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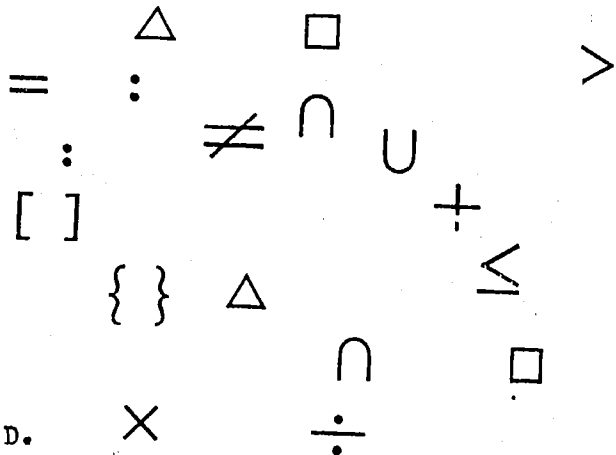

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BASIC CONCEPTS OF MATHEMATICS

volume one

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ENTEBBE MATHEMATICS SERIES

BASIC CONCEPTS OF MATHEMATICS

VOLUME ONE

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FOREWORD TO THE PRELIMINARY EDITION

This experimental text has been produced by the 1963 Entebbe Mathematics Workshop for use in training colleges in English-speaking Africa. The Entebbe programme is a comprehensive one. Its general purpose is to write texts which reflect recent thinking about mathematical education and which adapt this thinking to African conditions.

Rapid progress in science and technology and in mathematics itself has stimulated major efforts to improve mathematical education elsewhere. Principal emphasis has been put on understanding the ideas that have unified and simplified modern mathematics. In addition, methods have been adopted which lead the student to discover things for himself. New facts are established either from first principles or from facts already known, so that undue reliance on rote-learning is eliminated. A general discussion of these points will be found in the introductory section "Why Change Our Mathematics Teaching?"

The Entebbe Mathematics Workshop, comprising mathematicians and educators drawn from Africa, the United Kingdom and the United States of America, has produced experimental texts for use in Primary One and Two and in Secondary One and Two. These texts are being tried out in East and West Africa and will be revised in the light of experience in their use. Tests have also been devised by the Workshop to measure the effectiveness of the material under actual teaching conditions. Further texts are planned to follow those so far produced.

This experimental text is designed for use in training colleges for primary teachers. It aims to give teachers in training the kind of background of understanding which will help prepare them to teach the Primary texts produced by the Workshop, or other texts which are written to achieve the same purpose.

It is hoped that this text will be of interest to all those who are concerned with the new approaches to mathematics teaching.

This text is written so as to stimulate the discovery of central concepts by consideration of concrete examples. Problems are provided, both to deepen the understanding of the teacher and to assist in classroom teaching.

After the introductory section, the text is planned in two parts: "Structure of Arithmetic" and "Introduction to Geometry". At the 1963 Entebbe Mathematics Workshop the first four units out of a projected seven on "Structure of Arithmetic" were written by the Teacher Training Writing Group: basic concepts and language of sets, the whole numbers, the number line, and fractions. For expositions of negative numbers, the rational numbers, and the real numbers, which are the subjects of the units not yet written, appropriate portions of the Entebbe Secondary One and Two Student Texts and Teachers' Guides may be consulted. These presentations, however, are not necessarily organized in the same way as they may be written later for teachers. As for "Introduction to Geometry", indications of the directions the text for teacher training may take are to be found in Part 3: Content Outline for Primary I–III and the Appendix: Projected Content Outline for Primary IV–VI of the *Entebbe Mathematics Teachers' Handbook* prepared by the Primary Writing Group at the 1963 Entebbe Mathematics Workshop.

To enable both tutors and teachers in training to pursue the subject further, a bibliography has been provided at the end of the book. Tutors should encourage their students to read widely in order to further their knowledge of the new approach to mathematics.

FOREWORD TO THE REVISED EDITION

This edition of *Basic Concepts of Mathematics, Volume I,* is a revised version of the preliminary edition which was prepared during the summer of 1963 at Entebbe, Uganda. When the preliminary edition was published it included a statement acknowledging the existence of editorial and production defects occasioned by the speed with which it was produced. Nevertheless, it was felt that this was a small price to pay in return for having a much needed text in this subject available for the 1963-64 academic year. The intervening years have vindicated that decision: the book was warmly received in African Teacher Training Colleges, and is in ever-increasing demand today.

In view of this success a new edition has been prepared. Almost every chapter of the original text has been extensively rewritten, and a large number of new figures have been prepared. In particular, the chapters on sets and fractions received special attention and were substantially changed. Another significant change occurs in the treatment of the number line, which is now introduced early in the text (Chapter 5) and is used throughout the discussion of arithmetic much as the teacher might use it in the classroom. New problems have been added to most chapters, and old problems have been rewritten. A selected list of answers to problems has also been prepared and appears at the end of the book, together with an updated bibliography.

Of course, books are rarely, if ever, free from errors, and this one lays no claim to being an exception. Thus, in presenting this book for use in Africa, the African Education Programme can only repeat the hope that those who use the book will be tolerant of the imperfections that still remain.

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This experimental teacher training text was prepared during the summer of 1963 at Entebbe, Uganda, as part of a programme of curriculum revision being conducted by the African Education Programme of Educational Services Incorporated.

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CONTENTS

		Page
Introduction: Why Change our Mathematics Teaching?		xv
Basic Concepts and Language of Sets Unit I		1
CHAPTER 1	*Sets and Subsets*	2
CHAPTER 2	*Operations on Sets*	24
The Whole Numbers Unit II		35
CHAPTER 3	*Number*	36
CHAPTER 4	*The Set of Whole Numbers*	46
CHAPTER 5	*Introduction to the Number Line*	52
CHAPTER 6	*Order Properties on the Number Line*	58
CHAPTER 7	*Representation of Number*	62
CHAPTER 8	*Number Bases in the Hindu–Arabic System*	78
CHAPTER 9	*Addition*	86
CHAPTER 10	*Properties of Addition*	98
CHAPTER 11	*Subtraction*	112
CHAPTER 12	*Multiplication*	124
CHAPTER 13	*Properties of Multiplication*	133
CHAPTER 14	*Division*	147
CHAPTER 15	*A Summary of Whole Number Arithmetic*	160
CHAPTER 16	*Procedures for Addition and Subtraction*	164
CHAPTER 17	*Procedures for Multiplication and Division*	186
Fractions Unit III		199
CHAPTER 18	*Introduction to Fractions*	200
CHAPTER 19	*Properties of Fractions*	220
CHAPTER 20	*Addition and Subtraction of Fractions*	240
CHAPTER 21	*Multiplicaton and Division of Fractions*	264
Answers to Selected Problems		285
Bibliography		307
African Education Programme Mathematics Steering Committee		309
Index of Main Sub-headings		311

INTRODUCTION: WHY CHANGE OUR MATHEMATICS TEACHING?

We live in a rapidly changing world. The younger nations, like the older ones, face challenges which call for greater knowledge and greater willingness to learn new ways. This is why education is more important than ever before.

Everyone agrees that mathematics is important and that it should be taught in our primary and secondary schools. A knowledge of arithmetic is necessary for everyone in the modern world. A much greater knowledge of mathematics is required for those who take an active part in such things as harnessing water power to the needs of Africa or handling the financial problems of a nation. So we must teach mathematics well in our schools whether our students are to become ordinary citizens or leaders of their countries. But do we need to change the kind of mathematics we teach or the way that we teach it?

Most of those concerned with the teaching of mathematics feel dissatisfied with the job that has been done in the past. In spite of the importance of the subject, very few students have thought of mathematics as alive, exciting and interesting. This is not true of other subjects. There are many people who read history for fun. How many read books on mathematics for fun? Not very many! Everyone will agree that mathematics has not been a very popular subject. This points to a weakness in the way in which it has been taught. In any case, the methods of the past do not meet the challenges of the present and future.

But something new has happened. It has been found that students can actually get excited about mathematics and enjoy it tremendously. This was a fortunate discovery. If it is important to know mathematics, it is surely also important that students should find it interesting. The change which has come about is the result of a big effort to teach mathematics in a new way.

What is this new way? In brief, it is to get the student to *understand why* things work the way they do to a much greater extent than has been customary. More important is the fact that to interest the student we must get him to take an active part in

learning. He must be led to *discover* things for himself. This is true in all subjects; it is true in mathematics also. Mathematics is not a strange subject in which everything is different—where, for example, it is unnecessary to interest the student; where it is sufficient to drill him so that he always gets the right answer.

It is widely believed that there is only one right way to do a problem in mathematics, so that all we need to do is to show the student this right way and give him lots of practice in doing it. This idea leads the student to think that the only way he can be original is to be wrong. Of course, he is glad to oblige us! But, seriously, if we try to teach students to act like machines, we should not be surprised to find that those with independence and originality will rebel.

In fact, there are many ways to solve a mathematical problem correctly. Some may be shorter than others. Some may be longer but more illuminating. One way may seem more natural to one student than to another. Not all students are alike. The important thing is to find ways to solve problems that students can think out for themselves. In this case they will have a better chance of remembering them. Moreover the subject will make sense to them and catch their interest.

Every teacher imagines, we suppose, that he teaches the student to understand the subject. In a way this is true. It all depends on what you mean by "understand". We can, if we wish, say that a student understands how to divide one fraction by another if he can always apply the "invert and multiply" rule when he is given two fractions to divide. Thus if we ask him to divide $\frac{2}{3}$ by $\frac{3}{4}$ he will invert the $\frac{3}{4}$ to $\frac{4}{3}$ and multiply $\frac{2}{3}$ by $\frac{4}{3}$ to get $\frac{8}{9}$. In the process he also applies correctly the rule for multiplying two fractions: "Multiply the numerators and multiply the denominators." To repeat, we can say that the student understands a rule if he knows how to apply it with confidence and success. In the same way, many of us can truly say that we understand how to drive a car. But few of us understand *why* it operates the way it does.

There is therefore a deeper meaning of "understand". Suppose that the student asks, "Why do we invert and multiply when we wish to divide one fraction by another?" To answer this question it is not enough to say, "That is the rule". What we have to do is to give a reason for the rule. We have to lead the

student to see that the rule expresses what *ought* to be done. We have to *explain* the rule. An explanation always has to be given in terms of something else—something which is already known.

Later we shall return to this example and show what kind of an explanation could be given. We are not quite ready for it now. The point is that mathematics hangs together. It has a plot like a novel. Knowledge has to be built up in stages. You have to know something about what happened in the earlier chapters. In this respect, mathematics is again like other subjects. You cannot understand modern physics without first learning something about earlier discoveries and ideas, nor modern history without some knowledge of its background.

We have said that we should teach mathematics so that the student can discover things for himself and can understand why he does what he does. This is what makes learning exciting.

But there is another difference between the newer approach to the teaching of mathematics and the older one. The difference is particularly important for the students who will go on to more advanced mathematics.

Mathematics itself has changed. It has surprised many people to learn this. They know that science and technology do not stand still. They expect a man to be sent to the moon one of these days. New discoveries are made and science grows and renews itself like a living thing. New knowledge cannot be added to the old like a new room built on to a house. The whole subject must be rebuilt from time to time to take account of new and better ways of thinking about old facts. New ways of talking about the older knowledge are invented which help us to understand it better and to connect it with recent discoveries.

Few people think of mathematics in the same way. Almost everyone tends to think of it as somehow finished and complete, so that what we know was discovered long, long ago and has not been added to since. This is simply not true. Mathematical knowledge is growing faster than ever before. New mathematics is constantly being created to answer questions that the scientist and engineer need to answer so that they can forge ahead in new directions.

The situation is exactly the same as with all other subjects of knowledge. There are many new important things that have to be added to existing knowledge if we are to have the tools we

need to solve our problems. There is only so much time. Something has to be done to make room for the new. There are two things which can be done:

(1) Rework the older material so that it hangs together better, is more understandable and ties up with the new.
(2) Leave out some things which no longer seem as important as they once did.

Both these have been done. If we think about it, it seems obvious that this should be so. Mathematics is not something apart, unconnected with other human concerns. Throughout history it has been developed to solve problems which mankind needed to solve. As human needs have changed with changing conditions, mathematics has spread out in new directions and set itself to new kinds of problems. Like everything alive, it must meet new challenges or die.

We have said that to make room for newer mathematics we have to look at the older mathematics in a different way. We arrange the knowledge in new patterns. It is the same with technology. We are continually finding new and more suitable ways of doing old things. Africa is able to take advantage of the experience of the Western nations over the past century or more. It is unnecessary to go through all the stages over again. We can profit by the experience of others. In mathematics, too, we have learned by experience. Quicker and better ways of doing things have been found. The Greeks made wonderful discoveries in geometry 2,000 years ago but mankind has not been idle all these years: for example, the volume of a sphere can now be found by much simpler methods than the Greeks knew about. This does not mean that they were wrong. It does mean that a good deal has happened since their time. We do make progress.

It is like the opening up of a new territory. A fertile valley may have been reached by a very roundabout route. Once discovered it is possible to get there by a shorter route. In the end paved roads are built which help us to get where we want to go quickly and comfortably. We do not have to follow the old country roads. Mathematicians have been busy building wide straight roads so that they can get to the limits of the known fairly quickly. They are not worn out by the time they reach the frontier. This is lucky, because there is too much mathematics for any one person to know in detail.

The newer programmes of instruction have been worked out with the advice of mathematicians. They have tried to save time and labour for the students who are coming on, and so to make their paths easier and more comfortable. They have hoped that the student will reach places where he can look over the landscape and enjoy the view without losing himself in the bush.

What kind of understanding do we hope that the student will reach? What sort of views do we expect him to get on this journey through mathematics? We surely want him to think of mathematics as more than a collection of unrelated facts to learn by heart. We hope that he will see how the facts fall into patterns so that they make sense to him. For one thing, when he discovers these patterns he will not have to remember so much. If he should forget something, it is not lost for ever. He can work out for himself what he needs to know. He will not be like a man lost in a rain forest.

To take a simple example, suppose that the pupil has forgotten how to add fractions, say to find $\frac{2}{5} + \frac{3}{2}$. This is a matter which puzzles many adults. If he multiplies $\frac{2}{5} + \frac{3}{2}$ by 10, thus getting $10 \times (\frac{2}{5} + \frac{3}{2})$, he easily gets $10 \times \frac{2}{5} + 10 \times \frac{3}{2}$ which is $4 + 15 = 19$. If 10 times the required answer is 19, that answer must be $\frac{19}{10}$.

Again if the pupil wishes to multiply 82 by 98 he can of course multiply in the well-known way. If, however, he notices that $98 = 100 - 2$, he can multiply 82 by 100, which is easy, then multiply 82 by 2 which is also easy, and do a simple subtraction. This makes arithmetic more fun. It changes it from a dull routine into something more like a game. It gives the student a chance to use his mind instead of operating like a machine. A machine does not *notice* anything it has not been instructed to notice. It does not discover patterns for itself.

It sometimes happens that students who have been encouraged to work things out for themselves discover new ways of doing things. For example, a class was asked to subtract 28 from 42. Of course the standard way to do this is to say that 8 cannot be subtracted from 2. Therefore we must borrow from

$$\begin{array}{r} 42 \\ 28 \\ \hline 14 \end{array}$$

the next column and subtract 8 from 12 giving 4 in the ones place and 1 in the tens place. One student did this in quite a different way. The class had some familiarity with negative numbers. The boy said that $2 - 8 = -6$ and $40 - 20 = 20$. He reasoned that the answer must be the sum of $20 - 6$, that is 14. We do not say that this is a better way to do this exercise. But it is obvious that to be able to invent this new method is to add greatly to one's understanding. It also gives the student the very pleasant feeling of creating something new.

$$\begin{array}{r} 42 \\ 28 \\ \hline -6 \\ 20 \\ \hline 14 \end{array}$$

This is an important feeling for the student to have. It gives him confidence in using his own mind. In a changing world, we meet new situations. We cannot meet all of our problems by following rules. We have to invent new ways of doing things. This means that we must look for new ways of thinking. The most important thing which any teacher can do is to encourage any sign of originality in his students. If one of them has a new way of doing a problem, do not tell him that he must do it in the textbook way. Let him discover for himself that the textbook way is better, if it is. Meanwhile give him the pleasure of using his own mind.

Basic Concepts and Language of Sets UNIT I

Introduction

A flock of goats, a herd of cattle, a class of pupils are all examples of *sets*. Each of the words "flock", "herd", "class", implies that we are thinking about a collection of things. Every language has a large number of words with this meaning. A few more in English are:

bunch—a bunch of bananas
bundle—a bundle of sticks
family—a family of leopards, or a family of people
pack—a pack of cards, or a pack of wolves

In mathematics a single word is used to talk about a collection of things. That word is SET. The concept of a set is the basic idea in mathematics. In this unit you will learn some of the elementary facts about sets. In the later units these facts will be used to study numbers and the arithmetic taught in school.

Chapter 1

SETS AND SUBSETS

1-1 Sets and elements

Since we shall be talking about sets constantly from now on, we define the word again:

A SET is any collection of things

The important point to remember here is that when we talk about a set we are not talking about the particular things which make up the set. We are talking about the collection as a whole.

It is easy to give examples of sets. For instance:

A. The set consisting of the numbers 1, 2, 3, 4, 5
B. The set of letters in the English alphabet
C. The set of pupils in your school

You can certainly think of many more such examples.

In describing a set there is only one rule that must be followed. It is this. A set must be described so that there is no doubt about which objects are in it and which are not. Thus a phrase such as "a set of pupils" does *not* describe a specific set, because it does not tell us which pupils belong to the set and which do not. On the other hand, the phrase "the set of all pupils in Primary 1 in Uganda" does describe a set. For in this case we can tell whether a particular pupil is in the set or not.

Suppose you are given a set. Any set at all. Then each of the objects in that set is called an ELEMENT or MEMBER of the set. We say that an element BELONGS to a set. Thus the number 1 is an element of the set in Example A above. It belongs to that set. So do the numbers 2, 3, 4 and 5. All of them are elements of the set.

Note that to describe a set you must say precisely what are its elements.

EXERCISE 1-1

1. What are the elements which belong to each of the following sets?

a. The set of days in the week whose English names begin with the letter S.
b. The set of the last six letters in the English alphabet.
c. The set of months in the year whose English names begin with the letter J.
d. The set of letters in your name.
e. The set of persons who live in your home.

2. Here are several phrases. Some of them describe sets; some do not. Decide which do and which do not.
a. The set of all African countries.
b. The set of summer months in the year.
c. The set of all small whole numbers.
d. The set of all books in your school library.
e. The set of all Nigerian citizens.
f. The set of all numbers less than 100.
g. The set of all mountains in the world more than 10,000 feet high.
h. The set of all children.
i. The set of all large islands.

1-2 Describing sets

We have seen how sets can be described in words by saying such things as "the set of all letters in the English alphabet". In fact, we constantly give such descriptions when we talk to one another, even though we may not use the word "set". Everyone says such things as "my family", "our football team", "the population of Africa". And when he does, he has described a set.

However, there is another way of describing sets which each of us also uses almost every day. It consists of listing the elements in the set. Every teacher does this when he writes out a list of the pupils in his class. This list describes the set of pupils in his class. A newspaper does the same thing when it lists the names of the players in a football match. It is describing the set of people who played in that match.

In mathematics we also describe sets by listing their elements. When this is done we always enclose the list in curly brackets. The brackets are used to show that we are thinking about the

list as a set. Thus the set consisting of the numbers 1, 2, 3, 4, 5 would be described by writing

$$\{1, 2, 3, 4, 5\}$$

We read this as "the set whose elements are the numbers 1, 2, 3, 4 and 5". The commas are used to separate the elements in the set. Other examples of sets described by lists are

$$\{\text{Ghana, Kenya, Nigeria}\}$$
$$\{a, b, c, 1, 2\}$$

The first of these is the set whose elements are the nations Ghana, Kenya and Nigeria. The second is the set whose elements are the letters a, b, c and numbers 1, 2. This last example shows that the elements of a set do not all have to be of the same type. That is, all of them do not have to be letters, or all of them numbers, etc. There may be elements of many different kinds in the same set.

Now consider the following example. At a party two girls wore green dresses, one wore a white dress and three wore red dresses. The colours of the dresses form a set which we might write as

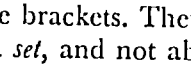
$$\{\text{green, green, white, red, red, red}\}$$

But the only colours which belong to this set are green, white and red. Thus it is enough to write

$$\{\text{green, white, red}\}$$

to describe the set. We do not have to repeat any of the colours. From now on we agree not to repeat elements inside the curly brackets when describing a set by a list since listing an element several times does not tell us more than we already know.

In teaching young children about sets it is often helpful to draw pictures. For instance, you might want to talk about a set whose elements are a circle, a triangle and a square. One way of doing this is to draw a picture such as



Note that we still use brackets. They tell the children that we are thinking about a *set*, and not about three separate things. As you draw more and more of these pictures, using brackets

each time, your pupils will learn to think about sets and to use brackets correctly.

1-3 Describing sets (continued)

At times it is unreasonable or impossible to list all of the elements in a set. This happens whenever the set has a large number of elements. However, we may still want to describe the set by a list. To do this we leave out part of the list, and put three dots in its place.

For example, suppose we wanted to describe the set of letters in the English alphabet. If we wished we could write them all down. But the list would be rather long. A much better idea would be to write something like this:

$$\{a, b, c, \ldots, z\}$$

The three dots take the place of the missing letters. We imagine that the letters are still there, even though they are not written down. In the same way

$$\{2, 4, 6, 8, \ldots, 20\}$$

is just an easy way of writing

$$\{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$$

Both stand for the same set.

When you replace part of a list by three dots you must make sure that enough of the list is left so that everyone knows what you want in place of the dots. Thus, if you write

$$\{2, 4, 6, 8, \ldots, 32\}$$

no one will misunderstand you. However,

$$\{2, 4, \ldots, 32\}$$

is not enough, because someone might think you meant

$$\{2, 4, 8, 16, 32\}$$

Sometimes in making a list we have no choice. We *must* use three dots. For instance, to describe the set consisting of the numbers 1, 2, 3, 4, *and so on*, we write

$$\{1, 2, 3, 4, \ldots\}$$

Here the three dots stand for the words "and so on". They mean that the list goes on and on, without stopping. Of course, we could also describe this set by writing

$$\{1, 2, 3, \ldots\}$$

or

$$\{1, 2, 3, 4, 5, \ldots\}$$

The important point is to write out enough of the list so that everyone will know what you have in mind.

Other examples of this sort are

$$\{2, 4, 6, 8, \ldots\}$$
$$\{1, 3, 5, 7, \ldots\}$$
$$\{5, 10, 15, \ldots\}$$

Do these lists tell you enough for you to write the next element in each of them? The next three? The next one hundred? If they do not, more elements should be listed.

EXERCISE 1-3

1. Describe the following sets in words.
   a. {Thursday, Tuesday}
   b. $\{c, a, b, e, d\}$
   c. {January, February, March}
   d. {2, 4, 6, 8, 10}
   e. {25, 20, 15, 10, 5}
2. What are the next three elements in the lists for the following sets?
   a. $\{10, 20, 30, \ldots\}$
   b. $\{2, 5, 8, 11, \ldots\}$
   c. $\{100, 96, 92, 88, \ldots, 0\}$
   d. $\{1, 10, 100, 1000, \ldots\}$
   e. $\{1, 4, 9, 16, 25, \ldots\}$
   f. $\{2, 22, 222, \ldots\}$
   g. $\{99, 88, 77, \ldots\}$
3. For which of the following sets is it possible to list all of the elements, provided you had the necessary information?
   a. The set of all whole numbers from 1 to 1,000,000.
   b. The set of all whole numbers greater than 10 which end in a 3.
   c. The set of all words in the English language.

d. The set of all living human beings.
e. The set of all whole numbers exactly divisible by four.

4. Use the three-dot notation to describe the following sets:
a. The set of whole numbers between 1 and 1,100 which are exactly divisible by 3.
b. The set of whole numbers from 9 to 100.
c. The set of whole numbers ending in 5.
d. The set of whole numbers greater than 3 which leave a remainder of 1 when divided by 3.
e. The set of all whole numbers which can be written using only the digit 7.
f. The set of all whole numbers ending in 5 or 6.

1-4 Special sets

We have given several examples of sets, and you have found more in the exercises. In this section we shall describe a number of special sets which will be used frequently from now on.

I. The first of these special sets is the set containing no elements at all. It is called the EMPTY SET. But what do we mean by speaking about a collection when no elements belong to the collection? Surely this is never done in everyday life. Or is it? Let's take a closer look.

Suppose you happen to meet a tourist who knows very little about Africa, and suppose he asks you to think about the set of all cities in Africa with a population of more than 5 million. Has he described a set? Yes, because we can tell whether any particular African city belongs to this set or not. But this set has no members at all. It is empty.

Another example. Consider the set of all people in a certain classroom at your school. At nine o'clock on a schoolday morning this set will have many elements. But what would you say about the set at midnight? Again, the set has no elements since no one is in the room. It is the empty set. The empty set is very important in mathematics. We shall meet it again when we talk about the number 0.

We shall represent the empty set by a pair of brackets with nothing inside:

$$\{\qquad\}$$

II. We have said that in mathematics the term "set" is used to refer to a collection of things. Any collection whatever. Thus a collection of things which has only one member is a set. Such sets are very common. For example, the set

$$\{3\}$$

is a set whose only member is the number 3. Similarly, the set of all headmasters of your school, and the set of all children in a family with only one child are sets of this type. Try to give more such examples.

Our reason for calling attention to these sets is to emphasize that a *set* containing only one element is something quite different from the element itself. The first is a collection; the second is a particular individual.

III. Finally, we introduce two special sets which arise constantly in arithmetic. The first of these is the familiar set

$$\{1, 2, 3, 4, \ldots\}$$

Its elements are the numbers 1, 2, 3, 4, and so on, which we use for counting things. For this reason these numbers are all called COUNTING NUMBERS, or NATURAL NUMBERS, and the set itself is known as the set of counting numbers, or the set of natural numbers.

In arithmetic we also need the set whose elements are the counting numbers together with the number zero. In order to distinguish this set from the set of counting numbers we call its elements WHOLE NUMBERS. Thus the set of whole numbers is the set.

$$\{0, 1, 2, 3, \ldots\}$$

The only difference between this set and the set of counting numbers is that it contains the number 0.

EXERCISE 1-4

1. Use the bracket notation to describe the following sets:
   a. The set of months of the year whose English names begin with the letter N.
   b. The set of days of the week whose English names begin with the letter N.
   c. The set of all counting numbers less than 1.

d. The set of all counting numbers less than 100 which are exactly divisible by 10 *and* by 11.

e. The set of all counting numbers less than 50 which end in 2 and are exactly divisible by 7.

2. Is the set consisting of just the whole number 0 empty? Explain your answer.

1-5 Equal sets

When you say that two plus two equals four, and write

$$2 + 2 = 4$$

you mean that $2 + 2$ and 4 stand for the same number. In other words, what you write on one side of the sign $=$ and what you write on the other side are just different names for the same number.

This is the point of view which we want to adopt. The statement "$a = b$" means that the thing named "a" and the thing named "b" are the same. For instance, when we write $S = \{1, 2, 3, 4, 5\}$ we mean that S and $\{1, 2, 3, 4, 5\}$ stand for the same set.

Here are some more examples:

Africa = the second largest continent
Tokyo = the capital of Japan
$(5 + 3) + 4 = 12$
$\{1, 2, 3\}$ = the set of counting numbers less than 4

Let us apply this to the following sets:

A = $\{1, 3, 4, 5, 2\}$
B = $\{2, 4, 1, 3, 5\}$
C = the set containing the first five counting numbers

If you compare set A with set B you will find that they contain exactly the same elements.

The elements of set A are the numbers 1, 2, 3, 4, 5.

The elements of set B are the numbers 1, 2, 3, 4, 5.

The only difference between A and B is the order in which their elements were listed. But, as you will recall, the definition of a set makes no mention of any order for elements. Order

doesn't matter. It follows that A and B are different names for the same set, and we write A = B.

Now look at set C. It too has the numbers 1, 2, 3, 4, 5 as elements. Since these are the same elements which belong to A and to B, we can say that B = C and A = C. Altogether we have A = B = C.

This example illustrates the principle of equality for sets:

Two sets are EQUAL when they contain the same elements

Thus whenever we write A = B for sets we mean that every element of A is also an element of B, and every element of B is also an element of A.

EXERCISE 1-5

1. Which of the following pairs of sets are equal?

a. A = $\{a, c, e, f\}$
B = $\{c, f, a, e\}$

b. A = $\{a, c, e, f\}$
B = $\{a, c, f\}$

c. A = the set of different letters in the word "banana"
B = $\{a, b, n\}$

d. A = the set of pupils in a class containing 10 children
B = $\{1, 2, 3, \ldots, 10\}$

e. A = the set of pupils in a class containing 10 children
B = $\{10\}$

f. A = $\{0\}$
B = $\{\ \}$

g. A = the set of counting numbers which end in 5 and are exactly divisible by 4
B = $\{\ \}$

h. A = the set of counting numbers less than 99 which are exactly divisible by 9
B = $\{10\text{-}1, 20\text{-}2, 30\text{-}3, \ldots, 100\text{-}10\}$

1-6 Subsets

Consider the following pairs of sets:

(*a*) A = $\{1, 2\}$ B = $\{1, 2, 3, 4\}$
(*b*) A = $\{a, e, i, o, u\}$ B = $\{a, b, c, \ldots, z\}$
(*c*) A = {book} B = {pin, tree, book}

In each case you will find that every element of set A is also an element of the corresponding set B. When this happens we say that A is a *subset* of B. Thus in the first pair of sets (*a*) the elements of A are the numbers 1, 2, and both of these numbers belong to B. Similarly, in (*b*) A contains the letters a, e, i, o, u, and each of these letters belongs to set B because B contains all the letters in the English alphabet. Finally, in (*c*) the only element of A is "book", and this element also belongs to B. In each case we say that set A is a SUBSET of the corresponding set B.

In general, the term "subset" is defined as follows:

If each element of set A is also an element of set B, then set A is called a SUBSET of set B.

Sets which are subsets of other sets appear continually in mathematics. As illustrations we give some examples from arithmetic.

(*a*) The set

$$A = \{0, 2, 4, 6, \ldots\}$$

is called the set of EVEN WHOLE NUMBERS. Every number which belongs to this set is said to be an even number. Thus a whole number is even if it is exactly divisible by 2. Remember that 0 is exactly divisible by 2 since $0 \div 2 = 0$, with no remainder. The set of even whole numbers is a subset of the set of all whole numbers.

(*b*) The set

$$A = \{1, 3, 5, 7, \ldots\}$$

is called the set of ODD WHOLE NUMBERS. Every number which belongs to this set is said to be an odd number. Thus a whole number is odd if it is not exactly divisible by 2. The set of odd numbers is a subset of the set of all whole numbers. It is also a subset of the set of all counting numbers.

(*c*) Another interesting subset of the set of whole numbers is the set of prime numbers. A whole number is said to be a PRIME NUMBER if it is greater than 1 and is exactly divisible only by itself and 1. Thus 2 is a prime number, since it is greater than 1 and is exactly divisible only by 2 and 1. The number 3 is also prime. However, 4 is not, since it is divisible by

2. You should note that 1 is *not* a prime number. The first ten prime numbers are

$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29$$

The set of all prime numbers is a subset of the set of all whole numbers.

It is an interesting problem to determine how many numbers there are in the set of all prime numbers. What do you think? Does the list go on and on, or is there a largest prime number? This is not an easy question to answer. However, mathematicians have shown that the list of prime numbers goes on and on without stopping. In other words, there are infinitely many prime numbers. This fact was first proved by Euclid in about 300 B.C.

EXERCISE 1-6

1. Let $X = \{1, 2, 4, 6, 7, 9, 11, 12\}$
   a. Which of the following sets are subsets of X?
   (i) $\{4, 7, 11\}$ (ii) $\{11, 2, 6, 1\}$
   (iii) $\{7, 3, 9\}$ (iv) $\{11\}$
   (v) The set made up of the first three even counting numbers.
   (vi) The set made up of the first two prime numbers.
   b. What is the subset of X made up of all the even numbers in X?
   c. What is the subset of X made up of all the odd numbers in X?
   d. What is the subset of X made up of all the prime numbers in X?
   e. What is the subset of X made up of all the elements of X such that twice the element is not in X?
2. Which of the following sets contain only prime numbers?
   a. $\{2, 19, 59, 71\}$ **b.** $\{1, 5, 17\}$
   c. $\{47, 89, 101, 103\}$ **d.** $\{79, 93, 111\}$
3. List the members of the set of all prime numbers less than 50.
4. Use the bracket notation to describe the following sets:
   a. The set of even numbers between 35 and 50.
   b. The set of whole numbers which end in 1, 3, 5, 7 or 9.
   c. The set of even prime numbers.

d. The set of prime numbers exactly divisible by 4.
e. The set of odd whole numbers exactly divisible by 3.
f. The set of even whole numbers exactly divisible by 3.

5. Let X, Y and Z be sets. Suppose that X is a subset of Y, and that Y is a subset of Z. Is X a subset of Z? Why?

1-7 More about subsets

Consider the following three sets:

A = the set of all pupils in your school
B = the set of all boys in your school
G = the set of all girls in your school

Then B and G are subsets of A, since every member of B belongs to A and every member of G also belongs to A.

Keeping this in mind, suppose your school is a boys' school. Then every member of set A will be a boy, and the subset B will be the same as A. In other words B = A. This says that B is just another name for the set A. But B is a subset of A. Thus A is a subset of itself.

This example illustrates the following general fact about the subsets of a set:

For any set A, A *is a subset of itself*

Now look at the subset G—the girls in your school. If the entire school consists of boys, then G will be the empty set. It will not have any members at all. Yet it still makes sense to speak about the subset of girls in your school. For this reason we agree that G is a subset of A even when G is empty.

This illustrates a second fact about the subsets of a set:

The empty set is a subset of every set

These two facts about subsets may be a little disturbing at first. So we shall say more about them.

Let A be a set, any set at all. Then a set B is a subset of A if every element of B belongs to A. Since every element of A clearly belongs to A, our definition of the term "subset" forces us to conclude that A is a subset of itself.

Now suppose E is the empty set. Does our definition also imply that E is a subset of A? In other words, does every

element of E belong to A? You may be puzzled about the answer to this question since E does not contain any elements. To see that the answer should be yes, it is helpful to turn things around, as follows. E will be a subset of A if there are no elements of E which do *not* belong to A. (If you study the last sentence carefully you will see that it says that every element of E belongs to A.) But since E is empty there are no elements of E which do not belong to A. Hence E is a subset of A.

Using what we have just learned, it is easy to find all of the subsets of a set A, provided A does not have too many elements. For instance, let

$$A = \{1, 2\}$$

Then the subsets of A are

$$\{\ \}, \{1\}, \{2\}, \{1, 2\}$$

Similarly, if

$$A = \{1, 2, 3\}$$

the subsets of A are

$$\{\ \}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$$

Note that we do not list both $\{1, 2\}$ and $\{2, 1\}$ as subsets of A. Can you tell why?

EXERCISE 1-7

1. Write down all subsets of the following sets.

   a. $\{x, y\}$ **b.** {▲, ●, ▨} **c.** $\{a\}$

2. Let A and B be sets. Suppose that A is a subset of B. Under what condition is B a subset of A?

3. Let $A = \{\ \}$ be the empty set. Find all the subsets of A. How many are there?

4. Let $A = \{1\}$. Show that A has exactly two subsets. Do you think that every set containing a single element has exactly two subsets? Why?

5. Let A be any set containing two elements. Show that A has exactly four subsets.

6. Let A be any set containing three elements. Show that A has exactly eight subsets.

7. If you worked Exercises 3, 4, 5 and 6 you discovered the following facts:

Number of elements in set A	*Number of subsets of set A*
0	1
1	2
2	4
3	8

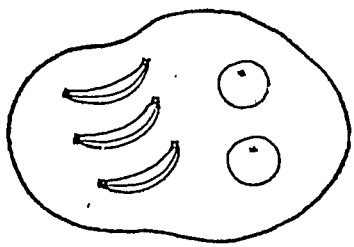
a. On the basis of this table, how many subsets would you expect to find when A contains four elements? Verify your answer by writing down all subsets of the set {1, 2, 3, 4}.

b. How many subsets would you expect to find when A contains five elements? How many when A contains six elements?

1-8 Picturing sets and subsets

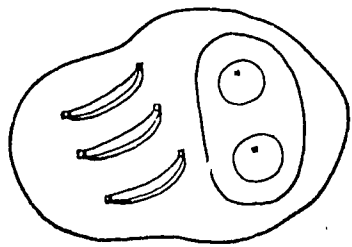
When teaching children about sets, and when studying them ourselves, it is often helpful to draw a picture of the set.

For example, suppose you wanted to talk to your class about a set containing three bananas and two oranges. The children might find it easier to understand what you say if you draw a picture like this:



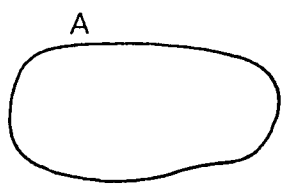
You could then illustrate the idea of a subset for them by

drawing a curve around the subset of oranges as shown below.

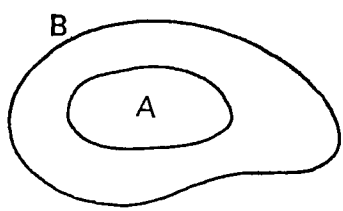


Then you might get them to use the same method to find subsets containing two bananas and one orange, and so forth. In this way children rapidly learn what the term "subset" means.

When mathematicians think about sets, they usually do not draw pictures like those shown above. Instead, they simply draw a curve such as the one shown below, and imagine that the set is represented by the region inside the curve. The shape of the curve makes no difference. Any curve will do. Sometimes letters or words are attached to the picture so that we can keep track of which set is being shown. Thus the picture below shows a set A. Remember, the region inside the curve represents the set.



If A and B are sets, and A is a subset of B, we draw another picture as follows:



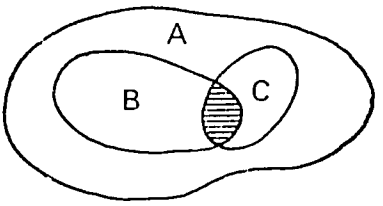
This shows A lying entirely inside B, and means that everything in A also belongs to B.

Example

Draw a picture to illustrate the sets A, B, C when

A = the set of all people in Nigeria
B = the set of all males in Nigeria
C = the set of all Nigerian children

Here both B and C are subsets of A. Thus in our picture B and C should lie inside A. Moreover, B and C have some members in common, since any child who is also a male is a member of B and of C. However, neither of these sets is a subset of the other. (Why?) Since our picture should show all of these facts it must look like this:

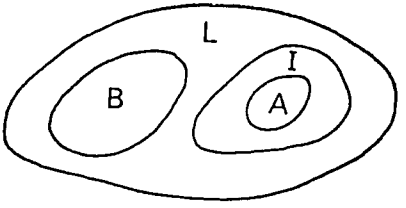


The shaded region shows that part of A which belongs to *both* B and C.

As we continue, you will find that pictures like this are very helpful in studying sets.

EXERCISE 1-8

1. Let A = $\{a, b, c, e, f\}$
   a. Draw a picture of set A.
   b. In your picture show a subset B of A.
   c. Show a subset C of B. Is C a subset of A?
2. In the following picture
   L = the set of all living things
   B = the set of all birds
   I = the set of all insects
   A = the set of all ants



Use the picture to decide whether the following statements are true or false:

a. All ants are insects.
b. Some insects are birds.
c. Some insects are ants.
d. All birds are living things.
e. Every living thing is a bird or an insect.
f. No ants are birds.

3. **a.** Draw a picture to show the following sets:
   U = the set of all people in the world
   A = the set of all Africans
   G = the set of all Ghanaians
   W = the set of all women in the world

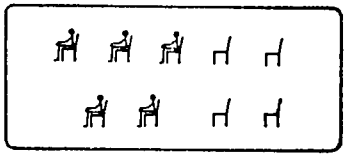
   b. Use your picture to decide whether the following statements are true or false:
   (i) All Africans are women.
   (ii) Some Ghanaians are not women.
   (iii) All Africans are people.
   (iv) All Ghanaian women are Africans.
   (v) No African women are Ghanaians.
   (vi) Some Ghanaians are not African women.

4. Draw a picture to show the following sets:
   C = the set of all counting numbers
   E = the set of all even counting numbers
   O = the set of all odd counting numbers
   F = the set of all counting numbers exactly divisible by 4
   P = the set of all prime numbers greater than 2
   Q = the set of all prime numbers greater than 100
   R = the set of all odd numbers greater than 50

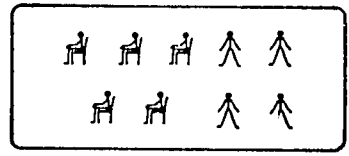
1-9 Equivalent sets

Imagine a room of children who cannot count. Suppose there are some chairs in the room, and suppose you ask the children to find out whether there are more chairs in the room than children, more children than chairs or the same number of each. How can the children answer your question? It is easy. They simply sit on the chairs; one child to one chair. If, when

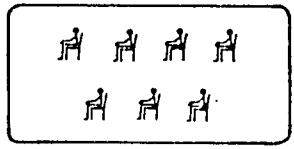
this is done, every child is seated and some chairs are still empty, there are more chairs than children. If every chair is filled and some children are still standing, there are more children than chairs. If every chair is filled and every child is seated, the number of chairs and children is the same. When this happens we say that the set of chairs and set of children *match exactly*.



More chairs than children



More children than chairs



Chairs and children match exactly

This example shows how we can tell when two sets have the same number of elements *without counting*. We simply try to match the elements in the sets and watch what happens. If we can make the sets match exactly they have the same number of elements. Otherwise they do not. Surprising as it may seem, this is one of the most important ideas in arithmetic. For this reason we have a special word to describe sets whose elements match exactly. We say that the sets are *equivalent*. Thus:

Two sets are EQUIVALENT if their elements match exactly

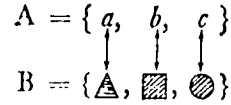
We now look at some examples.

Example 1

Let

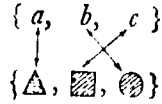
$$A = \{a, b, c\}$$
$$B = \{\qquad\qquad\}$$

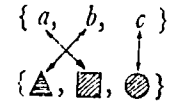
These sets are equivalent. To show this we must make the elements of A and B match exactly. In other words, for each element in set A we must find a partner in set B. Moreover, we must do this in such a way that each element in set B also has a partner in set A. The following picture shows how this can be done.



The double arrows in the picture show how the elements in the two sets can be matched with one another. Notice that every element in set A is matched with exactly one element in set B, *and* that every element in set B is matched with exactly one element in set A. The two sets match exactly. Thus they are equivalent.

The elements of A and B can also be matched in several other ways. Two other ways are shown below.

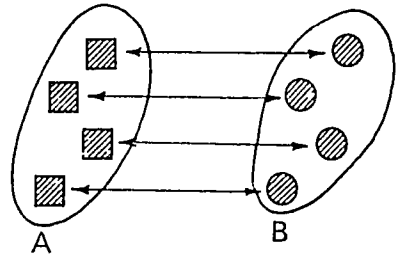


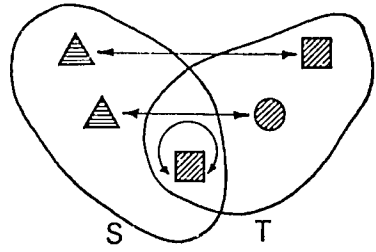


Each of these pictures shows that A and B are equivalent sets. The easiest way to show that two sets are equivalent is to draw pictures like these.

Example 2

The following pictures show pairs of equivalent sets. In each picture we have matched elements by drawing double arrows.





1-10 More about equivalent sets

We have said that two sets are equivalent if their elements can be matched exactly. You must be careful not to confuse equivalent sets with equal sets. Sets are equal only when they have the same elements. Sets are equivalent when their elements match exactly. For instance, the sets

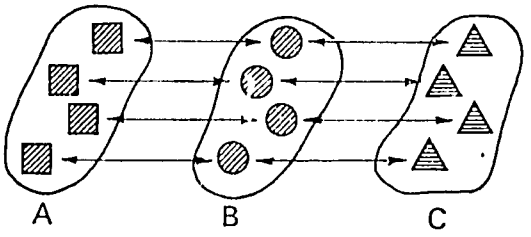
$$\{a, b, c, d\} \quad \text{and} \quad \{c, b, a, d\}$$

are equal,while the sets

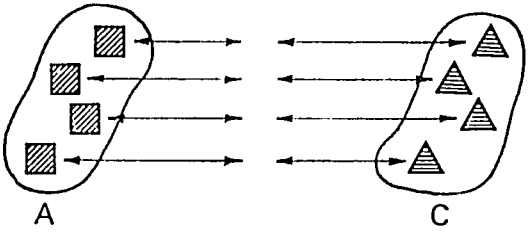
$$\{a, b, c, d\} \quad \text{and} \quad \{w, x, y, z\}$$

are equivalent but *not* equal Thus sets can be equivalent without being equal. However, it is easy to show that equal sets are always equivalent. Can you?

Now let us look at three sets, A, B and C. Suppose that A is equivalent to B, and that B is equivalent to C. We can show this by drawing a picture of the following type:



The double arrows from A to B show that the elements of A and B match exactly. Thus A and B are equivalent sets. Similarly B and C are equivalent sets. Now suppose set B is removed from the picture. We then have



From this we see that A and C are also equivalent sets. Do you think that this result is true for any three equivalent sets?

Pictures such as the ones we have just drawn show that it must be true. Thus:

Given sets A, B, C with A and B equivalent, and B and C equivalent,
then
A and C are equivalent

This is a very important fact about equivalent sets.

The idea of equivalence for sets is just another way of expressing the fact that two sets have the "same number" of elements. However, it is important to notice that we have introduced this idea without actually counting anything. Indeed, as we shall see later, we must know about equivalence for sets *before* we can learn to count. (Think of the example of the children and the chairs given earlier.)

In all of our examples up to now it was easy to see that the sets involved were equivalent. This was true because we could tell that they had the same number of elements just by looking at them. We now give an example where this does not happen.

Let A be the set of counting numbers, and let B be the set of even counting numbers. Thus

$$A = \{1, 2, 3, 4, 5, 6, \ldots\}$$
$$B = \{2, 4, 6, 8, 10, 12, \ldots\}$$

Do you think that A and B are equivalent? The first answer almost everyone gives is NO. After all, every element of B is already in A, and there are elements of A left over. Nevertheless, this answer is wrong. Sets A and B *are* equivalent. The following picture shows why.

$$\begin{array}{cccccccc} \{1, & 2, & 3, & 4, & 5, & 6, & \ldots\} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \ldots \\ \{2, & 4, & 6, & 8, & 10, & 12, & \ldots\} \end{array}$$

As usual the three dots mean that the picture goes on and on without stopping. The picture shows how we can match the elements of these sets exactly. Each number in A is matched with two times that number. This gives a number in B. Thus the number 11 in A is matched with 22 in B, the number 50 in A with 100 in B, and so on. In this way each element in B also has

a partner in A. This shows that the elements of A and B match exactly. Hence A and B are equivalent sets.

EXERCISE 1-10

1. Which of the following pairs of sets are equivalent? In each case where the sets are equivalent illustrate a matching of the elements.
   a. A = {2, 4, 6, 8, 10}
   B = {Ghana, Kenya, Nigeria, Sierra Leone, Uganda}
   b. A = the set of even whole numbers less than 20
   B = the set of even whole numbers greater than 40 and less than 60
   c. A = $\{a, b, c, \ldots, i\}$
   B = $\{a, c, e, \ldots, q\}$
   d. A = the set of prime numbers less than 30
   B = the set of whole numbers less than 100 exactly divisible by 10
   e. A = the set of all fathers in the world
   B = the set of all children in the world
2. Let A be the set of all counting numbers. Determine which of the following sets are equivalent to A. When the sets are equivalent illustrate a matching of the elements.
   a. B = {0, 5, 10, 15, . . .}
   b. C = {3, 6, 9, . . ., 3000}
   c. D = {0, 1, 2, 3, . . .}
   d. E = {1, 4, 9, 16, . . .}
3. Let A and B be equal sets. Show that A is equivalent to B.
4. Is it possible to match exactly the set of all counting numbers and the set of all counting numbers greater than 200? If your answer is "yes", show how this could be done. If your answer is "no" explain why it is "no".

Chapter 2

OPERATIONS ON SETS

In the last chapter we learned about sets and subsets. In this chapter we shall study several ways of making new sets out of old ones. As we shall see, these ideas are important in the study of arithmetic.

2-1 Union of sets

The easiest way to make a new set out of old ones is to combine the elements of the old sets to form one large set. For example, if

$$A = \{1, 3, 5\}$$
$$B = \{0, 2, 4\}$$

we can combine the elements of A and B to form the set

$$\{0, 1, 2, 3, 4, 5\}$$

This set is called the *union* of set A and set B because it was formed by "uniting" the elements of A with the elements of B. We write the union of A and B in symbols as

$$A \cup B$$

(which you should read as "A union B"). Thus when $A = \{1, 3, 5\}$ and $B = \{0, 2, 4\}$

$$A \cup B = \{0, 1, 2, 3, 4, 5\}$$

It is clear that any two sets can be combined in this way. We simply "put the sets together" to make one large set. If the original sets were A and B, the new set is called the *union* of A and B. Thus:

The UNION of set A and set B is the set consisting of *all* the elements in A together with *all* the elements in B. The union of A and B is always written $A \cup B$.

The following examples illustrate this definition.

Example 1

$$B = \text{the set of all boys in your school}$$
$$G = \text{the set of all girls in your school}$$
$$B \cup G = \text{the set of all pupils in your school}$$

Example 2

$$A = \{2, 3, 6\}$$
$$B = \{0, 3, 6\}$$
$$A \cup B = \{0, 2, 3, 6\}$$

Note that the numbers 3 and 6 belong to both A and B, but appear only once in the union of A and B. Can you explain this?

Example 3

$$A = \{0, 2, 4, 6, \ldots\}$$
$$B = \{1, 3, 5, 7, \ldots\}$$

The union of A and B is the set of all whole numbers;

$$A \cup B = \{0, 1, 2, 3, \ldots\}$$

Example 4

$$A = \{a, b, c, d\}$$
$$B = \{\quad\}$$
$$A \cup B = \{a, b, c, d\}$$

Example 5

$$A = \{a, b, c, d\}$$
$$B = \{b, c\}$$
$$A \cup B = \{a, b, c, d\}$$

Did you notice that in the last two examples $A \cup B = A$? This happened because B was a subset of A. Indeed, it is easy to see that $A \cup B$ must equal A when B is a subset of A. For then every element of B already belongs to A. So when we put the sets together we just get A.

Here is another fact about the union of two sets: Both A and B are subsets of $A \cup B$. Can you explain why this is true?

EXERCISE 2-1

1. Use the bracket notation to write the union of each of the following pairs of sets:

   a. $A = \{7, 2, 9, 5\}$ $\quad B = \{1, 5, 3, 2\}$

b. A = {a, e, i, o, u} B = the set of letters in the word "ocean"

c. A = {2, 4, 6, . . ., 20} B = {3, 6, 9, . . ., 21}

d. A = the set of prime numbers less than 20
B = the set of odd numbers less than 20

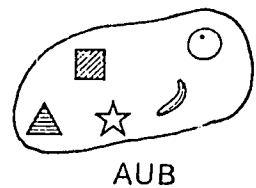
e. A = {0, 5, 10, 15, . . ., 100}
B = the set of all whole numbers less than 100 which are exactly divisible by 15

2. Let A and B be sets. Suppose A contains 5 elements, and B contains 10 elements.
   a. What is the *largest* possible number of elements in $A \cup B$? Explain your answer.
   b. What is the *smallest* possible number of elements in $A \cup B$? Explain your answer.
3. Let A and B be sets. Is $A \cup B = B \cup A$? Why?
4. **a.** Let A be the empty set. Let B be any set. Is A a subset of $A \cup B$? Why?
   b. What is $A \cup B$ when both A and B are empty?
5. Explain why A and B are subsets of $A \cup B$.

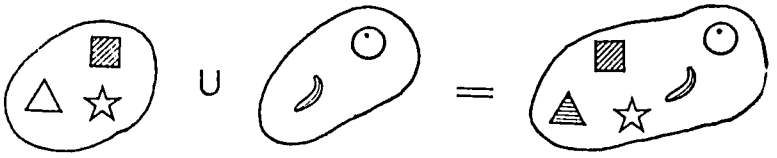
2-2 Picturing the union of sets

It is easy to illustrate the union of two sets by pictures. For instance, let A be a set containing a square, a triangle and a star. Let B be a set containing a banana and an orange. Then A and B can be represented by the pictures

To get a picture for $A \cup B$ we simply combine these pictures as follows:



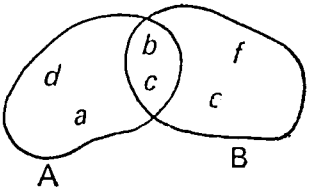
We could also show the same thing by the picture



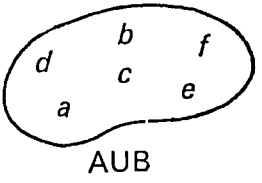
In the above example sets A and B had no elements in common. Thus we did not let the pictures for A and B overlap. The following example shows what happens when A and B do have elements in common.

Let
$$A = \{a, b, c, d\}$$
$$B = \{b, c, e, f\}$$

Since the elements named b and c belong to both A and B we have

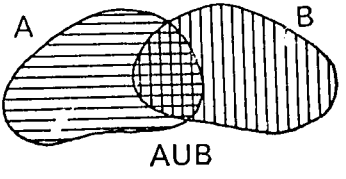


This time the picture for $A \cup B$ is



Notice that these pictures show why elements common to A and B appear only once in $A \cup B$.

Pictures like those above are extremely helpful in teaching children about the union of sets. There may be times, however, when you do not want to show individual elements in your pictures. When that happens you may draw a picture like this:



Remember that in such a picture set A is represented by the region inside the curve labelled A. It has been shaded ≡. Set B is represented by the region inside the curve labelled B. It has been shaded |||||. When these sets are put together we get the region inside both curves. Thus $A \cup B$ is the shaded region inside the heavy outer curve. It is best to draw such pictures with A and B overlapping unless you know that A and B have no elements in common.

EXERCISE 2-2

Draw pictures to illustrate A, B and $A \cup B$ for each of the following pairs of sets.

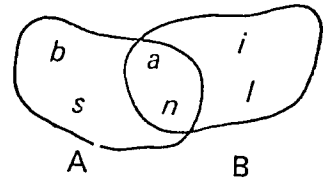
1. A = {▲, ●, ▨, ☆} B = {▭, D, ◠}
2. A = {0, 2, 4, 6} B = {0, 3, 6, 9}
3. A = {a, e, i, o, u} B = the set of letters in the word "ocean"
4. A = the set of prime numbers less than 20
   B = the set of odd numbers less than 10
5. A = {0, 5, 10, . . ., 30}
   B = {0, 10, 20, 30}
6. A = {2 + 3, 4 + 2, 7 + 5}
   B = {12, 6, 5}

2-3 Intersection of sets

Let A be the set of letters in the word "bananas". Let B be the set of letters in the word "nail". Then

$$A = \{a, b, n, s\}$$
$$B = \{a, i, l, n\}$$

These sets have the letters a and n in common. Hence if we wanted to draw a picture of A and B our picture would have to look like this:



The two sets overlap. The set consisting of the letters a and n is

called the *intersection* of set A and set B. The members of the intersection are the elements which belong both to A *and* to B. We write the intersection of A and B in symbols as

$$A \cap B$$

(which you should read "A intersect B"). Thus when

$$A = \{a, b, n, s\} \quad \text{and} \quad B = \{a, i, l, n\}$$
$$A \cap B = \{a, n\}$$

This example illustrates a second way to make a new set out of old ones. We simply form the set containing all the elements which the old sets have in common. This new set is called the *intersection* of the old sets. Thus:

The INTERSECTION of set A and set B is the set consisting of all the elements which belong both to A and to B. The intersection of A and B is always written $A \cap B$.

But can we always form the intersection of two sets? What if A and B have no elements in common? The answer is easy. According to the above definition the intersection of A and B then has no elements. It is the empty set. For instance, if

$$A = \{a, b, n, s\} \quad \text{and} \quad B = \{d, g, b, o\},$$

then

$$A \cap B = \{\ \}$$

This example shows why we really need such a thing as the empty set around. For without it we could not talk about the intersection of two sets with no elements in common.

Two sets with no elements in common are said to be DISJOINT. Thus the intersection of two disjoint sets is the empty set.

The following examples illustrate the intersection of sets:

Example 1

$$A = \{2, 3, 6\}$$
$$B = \{0, 3, 6\}$$
$$A \cap B = \{3, 6\}$$

Example 2

B = the set of all boys in your school
G = the set of all girls in your school
$B \cap G = \{\ \}$

Thus B and G are disjoint sets.

Example 3

$$A = \{a, c, d, f\}$$
$$B = \{c, d\}$$
$$A \cap B = \{c, d\}$$

Example 4

$$A = \{1, 2, 3, 4, \ldots\}$$
$$B = \{2, 4, 6, 8, \ldots\}$$
$$A \cap B = \{2, 4, 6, 8, \ldots\}$$

Did you notice that in the last two examples $A \cap B = B$? This happened because B was a subset of A. It is easy to see that $A \cap B$ must equal B when B is a subset of A because the elements which A and B have in common are precisely the elements of B when every element of B belongs to A.

Here is another fact about the intersection of sets: $A \cap B$ is a subset of A; it is also a subset of B. You should be able to explain why this is true.

EXERCISE 2-3

1. Use the bracket notation to write the intersection of each of the following pairs of sets.
   a. $A = \{1, 5, 8, 9, 11\}$ $B = \{0, 3, 4, 5, 8\}$
   b. A = the set of letters in the word "Entebbe"
   B = the set of letters in the word "African"
   c. $A = \{4, 8, 12, \ldots, 40\}$
   $B = \{3, 6, 9, \ldots, 39\}$
   d. $A = \{5, 10, 15, \ldots\}$
   $B = \{4, 8, 12, \ldots\}$
   e. $A = \{1, 4, 9, 16, \ldots\}$
   B = the set of all prime numbers
2. Let A and B be sets. Suppose A contains 5 elements, and B contains 10 elements.
   a. What is the *largest* possible number of elements in $A \cap B$? Explain your answer.
   b. What is the *smallest* possible number of elements in $A \cap B$? Explain your answer.
3. Let A and B be sets. Is $A \cap B = B \cap A$? Why?
4. Suppose $A \cap B = A \cup B$. What can you say about A and B? Why?

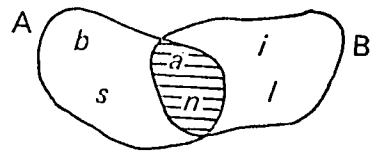
5. Explain why $A \cap B$ is a subset of A and a subset of B.
6. Is $A \cap B$ a subset of $A \cup B$? Why?

2-4 Picturing the intersection of sets

It is easy to illustrate the intersection of two sets by pictures. For example, if

$$A = \{a, b, n, s\}$$
$$B = \{a, i, l, n\}$$

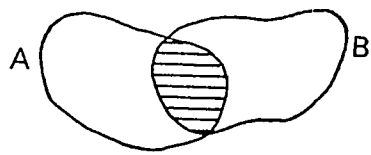
we have the picture



$A \cap B$

The elements in the shaded part of the picture make up $A \cap B$. Thus $A \cap B = \{a, n\}$

As usual, we can draw a picture of the intersection of two sets without showing the individual elements in the sets. The picture then looks like this:



$A \cap B$

The shaded part of the picture represents A intersect B.

Notice that the region representing $A \cap B$ in this picture lies inside set A. This shows that $A \cap B$ is a subset of A. Do you see why $A \cap B$ is also a subset of B?

