex, then $[\frac{\partial^2 \tilde{J}}{\partial u_\ell^2}]_i$ should be replaced by a positive number (unity could be a choice). In this case, $w_i \simeq 0$ implies a point satisfying the first order necessary conditions for optimality (see Equation (4.6.25)).

If w_i is not negligible, continue to step h.

h. Determine the binding constraint set $A^{++}(\underline{u}^i)$ from

$$A^{++}(\underline{u}^i) = \{j/u_j^{min} \leq u_j^i \leq u_j^{min} + \varepsilon_i \text{ and } \left(\frac{\partial \tilde{J}}{\partial u_j}\right)_i > 0$$

$$\text{or } u_j^{max} - \varepsilon_i \leq u_j^i \leq u_j^{max} \text{ and } \left(\frac{\partial \tilde{J}}{\partial u_j}\right)_i < 0$$

$$j = k, \ldots, T-1\} \tag{5.4.98}$$

where $\varepsilon_i = \min\{\varepsilon, w_i\}$, with ε being a small positive number.

i. Compute the two parts (nonbinding and binding) of the Newton's direction as follows:

For the nonbinding control variables (i.e. $\ell \notin A^{++}(\underline{u}^i)$)

$$d_\ell^i = \frac{(N_{us\ell}^i + B_\ell^i K_{\ell+1}^i \phi_\ell^i)\, \delta \bar{s}_{\ell/k}^i + N_{u\ell}^i + B_\ell^i k_{\ell+1}^i}{N_{uu\ell}^i + (B_\ell^i)^2 K_{\ell+1}^i} \tag{5.4.99}$$

where

$$K_T^i = N_{ssT}^i \tag{5.4.100}$$

$$k_T^i = N_{sT}^i \tag{5.4.101}$$

and for $\ell=T-1, \ldots, k+1$, if $\ell \notin A^{++}(\underline{u}^i)$

$$K_\ell^i = N_{ss\ell}^i + (\phi_\ell^i)^2 K_{\ell+1}^i - \frac{[N_{us}^i + B_\ell^i K_{\ell+1}^i \phi_\ell^i]^2}{N_{uu\ell}^i + (B_\ell^i)^2 K_{\ell+1}^i} \quad (5.4.102)$$

$$k_\ell^i = N_\ell^i + \phi_\ell^i \; {}_{\ell+1}^i - \frac{(N_{us\ell}^i + B_\ell^i K_{\ell+1}^i \phi_\ell^i)(N_{u\ell}^i + B_\ell^i k_{\ell+1}^i)}{N_{uu\ell}^i + (B_\ell^i)^2 K_{\ell+1}^i} \quad (5.4.103)$$

while if $\ell \in A^{++}(\underline{u}^i)$

$$K_\ell^i = N_{ss\ell}^i + (\phi_\ell^i)^2 K_{\ell+1}^i \quad (5.4.104)$$

$$k_\ell^i = N_{s\ell}^i + \phi_\ell^i k_{\ell+1}^i. \quad (5.4.105)$$

The state deviations $\delta\bar{s}_{\ell/k}^i$ in (5.4.99) are obtained from

$$\delta\bar{s}_{\ell+1/k}^i = \phi_\ell^i \; \delta\bar{s}_{\ell/k}^i + B_\ell^i d_\ell^i \quad (5.4.106)$$

$$\ell=k, \ldots, T-1$$

with $\delta\bar{s}_{k/k}^i = 0$ and d_ℓ^i as given by (5.4.99) if $\ell \in A^{++}(\underline{u}^i)$ and zero otherwise.

The binding Newton's direction (i.e. when $\ell \quad A^{++}(\underline{u}^i)$) is given by

$$d_\ell^i = -\left(\frac{\partial\tilde{J}}{\partial u_\ell}\right)\left(\frac{\partial^2\tilde{J}}{\partial u_\ell^2}\right)_i^{-1}. \quad (5.4.107)$$

(In nonconvex problems $\frac{\partial^2\tilde{J}}{\partial u_\ell^2}$ should be replaced by some positive number.)

j. Select a setpsize α_i from the following Armijo stepsize selection rule:

$$\alpha_i = \beta^{mi} \quad (5.4.108)$$

where m_i is the first nonnegative integer m for which

$$\tilde{J}(\underline{0}) - \tilde{J}([\beta^m \underline{d}^i]^{++}) \geq$$

$$-\sigma \left[\beta^m \sum_{\ell \notin A^{++}(\underline{u}^i)} \left(\frac{\partial \tilde{J}}{\partial u_\ell}\right)_i d_\ell^i + \sum_{\ell \in A^{++}(\underline{u}^i)} \left(\frac{\partial \tilde{J}}{\partial u_\ell}\right)_i [\beta^m d_\ell^i]^{++}\right] \tag{5.4.109}$$

and $\beta \varepsilon (0,1)$, $\sigma \varepsilon (1, \frac{1}{2})$

$$[\beta^m d_\ell^i]^{++} = \begin{cases} u_\ell^{min} - u_\ell^i, \text{ if } u_\ell^i + \beta^m d_\ell^i < u_\ell^{min}, \\ u_\ell^{max} - u_\ell^i, \text{ if } u_\ell^i + \beta^m d_\ell^i > u_\ell^{max}, \\ \beta^m d_\ell^i, \text{ otherwise} \end{cases} \tag{5.4.110}$$

k. Perform the iteration

$$u_\ell^{i+1} = u_\ell^i + \alpha_i d_\ell^i \tag{5.4.111}$$

$$\ell = k, \ldots, T-1$$

and repeat steps to k until an optimal trajectory $\{u_\ell^*\}_{\ell=k}^{T-1}$ (according to the criterion in Step g) is identified. (Care should be taken so that the specified probabilistic levels $\{\gamma_\ell^{max}, \gamma_\ell^{min}\}_{\ell=k+1}^{T}$ allow for a nonempty feasible solution set.)

l. Apply the optimal action u_k^* and repeat the previous operations at time k+1.

Before presenting computational experience with the above algorithm some variations will be noted:

1. In the version outlined the method at each iteration simultaneously accounts for both control and state constraints. This is expected to be effective for convex problems with linear or mildly nonlinear dynamics and convex cost functionals (as, for example, the HAD operation case). The theoretically correct approach (see Section 4.7.1) is to completely solve the control constrained problem for each new increase of the penalty parameter. Although this will add computational load, it will also be a safer route to follow.

2. In the event of "ill-conditioning" where C_i has been raised to numerically problematic levels and there are still more iterations needed for the algorithm to converge, we have found it reliable to restart the C_i increment cycles until convergence is induced. The idea is that increasingly better nominal trajectories are being identified and used as initial choices. However, this may be an issue in problems of very long control horizons (> 1000 time steps).

3. A modification which in certain cases can be particularly effective refers to second guessing the binding control constraint set $A^{++}(\underline{u}^i)$. Notice that the Newton's directions computed in Step i are specified according to the set $A^{++}(\underline{u}^i)$ determined in the previous Step h. However, this set is identified based on steepest descent considerations which may under- or overestimate the Newton's direction steps. Thus, after having computed the directions $\{d_\ell^i\}_{\ell=k}^{T-1}$, the idea is to go back to Step h and second guess the binding constraint set $A^{++}(\underline{u}^i)$ from

$$A^{++}(\underline{u}^i) = \{j/u_j^{min} \leq u_j^i \leq u_j^{min} + \varepsilon_i \text{ and } u_j^i + d_j^i \leq u_j^{min} + \varepsilon_i$$

$$\text{or } u_j^{max} - \varepsilon_i \leq u_j^i \leq u_j^{max} \text{ and } u_j^{max} - \varepsilon_i \leq u_j^i + d_j^i$$

$$j = k, \ldots, T-1\}. \tag{5.4.112}$$

At first this may seem to impose more computational load, yet it may save iterations especially when there are many loosely binding constraints at the solution. The above can be repeated until the same binding set $A^{++}(\underline{u}^i)$ is determined in two subsequent iterations.

4. At the end of Section 4.7 a suboptimality of the previous algorithm was mentioned concerning the use of the variance $P_{s\ell/k}$ in the specification of the state bounds. In essence the problem is created because the Open Loop Feedback controller at time k assumed no future decision adjustments and specifies the optimal trajectory for the entire control horizon. However, in reality only u_k^* will be used and the decision procedure will be reinitiated to identify what is to be applied next. Hence, attempting to maintain

$$s_\ell^{min} \leq \bar{s}_{\ell/k} - z_\ell^{min} \sqrt{P_{s\ell/k}} \text{ and } \bar{s}_{\ell/k} + z_\ell^{max} \sqrt{P_{s\ell/k}} \leq s_\ell^{max} \tag{5.4.113}$$

$$\ell = k+1, \ldots, T$$

ignores that at times $k+1, \ldots, T-1$ there exists the option to modify the decision considered optimal at time k. On the other hand, the allowable modification is contingent upon the control bounds and it may not be possible. An

idea attempting to remedy the situation is illustrated in Figure 5.4.5. Consider the optimal two-period control sequence (AB_1), (B_2C) producing the (ABC) mean state trajectory with (ADE) the associated upper uncertainty region $(\bar{s}_\ell + z_\ell^{max} \sqrt{P_{s\ell/k}}, \ell = k+1, k+2)$. (These trajectories result from integrating the corresponding differential equations and only for simplicity are here portrayed by straight lines. The lower uncertainty region is not shown.) According to the OLFC philosophy, the mean state value at time k+2 is confined to be lower or equal to C since $C + z_{k+2}^{max} \sqrt{P_{sk+2/k}}$ touches the upper state bound. Next, consider that during period $[t_k, t_{k+1}]$, control (AB_1) is applied and the inflow realization brings the state to D. If control (B_2C) determined at time k is then used, the new state trajectory would be (DG) and (DE) would be the corresponding uncertainty region. However, at time k+1 there exists the option of readjusting u_{k+1} within its bounds (which in this case it is suggested since (B_2C) was optimal when the mean state trajectory was passing from B) including the choice of the maximum release (DH). If control (DH) is used, the mean state trajectory would be (DH) and the associated uncertainty region would be (DF). Hence although at time k it was required that the mean state value at time k+2 be at least (EC) distance away from the upper bound, the probabilistic constraint could still be met even if the previous distance was reduced to (FC). This implies that the feasible state region can safely be augmented by considering the trajectory changes which the unused control portion $(DHCB_2)$ could, if necessary, bring about. However, notice that no state shift would be possible if the time k nominal control sequence was (AB_1), (LH). Since these

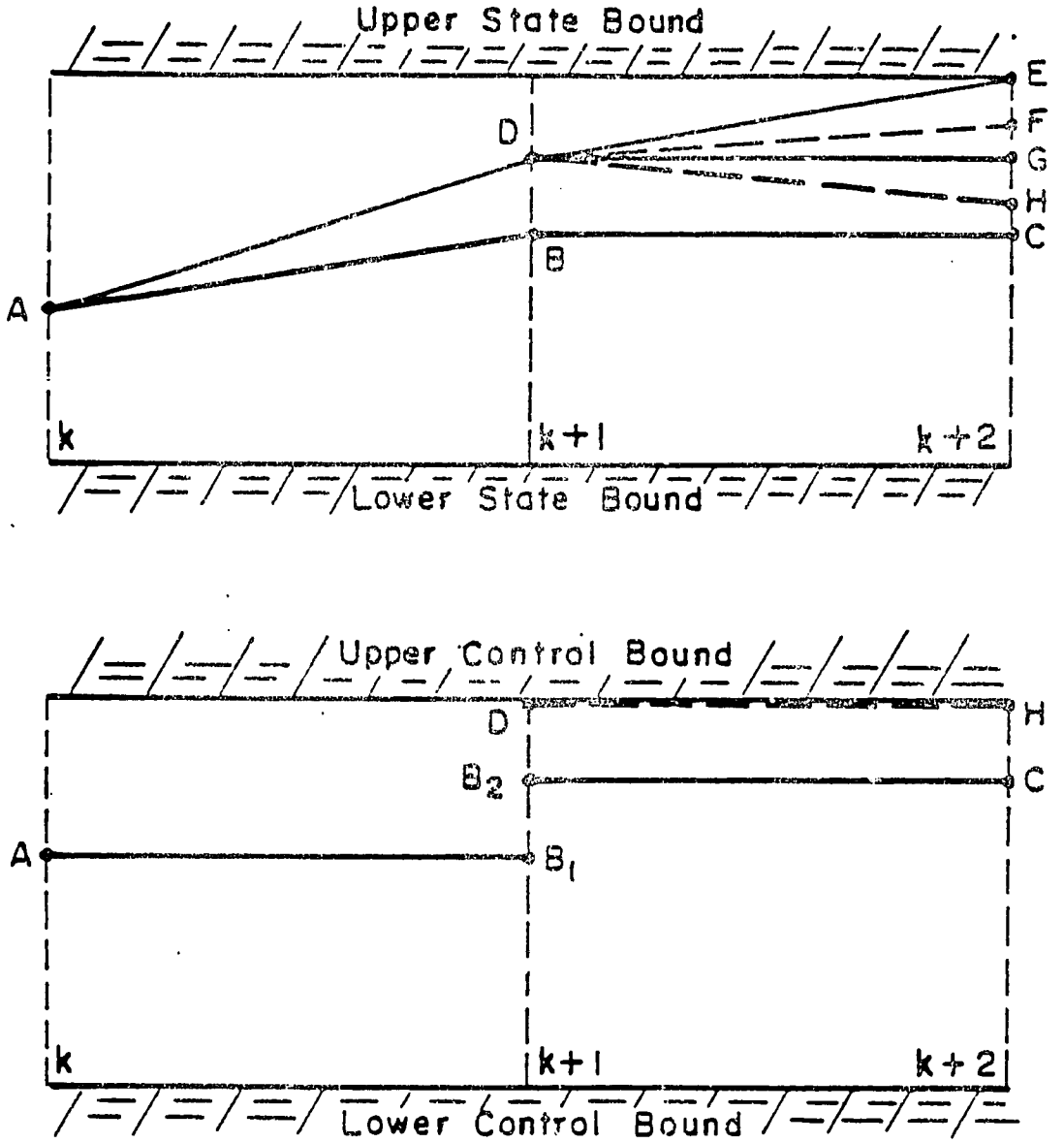

FIGURE 5.4.5: MODIFYING THE VARIANCE OF THE OLFC PROCEDURE

trajectory changes are precomputable, the following modification of the state bounds is suggested: Suppose that at a particular time ℓ the minimum allowable distance of the mean state trajectory from its upper bound has been specified to be x_ℓ^{max}. Determine the state variance $P_{s\ell/k}^{max}$ which based on the probabilistic allowance γ_ℓ^{max} corresponds to x_ℓ^{max}; namely

$$P_{s\ell/k}^{max} = \left(\frac{x_\ell^{max}}{z_\ell^{max}} \right)^2 \tag{5.4.114}$$

where z_ℓ^{max} is as in Section 5.4.3. Assume, further, that the current nominal control and state trajectories are denoted by $\{\bar{s}_{\ell/k}^{i}\}_{\ell=k}^{T}$, $\{u_\ell^i\}_{\ell=k}^{T-1}$. Setting the control variable $u(\tau)$, $\tau \in [t_\ell, t_{\ell+1}]$ at its upper bound, integrate the state's mean and variance equations

$$\frac{d\bar{s}(\tau)}{d\tau} = f(\bar{s}(\tau),\tau) - u^{max}(\tau) + \bar{w}(\tau) \tag{5.4.115}$$

$$\frac{dP_s(\tau)}{d\tau} = 2f_1(\bar{s}(\tau),\tau)\, P_s(\tau) + Q_w(\tau) \tag{5.4.116}$$

over the interval $[t_\ell, t_{\ell+1}]$ with initial conditions $\bar{s}_{\ell/k}^{i}$ and $P_{s\ell/k}^{max}$ respectively. Denote $\bar{s}_{\ell+1/k}^{max}$ and $P'_{s\ell+1/k}$ as the resulting end of the period quantities. If

$$\bar{s}_{\ell+1/k}^{max} + z_{\ell+1}^{max} \sqrt{P'_{s\ell+1/k}} \geq \bar{s}_{\ell+1/k}^{i} \tag{5.4.117}$$

set $$x_{\ell+1}^{max} = \bar{s}_{\ell+1/k}^{max} + z_{\ell+1}^{max} \sqrt{P'_{s\ell+1/k}} - \bar{s}_{\ell+1/k}^{i}, \tag{5.4.118}$$

otherwise set

$$x_{\ell+1}^{max} = 0 \tag{5.4.119}$$

and continue with the next time period until the end of the control horizon. Case (5.4.119) corresponds to enough evailable control flexibility to reduce the probabilistic state bound as far as the mean state trajectory. Clearly, no further reduction is possible. Similar considerations govern the modification of the lower state bounds. The procedure should start from time k+1 since this is the first time when any control adjustments can take place. Notice that the proposed modifications depend on the current state and control trajectories and may require some more iterations before convergence is achieved. However, it will be seen that using the OLFC variance $\{P_{s\ell/k}\}_{\ell=k+1}^{T}$ can lead to substantial suboptimalities which justifies the additional effort. If the nominal control trajectory lies on any of its boundaries, the proposed procedure becomes identical to the OLFC uncertainty bound specification approach.

A typical ELQG iteration cycle is shown on Figures 5.4.6 - 5.4.17 and Tables 5.4.1 - 5.4.6. The application seeks to determine the optimal HAD release trajectory over a 3-year period (36 months). The Nile inflows to Lake Nasser are assumed to follow their historical monthly distributions with the means and variances shown on Table 5.2.1 (a priori statistics in the terminology of Section 5.4.2). The monthly means are reduced by the monthly Sudan abstractions having the Figure 5.2.7 distribution and a 16.5×10^9 m^3 yearly total. The monthly evaporation rates and downstream water supply requirements are those of Figures 5.2.4 and 5.2.5. The reservoir storage at time 0 (beginning of the control horizon) is set at 95×10^9 m^3 and at any time it is not to exceed 137.72×10^9 m^3 (Toshka operating threshold)

or fall below 32.72×10^9 m^3 (dead storage zone ceiling) at reliability $\gamma = 0.975$. The maximum allowable downstream release is set at 7.59×10^9 m^3 per month. The objective is to maximize the expected energy generation given the previous constraints and the requirement to leave the expected terminal reservoir storage (beginning of the 37th month) at 95×10^9 m^3. The initial nominal control trajectory is taken equal to the downstream water supply requirements and is shown on Figure 5.4.6 together with the upper release bound. Figure 5.4.7 shows the corresponding mean state trajectory (under dotted line) along with the associated modified uncertainty bound. Notice that since the nominal control trajectory equals its lower release bound, there can be no upward state shift and hence the lower uncertainty region equals that of the OLFC procedure. On the contrary, the upper uncertainty region is substantially reduced as a result of the existing capability to release at higher than the nominal rates. The pronounced storage fluctuation is a combined effect of the marked intra-year mandatory release and expected inflow variation. Table 5.4.1 gives the values of some parameters used and some representative quantities computed at each iteration. The "HYDROPOWER" gives the energy in GWH which will be produced if the nominal control and mean state trajectories are realized. The associated value of the penalty cost function actually minimized is given under "PENALTY". Apart from penalizing energy production lower than 1280 GWH per month, this function also penalizes terminal mean storages away from 95×10^9 m^3 according to $\tilde{\ell}(\bar{s}_{T/k}) = (\bar{s}_{T/k}-95)^2 \times 10^8$. "Z" stands for parameter z corresponding to the reliability level γ and "C" represents the penalty parameter C which is increased at each

iteration i according to $C_i = C^i$. "BETA" and "SIGMA" are the parameters β and σ of the Armijo stepsize selection rule and "EI" specifies the value of $\varepsilon_i = \min\{\varepsilon, w_i\}$ introduced in Step h of Algorithm A5.4.1. "M" and "ALPHA" are the integer m_i and the corresponding stepsize $\alpha_i = \beta^{m_i}$ satisfying the Armijo stepsize rule test and "WI" is the Euclidean norm w_i of the optimality criterion. The columns from left to right display the months, mean storage values, nominal releases, first derivatives of the cost function with respect to the controls at the nominal sequences, second cost function derivatives with respect to the controls at the nominal sequences, upper uncertainty bound, lower uncertainty bounds, and the binding contraint set. The positive second derivatives $(\frac{\partial^2 \tilde{J}}{\partial u^2})_i$ (="DUUJ") give an indication of the problem's convexity while the negative first derivative $(\frac{\partial \tilde{J}}{\partial u})_i$ (="DUJ") show that the cost function can be decreased by increasing the nominal releases. A binding lower control constraint is indicated by "-1", a binding upper control constraint by "1", and a nonbinding constraint by "0". Notice that since $(\frac{\partial \tilde{J}}{\partial u})_i$ are found negative, the procedure does not indicate binding control constraints. The high value of "WI" implies that the initial nominal control trajectory is far from being optimal. The first iteration produces the nominal control and state trajectories shown on Figure 5.4.8, 5.4.9, and Table 5.4.2. The releases are now set higher because over a short time horizon (as the one considered here) it is in the interest of the energy generation to pass high discharges through the turbines rather than maintain High hydraulic head. This can also be deduced from Figures 5.4.3 and 5.4.4 where it is seen that the energy production function is much more sensitive to the reservoir

releases than to storage changes. After cutting down the original stepsize by a factor of 2, this iteration has reduced the penalty cost function, increased the energy generation, and brought the terminal storage from 107.1×10^9 m^3 to 93.2×10^9 m^3 (i.e. substantially closer to the desirable value 95×10^9 m^3). Notice the reduction of the lower uncertainty region as a result of the control trajectory displacement away from the lower bound. The new binding constraint set is drastically different and the magnitude of w_i has decreased but remains significant. No state constraint is binding. Iteration 2 is shown on Figure 5.4.10, 5.4.11, and Table 5.4.3. It places the terminal storage at 94.2×10^9 m^3 which further reduces the cost function although energy generation is now decreased and the associated penalty increased. All binding constraints at Iteration 1 are also binding here but the set $A^{++}(\underline{u}^2)$ is now larger. Iteration 3 (Figure 5.4.12, 5.4.13, and Table 5.4.4) reduces even more the cost function by building up higher terminal storage and identifying one more binding constraint. The following Iteration 4 (Figure 5.4.14, 5.4.15, and Table 5.4.5) shares the same binding constraint set with Iteration 3, and as theoretically predicted, materializes on impressive convergence rate essentially terminating the search. This is indicated by the negligible w_i value and is readily concluded by comparing its results with those of Iteration 5 (Figure 5.4.16, 5.4.17, and Table 5.4.6). (Notice, in particular, the coincidence of the last two iterations' nominal control trajectories and the terminal storage value of 94.999×10^9 m^3.) The iterations may continue but the additional improvements are marginal. The

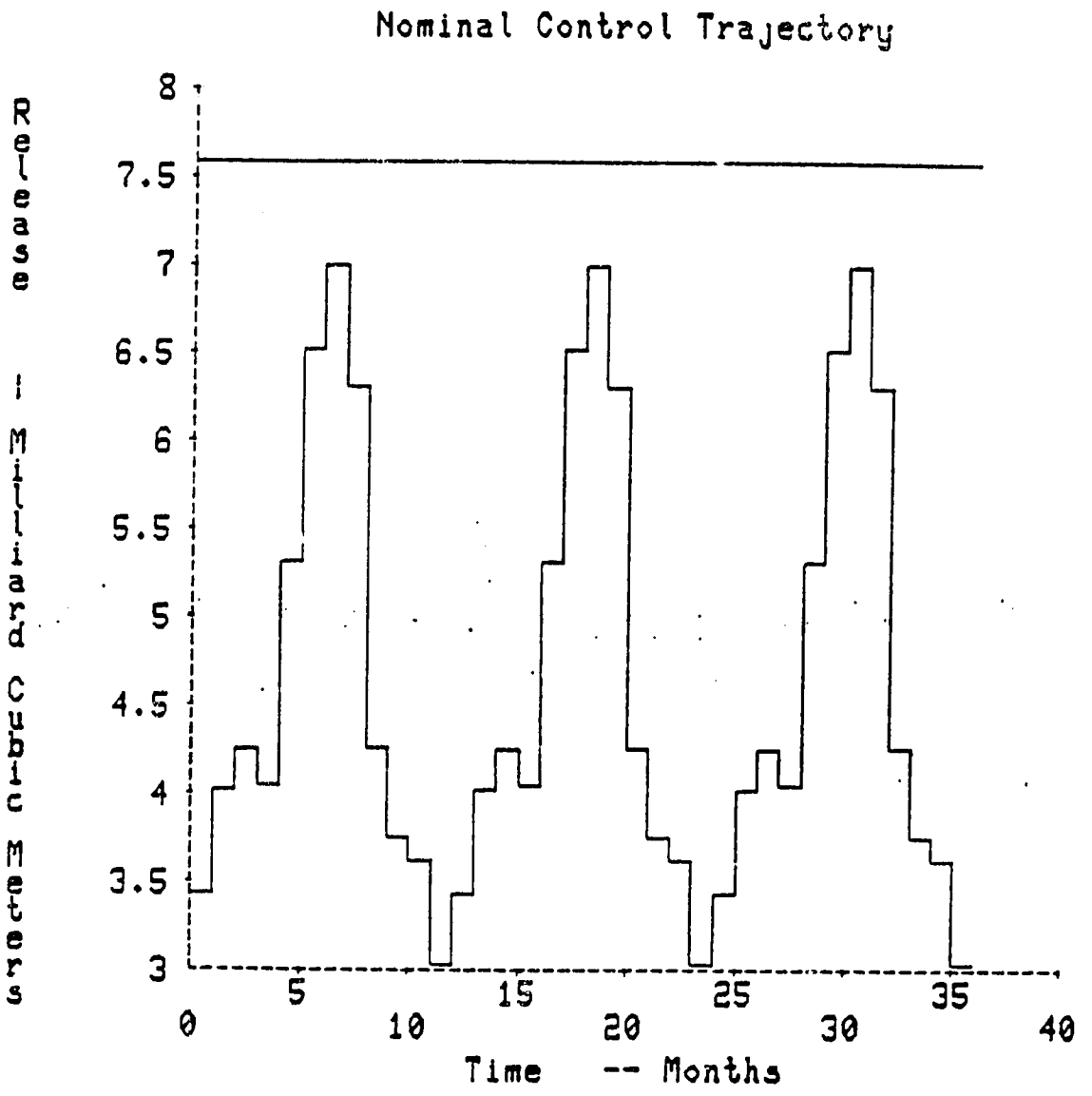

FIGURE 5.4.6: A TYPICAL ELQG ITERATION CYCLE: INITIAL NOMINAL CONTROL TRAJECTORY

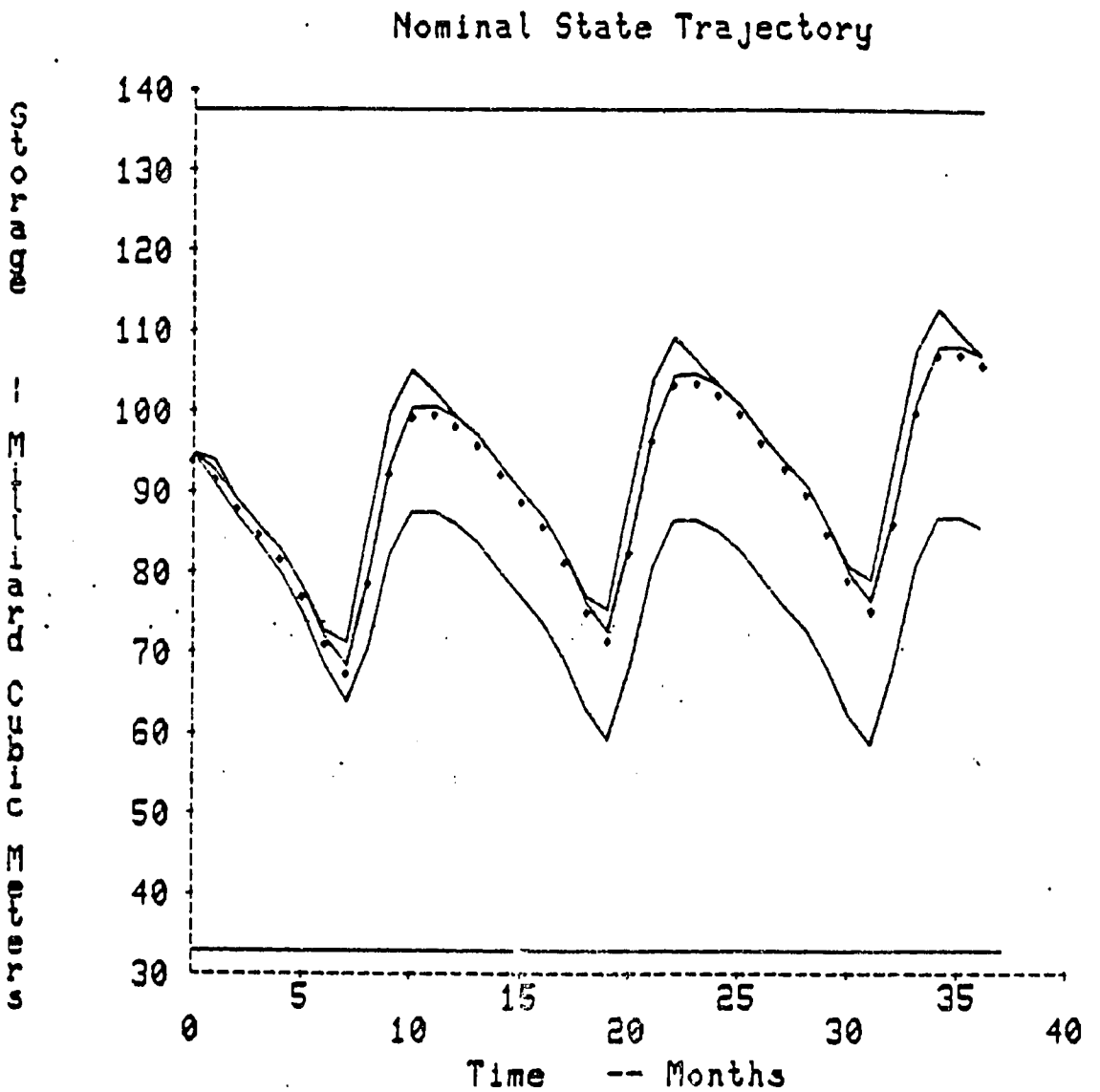

FIGURE 5.4.7: A TYPICAL ELQG ITERATION CYCLE: INITIAL NOMINAL STATE TRAJECTORY

INITIAL TRAJECTORIES

HYDROPOWER (GWH) = 0.20763734E+06
PENALTY = 0.14743756E+11

Z = 0.1960E+01
C = 0.1000E+02
BETA = 0.5000E+00
SIGMA = 0.1000E-03
EI = 0.4998E-02
A = 0
ALPHA = 0.0000E+00
UI = 0.1935E+02

L	E(S)	U	DUJ	DUUJ	STDEU	STDEL	A
1	95.800	3.438	-.17919E+10	0.54521E+08	0.00000E+00	0.00000E+00	0
2	92.581	4.018	-.18824E+10	0.55164E+08	0.73583E+00	0.73583E+00	0
3	89.047	4.240	-.18105E+10	0.55663E+08	0.00000E+00	0.10039E+01	0
4	85.715	4.040	-.18195E+10	0.56217E+08	0.00000E+00	0.12035E+01	0
5	82.557	5.312	-.18319E+10	0.56985E+08	0.00000E+00	0.13327E+01	0
6	77.871	6.521	-.18478E+10	0.57987E+08	0.00000E+00	0.15677E+01	0
7	71.913	6.999	-.18674E+10	0.59221E+08	0.32495E+00	0.17704E+01	0
8	68.306	6.310	-.18899E+10	0.60656E+08	0.13777E+01	0.24117E+01	0
9	79.381	4.251	-.19136E+10	0.62177E+08	0.33790E+01	0.45137E+01	0
10	93.386	3.741	-.19362E+10	0.63667E+08	0.32836E+01	0.58042E+01	0
11	100.538	3.619	-.19566E+10	0.65041E+08	0.23743E+01	0.66295E+01	0
12	100.710	3.819	-.19737E+10	0.66141E+08	0.34549E+00	0.68125E+01	0
13	99.422	3.438	-.19879E+10	0.67094E+08	0.00000E+00	0.68358E+01	0
14	96.974	4.018	-.19993E+10	0.67873E+08	0.00000E+00	0.68518E+01	0
15	93.418	4.240	-.20082E+10	0.68478E+08	0.00000E+00	0.68521E+01	0
16	90.068	4.040	-.20181E+10	0.69149E+08	0.00000E+00	0.68572E+01	0
17	86.884	5.312	-.20315E+10	0.70080E+08	0.00000E+00	0.68531E+01	0
18	82.166	6.521	-.20488E+10	0.71235E+08	0.00000E+00	0.68434E+01	0
19	76.167	6.999	-.20701E+10	0.72771E+08	0.32500E+00	0.68323E+01	0
20	72.512	6.310	-.20946E+10	0.74497E+08	0.13778E+01	0.69574E+01	0
21	83.538	4.251	-.21203E+10	0.76332E+08	0.33795E+01	0.78707E+01	0
22	97.490	3.741	-.21461E+10	0.78124E+08	0.32844E+01	0.86184E+01	0
23	104.595	3.619	-.21673E+10	0.79753E+08	0.23754E+01	0.91818E+01	0
24	104.730	3.819	-.21861E+10	0.81136E+08	0.34687E+00	0.92915E+01	0
25	103.411	3.438	-.22016E+10	0.82292E+08	0.00000E+00	0.93712E+01	0
26	100.937	4.018	-.22141E+10	0.83235E+08	0.00000E+00	0.92625E+01	0
27	97.361	4.240	-.22238E+10	0.83967E+08	0.00000E+00	0.92425E+01	0
28	93.998	4.040	-.22346E+10	0.84789E+08	0.00000E+00	0.92390E+01	0
29	90.790	5.312	-.22493E+10	0.85904E+08	0.00000E+00	0.92040E+01	0
30	86.043	6.521	-.22692E+10	0.87355E+08	0.00000E+00	0.91669E+01	0
31	80.807	6.999	-.22913E+10	0.89147E+08	0.32502E+00	0.91219E+01	0
32	76.310	6.310	-.23179E+10	0.91224E+08	0.13798E+01	0.91740E+01	0
33	87.289	4.251	-.23469E+10	0.93432E+08	0.33799E+01	0.93420E+01	0
34	101.198	3.741	-.23731E+10	0.95607E+08	0.32862E+01	0.10412E+02	0
35	108.201	3.819	-.23974E+10	0.97539E+08	0.23783E+01	0.10569E+02	0
36	108.382	3.819	-.24198E+10	0.99256E+08	0.34766E+00	0.10923E+02	0
37	107.136	0.000	0.00000E+00	0.00000E+00	0.00000E+00	0.10962E+02	0

TABLE 5.4.1: A TYPICAL ELQG ITERATION CYCLE

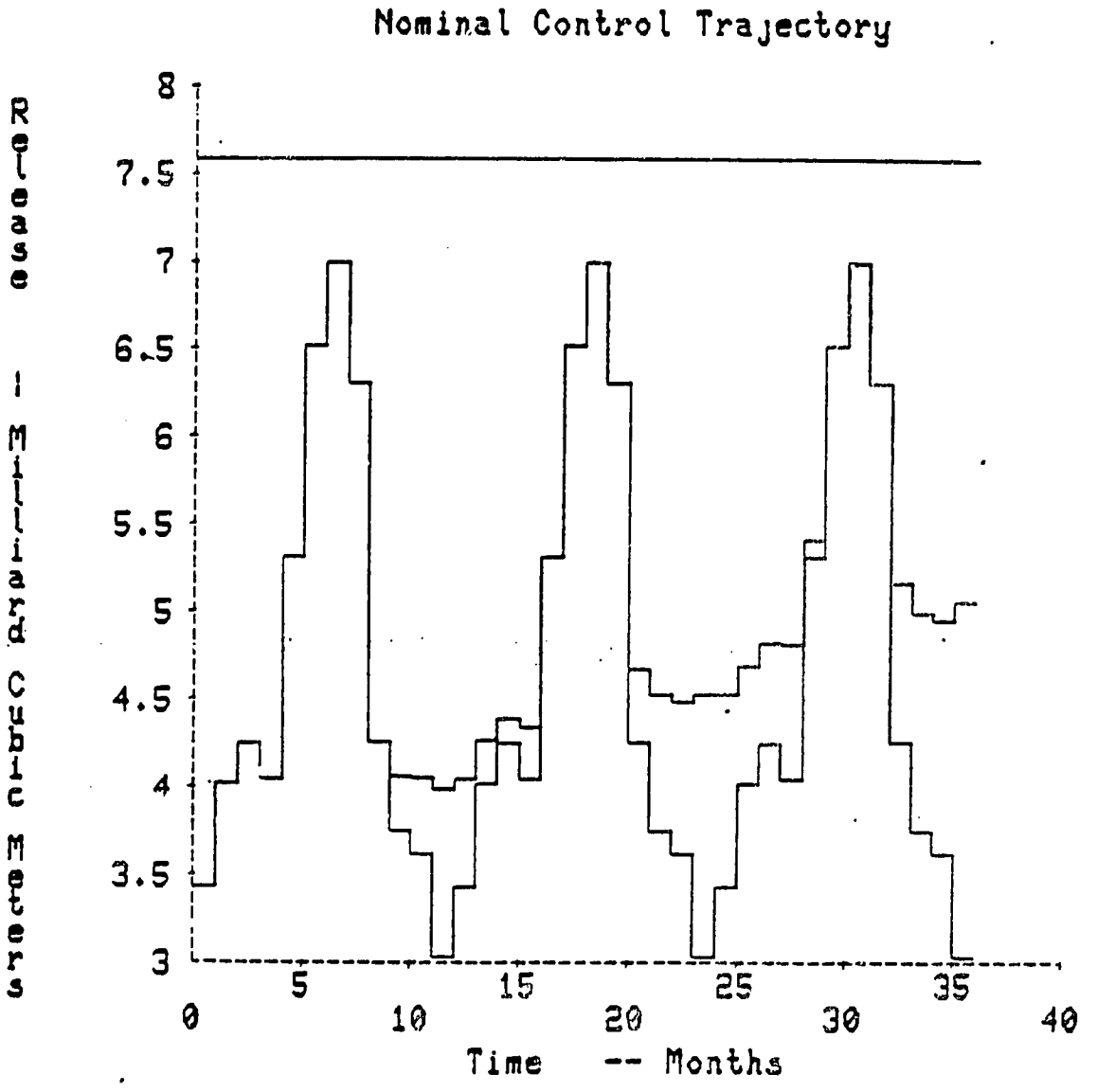

FIGURE 5.4.8: A TYPICAL ELQG ITERATION CYCLE: ITERATION #1

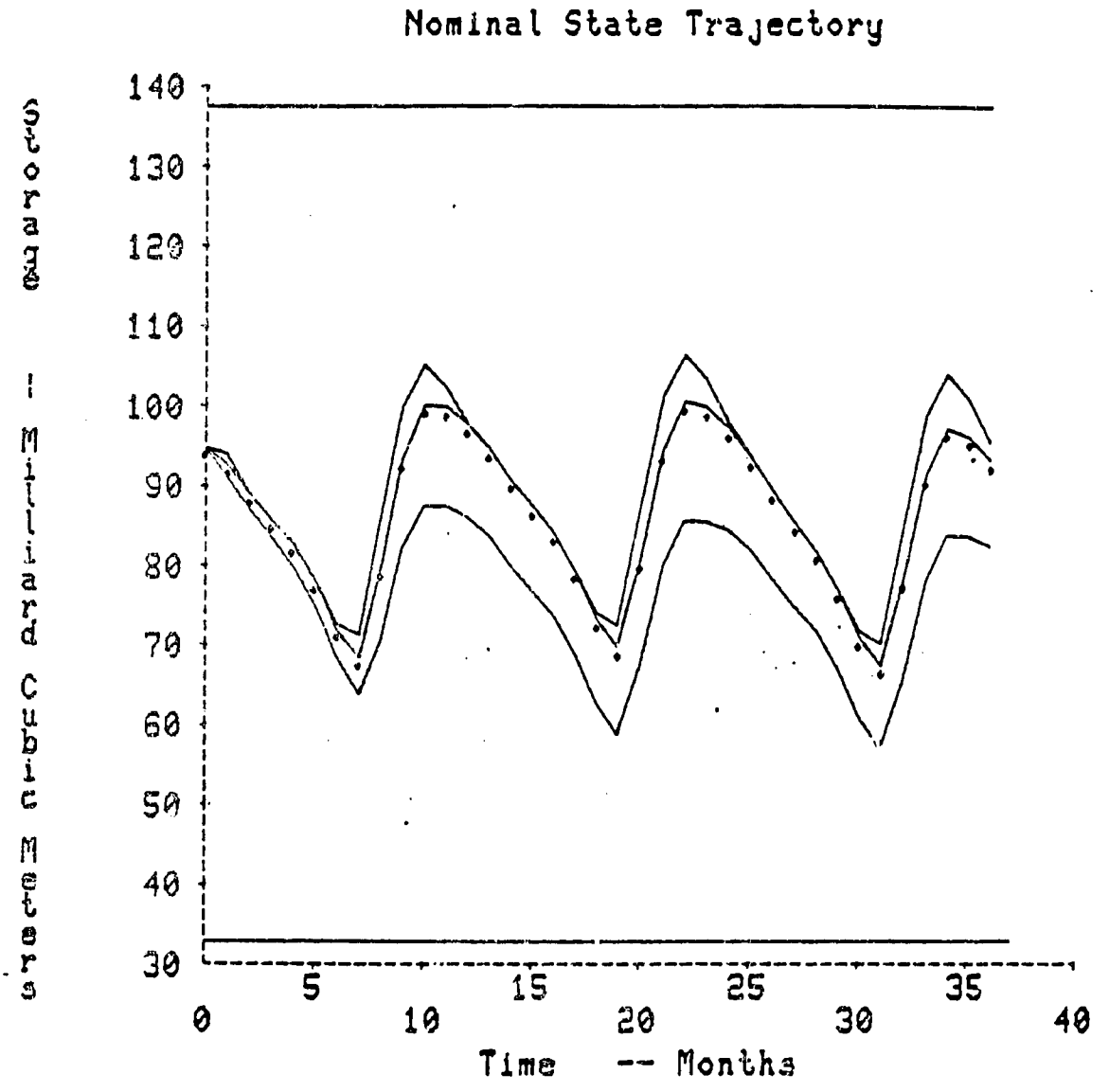

FIGURE 5.4.9: A TYPICAL ELQG ITERATION CYCLE: ITERATION #1

ITERATION # 1

HYDROPOWER (GWH) = 0.22604407E+06
PENALTY = 0.33383595E+03

Z = 0.1960E+01
C = 0.1000E+02
BETA = 0.5000E+00
SIGMA = 0.1000E-03
EI = 0.4998E-02
M = 1
ALPHA = 0.5000E+00
WI = 0.4053E+01

L	E(S)	U	DUJ	DUUJ	STDEU	STDEL	A
1	95.000	3.438	0.27241E+09	0.53983E+03	0.00000E+00	0.00000E+00	-1
2	92.501	4.013	0.27402E+09	0.54624E+03	0.73583E+00	0.73583E+00	-1
3	89.047	4.240	0.27526E+09	0.56119E+03	0.00000E+00	0.18039E+01	-1
4	85.715	4.040	0.27663E+09	0.55667E+03	0.00000E+00	0.12035E+01	-1
5	82.557	5.312	0.27055E+09	0.58433E+03	0.00000E+00	0.13327E+01	-1
6	77.871	6.521	0.29102E+09	0.57419E+03	0.00000E+00	0.15677E+01	-1
7	71.913	6.999	0.29401E+09	0.59841E+03	0.32492E+00	0.17794E+01	-1
8	68.386	6.310	0.28740E+09	0.60062E+03	0.13777E+01	0.24117E+01	-1
9	79.381	4.251	0.29092E+09	0.61563E+03	0.33790E+01	0.45137E+01	-1
10	93.364	4.058	0.29435E+09	0.63035E+03	0.32949E+01	0.57929E+01	0
11	100.194	4.045	0.29744E+09	0.84368E+03	0.25456E+01	0.84564E+01	0
12	99.928	3.976	0.30005E+09	0.85503E+03	0.12103E+01	0.64207E+01	0
13	97.705	4.039	0.30221E+09	0.86460E+03	0.00000E+00	0.53692E+01	0
14	94.674	4.258	0.30397E+09	0.87225E+03	0.00000E+00	0.58937E+01	0
15	90.897	4.383	0.30534E+09	0.87323E+03	0.00000E+00	0.55831E+01	0
16	87.405	4.337	0.30634E+09	0.88500E+03	0.00000E+00	0.56275E+01	0
17	83.962	5.312	0.30895E+09	0.89431E+03	0.00000E+00	0.53988E+01	-1
18	79.285	6.521	0.31167E+09	0.70640E+03	0.00000E+00	0.54199E+01	-1
19	73.294	6.999	0.31496E+09	0.73134E+03	0.32492E+00	0.54264E+01	-1
20	68.671	6.310	0.31370E+09	0.73870E+03	0.13777E+01	0.58162E+01	-1
21	80.714	4.874	0.32250E+09	0.75714E+03	0.33873E+01	0.67355E+01	0
22	94.256	4.523	0.32539E+09	0.77512E+03	0.35165E+01	0.73937E+01	0
23	100.317	4.487	0.32982E+09	0.79148E+03	0.29428E+01	0.78533E+01	0
24	98.984	4.623	0.33271E+09	0.80544E+03	0.17889E+01	0.73461E+01	0
25	97.138	4.623	0.33611E+09	0.81710E+03	0.38146E+00	0.66033E+01	0
26	93.631	4.693	0.33707E+09	0.82668E+03	0.00000E+00	0.60745E+01	0
27	89.424	4.832	0.33859E+09	0.83411E+03	0.00000E+00	0.57484E+01	0
28	85.498	4.811	0.34023E+09	0.84244E+03	0.00000E+00	0.54658E+01	0
29	81.812	5.407	0.34262E+09	0.85395E+03	0.00000E+00	0.50995E+01	0
30	78.583	6.521	0.34566E+09	0.86902E+03	0.00000E+00	0.54716E+01	-1
31	78.304	6.000	0.34936E+09	0.33762E+03	0.32498E+00	0.53942E+01	-1
32	67.308	6.318	0.35356E+09	0.84927E+03	0.13776E+01	0.53025E+01	-1
33	78.342	5.138	0.35798E+09	0.93223E+03	0.33964E+01	0.64727E+01	0
34	91.444	4.908	0.36231E+09	0.96462E+03	0.37717E+01	0.69191E+01	0
35	97.373	4.844	0.36804E+09	0.97493E+03	0.33710E+01	0.68050E+01	0
36	96.234	5.053	0.36930E+09	0.99233E+03	0.23741E+01	0.64674E+01	0
37	93.145	0.000	0.00000E+00	0.00000E+00	0.11130E+01	0.55625E+01	0

