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Risk in Market Equilibrium Models for Agriculture

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LINEAR PROGRAMMING MODELS are gaining increasing acceptance as tools for analysis of agricultural supply response and agricultural investment programs at both the regional and sectoral levels. One difficult problem in specifying aggregate models, however, lies with ways of incorporating the considerable price, yield, and resource uncertainties that confront producers.

There is now considerable literature that attests to the fact that farmers are not risk neutral. Notable examples are Officer and Halter (1968) and O'Mara (chapter 9 of this volume), both of which include estimates of farmer's utility functions. These functions typically show risk aversion in the relevant range of values of farm income. A direct consequence of this risk aversion is that omission of risk considerations in programming models is likely to lead to an overestimate of the supply response for farm enterprises with high variance in yields, prices, or both. Furthermore, since these are often high-value enterprises, omission of risk is likely to lead to an overstatement of the returns to investment. These biases may be particularly large in models of low-income agriculture, in which risk aversion is likely to be greatest.

Overview

Methodologies for handling risk at the individual farm level are well developed in the literature for a wide range of decision criteria. Two of the more appealing of these, when information about the probability distributions of stochastic components is available, use the E, V decision criterion (Freund 1956; Heady and Candler 1958; Markowitz 1959; McFarquhar

Note: This chapter is a revised and extended version of Hazell and Scandizzo (1974). Permission of the original publisher to use unrevised material here is gratefully acknowledged.

1961; Stovall 1966*b*) or the related E,σ measure (Baumol 1963; Hazell and Scandizzo 1974).

Such models are generally more descriptive of individual farm behavior than linear programming models that maximize expected income, and therefore provide a more useful starting point for the construction of aggregate models. At the aggregate level, however, problems arise when market demand structures are interfaced with the supply model to obtain simultaneous determination of the equilibrium levels of production and prices under the usual assumption of a perfectly competitive agricultural industry. The problem then is to specify the objective function of the aggregate model in such a way as to simulate the results that would be obtained if each farmer operated in a competitive environment according to the E,V or E,σ decision criterion.

Methods of simulating competitive market equilibrium through linear programming are well developed for the deterministic case. Samuelson (1952) showed that the appropriate objective function in the aggregate model is the maximization of net social payoff (the sum of consumers' and producers' surplus). He developed this result in the context of spatial equilibrium models, and Takayama and Judge (1964, 1971) further developed this objective function to obtain a quadratic programming formulation for multiproduct models. Duloy and Norton (1971, 1973; chapter 3 of this book) subsequently applied the method to agricultural sector models using linear programming approximations.

The purpose of this chapter is to provide a modification of the Duloy-Norton method when production is risky and individual farmers maximize E,σ utility instead of expected profits. A crucial issue in the development of such modifications lies in appropriate specification of the equilibrium solution to be simulated. The nature of market equilibria under risk conditions is a complex subject that has received scant attention in the economic literature. Furthermore, results are known to depend very much on assumptions made about the dynamics of market adjustment and on the nature of the stochastic components involved. In this chapter we assume that the initial source of risk lies in yields. Stochastic yields also lead to stochastic costs for those costs which are related to production rather than numbers of hectares or livestock. Under these assumptions, and assuming lagged behavior in supply response, useful results can be developed about market equilibria (Bergendorff, Hazell, and Scandizzo 1974; Hazell and Scandizzo 1973; Turnovsky 1968).

Some systematic experience with numerical implementation of this approach are reported for various Mexican producing areas in the next chapter of this book. In addition, the aproach has been incorporated in the models of chapters 12, 13, and 15.

In the following sections, we briefly review methodology for the deterministic case, and pose the problem for the risk situation. Relevant market

equilibrium results are then developed, and a methodology for obtaining market equilibrium solutions assuming an E,σ decision criterion at the farm level is developed. Finally, linear programming approximations are provided and illustrated with a schematic tableau.

The Deterministic Model

The deterministic model is premised on the assumption that individual farmers are profit maximizers, and that they compete in a perfectly competitive way. The latter assumption implies, in particular, that farmers plan on the basis of constant anticipated prices.

Define

$\hat{p}$ = An $n \times 1$ vector of anticipated product prices

c = An $n \times 1$ vector of unit costs

x = An $n \times 1$ vector of enterprise levels

M = An $n \times n$ diagonal matrix of enterprise yields with jth diagonal entry m_j

$y = Mx$ is the $n \times 1$ vector of total outputs.

Then the objective function for an individual farm problem is:

$$\max \pi = \hat{p}' y - c' x, \tag{7.1}$$

and this is to be maximized over some set of constraints that are usually specified to be linear.

Now if the product markets attain an equilibrium, then regardless of the way in which the anticipated prices $\hat{p}$ are formed over time, the equilibrium is unique. Furthermore, the market equilibrium prices and outputs occur at the points where the demand and implicit model supply functions intersect. This fact provides the basis of the solution procedure.

Let X, Y, C, and W be some appropriate aggregates[1] of the individual farms x, y, c and M matrixes, and P be the vector of unknown market prices. Then, assuming the linear demand structure

$$P = A - BY, \tag{7.2}$$

where B is a symmetric matrix of demand coefficients,[2] the Duloy-Norton aggregate model objective function is:

1. Aggregation should be exact to avoid biased results in the sector model. The usual approach to this problem is through appropriate classification of farms into homogeneous groups (Day 1963; Stovall 1966*a*). To simplify notation, however, it is assumed throughout this chapter that there is only one homogeneous group of farms in the sector.

2. The condition of symmetry is necessary to ensure the existence of the potential function in equation (7.3). An analysis of the implications of such an assumption is contained in Zusman (1969).