EXERCISE 2-4

1. Draw pictures to illustrate A, B and $A \cap B$ for each of the following pairs of sets:
   a. $A = \{1, 5, 9, 8, 4\}$ $B = \{3, 0, 1, 6, 9\}$
   b. $A = \{2, 4, 6, \ldots, 20\}$ $B = \{3, 6, 9, \ldots, 21\}$
   c. A = {△, ●, ▨, ☆} B = {▭, ◗, ◠}

d. A = the set of distinct letters in the word "Entebbe"
B = the set of distinct letters in the word "African"

2. There are 10 elements in set A, 7 elements in set B and 3 elements in $A \cap B$.
   a. How many elements are in $A \cup B$?
   b. Draw a picture to show A and B.
3. There are 15 elements in $A \cup B$, 8 elements in A, and A and B are disjoint. How many elements are in B?
4. There are 10 elements in $A \cup B$, 5 elements in $A \cap B$, and 2 elements in A which are *not* in B.
   a. Draw a picture to show A and B.
   b. How many elements are in A?
   c. How many elements are in B?
   d. How many elements are in B which are *not* in A?
5. There are 12 elements in $A \cup$, 6 elements in A which are *not* in B, and 3 elements in B which are *not* in A.
   a. Draw a picture to show A and B.
   b. How many elements are in A?
   c. How many elements are in B?
   d. How many elements are in $A \cap B$?
6. Let A, B and C be sets. Suppose that C is a subset of A *and* C is a subset of B. What is the relation between C and $A \cap B$? Illustrate your answer with a picture.

2-5 More about unions and intersections

Now that we know how to form unions and intersections the study of sets becomes quite interesting. This happens because we can use these two ideas together to make lots of new sets.

For example, suppose we are given three sets, A, B and C. We can form the union of B and C, and then take the intersection of set A with this union. In other words, we can form a new set in two steps, as follows:

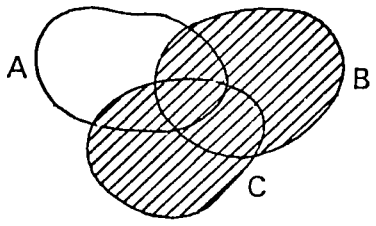
Step 1: Form $B \cup C$
Step 2: Form the intersection of A and $(B \cup C)$

We write the result as

$$A \cap (B \cup C)$$

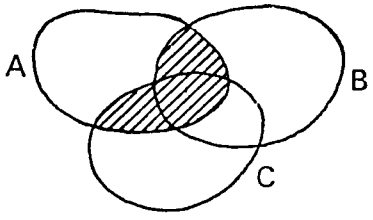
This symbol tells you to take the intersection of A with the set inside the brackets. Since the set inside the brackets is B union C, you must form the intersection of A and $B \cup C$. The following pictures show what happens.

Step 1: Form $B \cup C$



BUC shaded

Step 2: Form A intersect $B \cup C$



A∩B(BUC) shaded

In passing from Step 1 to Step 2 we simply kept the part of $B \cup C$ which lies in A. This is the shaded region in the last picture.

If you look carefully at the picture above you should be able to find another way of writing the set $A \cap (B \cup C)$. Do you see what it is? Locate the part of A which is in B. This represents the set $A \cap B$. Notice that it is shaded. Similarly the region representing $A \cap C$ is shaded. Moreover, the entire shaded region in our picture can be written as the union of these two shaded regions. This shows that the set $A \cap (B \cup C)$ is the union of $A \cap B$ and $A \cap C$. In other words these sets are equal, and we have

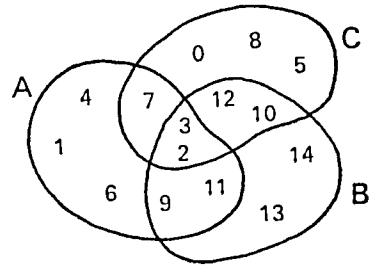
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

The moral here is that pictures can be used to discover relations between sets. If you have never done this sort of thing

before you will probably want to try some of the exercises below.

EXERCISE 2-5

1. Use the following picture to complete each of the statements given below.



a. $A \cap C =$ **b.** $A \cup C =$
c. $B \cap C =$ **d.** $A \cap (B \cap C) =$
e. $(A \cup B) \cup C =$ **f.** $(A \cap B) \cap C =$
g. $A \cap (B \cup C) =$ **h.** $(A \cap B) \cup (A \cap C) =$
i. $A \cup (B \cup C) =$ **j.** $(A \cup B) \cap (A \cup C) =$
k. $B. \cup (A \cap C) =$ **l.** $A \cup (B \cap C) =$

2. Which of the sets listed in Question 1 are equal to one another?
3. Let A, B and C be sets. Draw pictures to illustrate the sets $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$. What do you conclude about these sets?
4. There are 10 boys in all; 8 boys are playing ball and 6 are wearing jackets. How many boys must be punished for playing ball in their jackets?
5. True or false? There are 100 people in all. 75 drink coffee. 50 drink tea. 40 drink both. 10 do not drink either.
   Draw a picture to explain your answer.
6. There are 10 elements in set A
   6 elements in set B
   [illegible] elements in set C
   3 elements in both A and B
   2 elements in both A and C
   2 elements in both B and C
   1 element in all three
   How many elements altogether? Draw a picture to explain your answer.

The Whole Numbers UNIT II

Chapter 3

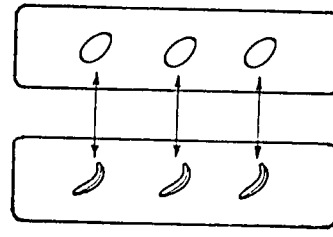
NUMBER

In the previous unit we studied sets and learned how to work with them. In this unit we are going to use sets to develop the idea of number. Once we know what numbers are we shall consider different ways of representing them. Then we shall study the number line and the relations of order in the set of whole numbers. Finally, we shall study arithmetic, and learn how addition, subtraction, multiplication and division are defined.

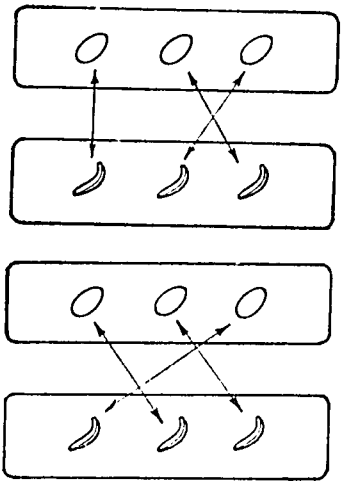
3-1 Comparison of sets by matching

It is interesting to consider the way pupils first develop the idea of number. It goes something like this.

Consider the set of pawpaws and the set of bananas in the following picture.

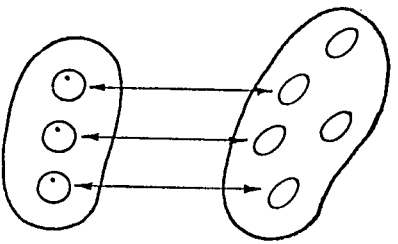


In this picture we have matched each pawpaw with exactly one banana just as children might do. In so doing each banana was matched with exactly one pawpaw. Recall that the two sets are then said to match exactly, or to be equivalent. From this we conclude that there are JUST AS MANY bananas as there are pawpaws, and JUST AS MANY pawpaws as there are bananas. Notice that any other matching of the elements of these sets will give the same relationship: the sets will match exactly. Two other matchings of these sets are shown in the following picture. Each of these matchings shows that the two sets are equivalent.



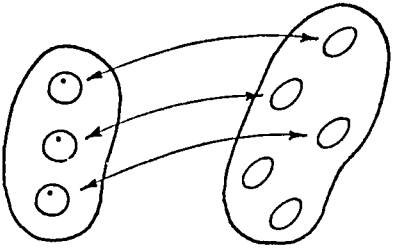
These sets can be matched in other ways as well. Can you find some of them?

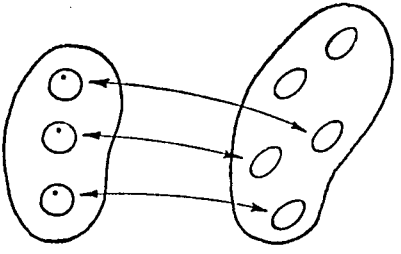
Now look at the following sets:



Double arrows have been drawn to match each orange with exactly one pawpaw. But this time there are some pawpaws left over. We describe this by saying that the set of pawpaws has MORE ELEMENTS than the set of oranges. We also say that the set of oranges has FEWER ELEMENTS than the set of pawpaws.

As in the case of an exact matching, there are other ways of matching the oranges and pawpaws in these two sets. But no matter what matching is used there will always be some pawpaws left over. The following picture shows some of these matchings. Each of them shows that the set of pawpaws has more elements than the set of oranges. They also show that the set of oranges has fewer elements than the set of pawpaws.





From the preceding illustrations we see that when we match the elements of a set A with the elements of a set B one of three things will happen:

1. They will match exactly.
2. There will be some elements of A left over.
3. There will be some elements of B left over.

When the elements match exactly we say that set A has *just as many elements as* set B. When there are elements of A left over we say that set A has *more elements than* set B. When there are elements of B left over we say that set A has *fewer elements than* set B.

EXERCISE 3-1

1. Which of the following sets are equivalent? Illustrate your answer by drawing pictures with matching arrows.

$A = \{a, b\}$ $B = \{x\}$ $C = \{$▲, ●$\}$
$D = \{1, 5, 7\}$ $E = \{\ \}$ $F = \{\ \}$

G = the set of distinct letters in the word "banana"
H = the set of all even prime numbers
I = {Ghana, Nigeria, Togo}

2. By drawing pictures with matching arrows show that the set $\{x, y\}$ is equivalent to three different subsets of the set {▲, ▨, ◍}.

3. Let A be the set of all whole numbers, and let B be the set of all even whole numbers. Consider the following pictures:

$$\begin{array}{l} A : \{0, 1, 2, 3, 4, \ldots\} \\ \quad\;\; \downarrow \quad\;\; \downarrow \quad\;\; \downarrow \quad \cdots \\ B : \{0, \quad 2, \quad 4, \ldots\} \end{array}$$

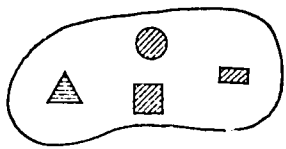
$$\begin{array}{l} A : \{0, 1, 2, 3, 4, \ldots\} \\ \quad\;\; \updownarrow \;\; \updownarrow \;\; \updownarrow \;\; \updownarrow \;\; \updownarrow \quad \cdots \\ B : \{0, 2, 4, 6, 8, \ldots\} \end{array}$$

The first picture seems to show that A has *more* elements than B. The second picture seems to show that A has as many elements as B. Which do you think is correct? What are the reasons for your answer?

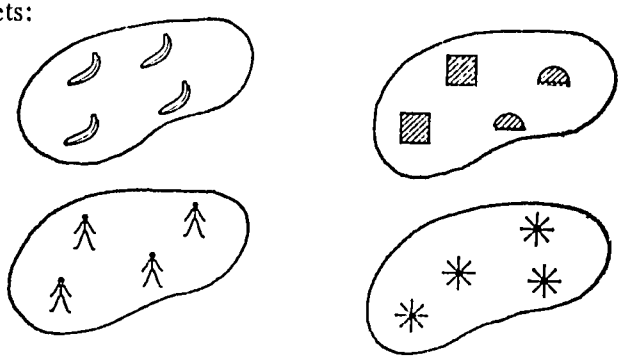
4. There is one set which is equivalent to a subset of *every* set. What is it?

3-2 Number as a property of sets

Suppose we start with a set such as the set pictured here:



Think of sets which are equivalent to this set; that is, sets which can be matched exactly with it. Here are pictures of several such sets:

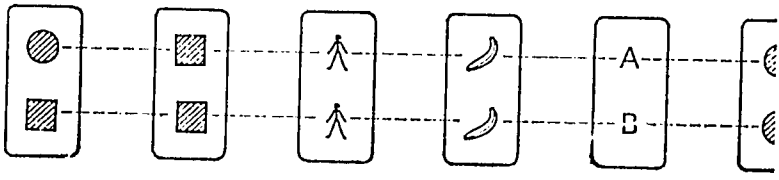


Of course, it is possible to form many other sets besides these which are equivalent to the original one. As we continue to form these sets we are led to consider ALL sets which are equivalent to the original set. As we shall see, this is the idea behind the concept of number.

EXERCISE 3-2a

1. Make up a few sets which match {◍, ▨} exactly.
2. Make up a few sets which match {A, B, C} exactly.

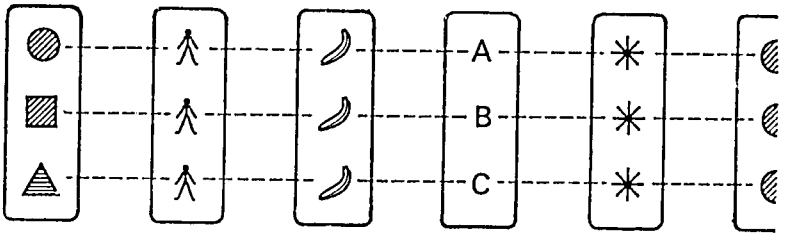
Think about *all* the sets which are equivalent to the set in the first exercise above. Some of them are shown below.



Of course, there are many, many more, so many that it is impossible to draw all of them. However we can *think* about all of them, and as we do we discover that there is only one property which they have in common. Do you see what that property is? Each of these sets has precisely as many elements as any other since they are all equivalent. We describe this property by saying that all of these sets have the same NUMBER of elements.

Thus the word "number" denotes the property shared by ALL sets equivalent to a given set. It is convenient to assign a name to the number property possessed by all sets equivalent to a given set. This name refers to the number of elements in the sets. For example, the name "two" is used to describe the property of "two-ness" shared by ALL sets equivalent to {◍, ▨}. We say that there are two elements in {◍, ▨}. The number called "two" is also attached to all the other sets in the picture above. We speak of *two* stickmen, *two* letters, *two* pieces of fruit, etc. The number *two* is also attached to every other set that is equivalent to {◍, ▨}. ALL these sets have something in common: it is the property of "two-ness" which we name by the number *two*.

Similarly, the name "three" is assigned to the number property of every set which is equivalent to {◍, ▨, △}. Some of these sets are illustrated below:



As you know, every number has a symbol which is used when we write the number. The symbol "2" is used to write the number named "two". The symbol "3" is used to write the number named "three". A symbol used to write a number is called a NUMERAL.

We have now described several important concepts for pupils to think about and learn in the classes you will teach. One is the concept of a *set* of objects. Another is the concept of *number* which tells how many elements the set contains. Each number has a *name*, and is represented by a symbol which is called a *numeral*.

EXERCISE 3-2b

1. Let A, B and C be sets. Suppose all three sets have the same number of elements. What can you say about these sets?
2. Let A, B and C be sets. Suppose A is equivalent to a subset of B, and B is equivalent to a subset of C. What can you say about the number of elements in A and in C? Illustrate your answer with pictures.
3. Let A and B be sets containing a finite number of elements. Suppose that $A \cup B = A$ but that A is not equal to B. What statement can you make about the number of elements in the sets A and B?
4. Let $A = \{a, b, c\}$
   a. (i) List all the subsets of A containing a single element.
   (ii) List all the subsets of A containing 2 elements.

(iii) Match each 2-element subset of A with the 1-element subset containing the third element of A; that is, $\{a, b\} \longleftrightarrow \{c\}$, and so on.

(iv) What can you say about the number of 2-element subsets of A and the number of 1-element subsets of A?

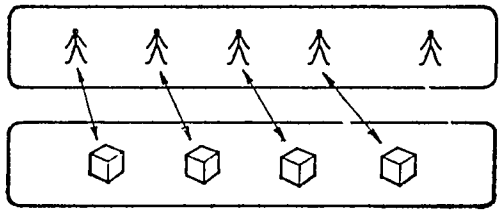
b. Let $A = \{a, b, c, d\}$
Use the idea explained in *a* above to show that the number of 3-element subsets of A is the same as the number of 1-element subsets of A.

c. Suppose A is a set containing 10 elements. How many different 9-element subsets does A have? Explain your answer.

d. How many different subsets of 51 cards can be chosen from a pack of 52 cards?

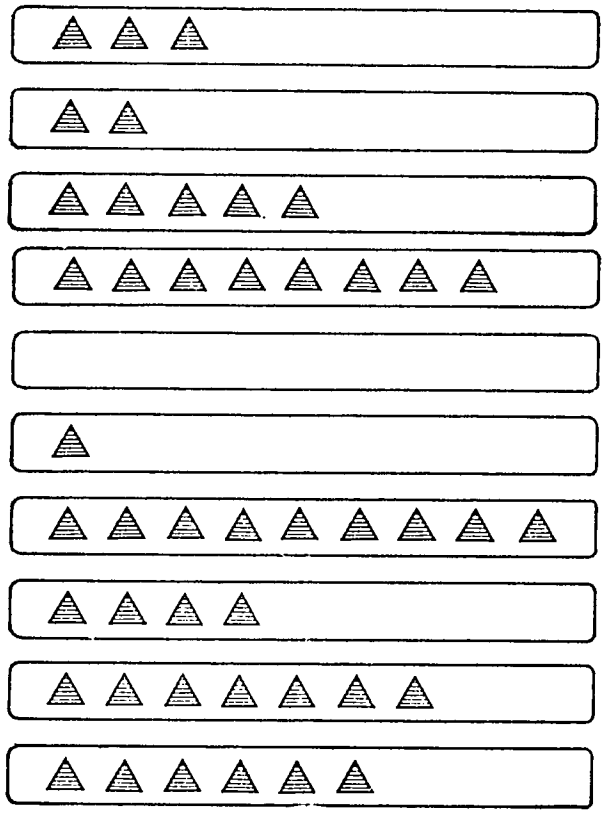
3-3 Sets in natural order

Let us now compare the following sets of stickmen and boxes:

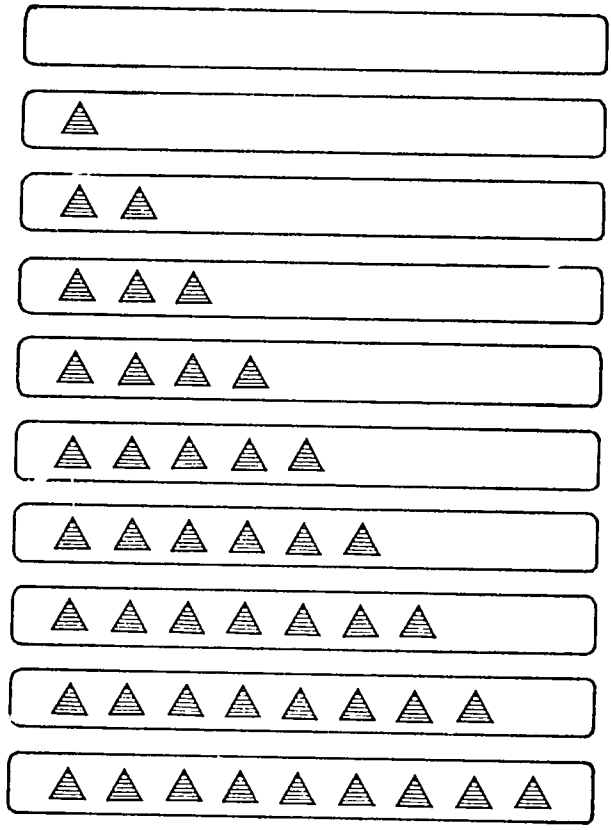


If we look at the set of stickmen left over, we see that this set has the property of "one-ness". There is one stickman left over. We then say that the set of stickmen has *one element more than* the set of boxes. This "one-more-than" relation provides the basis for placing sets in NATURAL ORDER.

Look at the following sets of triangles:



No set of triangles matches any other set exactly. However, we can arrange these sets in a one-more-than relation, starting with the set that has fewest elements. We note that there is an empty set. Because the empty set has the fewest elements of all, it comes at the beginning of the new arrangement. The set consisting of a single triangle has one more element than the empty set, so it comes next. By continuing to choose the set which has one more element than the preceding set, we place the sets in natural order as shown below:



EXERCISE 3-3

1. Let A, B, C, D and E be sets. Suppose that
   (i) A has one more element than E
   (ii) C has one more element than A

(iii) D has one more element than B
(iv) E has one more element than D
Arrange these sets in natural order. Which set has the most elements? Which set has the fewest elements?

2. Set A has one more element than set B. Set C is equivalent to set B. What can you say about sets A and C? Why?

3. Suppose that sets A, B, C, D can be arranged in natural order with A first, B second, C third and D last. Suppose set S is equivalent to set A, and that set T is equivalent to set D. Arrange B, C, S, T in natural order.

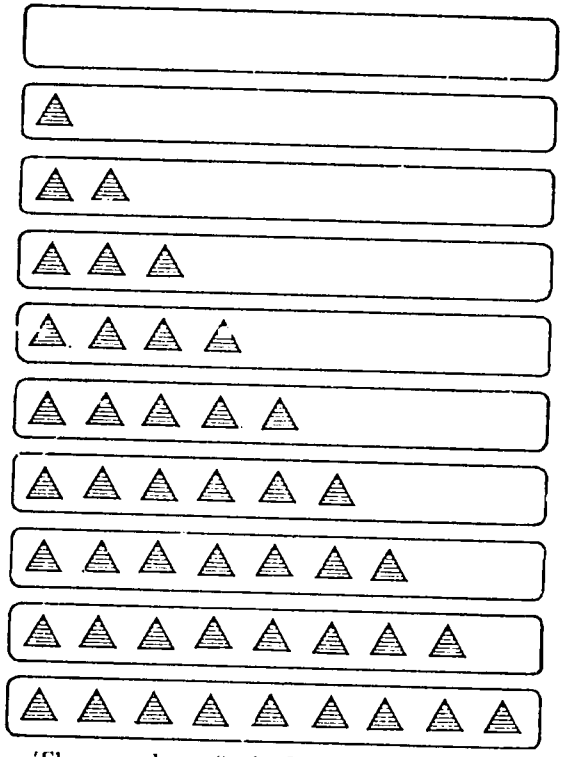
4. Set A has one more element than sets B and C. Set B has one more element than sets D, E and F. List *all* possible arrangements that are in natural order which can be made from these sets.

Chapter 4

THE SET OF WHOLE NUMBERS

4-1 Identifying the whole numbers

You have seen how to place sets in natural order. Using this natural order, we give a name to the number of elements in each set, and we write the numeral which represents the number.



Set	*Number*	*Numeral*
	zero	0
	one	1
	two	2
	three	3
	four	4
	five	5
	six	6
	seven	7
	eight	8
	nine	9

The numbers 0, 1, 2, 3, 4, 5 and so on are called WHOLE NUMBERS. The set $\{0, 1, 2, 3, 4, 5, \ldots\}$ is the set of whole numbers. We give these numbers a special name because as you will see later there are other kinds of numbers.

4-2 Order in the set of whole numbers

When *sets* are placed in natural order—as above—we also say that the corresponding *whole numbers* are then in natural order. Thus 0, 1, 2, 3 are in natural order. On the other hand 3, 6, 5, 4 are *not* in natural order; the natural order is 3, 4, 5, 6.

We have already seen how a set of objects has a whole number attached to it. Suppose that the number of elements in set A is m, and the number of elements in set B is n. Then:

(i) We say that m is EQUAL to n if set A has *just as many* elements as set B.
(ii) We say that m is LESS THAN n if set A has *fewer* elements than set B.
(iii) We say that m is GREATER THAN n if set A has *more* elements than set B.

Since these are the only three possibilities it follows that for any two whole numbers m and n one and only one of the following statements is true:

(i) m is *equal* to n
(ii) m is *less than* n
(iii) m is *greater than* n

There are special symbols in mathematics to express each of these facts.

(i) If m is *equal* to n we write $m = n$
(ii) If m is *less than* n we write $m < n$
(iii) If m is *greater than* n we write $m > n$

Using these symbols we can say that for any two whole numbers m and n one and only one of the following statements is true:

$$m = n \quad m < n \quad m > n$$

Before we go on, two things should be made clear.

First, if m and n are whole numbers with $m < n$, then we also have $n > m$. Thus since $2 < 5$, we also have $5 > 2$. Similarly, if $n > m$, then we also have $m < n$. In short, the statements

$$m < n \quad \text{and} \quad n > m$$

say the same thing. If one of them is true, the other is also true.

Second, to say that the whole number m is greater than the whole number n amounts to the same thing as saying that m comes after n in the natural order of the whole numbers. Thus, since 3, 4, 5, 6 are in natural order, we can write

$$6 > 5 \quad 6 > 4 \quad 6 > 3$$
$$5 > 4 \quad 5 > 3$$
$$4 > 3$$

EXERCISE 4-2

1. Here are four sets:
   {A, B, C, D}
   {●, ▲, ▭ }
   { }
   {B, C, E, F}
   Find the number of members in each. Now write all the statements you can, using these numbers and the symbols $<$, $=$, $>$.
2. Complete each of the following statements by inserting the correct symbol: $=$, $<$, $>$.
   a. 5 —— 7
   b. 12 —— 0
   c. $2 + 2$ —— 4
   d. $2 + 2$ —— $3 + 2$
   e. $9 + 7$ —— $8 + 7$
   f. $14 + 12$ —— $13 + 13$
   g. $6 + 8 + 10$ —— $5 + 7 + 9$
3. Use the bracket notation to describe the set of whole numbers n which make each of the following statements true:
   a. $n < 6$
   b. $25 < n$
   c. $12 < n$ and $n < 18$
   d. $n < 20$ and n is exactly divisible by 3
   e. $10 < n$, $n < 30$ and n is even
4. Is there a whole number m such that $m < n$ for *all other* whole numbers n? Explain your answer.
5. Is there a whole number m such that $m > n$ for *all* other whole numbers n? Explain your answer.

6. Is it true that $a > b$ and $b > c$ imply that $a > c$? With sets having 4 elements, 3 elements and 2 elements, show this idea by matching elements.
7. Suppose that A is a subset of B but is not equal to B. What statement can be made about the number of elements in A and the number of elements of B when A and B each have a finite number of elements.

4-3 Counting and the counting numbers

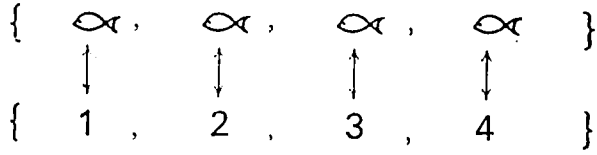
In section 4-1, we showed sets of triangles in natural order. Each set has one more triangle than the preceding set. Just as we placed these sets of triangles in natural order, we can also form sets of whole numbers and place them in natural order. Thus $\{1\}$ is the set whose *one* element is the number 1; $\{1, 2\}$ is the set containing the *two* numbers 1 and 2 in their natural order; $\{1, 2, 3\}$ is the set containing the *three* numbers 1, 2 and 3 in natural order, and so on. We come in this way, for example, to the set $\{1, 2, 3, \ldots, 27\}$, which is the set containing the *twenty-seven* numbers 1, 2, 3, and so on up to 27, in natural order. The last number in the set is 27, and 27 is also the number of elements in the set. Similarly, the set $\{1, 2, 3, \ldots, n\}$ is the set of whole numbers 1, 2, 3, and so on up to n in natural order. The last number in the set is n, and n is also the number of elements in the set. These sets of numbers are called COUNTING SETS. The numbers in each counting set are counting numbers in natural order starting from 1. The complete set of counting numbers is the set $\{1, 2, 3, 4, 5, \ldots\}$. Counting sets are represented below.

Counting sets	*Number of members*
$\{1\}$	1
$\{1, 2\}$	2
$\{1, 2, 3\}$	3
$\{1, 2, 3, 4\}$	4
$\{1, 2, 3, 4, 5\}$	5
$\{1, 2, 3, 4, 5, 6\}$	6
$\{1, 2, 3, 4, 5, 6, 7\}$	7
$\{1, 2, 3, 4, 5, 6, 7, 8\}$	8
$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$	9
$\{1, 2, 3, 4, 5, 6, 7, 8, 9, \ldots, n\}$	n

Corresponding to each counting set is the number of elements in that set. The last number in a counting set tells the number of elements in that counting set.

4-4 Finding the number of elements in a set

When we use the counting sets we follow the natural order of numbers; that is, we count: "one, two, three, . . .". Now you may ask, "How do we find the number of elements in any given set?" For example, take the set of fish shown below.



Match the fish with the numbers in a counting set starting with 1, proceeding in natural order. The set of fish is exhausted when the last fish is matched with the last number in a counting set. This number tells us the number of fish. It is the largest number in the counting set.

When we teach children how to count, we teach them to use a set which they describe by the spoken words "one", "two", "three", "four", and so on. It is important to note that as they say "one", they are matching the number 1 with an object; as they say "two", they are matching the number 2 with another object, etc. The last number they name is matched with the last object remaining in the set. This last number tells the number of objects in the set.

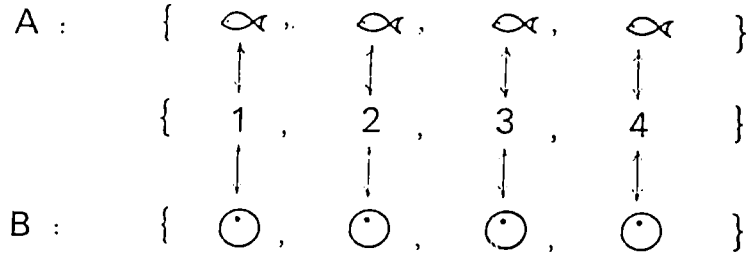
In counting, it is important to attach just one number to each member of a set. In the preceding example of counting fish, the order in which we take the fish is not important. What *is* important is that each fish is counted just once. When we count the elements of a set, no matter how we do the counting, we always find the same number of elements. When you count the sides of a square, for example, you always get 4.

4-5 Equivalence using counting sets

You can use counting sets to tell whether two sets are equiva-

lent. Recall that earlier in this unit we said that sets A and B are equivalent when there is an exact matching of the members of set A with the members of set B. We can now restate the idea of equivalence of sets using counting sets.

Let us suppose we have a set A of fish and a set B of oranges. In order to tell whether the two sets are equivalent find the counting set which matches set A exactly. Then see whether set B also matches the same counting set exactly. If it does, set A is equivalent to the counting set, and set B is equivalent to the counting set, and we can conclude that set A is equivalent to set B. Can you tell why?



In the picture above we see that sets A and B are equivalent to the counting set {1, 2, 3, 4}. Thus A and B are equivalent to one another.

EXERCISE 4-5

1. Find, by counting, whether A and B are equivalent sets.
   a. A = {A, ◎, ▨, ▭, ☆, ✳}
   B = the set of fingers on your left hand
   b. A = the set of consonants in the word *relation*
   B = the set of sides in the figure ▨

2. A and B are equivalent sets. There are 10 elements in A. How many elements are in B?

3. A and B are equivalent sets. There are 7 elements in B, and 3 elements in $A \cap B$. How many elements are there in A which are not in B?

4. A and B are equivalent sets. $A \cap B$ is empty, C is equivalent to $A \cup B$ and there are 12 elements in C. How many elements are there in A? How many in B?

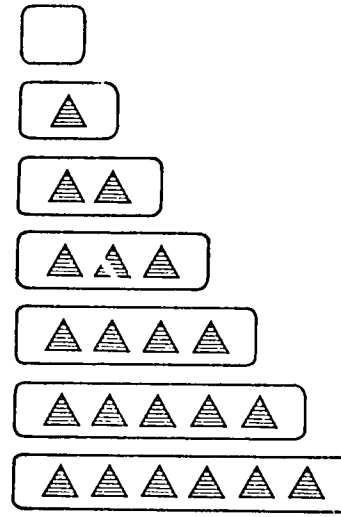
Chapter 5

INTRODUCTION TO THE NUMBER LINE

5-1 Introduction

There is another good way to introduce children to the number concept and counting. It makes use of the number line. Each of the different ways of presenting the number concept has its own advantages. Thus it is a good idea to use several different approaches. In this section, we illustrate how one can introduce the number line.

In the previous chapter you have seen how to put sets in natural order. For example, the following sets are in natural order:

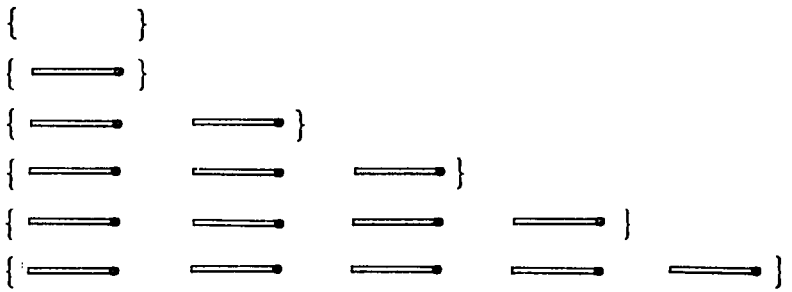


(and so on)

Each of the sets has one more member than the set above it. There is, of course, no end to the sets that you can build in this way.

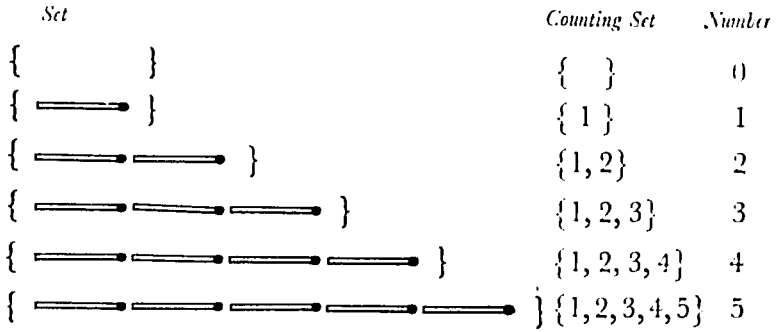
In order to introduce the number line we now use sets of

matches to illustrate sets in natural order. We take matches that are the same size and arrange them in sets as follows:



(and so on)

As you know, each of these sets can be matched with a counting set. Remember that this is just another way of describing how we count the elements of a set. The sets pictured above are matched with counting sets as follows:

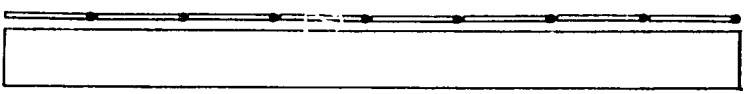


Set	*Counting Set*	*Number*
{ }	{ }	0
{ — }	{ 1 }	1
{ — — }	{1, 2}	2
{ — — — }	{1, 2, 3}	3
{ — — — — }	{1, 2, 3, 4}	4
{ — — — — — }	{1, 2, 3, 4, 5}	5

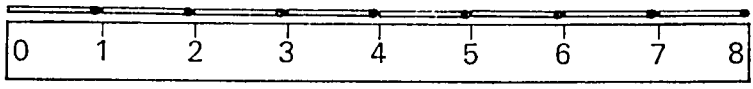
(and so on)

In this way the sets are given numbers, and we say we have counted the sets.

There is a convenient way of showing how each of these sets is matched with its corresponding counting set. To do this we take one of the sets and arrange the matches in it end-to-end. We then lay a strip of paper beside the set of matches as shown below:



At the left-hand end of the strip, mark 0 for the number of matches in the empty set. At the right-hand end of each match, mark the proper numeral on the strip of paper so that the strip is a counting set for that set of matches. Below you can see the example finished.

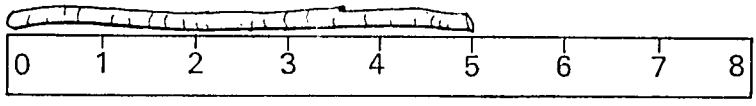


EXERCISE 5-1

When you teach this concept to pupils, you will want to get them to make such strips of paper with the counting numbers marked on them. Can you think of some sets that you can use instead of sets of matches in teaching this material? What properties do the elements of the sets have to have in common in order to serve this purpose?

5-2 Using counting sets

What would you do if you wished to make more sets of matches like those you made above, but had run out of matches? One answer would be to use the counting sets marked on the strips of paper to help you cut from a stick of wood more pieces of the same length as the matches. If you had a thin piece of bamboo or a long twig from a tree, you could cut it into pieces of the same size as the matches by laying one of your counting sets against the bamboo or the twig. If your paper counting set was long enough, you could tell how many match-lengths you would get from any stick. For example, there are 5 match-lengths in the stick shown below.



5-3 Counting sets used to build the number line

By now you have seen that these numbered strips of paper, these sets which help you to count the number of match-lengths in things, are very much like rulers. You are really

using them to "measure" things when you count the number of match-lengths. If you have a long enough strip of paper, with enough numbers on it, you can use it to measure very many things. In this way you can tell the number of match-lengths that are in any of these things.

If you had enough paper and enough time you could make such a strip of paper as long as you wish. In your mind you can think of a paper counting set which goes on and on without stopping. The edge of such a paper counting set illustrates the NUMBER LINE. All the whole numbers are on it in their natural order.

EXERCISE 5-3

You cannot draw the whole number line, since there is no end to it. But you can draw a piece of it. Make such a piece of the number line on a straight-edged piece of cardboard or flat stick, and use it to count the number of lengths (of the size you chose) in different objects in the classroom.

5-4 Size of the unit piece on the number line

In the preceding exercise you made a part of the number line, and used it for counting the number of lengths of a given size in things. In making it, you did one thing which is very important, and if you did not do it, you must work the exercise again. You should have labelled your line in such a way that each point labelled by a numeral was *the same distance* from the next point labelled by a numeral. When you arranged your sets of matches you *had* to do this, because the matches were all the same size. And when you made your counting sets on strips of paper you *had* to number your line in this way, because these strips were paired with the sets of matches. You must *always* do this in making a number line, since only in this way can you count pieces of a given size.

To build a number line, you can use any size you wish as your unit piece. But you must always keep the same unit size for as much of the number line as you build.

EXERCISE 5-4

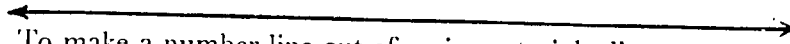
Make two different number lines on pieces of cardboard or flat sticks. One should have small unit pieces, and the other

should have larger unit pieces. Use the two number lines to count the number of unit lengths in different objects in the classroom. Compare the results you get using the two number lines on each object.

5-5 Drawing the number line

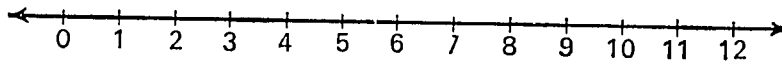
In the last exercise you saw that you can put many numerals on a section of the number line when each unit piece is small. You probably wondered how many numerals can go on a number line. The answer is easy. The line can be extended further than any part of it you can draw. Someone else can extend your section of line. Then you can go on adding pieces to the end of his line. In fact, pieces can be added to either end of your section of line, so that the line can be extended in either direction.

When you draw a line without any numbers on it, you show that it can be extended in both directions by putting arrows at both ends of your line. Your line will look like this:



To make a number line out of a given straight line, you mark some point as the starting point, or 0 point.

Choose some length as your unit length, and then keep putting down succeeding numerals as you attach more and more of these units to each other end-to-end. You could add unit pieces in either direction from the 0 point, but only the pieces going *to the right* from the 0 point are numbered. Thus you will end up with a line which looks like this:



You can put your 0 point anywhere, and you can make your unit length any length you wish. But once you have decided on them you should draw your number line as shown above.

On the number line there is a point for *every* whole number. Between the points for one whole number and the next whole number, there is one unit of length.

EXERCISE 5-5

1. Draw a number line on a piece of paper. Find these numerals on it: 7, 3, 0, 12, 1, 10.

2. On the number line, what number is four units to the right of 9? What number is four units to the left of 9? Make up other exercises like this that you could use in teaching addition and subtraction.
3. On a number line, mark the whole numbers 8 to 14 in red. How many whole numbers are marked in red?
4. If a match-stick is taken to represent the unit length on a number line, how many match-sticks could be placed end-to-end between the points marked 81 and 111?
5. A man takes 6 steps east, 3 steps west and then 4 more steps east. How many steps is he away from his starting point? Show the steps on the number line.
6. A man takes 3 steps forward, 1 step backward, 5 steps forward, another 2 steps forward and then 4 steps backward. How many steps must he take to return to his starting point? Mark the steps on a number line.
7. Suppose you start at the point 2 on the number line and in each successive step move 3 units to the right. What point do you reach

   a. after 2 steps? **b.** after 7 steps? **c.** after n steps?

Chapter 6

ORDER PROPERTIES ON THE NUMBER LINE

6-1 "Less than" on the number line

When you put counting sets in natural order, a counting set with fewer elements comes before a counting set with more elements. For example, compare the counting sets for 3 and 5, as shown below:

$$\{1, 2, 3\}$$
$$\{1, 2, 3, 4, 5\}$$

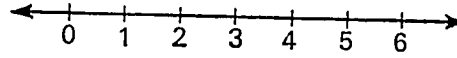
The counting set for 3 has fewer elements than the counting set for 5. In this case, you write

3 is less than 5

or, in symbols,

$$3 < 5$$

Now let us look at this on the number line.



The section of the number line representing the counting set for 5 goes all the way from 0 to 5. The section of the number line representing the counting set for 3 goes from 0 to 3, and thus stops before 5. You see one more important fact here. The numeral 3 is to the left of the numeral 5 on the number line. Since the counting set for 3 is completed before the counting set for 5, 3 is to the left of 5.

EXERCISE 6-1

Make a general statement for whole numbers, so that you can tell from their places on the number line which of two numbers is less than the other.

6-2 "Greater than" on the number line

Just as you can see if a whole number is less than another on the number line, so you can see when a whole number is greater

than another by looking at their places on the number line. You know that 9 is greater than 4. You write this as follows:

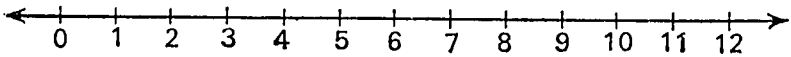
9 is greater than 4

or, in symbols,

$$9 > 4$$

On the number line 9 is to the right of 4.

If you did the exercise above correctly, you learned that a number m is *less* than another number n if m is to the *left* of n on the number line. Likewise, the number p is *greater* than the number q, if p is to the *right* of q on the number line. Look at this number line.



You can see that 12 is to the right of 10. Thus 12 is greater than 10. Also 7 is to the left of 10, and so 7 is less than 10. You can write these two facts in this way:

$12 > 10$ 12 is greater than 10
$7 < 10$ 7 is less than 10

EXERCISE 6-2

1. Draw a number line and show by a picture which of the following statements are true:

   a. $5 < 9$ **c.** $12 < 11$ **e.** $27 < 20$
   b. $0 > 1$ **d.** $45 > 54$

2. If $a > 5$ and $5 < c$, can you find numbers a and c such that:

   a. $a < c$?
   b. $a = c$?
   c. $a > c$?

   If your answer is yes, find such numbers.

3. Arrange the numbers a, b, c and d on the number line so that all of the following relations are satisfied:

$$a < b, c < a, b < d$$

4. Arrange the numbers d, f, n, p, and r on the number line using the following relations:

$$n < p, f < r, d < f, p < d, d < r$$

5. a. Arrange the following numbers on the number line using the following relations:

$$a > b,\ n < b,\ p < q,\ p > b,\ q < a$$

b. Use the number line you have just drawn above to find the relation between the following pairs of numbers:

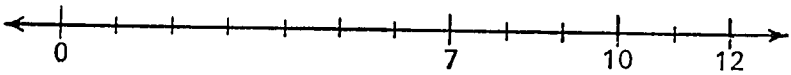
$$p, a \quad b, p \quad q, n$$

6-3 "Between" on the number line

Suppose you are given three whole numbers a, b, c arranged in natural order. For example, a might be 7, b might be 10 and c might be 12. Since these numbers are in natural order we know that a is less than b, and b is less than c; in symbols,

$$a < b \quad \text{and} \quad b < c$$

Thus, on the number line a is to the left of b, and b is to the left of c. In other words, b is *between* a and c on the number line. The following picture shows this for the numbers 7, 10 and 12:



Whenever three different numbers are shown on the number line, one of them must be between the other two. If the numbers are a, b and c, and if b is between a and c, with a to the left of b, then we can make the following statements about these numbers

$$a < b \quad b < c$$

Of course, we could also say the same thing by stating that b is greater than a, and c is greater than b. Either way, b is between a and c. This shows that the idea of "betweenness" for points on the number line tells us about the order of the numbers represented by the points.

EXERCISE 6-3

1. Using the number line, put the three numbers in each of the following sets into natural order, and tell which number is between the other two:

a. 1, 0, 3 **b.** 5, 7, 8
c. 6, 2, 3 **d.** 0, 1, 9
e. 11, 16, 14 **f.** 25, 14, 13

2. How many whole numbers on the number line lie between the following?

   a. 3 and 7 **b.** 30 and 39

   c. a and b when a and b are whole numbers such that $a < b$

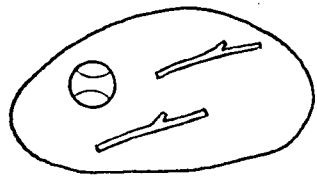
Chapter 7

REPRESENTATION OF NUMBER

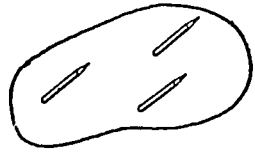
7-1 Forming the number concept. Abstraction

Children are not born with the idea of number. It takes them some years to learn it.

We say that a set of objects like

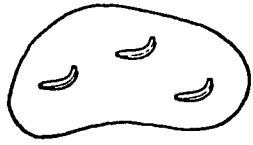


has three members. But at first a child sees only the objects or notices that they are together on a table. Even after he has learned the number names one, two, three, four, . . . , he may not connect the word "three" with the set shown. Before he can do this, he must see many sets of three objects. He must learn what these sets have in common. He must see them as sets of three things. To do this he must, as we say, *abstract* from the nature of the objects. That is, he must learn not to pay attention to what the things are. For example, he must not pay attention to the fact that



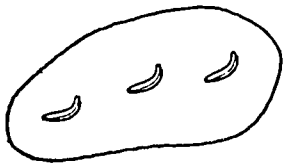
is a set of *pencils*

and

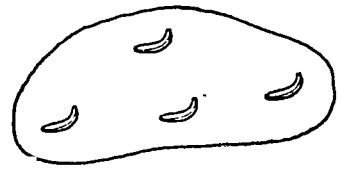


is a set of *bananas.*

At first these are the most obvious features he notices about these sets. He must try to push them out of his mind so that he sees something *alike* about the two sets even if their elements are so *different*. These sets are alike in a way that the two sets



and



are not alike, even though these two sets have elements of the same kind.

In learning to see the likeness between the set of pencils and the set of bananas (in spite of their obvious difference) he is learning to abstract. At an earlier age he has already made a beginning in this process of abstraction. He has, for example, learned to use the word "banana" for any one of a great many objects which look somewhat alike but not exactly alike. Some of these objects may be green and some may be yellow, but he has learned that this does not matter. They are bananas just the same.

When making the abstraction necessary for the number concept, one must ignore even more features of the objects than when naming them. Usually a child has to be about six years old before he can understand what number words mean in his own experience. Teaching this understanding takes great patience. As we have seen, it can be done by showing children sets of many sorts, by matching them with each other, and by learning the proper number words and numerals.

7-2 The need for symbols

The number four is an idea which indicates the "fourness" property of sets. Since the number four is an idea, it cannot be seen like a stone or a banana. To discuss and use this idea, we must represent it by a word or a symbol. Thus we can use the word "four" or the symbol "4", which we call a numeral. Just

as there are different words for the number four in different languages, there are different systems of numerals which have been used. The ancient Egyptians would have written IIII instead of 4. The Romans would have written IV.

While numbers cannot be seen, numerals can. They can be written on paper or on the blackboard. They seem more like real things. This is good. However, there is a danger. Numerals can become just marks on the paper which the student learns to write without connecting them with sets of objects. A little later on he may learn to put down 5 when he sees 2 + 3 without having any idea why he can do this or what 2 + 3 means in his experience. Then arithmetic becomes a meaningless game with symbols. To prevent this from happening the teacher should always try to keep the ideas behind arithmetic clear to the pupil. He must teach him that the facts of arithmetic do not depend on the names which we give to numbers. For example

$$2 + 3 \text{ is } 5$$

says the same thing as

$$\text{II} + \text{III is V}$$

The first statement is written in Hindu-Arabic numerals. The second is written in Roman numerals. They both mean the same thing.

7-3 Ways of representing numbers (Egyptian, Babylonian)

Among the earliest symbols used to represent numbers were model matching sets such as stroke (I) for one, the wings of a bird (𓅰) for two, three-leafed clover ( ) for three, the four legs of an animal ( ) for four, and so on. This principle was adopted by the Egyptians around 3500 B.C., possibly as an extension of a system of tallying, as follows:

1 2 3 4 . . . 9

I II III IIII IIIIIIIII

Clearly such systems are of little use for larger numbers. We cannot keep on creating and remembering new symbols and

names for numbers. Some kind of grouping becomes necessary with special symbols to represent the groups. The early Egyptians built up their system in groups of ten. Their single numerals were the following.

Number	*Egyptian numeral*	*Object represented by the numeral*
One	I	Stroke or staff
Ten	∩	Heel bone
One hundred	ᘓ	Scroll or coiled rope
One thousand	𓆼	Lotus flower
Ten thousand	𓂭	Pointing finger
Hundred thousand	𓆐	Polliwog
One million	𓁨	Astonished man

In the Egyptian system, the order in which the symbols are arranged does not matter. Thus ∩∩I, I∩∩, ∩I∩ are all numerals for 21. Other examples of Egyptian numerals are as follows:

	7	23	456	1,821
Egyptian numerals	IIIIIII	∩∩III	ᘓᘓᘓᘓIII ∩∩∩∩∩ III	𓆼ᘓᘓᘓᘓᘓ ∩∩I ᘓᘓᘓ

An entirely different system for representing numbers was invented by the Babylonians about 4,000 years ago. (The Babylonians lived in what is now the country of Iraq.) Their system used only two symbols 𒁹 and 𒌋. 𒁹 was used to represent 1, and 𒌋 was used to represent 10. For numbers from 1 to 59 the Babylonian system worked just like the Egyptian system except that the tens symbol was always written first in any numeral. Here are some examples of Babylonian numerals:

6	21	34	48
𒁹𒁹𒁹 𒁹𒁹𒁹	𒌋𒌋𒁹	𒌋𒌋𒌋𒁹𒁹𒁹 𒁹	𒌋𒌋𒌋𒌋𒁹𒁹𒁹 𒁹𒁹𒁹 𒁹𒁹𒁹

These symbols are rather difficult for us to draw, and you may wonder why the Babylonians didn't use something simpler. The reason is that the Babylonians did their writing on pieces of soft clay which they later hardened by placing in the sun. And for this kind of writing it is much easier to make wedge-shaped symbols like ▼ and ◀ with the end of a stick than to write symbols of the sort we use. Indeed, all Babylonian writing used wedge-shaped symbols, and for this reason it is called cuneiform, which means "wedge-shaped writing".

For numbers beyond 59 the Babylonian system is rather complicated. For 60 the Babylonians would write ▼, using the symbol for 1 again, but leaving a space after it. 61 would be written ▼ ▼, with a space, 70 would be written ▼◀, and so forth, up to 119. 120 would be written ▼▼, with a space after it. The system keeps going in this way until it reaches 360, which one would expect to be written ▼▼▼ ▼▼▼. But instead the Babylonians wrote ▼, using the symbol for 1 again. You can see how confusing this is. Learning arithmetic for the Babylonians must have been even harder than it is today.

EXERCISE 7-3

1. Write Egyptian and Babylonian numerals for each of the following:

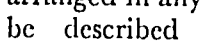
   12, 26, 52, 113, 201, 349

7-4 The Hindu-Arabic system of numeration

Nowadays we use a system of numerals which is called the Hindu-Arabic system. This system has ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9 which are called digits. We are able to represent any number, however large, by using these digits together with an especially clever idea, that of *place value*. As you know, in the Hindu-Arabic system we write digits in the ones place, the tens place, the hundreds place, the thousands place, and so on. The value of any digit depends both on the digit *and* on the place it occupies in the row of digits. Thus, in the numeral 3,234, the first 3 represents 3 thousands and the second 3 stands for 3 tens.

Hundred thousands	*Ten thousands*	*Thousands*	*Hundreds*	*Tens*	*Ones*
		3	2	3	4

This number would be written in the Egyptian system as



where the symbols could be arranged in any order. This could be described as an additive method and it is interesting to compare it with the Hindu-Arabic way of writing numerals.

You will notice that both systems work with groupings of ten. We are told that early man chose this particular grouping because he used his ten fingers for counting. When counting or matching sets of more than ten objects, his ten fingers would soon be used up. He would have to record this in some way or another. Perhaps he put a stone in his pocket or in a sack. Then he could continue using his fingers again. Each stone would then stand for ten. If he got a large number of stones in his pocket, he could replace each ten of them by a larger stone, again using ten as the natural group Each larger stone would then stand for ten tens, i.e. for one hundred or 10×10, often written as 10^2 for short. And so he could continue. Ten of these larger stones would form a new group of one thousand, i.e. $10 \times 10 \times 10$, written 10^3 for short. The next group would consist of ten thousands or $10 \times 10 \times 10 \times 10 = 10^4$. When we write numerals in this way, the raised numeral (the 4 in 10^4, for example) is called an *index* or *power*. We speak of 4 as the power to which 10 must be raised to give 10,000, or as the index of 10. Note that it is exactly equal to the number of zeros following the one in the numeral 10,000.

7-5 Grouping

The idea of grouping is fundamental in recording numbers. We must see that our pupils understand it thoroughly. They must realize the necessity for grouping. Our position is similar to that of a messenger boy of long ago who had to report to his chief how many people there were in a village. For each person he put a small stone in his sack. He intended to carry the sack to the chief and say, "Behold! There are as many people as

stones." This would have been a good method if there had been only a few people in the village. But the village was large and the sack soon became too heavy to carry. The messenger, therefore, used stones of two sizes, with the larger size representing ten of the smaller ones but weighing much less. He made the sack lighter by taking out ten small stones and putting in one large stone. He used ten because he had ten fingers on his hands. He could easily tell whether a set of small stones corresponded to a large stone by matching them with his fingers. He now said to his chief, "Behold! For each large stone there are as many people as fingers on my hands. For each small stone there is one person." The messenger needed no more than nine small stones. Instead of ten small stones, he could use one large stone.

In the above story the messenger was not counting, but was matching. Before people had developed the idea of counting and had systems of numeration, they were able to keep account of large numbers of people or cattle by tallying. This is a process of matching equivalent sets. If the messenger had cut notches in a stick to match the number of people, he could have cut a deeper notch to represent each group of ten. Thus he could have simplified his task. Children need to be given much practice in this kind of work, putting sticks into bundles of ten, arranging beans in groups of ten and so on. Some people carry out the process by writing nine strokes and a final one which crosses all the others (~~IIIIIIIII~~) for each ten. Sometimes people work in fives in this way. It makes the final counting of a large number of strokes much easier than it would be otherwise. Matching sets of things with sets of strokes or notches is called tallying.

7-6 A symbol for zero

When you are comparing the Egyptian system with the Hindu-Arabic system did you notice that the Egyptian system had no symbol for zero? Can you explain this? Why, then, does the Hindu-Arabic system need such a symbol? The reason is that the symbol for zero is absolutely essential to keep the digits in their correct position when there are empty places. For example, if there is no group of ten in a particular numeral we

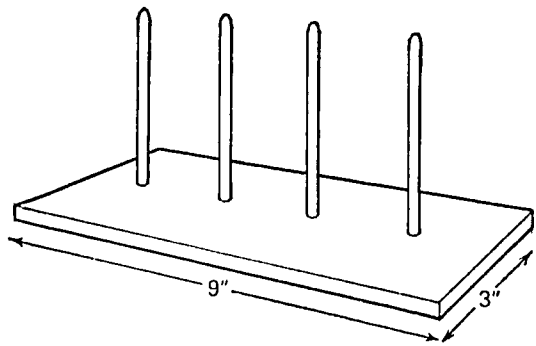
must have a symbol to show this. Take the numeral 306. Without the symbol 0, the 3 or the 6 could easily be written in the tens place and the numeral would look like 36. This, for instance, is just what happened in the Babylonian system. For a long time the Babylonians put up with this, but late in their history they invented a symbol to show that a place in one of their numerals was empty. The symbol they usually used was ●. From them the Hindus developed the symbol 0, to show that a place in a numeral is empty. This is the same symbol we use today, and it makes possible our system of numeration.

7-7 Representing numbers on the abacus

One way of giving your pupils experience in building numbers and then writing them is to use an *abacus*. This is really an aid to counting and is simply a set of sticks on which rings or beads are put to stand for numbers. The sticks are mounted on a stand so that they can be used more easily. There are several ways of making an abacus. Here are two for you to try. Make one of them to use for the exercises which follow.

Method 1

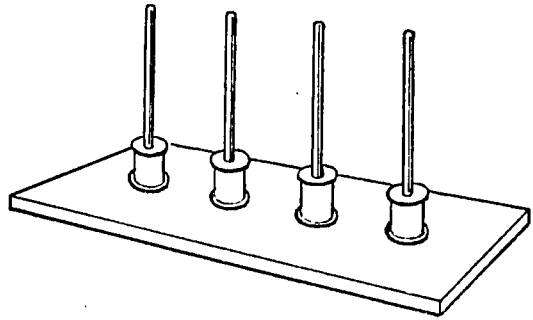
Use a piece of wood about 9 inches long, 3 inches wide and 1 inch thick. Mark 4 points along the middle of the wood about 2 inches apart. Knock a very long nail right through the wood at each of the marked points until it cannot go any further. File the ends of the nails if they are sharp. Your abacus is now made.

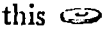


Method 2

You will need 4 reels or spools such as are used for sewing-thread. Push a stick or pencil or used ball-point pen into each

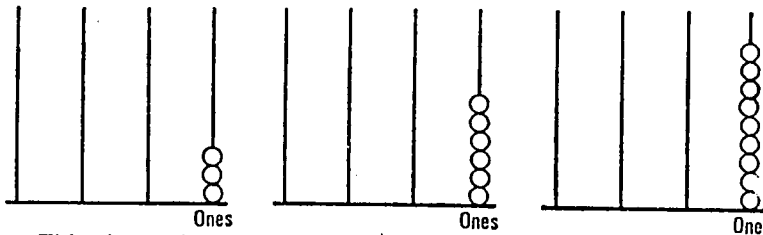
hole. Now you need some way to keep the 4 reels together in line. You can put them into a box of suitable size, or glue them to a piece of cardboard, or nail a piece of wood on to their bases. Your abacus should look like this.



Now you will need some beads, or rings to slip over the rods to show the numbers. These should look like this  or this . They can be made from rings of grass stems, or twisted grass or bamboo slices, or cardboard from a circular sweet packet like this.

Cut this along the dotted lines. Your rods must be long enough for 10 rings to fit on each rod.

To show numbers from 1 to 9 you put beads on the first rod from the right-hand side. This is the *ones* rod and every bead on the ones rod stands for *one*.



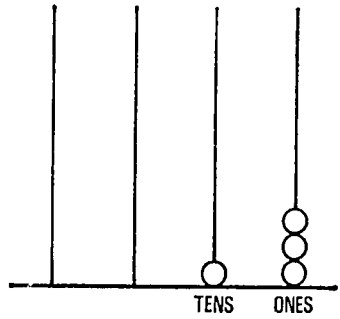
This shows 3 This shows 6 This shows 9

Your pupils can use an abacus for scoring games. Sooner or later they will want to show a score of 10. This is where they must learn the rule for using an abacus.

Rule of Procedure

Whenever there are 10 beads on a rod you must take them off and replace them by 1 bead on the next rod to the left.

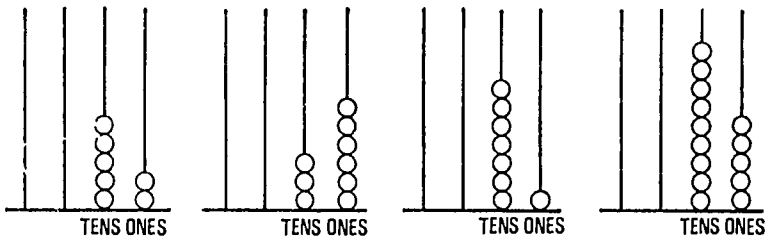
Since every bead on the second rod from the right represents ten, it is called the *tens* rod. Here is 13 shown on the abacus.



13 is 1-ten and 3-ones and so we have 1 bead on the tens rod and 3 beads on the ones rod.

EXERCISE 7-7A

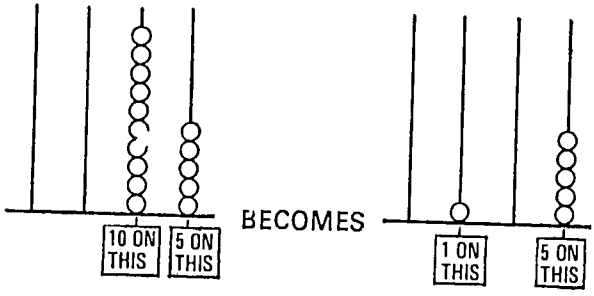
Write the numbers which are represented by each of the following pictures:



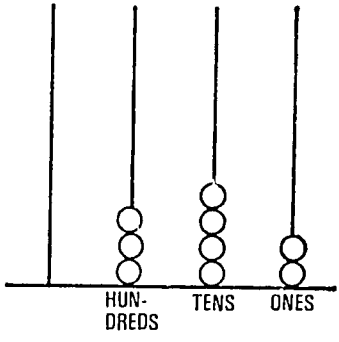
Here is the first one written out for you: There are 5 tens and 2 ones: this is 50 + 2 which is 52.