TABLE 5.4.2: A TYPICAL ELQG ITERATION CYCLE

Best Available Document

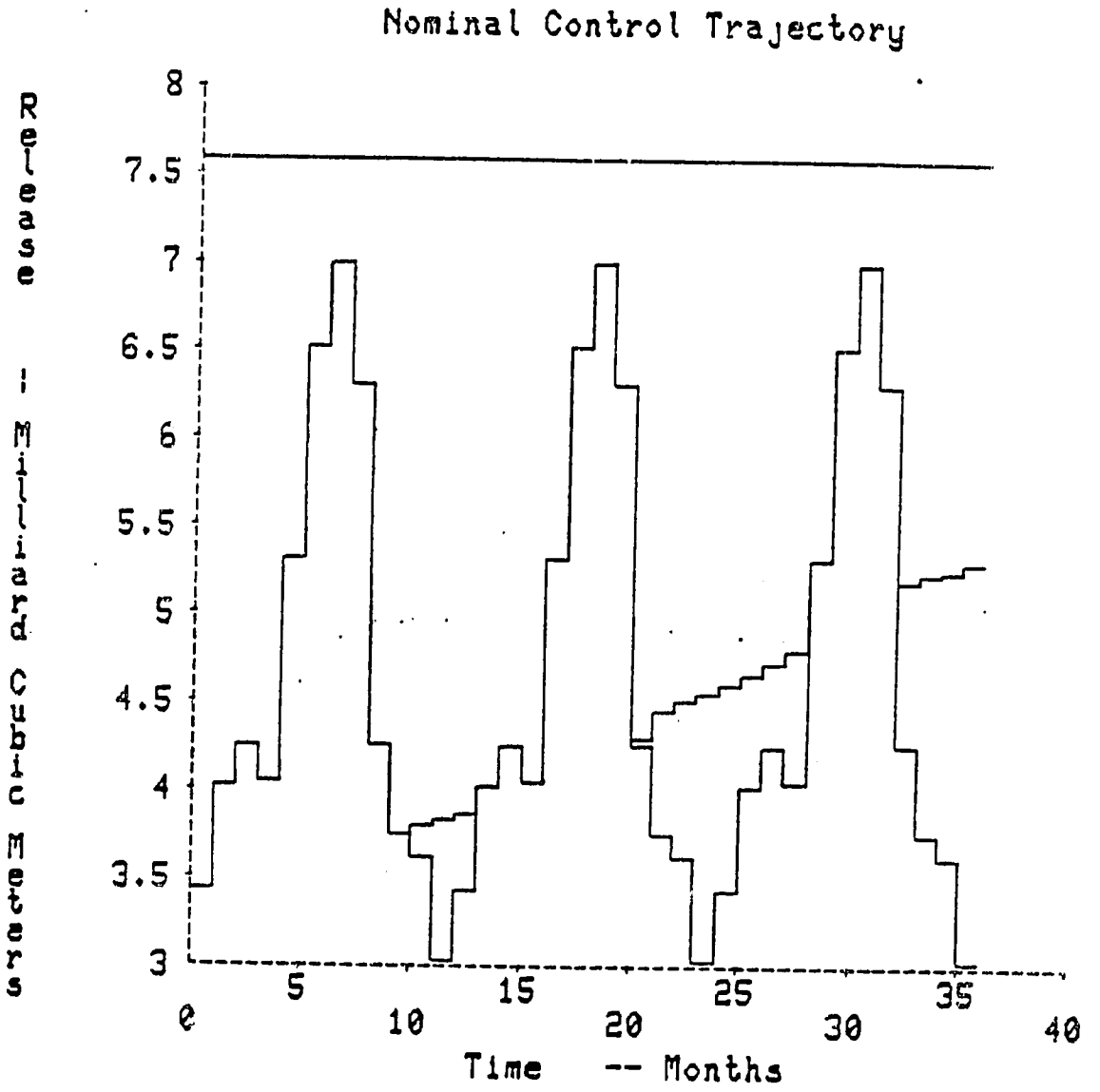

FIGURE 6.4.10: A TYPICAL ELQG ITERATION CYCLE: ITERATION #2

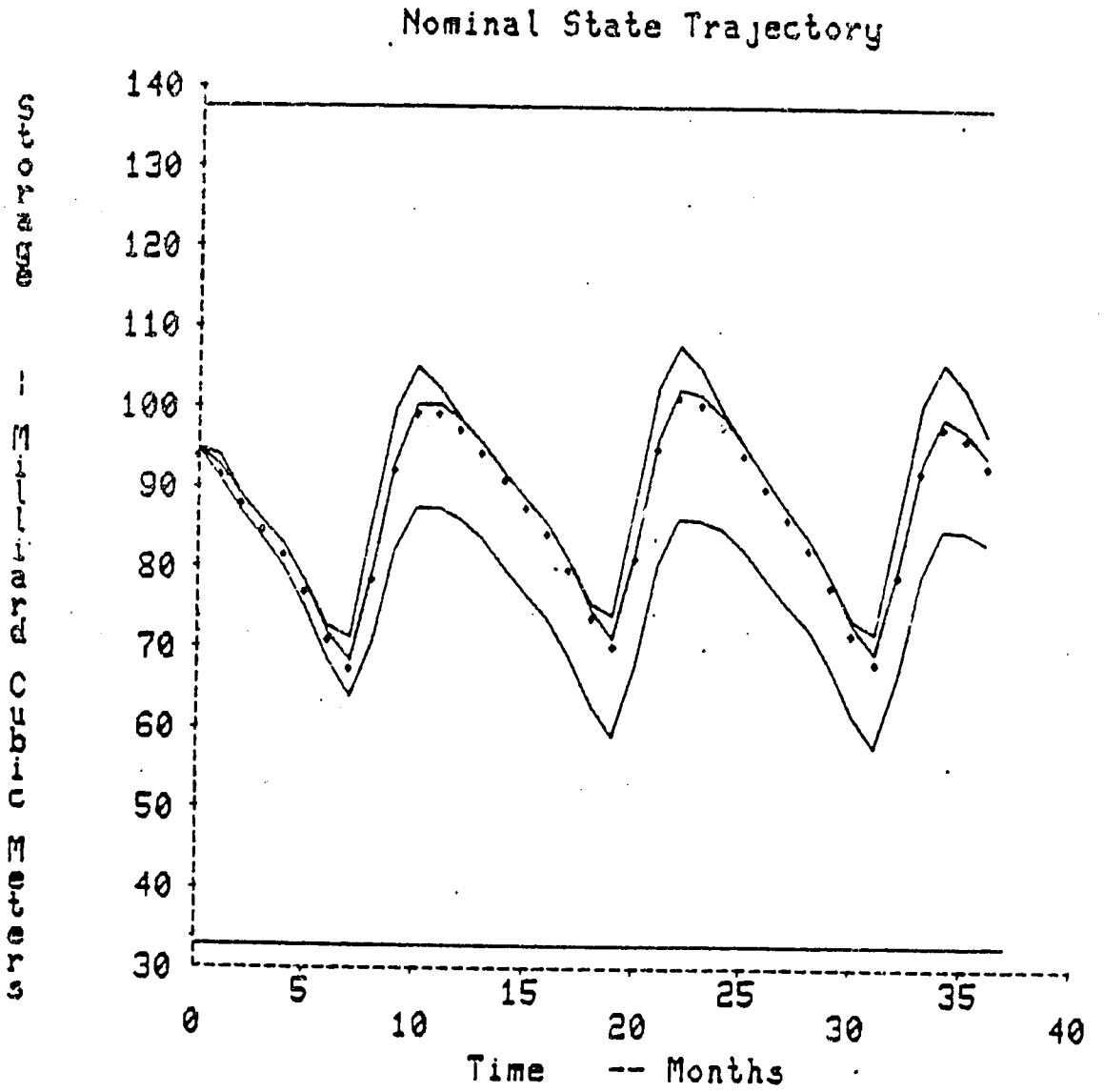

FIGURE 5.4.11: A TYPICAL ELQG ITERATION CYCLE: ITERATION #2

I T E R A T I O N # 2

HYDROPOWER (GWH) = 0.22553737E+06
PENALTY = 0.88703550E+03

Z = 0.1968E+01
C = 0.1000E+02
BETA = 0.5000E+00
SIGMA = 0.1000E-03
EI = 0.4983E-02
M = 0
ALPHA = 0.1000E+01
WI = 0.3964E+01

L	E(S)	U	DUJ	DUUJ	STDEU	STDEL	A
1	95.000	3.430	0.12543E+09	0.54139E+08	0.00000E+00	0.00000E+00	-1
2	92.581	4.918	0.12618E+09	0.54777E+08	0.73583E+00	0.73583E+00	-1
3	89.047	4.248	0.12675E+09	0.55272E+08	0.00000E+00	0.10039E+01	-1
4	85.715	4.840	0.12738E+09	0.55823E+08	0.00000E+00	0.12035E+01	-1
5	82.557	5.312	0.12828E+09	0.56589E+08	0.00000E+00	0.13927E+01	-1
6	77.871	6.521	0.12944E+09	0.57580E+08	0.00000E+00	0.15677E+01	-1
7	71.913	6.999	0.13083E+09	0.58806E+08	0.32499E+00	0.17704E+01	-1
8	68.306	6.310	0.13237E+09	0.60231E+08	0.13777E+01	0.24117E+01	-1
9	79.381	4.251	0.13396E+09	0.61741E+08	0.33790E+01	0.46137E+01	-1
10	93.336	3.741	0.13562E+09	0.63210E+08	0.32836E+01	0.58042E+01	-1
11	100.530	3.793	0.13694E+09	0.64546E+08	0.23774E+01	0.66284E+01	0
12	100.568	3.829	0.13814E+09	0.65882E+08	0.24794E+00	0.67096E+01	0
13	98.422	3.859	0.13913E+09	0.66630E+08	0.00000E+00	0.63290E+01	0
14	95.574	4.818	0.13994E+09	0.67484E+08	0.00000E+00	0.61416E+01	-1
15	92.025	4.240	0.14057E+09	0.68007E+08	0.00000E+00	0.61493E+01	-1
16	88.682	4.840	0.14126E+09	0.68877E+08	0.00000E+00	0.61689E+01	-1
17	85.506	5.312	0.14225E+09	0.69807E+08	0.00000E+00	0.61645E+01	-1
18	80.793	6.521	0.14351E+09	0.70811E+08	0.00000E+00	0.61645E+01	-1
19	74.811	6.999	0.14503E+09	0.72296E+08	0.32500E+00	0.61651E+01	-1
20	71.178	6.310	0.14672E+09	0.74022E+08	0.13778E+01	0.63182E+01	-1
21	82.311	4.282	0.14846E+09	0.75857E+08	0.33800E+01	0.73245E+01	0
22	96.092	4.445	0.15029E+09	0.77649E+08	0.33270E+01	0.80920E+01	0
23	102.511	4.598	0.15178E+09	0.79279E+08	0.27688E+01	0.83328E+01	0
24	101.781	4.546	0.15309E+09	0.80689E+08	0.16295E+01	0.79943E+01	0
25	98.658	4.596	0.15419E+09	0.81931E+08	0.27077E+00	0.72301E+01	0
26	96.308	4.659	0.15502E+09	0.82732E+08	0.00000E+00	0.65587E+01	0
27	91.200	4.729	0.15577E+09	0.83623E+08	0.00000E+00	0.63448E+01	0
28	87.341	4.739	0.15656E+09	0.84352E+08	0.00000E+00	0.61000E+01	0
29	82.439	5.312	0.15792E+09	0.85499E+08	0.00000E+00	0.57451E+01	-1
30	73.797	6.521	0.15983E+09	0.86900E+08	0.00000E+00	0.57513E+01	-1
31	74.829	6.999	0.16072E+09	0.83334E+08	0.32499E+00	0.57596E+01	-1
32	69.212	6.310	0.16262E+09	0.90977E+08	0.13777E+01	0.60215E+01	-1
33	88.341	5.190	0.16400E+09	0.93256E+08	0.33971E+01	0.69839E+01	0
34	93.254	5.229	0.16656E+09	0.96479E+08	0.37913E+01	0.73404E+01	0
35	92.933	5.247	0.16231E+09	0.97594E+08	0.35302E+01	0.72462E+01	0
36	97.489	5.297	0.16989E+09	0.99237E+08	0.25462E+01	0.65668E+01	0
37	94.147	0.000	0.00000E+00	0.00000E+00	0.14906E+01	0.55360E+01	0

TABLE 5.4.3: A TYPICAL ELQG ITERATION CYCLE

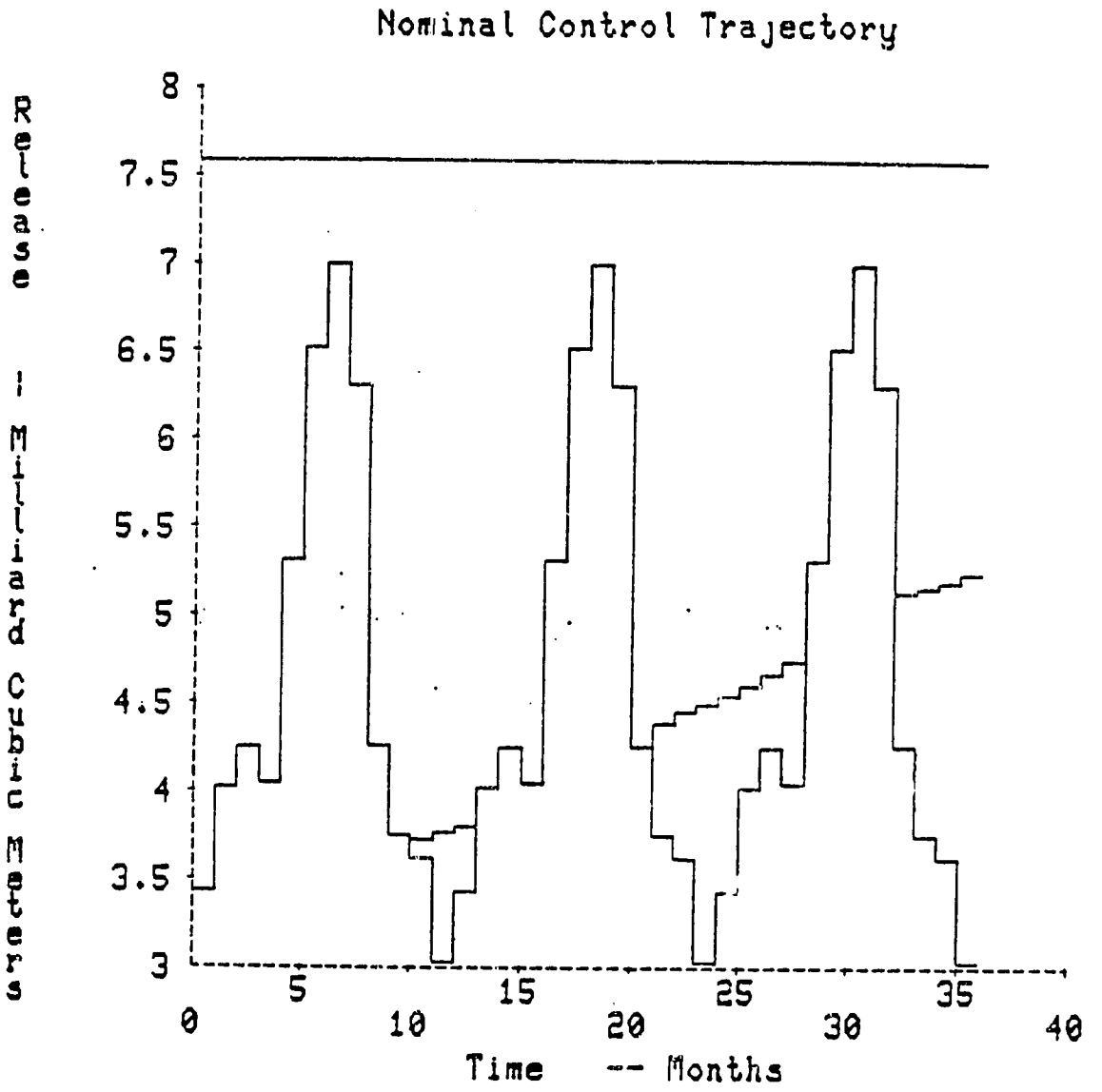

FIGURE 5.4.12: A TYPICAL ELQG ITERATION CYCLE: ITERATION #3

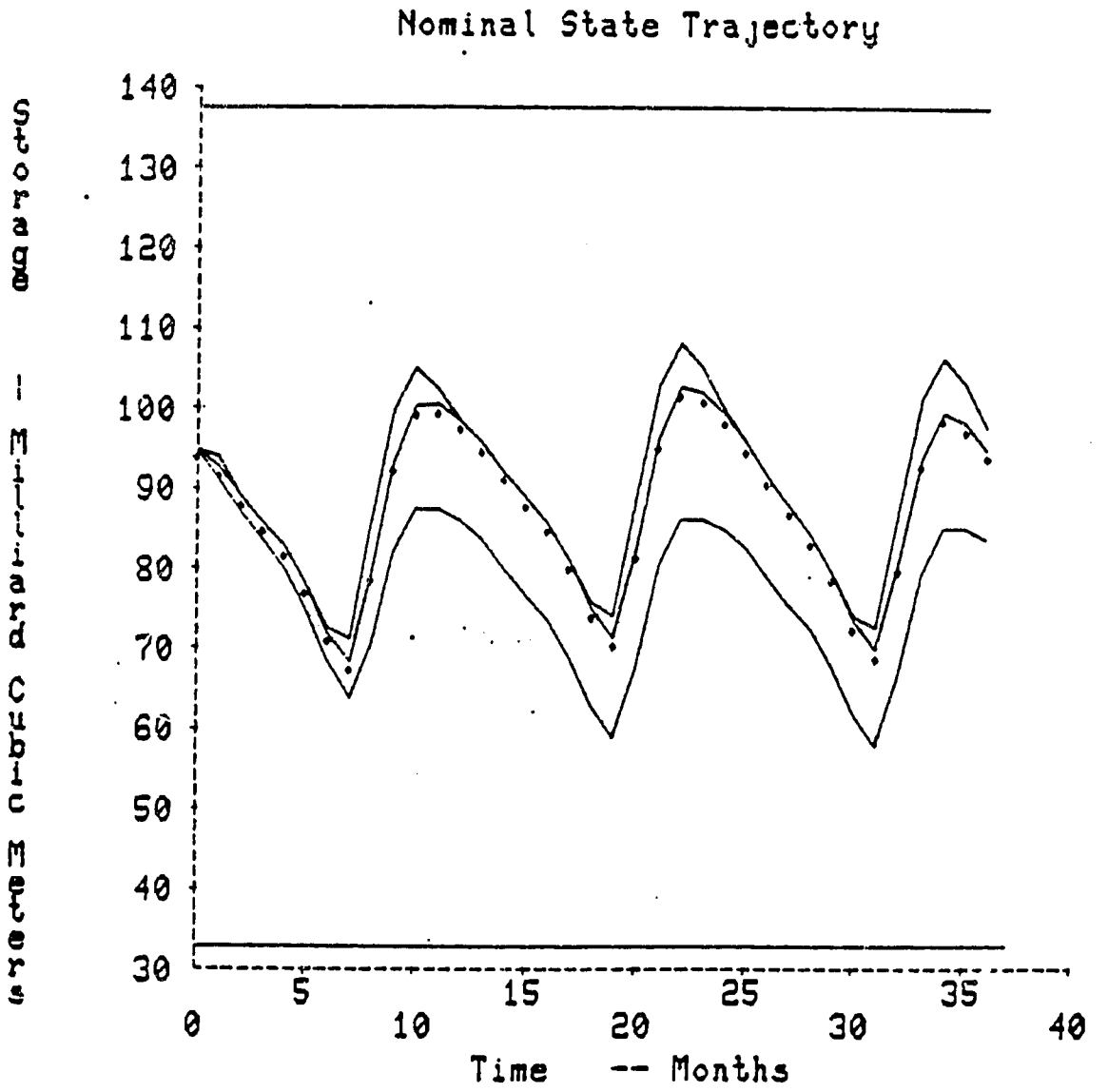

FIGURE 5.4.13: A TYPICAL ELQG ITERATION CYCLE: ITERATION #3

ITERATION # 3

HYDROPOWER (GWH) = 0.22451278E+05
PENALTY = 0.16298486E+08

Z = 0.1969E+01
C = 0.1000E+02
BETA = 0.5000E+00
SIGMA = 0.1000E-03
EI = 0.4998E-02
N = 0
ALPHA = 0.1000E+01
UI = 0.3325E+00

L	E(S)	U	DUJ	DUUJ	STDEU	STDEL	A
1	95.000	3.430	0.58973E+07	0.54176E+08	0.00000E+00	0.00000E+00	-1
2	92.501	4.018	0.59485E+07	0.54815E+08	0.73583E+00	0.73583E+00	-1
3	89.047	4.240	0.59809E+07	0.55311E+08	0.00000E+00	0.10039E+01	-1
4	85.715	4.040	0.60027E+07	0.55862E+08	0.00000E+00	0.12035E+01	-1
5	82.557	5.312	0.60839E+07	0.56627E+08	0.00000E+00	0.13827E+01	-1
6	77.871	6.521	0.61794E+07	0.57620E+08	0.00000E+00	0.15677E+01	-1
7	71.913	6.999	0.62559E+07	0.58846E+08	0.32499E+00	0.17704E+01	-1
8	68.306	6.310	0.62994E+07	0.60272E+08	0.13777E+01	0.24117E+01	-1
9	79.381	4.251	0.63084E+07	0.61784E+08	0.33790E+01	0.45137E+01	-1
10	93.386	3.741	0.63595E+07	0.63254E+08	0.32836E+01	0.58042E+01	-1
11	100.532	3.719	0.64211E+07	0.64590E+08	0.23761E+01	0.66277E+01	0
12	100.584	3.760	0.64774E+07	0.65727E+08	0.90925E+00	0.67485E+01	0
13	98.572	3.783	0.65239E+07	0.66675E+08	0.00000E+00	0.64022E+01	0
14	95.786	4.018	0.65682E+07	0.67458E+08	0.00000E+00	0.62488E+01	-1
15	92.236	4.240	0.66034E+07	0.68053E+08	0.00000E+00	0.62553E+01	-1
16	88.891	4.040	0.66277E+07	0.68723E+08	0.00000E+00	0.62658E+01	-1
17	85.714	5.312	0.67145E+07	0.69652E+08	0.00000E+00	0.62682E+01	-1
18	81.885	6.521	0.68168E+07	0.70856E+08	0.00000E+00	0.62667E+01	-1
19	75.016	6.999	0.68999E+07	0.72341E+08	0.32500E+00	0.62655E+01	-1
20	71.374	6.310	0.69488E+07	0.74066E+08	0.13778E+01	0.64140E+01	-1
21	82.412	4.251	0.69621E+07	0.75900E+08	0.33794E+01	0.74061E+01	-1
22	96.335	4.387	0.70423E+07	0.77691E+08	0.33072E+01	0.81833E+01	0
23	102.803	4.451	0.71157E+07	0.79321E+08	0.27228E+01	0.84466E+01	0
24	102.105	4.491	0.71778E+07	0.80710E+08	0.15651E+01	0.81326E+01	0
25	99.362	4.538	0.72292E+07	0.81871E+08	0.18668E+00	0.73945E+01	0
26	95.819	4.594	0.72711E+07	0.82821E+08	0.00000E+00	0.62510E+01	0
27	91.785	4.662	0.73035E+07	0.83582E+08	0.00000E+00	0.65636E+01	0
28	87.922	4.738	0.73397E+07	0.84390E+08	0.00000E+00	0.63463E+01	0
29	84.104	5.312	0.74074E+07	0.85534E+08	0.00000E+00	0.60178E+01	-1
30	79.407	6.521	0.75150E+07	0.87022E+08	0.00000E+00	0.60128E+01	-1
31	73.434	6.999	0.76455E+07	0.88861E+08	0.32499E+00	0.60229E+01	-1
32	69.810	6.310	0.76640E+07	0.90999E+08	0.13777E+01	0.61820E+01	-1
33	80.834	5.127	0.77163E+07	0.93271E+08	0.33960E+01	0.71937E+01	0
34	93.912	5.162	0.78074E+07	0.95489E+08	0.37583E+01	0.75559E+01	0
35	99.638	5.134	0.78995E+07	0.97510E+08	0.34506E+01	0.74761E+01	0
36	98.241	5.237	0.79582E+07	0.99239E+08	0.25656E+01	0.68293E+01	0
37	94.969	0.000	0.00000E+00	0.00000E+00	0.13331E+01	0.58155E+01	0

TABLE 5.4.4: A TYPICAL ELQG ITERATION CYCLE

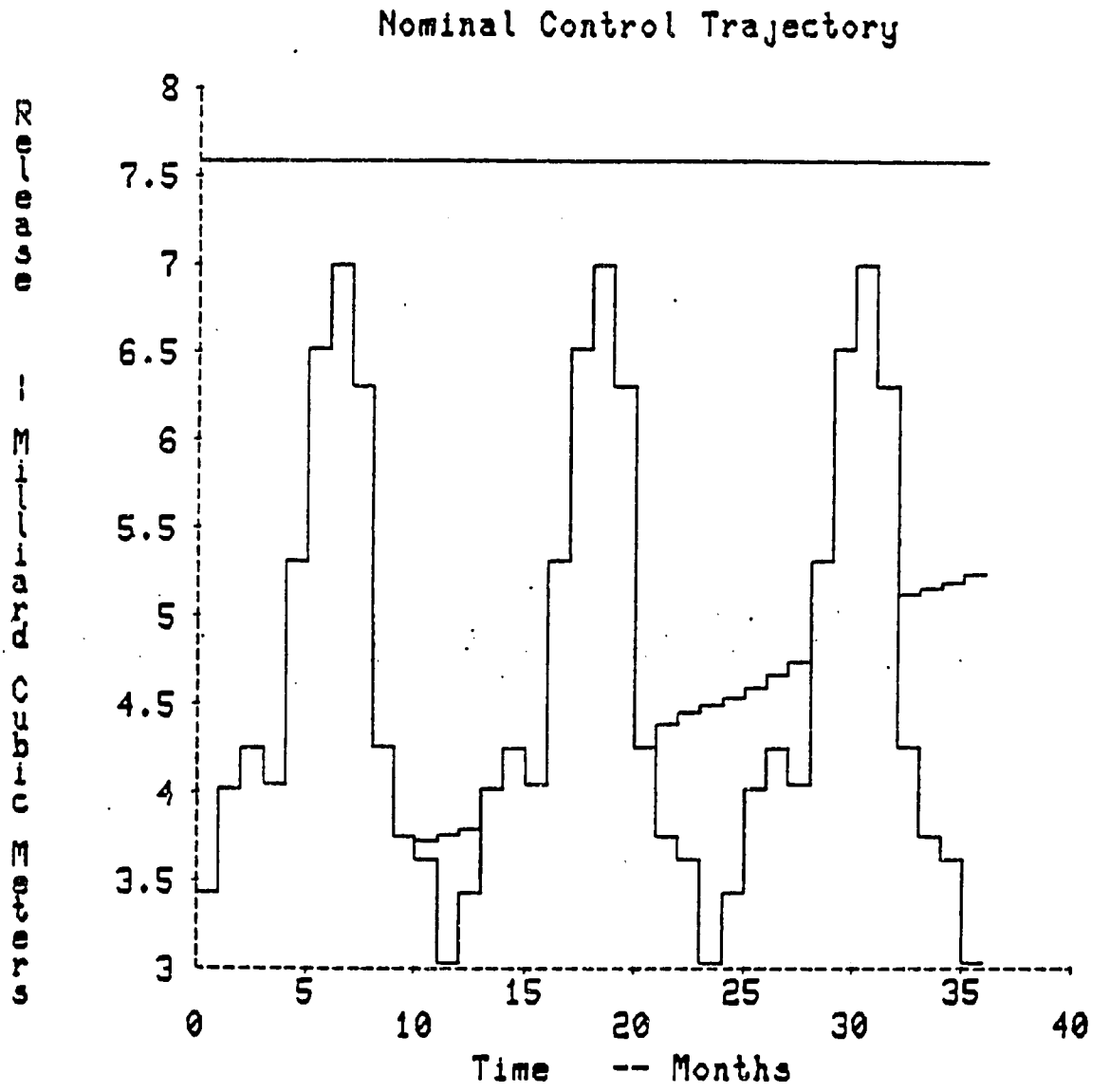

FIGURE 5.4.14: A TYPICAL ELQG ITERATION CYCLE: ITERATION # 4

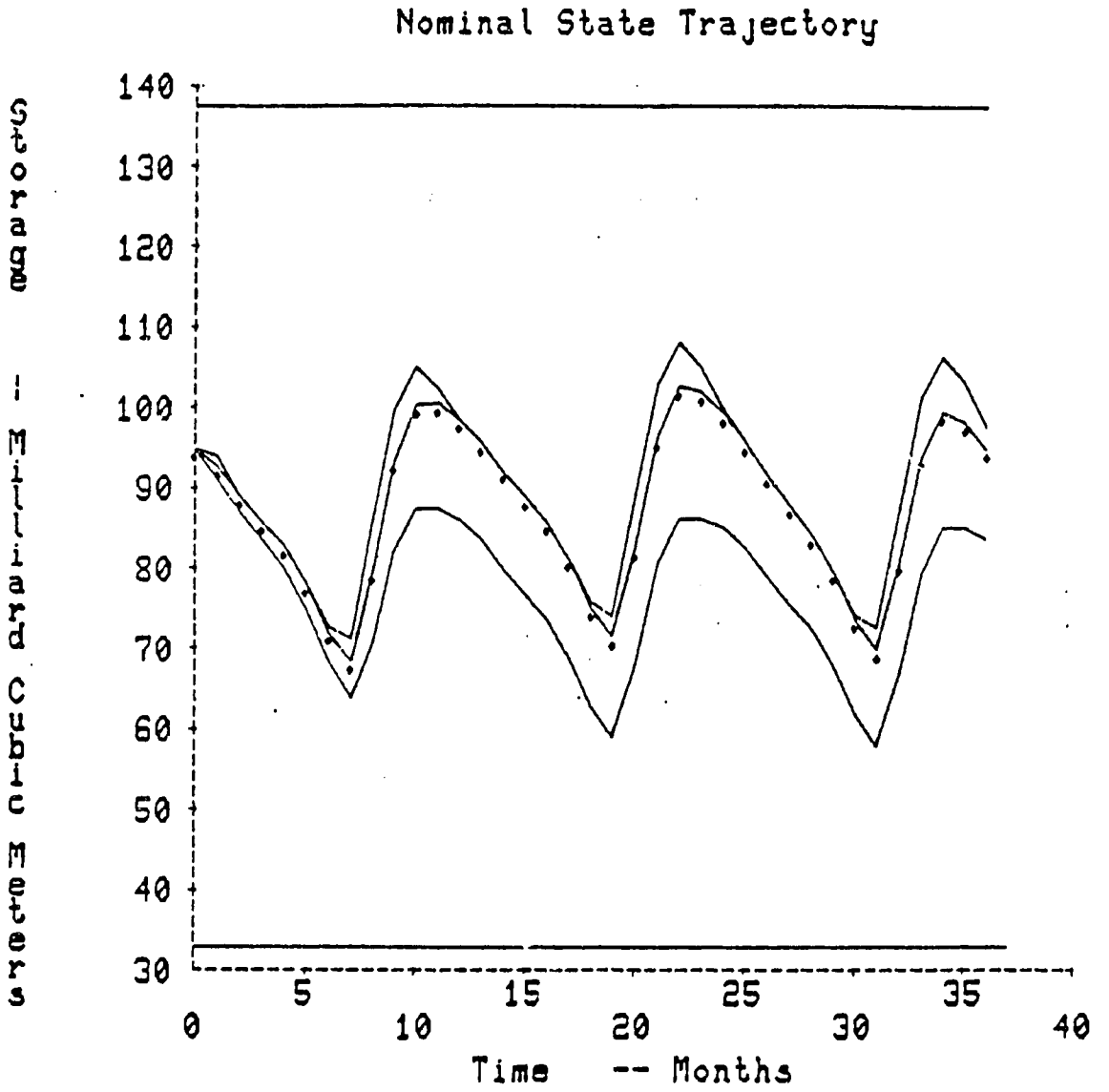

FIGURE 5.4.15: A TYPICAL ELQG ITERATION CYCLE: ITERATION # 4

I T E R A T I O N # 4

HYDROPOWER (GWH) = 0.22446444E+05
PENALTY = 0.16138858E+08

Z = 0.1960E+01
C = 0.1000E+02
BETA = 0.5000E+00
SIGMA = 0.1000E-03
EI = 0.8683E-04
M = 0
ALPHA = 0.1000E+01
UI = 0.8683E-04

L	E(S)	U	DUJ	DUUJ	STDEU	STDEL	A
1	95.000	3.430	0.18230E+05	0.54179E+08	0.00000E+00	0.00000E+00	-1
2	92.581	4.018	0.35943E+05	0.54817E+08	0.73583E+00	0.73583E+00	-1
3	89.047	4.240	0.40744E+05	0.55313E+08	0.00000E+00	0.10039E+01	-1
4	85.715	4.040	0.32959E+05	0.55864E+08	0.00000E+00	0.12035E+01	-1
5	82.557	5.312	0.73562E+05	0.56629E+08	0.00000E+00	0.13827E+01	-1
6	77.871	6.521	0.11668E+06	0.57622E+08	0.00000E+00	0.15677E+01	-1
7	71.913	6.999	0.12891E+06	0.58849E+08	0.32499E+00	0.17704E+01	-1
8	68.306	6.310	0.98513E+05	0.60275E+08	0.13777E+01	0.24117E+01	-1
9	79.381	4.251	0.30024E+05	0.61786E+08	0.33790E+01	0.45137E+01	-1
10	93.388	3.741	0.69442E+04	0.63257E+08	0.32836E+01	0.58042E+01	-1
11	100.532	3.714	0.16780E+04	0.64592E+08	0.23760E+01	0.66278E+01	0
12	100.589	3.755	0.16927E+04	0.65730E+08	0.90675E+00	0.67510E+01	0
13	98.582	3.783	0.17048E+04	0.66677E+08	0.00000E+00	0.64070E+01	0
14	95.800	4.018	0.83776E+04	0.67452E+08	0.00000E+00	0.62559E+01	-1
15	92.250	4.240	0.14295E+05	0.68055E+08	0.00000E+00	0.62623E+01	-1
16	88.905	4.040	0.61344E+04	0.68725E+08	0.00000E+00	0.62728E+01	-1
17	85.728	5.312	0.48542E+05	0.69655E+08	0.00000E+00	0.62751E+01	-1
18	81.018	6.521	0.93579E+05	0.70858E+08	0.00000E+00	0.62735E+01	-1
19	75.030	6.999	0.10659E+06	0.72343E+08	0.32530E+00	0.62721E+01	-1
20	71.388	6.310	0.74666E+05	0.74069E+08	0.13778E+01	0.64204E+01	-1
21	82.426	4.251	0.32899E+04	0.75902E+08	0.33794E+01	0.74115E+01	-1
22	96.348	4.383	0.18392E+04	0.77694E+08	0.33071E+01	0.21082E+01	0
23	102.820	4.448	0.13583E+04	0.79323E+08	0.27208E+01	0.84530E+01	0
24	102.125	4.428	0.13744E+04	0.80713E+08	0.15618E+01	0.81404E+01	0
25	99.375	4.534	0.13878E+04	0.81873E+08	0.18219E+00	0.74039E+01	0
26	95.845	4.581	0.13986E+04	0.82924E+08	0.00000E+00	0.68619E+01	0
27	91.734	4.659	0.19070E+04	0.83564E+08	0.00000E+00	0.65761E+01	0
28	87.956	4.734	0.19163E+04	0.84392E+08	0.00000E+00	0.63603E+01	0
29	84.139	5.312	0.20194E+05	0.85536E+08	0.00000E+00	0.60332E+01	-1
30	79.442	6.521	0.63914E+05	0.87024E+08	0.00000E+00	0.60349E+01	-1
31	73.468	6.999	0.76060E+05	0.88862E+08	0.32499E+00	0.60377E+01	-1
32	69.844	6.310	0.44444E+05	0.91900E+08	0.13777E+01	0.61962E+01	-1
33	80.868	5.124	0.20130E+04	0.93272E+08	0.33959E+01	0.72956E+01	0
34	93.848	5.160	0.20366E+04	0.95490E+08	0.37571E+01	0.75676E+01	0
35	99.675	5.132	0.20578E+04	0.97511E+08	0.34486E+01	0.74877E+01	0
36	98.279	5.236	0.20757E+04	0.99239E+08	0.25625E+01	0.62325E+01	0
37	94.989	0.000	0.00000E+00	0.00000E+00	0.13790E+01	0.52283E+01	0

TABLE 5.4.5: A TYPICAL ELQG ITERATION

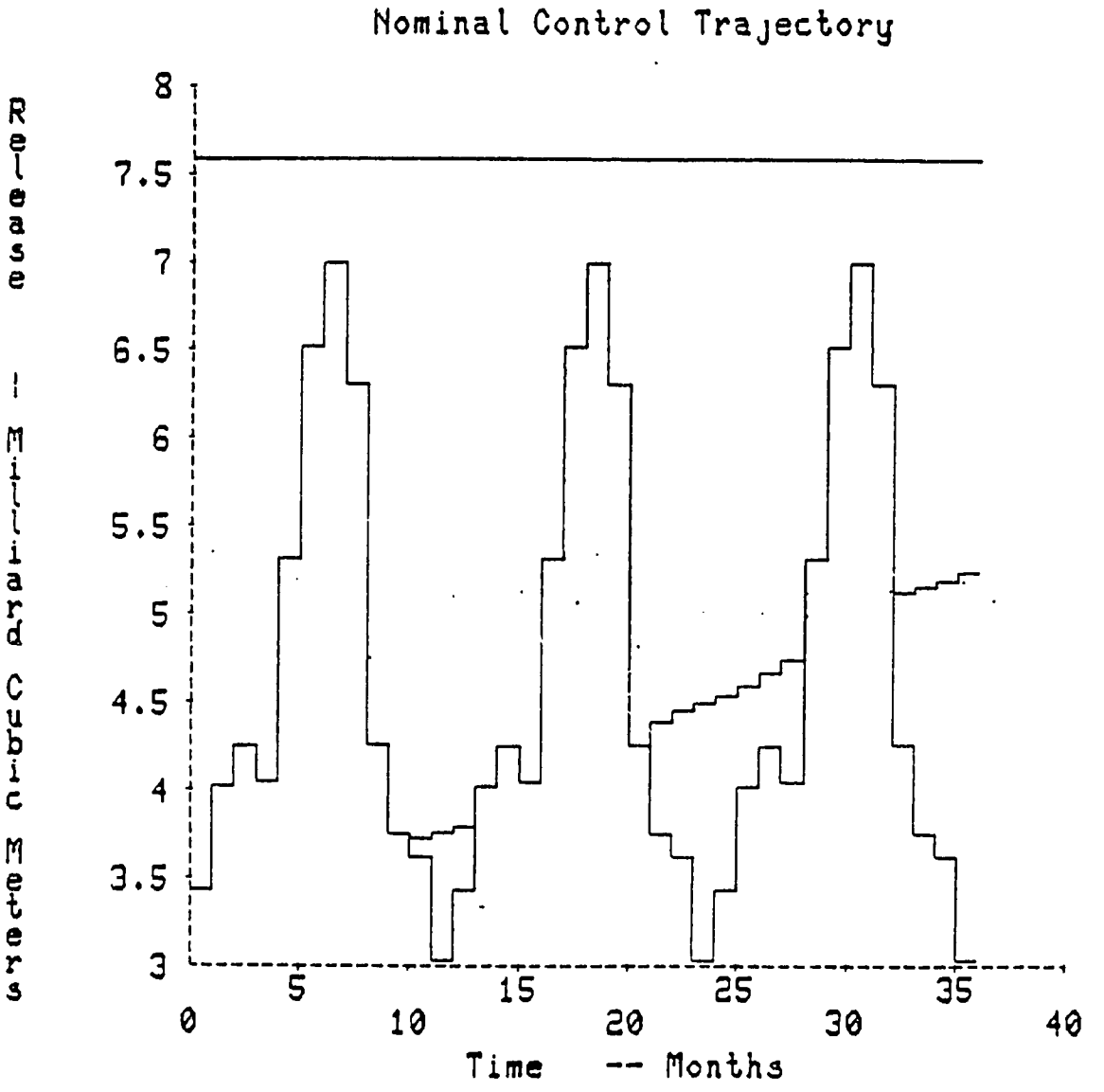

FIGURE 5.4.16: A TYPICAL ELQG ITERATION CYCLE
ITERATION #5

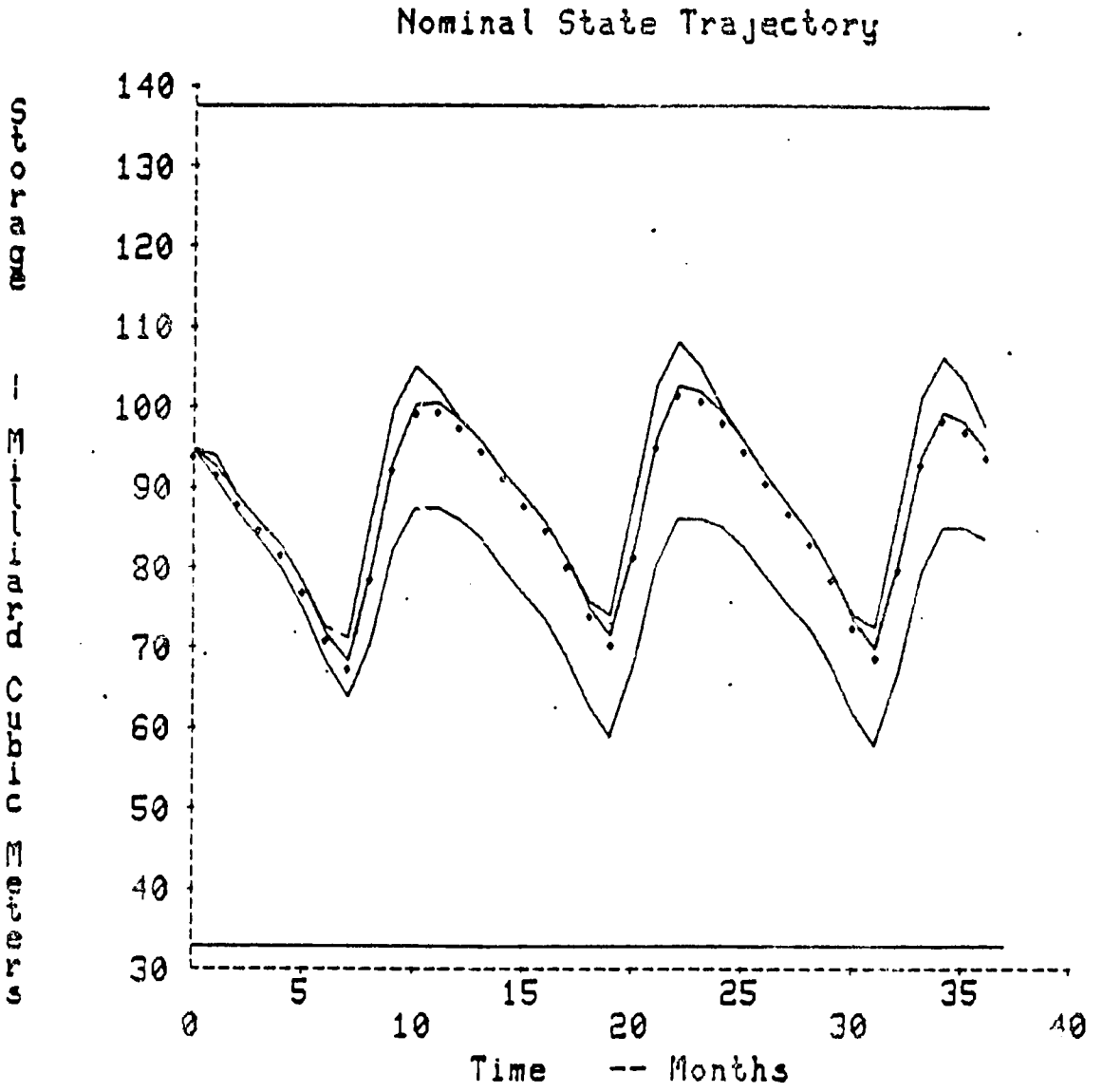

FIGURE 5.4.17: A TYPICAL ELQG ITERATION CYCLE
ITERATION #5

ITERATION 8 5

HYDROPOWER (GWH) = 0.22446448E+05
PENALTY = 0.16138857E+08

Z = 0.1960E+01
C = 0.1000E+02
BETA = 0.5000E+00
SIGMA = 0.1000E-03
EI = 0.4715E-04
M = 1
ALPHA = 0.5000E+00
WI = 0.4715E-04

L	E(S)	U	DUJ	DUUJ	STDEU	STDEL	A
1	95.000	3.430	0.17528E+05	.54179E+08	0.00000E+00	0.00000E+00	-1
2	92.581	4.018	0.34337E+05	.54817E+08	0.73583E+00	0.73583E+00	-1
3	89.847	4.240	0.40035E+05	.55313E+08	0.00000E+00	0.10039E+01	-1
4	85.715	4.040	0.32246E+05	0.55864E+08	0.00000E+00	0.12035E+01	-1
5	82.557	5.312	0.72854E+05	0.56629E+08	0.00000E+00	0.13827E+01	-1
6	77.871	6.521	0.11595E+06	0.57622E+08	0.00000E+00	0.15677E+01	-1
7	71.913	6.999	0.12818E+06	0.58349E+08	0.32499E+00	0.17704E+01	-1
8	68.306	6.310	0.97772E+05	0.60275E+08	0.13777E+01	0.24117E+01	-1
9	79.381	4.251	0.29275E+05	0.61786E+08	0.33790E+01	0.45137E+01	-1
10	93.386	3.741	0.61856E+04	0.63257E+08	0.32836E+01	0.53042E+01	-1
11	100.532	3.714	0.91078E+03	0.64592E+08	0.23769E+01	0.66279E+01	0
12	100.589	3.756	0.91875E+03	0.65739E+08	0.90673E+00	0.67510E+01	0
13	98.582	3.783	0.92533E+03	0.66677E+08	0.00000E+00	0.64071E+01	0
14	95.800	4.018	0.75943E+04	0.67452E+08	0.00000E+00	0.62559E+01	-1
15	92.250	4.240	0.13508E+05	0.68055E+08	0.00000E+00	0.62623E+01	-1
16	88.905	4.040	0.53438E+04	0.68725E+08	0.00000E+00	0.62728E+01	-1
17	85.728	5.312	0.47746E+05	0.69655E+08	0.00000E+00	0.62751E+01	-1
18	81.018	6.521	0.92777E+05	0.70858E+08	0.00000E+00	0.62735E+01	-1
19	75.030	6.999	0.10569E+06	0.72343E+08	0.32530E+00	0.62721E+01	-1
20	71.388	6.310	0.73846E+05	0.74069E+08	0.13778E+01	0.64204E+01	-1
21	82.426	4.251	0.24590E+04	0.75902E+08	0.33794E+01	0.74115E+01	-1
22	90.348	4.383	0.99834E+03	0.77694E+08	0.33071E+01	0.81882E+01	0
23	102.820	4.448	0.10087E+04	0.79323E+08	0.27208E+01	0.84530E+01	0
24	102.125	4.483	0.10175E+04	0.80713E+08	0.15618E+01	0.81404E+01	0
25	99.375	4.534	0.10247E+04	0.81973E+08	0.18213E+00	0.74039E+01	0
26	95.246	4.591	0.10306E+04	0.82824E+08	0.00000E+00	0.68619E+01	0
27	91.734	4.659	0.10351E+04	0.83564E+08	0.00000E+00	0.65761E+01	0
28	87.955	4.734	0.10402E+04	0.84392E+08	0.00000E+00	0.63604E+01	0
29	84.139	5.312	0.19312E+05	0.85536E+08	0.00000E+00	0.60332E+01	-1
30	79.442	6.521	0.63024E+05	0.87024E+08	0.00000E+00	0.60352E+01	-1
31	73.468	6.999	0.75161E+05	0.88362E+08	0.32499E+00	0.60377E+01	-1
32	69.344	6.310	0.43535E+05	0.91004E+08	0.13777E+01	0.61962E+01	-1
33	80.368	5.124	0.10923E+04	0.93872E+08	0.33959E+01	0.72058E+01	0
34	93.948	5.160	0.11065E+04	0.95490E+08	0.37571E+01	0.75675E+01	0
35	99.876	5.132	0.11171E+04	0.97511E+08	0.34486E+01	0.74077E+01	0
36	98.279	5.238	0.11258E+04	0.99239E+08	0.25625E+01	0.68325E+01	0
37	94.999	0.000	0.00000E+00	0.00000E+00	0.13800E+01	0.58283E+01	0

TABLE 5.4.6: A TYPICAL ELQG ITERATION CYCLE

computations required 25 seconds CPU time on a Honeywell 68/DPS digital computer. It it notable that only in the first and the last iteration it became necessary to cut in half the original stepsize and this is typical with this procedure (avoiding multiple cost function evalutations and adding to the overall computational efficiency). To this end using small σ values is instrumental. Lastly, note that a very good approximation of the optimal solution has already been obtained at the 3rd iteration.