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$$\max \Pi = X'W(A - 0.5BWX) - C'X, \tag{7.3}$$

where it is understood that $Y = WX$.

The term $X'W(A - 0.5BWX)$ is the sum of areas under the product demand functions. For example, in the single product case this would be

$$\int_0^Y (a - bt)\,dt = y(a - 0.5by) = wx(a - 0.5bxw).$$

The term $C'X$ is total production costs, or equivalently, the sum of areas under the product supply functions. Consequently, the difference between these two terms is the sum of producers' and consumers' surplus over all markets, and this reaches its maximum at the required intersections of supply and demand functions.

Introduction of Risk

The basic source of risk to be introduced is confined to yields. Thus, the vector of products for an individual farm now becomes

$$y = Nx,$$

where N is an $n \times n$ diagonal matrix of stochastic yields with jth diagonal element ϵ_j.

Stochastic yields imply stochastic supply functions, and hence lead to stochastic market prices. It is assumed, however, that input costs and the market demand structure remain nonstochastic, and that the farm linear programming constraints are not affected. The latter assumption can easily be relaxed, since several techniques are available to handle stochastic constraints that do not affect the farm model objective function (Charnes and Cooper 1959; Hillier and Lieberman 1967; Madansky 1962; Maruyama 1972).

It is further assumed that the individual farmers are risk averse, and that their behavior conforms to a single period E,σ specification. Consequently, the individual farm model objective function each year is

$$\max_x u = E(p'y) - c'x - \phi V(p'y)^{1/2}, \tag{7.4}$$

where E and V denote, respectively, the expectation and variance operators, and ϕ is a risk-aversion coefficient.

In order to enumerate equation (7.4) more precisely, it is necessary to make explicit assumptions about the nature of farmers' subjective expectations. These in part depend on the nature of perfect competition under risk.

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Perfect Competition under Risk

As a natural generalization of the deterministic concept of perfect competition, it is assumed that farmers continue to expect their outputs to not have any effect on the market. A set of assumptions regarding behavioral anticipations that are consistent with this are:

(A.1) $$E(\epsilon_j) = m_j$$

(A.2) $$V(\epsilon_j) = \sigma^2_{\epsilon_j}$$

(A.3) $$E(p_j) = \hat{p}_j$$

(A.4) $$V(p_j) = \sigma^2_{p_j}$$

(A.5) $$\text{cov}(p_i p_j) = \sigma_{p_{ij}}; \quad \text{cov}(\epsilon_i \epsilon_j) = \sigma_{\epsilon_{ij}}, \quad \text{for all } i \neq j$$

(A.6) $$\text{cov}(p_i y_i) = x_i \text{ cov}(p_i \epsilon_i) = 0, \quad \text{for all } i,$$

where all operators are now subjective expectations that may differ from the real-world parameters.

Assumption (A.3) states that farmers expect a constant mean price for each product, and by making the variance homoskedastic [assumption (A.4)] and the covariances between prices and outputs zero [assumption (A.6)], this implies that there is no expected relation between the output of the individual farm and the market.

It is important to note that these are a set of behavioral assumptions, and it is not required that farmers anticipate the true state of affairs. Market behavior is in part a reflection of what farmers anticipate, but this is basically no different from the deterministic case where, in the short run, the farmers' expectation of prices $\hat{p}$ can differ from the vector of market-clearing prices P.

Given the above set of assumptions, the components of equation (7.4) can be enumerated as follows:

$$E(p'y) = \hat{p}'Mx,$$

where $M = E(N)$,

$$V(p'y)^{1/2} = (x'\Omega x)^{1/2},$$

where Ω is an $n \times n$ covariance matrix of activity revenues with diagonal elements

$$\begin{aligned}\omega_{jj} &= V(p_j\,\varepsilon_j)\\ &= E(p_j^2\,\varepsilon_j^2) - \hat{p}_j\,m_j^2\\ &= E(p_j^2)\,E(\varepsilon_j^2) - \hat{p}_j^2\,m_j^2\\ &= \sigma_{p_j}^2\,E(\varepsilon_j^2) + \hat{p}_j^2\,\sigma_{\varepsilon_j}^2,\end{aligned}$$

and off-diagonal elements

$$\begin{aligned}\omega_{ij} &= \mathrm{cov}(p_i\varepsilon_i,\ p_j\varepsilon_j)\\ &= E(p_ip_j\varepsilon_i\varepsilon_j) - E(p_i\varepsilon_i)\,E(p_j\varepsilon_j)\\ &= E(p_ip_j)\,E(\varepsilon_i\varepsilon_j) - \hat{p}_i\hat{p}_j\,m_im_j\\ &= E(p_ip_j)\left[E(\varepsilon_i\varepsilon_j) - m_im_j\right] + m_im_j\left[E(p_ip_j) - \hat{p}_i\hat{p}_j\right]\\ &= \left[\sigma_{p_{ij}} + \hat{p}_i\hat{p}_j\right]\sigma_{\varepsilon_{ij}} + m_im_j\,\sigma_{p_{ij}}.\end{aligned}$$

The farm problem objective function is then

$$\max_x\ u = \hat{p}'Mx - c'x - \phi(x'\Omega x)_{1/2}. \tag{7.5}$$

Obviously, alternatives to assumptions (A.1) through (A.6) are possible. However, assumptions (A.1), (A.3), and (A.6) are retained, then the only effect of changing the assumptions is on the elements of Ω. This will affect the market behavior and the estimation of Ω in a quantitative way, but does not deter the development of qualitative results about farm behavior.