Look at the last picture in this exercise. Suppose we add another bead to the tens rod. This will make 10 beads. According to the rule, we must take them off and replace them by 1 bead on the next rod to the left. This bead will represent 10 tens which is 100. Thus the beads on the third rod are each

worth 100. We call this the *hundreds* rod. Every bead on the

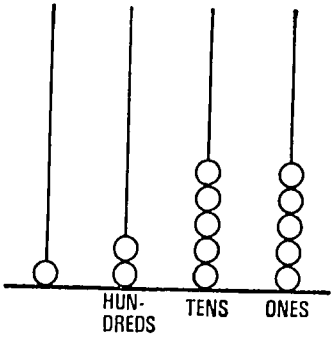


hundreds rod is worth 10 times as much as a bead on the tens rod. Every bead on the tens rod is worth 10 times as much as a bead on the ones rod. In general, every bead on *any* rod is worth 10 times as much as a bead on the rod next to the right.



This abacus shows 3 hundreds, 4 tens and 2 ones. This is 300 + 40 + 2, which is 342.

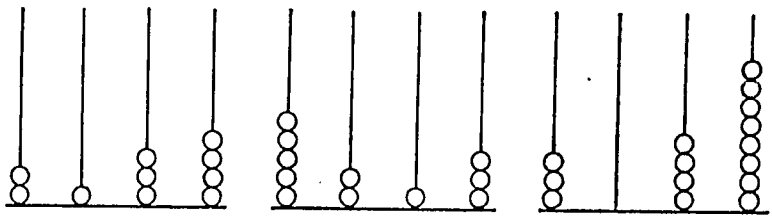
Now look at the following picture:



Here there is a bead on the fourth rod from the right. What is it worth? It is worth 10 beads on the rod to the right of it, which means that it is worth 10 hundreds. But you know that 10 hundreds are 1,000. Hence the fourth rod on the abacus is the thousands rod. Every bead on this rod is worth 1,000. Thus the above abacus shows 1 thousand, 2 hundreds, 5 tens and 5 ones which is the same as 1,000 + 200 + 50 + 5, or 1,255.

EXERCISE 7-7B

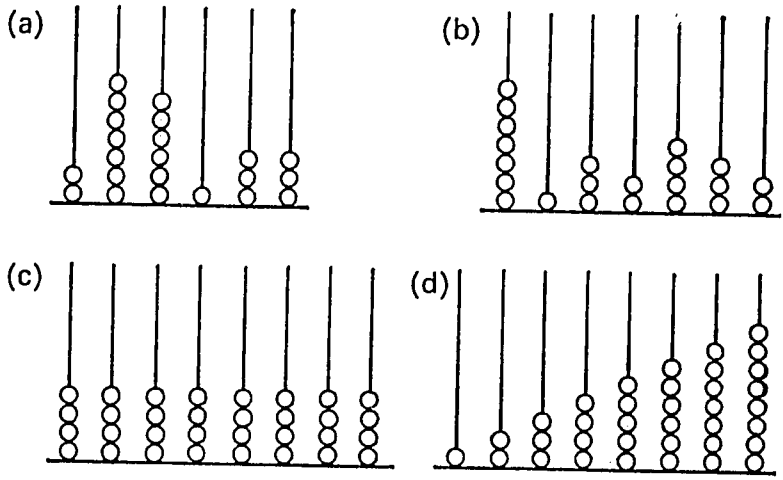
1. Show each of the following numbers on your abacus and make drawings of them.
   a. 8,324 **b.** 7,562 **c.** 6,666 **d.** 3,427
2. Write each of the numbers represented on the following abaci in (*a*) Hindu-Arabic numerals and (*b*) Egyptian numerals.



(i) (ii) (iii)

3. If you had an abacus with eight rods, what would a bead on
   a. the 5th rod from the right represent? **b.** the 6th rod?
   c. the 7th rod? **d.** the 8th rod?
   Draw a large abacus with 8 rods and label each rod.
4. Draw a picture of an abacus which shows one million.
5. Draw an abacus with 8 rods and draw some beads on *each* rod. Write the number which your picture represents. Represent 4 more numbers in this way, saying each time what the number is.
6. Draw a picture of an abacus representing the following number: 2 ten millions, 5 millions, 3 hundred thousands, 6 ten thousands, 1 thousand, 8 hundreds, 8 tens and 5 ones. This is the number: 25,361,885.

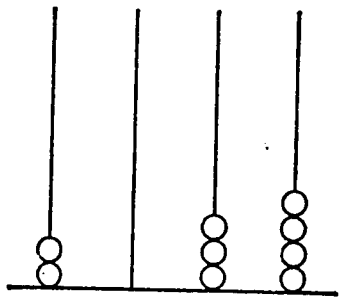
7. Write the numbers which are represented on these abaci, first the symbols, then in words.



7-8 The empty rod on the abacus

Some kind of abacus has been known for a very long time. It has been used in some form by most peoples from ancient Egyptian times to the present-day. The abacus is first used to make a record of a count. Its other use, in calculation, will be studied later in this book.

Suppose that you have counted all the people in a town and have made a record of your count on an abacus. Suppose that the abacus looks like this:



We now ask, "How shall we record in writing the number

represented above?" What symbols shall we use? In particular, how do we record the empty rod?

Different cultures have solved this problem in different ways. For example, an ancient Egyptian would record this number by writing

He did not need a symbol for the empty rod. An Egyptian symbol tells us which power of ten it represents, but it does not tell us how many there are. To show 30, that is 3 tens, an Egyptian had to write ∩∩∩. This is like writing ten ten ten. In writing the above numeral he could show that the hundreds rod was empty just by leaving out the symbol 𓍢 for one hundred.

The Hindu-Arabic system does not work this way. It is much more efficient. For one thing, it does not need to invent new symbols to represent larger and larger numbers. It uses the same symbols to represent the number of rings on any rod, and tells which rod they fill by the position or place of the symbol in the numeral. This is why the Hindu-Arabic system is called a *place value* system. This also explains why a special symbol is needed to represent an empty rod.

In the Hindu-Arabic system we tell how big a numeral is by its place value; that is, by the position of the digits used to write it. For example, since the tens place is the second place from the right we show 3 tens by putting the digit 3 in the second place in the numeral. But if we just wrote 3, and nothing more, it would be impossible to tell that we were talking about 3 *tens*. Thus we use a zero to show that the ones place is empty, and we represent 3 tens by writing 30. In the same way the number represented on the abacus in the last picture is written 2,034. The 0 shows that the hundreds rod is empty. Without this symbol we would have to do what the early Babylonians did, and write 234. And then everyone would read this incorrectly.

Finally, you should notice that this use of the symbol 0 fits in with what we said about sets. The set of rings on the empty rod is the empty set. The number of elements in the empty set is zero, and is written 0.

EXERCISE 7-8

1. Draw abaci to show these numbers:

a. 3,052	**b.** 2,308	**c.** 31,450
d. 700,103	**e.** 6,500	**f.** 5,000
g. 7,602,019	**h.** 9,999	

2. Add one to the number in (*h*), and draw the result.
3. Write the numbers represented on the abaci problem 2 in
   a. Egyptian numerals **b.** Hindu-Arabic numerals

7-9 The Roman system of numeration. Comparison with Hindu-Arabic system

Before the Hindu-Arabic system of numeration came into use in Europe, Roman numerals were used. The Roman system apparently came into use in about 300 B.C. and even today we still use Roman numerals on clocks and in many other ways. It took several centuries for people to give up the Roman system for the other. Why did the Roman system persist for so long? Why do we now prefer the Hindu-Arabic system? Let us take a look at the Roman system.

	1	5	10	50	100	500	1,000
Roman numerals	I	V	X	L	C	D	M

With the symbol I, we could write any number simply by writing I's in succession. For example, we could denote 7 by IIIIIII. By introducing the symbol V, to denote 5, we can replace IIIII in this string by V, to give us VII. This is the symbol for 7 in Roman numerals. Using VII instead of IIIIIII results in the saving of both space and effort.

The number 13 could be represented by VVIII. If we use the symbol X for 10, we can replace VV by X. This gives XIII which is the Roman numeral for 13.

The Romans made their system even simpler by agreeing that if a symbol for a smaller number is written to the left of a symbol for a larger number, the smaller number is to be subtracted from the larger number. Thus IV denotes 4, and IX

denotes 9. These are shorter than IIII and VIIII which we would have had to write otherwise. In writing Roman numerals we never put more than one symbol for a smaller number before a symbol for a larger number. Thus 9 is written IX whereas 8 is written VIII and *not* IIX. Similarly XC is the symbol for 90, while CX is the symbol for 110. This system is simple and easy to learn, which is one reason why it was used so long.

The main reason why people stopped using Roman numerals is that calculations with Roman numerals are much more cumbersome than with Hindu-Arabic numerals. For example, to add two numbers written in Roman numerals we first collect all of the symbols in the two numerals. Then we simplify by combining groups of symbols. Finally, we write the symbols in correct order. For example

$$\text{MCL} + \text{CCXI} = \text{MCCCLXI}$$
$$\text{XXVII} + \text{LXIV} = \text{LXXXV(IV)II} = \text{LXXXXI}$$

Multiplication and division are even more complicated in Roman numerals. In fact, they are so complicated that it would be a waste of time to describe all of the rules here. Thus we too shall do what was done in the past, and devote all of our attention from now on to the Hindu-Arabic system.

EXERCISE 7-9

Work out the following problems using Roman numerals.

1. CCLXV + DCCLVIII
2. MDCLX + MCCXLIV
3. LXXVII − XLIII
4. CCLXIV − CLXIX

Chapter 8

NUMBER BASES IN THE HINDU-ARABIC SYSTEM

8-1 Numeration in base five

We saw earlier that because man possesses ten fingers, ten became his natural counting group. For this reason, he chose to work in units, tens (10), hundreds (tens of tens $= 10 \times 10 = 10^2$), thousands (tens of tens of tens $= 10 \times 10 \times 10 = 10^3$) and so on. Some people in the past chose five, the number of fingers on one hand, as their counting group. They counted as follows: one, two, three, four, 1-five, 1-five and 1, 1-five and 2, 1-five and 3, 1-five and 4, 2-fives, 2-fives and 1, and so on, as in the following table:

Counting numbers in groups of five	*Numerals for counting numbers in base five*
one	1
two	2
three	3
four	4
1-five	10
1-five and 1	11
1-five and 2	12
1-five and 3	13
1-five and 4	14
2-fives	20
2-fives and 1	21
2-fives and 2	22
2-fives and 3	23
2-fives and 4	24
3-fives	30
4-fives and 3	43
1-twenty-five (5^2)	100 ($1 \times 5^2 + 0 \times 5 + 0 \times 1$)
1-twenty-five and one	101 ($1 \times 5^2 + 0 \times 5 + 1 \times 1$)

Notice that in base five, 213 means 2-twenty-fives and 1-five and 3-ones, which is 58 in the Hindu-Arabic system. In base-five counting we read the symbol 10 as "one-zero" and not as

10. Similarly we read 12 as "one-two" and not as 12. Notice that we need only five digits 0, 1, 2, 3 and 4 when counting in groups of five. When we are counting in groups of five as in the above table, we say we are using the *five-system of enumeration* or that we are working in *base five*. We refer to the numerals as *base-five numerals*.

EXERCISE 8-1

1. Copy the chart below for counting numbers from 1 to 100. Then complete the chart with numerals in base five:

1	2	3	4	10	11	12	13	14	20
21									
41									
								124	
						232			
311									

2. Use your chart to answer the following question. What are the largest numbers in base ten which are represented as one-digit, two-digit and three-digit numbers in your chart?

3. In the United States of America and in some other countries such as Liberia, the smallest unit of money is 1 cent. 5 cents make 1 nickel and 5 nickels make 1 quarter.

a. Express the following amounts in quarters, nickels and cents:

(i) 6 cents (iii) 26 cents (v) 33 cents
(ii) 10 cents (iv) 46 cents

b. How many cents are there in:

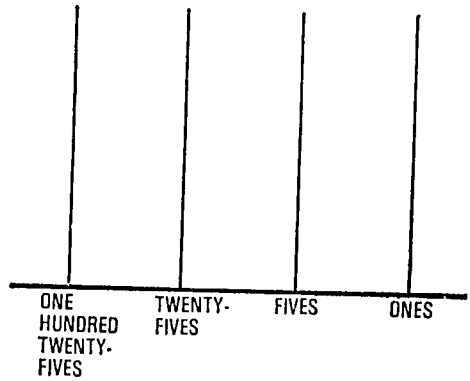
(i) 2 quarters, 1 nickel and 2 cents?

(ii) 3 quarters, 4 nickels and 3 cents?
(iii) 4 nickels and 4 cents?
(iv) 2 quarters, 2 nickels and 2 cents?
(v) 4 quarters and 2 nickels?

Notice that in question 3, we are working with the base-five system of enumeration.

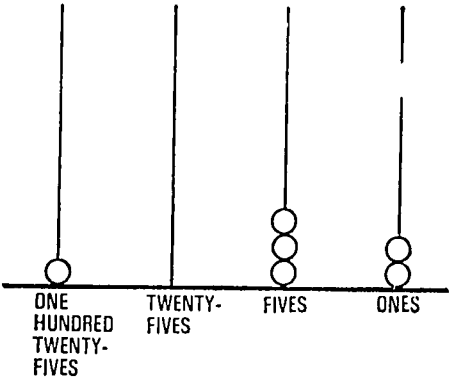
8-2 A base-five abacus

Just as we used the abacus to represent numbers expressed in base ten, we can also use an abacus to picture numbers expressed in base five. The first rod from the right will again register ones. But this time whenever *five* beads accumulate on the ones rod, we remove them and place 1 bead on the next rod to the left. Thus the rod to the left of the ones rod is called the fives rod and each bead on the fives rods represents 5 ones.



When 5 beads accumulate on the fives rod, we remove them and place 1 bead on the next rod to the left. Thus each bead on the third rod represents 5 fives. We call the third rod the twenty-fives rod, or more briefly, the 5^2 rod. Each time 5 beads accumulate on a rod, we replace them by 1 bead on the next rod to the left. Thus, the fourth rod is called the one hundred and twenty-fives rod or, briefly, the 5^3-rod. Each bead on it is equivalent to 5 beads on the twenty-fives rod.

The number represented by the beads on the abacus below can be written in base five very easily:



We note that there are

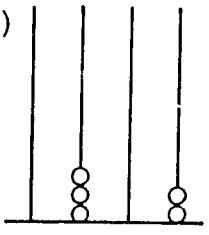
1 one hundred and twenty-fives	or 1×5^3
0 twenty-fives	or 0×5^2
3 fives	or 3×5
2 ones	or 2×1

on the abacus. Thus the number is written as $1{,}032_{five}$. We write "five" a little below the numeral to show that we are referring to a numeral in the base-five system.

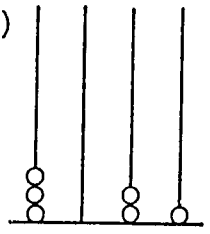
EXERCISE 8-2

1. What numbers are represented by the beads on each of the following base-five abaci. Express them first in base five and then in base ten.

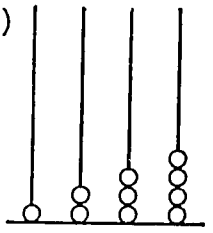
(a)



(b)



(c)



2. Draw pictures of base-five abaci to represent the following numbers.

a. 324_{five} **b.** $2{,}000_{five}$ **c.** twenty-five
d. thirty-five **e.** thirty-nine

8-3 Numeration base seven

There is an important discovery to be made in the above work which perhaps you have already made for yourself. We know now how to represent numbers with base-five and base-ten numerals. Has it occurred to you that numerals may be written in other bases, base two, base three, base four, . . ., base twelve, . . ., base twenty, . . .? In other words, there is nothing special about the decimal system. If you have realized this, you have done very well indeed. You have made a generalization of the kind which mathematicians frequently have to make and which is fundamental to the growth of mathematics. As your studies progress, you will realize that generalization is at the heart of mathematical thinking.

Check that you have made the generalization by working the following examples which use base-seven numerals.

EXERCISE 8-3

1. Complete the following chart for base-seven numerals from 1 to 100. How many symbols do base-seven numerals require?

1	2					10
						100

2. Describe how to use a base-seven abacus to represent numbers.

8-4 Grouping in twelves

Now that you can work in base seven, base five and base ten, you will find it easy to work in any base. The base-twelve system is worth special mention because we often group in twelves in

everyday life. We often work with dozens for instance. We may buy a dozen eggs and this is grouping in twelves. Your school may buy pencils and exercise books by the gross where a gross consists of twelve dozen, or twelve twelves, or 12^2. Twelve inches make one foot and twelve months make one year. There are twelve hours marked on the face of the clock and, in many countries, there are 12 pennies to a shilling. We shall therefore take a moment to see how we can use twelve as a number base.

In base-ten numeration, we need ten symbols; in base five, five symbols; in base seven, seven symbols. How many will we need to write numbers in base twelve? We will need twelve symbols, which is two more than in the base-ten system. The extra ones needed are for ten and eleven, which we may denote by *t* and *e* respectively. Counting then proceeds:

1, 2, 3, 4, 5, 6, 7, 8, 9, *t*, *e*, 10, 11, 12, . . ., 19, 1*t*, 1*e*, 20, 21, . . .

Here 10_{twelve} represents 1-twelve and 0-ones, 11_{twelve} represents 1-twelve and 1-one, and so on.

EXERCISE 8-4

Complete the following table for base-twelve numerals.

1	2	3	4	5	6	7	8	9	*t*	*e*	10
11											
											100

8-5 Base-two numerals

Base-two numerals are especially interesting because they use only two symbols, 0 and 1. Counting in the base-two system proceeds:

1, 10, 11, 100, 101, 110, 111, 1000, . . .

The places represent units, twos, fours (2^2), eights (2^3) and so on. We call such numerals *binary* numerals. In the base-two, binary system, the number 100 is written 1100100_{two}. This requires writing many digits, which is an apparent disadvantage of the binary system. The binary system does have a very important use, however, in electronic calculating machines. These machines calculate at lightning speed. They take only seconds to work problems which would take a man weeks and months to do. Binary arithmetic is used because it needs only two symbols, 0 and 1. These can correspond to current switched on and current switched off, or to a long and short buzz. Binary numerals can also be recorded on tape by means of a punched hole for the symbol 1 and no hole for the symbol 0.

EXERCISE 8-5

Copy and complete the chart below for base-two numerals from 1 to 50.

1	10	11	100				
1001							
							11001
101010							
						1,000,000	

In this chapter we have learned how to represent numbers in various bases. We shall return to this subject later when we study addition and multiplication. We shall then see how to use these numerals in calculations.

Chapter 9

ADDITION

9-1 An Example

Suppose we want to teach a child that $3 + 2 = 5$. What do we do? We start by showing him a set containing 3 elements, and another set containing 2 elements, and then we put the sets together to get one large set containing 5 elements. At this point we say, "Look! Three and two more make five." After doing this many times, and also letting the child try it for himself, we begin saying, "three plus two equals five". We then show him that this is written

$$3 + 2 = 5$$

Finally, we say that we are *adding* the numbers 3 and 2, and that the answer is 5. We tell him that this is a fact about numbers which he must learn. To make sure that he learns it we keep asking him, "What is three plus two?" We expect him to answer, "Five."

This example illustrates the two main points about addition of whole numbers:

First

Addition is related to forming the *union* of sets. In fact, when we mentioned taking a set containing 3 elements, and another set containing 2 elements, and putting them together to get one large set, we were forming the union of the two sets. For instance, if our sets were

$$\{\triangle, \triangle, \triangle\} \quad \text{and} \quad \{\bullet, \bullet\}$$

the five-element set obtained by putting these sets together would be

$$\{\triangle, \triangle, \triangle, \bullet, \bullet\}$$

Remember that we can show this by writing the set equation

$$\{\triangle, \triangle, \triangle\} \cup \{\bullet, \bullet\} = \{\triangle, \triangle, \triangle, \bullet, \bullet\}$$

Second

The sets used in this example were *disjoint*. That is, they had no elements in common. To see why we *must* use disjoint sets to illustrate that $3 + 2 = 5$, suppose we had used the sets

$$\{a, b, c\} \quad \text{and} \quad \{c, d\}$$

The first of these sets contains 3 elements, and the second contains 2 elements. However, when we form their union we get

$$\{a, b, c, d\}$$

because the element c belongs to both sets. And this set contains only four elements, not five.

9-2 Addition and the union of sets

In the last section we saw how to show that $3 + 2 = 5$ by forming the union of two disjoint sets. In this section we shall use this idea to describe how any two whole numbers are added.

Suppose A and B are (finite) sets. Also suppose that A and B are disjoint. For example, A and B might be the following sets:

A = {▨, ▨, ▨, ▨}
B = {◍, ◍, ◍}

Here A contains 4 elements. Hence it represents the number "four". B contains 3 elements. It represents the number "three". Now form the union of A and B. We get

A ∪ B = {▨, ▨, ▨, ▨, ◍, ◍, ◍}

This set also represents a whole number. Since it contains 7 elements, it represents the number "seven". We express this fact by writing $4 + 3 = 7$. We say that 7 is the SUM of 4 and 3, and we speak of ADDING 4 and 3 to get 7.

In short, we have

{▨, ▨, ▨, ▨} ∪ {◍, ◍, ◍} = {▨, ▨, ▨, ▨, ◍, ◍, ◍}

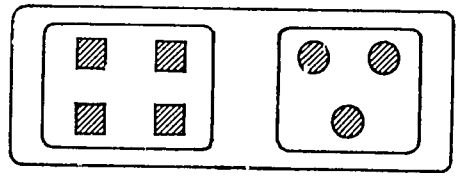
We write:

4	+	3	=	7

We say:

"four plus three equals seven"

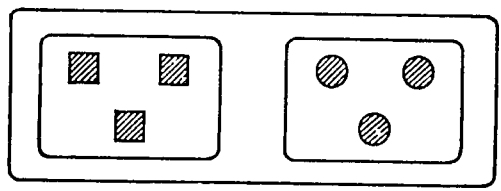
We can also show the same facts by drawing the following picture:



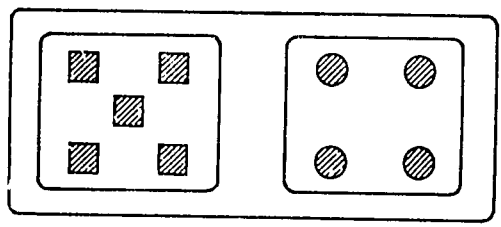
$4 + 3 = 7$

This time we have enclosed sets A and B with one large curve to show that we are thinking about their union. The statement $4 + 3 = 7$ beneath the picture tells us the number of elements in each of the sets in the picture.

There was nothing special about this example. Any two (finite) sets A and B would have worked just as well, provided they had no elements in common. Of course, we would not have obtained the same addition equation at the end if our sets had numbers of elements different from 4 and 3. The following pictures provide additional examples.



$3 + 3 = 6$



$5 + 4 = 9$

Children find such pictures very helpful when learning addition.

By now we have looked at enough examples to describe how the sum of *any* two whole numbers is defined. It goes as follows:

Let a and b be whole numbers. Choose a set A which contains a elements. Choose a set B which contains b elements. Make

sure that A and B are *disjoint*. Form $A \cup B$. We then say that the number of elements in $A \cup B$ is $a + b$.

Thus, if $a = 3$ and $b = 2$, set A must contain 3 elements, and set B 2 elements. Then $A \cup B$ contains 5 elements and we say that $3 + 2 = 5$.

The number $a + b$ is called the SUM of a and b. It is obtained by ADDING a and b.

EXERCISE 9-2

1. Find the numerals to put into the boxes to make each of the following equations true.

   a. $5 + 3 = \square$ **b.** $3 + 5 = \square$

   c. $2 + 8 = \square$ **d.** $5 + 5 = \square$

2. Draw a picture to illustrate each of the equations in Question 1.

3. When we studied the union of sets we saw that $A \cup B$ is the same set as $B \cup A$. What does this fact tell you about the sums $a + b$ and $b + a$ when a and b are whole numbers?

4. Suppose that a, b and c are whole numbers such that

$$a + b = c$$

   a. What can you say about b if $a = c$?

   b. What can you say about b if $a < c$?

5. Suppose that a, b and c are whole numbers. What can you say about a, b and c if $a + b = c$ and $a + c = b$?

9-3 Addition using counting sets

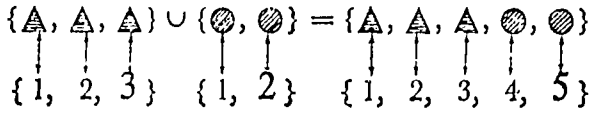
We have seen that the sum of two whole numbers is defined by forming the union of disjoint sets. Thus when we make the statement $3 + 2 = 5$ we are saying something about *all* unions of a set of 3 elements and a set of 2 elements when these sets are disjoint. Since this statement refers to the number of elements in these sets we should be able to illustrate it by using counting sets. We can, as follows.

Consider the sets

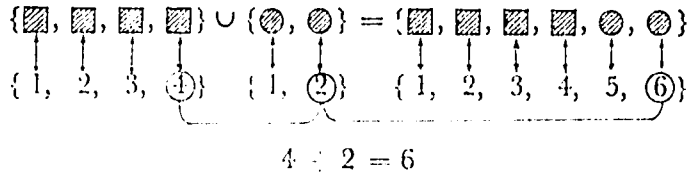
$$\{\triangle, \triangle, \triangle\} \quad \text{and} \quad \{\bullet, \bullet\}$$

The first contains 3 elements, the second contains 2 elements,

and their union contains 5 elements. The picture below shows all of these facts



In this picture we have matched the elements in our sets with the elements in counting sets. In other words, we have counted the elements in each of the sets. Now look at the last number in each counting set. These numbers are 3, 2 and 5. They tell us again that $3 + 2 = 5$. Pictures like these also help children to learn addition. One more is shown below.



EXERCISE 9-3

1. Draw pictures using counting sets to illustrate each of the following statements.

a. $3 + 4 = 7$ **b.** $2 + 2 = 4$
c. $2 + 8 = 10$ **d.** $5 + 5 = 10$

9-4 Many names for one number

Notice that the sums in problems (*c*) and (*d*) above are the same.

$$2 + 8 = 10 \quad \text{and} \quad 5 + 5 = 10$$

In problem (*c*) we can say that the sum of the number of elements in the two sets is $(2+8)$. But, as we know, $2+8=10$. Thus we have two ways of naming the same number; as 10 and as $2 + 8$. Moreover, problem (*d*) shows that $5 + 5$ gives us another name for 10. Thus $2 + 8$, 10, and $5 + 5$ are all symbols or numerals which represent the same number. So do $3 + 7$, $7 + 3$, $9 + 1$, and so on. When we write "=" between two numerals we mean that these two numerals are names for *the same number.*

EXERCISE 9-4

Write four different symbols which name each of the following numbers: 6, 13, 22, 8.

9-5 Mathematical sentences and the use of the box

In grammar, a sentence or a statement is a group of words which conveys a completed thought. In mathematics we also express our thoughts by sentences. An example of a number sentence or number statement is "$3 + 5 = 8$". This expresses the fact that "$3 + 5$" and "8" are different names for the same number.

The sentence "Pawpaws are green" expresses a true fact and we say that it is a *true sentence*. On the other hand, the sentence "Elephants are green" is clearly false. Number sentences may also be true or false. For example, "$3 + 5 = 8$" is a true sentence, whereas "$3 + 5 = 7$" is a false sentence.

Consider the sentence "It is green". Here it is impossible to say whether this sentence is true or false until we know to what the word "it" refers. If "it" refers to "pawpaws", the sentence is true. If "it" refers to "elephants", the sentence is false. We shall call such a sentence an *open sentence* or *open statement*. An open sentence contains a pronoun such as "it", "he", "they", etc., and we cannot tell whether the sentence is true or false until we know to what the pronoun refers.

In the same way, we can talk about open number sentences. You have already seen such open number sentences when you did Exercise 9-2. You solved the problem "$5 + 3 = \square$" by writing "8" in the box. You then had the true sentence, "$5 + 3 = 8$". Had you written "6" in the box, the sentence "$5 + 3 = 6$" would have been false. Thus the box plays the role of a pronoun in an open number sentence.

The box notation is a very suggestive way of writing an open number sentence, and children readily understand its use. The teacher can help the child find the numeral which must be put into the box to make the sentence true by using sets of stones or sticks.

After the pupil learns how to use the box, he can also use different symbols to represent the pronoun in the open number sentence. For example, he could write "$5 + 2 = \bigcirc$" or

"$5 + 2 = \triangle$" and then fill the circle or triangle with the number which makes the sentence true. Eventually, it will become more convenient to use a letter of the alphabet instead of a box, or circle, or triangle. We then write "$4 + 3 = N$" or "$5 + 2 = K$", and ask the student to find the number which should replace N or K to make these sentences true. Since this book is designed to help the teacher prepare for teaching young pupils these basic concepts, we shall use the box notation more than any other. As we continue you will see its advantages as a teaching device.

We have seen that an open number sentence becomes true or false when a symbol for a number, that is, a numeral, is written in the box. Thus when we state a problem involving an open number sentence such as $2 + \square = 7$ we should say, "Find the number whose numeral should be written in the box to make the open sentence $2 + \square = 7$ true." But to state the problem this way takes rather long and we shall usually shorten the statement to, "Find the number which makes the sentence $2 + \square = 7$ true". In doing this we are following the common practice of using the word "number" in place of "numeral". This will never cause any difficulty provided you remember the difference between these words, and realize that when we say "number" we often really mean "numeral".

EXERCISE 9-5

1. Give several examples of open sentences containing the pronoun "it". In each case give two examples of words that make the sentence true, and two examples of words that make the sentence false.
2. **a.** Give an example of a true number sentence using the symbol "$<$".
   b. Give an example of a false number sentence using the symbol "$<$".
   c. Find two numbers which make the sentence $\square < 5$ true.
   d. Find two numbers which make the sentence $\square > 5$ true.

9-6 Missing numbers in addition equations

The box may be used in different positions in a number

sentence. We can write "$4 + \square = 7$". The box is to be filled by a number which makes this sentence true. The number which belongs in the box is the number of elements in a set which must be combined with a set of 4 elements to give a set of 7 elements. This kind of addition problem is very useful for pupils. It helps them understand how one number can be made up by adding many different pairs of numbers. It therefore helps them to realize that one number can have many different names. It also prepares them for subtraction.

Here, for example, are some open number sentences involving the number four:

$$\square + 4 = 4, \quad \square + 2 = 4, \quad \square + 1 = 4, \quad 3 + \square = 4$$

When the missing numbers in these sentences are filled in we obtain the following true sentences:

$$\boxed{0} + 4 = 4 \quad \boxed{2} + 2 = 4 \quad \boxed{3} + 1 = 4 \quad 3 + \boxed{1} = 4$$

EXERCISE 9-6A

Find the numbers to put into the boxes to make each of the following sentences true:

1. $\square + 2 = 4$

2. $3 + \square = 10$

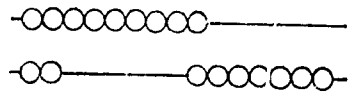
3. $4 + 7 = \square$

4. $0 + \square = 8$

5. $4 + 3 = \square + 5$

6. $\square + 6 = 5 + 5$

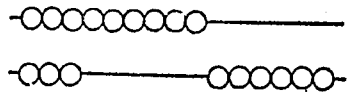
At this point you may be wondering how your pupils will be able to find the missing numbers in problems such as these. The answer is that they should work them out with sets. For instance, you may ask them to find the missing number for the open sentence $\square + 7 = 9$. To solve this problem they can use a single-bar abacus with nine beads, or any set of nine objects. If they use an abacus they should start with all of the beads on one side of the abacus, and move seven of the beads to the other side. The number of beads which were not moved will be the number which makes the sentence true. The number, of course, is 2.



Similarly, if the sentence is

$$6 + \square = 9$$

their solution will look like this:

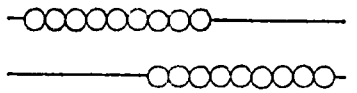


Hence, in this problem the box is holding a place for the number 3.

Here is another example. What number should be put in the box to make the following sentence true?

$$\square + 9 = 9$$

You know that the answer is 0. However, your pupils may have trouble with this. It will help them to solve the problem on an abacus, like this:



This time there are no beads on the extreme left of the abacus. Thus the box must be filled with the number 0 to make the sentence true.

We therefore write $\boxed{0} + 9 = 9$

Using the abacus in this way it is easy to show that each of the following is a true sentence:

$0 + 9 = 9$	$5 + 4 = 9$
$1 + 8 = 9$	$6 + 3 = 9$
$2 + 7 = 9$	$7 + 2 = 9$
$3 + 6 = 9$	$8 + 1 = 9$
$4 + 5 = 9$	$9 + 0 = 9$

Thus all of the sums $0 + 9$, $1 + 8$, $2 + 7$, etc., are different names for the number 9.

EXERCISE 9-6B

1. Plan a lesson for small children in which you would use an abacus or other means to find all sums of pairs of numbers that give 5.
   Do the same for 7 and 8.

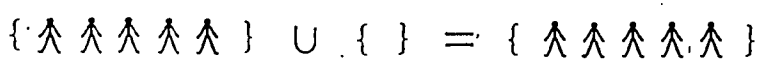
2. I am thinking of a number. This number plus 5 is 11. What is the number?
3. **a.** Each of twelve children in a class is assigned a different number from 1 to 12. If the children form pairs so that the sum of the numbers assigned to each pair is 13, which children are left without partners?
   b. Answer the same question when the sum is 14, and when the sum is 16.
   c. Answer the same question when the sum is 20, 23 and 24.
4. Imagine that you are given two dice. Each dice has six sides and each side shows one of the numbers 1, 2, 3, 4, 5 or 6. Throw both dice on a table at the same time. Make a list of all possible pairs of numbers that could turn up, and list their sums. Which sum do you think is most likely to turn up. Why?
5. Let a, b and c be whole numbers and suppose that $a + b = c$. How many such equations are there when
   a. $c = 5$ **b.** $c = 10$
   c. $c = 1{,}000$ **d.** $c = 0$
6. Let a and b be whole numbers such that

$$a + a = b$$

What can you say about b?

9-7 The property of zero in addition

If there is a set of 5 boys and 0 girls we can represent the union of these sets by the picture



The addition equation corresponding to this picture is

$$5 + 0 = 5$$

Similarly, if we form the union of any set with the empty set we get the original set back again. Thus we have the following true sentences, or equations:

$0 + 0 = 0$	$0 + 0 = 0$
$1 + 0 = 1$	$0 + 1 = 1$
$2 + 0 = 2$	$0 + 2 = 2$
$3 + 0 = 3$	$0 + 3 = 3$
$4 + 0 = 4$	$0 + 4 = 4$

These equations show that *when zero is added to a number the number is left unchanged.* This special property of zero is described by saying that zero is the IDENTITY for addition. The word "identity" is used because every number remains the same when zero is added to it. We can express this property of zero by writing the open sentences

$$0 + \square = \square \quad \text{and} \quad \square + 0 = \square$$

These sentences are true no matter what number is put in the boxes. You can choose any number you please so long as the same number is used in both boxes of each equation.

EXERCISE 9-7

1. Use pictures of sets to show that $0 + 0 = 0$ is a true statement.
2. Find the numbers which make each of the following statements true.

   a. $8 + \square = 8$ **b.** $479 + \square = 479$
   c. $\square < 1$ **d.** $\square + 1357 = 1357$
   e. $235 = \square + 235$ **f.** $35 = 35 + \square$
3. Suppose that I is a whole number such that

$$\square + I = \square$$

   is a true statement no matter what number is put in the box. Show that $I = 0$. [*Hint:* Put 0 in the boxes in this statement and then use the property of zero.]

9-8 Addition table

When your pupils thoroughly understand how to find the sum of two numbers by forming the union of disjoint sets, they can build an addition table.

First make a square with 121 small squares in it. That means that there will be 11 rows with 11 squares in each row. In the top left hand square put the + sign to show that you are adding. Follow this by the symbols 0, 1, 2, . . ., 9, one in each square along the top row. Similarly, write the symbols 0, 1,

2, . . ., 9, in the squares down the left-hand side. You now have a square like this:

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2							
1										
2										
3										
4										
5										
6					10					
7										
8										
9										

Begin with the first empty square on the second row. Put into this square the number which is the sum of the numbers you have already shown in the outside spaces at the left end of the row and the top of the column. This sum is 0 + 0, so we put 0 into the first empty space. The space next to this on the right is for the sum of 0 + 1. Thus 1 is put into this space. The next square along the second row has 0 and 2 at the ends of its row and column and so 2 is put here.

Where do we show the sum 6 + 4 = 10? We find the row which begins with 6 and the column which begins with 4. We then find the square where this row and column intersect and write 10 in this square.

EXERCISE 9-8

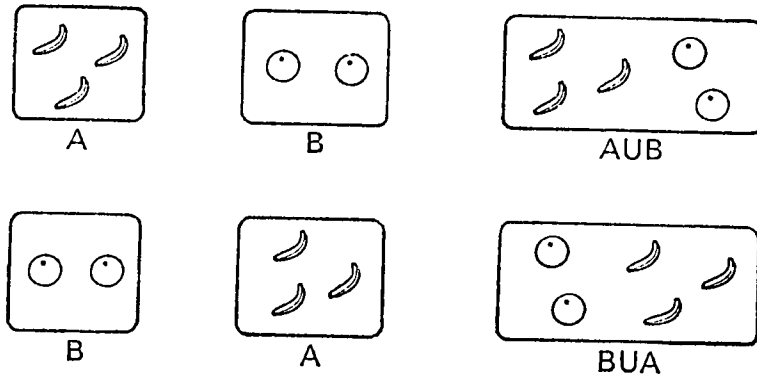
1. Prepare a table similar to the one described and fill in all the sums for pairs of numbers from 0 + 0 to 9 + 9.
2. Use the table to find six pairs of numbers whose sum is 8, and write the addition equations which show these facts.

Chapter 10

PROPERTIES OF ADDITION

10-1 Commutative property of the union of sets

When you form the union of two sets you know that it does not matter which set you take first. The union of a set of 3 bananas and a set of 2 oranges is the same as the union of the set of 2 oranges and the set of 3 bananas. The order does not make any difference to the result. The following picture illustrates this.



We see that the sets $A \cup B$ and $B \cup A$ have the same elements. They are equal sets. Therefore $A \cup B$ and $B \cup A$ are names for the same set, and we write

$$A \cup B = B \cup A$$

The fact that $A \cup B = B \cup A$ is called the COMMUTATIVE PROPERTY *of the union of sets.*

EXERCISE 10-1

1. **a.** Write the addition equations suggested by the sets in the picture above.
   b. Does this picture show that $3 + 2 = 2 + 3$? Explain.
2. Draw a picture like the one above to show that

$$1 + 2 = 2 + 1.$$

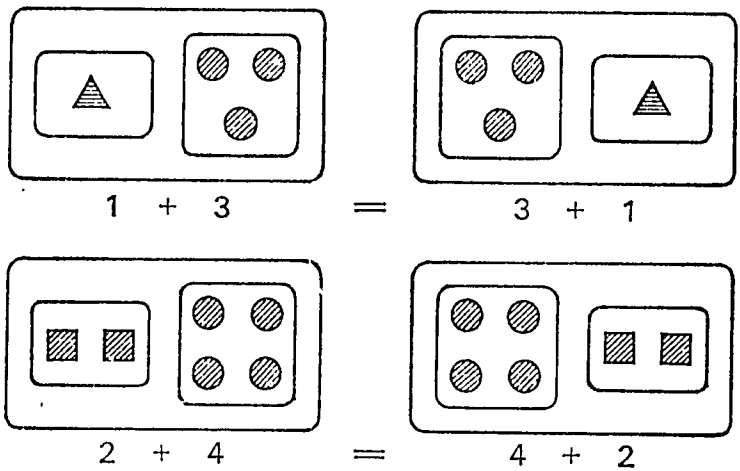
10-2 Commutative property of addition

In the last chapter we defined addition for whole numbers by counting elements in the union of disjoint sets. For this reason the commutative property is also a property of addition of whole numbers. For instance, when we write the addition equations for the union of the set of 3 bananas and the set of 2 oranges shown above we have

$$3 + 2 = 5$$
$$2 + 3 = 5$$

Therefore $3 + 2 = 2 + 3$

Our experience with the union of sets tells us that a similar statement is true for the sum of *any* two whole numbers. This special property of addition is known as the COMMUTATIVE PROPERTY of addition. The following pictures illustrate the commutative property for the number pairs 1, 3 and 2, 4.



Pictures like these help children understand why addition is commutative.

The children you teach will have had lots of experience in forming the union of two disjoint sets. This should help them to discover the commutative property of addition for themselves. Thus when they have found the number which is represented

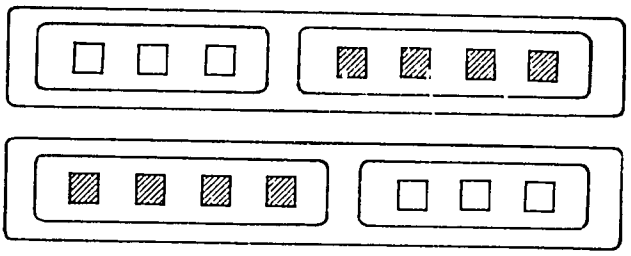
by $3 + 4$, they should realize that this number is also represented by $4 + 3$. You can help them understand this by encouraging them to find such "twin facts" as

$$3 + 2 = 5 \quad \text{and} \quad 2 + 3 = 5$$
$$6 + 4 = 10 \quad \text{and} \quad 4 + 6 = 10$$

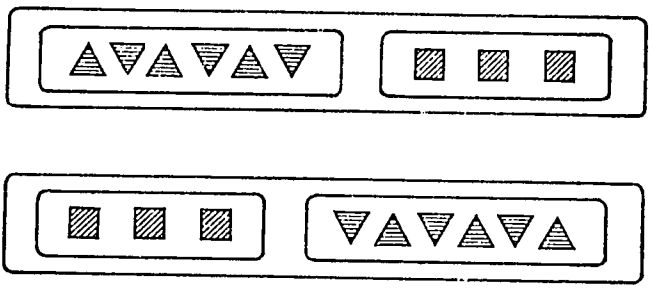
EXERCISE 10-2

1. Write addition equations suggested by each of the following pairs of pictures.

a.

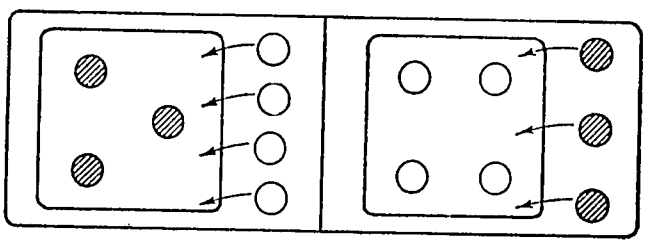


b.

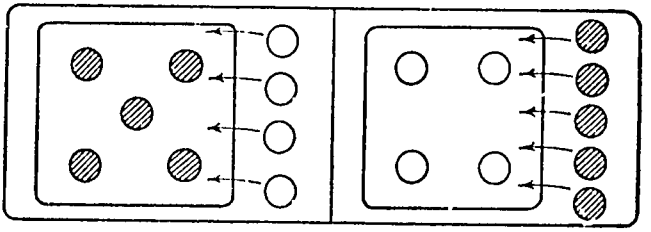


2. What statements about whole numbers do the following pictures suggest?

a.



b.



3. Draw pictures using sets to illustrate the following statements.

   a. $2 + 1 = 1 + 2$
   b. $3 + 5 = 5 + 3$
   c. $6 + 2 = 2 + 6$

4. Find numbers to put into the boxes which make each of the following sentences true.

   a. $\square + 3 = 5 \quad 3 + \square = 5$
   b. $11 + \square = 20 \quad \square + 11 = 20$
   c. $\square + 6 = 7 + 6$
   d. $\square + 5 = 5 + 4$
   e. $3 + \square = 10 + 3$
   f. $\square + 20 = 20$

5. What number can be put into the boxes to make the following sentences true?

   a. $\square + 3 = 3 + \square$
   b. $7 + \square = \square + 7$

10-3 The general notion of a variable

Question 5 at the end of the last section was very different from Question 4. Do you see why? For how many numbers is it true that $\square + 3 = 3 + 2$? Only for the single number two. For how many numbers is it true that

$$\square + 3 = 3 + \square\,?$$

Here the answer is "all numbers". Indeed, we have

$$0 + 3 = 3 + 0$$
$$1 + 3 = 3 + 1$$
$$2 + 3 = 3 + 2$$

and so on. Of course, whatever number is put in the left-hand

box must also be put in the right-hand box. Otherwise we would not get a true sentence.

We speak of a sentence such as

$$\square + 3 = 3 + \square$$

as an IDENTITY. In general, an *identity* is an equation which is true for *all* things under consideration—here for all whole numbers.

Actually we can go further than this since all of the following are also true identities:

$$\square + 0 = 0 + \square$$
$$\square + 1 = 1 + \square$$
$$\square + 2 = 2 + \square$$
$$\square + 3 = 3 + \square$$

and so on and on. We can put all of these identities together into a single one, as follows:

$$\square + \triangle = \triangle + \square$$

In writing this it is important to understand that the two boxes must be filled with the same number, and that the two triangles must be filled with the same number. However, the number in the boxes need not be the same as the number in the triangles.

For example, $3 + 3 = 3 + 3$

is true. So are $7 + 5 = 5 + 7$

and $9 + 8 = 8 + 9$

In general, $\square + \triangle = \triangle + \square$

is true for *all* numbers. This identity expresses what we call the *commutative property of addition.* This property says that the order in which two whole numbers are added does not matter.

It is customary in mathematics to use letters instead of boxes and triangles to express identities. When we do this the commutative property of addition of whole numbers reads as follows:

$$a + b = b + a$$

for *all* whole numbers a and b.

A letter which is used to stand for any one of a set of numbers

is called a VARIABLE. Thus in the identity above, a and b are variables. We use variables to help us write down and remember properties such as the commutative property of addition.

EXERCISE 10-3

1. Fill the boxes in 6 different ways so that

$$\square + \triangle = \triangle + \square$$

becomes a true sentence.

2. What number or numbers can be put in the boxes to make the following sentence true?

$$\square + \square = 18$$

Is this statement an identity for all whole numbers? Explain.

3. Is

$$\square + \bigcirc = \square$$

an identity for all whole numbers? Explain.

4. What does the commutative property of addition tell us if

a. $a = 3 \quad b = 1$? **b.** $a = 5 \quad b = 2$?
c. $a = 0 \quad b = 4$? **d.** $a = 27 \quad b = 53$?

5. Suppose that a and b are whole numbers such that

$$a = b + 2$$

What is a when

a. $b = 6$? **b.** $b = 9$? **c.** $b = 0$?

10-4 Associative property of the union of sets

In Chapter 2 we talked about forming the union of three sets A, B, C. We saw that this could be done in two ways. One way is to form $A \cup B$ first, and then take the union of this set with C. This gives the set

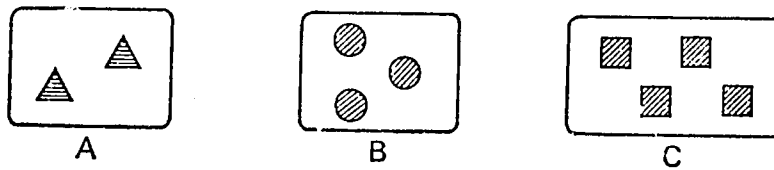
$$(A \cup B) \cup C$$

The second way is to form $B \cup C$ first, and then take the union of this set with A. This way we get the set

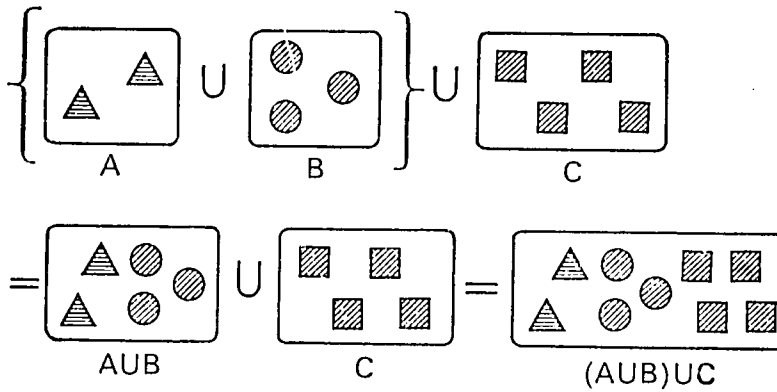
$$A \cup (B \cup C)$$

Note that we put brackets (or parentheses) around the pair of sets whose union is being taken first.

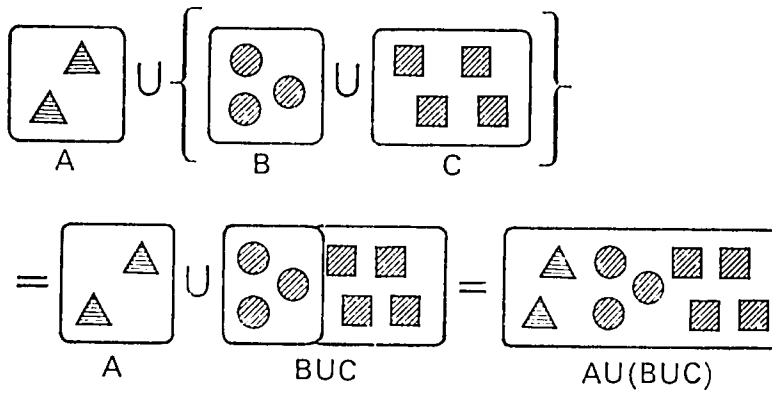
For example, let A, B and C be the sets shown below.



To form $(A \cup B) \cup C$ we proceed as follows:



Now suppose we form $A \cup (B \cup C)$. We get



Each time we get the same set in the end.

This example illustrates another important property of the union of sets. It shows that for any three sets A, B, C the sets

$(A \cup B) \cup C$ and $A \cup (B \cup C)$ contain the same elements. Therefore they are equal, and we can write.

$$(A \cup B) \cup C = A \cup (B \cup C)$$

This equation says that it does not matter how we group sets when we form their union. We call this property the ASSOCIATIVE PROPERTY *of the union of sets.*

10-5 Associative property of addition

By now you have probably guessed that the associative property is also a property of addition. It is. For example, when we write the addition equations which go with the pictures in the last section we have

$$(2 + 3) + 4 = 5 + 4 = 9$$
$$2 + (3 + 4) = 2 + 7 = 9$$

Thus $$(2 + 3) + 4 = 2 + (3 + 4)$$

In general, for any three whole numbers a, b, c we have

$$(a + b) + c = a + (b + c)$$

We call this special property the ASSOCIATIVE PROPERTY of addition. It tells us that in finding the sum of three whole numbers it does not matter how we group the numbers in pairs to add them.

The associative property is useful because it sometimes allows us to make an addition problem easier. For example, suppose we were asked to find the sum

$$19 + 6 + 4$$

One way to do this would be to add 19 and 6 first, and then add 4 to the result. However, the associative property tells us that if we wish we can first add $6 + 4$, and then add the result to 19. This is easier. Thus we write

$$19 + (6 + 4) = 19 + 10 = 29$$

Remember that the brackets here tell us which pair of numbers we are adding first.

Together, the associative property and commutative property allow us to make many addition problems easier. For

instance, suppose we were asked to find the sum of 4, 7 and 6. We could write this as

$$4 + (7 + 6)$$

But by the commutative property we can replace $7 + 6$ by $6 + 7$ to get

$$4 + (6 + 7)$$

We now finish the problem like this:

$$\begin{aligned} 4 + (6 + 7) &= (4 + 6) + 7 \quad \text{(associative property)} \\ &= 10 + 7 \\ &= 17 \end{aligned}$$

In this case we changed both the order and the grouping to take advantage of the fact that $6 + 4 = 10$.

EXERCISE 10-5

1. What does the associative property of addition tell us if
   a. $a = 1 \quad b = 2 \quad c = 1$?
   b. $a = 4 \quad b = 3 \quad c = 0$?
   c. $a = 5 \quad b = 5 \quad c = 2$?
   d. $a = 3 \quad b = 7 \quad c = 6$?
2. Find the number to put in place of a to make each of the following sentences true.
   a. $(a + 3) + 4 = 5 + (3 + 4)$
   b. $(2 + a) + 1 = 2 + (4 + 1)$
   c. $6 + (3 + a) = (6 + 3) + 2$
   d. $(a + 2) + 4 = 4 + 4$
   e. $(3 + a) + 2 = 10 + 2$
3. Find the number (or numbers) to put in place of a to make each of the following sentences true.
   a. $(a + 7) + 3 = 10$
   b. $(2 + a) + 9 = 2 + (a + 9)$
   c. $(a + 1) + (a + 2) = 7 + (a + 2)$
   d. $a + (6 + 4) = 10 + a$
   e. $(a + 3) + (2 + a) = 9$
4. Which of the statements in Question 3 are identities? Explain your answer.

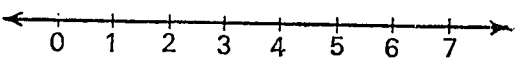
5. Put brackets (parentheses) in the following sums to show how you would group them to make the addition easier. Then find the answers.

a. $9 + 7 + 3$
b. $6 + 14 + 5$
c. $8 + 9 + 21$
d. $9 + 15 + 5$
e. $24 + 36 + 18$

10-6 Addition on the number line

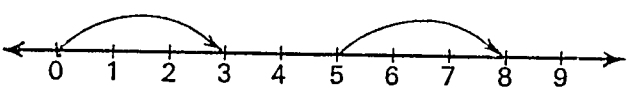
We have seen how to represent numbers on the number line. First we draw a line with arrows on each end to indicate that the line can be extended in both directions.

Then we mark some point "o" and choose some length as a unit length. Next we mark off points to the right which are



equally spaced so that the distance between successive points is the unit length. These points are labelled 1, 2, 3, 4, . . ., in order. We now have the number line.

The part of the number line from one number to the next is called a *segment*. The segment from 0 to 1 is of unit length. A segment of unit length will be called a *unit segment*. The segment from 0 to 3 is made up of three unit segments and has a length of 3 units. What is the length of the segment from 5 to 8? A look at the number line shows that this segment also contains

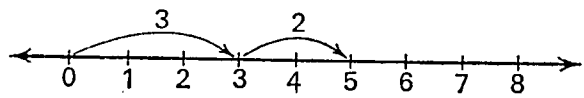


three unit segments and has a length of 3 units. Thus the segment from 0 to 3 and the segment from 5 to 8 have the same length.

EXERCISE 10-6A

1. **a.** Compare the lengths of the segment from 0 to 5 and the segment from 4 to 9.
   b. Compare the lengths of the segment from 2 to 5 and the segment from 9 to 12.
2. **a.** What segment starting at 0 has the same length as the segment from 6 to 9?
   b. What segment starting from 0 has the same length as the segment from 4 to 8?
   c. Does the segment in part (*a*) or in part (*b*) have the greater length?
   d. State how you can decide which of two segments is the larger or whether they are of equal length.

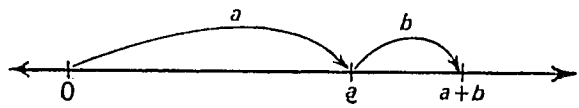
The segment from 0 to 5 is made up of two segments, one from 0 to 3, and the other from 3 to 5. Thus the segment consisting of five unit pieces can be broken down into two pieces, one of 3 units, and one of 2 units. You can see what we are leading up to in this way—and if you guide children correctly they will see it too. We are leading up to addition, shown on the number line. The sum $3 + 2$ looks like this when shown on the number line:



$$3 + 2 = 5$$

Any addition problem can be shown in this way, as you can plainly see.

In general we may picture $(a + b)$ on the number line as follows:



Remember that when you add on the number line you move to the right.

EXERCISE 10-6B

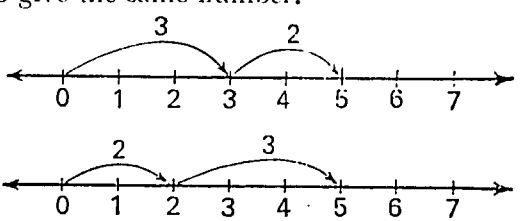
1. Show the following additions on the number line:

a. $2 + 4$	**b.** $3 + 3$
c. $4 + 2$	**d.** $5 + 1$
e. $6 + 0$	**f.** $12 + 7$
g. $19 + 0$	**h.** $6 + 11$
i. $11 + 6$	**j.** $0 + 6$

2. While Bandele sat at his desk looking at the number line in his arithmetic book, a cricket jumped on the book, landing at 0. First it jumped to 7, and then it jumped on to 11. This very smart cricket was really doing an addition problem. What was the problem?
3. Tell how you can use examples of the kind in the exercise above to help children understand the number line. Make up games and stories to explain each of the addition problems given above.
4. If x is a whole number between 3 and 6 and y is a whole number between 2 and 7, illustrate on the number line the set of all possible points corresponding to $x + y$.
5. Starting from the point 0 on the number line a boy lays match-sticks in the following order: 1 stick, then 2 sticks, then 3 sticks and so on. How many sticks would he have used after laying the fifth set of sticks?
6. A child jumping along a straight track marked in feet made jumps of 3 feet, 4 feet, 3 feet, 4 feet, and so on. If he started at the point 0 with a jump of 3 feet, after what jump would he land on the point marked 31 feet?

10-7 Commutative and associative properties of addition

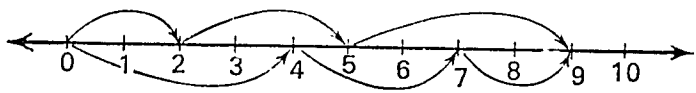
You can see from the number line that such sums as $3 + 2$ and $2 + 3$ give the same number.



In illustrating the operation of addition it does not matter which of the moves on the number line is taken first. If we first take 3 unit steps followed by 2 unit steps, we come to 5. If we had first taken 2 unit steps followed by 3 unit steps, we would have again come to 5.

EXERCISE 10-7

1. What is the name of the property of addition just described, and what is the most general way to state it? If you do not remember, refer to the section on properties of addition.
2. Prepare word problems which will help children to understand this property by using the number line.
3. Name another property of addition. Give examples on the number line to help children understand the property.
4. Fill in the missing blanks in the sentence below.



   a. The above diagram on the number line shows that
   —— = ——

   b. The diagram illustrates the —— property and also the —— property of addition.

10-8 Generalized properties of addition

The following exercises will show you that you can add numbers in any order and in any grouping that you like. This means that if you have to find the sum of several numbers you can begin with pairs whose sum is easy to find, and that you can re-arrange the numbers to bring the pairs together.

EXERCISE 10-8

1. Name the property of addition which makes each sum shown below equal to the following one:

$$(4 + 3) + 2 = 4 + (3 + 2) = 4 + (2 + 3)$$
$$= (2 + 3) + 4 = 2 + (3 + 4)$$

2. In adding three numbers the associative property of addition makes it possible to group the numbers any way you please.

If you add four numbers, can you group them in any way you please? Explain. Is this also true for more numbers?

3. How many different combinations of five or fewer *different* numbers from 1 to 10 add up to 15? (Note that 10 and 5 is the same combination as 5 and 10 and should only be counted once. Also $3 + 3 + 9$ uses the number 3 twice and this is not allowed.)

4. The game of "31" is played by two players. The first player announces a number from 1 to 5. The players then take turns, each adding a number from 1 to 5 to the previous result and announcing the new result. The player who announces 31 is the winner. Explain how the player who goes first can be sure of winning.

Chapter 11

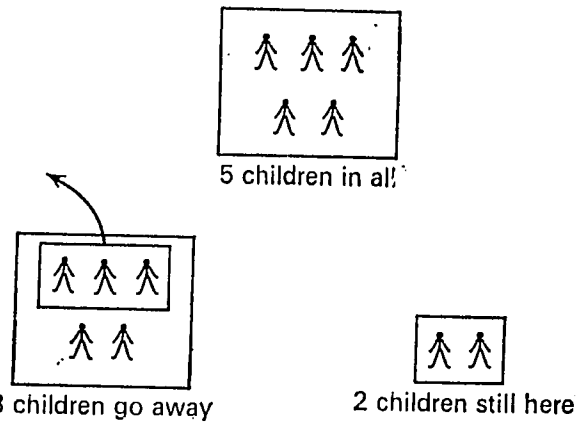
SUBTRACTION

11-1 Finding the missing addend

Consider the following problem:

There are 5 children in a room. 3 go away. How many are left in the room?