To test if the method performs as well in longer control horizons, the previous problem's control horizon was extended to 360 months. The initial nominal control trajectory was again taken equal to the downstream water supply requirements, and it is shown in Figure 5.4.18. Figure 5.4.19 displays the corresponding nominal state trajectory. The Toshka spillway is not assumed operational and no probabilistic constraints are imposed. (i.e., the bounds shown on Figure 5.4.19 at 137.72×10^9 m^3 and 32.72×10^9 m^3 reservoir storage are not active, C = 0.00). The coefficient of the terminal storage quadratic penalty was increased to 10^{10}. Figures 5.4.20 and 5.4.21 show the nominal trajectories of the 1st iteration. ELQG converges to the trajectories shown in Figure 5.4.22 and 5.4.23 at the 6th iteration. Some characteristic quantities of the 6 iterations are reported on Table 5.4.7. As in the previous experiments the algorithm locks on the optimal trajectory in one iteration after the final binding constraint set is identified. Also notice the progression of the terminal storage towards 95.000×10^9 m^3 and the use of unity stepsize for most

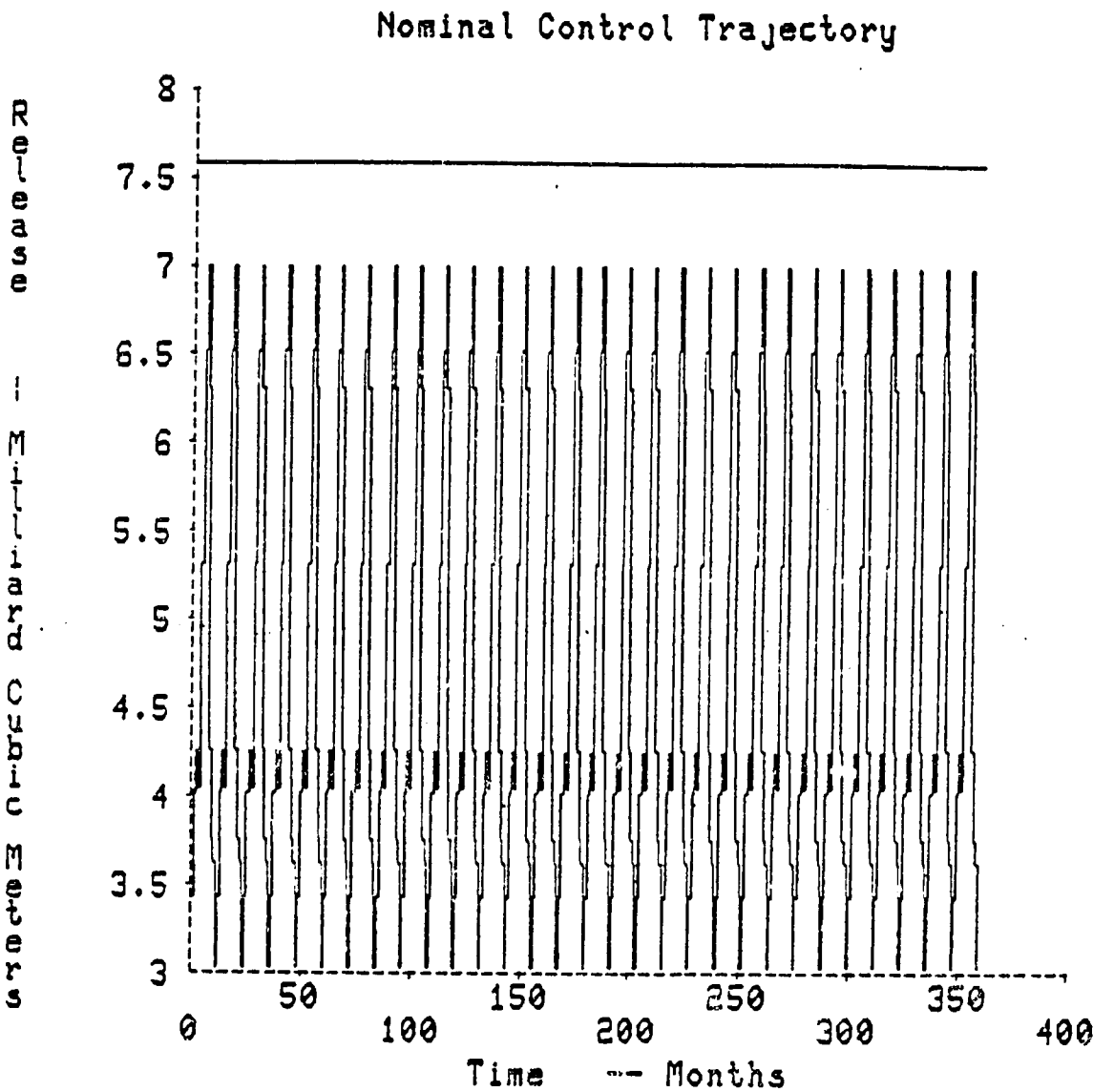

FIGURE 5.4.18: LONG RANGE ELQG CONTROL: INITIAL NOMINAL CONTROL TRAJECTORY

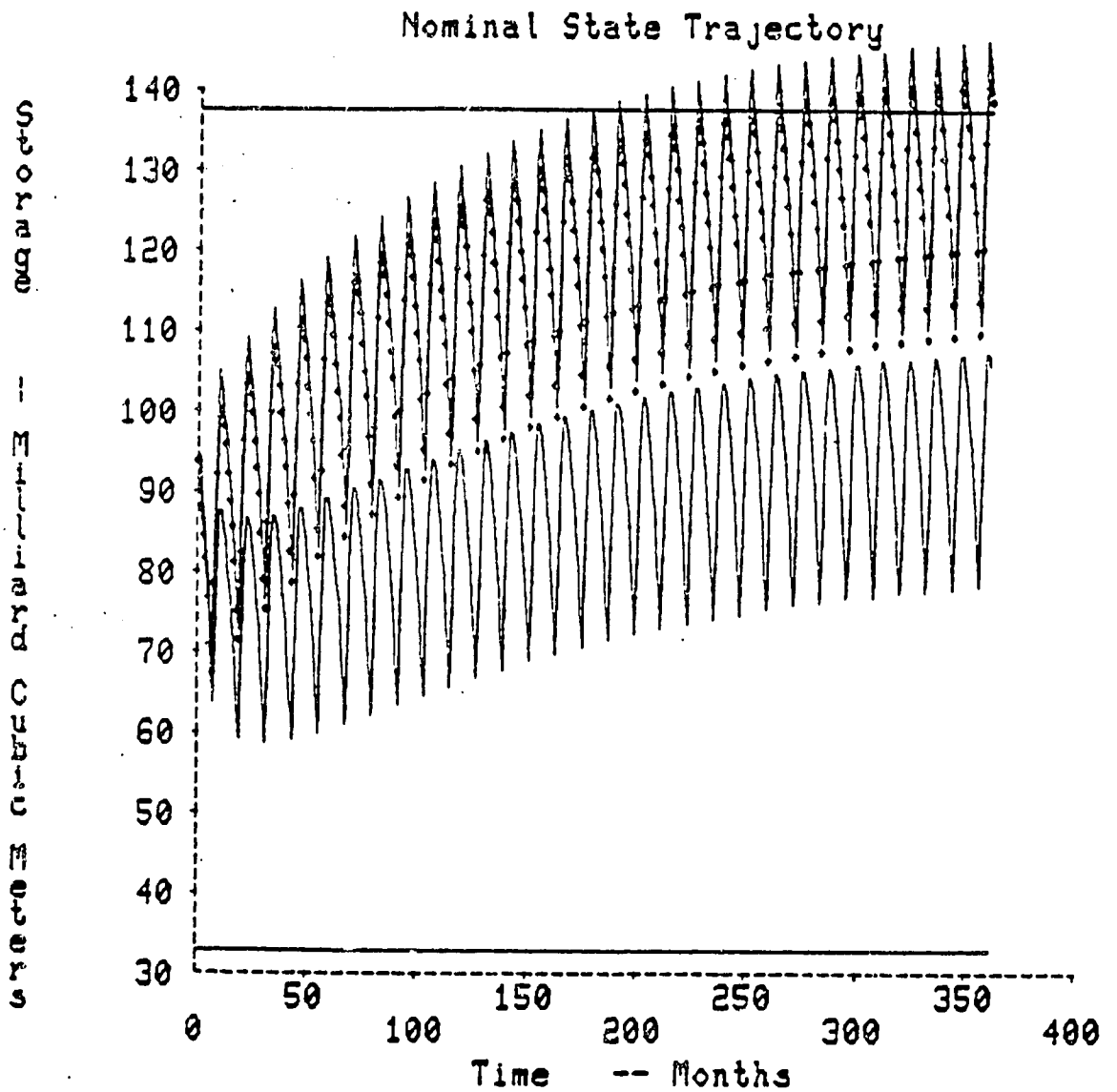

FIGURE 5.4.19: LONG RANGE ELQG CONTROL: INITIAL NOMINAL STATE TRAJECTORY

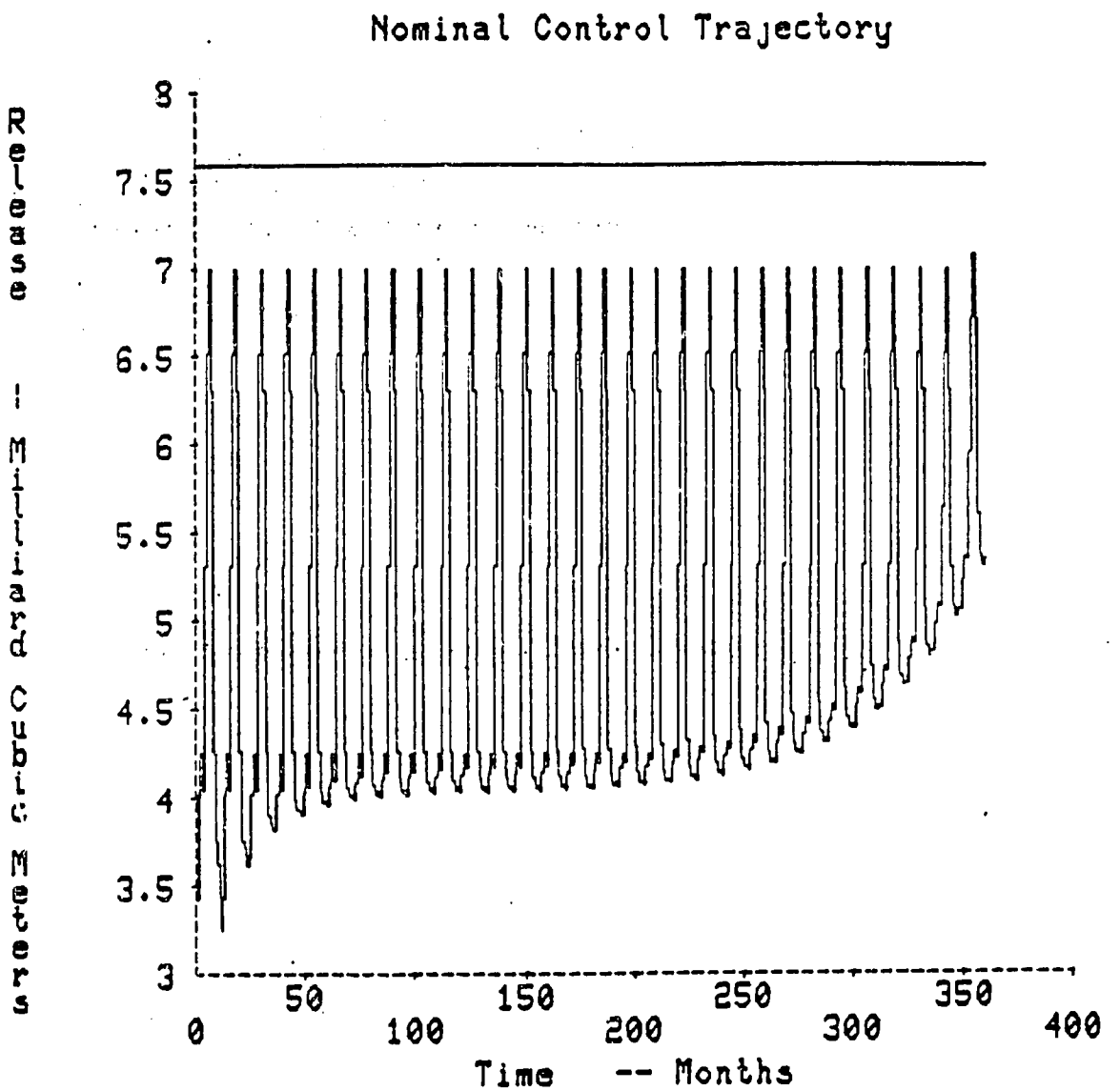

FIGURE 5.4.20: LONG RANGE ELQG CONTROL: ITERATION #1

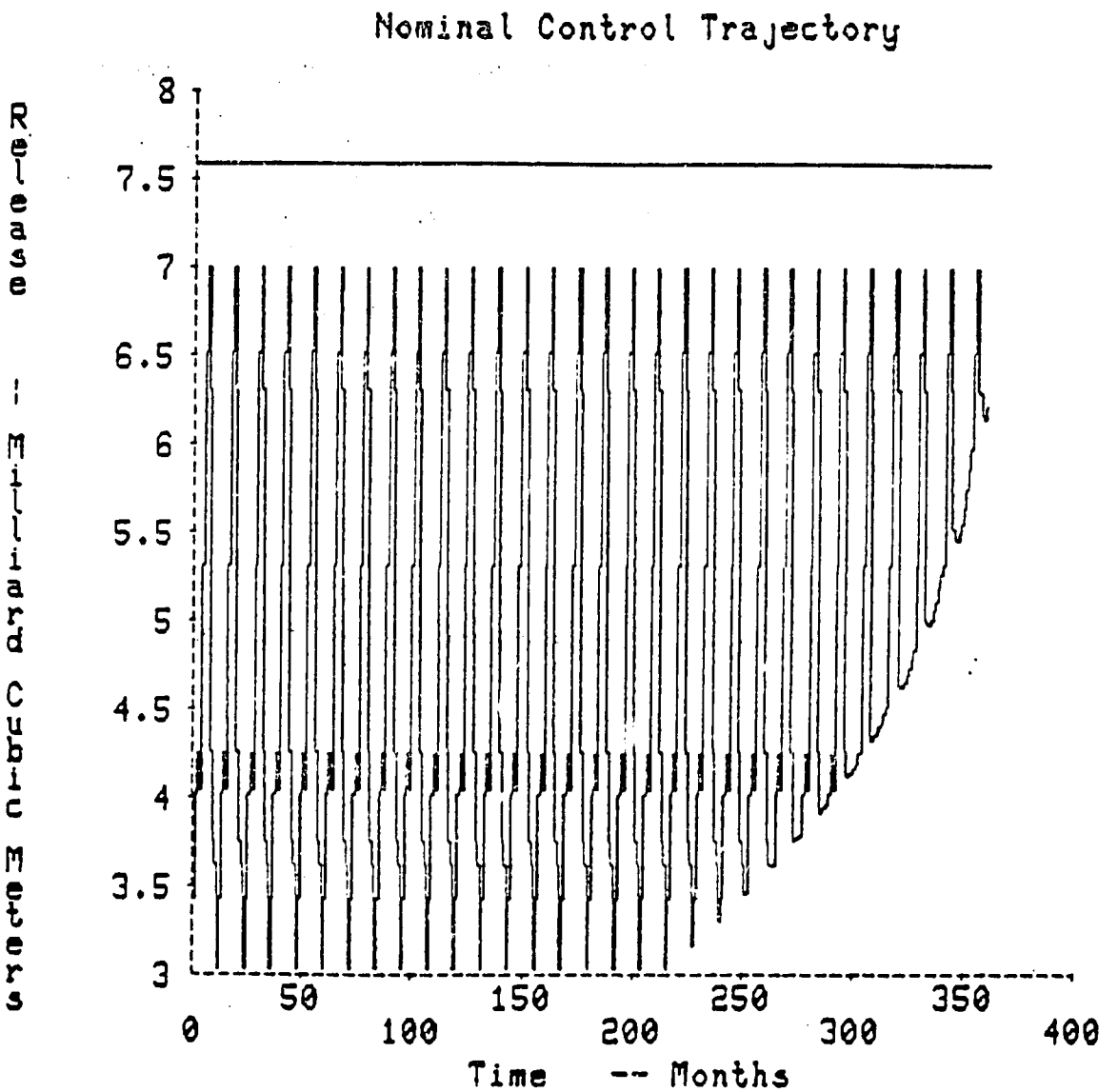

FIGURE 5.4.22: LONG RANGE ELQG CONTROL: ITERATION #6

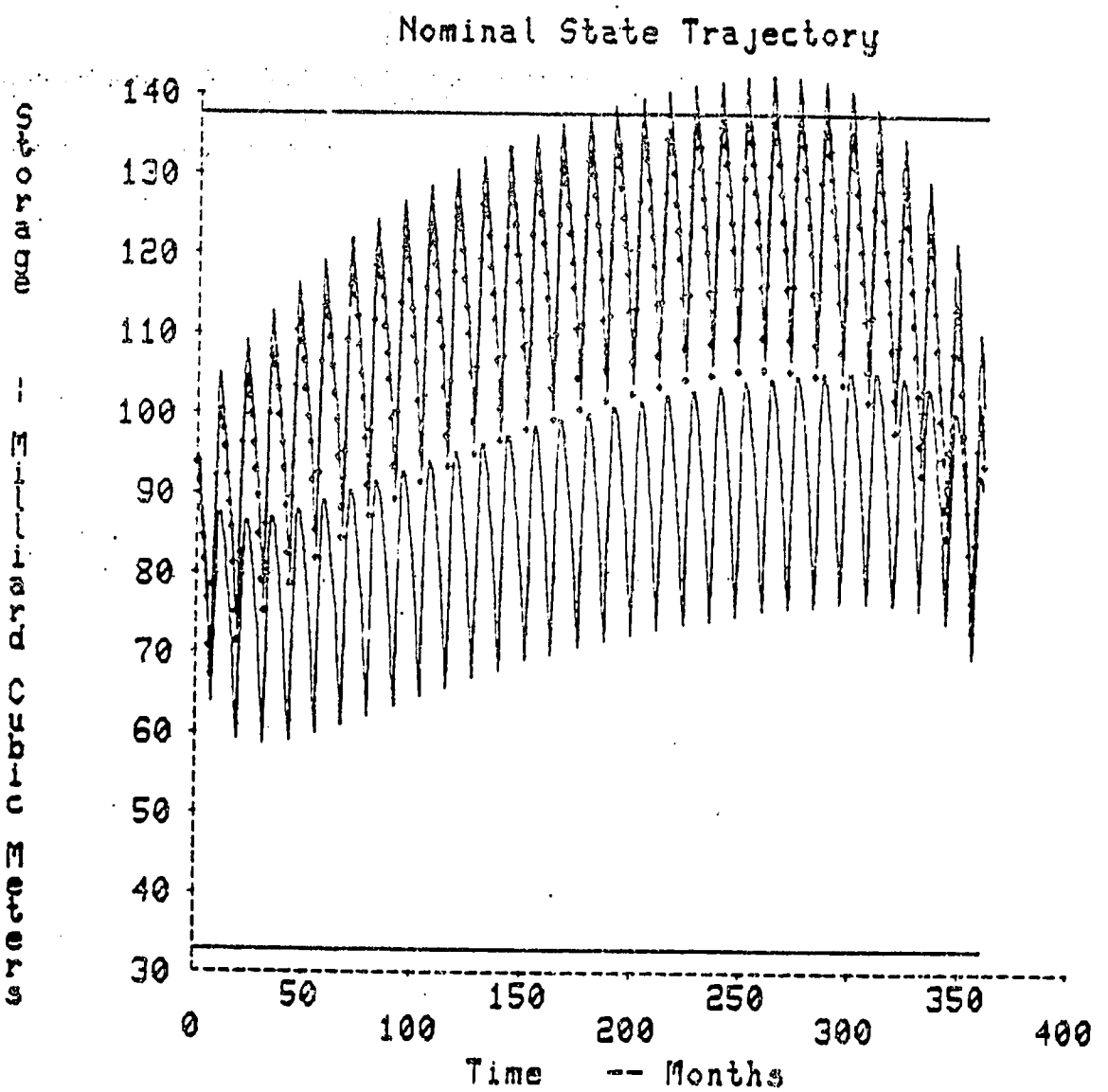

FIGURE 5.4.23: LONG RANGE ELQG CONTROL: ITERATION #6

Z ($\times 10^9$)	= 1.96
C	= 0.00
BETA	= 0.5
SIGMA	= 0.0001
T-1 (months)	= 360

	INITIAL TRAJECTORIES	ITERATION #1	ITERATION #2	ITERATION #3	ITERATION #4	ITERATION #5	ITERATION #6
EI ($\times 10^9 m^3$/mon)		0.4998×10^{-2}	0.4998×10^{-2}	0.4998×10^{-2}	0.4998×10^{-2}	0.4998×10^{-2}	0.8605×10^{-4}
H		1	0	0	0	0	0
ALPHA		0.5	1	1	1	1	1
WI ($\times 10^9 m^3$/mon)	61.20	10.15	9.994	9.421	6.408	0.5[illegible]35	0.8605×10^{-4}
HYDROPOWER (GWH)	0.23379759×10^6	0.23227854×10^6	0.23816678×10^6	0.23950362×10^6	0.23964396×10^6	0.23964794×10^6	0.23964871×10^6
PENALTY	$0.20365898 \times 10^{14}$	$0.14327061 \times 10^{13}$	$0.58198997 \times 10^{12}$	$0.74253168 \times 10^{11}$	0.26455934×10^9	0.15440078×10^9	0.14781579×10^9
NUMBER OF BINDING CONTROL CONSTR.	0	159	253	290	300	304	304
MEAN TERMINAL STORAGE ($\times 10^9 m^3$)	140.178	83.031	87.372	92.278	94.500	94.974	95.000

TABLE 5.4.7. LONG RANGE ELGG CONTROL

of the iterations. These computations required 4 minutes and 35 seconds CPU time of a Honeywell 68/DPS computer. By contrast, a straightforward Dynamic Programming solution of the same problem (Algorithm A4.4.2 or the D.P. procedures in Section 5.3) would require hours of CPU time. Focusing on the optimal state trajectory on Figure 5.4.23, we observe an intuitive result. Namely, from a long-run energy perspective, it is best to build up reservoir storage and operate under high hydraulic head rather than raising the release rate. This is accomplished by releasing the downstream water supply requirements (Figure 5.4.22) during most of the time (20 years) and thereafter gradually raising the releases until the end of the control horizon. (Also compare the energy generation at the trajectories of the first iteration with that of the sixth in Table 5.4.7). The fast portion of the control horizon where the releases are higher shows how long the terminal storage requirement remains influential.

Following is an experimental run where the problem includes probabilistic state constraints. The additional requirement is to keep the reservoir storage lower than 137.72 x 10^9 m^3 and higher than 32.72 x 10^9 m^3 at reliability 0.975 (z=1.96). The optimal trajectories of the previous experiment (Figure 5.4.22, 5.4.23) are taken as initial nominal trajectories. The new optimal trajectories are shown in Figures 5.4.24, 5.4.25. There are 11 state and 284 binding control constraints and the energy production has now dropped (with respect to the state unconstrained optimum) to 0.23900460 x 10^6 GWH. The terminal storage is again 95.000 x 10^9 m^3. The penalty parameter

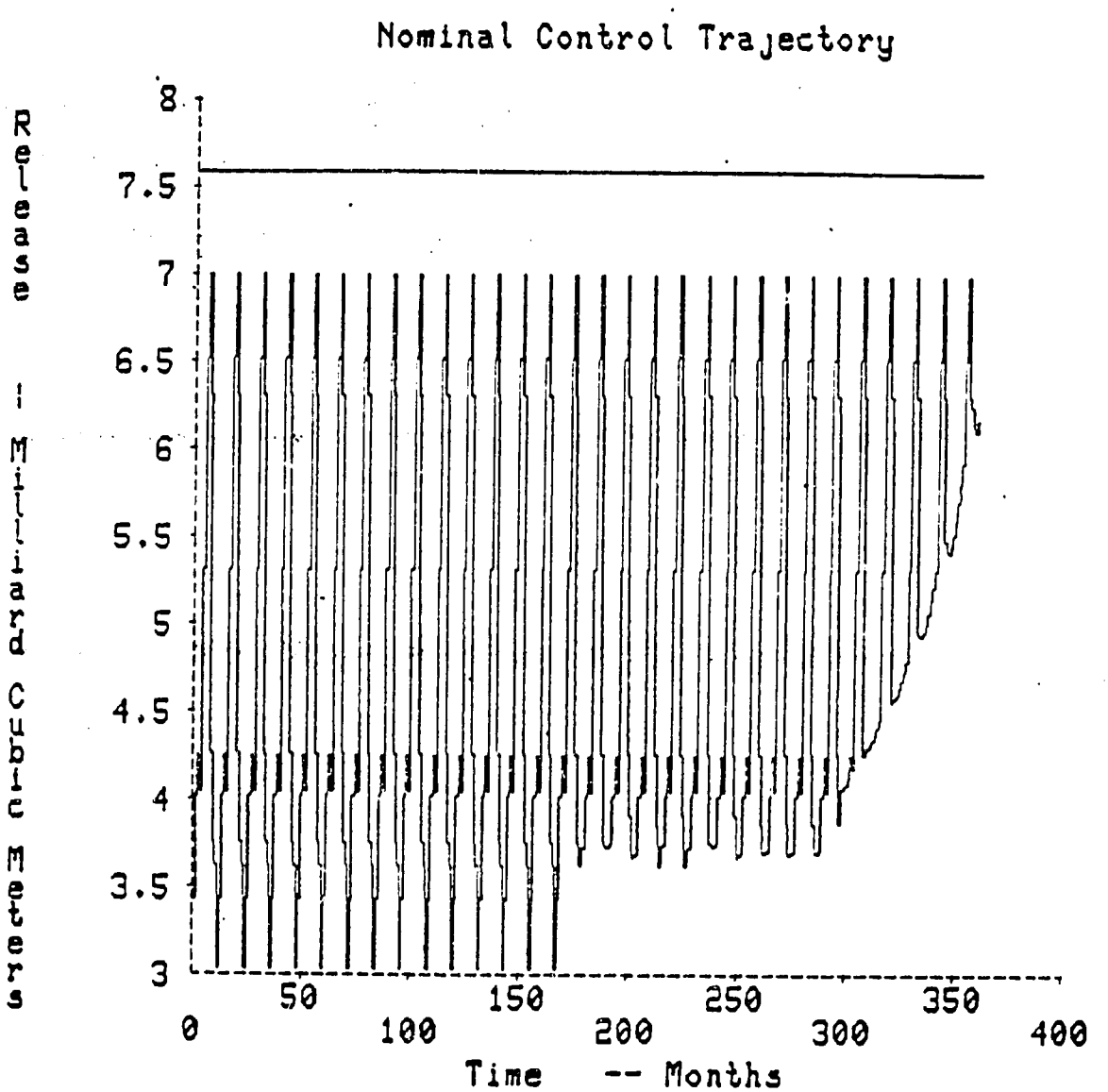

FIGURE 5.4.24: ELQG CONTROL WITH STATE CONSTRAINTS

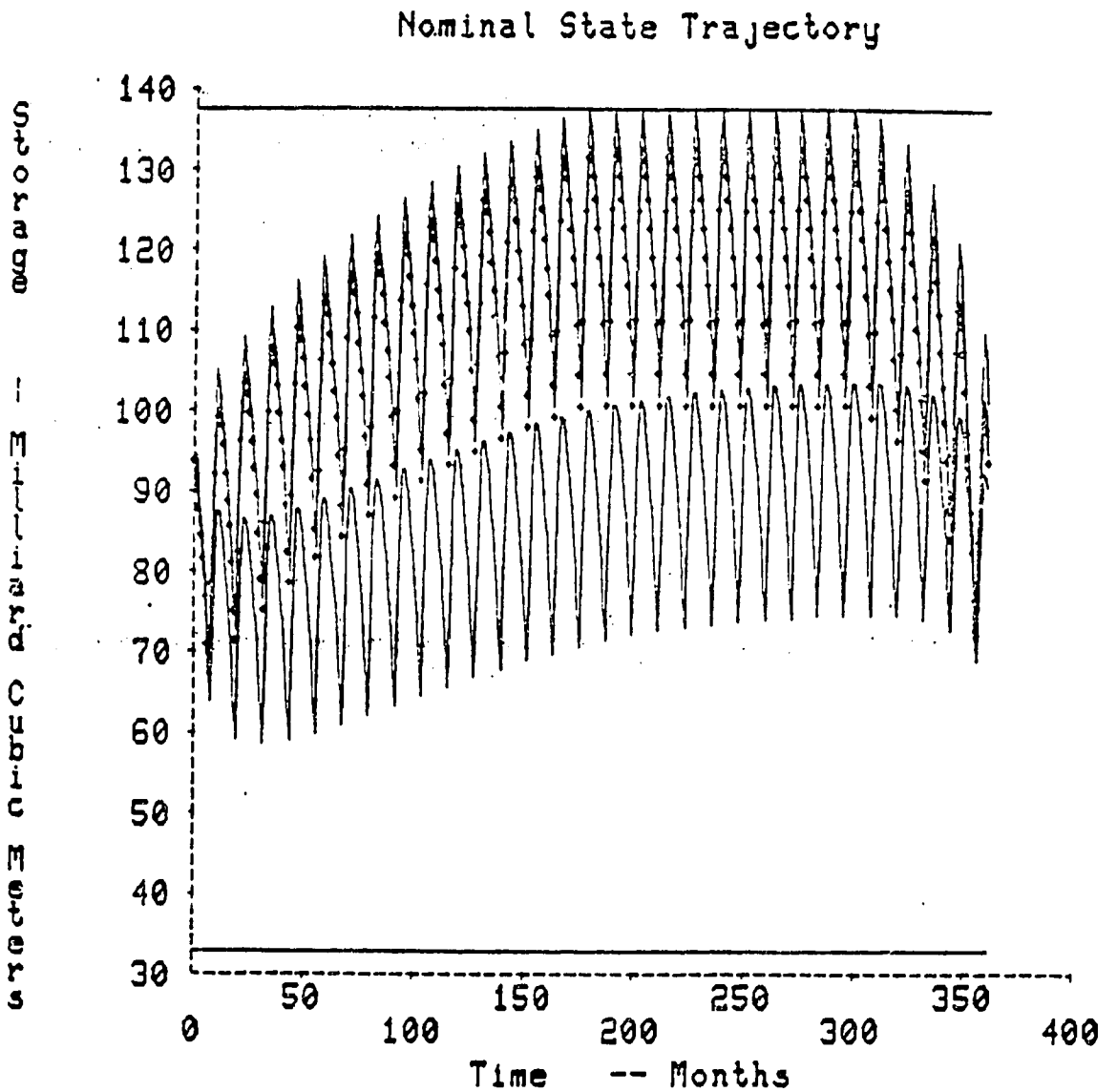

FIGURE 5.4.25: ELQG CONTROL WITH STATE CONSTRAINTS

was taken equal to 10. However, the procedure in Comment 2 following Algorthm A5.4.1 had to be employed, and 2 iteration cycles were necessary to yield the trajectories shown. Although reliable overall, the handling of state constraints via the penalty function method is, in general, less efficient than the treatment of the control constraints via the Projected Newton's method. Notice again that the optimal release trajectory follows the minimum requirements until the reservoir storage rises to its upper bound. Subsequently, the controls are adjusted to maintain the highest feasible storage and as before the last part draws the reservoir down to 95.000 x 10^9 m^3,

By contrast if the unmodified OLFC variance is used, the probabilistic state trajectory in Figure 5.4.25 looks like Figure 5.4.26. The drastically larger uncertainty regions are due to the high OLFC standard deviation which in time stabilizes at about 15 x 10^9 m^3. This effect is caused by the negative evaporation term in the system dynamics (see footnote in Section 4.7). If it were not for this term, the variance would grow unbounded which brings up the necessity of a modification scheme. Figures 5.4.27 and 5.4.28 show the optimal trajectories of the OLFC variance case where there are 25 state and 183 binding control constraints. The energy generation is reduced here to 0.22908150 x 10^6 GWH, considerably lower production with respect to the previous experiment.

These computational runs were performed to demonstrate the efficiency of the ELQG control method. Regarding the HAD operation problem, we intend to establish a sequential control shceme with a much shorter control horizon. The discusion in Section 4.7.2 is

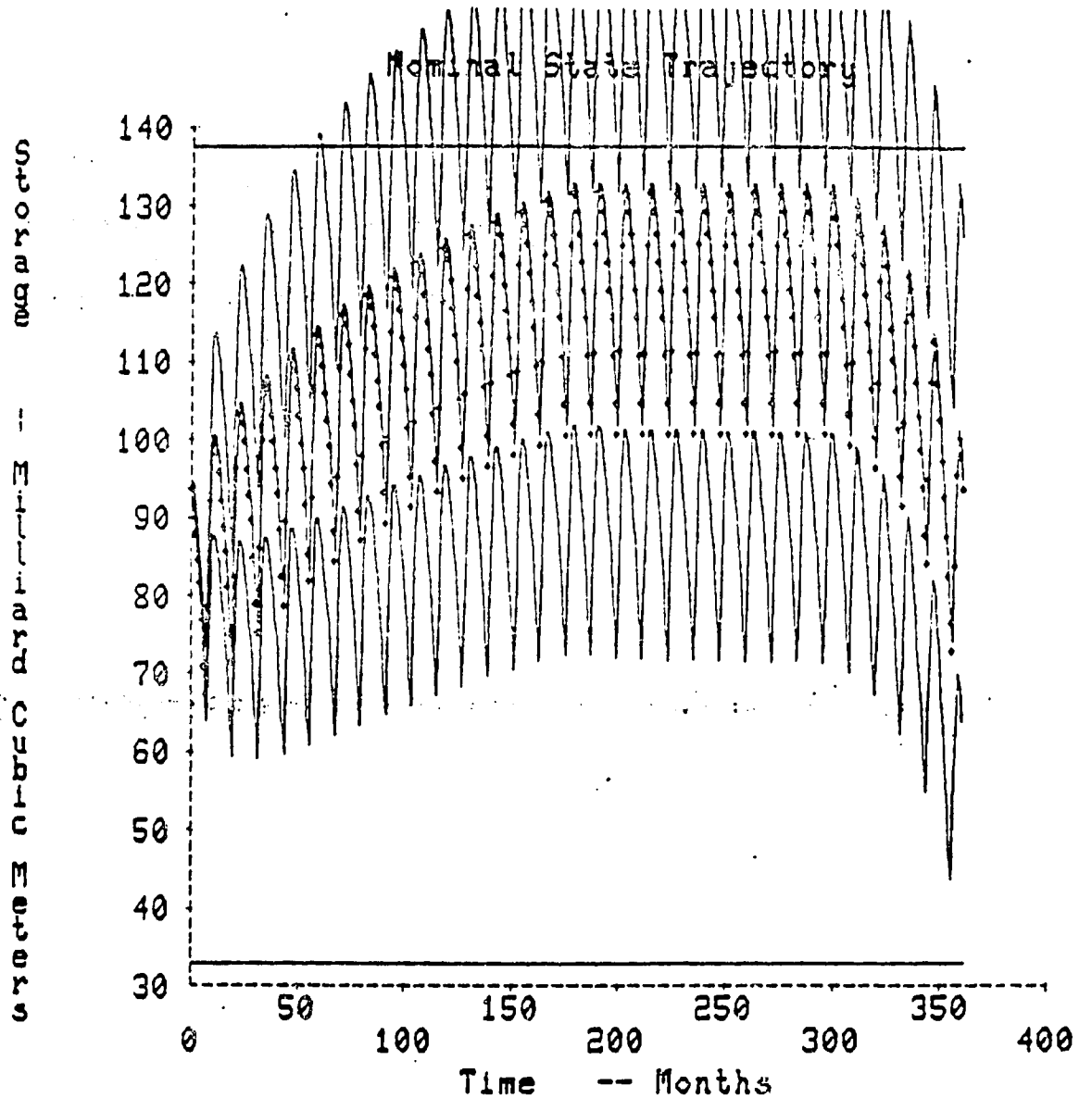

FIGURE 5.4.26: OLFC VARIANCE: INITIAL TRAJECTORY

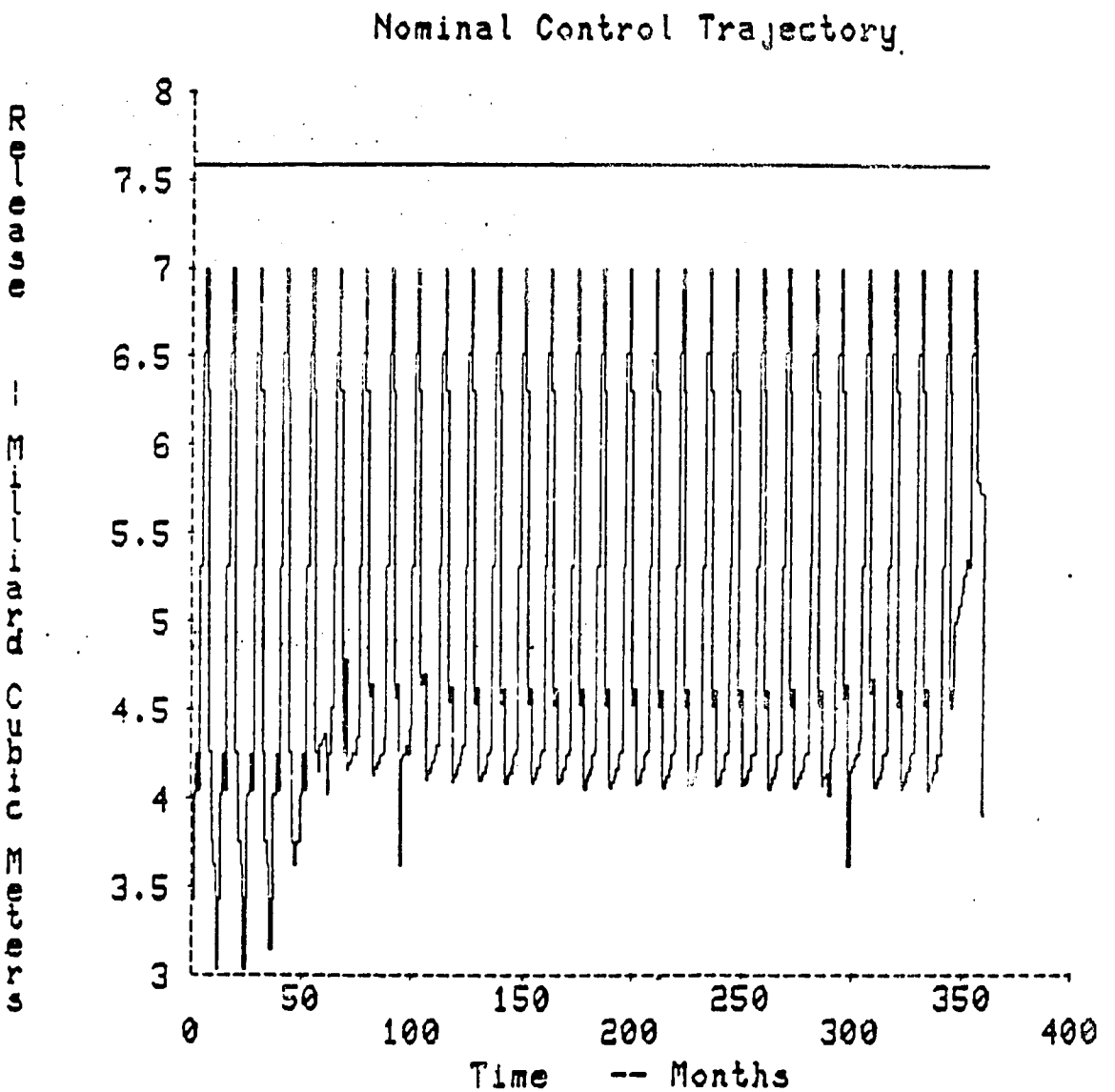

FIGURE 5.4.27: OLFC VARIANCE: OPTIMAL CONTROL TRAJECTORY

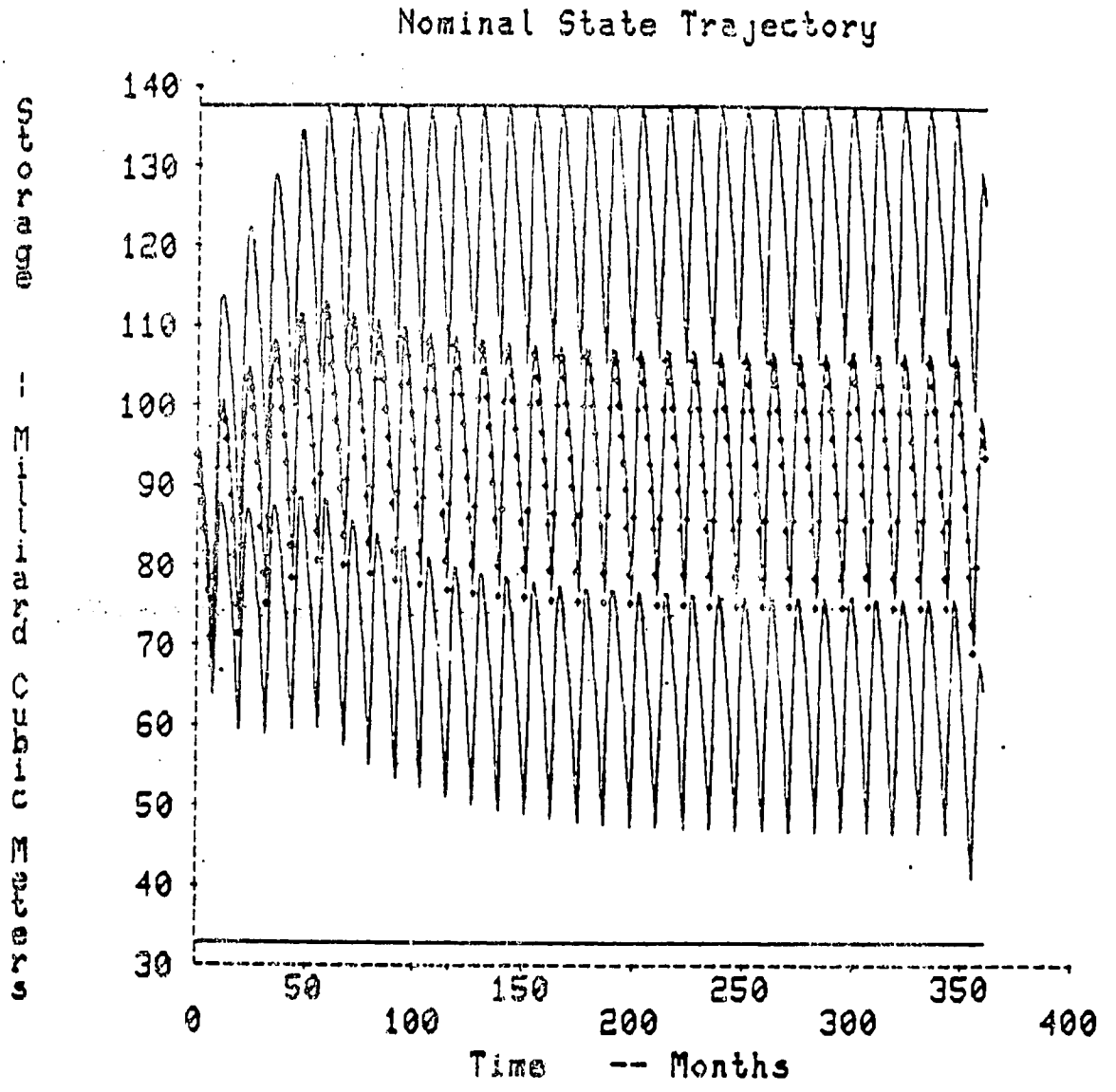

FIGURE 5.4.28: OLFC VARIANCE: OPTIMAL STATE TRAJECTORY

relevant here, and the question becomes one of properly specifying the terminal state penalty term. (The crucial influence of this term on the optimal trajectories is apparent in all the previous experimental runs.) To this end, it was noted that in the long run it pays (in energy generation) to build up high reservoir elevations by releasing the minimum downstream requirements (see, for instance, Figures 5.4.22, 5.4.23, and 5.4.24, 5.4.25). The releases are modified due to either the terminal penalty term influence or the upper storage bound being reached. It turns out that this behavior is independent of the initial storage, and therefore, in an infinite horizon problem it leads to the following operation policy: Release the minimum requirement until the upper state bound is reached. Thereafter, optimize the releases to maintain the highest feasible reservoir storage and maximize the energy generation. As another example verifying the above, see Figures 5.4.29 and 5.4.30 where a 720 month control problem was solved. Here $\gamma = 0.50$ ($z = 0.00$) implying expected value (deterministic optimization) with initial reservoir storage at 125×10^9 m^3 and terminal storage as great as feasible. The active state bound is at 137.72×10^9 m^3. Clearly it is optimal to release minimal amounts as long as storage remains feasible. The same conclusion was drawn when (1) the mean monthly inflow levels and/or the monthly water supply requirement were set higher (to consider future conditions) and (2) when only the first year's inflows were increased with the upcoming years/ inflows at their historical levels. (The second set of experiments was run to simulate the real-time operating conditions when, based on the forecasting models in Section 5.4.2, it is not realistic to make overyear predictions other