Let the linear programming constraints for the farm model be denoted by $Dx \le b$, $x \ge 0$, then the Lagrangian function for maximizing equation (7.5) over this set is:

$$L = \hat{p}'Mx - c'x - \phi(x'\Omega x)^{1/2} + \nu'(b - Dx), \tag{7.6}$$

where ν is a vector of dual values. An optimal solution to the problem is then a "saddle point," and necessary and sufficient conditions for any (x,ν) to be this saddle point, are obtainable from the Kuhn-Tucker conditions. The necessary conditions are:

$$\frac{\partial L}{\partial x} \le 0, \quad \frac{\partial L}{\partial \nu} \ge 0, \tag{7.7}$$

$$x\frac{\partial L}{\partial x} = 0, \quad \nu\frac{\partial L}{\partial \nu} = 0. \tag{7.8}$$

Of these, the requirements in equation (7.8) are the complementary slackness requirements that an activity cannot be active and at the same time have a nonzero opportunity cost and that a resource cannot be slack and at the same time have a nonzero dual value. Sufficient conditions for a

saddle point can be derived, but they reduce to the requirement that Ω be a positive semidefinite matrix (Takayama and Judge 1971, p. 19).

Applying the necessary requirements in equation (7.7) to equation (7.6) gives

$$\frac{\partial L}{\partial x} = P'M - c' - \phi x'\Omega(x'\Omega x)^{-1/2} - \nu' D \leq 0, \tag{7.9}$$

$$\frac{\partial L}{\partial \nu} = b - Dx \geq 0. \tag{7.10}$$

Equation (7.10) is merely the feasibility requirement, but equation (7.9) contains the risk counterpart to the classical marginality rules for output determination in a deterministic firm. Taking the jth element of the vector $\partial L/\partial x$, and rearranging terms and dividing by m_j, we obtain

$$\hat{p}_j \leq \frac{1}{m_j}\left[\sum_k \nu_k\, d_{kj} + c_j + \phi(x'\Omega x)^{-1/2}\sum_i \omega_{ji}x_i\right]. \tag{7.11}$$

This states that for each product the expected marginal cost per unit of output must be equal to or greater than the expected price. The expected marginal cost comprises expected own marginal cost c_j/m_j, plus a marginal risk factor

$$\frac{\phi}{m_j}\frac{\partial V(p'\gamma)^{1/2}}{\partial x_j} = \frac{1}{m_j}\phi(x'\Omega x)^{-1/2}\sum_i \omega_{ji}x_i,$$

plus expected opportunity costs

$$\frac{1}{m_j}\sum_k \nu_k\, d_{kj}$$

as reflected in the dual values of the resources used by that activity. This differs from the comparable requirements of a deterministic model primarily in that a risk term has been introduced. This is quite reasonable because the risk term is really nothing but a new cost; namely, the additional expected return demanded by farmers as compensation for taking risk. This is even clearer when farmers can participate in a crop insurance program, for then the risk term is the marginal premium a farmer would be willing to pay to insure against risk (that is, a certainty equivalent cost).

This result is not new and is consistent with the results obtained for single-product firms in economic analysis (for example, see Magnusson 1979). Further, the appearance of the risk factor as a marginal cost provides the rationale for the expectation that deterministic models overestimate the supply response of high-risk crops. This is because $\Sigma_i\, \omega_{ji}x_i$ will then be positive, hence the marginal cost curve must lie above the mar-

ginal cost curve that would be obtained from a deterministic (or risk-neutral) model.

Although equation (7.11) is a necessary condition, it is clear from the complementary slackness condition $x(\partial L/\partial x)$ in equation (7.8) that the condition will always be satisfied as an equality in an optimal solution for all activities that enter the basis; that is, $x_j > 0$. Consequently, for all basic activities the risk counterpart to the price-equals-marginal-cost rule can be written as:

$$\hat{p}_j = \frac{1}{m_j}\left[\sum_k v_k\, d_{kj} + c_j + \phi(x'\Omega x)^{-1/2}\sum_i \omega_{ji} x_i\right]. \tag{7.12}$$

The right-hand side of equation (7.12) is then the short-run supply function for the farm as implicitly embedded in the programming model. This is a basic behavioral relation, and expresses the farmer's determination of x_j given his expectations about yields and prices. That is, $x_j = f(M, \hat{p}, \Omega)$, with everything else constant. Multiplying by the mean yield m_j, a conditionally expected supply function is immediately obtained:

$$E(y_j|x_j) = m_j x_j = m_j\, f(M, \hat{p}, \Omega). \tag{7.13}$$

Since all the expectations involved are subjective anticipations, it is useful to denote equation (7.13) as the "anticipated supply function" for the farm to distinguish it from a true statistical relation.

Market Equilibrium under Risk

By summing the anticipated supply functions over all farms, an aggregate anticipated supply function can be obtained as a basic behavioral relation in the market. Ignoring aggregation problems for now, the jth supply function can be written as