The answer, as you know, is 2. Why? Well, one way to show this is to draw pictures like these:



We can also solve this problem by writing an equation. We start with 5 children. Three go away. To find out how many are left we must find the number to put in the box to make the following sentence true:

$$5 = 3 + \square$$

Since $5 = 3 + 2$, the number 2 must be put in the box.

In a sentence such as $3 + 2 = 5$ the numbers 3 and 2 are called *addends*. The number 5 is called the *sum* of these addends. In the above problem we knew the sum and *one* of the addends. To solve the problem we had to find the *missing addend*.

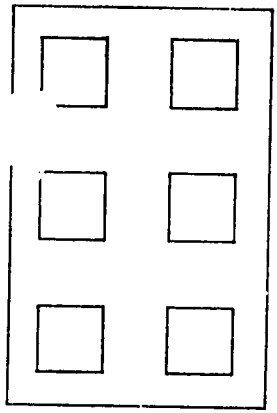
Here is another problem of the same kind.

There are 9 children in class. Each must have a book. There are 6 books in the room. How many more do you need?

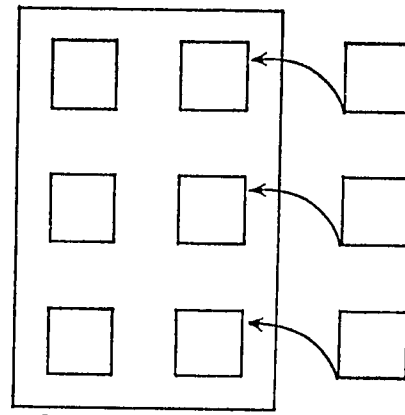
This time the addition equation is

$$6 + \square = 9$$

To solve the problem we must find the number which when added to 6 gives 9. This number is the missing addend. It is 3 because $6 + 3 = 9$. The following pictures show this:



6 books
in class



3 more books give
9 books in all

EXERCISE 11-1

Find the missing addend:

1. $7 + \square = 11$
2. $\square + 9 = 15$
3. $\square + 6 = 18$
4. $9 + \square = 14$
5. $6 + \square = 6$
6. $21 = \square + 11$
7. $19 = 5 + \square$
8. $7 + \square = 15$
9. $\square + 12 = 30$
10. $29 = \square + 17$

Write an addition equation for each of the following problems. Then find the missing addend.

11. There are 10 birds in a tree. 2 fly away. How many are still in the tree?
12. I have 7 bananas. I eat 2. How many are left?
13. There are 12 children in all. 5 are girls. How many are boys?

14. I need 25 chairs, and have 12. How many more must be found?

15. I have 4 shillings and need 17. How many more must I [illegible]

11-2 Subtraction

When we find the missing addend in an addition equation we say that we are doing SUBTRACTION. We call the missing addend the DIFFERENCE between the sum and the addend which is known. Thus, to solve the addition equation

$$3 + \square = 5$$

we must subtract 3 from 5. We show this by using a minus sign "$-$" and writing

$$\square = 5 - 3$$

The number which makes this a true sentence is the same number which makes $3 + \square = 5$ a true sentence. Since this number is 2 we write

$$2 = 5 - 3$$

We call 2 the *difference* between 5 and 3.

It is important to notice that the sentences

$$3 + 2 = 5 \quad \text{and} \quad 2 = 5 - 3$$

say exactly the same thing. In other words:

When you know the answer to this $\rightarrow 3 + \square = 5$

you also know the answer to this $\rightarrow \square = 5 - 3$

We call the equation $\square = 5 - 3$

the *subtraction equation* corresponding to the addition equation

$$3 + \square = 5$$

Similarly, the addition equation

$$\square + 6 = 9$$

can be written as the subtraction equation

$$\square = 9 - 6$$

To solve the first equation we must find the missing addend. To solve the second equation we must subtract 6 from 9. These are just different ways of saying the same thing. In each case 3 must be put in the box to give a true sentence:

$$\boxed{3} + 6 = 9$$
$$\boxed{3} = 9 - 6$$

EXERCISE 11-2

1. Write the subtraction equation corresponding to each of the following addition equations:

   a. $5 + \square = 7$ **b.** $9 = \square + 5$
   c. $26 = 17 + \square$ **d.** $\square + 11 = 20$
   e. $12 = \square + 12$ **f.** $19 = 8 + \square$

2. Write an addition equation corresponding to each of the following subtraction equations:

   a. $8 - 6 = \square$ **b.** $\square = 4 - 4$
   c. $9 - 2 = \square$ **d.** $\square = 8 - 7$
   e. $\square = 15 - 9$ **f.** $31 - 14 = \square$

3. Write addition and subtraction equations for each of the following problems, and then solve the problems.
   a. 8 children are running. 3 fall down. How many are still running?
   b. 15 people are coming. 6 are already here. How many more are to come?
   c. 12 birds are in a tree. 7 fly away. How many are still in the tree?
   d. 27 children are in class. 9 go outside. How many are still in class?
   e. 32 pencils are needed. 15 are here. How many more are needed?

11-3 Subtraction facts from addition facts

Consider the sentence $4 + 2 = 6$. What subtraction facts does it tell us? Suppose we cover up the 4. We then have the

addition equation $\square + 2 = 6$. The corresponding subtraction is $\square = 6 - 2$. We can also cover up the 2 and get the addition equation $4 + \square = 6$. Here the corresponding subtraction equation is $\square = 6 - 4$. Thus

$$4 + 2 = 6$$

gives two subtraction equations:

$$\square = 6 - 2 \quad \text{and} \quad \square = 6 - 4$$

Notice that each of these equations corresponds to one of the addends in the original sentence $4 + 2 = 6$.

EXERCISE 11-3

What subtraction equations can you get from each of the following?

1. $7 + 6 = 13$

2. $5 + 9 = 14$

3. $6 = 6 + 0$

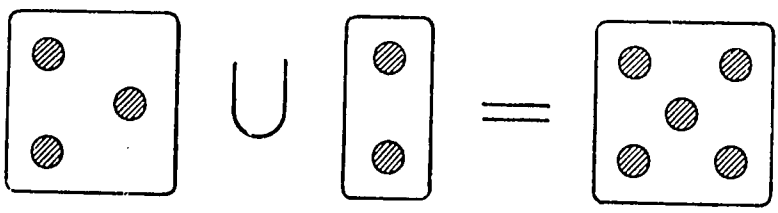
4. $12 = 9 + 3$

5. $7 + 7 = 14$

6. $15 + 0 = 15$

11-4 Subtraction as separating a set into subsets

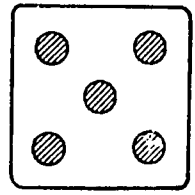
We have seen that addition is based on the idea of forming the union of two disjoint sets. For instance, the sentence $3 + 2 = 5$ tells us that the union of a set of 3 elements and a set of 2 elements always contains 5 elements whenever the original sets are disjoint. The following picture shows this:



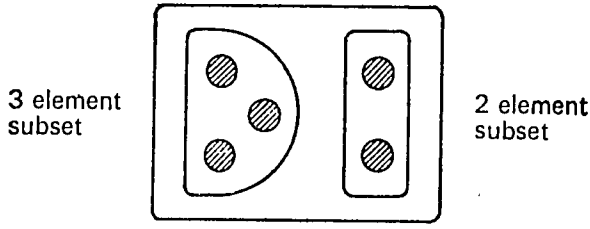
$$3 + 2 = 5$$

We can also illustrate subtraction by using sets. We do this by separating a set into disjoint subsets. For example, suppose

we wish to show that $2 = 5 - 3$. First we draw a set containing 5 elements:

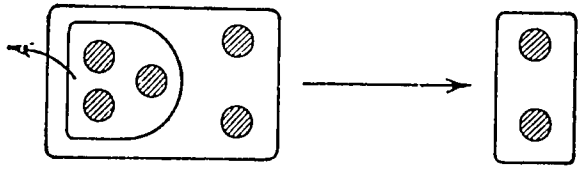


Then we divide this set into two subsets, one of which contains 3 elements. The number of elements in the other subset is $5 - 3$.



The picture shows that $5 - 3 = 2$.

The following picture gives another way of showing this. Here we start with a 5-element set and remove a 3-element subset. The number of elements left is $5 - 3$.



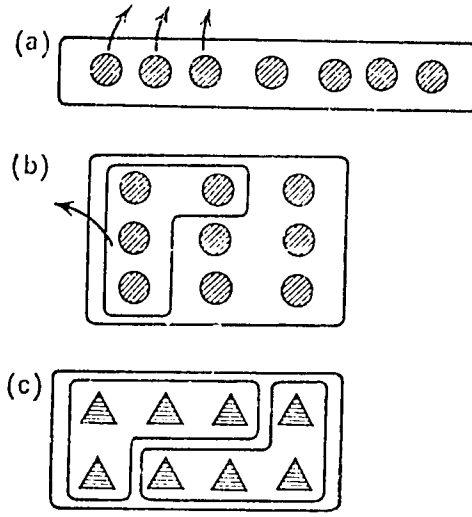
$5 - 3 = 2$

Pictures like this are very helpful when teaching children subtraction. They show how subtraction problems can be solved by removing elements from a set. They also show why addition and subtraction are closely related since they show that

> addition corresponds to forming the union of disjoint sets, while subtraction corresponds to separating a set into disjoint subsets.

EXERCISE 11-4

1. What subtraction facts do the following pictures illustrate?



2. Illustrate each of the following problems with a picture, and then solve the problem.
   a. There are 12 people in a room. 6 are seated and the rest are standing. How many are standing?
   b. Kwame found 8 eggs, but 2 were broken. How many were not broken?
   c. Esi puts 7 cups on a table. 1 cup is large and the rest are small. How many small cups are there?
3. If A and B are sets we let A − B be the set consisting of the elements in A which are *not* in B.
   a. Find A − B when $A = \{a, b, c, d, e, f, g\}$
   $B = \{b, d, f\}$

   Interpret this in terms of the number of elements in each set. (Notice that B is a subset of A.)

   b. Find A − B when $A = \{a, b, c, d, e\}$
   $B = \{b, f\}$

   Interpret this in terms of the number of elements in each set. Can you explain what happened here?

11-5 Subtraction problems which cannot be solved with whole numbers

When we talked about addition we saw that we can *always* find the sum of two whole numbers. This is not true of subtraction. For instance, if someone tells you to separate a set of 8 stones into a subset of 9 stones and another unknown subset, you cannot do it. The reason of course is that 9 is greater than 8. Thus the set of 8 stones will be completely used up before 9 stones have been removed. The addition and subtraction equations for this problem are:

$$8 = 9 + \square \quad \text{and} \quad 8 - 9 = \square$$

However, there is no whole number which makes these equations true. In other words, we cannot subtract 9 from 8 and get a whole number answer.

In this example there was nothing special about 8 and 9. Indeed, the same thing happens for any two whole numbers a and b whenever a is greater than b. In general, if a is greater than b, then $b - a = \square$ does not have a whole number answer.

EXERCISE 11-5

Find the whole number which makes each of the following equations true whenever it is possible to do so.

1. $7 + \square = 7$

2. $9 - 2 = \square$

3. $\square = 1 - 8$

4. $\square + 2 = 3$

5. $5 = \square + 8$

6. $\square = 6 - 6$

11-6 Subtraction as the inverse of addition

Suppose you have 3 books, and put 2 more with them. Now remove 2 of the books. How many remain? 3; just the number you started with. We can state this as follows:

$$(3 + 2) - 2 = 3$$

Similarly, suppose you start with 3 coins, and remove 2 of them. Now put 2 coins back. How many coins do you have?

Once again you have 3. This time we can show what happened by writing

$$(3 - 2) + 2 = 3$$

The same thing happens with other pairs of numbers. Thus

$$\begin{cases}(10 + 4) - 4 = 10 \\ (10 - 4) + 4 = 10\end{cases} \qquad \begin{cases}(19 + 13) - 13 = 19 \\ (19 - 13) + 13 = 19\end{cases}$$

These pairs of equations show that subtraction undoes the work that addition has done, and that addition undoes the work that subtraction has done. We express this in mathematical language by saying that addition and subtraction are INVERSE operations. Subtraction is the inverse of addition, and addition is the inverse of subtraction. These two statements can be summarized as follows:

$$(a + b) - b = a$$
$$(a - b) + b = a$$

(Naturally we have to be sure that $a - b$ is a whole number in the second equation.)

EXERCISE 11-6

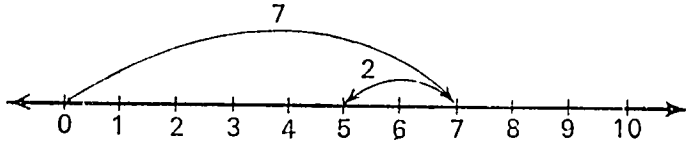
1. What do the above statements become for the following pairs of numbers?
   a. $a = 12 \quad b = 4$ **b.** $a = 5 \quad b = 0$
   c. $a = 0 \quad b = 0$
2. Let A and B be sets.
   a. When is $(A - B) \cup B = A$?
   b. When is $(A \cup B) - B = A$?
   (See Question 3, Exercise 11-4 for the definition of $A - B$.)

11-7 Subtraction on the number line

The number line can also be used to illustrate subtraction. We have seen that to find $3 + 2$ on the number line, we start at 0 and first move 3 units to the right. Then we move 2 units to the right. This puts us at 5. Thus $3 + 2 = 5$.

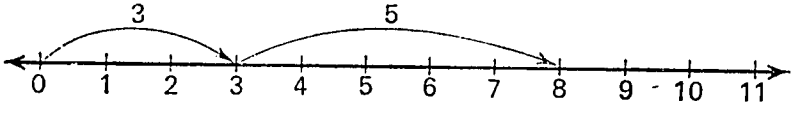
It is not always necessary to move to the right on the number

line. Suppose a cricket jumps from 0 to 7 on the number line, and then jumps back 2 units. He will land on 5. You can see his moves on the number line below:



What operation do you think the cricket is performing when he jumps to the left on the number line? From the picture you see that the cricket is subtracting when he jumps to the left. He has solved the subtraction problem $7 - 2$.

You can also think of subtraction on the number line as finding a missing jump to the right. For instance, if the cricket went from 0 to 3 on his first jump, and wanted to land on 8 the next time, how far would he have to jump to the right? The following picture shows that the answer is 5.



This time the cricket has solved the problem

$$3 + \square = 8$$

This explanation shows how you can use the device of a cricket jumping on the number line in teaching subtraction to children. It is often possible to make up similar stories and games to make learning arithmetic more enjoyable.

EXERCISE 11-7

1. Find the answer to each of the following problems by drawing pictures of the number line:

   a. $4 - 2 = \square$ **b.** $8 + \square = 12$

   c. $6 + \square = 9$ **d.** $7 - 4 = \square$

   e. $2 - 0 = \square$ **f.** $17 - 8 = \square$

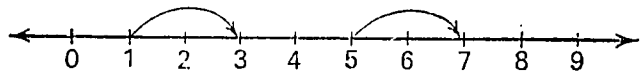
   g. $11 + \square = 19$ **h.** $10 + \square = 10$

2. Suppose that a is a whole number between 8 and 12, and b is a whole number between 1 and 7. Show on the number line the set of all possible points corresponding to $a - b$.

3. If you begin with 12 and repeatedly subtract 2, after how many steps will the process end? Draw a picture of the number line to show this.
4. Starting from the point marked 35 on the number line, every third point to the left is marked with a cross. What is the last point marked with a cross? How many points are marked with a cross?
5. We have seen that addition is commutative. Do you think that subtraction is also commutative? (Hint: Find $5 - 3$. Now try to find $3 - 5$.)
6. Find the points $4 - (3 - 1)$ and $(4 - 3) - 1$ on the number line. What does this show about the associative property for subtraction?

11-8 Order, and addition and subtraction

Let us locate 1 and 5 on the number line.



Now add 2 to each of them. Each point has been replaced by the point 2 units to the right. Thus since 1 is to the left of 5, $1 + 2$ is to the left of $5 + 2$. In terms of inequalities we can write

$$1 < 5$$
$$1 + 2 < 5 + 2 \quad \text{or} \quad 3 < 7$$

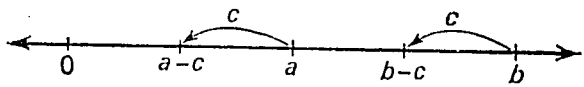
From the true sentence $1 < 5$ we get the new true sentence $3 < 7$ by adding the number 2 to both sides.

In general, let a and b be whole numbers with $a < b$. Add the whole number c to a and to b. This gives $a + c$ and $b + c$. But $a + c$ is c units to the right of a on the number line, while $b + c$ is c units to the right of b. We have moved equal distances to the right of a and b. This does not change the order. Thus: If $a < b$ is true, then $a + c < b + c$ is also true.

For example, from $4 < 6$
we conclude that $8 < 10$

We did this by adding 4 to both sides of the first inequality.

Again let a and b be whole numbers with $a < b$. Let us subtract the whole number c from a and from b. This means that we move c units to the left on the number line.



Again the order does not change. Thus

$$a - c < b - c$$

whenever $a < b$. Of course, if c is too large, $a - c$ will not be a whole number. Can you say what the largest possible value of c can be?

EXERCISE 11-8

1. We know that $0 < 2$. Use this inequality and the numbers 1, 2, 3, 4, 5 to write five new inequalities.
2. Starting with the inequality $4 < 9$ get a new inequality with 0 on the left side by subtracting a number from both sides. Show what you have done by using arrows on the number line.
3. **a.** Show that if a and b are whole numbers with $a < b$, then $0 < b - a$.
   b. Also show that if $0 < b - a$, then $a < b$.

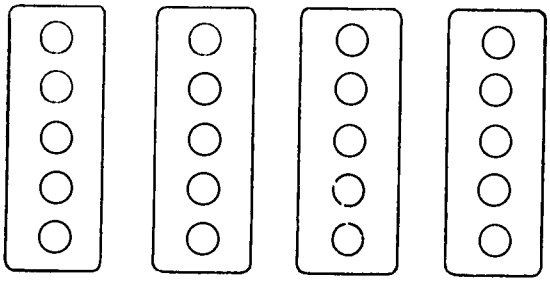
Chapter 12

MULTIPLICATION

12-1 Repeated addition

We have seen that addition is based on counting the number of elements in a union of disjoint sets. We shall now use this idea to solve problems in which addition is repeated several times.

Suppose a man earns 5 shillings a day and works for 4 days. We can find how much he earns by taking 4 sets of 5 shillings each and putting them together. We get



4 sets of 5 elements each

Each of the circles in this picture represents a shilling. Each of the sets contains 5 circles, and shows how much the man earns in one day. To find how much the man earns altogether we count the number of elements in all four sets. We find that the answer is 20.

We can also solve this problem by writing an addition equation. The equation is

$$5 + 5 + 5 + 5 = \square$$

Adding these 4 fives we find that the number 20 should be put in the box to give a true sentence. This again shows that the man earns 20 shillings.

We can describe the answer to this problem in several different ways:

The union of 4 disjoint sets of 5 elements has 20 elements

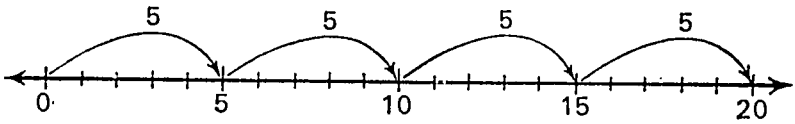
$$5 + 5 + 5 + 5 = 20$$

5 added 4 times equals 20

4 fives are 20

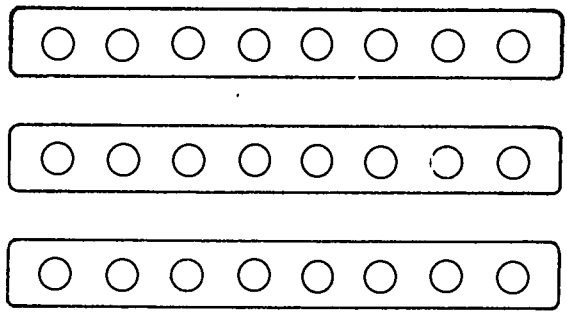
All of these statements say the same thing.

We can also show this on the number line:



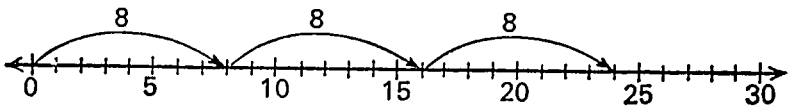
Here each jump moves 5 units to the right. The jump shows what the man earns in one day. From this picture we see that at the end of the first day the man has earned 5 shillings, at the end of the second day 10 shillings, at the end of the third day 15 shillings, and at the end of the fourth day 20 shillings. This is the total amount earned.

All problems in repeated addition can be solved like this. As another example, suppose we are asked to find $8 + 8 + 8$. We can show this sum by drawing 3 disjoint sets each containing 8 elements:



By counting the elements in the union of these sets we see that $8 + 8 + 8 = 24$.

We can also solve this problem on the number line:



After 3 jumps to the right of 8 units each we are at 24 on the number line. Thus $8 + 8 + 8 = 24$.

This time we describe the result as follows:

The union of 3 disjoint sets of 8 elements each has 24 elements

$$8 + 8 + 8 = 24$$

8 added 3 times equals 24

3 eights are 24

When you teach repeated addition to children you should draw pictures like those we have drawn here. By using sets of stones or bottle caps children can work out the answers for themselves. Afterwards you should write the four sentences we have written in our examples and explain what they mean to your class.

EXERCISE 12-1

1. Draw pictures using sets and the number line to illustrate each of the following sums. Then write the four sentences which describe the answer to each.

   a. $3 + 3 + 3 + 3 + 3 + 3 = \square$
   b. $7 + 7 + 7 = \square$
   c. $9 + 9 + 9 + 9 + 9 = \square$
   d. $4 = \square$

2. Make up word problems which you could use in class to fit each of the problems in the above exercise.
3. Suppose a man earns 4 shillings a day, and works for 5 days. How much does he earn in all? Illustrate his earnings by drawing a picture of sets, with each set of 4 shillings arranged in a row like this: $\boxed{\text{OOOO}}$. Compare this picture with the picture on page 124. What do you conclude?
4. Use pictures of sets to show that 3 eights is the same as 8 threes. Now show the same thing on the number line.

12-2 Multiplication

You can see what this is leading to. Moreover, if you do this carefully in the classroom your children will see where you are leading them. After they have done many problems in repeated addition they will want to write such problems in an easier way.

When they do you should tell them that repeated addition is so common that it is given a special name. That name is MULTIPLICATION.

For instance, if we want to say that $5 + 5 + 5 + 5 = 20$, we write the short and easy sentence:

$$4 \times 5 = 20$$

The "$\times$" here is read "times". Thus this sentence is read "4 times 5 equals 20". It tells us that if we form the union of 4 disjoint sets of 5 elements each and count the resulting set, we will find that the set contains 20 elements. Thus we now have five statements, all of which say the same thing:

The union of 4 disjoint sets of 5 elements each has 20 elements

$5 + 5 + 5 + 5 = 20$

5 added 4 times equals 20

4 fives are 20

$4 \times 5 = 20$

When we write the sentence $4 \times 5 = 20$ we say that 20 is the PRODUCT of 4 and 5.

EXERCISE 12-2

Solve each of the following problems. Illustrate your answers by drawing pictures of sets.

1. $9 \times 3 = \square$

2. $2 \times 5 = \square$

3. $5 \times 2 = \square$

4. $1 \times 7 = \square$

5. $7 \times 1 = \square$

6. $2 \times 10 = \square$

7. $3 \times 10 = \square$

8. $4 \times 10 = \square$

12-3 Multiplication as mixing sets

Here is another way to look at multiplication.

Suppose you have two boxes with cards in each box. In the first box there are two cards marked 1 and 2. In the second box there are three cards marked *a*, *b* and *c*. How many different combinations of cards can you get if you pick one card from each box?

We solve this problem as follows: Pick a card from the first box. Suppose it is card 1. Then when we pick from the second box we get one of the cards a, b or c. This gives three possible combinations:

$$(1, a) \quad (1, b) \quad (1, c)$$

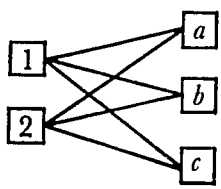
On the other hand, if we start with card 2 from the first box we get three more possible combinations:

$$(2, a) \quad (2, b) \quad (2, c)$$

Altogether there are 6 combinations. They can be listed in a chart as follows:

	a	b	c
1	$(1, a)$	$(1, b)$	$(1, c)$
2	$(2, a)$	$(2, b)$	$(2, c)$

We can also show the same thing with a picture like this:



The number of lines in this picture is the number of possible combinations.
Every combination is connected by a line. Since there are 6 lines there are 6 combinations in all.

What we are doing in a problem like this is mixing one set with another set. We get a new set whose members are pairs made up of one element from each of the original sets. In the above problem the two sets are sets of cards. The elements of the new set are all the possible combinations of cards, one from each box. We saw that once the first card was chosen there were

three choices for the second card. Thus there are 2 sets of 3 possible choices. The addition equation for this is

$$3 + 3 = \square$$

We can also write this as the multiplication equation

$$2 \times 3 = \square$$

This equation tells us that the number of pairs in the new set is found by multiplying the number of elements in the first set and the number of elements in the second set. Thus by mixing sets we get another way of looking at multiplication. Sometimes it is easier to think of multiplication in terms of unions of sets. At other times it is easier to think of it in terms of mixing sets. The two ways give the same result, and so you should learn both. Furthermore you can use both in teaching children.

EXERCISE 12-3

1. Three people are having dinner in a restaurant. They can order fish, beef or curry. Find all possible combinations of a person and a meal. How many combinations are there? Make a chart which shows all of them.
2. Every road from Aras to Cona passes through Buka. There are 5 roads from Aras to Buka and 7 roads from Buka to Cona. How many roads are there from Aras to Cona?
3. Illustrate each of the following multiplication equations by using the idea of mixing two sets:

   a. $2 \times 4 = \square$ **b.** $5 \times 7 = \square$

   c. $1 \times 8 = \square$ **d.** $5 \times 3 = \square$

12-4 Products of the type $a \times 0$ and $0 \times a$

Consider the multiplication equation $5 \times 0 = \square$. To solve this problem we remember that this equation says the same thing as the addition equation

$$0 + 0 + 0 + 0 + 0 = \square$$

But 0 is the number of elements in the empty set. Thus this equation tells us to form the union of five empty sets and then

count the elements in that union. How many will there be? Zero, because the union of 5 empty sets is still empty. Thus $5 \times 0 = 0$.

In the same way we see that

$$\begin{aligned} 0 \times 0 &= 0 \\ 1 \times 0 &= 0 \\ 2 \times 0 &= 0 \\ 3 \times 0 &= 0 \\ 4 \times 0 &= 0 \end{aligned}$$

and so on. In general we have the identity

$$\square \times 0 = 0$$

where any whole number can be put in the box. We can also express this result as

$$a \times 0 = 0$$

where a is any whole number.

In much the same way we can show that

$$0 \times a = 0$$

for any whole number a. Take 0×4 for example. To find 0×4 we think of multiplication as repeated addition. The product 0×4 tells us to add 4 to itself a certain number of times. How many times? Zero times. In other words, we do not add any fours at all. The answer then must be zero.

We can also see this by thinking about sets. Do you remember how this is done? Think of 2×4 and 1×4. In the case of 2×4 we form the union of 2 disjoint sets with 4 elements in each. In the case of 1×4 we form the union of 1 set of 4 elements. Finally, to find 0×4 using sets we form the union of 0 sets of 4 elements. How many elements do we have? Zero. Thus we again see that $0 \times 4 = 0$.

In the same way we see that

$$\begin{aligned} 0 \times 0 &= 0 \\ 0 \times 1 &= 0 \\ 0 \times 2 &= 0 \\ 0 \times 3 &= 0 \end{aligned}$$

and so on. In general we have the identity

$$0 \times \square = 0$$

where any whole number can be put in the box. This can also be expressed as

$$0 \times a = 0$$

where a is any whole number.

EXERCISE 12-4

1. Make up examples that you could use to show children that

$$a \times 0 = 0 \quad \text{and} \quad 0 \times a = 0$$

for any whole number a.

12-5 The multiplication table

You are now ready to work out the multiplication table for yourself. At first take only the numbers from 0 to 9. Later you will learn how to work with numbers that are greater than 9. But this needs special methods which are better learned by themselves. Although you already know the multiplication table you may not have thought much about it before. But now that you know how to work it out you should do it for yourself. In the exercise below we have written out a multiplication table. You should find each product and put it in the proper place in the table. Remember that there are several ways of finding the product of two numbers: repeated union of disjoint sets, repeated addition of numbers, mixing sets and the number line. Any way you do it is correct, so long as you satisfy yourself that the answers you learned as a child are right. In this way you will be getting ready to teach children what multiplication really means. Thus *your* classes will never have to sing the multiplication tables without knowing what the words mean.

In doing this work set up a table like the one below. One example—to find 6×4—is already done. The 6 in the column at the left is multiplied by the 4 in the top row. The answer, 24, has been put in its place in the table.

EXERCISE 12-5

Complete the following table using the methods given in this chapter.

	0	1	2	3	4	5	6	7	8	9
0										
1										
2										
3										
4										
5										
6					24					
7										
8										
9										

12-6 The property of 1

Look at the multiplication table you completed above. Do you see what happens when one of the numbers in the product is 1? Notice, for instance, that

$$1 \times 3 = 3 \qquad 1 \times 4 = 4 \qquad 1 \times 7 = 7$$
$$3 \times 1 = 3 \qquad 4 \times 1 = 4 \qquad 7 \times 1 = 7$$

Do you think that this will always happen? Why? Using the ideas in this chapter you should be able to show that

$$1 \times a = a \quad \text{and} \quad b \times 1 = b$$

for any whole numbers a and b.

EXERCISE 12-6

1. Use pictures to show that each of the following products is correct:

   a. $1 \times 9 = 9$ **b.** $9 \times 1 = 9$
   c. $3 \times 1 = 3$ **d.** $1 \times 7 = 7$

2. Show that $1 \times a = a$ and $b \times 1 = b$ for any whole numbers, a and b. [*Hint:* Remember that multiplication is repeated addition.]

Chapter 13

PROPERTIES OF MULTIPLICATION

13-1 Multiplication by 0 and by 1

In the last chapter we discovered two special properties of multiplication. The first was that for any whole number a,

$$a \times 0 = 0 \quad \text{and} \quad 0 \times a = 0$$

We can describe this property in words by saying that the product of two whole numbers is zero whenever one of the numbers is zero.

Do you think that the converse of this statement is true? That is, do you think that the only time the product of two whole numbers is zero is when one of the numbers is zero? Actually it is easy to see that this is true. You have been asked to show that it is in Question 1 below.

When these statements about 0 are combined they give the following important property of multiplication:

The product of two whole numbers is zero whenever one of the numbers is zero. Moreover, this is the only time a product of whole numbers is zero.

Here is another way of saying the same thing.

Let a *and* b *be whole numbers. Then* $a \times b = 0$ *if and only if* $a = 0$, *or* $b = 0$, *or both.*

This property will be very important when we study division in the next chapter.

The second property of multiplication that we discovered in the last chapter said that when we multiply a whole number by 1 the number remains unchanged. We express this fact by writing

$$1 \times a = a \quad \text{and} \quad a \times 1 = a$$

Each of these statements is true for any whole number a. This special property of the number 1 is often described by saying that 1 is the IDENTITY for multiplication.

EXERCISE 13-1

1. Use the description of multiplication as a union of disjoint sets each containing the same number of elements to show that $a \times b$ is different from zero whenever *a and b* are whole numbers different from zero. (This shows that the only time a product of two whole numbers is zero is when one of the numbers is zero.)
2. Consider the equation

$$2 \times (\square - 3) = 0$$

This equation says that the number in brackets is to be multiplied by 2. What number makes this equation a true sentence?

3. Solve the equation

$$5 \times (12 - \square) = 0$$

4. In this section we said that the number 1 is the identity for multiplication. Earlier we said that zero is the identity for addition. Explain why the same word is used both times.

13-2 The commutative property of multiplication

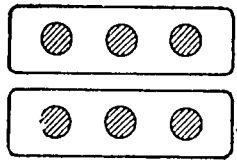
When you constructed the multiplication table in the last chapter you found, for instance, that $2 \times 5 = 5 \times 2$, that $3 \times 7 = 7 \times 3$, and so on. In fact if a and b are any pair of whole numbers whose product appears in your table you will find that

$$a \times b = b \times a$$

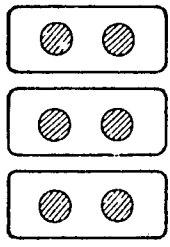
Actually this property holds for *all* pairs of whole numbers a and b. It tells us that when we multiply two whole numbers the order of multiplication does not matter. We get the same answer either way. We call this property the COMMUTATIVE PROPERTY of multiplication of whole numbers. We also refer to it by saying that multiplication of whole numbers is *commutative*. We shall now show why this must be true.

If we consider multiplication as repeated addition, we think of 2×3 as the number of elements in the union of 2 disjoint

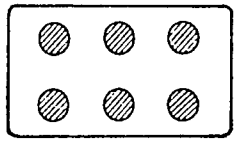
sets, each of which has 3 elements. We can picture this as



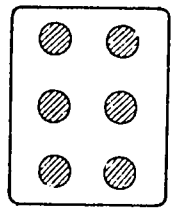
On the other hand, we think of 3×2 as the number of elements in the union of 3 disjoint sets, each of which has 2 elements. This is pictured as



In the first case the union can be drawn as follows:

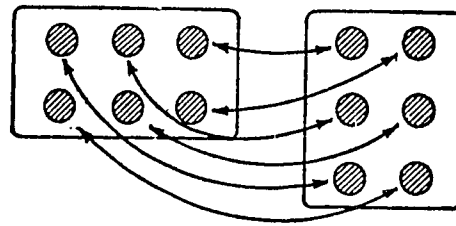


In the second case, the union can be drawn in this way:



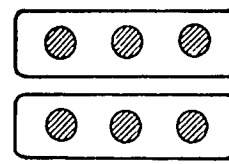
Do you see an easy way to show that these two unions have the same number of elements? Simply turn the second one on its side. It then looks exactly like the first one. In this way you can see immediately that there is a one-to-one matching of the

elements in these two sets. Each element of the second set



corresponds to the element of the first into which it rotates when we turn the second set on its side. And turning a set on its side does not change the number of elements it contains.

The same idea can be used to illustrate the commutative property for any pair of whole numbers. It gives a clear picture of why the commutative property must hold. It is also very easy to show this to your pupils. For instance, to show that $2 \times 3 = 3 \times 2$ you merely draw the following picture to represent 2×3:



Then turn the picture on its side. The picture will now show 3 rows each having 2 elements. Hence it now represents 3×2. But the number of elements has not changed. Thus

$$2 \times 3 = 3 \times 2$$

and the commutative property has been demonstrated in this case.

EXERCISE 13-2

1. Illustrate 3×5 by drawing 3 rows of 5 dots each on a sheet of paper as you would to demonstrate the commutative property to your pupils. Turn the sheet on its side. How many rows does the drawing now have? How many dots are in each row? What product does this drawing represent?
2. Describe the same procedure when there are a rows with b elements in each row, where a and b represent whole numbers. What do you conclude?

13-3 The commutative property (continued)

The commutative property of multiplication can also be illustrated by mixing sets. For example, suppose we wish to show that $2 \times 3 = 3 \times 2$. We mix two sets, as follows. Let $A = \{1, 2\}$ and $B = \{a, b, c\}$. Then A has 2 elements and B has 3. We begin by mixing them as shown in the first picture below. Altogether there are 6 pairs in this picture. The picture therefore shows that $2 \times 3 = 6$

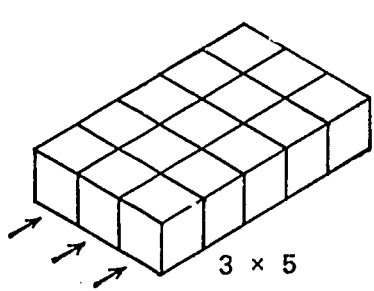
	a	b	c
1	$(1, a)$	$(1, b)$	$(1, c)$
2	$(2, a)$	$(2, b)$	$(2, c)$

Now suppose we mix the sets in the opposite order. We then get the following picture:

	1	2
a	$(a, 1)$	$(a, 2)$
b	$(b, 1)$	$(b, 2)$
c	$(c, 1)$	$(c, 2)$

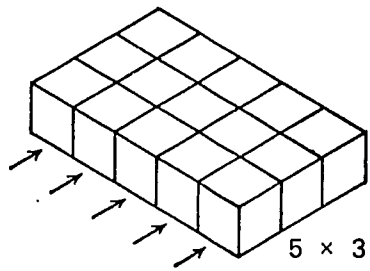
Notice that the pairs in this picture are the same as the pairs in the first picture. The only difference is that the letters now come first and the numbers second. The total number of pairs is the same both times. This shows that $2 \times 3 = 3 \times 2$. Using this idea we can show that $a \times b = b \times a$ for any two whole numbers.

Another way to teach children the commutative property of multiplication is to use objects like blocks. For instance, to show that $3 \times 5 = 5 \times 3$ arrange 15 blocks on a table like this:



We have 3 sets of 5 blocks, or 3×5 blocks.

Now look at the blocks from the other side. Here is what you see:



There are now 5 sets of 3 blocks each. But the number of blocks has not changed. Thus $3 \times 5 = 5 \times 3$. By using this idea you can give many illustrations of the commutative principle to your pupils.

EXERCISE 13-3

1. Show that $2 \times 4 = 4 \times 2$ by mixing two sets. Also show the same thing by drawing pictures of blocks.
2. Suppose one of your pupils said, "I don't believe that $3 \times 5 = 5 \times 3$." What would you do to make him change his mind?

13-4 The associative property of multiplication

In our study of addition we saw that the way numbers are grouped makes no difference in finding their sum. In other words

$$(a + b) + c = a + (b + c)$$

for any three whole numbers. We described this property by saying that addition is associative. In this section we shall see that the same property holds for multiplication, that is

$$(a \times b) \times c = a \times (b \times c)$$

When we have shown this we will have shown that *multiplication of whole numbers is associative.* (The letters in brackets should be multiplied first.)

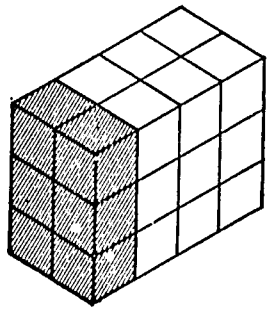
Let us start with an example. Suppose $a = 2$, $b = 3$, $c = 4$. Then

$$\begin{aligned}(a \times b) \times c &= (2 \times 3) \times 4 \\ &= 6 \times 4 \\ &= 24\end{aligned}$$

On the other hand, $$\begin{aligned}a \times (b \times c) &= 2 \times (3 \times 4) \\ &= 2 \times 12 \\ &= 24\end{aligned}$$

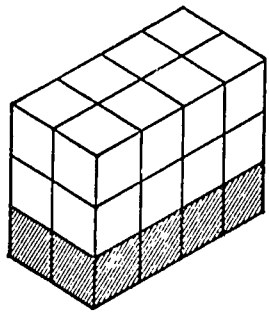
Thus in this case $(a \times b) \times c = a \times (b \times c)$.

We now show how this can be illustrated by pictures. Arrange a set of blocks as shown below.



The shaded part of the picture contains 3×2 blocks, and there are 4 sets of blocks exactly like the shaded set. Thus the entire picture illustrates the product $(3 \times 2) \times 4$.

Now look at the same set of blocks in the following way:



This time the shaded part of the picture contains 2×4 blocks, and there are 3 sets of blocks exactly like the shaded set. Thus

the picture now illustrates the product $3 \times (2 \times 4)$. Since we have the same set of blocks each time we conclude that $(3 \times 2) \times 4 = 3 \times (2 \times 4)$.

By using sets of blocks in this way we can show that $(a \times b) \times c = a \times (b \times c)$ for any three whole numbers. This is valuable because it helps children understand why the associative property of multiplication is true.

EXERCISE 13-4

1. Use pictures to show that $(2 \times 2) \times 4 = 2 \times (2 \times 4)$.
2. Verify that each of the following equations is true:
   a. $(1 \times 2) \times 4 = 1 \times (2 \times 4)$
   b. $(3 \times 0) \times 5 = 3 \times (0 \times 5)$
   c. $(4 \times 6) \times 5 = 4 \times (6 \times 5)$

13-5 The distributive property

We have seen that multiplication of whole numbers can be considered as repeated addition. This is one way in which multiplication and addition are related. We shall now show that they are related in another way.

Consider a class which has 3 girls and 4 boys. Suppose there are 2 such classes. How many children are there in both classes together? We can answer this question in two ways, as follows:

First
Find the total number of girls in both classes and the total number of boys. Add these numbers together to get the total number of children. Thus we have computed

$$(2 \times 3) + (2 \times 4)$$

Second
Find the total number of children in one class, and then multiply by 2 to get the total in both classes. This time we computed

$$2 \times (3 + 4)$$

Since both answers are the same we conclude that

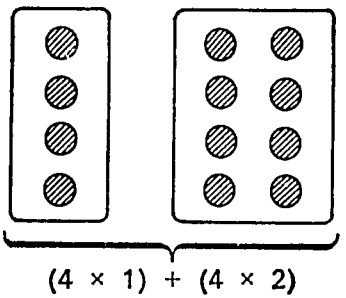
$$(2 \times 3) + (2 \times 4) = 2 \times (3 + 4)$$

This equality holds for any three whole numbers, and is known as the *distributive property*. In general the distributive property says that

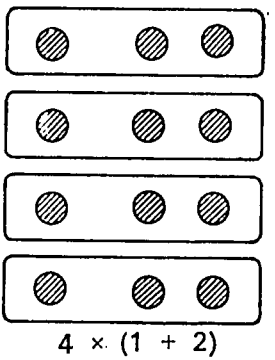
$$(a \times b) + (a \times c) = a \times (b + c)$$

for whole numbers a, b, c.

The distributive property is easy to illustrate with pictures. For instance $(4 \times 1) + (4 \times 2)$ can be shown as follows:



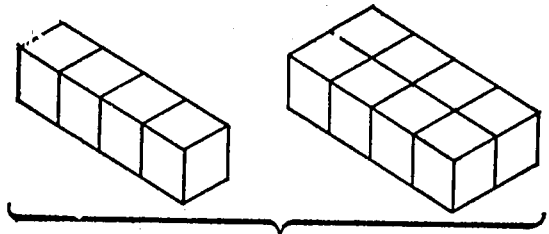
To show $4 \times (1 + 2)$ we group the same dots like this



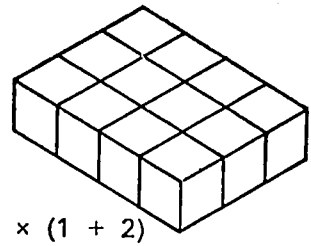
Since the number of dots has not changed we have

$$(4 \times 1) + (4 \times 2) = 4 \times (1 + 2)$$

Pictures like this help children to understand the distributive property. You can also illustrate the distributive property in your class by using blocks as in the following pictures:



$(4 \times 1) + (4 \times 2)$



$4 \times (1 + 2)$

EXERCISE 13-5

1. Verify each of the following equalities:
   a. $(2 \times 2) + (2 \times 5) = 2 \times (2 + 5)$
   b. $(7 \times 6) + (8 \times 6) = (7 + 8) \times 6$
   c. $(3 \times 4) + (3 \times 5) = 3 \times (4 + 5)$
2. Use a picture to illustrate the following equality:

$$(3 \times 2) + (3 \times 3) = 3 \times (2 + 3)$$

3. Solve the following equations:
   a. $(2 \times 3) + (2 \times \square) = 2 \times (3 + 4)$
   b. $(6 \times \square) + (6 \times 2) = 6 \times (4 + 2)$
   c. $(\square \times 3) + (\square \times 4) = 6 \times 7$
   d. $(2 \times 5) + (2 \times 4) = \square \times 9$
   e. $(3 \times \square) + (3 \times 2) = 3 \times 8$
4. Use the distributive property twice to write

$$(2 + 3) \times (6 + 4)$$

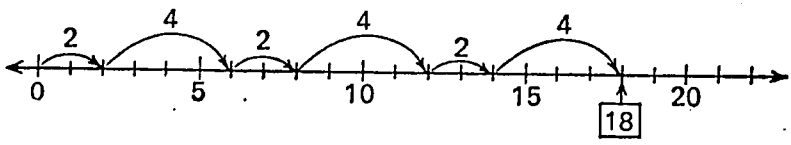
   as a sum of four products of two numbers each.
5. Generalize the preceding exercise to the case

$$(a + b) \times (c + d)$$

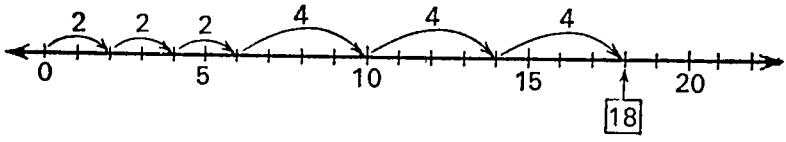
   when a, b, c, d are any whole numbers.

13-6 The distributive property on the number line

The number line can also be used to illustrate the distributive property. The drawing below shows $3 \times (2 + 4)$ on the number line.



Notice that we have repeated a jump of 2 followed by a jump of 4 three times. Altogether we took three jumps of 2 and three jumps of 4. If we make all three jumps of 2 first and follow them by all three jumps of 4 we come to the same point on the number line.



But the second drawing represents $(3 \times 2) + (3 \times 4)$. Thus

$$3 \times (2 + 4) = (3 \times 2) + (3 \times 4)$$

Using this idea you can easily show that the distributive property holds for any three whole numbers.

EXERCISE 13-6

1. Illustrate the following sentence on the number line:

$$(4 \times 3) + (4 \times 5) = 4 \times (3 + 5)$$

2. A newspaper boy sold 30 newspapers on a certain day and 60 on another day. If the cost of a newspaper is 2 pence which of the following statements can be used to work out his total sales on the two days?

a. 60×4 pence
b. 30×4 pence
c. $(30 \times 2) + (60 \times 2)$ pence
d. $(30 + 60) \times 4$ pence
e. $(30 + 60) \times 2$ pence
f. $2 \times (60 + 30)$ pence

3. If a set with 15 elements is separated into 3 disjoint subsets so that each successive subset has one more element than the preceding subset, what is the number of elements in each subset?
4. A group of children was divided into 4 teams of 3 children each. Each person on each team was assigned one of the numbers 1, 2 and 3 so that each of these numbers appeared on each team. What was the total sum of all the numbers assigned? Can you think of several ways to work this problem?
5. Towns A, B and C lie on a straight road with B between the other two. The distance from A to B is 5 miles and the distance from B to C is 4 miles. The distance from another town D to town A is 3 times the distance from town A to town C. What is the distance between town A and D? Illustrate these distances by a picture.
6. Find the number $(2 + 3) \times (4 + 5)$. Can you think of three or more different ways to do this? In each case state the properties of the operations you use.

13-7 Why the properties are important

Now that we have talked about the properties of addition and multiplication it is time to say why we have paid so much attention to them. What do you think the reason is? If you have never thought about this before you may answer as most people do: "These properties help to make addition and multiplication problems easier to solve." This sometimes happens, but it is not the real reason at all. The real reason is that these properties tell us how the arithmetic of whole numbers is put together.

Let us illustrate what we mean by an example from the real world. Suppose a man is about to build a house and has collected everything he needs to do the job. What will he have? A pile of sticks or boards, some bricks possibly, and other things as well. But he won't have a house. NOT YET. First he must do the building. He must make a *structure* out of the pile of things in front of him. Only then does he have a house.

Arithmetic is very much the same. First come the whole numbers which are like the sticks and bricks for the house. But

as yet they have no structure. No building has been done. This is where arithmetic comes in. It begins by putting the numbers in natural order, then introduces addition and multiplication. The properties of addition and multiplication are like the plans for the house. They tell us how everything fits together. This is why they are so important.

13-8 An illustration

We have said that the properties of addition and multiplication tell us how arithmetic is put together. We now give an illustration of how this happens.

Suppose you are at the point in your study of arithmetic where you know how to add whole numbers but not how to multiply them. Suppose you say, "I now want to define a new way of combining whole numbers which I will call multiplication. I want the property of 0 to hold, and I want 1 to be the identity for this multiplication. I also want the distributive property to hold." What happens?

Well, you have $a \times 0 = 0$ and $0 \times a = 0$
You also have $a \times 1 = a$ and $1 \times a = a$

And you have the distributive property:

$$(a \times b) + (a \times c) = a \times (b + c)$$

But you actually have a great deal more.

For instance, suppose you wanted to find 2×2. Remember no one has ever told you the answer. Can you do it? The surprising answer is that you can. As follows. First use the distributive property to write

$$\begin{aligned} 2 \times 2 &= 2 \times (1 + 1) \\ &= (2 \times 1) + (2 \times 1) \end{aligned}$$

Now use the fact that 1 is the identity for multiplication to get

$$(2 \times 1) + (2 \times 1) = 2 + 2$$

Finally, since you know how to add you know that $2 + 2 = 4$. Thus 2×2 *must* equal 4. The properties of multiplication give you no choice.

In much the same way you can use the distributive property

to show that *all* whole number products are just what you teach children they are. There is no other possible answer. The structure of your arithmetic makes it so.

We give one more example: Find 2×3.

Here we write
$$\begin{aligned} 2 \times 3 &= 2 \times (2 + 1) \\ &= (2 \times 2) + (2 \times 1) \\ &= 4 + 2 \\ &= 6 \end{aligned}$$

Notice how this was done. We first wrote 3 as $2 + 1$. We then used the distributive property. Next we used the fact that $2 \times 2 = 4$ and the property of 1 to compute the products 2×2 and 2×1. Finally we added 4 and 2 to get the answer 6.

EXERCISE 13-8

1. Use the method of this section to compute the following products. You may assume that you know all the products up to 5×5.

   a. 7×6 **b.** 9×8
   c. 9×9 **d.** 8×7
   e. 6×10 **f.** 11×11

Chapter 14

DIVISION

In this chapter we introduce division, the last of the four basic operations of arithmetic. Since children often have more trouble learning division than learning addition, subtraction or multiplication we shall discuss several different ways of teaching it. All of them are useful in the classroom and it is a good idea to be familiar with each.

14-1 The meaning of division

In the last two chapters we learned how multiplication is defined and what some of its properties are. We also learned how to solve multiplication problems. Such a problem always involves an equation of the form

$$a \times b = \square$$

where a and b are whole numbers. For instance, the problem of finding how many oranges are needed if we wish to give 6 children 2 oranges each can be written as a multiplication equation

$$6 \times 2 = \square$$

The solution of this equation—that is, the number which must be put in the box to give a true sentence—is 12 since $6 \times 2 = 12$. Remember that 12 is said to be the product of 6 and 2, and that 6 and 2 are called *factors* of 12. Besides 2 and 6, the number 12 has four other factors. Can you name them?

In everyday life we often encounter multiplication problems where the product and *one* of the factors are known, and we are asked to find the missing factor. For example, suppose there are 6 children and 12 oranges, and we are asked to find the number of oranges that can be given to each child if the oranges are shared equally among the children. In this case the problem can be described by the equation

$$6 \times \square = 12$$

Thus to solve this problem we must find the *missing factor* in a multiplication equation. The missing factor is the number which makes the above equation a true sentence. (Can you write another equation which also describes this problem? Why is your second equation really the same as the one written above it?)

Problems which involve finding a missing factor in a multiplication equation are called DIVISION PROBLEMS. Thus when we find the missing factor in $6 \times \square = 12$ we say that we DIVIDE 12 by 6. We then write $12 \div 6$ which, as you know, is read "twelve divided by six". Hence there are two different ways of writing the equation which describes the problem of sharing 12 oranges among 6 children:

$$6 \times \square = 12$$
$$\square = 12 \div 6$$

Both of these equations say the same thing. The same number makes each of them a true sentence. The first equation is written in terms of multiplication in which case we say that we must find the missing factor. The second equation is written in terms of division in which case we say that we must find the *quotient.* The number 2 makes both of these equations true. Thus 2 is the missing factor in $6 \times \square = 12$, and is the quotient of $12 \div 6$.

EXERCISE 14-1

1. Write each of the following as a division equation and then find the quotient.

   a. $\square \times 5 = 45$ **b.** $\square \times 3 = 27$
   c. $6 \times \square = 54$ **d.** $7 \times \square = 84$
   e. $9 \times \square = 36$

2. Is the equation $0 = 0 \div \square$ an identity for all counting numbers? Why?
3. Is the equation $\square = \square \div 1$ an identity for all whole numbers? Why?
4. For what whole numbers is $\square = 6 \div 0$ true? Why?
5. There are two division equations which correspond to the multiplication equation $4 \times 7 = 28$. What are they?
6. Find *all* the factors of 24, and write two division equations for each pair of factors whose product is 24.
7. Write a multiplication and a division equation for the fol-

lowing problem and then solve the problem: Three times a certain number is 18. What is the number?

8. What number when divided by 5 and then multiplied by 3 gives 6? Write an equation which describes this problem.
9. What number when divided by 3 three times in succession gives 2?

14-2 Division as separating a set into subsets

When we solve an equation such as $\square = 24 \div 4$ we say that we *divide* 24 by 4. But why do we use the word "divide" here? To answer this question we must go back to the definition of multiplication as the union of disjoint sets each containing the same number of elements. In particular, for the division equation

$$\square = 24 \div 4$$

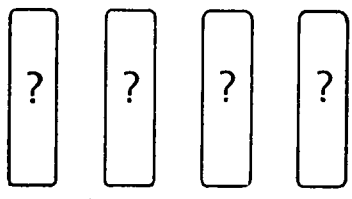
we must consider the corresponding multiplication equation

$$4 \times \square = 24$$

What does this equation say in terms of sets? Two interpretations are possible.

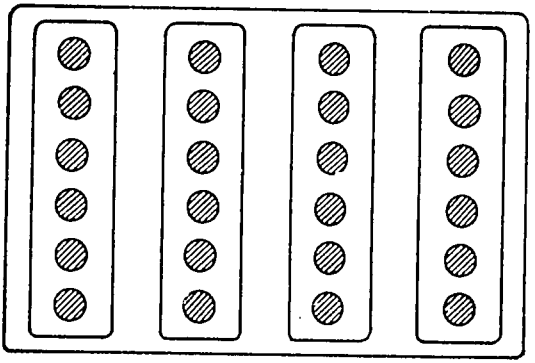
I. We are given 4 disjoint sets each containing the same number of elements; i.e. 4 equivalent disjoint sets. How many elements must be in each set in order that their union contains 24 elements?

This problem can be represented by a picture of the following type:



4 equivalent disjoint sets
24 elements in all
How many elements in each set?

One way to solve this problem is to take a set containing 24 elements, say a set of 24 stones, and then separate or divide the set into 4 subsets each containing the same number of stones. When this is done the answer can be found by counting the number of stones in any one of the subsets. As you know, each subset will contain 6 stones. Thus our picture will look like this:

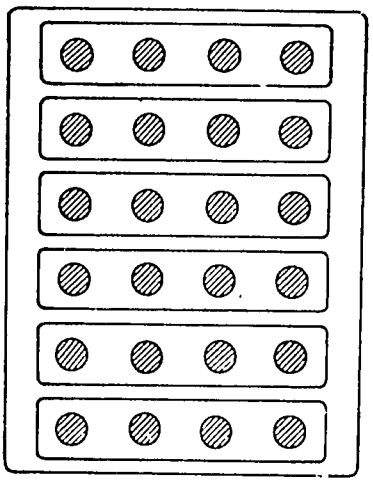


24 elements in all
4 equivalent disjoint subsets
6 elements in each subset

Notice how the problem was solved. We *divided* the large set into equivalent disjoint subsets. This explains why the word "division" is used when talking about a problem such as

$$\square = 24 \div 4$$

II. The second method for solving the problem $\square = 24 \div 4$ goes as follows. Start with a set containing 24 elements. Divide the set into as many 4-element subsets as possible and count the number of subsets obtained. The answer, of course, is 6, just as before, but this time the picture which shows the solution looks like this:



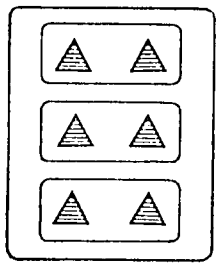
24 elements in all
4 elements in each subset
6 equivalent disjoint subsets

When you teach division to children you should draw pictures like those above and explain what the pictures say. It is also a good idea to let children work with sets of stones (or similar objects) in the classroom, using both of the above methods to solve division problems. As they practise they will learn the meaning of division and the way it is related to multiplication.

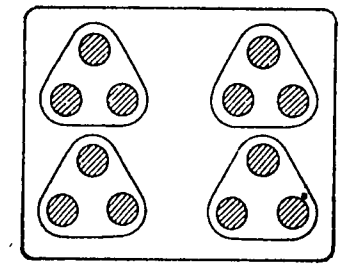
EXERCISE 14-2

Write the division equation which is suggested by each of the following pictures. Then draw a second picture which also describes the equation.

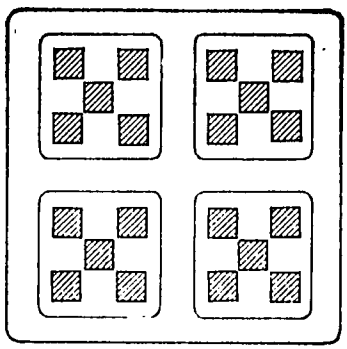
1.



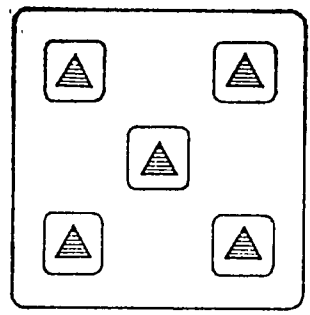
2.



3.



4.



14-3 Division as repeated subtraction

We know that multiplication can be viewed as repeated addition. Thus 4×6 is the same as $6 + 6 + 6 + 6$. Since the division problem $24 \div 6 = \square$ is exactly the same as the multiplication problem $\square \times 6 = 24$, this problem asks us to

find the number of sixes that must be added to give 24. If we did not know that 4 sixes added together give 24 we could find the answer by adding sixes until we reach 24. Thus $6 + 6 = 12$, $6 + 6 + 6 = 18$, $6 + 6 + 6 + 6 = 24$, and the answer is 4.

Instead of solving this problem by *adding* sixes until we reach 24 we could start with 24 and *subtract* sixes until we reach 0. This time we have

$$\begin{aligned} 24 - 6 &= 18 \\ 18 - 6 &= 12 \\ 12 - 6 &= 6 \\ 6 - 6 &= 0 \end{aligned}$$

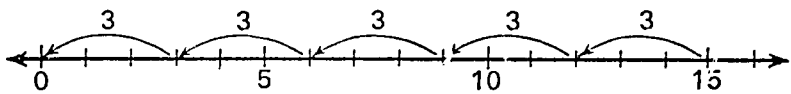
4 sixes subtracted from 24 give 0. But since subtraction is the inverse of addition this says that the sum of 4 sixes is 24. In other words $4 \times 6 = 24$, or $24 \div 6 = 4$.

The method of repeated subtraction can always be used to solve division problems. For example, to divide 15 by 3 your pupils can write

$$\begin{aligned} 15 - 3 &= 12 \\ 12 - 3 &= 9 \\ 9 - 3 &= 6 \\ 6 - 3 &= 3 \\ 3 - 3 &= 0 \end{aligned}$$

It took five subtractions to reach 0. Thus $15 \div 3 = 5$.

The method of repeated subtraction can be illustrated very clearly on the number line. Thus to show the solution of the preceding problem on the number line we start at 15 and move 3 units to the left each time until we reach 0. We get the following picture:



The picture shows that the quotient of 15 divided by 3 is 5.

EXERCISE 14-3

1. Use the method of repeated subtraction to solve each of the following division problems:

a. $\square = 26 \div 13$ **b.** $\square = 21 \div 7$
c. $\square = 42 \div 6$ **d.** $\square = 64 \div 8$
e. $\square = 5 \div 1$ **f.** $\square = 0 \div 3$

2. Illustrate each of the above problems on the number line.
3. Solve the following problem by the method of repeated subtraction and illustrate the solution on the number line. Kofi had 10 shillings and spent 2 every day. How many days did his money last?