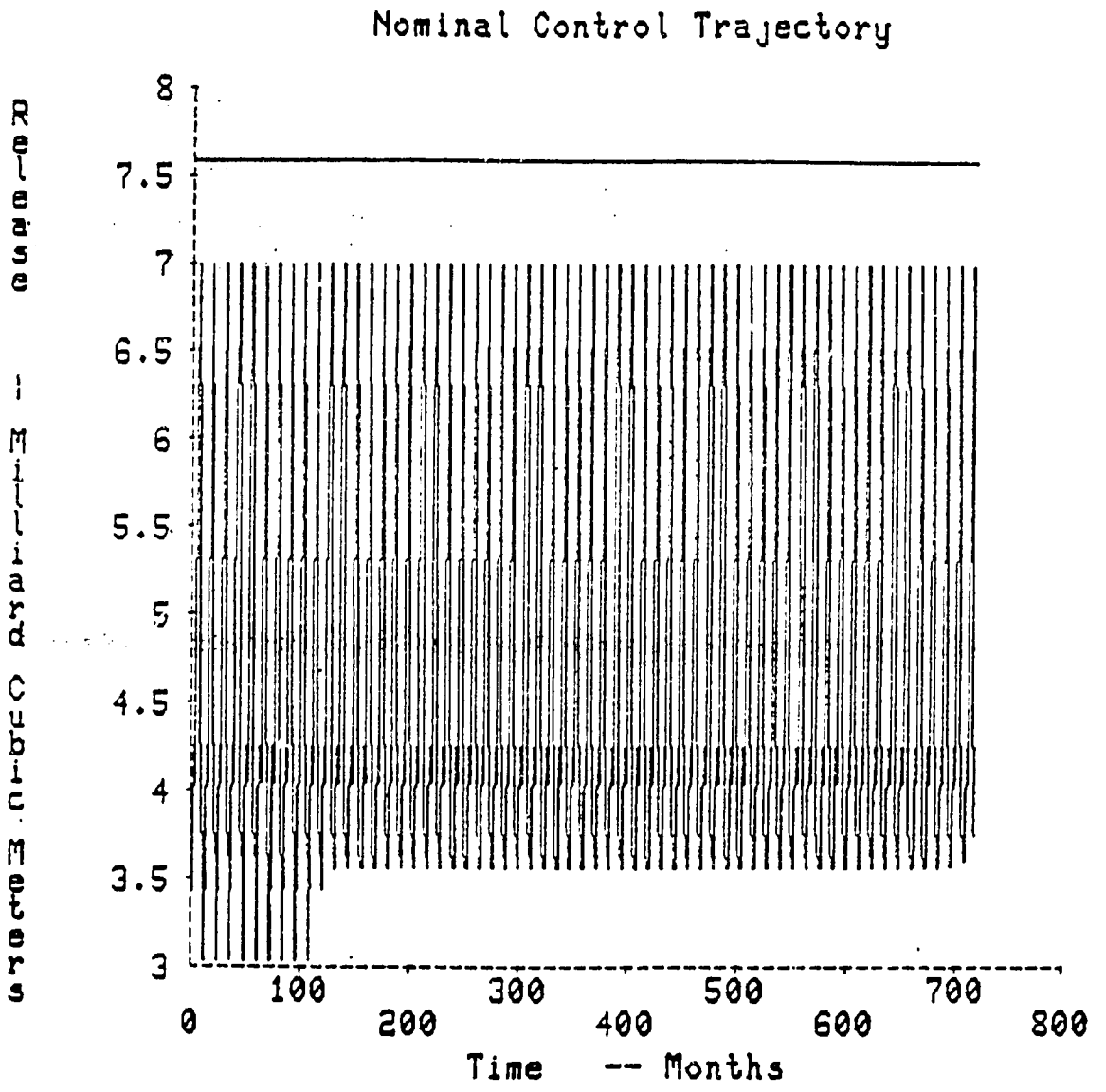

FIGURE 5.4.29: DETERMINISTIC ELQ LONG RANGE OPTIMIZATION

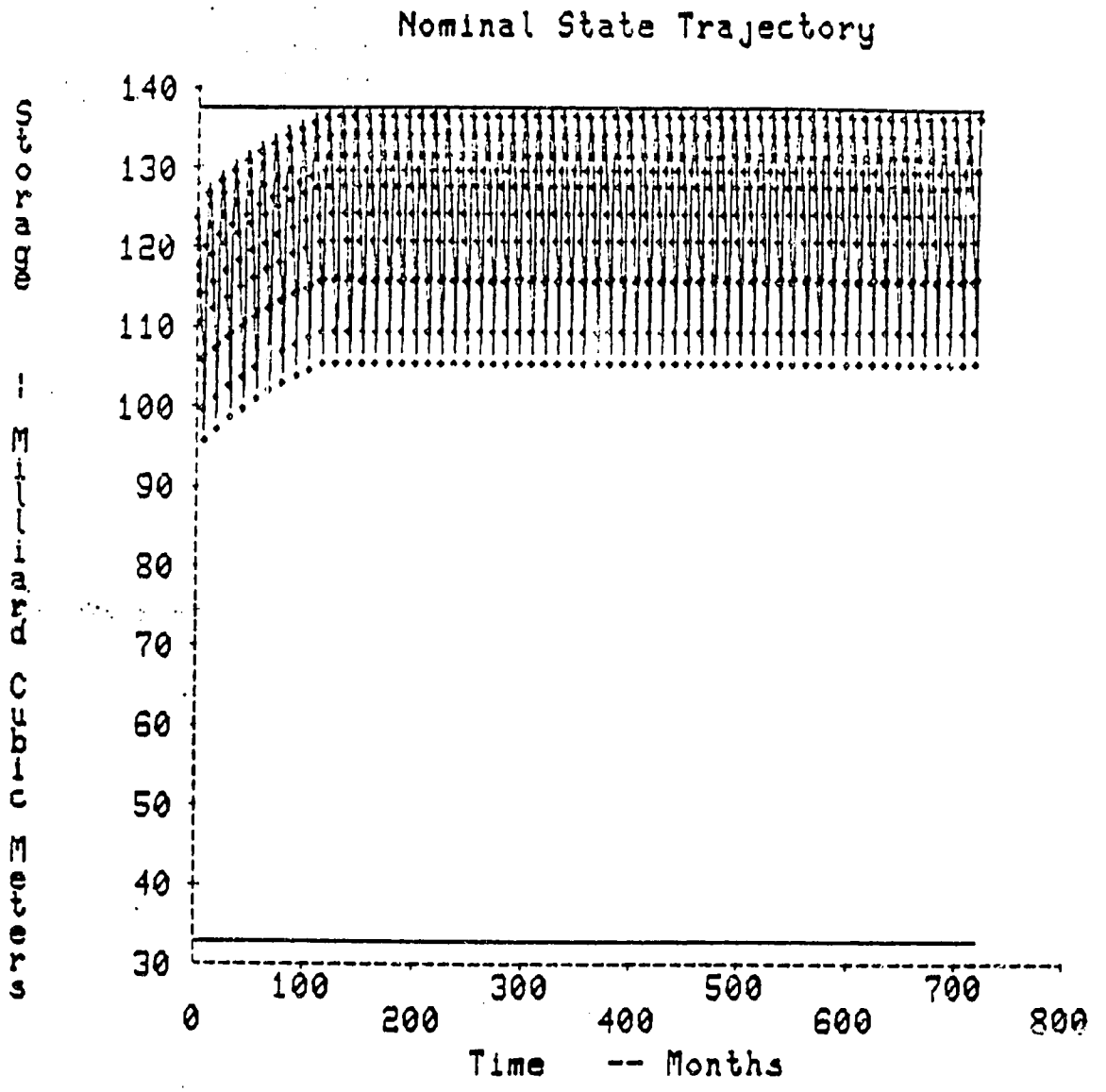

FIGURE 5.4.30: DETERMINISTIC ELQ LONG RANGE OPTIMIZATION

than the historical means at the historical variances.) Considering Toshka, due to the rapidly escalating losses when reservoir storage exceed 137.72×10^9 m^3 (see Figure 5.4.2), it is evident that the operation should avoid excessive spills by passing higher (yet feasible) discharges through the power plant turbines. Thus, the otherwise wasted water can be used in energy generation and other downstream water uses (e.g., using the available storage downstream of the HAD, the water in excess of the current water supply requirements can be used as a reserve supply to prevent reservoir drawdowns, to support land reclamation activities, or to help replenish the Nile Delta groundwater aquifer). Figures 5.4.31 and 5.4.32 show the solution of a 240-month control problem where Toshka is active. (The line at 137.72×10^9 m^3 reservoir storage has no boundary significance. The problem had no upper state constraint.) The initial reservoir storage was set at 130×10^9 m^3, and the terminal storage was required to be at 137.72×10^9 m^3. As expected, Toshka acts similarly to an upper bound and forces the optimal state trajectory lower by somewhat greater releases. The storages in excess of 137.72×10^9 m^3 make up for the energy loss from not establishing higher elevations by utilizing the water that would have been spilled.

From the previous considerations we can easily deduce what penalty term should be employed in a short horizon sequential control scheme; the term should seek to leave the reservoir at the highest feasible elevation. Note that it was necessary to look at the long-term system behavior to establish the upper storage bound and the terminal cost term. These specifications cannot be done with short-run investigations because over a short horizon it is energy-optimal to empty the reservoir.

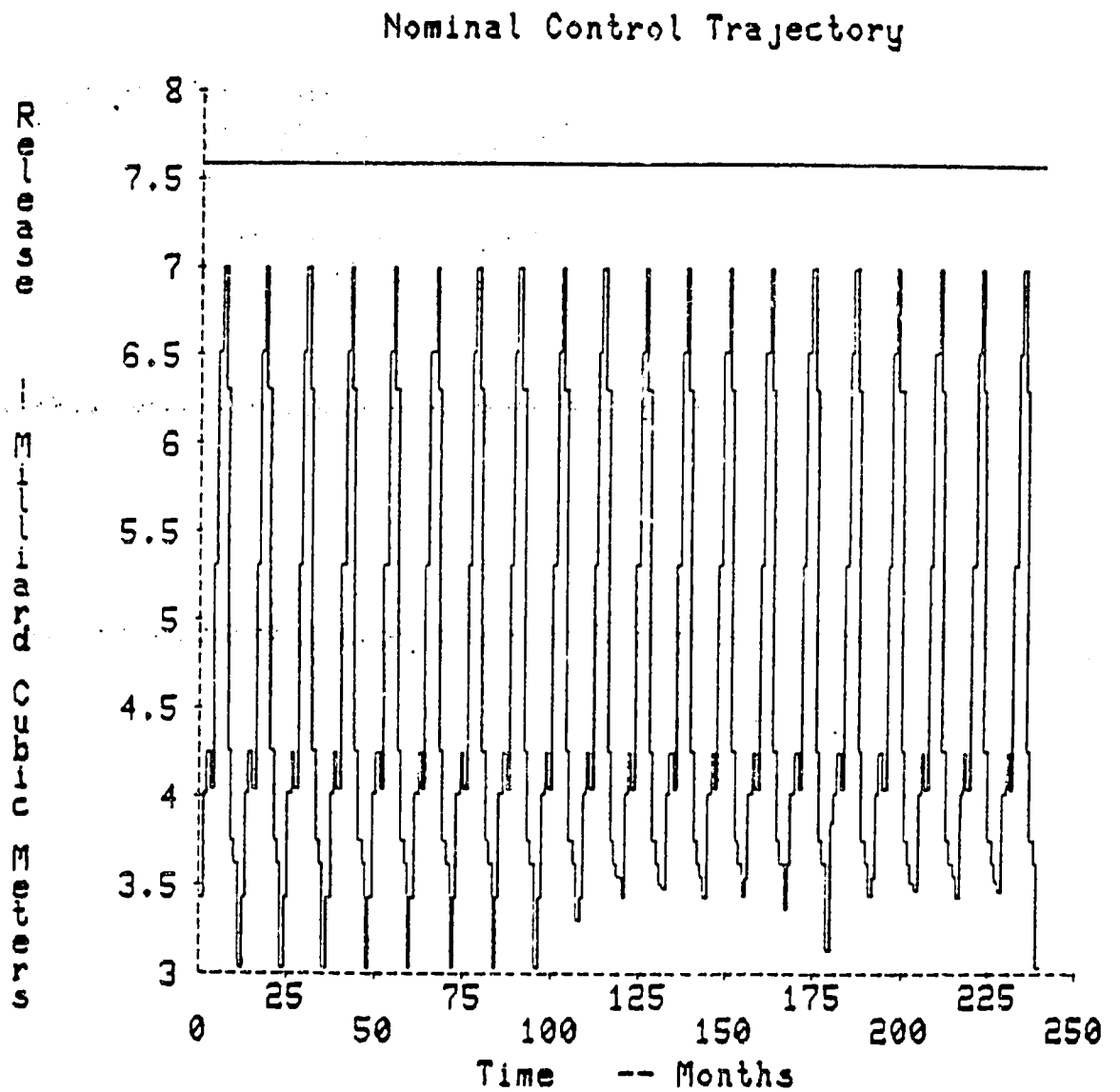

FIGURE 5.4.31: THE EFFECT OF TOSHKA

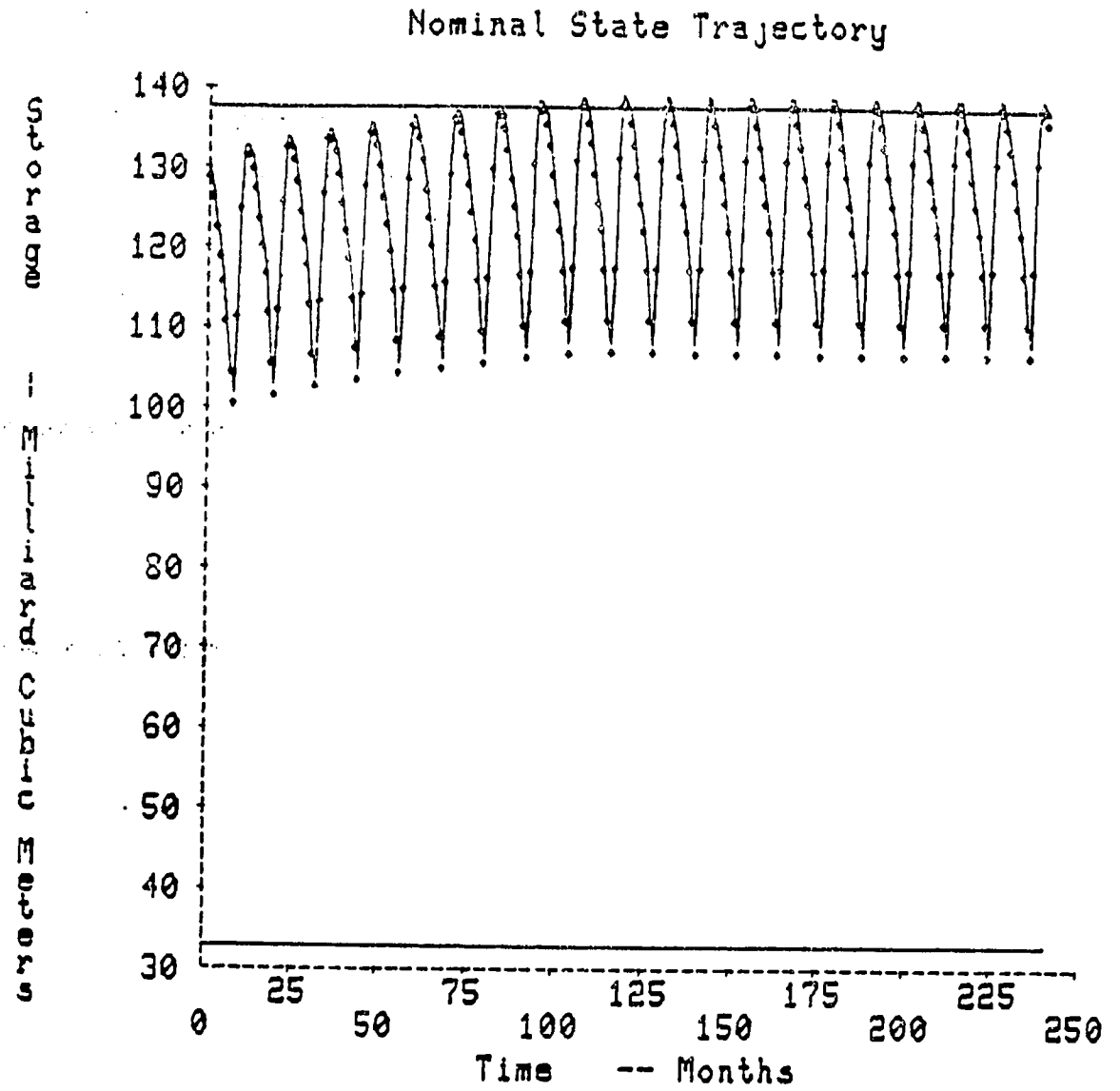

FIGURE 5.4.32: THE EFFECT OF TOSHKA

The value where, according to Figure 5.4.32, the upper bound should be placed is approximately $139.25 \times 10^9\ m^3$,

To further test ELQG in a real time operation and to compare its performance with other optimization procedures, extensive simulation experiments were run. The results are reported in the following section.

5.5 Simulation Analysis

The historical Nile flows at Wadi Halfa (entrance of the HAD reservoir) from January 1912 to December 1965 were the data base for the simulation experiments (see Appendix H). The recorded levels were adjusted by the estimated Sudan abstractions having the monthly distribution of Figure 5.2.7 and 16.5 x 10^9 m^3 yearly total. The downstream water supply requirements, channel degradation threshold, evaporation rates, etc., were as in the previous sections. (The parameters which differed from run to run will be separately mentioned.) In the ELQG simulation experiments, Algorithm A5.4.1 was implemented with a 12 month control horizon, C=10, and a terminal cost term penalizing any terminal storage deviation from the upper bound. The forecasting models of Section 5.4.2 are here denoted by "A-S" (a priori statistics), "T-F" (Thomas-Fiering lag-1 univariate seasonal autoregressive model), and "C-B" (multivariate seasonal autoregressive model, Curry and Bras, 1980). Results from using the Steady State Markov Dynamic Programming ("SSMDP") and Adaptive Markov Dynamic Programming ("AMDP") methods (see Section 5.3) in the same simulation experiments are also reported here from Buchanan and Bras, 1981, for purposes of comparison.

This section is organized in two basic parts: The first will discuss the ELQG's performance for various combinations of γ (reliability parameter) levels, different forecasting models and with respect to other control methods' performance. The second will employ ELQG in determining the tradeoff between two system objectives. The intention here is to verify the new method's potential in real time decision making. There is a plethora of issues related to the HAD operation which ELQG can

thoroughly investigate. Some of these possibilities will be mentioned at the end of this section.

Table 5.5.1 presents some results from 8 simulation runs (each column represents a 54 year simulation experiment). The quantities reported are the total volume of water supply deficits (i.e., the amount by which the models failed to meet the downstream water supply requirements), the total volume of water released in excess of 7.59×10^9 m^3 per month, the total volume of spills to Toshka, the mean annual energy production (GWH), the reservoir storage at the end of the simulation period, the corrected mean annual energy generation according to the terminal storage difference, and the mean annual evaporation losses. All Runs were started at 180 meters reservoir elevation (or at 149.55×10^9 m^3 reservoir storage). The correction of the mean annual energy generation was necessary for the energy comparisons since the model, which has maintained higher terminal stroage, could have alternatively generated more energy. The adjustment was as follows: The lowest terminal storage was subtracted from the rest, and the resulting amount was assumed to generate energy at maximum release for as long as it lasted. The evaporation rate was that of January. The energy produced was divided by 54 and added to the mean annual energy production. Terminal storage information was not available for the SSMDP and AMDP methods, and therefore no energy connection was attempted. In all ELQG runs the upper storage bound was placed at 139.25×10^9 m^3 (see previous section). It was found that all models met the downstream water supply requirements and prevented channel degradation at all times. The ELQG models caused significantly less Toshka

spills, and as a result produced on the average about 300 GWH more energy per year over the SSMDP and AMDP methods. Instrumental for this result is ELQG's ability to explicitly account for control and state constraints as well as its analytical structure. Owing to the latter, the ELQG models required about 30 minutes CPU computer time to complete the simulations. By contrast AMDP required 50 CPU hours (Buchanan and Bras, 1981) while SSMDP, 20 sec. (Concerning SSMDP, the above figure does not include the time needed to obtain the release policies which were used unchanged throughout the simulation period - SSMDP is not a sequential scheme.) Focusing on the performance of the various ELQG model combinations, it is seen that those with reliability constraint provisions incur less Toshka spills than their deterministic counterparts. (Compare, for instance, model (z=1.96, A-S) to (z=0, A-S) and (z=1.96, T-F) to (z=0, T-F).) More successful in this sense was the (z=1.96, C-B) model which had more accurate foresight and as a result was able to more effectively manage the incoming flows. In fact, the comparison was not exactly fair for the (z=1.96, C-B) combination because its forecasting model was not implemented but instead was simulated as follows: Based on the a priori inflow variance and the R^2 statistic of the Curry-Brass forecasting model, the forecast variances were obtained for the upcoming 12 months' inflows (see Equation (5.4.45), (5.4.46) and Table 5.3.2 in Section 5.4.2). Subsequently a Gaussian random number generator was used to derive a value from each of 12 probability densities having the actual inflows as means and the previously obtained forecast variances as second moments. These values were adopted as the expected values of the forecasts'

p.d.f.s. Notice that this procedure utilizes the process's correlation structure only in the variance specification and not in the forecast's mean; therefore, it is at a disadvantage with respect to the Thomas-Fiering procedure which was fitted to and forecasted the 54 year historical inflow data. This disadvantage appears somewaht in the energy generation where the (z=1.96, C-B) combination does only slightly better compared to the other stocahstic combinations. Model (z=0, Perfect Forecasts) was run so that on upper bound estimate of the sequential models' performance could be obtained. It had perfect knowledge of the 12 upcoming inflows, and consequently, it generated the most energy allowing for Toshka spills in accorance with the upper state constraint. However, notice that all ELQG models perform well despite their imprecise information concerning the future inflow sequence. As the performance differences are not signficant, another conclusion to be drawn is that, presently forecasted information is not of critical significance to the HAD operation planning. The reasons are (1) that most of the time the reservoir has enough available storage to accommodate the incoming flows and (2) that the presence of Toshka has eliminated any threat for severe flooding events. However, the importance of accurate forecasting is expected to rise at a future time when the problem will not be one of maximizing energy generation but rather one of striving to meet the growing water supply requirements. Another comment regarding Table 5.5.1 refers to the competing nature of energy maximization and Toshka spills prevention by forcing the reservoir to operate at lower heads (via the reliability constraints). For instance,

	UNITS
ENERGY	GWH
VOLUME	$x10^9$ m^3

			E L Q G					
	SSMDP	AMDP	z=0,A-S	z=0, T-F	PERFECT z=0,FORECASTS	z=1.96, A-S	z=1.96, T-F	z=1.96,C-B
WATER SUPPLY DEFICITS	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
EXCESSIVE RELEASES	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
TOSHKA SPILLS (OVER 54 YEARS)	100.700	100.700	5.073	3.298	2.584	3.188	2.370	0.898
MEAN ANNUAL ENERGY PRODUCTION	7,787.00	7,787.00	8,051.95	8,077.06	8,088.52	8,066.18	8,079.84	8,070.57
TERMINAL STORAGE	—	—	136.145	130.859	136.962	133.321	129.110	132.880
CORRECTED MEAN ANNUAL ENERGY PRODUCTION	—	—	8,070.94	8,081.75	8,109.73	8,077.50	8,079.84	8,080.70
MEAN ANNUAL EVAPORATION LOSSES	13.600	13.600	13.446	13.236	13.226	13.208	13.111	12.940

TABLE 5.5.1: SIMULATION RESULTS

ELQG model (z=0, T-F) manages to produce more power than the stochastic models by maintaining on the average higher hydraulic head (this is evident from the heavier mean annual evaporation losees), although it also causes higher Toshka spills. However, the same is not repeated in the (z=0, A-S) case. Owing to this model's naive inflow predictor, it cannot help spilling more substantial water volume thus limiting its potential for energy generation. To test ELGQ's performance with the OLFC rather than the modified variance, the model (z=1.96, T-F) was also run without the variance adjustment. The results were Toshka spills 0.705×10^9 m^3, energy production 8,019.079 GWH (mean annual) terminal storage 122.242×10^9 m^3, and evaporation losses 12.417×10^9 m^3 (mean annual). As has been noted, this procedure is overly conservative keeping the reservoir storage lower than actually needed but consequently generating less energy. Tables 5.5.2 and 5.5.3 show two years (1916, 1917) from the simulations of the models (z=0, A-S) and (z=1.96, C-B). These are two consecutive high flood years (especially the inflows in September and October are approximately 3 standard deviations above the historical means) and they demonstrate the anticipative ability of models with forecasted information. The first controller becomes alerted and starts releasing at maximum rate only after the flood has arrived and Toshka is already in action (October 1917). The second is aware of the probable emergency and suggests higher releases one year before the flood's arrival. As a result, it incurs minimal Toshka losses and generates 2,676 GWHs more compared to the first model's energy production over this two-year period. Table 5.5.4 gives more statistics from the results of the (Z=1.96, C-B) simulation experiment; namely, it reports

DEC.	3.019	3.107	3.019	98.763	170.29	365.34	0.000	0.000	0.948

* Y E A R : 1916 *

MONTH	SUG/ED REL.SE 10**9 C.M./M.	INFLOW 10**9 C.M./M.	ACTUAL REL/SE 10**9 C.M./M.	TERM/AL STORAGE 10**9 C.M.	ELEVAT. METERS	POWER GWH/M.	IRRIG. DEF/CT 10**9 C.M./M.	EXCES. REL/SE 10**9 C.M./M.	EVAP. 10**9 C.M./M
JAN.	3.430	1.799	3.430	96.361	169.74	422.19	0.000	0.000	0.771
FEB.	4.018	0.753	4.018	92.515	168.83	502.41	0.000	0.000	0.580
MAR.	4.240	0.525	4.240	88.372	167.81	525.06	0.000	0.000	0.428
APR.	4.040	0.698	4.040	84.449	166.81	485.16	0.000	0.000	0.581
MAY	5.312	0.660	5.312	79.072	165.37	655.04	0.000	0.000	0.726
JUN.	6.521	0.502	6.521	72.222	163.41	805.94	0.000	0.000	0.830
JUL.	6.999	4.691	6.999	69.009	162.43	847.97	0.000	0.000	0.906
AUG.	6.310	25.948	6.310	87.540	167.60	785.88	0.000	0.000	1.106
SEP.	4.251	29.722	4.251	111.585	173.07	549.13	0.000	0.000	1.426
OCT.	3.741	21.057	3.741	127.329	176.14	512.64	0.000	0.000	1.572
NOV.	3.619	10.062	3.619	132.311	177.05	511.48	0.000	0.000	1.460
DEC.	3.019	5.247	3.019	133.337	177.24	416.18	0.000	0.000	1.202

* Y E A R : 1917 *

MONTH	SUG/ED REL/SE 10**9 C.M./M.	INFLOW 10**9 C.M./M.	ACTUAL REL/SE 10**9 C.M./M.	TERM/AL STORAGE 10**9 C.M.	ELEVAT. METERS	POWER GWH/M.	IRRIG. DEF/CT 10**9 C.M./M.	EXCES. REL/SE 10**9 C.M./M.	EVAP. 10**9 C.M./M
JAN.	3.430	3.629	3.430	132.543	177.09	484.83	0.000	0.000	0.992
FEB.	4.018	2.713	4.018	130.478	176.72	582.94	0.000	0.000	0.760
MAR.	4.240	2.815	4.240	128.478	176.36	617.38	0.000	0.000	0.574
APR.	4.040	1.928	4.040	125.569	175.82	577.93	0.000	0.000	0.797
MAY	5.312	1.290	5.312	120.530	174.86	789.98	0.000	0.000	1.017
JUN.	6.521	1.262	6.521	114.074	173.58	988.11	0.000	0.000	1.196
JUL.	6.999	4.421	6.999	110.160	172.78	1053.69	0.000	0.000	1.336
AUG.	6.310	18.448	6.310	120.769	174.90	943.19	0.000	0.000	1.530
SEP.	4.251	29.522	4.251	144.165	179.11	624.67	0.000	0.000	1.843
OCT.	7.590	22.057	7.590	154.835	180.85	1280.00	0.000	0.000	3.552
NOV.	7.590	9.462	7.590	152.179	180.43	1280.00	0.000	0.000	4.238
DEC.	7.590	5.067	7.590	146.580	179.51	1280.00	0.000	0.000	2.834

TABLE 5.5.2: TWO HIGH FLOOD YEARS FROM THE ELQG (z=0, A-S) SIMULATION EXPERIMENT

DEC.	3.019	3.107	3.019	99.277	170.41	366.20	0.000	0.000	0.952

* Y E A R : 1916 *

MONTH	DEM/ED REL/SE 10**9 C.M./M.	INFLOW 10**9 C.M./M.	ACTUAL REL/SE 10**9 C.M./M.	TERM/AL STORAGE 10**9 C.M.	ELEVAT. METERS	POWER GWH/M.	IRRIG. DEF/CT 10**9 C.M./M.	EXCES. REL/SE 10**9 C.M./M.	EVAP. 10**9 C.M./M
JAN.	3.675	1.799	3.675	96.628	169.80	459.23	0.000	0.000	0.774
FEB.	4.018	0.753	4.018	92.780	168.89	503.06	0.000	0.000	0.582
MAR.	4.684	0.525	4.684	88.194	167.77	591.38	0.000	0.000	0.428
APR.	4.040	0.698	4.040	84.272	166.76	484.69	0.000	0.000	0.580
MAY	5.312	0.660	5.312	78.895	165.32	654.36	0.000	0.000	0.724
JUN.	6.521	0.502	6.521	72.048	163.35	805.03	0.000	0.000	0.828
JUL.	6.999	4.691	6.999	68.836	162.37	846.94	0.000	0.000	0.904
AUG.	6.310	25.948	6.310	87.370	167.56	785.03	0.000	0.000	1.104
SEP.	4.251	29.722	4.251	111.416	173.04	548.69	0.000	0.000	1.424
OCT.	5.305	21.057	5.305	125.606	175.82	775.43	0.000	0.000	1.562
NOV.	7.180	10.062	7.180	127.059	176.09	1145.45	0.000	0.000	1.429
DEC.	7.590	5.247	7.590	123.568	175.44	1219.61	0.000	0.000	1.247

* Y E A R : 1917 *

MONTH.	DEM/ED REL/SE 10**9 C.M./M.	INFLOW 10**9 C.M./M.	ACTUAL REL/SE 10**9 C.M./M.	TERM/AL STORAGE 10**9 C.M.	ELEVAT. METERS	POWER GWH/M.	IRRIG. DEF/CT 10**9 C.M./M.	EXCES. REL/SE 10**9 C.M./M.	EVAP. 10**9 C.M./M
JAN.	6.534	3.629	6.534	119.739	174.70	1006.63	0.000	0.000	0.824
FEB.	5.912	2.713	5.912	115.844	173.93	880.67	0.000	0.000	0.695
MAR.	7.590	2.815	7.590	110.554	172.86	1165.78	0.000	0.000	0.515
APR.	7.137	1.928	7.137	104.649	171.60	1058.76	0.000	0.000	0.697
MAY	6.846	1.290	6.846	98.223	170.17	980.82	0.000	0.000	0.869
JUN.	6.521	1.262	6.521	91.957	168.69	899.45	0.000	0.000	1.007
JUL.	6.999	4.421	6.999	88.263	167.78	953.50	0.000	0.000	1.116
AUG.	6.310	18.448	6.310	99.112	170.37	857.56	0.000	0.000	1.238
SEP.	4.251	29.522	4.251	122.824	175.30	577.19	0.000	0.000	1.558
OCT.	7.050	22.057	7.050	136.152	177.73	1132.50	0.000	0.000	1.678
NOV.	5.561	9.462	5.561	138.522	178.15	875.66	0.000	0.000	1.522
DEC.	6.173	5.067	6.173	136.177	177.74	991.99	0.000	0.000	1.228

TABLE 5.5.3: TWO HIGH FLOOD YEARS FROM THE ELQG (z=1.96, C-B) SIMULATION EXPERIMENT

MONTH	ELEVATION (METERS) MINIMUM	MAXIMUM	MEAN	ST.DEV.	STORAGE (10**9 C.M./M.) MINIMUM	MAXIMUM	MEAN	ST.DEV.
JAN.	167.231	179.795	174.403	2.899	86.083	142.069	118.939	14.129
FEB.	166.163	178.095	173.615	3.003	82.002	138.223	115.029	14.219
MAR.	165.389	177.412	172.361	3.062	79.148	134.329	111.266	14.128
APR.	164.614	176.712	172.160	3.127	76.375	130.433	106.056	14.100
MAY	163.167	176.108	171.105	3.279	71.428	127.139	103.256	14.218
JUN.	161.260	175.140	169.651	3.495	65.339	122.003	96.942	14.316
JUL.	159.457	174.457	168.592	3.630	60.007	118.481	92.545	14.239
AUG.	161.993	175.713	171.105	3.175	67.620	125.023	103.198	13.740
SEP.	165.591	177.675	174.100	2.908	79.884	135.816	117.408	13.769
OCT.	167.898	178.867	175.439	2.795	88.721	142.729	124.269	14.085
NOV.	168.188	178.261	175.324	2.759	89.889	139.179	123.642	13.836
DEC.	167.986	178.017	174.930	2.771	89.076	137.775	121.591	13.724

MONTH	TOSHKA LOSSES (10**9 C.M./M.) MIN.	MAX.	MEAN	ST.DEV.	# DATA	EVAPORATION (10**9 C.M./M.) MINIMUM	MAXIMUM	MEAN	ST.DEV.
JAN.	0.000	0.192	0.096	0.096	2	0.706	1.865	0.930	0.155
FEB.	0.089	0.089	0.089	0.000	1	0.526	0.861	0.691	0.071
MAR.	0.002	0.002	0.002	0.000	1	0.388	0.596	0.514	0.053
APR.	0.000	0.000	0.000	0.000	0	0.531	0.824	0.706	0.075
MAY	0.000	0.000	0.000	0.000	0	0.665	1.053	0.896	0.100
JUN.	0.000	0.000	0.000	0.000	0	0.759	1.256	1.048	0.126
JUL.	0.000	0.000	0.000	0.000	0	0.814	1.414	1.160	0.148
AUG.	0.000	0.000	0.000	0.000	0	0.925	1.597	1.332	0.159
SEP.	0.000	0.000	0.000	0.000	0	1.107	1.777	1.548	0.160
OCT.	0.000	0.057	0.015	0.018	8	1.179	1.830	1.585	0.152
NOV.	0.001	0.097	0.039	0.034	10	1.075	1.647	1.406	0.132
DEC.	0.001	0.034	0.015	0.012	7	0.870	1.231	1.124	0.103

MONTH	RELEASE (10**9 C.M./M.) MINIMUM	MAXIMUM	MEAN	ST.DEV.	HYDROPOWER (GWH/M.) MINIMUM	MAXIMUM	MEAN	ST.DEV.
JAN.	3.430	7.590	3.823	0.998	401.78	1280.00	532.48	136.01
FEB.	4.018	7.590	4.252	0.639	475.67	1209.46	593.25	122.90
MAR.	4.240	7.590	4.450	0.726	497.15	1212.35	616.78	135.32
APR.	4.040	7.590	4.163	0.629	461.34	1135.13	560.72	110.35
MAY	5.312	6.846	5.357	0.240	623.77	980.82	743.37	52.26
JUN.	6.521	6.521	6.521	0.000	766.77	1014.80	916.50	62.68
JUL.	6.999	6.999	6.999	0.000	798.21	1087.04	971.82	71.22
AUG.	6.310	6.310	6.311	0.000	711.67	965.89	871.75	59.84
SEP.	4.251	4.452	4.255	0.027	477.09	631.13	575.41	34.52
OCT.	3.741	7.590	4.013	0.365	437.17	1265.56	563.98	168.39
NOV.	3.619	7.590	4.242	1.077	431.99	1279.56	612.91	210.22
DEC.	3.019	7.590	3.650	1.209	348.32	1232.19	511.58	220.27

ANNUAL POWER STATISTICS (GWH/Y.)

MINIMUM	:	6430.937
MAXIMUM	:	11379.511
MEAN	:	8070.567
ST. DEV.	:	976.884

TABLE 5.5.4: SOME MONTHLY AND ANNUAL STATISTICS OF THE ELQG (z=1.96, C-B) SIMULATION EXPERIMENT RESULTS

monthly maximum and minimum values, mean values, and standard deviations for reservoir elevation, storage, Toshka losses, evaporation, release and energy generation. The same statistics are also computed for the annual energy generation.

It should be pointed out that a more conclusive analysis would require simulation experiments on synthetically generated inflow series as well. ELQG parameters which can be determined from such a study are the reliability levels (γ can change from month to month) and the control horizon length maximizing expected energy production. With respect to the control horizon, it should improve the model's performance if it is increased because of the peculiar Nile flow behavior to persist in high or low levels. In setting up the previous experiments, ELQG is not aware of such tendencies since it plans on historically average inflows for the upcoming years. To this end, an annual flow predictor should be useful.

As the simulation experiments have shown, HAD reservoir suffers heavy evaporation losses (approximately 13 billion cubic meters are lost per year). On the other hand, its storage capacity is more than adequate for the current water supply purposes. For instance, this can be seen in Table 5.5.4 where throughout the (z=1.96, C-B) simulation experiment the reservoir storage has not fallen below 60×10^9 m^3. If it were possible to maintain lower reservoir elevations while always meeting the water supply requirements, there would be some water gains from evaporation and some energy losses owing to the lower hydraulic head. This is the tradeoff we wish to determine in the second part of

this section. It has a practical interest because in the years to come water rather than hydropower availability will be the limiting factor for the development of the Egyptian economy. The issue may also become particularly crucial if the seismic activity affecting the HAD reservoir continues to intensify forcing operation at lower elevations. The procedure adopted here was to place the upper state bound at increasingly lower levels and perform simulation experiments. ELQG was used with (z=0, T-F). The initial storage was assumed equal to the corresponding upper bound. Table 5.5.5 reports some results of these experiments. The maximum allowable elevation was decreased from 178 m (137.72×10^9 m^3 storage) to the point where water supply deficits could not be avoided (at approximately 171 meters). (In fact, if a stochastic ELQG model had been used, the upper bound could have been safely decreased even further.) The results show that on the average about 2 milliard m^3 can be saved from evaporation at the expense of 350 GWH per year (maximum elevation at 173 m) or 0.9 milliard m^3 at 100 GWH (maximum elevation at 176 m). With Egypt and Sudan already involved in costly expenditures on water conservation projects (Egyptian Ministry of Irrigation, Main Report and TR 5, 1981), for fear of shortages, these results become particularly signficant. Note further that compared to the other control methods as well as the currently employed heuristic operation policy (see Buchanan and Bras, 1981) ELQG control results in over 2 milliard m^3 water benefits at 7,800 GWH annual energy production (compare also with Table 5.5.1 although in those runs the initial reservoir storage was equal to 149.55×10^9 m^3). Table 5.5.6

	UNITS
ENERGY	GWH
VOLUME	$x10^9 m^3$
ELEVATION	METERS

	UPPER STORAGE BOUND (ELEVATION)							
	137.72 (178.008)	132.00 (177.000)	126.50 (175.989)	121.25 (175.000)	116.25 (174.016)	111.50 (173.053)	106.50 (172.002)	102.00 (171.02)
WATER SUPPLY DEFICITS	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.874
EXCESSIVE RELEASES	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
TOSHKA SPILLS (OVER 54 YEARS)	3.843	1.098	0.404	0.048	0.000	0.000	0.000	0.000
MEAN ANNUAL ENERGY PROD.	8,036.405	7,997.292	7,932.810	7,856.111	7,777.291	7,689.793	7,590.068	7,484.844
MEAN ANNUAL EVAPORA-TION LOSSES	13.155	12.954	12.254	11.866	11.483	11.094	10.679	10.275
MINIMUM STORAGE (ELEVATION) (OVER 54 YEARS)	59.345 (159.22[illegible])	55.767 (157.917)	51.381 (156.210)	47.152 (154.432)	43.055 (152.565)	39.273 (150.691)	35.414 (148.596)	32.714 (147.000)

TABLE 5.5.5: WATER SAVINGS FROM EVAPORATION VS. ENERGY GENERATION REDUCTION

gives some additional statistics from the simulation experiment with upper storage bound at 111.50×10^9 m^3 (173 m elevation).

The vital importance of the High Aswan Dam Reservoir to the Egyptian economy has been emphasized earlier. In this chapter it was shown how the reservoir can become more energy efficient, if optimally operated at elevations below Toshka's threshold. Estimates of water savings and energy losses were obtained from operating at lower elevations, and a potentially attractive tradeoff to the Egyptian water planning authorities resulted. The ELQG control method can be used to analyze a variety of other related issues. For instance, optimal reservoir operation under varying hydrology conditions (such as resulting from upstream water conservation projects and the growing Sudan abstractions) and changes in the downstream water demand (consequence of the intense development programs) can be thoroughly investigated to assist in successful long-term planning. Gradual modification of the irrigation demand pattern (constituting the bulk of the downstream water supply requirements) may be studied aiming to achieve a better complementarity with the energy generation objective. Similarly, the appropriate reclamation rate of desert lands and their cropping pattern need to be determined in connection with agroeconomic studies. Apart from long-term planning issues, a hierarchy of ELQG models can be put together to dictate real time, day to day (or shorter interval) reservoir operation. These along with many other investigations can positively assist in the optimal management of the Nile waters.

With regard to the ELQG control method, this chapter has demonstrated that its theoretically predicted properties are indeed realized

MONTH	ELEVATION (METERS) MINIMUM	MAXIMUM	MEAN	ST.DEV.	STORAGE (10**9 C.M./M.) MINIMUM	MAXIMUM	MEAN	ST.DEV.
JAN.	162.647	175.115	170.375	2.976	69.720	121.876	99.824	12.192
FEB.	161.238	174.069	169.476	3.093	65.269	116.516	95.972	12.174
MAR.	159.814	173.205	168.629	3.179	61.031	112.242	92.452	12.063
APR.	158.447	172.321	167.830	3.260	57.193	108.000	89.222	11.952
MAY	156.492	171.326	166.615	3.482	53.085	103.381	84.571	12.114
JUN.	153.623	169.975	164.910	3.805	45.324	97.320	78.404	12.225
JUL.	150.591	169.207	163.669	4.101	39.273	94.098	74.202	12.504
AUG.	157.241	171.979	166.732	3.569	53.991	106.394	85.069	12.639
SEP.	161.616	174.935	170.249	3.306	66.438	120.936	99.434	12.512
OCT.	164.118	176.643	171.693	3.108	74.646	130.054	105.857	12.500
NOV.	164.233	176.718	171.484	2.934	75.392	130.462	104.629	12.350
DEC.	163.740	176.039	171.029	2.926	73.254	126.767	102.724	12.352

MONTH	TOSHKA LOSSES (10**9 C.M./M.) MIN.	MAX.	MEAN	ST.DEV.	% DATA	EVAPORATION (10**9 C.M./M.) MINIMUM	MAXIMUM	MEAN	ST.DEV.
JAN.	0.000	0.000	0.000	0.000	0	0.593	0.840	0.734	0.082
FEB.	0.000	0.000	0.000	0.000	0	0.435	0.702	0.596	0.083
MAR.	0.000	0.000	0.000	0.000	0	0.313	0.519	0.442	0.049
APR.	0.000	0.000	0.000	0.000	0	0.416	0.710	0.604	0.069
MAY	0.000	0.000	0.000	0.000	0	0.506	0.899	0.762	0.092
JUN.	0.000	0.000	0.000	0.000	0	0.552	1.053	0.881	0.115
JUL.	0.000	0.000	0.000	0.000	0	0.552	1.174	0.954	0.142
AUG.	0.000	0.000	0.000	0.000	0	0.684	1.341	1.118	0.156
SEP.	0.000	0.000	0.000	0.000	0	0.923	1.590	1.335	0.162
OCT.	0.000	0.000	0.000	0.000	0	1.016	1.617	1.386	0.153
NOV.	0.000	0.000	0.000	0.000	0	0.927	1.464	1.230	0.129
DEC.	0.000	0.000	0.000	0.000	0	0.745	1.171	0.932	0.101

MONTH	RELEASE (10**9 C.M./M.) MINIMUM	MAXIMUM	MEAN	ST.DEV.	HYDROPOWER (GWH/M.) MINIMUM	MAXIMUM	MEAN	ST.DEV.
JAN.	3.430	7.590	3.981	0.938	364.19	1215.30	519.34	175.24
FEB.	4.018	7.590	4.239	0.758	427.36	1192.88	555.08	145.46
MAR.	4.240	7.590	4.379	0.634	440.58	1171.23	557.20	123.34
APR.	4.040	7.590	4.176	0.648	401.94	1151.50	517.71	121.39
MAY	5.312	7.590	5.354	0.307	534.34	1130.36	673.93	77.32
JUN.	6.521	7.590	6.541	0.144	640.78	1104.08	824.92	74.30
JUL.	6.999	6.999	6.999	0.000	643.13	990.73	875.70	76.71
AUG.	6.310	6.310	6.311	0.000	593.16	876.41	789.73	65.49
SEP.	4.251	7.590	4.313	0.450	430.28	1167.16	539.68	93.64
OCT.	3.741	7.590	4.652	1.531	401.96	1211.82	636.02	293.29
NOV.	3.619	7.590	4.827	1.673	397.47	1240.52	674.52	311.25
DEC.	3.019	7.590	3.248	1.430	319.30	1233.60	510.96	260.62

ANNUAL POWER STATISTICS
(GWH/Y.)