$$E(Y_j|X_j) = w_j X_j = w_j\, g(W, \hat{p}, \Gamma), \tag{7.14}$$

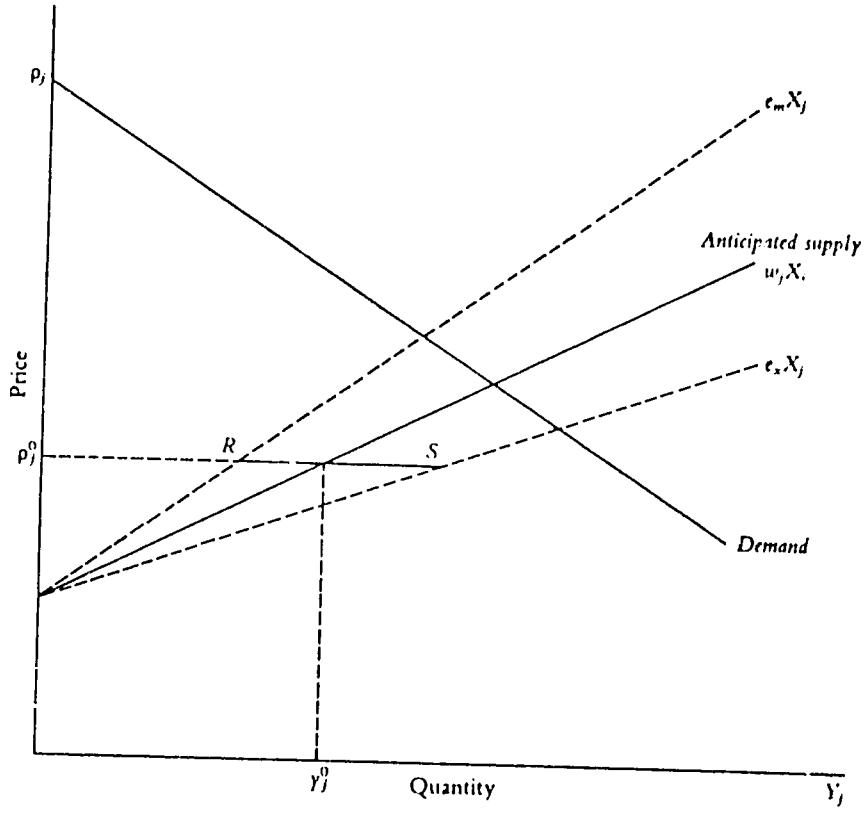
where X_j, Y_j, W, $\hat{p}$, and Γ are suitable aggregates of x_j, y_j, M, $\hat{p}$, and Ω, respectively, and w_j is the jth diagonal element of W.

Given X_j, actual supply is

$$Y_j|X_j = e_j X_j = e_j g(W, \hat{p}, \Gamma), \tag{7.15}$$

where e_j is a suitable aggregate of the farm ϵ_j's such that $E(e_j) = w_j$. Clearly, actual supply is stochastic with e_j and, furthermore, is of a specification in which the slope of the supply function is stochastic. Ignoring the unnecessary complication of a stochastic intercept term, and assuming that e_j is bounded on some positive interval $e_m \le e_j \le e_x$, the market situation can be portrayed as shown in figure 7-1.

Figure 7-1. *Market Situation under Aggregate Anticipated Supply Function*



If W and Γ remain constant, the anticipated supply function is fixed, and aggregate anticipated supply is determined by $\hat{p}$. Thus for given $\hat{p}$ with $\hat{p} = p_j^0$, farmers will plan their farms so that aggregate expected output is y_j^0. Yet, because yields are stochastic over the range e_m to e_x, actual supply can take on any value between R and S. More generally, the actual supply function can rotate around the anticipated supply function to any position contained in the funnel defined by $e_m X_j$ and $e_x X_j$. It follows that market price must always be stochastic and will fluctuate with both e_j as well as with X_j if the latter does not stabilize to some equilibrium amount. Further complications arise when W or Γ is not fixed, for then the whole "supply funnel" may shift structurally over time.

The question of what is a perfectly competitive and equilibrium solution to the market now arises, and, since it is obviously not a point solution, the further question of what characteristics can usefully be

derived with a mathematical programming model can be posed. The intuitive answer would be to view the market as stabilizing in its price distribution, and to seek the perfectly competitive solution for expected price and output at the intersection of the demand and anticipated supply schedules. Unfortunately, although equilibrium is appropriately viewed in terms of a stabilizing price distribution, the properties of this distribution do depend on the way in which farmers form their expectations about $\hat{\rho}$, W, and Ω. That is, the equilibrium is interdependent with the adjustment mechanism. Furthermore, only under certain conditions do the demand and anticipated supply schedules actually intersect at the equilibrium value of expected price. It is therefore necessary to explore these problems in a little more depth.

To initiate the analysis, the following assumptions are made:

(A.7) W and Γ are fixed.

(A.8) X_j is a linear function in $\hat{\rho}_j$ of the form $X = \lambda\hat{\rho}_j$, where λ is some appropriate function of W and Γ and is therefore a constant by assumption (A.7)—that is, the response function for X_j in a programming model is being approximated by a straight line; actual supply in the tth year is then $Y_{jt} = \lambda e_{jt}\hat{\rho}_{jt}$.

(A.9) Demand is linear and of the form $D_{jt} = a - bP_{jt}$.

(A.10) Farmers form their anticipated price forecasts each year as a weighted average of past prices.[3]

That is, from assumption (A.10),

$$\hat{\rho}_{jt} = \sum_{i=1}^{m} \gamma_i P_{jt-i} \quad \text{and} \quad \sum_{i=1}^{m} \gamma_i = 1. \tag{7.16}$$

Note that a "naive cobweb" formulation and the Nerlove-type adaptive expectation models are special cases of equation (7.16), so that the formulation is quite general. A further assumption is that

$$\text{cov}(c_{jt}, c_{jt-i}) = 0, \text{ for all } t. \tag{A.11}$$

That is, the yield of an individual activity is uncorrelated with itself over time.

Ignoring the jth subscript for convenience, we find that if the market clears each year, then market-clearing price is

3. It is to be noted that an E,σ decision specification states only which price and yield parameters are relevant each year; it does not state how anticipations about these parameters are formed over time.