14-4 Remainders

Suppose we try to divide 14 by 3 using the method of repeated subtraction. We have

$$\begin{aligned} 14 - 3 &= 11 \\ 11 - 3 &= 8 \\ 8 - 3 &= 5 \\ 5 - 3 &= 2 \end{aligned}$$

But this is as far as we can go since we cannot subtract 3 from 2 and get a whole number. We can only subtract 4 threes from 14, and when that is done we still have 2 left. For this reason we say that 2 is the *remainder* when 14 is divided by 3. Sometimes we also say that 14 divided by 3 is 4 plus a remainder of 2. By this we mean that $14 = (4 \times 3) + 2$.

Division problems involving remainders can be presented to your pupils in many ways. For instance, you might ask your pupils to arrange a set of 29 stones in rows of 8. When they try they will find that they cannot do it. They will be able to form 3 rows of 8 but there will be 5 stones left over. Thus a set containing 29 elements can be divided into 3 subsets of 8 and a subset of 5. We can express this by writing

$$29 = (3 \times 8) + 5$$

The number 5 is the remainder when 29 is divided by 8.

In problems of this type we speak of division with a remainder since there is no whole number quotient. In general, if there is no whole number which will make the equation $a \div b = \square$ true we have a division problem with a remainder. In this case we say that *a is not divisible by b*. If, on the other hand, there is such a number, we say that *a* is *divisible by b*. In that case we also say that *b divides into a*, or that *b* is a *divisor* of *a*.

EXERCISE 14-4

1. If possible, find the whole number solution for the following division equations. When the division is not possible find the remainder.

   a. $48 \div 8 = \square$ **b.** $65 \div 9 = \square$
   c. $47 \div 6 = \square$ **d.** $63 \div 9 = \square$
   e. $63 \div 8 = \square$ **f.** $42 \div 4 = \square$

2. Find *all* divisors of each of the following numbers:

   a. 12 **b.** 23 **c.** 28
   d. 36 **e.** 17 **f.** 100
   g. 1 **h.** 0

3. Bankole has 30 shillings with which to buy petrol. How many gallons can he buy if petrol costs 4 shillings a gallon? How many shillings will he have left?

4. A number is divided by 12. The quotient is 6 and the remainder is 3. What is the number?

5. Solve each of the following equations:

   a. $14 = (3 \times 4) + \square$ **b.** $\square = (4 \times 8) + 2$
   c. $41 = (\square \times 8) + 1$ **d.** $54 = (12 \times \square) + 6$
   e. $49 = (7 \times 7) + \square$ **f.** $38 = (\square \times \square) + 2$

14-5 Division by zero

Suppose you were asked to solve the division problem $2 \div 0 = \square$. What would you do? You would write the corresponding multiplication equation

$$\square \times 0 = 2$$

and try to find a number which makes this equation a true sentence. But when we studied multiplication we saw that

$$a \times 0 = 0$$

for every whole number a. Thus there is no whole number which will make the equation $\square \times 0 = 2$ true. This shows that the division problem $2 \div 0 = \square$ has no solution. In other words *we cannot divide 2 by zero.*

The same thing happens if we try to solve any division equation of the form $a \div 0 = \square$ where a is a counting number;

i.e. a is one of the numbers 1, 2, 3, 4, ... Here the corresponding multiplication equation is $\square \times 0 = a$, and since $\square \times 0$ is always equal to 0 this equation has no solution if a is different from 0. *Thus we cannot divide any counting number by zero.*

Finally, we ask what happens when $a = 0$. In other words, can we solve the division equation

$$0 \div 0 = \square?$$

This time the corresponding multiplication equation is

$$\square \times 0 = 0$$

But this equation is an *identity*. It is true for *all* whole numbers. Hence we cannot say that the division equation $0 \div 0 = \square$ has *a solution* in the sense that there is a *single* whole number which makes the corresponding multiplication equation true. Since we do not want a division problem to have more than one answer we are forced to say that $0 \div 0$ has no meaning.

Putting all of this together we have the following important rule:

DIVISION BY ZERO IS IMPOSSIBLE

When you teach division you should make certain that your pupils understand this. For if they do not they will always have trouble with division.

14-6 Division as the inverse of multiplication

We have seen that the division equation $a \div b = \square$ says the same thing as the multiplication equation $a = b \times \square$. By this we mean that if there is a whole number which makes one of these equations a true sentence then the same whole number will make the other equation a true sentence. Thus, for example, each equation in the following pairs says the same thing:

$$\begin{cases} 26 \div 2 = \square \\ 26 = 2 \times \square \end{cases} \qquad \begin{cases} 33 \div 11 = \square \\ 33 = 11 \times \square \end{cases}$$

In the first pair the number 13 makes both equations true; in the second the number 3 makes both true. Pairs of equations

such as these show how multiplication and division are related.

For instance, consider the pair

$$26 = 2 \times 13$$
$$26 \div 2 = 13$$

The first of these equations tells us that the union of 2 disjoint sets of 13 elements each contains 26 elements. The second equation tells us that if we divide a set containing 26 elements into 2 disjoint subsets each subset will contain 13 elements. Together these equations show that if we multiply 13 by 2 and then divide the product by 2 we get 13 back again. In other words

$$(13 \times 2) \div 2 = 13$$

The same thing happens for any other pair of whole numbers as long as the second number is not zero. Thus $11 \times 3 = 33$ and $33 \div 3 = 11$ together give

$$(11 \times 3) \div 3 = 11$$

Here we multiplied 11 by 3 and then divided by 3. The result was the number we started with, 11. In general we have the following fact:

If a *and* b *are whole numbers and* b *is different from zero, then*

$$(a \times b) \div b = a.$$

(Why must we say that *b* is different from zero here?) This equation tells us that division undoes the work of multiplication. Mathematicians express this fact by saying that *division is the* INVERSE *of multiplication.*

The same thing happens if we divide first and then multiply. For instance,

$$63 \div 9 = 7 \quad \text{and} \quad 7 \times 9 = 63$$

Thus $$(63 \div 9) \times 9 = 63$$

Here we began by dividing 63 by 9. We then multiplied the quotient by 9 and got 63 back again. In general we can make the following statement:

Let a *and* b *be whole numbers and suppose that* a *is divisible by* b. *Then* $(a \div b) \times b = a$.

Mathematicians describe this fact by saying that *multiplication is the* INVERSE *of division.*

14-7 Factors and prime numbers

The number 30 is the product of the two factors 5 and 6, i.e. $5 \times 6 = 30$. But 6 itself is the product of 3 and 2 so that

$$30 = 5 \times 3 \times 2$$

None of the numbers 5, 3 or 2 is the product of two whole numbers (other than 1 and itself). You may recall that such numbers are said to be *prime numbers.* Specifically, a whole number is said to be prime if it is greater than 1 and has only 1 and itself as factors. Thus we have displayed 30 as a product of prime factors.

Can we do this with every whole number greater than 1? The answer is yes. We just continue to write each factor of the number as a product of its factors until we cannot continue any further.

EXERCISE 14-7

Write each of the following numbers as a product of prime factors:

1. 36	**2.** 29	**3.** 8	**4.** 100
5. 91	**6.** 64	**7.** 60	**8.** 500
9. 3,000	**10.** 102		

14-8 Greatest common factors

A whole number a is said to be a *divisor* or *factor* of a whole number b if $b \div a$ is a whole number. Thus, for example, 2 is a factor of 6 because $6 \div 2 = 3$. Similarly, 5 is a factor of 20 because $20 \div 5 = 4$.

Consider the numbers 6 and 8. Since both of these numbers are even, 2 is a factor of each of them. We therefore say that 2 is a *common factor* of 6 and 8. Similarly 3 is a common factor of 12 and 15, since 12 and 15 are both divisible by 3. In general, the whole number a is said to be a COMMON FACTOR of the

whole numbers b and c if a divides into b *and* a divides into c. Other examples of common factors are

1—a common factor of 2 and 3
4—a common factor of 12 and 16
6—a common factor of 18 and 42
7—a common factor of 28 and 98

Notice that every pair of whole numbers has at least one common factor. That factor is 1, since every whole number is divisible by 1.

Now consider the numbers 18 and 42. What are their common factors? As we have just said, 1 is a common factor. So is 2, because both numbers are even. Moreover, 3 and 6 are also common factors of 18 and 42. However, no number larger than 6 is a common factor. Thus the complete list of common factors of 18 and 42 is 1, 2, 3, 6. The largest number on this list is 6. For this reason 6 is called the *greatest common factor* of 18 and 42.

In general, if a is the *largest* common factor of the whole numbers b and c, a is called the GREATEST COMMON FACTOR of b and c. For example:

1 is the greatest common factor of 5 and 9
4 is the greatest common factor of 8 and 12
8 is the greatest common factor of 16 and 24
14 is the greatest common factor of 28 and 98

The easiest way to find the greatest common factor of two whole numbers is to write each number as a product of prime numbers as you did in the exercises at the end of the last section. Then find all of the prime factors which are common to *both* numbers. The product of these factors then will be the greatest common factor of the two numbers. Can you explain why this is true?

For example, to find the greatest common factor of 28 and 98 we write 28 and 98 as products of prime numbers:

$$28 = 2 \times 2 \times 7$$
$$98 = 2 \times 7 \times 7$$

Thus 2, 7 and $2 \times 7 = 14$ are the only common factors of 28

and 98 other than 1. Therefore the greatest common factor of these numbers is 14.

Similarly, since

$$300 = 2 \times 2 \times 3 \times 3 \times 5 \times 5$$

and

$$924 = 2 \times 2 \times 3 \times 7 \times 11$$

we see that 1, 2, 3, $2 \times 2 = 4$, $2 \times 3 = 6$ and $2 \times 2 \times 3 = 12$ are the only common factors of 300 and 924. Hence the greatest common factor of 300 and 924 is 12.

EXERCISE 14-8

Find *all* common factors of each of the following pairs of numbers.

1. 6, 15
2. 12, 42
3. 40, 120
4. 0, 12
5. 99, 126
6. 84, 980
7. 637, 434
8. 42, 715
9. 275, 374
10. 60, 540

Chapter 15

A SUMMARY OF WHOLE NUMBER ARITHMETIC

15-1 The whole numbers; natural order

Arithmetic begins by constructing the set of whole numbers $\{0, 1, 2, 3, \ldots\}$. The three dots used here indicate that this list goes on and on without end and tell us that the set of whole numbers is infinite.

We have seen that the whole numbers can be put in *natural order*. The first number in this order is 0. Next comes 1, then 2, 3, 4, and so on. This is the order which is familiar to everyone, and was used when we wrote $\{0, 1, 2, 3, \ldots\}$. Given two whole numbers, such as 7 and 5, the one that appears first in the list $0, 1, 2, 3, \ldots$, is said to be *less than* the one that appears second. In the case of 7 and 5, 5 comes first and is therefore less than 7. We express this by writing $5 < 7$. We also say that 7 is *greater than* 5, which we write as $7 > 5$. The most important property of the natural ordering of the set of whole numbers is the following one.

If a *and* b *are any two whole numbers then one and only one of the following statements is true:*

$$a < b, \quad a = b, \quad a > b$$

The order properties of the set of whole numbers are also related to addition and multiplication. These relations are as follows:

Let a *and* b *be whole numbers and suppose that* $a < b$. *Then for any whole number* c,

$$a + c < b + c$$

Moreover, if c *is different from zero then*

$$a \times c < b \times c$$

(*Question:* What happens to the last statement when $c = 0$?)

For example since $5 < 7$ we have

$$5 + 4 < 7 + 4 \quad \text{and} \quad 5 \times 4 < 7 \times 4$$

Similarly since $23 < 36$ we have

$$23 + 15 < 36 + 15 \quad \text{and} \quad 23 \times 15 < 36 \times 15$$

Notice however that $23 \times 0 = 36 \times 0$ even though $23 < 36$. This is a consequence of the fact that $a \times 0 = 0$ for *all* whole numbers.

15-2 Addition and multiplication

Whole numbers can be added and multiplied. Thus if we are given any two whole numbers a and b their *sum* $a + b$ and *product* $a \times b$ are again whole numbers. Addition and multiplication have several special properties which we now list.

I. *The commutative property.* The order in which two (or more) whole numbers are added or multiplied does not affect the answer. For instance

$$3 + 8 = 8 + 3 \quad \text{and} \quad 3 \times 8 = 8 \times 3$$

We describe this by saying that addition and multiplication of whole numbers are *commutative.* When put in general form by letting a and b stand for any two whole numbers these commutative properties read

$$a + b = b + a \quad \text{and} \quad a \times b = b \times a$$

II. *The associative property.* We have seen that when three whole numbers are added or multiplied the way in which the numbers are grouped does not affect the answer. For instance

$$(2 + 3) + 5 = 2 + (3 + 5) \text{ and } (2 \times 3) \times 5 = 2 \times (3 \times 5)$$

We describe this by saying that addition and multiplication of whole numbers are *associative.* Using a, b and c to represent any three whole numbers these associative properties read

$$(a + b) + c = a + (b + c) \quad \text{and} \quad (a + b) \times c = a \times (b \times c)$$

III. *The distributive property.* Multiplication and addition of

whole numbers are related by a property known as the *distributive property*. In its general form this property states that if a, b and c are whole numbers then

$$a \times (b + c) = (a \times b) + (a \times c)$$

For example, when $a = 5$, $b = 4$ and $c = 2$ we have

$$5 \times (4 + 2) = (5 \times 4) + (5 \times 2)$$

Here it is easy to verify that the equation is correct since

$$5 \times (4 + 2) = 5 \times 6 = 30$$

and $$(5 \times 4) + (5 \times 2) = 20 + 10 = 30$$

The distributive property is often described more precisely by saying that *multiplication is distributive over addition*. The reason for this is that distributivity does not work the other way around; addition is *not* distributive over multiplication. For if it were we would have

$$a + (b \times c) = (a + b) \times (a + c)$$

for any three whole numbers a, b and c. But when $a = 1$, $b = 2$ and $c = 3$ we find that

$$1 + (2 \times 3) = 1 + 6 = 7$$

while $$(1 + 2) \times (2 + 3) = 3 \times 4 = 12$$

Hence the above equality fails.

However, it is possible to show that multiplication is distributive over subtraction; i.e. that

$$a \times (b - c) = (a \times b) - (a \times c)$$

(Of course, we must assume that b is at least as large as c here. Otherwise $b - c$ would not be a whole number.) For instance, when $a = 2$, $b = 5$, $c = 3$, we have

$$a \times (b - c) = 2 \times (5 - 3) = 2 \times 2 = 4$$

and

$$(a \times b) - (a \times c) = (2 \times 5) - (2 \times 3) = 10 - 6 = 4$$

Thus $$2 \times (5 - 3) = (2 \times 5) - (2 \times 3)$$

Similarly $4 \times (6 - 3) = (4 \times 6) - (4 \times 3)$

because $4 \times (6 - 3) = 4 \times 3 = 12$

and $(4 \times 6) - (4 \times 3) = 24 - 12 = 12$

EXERCISE 15-2

1. Is subtraction of whole numbers associative; i.e. is

$$(a - b) - c = a - (b - c)$$

true for all whole numbers a, b, c?

2. Is subtraction of whole numbers commutative?

3. Is division of whole numbers associative; i.e. is

$$(a \div b) \div c = a \div (b \div c)$$

for all whole numbers a, b, c, for which these quotients are defined? [*Hint:* Try $a = 8$, b, $= 4$, $c = 2$.]

15-3 The properties of 0 and 1

The last of the basic properties of arithmetic for whole numbers concerns the numbers 0 and 1. They have all been discussed in earlier chapters and will just be listed here.

First, 0 is the *identity* for addition, and 1 is the *identity* for multiplication. This is another way of saying that

$$a + 0 = a \quad \text{and} \quad a \times 1 = a$$

for every whole number a.

Second, if a is any whole number then

$$a \times 0 = 0$$

In other words the product of any whole number and 0 is 0. The converse of this statement is also true, and is also important. It says that if the product of two whole numbers is zero then at least one of the numbers must be zero. (Of course both of the numbers might be zero. The words "at least" allow for this possibility.) We can say this in symbols, as follows:

If $a \times b = 0$ then either $a = 0$, or $b = 0$, or both.

Finally, as a consequence of the fact that $a \times 0 = 0$ for all whole numbers a we conclude that *division by zero is impossible.*

Chapter 16

PROCEDURES FOR ADDITION AND SUBTRACTION

16-1 Basic addition facts and their consequences

In our earlier work we considered the meaning of addition as an operation on whole numbers. Two numbers were chosen and an operation was performed on them which gave a third whole number called the sum of the two numbers. The sum was defined in terms of an operation on sets. Do you remember what that operation was? Could you, for example, explain how the sum $7 + 2$ is defined by using sets?

In our earlier work we also studied the properties of addition. For instance, we discovered that addition is commutative and associative; that is for all whole numbers a, b, c

$$a + b = b + a \quad \text{and} \quad (a + b) + c = a + (b + c)$$

We also discovered that 0 is the identity for addition. By this we mean that for every whole number a

$$a + 0 = a \quad \text{and} \quad 0 + a = a$$

We shall use these properties throughout this chapter.

Now that we have explained the meaning of addition and know what its basic properties are we shall consider how addition is performed. In other words we want to learn how to *compute* sums of whole numbers. As we shall see the procedures for adding whole numbers depend upon an understanding of how the number system is used to rename numbers, and also on the properties of addition mentioned above.

We have already dealt with sums of whole numbers less than 10. These sums are known as the *basic addition facts*. Everyone must know them thoroughly before going on to sums of larger numbers because these basic facts are used to find all other sums. For example, starting with the basic addition fact $5 + 3 = 8$

we find that $15 + 3 = 18$. This follows from the associativity of addition since

$$\begin{aligned} 15 + 3 &= (10 + 5) + 3 \\ &= 10 + (5 + 3) \\ &= 10 + 8 \\ &= 18 \end{aligned}$$

Similarly $25 + 3 = 28$, $35 + 3 = 38$, and so on. Thus starting with the basic addition fact $5 + 3 = 8$ we obtain

$$\begin{aligned} 15 + 3 &= 18 \\ 25 + 3 &= 28 \\ 35 + 3 &= 38 \\ \\ 95 + 3 &= 98 \\ \\ 145 + 3 &= 148 \end{aligned}$$

It is important that children understand why we know all these sums as soon as we know that $5 + 3 = 8$. Otherwise they will never learn how to add properly.

In the same way, as soon as a child has mastered the basic fact $8 + 6 = 14$ he should also see that it implies

$$\begin{aligned} 18 + 6 &= 24 \\ 28 + 6 &= 34 \\ \\ 178 + 6 &= 184 \end{aligned}$$

Here again these results follow from the associativity of addition. For instance,

$$\begin{aligned} 18 + 6 &= (10 + 8) + 6 \\ &= 10 + (8 + 6) \\ &= 10 + 14 \\ &= 10 + (10 + 4) \\ &= (10 + 10) + 4 \\ &= 20 + 4 \\ &= 24 \end{aligned}$$

Of course, we do not want children to have to go through this chain of steps to conclude that $18 + 6 = 24$. They should be able to say "$18 + 6 = 24$" automatically, and need to be given

enough practice so that they can do so. However, if asked they ought to be able to say *why* $18 + 6 = 24$; in other words they should know how this result follows from the basic fact $8 + 6 = 14$ and the associativity of addition.

These ideas are applied in working problems in column addition. For instance, consider the problem of finding the sum

$$\begin{array}{r} 9 \\ 9 \\ +\,6 \end{array}$$

If a child solves this problem by adding downwards he must automatically know that $9 + 9 = 13$ and that $18 + 6 = 24$. When doing the addition he should think: "9, 18, 24". Although we did this example by adding downwards, the direction is not important. The commutative and associative properties of addition tell us that we will get the same answer no matter which direction we use. When teaching column addition it is a good idea to get children to learn to add in both directions so that they can check their work. Encourage them to do problems of this kind twice, once in each direction, until they get the same answer both times. When they do, their answer will probably be correct.

EXERCISE 16-1

1. Use the basic addition fact $7 + 8 = 15$ to show that
   a. $37 + 8 = 45$
   b. $97 + 8 = 105$
   c. $497 + 8 = 505$
2. We have seen that in base-five addition

$$4_{five} + 2_{five} = 11_{five}$$

   Use this basic addition fact to show that
   a. $24_{five} + 2_{five} = 31_{five}$
   b. $234_{five} + 2_{five} = 241_{five}$
   c. $144_{five} + 2_{five} = 201_{five}$
   d. $4444_{five} + 2_{five} = 50{,}001_{five}$

[In solving this problem you may find it helpful to draw pictures on a base-five abacus.]

16-2 Addition of multiples of 10

Once children have learned the basic addition facts and how to use them to find sums such as 15 + 3, 28 + 6, and so on, they are ready to start adding two-digit numbers. Here it is best to begin with sums involving multiples of 10 such as 20 + 30 70 + 50, etc. In teaching children how to find such sums you will want to revise the meaning of two-digit numbers. You should also make sure that they can count by tens with understanding. Finally they should know that 20 can be represented by 2 bundles of sticks with 10 sticks in each bundle, that 30 can be represented by 3 bundles of sticks with 10 sticks in each bundle, and so forth. If they know this they know that 20 is 2 tens, that 30 is 3 tens, etc. Using these ideas children will be able to discover for themselves that

$$\begin{aligned} \text{2 bundles of 10} + \text{3 bundles of 10} &= \text{5 bundles of 10} \\ \text{2 tens} + \text{3 tens} &= \text{5 tens} \\ 20 + 30 &= 50 \end{aligned}$$

This problem should also be written in vertical form as

$$\begin{array}{r} \text{2 tens} \\ +\ \text{3 tens} \\ \hline \text{5 tens} \end{array} \qquad \begin{array}{r} 20 \\ +\ 30 \\ \hline 50 \end{array}$$

Notice that to solve such problems children must know the meaning of the numbers 20, 30, 40, . . . They must know that

20 is the same as 2 tens
30 is the same as 3 tens
40 is the same as 4 tens

and so on.

16-3 Renaming numbers in the decimal system

The next step in teaching addition is to find the sum of *any* two two-digit numbers. Once children have learned how to find such sums it is easy for them to go on to sums involving larger numbers and sums involving more than two numbers. And once they can handle such problems they have completed the task of learning how to add whole numbers.

In order to find sums of two-digit numbers your pupils will have to know how numbers are renamed on our number system. In our earlier work we have seen that there are many ways of naming a whole number. For instance, 46 is also named as $48 - 2$, $47 - 1$, $36 + 10$, $92 \div 2$, and so on. When working addition problems two of these names are more useful than others. They are

$$40 + 6 \quad \text{and} \quad 30 + 16$$

In writing each of these names we have expressed 46 as (so many) tens + (so many) ones. We say that we have *renamed* or *regrouped* the number 46.

The same thing can be done for larger numbers. For example, 376 may be written

$$300 + 70 + 6$$

which means 3 hundreds + 7 tens + 6 ones
But this number can also be written

$$300 + 60 + 16$$

3 hundreds + 6 tens + 16 ones

and

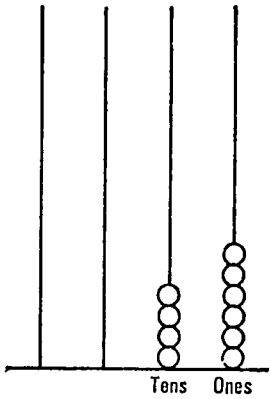
$$200 + 170 + 6$$

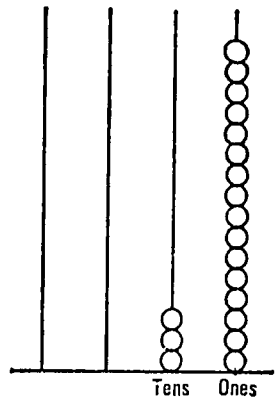
2 hundreds + 17 tens + 6 ones

Can you think of two other ways of renaming 376 which are similar to the ones given above?

The representations of 46 and 376 given above show that our number system uses groupings by tens. In other words the *base* of our number system is ten. This system is called the *decimal system* from the Latin word "decem" meaning ten.

In addition to representing numbers by bundles of sticks we can also use an abacus. The abacus is a particularly valuable teaching aid at this point because it shows how we rename the numbers on our number system when working addition and subtraction problems. As an example, the abacus on the left below shows 46 as 4 tens + 6 ones while the abacus on the right shows 46 as 3 tens + 16 ones.





Notice that there is a great difference between the representations of numbers on an abacus and the representations of numbers by bundles of sticks. With bundles of sticks we actually use 10 sticks tied together to form 1 ten. Hence 3 bundles containing 10 sticks each represent the number 30. Here it is possible for a child to count 1, 2, 3, . . ., 30, and actually see 30 single sticks in these bundles. On the abacus, however, we do not usually have 30 rings. Instead, we just have 3 rings on the tens rod. The *position* of the rods on the abacus shows whether a ring represents the number 1, or 10, or 10×10, and so on. For this reason the abacus gives a much better picture of our number system and the way it works than do bundles of sticks.

EXERCISE 16-3

1. Draw pictures which show each of the following numbers represented on an abacus in three different ways.

   a. 271 **b.** 502 **c.** 3,010 **d.** 2,000

2. Draw pictures which show each of the following numbers represented on a base-five abacus in three different ways.

   a. 243_{five} **b.** 441_{five} **c.** $4{,}002_{five}$ **d.** $2{,}000_{five}$

16-4 Addition without regrouping

In the work which follows we shall show the transition from the stage where a child does addition problems by using bundles of sticks to the stage where he uses an abacus and the position of rings on its rods.

Let us take as an example the problem of finding the sum of 46 and 12. What is 46 + 12? Because you as a teacher have skill with the basic operations of arithmetic you follow the systematic procedure we were all taught as children to get the answer 58. You know this procedure so well that you follow it automatically. Your work is mechanical. It requires very little thought. This is an advantage, of course, but when you teach children you must realize that the mechanical procedure you use as an adult is the result of a long period of development. It is the *final* stage in a carefully planned sequence in which the *meaning* of each step should be taught. As teachers we must always try to teach in such a way that children discover basic meanings and important relationships for themselves. If they do they will gain a much greater understanding of mathematics and its applications. Thus, in teaching children how to add 46 and 12 a particularly *bad* beginning would be to write 46 + 12 on the blackboard and then proceed to say: "6 + 2 = 8. Write 8 under the 6 and 2. 4 + 1 = 5. Write 5 under the 4 and 1. Thus the answer is 58." This is a poor way to *begin* teaching addition since it tells children *what* to do but not *why* they do it. As teachers we must help children discover why certain procedures lead to correct answers. For only then will they be likely to remember what they learn. Thus we shall discuss the problem of finding 46 + 12 in detail, starting with the method children should use when they *first* try to find such sums. From there we go on to an intermediate stage in the learning process, and then to the method we use as adults.

Stage I. (An introductory stage in the learning process)

At this stage 46 is represented by 4 bundles of 10 sticks each + 6 loose sticks. We call the loose sticks ones. Similarly 12 is represented by 1 bundle of 10 sticks + 2 ones. Thus we have 6 + 2 = 8 ones and 4 + 1 = 5 bundles of 10 sticks each. Altogether there are 50 + 8 = 58 sticks. Hence 46 + 12 = 58.

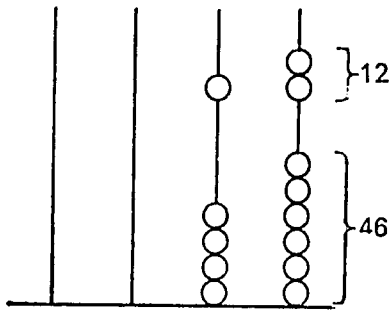
This solution can be recorded more briefly as follows:

46 is renamed	40 + 6
12 is renamed	10 + 2
TOTAL	50 + 8

Thus $46 + 12 = 50 + 8$, and since $50 + 8 = 58$ we have $46 + 12 = 58$.

Stage II. (*An intermediate stage in learning*)

At this stage we use an abacus to find $46 + 12$. We begin by placing rings on the abacus to represent each addend separately as shown in the following picture:



Now show the result of adding the ones, and the result of adding the tens. The number then represented on the abacus is 5 tens + 8 ones, or 58. Thus $46 + 12 = 58$.

This time the procedure leading to the solution can be written down as follows:

$$\begin{array}{r} 4 \text{ tens} + 6 \text{ ones} \\ 1 \text{ ten} + 2 \text{ ones} \\ \hline 5 \text{ tens} + 8 \text{ ones} = 58 \end{array}$$

The work described here as Stages I and II provides children with the kind of meaningful practice which is essential in learning addition. Once children have mastered these ideas they are ready for the final stage which is nothing more than a short form of writing this down.

Stage III. (*A final stage in learning*)

At this stage we write the problem and its solution together

without renaming any of the numbers. It looks like this:

	Tens	Ones
	4	6
	+1	2
TOTAL	5	8

As we do the problem we now think in the following way:

Ones: $6 + 2 = 8$. Write an 8 in the ones place.

Tens: $4 + 1 = 5$. Write a 5 in the tens place.

Thus the answer is 58.

Stage I with bundles of sticks is the most important stage in learning addition since it is based on the idea of forming unions of disjoint sets. And this, after all, is how addition is defined. The next two stages are just efficient ways of computing sums and recording the answers. Of course, all three stages are familiar to us as teachers. However it is important that we lead children through each of them gradually so that they really understand how they arrive at the final stage.

EXERCISE 16-4

Write out the solution of each of the following problems in Stages I, II and III as in the text.

1. $14 + 23 = \square$ **2.** $32 + 27 = \square$ **3.** $243 + 154 = \square$
4. $475 + 204 = \square$ **5.** $1{,}036 + 2{,}003 = \square$

16-5 Addition with regrouping

We now consider addition problems which require regrouping in order to find the answer. The following problem is typical:

Problem

Find the sum 35 + 27.

This time the three stages in the solution go as follows:

Stage I

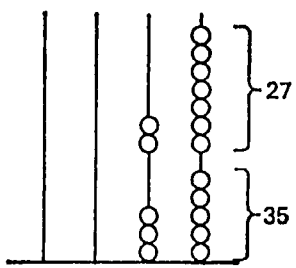
Using bundles of sticks we see that

35 can be represented by 3 bundles of 10 + 5 ones
27 can be represented by 2 bundles of 10 + 7 ones

Altogether there are 5 bundles of 10 + 12 loose sticks (ones). The 12 loose sticks can be combined to give 1 bundle of 10 + 2 loose sticks. Thus there are 5 + 1 = 6 bundles of 10 + 2 loose sticks. This collection of sticks represents the number 62. Hence 35 + 27 = 62.

Stage II

Here the problem looks like this when recorded on an abacus:



From this picture we see that the solution is obtained as follows

$$\begin{aligned}
35 &= 3 \text{ tens} + 5 \text{ ones} \\
27 &= 2 \text{ tens} + 7 \text{ ones} \\
\hline
35 + 27 &= 5 \text{ tens} + 12 \text{ ones} = 5 \text{ tens} + (1 \text{ ten} + 2 \text{ ones}) \\
&= (5 \text{ tens} + 1 \text{ ten}) + 2 \text{ ones} \\
&= 6 \text{ tens} + 2 \text{ ones} \\
&= 62
\end{aligned}$$

Do you see the step where the associative property of addition was used?

Stage III

Again we record the problem and its solution together. We have

	Tens	Ones
	3	5
	+2	7
TOTAL	6	2

When we do the problem this way we think as follows:

Ones: $5 + 7 = 12$. But 12 is 1 ten plus 2 ones. Write a 2 in the ones column and remember that 1 ten must be added to the tens column. (The best way to avoid mistakes is to begin the addition in the tens column with the 1 ten.)

Tens: $1 + 3 + 2 = 6$. Write a 6 in the tens column.
Thus $35 + 27 = 62$.

In the above example we had to regroup 10 ones as 1 ten. As you know it is sometimes necessary to regroup further using

1 hundred in place of 10 tens
1 thousand in place of 10 hundreds

and so on.

Example

Solve the following problem:

$$848 + 537 + 192 = \square$$

For such a problem Stage I is not worth considering since no one would go to the trouble of gathering such a large number of sticks, let alone counting them. Thus we start with the second stage where the problem and its solution can be recorded as follows:

848 = 8 hundreds + 4 tens + 8 ones
537 = 5 hundreds + 3 tens + 7 ones
192 = 1 hundred + 9 tens + 2 ones

14 hundreds + 16 tens + 17 ones
= (10 hundreds + 4 hundreds) + (10 tens + 6 tens) + (1 ten + 7 ones)
= 1 thousand + (4 hundreds + 1 hundred) + (6 tens + 1 ten) + 7 ones
= 1 thousand + 5 hundreds + 7 tens + 7 ones
= 1,577

A somewhat easier way of writing all this is

848 = 800 + 40 + 8
537 = 500 + 30 + 7
192 = 100 + 90 + 2

1,400 + 160 + 17 = (1,000 + 400) + (100 + 60) + (10 + 7)
= 1,000 + (400 + 100) + (60 + 10) + 7
= 1,000 + 500 + 70 + 7
= 1,577

Stage III

	Thous.	Hund.	Tens	Ones
		8	4	8
		5	3	7
	+	1	9	2
TOTAL	1	5	7	7

As before, we think like this:

Ones: 8 + 7 + 2 = 17. Write a 7 in the ones column and add 1 ten to the tens column.

Tens: 1 + 4 + 3 + 9 = 17. Write a 7 in the tens column and add 1 hundred to the hundreds column.

Hundreds: $1 + 8 + 5 + 1 = 15$. Write a 5 in the hundreds column and a 1 in the thousands column.

The answer is 1,577.

EXERCISE 16-5

Write out the solutions of each of the following problems in Stages II and III as in the text.

1. $36 + 29 = \square$ **2.** $65 + 87 = \square$

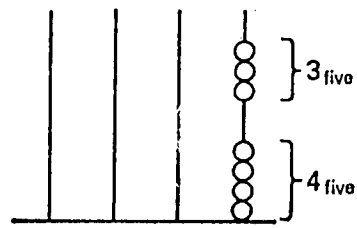
3. $123 + 45 + 82 = \square$ **4.** $437 + 263 = \square$

5. $1{,}532 + 2{,}479 = \square$ **6.** $135 + 221 + 644 = \square$

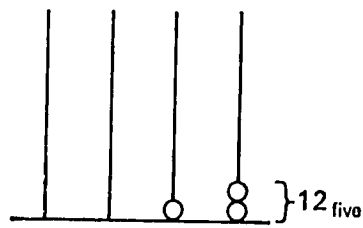
16-6 Addition in other bases

To add numbers expressed in other bases we use *exactly* the same principles that are used to add numbers base ten (that is, the decimal system). The only change is that the *basic addition facts* are different in different bases. In the present section we shall illustrate this by discussing addition of numbers expressed in base five.

For instance, suppose we wish to add 4_{five} and 3_{five}. Using a base-five abacus we put four beads on the ones rod to represent 4_{five}. We then put three more beads on the ones rod to represent 3_{five}. Our abacus then looks like this:



But whenever we have a group of five beads on any rod of a base-five abacus we remove them and put one bead on the next rod. When we do this in the present case we have the following situation:



Finally, since the number represented on this abacus is 12_{five} we have

$$4_{five} + 3_{five} = 12_{five}$$

EXERCISE 16-6A

Use the method described above to complete the following addition table for base five:

+	0	1	2	3	4
0	0			3	
1					
2					
3			10		
4				12	

The above table gives all the *basic addition facts* for numbers base five.

We now use these facts to solve addition problems base five involving numbers having several digits.

Example

Solve the following problem:

$$321_{five} + 122_{five} = \square_{five}$$

Stage I

We give the solution as it would appear on a base-five abacus. If necessary you should draw a picture to illustrate the problem.

$$321_{five} = 3 \text{ twenty-fives} + 2 \text{ fives} + 1 \text{ one}$$
$$122_{five} = 1 \text{ twenty-five} + 2 \text{ fives} + 2 \text{ ones}$$

$$321_{five} + 122_{five} = 4 \text{ twenty-fives} + 4 \text{ fives} + 3 \text{ ones} = 443_{five}$$

This solution can also be written in the following way:

$$321_{five} = 300_{five} + 20_{five} + 1_{five}$$
$$122_{five} = 100_{five} + 20_{five} + 2_{five}$$

$$321_{five} + 122_{five} = 400_{five} + 40_{five} + 3_{five} = 443_{five}$$

We now consider an example which requires regrouping.

Again you may find it helpful to draw a picture of an abacus as you work the problem.

Example

Find the sum $432_{five} + 414_{five}$.

Stage II

Here we have

$$\begin{aligned} 432_{five} &= 400_{five} + 30_{five} + 2_{five} \\ 414_{five} &= 400_{five} + 10_{five} + 4_{five} \end{aligned}$$

$$\begin{aligned} 432_{five} + 444_{five} &= 1{,}300_{five} + 40_{five} + 11_{five} \\ &= (1{,}000_{five} + 300_{five}) + 40_{five} + (10_{five} + 1_{five}) \\ &= 1{,}000_{five} + 300_{five} + (40_{five} + 10_{five}) + 1_{five} \\ &= 1{,}000_{five} + 300_{five} + 100_{five} + 1_{five} \\ &= 1{,}000_{five} + 400_{five} + 1_{five} \\ &= 1{,}401_{five} \end{aligned}$$

Finally, we show how the solution of this problem would proceed in Stage III.

Stage III

	One hundred twenty-fives	Twenty-fives	Fives	Ones
		4	3	2
	+	4	1	4
TOTAL	1	4	0	1

Here the solution was obtained as follows:

Ones: $2 + 4 = 11_{five}$. Write a 1 in the ones column and add 1 five to the fives column since $11_{five} = 1_{five} + 1$.

Fives: $1 + 3 + 1 = 10_{five}$. Write a 0 in the fives column and add 1 twenty-five to the twenty-fives column.

Twenty-fives: $1 + 4 + 4 = 14_{five}$. Write a 4 in the twenty-fives column and a 1 in the one hundred and twenty-fives column.

Thus the answer is $1{,}401_{five}$.

These examples illustrate how the procedure for finding sums is the same in base five as it is in base ten. In fact, it is

the same in *every* base since it only relies on the place value system for renaming numbers and the associative property of addition. In the following exercises we have given problems in base two and base seven to further illustrate this important fact.

EXERCISE 16-6B

1. Write out the solution of each of the following problems in Stages II and III as in the text.
   a. $134_{five} + 310_{five} = \square_{five}$
   b. $213_{five} + 434_{five} = \square_{five}$
   c. $1{,}403_{five} + 2142_{five} = \square_{five}$
   d. $211_{five} + 324_{five} + 121_{five} = \square_{five}$
   e. $32_{five} + 2{,}002_{five} + 421_{five} = \square_{five}$
2. Complete the following table for addition base two.

+	0	1
0		
1		

3. Write out the solution of each of the following problems in Stages II and III as in the text. Use the addition table you constructed in Question 2 above for the basic addition facts.
   a. $1{,}001_{two} + 110_{two} = \square_{two}$
   b. $11{,}010_{two} + 11{,}010_{two} = \square_{two}$
   c. $1{,}111_{two} + 1{,}111_{two} = \square_{two}$
4. Complete the following table for addition base 7.

+	0	1	2	3	4	5	6
0							
1							10
2							
3							
4				10			
5					12		
6							15

5. Write out the solution of each of the following problems in Stages II and III as in the text. Use the addition table you constructed in Question 4 above for the basic addition facts.

   a. $45_{\text{seven}} + 21_{\text{seven}} = \square_{\text{seven}}$
   b. $3{,}062_{\text{seven}} + 1{,}345_{\text{seven}} = \square_{\text{seven}}$
   c. $521_{\text{seven}} + 364_{\text{seven}} + 142_{\text{seven}} = \square_{\text{seven}}$
   d. $6{,}001_{\text{seven}} + 666_{\text{seven}} = \square_{\text{seven}}$
   e. $4{,}132_{\text{seven}} + 1{,}354_{\text{seven}} + 2{,}652_{\text{seven}} = \square_{\text{seven}}$

16-7 Procedures for subtraction

We have seen that the operation of subtraction is the inverse of addition. This means that a subtraction problem such as

$$97 - 34 = \square$$

is the same as the addition problem

$$34 + \square = 97$$

The number which makes this last sentence true is called the *missing addend.*

Since addition and subtraction are so closely related the procedure for working subtraction problems is similar to that used in addition problems. Again the essential step consists of using the place value system when necessary to rename or regroup numbers. The following examples illustrate this. As before we give solutions at three different stages, and begin with a problem which does not require regrouping.

Example

Solve the subtraction problem

$$97 - 34 = \square$$

Stage I

We begin by representing 97 by 9 bundles of 10 sticks each + 7 loose sticks (ones). Then to show the result of subtracting 34 from 97 we remove 4 loose sticks (ones) and 3 bundles of 10. This leaves $7 - 4 = 3$ ones and $9 - 3 = 6$ bundles of 10. Altogether there are $60 + 3 = 63$ sticks remaining, and it follows that $97 - 34 = 63$.

This solution can be recorded more briefly as follows:

97 is represented by 9 bundles of 10 + 7 ones
34 is represented by 3 bundles of 10 + 4 ones

97 − 34 is represented by (9 − 3) bundles of 10 + (7 − 4) ones

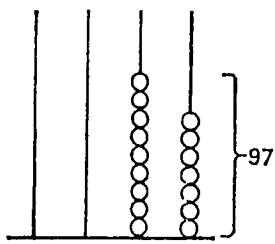
= 6 bundles of 10 + 3 ones
= 63

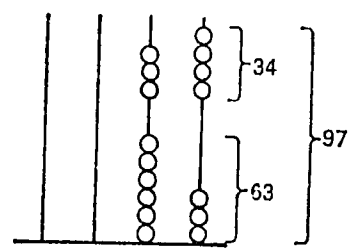
We can also write this result as

$$\begin{array}{rl} 97 & = 90 + 7 \\ -\ 34 & = 30 + 4 \\ \hline 97 - 34 & = (90 - 30) + (7 - 4) \\ & = 60 + 3 \\ & = 63 \end{array}$$

Stage II

On an abacus the problem 97 − 34 = □ is solved as follows. First represent 97 by placing 7 beads on the ones rod and 9 beads on the tens rod as shown on the abacus to the left below. Now separate 4 ones from the 7 ones and 3 tens from the 9 tens to show the result of subtracting 34 from 97. As you can see by the figure on the right the number now represented on the abacus is 63. Thus 97 − 34 = 63.





The solution shown here can be recorded in the following way:

$$\begin{array}{rl} 97 & = 9 \text{ tens} + 7 \text{ ones} \\ -\ 34 & = 3 \text{ tens} + 4 \text{ ones} \\ \hline 97 - 34 & = (9 \text{ tens} - 3 \text{ tens}) + (7 \text{ ones} - 4 \text{ ones}) \\ & = 6 \text{ tens} + 3 \text{ ones} \\ & = 60 + 3 \\ & = 63 \end{array}$$

Finally, we pass to Stage III and write the solution in the standard way.

Stage III

	Tens	Ones
	9	7
	−3	4
TOTAL	6	3

When we solve the problem this way we think as follows:

Ones: $7 - 4 = 3$. Write a 3 in the ones column.

Tens: $9 - 3 = 6$. Write a 6 in the tens column.

Thus the answer is 63.

The above example did not involve any regrouping. We now consider one which does.

Example

Solve the subtraction problem

$$62 - 25 = \square$$

Stage I

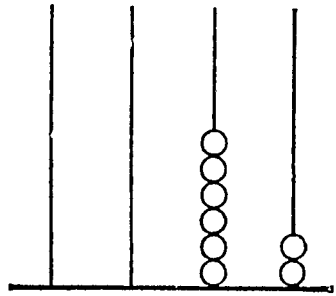
If we start by representing 62 with 6 bundles of 10 + 2 ones and try to remove 2 bundles of 10 + 5 ones we find that we do not have enough loose sticks. Thus we must regroup. We do this by separating one bundle of 10 into 10 loose sticks. In other words we represent 62 by 5 bundles of 10 + 12 loose sticks. We now remove 2 bundles of 10 + 5 ones. This leaves 3 bundles of 10 + 7 ones. Hence $62 - 25 = 37$.

Here the solution can be written as follows:

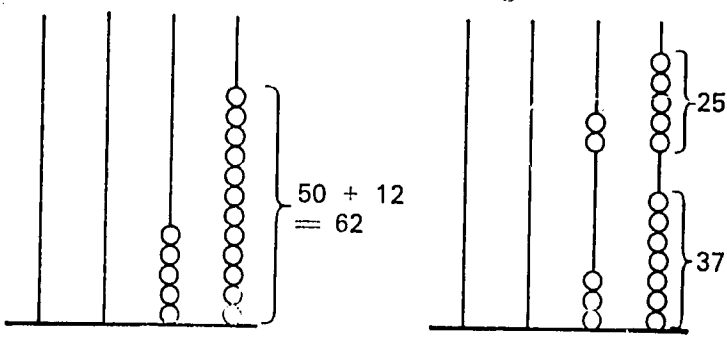
$$\begin{array}{rl} 62 & = 50 + 12 \\ -\ 25 & = 20 + 5 \\ \hline 62 - 25 & = (50 - 20) + (12 - 5) \\ & = 30 + 7 \\ & = 37 \end{array}$$

Stage II

We begin by representing 62 on an abacus as shown below.



We then try to remove 5 beads from the ones rod and 2 beads from the tens rod to show the result of subtracting 25 from 62. But there are not enough beads on the ones rod. Thus we regroup by removing one bead from the tens rod and placing 10 more beads on the ones rod. Our abacus now appears as shown on the left below, and the problem can be solved. The solution is pictured on the abacus to the right.



This time the written solution goes like this:

$$62 = 6 \text{ tens} + 2 \text{ ones}$$
$$25 = 2 \text{ tens} + 5 \text{ ones}$$

Since there are more ones in 25 than in 62 we must regroup. Thus we write

$$\begin{aligned} 62 &= 5 \text{ tens} + 12 \text{ ones} \\ -\ 25 &= 2 \text{ tens} + 5 \text{ ones} \\ \hline 62 - 25 &= (5 \text{ tens} - 2 \text{ tens}) + (12 \text{ ones} - 5 \text{ ones}) \\ &= 3 \text{ tens} + 7 \text{ tens} \\ &= 37 \end{aligned}$$

Stage III

	Tens	Ones
	6	2
	− 2	5
TOTAL	3	7

Here we think as follows:

Ones: We must rewrite 62 so that we have more ones. Thus we mentally rename 62 as 5 tens plus 12 ones. Then $12 - 5 = 7$. Write a 7 in the ones column.

Tens: $6 - 2 = 3$.

Thus the answer is 37.

Problems involving subtraction of numbers expressed in other bases proceed in the same way. For instance, the problem

$$421_{five} - 234_{five}$$

can be solved as follows:

$$\begin{aligned} 421_{five} &= 400_{five} + 20_{five} + 1_{five} \\ &= 300_{five} + 120_{five} + 1_{five} \\ &= 300_{five} + 110_{five} + 11_{five} \end{aligned}$$

Thus

$$\begin{array}{rl} 421_{five} &= 300_{five} + 110_{five} + 11_{five} \\ -\ 234_{five} &= 200_{five} + 30_{five} + 4_{five} \\ \hline (421_{five} - 234_{five}) &= 100_{five} + 30_{five} + 2_{five} \\ &= 132_{five} \end{array}$$

EXERCISE 16-7

Write out the solution of each of the following problems in Stages II and III as in the text.

1. $463 - 258 = \square$

2. $372 - 173 = \square$

3. $2{,}006 - 948 = \square$
4. $4{,}321_{\text{five}} - 1{,}234_{\text{five}} = \square_{\text{five}}$
5. $3{,}042_{\text{five}} - 2{,}144_{\text{five}} = \square_{\text{five}}$
6. $110{,}110_{\text{two}} - 101{,}010_{\text{two}} = \square_{\text{two}}$
7. $10{,}001_{\text{two}} - 1{,}111_{\text{two}} = \square_{\text{two}}$
8. $6{,}435_{\text{seven}} - 1{,}226_{\text{seven}} = \square_{\text{seven}}$
9. $2{,}003_{\text{seven}} - 1{,}024_{\text{seven}} = \square_{\text{seven}}$
10. $3{,}002_{\text{seven}} - 3{,}002_{\text{seven}} = \square_{\text{seven}}$

Chapter 17

PROCEDURES FOR MULTIPLICATION AND DIVISION

17-1 Introduction

We have seen that multiplication of whole numbers can be thought of as repeated addition. Thus to find a product such as 4×3 we can replace the 4×3 by $3 + 3 + 3 + 3$ and add. Since the sum of 4 threes is 12 we write $4 \times 3 = 12$. This method of finding products works well when the numbers are small. But when we come to larger numbers such as 14×39 the method of repeated addition is too tedious. Thus we look for faster ways of doing multiplication.

As teachers you know how products such as 14×39 are found. However, as with anything in mathematics, it is necessary to know *why* the method which you use actually works. This is the purpose of the present chapter: we shall show why the method which everyone uses for multiplication gives the correct answer. Once this has been done we will be able to do the same thing for division, since division is the inverse of multiplication.

17-2 Multiplication by multiples of 10

Consider the following multiplication problem:

$$8 \times 10 = \square$$

To solve this problem we think: $8 \times 10 = 8$ tens. But in our number system 8 tens is written as 80. Thus $8 \times 10 = 80$. Similarly

$$\begin{aligned} 9 \times 10 &= 9 \text{ tens} = 90 \\ 13 \times 10 &= 13 \text{ tens} = 130 \\ 26 \times 10 &= 26 \text{ tens} = 260 \\ 95 \times 10 &= 95 \text{ tens} = 950 \end{aligned}$$

and so on. Notice that in each case the answer is obtained by

writing a single zero after the number which is being multiplied by 10. This happens because we use the place value system to write numerals.

The same reasoning can be used to find the product of two whole numbers when one of the numbers is 100, 1,000, and so on. For instance.

$$8 \times 100 = 8 \text{ hundreds} = 800$$
$$8 \times 1{,}000 = 8 \text{ thousands} = 8{,}000$$
$$8 \times 10{,}000 = 8 \text{ ten thousands} = 80{,}000$$

Here too the answers were obtained by writing the correct number of zeros after the original number; two when we multiplied by 100, three when we multiplied by 1,000, four when we multiplied by 10,000.

Using these results it is easy to compute products when one of the factors is a multiple of 10 less than 100, or a multiple of 100 less than 1,000, etc. The following examples show how this is done:

$8 \times 20 = 8 \times (2 \times 10)$	since $20 = 2$ tens $= 2 \times 10$
$= (8 \times 2) \times 10$	since multiplication of
$= 16 \times 10$	whole numbers is associative
$= 160$	

It is important to observe how the associative property of multiplication was used here to get the answer. In exactly the same way we find that

$$\begin{aligned} 5 \times 300 &= 5 \times (3 \times 100) \\ &= (5 \times 3) \times 100 \\ &= 15 \times 100 \\ &= 1{,}500 \end{aligned}$$

EXERCISE 17-2

Solve each of the following equations.

1. $235 \times 10 = \square$
2. $48 \times \square = 480$
3. $100 \times 37 = \square$
4. $\square \times 730 = 73{,}000$
5. $\square \times 100 = 2{,}100$
6. $1{,}800 = \square \times 200$
7. $92{,}000 = 100 \times \square$
8. $500 \times 200 = \square$
9. $80 \times \square = 3{,}200$
10. $8{,}100 = \square \times 90$

11. a. Find each of the following base-five products.

(i) $4_{five} \times 10_{five}$
(ii) $12_{five} \times 10_{five}$
(iii) $234_{five} \times 10_{five}$
(iv) $3_{five} \times 100_{five}$
(v) $31_{five} \times 100_{five}$
(vi) $2_{five} \times 1{,}000_{five}$
(vii) $121_{five} \times 1{,}000_{five}$

b. Explain why the same rule of adding zeros applies in base five when multiplying 10_{five}, 100_{five}, etc., as applies in base ten when multiplying by 10, 100, etc.

c. Do you think this result will hold in every base? Why?

17-3 The general procedure for multiplication

At the end of Chapter 13 we showed how the properties of addition and multiplication can be used to compute products. For instance, if we know that $2 \times 2 = 4$ and that $2 \times 1 = 2$ we can find 2×3 as follows:

$$\begin{aligned} 2 \times 3 &= 2 \times (2 + 1) \\ &= (2 \times 2) + (2 \times 1) \\ &= 4 + 2 \\ &= 6 \end{aligned}$$

The essential step here was provided by the distributive property which allowed us to write $2 \times (2+1) = (2 \times 2) + (2 \times 1)$. Thus we were able to break the original problem down into simpler problems whose answers were known. This is the method we are now going to use to find products of numbers involving several digits. In using this method we shall assume that all products up to 9×9 are known. These products are called the *basic multiplication facts*. They are the multiplication facts which children must learn thoroughly in order to multiply efficiently.

We begin with the following problem:

$$7 \times 56 = \square$$

To solve this problem we write

$$\begin{aligned} 7 \times 56 &= 7 \times (50 + 6) \\ &= (7 \times 50) + (7 \times 6) \end{aligned}$$

But since $7 \times 5 = 35$, we know that $7 \times 50 = 350$. Thus

$$\begin{aligned}(7 \times 50) + (7 \times 6) &= 350 + 42 \\ &= 392\end{aligned}$$

It now follows from the distributive property that

$$7 \times 56 = 392$$

Notice how this problem was solved: we reduced it to a problem involving addition, basic multiplication facts and the rule for multiplying by a multiple of 10. *All multiplication problems are solved in this way.* Here are some further examples.

Example

Find the product 6×234

$$\begin{aligned}6 \times 234 &= 6 \times (200 + 30 + 4) \\ &= (6 \times 200) + (6 \times 30) + (6 \times 4) \\ &= 1{,}200 + 180 + 24 \\ &= 1{,}404\end{aligned}$$

You should be able to give reasons for each step in these computations.

Example

Solve the equation $14 \times 39 = \square$

Here both factors in the product have two digits. However, the method introduced above still works. Indeed, by the distributive property we have

$$\begin{aligned}14 \times 39 &= 14 \times (30 + 9) \\ &= (14 \times 30) + (14 \times 9)\end{aligned}$$

To compute 14×30 and 14×9 we again use the distributive property:

$$\begin{aligned}14 \times 30 &= (10 + 4) \times 30 \\ &= (10 \times 30) + (4 \times 30) \\ &= 300 + 120 \\ &= 420 \\ 14 \times 9 &= (10 + 4) \times 9 \\ &= (10 \times 9) + (4 \times 9) \\ &= 90 + 36 \\ &= 126\end{aligned}$$

Thus $\quad 14 \times 39 = 420 + 126 = 546$

Example

Find the product 23×42

$$\begin{aligned} 23 \times 42 &= 23 \times (40 + 2) \\ &= (23 \times 40) + (23 \times 2) \\ &= [(20 + 3) \times 40] + [(20 + 3) \times 2] \\ &= (20 \times 49) + (3 \times 40) + (20 \times 2) + (3 \times 2) \\ &= 800 + 120 + 40 + 6 \\ &= 966 \end{aligned}$$

Again you should be able to give reasons for each of the above steps.

EXERCISE 17-3

Use the method of this section to find each of the following products.

1. 8×76 **2.** 59×7 **3.** 83×6

4. 26×17 **5.** 67×73 **6.** 45×38

7. 7×425 **8.** 26×308 **9.** 31×527

10. 604×195

11. Here is the basic five multiplication table for all products up to $4_{five} \times 4_{five}$. (These are the basic multiplication facts for base-five arithmetic.)

×	1	2	3	4
1	1	2	3	4
2	2	4	11	13
3	3	11	14	22
4	4	13	22	31

Use the above table and the method of this section to find each of the following products:

a. $3_{five} \times 42_{five}$ **b.** $4_{five} \times 31_{five}$

c. $21_{five} \times 30_{five}$ **d.** $41_{five} \times 32_{five}$

e. $304_{five} \times 13_{five}$

17-4 The short form for multiplication

In the last section we learned how to solve multiplication problems by using the distributive property and the basic multiplication facts.

However, as some of the problems at the end of the section showed, this method can be long and tedious if every step is completely written out. Thus we seek ways of shortening these computations. But before we begin we point out that the computations themselves will be *exactly* the same as before. The only changes will be in the way they are written out.

Consider the problem of finding 7×56. We solved this problem in the last section by writing

$$\begin{aligned} 7 \times 56 &= 7 \times (50 + 6) \\ &= (7 \times 50) + (7 \times 6) \\ &= 350 + 42 \\ &= 392 \end{aligned}$$

The first step in shortening these computations is to rewrite them in vertical form as follows:

$$\begin{array}{rl} 56 & \\ \times 7 & \\ \hline 42 & (7 \times 6) \\ 350 & (7 \times 50) \\ \hline 392 & \end{array}$$

Notice that this arrangement eliminates two of the steps in the first computation. This was done by renaming 56 as $50 + 6$ and applying the distributive property automatically.

We can do the same thing with products where both factors involve several digits. For instance, to find 29×36 we can write

$$\begin{array}{rl} 29 & \\ \times 36 & \\ \hline 54 & (6 \times 9) \\ 120 & (6 \times 20) \\ 270 & (30 \times 9) \\ 600 & (30 \times 20) \\ \hline 1{,}044 & \end{array}$$

This time we thought of 29 as $20 + 9$ and 36 as $30 + 6$, and applied the distributive property.

As you know, we can simplify these computations even

further. For example, we can find 7×56 in just two steps, as follows:

Step 1. Since $7 \times 6 = 42$ we record a 2 in the ones column and remember 4 tens for the tens column.

Step 2. Since $7 \times 5 = 35$, we have $35 + 4 = 39$ in the tens column. Thus the answer is 392.

When written out this solution now appears as

$$\begin{array}{r} 56 \\ \times 7 \\ \hline 392 \end{array}$$

This method also works in general. For instance, to find 29×36 we proceed as follows:

Step 1. $6 \times 9 = 54$. Record a 4 in the ones column and remember 5 tens for the tens column.

Step 2. Since $6 \times 2 = 12$ we have $12 + 5 = 17$ in the tens column.

With this we have multiplied 29 by 6 and the problem is half solved. The computations look like this:

$$\begin{array}{r} 29 \\ \times 36 \\ \hline 174 \end{array}$$

We now continue, multiplying 29 by 30 in two steps:

Step 1. Since $3 \times 9 = 27$ we record a 7 in the tens column and remember 2 hundreds for the hundreds column.

Step 2. Since $3 \times 2 = 6$ we have $6 + 2 = 8$ in the hundreds column. Thus $29 \times 30 = 870$, and we have

$$\begin{array}{r} 29 \\ \times 36 \\ \hline 147 \\ 87 \\ \hline \end{array}$$

Finally we complete the solution by adding. We get

$$\begin{array}{r} 29 \\ \times 36 \\ \hline 174 \\ 87 \\ \hline 1{,}044 \end{array}$$

EXERCISE 17-4

Find each of the following products in two ways; first by the method given at the beginning of this section, and second by the method given at the end of the section. Use the multiplication table in the preceding set of exercises to solve the problems in base five.

1. 46×72	**2.** 308×56	**3.** 97×201
4. 316×24	**5.** 708×509	**6.** 199×512
7. $34_{five} \times 12_{five}$	**8.** $214_{five} \times 31_{five}$	**9.** $402_{five} \times 43_{five}$
10. $142_{five} \times 314_{five}$		

17-5 Division by the method of repeated subtraction

Since multiplication can be viewed as repeated addition its inverse, division, can be viewed as repeated subtraction. You should be sure you understand why this is true because it is the idea behind the procedures for division which are taught in school. We now show how these procedures are developed by considering several examples.