MINIMUM	: 5651.173
MAXIMUM	: 11545.351
MEAN	: 7688.795
ST. DEV.	: 1267.774

TABLE 5.5.6: SOME MONTHLY AND ANNUAL STATISTICS FROM THE (z=0,T-F) ELQG SIMULATION EXPERIMENT WITH UPPER BOUND AT 173 m.

in practice. The following chapter will briefly discuss an application to a three reservoir system and will conclude this work. Case studies on dimensionally higher systems will be reported in separate publications.

Chapter 6

A THREE RESERVOIR CASE STUDY

6.1 Introduction and Overview

This chapter's objective is to assure that the method's previously demonstrated efficiency is also present in high order system applications. The second example is a three lake system also located in the Nile River basin. After presenting a model for this system's dynamics, a control problem will be set up and solved. Due to the limited data base, this case study will not involve simulation experiments. However, further usage of the method along the lines of the previous chapter is straightforward.

6.2 The Equatorial Lake System

The River Nile emanates at the Equatorial (or Great) Lake region in the Eastern part of Africa (see Figure 5.2.1). Figure 6.2.1 focuses on this area's three major lakes, namely, Lakes Victoria, Kyoga, and Albert (or Mobutu Sese Seco). Connected by the Victoria and Kyoga Nile branches, these lakes form a cascaded system containing enormous quantities of fresh water (about 3,200 km^3). The outflow of this system (approximately 2.3×10^9 m^3 per month at the mouth of Lake Albert) eventually joins the White Nile waters after first passing through the Sudd Swamps (Figure 5.2.1), where half of it is lost to evaporation. The Lakes (especially Victoria and Albert) are natural reservoirs of large storage capability and can be regulated for the benefit of the riparian countries (Kenya, Tanzania, and Uganda) as well as the Sudan and Egypt. The regulation studies are still at a preliminary phase (regulation structures have not

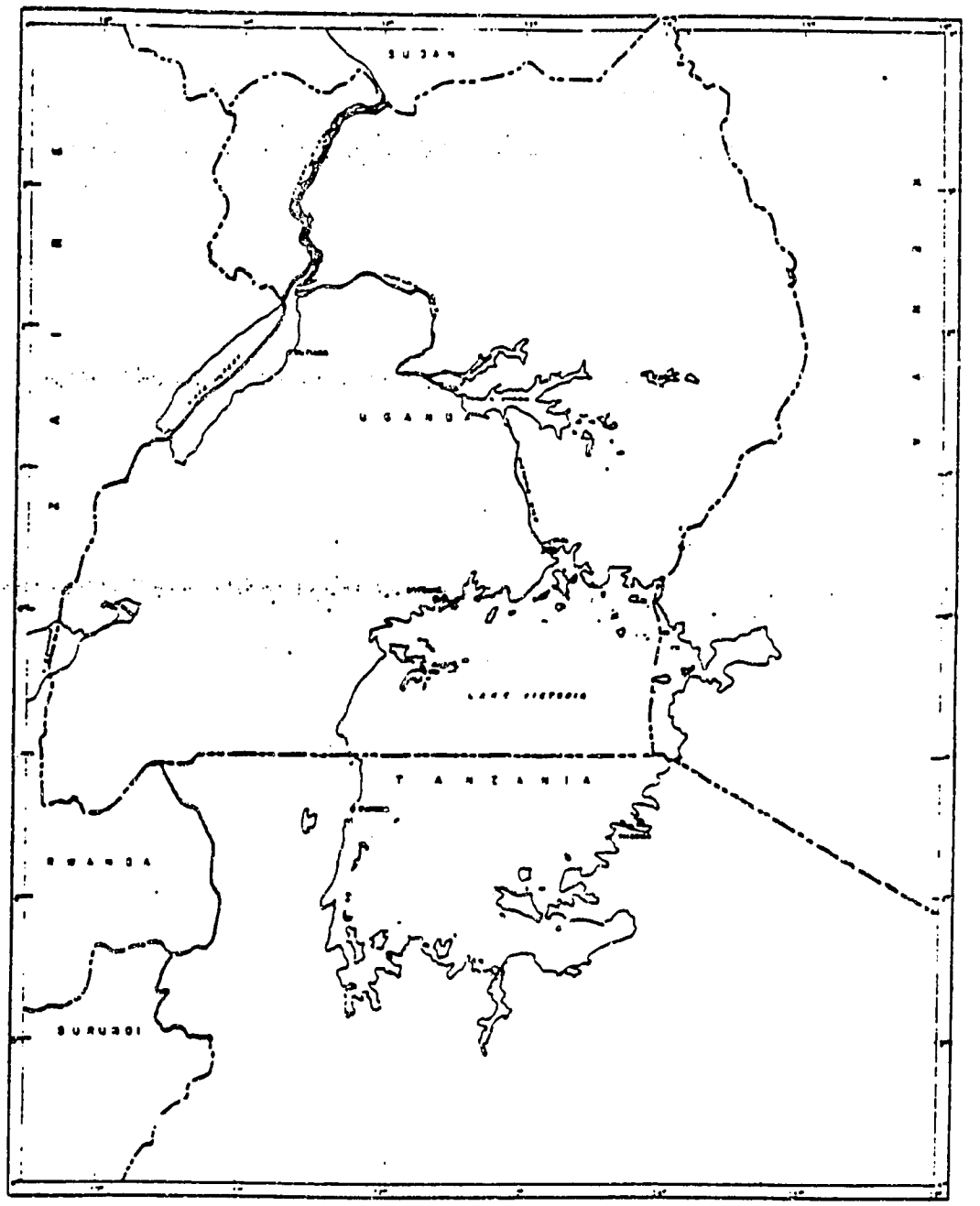

FIGURE 6.2.1: A MAP OF THE EQUATORIAL LAKES

yet been constructed), but they gain increasing importance as the Jonglei Canal Project (a channel to bypass the swampy region) is well under construction. The Jonglei Project will strengthen the hydrologic coupling of the Equatorial Lakes with the rest of the Nile system and will facilitate the coordinated operation of all the major Nile control projects (Ministry of Irrigation, Water Master Plan, TR 15).

Here the Equatorial Lake system will provide an example of a short ELQG case study. The objective is to test the new method's performance in handling systems of more than one reservoir. Potential comprehensive usage of the method in the particular system will also be discussed.

The hydrology of the Lakes' area is thoroughly discussed in the four volumes of the "Hydrometeorological Survey of the Catchments of Lakes Victoria, Kyoga, and Albert," World Meteorological Organization, 1974. Most of the data used here are taken from that study and from communication with Engineer A. Fahmy of the Egyptian Ministry of Irrigation (1982). Table 6.2.1 gives monthly average evaporation and rainfall depths over the three lakes, and Table 6.2.2 gives the monthly mean and standard deviations of the net basin contributions to each lake. The net basin contributions have been calculated from water balance considerations of historical data over the period 1948-1958 as follows: Let s_ℓ denote a lake's storage at the beginning of month ℓ, $A(s_\ell)$ the surface area corresponding to s_ℓ, e_ℓ the evaporation minus rainfall rate from Table 6.2.1, Q_ℓ the lake's outflow minus any upstream lake's inflow (for Kyoga and Albert), and w_ℓ the net basin contribution. Then, the following water balance equation can be written for month ℓ:

$$s_{\ell+1} = s_\ell - e_\ell A((s_\ell + s_{\ell+1})/2) - Q_\ell + N_\ell . \qquad (6.2.1)$$

TABLE 6.2.1: MONTHLY EVAPORATION AND RAINFALL RATES FOR LAKES VICTORIA, KYOGA, AND ALBERT

	L. VICTORIA		L. KYOGA		L. ALBERT	
MONTH	Evap. (mm)	Rain. (mm)	Evap. (mm)	Rain. (mm)	Evap. (mm)	Rain. (mm)
JAN.	96	105	110	28	160	17
FEB.	106	113	141	46	138	27
MAR.	135	197	146	107	149	60
APR.	133	287	161	178	118	108
MAY	136	226	171	164	112	71
JUN.	156	88	117	94	116	44
JUL.	143	60	91	85	107	58
AUG.	119	71	123	130	238	66
SEP.	106	83	113	120	67	66
OCT.	106	103	109	115	93	91
NOV.	128	180	130	96	93	77
DEC.	109	178	104	37	155	24
ANNUAL	1473	1691	1516	1220	1546	709

TABLE 6.2.2: MONTHLY MEANS AND ST. DEVIATIONS OF THE NET BASIN CONTRIBUTIONS (x $10^9 m^3$)

	L. VICTORIA		L. KYOGA		L. ALBERT	
MONTH	Mean	St. Dev.	Mean	St. Dev.	Mean	St. Dev.
JAN.	-0.209	1.763	-0.299	0.409	0.470	0.281
FEB.	0.394	3.005	-0.062	0.332	0.199	0.315
MAR.	-1.225	3.542	-0.234	0.263	0.233	0.188
APR.	0.885	3.952	-0.004	0.196	-0.056	0.738
MAY	2.228	4.592	-0.273	0.720	0.741	0.592
JUN.	1.151	2.688	0.258	0.916	0.388	0.261
JUL.	1.290	2.402	-0.041	0.325	0.450	0.354
AUG.	0.773	0.981	0.007	0.305	1.308	0.283
SEP.	0.394	2.907	0.277	0.546	0.478	0.349
OCT.	0.004	2.374	-0.255	0.485	0.434	0.248
NOV.	-2.151	4.955	-0.159	0.313	0.276	0.597
DEC.	0.132	6.508	-0.437	0.483	0.856	0.522
ANNUAL	3.666		-1.222		5.777	

(For better approximation $A((s_\ell + s_{\ell+1})/2)$ was used rather than $A(s_\ell)$ or $A(s_{\ell+1})$.) Table 6.2.3 gives adequate analytical approximations of available storage-elevation and storage-area data. These functions are used to transform historical lake water elevation measurements into storage and surface area sequences which finally yielded net basin contribution estimates via Eq. (6.2.1) and historical lake outflow data. The assumption was made that a lake's outflow was also inflow to the downstream lake with any channel losses or gains included in the downstream lake's net basin contribution. The procedure did not consider measurement errors. Notice that some of the net basin contributions were negative as a result of using the (evaporation-rainfall) rates reported in Table 6.2.1 throughout the period 1948-1958. Another consequence of this assumption is the high net basin contribution variance for Lakes Victoria and Albert.

Table 6.2.4 specifies maximum and minimum lake storage and outflow values based on assumptions of previous regulation studies (see Ibrahim et al., 1981).

In the following section a model for the Equatorial Lakes is presented and, subsequently, used in experimental ELQG computer runs.

6.3 Extended Linear Quadratic Gaussian Control

Based on the previous section's assumptions, the three lake system can be represented by

$$\left.\begin{aligned}
\frac{ds_V(t)}{dt} &= -e_V(t)\ A_V(s_V(t)) - u_V(t) + w_V(t) \\
\frac{ds_K(t)}{dt} &= -e_K(t)\ A_K(s_K(t)) + u_V(t) - u_K(t) + w_K(t) \\
\frac{ds_A(t)}{dt} &= -e_A(t)\ A_A(s_A(t)) + u_K(t) - u_A(t) + w_A(t)
\end{aligned}\right\} . \qquad (6.3.1)$$

TABLE 6.2.3: ELEVATION E(m), STORAGE s(x $10^9 m^3$), AND SURFACE AREA A(x $10^9 m^2$) RELATIONSHIPS

LAKE VICTORIA (Water Elevations at Jinja Gauge)

$$E = 932.74105 + 0.00739997 \cdot s + 22.426219 \cdot \ln s$$

$$s = -8.9288208 + 23.280746 \cdot E + 662.33163 \cdot \ln E - 32394526./E + 351.29970 \cdot \tan\left(\frac{\pi}{180} \cdot E\right)$$

$$A = 719.47935 + 0.06934627 \cdot s - 517065.27/s - -10.231907/(0.00739997 + 22.426219/s)$$

LAKE KYOGA (Water Elevations at Bugondo Gauge)

$$E = 1027.5663 + 0.15630936 \cdot s + 1.1132319 \cdot \ln s$$

$$s = 90092.272 + 19.404901 \cdot E - 15867.164 \cdot \ln E$$

$$A = 84.83482 + 1.4969286 \cdot s - 33.946833 \cdot \ln s - 171.76682/s$$

LAKE ALBERT (Water Elevations at Butiaba Gauge)

$$E = 67.459566 - 1.4489289 \cdot s + 0.00226650 \cdot s^2 + 143.3840714 \cdot \ln s$$

$$s = 15016.319 - 40.183467 \cdot E - 0.02335817 \cdot E^2 + 0.00006388 \cdot E^3 + 591.63559 \cdot \ln E$$

$$A = 45.69575 + 0.5773356 \cdot s - 0.0013191016 \cdot s^2 - 19.405298 \cdot \ln s$$

TABLE 6.2.4: STORAGE AND OUTFLOW BOUNDS

	L. VICTORIA	L. KYOGA	L. ALBERT
MAX. STORAGE (x 10^9 m^3)	3,100	30.0	175.0
MIN. STORAGE (x 10^9 m^3)	2,915	5.5	145.5
MAX. OUTFLOW (x 10^9 m^3/MONTH)	4.666	4.666	3.888
MIN. OUTFLOW (x 10^9 m^3/MONTH	1.037	1.037	1.296

Each variable or function is defined as in the previous section and is indexed after the corresponding lake. In vector notation system (6.3.1) can be expressed as follows:

$$\frac{d\underline{s}(t)}{dt} = \underline{F}(\underline{s}(t),t) + \underline{\underline{L}}\,\underline{u}(t) + \underline{w}(t) \tag{6.3.2}$$

where

$$\underline{s}(t) = \begin{bmatrix} s_V(t) \\ s_K(t) \\ s_A(t) \end{bmatrix}, \quad \underline{u}(t) = \begin{bmatrix} u_V(t) \\ u_K(t) \\ u_A(t) \end{bmatrix}, \quad w(t) = \begin{bmatrix} w_V(t) \\ w_K(t) \\ w_A(t) \end{bmatrix}$$

$$\underline{F}(\underline{s}(t),t) = \begin{bmatrix} -e_V(t)\ A_V(s_V(t)) \\ -e_K(t)A_K(s_K(t)) \\ -e_A(t)A_A(s_A(t)) \end{bmatrix}, \qquad \underline{L} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

System (3.6.2) can next be converted into the continuous time linearized perturbation version and then into the discrete time formulation by following the general procedure outlined in Chapter 4. (There is no observation equation - perfect state information is again assumed). The resulting model is of the following form:

$$\delta\underline{s}_{\ell+1} = \underline{\underline{\phi}}_\ell\, \delta\underline{s}_\ell + \underline{\underline{B}}_\ell\, \delta\underline{u}_\ell + \underline{\xi}_\ell \tag{6.3.3}$$

$$\ell = 0, \ldots, T-1$$

where

$$\delta\underline{s}_\ell = \delta\underline{s}(t_\ell) = \underline{s}(t_\ell) - \bar{s}(t_\ell) \quad ,$$

$$\delta\underline{u}_\ell = \delta\underline{u}(t_\ell) = \underline{u}(t_\ell) - \bar{\underline{u}}(t_\ell) \quad ,$$

$$\underline{\underline{\phi}}_\ell = \underline{\underline{\phi}}(t_{\ell+1}, t_\ell) \quad ,$$

$$\frac{d\,\underline{\underline{\phi}}(\tau,t_\ell)}{d\tau} = \underline{\underline{F}}_1(\bar{\underline{s}}(\tau),\tau)\underline{\underline{\phi}}(\tau,t_\ell) \quad , \qquad \underline{\underline{\phi}}(t_\ell,t_\ell) = \underline{\underline{I}}$$

$$\underline{\underline{F}}_1(\bar{s}(\tau),\tau) = \begin{bmatrix} -e_V(t)\dfrac{\partial A_V(\bar{s}_V(t))}{\partial s(t)} & 0 & 0 \\ 0 & -e_K(t)\dfrac{\partial A_K(\bar{s}_K(t))}{\partial s(t)} & \\ 0 & 0 & -e_A(t)\dfrac{\partial A_A(\bar{s}_A(t))}{\partial s(t)} \end{bmatrix}$$

$$\underline{\underline{B}}_\ell = \left[\int_{t_\ell}^{t_{\ell+1}} \phi(t_{\ell+1},\tau)d\tau\right]\underline{\underline{L}}$$

$$E\left\{\underline{\xi}_\ell\right\} = \underline{0} \ ,$$

$$E\left\{\underline{\xi}_\ell\,\underline{\xi}_\mu^T\right\} = \begin{cases} \underline{\underline{0}} & \text{if } \ell \neq \mu \\ \displaystyle\int_{t_\ell}^{t_{\ell+1}} \underline{\underline{\phi}}(t_{\ell+1},\tau)\,\underline{\underline{Q}}_w(\tau)\,\underline{\underline{\phi}}^T(t_{\ell+1},\tau)d\tau & \text{if } \ell = \mu \end{cases}$$

and $\bar{\underline{s}}^T(t) = [\bar{s}_V(t)\ \bar{s}_K(t)\ \bar{s}_A(t)]$ is obtained from the integration of

$$\frac{d\bar{\underline{s}}(t)}{dt} = \underline{F}(\bar{s}(t),t) + \underline{\underline{L}}\,\bar{\underline{u}}(t) + \bar{\underline{w}}(t) \tag{6.3.4}$$

where $\bar{\underline{w}}(t)$ is the mean net basin contributing vector. The elements of this vector as well as those of the matrix

$$\underline{\underline{Q}}_w(t) = \begin{bmatrix} Q_{w_V}(t) & 0 & 0 \\ 0 & Q_{w_K}(t) & 0 \\ 0 & 0 & Q_{w_A}(t) \end{bmatrix}$$

are obtained from the discrete time values reported in Table 6.2.2

according to the procedure discussed in Section 5.4.2 of the Single Reservoir Case Study. The nominal control trajectory $\bar{\underline{u}}(t)$, $t\varepsilon[t_0,t_T]$, is piecewise constant (monthly intervals):

$$\left\{ \bar{\underline{u}}(\tau) = \bar{\underline{u}}(t_\ell) \quad , \qquad \tau\varepsilon[t_\ell, t_{\ell+1}) \right\} \quad .$$

The differential equation for the state covariance $\underline{\underline{P}}_s(t)$ is as follows (compare with (5.4.27)):

$$\frac{d\underline{\underline{P}}s(t)}{dt} = \underline{\underline{F}}_1(\bar{\underline{s}}(t),t)\ \underline{\underline{P}}_s(t) + \underline{\underline{P}}_s(t)\ \underline{\underline{F}}_1^T(\bar{\underline{s}}(t),t) + \underline{\underline{Q}}_w(t) \qquad (6.3.5)$$

Since the initial condition is the zero matrix and $\underline{\underline{Q}}_w(t)$, $\underline{\underline{F}}_1(\bar{\underline{s}}(t),t)$ are diagonal, it follows that $\underline{\underline{P}}_s(t)$ is also diagonal and the system (6.3.5) can be easily solved by the separate integration of three differential equations.

The regulation of the Equatorial Lakes is a multiobjective problem. It involves energy generation (there exists a power plant at Jinja), navigation and flood control requirements (imposing constraints on the lakes' fluctuation and outflow), and meeting the downstream demand (water supply, flood protection, energy generation, etc.). However, detailed data are not presently available to this author. To continue with the case study, it will be assumed that the objective is to have the lakes track certain storage and outflow (release) trajectories subject to the constraints in Table 6.2.4.

As an example, consider a 12 month control problem with the lake storages at the following initial values:

$$s_{V0} = 3{,}020 \times 10^9\ m^3$$

$$s_{K0} = 10 \times 10^9\ m^3$$

$$s_{A0} = 150 \times 10^9 \ m^3$$

Suppose that the objective is to

$$\min_{\underline{u}_\ell} \left\{ J = E_{\underline{w}_\ell} \left\{ \sum_{\ell=0}^{12} \frac{1}{2} \left[(\underline{s}_\ell - \underline{s}_\ell^*)^T \underline{\underline{N}}_{ss\ell} (\underline{s}_\ell - \underline{s}_\ell^*) + (\underline{u}_\ell - \underline{u}_\ell^*)^T \underline{\underline{N}}_{uu\ell} (\underline{u}_\ell - \underline{u}_\ell^*) \right] \right. \right.$$

$$\ell = 0, \ldots, 12$$

$$\left. \left. + \frac{1}{2} (\underline{s}_{13} - \underline{s}_{13}^*)^T \underline{\underline{N}}_{ss13} (\underline{s}_{13} - \underline{s}_{13}^*) \right\} \right\} \qquad (6.3.6)$$

subject to the dynamical Equation (6.3.3) and the other system constraints (Table 6.2.4). The target storages and releases are given in Table 6.3.1 along with the diagonal matrices $\left\{\underline{\underline{N}}_{ss\ell}\right\}_{\ell=0}^{13}$, $\left\{\underline{\underline{N}}_{uu\ell}\right\}_{\ell=0}^{12}$. The elements of the coefficient matrices were taken equal to

$$\left.\begin{aligned} N_{ss\ell} &= \frac{1}{(s^{max} - s^{min})^2}, \quad \ell = 0, \ldots, 12 \\ N_{uu\ell} &= \frac{1}{(u^{max} - u^{min})^2}, \quad \ell = 0, \ldots, 12 \end{aligned}\right\} \qquad (6.3.7)$$

for all three lakes, while the coefficient of the terminal storage term was additionally multiplied by 10^9 (to make sure that the terminal storages will be close to the target values at the end of the control horizon). The relationships (6.3.7) were used to comparably penalize the various variables' deviations from their target values. The target releases were taken equal to the corresponding lake's average water gain over the period 1948-1958 (based on the previous section's modeling assumptions).

Tables 6.3.2, 6.3.3, and 6.3.4 present some results of the iterations required by the ELQG control algorithm to solve the above problem.

TABLE 6.3.1 : DATA FOR THE COMPUTATIONAL EXPERIMENT

	L. VICTORIA	L. KYOGA	L. ALBERT
s_ℓ^* (x 10^9 m^3) $\ell = 0, \ldots, 13$	3,000.	20.	160.
u_ℓ^* (x 10^9 m^3/month) $\ell = 0, \ldots, 12$	1.532	1.324	1.400
$N_{ss\ell}$ $\ell = 0, \ldots, 12$	0.292×10^{-4}	0.16659×10^{-2}	0.11491×10^{-2}
N_{ss13}	0.292×10^{5}	0.16659×10^{7}	0.11491×10^{7}
$N_{uu\ell}$ $\ell = 0, \ldots, 12$	0.759406×10^{-1}	0.759406×10^{-1}	0.1488435

"OBJ. FUNCTION 1" reports the performance index value (6.3.6) neglecting the covariance terms. "OBJ. FUNCTION 2" additionally includes the contributions from any state constraint violations. "WI", "M", and "ALPHA" are as in the single reservoir case. The number of the binding control constraints is given next. The tables report the trajectories of the mean storage values ("E{S}"), St. Deviations ("ST.DEV."), releases ("U"), first ("DUJ"), and second ("DUUJ") objective function control derivatives. Under "A", the control constraints are signified by "-1" if they are lower binding, "1" if they are upper binding, and "0" if they are not active. The procedure starts at nominal release trajectories which are equal to each lake's target releases. To test the method's convergence properties, the linearization of the dynamics is performed only once at the initial nominal sequences, and the resulting linear model is also used at the subsequent iterations. Convergence is practically achieved in two iterations as can be seen by the Gradient elements and the "WI" value on Table 6.3.4. In fact, the algorithm converges in one iteration after the binding control constraint set is identified. The terminal storages equal the desired targets. This experiment did not involve probabilistic state constraints. (The St. Deviations are computed according to the Open Loop procedure.) The iterations required approximately 8 seconds CPU time on a 68/DPS Honeywell computer. Computational experience with variations of the above problem (longer control horizons, different target values and feasible constraint ranges, etc.) suggests the following:

1. After identifying the binding constraint sets, the algorithm converges in very few iterations (2 ~ 3 for nonlinear dynamics, 1 for linear).

INITIAL TRAJECTORIES

OBJ. FUNCTION 1 = 0.14356832E+09
OBJ. FUNCTION 2 = 0.14356832E+09
UI = 0.10906595E+02
M = 0
ALPHA = 0.10000000E+01
BINDING CONSTR. = 0

	LAKE VICTORIA				LAKE KYOGA				LAKE ALBERT			
L	E(S)	ST.DEV.	U	A	E(S)	ST.DEV.	U	A	E(S)	ST.DEV.	U	A
1	3020.000	0.000	1.532	0	10.000	0.000	1.324	0	150.000	0.000	1.400	0
2	3018.834	2.772	1.532	0	9.536	0.499	1.324	0	149.622	0.231	1.400	0
3	3019.292	4.103	1.532	0	9.261	0.533	1.324	0	149.156	0.418	1.400	0
4	3029.813	5.434	1.532	0	9.072	0.597	1.324	0	148.243	0.475	1.400	0
5	3029.876	6.743	1.532	0	9.340	0.640	1.324	0	148.680	0.874	1.400	0
6	3036.099	8.135	1.532	0	9.259	0.970	1.324	0	149.198	1.048	1.400	0
7	3031.083	8.558	1.532	0	9.613	1.328	1.324	0	149.042	1.077	1.400	0
8	3026.391	8.872	1.532	0	9.758	1.369	1.324	0	149.161	1.130	1.400	0
9	3021.333	9.935	1.532	0	10.011	1.403	1.324	0	149.476	1.155	1.400	0
10	3018.675	9.389	1.532	0	10.509	1.503	1.324	0	149.871	1.206	1.400	0
11	3018.959	9.718	1.532	0	10.485	1.577	1.324	0	150.210	1.235	1.400	0
12	3016.947	10.943	1.532	0	10.375	1.619	1.324	0	150.322	1.369	1.400	0
13	3020.107	12.693			9.928	1.703			150.381	1.454		

L	DUJ	DUUJ	DUJ	DUUJ	DUJ	DUUJ
1	-.18126D+08	0.18400D+07	0.6925D+07	0.2879D+07	0.1061D+08	0.1059D+07
2	-.17858D+08	0.17935D+07	0.6585D+07	0.2838D+07	0.1068D+08	0.1073D+07
3	-.17669D+08	0.17555D+07	0.6341D+07	0.2811D+07	0.1074D+08	0.1085D+07
4	-.17644D+08	0.17504D+07	0.6239D+07	0.2812D+07	0.1077D+08	0.1089D+07
5	-.17654D+08	0.17527D+07	0.6295D+07	0.2817D+07	0.1078D+08	0.1094D+07
6	-.17810D+08	0.17437D+07	0.6286D+07	0.2815D+07	0.1082D+08	0.1100D+07
7	-.17569D+08	0.17355D+07	0.6131D+07	0.2814D+07	0.1085D+08	0.1108D+07
8	-.17572D+08	0.17361D+07	0.6067D+07	0.2328D+07	0.1092D+08	0.1121D+07
9	-.17594D+08	0.17404D+07	0.6041D+07	0.2342D+07	0.1097D+08	0.1131D+07
10	-.17614D+08	0.17444D+07	0.6060D+07	0.2846D+07	0.1097D+08	0.1131D+07
11	-.17587D+08	0.17350D+07	0.6082D+07	0.2038D+07	0.1097D+08	0.1132D+07
12	-.17442D+08	0.17100D+07	0.5839D+07	0.2922D+07	0.1102D+08	0.1141D+07

TABLE 6.3.2: THREE RESERVOIR ELQG CONTROL

I T E R A T I O N 0 1

OBJ. FUNCTION 1 = 0.30496794E+05
OBJ. FUNCTION 2 = 0.30496794E+05
VI = 0.67054079E+00
R = 0
ALPHA = 0.10000000E+01
BINDING CONSTR. = 3

	LAKE VICTORIA				LAKE KYOGA				LAKE ALBERT			
L	E(S)	ST.DEV.	U	A	E(S)	ST.DEV.	U	A	E(S)	ST.DEV.	U	A
1	3020.000	0.000	3.233	0	10.000	0.000	2.087	0	150.000	0.000	1.296	-1
2	3016.523	2.772	3.698	0	11.155	0.409	2.137	0	150.438	0.281	1.296	-1
3	3013.755	4.103	3.527	0	12.259	0.533	2.169	0	150.929	0.418	1.296	-1
4	3013.487	5.434	3.371	0	13.242	0.597	2.193	0	151.554	0.475	1.348	0
5	3021.297	6.743	3.237	0	14.469	0.640	2.292	0	152.310	0.874	1.405	0
6	3026.008	8.135	3.125	0	15.222	0.970	2.211	0	153.591	1.348	1.453	0
7	3019.408	8.558	3.041	0	16.306	1.323	2.207	0	154.350	1.077	1.495	0
8	3012.118	8.872	2.974	0	17.033	1.369	2.201	0	155.241	1.138	1.531	0
9	3006.724	8.935	2.925	0	17.893	1.493	2.194	0	156.246	1.155	1.559	0
10	3002.677	9.389	2.899	0	18.903	1.503	2.183	0	157.351	1.206	1.577	0
11	2999.634	9.718	2.888	0	19.330	1.577	2.172	0	158.571	1.235	1.583	0
12	2998.215	10.943	2.892	0	19.833	1.619	2.156	0	159.134	1.369	1.594	0
13	3000.000	12.693			20.000	1.703			159.770	1.454		

L	DUJ	DUUJ	DUJ	DUUJ	DUJ	DUUJ
1	0.51508D+00	0.18490D+07	-.2542D+06	0.2879D+07	0.2542D+06	0.1059D+07
2	0.51815D+00	0.17935D+07	-.2559D+06	0.2838D+07	0.2559D+06	0.1073D+07
3	0.50668D+00	0.17555D+07	-.2572D+06	0.2811D+07	0.2572D+06	0.1085D+07
4	0.50606D+00	0.17504D+07	-.2579D+06	0.2812D+07	0.2579D+06	0.1090D+07
5	0.50590D+00	0.17527D+07	-.2583D+06	0.2817D+07	0.2583D+06	0.1094D+07
6	0.50608D+00	0.17437D+07	-.2581D+06	0.2815D+07	0.2581D+06	0.1100D+07
7	0.50463D+00	0.17355D+07	-.2599D+06	0.2814D+07	0.2599D+06	0.1108D+07
8	0.50470D+00	0.17361D+07	-.2615D+06	0.2829D+07	0.2615D+06	0.1121D+07
9	0.50517D+00	0.17404D+07	-.2626D+06	0.2842D+07	0.2626D+06	0.1131D+07
10	0.50565D+00	0.17444D+07	-.2629D+06	0.2846D+07	0.2629D+06	0.1131D+07
11	0.50464D+00	0.17350D+07	-.2628D+06	0.2838D+07	0.2628D+06	0.1132D+07
12	0.50224D+00	0.17100D+07	-.2638D+06	0.2822D+07	0.2638D+06	0.1141D+07

TABLE 6.3.3: THREE RESERVOIR ELQG CONTROL

I T E R A T I O N # 2

OBJ. FUNCTION 1 = 0.23107253E+01
OBJ. FUNCTION 2 = 0.23107253E+01
VI = 0.24073563E-06
M = 0
ALPHA = 0.10000000E+01
BINDING CONSTR. = 3

	LAKE VICTORIA				LAKE KYOGA				LAKE ALBERT			
L	E(S)	ST.DEV.	U	A	E(S)	ST.DEV.	U	A	E(S)	ST.DEV.	U	A
1	3020.000	0.000	3.896	0	10.000	0.000	2.896	0	150.000	0.000	1.296	-1
2	3016.520	2.772	3.790	0	11.150	0.409	2.144	0	150.494	0.231	1.296	-1
3	3013.760	4.103	3.622	0	12.248	0.533	2.174	0	150.945	0.418	1.296	-1
4	3013.481	5.434	3.372	0	13.226	0.597	2.198	0	151.576	0.479	1.318	0
5	3021.290	6.743	3.233	0	14.451	0.640	2.202	0	152.364	0.874	1.377	0
6	3025.999	8.136	3.125	0	15.294	0.970	2.210	0	153.624	1.048	1.426	0
7	3019.399	8.558	3.041	0	16.289	1.328	2.205	0	154.452	1.077	1.469	0
8	3012.109	8.872	2.973	0	17.062	1.369	2.198	0	155.372	1.130	1.505	0
9	3006.712	8.935	2.924	0	17.881	1.403	2.190	0	156.397	1.155	1.534	0
10	3002.672	9.389	2.898	0	18.893	1.503	2.178	0	157.524	1.296	1.553	0
11	2999.638	9.718	2.886	0	19.374	1.577	2.167	0	158.564	1.235	1.564	0
12	2998.313	10.943	2.890	0	19.930	1.619	2.151	0	159.346	1.369	1.570	0
13	3000.000	12.693			20.000	1.783			160.000	1.454		

L	DUJ	DUUJ	DUJ	DUUJ	DUJ	DUUJ
1	0.89872D-01	0.18490D+07	0.6730D-01	0.2379D+07	0.5154D-01	0.1059D+07
2	0.10136D+00	0.17935D+07	0.6573D-01	0.2938D+07	0.4104D-01	0.1073D+07
3	0.10226D+00	0.17565D+07	0.6462D-01	0.2811D+07	0.3091D-01	0.1085D+07
4	0.10214D+00	0.17504D+07	0.6443D-01	0.2812D+07	0.2467D-01	0.1090D+07
5	0.10132D+00	0.17527D+07	0.6446D-01	0.2317D+07	0.2471D-01	0.1094D+07
6	0.10204D+00	0.17437D+07	0.6414D-01	0.2315D+07	0.2478D-01	0.1100D+07
7	0.10242D+00	0.17355D+07	0.6385D-01	0.2314D+07	0.2486D-01	0.1108D+07
8	0.10255D+00	0.17381D+07	0.6371D-01	0.2928D+07	0.2502D-01	0.1121D+07
9	0.10251D+00	0.17404D+07	0.6372D-01	0.2842D+07	0.2512D-01	0.1131D+07
10	0.10244D+00	0.17444D+07	0.6382D-01	0.2846D+07	0.2513D-01	0.1131D+07
11	0.10282D+00	0.17350D+07	0.8356D-01	0.2338D+07	0.2514D-01	0.1132D+07
12	0.10314D+00	0.17100D+07	0.8281D-01	0.2822D+07	0.2524D-01	0.1141D+07

TABLE 6.3.4: THREE RESERVOIR ELQG CONTROL

2. The number of iterations until the binding constraint set is found depends on how restrictive these constraints are. Namely, many loosely active constraints may slow down the convergence rate.

3. Generally, the Armijo test is passed at low m values.

4. Before increasing the penalty terms for any state constraint violations, convergence or near convergence of the control constrained algorithm is first recommended. (This way ill conditioning problems are avoided.)

5. Computer time and storage requirements are impressively low.

Under a more extensive data base, ELQG can be used in a comprehensive investigation of the Equatorial Lake Project. Different regulation alternatives can be evaluated and assist in international agreements. For example, an objective of the previous regulation studies has been to operate the lakes so that Lake Albert's outflow will always be equal to its historical mean. This all-the-year-round steady outflow regulation aims at providing downstream users with safe water supply. The question is how much lake storage is necessary to achieve this goal and what are the costs involved. Regarding the associated costs, they refer not only to the explicit costs of the regulation structures but also to implicit costs related to the navigability of the existing lake ports, possible reductions in energy generation, and the effects on other activities (fisheries, etc). Not suffering from dimensionality problems, ELQG can also investigate the benefits of coordinating the Equatorial Lake system with the High Aswan Dam. Towards this end, the lakes can serve two purposes: First, as an overyear assignment they provide a dependable reserve water supply in periods of severe Blue Nile droughts,

and secondly, on a seasonal basis, they can help diminish the HAD's level fluctuations caused by Blue Nile floods (August, September, October) and the downstream water demand peak (May, June, July, August). Based on our experience with the first case study, the second of the above tasks will help increase the HAD power production while the first will add to the reliability of meeting the water supply requirements. This second regulation strategy, however, may require substantial lake storage volumes which may be in conflict with the interests of shoreline residents. However, by establishing the true tradeoffs among the various objectives, fair grounds can be established for the lake regulation agreements.

Chapter 7

CONCLUSIONS AND FURTHER RESEARCH RECOMMENDATIONS

7.1 Summary of Results

This work has used stochastic control theory background to design a method for the operational management of reservoir systems.

The actual reservoir and river segment response was modelled by a set of coupled nonlinear differential equations in continuous time. Potential parameter identifiability problems were discussed, and an identification procedure for separately treating reservoirs and river segments was suggested. The system model was linearized around nominal control and state trajectories and subsequently converted into a discrete time linear perturbation model where the discretization intervals can be arbitrary.

The various system objectives and operational characteristics were seen to induce control and probabilistic state constraints. The goal assigned to the control procedure was to generate points on the properly defined tradeoff surface. When treated according to the Open Loop Feedback control philosophy, the resulting stochastic control problem was seen to be equivalent to a deterministic problem in the space of the state's probability density. Due to the linearizations employed, this function was locally approximated by a Gaussian density. The algorithm designed to solve this problem was named Extended Linear Quadratic Gaussian (ELQG) controller, and it is of the trajectory iteration type. It successively treats the unconstrained problem, the problem with control constraints, and the problem with both control and state constraints. For these operations it respectively uses a Newton, a Projected Newton,

and a Penalty Function method. ELQG iterates along directions obtained via analytical considerations. It identifies locally optimal trajectories, and it is expected to display reliability and a fast-convergence rate. By its analytical structure, it does not suffer from dimensionality limitations.

The method was tested in case studies of two actual reservoir systems. The first case study concerned the control of the High Aswan Dam in Egypt. The problem was to maximize expected energy generation subject to release and probabilistic storage constraints resulting from other operational requirements (water supply and flood control objectives). ELQG was employed in several computational experiments and performed impressively even for very long control horizons. Control constraints were accounted for within very few iterations (5 or 6), while the handling of the probabilistic state constraints was satisfactory overall, although less efficient. Compared with state-of-the-art procedures, the method exhibited superior performance. The second case study concerned the control of the Equatorial three lake system located in northeast Africa. It was shown that the previously identified ELQG properties also characterize the method's performance in multireservoir system problems.

7.2 Further Research Recommendations

Further research work can proceed along several directions:

1. Although local linearization of the system dynamics was seen in both case studies to be overall adequate, the possibility of second order approximations can be investigated for strongly nonlinear systems. In such a procedure the method's iterative steps will utilize the Newton's direction for the nonlinear control problem (and not of its linearized

version as was the case here) which will result in faster convergence rate.

2. Theoretically, multiplier methods are characterized by better convergence properties compared to the Penalty Function approach. If the initial multiplier guess problem is alleviated, these methods will add to the overall ELQG efficiency.

3. Computational experience with large systems should be acquired, and the system order up to which the new method can be advantageously employed should be determined. For higher order systems, research efforts can turn to ELQG decomposition schemes. The problem consists of two separate parts: (1) decomposition of the probabilistic state dynamics and (2) decomposition of the control gain equations. Toward this end, the serial reservoir system nature may be usefully exploited.

4. Regarding practical applications, the method can be employed in studies of various actual systems. Cases with imperfect state information as well as ELQG models with on line system identification procedures can be implemented. Care should be taken to properly account for the particular problem's characteristics. The method cannot substitute for intuitive system understanding.

5. Hierarchical control structures can be designed to consist of ELQG models differing in the discretization intervals, the performance index, and possibly the input process description.

As a concluding remark, ELQG can be rewarding to both theoretical research and to actual reservoir systems operation applications.

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Appendix A

MATHEMATICAL REVIEW

Some mathematical definitions and results are collected here to supplement the presentation of the main text.

Consider the scalar function $f(\underline{x})$ of the n dimensional vector $\underline{x}$. The Gradient vector of $f(\underline{x})$ with respect to $\underline{x}$ is denoted as $\nabla_x f(\underline{x})$ and is given by

$$\nabla_x f(\underline{x}) = \begin{bmatrix} \frac{\partial f(\underline{x})}{\partial x_1} \\ \vdots \\ \frac{\partial f(\underline{x})}{\partial x_n} \end{bmatrix} . \tag{A.1}$$

The matrix of the second $f(\underline{x})$ derivatives with respect to the vector $\underline{x}$ is called the Hessian of $f(\underline{x})$ and is defined as follows:

$$\nabla_{xx} f(\underline{x}) = \begin{bmatrix} \frac{\partial f(\underline{x})}{\partial x_1 \partial x_1} & \cdots & \frac{\partial f(\underline{x})}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{f(\underline{x})}{\partial x_n \partial x_1} & \cdots & \frac{f(\underline{x})}{\partial x_n \partial x_n} \end{bmatrix} . \tag{A.2}$$

The function $f(\underline{x})$ can be approximated around some vector $\underline{x}_0$ by the following 2nd order Taylor series expression:

$$f(\underline{x}) \simeq f(\underline{x}_0) + \nabla_x^T f(\underline{x}_0)(\underline{x}-\underline{x}_0) + \frac{1}{2}(\underline{x}-\underline{x}_0)^T \nabla_{xx} f(\underline{x}_0)(\underline{x}-\underline{x}_0) \tag{A.3}$$

("T" denotes transpose.)

Higher order Taylor series expressions of $f(\underline{x})$ can be obtained from

$$f(\underline{x}) \approx \sum_{m=0}^{m-1} \frac{1}{m!} \left[(x_1 - x_{01}) \cdot \frac{\partial}{\partial x_1} + \ldots + (x_n - x_{n0}) \cdot \frac{\partial}{\partial x_n}\right]^m \cdot$$

$$\cdot\ f(\underline{x}_0) \tag{A.4}$$

where the expression in the brackets is an operator applied to $f(\underline{x})$, and the resulting expression is evaluated at $\underline{x}_0$.

Consider an m dimensional real function $\underline{f}(\underline{x})$: $\mathbb{R}^n \to \mathbb{R}^m$ ($\mathbb{R}^j$ denotes the space of j dimensional real vectors). The Gradient matrix of $\underline{f}(\cdot)$ with respect to $\underline{x}$ is defined by

$$\nabla_x(\underline{f})(\underline{x}) = \begin{bmatrix} \frac{\partial f_1(\underline{x})}{\partial x_1} & \cdots & \frac{\partial f_m(\underline{x})}{\partial x_1} \\ \vdots & & \vdots \\ \frac{\partial f_1(\underline{x})}{\partial x_n} & \cdots & \frac{\partial f_m(\underline{x})}{\partial x_n} \end{bmatrix} . \tag{A.5}$$

If $\underline{f}(\underline{x}) = \underline{h}(\underline{g}(x))$ (A.6)

where $\underline{h}$: $\mathbb{R}^r \to \mathbb{R}^m$ and $\underline{g}$: $\mathbb{R}^n \to \mathbb{R}^r$, there holds

$$\nabla_x \underline{f}(\underline{x}) = \nabla_x\ \underline{g}(\underline{x})\ \nabla_{\underline{g}} \underline{h}\ (\underline{g}(x)) \tag{A.7}$$

(chain rule of differentiation).

Using this result we obtain the following expressions:

$$\nabla_x(\underline{a}^T \underline{x}) = \nabla_x(\underline{x}^T\ \underline{a}) = \underline{a} \tag{A.8}$$

$$\nabla_x(\underline{x}^T\ \underline{\underline{A}}\ \underline{x}) = \underline{\underline{A}}\ \underline{x} + \underline{\underline{A}}^T\ \underline{x} \tag{A.9}$$

where $\underline{a}$ is an n-vector and $\underline{\underline{A}}$ is an n x n matrix.

APPENDIX B

ANALYTICAL APPROXIMATION OF THE REFORMULATED COST TERMS

We wish to obtain analytical approximations of the cost terms $\hat{\ell}_\ell(\cdot,\cdot,\cdot)$ or $\tilde{\ell}_\ell(\cdot,\cdot)$ defined by

$$\hat{\ell}_\ell(\delta\bar{\underline{s}}_{\ell/k}, \underline{\underline{P}}_{s\ell/k}, \delta\underline{u}_\ell) = \tilde{\ell}_\ell(\delta\bar{\underline{s}}_{\ell/k}, \delta\underline{u}_\ell) = \underset{\delta\underline{s}_\ell}{E} \{\ell_\ell(\delta\underline{s}_\ell, \delta\underline{u}_\ell)\} \qquad (B.1)$$

where $(\cdot,\cdot)$ is a given real valued function, δs_ℓ is a Gaussian n_s dimensional vector, and $\delta\underline{u}_\ell$ is the n_u dimensional control vector.