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$$P_t = \frac{a}{b} - \frac{\lambda}{b} e_t \hat{p}_t \tag{7.17}$$

$$= \frac{a}{b} - \frac{\lambda}{b} e_t \sum_{i=1}^{m} \gamma_i P_{t-i}.$$

Equilibrium can now be defined according to the convergent properties of P_t over time. There are a number of alternative properties to choose from (Bergendorff, Hazell, and Scandizzo 1974; Turnovsky 1968), but all basically are variants of the concept of convergence in the probability density function of p_t and its various central moments. Consider first the convergence of expected price:

$$E(P_t) = \frac{a}{b} - \frac{\lambda}{b} E(e_t \hat{p}_t) \tag{7.18}$$

$$= \frac{a}{b} - \frac{\lambda}{b} w\, E(\hat{p}_t)$$

$$= \frac{a}{b} - \frac{\lambda}{b} w \sum_{i=1}^{m} \gamma_i E(P_{t-i}).$$

This is an mth order difference equation and, if convergence occurs,[4] has the particular solution

$$\lim_{t \to \infty} E(P_t) = \frac{a}{b + \lambda w}. \tag{7.19}$$

Further, it is the same for any choice of γ weights satisfying equation (7.16).

Solving now for the intersection price (P^*) of demand and anticipated supply $(Y_t | X_t = \lambda w_t\ \hat{p}_t)$, we obtain $a - bP^* = \lambda w P^*$; hence $P^* = a/(b + \lambda w)$, which is identical to $\lim E(P_t)$.

Thus, under assumptions (A.7) through (A.11), the asymptotic expectation of market price is the same regardless of the specific price-learning model and, furthermore, corresponds to the desired intersection of the demand and anticipated functions. It is also clear from the derivation of anticipated supply that at this point $\lim E(P_t) = E(\text{marginal costs})$, which provides an acceptable equivalence to the equilibrium point of a competitive but deterministic market. These results also hold for the multiproduct case, as is shown in the appendix to this chapter.

Turning now to other properties of the equilibrium, we can see that.

4. Necessary conditions for convergence depend on the characteristic roots of equation (7.18), but a sufficient condition is simply $\lambda/b < 1/w$, and which is also the necessary condition for the naive cobweb (Bergendorff, Hazell, and Scandizzo 1974).

even under assumptions (A.7) through (A.11), the variance and probability density function of price *do not* converge to the same limits for alternative choices of the γ weights (Bergendorff, Hazell, and Scandizzo 1974; Hazell and Scandizzo 1973). These properties of the equilibrium do depend on the way farmers form their price anticipations each year and, consequently, can only be enumerated through simulation-type procedures given explicit assumptions about the behavior involved. A similar result pertains for the stochasticity of input decisions, and hence X_j.

The results for $E(P_t)$ are clearly quite useful, but what happens if any of the assumptions (A.7) through (A.11) are relaxed?

- If either assumption (A.8) or (A.9) is relaxed to permit nonlinearities, then lim $E(P_t)$ usually becomes dependent on the γ weights in equation (7.16) and no longer corresponds to the intersection of demand and anticipated supply. Under these conditions the intersection price is only a linear approximation to the asymptotic expectation of price.
- Relaxation of assumption (A.7) leads to a situation in which the slope of the supply formation (λ) varies structurally over time, and the location of the anticipated supply function will no longer be fixed. However, if W and Γ converge,[5] then a stable equilibrium is attained with the above properties, though the length and nature of the adjustment path in attaining this equilibrium may be quite different. A similar result pertains when assumption (A.10) is relaxed providing lim $E(\hat{p}_t)$ = lim $E(P_t)$, otherwise approximations are again involved.
- Assumption (A.11) is only a simplifying assumption, and can be relaxed to consider autocorrelated yields providing the stochastic residual of the process satisfies assumption (A.11), that is, $e_t = \mu e_{t-1} + \mu_t$ where $cov(u_t, u_{t-1}) = 0$, and μ is a constant. In this case the basic equilibrium properties still hold, and the main effects are on the length and nature of the adjustment path.

If it is accepted that assumptions (A.7) through (A.11) and the possible modifications stated above are quite reasonable, then any mathematical programming model that could provide the intersection solutions for demand and anticipated supply would generate results that have a direct and relevant economic interpretation and that would be reasonably general with respect to the way in which farmers form their price and yield anticipations over time. The task of providing a modification to the Duloy-Norton objective function that solves this problem in an aggregate model remains.

5. Note that if assumptions (A.1) through (A.6) are a true description of competitive behavior, then Γ is unlikely to converge to its real statistical value, for there will almost certainly be nonzero correlations between individual farm outputs and market prices.

Solving for the Asymptotic Expectation of Price

X, Y, W, C, $\hat{\rho}$, and Γ have already been defined as aggregates of the farm x, y, M, c, $\hat{p}$, and Ω matrixes. Also, let Φ be an aggregate of the farm risk parameters ϕ.

There are definite problems in forming these aggregates that are closely related to the problem of establishing criteria of farm classification for exact aggregation in quadratic models. This problem lies beyond the scope of this chapter, and it is merely noted that the aggregate variables Γ and Φ must be chosen so that

$$\Phi(X'\Gamma X)^{1/2} = \sum_k \phi_k(X_k' \Omega_k x_k)^{1/2}, \tag{7.20}$$

where k denotes the kth individual farm. This equation states that the aggregate level of risk calculated for the model must be equal to the sum of individual risks over all farms. Without this condition, covariance relations between farms could be exploited in the aggregate model in seeking efficient diversification, and this would be inconsistent with assumed competitive behavior. For example, if all the farms were identical, a suitable choice of the aggregate variables would be $\Phi = (1/k)\phi$ and $\Gamma = \Omega$, so that

$$\begin{aligned}\sum_k \phi_k(x_k' \Omega_k x_k)^{1/2} &= K\,\phi(x'\,\Omega\,x)^{1/2}\\ &= \Phi(kx'\,\Gamma\,kx)^{1/2} = \Phi(X'\,\Gamma\,X)^{1/2},\end{aligned}$$

where $X = Kx = \Sigma_k x_k$.