For our first example we take the easy problem

$$48 \div 2 = \square$$

To solve this problem we have to determine how many twos must be subtracted from 48 to reach 0. Since it would take a long time to subtract the twos one at a time we use our knowledge of multiplication to shorten our work. For instance, we know that $2 \times 10 = 20$, so that if we subtract 20 from 48 we have actually subtracted two 10 times. After doing this we have 28 left. Thus we can subtract 10 twos again to get 8. Finally since $2 \times 4 = 8$ we subtract 4 more twos to reach 0. Our work can be written down like this:

	No. of twos subtracted
48	
−20	10
28	
−20	10
8	
−8	4
0	24 = total number of twos subtracted

To find the quotient $48 \div 2$ we now add the number of twos that were subtracted from 48. Since this number is 24 we have

$$48 \div 2 = 24$$

The work in solving this problem could have been shortened if we had noticed at the beginning that $2 \times 20 = 40$, and had subtracted 40 from 48. This gives

	No. of twos subtracted	
48		
−40	20	
8		
− 8	4	
0	24	= total number of twos subtracted

When we solve division problems this way we usually try to organize our work more efficiently. One way of doing this is as follows:

$$\begin{array}{rl} 2\overline{)48} & \\ -40 & 20 \\ \hline 8 & \\ -8 & 4 \\ \hline 0 & 24 \end{array}$$

Here we wrote the number of twos subtracted to the right of each subtraction. We then added these numbers to obtain the quotient. Another way of organizing this work is to write the number of twos subtracted at each step above the 48. We then have

$$\begin{array}{r} \left.\begin{array}{r} 4 \\ 20 \end{array}\right\} 24 \\ \hline 2\overline{)48} \\ -40 \\ \hline 8 \\ -8 \\ \hline 0 \end{array}$$

Finally, we can shorten this process one more step by making use of the place value system. We agree that the places directly above the digits in 48 will have the same place value as they have in the number 48. Thus the place above the 4 is the tens

place and the place above the 8 is the ones place. Then instead of writing 20 above the 48 we just write a 2 in the tens place above the 4. When we do this we get the familiar pattern

$$\begin{array}{r} 24 \\ 2\overline{)48} \\ \underline{-40} \\ 8 \\ \underline{-8} \\ 0 \end{array}$$

As our second example we consider the problem

$$960 \div 30 = \square$$

A pupil just learning division might use the fact that $10 \times 30 = 300$ to solve this problem in the following way:

	No. of thirties subtracted
960	10
−300	
660	10
−300	
360	10
−300	
60	2
− 60	
0	32 = total number of thirties subtracted

There is absolutely nothing wrong with this solution. However, it is longer than it need to be. If the pupil happened to notice that since $3 \times 3 = 9$, $30 \times 30 = 900$ he could have got the answer much faster. For then he could write

	No. of thirties subtracted
960	30
−900	
60	
−60	2
0	32 = total number of thirties subtracted.

When you teach division to children you should not expect them to begin by solving problems in the fewest steps possible. At first they will be cautious and take many steps. This is actually good since it helps them to learn that division can be considered as repeated subtraction. But as they continue working problems they will become more confident and get answers faster. Only when this happens should they be made to organize their work in the short form using the fewest number of steps. For instance, the pupil who solves $960 \div 30 = \square$ by writing

$$\begin{array}{r} 960 \\ -900 \\ \hline 60 \\ -60 \\ \hline 0 \end{array} \qquad \begin{array}{r} 30 \\ \\ 2 \\ \hline 32 \end{array}$$

is ready to put his work in the form

$$\begin{array}{r} \left.\begin{array}{r} 2 \\ 30 \end{array}\right\} 32 \\ \hline 30\overline{)960} \\ -900 \\ \hline 60 \\ -60 \\ \hline 0 \end{array}$$

Once he completely understands this method he can go on to the final form

$$\begin{array}{r} 32 \\ \hline 30\overline{)960} \\ -900 \\ \hline 60 \\ -60 \\ \hline 0 \end{array}$$

EXERCISE 17-5

Solve each of the following division problems by the method of repeated subtraction. Then write the solution in the short form using the fewest steps possible.

1. $273 \div 3$ **2.** $445 \div 5$ **3.** $644 \div 7$

4. $840 \div 20$ **5.** $50{,}250 \div 50$ **6.** $41{,}200 \div 40$

7. $323_{five} \div 4_{five}$ **8.** $12{,}430_{five} \div 30_{five}$ **9.** $2{,}420_{five} \div 20_{five}$
10. $214{,}200_{five} \div 300_{five}$

17-6 Division by the method of repeated subtraction (continued)

In the last section we used the method of repeated subtraction to solve easy division problems. We now illustrate how this method can be used to solve harder problems as well.

Example

Find the quotient $96{,}369 \div 21$.

When solving problems like this a few mental computations can often save a great deal of work. Thus, in the present case, if we notice that $21 \times 4{,}000 = 84{,}000$ while $21 \times 5{,}000 = 105{,}000$ we can begin the solution by subtracting $21 \times 4{,}000$ from 96,369. This gives

```
21)96,369
  -84 000   4,000
   12,369
```

Next we notice that $21 \times 500 = 10{,}500$ while $21 \times 600 = 12{,}600$. Since 12,369 is less than 12,600 but greater than 10,500 we now subtract 21×500 from 12,369. We then have

```
21)96,369
  -84 000   4,000
   12 369
  -10 500     500
    1 869
```

Continuing in this way we obtain

```
21)96,369
  -84 000   4,000
   12 369
  -10 500     500
    1 869
  -1 680       80
     189
   - 189        9
       0    4,589
```

Thus the quotient is 4,589. When these computations are written using the short form they appear as follows:

```
     4 589
21)96 369
  -84 000
   12 369
  -10 500
    1 869
   -1 680
      189
     -189
        0
```

At this point it is a good idea to check the computations by multiplication. The above result says that

$$96{,}369 \div 21 = 4{,}589$$

But since division is the inverse of multiplication this equality will be true if and only if

$$96{,}369 = 4{,}589 \times 21$$

Thus to see if the answer is correct we simply multiply 4,589 by 21. We get

```
  4 589
   ×21
  4 589
 91 78
 96,369
```

Thus all is well. When you teach division you should strongly encourage your pupils to check their answers this way.

EXERCISE 17-6

Solve each of the following division problems and check the answers by multiplication.

1. 6,528 ÷ 17

2. 60,974 ÷ 86

3. 69,174 ÷ 54

4. 74,672 ÷ 359

5. 351,203 ÷ 73

Fractions UNIT III

Chapter 18

INTRODUCTION TO FRACTIONS

18-1 Introduction

Every day each of us uses numbers which are not whole numbers. We talk about such things as

(*a*) *one-half* of a loaf of bread,
(*b*) *three-quarters* of a pound of meat,
(*c*) *two-thirds* of a cup of sugar.

The words "one-half", "three-quarters", "two-thirds" are names of numbers. These numbers are called *fractions*. In this unit we shall learn what fractions are and how they are used in arithmetic.

As you know, many pupils find fractions difficult to understand. Often they can solve problems involving fractions only by using complicated rules which they have memorized. Actually there is no need for this since the arithmetic of fractions can be explained just as clearly as the arithmetic of whole numbers. But to do so it is very important that pupils first become really familiar with the idea of a fraction. They must understand what a fraction is, and what its properties are *before* they begin to add, subtract, multiply and divide fractions. Thus we shall spend the next two chapters discussing the ideas needed to understand fractions. Once this has been done the arithmetic of fractions becomes quite easy.

18-2 Names and symbols for fractions

In English there are special names such as "one-half", "one-third", and so on to describe parts of objects. Most primary school children know some of these names. For instance, they talk about one-half of a piece of sugar cane, or one-half of an orange, and know what this means. They realize that the word "one-half" refers to *one* part of an object that has been divided into *two* EQUAL parts. For them "one-half" means *one* of *two*

equal parts, or, briefly, *one* of *two*. In the same way they may use the word "one-third" to speak about *one* part of an object that has been divided into *three* equal parts, or, *one* of *three*. Before we go on to talk about the number concepts behind these names we take a closer look at the names themselves.

We have said that the name "one-half" stands for the idea of choosing *one* of *two* equal parts. The following picture illustrates this idea:

One of *two*

One-half

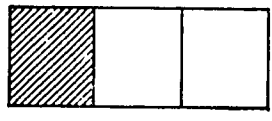
In the picture we have shown a rectangle divided into two *equal* parts, one of which has been chosen. The chosen part has been shaded.

At this point we ask if there is an easy way of putting into symbols the idea represented by this picture. As you know, there is. We simply write the numerals 1 and 2, and separate them by a stroke. This gives $\frac{1}{2}$ or 1/2. Both of these symbols are used since both make us think of the basic idea: *one* of *two* equal parts.

To summarize:

We think: *one* of *two* equal parts
We say: *one-half*
We write: $\frac{1}{2}$ or 1/2

Now suppose we think of an object divided into three equal parts. We then say that each part is *one-third* of the object, and we write $\frac{1}{3}$. This time we have the following picture:



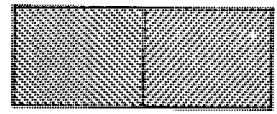
One of *three*
One-third
$\frac{1}{3}$

Continuing in this way we think of objects divided into 4 equal parts, 5 equal parts, and so on. In the case of

4	equal parts each part is called	*one-fourth*
5	,, ,, ,, ,, ,, ,,	*one-fifth*
6	,, ,, ,, ,, ,, ,,	*one-sixth*

We represent these names by the symbols $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, etc. Finally, in speaking about one-fourth the special name one-quarter is sometimes used.

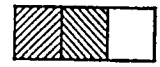
We also want to be able to talk about choosing more than one part of an object that has been divided into equal parts. For example, suppose a piece of sugar cane had been divided into two equal pieces to be shared between two children and then one of the children ate *both* pieces. He ate *two-halves* of the sugar cane. Thus when we choose *two* parts of an object that has been divided into *two* equal parts, we have chosen *two-halves* of the object. We write this as $\frac{2}{2}$, and represent it by the picture



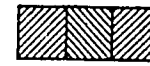
Two of *two*
Two-halves
$\frac{2}{2}$

By using objects divided into more and more parts we can go on like this as long as we please. The following pictures show what happens:

One of *three*
One-third
$\frac{1}{3}$



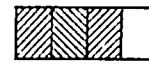
Two of *three*
Two-thirds
$\frac{2}{3}$



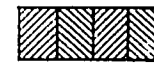
Three of *three*
Three-thirds
$\frac{3}{3}$

One of *four*
One-fourth
$\frac{1}{4}$

Two of *four*
Two-fourths
$\frac{2}{4}$



Three of *four*
Three-fourths
$\frac{3}{4}$

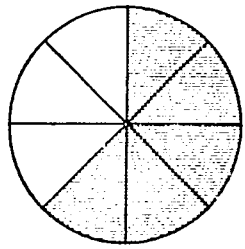


Four of *four*
Four-fourths
$\frac{4}{4}$

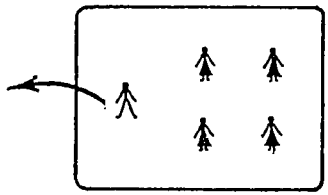
18-3 Pictures that show fractions

When children first learn about fractions they should be shown many pictures. Here are some that you might use in your class.

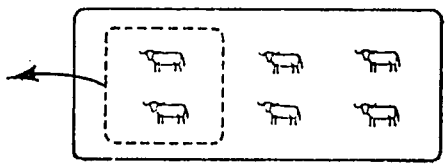
A cake is cut into 8 equal pieces and I take 5 of them. I have $\frac{5}{8}$ of the cake.



There are 4 girls and 1 boy. The boy goes away so $\frac{1}{5}$ of the children have gone away.



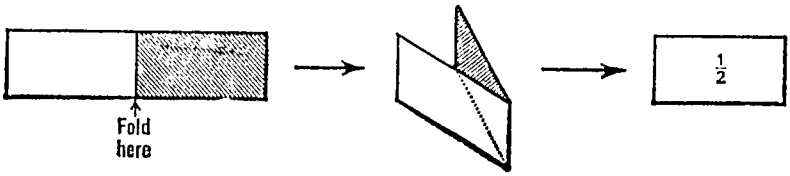
There is a set of 6 buffaloes. Some lions chase 2 away so that $\frac{2}{6}$ of the buffaloes have gone away.



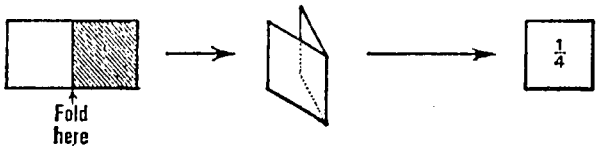
There are 27 children in the class. One child is away so $\frac{1}{27}$ of the class is absent. Yesterday no children were away so there were $\frac{0}{27}$ absent.

Children can picture fractions for themselves by folding paper. One piece of paper can be folded to show $\frac{1}{2}$ and $\frac{1}{4}$ and $\frac{1}{8}$. In each picture below the shaded part is being folded along the

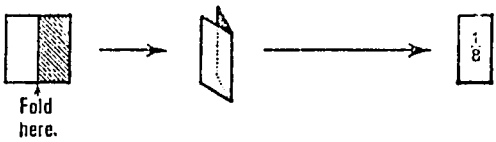
dotted line to fit behind the unshaded part. First the whole is folded into 2 equal parts.



Then the $\frac{1}{2}$ piece is folded again to show $\frac{1}{4}$ of the whole piece.



The $\frac{1}{4}$ piece is folded again to show $\frac{1}{8}$ of the whole piece.



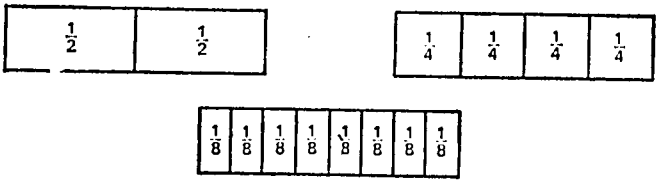
If you open the paper out again after each folding you can show that

2 halves equal 1 whole
4 fourths equal 1 whole
8 eighths equal 1 whole

In other words,

$$2 \times \tfrac{1}{2} = \tfrac{2}{2} = 1$$
$$4 \times \tfrac{1}{4} = \tfrac{4}{4} = 1$$
$$8 \times \tfrac{1}{8} = \tfrac{8}{8} = 1$$

The picture looks like this:

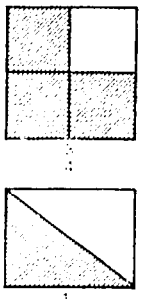


Notice how well these pictures illustrate the following general rule:

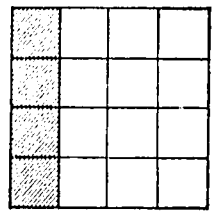
$$n \times \frac{1}{n} = 1 \text{ or } \frac{n}{n} = 1$$

for any counting number n. In words these equations say that if an object is divided into n equal parts so that each part is $\frac{1}{n}$ of the whole, then all n parts together give us the entire object. By folding pieces of paper and drawing pictures you can help pupils to discover this rule for themselves. It is important and will be used later when we discuss multiplication of fractions.

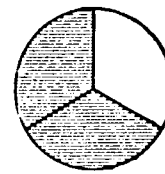
It is also helpful for children to see that fractions can be shown by using many different kinds of pictures. Here are some shapes to show various fractions.



$\frac{1}{2}$



$\frac{4}{16}$



$\frac{2}{3}$

Pictures and paper are useful aids to understanding, but real objects such as sets of bottle tops, beans or berries are better still. Or you can cut up clay cakes or sugar cane to show fractions to your pupils.

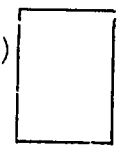
EXERCISE 18-3

1. Shade each shape to show the fraction written by it.

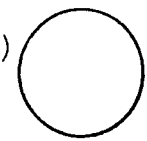
(a) $\frac{3}{5}$

(b) $\frac{2}{3}$

(c) $\frac{7}{8}$



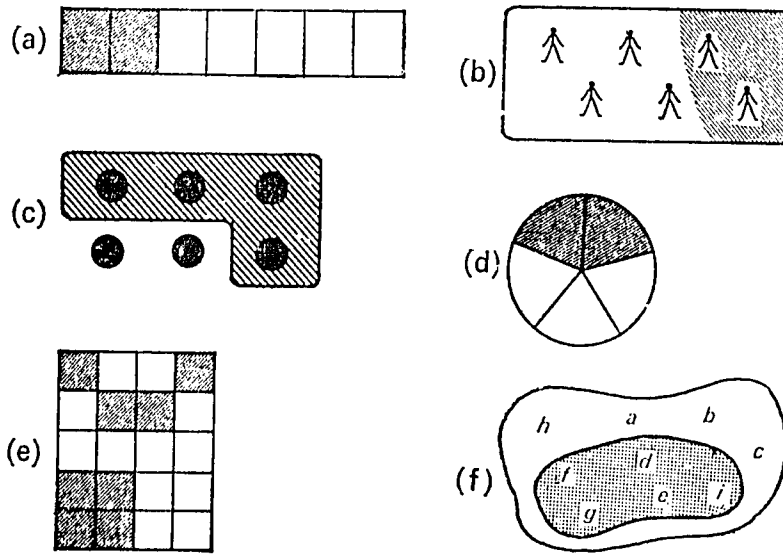
(d) $\frac{3}{4}$



2. Draw pictures of sets to show the following fractions:

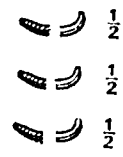
a. $\frac{1}{2}$ of 8 oranges **b.** $\frac{2}{3}$ of 6 chickens **c.** $\frac{3}{5}$ of 10 buns

3. What part is shaded in each of these pictures?



18-4 Names of fractions (continued)

We have seen that the name "one-half" stands for *one* of *two* equal parts, and is written in symbols as $\frac{1}{2}$. In the following picture we have divided each of three bananas into two equal parts and have shaded one of these parts. Each shaded part is one-half of one banana.



Suppose someone gave you all three shaded pieces. You would then have three half bananas, or simply, *three-halves*. Just as we use the symbol $\frac{1}{2}$ to represent *one*-half, and the symbol $\frac{2}{2}$ to represent *two*-halves, we use the symbol $\frac{3}{2}$ to represent *three*-halves. The number above the line tells how many halves you have. It is called the NUMERATOR. The number below

the line tells that there are two equal pieces in each whole object. It is called the DENOMINATOR.

We can also represent the idea of choosing *three-halves* by a picture of the sort used in the first section of this chapter. This time, however, we start with *two* rectangles, both of the same size. Divide each into two equal parts so that each part is one-half of a whole rectangle. Now take *three* of these parts, as shown below, to get three-halves.

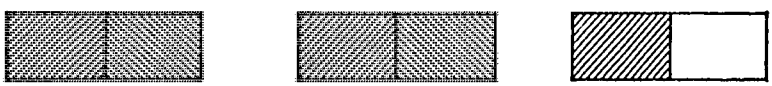
Three-halves

$\frac{3}{2}$

By using more and more rectangles, and shading more and more pieces we can represent four-halves, five-halves, and so on.

Four-halves

$\frac{4}{2}$



Five-halves

$\frac{5}{2}$

Fractions such as $\frac{1}{2}$, $\frac{4}{7}$, $\frac{100}{101}$ whose numerators are *less* than their denominators are called PROPER FRACTIONS because they are always less than 1 whole. Fractions such as $\frac{3}{2}$, $\frac{6}{5}$, $\frac{10}{7}$ whose numerators are *greater* than their denominators are called IMPROPER FRACTIONS because they are always greater than 1 whole. Actually there is nothing "improper" about them at all. However, since a single object cannot be divided into more than 2 halves, or more than 5 fifths, etc., people originally felt that such fractions were a little strange. But we have already seen that by dividing two or more objects into parts we can show improper fractions just as easily as proper ones.

Every improper fraction can be written another way. For instance, the fraction $\frac{3}{2}$ represents 1 whole plus one-half; that is, $\frac{3}{2} = 1 + \frac{1}{2}$. This is shortened by writing $1\frac{1}{2}$ in place of $1 + \frac{1}{2}$. In this form we speak of the MIXED NUMBER or MIXED FORM of the fraction $1\frac{1}{2}$ because it consists of a whole number and a proper fraction. Similarly $\frac{6}{5}$ can be written as the mixed number $1\frac{1}{5}$ since $1 = \frac{5}{5}$.

EXERCISE 18-4

1. Draw pictures that show each of the following fractions:
   a. $\frac{4}{3}$ **b.** $\frac{9}{5}$ **c.** $\frac{14}{10}$ **d.** $\frac{18}{4}$
2. Write each of the fractions in Question 1 as a mixed number.
3. Draw pictures that show each of the following mixed numbers:
   a. $1\frac{3}{4}$ **b.** $2\frac{4}{5}$ **c.** $1\frac{5}{8}$ **d.** $3\frac{2}{3}$
4. Write each of the mixed numbers in Question 3 as an improper fraction.

18-5 Fractions and division

In this section we shall show how fractions arise when we try to solve division problems. If you think about it, this is not surprising. For, as we have seen, the idea behind fractions is related to the idea of separating or *dividing* objects into parts of equal size.

We begin with the simple division problem

$$6 \div 2 = \square$$

One way to solve this problem is to rewrite the equation as

$$2 \times \square = 6$$

and then ask what number must be put in the box to get a true sentence. We do this so that the pupil who knows only about multiplication can find the answer. When he says that the answer is 3 because $2 \times 3 = 6$, we tell him that in finding this answer he has *divided* 6 by 2. We then write the equation

$$6 \div 2 = 3$$

and draw pictures to show the pupil what this means. One of the pictures we draw looks like this:

This picture shows that if we remove elements *two at a time* from a set of 6 elements we will get 3 subsets of 2 elements each. Thus $6 \div 2 = 3$.

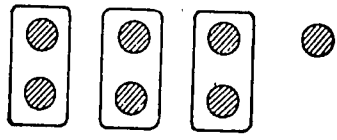
In this example the problem had a whole number answer. But suppose we change the problem to read

$$7 \div 2 = \square$$

What do we do then? If the pupil now tries to solve the corresponding multiplication problem

$$2 \times \square = 7$$

he will quickly find that he cannot give a whole number answer. For if he starts with a set containing 7 elements and begins removing them 2 at a time he will get the following picture:



There are 3 sets of 2, and 1 element remains at the end.

Children first learn about problems like this by using remainders. They say

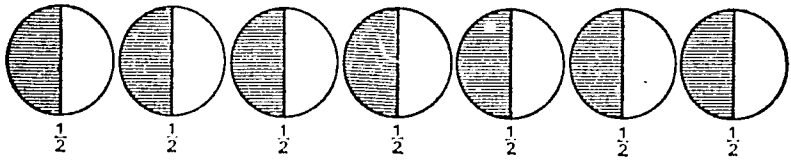
$$7 \div 2 = 3 \text{ remainder } 1$$

But when they do this they have not really solved the problem. What they have done is to divide 6 by 2 to get 3 and then stopped. The 1 was left undivided. Although there is nothing wrong with this it still does not tell what *seven* divided by two is.

To bring the pupil to the point where he can solve this problem completely we must get him to look at it differently. As long as he keeps trying to remove *whole* objects 2 at a time from a set of 7 he will never be able to remove them all. Thus

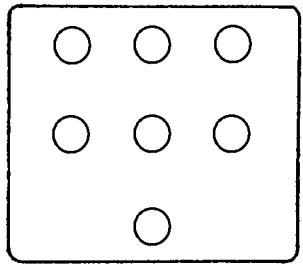
he will never finish the problem. To help him change his point of view it is a good idea to change the problem a little, as follows. Ask him to divide 7 mangoes equally between himself and another pupil, and tell you how many mangoes he gets. Now he is almost certain to produce the answer. He will say, "I get 3 whole mangoes and one-half a mango". He thinks of cutting the last mango in half and giving himself one of the pieces. He does this because he knows that *mangoes* can be cut in half. He didn't do it with the original problem $7 \div 2$ because no one had told him that whole numbers can be "cut in half".

To help him, draw a picture like this:

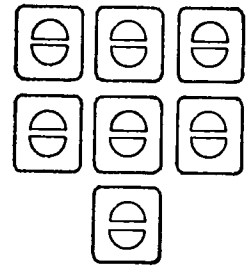


There are 7 objects, and he is to get one-half of each. How many does he get? Seven halves.

You can also draw the following picture:



One set of 7



7 sets of 2-halves each

This shows that to divide a set containing 7 elements into 2 equal subsets we can remove *halves* two at a time. We get 7 subsets of 2-halves each. Thus $7 \div 2 = \frac{7}{2}$.

We now show this with equations. We begin with the division equation

$$7 \div 2 = \square$$

and the corresponding multiplication equation

$$2 \times \square = 7$$

This equation tells us to put a number in the box so that twice that number equals 7. Since we are multiplying by 2 we try to fill the box with a certain number of *halves*. We can show this by writing the equation

$$2 \times \triangle\text{-halves} = 7$$

But when 7 objects are divided into halves we get 14 halves. Thus we can replace the 7 in this equation by 14-halves. This gives

$$2 \times \triangle\text{-halves} = 14\text{-halves}$$

Our problem now looks like

$$2 \times \triangle = 14$$

The only difference is that the $\triangle$ and the 14 now refer to *halves*. Since we must put 7 in the $\triangle$ to get a true sentence, we must put $\frac{7}{2}$ in the box to make $2 \times \square = 7$ a true sentence. This shows that

$$7 \div 2 = \tfrac{7}{2}$$

Notice that now there is no remainder because the division is complete.

We have gone through this problem very slowly because you will have to go very slowly when you explain it to your pupils. You will also have to give many problems like this until you are sure the idea is understood. Here are a few more problems of this type.

Problem

8 pieces of candy are to be shared between 3 children. What is each child's share?

This time the answer must be in *thirds* because there are 3 children. The division equation is

$$8 \div 3 = \square$$

The corresponding multiplication equation is

$$3 \times \square = 8$$

We now write this equation in thirds as

$$3 \times \triangle\text{-thirds} = 24\text{-thirds}.$$

To make this a true sentence we must put 8 in the triangle. This shows that $\square = \frac{8}{3}$. Thus

$$8 \div 3 = \tfrac{8}{3}$$

Each child gets $\frac{8}{3}$ or $2\frac{2}{3}$ pieces of candy.

Problem

Kwese has 12 crayons. How many children can get 5 crayons each?

The equations are

$$12 \div 5 = \square \quad \text{or} \quad 5 \times \square = 12$$

$$5 \times \triangle\text{-fifths} = 12$$

$$5 \times \underline{\overset{\triangle}{12}}\text{-fifths} = 12$$

Thus $\qquad 5 \times \frac{12}{5} = 12$

and $\qquad 12 \div 5 = \frac{12}{5}$

The question was, how many children can get 5 crayons each? The answer is $\frac{12}{5}$ or $2\frac{2}{5}$ children. This is not a very realistic problem, and here the answer would be better with a remainder. Two children can have 5 crayons each and there will be 2 crayons left over. This shows that it is better to use sharing problems to explain fractions when you first introduce them to children. You can cut up a cake or a banana but not a child.

By now you will see the pattern that links the answer with the question. Here are some more examples of division equations

$$2 \div 7 = \tfrac{2}{7}$$
$$2 \div 8 = \tfrac{2}{8}$$
$$8 \div 5 = \tfrac{8}{5}$$
$$100 \div 101 = \tfrac{100}{101}$$

In general, if a and b are counting numbers, then $a \div b = \dfrac{a}{b}$.

EXERCISE 18-5

Solve the following division problems giving each answer in 2 ways, as a fraction and as a mixed number.

1. $72 \div 7$ **2.** $13 \div 4$ **3.** $96 \div 6$

4. $147 \div 24$ **5.** $2{,}176 \div 322$ **6.** $625 \div 10$

7. $83 \div 83$ **8.** $1{,}215 \div 100$

9. Give a word problem for 1 and 2 above. Make your problems realistic.
10. Explain why $1 \div 3 = \frac{1}{3}$
11. Explain why $8 \div 5 = \frac{8}{5}$

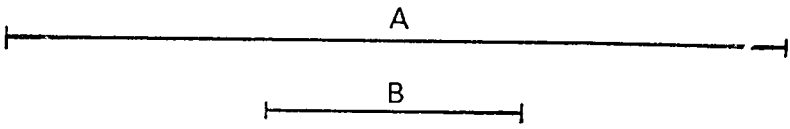
Draw a picture to show the meaning of the following problems and write the answer to each in words:

12. How many oranges will you receive if you share 13 oranges among yourself and 3 friends?
13. Kwese gives out 12 pencils to some children. How many children can have 5 pencils each?

18-6 An Example

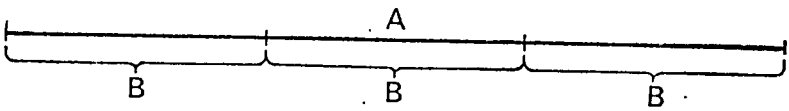
We now consider another type of problem which leads to the idea of a fraction. This time the problem involves comparing the lengths of line segments.

Suppose we are given the line segments shown below. We can imagine that these segments represent two sticks.



Suppose someone tells us that segment B is 1 unit long, and then asks us to find the length of A. For instance, B could be a ruler 1 foot long, or a ruler 1 yard long. The unit of length we use really does not matter.

To solve the problem we must find the number of segments of length B that are needed to get A. Thus we start by laying B off along A as many times as possible. The following picture shows what happens:

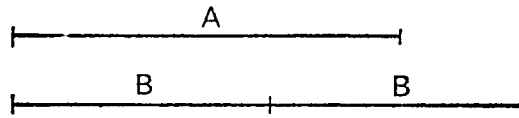


The picture shows that A is three times as long as B. Thus

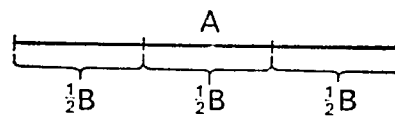
$$A = B + B + B \quad \text{or} \quad A = 3B$$

We say that A is 3 B-units long, or that the length of A is 3.

All this is simple enough as long as we can measure A *exactly* by laying B along A a certain number of times. But suppose this does not happen. Suppose, for instance, that A is the segment shown in the following picture:

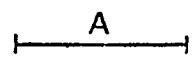


Here A is longer than B, but shorter than 2B. We cannot measure A exactly by using units of length B. However, we can say that the length of A is between 1 and 2. This suggests that we try to measure A by using *part* of the B segment instead of all of it. But what part of B should we use? The first thing to try is to cut B into two equal parts, and use one of them to try to measure A. In other words we now try to use one-half of B as our unit of length. If you try this with the A segment shown above you will find that A can be measured exactly by laying one-half of B 3 times along A.

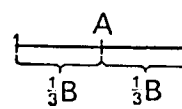


Thus A is 3 times as long as $\frac{1}{2}$B. In this case we say that the length of A is $\frac{3}{2}$.

It is easy to give examples like this. For instance, in the following picture A is shorter than B. It is also longer than $\frac{1}{2}$B. This



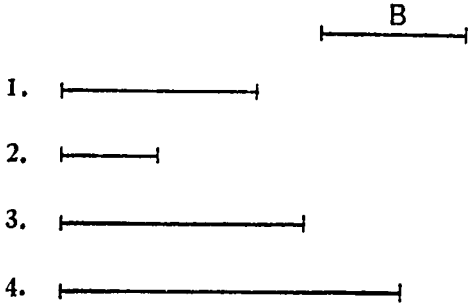
time we cannot measure A exactly by using one-half of B as our unit of length. We must divide B into more pieces—all of equal length—to have a smaller piece of B to measure with. We try dividing B into three equal pieces. Then each piece is one of three equal parts of B, or *one-third* of B. Using one of these pieces A can be measured exactly by laying it two times along A:



Thus A is *two times* as long as $\frac{1}{3}$B. We say that the length of A is $\frac{2}{3}$.

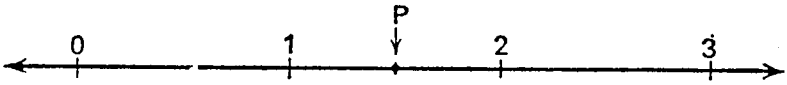
EXERCISE 18-6

Measure each of the following segments in terms of

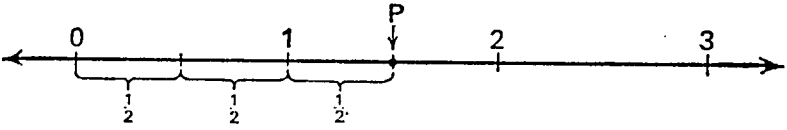


18-7 The Number Line

Suppose you wanted to describe the location of point P on the number line shown below.



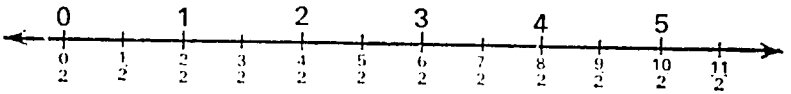
P lies somewhere between 1 and 2. To locate P precisely we must measure the segment from 0 to P in terms of the unit segment from 0 to 1. But this is the very problem we considered a moment ago. To solve it we divide the segment from 0 to 1 into parts of equal length and use one of them to measure with. Suppose we take one-half of the segment from 0 to 1. The following picture shows what happens:



We reach point P in three steps. Thus, exactly as before, we say that P is the point "three-halves", which we write $\frac{3}{2}$. This tells us precisely where P is found on the number line.

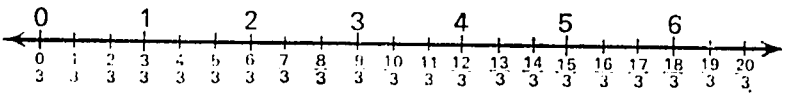
By using one-half of the unit segment from 0 to 1 this way we

can locate and name many new points on the number line. The following picture shows some of these points:



In each of the symbols $\frac{0}{2}$, $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$, etc., the number on top, the numerator, tells how many units of length one-half must be used to get to that point, starting from 0. For instance, $\frac{0}{2}$ is the point *zero* units of length $\frac{1}{2}$ from 0. This is just the point 0 again. Thus $\frac{0}{2}$ is another name for 0. Notice that $\frac{2}{2}$ is the same point as 1, that $\frac{4}{2}$ is the same point as 2, and so forth. Can you explain why this happens?

If we now do the same thing, using one-third of the original unit of length instead of one-half, we can locate and name still more points on the number line. This time our picture is



Continuing in this way using one-fourth of the original unit, one-fifth, etc., we can locate and name more and more points on the number line.

EXERCISE 18-7

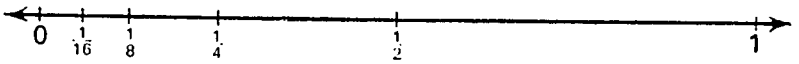
1. Draw the number line and mark the points from 0 to 4. Divide your unit of length into fourths, and locate as many points as you can on your number line.
2. Show each of the following points on the number line you drew for Question 1.

   a. $A = 1\frac{3}{4}$ **b.** $B = 3\frac{1}{4}$ **c.** $C = 2\frac{1}{2}$
   d. $D = \frac{1}{2}$

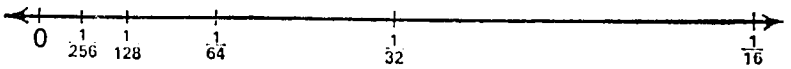
18-8 The number line (continued)

We can use the number line to show two very important properties of fractions. Draw a number line about 5 inches long and let your unit of length be 4 inches. Mark the points $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$.

Your picture should look like this:



Now draw another number line of the same length, but this time let 4 inches represent the distance from 0 to $\frac{1}{16}$. Mark the points $\frac{1}{32}$, $\frac{1}{64}$, $\frac{1}{128}$, $\frac{1}{256}$. Each of these points is half way between 0 and the last point marked on your number line. Thus your picture is



Now suppose you continued doing this. You find that you can keep on going as long as you wish. Every time you locate a point you can go on to locate another point. This new point is always half way between 0 and the last point you located. Notice that you *never* reach 0 this way. This shows that THERE IS NO SMALLEST FRACTION. Indeed if you are given any fraction different from zero you can always locate it on the number line and then find a smaller fraction by moving half way to 0.

Actually, we can say more than this. For suppose we are given *any* two fractions. Locate them on the number line, and then find the point half way between them. When this point is named it will give us a fraction *between* the two fractions we started with. Thus

BETWEEN EVERY TWO FRACTIONS THERE IS ANOTHER FRACTION.

In Chapter 20 you will learn how to find such a fraction.

From this we see that the set of fractions is non-ending. There is no last element in the set, and between any two fractions there is a non-ending set of intermediate fractions.

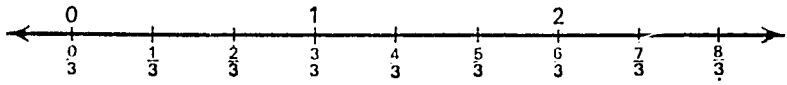
EXERCISE 18-8

1. Find three fractions smaller than $\frac{1}{256}$. Show these fractions on the number line.
2. Find five fractions which are different from 0 and smaller than $\frac{1}{3}$. Show these fractions on the number line.
3. Find five fractions between $\frac{1}{2}$ and 1. Show these fractions on the number line.

18-9 Order properties

When you learned about the number line earlier, you saw that it shows the whole numbers in their natural order, starting with 0. You saw that the statement $5 < 8$ means 5 is to the left of 8 on the number line. Another reason why 5 is less than 8 is, of course, that a set of 5 things has fewer members than a set of 8 things.

You can do the same sort of thing with fractions. Look at the number line drawn below, where each unit piece is divided into three equal-sized parts, and the fractions are labelled



From this you can say that $\frac{1}{3} < \frac{2}{3}$, because $\frac{1}{3}$ is to the left of $\frac{2}{3}$ on the line. Notice that $\frac{1}{3} < \frac{2}{3}$ can also be pictured in terms of sets. If a thing is divided into thirds, the set of 1 of those thirds has fewer members than the set of 2 of those thirds.

In a similar way you can say that one fraction is greater than another. On the number line you can see that $\frac{7}{3} > \frac{4}{3}$, because $\frac{7}{3}$ is to the right of $\frac{4}{3}$. Moreover, by looking at the number line, you can see that 5 is greater than 4 and less than $\frac{16}{3}$, and thus 5 lies between 4 and $\frac{16}{3}$.

Exercise 18-9

In Questions 1, 2, 3 draw a number line using 2 inches as the unit of distance.

1. Mark the points which represent $\frac{5}{6}$ and $\frac{3}{4}$, and then say which of the following statements is true:

$$\frac{5}{6} < \frac{3}{4} \quad \frac{5}{6} = \frac{3}{4} \quad \frac{5}{6} > \frac{3}{4}$$

[*Hint:* Mark the line to show twelfths.]

2. Mark the points which represent $\frac{5}{2}$ and $\frac{5}{3}$, and then say which of the following statements is true:

$$\frac{5}{2} < \frac{5}{3} \quad \frac{5}{2} = \frac{5}{3} \quad \frac{5}{2} > \frac{5}{3}$$

3. Mark the points which represent $\frac{2}{3}$ and $\frac{8}{12}$, and then put the correct sign in the box:

$$\frac{2}{3} \square \frac{8}{12}$$

4. Which is greater

a. $\frac{1}{2}$ or $\frac{1}{3}$? **b.** $\frac{3}{2}$ or $\frac{3}{3}$?

c. $\frac{1}{4}$ or $\frac{1}{5}$? **d.** $\frac{9}{4}$ or $\frac{9}{5}$?

e. $\frac{1}{100}$ or $\frac{1}{137}$? **f.** $\frac{300}{100}$ or $\frac{300}{137}$?

5. Let $\frac{a}{b}$ and $\frac{a}{c}$ be fractions with a difference from 0 and $b < c$. Which of the following statements is true?

$$\frac{a}{b} < \frac{a}{c} \quad \frac{a}{b} = \frac{a}{c} \quad \frac{a}{b} > \frac{a}{c}$$

6. Explain how you could make a number line to demonstrate fractions in a classroom without a blackboard.

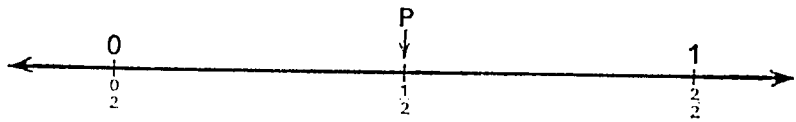
Chapter 19

PROPERTIES OF FRACTIONS

19-1 Equivalent fractions

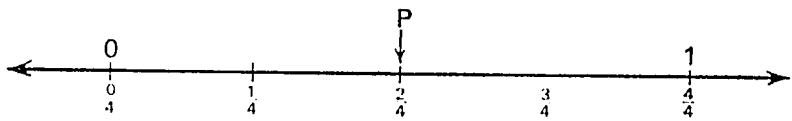
In the last chapter you probably noticed that there are different ways of writing the same fraction. We can show why this happens by looking at the number line.

Consider the point P in the following picture:



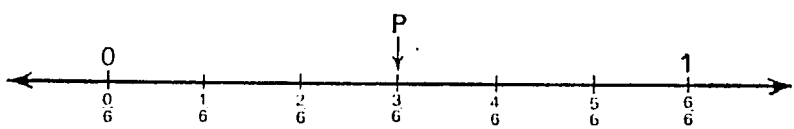
P is halfway between 0 and 1. Therefore it is the point "one-half" on the number line, and has been labelled $\frac{1}{2}$.

But P can be named in many other ways as well. For instance, if we divide the segment from 0 to 1 into four equal parts and label the points starting at 0 we have the picture



This time P is the point "two-fourths". But P has not changed. Only its name has changed. Thus "one-half" and "two-fourths" are different names for the same point on the number line. We describe this by saying that $\frac{1}{2}$ and $\frac{2}{4}$ are *equivalent fractions.*

Our next picture shows that P can also be labelled $\frac{3}{6}$:



This gives us three names for P: $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}$. We say that these three fractions are equivalent.

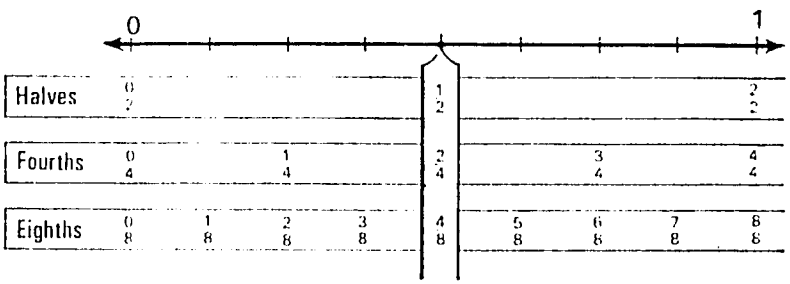
If we continue like this, dividing the segment between 0 and 1

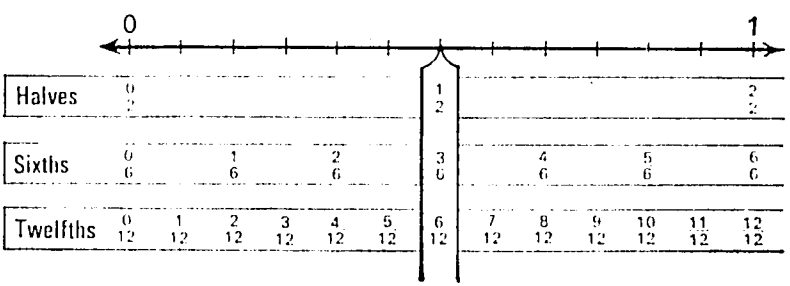
into more and more pieces all of the same size, we can label P as $\frac{4}{8}$, $\frac{5}{10}$, $\frac{6}{12}$, and so on. There is no end to the list of fractions which name the point P. If we collect all these fractions together we get the set

$$\{\tfrac{1}{2}, \tfrac{2}{4}, \tfrac{3}{6}, \tfrac{4}{8}, \ldots\}$$

All of the fractions in this set name the same point on the number line. We therefore say that they are *equivalent fractions.*

The following pictures show several of these fractions at once. Such pictures help children understand the idea behind equivalent fractions:





Two fractions are EQUIVALENT when they name the *same* point on the number line.

Since $\frac{1}{2}$ and $\frac{2}{4}$ are at the same point on the number line we write $\frac{1}{2} = \frac{2}{4}$. For the same reason we write $\frac{1}{2} = \frac{3}{6}$, $\frac{1}{2} = \frac{4}{8}$, and so on. Thus

$$\tfrac{1}{2} = \tfrac{2}{4} = \tfrac{3}{6} = \tfrac{4}{8} = \ldots$$

Now that we have seen why the point "one-half" on the number line has many different names it is easy to show that the same thing must happen at *every* point on the number line.

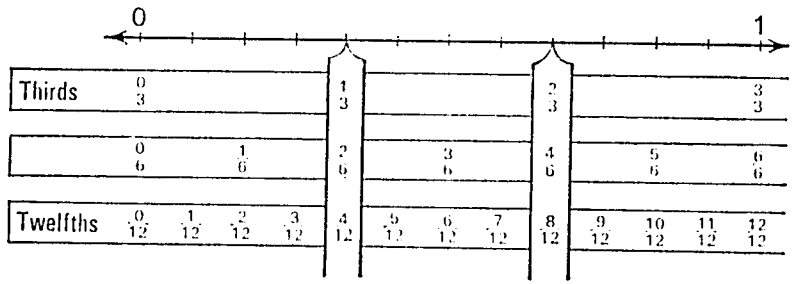
For instance, $\frac{1}{3}, \frac{2}{6}, \frac{3}{9}, \frac{4}{12}, \ldots$ all describe the same point on the number line. We therefore write

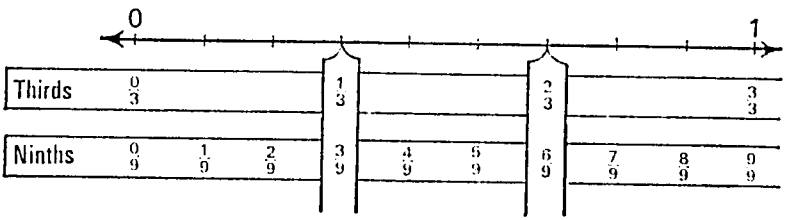
$$\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{4}{12} = \ldots$$

We say that all of these fractions are *equivalent*. Similarly, the fractions in the set

$$\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots\right\}$$

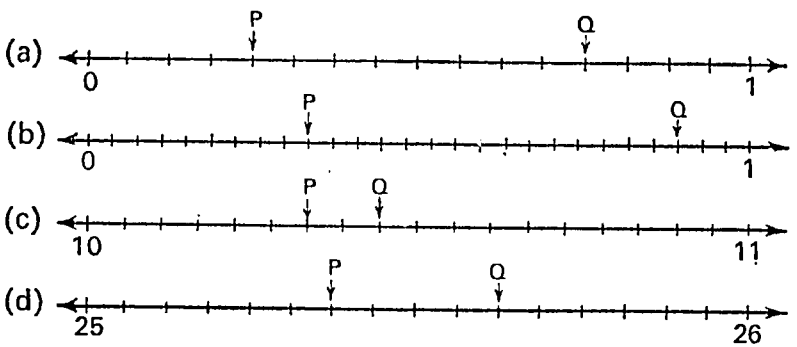
are equivalent because they all name the same point on the number line.





EXERCISE 19-1

1. Find at least two fractions which name the points P and Q on each of the following number lines:



2. **a.** Draw a picture of the number line which shows that the following fractions are equivalent: $\frac{2}{5}$, $\frac{4}{10}$, $\frac{6}{15}$.
   b. Without showing any more points on your number line find three more fractions which are equivalent to $\frac{2}{5}$.
3. **a.** Draw a number line from 0 to 1 and mark it to show halves, thirds, and sixths. Write down all sets of equivalent fractions you find on this line.
   b. Without showing any more points find a fraction with denominator 12 in each set of equivalent fractions on this line.
4. Find five fractions which are equivalent to each of the following fractions:
   a. $\frac{2}{3}$ **b.** $\frac{3}{4}$ **c.** $\frac{1}{6}$ **b.** $\frac{3}{2}$
5. We saw above that all of the fractions in the set

$$\left\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \ldots\right\}$$

   are equivalent. What are the next five fractions in this set? Which of the following fractions belong to this set?
   a. $\frac{25}{50}$ **b.** $\frac{201}{420}$ **c.** $\frac{968}{1936}$ **d.** $\frac{3000}{6000}$

19-2 Equivalent fractions (continued)

We can also illustrate the idea of equivalent fractions by looking at problems from everyday life. Consider the following example.

A man had a piece of sugar cane which he cut into 4 equal pieces for his children. But only 2 of the children came to get some. Thus each child got 2 pieces of sugar cane. We can write this as $\frac{2}{4}$ since each got 2 fourth parts. The following picture shows this:

When you look at this picture you see that each child got the same amount of sugar cane as he would if the man had cut the piece into two equal parts and had given each child one of them. In that case the picture would have been

This shows that the fractions $\frac{1}{2}$ and $\frac{2}{4}$ both tell how much sugar cane each child was given. They describe the same amount of sugar cane. Thus they are equivalent fractions and we write $\frac{1}{2} = \frac{2}{4}$.

Now suppose that the man had 6 children instead of 4. Then he would have divided the sugar cane into 6 equal pieces, and the 2 children who came to get some would have been given 3 pieces each. This time the picture is

But each child's share of the sugar cane is still the same. Therefore $\frac{3}{6}$ is also equivalent to $\frac{1}{2}$ and to $\frac{2}{4}$, and we write

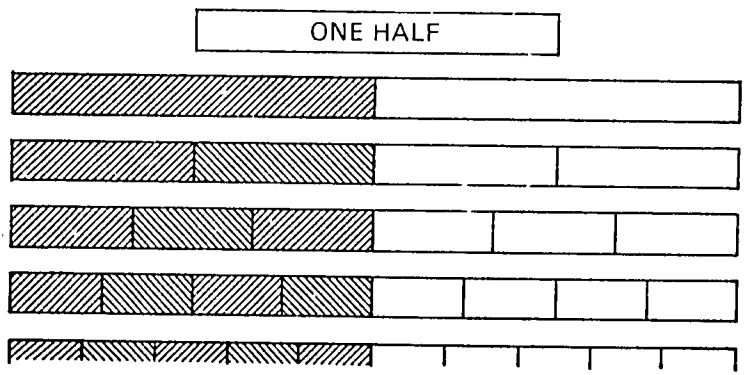
$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6}$$

Notice that although the total amount of sugar cane given to each child was the same both times, in the first case each received two pieces while in the second case each received three somewhat smaller pieces. In the first case each received 2 fourth parts; in the second case each received 3 sixth parts. This is one reason why we say that the fractions $\frac{2}{4}$ and $\frac{3}{6}$ are "equivalent" rather than "equal".

In exactly the same way we can show that $\frac{4}{8}$ is equivalent to $\frac{1}{2}$; that $\frac{5}{10}$ is equivalent to $\frac{1}{2}$; and so on. In other words all of the fractions in the set

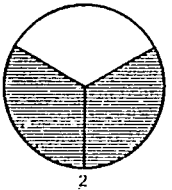
$$\{\tfrac{1}{2}, \tfrac{2}{4}, \tfrac{3}{6}, \tfrac{4}{8}, \ldots\}$$

are equivalent. The following picture shows all of this at once.

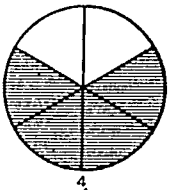


By dividing objects into parts like this we can show that every fraction has many other fractions which are equivalent to it. Moreover we can also find a rule for changing a fraction into any one of its equivalent fractions.

Take the fraction $\frac{2}{3}$ for example. By cutting a circle into 3 equal pieces and shading 2 of the pieces we get a picture which represents $\frac{2}{3}$.



$\frac{2}{3}$



$\frac{4}{6}$

Now suppose each piece of this article is cut into 2 equal pieces. Then there will be 6 pieces in all, and 4 will be shaded. Thus the shaded portion of the circle now represents the fraction $\frac{4}{6}$. This shows that $\frac{2}{3}$ and $\frac{4}{6}$ are equivalent fractions. Notice that in going from $\frac{2}{3}$ to $\frac{4}{6}$ both the numerator and denominator in $\frac{2}{3}$ were multiplied by 2. Hence this pair of equivalent fractions can be written

$$\frac{2}{3} = \frac{2 \times 2}{3 \times 2}$$

If we do the same thing again, this time cutting each piece of the original circle into thirds, there will be 9 pieces in all, 6 of them shaded. Thus $\frac{6}{9}$ is equivalent to $\frac{2}{3}$, and we have

$$\frac{2}{3} = \frac{2 \times 3}{3 \times 3}$$

As we keep cutting the original pieces further and further we get more and more fractions equivalent to $\frac{2}{3}$. For instance, when each piece is cut into fourths we get

$$\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12}$$

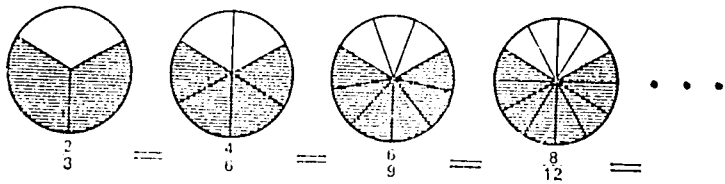
When each piece is cut into fifths we get

$$\frac{2}{3} = \frac{2 \times 5}{3 \times 5} = \frac{10}{15}$$

In general, if n is any counting number we have

$$\frac{2}{3} = \frac{2 \times n}{3 \times n}$$

The number n tells us how many pieces we have made out of each original piece of the circle. The following picture shows what happens:



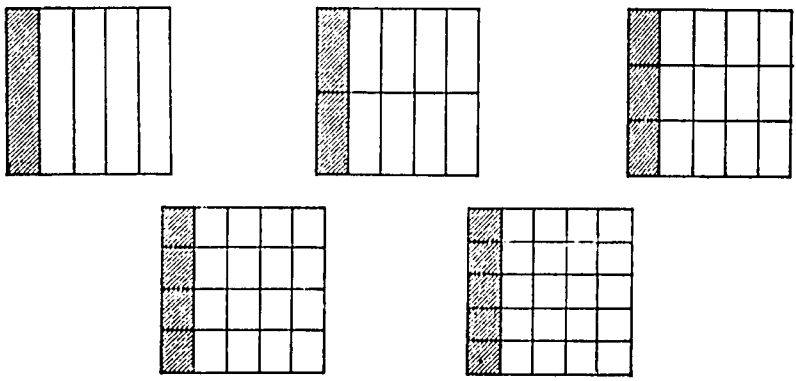
EXERCISE 19-2

1. Draw circles divided into pieces to show that the following fractions are equivalent:

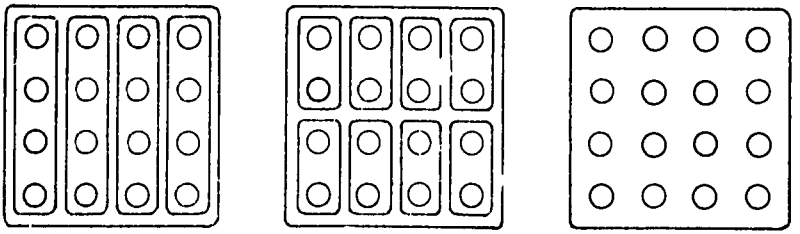
$$\tfrac{1}{4}, \tfrac{2}{8}, \tfrac{3}{12}$$

   Also show this on the number line.

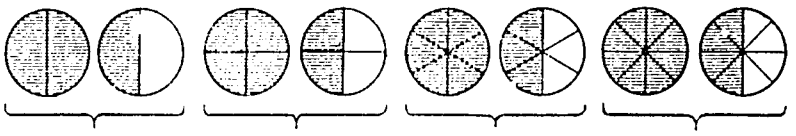
2. What equivalences for fractions are shown in the following pictures?



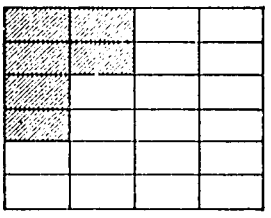
3. What equivalences for fractions are shown in the following pictures?



4. What equivalences for fractions are shown by the following pictures?



5. Consider the following picture.



a. Find all equivalent fractions suggested by this picture when you compare the shaded pieces with the entire rectangle. [*Hint:* There are 4 fractions in all.]

b. If the area of the entire rectangle is 70 square units, what is the area of the shaded part?

c. If the area of the shaded part is 25 square units, what is the area of the entire rectangle?

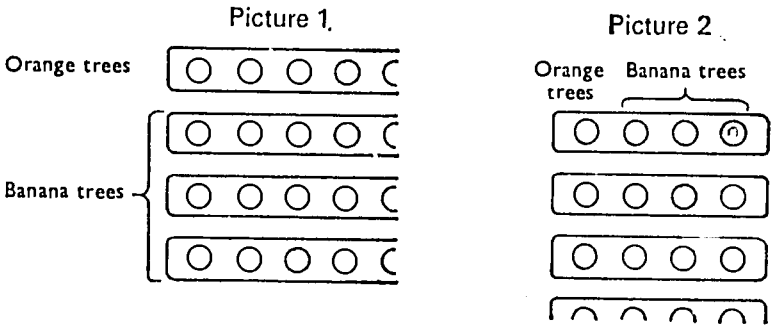
6. Find five fractions which are equivalent to each of the following:

a. $\frac{2}{9}$ **b.** $\frac{7}{8}$ **c.** $1\frac{3}{4}$ **d.** $5\frac{1}{2}$ **e.** $\frac{21}{6}$

7. Which of the following fractions are equivalent to $\frac{4}{3}$, and which are not? Explain your answers.

$$\frac{8}{6},\ \frac{12}{6},\ \frac{24}{15},\ \frac{28}{21},\ \frac{44}{33},\ \frac{4{,}000{,}000}{3{,}000{,}000}$$

8. Suppose you are told that on a certain farm one-fourth of the trees are orange trees, and the rest are banana trees. You could show this by drawing either one of the following pictures.



In the first picture you say to yourself: "There are four sets of trees with the same number of trees in each set. I do not know how many trees are in any set, but one set consists of orange trees and the other three sets consist of banana trees."

In the second picture you say to yourself: "I do not know how many sets of 4 trees there are, but in each set of 4 one is an orange tree and 3 are banana trees."

a. Suppose there were 100 trees in all. How many are orange trees? How many are banana trees? What fraction of the trees are orange trees? What fraction are banana trees? (Express your answers in hundredths.)

b. Answer the questions in (*a*) when there are 600 trees. Express your answers this time in six-hundredths.

c. Could there be 750 trees on the farm? Why?

d. What equivalent fractions are shown by this example? Explain why these fractions are equivalent.

19-3 The equivalence property of fractions

In the last section we saw that if the numerator and denominator of a fraction are both multiplied by the same counting number we get an equivalent fraction. For example, $\frac{2}{6}$ is equivalent to $\frac{1}{3}$ because

$$\frac{2}{6} = \frac{1 \times 2}{3 \times 2}.$$

Similarly $\frac{1 \times 3}{3 \times 3} = \frac{3}{9}$, $\frac{1 \times 4}{3 \times 4} = \frac{4}{12}$, $\frac{1 \times 5}{3 \times 5} = \frac{5}{15}, \ldots$

are all equivalent to $\frac{1}{3}$.

This rule works in the other direction as well;

$$\frac{6}{18} = \frac{6 \div 6}{18 \div 6} = \frac{1}{3}$$

$$\frac{9}{27} = \frac{9 \div 9}{27 \div 9} = \frac{1}{3}$$

and so on. Here we simply divide the numerator and denominator by a counting number which is a factor of both.

This property of fractions is known as the equivalence property, and we now write it out in full.

THE EQUIVALENCE PROPERTY OF FRACTIONS

If the numerator and denominator of a fraction are both multiplied or divided by the same counting number the result is equivalent to the original fraction.

This property is useful because it tells us how to find all fractions which are equivalent to a given fraction, $\frac{a}{b}$. This is done in two steps, as follows:

Step 1. Divide the numerator and denominator of $\frac{a}{b}$ by their greatest common factor. This gives a fraction which is equivalent to $\frac{a}{b}$ but with numerator and denominator as small as possible. We say that such a fraction is in LOWEST TERMS.

Step 2. Starting with the fraction in lowest terms multiply its numerator and denominator by 2, 3, 4, and so on. This gives the entire set of fractions equivalent to $\frac{a}{b}$.

Example

Let $\frac{a}{b}$ be $\frac{28}{42}$. To put this fraction in lowest terms we start by

dividing 28 and 42 by 2. This gives the equivalent fraction $\frac{14}{21}$. But the numerator and denominator of $\frac{14}{21}$ can both be divided by 7. Doing this we get $\frac{2}{3}$. This is as far as we can go since 2 and 3 have no common factor greater than 1. Thus $\frac{2}{3}$ is the fraction in lowest terms which is equivalent to $\frac{28}{42}$.

We now multiply the numerator and denominator of $\frac{2}{3}$ by 2, 3, 4, and so on. This gives

$$\frac{2 \times 2}{3 \times 2} = \frac{4}{6}, \quad \frac{2 \times 3}{3 \times 3} = \frac{6}{9}, \quad \frac{2 \times 4}{3 \times 4} = \frac{8}{12}, \ldots$$

Thus the set of fractions equivalent to $\frac{28}{42}$ is

$$\left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots, \frac{28}{42}, \ldots\right\}.$$

Sometimes it is difficult to tell whether the numerator and denominator of a fraction have a common factor. When this happens you must break each number up into its prime factors. Once this has been done it is easy to find the greatest common factor.