To illustrate the approach we shall begin with the case of scalar state and control. In fact, this analysis will suffice when $\ell_\ell(\cdot,\cdot)$ is separable into terms depending only on one state and control variable (compare with performance index (4.2.14)). The idea is to represent $\ell_\ell(\delta s_\ell, \delta u_\ell)$ by a finite number of Taylor series terms and analytically evaluate the expectation in (B.1):

$$\ell_\ell(\delta s_\ell, \delta u_\ell) \simeq \sum_{m=0}^{M-1} \frac{1}{m!}\left[\delta s_\ell \frac{\partial}{\partial \delta s_\ell} + \delta u_\ell \frac{\partial}{\partial \delta u_\ell}\right]^m \left[\ell_\ell(\delta s_\ell, \delta u_\ell)\right]_{\substack{\delta s_\ell=0 \\ \delta u_\ell=0}} \qquad (B.2)$$

where the expansion is around the current nominal state and control trajectories. In the notation of Equation (B.2) the quantity in brackets is an operator applied on $\ell_\ell(\cdot,\cdot)$. Expanding the powers in (B.2), we find

$$\ell_\ell(\delta s_\ell,\delta u_\ell) \simeq \sum_{m=0}^{M-1} \frac{1}{m!} \left\{ \sum_{\substack{m_1,m_2 \\ m_1+m_2=m}} \frac{m!}{m_1!m_2!} (\delta s_\ell)^{m_1} (\delta u_\ell)^{m_2} \cdot \right.$$

$$\left. \frac{\partial^m [\ell_\ell(0,0)]}{(\partial\delta s_\ell)^{m_1} (\partial\delta u_\ell)^{m_2}} \right\} \tag{B.3}$$

where m_1, m_2 are all non-negative integers for which there holds $m_1+m_2 = m$. Taking expectations in (B.3) we find

$$E\{\ell_\ell(\delta s_\ell,\delta u_\ell)\} \simeq \sum_{m=0}^{M-1} \frac{1}{m!} \left\{ \sum_{\substack{m_1,m_2 \\ m_1+m_2=m}} \frac{m!}{m_1!m_2!} \frac{\partial^m [\ell_\ell(0,0)]}{(\partial\delta s_\ell)^{m_1} (\partial\delta u_\ell)^{m_2}} \right.$$

$$\left. E\{(\delta s_\ell)^{m_1}\} (\delta u_\ell)^{m_2} \right\} . \tag{B.4}$$

Equation (B.4) can provide an analytical approximation of $\hat{\ell}_\ell(\delta\bar{s}_{\ell/k}, P_{s\ell/k}, \delta u_\ell)$ if $E\{(\delta s_\ell)^{m_1}\}$ can be expressed in terms of the mean value $\delta\bar{s}_{\ell/k}$ and the variance $P_{s\ell/k}$. Toward this end we shall make use of the "Gaussian moment factoring", a simplifying property characterizing Gaussian random variables. For a Gaussian variable x with mean value $\bar{x}$ and variance σ, there holds (Jazwinski, 1970, Section 2.2)

$$E\{(x-\bar{x})^m\} = \left\{ \begin{array}{l} 0, \text{ all odd } m \geq 1 \\ \\ 1.3.5\ldots(m-1)\sigma^m, \text{ all even } m \geq 2. \end{array} \right\} \tag{B.5}$$

Result (B.5) can be easily derived from the properties of the associated "characteristic function". Using (B.5) one can also evaluate any statistical moment $E\{x^m\}$ by induction:

$$m=2: \quad E\{(x-\bar{x})^2\} = E\{x^2\}-2E\{x\}\bar{x} + \bar{x}^2 = \sigma^2, \text{ or}$$

$$E\{x^2\} = \bar{x}^2 + \sigma^2.$$

$$m=3: \quad E\{(x-\bar{x})^3\} = E\{x^3\}-3E\{x^2\}\bar{x} + 3E\{x\}\bar{x}^2 - \bar{x}^3 = 0, \text{ or}$$

$$E\{x^3\} = \bar{x}^3 + 3\bar{x}\sigma^2,$$

$$m=4: \quad E\{x^4\} = \bar{x}^4 + t\bar{x}^2 + 3\ ^4,$$

etc.

These expressions can also be obtained from the characteristic function

$$q_x(z) = e^{iz\bar{x}-\frac{1}{2}z^2\sigma^2}, \quad i^2 = -1, \tag{B.6}$$

and the fact that

$$E\{x^m\} = \frac{1}{i^m}\left[\frac{d^m q_x(z)}{dz^m}\right]_{z=0}. \tag{B.7}$$

Hence, $E\{(\delta s_p)^{m_1}\}$ can be expressed in terms of $\delta\bar{s}_{\ell/k}$, $P_{s\ell/k}$ and Equation (B.9) can be used to approximate $\hat{\ell}_\ell(\cdot,\cdot,\cdot) = \tilde{\ell}_\ell(\cdot,\cdot)$ at any desirable accuracy.

Regarding the multidimensional case where $\delta\underline{s}_\ell$, $\delta\underline{u}_\ell$ are n_s, n_u dimensional vectors respectively the previous equations become

$$\ell_\ell(\delta\underline{s}\ ,\delta\underline{u}_\ell) \simeq \sum_{m=0}^{M-1} \frac{1}{m!} \left[\delta s_{1\ell} \frac{\partial}{\partial\delta s_{1\ell}} + \ldots + \delta s_{n_s\ell} \frac{\partial}{\partial\delta s_{n_s\ell}} + \frac{\partial}{\partial\delta u_{1\ell}} + \ldots \delta u_{n_u\ell} \frac{\partial}{\partial\delta u_{n_u\ell}}\right]$$

$$\cdot \left[\ell_\ell(\delta\underline{s}_\ell,\delta\underline{u}_\ell)\right]_{\substack{\delta\underline{s}_\ell=\underline{0} \\ \delta\underline{u}_\ell=0}} , \qquad \text{(B.8)}$$

$$E\{\ell_\ell(\delta\underline{s}_\ell,\delta\underline{u}_\ell)\} \simeq \sum_{m=0}^{M-1} \frac{1}{m!}$$

$$\cdot \left\{ \sum_{\substack{m_{s1},\ldots,m_{sn_s},m_{u1},\ldots,m_{un_u} \\ m_{s1}+\ldots+m_{sn_s}+m_{u1}+\ldots+m_{un_u}=m}} \frac{m!}{m_{s1}!\ldots m_{sn_s}!m_{u1}!\ldots m_{un_u}!} \cdot \right.$$

$$\cdot \frac{\partial^m[\ell_\ell(\underline{0},\underline{0})]}{(\partial\delta s_{1\ell})^{m_{s1}}\ldots(\partial\delta s_{n_s\ell})^{m_{sn_s}}(\partial\delta u_{1\ell})^{m_{u1}}\ldots(\partial\delta u_{n_u\ell})^{m_{un_u}}} \cdot$$

$$\left. \cdot\ E\ \{\delta s_{1\ell})^{m_{s1}}\ldots(\delta s_{n_s\ell})^{m_{sn_s}}\}\ (\delta u_{1\ell})^{m_{u1}}\ldots(\delta u_{n_u\ell})^{m_{un_u}} \quad . \right\} \qquad \text{(B.9)}$$

Again, $E\{(\delta s_{1\ell})^{m_{s1}}\ldots(\delta s_{n_s\ell})^{m_{sn_s}}\}$ can be obtained in terms of the elements of the mean vector $\delta\bar{\underline{s}}_{\ell/k}$ and the covariance matrix $\underline{\underline{P}}_{s\ell/k}$ by the "Gaussian moment factoring" result, which in this case takes the following form (Sandell and Shapiro, 1976): Let $\{x_1,x_2,\ldots x_{n_s}\}$ be a set of jointly Gaussian random variables with mean values

$\{\bar{x}_i\}_{i=1}^{n_s}$ and covariances σ_{ij}, $1 < i,j \leq n_s$. Let $\{k_1, k_2, \ldots, k_L\}$ be a set of integers selected from $\{1,2,\ldots,n_s\}$ with repetition allowed. It holds that

$$E\{(x_{k_1}-\bar{x}_{k_1})\ldots(x_{k_L}-\bar{x}_{k_L}) = \begin{cases} 0, & L \text{ odd} \\ \sum \sigma_{\ell_1\ell_2}\sigma_{\ell_3\ell_4}\cdots\sigma_{\ell_{L-1}\ell_L}, & L \text{ even} \end{cases} \quad \text{(B.10)}$$

where the summation in (B.10) is over all distinct pairs of subscripts $\{\ell_1,\ell_2,\ldots \ell_L\}$ which are permutations of $\{k_1,k_2,\ldots,k_L\}$. Although more complicated, this result can be used inductively to provide the desirable expressions.

Alternatively, the same results can be obtained through the joint characteristic function which, in the case of the $\{x_1,\ldots,x_{n_s}\}$ jointly Gaussian variables becomes

$$q_x(\underline{z}) = \exp\{i\underline{z}^T\bar{\underline{x}} - \frac{1}{2}\underline{z}\,\underline{\underline{P}}_x\,\underline{z}\} \quad \text{(B.11)}$$

where $\underline{z}$ is on n_s dimensional vector, $\bar{\underline{x}}^T = [\bar{x}_1\ \bar{x}_2\ldots\ \bar{x}_{n_s}]$ and $\underline{\underline{P}}_x$ is the covariance matrix of $\underline{x}^T = [x_1\ldots x_{n_s}]$. Then,

$$E\{x_1\ x_2\ \ldots\ x_{n_s}\} = \frac{1}{i^{n_s}}\,[\nabla_z\ q_x(\underline{z})]_{\underline{z}=\underline{0}} \quad \text{(B.12)}$$

where $\nabla_z q_x(\underline{z})$ is the gradient of $q_x(\underline{z})$. Moments such as $E\{x_1^{m_1}\ x_2^{m_2}\ldots x_{n_s}^{m_{ns}}\}$ can also be obtained by repeating m_i of the variables x_i, $i=1, \ldots, n_s$, in the jointly Gaussian set.

Although the accuracy of the analytical approximation suggested here can be improved by retaining more Taylor expansion terms, in practice the first few terms will generally suffice.

APPENDIX C

EVALUATION OF GRADIENT AND HESSIAN

Consider the following two optimal control problems reproduced here from Section 4.5:

$$\left.\begin{aligned}
&\underset{\{\delta \underline{u}_\ell\}_{\ell=k}^{T-1}}{\text{Minimize}} \left\{ \tilde{J} = \sum_{\ell=k}^{T-1} \tilde{\ell}_\ell(\delta \bar{\underline{s}}_{\ell/k}, \delta \underline{u}_\ell) + \tilde{\ell}_T(\delta \bar{\underline{s}}_{T/k}) \right\} \\
&\text{subject to} \\
&\delta \bar{\underline{s}}_{\ell+1/k} = \underline{\underline{\phi}}_\ell \, \delta \bar{\underline{s}}_{\ell/k} + \underline{\underline{B}}_\ell \, \delta \underline{u} \,, \quad \bar{\underline{s}}_{k/k} = \underline{0}, \\
&\ell = k, \ldots, T-1
\end{aligned}\right\} \qquad (C.1)$$

and

$$\left.\begin{aligned}
&\underset{\{\delta \underline{u}_\ell\}_{\ell=k}^{T-1}}{\text{minimize}} \quad \bar{J} = \left\{ \sum_{\ell=k}^{T-1} [\underline{N}_{s\ell}^T \delta \bar{\underline{s}}_{\ell/k} + \underline{N}_{u\ell}^T \delta \underline{u}_\ell + \tfrac{1}{2} \delta \bar{\underline{s}}_{\ell/k}^T \underline{\underline{N}}_{ss\ell} \delta \bar{\underline{s}}_{\ell/k} + \right. \\
&\qquad + \tfrac{1}{2} \delta \underline{u}_\ell^T \underline{\underline{N}}_{uu\ell} \delta \underline{u}_\ell + \delta \underline{u}_\ell^T \underline{\underline{N}}_{us\ell} \delta \bar{\underline{s}}_{\ell/k}] + \\
&\qquad \left. + \underline{N}_{sT}^T \delta \bar{\underline{s}}_{T/k} + \tfrac{1}{2} \delta \bar{\underline{s}}_{T/k}^T \underline{\underline{N}}_{ssT} \delta \bar{\underline{s}}_{T/k} \right\} \\
&\text{subject to} \\
&\delta \bar{\underline{s}}_{\ell+1/k} = \underline{\underline{\phi}}_\ell \, \delta \bar{\underline{s}}_{\ell/k} + \underline{\underline{B}}_\ell \delta \underline{u}_\ell, \quad \delta \bar{\underline{s}}_{k/k} = \underline{0}, \\
&\ell = k, \ldots, T-1 \\
&\text{where} \quad \underline{N}_{s\ell} = \nabla_{\delta \bar{\underline{s}}_{\ell/k}} \tilde{\ell}_\ell(\underline{0}, \underline{0}), \\
&\qquad \underline{N}_{u\ell} = \nabla_{\delta \underline{u}_\ell} \tilde{\ell}_\ell(\underline{0}, \underline{0}), \\
&\qquad \underline{\underline{N}}_{ss\ell} = \nabla^2_{\delta \bar{\underline{s}}_{\ell/k} \delta \bar{\underline{s}}_{\ell/k}} \tilde{\ell}_\ell(\underline{0}, \underline{0}), \text{ (symmetric)} \\
&\qquad \underline{\underline{N}}_{uu\ell} = \nabla^2_{\delta \underline{u}_\ell \delta \underline{u}_\ell} \tilde{\ell}_\ell(\underline{0}, \underline{0}), \text{ (symmetric)} \\
&\qquad \underline{\underline{N}}_{us\ell} = [\nabla^2_{\delta \underline{u}_\ell \delta \bar{\underline{s}}_{\ell/k}} \tilde{\ell}_\ell(0,0)]^T.
\end{aligned}\right\} \qquad (C.2)$$

We wish to show

$$\nabla_u \tilde{J}(\delta \underline{u}_k = \underline{0}, \ldots, \delta \underline{u}_{T-1} = 0) = \nabla_u \bar{J}(\delta \underline{u}_k = \underline{0}, \ldots, \delta \underline{u}_{T-1} = \underline{0}), \tag{C.3}$$

$$\nabla^2_{uu} \tilde{J}(\delta \underline{u}_k = \underline{0}, \ldots, \delta \underline{u}_{T-1} = 0) = \nabla^2_{uu} \bar{J}(\delta \underline{u}_k = \underline{0}, \ldots, \delta \underline{u}_{T-1} = \underline{0}) \tag{C.4}$$

where

$$\nabla_u \tilde{J} = \begin{bmatrix} \nabla_{\delta \underline{u}_k} \tilde{J} \\ \vdots \\ \nabla_{\delta \underline{u}_{T-1}} \tilde{J} \end{bmatrix}, \quad \nabla^2_{uu} \tilde{J} = \begin{bmatrix} \nabla^2_{\delta \underline{u}_k \delta \underline{u}_k} \tilde{J} & \ldots & \nabla^2_{\delta \underline{u}_k \delta \underline{u}_{T-1}} \tilde{J} \\ \vdots & & \vdots \\ \nabla^2_{\delta \underline{u}_{T-1} \delta \underline{u}_k} \tilde{J} & \ldots & \nabla^2_{\delta \underline{u}_{T-1} \delta \underline{u}_{T-1}} \tilde{J} \end{bmatrix} \tag{C.5}$$

and similarly for $\nabla_u \bar{J}$, $\Delta^2_{uu} \bar{J}$.

As a proof we shall directly evaluate representative elements of the above Gradients and Hessians and show that along the nominal sequences (: $\{\delta \underline{u}_\ell = 0\}_{\ell=k}^{T-L}, \{\delta \bar{\underline{s}}_{\ell/k} = \underline{0}\}_{\ell=k}^{T}$,) they are equal. To this end we shall need the following fact:

$$\delta \bar{\underline{s}}_{j+1/k} = \underline{\underline{\phi}}(j+1,k)\, \delta \bar{\underline{s}}_{k/k} + \sum_{i=k}^{j} \underline{\underline{\phi}}(j+1,\, i+1)\, \underline{\underline{B}}_i\, \delta \underline{u}_i \tag{C.6}$$

where $\underline{\underline{\phi}}(j,k) \triangleq \underline{\underline{\phi}}_{j-1}\, \underline{\underline{\phi}}_{j-2} \cdots \underline{\underline{\phi}}_k$, $\underline{\underline{\phi}}(j,j) = \underline{\underline{I}}$, (C.7)

for any $j > k$.

The validity of (C.6) can be established by recursing the dynamics from time k to time j+1:

$$\delta \bar{\underline{s}}_{k+1/k} = \underline{\underline{\phi}}_k \, \delta\bar{\underline{s}}_{k/k} + \underline{\underline{B}}_k \, \delta\underline{u}_k,$$

$$\delta\bar{\underline{s}}_{k+2/k} = \underline{\underline{\phi}}_{k+1}[\underline{\underline{\phi}}_k \quad \delta\bar{\underline{s}}_{k/k} + \underline{\underline{B}}_k \delta\underline{u}_k] + \underline{\underline{B}}_{k+1} \, \delta\underline{u}_{k+1} =$$

$$= \underline{\underline{\phi}}_{k+1} \, \underline{\underline{\phi}}_k \, \delta\bar{\underline{s}}_{k/k} + \underline{\underline{\phi}}_{k+1} \, \underline{\underline{B}}_k \, \delta\underline{u}_k + \underline{\underline{B}}_{k+1} \, \delta\underline{u}_{k+1} =$$

$$= \underline{\underline{\phi}}(k+2,k) \, \delta\bar{\underline{s}}_{k/k} + \sum_{i=k}^{k+1} \underline{\underline{\phi}}(k+2,i+1) \, \underline{\underline{B}}_i \, \delta\underline{u}_i,$$

$$\vdots \qquad\qquad \vdots$$

which by induction leads to (C.6).

Now consider taking the derivative of J with respect to the control vector at time ℓ:

$$\nabla_{\delta\underline{u}_\ell} \tilde{J} = \nabla_{\delta\underline{u}_\ell} [\tilde{\ell}_k(\delta\bar{\underline{s}}_{k/k},\delta\underline{u}_k) + \ldots + \tilde{\ell}_\ell(\delta\bar{\underline{s}}_{\ell/k},\delta\underline{u}_\ell) +$$

$$+ \tilde{\ell}_{\ell+1}(\delta\bar{\underline{s}}_{\ell+1/k},\delta\underline{u}_{\ell+1}) + \ldots + \tilde{\ell}_T(\delta\bar{\underline{s}}_{T/k})]. \qquad (C.8)$$

Since the cost terms prior to time ℓ are not affected by the control $\delta\underline{u}_\ell$, they vanish after the differentiation. The only cost term explicitly involving $\delta\underline{u}_\ell$ is $\tilde{\ell}_\ell(\delta\bar{\underline{s}}_{\ell/k},\delta\underline{u}_\ell)$ while the terms $\tilde{\ell}_j(\delta\bar{\underline{s}}_{j/k},\delta\underline{u}_j)$ for $j > \ell$ are implicitly affected through the state vectors. In view of the above

$$\nabla_{\delta\underline{u}_\ell} \tilde{J} = \nabla_{\delta\underline{u}_\ell} \tilde{\ell}_\ell(\delta\bar{\underline{s}}_{\ell/k},\delta\underline{u}_\ell) + \sum_{j=\ell+1}^{T-L} \nabla_{\delta\underline{u}_\ell} \delta\bar{\underline{s}}_{j/k} \cdot \nabla_{\delta\bar{\underline{s}}_{j/k}} \tilde{\ell}_j(\delta\bar{\underline{s}}_{j/k},\delta\underline{u}_j)$$

$$+ \nabla_{\delta\underline{u}_\ell} \delta\bar{\underline{s}}_{T/k} \cdot \nabla_{\delta\bar{\underline{s}}_{T/k}} \tilde{\ell}_T(\delta\bar{\underline{s}}_{T/k})]. \qquad (C.9)$$

The expressions in Equation (C.9) result from application of the chain rule for vector differentiation according to which (see Appendix A):

$$\nabla_{\delta \underline{u}_\ell} \tilde{\ell}_j(\delta \bar{\underline{s}}_{j/k}, \delta \underline{u}_j) = \nabla_{\delta \underline{u}_\ell} \delta \bar{\underline{s}}_{j/k} \, \nabla_{\delta \bar{\underline{s}}_{j/k}} \tilde{\ell}_j(\delta \bar{\underline{s}}_{j/k}, \delta \underline{u}_j). \tag{C.10}$$

To evaluate $\nabla_{\delta \underline{u}_\ell} \delta \bar{\underline{s}}_{j/k}$, $j=\ell+1, \ldots, T$, we next invoke the result (C.6):

$$\nabla_{\delta \underline{u}_\ell} \delta \bar{\underline{s}}_{j/k} = \nabla_{\delta \underline{u}_\ell} [\underline{\underline{\phi}}(j,k) \delta \bar{\underline{s}}_{k/k} + \sum_{i=k}^{j=1} \underline{\underline{\phi}}(j,i+1) \, \underline{\underline{B}}_i \, \delta \underline{u}_i] =$$

$$= \nabla_{\delta \underline{u}_\ell} [\underline{\underline{\phi}}(j,\ell+1) \, \underline{\underline{B}}_\ell \, \delta \underline{u}_\ell] = \underline{\underline{B}}_\ell^T \, \underline{\underline{\phi}}^T(j, \ell+1) \tag{C.11}$$

where the following vector differentiation rule was also used (see Appendix A):

$$\nabla_{\delta \underline{u}_\ell} [\underline{\underline{\phi}}(j,\ell+1) \, \underline{\underline{B}}_\ell \, \delta \underline{u}_\ell] = [\underline{\underline{\phi}}(j,\ell+1) \, \underline{\underline{B}}_\ell]^T.$$

Substituting (C.11) in (C.9) we have

$$\nabla_{\delta \underline{u}_\ell} \tilde{J} = \nabla_{\delta \underline{u}_\ell} \tilde{\ell}_\ell(\delta \bar{\underline{s}}_{\ell/k}, \delta \underline{u}_\ell) + \underline{\underline{B}}_\ell^T \left[\sum_{j=\ell+1}^{T-1} \underline{\underline{\phi}}^T(j,\ell+1) \nabla_{\delta \bar{\underline{s}}_{j/k}} \tilde{\ell}_j(\delta \bar{\underline{s}}_{j/k}, \delta \underline{u}_j) + \right.$$

$$\left. + \underline{\underline{\phi}}^T(T,\ell+1) \, \nabla_{\delta \bar{\underline{s}}_{T/k}} \tilde{\ell}_T(\delta \bar{\underline{s}}_{T/k}) \right]. \tag{C.12}$$

Since we wish to evaluate the above Gradient at the nominal trajectories, we set $\{\delta \underline{u}_j = 0\}_{j=k}^{T-1}$, $\{\delta \bar{\underline{s}}_{j/k} = \underline{0}\}_{j=k}^{T}$ (the zero state sequence results from the zero control sequence, the dynamics, and the zero initial state condition) to arrive at

$$\nabla_{\delta \underline{u}_\ell} \tilde{J}(\delta \underline{u}_k = \underline{0}, \ldots, \underline{u}_{T-1} = \underline{0} = \underline{N}_{u\ell} + \underline{\underline{B}}_\ell^T \sum_{j=\ell+1}^{T} \phi^T(j,\ell+1)\underline{N}_{sj}] \tag{C.13}$$

where the coefficients $\underline{N}_{u\ell}$, $\{\underline{N}_{sj}\}_{j=\ell+1}^{T}$ have been defined in (C.2).

To prove equality of Gradients for problems (C.1) and (C.2), we need to show that the righthand side of (C.13) also represents $\nabla_{\delta \underline{u}_\ell} \bar{J}(\delta \underline{u}_k = \underline{0}, \ldots, \delta \underline{u}_{T-1} = 0)$. To this end we observe that all operations invoked to derive (C.12) are valid for problem (C.2) as well, the only difference being the form of the cost terms. If we denote the quadratic costs by $\{\bar{\ell}_j(\delta \bar{\underline{s}}_{j/k}, \delta \underline{u}_j)\}_{j=0}^{T-1}$, $\bar{\ell}_T(\delta \bar{\underline{s}}_{T/k})$, we find the following expressions for the derivatives indicated in (C.12):

$$\left.\begin{aligned}
&\nabla_{\delta \underline{u}_\ell} \bar{\ell}_\ell(\delta \bar{\underline{s}}_{\ell/k}, \delta \underline{u}_\ell) = \underline{N}_{u\ell} + \underline{\underline{N}}_{uu\ell}\, \delta \underline{u}_\ell + \underline{\underline{N}}_{us\ell}\, \delta \bar{\underline{s}}_{\ell/k}, \\
&\nabla_{\delta \bar{\underline{s}}_{j/k}} \bar{\ell}_j(\delta \bar{\underline{s}}_{j/k}, \delta \underline{u}_j) = \underline{N}_{sj} + \underline{\underline{N}}_{ssj}\, \delta \bar{\underline{s}}_{j/k} + \underline{\underline{N}}_{usj}^T\, \delta \underline{u}_j, \\
&\quad j = \ell+1, \ldots, T-1 \\
&\nabla_{\delta \bar{\underline{s}}_{T/k}} \bar{\ell}_T(\delta \bar{\underline{s}}_{T/k}) = \underline{N}_{sT} + \underline{\underline{N}}_{ssT}\, \delta \bar{\underline{s}}_{T/k}.
\end{aligned}\right\} \tag{C.14}$$

To carry out these differentiations, we have used the fact that for two real vectors $\underline{x}$, $\underline{y}$ and a symmemtric matrix $\underline{\underline{A}}$ of compatible dimensions there holds

$$\nabla_{\underline{x}}(\underline{y}^T\underline{x}) = \nabla_{\underline{x}}(\underline{x}^T\underline{y}) = y$$
$$\nabla_{\underline{x}}(\underline{x}^T\underline{\underline{A}}\underline{x}) = \underline{\underline{A}}\,\underline{x} + \underline{\underline{A}}^T\underline{x} = 2\underline{\underline{A}}\,\underline{x}.$$

In view of (C.14) and (C.12) we obtain

$$\nabla_{\delta \underline{u}_\ell} \bar{J} = \underline{N}_{u\ell} + \underline{\underline{N}}_{uu\ell}\delta \underline{u}_\ell + \underline{\underline{N}}_{us\ell}\delta \bar{\underline{s}}_{\ell/k} + \underline{\underline{B}}_\ell^T \Bigg[\sum_{j=\ell+1}^{T-1} \underline{\underline{\phi}}^T(j,\ell+1) \cdot$$

$$\cdot \quad [\underline{N}_{sj} + \underline{\underline{N}}_{ssj}\delta \bar{\underline{s}}_{j/k} + \underline{\underline{N}}_{usj}^T \delta \underline{u}_j] + \underline{\underline{\phi}}^T(T,\ell+1)[\underline{N}_{st} + \underline{\underline{N}}_{ssT}\delta \bar{\underline{s}}_{T/k}] \Bigg]. \quad (C.15)$$

Finally, setting $\{\delta \underline{u}_j = \underline{0}\}_{j=k}^{T-1}$, $\{\delta \bar{\underline{s}}_{j/k} = \underline{0}\}_{j=k}^{T}$ in (C.15) we find that $\nabla_{\delta \underline{u}} \bar{J}(\delta \underline{u}_k = \underline{0}, \ldots, \delta \underline{u}_{T-1} = \underline{0})$ is given by

$$\nabla_{\delta \underline{u}_\ell} \bar{J}(\delta \underline{u}_k = \underline{0}, \ldots, \delta \underline{u}_{T-1} = \underline{0}) = \underline{N}_{u\ell} + \underline{\underline{B}}_\ell^T \Big[\sum_{j=\ell+1}^{T} \underline{\underline{\phi}}^T(j,\ell+1)\, \underline{N}_{sj} \Big] . \quad (C.16)$$

Since equations (C.16) and (C.13) hold for any time $\ell = k, \ldots, T-1$, their equality proves the claim that along the nominal sequences problems (C.1) and (C.2) have equal Gradients.

We turn now to the evaluation of the Hessians. Regarding problem (C.1) we shall start by differentiating Equation (C.12) with respect to a control vector $\delta \underline{u}_m$ where we assume $k < m < \ell$:

$$\nabla^2_{\delta \underline{u}_\ell \delta \underline{u}_m} \tilde{J} = \nabla_{\delta \underline{u}_m} \delta \bar{\underline{s}}_{\ell/k} \cdot \nabla^2_{\delta \underline{u}_\ell \delta \bar{\underline{s}}_{\ell/k}} \tilde{\ell}_\ell(\delta \bar{\underline{s}}_{\ell/k}, \delta \underline{u}_\ell) +$$

$$+ \Bigg[\sum_{j=\ell+1}^{T-1} \nabla_{\delta \underline{u}_m} \delta \bar{\underline{s}}_{j/k} \cdot \nabla^2_{\delta \bar{\underline{s}}_{j/k} \delta \bar{\underline{s}}_{j/k}} \tilde{\ell}_j(\delta \bar{\underline{s}}_{j/k}, \delta \underline{u}_j) \underline{\underline{\phi}}(j,\ell+1) +$$

$$+ \nabla_{\delta \underline{u}_m} \delta \bar{\underline{s}}_{T/k} \nabla^2_{\delta \bar{\underline{s}}_{T/k} \delta \bar{\underline{s}}_{T/k}} \tilde{\ell}_T(\delta \bar{\underline{s}}_{T/k}) \underline{\underline{\phi}}(T,\ell+1) \Bigg] \underline{\underline{B}}_\ell . \quad (C.17)$$

Equation (C.17) results solely by use of the chain rule of differentiation since all terms in (C.12) are implicity functions of $\delta \underline{u}_m$ through the state vectors and the dynamics. Similarly to (C.11) we find

$$\nabla_{\delta \underline{u}_m} \delta \bar{\underline{s}}_{j/k} = \underline{\underline{B}}_m^T \; \underline{\underline{\phi}}^T(j,m+1) \tag{C.18}$$

and so

$$\nabla^2_{\delta \underline{u}_\ell \, \delta \underline{u}_m} \tilde{J} = \underline{\underline{B}}_m^T \, \underline{\underline{\phi}}^T(\ell,m+1) \; \nabla^2_{\delta \underline{u}_\ell \delta \bar{\underline{s}}_{\ell/k}} \; \tilde{\ell}_\ell(\delta \bar{\underline{s}}_{\ell/k}, \delta u_\ell) +$$

$$+ \underline{\underline{B}}_m^T \Bigg[\sum_{j=\ell+1}^{T-1} \underline{\underline{\phi}}^T(j,m+1) \; \nabla^2_{\delta \bar{\underline{s}}_{j/k} \delta \bar{\underline{s}}_{j/k}} \; \tilde{\ell}_j(\delta \bar{\underline{s}}_{j/k}, \delta \underline{u}_j) \; \underline{\phi}(j,\ell+1) +$$

$$+ \underline{\underline{\phi}}^T(T,m+1) \; \nabla^2_{\delta \bar{\underline{s}}_{T/k} \delta \bar{\underline{s}}_{T/k}} \; \tilde{\ell}_T(\delta \bar{\underline{s}}_{T/k}) \; \underline{\phi}(T,\ell+1) \Bigg] \underline{\underline{B}}_\ell . \tag{C.19}$$

Evaluating this expression along the nominal sequences we get

$$\nabla^2_{\delta \underline{u}_\ell \delta \underline{u}_m} \tilde{J}(\delta \underline{u}_k = \underline{0}, \; \ldots, \; \delta \underline{u}_{T-1} = 0) = \underline{\underline{B}}_m^T \, \underline{\underline{\phi}}^T(\ell,m+1) \; \underline{\underline{N}}^T_{us\ell} +$$

$$+ \underline{\underline{B}}_m^T \Bigg[\sum_{j=\ell+1}^{T} \underline{\underline{\phi}}^T(j,m+1) \; \underline{\underline{N}}_{ssj} \; \underline{\underline{\phi}}(j,\ell+1) \Bigg] \underline{\underline{B}}_\ell . \tag{C.20}$$

Regarding the Hessian of problem (C.2), we differentiate Equation (C.15) with respect to $\delta \underline{u}_m$ to obtain

$$\nabla^2_{\delta \underline{u}_\ell \delta \underline{u}_m} \bar{J} = \nabla_{\delta \underline{u}_m} \delta \bar{\underline{s}}_{\ell/k} \; \underline{\underline{N}}^T_{us\ell} + \Bigg[\sum_{j=\ell+1}^{T-1} \nabla_{\delta \underline{u}_m} \delta \bar{\underline{s}}_{j/k} \; \underline{\underline{N}}_{ssj} \; \underline{\underline{\phi}}(j,\ell+1) +$$

$$+ \Delta_{\delta \underline{u}_m} \delta \bar{\underline{s}}_{T/k} \; \underline{\underline{N}}_{ssT} \; \underline{\underline{\phi}}(T,\ell+1) \Bigg] \underline{\underline{B}}_\ell . \tag{C.21}$$

Invoking (C.18) the above becomes

$$\nabla^2_{\delta \underline{u}_\ell \delta \underline{u}_m} \bar{J} = \underline{\underline{B}}_m^T \underline{\underline{\phi}}^T(\ell, m+1) \; \underline{\underline{N}}_{us\ell}^T +$$

$$+ \underline{\underline{B}}_m^T \left[\sum_{j=\ell+1}^{T} \underline{\underline{\phi}}^T(j,m+1) \; \underline{\underline{N}}_{ssj} \; \underline{\underline{\phi}}(j,\ell+1) \right] \underline{\underline{B}}_\ell \tag{C.22}$$

which by (C.20) is equal to $\nabla^2_{\delta \underline{u}_\ell \, \delta \underline{u}_m} \tilde{J}(\delta \underline{u}_k = \underline{0}, \ldots, \delta \underline{u}_{T-1} = 0)$.

Similarly, for $\ell = m$ we can show that

$$\nabla^2_{\delta \underline{u}_\ell \delta \underline{u}_\ell} \tilde{J}(\delta \underline{u}_k = \underline{0}, \ldots, \delta \underline{u}_{T-1} = 0) = \underline{\underline{N}}_{uu\ell} + \underline{\underline{B}}_\ell^T \left[\sum_{j=\ell+1}^{T} \underline{\underline{\phi}}^T(j,\ell+1) \; \underline{\underline{N}}_{ssj} \cdot \right.$$

$$\left. \cdot \; \underline{\underline{\phi}}(j,\ell+1) \right] \underline{\underline{B}}_\ell + \nabla^2_{\delta \underline{u}_\ell \delta \underline{u}_\ell} \bar{J} \tag{C.23}$$

and for $\ell < m \leq T-1$

$$\nabla^2_{\delta \underline{u}_\ell \delta \underline{u}_m} \tilde{J}(\delta \underline{u}_k = \underline{0}, \ldots, \delta \underline{u}_{T-1} = 0) = \underline{\underline{N}}_{usm} \; \underline{\underline{\phi}}(m,\ell+1) \; \underline{\underline{B}}_\ell +$$

$$+ \underline{\underline{B}}_m^T \left[\sum_{j=m+1}^{T} \underline{\underline{\phi}}^T(j,m+1) \; \underline{\underline{N}}_{ssj} \; \underline{\underline{\phi}}(j,\ell+1) \right] \underline{\underline{B}}_\ell = \nabla^2_{\delta \underline{u}_\ell \delta \underline{u}_m} \bar{J} \tag{C.24}$$

proving that problems (C.1) and C.2) also have equal Hessians along the nominal sequences.

Before concluding this appendix we shall present a result to be used later in Chapter 4. The purpose is to provide efficient ways to compute the Gradient $\nabla_{\delta \underline{u}_\ell} \tilde{J}(\underline{0}, \ldots, \underline{0})$ and the Hessian $\nabla^2_{\delta \underline{u}_\ell \delta \underline{u}_\ell} \tilde{J}(\underline{0}, \ldots, \underline{0})$ at any time $\ell = k, \ldots, T-1$. Consider the sequence of vectors $\{\underline{p}_i\}_{i=k}^T$ and matrices $\{\underline{\underline{G}}_i\}_{i=k}^T$ obtained recursively by

$$\left.\begin{aligned} &\underline{p}_T = \underline{N}_{sT} \\ &\underline{p}_i = \underline{\underline{\phi}}_i^T \underline{p}_{i+1} + \underline{N}_{si} \\ &i = T-1,\ T-2,\ \ldots, k, \end{aligned}\right\} \tag{C.25}$$

and

$$\left.\begin{aligned} &\underline{\underline{G}}_T = \underline{\underline{N}}_{ssT} \\ &\underline{\underline{G}}_i = \underline{\underline{\phi}}_i^T\ \underline{\underline{G}}_{i+1}\phi_i + \underline{\underline{N}}_{ssi} \\ &i = T-1,\ T-2,\ \ldots,\ k. \end{aligned}\right\} \tag{C.26}$$

We claim that for any $\ell=k,\ \ldots,\ T-1$,

$$\nabla_{\delta\underline{u}_\ell} \tilde{J}(\underline{0},\ldots,\underline{0}) = \underline{N}_{u\ell} + \underline{\underline{B}}_\ell^T \underline{p}_{\ell+1}, \tag{C.27}$$

$$\nabla^2_{\delta\underline{u}_\ell\delta\underline{u}_\ell} \tilde{J}(\underline{0},\ldots,\underline{0}) = \underline{\underline{N}}_{uu\ell} + \underline{\underline{B}}_\ell^T\ \underline{\underline{G}}_{\ell+1}\ \underline{\underline{B}}_\ell. \tag{C.28}$$

To see why this is so we shall prove that (C.27) and (C.28) are respectively equal to the expressions (C.13) and (C.23) derived earlier.

Recursing (C.25) backwards we get

$$\underline{p}_T = \underline{N}_{sT},$$

$$\underline{p}_{T-1} = \underline{\underline{\phi}}_{T-1}^T \underline{p}_T + \underline{N}_{sT-1} = \underline{\underline{\phi}}_{T-1}^T \underline{N}_{sT} + \underline{N}_{sT-1},$$

$$\begin{aligned} \underline{p}_{T-2} &= \underline{\underline{\phi}}_{T-2}^T \underline{p}_{T-1} + \underline{N}_{sT-2} = \underline{\underline{\phi}}_{T-2}^T = \underline{\underline{\phi}}_{T-2}^T\ \underline{\underline{\phi}}_{T-1}^T \underline{N}_{sT-1} + N_{sT-2} \\ &= \underline{\underline{\phi}}^T(T,T-2)\ \underline{N}_{sT} + \underline{\underline{\phi}}^T(T-1,T-2)\ \underline{N}_{sT-1} + \underline{\underline{\phi}}^T(T-2,T-2)\underline{N}_{sT-2} \\ &= \sum_{j=T-2}^{T} \underline{\underline{\phi}}^T(j,T-2)\ \underline{N}_{sj}, \end{aligned}$$

where use was made of the definition (C.7).

Continuing the induction up to time $\ell+1$, we find

$$\underline{P}_{\ell+1} = \sum_{j=\ell+1}^{T} \underline{\underline{\phi}}^T(j,\ell+1)\ \underline{N}_{sj}. \tag{C.29}$$

Substituting (C.29) in (C.27) we obtain

$$\nabla_{\delta \underline{u}_\ell} \tilde{J}(\underline{0},\ldots,\underline{0}) - \underline{N}_{u\ell} + \underline{\underline{B}}_\ell^T \left[\sum_{j=\ell+1}^{T} \underline{\underline{\phi}}^T(j,\ell+1)\ \underline{N}_{sj} \right] \tag{C.30}$$

which is the same expression of $\nabla_{\delta \underline{u}_\ell} \tilde{J}(\underline{0},\ldots,\underline{0})$ as the one in Equation (C.13).

Similarly, regarding the diagonal submatrix of the Hessian, we recurse Equation (C.26) to find

$$\underline{\underline{G}}_T + \underline{\underline{N}}_{ssT}$$

$$\underline{\underline{G}}_{T-1} + \underline{\underline{\phi}}_{T-1}^T\ \underline{\underline{G}}_T\ \underline{\underline{\phi}}_{T-1} + \underline{\underline{N}}_{ssT-1} = \underline{\underline{\phi}}_{T-1}^T\ \underline{\underline{N}}_{ssT}\ \underline{\underline{\phi}}_{T-1} + \underline{\underline{N}}_{ssT-1}$$

$$\begin{aligned}
\underline{\underline{G}}_{T-2} &= \underline{\underline{\phi}}_{T-2}^T\ \underline{\underline{G}}_{T-1}\ \underline{\underline{\phi}}_{T-2} + \underline{\underline{N}}_{ssT-2} = \\
&= \underline{\underline{\phi}}_{T-2}^T\ \underline{\underline{\phi}}_{T-1}^T\ \underline{\underline{N}}_{ssT}\ \underline{\underline{\phi}}_{T-1}\ \underline{\underline{\phi}}_{T-2} + \underline{\underline{\phi}}_{T-2}^T\ \underline{\underline{N}}_{ssT-1}\ \underline{\underline{\phi}}_{T-2} + \underline{\underline{N}}_{ssT-2} = \\
&= \underline{\underline{\phi}}^T(T,T-2)\ \underline{\underline{N}}_{ssT}\ \underline{\underline{\phi}}(T,T-2) + \underline{\underline{\phi}}^T(T-1,T-2)\ \underline{\underline{N}}_{ssT-1}\ \underline{\underline{\phi}}(T-1,T-2) + \\
&+ \underline{\underline{\phi}}^T(T-2,T-2)\ \underline{\underline{N}}_{ssT-2}\ \underline{\underline{\phi}}(T-2,T-2) = \\
&= \sum_{j=T-2}^{T} \underline{\underline{\phi}}^T(j,T-2)\ \underline{\underline{N}}_{ssj}\ \underline{\underline{\phi}}(j,T-2).
\end{aligned}$$

which by induction leads to

$$\underline{\underline{G}}_{\ell+1} = \sum_{j=\ell+1}^{T} \underline{\underline{\phi}}^T(j,\ell+1)\ \underline{\underline{N}}_{ssj}\ \underline{\underline{\phi}}(j,\ell+1). \tag{C.31}$$

Substituting this result into (C.28) we find

$$\nabla^2_{\delta \underline{u}_\ell \delta \underline{u}_\ell} \tilde{J}(\underline{0}, \ldots, \underline{0}) = \underline{\underline{N}}_{uu\ell} + \underline{\underline{B}}_\ell^T [\sum_{j=\ell+1}^{T} \underline{\underline{\phi}}^T(j,\ell+1) \, \underline{\underline{N}}_{ssj} \, \underline{\underline{\phi}}(j,\ell+1)] \underline{\underline{B}}_\ell \tag{C.32}$$

which by comparison with Equation (C.28) proves the claim.

APPENDIX D

COMPUTATION OF THE NEWTON'S DIRECTION

We wish to determine the control sequence $\{\delta \underline{u}_\ell^*\}_{\ell=k}^{T-1}$ which

$$\left.\begin{array}{l}
\text{minimizes}\left\{\bar{J} = \sum_{\ell=k}^{T-1}[\underline{N}_{s\ell}^T\,\bar{\underline{s}}_{\ell/k} + \underline{N}_{u_\ell}^T\,\delta\underline{u}_\ell + \frac{1}{2}\,\delta\bar{\underline{s}}_{\ell/k}^T\,\underline{\underline{N}}_{ss\ell}\delta\bar{\underline{s}}_{\ell/k} + \right. \\
\qquad + \frac{1}{2}\,\delta\underline{u}_\ell^T\,\underline{\underline{N}}_{uu\ell}\,\delta\underline{u}_\ell + \delta\underline{u}_\ell^T\,\underline{\underline{N}}_{us\ell}\delta\bar{\underline{s}}_{\ell/k}] + \\
\qquad \left. + \underline{N}_{sT}^T\,\delta\bar{\underline{s}}_{T/k} + \frac{1}{2}\delta\bar{\underline{s}}_{T/k}^T\,\underline{\underline{N}}_{ssT}\,\delta\bar{\underline{s}}_{T/k}\right\} \\
\text{subject to} \\
\quad \delta\bar{\underline{s}}_{\ell+1/k} \quad \underline{\underline{\phi}}_\ell\delta\bar{\underline{s}}_{\ell/k} + \underline{\underline{B}}_\ell\delta\underline{u}_\ell,\ \delta\bar{\underline{s}}_{k/k} = \underline{0}, \\
\quad \ell=k,\ \ldots,\ T-1.
\end{array}\right\} \quad (D.1)$$

To guarantee uniqueness of the solution, we additionally require that $\{\underline{\underline{N}}_{ss\ell}\}_{\ell=k}^{T-1}$ are positive semidefinite and that $\{\underline{\underline{N}}_{uu\ell}\}_{\ell=k}^{T-1}$ are positive definite matrices. These assumptions are expected to be valid due to the form of the objective function discussed in Section 4.2; yet, if they are not there are still ways to remedy the situion. One would be to convexify the performance index by placing higher exponent penalties on the power target deviations. Others will be mentioned as the discussion progresses.

The solution of problem (D.1) will proceed according to the Dynamic Programming Methodology outlined by Algorithm A4.4.1.