Given the necessary aggregate variables and parameters, it is now possible to modify the Duloy-Norton objective function to obtain the solution corresponding to the intersection of demand and anticipated supply schedules. This modified function is:

$$\max\ U = X'W(A - 0.5BWX) - C'W - \Phi(X'\Gamma X)^{1/2}, \tag{7.21}$$

where $X'W(A - 0.5BWX)$ is now the sum of expected areas under the demand curves and $C'X + \Phi(X'X)^{1/2}$ is a revised sum of areas under the supply curves.

To verify that equation (7.21) gives the desired market equilibrium solution, form the Lagrangian function

$$L = X'W(A - 0.5BWX) - C'X - \Phi(X'\Gamma X)^{1/2} + \nu'(b - DX),$$

where b and D denote the aggregate constraints and ν is a vector of dual

values. Apart from complementary slackness requirements, the necessary Kuhn-Tucker conditions are:[6]

$$\frac{\partial L}{\partial X} = WA - WBWX - C - \Phi\,\Gamma X(X'\Gamma X)^{-1/2} - D'\nu \leq 0 \tag{7.22}$$

and

$$\frac{\partial L}{\partial \nu} = b - DX \geq 0. \tag{7.23}$$

Equation (7.23) is the feasibility requirement, and equation (7.22) can be rearranged as:

$$(A - BWX) \leq W^{-1}\left[C + \Phi\,\Gamma X(X'\Gamma X)^{-1/2} + D'\nu\right]. \tag{7.24}$$

Now WX is the vector of anticipated supplies $E(Y|X)$, and $A - BWX$ is the corresponding vector of market prices. Further, the right-hand side of equation (7.24) is the sum of expected marginal cost curves over all farms [the jth component is the sum of the right-hand sides of equations such as (7.11)]. That is, it is the vector of aggregate anticipated supply functions. The inequality in equation (7.24) states that, in aggregate, farmers must operate around some expected point on the anticipated supply functions that lie at or above the intersections with demand. Because of the complementary slackness condition $X(\partial L/\partial X) = 0$, optimality will occur at the intersection point for all nonzero activities in the solution, and then $A - BWX$ is the intersection price vector. Conditions, however, have already been established for this to be an approximation to the vector of asymptotically expected prices, and if these conditions are met, the perfectly competitive solution for $\lim E(P) = E$ (marginal costs) will have been obtained.

Linear Programming Approximations

The aggregate model with the objective function defined in equation (7.21) is a quadratic programming problem. Because of the large dimensions of any realistic sector model, and the difficulties that still exist with computer codes for quadratic programming, it is clearly desirable to linearize this problem.

6. In order for the Kuhn-Tucker conditions to be necessary and sufficient, it is required that the matrix $(WBX + \Gamma)$ be positive semidefinite. Since W is a diagonal matrix, it is sufficient that the individual matrices B and Γ be positive semidefinite. This condition is invariably satisfied for covariance (Γ) matrixes and is generally satisfied for demand coefficient (B) matrixes.

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Duloy and Norton (1971, 1973) have shown how the term $Y'(A - 0.5BY)$, where $Y = WX$, can be linearized. To illustrate their method, consider the simplest case when B is diagonal, implying that the product demands are independent. Then, letting V_j denote the area under the demand curve (the definite integral) from 0 to Y_j for the jth product,

$$Y'(A - 0.5BY) = \sum_{j=1}^{n} V_j.$$

Now V_j is a quadratic, concave function when plotted against Y_j, and, since the programming model is a maximization problem, V_j can be approximated by a series of linear segments using conventional computer codes for linear programming. Duloy and Norton introduce segment weighting activities, V_{ij}, $0 \leq V_{ij} \leq 1$, $i = 1$ to k, for each V_j; assign segment upper bounds g_{ij} on Y_j, over which interval V_{ij} is relevant; and assign a single value of V_j—say, d_{ij}—which is to approximate V_j over the interval $Y_j \leq g_{ij}$. They then suggest that the part of the programming problem involving Max $Y'(A - 0.5BY)$, with $Y = WX$, be replaced by the linear programming problem:

$$\max \sum_j \sum_{i=1}^{k} d_{ij} V_{ij}, \tag{7.25}$$

such that

$$X_j m_j - \sum_i g_{ij} V_{ij} \geq 0, \quad \text{for all } j, \tag{7.26}$$

$$\sum_i V_{ij} \leq 1, \quad \text{for all } j. \tag{7.27}$$

This method adds only two rows for each product, but permits inclusion of as many V_{ij} activities as necessary to increase the accuracy of the approximation to any desired degree of precision.

When Γ is estimated on the basis of time-series data, an efficient way of linearizing the remaining quadratic term $\Phi\,(X'\Gamma X)^{1/2}$ is to use the mean absolute deviation (MAD) method proposed by Hazell (1971).