Take $\frac{225}{252}$ for example. We first find the prime factors of 225. Since 225 is not even, 2 is not a factor. Thus we begin with 3:

$$\begin{aligned} 225 \div 3 &= 75 \quad \text{so} \quad 225 = 3 \times 75 \\ 75 \div 3 &= 25 \quad \text{so} \quad 225 = 3 \times 3 \times 25 \\ 25 \div 5 &= 5 \quad \text{so} \quad 225 = 3 \times 3 \times 5 \times 5 \end{aligned}$$

Since all of these factors are prime numbers, we have finished with 225. As for 252, we have

$$\begin{aligned} 252 \div 2 &= 126 \quad \text{so} \quad 252 = 2 \times 126 \\ 126 \div 2 &= 63 \quad \text{so} \quad 252 = 2 \times 2 \times 63 \\ 63 \div 3 &= 21 \quad \text{so} \quad 252 = 2 \times 2 \times 3 \times 21 \\ 21 \div 3 &= 7 \quad \text{so} \quad 252 = 2 \times 2 \times 3 \times 3 \times 7 \end{aligned}$$

Thus
$$\frac{225}{252} = \frac{3 \times 3 \times 5 \times 5}{2 \times 2 \times 3 \times 3 \times 7}$$

Looking at the numerator and denominator of this last fraction we see that 3 is the only factor which occurs in both. Since it occurs twice, 3×3 is the greatest common factor of 225 and 252. Dividing by it we get

$$\frac{5 \times 5}{2 \times 2 \times 7} = \frac{25}{28}$$

Thus $\frac{25}{28}$ is the fraction in lowest terms which is equivalent to $\frac{225}{252}$.

EXERCISE 19-3

1. Find the set of fractions equivalent to each of the following:
   a. $\frac{1}{5}$ **b.** $\frac{2}{7}$ **c.** $\frac{5}{4}$ **d.** $\frac{4}{9}$ **e.** $\frac{31}{6}$
2. **a.** Find the fraction in lowest terms which is equivalent to each of the following:
   (i) $\frac{26}{39}$ (ii) $\frac{72}{36}$ (iii) $\frac{315}{210}$ (iv) $\frac{91}{247}$ (v) $\frac{3,003}{1,683}$
   b. Find the set of fractions equivalent to each of these fractions.
3. **a.** Find the fraction in lowest terms which is equivalent to $\frac{4}{2}$.
   b. Find all fractions equivalent to $\frac{4}{2}$.
   c. Show the point on the number line named by these fractions. What is another name for this point?
4. **a.** Find all fractions equivalent to $\frac{0}{1}$.
   b. Show the point on the number line named by these fractions. What is another name for this point?
5. Find the number to put in the box to make each of the following sentences true:
   a. $\frac{2}{5} = \frac{\square}{10}$ **b.** $\frac{5}{12} = \frac{\square}{60}$
   c. $\frac{5}{12} = \frac{35}{\square}$ **d.** $\frac{24}{64} = \frac{\square}{256}$
   e. $\frac{8}{12} = \frac{\square}{9}$ **f.** $\frac{12}{28} = \frac{\square}{35}$
6. Is it possible to find a counting number which makes each of the following sentences true? If it is, find the number. If it is not, explain why.
   a. $\frac{8}{12} = \frac{\square}{5}$ **b.** $\frac{8}{12} = \frac{\square}{15}$
   c. $\frac{15}{21} = \frac{65}{\square}$ **d.** $\frac{15}{21} = \frac{48}{\square}$

19-4 Tests for equivalent fractions

Two fractions are equivalent if they name the same point on the number line. For instance, $\frac{2}{4}$ and $\frac{3}{6}$ are equivalent because they both name the point $\frac{1}{2}$ on the number line. One way to show this is to put both fractions in lowest terms:

$$\frac{2}{4} = \frac{2 \div 2}{4 \div 2} = \frac{1}{2}$$

$$\frac{3}{6} = \frac{3 \div 3}{6 \div 3} = \frac{1}{2}$$

Since the result is the same both times, the fractions are equivalent. Similarly $\frac{9}{12}$ and $\frac{21}{28}$ are equivalent since

$$\frac{9}{12} = \frac{9 \div 3}{12 \div 3} = \frac{3}{4}$$

$$\frac{21}{28} = \frac{21 \div 7}{28 \div 7} = \frac{3}{4}$$

When written in lowest terms both of these fractions have the same point in the number line. Thus they are equivalent.

This method works for any two fractions, and gives us a way of testing fractions for equivalence:

Two fractions are equivalent if they are the same when written in lowest terms. If the fractions are not the same when written in lowest terms, then they are not equivalent.

Although this test can always be used to tell whether or not two fractions are equivalent it is difficult to apply when the fractions involve large numbers. For instance, the fractions $\frac{182}{418}$ and $\frac{637}{4{,}389}$ are equivalent, but it would be hard to show this by the above test. For this reason we look for an easier way to test fractions for equivalence.

Consider the pair of equivalent fractions $\frac{2}{4}$ and $\frac{3}{6}$. If we multiply the numerator and denominator of $\frac{2}{4}$ by 6 we get the equivalent fraction $\frac{12}{24}$. If we multiply the numerator and denominator of $\frac{3}{6}$ by 4 we get the equivalent fraction $\frac{12}{24}$. Thus $\frac{2}{4}$ and $\frac{3}{6}$ are both equivalent to $\frac{12}{24}$. They are therefore equivalent to each other.

Why did we select 6 and 4 as the multipliers here? This was done to obtain two fractions with the *same* denominator. For once their denominators are equal two fractions will be equivalent only when their numerators are also equal. In this case $2 \times 6 = 3 \times 4$, so the fractions $\frac{2}{4}$ and $\frac{3}{6}$ are equivalent. This gives us a very easy test to determine when two fractions are equivalent. We just multiply the numerator of the first fraction by the denominator of the second fraction. Then we multiply the numerator of the second fraction by the denominator of the first fraction. If these two products are equal the fractions are equivalent. For example,

$$\tfrac{2}{4} \times \tfrac{3}{6} : 2 \times 6 = 12, \quad 3 \times 4 = 12$$

Thus the fractions $\frac{2}{4}$ and $\frac{3}{6}$ are equivalent.

Suppose we try this on some other fractions in the set

$$\{\tfrac{1}{2}, \tfrac{2}{4}, \tfrac{3}{6}, \tfrac{4}{8}, \ldots\}.$$

$$\tfrac{1}{2} \times \tfrac{3}{6} : 1 \times 6 = 6, \quad 3 \times 2 = 6$$

$$\tfrac{2}{4} \times \tfrac{4}{8} : 2 \times 8 = 16, \quad 4 \times 4 = 16$$

$$\tfrac{3}{6} \times \tfrac{15}{30} : 3 \times 30 = 90, \quad 15 \times 6 = 90$$

Both products are the same every time.

On the other hand, if we test $\frac{2}{3}$ and $\frac{3}{4}$ this way we get

$$\tfrac{2}{3} \times \tfrac{3}{4} : 2 \times 4 = 8, \quad 3 \times 3 = 9$$

Thus $\frac{2}{3}$ is not equivalent to $\frac{3}{4}$. In fact, $\frac{2}{3}$ is equivalent to $\frac{8}{12}$ and $\frac{3}{4}$ is equivalent to $\frac{9}{12}$. But $\frac{8}{12}$ is certainly not equivalent to $\frac{9}{12}$. Thus $\frac{2}{3}$ is not equivalent to $\frac{3}{4}$.

We have therefore devised a second method for testing fractions for equivalence:

Two fractions $\frac{a}{b}$ *and* $\frac{c}{d}$ *are equivalent if* $a \times d = b \times c$. *On the other hand, if* $a \times d$ *is not equal to* $b \times c$, *the fractions are not equivalent.*

EXERCISE 19-4

1. Use both of the tests given above to show that the following pairs of fractions are equivalent:

   a. $\frac{3}{9}, \frac{15}{45}$ **b.** $\frac{14}{40}, \frac{18}{63}$
   c. $\frac{8}{6}, \frac{20}{15}$ **d.** $\frac{99}{55}, \frac{117}{65}$
   e. $\frac{25}{8}, \frac{168}{64}$

2. Test the following pairs of fractions for equivalence:

   a. $\frac{7}{5}, \frac{12}{8}$ **b.** $\frac{2}{7}, \frac{24}{72}$
   c. $\frac{31}{54}, \frac{65}{93}$ **d.** $\frac{54}{72}, \frac{75}{100}$
   e. $\frac{156}{91}, \frac{180}{105}$

3. Show that $\frac{182}{418}$ and $\frac{637}{4,389}$ are equivalent fractions.

4. Is there a fraction which is equivalent to $\frac{5}{9}$ *and* $\frac{4}{11}$? Explain your answer.

19-5 Fractions

We have seen that every point on the number line can be given many different names and can be represented by many different

symbols. For instance, every one of these symbols in the set

$$\{\tfrac{1}{2}, \tfrac{2}{4}, \tfrac{3}{6}, \tfrac{4}{8}, \ldots\}$$

represents the same point on the number line. But every point on the number line represents a single number. Thus all of the symbols in the above set represent different names for the same number. This number is called a FRACTIONAL NUMBER, or for short, a FRACTION. Some of the names for this fraction are "one-half", "two-fourths", "three-sixths". But it also has many, many more.

In the same way "one-third", "two-sixths", "three-ninths", etc., all name a single fractional number since they all describe the same point on the number line. This number is the fraction represented by the symbols

$$\tfrac{1}{3}, \tfrac{2}{6}, \tfrac{3}{9}, \tfrac{4}{12}, \ldots$$

It is important to realize that a fraction has many different names and that each of these names is represented by a different symbol which we call a *fractional numeral.* Thus we have

ONE FRACTION
but
MANY NAMES
and
MANY SYMBOLS

At first it may seem strange to have so many names and symbols for a single fraction. But as we have already seen, the same thing happens for whole numbers as well. For instance, the number 10 can be represented by any one of the symbols

$$1 + 9$$
$$2 + 8$$
$$3 + 7$$

and so forth. It can also be represented by the symbols

$$11 - 1$$
$$12 - 2$$
$$13 - 3$$

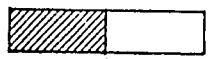
Thus here too we find many different names and symbols for a single number.

In talking about fractions we are usually a little careless and say such things as "the fraction one-half" when we really should say "the fraction *one of whose names* is one-half". This causes no difficulty so long as we realize that "the fraction one-half" has many other names as well. It is the same as "the fraction two-fourths", "the fraction four-eighths", and so forth. We only get into trouble if we believe that because these names are different the numbers they name are different. This, as we have seen, is *false*. In teaching children about fractions you must keep them from making this mistake. You can let them talk about the fraction one-half, and the fraction two-fourths, etc. But you must be sure that they realize they are talking about the same fraction each time.

When we talk about the symbols which represent fractions we are also a little careless. Thus almost everyone says "Consider the fraction $\frac{1}{2}$" instead of "Consider the fraction represented by the symbol $\frac{1}{2}$." Here again there is no difficulty so long as you remember that different symbols can represent the same fraction. Thus $\frac{1}{2}$ and $\frac{2}{4}$ are not different fractions; they are different symbols for the same fraction. In fact, as we have seen, every fraction can be represented by an infinite set of symbols.

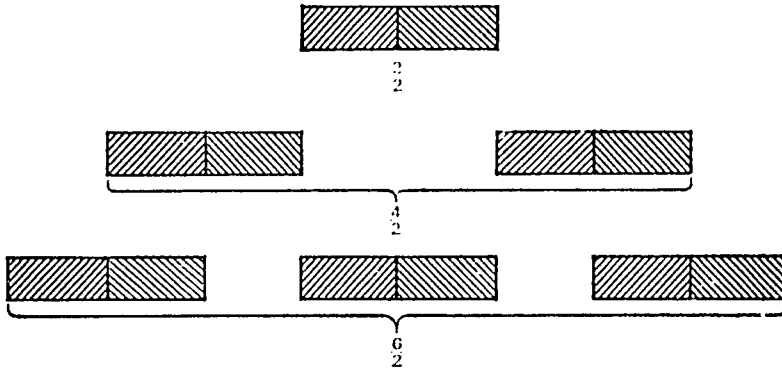
19-6 Whole numbers as fractions

At the beginning of the last chapter we introduced the names for fractions by looking at objects divided into equal parts. For instance we said that "one-half" stands for the idea of choosing one part of an object that has been divided into two equal parts. We illustrated this by the picture



$\frac{1}{2}$

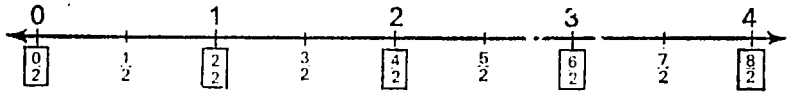
At the same time we introduced such numbers as $\frac{2}{2}$, $\frac{4}{2}$, $\frac{6}{2}$, etc. Here the pictures are as follows:



These pictures show that

2 halves are the same as 1 whole
4 halves are the same as 2 wholes
6 halves are the same as 3 wholes

Continuing in this way we find that every whole number can be given a fractional name. The following picture shows this on the number line:



It shows that the fractions $\frac{0}{2}, \frac{2}{2}, \frac{4}{2}, \frac{6}{2}, \ldots$, name the whole numbers on the number line.

We have seen that every fraction has many different names. For example, consider $\frac{2}{2}$. The first thing we notice about this fraction is that it is not in lowest terms. Both its numerator and denominator are divisible by 2. If we do this division we get the fraction $\frac{1}{1}$. Notice that $\frac{1}{1}$ is in lowest terms. Now suppose we multiply the numerator and denominator of $\frac{1}{1}$ by 2, 3, 4, and so on. This gives the set of equivalent fractions

$$\left\{\frac{1}{1}, \frac{2}{2}, \frac{3}{3}, \frac{4}{4}, \ldots\right\}$$

All of these fractions name the whole number 1 on the number line. They are *fractional names* for 1, and we write

$$1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \ldots$$

The same thing happens for the fractions $\frac{4}{2}$, $\frac{6}{2}$, and so forth.

For instance, if we divide the numerator and denominator of $\frac{4}{2}$ by 2 we get the fraction $\frac{2}{1}$ which is in lowest terms. Using it we build the set of equivalent fractions

$$\{\tfrac{2}{1}, \tfrac{4}{2}, \tfrac{6}{3}, \tfrac{8}{4}, \ldots\}$$

All of these fractions name the whole number 2 on the number line. Thus they are simply different names for 2, and we write

$$2 = \tfrac{2}{1} = \tfrac{4}{2} = \tfrac{6}{3} = \ldots$$

Similarly,

$$3 = \tfrac{3}{1} = \tfrac{6}{2} = \tfrac{9}{3} = \ldots$$
$$4 = \tfrac{4}{1} = \tfrac{8}{2} = \tfrac{12}{3} = \ldots$$

and, in general,

$$n = \frac{n}{1} = \frac{2n}{2} = \frac{3n}{3} = \ldots$$

for any whole number n.

Notice that in each of the above lists the fraction in lowest terms has a 1 in the denominator. These are the fractions $\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \frac{4}{1}, \ldots$ In writing such fractions we usually omit the 1 in the denominator. And when this is done these symbols look exactly like the ones we ordinarily use for the whole numbers.

19-7 The question of zero

Consider the point 0 on the number line. By what we have just said we can write

$$0 = \tfrac{0}{1} = \tfrac{0}{2} = \tfrac{0}{3} = \ldots$$

In other words, each of the fractions in the set $\{\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \ldots\}$ is a fractional name for 0. Let us see why this is so.

Consider the fraction $\frac{0}{2}$. Among other things this fraction represents the idea of choosing 0 parts of an object divided into 2 equal parts. How many parts do we have? Zero. The set of parts chosen is the empty set. Thus $\frac{0}{2}$ is just another way of naming the number 0. In the same way $\frac{0}{1}, \frac{0}{3}, \frac{0}{4}, \ldots$ all name the number 0.

We can also show this by looking at problems in division. In

general, we have seen that the fraction $\frac{a}{b}$ is the solution of the division problem

$$a \div b = \square$$

It is also the solution of the multiplication problem

$$b \times \square = a$$

Thus $\frac{3}{2}$ solves the equations

$$3 \div 2 = \square \quad \text{and} \quad 2 \times \square = 3$$

$\frac{4}{9}$ solves the equations

$$4 \div 9 = \square \quad \text{and} \quad 9 \times \square = 4$$

and so on.

Now suppose we consider the equation

$$0 \div 2 = \square$$

This time the corresponding multiplication equation is

$$2 \times \square = 0$$

Since the only number which can be put into the box to make this a true sentence is 0, we see that

$$0 \div 2 = 0$$

Hence we again have $\frac{0}{2} = 0$. Similarly, $\frac{0}{3} = 0$, $\frac{0}{4} = 0$, etc. In general, if b is any counting number,

$$\frac{0}{b} = 0$$

Finally, we ask whether a fraction can ever have 0 in its denominator. Does $\frac{a}{0}$ make sense for a counting number a? In the first place, if we interpret $\frac{a}{0}$ as meaning "choose a pieces of an object that has been divided into zero pieces" we see that we are talking nonsense. For if you have an object—any object at all—and divide it into pieces of equal size you will always have at least one piece. Thus the symbol $\frac{a}{0}$ is meaningless.

We can reach the same conclusion if we look at the multiplication and division equations which would have to go with the symbol $\frac{a}{0}$ if this symbol has any meaning. The division equation is

$$a \div 0 = \square$$

the multiplication equation is

$$0 \times \square = a$$

What number can we put in the box to make this a true sentence? Since the number a is a counting number it is different from 0. Thus no number will work because 0 times any number is 0.

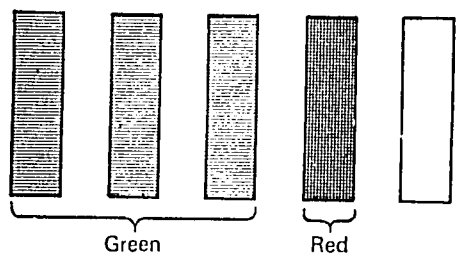
Chapter 20

ADDITION AND SUBTRACTION OF FRACTIONS

20-1 Addition of fractions

If fractions are to act like whole numbers we must be able to add, subtract, multiply and divide them. The first thing to think about is what it means to add fractions.

Suppose you have a cardboard square which has been cut into 5 strips of equal size as shown in the following picture:

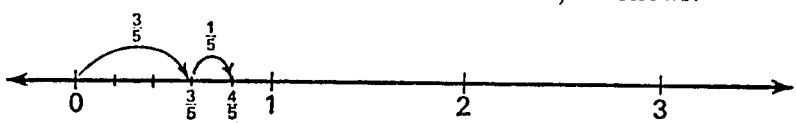


Suppose 3 of the strips are coloured green and 1 is coloured red. Then $\frac{3}{5}$ of the cardboard is green and $\frac{1}{5}$ is red. Altogether $\frac{4}{5}$ of the cardboard is coloured. We express this fact by writing

$$\frac{3}{5} + \frac{1}{5} = \frac{4}{5}$$

This equation says that if we add the $\frac{3}{5}$ of the cardboard which is green and the $\frac{1}{5}$ which is red we get the $\frac{4}{5}$ which is coloured.

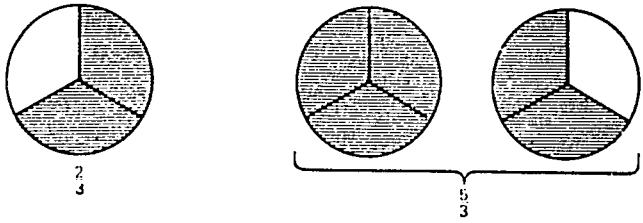
We can also show this on the number line, as follows:



This picture shows that if we start with three-fifths and add one more fifth we get a total of four-fifths. In other words,

$$\frac{3}{5} + \frac{1}{5} = \frac{4}{5}$$

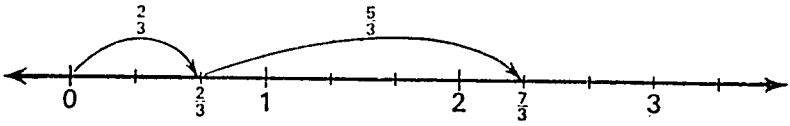
Next let us try to find the sum of $\frac{2}{3}$ and $\frac{5}{3}$:



Counting the number of thirds which are shaded in the above picture we find that there are 7 in all. This shows that

$$\tfrac{2}{3} + \tfrac{5}{3} = \tfrac{7}{3}$$

The following picture shows the same thing on the number line:



You will recall that addition on the number line is shown by marking off from 0 the distance corresponding to one of the numbers, and then continuing to the right for a distance corresponding to the other number. In the above picture we marked the distance from 0 to $\frac{2}{3}$ first. Then, starting at $\frac{2}{3}$ we marked off an additional $\frac{5}{3}$. The final point reached is $\frac{7}{3}$. Thus

$$\tfrac{2}{3} + \tfrac{5}{3} = \tfrac{7}{3}$$

(Do you recall the story of the cricket that did addition problems for whole numbers by jumping along the number line? The above picture shows that you can use this story to illustrate addition of fractions as well as addition of whole numbers.)

Notice that the addition problem $\frac{2}{3} + \frac{5}{3} = \frac{7}{3}$ can also be written using mixed numbers. It then reads

$$\tfrac{2}{3} + 1\tfrac{2}{3} = 2\tfrac{1}{3}$$

Children often find equations like this hard to understand. For this reason you should always change mixed numbers to improper fractions before you add. After the addition is completed you can write the answer as a mixed number if you wish.

For example, to add $1\frac{2}{5}$ and $2\frac{4}{5}$, we first write $1\frac{2}{5}$ as $\frac{7}{5}$, and $2\frac{4}{5}$ as $\frac{14}{5}$. Thus

$$1\tfrac{2}{5} + 2\tfrac{4}{5} = \tfrac{7}{5} + \tfrac{14}{5}$$
$$= \tfrac{21}{5}$$

Since $\frac{21}{5}$ is the same as $4\frac{1}{5}$, we can also write

$$1\tfrac{2}{5} + 2\tfrac{4}{5} = 4\tfrac{1}{5}$$

Either answer is correct.

By now you see what the sum of two fractions should be when the fractions have the same denominator. In the general case we have

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

In other words, to add two fractions with the same denominator we add their numerators and leave the denominator unchanged.

When teaching children how to add fractions according to this rule it is a good idea to use word problems as well as pictures to help them understand what happens. Here are two examples of problems you could use.

Example

A child sleeps 9 hours a day and plays for 4 hours. What part of the day does he spend sleeping, what part playing, and what part sleeping and playing? *Answer*: He sleeps $\frac{9}{24}$ of the day and plays $\frac{4}{24}$ of the day. Since $\frac{9}{24} + \frac{4}{24} = \frac{13}{24}$, he spends $\frac{13}{24}$ of the day sleeping and playing.

Example

A teacher has a set of 15 crayons. One child is using 6 of them and another child is using 5 of them. What part of the set is being used by each child, and what part by both together? *Answer*: The first child is using $\frac{6}{15}$ of the crayons; the second child is using $\frac{5}{15}$ of the crayons. Together they are using $\frac{6}{15} + \frac{5}{15} = \frac{11}{15}$ of the crayons. Thus $\frac{11}{15}$ of the set is being used.

EXERCISE 20-1

1. Draw pictures using squares, circles, and the number line to show each of the following additions. Then do the problems and compare your answers with your pictures.

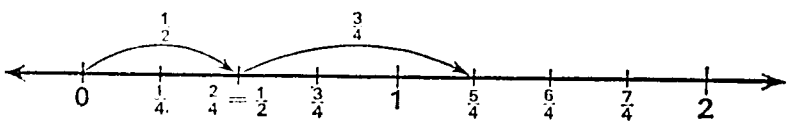
a. $\frac{2}{5} + \frac{1}{5} = \square$ **b.** $\frac{3}{10} + \frac{4}{10} = \square$
c. $\frac{1}{3} + \frac{4}{3} = \square$ **d.** $\frac{0}{4} + \frac{2}{4} = \square$
e. $1\frac{3}{5} + 1\frac{2}{5} = \square$ **f.** $2\frac{1}{4} + 1\frac{1}{4} = \square$

2. Make up two word problems which you could use with children to illustrate each of the sums in the above exercise.

20-2 Addition of fractions (continued)

We have seen that it is easy to add fractions when their denominators are the same. However, when the denominators are different the problem is harder. For instance, suppose we want to find the sum of $\frac{1}{2}$ and $\frac{3}{4}$. What do we do?

One thing we can do is look at the number line. The following picture shows what happens:



Notice that we divided the number line into *fourths*. We then showed $\frac{1}{2} + \frac{3}{4}$ by first marking the point $\frac{1}{2}$ and then going beyond that point a distance $\frac{3}{4}$. The second jump ends at the point $\frac{5}{4}$. Thus

$$\frac{1}{2} + \frac{3}{4} = \frac{5}{4}$$

We now ask if there is any way of finding sums such as $\frac{1}{2} + \frac{3}{4}$ by using what we have already learned about adding fractions with the same denominator. To do this we must write $\frac{1}{2}$ and $\frac{3}{4}$ as fractions having the same denominator. But, as we know, $\frac{1}{2}$ is equivalent to $\frac{2}{4}$. In other words, $\frac{1}{2}$ and $\frac{2}{4}$ are different names for the *same* number, and so we can find the sum of $\frac{1}{2}$ and $\frac{3}{4}$ by adding $\frac{2}{4}$ and $\frac{3}{4}$. Thus

$$\begin{aligned}\frac{1}{2} + \frac{3}{4} &= \frac{2}{4} + \frac{3}{4}\\ &= \frac{5}{4}\end{aligned}$$

The problem is easy once the fractions have been written with the same denominator.

All addition problems for fractions with different denominators are solved this way. For instance, to add $\frac{1}{3}$ and $\frac{3}{5}$ we

look at the set of equivalent fractions for each and find replacements having the same denominator. The two sets are

$$\left\{\tfrac{1}{3}, \tfrac{2}{6}, \tfrac{3}{9}, \tfrac{4}{12}, \boxed{\tfrac{5}{15}}, \tfrac{6}{18}, \ldots\right\}$$

$$\left\{\tfrac{3}{5}, \tfrac{6}{10}, \boxed{\tfrac{9}{15}}, \tfrac{12}{20}, \tfrac{15}{25}, \tfrac{18}{30}, \ldots\right\}$$

We see that $\frac{1}{3}$ is equivalent to $\frac{5}{15}$, and that $\frac{3}{5}$ is equivalent to $\frac{9}{15}$. Thus

$$\begin{aligned}\tfrac{1}{3} + \tfrac{3}{5} &= \tfrac{5}{15} + \tfrac{9}{15}\\ &= \tfrac{14}{15}\end{aligned}$$

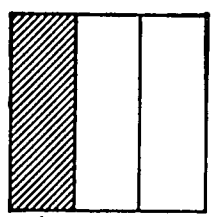
When we replace the fractions in an addition problem by equivalent fractions having the same denominator we say that we have put the original fractions over a COMMON DENOMINATOR. Thus 15 is a common denominator for $\frac{3}{5}$ and $\frac{1}{3}$, and 4 is a common denominator for $\frac{1}{2}$ and $\frac{3}{4}$.

We can now describe how to add fractions with different denominators. The method is as follows:

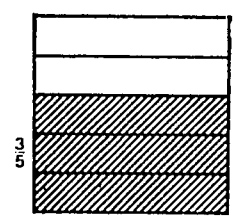
To add two fractions with different denominators find equivalent fractions for each so that the replacements have a common denominator. Then add these fractions by adding their numerators.

Thus the real problem in adding fractions with different denominators is to find a common denominator for the fractions. To see how this can be done we again look at the problem $\frac{1}{3} + \frac{3}{5}$. But this time we shall solve the problem by using pictures.

Our first picture shows $\frac{1}{3}$ and $\frac{3}{5}$ as shaded portions of two squares of equal size. It represents the original problem of finding $\frac{1}{3} + \frac{3}{5}$. However, we cannot solve the problem by using

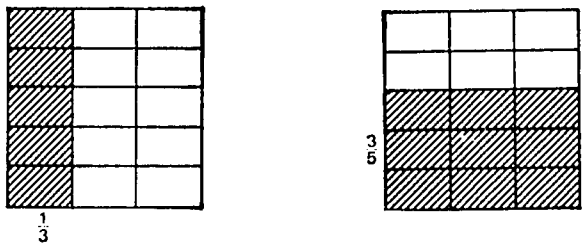


$\frac{1}{3}$



$\frac{3}{5}$

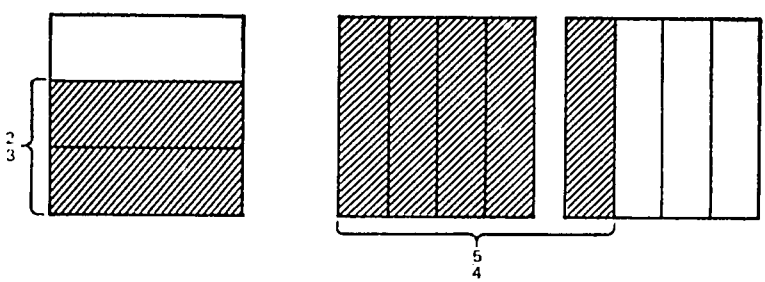
this picture since the squares are divided into pieces of *different* size. We have no way of finding the total shaded area. But if we could divide the squares into a larger number of pieces so that the new pieces were the same size in both squares we then could find the total shaded area by counting shaded pieces. How can we do this? One way is to *divide each square the way the other is already divided.* The following picture shows what happens:



Both squares are now divided into 15 parts of equal size. Each part therefore represents $\frac{1}{15}$ of the whole square. In the first square 5 of these parts are shaded; in the second square 9 are shaded. Altogether there are 14 shaded parts, and we can write

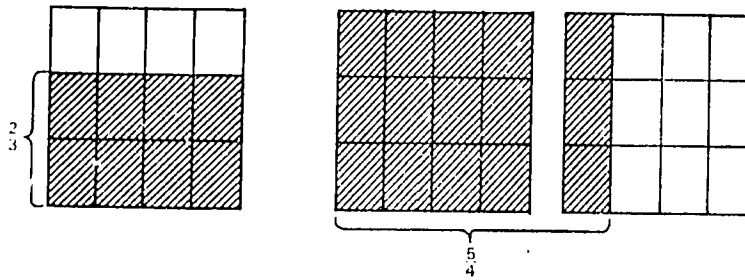
$$\begin{aligned}\tfrac{1}{3} + \tfrac{3}{5} &= \tfrac{5}{15} + \tfrac{9}{15}\\ &= \tfrac{14}{15}\end{aligned}$$

Let us look at another example this way. Suppose we wish to add $\frac{2}{3}$ and $\frac{5}{4}$. First we show each fraction separately, as follows:



Next we divide the first square into fourths by vertical lines,

and divide the last two squares into thirds by horizontal lines. Our picture then becomes



Each square is now divided into 12 equal parts. The shaded portion of the first square is $\frac{8}{12}$. The shaded portion of the last two squares is $\frac{15}{12}$. Altogether 23 parts of size $\frac{1}{12}$ are shaded. Thus

$$\begin{aligned}\tfrac{2}{3} + \tfrac{5}{4} &= \tfrac{8}{12} + \tfrac{15}{12}\\ &= \tfrac{23}{12}\end{aligned}$$

We now ask if there is a shorter way to do these additions. Let us look at the results of the last two examples:

$$\tfrac{1}{3} + \tfrac{3}{5} = \tfrac{14}{15}, \qquad \tfrac{2}{3} + \tfrac{5}{4} = \tfrac{23}{12}$$

In the first the answer is in fifteenths. In the second it is in twelfths. Do you see how denominators in these answers are related to the denominators of the fractions being added? Notice that $15 = 3 \times 5$, and that $12 = 3 \times 4$. In both examples the denominator in the answer is the product of the denominators of the fractions being added. We shall now show that this *always* works.

Suppose we wish to add the fractions $\frac{a}{b}$ and $\frac{c}{d}$. For instance, we might have $\frac{a}{b} = \frac{3}{7}$ and $\frac{c}{d} = \frac{9}{8}$. We know that every fraction of the form $\frac{3 \times n}{7 \times n}$ is equivalent to $\frac{3}{7}$ when n is a counting number. In other words

$$\frac{3}{7} = \frac{3 \times n}{7 \times n}$$

In particular, when $n = 8$ we have

$$\frac{3}{7} = \frac{3 \times 8}{7 \times 8}$$

Similarly, every fraction of the form $\frac{9 \times n}{8 \times n}$ is equivalent to $\frac{9}{8}$ when n is a counting number. Thus, if we let $n = 7$ we have

$$\frac{9}{8} = \frac{9 \times 7}{8 \times 7}$$

The two fractions $\frac{3 \times 8}{7 \times 8}$ and $\frac{9 \times 7}{8 \times 7}$ have the same denominator. Moreover, this denominator is the product of the denominators of $\frac{3}{7}$ and $\frac{9}{8}$. Thus

$$\begin{aligned} \frac{3}{7} + \frac{9}{8} &= \frac{3 \times 8}{7 \times 8} + \frac{9 \times 7}{8 \times 7} \\ &= \frac{24}{56} + \frac{63}{56} \\ &= \frac{87}{56} \end{aligned}$$

In the general case where we want to find the sum $\frac{a}{b} + \frac{c}{d}$ we write

$$\frac{a}{b} = \frac{a \times d}{b \times d} \qquad \frac{c}{d} = \frac{c \times b}{d \times b}$$

The fractions $\frac{a \times d}{b \times d}$ and $\frac{c \times b}{d \times b}$ have a common denominator. This denominator is the product of the original denominators b and d. Thus

$$\begin{aligned} \frac{a}{b} + \frac{c}{d} &= \frac{a \times d}{b \times d} + \frac{c \times b}{d \times b} \\ &= \frac{(a \times d) + (c \times b)}{b \times d} \end{aligned}$$

This formula is often abbreviated as follows:

$$\boxed{\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}}$$

EXERCISE 20-2

1. Draw pictures using squares to illustrate each of the following addition problems:

   a. $\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$ **b.** $1\frac{2}{3} + \frac{4}{7} = \frac{27}{21}$
   c. $2\frac{1}{2} + 1\frac{4}{5} = 4\frac{3}{10}$ **d.** $\frac{3}{4} + \frac{3}{5} = \frac{27}{20}$
   e. $\frac{5}{6} + \frac{1}{3} = \frac{21}{18}$

2. Illustrate each of the following addition problems on the number lines, and then find the sum.

   a. $\frac{3}{4} + \frac{5}{2} = \square$ **b.** $\frac{2}{3} + \frac{3}{2} = \square$
   c. $\frac{1}{2} + \frac{5}{6} = \square$ **d.** $\frac{5}{8} + \frac{3}{4} = \square$
   e. $1\frac{1}{2} + 1\frac{3}{8} = \square$

3. Use the formula $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ to find each of the following sums:

   a. $\frac{5}{12} + \frac{3}{4}$ **b.** $\frac{7}{8} + \frac{4}{3}$
   c. $\frac{1}{3} + \frac{1}{4}$ **d.** $\frac{4}{5} + \frac{1}{3}$
   e. $\frac{3}{10} + \frac{5}{3}$ **f.** $1\frac{2}{3} + 3\frac{1}{8}$
   g. $5\frac{1}{2} + 3\frac{2}{5}$ **h.** $2 + 4\frac{2}{7}$
   i. $3\frac{1}{4} + 2\frac{3}{5}$ **j.** $7\frac{1}{9} + 5\frac{3}{10}$

4. One man can plough a field in 2 days and another can plough the same field in 3 days. If they work together, what part of the field can they plough in 1 day?

5. Two men share a basket of maize. The first takes $\frac{3}{8}$ of the maize and the second takes $\frac{5}{12}$ of it. What part of the maize have they taken?

6. Suppose someone told you, "$\frac{5}{8}$ of a piece of paper is coloured red and $\frac{4}{7}$ is coloured green." Would you believe him? Why?

7. A student adds fractions as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{a + c}{b + d}$$

Will he ever get the right answer? Why?

8. Is $\frac{1}{p} + \frac{1}{q} = \frac{1}{p+q}$ ever a true statement when p and q are counting numbers? Why?

20-3 The least common denominator

Consider the problem of adding $\frac{5}{6}$ and $\frac{3}{4}$. One way to find this sum is to use the formula in the last section. This formula tells us to replace $\frac{5}{6}$ and $\frac{3}{4}$ by equivalent fractions having $6 \times 4 = 24$ as their denominator. The number 24 works because the set of fractions equivalent to $\frac{5}{6}$ and the set of fractions equivalent to $\frac{3}{4}$ *both* contain a fraction with denominator 24:

$$\left\{\frac{5}{6}, \boxed{\frac{10}{12}}, \frac{15}{18}, \boxed{\frac{20}{24}}, \ldots\right\}$$

$$\left\{\frac{3}{4}, \frac{6}{8}, \boxed{\frac{9}{12}}, \frac{12}{16}, \frac{15}{20}, \boxed{\frac{18}{24}}, \ldots\right\}$$

But both of these sets also contain a fraction expressed in twelfths. Thus we can use 12 as a common denominator for $\frac{5}{6}$ and $\frac{3}{4}$ in place of 24. When we do we get

$$\begin{aligned} \frac{5}{6} + \frac{3}{4} &= \frac{10}{12} + \frac{9}{12} \\ &= \frac{19}{12} \end{aligned}$$

In this problem we were able to use a common denominator that was smaller than the product of the denominator of the given fractions. Notice that 12 is the *smallest* number which can be used as a common denominator in adding $\frac{5}{6}$ and $\frac{3}{4}$. For this reason 12 is called the LEAST COMMON DENOMINATOR for the fractions $\frac{5}{6}$ and $\frac{3}{4}$. The term "least common denominator" is often abbreviated to LCD.

Now let us look at another example. Suppose we are asked to find $\frac{1}{6} + \frac{5}{9}$. If we use $6 \times 9 = 54$ as a common denominator for these fractions we get

$$\begin{aligned} \frac{1}{6} + \frac{5}{9} &= \frac{1 \times 9}{6 \times 9} + \frac{5 \times 6}{9 \times 6} \\ &= \frac{9}{54} + \frac{30}{54} \\ &= \frac{39}{54} \end{aligned}$$

But 54 is not the least common denominator for $\frac{1}{6}$ and $\frac{5}{9}$. To see this we look at the sets of fractions equivalent to $\frac{1}{6}$ and $\frac{5}{9}$:

$$\left\{\frac{1}{6}, \frac{2}{12}, \frac{3}{18}, \ldots\right\}$$

$$\left\{\frac{5}{9}, \frac{10}{18}, \frac{15}{27}, \ldots\right\}$$

Both of these sets contain a fraction having 18 as a denominator. Moreover, 18 is the smallest number with this property. Thus 18 is the least common denominator for $\frac{1}{6}$ and $\frac{5}{9}$. Using it we can write

$$\begin{aligned}\frac{1}{6} + \frac{5}{9} &= \frac{3}{18} + \frac{10}{18}\\ &= \frac{13}{18}\end{aligned}$$

Notice that we now have two answers to the problem $\frac{1}{6} + \frac{5}{9}$:

$$\frac{1}{6} + \frac{5}{9} = \frac{39}{54} \quad \text{and} \quad \frac{1}{6} + \frac{5}{9} = \frac{13}{18}$$

However, the fractions $\frac{39}{54}$ and $\frac{13}{18}$ are equivalent since

$$\frac{13}{18} = \frac{13 \times 3}{18 \times 3} = \frac{39}{54}$$

This shows that the two answers are really the same since they both name the same number. In most cases the answer to this problem would be written $\frac{13}{18}$, not $\frac{39}{54}$, because $\frac{13}{18}$ is in lowest terms. In general, answers to problems involving fractions should be written in lowest terms.

When adding fractions with different denominators it is helpful to put the fractions over their least common denominator. This makes the work easier because the numbers in the problem are then smaller.

But how do we find the LCD for a pair of fractions? Consider $\frac{1}{6}$ and $\frac{5}{9}$ for example. We have seen that the LCD for these fractions is 18. But 18 is a multiple of 6 *and* 9. In other words it is a *common multiple* of 6 and 9. Moreover, it is the least, or smallest common multiple of these numbers. This illustrates the following general fact:

The least common denominator for two fractions is the least common multiple of their denominators.

We can use this fact together with what we know about factoring numbers to find the LCD of any two fractions. Take $\frac{11}{42}$ and $\frac{31}{105}$ for example. To find the LCD for these fractions we must find the least common multiple of 42 and 105. To do this we first factor each of these numbers:

$$42 = 2 \times 21 \qquad 105 = 3 \times 35$$
$$42 = 2 \times 3 \times 7 \quad 105 = 3 \times 5 \times 7$$

We now recall that *every* common multiple of two numbers must contain *all* of the prime factors of *each* number. Thus every common multiple of 42 and 105 must contain the factors 2, 3, 5, 7. Since the smallest number with these factors is

$$2 \times 3 \times 5 \times 7 = 210$$

we see that the LCD of $\frac{11}{42}$ and $\frac{31}{105}$ is 210. (Notice that this number is much smaller than the product $42 \times 105 = 4{,}410$.)

Now we have found the LCD of $\frac{11}{42}$ and $\frac{31}{105}$ we can add these fractions. We get

$$\frac{11}{42} + \frac{31}{105} = \frac{11 \times 5}{42 \times 5} + \frac{31 \times 2}{105 \times 2}$$
$$= \frac{55 + 62}{210}$$
$$= \frac{117}{210}$$

The 5 and the 2 used here as multipliers are also found by looking at the prime factorization of 42 and 105. For the first fraction we must multiply 42 by a number to get 210 as the product. Looking at the factorizations for 42 and for 210 we have

$$42 = 2 \times 3 \times 7 \quad \text{and} \quad 210 = 2 \times 3 \times 5 \times 7$$

We ask which of the factors of 210 are missing from 42. The factor 5 is the only one missing so we multiply the numerator and denominator of $\frac{11}{42}$ by 5. In the second fraction we have $105 = 3 \times 5 \times 7$ in the denominator. Comparing this with

$210 = 2 \times 3 \times 5 \times 7$ we see that the factor 2 is missing. Thus we must multiply the numerator and denominator of the second fraction by 2.

In many problems we can find the multipliers just by looking at the denominators. But with larger numbers the idea of finding the missing factors is useful.

There is one other way in which factorizations are helpful in the addition of fractions. Look at the answer to the last problem, $\frac{117}{210}$. Is this answer in lowest terms or not? We know that the prime divisors of the denominator are 2, 3, 5 and 7. Thus any prime factor of the numerator *and* denominator of $\frac{117}{210}$ would have to be one of these numbers. If we try them we find that only the 3 divides into 117. Thus

$$\frac{117}{210} = \frac{117 \div 3}{210 \div 3} = \frac{29}{70}$$

Since the fraction $\frac{29}{70}$ is in lowest terms we write

$$\frac{11}{42} + \frac{31}{105} = \frac{29}{70}$$

EXERCISE 20-3

1. Find the least common denominator for each of the following pairs of fractions and then add the fractions:

   a. $\frac{2}{5}$ and $\frac{3}{10}$ **b.** $\frac{5}{6}$ and $\frac{3}{8}$
   c. $\frac{7}{15}$ and $\frac{4}{25}$ **d.** $\frac{7}{12}$ and $\frac{13}{30}$
   e. $\frac{7}{15}$ and $\frac{23}{35}$ **f.** $\frac{17}{12}$ and $\frac{11}{40}$
   g. $\frac{29}{126}$ and $\frac{32}{105}$ **h.** $\frac{5}{108}$ and $\frac{50}{462}$
   i. $2\frac{3}{84}$ and $3\frac{41}{525}$ **j.** $1\frac{3}{10}$ and $2\frac{5}{12}$

2. Find the fraction in lowest terms which is equivalent to each of the following fractions:

   a. $\frac{24}{32}$ **b.** $\frac{21}{66}$
   c. $\frac{117}{36}$ **d.** $\frac{231}{308}$
   e. $\frac{315}{189}$

3. Each of the fractions $\frac{5}{6}$, $\frac{7}{24}$, $\frac{9}{25}$, $\frac{31}{30}$, $\frac{101}{150}$ is equivalent to a fraction with denominator 600. Find these fractions.

4. Find all fractions of the form $\frac{1}{p}$ which are equivalent to a fraction with denominator 24.

20-4 The commutative property of addition

When we studied addition of whole numbers we saw that $a + b = b + a$ for any two whole numbers a and b. For instance $2 + 3 = 3 + 2$, $5 + 4 = 4 + 5$, and so on. This property is described by saying that the addition of whole numbers is *commutative*. We shall now show that the same property holds for addition of fractions. In other words, if $\frac{a}{b}$ and $\frac{c}{d}$ are fractions, then

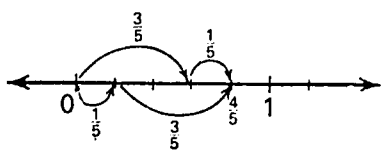
$$\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}$$

We begin with an example. Consider the fractions $\frac{3}{5}$ and $\frac{1}{5}$. Does $\frac{3}{5} + \frac{1}{5} = \frac{1}{5} + \frac{3}{5}$? In this case it is easy to see that the answer is yes, since

$$\frac{3}{5} + \frac{1}{5} = \frac{3 + 1}{5} = \frac{4}{5}$$

$$\frac{1}{5} + \frac{3}{5} = \frac{1 + 3}{5} = \frac{4}{5}$$

The following picture shows this on the number line:



Notice that the real reason why these two answers are the same is that $3 + 1 = 1 + 3$. Because of this we have

$$\frac{3 + 1}{5} = \frac{1 + 3}{5}$$

Thus we can say that $\frac{3}{5} + \frac{1}{5} = \frac{1}{5} + \frac{3}{5}$ because addition of whole numbers is commutative.

In exactly the same way we can show that addition of fractions is commutative whenever the fractions have the *same*

denominator. Indeed, suppose the fractions are $\frac{a}{b}$ and $\frac{c}{b}$. Then

$$\begin{aligned}\frac{a}{b} + \frac{c}{b} &= \frac{a+c}{b} \\ &= \frac{c+a}{b} \qquad (\text{since } a + c = c + a) \\ &= \frac{c}{b} + \frac{a}{b}\end{aligned}$$

This shows that

$$\frac{a}{b} + \frac{c}{b} = \frac{c}{b} + \frac{a}{b}$$

Thus, for example,

$$\tfrac{1}{2} + \tfrac{2}{3} = \tfrac{2}{3} + \tfrac{1}{2}, \quad \tfrac{9}{8} + \tfrac{11}{15} = \tfrac{11}{15} + \tfrac{9}{8}$$

and so on.

Finally, let us consider what happens when the fractions have different denominators. In this case we put the fractions over a common denominator and then add. But, by what we have just said, addition of fractions with a common denominator is commutative. Thus the addition of the original fractions is also commutative. This shows that

$$\boxed{\frac{a}{b} + \frac{c}{d} = \frac{c}{d} + \frac{a}{b}}$$

for any two fractions $\frac{a}{b}$ and $\frac{c}{d}$.

EXERCISE 20-4

1. Use squares, properly subdivided, to show that $\frac{3}{4} + \frac{5}{6}$ is the same as $\frac{5}{6} + \frac{3}{4}$.
2. Use the number line to show that the two sums in Question 1 are the same.
3. **a.** Repeat problem 1 for the sums $\frac{2}{3} + \frac{4}{5}$ and $\frac{4}{5} + \frac{2}{3}$.
   b. Show that $\frac{2}{3} + \frac{4}{5} = \frac{4}{5} + \frac{2}{3}$ on the number line.

4. Show, by computing the sums, that each of the following statements is true.

a. $\frac{7}{10} + \frac{4}{5} = \frac{4}{5} + \frac{7}{10}$ **b.** $1\frac{7}{8} + 9\frac{2}{3} = 9\frac{2}{3} + 1\frac{7}{8}$

c. $3\frac{5}{12} + 4\frac{3}{8} = 4\frac{3}{8} + 3\frac{5}{12}$ **d.** $\frac{a}{2} + \frac{c}{3} = \frac{c}{3} + \frac{a}{2}$

e. $\frac{4}{b} + \frac{c}{5} = \frac{c}{5} + \frac{4}{b}$

20-5 The associative property of addition

The second important property of addition of whole numbers is the *associative property*. This property tells us that when adding whole numbers we can group the numbers any way we like. The answer is always the same. We express the fact that addition of whole numbers is associative by writing

$$(a + b) + c = a + (b + c)$$

Remember that the brackets tell us how the numbers in each sum are grouped together as we add them.

We shall now show that the same property holds for addition of fractions. For instance, consider the fractions $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{3}{8}$. We have

$$\begin{aligned}(\tfrac{1}{2} + \tfrac{1}{4}) + \tfrac{3}{8} &= (\tfrac{2}{4} + \tfrac{1}{4}) + \tfrac{3}{8}\\ &= \tfrac{3}{4} + \tfrac{3}{8}\\ &= \tfrac{6}{8} + \tfrac{3}{8}\\ &= \tfrac{9}{8}\end{aligned}$$

Adding the other way we get

$$\begin{aligned}\tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{3}{8}) &= \tfrac{1}{2} + (\tfrac{2}{8} + \tfrac{3}{8})\\ &= \tfrac{1}{2} + \tfrac{5}{8}\\ &= \tfrac{4}{8} + \tfrac{5}{8}\\ &= \tfrac{9}{8}\end{aligned}$$

Both answers are the same. Thus

$$(\tfrac{1}{2} + \tfrac{1}{4}) + \tfrac{1}{8} = \tfrac{1}{2} + (\tfrac{1}{4} + \tfrac{1}{8})$$

Let us see why this must always happen. As usual we first look at fractions having the same denominator. Suppose the fractions are $\frac{a}{b}, \frac{c}{b}, \frac{e}{b}$.

Then
$$\left(\frac{a}{b} + \frac{c}{b}\right) + \frac{e}{b} = \frac{a + c}{b} + \frac{e}{b}$$
$$= \frac{(a + c) + e}{b}$$

On the other hand,

$$\frac{a}{b} + \left(\frac{c}{b} + \frac{e}{b}\right) = \frac{a}{b} + \frac{c + e}{b}$$

But these answers are the same because we know that $(a + c) + e = a + (c + e)$. This shows that

$$\left(\frac{a}{b} + \frac{c}{b}\right) + \frac{e}{b} = \frac{a}{b} + \left(\frac{c}{b} + \frac{e}{b}\right)$$

Finally, when the denominators of the fractions are not the same we put the fractions over a common denominator and then add. Since we now know that addition of fractions with the same denominator is associative, we can say that the addition of the original fractions is also associative. Thus in general we have

$$\boxed{\left(\frac{a}{b} + \frac{c}{d}\right) + \frac{e}{f} = \frac{a}{b} + \left(\frac{c}{d} + \frac{e}{f}\right)}$$

EXERCISE 20-5

Show, by computing the sums, that each of the following statements is true.

1. $\frac{2}{3} + (\frac{3}{2} + \frac{5}{8}) = (\frac{2}{3} + \frac{3}{2}) + \frac{5}{8}$
2. $(1\frac{1}{8} + 1) + \frac{1}{5} = 1\frac{1}{8} + (1 + \frac{1}{5})$
3. $(\frac{1}{6} + \frac{3}{4}) + \frac{7}{8} = \frac{1}{6} + (\frac{3}{4} + \frac{7}{8})$
4. $\frac{12}{7} + (\frac{2}{5} + \frac{11}{21}) = (\frac{12}{7} + \frac{2}{5}) + \frac{11}{21}$
5. $\dfrac{a}{8} + \left(\dfrac{b}{12} + \dfrac{c}{16}\right) = \left(\dfrac{a}{8} + \dfrac{b}{12}\right) + \dfrac{c}{16}$

20-6 The property of zero

When we studied addition of whole numbers we saw that the number 0 has the following special property: when zero is

added to any whole number the number is left unchanged. That is,

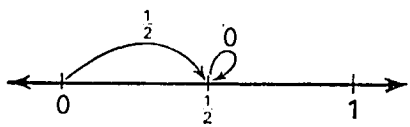
$$a + 0 = a \quad \text{and} \quad 0 + a = a$$

for any whole number a.

We can now show that the same thing happens when we add 0 to a fraction. Consider the fraction $\frac{1}{2}$ for example. We have

$$\begin{aligned} \frac{1}{2} + 0 &= \frac{1}{2} + \frac{0}{2} \\ &= \frac{1 + 0}{2} \\ &= \frac{1}{2} \end{aligned}$$

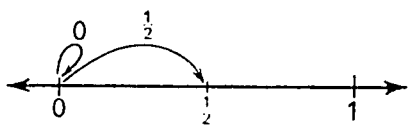
Notice that in adding 0 to $\frac{1}{2}$ we thought of 0 as a fraction with denominator 2. Thus we wrote 0 as $\frac{0}{2}$. The following picture shows this addition problem on the number line:



Next, we recall that addition of fractions is commutative. This tells us that $\frac{1}{2} + 0 = 0 + \frac{1}{2}$. Thus since $\frac{1}{2} + 0 = \frac{1}{2}$ we can also write

$$0 + \frac{1}{2} = \frac{1}{2}$$

The picture on the number line now looks like this:



In general, if $\frac{a}{b}$ is any fraction at all we have

$$\frac{a}{b} + 0 = 0 + \frac{a}{b} = \frac{a}{b}$$

You should be able to show this the same way that we showed that $\frac{1}{2} + 0 = 0 + \frac{1}{2} = 0$.

The above equations tell us that when 0 is added to any fraction the fraction is left unchanged. We describe this property of zero by saying that *the number zero is the* IDENTITY *for addition of fractions.*

20-7 Subtraction of fractions

Now that we know how to add fractions we can also subtract them by thinking of subtraction as the inverse of addition. For instance, consider the following problem:

> A man wishes to pick three baskets of oranges. After picking for a while he finds that he has enough oranges to fill the basket $1\frac{2}{3}$ times. How many more baskets of oranges should he pick?

To solve this problem we must find the number to put in the box to make the equation

$$1\tfrac{2}{3} + \square = 3$$

a true sentence. The number which goes in the box is called the *missing addend.* To find it we first write $1\frac{2}{3}$ as an improper fraction: $1\frac{2}{3} = \frac{5}{3}$. Next we recall that 3 can be written as $\frac{3}{1}$, or as $\frac{9}{3}$. Thus the above equation is the same as

$$\tfrac{5}{3} + \square = \tfrac{9}{3}$$

Everything is now written in thirds. Suppose we write this equation without using the word "thirds". It then becomes the whole number equation.

$$5 + \triangle = 9$$

As you know, this can also be written as a subtraction equation:

$$\triangle = 9 - 5$$

In either case the number 4 must be put in the triangle to give a true sentence. If we now change back to thirds we get the true sentences:

$$\text{5-thirds} + \text{4-thirds} = \text{9-thirds}$$
$$\text{4-thirds} = \text{9-thirds} - \text{5-thirds}$$

In fractions these sentences read

$$\tfrac{5}{3} + \tfrac{4}{3} = \tfrac{9}{3}$$
$$\tfrac{4}{3} = \tfrac{9}{3} - \tfrac{5}{3}$$

They show that the man must pick $\frac{4}{3}$ or $1\frac{1}{3}$ more baskets of oranges.

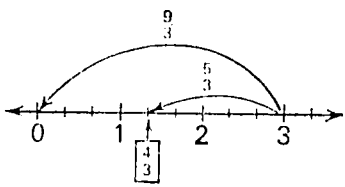
In the above problem we saw that the addition equation

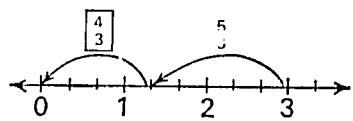
$$\tfrac{5}{3} + \square = \tfrac{9}{3}$$

can also be written as the subtraction equation

$$\square = \tfrac{9}{3} - \tfrac{5}{3}$$

These equations are just two different ways of asking the same question. The first asks us to find the missing addend which must be added to $\frac{5}{3}$ to give $\frac{9}{3}$. The second asks us to *subtract* $\frac{5}{3}$ from $\frac{9}{3}$. Either way the answer is $\frac{4}{3}$. The following pictures show this on the number line:





What do we do if a missing addend problem has fractions with different denominators? The answer is easy. We simply put the fractions over a common denominator and then proceed as in the problem just solved. Here is an example:

Solve the equation

$$\tfrac{5}{4} + \square = \tfrac{8}{3}$$

We first find the LCD for the fractions $\frac{5}{4}$ and $\frac{8}{3}$. Since 3 is a prime number and $4 = 2 \times 2$, we see that the LCD is $3 \times 2 \times 2$. Moreover,

$$\frac{5}{4} = \frac{5 \times 3}{4 \times 3} = \frac{15}{12}$$

$$\frac{8}{3} = \frac{8 \times 4}{3 \times 4} = \frac{32}{12}$$

Thus the above equation can be written

$$\tfrac{15}{12} + \square = \tfrac{32}{12}$$

Since the fractions in this equation are now in twelfths we know that the box must be filled with a fraction in twelfths to give a true sentence. Suppose we write

$$\square = \tfrac{\triangle}{12}$$

Then our equation becomes

$$\tfrac{15}{12} + \tfrac{\triangle}{12} = \tfrac{32}{12}$$

This shows that

$$\frac{15 + \triangle}{12} = \frac{32}{12}$$

But $15 + 17 = 32$. Thus $\triangle = 17$, and we have

$$\square = \tfrac{17}{12}$$

We can also solve this problem by writing everything in terms of subtraction. When we do it reads as follows:

$$\begin{aligned}\square &= \frac{3}{8} - \frac{5}{4}\\ &= \frac{32}{12} - \frac{15}{12}\\ &= \frac{32 - 15}{12}\\ &= \frac{17}{12}\end{aligned}$$

Again we have $\square = \tfrac{17}{12}$.

Notice that each step in the second solution of this problem has a corresponding step in the first solution. This becomes very clear if we write both solutions side-by-side as follows:

$\frac{5}{4} + \square = \frac{8}{3}$	$\square = \frac{8}{3} - \frac{5}{4}$
$\frac{15}{12} + \square = \frac{32}{12}$	$\square = \frac{32}{12} - \frac{15}{12}$
$\frac{15 + \triangle}{12} = \frac{32}{12}$	$\square = \frac{32 - 15}{12}$
$\frac{15 + 17}{12} = \frac{32}{12}$	$\square = \frac{17}{12}$
$\square = \frac{17}{12}$	

Now suppose we wish to solve the equation

$$\square = \frac{a}{b} - \frac{c}{d}$$

We first put these fractions over the common denominator $b \times d$, and then proceed as in the examples. We get

$$\square = \frac{a \times d}{b \times d} - \frac{c \times b}{d \times b}$$

$$= \frac{(a \times d) - (c \times b)}{b \times d}$$

Thus we have the following statement:

$$\boxed{\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}}$$

Notice, however, that this makes sense only if $\frac{a}{b}$ is greater than or equal to $\frac{c}{d}$. If you draw a picture on the number line you can see why.

EXERCISE 20-7

1. Draw squares, properly subdivided, to show that

$$\tfrac{8}{3} - \tfrac{5}{4} = \tfrac{17}{12}$$

2. Illustrate the subtraction equation in Question 1 on the number line.

3. Find the missing addends in each of the following equations. Rewrite the equations as subtraction equations.

a. $\frac{2}{3} + \square = \frac{5}{3}$ **b.** $\square + 7 = \frac{31}{2}$

c. $\square \times \frac{23}{7} = \frac{41}{4}$ **d.** $1\frac{5}{8} + \square = 2\frac{7}{12}$

e. $\square + 4\frac{5}{16} = 5\frac{3}{10}$

4. Solve each of the following equations:

a. $\square = \frac{3}{7} - \frac{2}{9}$ **b.** $\square = \frac{11}{18} - \frac{2}{21}$

c. $\square = 3\frac{5}{9} - 2\frac{4}{5}$

5. A man has a piece of sugar cane $21\frac{1}{2}$ inches long. If he cuts off a piece $2\frac{3}{8}$ inches long, how much remains?
6. A girl had $\frac{5}{8}$ of a cake. She gave a piece equal to $\frac{5}{12}$ of the whole cake to her friend. How much did she have left?
7. Two girls are baking a cake. They have $1\frac{1}{3}$ cups of sugar, and they use $\frac{3}{4}$ of a cup in their baking. How much sugar do they have left?
8. A man working alone could pick the fruit from some trees in 5 days. Another man works with him and together they pick the fruit in 2 days. What part of the total could the second man, working alone, pick in one day?
9. Let $\frac{a}{b}$ and $\frac{c}{d}$ be fractions. Can the problem $\frac{a}{b} - \frac{c}{d} = \square$ always be solved with fractions? Illustrate your answer with examples.