At the terminal time T there is no control variable and we simply set

$$\bar{J}_T[\delta\bar{\underline{s}}_{T/k}] = \underline{N}_{sT}^T\ \delta\bar{\underline{s}}_{T/k} + \frac{1}{2}\ \delta\bar{\underline{s}}_{T/k}^T\ \underline{\underline{N}}_{ssT}\ \delta\bar{\underline{s}}_{T/k}. \tag{D.2}$$

Stepping backward, we want to select the feedback control law $\delta\underline{u}_{T-1}^* = \underline{\mu}_{T-1}^*(\delta\bar{\underline{s}}_{T-1/k})$ so that

$$\underline{\mu}_{T-1}^*(\delta\bar{\underline{s}}_{T-1/k}) = \arg\min_{\delta\underline{u}_{T-1}} \Big\{ \bar{J}_{T-1} = \underline{N}_{sT-1}^T\ \delta\bar{\underline{s}}_{T-1/k} + \underline{N}_{uT-1}^T\ \delta\underline{u}_{T-1}$$

$$+ \frac{1}{2}[\delta\bar{\underline{s}}_{T-1/k}^T\ \underline{\underline{N}}_{ssT-1}\delta\bar{\underline{s}}_{T-1/k} + \delta\underline{u}_{T-1}^T\ \underline{\underline{N}}_{uuT-1}\ \delta\underline{u}_{T-1}]$$

$$+ \delta\underline{u}_{T-1}^T\ \underline{\underline{N}}_{usT-1}\ \delta\bar{\underline{s}}_{T-1/k} + \bar{J}_T(\delta\bar{\underline{s}}_{T/k}) \Big\}. \tag{D.3}$$

Substituting for $\delta\bar{\underline{s}}_{T/k}$ from the dynamics

$$\delta\bar{\underline{s}}_{T/k} = \underline{\underline{\phi}}_{T-1}\ \delta\bar{\underline{s}}_{T-1/k} + \underline{\underline{B}}_{T-1}\ \delta\underline{u}_{T-1}$$

and using (D.2), we can restate (D.3) as follows:

$$\underline{\mu}_{T-1}^*(\delta\bar{\underline{s}}_{T-1/k}) = \arg\min_{\delta\underline{u}_{T-1}} \Big\{ \bar{J}_{T-1} = [\underline{N}_{sT}^T\ \underline{\underline{\phi}}_{T-1} + \underline{N}_{sT-1}^T]\ \delta\bar{\underline{s}}_{T-1/k} +$$

$$+[\underline{N}_{sT}^T\ \underline{\underline{B}}_{T-1} + \underline{N}_{uT-1}^T]\delta\underline{u}_{T-1} + \frac{1}{2}\delta\bar{\underline{s}}_{T-1/k}^T[\underline{\underline{\phi}}_{T-1}^T\ \underline{\underline{N}}_{ssT}\ \underline{\underline{\phi}}_{T-1}] +$$

$$+ \underline{\underline{N}}_{ssT-1}]\ \delta\bar{\underline{s}}_{T-1/k} + \frac{1}{2}\ \delta\underline{u}_{T-1}^T\ [\underline{\underline{B}}_{T-1}^T\ \underline{\underline{N}}_{ssT}\ \underline{\underline{B}}_{T-1} + \underline{\underline{N}}_{uuT-1}]\delta\underline{u}_{T-1} +$$

$$+ \delta\underline{u}_{T-1}^T\ [\underline{\underline{B}}_{T-1}^T\ \underline{\underline{N}}_{ssT}\ \underline{\underline{\phi}}_{T-1} + \underline{N}_{usT-1}]\ \delta\bar{\underline{s}}_{T-1/k} \Big\}. \tag{D.4}$$

Because of the positive (semi)definiteness assumptions on $\underline{\underline{N}}_{ssT}$ and $\underline{\underline{N}}_{uuT-1}$,

it follows that $\underline{\underline{B}}_{T-1}^T \underline{\underline{N}}_{ssT} \underline{\underline{B}}_{T-1} + \underline{\underline{N}}_{uuT-1}$ is also positive definite (as a basic fact from matrix algebra, Strang, 1980) proving that $\bar{J}_{T-1}$ is a convex function of $\delta \underline{u}_{T-1}$. To find the unique global minimum, we differentiate with respect to δu_{T-1} and set the result equal to zero.

$$\underline{\underline{B}}_{T-1}^T \underline{N}_{sT} + \underline{N}_{uT-1} + [\underline{\underline{B}}_{T-1}^T \underline{\underline{N}}_{ssT} \underline{\underline{B}}_{T-1} + \underline{\underline{N}}_{uuT-1}]\delta \underline{u}_{T-1}$$

$$+ [\underline{\underline{B}}_{T-1}^T \underline{\underline{N}}_{ssT} \underline{\underline{\phi}}_{T-1} + \underline{\underline{N}}_{usT-1}]\delta \bar{\underline{s}}_{T-1/k} = \underline{0}. \tag{D.5}$$

In the derivation of (D.5) we have used that for the vectors $\underline{x}$, $\underline{y}$ and the symmetric matrix $\underline{\underline{A}}$ of compatible dimensions there holds

$$\nabla_x[\underline{y}^T\underline{x}] = \underline{y} = \nabla_x[\underline{x}^T\underline{y}],$$

$$\nabla_x[\underline{x}^T\underline{\underline{A}}\underline{x}] = \underline{\underline{A}}\underline{x} + \underline{\underline{A}}^T\underline{x} = 2\underline{\underline{A}}\underline{x}.$$

Equation (D.5) can be solved for $\delta \underline{u}_{T-1}$ to yield:

$$\delta \underline{u}_{T-1}^* = \mu^*(\delta \bar{\underline{s}}_{T-1/k}) = -[\underline{\underline{B}}_{T-1}^T \underline{\underline{N}}_{ssT} \underline{\underline{B}}_{T-1} + \underline{\underline{N}}_{uuT-1}]^{-1} \cdot$$

$$\cdot [(\underline{\underline{B}}_{T-1}^T \underline{\underline{N}}_{ssT} \underline{\underline{\phi}}_{T-1} + \underline{\underline{N}}_{usT-1})\delta \bar{\underline{s}}_{T-1/k} + \underline{\underline{B}}_{T-1}^T \underline{N}_{sT} + \underline{N}_{uT-1}]. \tag{D.6}$$

Calling $\underline{\underline{D}}_{T-1} = [\underline{\underline{B}}_{T-1}^T \underline{\underline{N}}_{ssT} \underline{\underline{B}}_{T-1} + \underline{\underline{N}}_{uuT-1}]^{-1}$. (D.7)

$$\underline{\underline{L}}_{T-1} = \underline{\underline{B}}_{T-1}^T \underline{\underline{N}}_{ssT} \underline{\underline{\phi}}_{T-1} + \underline{\underline{N}}_{usT-1}, \tag{D.8}$$

$$\underline{\Lambda}_{T-1} = \underline{\underline{B}}_{T-1}^T \underline{\underline{N}}_{sT} + \underline{N}_{uT-1}, \tag{D.9}$$

we see that the optimal control law is the following linear function of

$$\delta \underline{u}^*_{T-1} = \underline{\mu}^*_{T-1}(\delta \bar{\underline{s}}_{T-1/k}) = - \underline{\underline{D}}_{T-1}[\underline{\underline{L}}_{T-1}\ \delta \bar{\underline{s}}_{T-1/k} + \underline{\Lambda}_{T-1}]. \tag{D.10}$$

In case $\underline{\underline{D}}^-_{T-1}$ does not exist, one can still solve (D.5) using generalized inverses (Bellman, 1957) but the solution is not necessarily unique.)

Given that at time T-1 the control law (D.10) will be applied, the optimal "cost to go" $\bar{J}_{T-1}[\delta \bar{\underline{s}}_{T-1/k}]$ becomes

$$\bar{J}_{T-1}(\delta \bar{\underline{s}}_{T-1/k}) = \frac{1}{2}\delta \bar{\underline{s}}^T_{T-1/k}\delta \underline{\underline{K}}_{T-1}\ \delta \bar{\underline{s}}_{T-1/k} +$$

$$+ \underline{k}^T_{T-1}\ \delta \bar{\underline{s}}_{T-1/k} + c_{T-1}, \tag{D.11}$$

where

$$\underline{\underline{K}}_{T-1} = \underline{\underline{N}}_{ssT-1} + \underline{\underline{\phi}}^T_{T-1}\ \underline{\underline{N}}_{ssT}\ \underline{\underline{\phi}}_{T-1} +$$

$$+ \underline{\underline{L}}^T_{T-1}\ \underline{\underline{D}}_{T-1}\ [\underline{\underline{B}}^T_{T-1}\ \underline{\underline{N}}_{ssT}\ \underline{\underline{B}}_{T-1} + \underline{\underline{N}}_{uuT-1}]\ \underline{\underline{D}}_{T-1}\ \underline{\underline{L}}_{T-1} -$$

$$- 2\ \underline{\underline{L}}^T_{T-1}\ \underline{\underline{D}}_{T-1}\ [\underline{\underline{B}}_{T-1}\ \underline{\underline{N}}_{ssT}\ \underline{\underline{\phi}}_{T-1} + \underline{\underline{N}}_{usT-1}], \tag{D.12}$$

$$\underline{k}_{T-1} = \underline{N}_{sT-1} + \underline{\underline{\phi}}^T_{T-1}\ \underline{N}_{sT} - \underline{\underline{L}}_{T-1}\ \underline{\underline{D}}_{T-1}[\underline{\underline{B}}^T_{T-1}\underline{N}_{sT} + \underline{N}_{uT-1}] +$$

$$+ \underline{\underline{L}}^T_{T-1}\ \underline{\underline{D}}_{T-1}\ [\underline{\underline{B}}^T_{T-1}\ \underline{\underline{N}}_{ssT}\ \underline{\underline{B}}_{T-1} + \underline{\underline{N}}_{uuT-1}]\ \underline{\underline{D}}_{T-1}\ \underline{\Lambda}_{T-1} -$$

$$- [\underline{\underline{\phi}}^T_{T-1}\ \underline{\underline{N}}_{ssT}\ \underline{\underline{B}}_{T-1} + \underline{\underline{N}}^T_{usT-1}]\ \underline{\underline{D}}_{T-1}\ \underline{\Lambda}_{T-1}\ . \tag{D.13}$$

$$c_{T-1} = -[\underline{N}^T_{sT}\ \underline{\underline{B}}_{T-1} + \underline{N}^T_{uT-1}]\ \underline{\underline{D}}_{T-1}\ \underline{\Lambda}_{T-1} +$$

$$+ \frac{1}{2}\ \underline{\Lambda}^T_{T-1}\ \underline{\underline{D}}_{T-1}\ [\underline{\underline{B}}^T_{T-1}\ \underline{\underline{N}}_{ssT}\ \underline{\underline{B}}_{T-1} + \underline{\underline{N}}_{uuT-1}]\ \underline{\underline{D}}_{T-1}\ \underline{\Lambda}_{T-1}. \tag{D.14}$$

The above relationships result by substituting (D.10) in the expression (D.4) for $\bar{J}_{T-1}$ and use of the fact that $\underline{\underline{N}}_{ssT}$, $\underline{\underline{N}}_{ssT-1}$, $\underline{\underline{N}}_{uuT-1}$, $\underline{\underline{D}}_{T-1}$,

$\underline{\underline{B}}_{T-1}^T \underline{\underline{N}}_{ssT} \underline{\underline{B}}_{T-1} + \underline{\underline{N}}_{uuT-1}$ are symmetric.

Equivalently, Equation (D.12), (D.13), (D.14) can be rewritten as follows after invoking (D.7), (D.8), (D.9):

$$\underline{\underline{K}}_{T-1} = \underline{\underline{N}}_{ssT-1} + \underline{\underline{\phi}}_{T-1}^T \underline{\underline{N}}_{ssT} \underline{\underline{\phi}}_{T-1} - [\underline{\underline{B}}_{T-1}^T \underline{\underline{N}}_{ssT} \underline{\underline{\phi}}_{T-1} + \underline{\underline{N}}_{usT-1}]^T \cdot$$

$$\cdot [\underline{\underline{B}}_{T-1}^T \underline{\underline{N}}_{ssT} \underline{\underline{B}}_{T-1} + \underline{\underline{N}}_{uuT-1}]^{-1} [\underline{\underline{B}}_{T-1}^T \underline{\underline{N}}_{ssT} \underline{\underline{\phi}}_{T-1} + \underline{\underline{N}}_{usT-1}] \quad (D.15)$$

$$\underline{k}_{T-1} = \underline{N}_{sT-1} + \underline{\underline{\phi}}_{T-1}^T \underline{N}_{sT} - [\underline{\underline{B}}_{T-1}^T \underline{N}_{ssT} \underline{\underline{\phi}}_{T-1} + \underline{\underline{N}}_{usT-1}]^T \cdot$$

$$\cdot [\underline{\underline{B}}_{T-1}^T \underline{\underline{N}}_{ssT} \underline{\underline{B}}_{T-1} + \underline{N}_{uuT-1}]^{-1} [\underline{\underline{B}}_{T-1}^T \underline{N}_{s} + \underline{N}_{uT-1}] \cdot \quad (D.16)$$

$$c_{T-1} = -\frac{1}{2} [\underline{\underline{B}}_{T-1}^T \underline{N}_{sT} + \underline{N}_{uT-1}]^T [\underline{\underline{B}}_{T-1}^T \underline{\underline{N}}_{s} \underline{\underline{B}}_{T-1} + \underline{\underline{N}}_{uuT-1}]^{-1} \cdot$$

$$\cdot [\underline{\underline{B}}_{T-1}^T \underline{N}_{sT} + \underline{N}_{uT-1}] \quad (D.17)$$

Having determined the form of the optimal "cost to go" $\bar{J}_{T-1}[\bar{\underline{s}}_{T-1/k}]$ by Equation (D.11), (D.15), (D.16), (D.17), we may now step backward at time T-2 and select

$$\underline{\mu}_{T-2}^*(\delta\bar{\underline{s}}_{T-2/k}) = \arg\min_{\delta\underline{u}_{T-2}} \Big\{ \bar{J}_{T-2} = N_{sT-2}^T \delta\bar{\underline{s}}_{T-2/k} + \underline{N}_{uT-2}^T \cdot \delta\underline{u}_{T-2} +$$

$$+ \frac{1}{2}[\delta\bar{\underline{s}}_{T-2/k} \underline{\underline{N}}_{ssT-2} \delta\bar{\underline{s}}_{T-2/k} + \delta\underline{u}_{T-2}^T \underline{\underline{N}}_{uuT-2} \delta\underline{u}_{T-2} +$$

$$+ \delta\underline{u}_{T-2}^T \underline{\underline{N}}_{usT-2} \delta\bar{\underline{s}}_{T-2/k} + \bar{J}_{T-1}(\delta\bar{\underline{s}}_{T-1/k}) \Big\} . \quad (D.18)$$

where

$$\underline{\underline{D}}_{T-2} = [\underline{\underline{B}}^T_{T-2}\ \underline{\underline{K}}_{T-1}\ \underline{\underline{B}}_{T-2} + \underline{\underline{N}}_{uuT-2}]^{-1}, \tag{D.22}$$

$$\underline{\underline{L}}_{T-2} = \underline{\underline{B}}^T_{T-2}\ \underline{\underline{K}}_{T-1}\ \underline{\underline{\phi}}_{T-2} + \underline{\underline{N}}_{usT-2}, \tag{D.23}$$

$$\underline{\Lambda}_{T-2} = \underline{\underline{B}}^T_{T-2}\ \underline{k}_{T-1} + \underline{N}_{uT-2}. \tag{D.24}$$

Putting $\delta\underline{u}^*_{T-2}$ from (D.21) into the expression for $\bar{J}_{T-2}$ and regrouping the various terms, we obtain

$$\bar{J}_{T-2}(\delta\bar{\underline{s}}_{T-2/k}) = \frac{1}{2}\ \delta\bar{\underline{s}}_{T-2/k}\ \underline{\underline{K}}_{T-2}\ \delta\bar{\underline{s}}_{T-2/k} + \\ + \underline{k}^T_{T-2}\ \delta\bar{\underline{s}}_{T-2/k} + c_{T-2} \tag{D.25}$$

$$\underline{\underline{K}}_{T-2} = \underline{\underline{N}}_{ssT-2} + \underline{\underline{\phi}}^T_{T-2}\ \underline{\underline{K}}_{T-1}\ \underline{\underline{\phi}}_{T-2} + \\ + \underline{\underline{L}}^T_{T-2}\ \underline{\underline{D}}_{T-2}\ [\underline{\underline{B}}^T_{T-2}\ \underline{\underline{K}}_{T-1}\ \underline{\underline{B}}_{T-2} + \underline{\underline{N}}_{uuT-2}]\underline{\underline{D}}_{T-2}\ \underline{\underline{L}}_{T-2} - \\ - 2\ \underline{\underline{L}}^T_{T-2}\ \underline{\underline{D}}_{T-2}\ [\underline{\underline{B}}^T_{T-2}\ \underline{\underline{K}}_{T-1}\ \underline{\underline{\phi}}_{T-2} + \underline{\underline{N}}_{usT-2}], \tag{D.26}$$

$$\underline{k}_{T-2} = \underline{N}_{sT-2} + \underline{\underline{\phi}}^T_{T-2}\ \underline{k}_{T-1} - \underline{\underline{L}}^T_{T-2}\ \underline{\underline{D}}_{T-2}[\underline{\underline{B}}^T_{T-2}\ \underline{k}_{T-2} + \underline{N}_{uT-2}] + \\ + \underline{\underline{L}}_{T-2}\ \underline{\underline{D}}_{T-2}\ [\underline{\underline{B}}^T_{T-2}\ \underline{\underline{K}}_{T-1}\ \underline{\underline{B}}_{T-2} + \underline{\underline{N}}_{uuT-2}]\ \underline{\underline{D}}_{T-2}\ \ {}_{T-2} - \\ - [\underline{\underline{\phi}}^T_{T-2}\ \underline{\underline{K}}_{T-1}\ \underline{\underline{B}}_{T-2} + \underline{\underline{N}}^T_{usT-2}]\underline{\underline{D}}_{T-2}\ \ \underline{\Lambda}_{T-2}, \tag{D.27}$$

$$c_{T-2} = c_{T-1} - [\underline{k}^T_{T-1}\ \underline{\underline{B}}_{T-2} + \underline{N}^T_{uT-2}]\ \underline{\underline{D}}_{T-2}\ \underline{\Lambda}_{T-2} + \\ + \frac{1}{2}\ \underline{\Lambda}^T_{T-2}\ \underline{\underline{D}}_{T-2}\ [\underline{\underline{B}}^T_{T-2}\ \underline{\underline{K}}_{T-1}\ \underline{\underline{B}}_{T-2} + \underline{\underline{N}}_{uuT-2}]\ \underline{\underline{D}}_{T-2}\ \underline{\Lambda}_{T-2}. \tag{D.28}$$

Lastly, using (D.22), (D.23), and (D.24) we can refine the above expressions as shown below:

As before, we next substitute $\bar{J}_{T-1}(\cdot)$ from (D.11) and $\delta\bar{\underline{s}}_{T-1/k}$ from

$$\delta\bar{\underline{s}}_{T-1/k} = \underline{\underline{\phi}}_{T-2}\ \delta\bar{\underline{s}}_{T-2/k} + \underline{\underline{B}}_{T-2}\ \delta\underline{u}_{T-2}$$

to obtain

$$\underline{\mu}^*_{T-2}(\delta\bar{\underline{s}}_{T-2/k}) = \arg\min_{\delta\underline{u}_{T-2}} \Big\{ \bar{J}_{T-2} = [\underline{k}^T_{T-1}\ \underline{\underline{\phi}}_{T-2} + \underline{N}^T_{sT-2}]\delta\bar{\underline{s}}_{t-2/k} +$$

$$+[\underline{k}^T_{T-1}\ \underline{\underline{B}}_{T-2} + \underline{N}^T_{uT-2}]\ \delta\underline{u}_{T-2} + \frac{1}{2}\delta\bar{\underline{s}}^T_{T-2/k}[\underline{\underline{\phi}}^T_{T-2}\ \underline{\underline{K}}_{T-1}\ \underline{\underline{\phi}}_{T-1} +$$

$$+ \underline{\underline{N}}_{ssT-2}]\delta\bar{\underline{s}}_{T-2/k} + \frac{1}{2}\delta\underline{u}^T_{T-2}[\underline{\underline{B}}^T_{T-2}\ \underline{\underline{K}}_{T-1}\ \underline{\underline{B}}_{T-2} + \underline{\underline{N}}_{uuT-2}]\delta\underline{u}_{T-2} +$$

$$+\delta\underline{u}^T_{T-2}[\underline{\underline{B}}^T_{T-2}\ \underline{\underline{K}}_{T-1}\ \underline{\underline{\phi}}_{T-2} + \underline{\underline{N}}_{usT-2}]\delta\bar{\underline{s}}_{T-2/k} + c_{T-1} \Big\}. \quad (D.19)$$

$\underline{\underline{K}}_{T-1}$ can be seen by Equation (D.15) to be positive semidefinite which guarantees that $\underline{\underline{B}}^T_{T-1}\ \underline{\underline{K}}_{T-1}\ \underline{\underline{B}}_{T-2} + \underline{\underline{N}}_{uuT-2}$ is positive definite and that $\bar{J}_{T-2}$ is convex with respect to $\delta\underline{u}_{T-2}$.

Differentiating $\bar{J}_{T-2}$ and setting the derivative equal to the zero vector we obtain

$$\underline{\underline{B}}^T_{T-2}\ \underline{k}_{T-1} + \underline{N}_{uT-2} + [\underline{\underline{B}}^T_{T-2}\ \underline{\underline{K}}_{ssT-1}\ \underline{\underline{B}}_{T-2} + \underline{\underline{N}}_{uuT-2}]\delta\underline{u}_{T-2} +$$

$$+ [\underline{\underline{B}}^T_{T-2}\ \underline{\underline{K}}_{T-1}\ \ \underline{\underline{\phi}}_{T-2} + \underline{N}_{usT-2}]\ \delta\bar{\underline{s}}_{T-2/k} = \underline{0}, \quad (D.20)$$

$$\text{or}\quad \delta\underline{u}^*_{T-2} = \underline{\mu}^*_{T-2}(\delta\bar{\underline{s}}_{T-2/k}) = -\underline{\underline{D}}_{T-2}[\underline{\underline{L}}_{T-2}\ \delta\bar{\underline{s}}_{T-2/k} + \underline{\Lambda}_{T-2}] \quad (D.21)$$

$$\underline{\underline{K}}_{T-2} = \underline{\underline{N}}_{ssT-2} + \underline{\underline{\phi}}^T_{T-2}\, \underline{\underline{K}}_{T-1}\, \underline{\underline{\phi}}_{T-2} - [\underline{\underline{B}}^T_{T-2}\, \underline{\underline{K}}_{T-1}\, \underline{\underline{\phi}}_{T-2} + \underline{\underline{N}}_{usT-2}]^T$$

$$[\underline{\underline{B}}^T_{T-2}\, \underline{\underline{K}}_{T-1}\, \underline{\underline{B}}_{T-2} + \underline{\underline{N}}_{uuT-2}]^{-1}\, [\underline{\underline{B}}^T_{T-2}\, \underline{\underline{K}}_{T-1}\, \underline{\underline{\phi}}_{T-2} + \underline{\underline{N}}_{usT-2}], \tag{D.29}$$

$$\underline{k}_{T-2} = \underline{N}_{sT-2} + \underline{\underline{\phi}}^T_{T-2}\, \underline{k}_{T-1} - [\underline{\underline{B}}^T_{T-2}\, \underline{\underline{K}}_{T-1}\, \underline{\underline{\phi}}_{T-2} + \underline{\underline{N}}_{usT-2}]^T$$

$$[\underline{\underline{B}}^T_{T-2}\, \underline{\underline{K}}_{T-1}\, \underline{\underline{B}}_{T-2} + \underline{\underline{N}}_{uuT-2}]^{-1}\, [\underline{\underline{B}}^T_{T-2}\, \underline{k}_{T-1} + \underline{N}_{uT-2}] \tag{D.30}$$

$$c_{T-2} = C_{T-1} - \frac{1}{2}[\underline{\underline{B}}^T_{T-2}\, \underline{k}_{T-1} + \underline{N}_{uT-2}]^T [\underline{\underline{B}}^T_{T-2}\, \underline{\underline{K}}_{T-1}\, \underline{\underline{B}}_{T-2} + \underline{\underline{N}}_{uuT-2}]$$

$$\cdot\, [\underline{\underline{B}}^T_{T-2}\, \underline{k}_{T-1} + \underline{N}_{uT-1}]. \tag{D.31}$$

Comparing the expressions for $\delta\underline{u}^*_\ell$, $\underline{\underline{D}}_\ell$, $\underline{\underline{L}}_\ell$, $\underline{\Lambda}_\ell$, $\bar{J}_\ell(\delta\bar{\underline{s}}_{\ell/k})$, $\underline{\underline{K}}_\ell$, $\underline{k}_\ell$, c_ℓ at ℓ = T-1, T-2, we can easily recognize a well defined recursive solution pattern. The induction may be continued by deriving the optimal control law δu^*_{T-3}, but since the form of $\bar{J}_{T-2}(\delta\bar{\underline{s}}_{T-2/k})$ is the same as the form of $\bar{J}_{T-1}(\delta\bar{\underline{s}}_{T-1/k})$, the results will be analogous to the ones obtained at time T-2. Thus, it can be stated that the optinal solution of problem (D.1) is as follows:

For ℓ = k, k+1, ..., T-1,

$$\delta\underline{u}^*_\ell = \underline{\mu}^*(\delta\bar{\underline{s}}_{\ell/k}) = -\underline{\underline{D}}_\ell[L_\ell\, \delta\bar{\underline{s}}_{\ell/k} + \underline{\Lambda}_\ell], \tag{D.32}$$

$$\underline{\underline{D}}_\ell = [\underline{\underline{B}}^T_\ell\, \underline{\underline{K}}_{\ell+1}\, \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell}]^{-1}, \tag{D.33}$$

$$\underline{\underline{L}}_\ell = \underline{\underline{B}}^T_\ell\, \underline{\underline{K}}_{\ell+1}\, \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{us\ell}, \tag{D.34}$$

$$\underline{\Lambda}_\ell = \underline{\underline{B}}^T_\ell\, \underline{k}_{\ell+1} + \underline{N}_{u\ell}, \tag{D.35}$$

where the positive semidefinite matrices $\{\underline{\underline{K}}_\ell\}_{\ell=k}^T$ are obtained recursively by the following matrix Riccati equation:

$$\left.\begin{aligned}
\underline{\underline{K}}_T &= \underline{\underline{N}}_{ssT} \\
\underline{\underline{K}}_\ell &= \underline{\underline{N}}_{ss\ell} + \underline{\underline{\phi}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell - [\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{us\ell}]^T \cdot \\
&\quad \cdot [\underline{\underline{B}}_\ell^T \underline{\underline{k}}_{\ell+1} \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell}]^{-1} [\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{us\ell}] \\
&= T-1, T-2, \ldots, k,
\end{aligned}\right\} \qquad (D.36)$$

and the vectors $\{\underline{k}_\ell\}_{\ell=k}^T$ are obtained from

$$\left.\begin{aligned}
\underline{k}_T &= \underline{N}_{sT} \\
\underline{k}_\ell &= \underline{N}_{s\ell} + \underline{\underline{\phi}}_\ell^T \underline{k}_{\ell+1} - [\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell + N_{us\ell}]^T \cdot \\
&\quad \cdot [\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell}]^{-1} [\underline{\underline{B}}_\ell^T \underline{k}_{\ell+1} + \underline{N}_{u\ell}] \\
\ell &= T-1, T-2, \ldots, k.
\end{aligned}\right\} \qquad (D.37)$$

The optimal "costs to go" are given by

$$\bar{J}_\ell(\delta\bar{\underline{s}}_{\ell/k}) = \frac{1}{2}\, \delta\bar{\underline{s}}_{\ell/k}^T \underline{\underline{K}}_\ell\, \delta\bar{\underline{s}}_{\ell/k} + \underline{k}_\ell^T\, \delta\bar{\underline{s}}_{\ell/k} + c_\ell, \qquad (D.38)$$

where $\underline{\underline{K}}_\ell$, $\underline{k}_\ell$ as above and

$$\left.\begin{aligned}
c_T &= 0, \\
c_\ell &= c_{\ell+1} - \frac{1}{2}[\underline{\underline{B}}_\ell^T \underline{k}_{\ell+1} + \underline{N}_{u\ell}]^T [\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell}]^{-1} [\underline{\underline{B}}_\ell^T \underline{k}_{\ell+1} + \underline{N}_{u\ell}] \\
\ell &= T-1, T-2, \ldots, k.
\end{aligned}\right\} \qquad (D.39)$$

Lastly, the optimal value of the problem is

$$\bar{J}_k(\delta\bar{\underline{s}}_{k/k}) = \frac{1}{2}\,\delta\bar{\underline{s}}^T_{k/k}\,\underline{\underline{K}}_k\,\delta\bar{\underline{s}}_{k/k} + \underline{k}^T_k\,\delta\bar{\underline{s}}_{k/k} + c_k \tag{D.40}$$

which for problem (D.1) is equal to c_k since $\delta\bar{\underline{s}}_{k/k} = \underline{0}$.

Further discussion of this solution's properties and implementation aspects is offered in Section 4.5.

APPENDIX E

ENFORCING POSITIVE DEFINITENESS

The following procedure enforces the positive definiteness property on an nxn symmetric matrix $\underline{\underline{A}}$. This is accomplished by forming the Cholesky factorization of $\underline{\underline{A}}$ (Bertsekas, 1978) while sequentially adding the elements of a positive definits diagonal matrix $\underline{\underline{M}}$. The final results is given as a prduct of a lower triangular and an upper triangular matrix:

$$\underline{\underline{L}}\ \underline{\underline{L}}^T = \underline{\underline{A}} + \underline{\underline{M}} \tag{E.1}$$

The procedure is as follows: Let $\mu > 0$ be a fixed scalar and denot $_{ij}$ the elements of $\underline{\underline{A}}$. Consider the i x i lower triangular matricies $\underline{\underline{L}}_i$ recursively defined by

$$\underline{\underline{L}}_i = \begin{Bmatrix} \sqrt{\alpha_{11}}, & \text{if} \quad \sqrt{\alpha_{11}} > \mu \\ \mu\ , & \text{otherwise} \end{Bmatrix} \tag{E.2}$$

$$\underline{\underline{L}}_i = \begin{pmatrix} \underline{\underline{L}}_{i-1} & 0 \\ \underline{\ell}_i^T & \lambda_{ii} \end{pmatrix}, i = 2, \ldots , n \tag{E.3}$$

$$\text{where} \quad \underline{\ell}_i = [\underline{\underline{L}}_{i-1}]^{-1}\, \alpha_i, \quad \alpha_i = \begin{pmatrix} \alpha_{1i} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{i-1i} \end{pmatrix} \tag{E.4}$$

$$\lambda_{ii} = \begin{cases} \sqrt{\alpha_{ii} - \underline{\ell}_i^T \underline{\ell}_i} & , \text{ if } \sqrt{\alpha_{ii} - \underline{\ell}_i^T \underline{\ell}_i} > \mu \\ \mu & , \text{ otherwise} \end{cases} \quad . \tag{E.5}$$

Then the transformed positive definite matrix is given by $\underline{\underline{L}}_n \underline{\underline{L}}_n^T$. Parameter μ should be large enough so that the resulting matrix is not nearly singular.

APPENDIX F

COMPUTATION OF THE NONBINDING PROJECTED NEWTON DIRECTION

Recall that in Section 4.5 the Newton direction for the unconstrained nonquadratic problem was obtained by solving a quadratic. At each time ℓ that solution was determined via Dynamic Programming considerations (See Appendix D) as follows:

$$\underline{\mu}_\ell^* (\delta\bar{\underline{s}}_{\ell/k}) = \arg\min_{\delta\underline{u}_\ell} \Big\{ \bar{J}_\ell = \underline{N}_{s\ell}^T \, \delta\bar{\underline{s}}_{\ell/k} + \underline{N}_{u\ell}^T \delta\underline{u}_\ell + \frac{1}{2}\delta\bar{\underline{s}}_{\ell/k}^T \underline{\underline{N}}_{ss\ell} \delta\bar{\underline{s}}_{\ell/k} + \frac{1}{2}\,\delta\underline{u}_\ell^T \underline{\underline{N}}_{uu\ell}\delta\underline{u}_\ell + \delta\underline{u}_\ell^T \underline{\underline{N}}_{us\ell}\delta\bar{\underline{s}}_{\ell/k} + \bar{J}_{\ell+1}(\delta\bar{\underline{s}}_{\ell+1/k}) \Big\}, \tag{F.1}$$

of, after substituing for $\bar{J}_{\ell+1}(\delta\bar{\underline{s}}_{\ell+1/k})$ from

$$\bar{J}_{\ell+1}(\delta\bar{\underline{s}}_{\ell+1/k}) = \frac{1}{2}\,\delta\bar{\underline{s}}_{\ell+1/k}^T \underline{\underline{K}}_{\ell+1}\,\delta\bar{\underline{s}}_{\ell+1/k} + \underline{k}_{\ell+1}^T\,\delta\bar{\underline{s}}_{\ell+1/k} + c_{\ell+1} \tag{F.2}$$

and for $\delta\bar{\underline{s}}_{\ell+1/k}$ from

$$\delta\bar{\underline{s}}_{\ell+1/k} = \underline{\underline{\phi}}_\ell \;\delta\bar{\underline{s}}_{\ell/k} + \underline{\underline{B}}_\ell\,\delta\underline{u}_\ell, \tag{F.3}$$

$$\underline{\mu}_\ell^*(\delta\bar{\underline{s}}_{\ell/k}) = \arg\min_{\delta\underline{u}_\ell} \Big\{ \bar{J}_\ell = [\underline{k}_{\ell+1}^T \underline{\underline{\phi}}_\ell + \underline{N}_{s\ell}^T]\delta\underline{s}_{\ell/k} + [\underline{k}_{\ell+1}^T \underline{\underline{B}}_\ell + \underline{N}_{u\ell}^T]\delta u_\ell + \frac{1}{2}\,\delta\bar{\underline{s}}_{\ell/k}^T[\underline{\underline{\phi}}_\ell^T \underline{\underline{K}}_{\ell+1}\underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{ss\ell}]\delta\bar{\underline{s}}_{\ell/k} + \frac{1}{2}\,\delta\underline{u}_\ell^T[\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1}\underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell}]\delta\underline{u}_\ell + \delta\underline{u}_\ell^T[\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1}\underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{us\ell}]\delta\bar{\underline{s}}_{\ell/k} \Big\}. \tag{F.4}$$

Now assume that the nominal value for the jth control component of $\delta \underline{u}_\ell$ is currently equal to the upper control bound $u_{j\ell}^{max}$ and that $\frac{\partial \tilde{J}(\underline{0},\ldots,\underline{0})}{\partial \delta \underline{u}_{j\ell}} < 0$. Under these conditions a "movement" of $u_{j\ell}$ beyond the bound $u_{j\ell}^{max}$ would reduce the objective function's value, but it would be infeasible. Then the optimal constrained value for the control $\delta u_{j\ell}$ is zero and the same is true for any control element at the lower bound with positive Gradient. (It is assumed that $\underline{\underline{K}}_{\ell+1}$, $\underline{\underline{N}}_{uu\ell}$ are positive semidefinite and definite respectively so as $\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell}$ to be positive definite and $\bar{J}_\ell$ convex.)

Substituting these values for the binding control elements we find that $\bar{J}_\ell$ becomes

$$\begin{aligned}\bar{J}_\ell = {} & [\underline{k}_{\ell+1}^T \underline{\underline{\phi}}_\ell + \underline{N}_{s\ell}^T]\delta \bar{\underline{s}}_{\ell/k} + [\underline{k}_{\ell+1}^T \underline{\underline{B}}_\ell + \underline{N}_{u\ell}^T]^c \delta \underline{u}_\ell + \\ & + \frac{1}{2} \delta \bar{\underline{s}}_{\ell/k}^T [\underline{\underline{\phi}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{ss\ell}]\delta \bar{\underline{s}}_{\ell/k} + \\ & + \frac{1}{2} \delta \bar{\underline{u}}_\ell^T [\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell}]^{rc} \delta \bar{\underline{u}}_\ell + \\ & + \delta \bar{\underline{u}}_\ell^T [\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell + \underline{N}_{us\ell}]^r \delta \bar{\underline{s}}_{\ell/k} \qquad (F.5)\end{aligned}$$

where $\delta \bar{\underline{u}}_\ell$ denotes the nonbinding control elements, $[\underline{k}_{\ell+1}^T \underline{\underline{B}}_\ell + \underline{N}_{u\ell}^T]^c$ implies that from the vector $\underline{k}_{\ell+1}^T \underline{\underline{B}}_\ell + \underline{N}_{u\ell}^T$ the columns corresponding to binding controls have been deleted, and the same is true for the respective rows and columns of the matrix $\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell}$ and the rows of the matrix $\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{us\ell}$.

Now we can differentiate (F.5) with respect to $\delta \bar{\underline{u}}_\ell$, see the result equal to zero, and solve the the minimizing control to find

$$\delta \bar{\underline{u}}_\ell^* = \underline{\mu}_\ell^*(\delta \bar{\underline{s}}_{\ell/k}) = -\underline{\underline{D}}_\ell [\underline{\underline{L}}_\ell \, \delta \bar{\underline{s}}_{\ell/k} + \underline{\Lambda}_\ell] \tag{F.6}$$

where

$$\underline{\underline{D}}_\ell = [(\underline{\underline{B}}_\ell^T \, \underline{\underline{K}}_{\ell+1} \, \underline{\underline{B}}_\ell \quad \underline{\underline{N}}_{us\ell})^{rc}]^{-1}, \tag{F.7}$$

$$\underline{\underline{L}}_\ell = (\underline{\underline{B}}_\ell^T \, \underline{\underline{K}}_{\ell+1} \, \underline{\underline{\phi}}_\ell \quad + \underline{\underline{N}}_{us\ell})^r, \tag{F.8}$$

$$\underline{\Lambda}_\ell = (\underline{\underline{B}}_\ell^T \quad \underline{k}_{\ell+1} + \underline{N}_{u\ell})^r. \tag{F.9}$$

Substituting this result into (F.5) and regrouping the corresponding terms we obtain

$$\bar{J}_\ell(\delta \bar{\underline{s}}_{\ell/k}) = \frac{1}{2} \, \delta \bar{\underline{s}}_{\ell/k}^T \, \underline{\underline{K}}_\ell \, \delta \bar{\underline{s}}_{\ell/k} + \underline{k}_\ell^T \, \delta \bar{\underline{s}}_{\ell/k} + c_\ell \tag{F.10}$$

where

$$\begin{aligned} \underline{\underline{K}}_\ell = {} & \underline{\underline{N}}_{ss\ell} + \underline{\underline{\phi}}_\ell^T \, \underline{\underline{K}}_{\ell+1} \, \underline{\underline{\phi}}_\ell + \\ & + \underline{\underline{L}}_\ell^T \, \underline{\underline{D}}_\ell [\underline{\underline{B}}_\ell^T \quad \underline{\underline{K}}_{\ell+1} \, \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell}]^{rc} \, \underline{\underline{D}}_\ell \, \underline{\underline{L}}_\ell - \\ & - 2 \, \underline{\underline{L}}_\ell^T \quad \underline{\underline{D}}_\ell [\underline{\underline{B}}_\ell^T \, \underline{\underline{K}}_{\ell+1} \, \underline{\underline{\phi}}_\ell \quad + \underline{\underline{N}}_{us} \,]^r, \end{aligned} \tag{F.11}$$

$$\begin{aligned} \underline{k}_\ell = {} & \underline{N}_{s\ell} + \underline{\underline{\phi}}_\ell^T \, \underline{k}_{\ell+1} - \underline{\underline{L}}_\ell^T \, D_\ell [\underline{\underline{B}}_\ell^T \, \underline{k}_{\ell+1} + \underline{N}_{u\ell}]^r + \\ & + \underline{\underline{L}}_\ell^T \quad \underline{\underline{D}}_\ell \quad [\underline{\underline{B}}_\ell^T \, \underline{\underline{K}}_{\ell+1} \, \underline{\underline{B}}_\ell + \underline{N}_{uu\ell}]^{rc} \, \underline{\underline{D}}_\ell \underline{\Lambda}_\ell - \\ & - [\underline{\underline{\phi}}^T \, \underline{\underline{K}}_{\ell+1} \, \underline{\underline{B}}_\ell \quad + \underline{\underline{N}}_{us\ell}^T]^c \, \underline{\underline{D}}_\ell \underline{\Lambda}_\ell \quad , \end{aligned} \tag{F.12}$$

$$\begin{aligned} c_\ell = {} & c_{\ell+1} - [\underline{k}_{\ell+1}^T \, \underline{\underline{B}}_\ell \quad + \underline{N}_{u\ell}^T]^c \, \underline{\underline{D}}_\ell \underline{\Lambda}_\ell \quad + \\ & + \frac{1}{2} \, \underline{\Lambda}_\ell^T \, \underline{\underline{D}}_\ell [\underline{\underline{B}}_\ell^T \quad \underline{\underline{K}}_{\ell+1} \, \underline{\underline{B}}_\ell \quad + \underline{\underline{N}}_{uu\ell}]^{rc} \, \underline{\underline{D}}_\ell \underline{\Lambda}_\ell . \end{aligned} \tag{F.13}$$

Using (F.7), (F.8), and (F.9), the above can be simplified as shown below:

$$\underline{\underline{K}}_\ell = \underline{\underline{N}}_{ss\ell} + \underline{\underline{\phi}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell - [(\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{us\ell})^r]^T \cdot$$

$$\cdot [(\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell})^{rc}]^{-1} [\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell + \underline{\underline{N}}_{us\ell}]^r. \tag{F.14}$$

$$\underline{k}_\ell = \underline{N}_{s\ell} + \underline{\underline{\phi}}_\ell^T \underline{k}_{\ell+1} - [(\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{\phi}}_\ell N_{us\ell})^r]^T \cdot$$

$$\cdot [(\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell})^{rc}]^{-1} [\underline{\underline{B}}_\ell^T \underline{\underline{k}}_{\ell+1} \underline{\underline{N}}_{u\ell}]^r, \tag{F.15}$$

$$c_\ell = c_{\ell+1} - \frac{1}{2}[(\underline{\underline{B}}_\ell^T \underline{k}_{\ell+1} + \underline{N}_{u\ell})^r]^T [(\underline{\underline{B}}_\ell^T \underline{\underline{K}}_{\ell+1} \underline{\underline{B}}_\ell + \underline{\underline{N}}_{uu\ell})^{rc}]^{-1} \cdot$$

$$\cdot [(\underline{\underline{B}}_\ell \underline{k}_{\ell+1} + \underline{N}_{u\ell})^r]. \tag{F.16}$$

These equations have the same form as the ones derived for the unconstrained problem in Appendix D except that now they are applicable to the nonbinding controls. In the event that no controls are binding, they assume the full form of the unconstrained case.

APPENDIX G

STATE STATISTICS' UPDATE

The problem of updating the estimate of the state's p.d.f. arises as follows: Suppose that the optimal control action $\underline{u}_k^*$ is applied during the interval $[t_k, t_{k+1}]$. Also let $\bar{\underline{s}}_{k/k}$, $\underline{\underline{P}}_{sk/k}$ denote the best estimates of the state's mean vector and covariance matrix based on observations up to and including time t_k. (That is, these are estimates based on the information set $I_k = \{\underline{z}_1, \ldots, \underline{z}_k, \underline{u}_o^*, \ldots, \underline{u}_{k-I}^*\}$.) Then, by means of the system dynamics we can obtain the a priori (prior to observation $\underline{z}_{k+1}$) estimates of the state statistics at time t_{k+1}. In fact, these have already been obtained at the last iteration of the OLF procedure. If $\underline{s}^{nom}(t_{k+1})$ is the nominal state trajectory (see Section 4.3) corresponding to the optimal control sequence and $\underline{\underline{\phi}}_k$ is the associated state transition matrix, then

$$\bar{\underline{s}}_{k+1/k} = \underline{s}^{nom}(t_{k+1})$$
$$\underline{\underline{P}}_{sk+1/k} = \underline{\underline{\phi}}_k \underline{\underline{P}}_{sk/k} \underline{\underline{\phi}}_k^T + \underline{\underline{Q}}_{\varepsilon k}. \tag{G.1}$$

At time t_{k+1} measurement $\underline{z}_{k+1}$ is taken. The question is how to obtain optimal a posteriori estimates $\bar{\underline{s}}_{k+1/k+1}$, $\underline{\underline{P}}_{sk+1/k+1}$ based on the new information set $I_{k+1} + I_k \cup \{\underline{z}_{k+1}, \underline{u}_k^*\}$. To this end one can employ the Extended Kalman Filter (EKF) procedure which basically consists of applying the Kalman Filter results on a linearized version of the

nonlinear problem. In our case the EKF procedure is as follows:

$$\bar{\underline{s}}_{k+1/k+1} = \bar{\underline{s}}_{k+1/k} + \underline{\underline{\Pi}}_{k+1}[\underline{z}_{k+1} - \underline{H}_0(\bar{\underline{s}}_{k+1/k})], \tag{G.2}$$

$$\underline{\underline{P}}_{sk+1/k+1} = [\underline{\underline{I}} - \underline{\underline{\Pi}}_{k+1}\,\underline{\underline{H}}_1(\bar{\underline{s}}_{k+1/k})]\,\underline{\underline{P}}_{sk+1/k}, \tag{G.3}$$

$$\underline{\underline{\Pi}}_{k+1} = \underline{\underline{P}}_{sk+1/k}\,\underline{\underline{H}}_1^T(\bar{\underline{s}}_{k+1/k})[\underline{\underline{H}}_1(\bar{\underline{s}}_{k+1/k})\underline{\underline{P}}_{sk+1/k}\,\underline{\underline{H}}_1^T(\bar{\underline{s}}_{k+1/k})+\underline{\underline{R}}_{vk+1}]^{-1}. \tag{G.4}$$

In the above equations the vector $\underline{H}_0(\cdot)$ and the matrix $\underline{\underline{H}}_1(\cdot)$ result from the linearization of the state observation function $\underline{H}(\cdot)$ about the a priori state estimate (compare with Equations (4.3.2), (4.3.9)). $\underline{\underline{I}}$ is the identify $n_s \times n_s$ matrix. If the system was linear, the above procedure would yield unbiased, minimum variance estimates (Jazwinski, 1970). In the nonlinear case these properties are no longer valid (unbiasness may be retained by employing statistical linearization) yet in practice the resulting estimates have been found to adequately perform.