Let $r_{jt} = P_{jt} m_{jt}$ denote the tth year's observation on the revenue of the jth activity X_j, $t = 1$ to T, and let $\bar{r}_j$ denote the sample mean revenue for the activity over the T years.[7] Then, the MAD estimator of the standard deviation of income[8] is:

$$\text{est}(X'\Gamma X)^{1/2} = \Delta \frac{1}{T} \sum_t \left| \sum_j (r_{jt} - \bar{r}_j) X_j \right|, \tag{7.28}$$

7. The raw data should first be analyzed for any trend and other systematic movements over time, and these components removed to obtain a random residual.

8. Since costs are nonstochastic by assumption, the variances of income and total revenue are identical.

where

$$\Delta = \left(\frac{T\pi}{2(T-1)}\right)^{1/2}$$

is Fisher's (1920) correction factor to convert the sample MAD to an estimate of the population standard deviation,[9] and π is the mathematical constant.

To obtain a linear programming formulation, define new variables $z_t \geq 0$ for all t such that

$$z_t + \sum_j (r_{jt} - \bar{r}_j)X_j \geq 0, \tag{7.29}$$

where $\Sigma_j(r_{jt} - \bar{r}_j)X_j$ measures the deviation in total revenue from the mean, $\Sigma_j \bar{r}_j X_j$, for the tth set of revenue outcomes. Now, if the z_t variables are selected in a minimal way, then for each t either $z_t = 0$ when $\Sigma_j(r_{jt} - \bar{r}_j)X_j$ is positive, or z_t measures the absolute value of the negative deviation in total revenue when $\Sigma_j(r_{jt} - \bar{r}_j)X_j$ is negative. Consequently, $\Sigma_t z_t$ measures the sum of the absolute values of the negative deviations. Since the sum of the negative deviation around the mean is always equal to the sum of the positive deviations for any random variable, it follows that $2\,\Sigma_t z_t$ is the sum of absolute deviations in total revenue, and hence

$$\text{est}(X'\Gamma X)^{1/2} = \left(\frac{2\Delta}{T}\right) \sum_t z_t.$$

In summary, the appropriate linear programming subproblem to minimize $\Phi\,(X'\Gamma X)^{1/2}$ is:

$$\min K \sum_t z_t, \tag{7.30}$$

such that $z_t + \Sigma_j(r_{jt} - \bar{r}_j)X_j \geq 0$ for all t, where $K = \Phi\,(2\Delta / T)$ is a constant.

The reliability of this method, compared with using quadratic programming directly on $\Phi\,(X'\Gamma X)^{1/2}$, has been discussed elsewhere (Hazell 1971*a*; 1971*b*). It basically depends on the relative efficiency of the sample MAD compared to the sample standard deviation as an estimator of the population standard deviation.[10] Surprisingly, the sample MAD may actually be better than the sample standard deviation for skewed income distributions (Hazell 1971*b*), but it is less efficient for normal distributions (Hazell 1971*a*).

9. Strictly speaking, this form of Fisher's correction factor only holds when income is normally distributed (that is, $\Sigma_j r_{jt}X_j \sim N$). For other distributions, appropriate correction factors should be used (see Fisher 1920).

10. More favorable results about the ability of the sample MAD to rank farm plans efficiently (Thomson and Hazell 1972) do not hold in the current problem, because the MAD estimate of $(X'\Gamma X)^{1/2}$ is required to appear in the weighted objective function defined in equation (7.21).

In order to show how all these approximations fit together, a complete linear programming tableau is formulated in table 7-1, which approximates the solution to the original quadratic programming problem. If the quadratic variables V_j, $j = 1$ to n, are each approximated by k segments, the linearized problem adds an additional $kn + T$ activities and $2n + T$ rows to the problem, but this is highly efficient computationally when solutions are obtained through a revised simplex algorithm.

Conclusions

In this chapter a methodology has been developed for simulating, with a linear programming model, the market equilibrium of a perfectly competitive but risky agriculture in which producers behave according to an E,σ decision criterion. Such results are useful for comparative static analysis of policy problems, and, according to the degree of risk involved, should be more descriptive than existing models that ignore risk.

The development of this methodology has pinpointed a number of difficult issues with respect to both the design and implementation of aggregate risk models.

First, market equilibrium is considerably more complicated under risk than in a deterministic setting, with interactive effects between the way farmers form their anticipations about prices and yields and the properties of an equilibrium if attained. Consequently, it is difficult to design a general programming model that will always provide meaningful economic answers. In this chapter, the problem was resolved by specifying a set of plausible assumptions under which the proposed model is appropriate, albeit with an obvious loss in generality.

Second, an aggregate model must be defined with variables that are inherently difficult to measure. This is partly because such variables are based on individual farmers' utility functions and subjective expectations about stochastic variables and are therefore difficult to observe, but partly because such variables involve aggregation procedures that have not been adequately explored in the literature. Although practical ways of overcoming these problems are demonstrated in other chapters of this book, more refined procedures really ought to be developed.

Appendix. Derivation of Market Equilibrium Results for the Multiproduct Case

In the multiproduct case it is likely that there will be interrelations on the demand side as well as contemporaneous correlations between yields. With this in mind, the market structure can be written as:

Table 7-1. *Layout of Tableau for the Linearized Problem*

Constraint and equation number in text	*Production activities*			*n sets of activities to linearize areas under demand functions*					*T negative deviation counters*	*RHS*
	X_1	$X_2 \cdots\cdots\cdots\cdots$	X_n	$V_{11} \cdots$	V_{1k}	$V_{21} \cdots$	$V_{2k} \cdots$	$V_{n1} \cdots V_{nk}$		
Objective function, (7.21), (7.24), and (7.29)	$-c_1$	$-c_2 \cdots\cdots\cdots\cdots$	$-c_n$	$d_{11} \cdots$	d_{1k}	$d_{21} \cdots$	$d_{2k} \cdots$	$d_{n1} \cdots d_{nk}$	$-K \cdots\cdots\cdots -K$	(Max)
n commodity balance constraints, (7.25)	m_1	m_2		$-g_{11} \cdots$	$-g_{1k}$	$-g_{21} \cdots$	$-g_{2k}$			
		$\ddots$				$\ddots$				≥ 0
			m_n					$-g_{n1} \cdots -g_{nk}$		
n convex combination constraints, (7.26)				$1 \cdots 1$		$1 \cdots 1$				
						$\ddots$				≤ 1
								$1 \cdots 1$		
T revenue constraints containing sample data, (7.28)	$r_{11} - \bar{r}_1$	$r_{21} - \bar{r}_2 \cdots\cdots r_{n1} - \bar{r}_n$							1	
	$\vdots$	$\vdots$							$\ddots$	≥ 0
	$r_{1T} - \bar{r}_1$	$r_{2T} - \bar{r}_2 \cdots\cdots r_{nT} - \bar{r}_n$							1	
Resource constraints		D Matrix								$\leq b$ Vector

RHS = Right-hand side.

$$Y_t = \Gamma_t\, G_t\,(L)\, P_t$$

$$D_t = A - BP_t,$$

where Y_t and D_t are $n \times 1$ vectors of quantities supplied and demanded respectively; P_t is the $n \times 1$ vector of prices; Γ_t is the $n \times m$ matrix with ijth element $\gamma_{ijt} = \lambda_{ij}\, e_{ijt}$; $G\,(L)$ is a polynomial form in the lagged operator L; and A and B are the demand coefficient matrixes defined earlier.

It is again assumed that yields are distributed independently over time, so that cov $(e_{ijt}, e_{ijt-1}) = 0$, for all i, j, and that anticipated prices $\hat{p}_t$ are linear lagged functions of past prices. In particular, let:

$$\hat{p}_t = G_t\,(L)\, P_t = \begin{bmatrix} \sum_{i=1}^{m} \delta_{1t-i}\,(L^i) & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & & & & \cdot \\ \cdot & & \cdot & & & \cdot \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & \cdot & \cdot \\ \cdot & & & & & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot & \sum_{i=1}^{m} \delta_{nt-1}\,(L^i) \end{bmatrix} \begin{bmatrix} P_{1t} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ P_{nt} \end{bmatrix},$$

where $\Sigma_{i=1}^{m}\, \delta_{jt-i} = 1$, for all j.

The expected market clearing price is then

$$E(P_t) = B^{-1}\, A - B^{-1}\, \bar{\Gamma}\, G_t\,(L)\, E(P_t), \tag{7.31}$$

where $\bar{\Gamma} = E(\Gamma_t)$ has ijth element $\gamma_{ij} = \lambda_{ij}\, w_{ij}$, and $w_{ij} = E(e_{ij})$.

Equation (7.31) defines a system of mth order difference equations, and if convergence occurs, has the particular solution

$$\lim_{t \to \infty} E(P_t) = [I + B^{-1}\, \bar{\Gamma}\, G]^{-1}\, B^{-1}\, A, \tag{7.32}$$

where we have assumed that $G = \lim_{t \to \infty} G_t$ exists.

Conditions for convergence can be developed as follows. Decomposing the G_t matrix in equation (7.31) and applying the lag operator gives the more explicit version:

$$EP_t = B^{-1}\, A - B^{-1}\, \bar{\Gamma}\, [G_{1t}\, EP_{t-1} + G_{2t}\, EP_{t-2} + \ldots + G_{mt} EP_{t-m}],$$

where G_{it}, $i = 1$ to m, are $n \times n$ diagonal matrixes with jth diagonal element δ_{jt-i}. Necessary and sufficient conditions for equation (7.32) to hold are that the roots of the polynomial equation

$$\lim_{t \to \infty} [G_{1t}\, p^m + G_{2t}\, P^{m-1} + \ldots + G_{mt}\, P_1 = 0,$$

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where P^i is an $n \times 1$ vector with jth element $[p^i]$, are all within the unit circle. Sufficient conditions for convergence can, however, be obtained by noting that attainment of equation (7.32) is guaranteed if

$$\lim_{t \to \infty} (B^{-1} \bar{\Gamma} G_i)^t = 0, \quad i = 1 \text{ to } m,$$

but

$$\lim_{t \to \infty} (B^{-1} \bar{\Gamma} G_i)^{+t} = \lim_{t \to \infty} (H_i \Psi_i H_i^{-1})^t = \underline{0},$$

where

$$G_i = \lim_{t \to \infty} G_{it},$$

Ψ_i is the diagonal matrix formed by the characteristic roots of $B^{-1} \bar{\Gamma} G_i$, and H_i is the appropriate orthogonal transformation matrix.[11] This result implies that all the roots of the matrix $B^{-1} \bar{\Gamma} G_i$ must be real, nonnegative, and equal or less than 1, a result that generalizes the single product condition $(\lambda / b) < (1 / w)$.[12]

Turning now to the interaction price vector (P^*) for demands and anticipated supplies, we find that this occurs where:

$$\bar{\Gamma} G P^* = A - BP^*.$$

Solving, we obtain $P^* = [I + B^{-1} \bar{\Gamma} G]^{-1} B^{-1} A$, which is the same as

$$\lim_{t \to \infty} E(P_t)$$

in equation (7.32).

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11. Note that B, $\bar{\Gamma}$, and G are assumed to be positive definite.

12. This result corresponds to the well-known stability requirement for the system of stochastic difference equations in equation (7.30). (See Dhrymes 1970, p. 519.)

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