APPENDIX H

HISTORICAL MONTHLY STREAMFLOW AT WADI HALFA (10**6 CUBIC METERS)

YEAR	JAN	FEB	MAR	APR	MAY	JUN	JUL	AUG	SEP	OCT	NOV	DEC
1912	3880	2540	1920	1450	1230	1080	5700	20800	19600	11100	6100	4580
1913	3650	2310	1570	1430	1360	1600	2230	7600	13400	7860	4140	3020
1914	2100	1420	1290	1110	1180	1140	2700	22900	22300	15000	10500	6260
1915	4190	2930	2050	1340	1290	1600	3040	12300	16400	15000	7840	4910
1916	3510	2170	1500	1140	1130	1320	6100	27100	31700	25200	12200	7060
1917	5340	4130	3790	2410	1760	2080	5930	19600	31530	24200	11600	6840
1918	5350	4350	4510	4540	4340	3460	4300	15600	18800	11900	6670	5010
1919	3700	2370	2070	1620	1470	1680	5070	19200	22400	13000	6260	4500
1920	3430	2050	1590	1310	1220	1970	6230	20700	19100	15800	8710	5180
1921	3800	2180	1470	1300	1120	1380	3300	17900	21500	15200	7140	4580
1922	3420	1960	1380	1050	980	1000	3820	19300	25500	17000	8130	5060
1923	3650	1970	1260	1130	1520	2370	5700	22500	23100	16700	6970	4790
1924	3730	2400	1550	1240	1400	1550	5730	19600	23300	14500	7600	5290
1925	3760	2470	1820	1370	1290	1630	4440	15700	17900	12800	5940	4090
1926	3390	2230	1820	1690	1760	2700	4960	21600	23100	14900	7500	4810
1927	3850	2740	1960	1590	1570	1500	4750	18300	20300	13700	5510	3670
1928	2590	1690	1460	1330	1720	2710	7520	19900	21000	12200	6610	4350
1929	3480	2230	1810	1430	1790	3770	10000	25400	25800	18700	9320	5210
1930	4210	2870	2210	1730	1740	1910	5590	19700	18500	11500	5230	3660
1931	2720	1640	1470	1260	1340	1200	3370	16900	21600	14800	7920	4420
1932	3440	2010	1670	1360	1390	1980	5070	20200	23400	15600	7300	4680
1933	4200	3387	2730	1870	1760	1930	3310	14400	22800	15100	8690	5660
1934	4190	2710	2010	1670	1660	1820	7600	22100	24400	16500	7400	5020
1935	4200	2450	2140	1730	1440	2680	8000	22900	24200	17200	7520	4950
1936	3950	2640	2230	1771	1730	1800	6230	20600	25600	15600	6890	4200
1937	3170	1900	1650	1450	1400	1800	4840	21600	23800	13500	5790	4310
1938	3350	2090	2210	1670	1470	1600	4520	23800	27000	19800	8650	5260
1939	4120	3160	2340	2140	2220	2170	4330	14000	18400	12400	7560	4360
1940	3160	2080	2500	1830	1350	1550	3430	16200	18200	9790	4290	2990
1941	2040	1620	2150	1470	1150	1930	4420	13300	15100	11100	7430	3990
1942	2990	2080	2540	2520	1810	1920	5450	22400	21000	14500	5520	3810
1943	3030	1850	2500	2220	1610	1560	3340	17000	24300	12900	6090	3530
1944	2670	1700	2570	2450	1800	2130	5190	18300	18500	12130	5040	3640
1945	2610	1880	2750	2240	1900	2010	3890	16000	19600	15900	8000	4690
1946	3700	2290	2510	2470	2130	1610	7320	26700	27700	14600	9250	5200
1947	3990	3040	3060	2600	2860	2920	5490	16400	22900	13900	5760	4060
1948	3410	2680	2500	2500	2200	1830	5860	17500	19200	15500	8690	4560
1949	3490	2590	2410	2520	2550	2010	6430	18500	20200	13200	6960	4380
1950	3750	2720	2540	2510	2720	2270	5430	22400	23300	14100	5910	3730
1951	3110	1890	2320	2420	1990	1300	3400	16500	18200	11900	7610	4420
1952	1060	2320	2760	2251	1590	1490	3930	15790	20450	11400	5300	3240
1953	2490	1830	2320	2420	1510	1740	5200	23000	20400	12700	5430	3620
1954	2560	1790	2550	2350	1650	1440	6970	25200	27600	19000	7370	4690
1955	3520	2580	2150	2440	2530	2390	5030	19900	22900	17200	6400	4090
1956	3410	2730	2470	2590	2720	2630	6060	20100	20830	17900	11100	5260
1957	3750	3020	2420	2920	3070	3390	4770	18600	19500	8530	4360	3020
1958	2320	1840	2170	2400	1770	1610	5920	25200	22700	15200	71[illegible]	4540
1959	3220	2310	2280	2540	2270	1860	5970	19500	27800	14800	8470	4350
1960	3120	2220	2230	2500	2500	1900	4480	17400	20400	13500	5370	3140
1961	2550	1730	2310	2600	2030	1510	6550	23200	26800	17400	7560	4920
1962	3670	2760	2590	2970	2980	3220	5000	17400	22600	15500	5830	4020
1963	3350	2820	2560	2520	3250	3590	6990	22100	19700	10600	6880	4560
1964	4300	3240	2860	2700	3630	4100	7920	24900	25700	16400	9450	5850
1965	5750	5080	4390	3550	4170	4520	5920	14800	15400	11000	6600	4910

Appendix I

IDENTIFICATION OF A SINGLE RESERVOIR MODEL STRUCTURE

The physically hypothesized structure of a single reservoir was seen in Section 3.3 to be (omitting all subscripts for notational convenience):

$$\frac{ds(t)}{dt} = I(t) - u(t) - e(t)\, A(s(t)) + g\, E(s(t)) \qquad (I.1)$$

(cf. Eq. (3.3.1) , (3.3.2)). Further, it will be assumed that observations of $I(t)$, $u(t)$, $E(s(t))$ are available at the discrete time instants t_ℓ, $\ell = 0, \ldots N$. The identification problem is to estimate the parameters $e(t)$, g so that the model (I.1) will generate input $(I(t), u(t))$ - output $(E(s(t)))$ sequences compatible with the previous observations.

The purpose of this appendix is to resolve the issue of parameter identifiability. The suggested approach is to follow an outline showing how parameter estimation can be practically performed up to the point where some conclusions regarding the previous problem can be drawn.

Assume that the release $u(t)$ is constant within the observation intervals; namely

$$\left\{u(t) = u(t_\ell), \quad t\varepsilon[t_\ell, t_{\ell+1}], \ \ell = 0, \ldots, N-1\right\} \qquad (I.2)$$

which is realistic because reservoir releases are not continuously adjusted. Also, the input $I(t)$ will be considered known from the corresponding measurements and the approximate relationship

$$I(t) = I(t_\ell) + \frac{I(t_{\ell+1}) - I(t_\ell)}{(t_{\ell+1} - t_\ell)} (t-t_\ell) \quad t\varepsilon[t_\ell, t_{\ell+1}] \quad , \qquad (I.3)$$

$$\ell = 0, \ldots, N-1$$

The possible errors in the $\left\{I(t_\ell)\right\}_{\ell=0}^{N}$ measurements will be assigned to a random term added at a later stage (see Schweppe, 1973, Section 3.2.15, 2nd case, for a justification of this assumption). The approximation (I.3) will, in general, suffice because the identification phases' purpose is to determine a valid model useable in discrete time optimal control (this will become clearer after the developments in Chapter 4). In fact, (I.3) may be replaced by a relationship similar to (I.2) and still be adequate.

To take advantage of parameter estimation techniques developed for linear systems, Eq. (I.1) can be linearized around the state trajectory $\bar{s}(t)$, $t\varepsilon[t_0, t_N]$ obtained by integrating

$$\frac{d\bar{s}(t)}{dt} = I(t) - u(t) - e(t)\ A(\bar{s}(t)) - g\ E\ (\bar{s}(t)) \tag{I.4}$$

with initial condition $\bar{s}(t_0) = s(t_0)$ corresponding to the measurement $E(s(t_0))$. (Some numerical integration routine can generally be employed.) To carry out the above integration, parameters $e(t)$, g are set equal to their current estimates. (The procedure is iterative.) Thus, (I.1) can be locally approximated by

$$\frac{ds(t)}{dt} = I(t) - u(t) - e(t)\ A(\bar{s}(t)) - e(t) \left[\frac{\partial A(s(t))}{\partial s(t)}\right]_{s(t)=\bar{s}(t)} (s(t) - \bar{s}(t))$$

$$- g\ E(\bar{s}(t)) - g \left[\frac{\partial E(s(t))}{\partial s(t)}\right]_{s(t)=\bar{s}(t)} s(t) - \bar{s}(t)) \tag{I.5}$$

which after invoking (I.4) and defining

$$x(t) = s(t) - \bar{s}(t) \tag{I.6}$$

becomes

$$\frac{dx(t)}{dt} = (a_1(t) + a_2(t))\ x(t) \tag{I.7}$$

where $$a_1(t) = -e(t) \left[\frac{\partial A(s(t))}{\partial s(t)}\right]_{s(t)=\bar{s}(t)} , \tag{I.8}$$

$$a_2(t) = -g \left[\frac{\partial E(s(t))}{\partial s(t)}\right]_{s(t)=\bar{s}(t)} . \tag{I.9}$$

The observation equation corresponding to the water elevation measurements

$$H(t_\ell) = E(s(t_\ell)) \tag{I.10}$$

can also be linearized around $\bar{s}(t_\ell)$ to give

$$H(t_\ell) = E(\bar{s}(t_\ell)) + \left[\frac{\partial E(s(t))}{\partial s(t)}\right]_{s(t)=\bar{s}(t_\ell)} (s(t_\ell)-\bar{s}(t_\ell)). \tag{I.11}$$

Defining

$$z(t_\ell) = H(t_\ell) - E(\bar{s}(t_\ell)), \tag{I.12}$$

$$a_3(t_\ell) = \left[\frac{\partial E(s(t))}{\partial s(t)}\right]_{s(t)=\bar{s}(t_\ell)} , \tag{I.13}$$

Eq. (I.11) can be equivalently expressed as follows:

$$z(t_\ell) = a_3(t_\ell)\, x(t_\ell) . \tag{I.14}$$

Next, it will be assumed that the coefficients $a_1(t)$ and $a_2(t)$ are approximately constant over the intervals $[t_\ell , t_{\ell+1}]$. This assumption, made here to facilitate the presentation, can also be argued on the basis of the following considerations: Generally, the water volume added or subtracted from a reservoir by the releases, input process, evaporation, seepage, and rainfall over the intervals $[t_\ell, t_{\ell+1}]$, $\ell = 0, \ldots, N-1$, is a small fraction of the volume $s(t_\ell)$ in store. Hence,

when $[t_\ell, t_{\ell+1}]$ is on the order of a few days, the derivations of the functions $A(s(t))$, $E(s(t))$ evaluated at $\bar{s}(t)$ will be approximately constant. Secondly, the parameter $e(t)$ is a lumped parameter depending on climatic conditions and can be considered constant over a season (and therefore constant over $[t_\ell, t_{\ell+1}]$). Setting

$$\left.\begin{aligned} a_1(t) &= a_1(t_\ell) \\ a_2(t) &= a_2(t_\ell) \\ t &\in [t_\ell, t_{\ell+1}), \qquad \ell = 0, 1, \ldots N-1 \end{aligned}\right\} \tag{I.15}$$

we can integrate (I.7) over $[t_\ell, t_{\ell+1}]$ to get

$$x(t_{\ell+1}) = e^{(a_1(t_\ell) + a_2(t_\ell))\Delta} x(t_\ell) \tag{I.16}$$

where $\Delta = t_{\ell+1} - t_\ell$. At this point we can add a discrete time white Gaussian noise term $\varepsilon(t_\ell)$, $\ell = 1, \ldots N$, to account for the input measurement errors and various approximations' errors (i.e., linearizations, modeling assumptions, etc.):

$$x(t_{\ell+1}) = \phi(t_\ell)\, x(t_\ell) + \varepsilon(t_\ell) \tag{I.17}$$

where $\varepsilon(t_\ell)$ is Gaussian with

$$E\left\{\varepsilon(t_\ell)\right\} = 0 \quad , \text{ (E\{·\} denotes expectation)} \tag{I.18}$$

$$E\left\{\varepsilon(t_\ell)\, \varepsilon(t_m)\right\} = \left\{\begin{matrix} 0 & \text{if } \ell \neq m \\ Q_\varepsilon(t_\ell), & \text{if } \ell = m \end{matrix}\right\} , \tag{I.19}$$

and $$\phi(t_\ell) = e^{(a_1(t_\ell) + a_2(t_\ell))\Delta} . \tag{I.20}$$

Variance $Q_\varepsilon(t_\ell)$ is another unknown parameter to be estimated and can be assumed constant or periodically constant (with a period of one year to account for nonuniform yearly variability of the evaporation and rainfall processes).

A discrete zero mean Gaussian noise process of small known variance may also be added to the observation equation to account for the approximations in $E(s(t))$ and its linearized version.

Summarizing, the hypothesized structure for an actual reservoir is as follows:

$$\left.\begin{array}{l}
x(t_{\ell+1}) = \phi(t_\ell)\, x(t_\ell) + \varepsilon(t_\ell) \\
z(t_\ell) = a_3(t_\ell)\, x(t_\ell) + v(t_\ell) \\
\ell = 0, 1, \ldots, N-1 \\
\phi(t_\ell) = e^{(a_1(t_\ell) + a_2(t_\ell))\,\Delta} \\
a_1(t_\ell),\ a_2(t_\ell):\ \text{unknown} \\
a_3(t_\ell):\ \text{known} \\
\varepsilon(t_\ell),\ v(t_\ell)\ \text{Gaussian random variables with} \\
E\{\varepsilon(t_\ell)\} = 0,\ \text{all } \ell \\
E\{\varepsilon(t_\ell)\varepsilon(t_m)\} = \begin{cases} 0, & \ell \neq m \\ Q_\varepsilon(t_\ell), & \ell = m \quad \text{(unknown)} \end{cases} \\
E\{v(t_\ell)\} = 0, \quad \text{all } \ell \\
E\{v(t_\ell)\ (t_m)\} \quad \begin{cases} 0, & \ell \neq m \\ R, & \ell = m \quad \text{(known)} \end{cases} \\
E\{v(t_\ell)\ \varepsilon(t_m)\} = 0, \quad \text{all } \ell, m.
\end{array}\right\} \qquad (I.21)$$

The initial state condition $x(t_0)$ may also be assumed as an unknown parameter to be estimated, or since a measurement $z(t_0)$ is available, it can be taken as a zero mean Gaussian random variable with $E\{x^2(t_0)\} = \frac{R}{a_3^2(t_0)}$. In the second case, $x(t_0)$ should be independent at $\varepsilon(t_\ell)$ and $v(t_\ell)$ for all ℓ.

Model structure (I.21) is now in a format which allows straightforward application of parameter estimation techniques such as the Maximum Likelihood or the Extended Kalman Filter. For example, the Maximum Likelihood parameter estimates are obtained from the solution of the following optimization problem:

$$\left.\begin{aligned}
&\underset{a_1(t_\ell),\ a_2(t_\ell),\ Q_\varepsilon(t_\ell)}{\text{Maximize}} \left\{\xi(N) = \frac{1}{2}\left[\xi_{bias}(N) + \xi_{obs}(N)\right]\right\} \qquad \ell = 0, 1, \ldots, N \\
&\text{where} \\
&\xi_{bias}(N) = -N\ln(2\pi) - \sum_{\ell=1}^{N} \ln\left|P_z(t_\ell/t_{\ell-1})\right| , \\
&\xi_{obs}(N) = -\sum_{\ell=1}^{N} \frac{\left[z(t_\ell) - a_3(t_\ell)\,\phi(t_{\ell-1})\,\hat{x}(t_{\ell-1}/t_{\ell-1})\right]^2}{P_z(t_\ell/t_{\ell-1})} \\
&P_z(t_\ell/t_{\ell-1}) = R + a_3^2(t_\ell)\left[\phi^2(t_{\ell-1})P_x(t_{\ell-1}/t_{\ell-1}) + Q_\varepsilon(t_{\ell-1})\right] , \\
&\hat{x}(t_\ell/t_\ell) = \phi(t_{\ell-1})\,\hat{x}(t_{\ell-1}/t_{\ell-1}) + \\
&+ \frac{\left[\phi^2(t_{\ell-1})P_x(t_{\ell-1}/t_{\ell-1}) + Q_\varepsilon(t_{\ell-1})\right]a_3(t_\ell)\left[z(t_\ell) - a_3(t_\ell)\phi(t_{\ell-1}\hat{x}(t_{\ell-1}/t_{\ell-1})\right]}{P_z(t_\ell/t_{\ell-1})} \\
&P_x(t_\ell/t_\ell) = \left[\phi^2(t_{\ell-1})P_x(t_{\ell-1}/t_{\ell-1})Q_\varepsilon(t_{\ell-1})\right]\frac{R}{P_z(t_\ell/t_{\ell-1})} , \\
&\ell = 1, \ldots M, \\
&\hat{x}(t_0/t_0 = E\left\{x(t_0)\right\} = 0 , \\
&P_x(t_0/t_0) = E\left\{x^2(t_0)\right\} = \frac{R}{a_3^2(t_0)}.
\end{aligned}\right\} \qquad \text{(I.22)}$$

This formulation will not be further explained here. For a complete discussion on the subject see Schweppe, 1973, Chapter 14. Notice, however, that due to the way the unknown parameters $a_1(t_\ell)$ and $a_2(t_\ell)$ enter the above optimization problem (I.22) it is not possible to identify each one individually. Rather, only their sum $a_1(t_\ell) + a_2(t_\ell)$ can be estimated. Given this situation, one can either assign some reasonable value to parameter g and account for any discrepancies by estimating e(t) or assume a sequence for e(t) (yearly periodic) from existing (even though questionable spatial) measurements and assign possible errors to the estimate of g. The choice also depends on the magnitude of the particular terms. For example, g may be completely disregarded in case the main losses are due to evaporation. (This happens with the High Aswan Dam reservoir as discussed in Chapter 5). Another possibility is to consider e(t) as a white Gaussian noise of known mean and unknown variance, but this complicates the analysis and is not suggested unless the previous alternatives fail.

It is generally advantageous to have the identified model in continuous time because then the controller can be adjusted to any control hierarchy level by discretization over the appropriate time intervals. Toward this end we need to define a continuous time white Gaussian noise which produces statistical effects consistent with the discrete time process $\varepsilon(t_\ell)$. If we denote this new process by q(t) there holds (Gelb, 1974)

$$\varepsilon(t_\ell) = \int_{t_\ell}^{t_{\ell+1}} \phi(t_{\ell+1}, \tau)\, q(\tau)\, d\tau \tag{I.23}$$

where

$$\phi(t_{\ell+1}, \tau) = e^{[a_1(t_{\ell+1}) + a_2(t_{\ell+1})](t_{\ell+1} - \tau)} \tag{I.24}$$

Assuming that $q(\tau)$ has constant mean $\bar{q}(\tau)$, $\tau\varepsilon[t_{\ell+1},t_\ell]$, we can take expectations in (I.23) to obtain:

$$0 = \bar{q}(\tau) \int_{t_\ell}^{t_{\ell+1}} \phi(t_{\ell+1},\tau)d\tau \tag{I.25}$$

Since $\phi(t_{\ell+1},\ \tau)$ is not of any special form (e.g. antisymmetric), it follows that $\bar{q}(\tau) = 0$. Regarding $q(\tau)$'s spectral density ($E\{q(\tau)q(\rho)\} = Q_q(\tau)\delta(\tau-\rho)$ where $\delta(\tau-\rho)$ is the dirac delta function), we again have from (I.23):

$$Q_\varepsilon(t_\ell) = E\left\{\left[\int_{t_\ell}^{t_{\ell+1}} \phi(t_{\ell+1}),\tau)q(\tau)d\tau\right]^2\right\} =$$

$$= \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^{t_{\ell+1}} \phi(t_{\ell+1},\tau)E\{q(\tau)q(\rho)\}\ \phi(t_{\ell+1},\rho)d\rho\, d\tau \tag{I.26}$$

Using the shifting property of the dirac delta function, we get

$$Q_\varepsilon(t_\ell) = \int_{t_\ell}^{t_{\ell+1}} \phi^2(t_{\ell+1},\tau)\ Q_q(\tau)d\tau \tag{I.27}$$

and taking $Q_q(\tau)$ constant over $[t_\ell,t_{\ell+1}]$, we finally find

$$Q_q(\tau) = \frac{Q_\varepsilon(t_\ell)}{\int_{t_\ell}^{t_{\ell+1}} \phi^2(t_{\ell+1},\tau)d\tau}\ ,\quad \tau\varepsilon[t_\ell,\ t_{\ell+1}]. \tag{I.28}$$

Thus, with the unknown parameters estimated, the single reservoir model is given by

$$\frac{ds(t)}{dt} = I(t) - u(t) - e(t)A(s(t)) - g\ E(s(t)) + q(t) \tag{I.29}$$

with $q(t)$ as above.

Appendix J

IDENTIFICATION OF A RIVER SEGMENT MODEL STRUCTURE

For reference ease let the system of differential Eq. (3.3.5) describing the response of the jth river segment be denoted by

$$\dot{\underline{s}}_j(t) = \underline{F}(\underline{s}_j(t)) + \underline{e}_1 I_{j1}(t) - \underline{e}_\ell w_{j\ell}(t) \tag{J.1}$$

where $\underline{F}(\cdot)$ is the r_j vector function of the special form shown in (3.3.5), $\underline{e}_k$ $k=1,\ell$, is an r_j vector of zeros with a 1 at the kth row, and $\dot{\underline{s}}_j(t)$ denotes the time derivative of $\underline{s}_j(t)$.

Measurements of only the segment's inputs $I_{j\ell}(t)$, $w_{j\ell}(t)$ and final outflow $Q_{jr_j}(t_\ell)$ are available at the discrete time instants t_ℓ, $\ell=0, 1, \ldots, N$:

$$Q_{jr_j}(t_\ell) = \alpha_{jr_j} s_{jr_j}^{m_{jr_j}}(t_\ell). \tag{J.2}$$

This appendix will discuss the estimation of the previous model's parameters. Some identifiability issues will be examined and will necessitate changes in the original model structure. The resulting model will be brought into a format directly amenable to the basic estimation techniques.

Consider a normal trajectory of the conceptual storage state vector $\bar{\underline{s}}_j(t)$ such that $(I_{j1}(t), w_{j\ell}(t)$ are assumed known as in Appendix I, Eq. (I.3), (I.2))

$$\dot{\bar{\underline{s}}}_j(t) = \underline{F}[\bar{\underline{s}}_j(t)] + \underline{e}_1 I_{j1}(t) - \underline{e}_\ell w_{j\ell}(t) \quad . \tag{J.3}$$

Next, linearize the system (J.1) around $\bar{\underline{s}}_j(t)$. If we define

$$\underline{x}_j(t) = \underline{s}_j(t) - \bar{\underline{s}}_j(t) \quad , \tag{J.4}$$

we easily get to the following result:

$$\left.\begin{aligned}
\dot{x}_{j1}(t) &= -\left[\lambda_{j1}(t) + \mu_{j1}(t)\right] x_{j1}(t) \\
\dot{x}_{j2}(t) &= \lambda_{j1}(t)\, x_{j1}(t) - \left[\lambda_{j2}(t) + \mu_{j2}(t)\right] x_{j2}(t) \\
&\vdots \\
\dot{x}_{j\ell}(t) &= \lambda_{j\ell-1}(t)\, x_{j\ell-1}(t) - \left[\lambda_{j\ell}(t) + \mu_{j\ell}(t)\right] x_{j\ell}(t) \\
&\vdots \\
\dot{x}_{jr_j}(t) &= \lambda_{jr_j-1}(t)\, x_{jr_j-1}(t) - \left[\lambda_{jr_j}(t) + \mu_{jr_j}(t)\right] x_{jr_j}(t)
\end{aligned}\right\} \quad (J.5)$$

where

$$\lambda_{ji}(t) = \alpha_{ji} m_{ji}\, \bar{s}_{ji}^{\,m_{ji-1}}(t) \quad , \quad \mu_{ji}(t) = K_{ji}\, \beta_{ji}\, \bar{s}_{ji}^{\,\beta_{ji-1}}(t)$$

Also

$$z_j(n) = Q_{jr_j}(n) - \alpha_{jr_j} \bar{s}_{jr_j}^{\,m_{jr_j}}(t_n) = \lambda_{jr_j}(n)\, x_{jr_j}(n) \quad . \qquad (J.6)$$

The previous equations specify a very general system model. The question is whether the parameters are identifiable. This is a crucial issue and should be resolved before a final structure is suggested for identification. Appendix K deals with this problem for a time invariant version of the above system. Assuming that the coefficients in (J.5) are slowly varying (which is a good approximation if $\bar{s}_j(t)$ corresponds to the uniform channel flow condition), the results carry over to the general case. The conclusion of that analysis (which can be used for any linear time invariant system) is that $\left\{\lambda_i, \mu_i\right\}_{i=1}^{r_j}$ can be identified only if the initial conditions $\underline{x}_j(t_0)$ are known. If $\underline{x}_j(t_0)$ is not known, only $\left\{\lambda_i + \mu_i\right\}_{i=1}^{r_j}$ can be estimated. In our case $\underline{x}_j(t_0)$ is not known, and this necessitates some more assumptions. Instead, consider the seepage losses known. Set $\beta_{ji} = 1$ for all i since this is the middle point of its

variation. Furthermore, by observing the inputs $(I_{j1}(t), w_{j\ell}(t))$ and the output $Q_{jr_j}(t)$ for a long enough time interval $[t_1, t_2]$ (longer than any system time constant) and comparing the total incoming volume of water with the outflow, we can have an estimate of the cumulative losses. Assuming the same $K_{ji} = K_j$ for all i the same segment, we can for each trajectory of linearization specify a K_j which will account for the above difference. In this way K_j can be assumed known, which makes the coefficients $\mu_{ji}(t)$ known as well. Computational experience with the coefficient m_{ji} shows (Georgakakos, K. and Bras, R., 1980, Restrepo Posada, P. and Bras, R., 1982) that for short river segments a value of 0.8 performs satisfactorily even in flash-flood conditions. In times of low flows, long reaches, or when the response of interest concerns average behavior over significant time intervals, it can be assumed that the river behaves linearly or that $m_{ji} = 1$ for all i.

Notice that if $m_{ji} = 1 = \beta_{ji}$ for all i, the original system is linear and no approximations are needed. Thus, to avoid identifiability problems and to keep the model's structure as simple as possible, we first suggest that the following linear model be estimated:

$$\left.\begin{aligned}
\dot{s}_{j1}(t) &= -\left[\alpha_{j1} + K_j\right] s_{j1}(t) + I_{j1}(t) \\
\dot{s}_{j2}(t) &= \alpha_{j1}\, s_{j1}(t) - \left[\alpha_{j2} + K_j\right] s_{j2}(t) \\
&\vdots \\
\dot{s}_{j\ell}(t) &= \alpha_{j\ell-1}\, s_{j\ell-1}(t) - \left[\alpha_{j\ell} + K_j\right] s_{j\ell}(t) - w_{j\ell}(t) \\
&\vdots \\
\dot{s}_{jr_j}(t) &= \alpha_{jr_j-1}\, s_{jr_j-1}(t) - \left[\alpha_{jr_j} + K_j\right] s_{jr_j}(t)
\end{aligned}\right\} \qquad (J.7)$$

$$z_j(t_\ell) = \alpha_{jr_j}\, s_{jr_j}(t_\ell) \qquad (J.8)$$

To construct a discrete time system model we need to find the transition matrix. The system in Eq. (3.7) can be written in matrix notation as follows:

$$\dot{\underline{s}}_j(t) = \begin{bmatrix} -\lambda_{j1} & 0 & & & 0 \\ \alpha_{j2} & -\lambda_{j2} & & & \\ 0 & \alpha_{j2} & -\lambda_{j3} & & \\ & & \ddots & \ddots & 0 \\ 0 & 0 & \alpha_{jr_j-1} & & -\lambda_{jr_j} \end{bmatrix} \underline{s}_j(t) + \begin{bmatrix} 1 & 0 \\ 0 & \vdots \\ \vdots & \\ & -1 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_{j1}(t) \\ w_{j\ell}(t) \end{bmatrix} \quad \ell^{th} \text{ row} \tag{J.9}$$

$$z(n) = \begin{bmatrix} 0 & \text{———} & 0 & \alpha_{jr_j} \end{bmatrix} \underline{s}_j(t_n) \tag{J.10}$$

where $\quad -\lambda_{ji} = -[\alpha_{ji} + K_j]$.

As is shown in Appendix K the system matrix, denote it $\underline{\underline{A}}$, can be diagonalized by a similarity transformation as follows:

$$\underline{\underline{A}} = \underline{\underline{V}} \begin{bmatrix} -\lambda_{j1} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & -\lambda_{jr_j} \end{bmatrix} \underline{\underline{V}}^{-1} \tag{J.11}$$

where $\underline{\underline{V}}$ is the matrix of the eigenvectors and $\underline{\underline{V}}^{-1}$ its inverse. $\underline{\underline{V}}$ and $\underline{\underline{V}}^{-1}$ can be explicitly otbained here due to the special structure of the matrix $\underline{\underline{A}}$ and were found to be of the form:

$$\underline{\underline{V}} = \begin{bmatrix} 1 & 0 & & 0 \\ & & & \\ & & & \\ \upsilon_{k\ell} = \prod_{m=\ell+1}^{k} \frac{\alpha_{m-1}}{\lambda_m - \lambda_\ell} & & & 0 \\ & & & 1 \end{bmatrix} \qquad (J.12)$$

$$k = 2, \ldots, r_j$$

$$\underline{\underline{V}}^{-1} = \begin{bmatrix} 1 & 0 & & 0 \\ & & & \\ & & & 0 \\ \upsilon^{-1}_{k\ell} = (-1)^{k+\ell} \prod_{m=\ell}^{k=1} \frac{\alpha_m}{\lambda_k - \lambda_m} & & & 1 \end{bmatrix} \qquad (J.13)$$

$$\ell = 1, \ldots, r_{j-1}$$

The system transition matrix $\underline{\underline{\phi}}(t,\tau)$ can then be found to be (Brockett, 1970, pg. 31-32)

$$\underline{\underline{\phi}}(t,\tau) = \underline{\underline{V}} \begin{bmatrix} e^{-\lambda_{j1}(t-\tau)} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & e^{-\lambda_{jr_j}(t-\tau)} \end{bmatrix} \underline{\underline{V}}^{-1} \qquad (J.14)$$

The solution of the system (3.1.3-13), (3.1.3-14), which we denote for convenince by

$$\begin{aligned} \dot{\underline{s}}_j(t) + \underline{\underline{A}}\ \underline{s}_j(t) &= \underline{\underline{B}}\ \underline{u}(t) \\ z(t_\ell) &= \underline{C}\ \underline{s}_j(t_\ell) \end{aligned} \qquad (J.15)$$

is in the interval $[t_\ell, t_{\ell+1}]$ as follows:

$$\underline{s}_j(t_{\ell+1}) = \underline{\underline{\phi}}(t_{\ell+1}, t_\ell)\underline{s}_j(t_\ell) + \int_{t_\ell}^{t_{\ell+1}} \underline{\underline{\phi}}(t_{\ell+1}, \tau)\, \underline{\underline{B}}\, \underline{u}(\tau)\, d\tau$$

$$z(t_\ell) = \underline{C}\, \underline{s}_j(t_\ell)$$

If we assume that $\underline{u}(\tau)$ is approximately constant over $[t_\ell, t_{\ell+1}]$, then the integral term above can be integrated explicitly. Adding white noise terms to account for possible model errors and uncertainty in $I_{j1}(t)$ we arrive at

$$\underline{s}_j(t_{\ell+1}) = \underline{\underline{V}} \begin{bmatrix} e^{-\lambda_{j1}\Delta} & & 0 \\ & \ddots & \\ 0 & & e^{-\lambda_{jr_j}\Delta} \end{bmatrix} \underline{\underline{V}}^{-1}\, \underline{s}_j(t_\ell) \quad +$$

$$+ \quad \underline{\underline{V}} \begin{bmatrix} \frac{1}{\lambda_{j1}}(1-e^{-\lambda_{j1}\Delta}) & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\lambda_{jr_j}}(1-e^{-\lambda_{jr_j}\Delta}) \end{bmatrix} \underline{\underline{V}}^{-1}\underline{\underline{B}}\, \underline{u}(t_\ell) + \varepsilon(t_\ell) \qquad (J.16)$$

$$z(t_\ell) = \underline{C}\, \underline{s}_j(t_\ell) + v(t_\ell)$$

$$\underline{s}_j(t_0): \text{ unknown}$$

where $\underline{\underline{B}}$, $\underline{\underline{V}}$, $\underline{\underline{V}}^{-1}$, $\underline{C}$ are as given above, and

$$E\left\{\underline{\varepsilon}(t_\ell)\right\} = \underline{0} \quad , \quad E\left\{\underline{\varepsilon}(t_\ell)\,\underline{\varepsilon}^T(t_m)\right\} = \begin{cases} \underline{\underline{0}} & , \quad \ell \neq m \\ \underline{\underline{Q}}(t_\ell) & , \quad \ell = m \quad \text{(unknown)} \end{cases}$$

$$E\left\{v(t_\ell)\right\} = 0 \quad , \quad E\left\{v(t_\ell)V(t_m)\right\} = \begin{cases} 0 & , \quad \ell \neq m \\ R & , \quad \ell = m \quad \text{(known)} \end{cases}$$

$v(t_\ell)$, $\underline{\varepsilon}(t_m)$: uncorrelated for all ℓ, m.

Model structure (J.16) can now be treated through Maximum Likelihood methods (Filtering or Smoothing algorithms, Schweppe, 1973) or the Extended Kalman Filter (Gelb 1974) for the estimation of its parameters. The number of the conceptual reservoirs r_j also needs to be determined, and this may be done by identifying and testing the validity of increasingly more conceptual reservoir. The first valid model can be adopted in the further considerations (in accordance with the principle of parsimony).

It is convenient for the control purposes to have the system model in continuous time, and for this reason it is desirable to identify a continuous time white Gaussian process corresponding to the previously introduced vector $\underline{\varepsilon}(t_\ell)$, $\ell = 0, \ldots, N$. We shall devote this new process by $\underline{q}(t)$ and we shall assume that it has constant mean vector and spectral density matrix over the intervals $[t_\ell, t_{\ell+1}]$. There holds

$$\underline{\varepsilon}(t_\ell) = \int_{t_\ell}^{t_{\ell+1}} \underline{\underline{\phi}}(t_{\ell+1},\tau)\,\underline{q}(\tau)\,d\tau \qquad \text{(J.17)}$$

where $\underline{\underline{\phi}}(t_{\ell+1},\tau)$ is the state transition matrix given by (J.14). Taking expectations of both sides in (J.17) we find

$$\underline{0} = \int_{t_\ell}^{t_{\ell+1}} \underline{\underline{\phi}}(t_{\ell+1},\tau)\ E\left\{\underline{q}(\tau)\right\} d\tau . \tag{J.18}$$

Since $E\left\{\underline{q}(\tau)\right\}$ is assumed constant over $[t_\ell, t_{\ell+1}]$, it follows that it should be zero because the state transition matrix $\underline{\underline{\phi}}\ (t_{\ell+1},\tau)$ is not of any special form. Regarding the spectral density matrix $\underline{\underline{Q}}_q(\tau)$, we have

$$\underline{\underline{Q}}_\varepsilon(t_\ell) = E\left\{\underline{q}(t_\ell)\underline{q}^T(t_\ell)\right\} = E\left\{\left[\int_{t_\ell}^{t_{\ell+1}} \underline{\underline{\phi}}(t_{\ell+1},\tau)\underline{q}(\tau)d\tau\right]\left[\int_{t_\ell}^{t_{\ell+1}} \underline{\underline{\phi}}(t_{\ell+1},\rho)\underline{q}(\rho)d\rho\right]^T\right\}$$

$$= \int_{t_\ell}^{t_{\ell+1}} \int_{t_\ell}^{t_{\ell+1}} \underline{\underline{\phi}}(t_{\ell+1},\tau)\ E\left\{\underline{q}\ (\tau)\underline{q}^T(\rho)\right\} \underline{\underline{\phi}}^T(t_{\ell+1},\rho)\ d\rho d\tau\ ,$$

Substituting $E\{\underline{q}(\tau)\underline{q}^T(\tau)$ by $\underline{\underline{Q}}_q(\tau)\delta(\tau-\rho)$ and using the shifting property of the dirac delta function, we obtain:

$$\underline{\underline{Q}}_\varepsilon\ (t_\ell) = \int_{t_\ell}^{t_{\ell+1}} \underline{\underline{\phi}}\ (t_{\ell+1},\tau)\ \underline{\underline{Q}}_q\ (\tau)\ \underline{\underline{\phi}}^T(t_{\ell+1},\tau)d\tau \tag{J.19}$$

Knowing $\underline{\underline{Q}}_\varepsilon(t_\ell)$, $\phi(t_{\ell+1},\tau)$ (cf. Eq. J.14), and assuming $\underline{\underline{Q}}_q(\tau)$ constant over $[t_\ell\ ,\ t_{\ell+1}]$ suffice to determine the elements of $\underline{\underline{Q}}_q(\tau)$ from (J.19). Alternatively, an approximate procedure to obtain $\underline{\underline{Q}}_q(\tau)$ from $\underline{\underline{Q}}_\varepsilon(t_\ell)$ is the following relationship: (Gelb, 1974)

$$\underline{\underline{Q}}_q(\tau) = \underline{\underline{Q}}_\varepsilon(t_\ell)\ \frac{1}{[t_{\ell+1} - t_\ell]} \tag{J.20}$$

Eq. (J.20) becomes increasingly accurate as $t_{\ell+1} - t$ gets smaller. Thus, after the estimation of its parameters, the river segment model takes the following form:

$$\dot{\underline{s}}_j(t) = \underline{F}(\underline{s}_j(t)) + \underline{e}_1\ I_{j1}(t) - \underline{e}_\ell\ w_{j\ell}(t) + \underline{q}(t) \tag{J.21}$$

Appendix K

IDENTIFIABILITY STUDY

The following concerns the parameter identifiability of Eq. (J.5) (J.6) time invariant version which consists of a system-model for a river segment response. However, the procedure may be used for a general time invariant linear system. Consider a 3-dimensional system structure:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\alpha_1 & 0 & 0 \\ \lambda_1 & -\alpha_2 & 0 \\ 0 & \lambda_2 & -\alpha_3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{K.1}$$

$$z(t) = \begin{bmatrix} 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{K.2}$$

where $\alpha_i = \lambda_i + \mu_i$ and $\underline{x}(t_0) = \underline{x}_0$.

The question is if it is possible to specify the parameters $\left\{\lambda_i, \mu_i\right\}_{i=1}^{3}$ by observing $z(t)$ with or without knowledge of $\underline{x}_0$.

First, we shall transform the system into a state-space form which is more convenient to work with, i.e., the Jordan Canonical form.

The matrix in (K.1) is lower triangular and its eigenvalues are the diagonal elements $-\alpha_1$, $-\alpha_2$, $-\alpha_3$ (Strang, 1980, pg. 187), which are assumed distinct; otherwise, say $\alpha_1 = \alpha_2$, there is no reason to distinguish between the two conceptual reservoirs 1 and 2, but rather they should be lumped together. (The assumption is that $z(t)$ is indeed generated by the hypothesized structure and the only problem is to identify

the parameters.) The associated eigenvectors can be easily found, due to the special structure of the matrix, from

$$\begin{bmatrix} -\alpha_1 & 0 & 0 \\ \lambda_1 & -\alpha_2 & 0 \\ 0 & \lambda_2 & -\alpha_3 \end{bmatrix} \begin{bmatrix} \upsilon_{i1} \\ \upsilon_{i2} \\ \upsilon_{i3} \end{bmatrix} = -\alpha_i \begin{bmatrix} \upsilon_{i2} \\ \upsilon_{i2} \\ \upsilon_{i3} \end{bmatrix} \tag{K.3}$$

for $i = 1, 2, 3$. They are

$$\underline{\upsilon}_1 = \begin{bmatrix} 1 \\ \dfrac{\lambda_1}{\alpha_2-\alpha_1} \\ \dfrac{\lambda_1 \lambda_2}{(\alpha_3-\alpha_1)(\alpha_2-\alpha_1)} \end{bmatrix}, \quad \underline{\upsilon}_2 = \begin{bmatrix} 0 \\ 1 \\ \dfrac{\lambda_2}{\alpha_3-\alpha_2} \end{bmatrix}, \quad \underline{\upsilon}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{K.4}$$

Consider next a similarity transformation which produces a new set of state variables:

$$\left.\begin{aligned} \underline{y}(t) &= \underline{\underline{V}}\, \underline{x}\,(t) \\ \text{where} \qquad \underline{\underline{V}} &= \left[\underline{\upsilon}_1 \mid \underline{\upsilon}_2 \mid \underline{\upsilon}_3\right] \end{aligned}\right\} \tag{K.5}$$

(The double underscore denotes matrices.)

Differentiating, we find

$$\dot{\underline{y}}(t) = \underline{\underline{V}}\, \dot{\underline{x}}(t) = \underline{\underline{V}}\, \underline{\underline{A}}\, \underline{x}(t) = \underline{\underline{V}}\, \underline{\underline{A}}\, \underline{\underline{V}}^{-1}\, \underline{y}(t) \tag{K.6}$$

where $\underline{\underline{A}}$ is the system matrix in (K.1) and $\underline{\underline{V}}^{-1}$ exists because $\underline{\upsilon}_1$, $\underline{\upsilon}_2$, $\underline{\upsilon}_3$ are linearly independent, being associated with distinct eigenvectors. Also,

$$z(t) = \begin{bmatrix} 0 & 0 & \lambda_3 \end{bmatrix} \underline{\underline{V}}^{-1} \underline{y}(t) \tag{K.7}$$

and

$$\underline{y}(0) = \underline{\underline{V}}\, \underline{x}_0 \tag{K.8}$$

From Strang, 1980, pg. 190, we have that

$$\underline{\underline{V}}\, \underline{\underline{A}}\, \underline{\underline{V}}^{-1} = \underline{\underline{\Lambda}} = \begin{bmatrix} -\alpha_1 & 0 & 0 \\ 0 & -\alpha_2 & 0 \\ 0 & 0 & -\alpha_3 \end{bmatrix} \tag{K.9}$$

and we also find (using $\underline{\underline{V}}\, \underline{\underline{V}}^{-1} = \underline{\underline{I}}$ or more direct formulas) that

$$\underline{\underline{V}}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{-\lambda_1}{(\alpha_2 - \alpha_1)} & 1 & 0 \\ \dfrac{\lambda_1 \lambda_2}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)} & \dfrac{-\lambda_2}{(\alpha_3-\alpha_2)} & 1 \end{bmatrix} \tag{K.10}$$

Combining the above, we arrive at the following new system form (Jordan Canonical Form):

$$\dot{\underline{y}}(t) = \begin{bmatrix} -\alpha_1 & 0 & 0 \\ 0 & -\alpha_2 & 0 \\ 0 & 0 & -\alpha_3 \end{bmatrix} \underline{y}(t) \tag{K.11}$$

$$z(t) = \begin{bmatrix} \dfrac{\lambda_1\lambda_2\lambda_3}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)} & \dfrac{-\lambda_2\lambda_3}{(\alpha_3 - \alpha_2)} & \lambda_3 \end{bmatrix} \underline{y}(t) \tag{K.12}$$

$$y_0 = \underline{\underline{V}}\, \underline{x}_0 \tag{K.13}$$

By the nonsingularity of $\underline{\underline{V}}$ we have that identifiability of the system implies identifiability of the original system and vice versa.

We can now solve (K.11) to get (decoupled differential equations of first order):

$$\underline{y}(t) = \begin{bmatrix} e^{-\alpha_1 t} & 0 & 0 \\ 0 & e^{-\alpha_2 t} & 0 \\ 0 & 0 & e^{-\alpha_3 t} \end{bmatrix} \underline{y}_9$$

$$z(t) = \frac{\lambda_1\lambda_2\lambda_3\, e^{-\alpha_1 t} y_{10}}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)} + \frac{-\lambda_2\lambda_3\, e^{-\alpha_2 t} y_{20}}{(\alpha_3-\alpha_2)} + \lambda_3\, e^{-\alpha_3 t} y_{30}. \qquad \text{(K.15)}$$

The objective is to determine if by knowing z(t) we can find the parameters of the right hand sides.

Studying the Laplace transform of z(t), we can express it in the form

$$\hat{z}(s) = \frac{b_0\, s^2 + b_1\, s + b_2}{(s + a_1)(s + a_2)(s + a_3)} \qquad \text{(K.16)}$$

The above holds true from the assumption that z(t) is generated by a third order time invariant system which ensures that its transfer function is proper and rational with the denominator being a cubic polynomial in s (Chen, 1970, pg. 151). Hence, by knowing the signal z(t), we can specify the poles $-a_1$, $-a_2$, $-a_3$ and the coefficients b_0, b_1, b_2. Next, taking the Laplace transform of Eq. (K.15), we find

$$\hat{z}(s) = \frac{\lambda_1\lambda_2\lambda_3 y_{10}}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)(s+\alpha_1)} - \frac{\lambda_2\lambda_3 y_{20}}{(\alpha_3-\alpha_2)(s+\alpha_2)} + \frac{\lambda_3 y_{30}}{(s+\alpha_3)} \qquad \text{(K.17)}$$

Or

$$\hat{z}(s) = \frac{\dfrac{\lambda_1\lambda_2\lambda_3 y_{10}(s+\alpha_2)(s+\alpha_3)}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)} - \dfrac{\lambda_2\lambda_3 y_{20}(s+\alpha_2)(s+\alpha_3)}{(\alpha_2-\alpha_2)} + \lambda_3 y_{30}(s+\alpha_1)s+\alpha_2)}{(s+\alpha_1)(s+\alpha_2)(s+\alpha_3)} \tag{K.18}$$

Comparing (A-18) with (A-16) we find $\alpha_i = a_i$, $i = 1,2,3$, and

$$b_0 = \frac{\lambda_1\lambda_2\lambda_3\, y_{10}}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)} - \frac{\lambda_2\lambda_3\, y_{20}}{(\alpha_3-\alpha_2)} + \lambda_3\, y_{30} \tag{K.19}$$

$$b_1 = \frac{\lambda_1\lambda_2\lambda_3 y_{10}(\alpha_2+\alpha_3)}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)} - \frac{\lambda_2\lambda_3 y_{20}(\alpha_1+\alpha_3)}{(\alpha_3-\alpha_2)} + \lambda_3 y_{30}(\alpha_1+\alpha_2) \tag{K.20}$$

$$b_2 = \frac{\lambda_1\lambda_2\lambda_3\, y_{10}\alpha_2\alpha_3}{(\alpha_3-\alpha_2)(\alpha_3-\alpha_1)} - \frac{\lambda_2\lambda_3 y_{20}\, \alpha_1\alpha_3}{(\alpha_3-\alpha_2)} + \lambda_3 y_{20}\, \alpha_1\alpha_2 \tag{K.21}$$

If we were able to specify the λ's from the above system, then all parameters $\left\{\lambda_i, \mu_i\right\}_{i=1}^{3}$ would be identifiable. This can indeed be done only if the initial conditions y_{10}, y_{20}, y_{30} are known. (We shall not go into the algebraic manipulation, but it is easy to see the logic of solution.)

In short, we came to the following conclusion: If the initial conditions $\underline{x}_0$ are assumed known, then all parameters $\left\{\lambda_i, \mu_i\right\}_{i=1}^{3}$ can be identified. If $\underline{x}_0$ is not known, then only $\lambda_i + \mu_i$, $i = 1, 2, 3$ can be determined. The above are also valid for higher dimensional systems of similar structure.