20-8 Subtraction as the inverse of addition

What do we mean when we say that addition and subtraction are inverses of one another? For whole numbers we mean that each undoes what the other does. In other words, if we start with a whole number, and add any whole number to it, and then subtract the same number we get the original number back again. The same thing happens if we subtract first and then add.

We can show that this also happens for addition and subtraction of fractions. For instance, the sentence

$$\tfrac{3}{4} + \tfrac{1}{8} = \tfrac{7}{8}$$

tells us that if $\frac{1}{8}$ is added to $\frac{3}{4}$ the sum is $\frac{7}{8}$. Suppose that after adding we subtract it. We get

$$(\tfrac{3}{4} + \tfrac{1}{8}) - \tfrac{1}{8} = \tfrac{7}{8} - \tfrac{1}{8}$$

or

$$(\tfrac{3}{4} + \tfrac{1}{8}) - \tfrac{1}{8} = \tfrac{3}{4}$$

This shows that if we add $\frac{1}{8}$ to $\frac{3}{4}$ and then subtract $\frac{1}{8}$ from the sum we get $\frac{3}{4}$ again.

Now suppose we start with a subtraction; for example

$$\tfrac{7}{8} - \tfrac{1}{8} = \tfrac{3}{4}$$

If we add $\frac{1}{8}$ to each side of this equality we have

$$\begin{aligned}(\tfrac{7}{8} - \tfrac{1}{8}) + \tfrac{1}{8} &= \tfrac{3}{4} - \tfrac{1}{8} \\ &= \tfrac{7}{8}\end{aligned}$$

Again we are back where we started.

In general we have the identities

$$\boxed{\begin{aligned}\left(\frac{a}{b} + \frac{c}{d}\right) - \frac{c}{d} &= \frac{a}{b} \\ \left(\frac{a}{b} - \frac{c}{d}\right) + \frac{c}{d} &= \frac{a}{b}\end{aligned}}$$

These identities tell us that addition and subtractions are inverses of one another for fractions as well as for whole numbers.

Chapter 21

MULTIPLICATION AND DIVISION OF FRACTIONS

21-1 Multiplication of fractions

Now that we have learned how to add and subtract fractions we consider what it means to multiply them. We begin by showing how to multiply a whole number and a fraction.

Suppose we are asked to find $4 \times \frac{2}{3}$. Let us try to discover the answer by using what we know about multiplication of whole numbers. You will recall that multiplication of whole numbers means repeated addition. Thus

$$4 \times 3 \quad \text{means} \quad 3 + 3 + 3 + 3$$
$$4 \times 2 \quad \text{means} \quad 2 + 2 + 2 + 2$$
$$4 \times 1 \quad \text{means} \quad 1 + 1 + 1 + 1$$

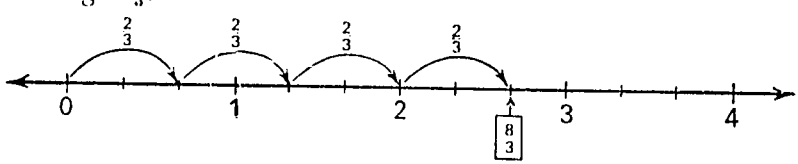
If we wish this pattern to continue when the second factor is $\frac{2}{3}$, we must agree that

$$4 \times \tfrac{2}{3} \quad \text{means} \quad \tfrac{2}{3} + \tfrac{2}{3} + \tfrac{2}{3} + \tfrac{2}{3}$$

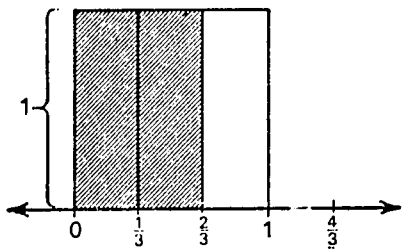
For this reason we write

$$\begin{aligned} 4 \times \tfrac{2}{3} &= \frac{2}{3} + \frac{2}{3} + \frac{2}{3} + \frac{2}{3} \\ &= \frac{2 + 2 + 2 + 2}{3} \\ &= \frac{8}{3} \end{aligned}$$

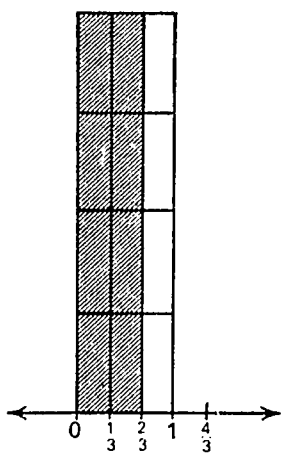
We can show this multiplication on the number line in the same way that we showed the multiplication of whole numbers. This time we start at 0 and make four jumps to the right each of length $\frac{2}{3}$.



We can also show the meaning of $4 \times \frac{2}{3}$ by building squares on the number line. We start with a square which has one side on the segment of the number line from 0 to 1. We then divide the square into three equal parts and shade two of them to represent the fraction $\frac{2}{3}$. Our picture looks like this:



How shall we show $4 \times \frac{2}{3}$? One way is to use this square to build a rectangle 4 units high as shown in the following picture.



In each square 2 of the 3 parts are shaded. Altogether there are 8 shaded parts, each of size $\frac{1}{3}$. Thus we again see that

$$4 \times \frac{2}{3} = \frac{8}{3}$$

This picture also shows that the product $4 \times \frac{2}{3}$ can be viewed as repeated addition. This time we add, or join together 4 unit squares, each shaded to represent the fraction $\frac{2}{3}$. (We call such squares *unit* squares because their sides are one unit in length. Notice that the area of such a square is $1 \times 1 = 1$ square unit.)

We now do exactly the same thing for the product of any

whole number and any fraction. Let a be the whole number, and let $\frac{m}{n}$ be the fraction. Then

$$a \times \frac{m}{n} = \frac{m}{n} + \frac{m}{n} + \ldots + \frac{m}{n}$$

(Here we imagine that the fraction $\frac{m}{n}$ is written a times.)

But $$\frac{m}{n} + \frac{m}{n} + \ldots + \frac{m}{n} = \frac{m + m + \ldots + m}{n}$$

Since the number of addends in the fraction is a we have

$$\frac{m + m + \ldots + m}{n} = \frac{a \times m}{n}$$

Thus $$\boxed{a \times \frac{m}{n} = \frac{a \times m}{n}}$$

We can describe this result very simply: To multiply a whole number and a fraction, multiply the numerator of the fraction by the whole number and leave the denominator the same.

Here are some examples:

$$3 \times \frac{1}{2} = \frac{3 \times 1}{2} = \frac{3}{2}$$

$$5 \times \frac{9}{4} = \frac{5 \times 9}{4} = \frac{45}{4}$$

$$7 \times 3\frac{3}{8} = 7 \times \frac{27}{8} = \frac{189}{8}$$

EXERCISE 21–1

1. Show the meaning of $4 \times \frac{2}{3}$ as repeated addition by using squares properly subdivided.
2. Show the meaning of $4 \times \frac{2}{3}$ as repeated addition on the number line.

Solve each of the following multiplication problems. Express your answer in lowest terms.

3. $7 \times \frac{5}{12}$ **4.** $4 \times \frac{3}{4}$ **5.** $9 \times 3\frac{2}{3}$

6. $5 \times 4\frac{3}{4}$ **7.** $83 \times \frac{27}{4}$ **8.** $32 \times \frac{5}{8}$

9. $13 \times \frac{5}{6}$ **10.** $2\frac{2}{5} \times 11$

11. If you need $1\frac{1}{3}$ cups of flour to bake one cake, how many cups of flour will you need for 5 cakes?

12. Eight children divide some chocolate bars equally. Each gets $2\frac{1}{2}$ bars. How many chocolate bars are there?

21-2 Making multiplication simple

In doing the problems above you may have noticed that in some of them you could make your work easier. Take Question 8 for example: $32 \times \frac{5}{8}$. If we use the formula

$$a \times \frac{m}{n} = \frac{a \times m}{n}$$

we get

$$32 \times \frac{5}{8} = \frac{32 \times 5}{8} = \frac{160}{8}$$

But, by the rules for equivalent fractions we see that

$$\frac{160}{8} = \frac{160}{8} \div \frac{8}{8} = \frac{20}{1} = 20$$

Thus when the answer to this problem is expressed in lowest terms we have $32 \times \frac{5}{8} = 20$. We can get this answer without actually doing this multiplication at all. In fact, we can write

$$32 \times \frac{5}{8} = \frac{32 \times 5}{8} = \frac{(8 \times 4) \times 5}{8 \times 1} = \frac{4 \times 5}{1} = 20$$

You should be able to say why each of these steps is correct.

As another example of the same sort consider the product $16 \times \frac{5}{12}$. Here we can write

$$16 \times \frac{5}{12} = \frac{16 \times 5}{12} = \frac{(4 \times 4) \times 5}{4 \times 3} = \frac{4 \times 5}{3} = \frac{20}{3}$$

Notice that when we proceed this way our answer is in lowest terms. Can you explain why this happens?

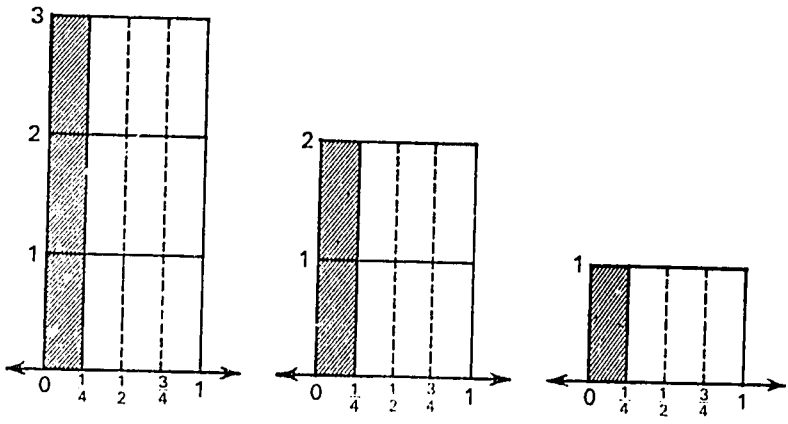
21-3 Multiplication of fractions (continued)

We now consider the problem of multiplying one fraction by another. For instance, what is $\frac{1}{2} \times \frac{1}{4}$? or $\frac{1}{2} \times \frac{3}{4}$? or $\frac{3}{5} \times \frac{3}{4}$? In general we want to know what $\frac{a}{b} \times \frac{c}{d}$ is when $\frac{a}{b}$ and $\frac{c}{d}$ are any two fractions.

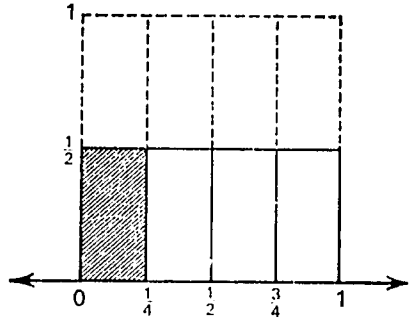
The easiest way to find answers for these questions is to use unit squares on the number line. We have seen how to draw pictures which represent such products as

$$3 \times \tfrac{1}{4},\ 2 \times \tfrac{1}{4},\ 1 \times \tfrac{1}{4}$$

Thus the picture on the left below represents $3 \times \frac{1}{4}$ since it shows 3 unit squares divided into 4 equal parts with one part of each square shaded. The shaded portion of this picture goes from 0 to $\frac{1}{4}$ along the number line and is 3 units high. The picture shows why we write $3 \times \frac{1}{4} = \frac{3}{4}$. Similarly, the picture in the middle shows that $2 \times \frac{1}{4} = \frac{2}{4}$. Here the shaded portion goes from 0 to $\frac{1}{4}$ and is 2 units high. Finally, in the third picture the shaded portion is only 1 unit high. Notice that the height of each picture is equal to the multiplier of $\frac{1}{4}$.



Suppose we try to keep this pattern when the multiplier of $\frac{1}{4}$ is a fraction instead of a whole number. Take the multiplier $\frac{1}{2}$ for instance. Our picture would then have to look like this:

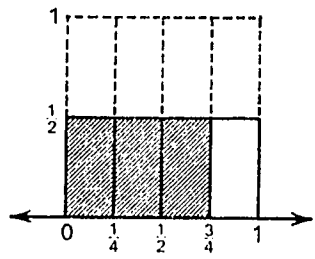


Here we have drawn one-half of the unit square and shaded the portion between 0 and $\frac{1}{4}$. As you can see from the picture, $\frac{1}{8}$ of the entire square is now shaded. If we agree that the shaded area represents the product $\frac{1}{2} \times \frac{1}{4}$, we must then write

$$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$

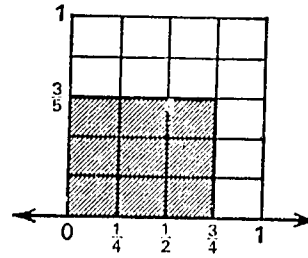
Using this idea we can draw pictures which represent products such as $\frac{1}{2} \times \frac{3}{4}$ and $\frac{3}{5} \times \frac{3}{4}$. For instance, the picture for $\frac{1}{2} \times \frac{3}{4}$ must show one-half of the unit square with the portion from 0 to $\frac{3}{4}$ shaded. Altogether, $\frac{3}{8}$ of the whole square is now shaded. Thus we write

$$\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$$



To draw a picture which represents the product $\frac{3}{5} \times \frac{3}{4}$ we divide the unit square into 4 equal parts along the number line and then divide the other side of the square into 5 equal parts. As the following picture shows, the square is now divided into 20 equal parts. Each part therefore represents the fraction $\frac{1}{20}$. If we shade the portion of the square from 0 to $\frac{3}{4}$ along the number line which lies below the line representing $\frac{3}{5}$ we find

that 9 parts of the square are shaded. Thus we write $\frac{3}{5} \times \frac{3}{4} = \frac{9}{20}$



Do you now see the general rule for multiplying fractions? Look at the results of our examples:

$$\frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$$
$$\frac{1}{2} \times \frac{3}{4} = \frac{3}{8}$$
$$\frac{3}{5} \times \frac{3}{4} = \frac{9}{20}$$

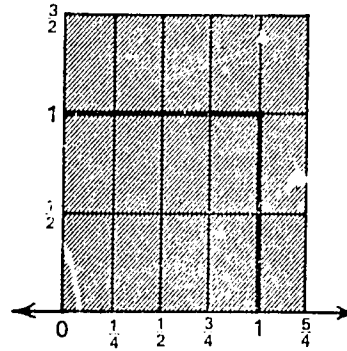
In each case the numerator of the product is the product of the numerators of the factors, and the denominator of the product is the product of the denominators. Thus, in general, we write

$$\boxed{\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}}$$

In all of the above examples the fractions were proper fractions. However, the pictures and rules work equally well if one or both of the factors are improper fractions. For example, suppose, we want to find the product $\frac{3}{2} \times \frac{5}{4}$. According to the above formula we have

$$\frac{3}{2} \times \frac{5}{4} = \frac{3 \times 5}{2 \times 4} = \frac{15}{8}$$

The following picture illustrates this result:



EXERCISE 21–3

Find each of the following products. Express your answers in lowest terms.

1. $\frac{4}{5} \times \frac{2}{3}$
2. $\frac{5}{3} \times \frac{3}{2}$
3. $\frac{1}{4} \times 2\frac{1}{2}$
4. $\frac{5}{3} \times \frac{3}{8}$
5. $\frac{2}{5} \times \frac{5}{4}$
6. $\frac{20}{21} \times \frac{27}{125}$
7. $4\frac{9}{16} \times 3\frac{5}{12}$
8. $2\frac{5}{24} \times 6\frac{11}{30}$
9. $\frac{12}{35} \times \frac{28}{33}$
10. $\frac{117}{385} \times \frac{55}{68}$
11. Draw pictures using squares to illustrate questions 1–5.
12. Five children share $\frac{3}{4}$ of a cake equally between them. What part of the cake does each get? Illustrate your answer on the number line.
13. Two brothers and a friend pick enough nuts to fill a basket $3\frac{1}{2}$ times. They then share the nuts equally. What is each boy's share? Illustrate your answer on the number line.

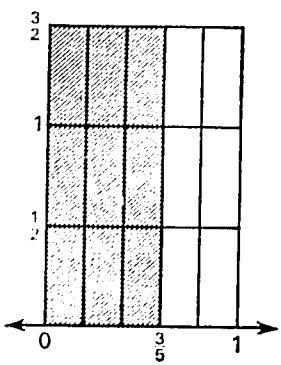
21-4 The commutative property of multiplication

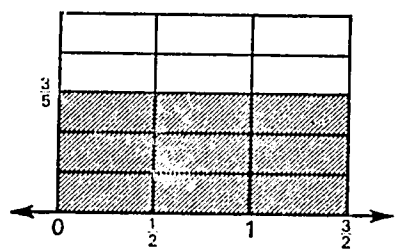
For any two whole numbers a and b, $a \times b = b \times a$, or as we usually write it $ab = ba$. We now ask if the same thing is true for fractions. Does $\frac{3}{2} \times \frac{3}{5} = \frac{3}{5} \times \frac{3}{2}$? Does $\frac{1}{2} \times \frac{3}{4} = \frac{3}{4} \times \frac{1}{2}$? In general, does $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$? We shall show that it does.

To answer these questions we must write equations. For instance, since

$$\frac{3}{2} \times \frac{3}{5} = \frac{9}{10} \quad \text{and} \quad \frac{3}{5} \times \frac{3}{2} = \frac{9}{10}$$

we see that $\frac{3}{2} \times \frac{3}{5} = \frac{3}{5} \times \frac{3}{2}$. The following pictures illustrate this result:





These pictures show the same rectangle in different positions. In each picture the shaded area contains 9 equal parts, each of which is $\frac{1}{10}$ of the unit square. Thus both pictures represent the fraction $\frac{9}{10}$, and we have

$$\tfrac{3}{2} \times \tfrac{3}{5} = \tfrac{3}{5} \times \tfrac{3}{2}$$

In exactly the same way we can show that $\frac{1}{2} \times \frac{3}{4} = \frac{3}{4} \times \frac{1}{2}$. We now consider the general case. Here the equations are

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d} \quad \text{and} \quad \frac{c}{d} \times \frac{a}{b} = \frac{c \times a}{d \times b}$$

But since multiplication of whole numbers is commutative we have $a \times c = c \times a$ and $b \times d = d \times b$. Thus

$$\frac{a \times c}{b \times d} = \frac{c \times a}{d \times b}$$

and it follows that

$$\boxed{\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}}$$

We describe this identity by saying that multiplication of fractions is *commutative*.

EXERCISE 21-4

1. What does the identity $\frac{a}{b} \times \frac{c}{d} = \frac{c}{d} \times \frac{a}{b}$ become in each of the following cases?
   a. When $a = 3$, $b = 5$, $c = 2$, $d = 7$
   b. When $a = 4$, $b = 3$, $c = 6$, $d = 11$
   c. When $a = 8$, $b = 9$, $c = 12$, $d = 25$
2. Check the commutative property of multiplication for each of the pairs of products in Question 1 above.
3. Draw pictures which illustrate the following equalities:
   a. $\frac{3}{5} \times \frac{7}{4} = \frac{7}{4} \times \frac{3}{5}$
   b. $1\frac{2}{3} \times 2\frac{1}{6} = 2\frac{1}{6} \times 1\frac{2}{3}$

21-5 The associative property of multiplication

We have seen that if a, b and c are whole numbers, then $(a \times b) \times c = a \times (b \times c)$. We now show that this property also holds for multiplication of fractions.

Look at the following example:

$$(\tfrac{1}{2} \times \tfrac{2}{3}) \times \tfrac{3}{4} = \tfrac{2}{6} \times \tfrac{3}{4} = \tfrac{1}{4}$$

Grouping the other way we get

$$\tfrac{1}{2} \times (\tfrac{2}{3} \times \tfrac{3}{4}) = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}$$

Both answers are the same. Thus

$$(\tfrac{1}{2} \times \tfrac{2}{3}) \times \tfrac{3}{4} = \tfrac{1}{2} \times (\tfrac{2}{3} \times \tfrac{3}{4})$$

The same thing happens in the general case. For them we have

$$\left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f} = \frac{a \times c}{b \times d} \times \frac{e}{f} = \frac{(a \times c) \times e}{(b \times d) \times f}$$

$$\frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right) = \frac{a}{b} \times \left(\frac{c \times e}{d \times f}\right) = \frac{a \times (c \times e)}{b \times (d \times f)}$$

Using the associativity of multiplication of whole numbers we can write

$$(a \times c) \times e = a \times (c \times e) \text{ and } (b \times d) \times f = b \times (d \times f)$$

Thus
$$\frac{(a \times c) \times e}{(b \times d) \times f} = \frac{a \times (c \times e)}{b \times (d \times f)}$$

and it follows that

$$\boxed{\left(\frac{a}{b} \times \frac{c}{d}\right) \times \frac{e}{f} = \frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right)}$$

We describe this identity by saying that multiplication of fractions is *associative*. It tells us that when we multiply fractions we can group the factors any way we please. The answer will always be the same.

EXERCISE 21-5

1. Check the following products to see that the associative property of multiplication holds.
   a. $(\frac{7}{3} \times \frac{15}{13}) \times \frac{10}{9} = \frac{7}{3} \times (\frac{15}{13} \times \frac{10}{9})$
   b. $(\frac{5}{12} \times \frac{3}{8}) \times \frac{4}{25} = \frac{5}{12} \times (\frac{3}{8} \times \frac{4}{25})$
   c. $2\frac{5}{9} \times (1\frac{4}{7} \times 3\frac{11}{12}) = (2\frac{5}{9} \times 1\frac{4}{7}) \times 3\frac{11}{12}$
2. What does the associative property of multiplication say in each of the following cases?

   a. $\frac{a}{b} = \frac{14}{23}$ $\quad \frac{c}{d} = \frac{3}{11}$ $\quad \frac{e}{f} = \frac{17}{5}$

   b. $\frac{a}{b} = \frac{37}{12}$ $\quad \frac{c}{d} = \frac{9}{17}$ $\quad \frac{e}{f} = \frac{26}{15}$

21-6 The distributive property

In our work with whole numbers we saw that addition and multiplication are related to one another by the distributive property. This property says that

$$a \times (b + c) = (a \times b) + (a \times c)$$

for any three whole numbers a, b and c. Here again the same property holds for fractions.

As an example, we show that

$$\tfrac{1}{2} \times (\tfrac{2}{3} + \tfrac{3}{4}) = (\tfrac{1}{2} \times \tfrac{2}{3}) + (\tfrac{1}{2} \times \tfrac{3}{4})$$

In the first place

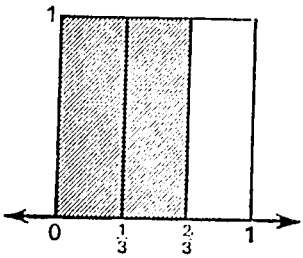
$$\begin{aligned} \tfrac{1}{2} \times (\tfrac{2}{3} + \tfrac{3}{4}) &= \tfrac{1}{2} \times (\tfrac{8}{12} + \tfrac{9}{12}) \\ &= \tfrac{1}{2} \times \tfrac{17}{12} \\ &= \tfrac{17}{24} \end{aligned}$$

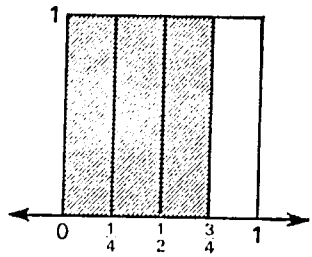
On the other hand

$$\begin{aligned} (\tfrac{1}{2} \times \tfrac{2}{3}) + (\tfrac{1}{2} \times \tfrac{3}{4}) &= \tfrac{1}{3} + \tfrac{3}{8} \\ &= \tfrac{8}{24} + \tfrac{9}{24} \\ &= \tfrac{17}{24} \end{aligned}$$

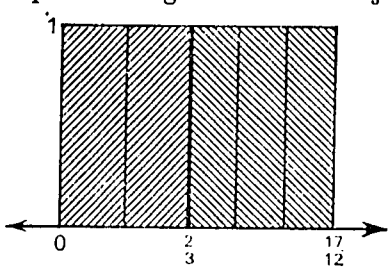
Since the two results are the same, the distributive property holds in this case.

We can also illustrate this result by pictures. We start by representing the fractions $\frac{2}{3}$ and $\frac{3}{4}$ as shown below.

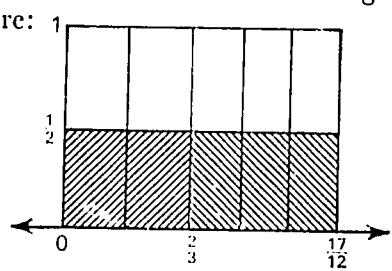




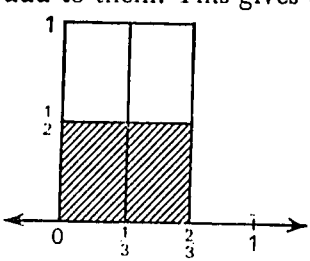
By putting these pictures together the sum $\frac{2}{3} + \frac{3}{4}$ appears as follows:

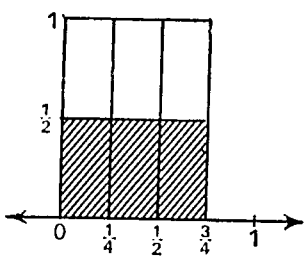


Finally, to show what happens when we multiply $\frac{2}{3} + \frac{3}{4}$ by $\frac{1}{2}$ we shade only the lower half of this rectangle. This gives the following picture:

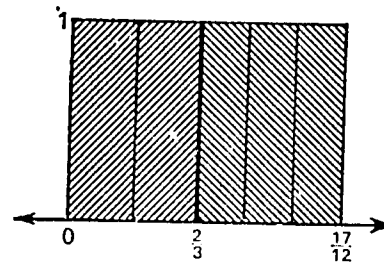


Now consider $(\frac{1}{2} \times \frac{2}{3}) + (\frac{1}{2} \times \frac{3}{4})$. This time we must shade the lower halves of the rectangles representing $\frac{2}{3}$ and $\frac{1}{2}$ before we add to them. This gives the pictures





We now add by putting these pictures side by side. The final picture is the same one that we had when we added first and then multiplied by $\frac{1}{2}$.



This again shows that $\frac{1}{2} \times (\frac{2}{3} + \frac{3}{4}) = (\frac{1}{2} \times \frac{2}{3}) + (\frac{1}{2} \times \frac{3}{4})$

This result is a special case of the following general formula:

$$\boxed{\frac{a}{b} \times \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} \times \frac{c}{d}\right) + \left(\frac{a}{b} \times \frac{e}{f}\right)}$$

This identity is known as the *distributive property* for fractions. Since the computations required to show this result are somewhat long we shall omit them. However, it would give you good practice in working with fractions to try them for yourself.

EXERCISE 21-6

1. Draw a picture which illustrates the following equality:

$$2 \times (\tfrac{2}{3} + \tfrac{3}{2}) = (2 \times \tfrac{2}{3}) + (2 \times \tfrac{3}{2})$$

2. By computing the sums and products show that the following equalities are true.

   a. $\frac{5}{3} \times (\frac{7}{2} + \frac{4}{9}) = (\frac{5}{3} \times \frac{7}{2}) + (\frac{5}{3} \times \frac{4}{9})$
   b. $\frac{9}{5} \times (\frac{3}{8} + \frac{5}{9}) = (\frac{9}{5} \times \frac{3}{8}) + (\frac{9}{5} \times \frac{5}{9})$
   c. $2\frac{3}{4} \times (1\frac{4}{5} + 5\frac{7}{8}) = (2\frac{3}{4} \times 1\frac{4}{5}) + (2\frac{3}{4} \times 5\frac{7}{8})$

3. By computing the sums and products show that the identity

$$\frac{a}{b} \times \left(\frac{c}{d} + \frac{e}{f}\right) = \left(\frac{a}{b} \times \frac{c}{d}\right) + \left(\frac{a}{b} \times \frac{e}{f}\right)$$

is true.

21-7 The property of 1

When we studied multiplication of whole numbers we found that the number 1 has the special property that when we multiply any whole number by it we get the original number back again. In other words,

$$a \times 1 = a$$

for any whole number a. We described this property by saying that 1 is the identity for multiplication of whole numbers. We now show that the same thing happens when we multiply a fraction by 1.

Let us start with an example: Find $\frac{3}{4} \times 1$.

To compute this product we replace 1 by the equivalent fraction $\frac{4}{4}$ and multiply. This gives

$$\frac{3}{4} \times 1 = \frac{3}{4} \times \frac{4}{4} = \frac{3 \times 4}{4 \times 4} = \frac{3}{4}$$

Thus the product of $\frac{3}{4}$ and 1 is $\frac{3}{4}$.

Similarly $$\frac{7}{5} \times 1 = \frac{7}{5} \times \frac{5}{5} = \frac{7 \times 5}{5 \times 5} = \frac{7}{5}$$

It is easy to show that this must always happen. For let $\frac{a}{b}$ be any fraction. Then

$$\frac{a}{b} \times 1 = \frac{a}{b} \times \frac{b}{b} = \frac{a \times b}{b \times b} = \frac{a}{b}$$

Moreover, since multiplication of fractions is commutative we also have $1 \times \frac{a}{b} = \frac{a}{b}$. Thus

$$\frac{a}{b} \times 1 = \frac{a}{b} \quad \text{and} \quad 1 \times \frac{a}{b} = \frac{a}{b}$$

This shows that 1 is the *identity* for multiplication of fractions.

21-8 Division of fractions

In an earlier chapter we studied multiplication problems such as

$$6 \times \square = 12$$

where we were asked to find the missing factor. As you know, such problems are called division problems and we write

$$\square = 12 \div 6$$

Problems like this also occur with fractions. For instance in the equation

$$\tfrac{2}{3} \times \square = \tfrac{3}{4}$$

we must find a missing factor. As before we say that the missing factor is found by *dividing* $\frac{3}{4}$ by $\frac{2}{3}$. We write this as

$$\square = \tfrac{3}{4} \div \tfrac{2}{3}$$

Thus the two equations

$$\tfrac{2}{3} \times \square = \tfrac{3}{4}$$
$$\square = \tfrac{3}{4} \div \tfrac{2}{3}$$

say the same thing. The same number makes each of them a true sentence.

To learn how to solve such equations let us start with the simple problem

$$\square = 1 \div \tfrac{2}{3}$$

Here the corresponding multiplication equation is

$$\frac{2}{3} \times \square = \frac{2}{3} \times \frac{a}{b} = \frac{2 \times a}{3 \times b}$$

Thus we must choose a and b so that $\dfrac{2 \times a}{3 \times b}$ is equivalent to 1.

But this will happen only if $2 \times a = 3 \times b$. One way to make this last sentence true is to let $a = 3$ and $b = 2$. For then $2 \times a = 2 \times 3 = 6$, and $3 \times b = 3 \times 2 = 6$. Thus

$$\square = \tfrac{3}{2}$$

is the solution of

$$\tfrac{2}{3} \times \square = 1$$

Do you notice anything special about the $\frac{3}{2}$? It is $\frac{2}{3}$ "upside down". We call $\frac{3}{2}$ the INVERSE or RECIPROCAL of $\frac{2}{3}$. Thus 1 divided by $\frac{2}{3}$ equals $\frac{3}{2}$, the inverse of $\frac{2}{3}$. The product of $\frac{2}{3}$ and

its inverse, $\frac{3}{2}$, is 1. Let us look at another example of this type. If we want to solve $\square = 1 \div \frac{5}{6}$ we can write

$$\frac{5}{6} \times \square = 1$$

Is there a fraction that will make this sentence true? Yes there is, and again it is the *inverse* $\frac{6}{5}$ because

$$\frac{5}{6} \times \frac{6}{5} = \frac{5 \times 6}{6 \times 5} = \frac{30}{30} = 1$$

Does this always happen? In other words, is the solution of $\square = 1 \div \frac{a}{b}$ always the inverse of $\frac{a}{b}$? Yes, PROVIDED a IS DIFFERENT FROM ZERO. (The reason for saying that a must be different from zero is that $\frac{b}{a}$ will not be a fraction if $a = 0$. In fact, as we have seen, the symbol $\frac{b}{0}$ is meaningless.) To show that $\frac{b}{a}$ is the solution of $\square = 1 \div \frac{a}{b}$ when a is different from zero we simply write

$$\frac{a}{b} \times \frac{b}{a} = \frac{a \times b}{b \times a} = \frac{a \times b}{a \times b} = 1$$

Thus we have proved the following important fact:

Every fraction different from zero has an INVERSE, *and the product of any fraction and its inverse is one.*

EXERCISE 21-8

1. Write the division equation which corresponds to each of the following multiplication equations:

   a. $\frac{4}{5} \times \square = \frac{5}{8}$ **b.** $\square \times \frac{9}{4} = \frac{7}{10}$

   c. $\square \times 1\frac{3}{5} = 2\frac{1}{2}$ **d.** $3\frac{3}{4} \times \square = 1\frac{7}{8}$

2. Write a multiplication equation which corresponds to each of the following division equations.

   a. $\square = \frac{7}{12} \div \frac{3}{5}$ **b.** $\square = \frac{4}{11} \div \frac{8}{3}$

   c. $\square = \frac{9}{17} \div \frac{12}{11}$ **d.** $\square = \frac{32}{2} \div \frac{10}{7}$

3. What are the inverses of the following fractions?

a. $\frac{5}{7}$ **b.** $\frac{4}{11}$ **c.** $\frac{21}{5}$ **d.** $\frac{01}{32}$
e. $2\frac{5}{8}$ **f.** $5\frac{6}{13}$ **g.** $7\frac{3}{4}$ **h.** 9
i. 5 **j.** 0

4. Solve each of the following equations:

a. $\square = 1 \div \frac{6}{25}$ **b.** $\square = 1 \div 4\frac{2}{3}$
c. $\square \times 5 = 1$ **d.** $6 \times \square = 1$
e. $\square \times \frac{29}{8} = 1$ **f.** $\square = 1 \div 37$
g. $\square = 1 \div 2\frac{5}{17}$

5. Explain why the division equation $\square = 1 \div \frac{0}{b}$ has no solution.

6. Will the division equation $\square = \frac{a}{b} \div \frac{0}{d}$ ever have a solution?

21-9 Division of fractions (continued)

Now that we know how to solve division equations of the type

$$\square = 1 \div \frac{a}{b}$$

when a is different from zero, it is easy to solve all division problems involving fractions. Again we start with an example.

Solve the equation

$$\square = \tfrac{1}{2} \div 3$$

We begin by writing the corresponding multiplication equation

$$3 \times \square = \tfrac{1}{2}$$

What number makes this sentence true? To answer this question we shall multiply *both* sides of the equation by *the same* number. What number shall we use as a multiplier? Let us try $\frac{1}{3}$, the inverse of 3. We then get

$$\tfrac{1}{3} \times (3 \times \square) = \tfrac{1}{3} \times \tfrac{1}{2}$$

Using the associative property of multiplication we can write this equation as

$$(\tfrac{1}{3} \times 3) \times \square = \tfrac{1}{3} \times \tfrac{1}{2}$$

Thus $$1 \times \square = \tfrac{1}{3} \times \tfrac{1}{2}$$

and we have $$\square = \tfrac{1}{3} \times \tfrac{1}{2} = \tfrac{1}{6}$$

This shows that $\frac{1}{2} \div 3 = \frac{1}{6}$.

To see that this answer is correct we go back to the equation $3 \times \square = \frac{1}{2}$ and replace the box by $\frac{1}{6}$. This gives

$$3 \times \square = 3 \times \tfrac{1}{6} = \tfrac{3}{6} = \tfrac{1}{2}$$

So the answer is correct.

Notice how this problem was solved. We started with

$$\square = \tfrac{1}{2} \div 3$$

and wrote the corresponding multiplication equation

$$3 \times \square = \tfrac{1}{2}$$

We then multiplied both sides of this equation by $\frac{1}{3}$, the inverse of 3. The left side of the equation then became $1 \times \square$, or just $\square$, while the right side became $\frac{1}{3} \times \frac{1}{2}$. Thus

$$\square = \tfrac{1}{3} \times \tfrac{1}{2}$$

It is not hard to see that this method will *always* work. For instance, let us solve

$$\square = \tfrac{2}{3} \div \tfrac{3}{5}$$

This time the corresponding multiplication equation is

$$\tfrac{3}{5} \times \square = \tfrac{2}{3}$$

We now multiply by $\frac{5}{3}$, the inverse of $\frac{3}{5}$. We get

$$\begin{aligned} \tfrac{5}{3} \times (\tfrac{3}{5} \times \square) &= \tfrac{5}{3} \times \tfrac{2}{3} \\ (\tfrac{5}{3} \times \tfrac{3}{5}) \times \square &= \tfrac{5}{3} \times \tfrac{2}{3} \\ 1 \times \square &= \tfrac{5}{3} \times \tfrac{2}{3} \\ \square &= \tfrac{10}{9} \end{aligned}$$

Thus $\frac{2}{3} \div \frac{3}{5} = \frac{10}{9}$. (You should be able to give reasons for each step of this solution.)

To check that this answer is correct we replace the box in $\frac{3}{5} \times \square = \frac{2}{3}$ by $\frac{10}{9}$. This gives

$$\frac{3}{5} \times \frac{10}{9} = \frac{3 \times 10}{5 \times 9} = \frac{30}{45} = \frac{2 \times 15}{3 \times 15} = \frac{2}{3}$$

Hence the answer is correct.

EXERCISE 21-9

1. Find the missing factors.

a. $5 \times \square = \frac{9}{4}$ **b.** $\frac{7}{3} = \square \times 4$
c. $\frac{5}{6} = \frac{1}{3} \times \square$ **d.** $\frac{4}{15} \times \square = \frac{8}{9}$
e. $\square \times 1\frac{1}{2} = 3\frac{1}{4}$ **f.** $\square \times 5\frac{3}{8} = 2\frac{7}{10}$

2. Solve the following division problems. Check your answers by multiplication.

a. $7\frac{1}{2} \div \frac{2}{5}$ **b.** $6 \div \frac{3}{5}$
c. $\frac{18}{7} \div \frac{3}{14}$ **d.** $2\frac{3}{10} \div 1\frac{4}{7}$
e. $\frac{11}{4} \div \frac{29}{7}$ **f.** $\frac{153}{39} \div \frac{47}{141}$
g. $\frac{57}{8} \div \frac{13}{8}$ **h.** $\frac{162}{35} \div \frac{162}{17}$
i. $7\frac{11}{16} \div 4\frac{5}{12}$ **j.** $3\frac{4}{13} \div 12\frac{2}{9}$

21-10 Division of fractions—the general rule

While working the problems at the end of the last section you probably discovered the general rule for solving division problems involving fractions. It is as follows:

Suppose $\frac{a}{b}$ *and* $\frac{c}{d}$ *are fractions with c different from zero. Then the solution of*

$$\square = \frac{a}{b} \div \frac{c}{d}$$

is

$$\frac{a}{b} \times \frac{d}{c}$$

In other words, *to divide one fraction by another multiply the first by the inverse of the second.* This is sometimes described very briefly as "invert and multiply".

How can we show that this rule is true? We just carry out the method used in the last section in the general case. This time we start with

$$\frac{c}{d} \times \square = \frac{a}{b}$$

and multiply by $\frac{d}{c}$. (Notice that c must be different from zero.

For otherwise $\frac{d}{c}$ would be meaningless.) We get

$$\frac{d}{c} \times \left(\frac{c}{d} \times \square\right) = \frac{a}{b} \times \frac{d}{c}$$

$$\left(\frac{d}{c} \times \frac{c}{d}\right) \times \square = \frac{a}{b} \times \frac{d}{c}$$

$$1 \times \square = \frac{a}{b} \times \frac{d}{c}$$

$$\square = \frac{a}{b} \times \frac{d}{c}$$

Thus the rule is true, and we now know how to divide any fraction by any other fraction provided the second fraction is different from zero.

There is one more point to notice about division of fractions. Do you see what it is? Remember that when we studied division of whole numbers we found that it is not always possible to divide one whole number by another and get a whole number answer. For instance the equation $\square = 7 \div 2$ does not have a whole number solution. But with fractions this does not happen. For we have just seen that when $\frac{a}{b}$ and $\frac{c}{d}$ are fractions $\frac{a}{b} \div \frac{c}{d}$ is again a fraction provided that c is different from zero. In other words *the equation*

$$\square = \frac{a}{b} \div \frac{c}{d}$$

can always be solved if c *is different from zero.* This is one of the most important properties of fractions.

EXERCISE 21-10

1. Verify that $\frac{a}{b} \div \frac{c}{d} = \frac{a \times d}{b \times c}$ in each of the following cases.

a. $\frac{a}{b} = \frac{7}{15}$ $\quad \frac{c}{d} = \frac{8}{11}$

b. $\frac{a}{b} = \frac{23}{15}$ $\quad \frac{c}{d} = \frac{5}{12}$

c. $\frac{a}{b} = \frac{9}{16}$ $\quad \frac{c}{d} = \frac{21}{38}$

2. If a man can plough $\frac{3}{10}$ of a field in one hour, how long will it take him to plough the whole field?
3. Five men can paint $\frac{2}{3}$ of a house in one day. How long will it take them to paint the entire house?
4. Three boys can pick $2\frac{1}{4}$ baskets of nuts in $1\frac{1}{8}$ hours. How much can they pick in one hour? How much in $2\frac{1}{2}$ hours?
5. Suppose that $\frac{a}{b}$ and $\frac{c}{d}$ are fractions with c different from zero. Show that

$$\left(\frac{a}{b} \times \frac{c}{d}\right) \div \frac{c}{d} = \frac{a}{b}$$

How would you describe this fact?

6. Show that

$$\left(\frac{a}{b} \div \frac{c}{d}\right) \times \frac{c}{d} = \frac{a}{b}$$

when c is different from zero. How would you describe this fact?

ANSWERS TO SELECTED PROBLEMS

CHAPTER 1

Exercise 1-1

1. **a.** Saturday, Sunday **b.** u, v, w, x, y, z
   c. January, June, July
2. **a, d, e, f, g** describe sets; the others do not.

Exercise 1-3

1. **a.** The set of days in the week whose English names begin with the letter T.
   b. The set made up of the first five letters of the English alphabet.
   c. The set made up of the first three months of the year.
   d. The set made up of the first five even numbers starting with 2.
   e. The set made up of the first five multiples of 5 starting with 5.
2. **a.** 40, 50, 60 **b.** 14, 17, 20
   c. 84, 80, 76 **d.** 10,000, 100,000, 1,000,000
   e. 36, 49, 64 **f.** 2,222, 22,222, 222,222
   g. 66, 55, 44
3. **a, c, d**
4. **a.** {3, 6, 9, 12, . . . , 1,098} **b.** {9, 10, 11, . . . , 99, 100}
   c. {5, 15, 25, 35, . . . } **d.** {4, 7, 10, 13, . . . }
   e. {7, 77, 777, . . . } **f.** {5, 6, 15, 16, 25, 26, . . . }

Exercise 1-4

1. **a.** {November} **b.** { } **c.** { }
   d. { } **e.** {42}
2. No, it contains the whole number 0 as an element.

Exercise 1-5

1. **a, c, g, h**

Exercise 1-6

1. **a.** (i), (ii), (iv), (v) **b.** {2, 4, 6, 12}
   c. {1, 7, 9, 11} **d.** {2, 7, 11}
   e. {4, 7, 9, 11, 12}
2. **a, c**
3. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47

4. a. {36, 38, 40, . . . , 50} **b.** {1, 3, 5, 7, 9, 11, . . . }
c. {2} **d.** { }
e. {3, 9, 15, 21, . . . } **f.** {6, 12, 18, 24, . . . }

5. Yes, because every element of X belongs to Z.

EXERCISE 1-7

1. a. { }, {x}, {y}, {x, y}

b. { }, {▲}, {●}, {▨}, {△, ●}, {△, ▨}, {●, ▨}, {△, ●, ▨}

c. { }, {a}

2. Where A = B

3. There is only one subset of { }. It is { }.

4. The subsets of A are { } and {1}. This shows that every set containing a single element has exactly two subsets.

5. See **1a** above.

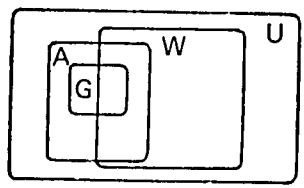
6. See **1b** above.

7. a. There are 16 subsets of {1, 2, 3, 4}. They are
{ }, {1}, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}.

b. When A contains 5 elements A has 32 subsets.
When A contains 6 elements A has 64 subsets.

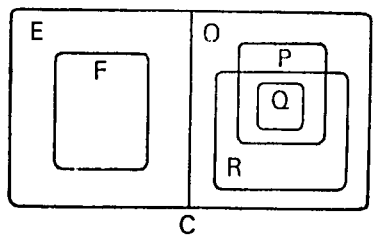
EXERCISE 1-8

3. a. The picture should look like this:



(Other pictures are possible)

4. The picture should look like this:



EXERCISE 1-10

1. The pairs in **a**, **c** and **d** are equivalent. The others are not.

2. B, D and E are equivalent to A. Exact matchings are

$$\begin{array}{l} A: \{1, 2, 3, 4, \ldots\} \\ \quad\;\; \uparrow\;\; \uparrow\;\; \uparrow\;\; \uparrow\;\; \cdots \\ B: \{0, 5, 10, 15, \ldots\} \end{array}$$

$$\begin{array}{l} A: \{1, 2, 3, 4, \ldots\} \\ \quad\;\; \uparrow\;\; \uparrow\;\; \uparrow\;\; \uparrow\;\; \cdots \\ B: \{0, 1, 2, 3, \ldots\} \end{array}$$

$$\begin{array}{l} A: \{1, 2, 3, 4, \ldots\} \\ \quad\;\; \uparrow\;\; \uparrow\;\; \uparrow\;\; \uparrow\;\; \cdots \\ B: \{1, 4, 9, 16, \ldots\} \end{array}$$

4. Yes. An exact matching is

$$\begin{array}{l} \{1, \;\; 2, \;\; 3, \;\; 4, \ldots\} \\ \quad\;\; \searrow\;\; \searrow\;\; \searrow\;\; \searrow \\ \{201, 202, 203, 204, \ldots\} \end{array}$$

CHAPTER 2

EXERCISE 2-1

1. **a.** $A \cup B = \{1, 2, 3, 5, 7, 9\}$ **b.** $A \cup B = \{a, c, e, i, n, o, u\}$
c. $A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21\}$
d. $A \cup B = \{1, 2, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$
e. $A \cup B = \{0, 5, 10, 15, \ldots, 100\}$

2. **a.** 15 elements. This happens when A and B have no elements in common.
b. 10 elements. This happens when A is a subset of B.

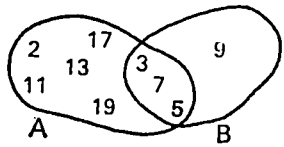
3. Yes. They contain the same elements.

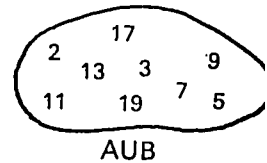
4. **a.** Yes, the empty set is a subset of every set.
b. $A \cup B = \{\ \}$

5. Every element of A belongs to $A \cup B$; every element of B belongs to $A \cup B$.

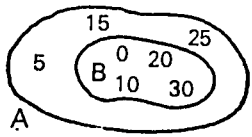
EXERCISE 2-2

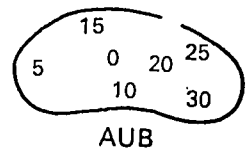
4.



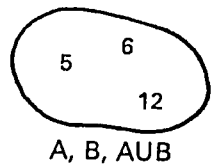


5.





6.



EXERCISE 2-3

1. a. $A \cap B = \{5, 8\}$ **b.** $A \cap B = \{n\}$
c. $A \cap B = \{12, 24, 36\}$ **d.** $A \cap B = \{20, 40, 60, \ldots\}$
e. $A \cap B = \{\ \}$

2. a. 5. This happens when A is a subset of B.
b. 0. This happens when A and B have no elements in common.

3. Yes. They both contain the same elements.

4. $A = B$

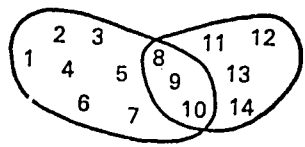
5. Every element of $A \cap B$ belongs to A and to B.

6. Yes. Every element of $A \cap B$ belongs to $A \cup B$.

EXERCISE 2-4

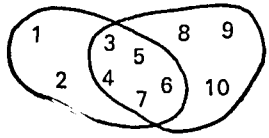
2. a. 14

b.



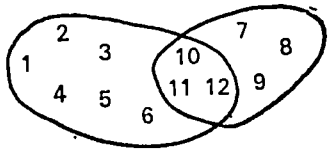
3. 7

4. a.



b. 7 **c.** 8 **d.** 3

5. a.



b. 9 **c.** 6 **d.** 3

6. C is a subset of $A \cap B$

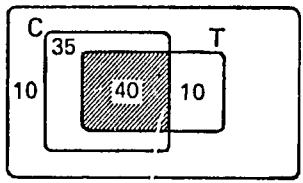
EXERCISE 2-5

1. a. $A \cap C = \{2, 3, 7\}$
b. $A \cup C = \{0, 1, 2, \ldots, 12\}$
c. $B \cap C = \{2, 3, 10, 12\}$
d. $A \cap (B \cup C) = \{2, 3, 7, 9, 11\}$
e. $A \cap (B \cap C) = \{2, 3\}$
f. $(A \cap B) \cap C = \{2, 3\}$
g. $B \cup (A \cap C) = \{2, 3, 7, 9, 10, 11, 12, 13, 14\}$
h. $(A \cap B) \cup (A \cap C) = \{2, 3, 7, 9, 11\}$
i. $(A \cup B) \cup C = \{0, 1, 2, \ldots, 14\}$
j. $(A \cup B) \cap (A \cup C) = \{1, 2, 3, 4, 6, 9, 10, 11, 12\}$
k. $A \cup (B \cap C) = \{1, 2, 3, 4, 6, 9, 10, 11, 12\}$
l. $A \cup (B \cup C) = \{0, 1, 2, \ldots, 14\}$

3. The sets are equal.

4. At least 4 must be punished.

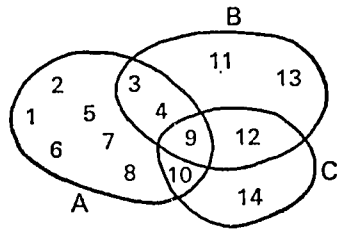
5.



The picture shows $35 + 40 = 75$ in set C,
$40 + 10 = 50$ in set T,
10 not in T or in C.

Altogether there are $10 + 35 + 40 + 10 = 95$. Thus the statement is false.

6.



There are 14 elements in all.

CHAPTER 3

Exercise 3-1

1. A and C are equivalent.
E and F are equivalent.
B and H are equivalent.
D, G and I are equivalent.

3. The second is correct because it shows that the sets match exactly.

4. The empty set.

Exercise 3-2B

1. The three sets are equivalent.

2. C has at least as many elements as A.

3. A has more elements than B.

4. **a.** (iv) They are the same.
c. 10
d. 52

Exercise 3-3

1. B, D, E, A, C. C has the most elements; B has the fewest.

2. A has one more element than C.

3. S, B, C, T

4. D, B, A
D, C, A
E, B, A
E, C, A
F, B, A
F, C, A

CHAPTER 4

Exercise 4-2

2. a. $5 < 7$ **b.** $12 > 0$ **c.** $2 + 2 = 4$
d. $2 + 2 < 3 + 2$ **e.** $9 + 7 > 8 + 7$ **f.** $14 + 12 = 13 + 13$
g. $6 + 8 + 10 > 5 + 7 + 9$

3. a. $\{0, 1, 2, 3, 4, 5\}$ **b.** $\{26, 27, 28, \ldots\}$
c. $\{13, 14, 15, 16, 17\}$ **d.** $\{0, 3, 6, 9, 12, 15, 18\}$
e. $\{12, 14, 16, \ldots, 28\}$

4. Yes. The number 0.

5. No, there is no largest whole number.

6. Yes.

7. A has fewer elements than B.

Exercise 4-5

2. 10 **3.** 4 **4.** 6 in each

CHAPTER 5

Exercise 5-5

2. 13 is four units to the right of 9.
5 is four units to the left of 9.

3. 7 **4.** 30 **5.** 7 **6.** 5

7. a. 8 **b.** 24
c. $3n + 2$; i.e. three times n plus 2.

CHAPTER 6

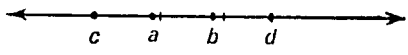
Exercise 6-1

1. The whole number m is less than the whole number n if m is to the left of n on the number line.

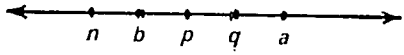
Exercise 6-2

2. a. Yes. For instance, $a = 6$ and $c = 7$.
b. Yes.
c. Yes. For instance, $a = 7$ and $c = 6$.

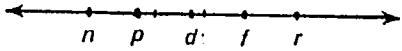
3.



4.



5. a.



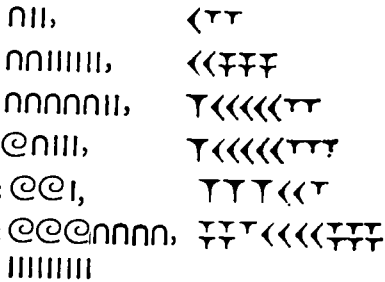
b. $p < a$, $b < p$, $n < q$

EXERCISE 6-3

2. a. 3 **b.** 8 **c.** $b - a - 1$

CHAPTER 7

EXERCISE 7-3

12: 

2. (i) 2,134,

(ii) 5,213,

(iii) 3,048,

3. a. ten-thousands **b.** hundred-thousands
c. millions **d.** ten-millions

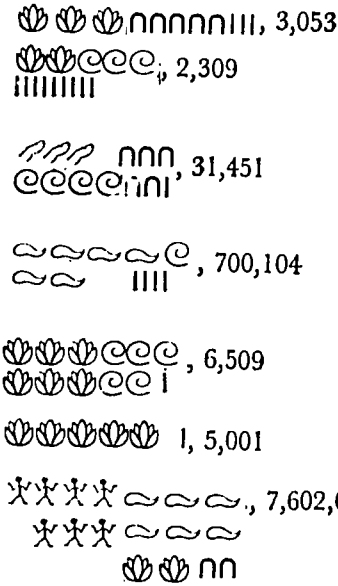
7. a. 276,133 **b.** 7,132,432
c. 44,444,444 **d.** 12,345,678

EXERCISE 7-7A

2. $30 + 6 = 36$ **3.** $70 + 1 = 71$ **4.** $90 + 5 = 95$

EXERCISE 7-7B

EXERCISE 7-8

3. 

EXERCISE 7-9

1. MCXIII **2.** MMCMIV **3.** XXXIV **4.** XCV

CHAPTER 8

EXERCISE 8-1

2. 4 is the largest one-digit number.
24 is the largest two-digit number.
100 is the largest three-digit number.

EXERCISE 8-2

1. a. 302_{five}, 77_{ten} **b.** $3{,}021_{five}$, 386_{ten}
c. $1{,}234_{five}$, 194_{ten}

CHAPTER 9

EXERCISE 9-2

1. a. 8 **b.** 8 **c.** 10 **d.** 10

3. They are the same; i.e., $a + b = b + a$.

4. a. $b = 0$ **b.** $b > 0$

5. $a = 0$ and $b = c$.

EXERCISE 9-6A

1. 2 **2.** 7 **3.** 11 **4.** 8 **5.** 2 **6.** 4

EXERCISE 9-6B

2. 6

3. **a.** None.

b. The children with numbers 1 and 7 are without partners when the sum is 14.
The children with numbers 1, 2, 3 and 8 are without partners when the sum is 16.

c. The children with numbers 1, 2, 3, 4, 5, 6, 7 and 10 are without partners when the sum is 20.
All the children except those with numbers 11 and 12 are without partners when the sum is 23.
All the children are without partners when the sum is 24.

4. The pairs are (1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)
The sums are 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12.
Seven is most likely to turn up because more of the sums are 7 than any other number.

5. **a.** 6 **b.** 11 **c.** 1,001 **d.** 1

6. *b* is an even number.

EXERCISE 9-7

2. All answers are 0.

3. From the property of 0 we know that $0 + \mathrm{I} = \mathrm{I}$. But, by putting 0 in the boxes in $\square + \mathrm{I} = \square$ we have $0 + \mathrm{I} = 0$. Together these two statements say that $\mathrm{I} = 0$.

CHAPTER 10

EXERCISE 10-2

1. **a.** $3 + 4 = 4 + 3$ **b.** $6 + 3 = 3 + 6$

2. **a.** $3 + 4 = 4 + 3$ **b.** $5 + 4 = 4 + 5$

4. **a.** 2 **b.** 9 **c.** 7 **d.** 4 **e.** 10 **f.** 0

5. **a.** and **b.** Any whole number.

EXERCISE 10-3

2. 9. It is not an identity since it is not true for all whole numbers.

3. Yes, because it is true for all whole numbers.

4. **a.** $3 + 1 = 1 + 3$ **b.** $5 + 2 = 2 + 5$
c. $0 + 4 = 4 + 0$ **d.** $27 + 52 = 52 + 27$

5. **a.** 8 **b.** 11 **c.** 2

EXERCISE 10-5

1. **a.** $(1 + 2) + 1 = 1 + (2 + 1)$
b. $(4 + 3) + 0 = 4 + (3 + 0)$
c. $(5 + 5) + 2 = 5 + (5 + 2)$
d. $(3 + 7) + 6 = 3 + (7 + 6)$

2. **a.** 5 **b.** 4 **c.** 2 **d.** 2 **e.** 7

3. **a.** 0 **b.** any whole number **c.** 6
d. any whole number **e.** 2

4. **b.** and **d.** are identities; the others are not.

EXERCISE 10-6A

1. **a.** and **b.** They have the same length.

2. **a.** The segment from 0 to 3.
b. The segment from 0 to 4.
c. The segment in **b** is larger.
d. The segment from a to b is longer than the segment from c to d if $b - a$ is greater than $d - c$. They are of equal length if $b - a = d - c$.

EXERCISE 10-6B

4. The whole numbers starting with 5 and ending with 13.

5. $1 + 2 + 3 + 4 + 5 = 15$

6. 9

EXERCISE 10-7

4. **a.** $(2 + 3) + 4 = (4 + 3) + 2$
b. associative, commutative

EXERCISE 10-8

1. associative, commutative, commutative, associative

2. Yes, for both questions.

3. There are 26 combinations. They are
$1 + 14$, $2 + 13$, $3 + 12$, $1 + 2 + 12$, $4 + 11$, $1 + 3 + 11$, $5 + 10$, $1 + 4 + 10$, $2 + 3 + 10$, $6 + 9$, $5 + 1 + 9$,

$4 + 2 + 9$, $3 + 2 + 1 + 9$, $7 + 8$, $6 + 1 + 8$, $5 + 2 + 8$, $4 + 3 + 8$, $4 + 2 + 1 + 8$, $7 + 6 + 2$, $7 + 5 + 3$, $7 + 5 + 2 + 1$, $7 + 4 + 3 + 1$, $6 + 5 + 4$, $6 + 5 + 3 + 1$, $6 + 4 + 3 + 2$, $5 + 4 + 3 + 2 + 1$

4. The first player announces the number 1, 7, 13, 19, 25, 31.

CHAPTER 11

Exercise 11-1

1. 4 **2.** 5 **3.** 12 **4.** 5 **5.** 0
6. 10 **7.** 14 **8.** 8 **9.** 18 **10.** 12
11. $2 + \square = 10$, $\square = 8$ **12.** $2 + \square = 7$, $\square = 5$
13. $5 + \square = 12$, $\square = 7$ **14.** $12 + \square = 25$, $\square = 13$
15. $4 + \square = 17$, $\square = 13$

Exercise 11-2

1. a. $\square = 7 - 5$ **b.** $\square = 9 - 5$ **c.** $\square = 26 - 17$
d. $\square = 20 - 11$ **e.** $\square = 12 - 12$ **f.** $\square = 19 - 8$

2. a. $6 + \square = 8$ or $\square + 6 = 8$
b. $4 + \square = 4$ or $\square + 4 = 4$
c. $2 + \square = 9$ or $\square + 2 = 9$
d. $7 + \square = 8$ or $\square + 7 = 8$
e. $9 + \square = 15$ or $\square + 9 = 15$
f. $14 + \square = 31$ or $\square + 14 = 31$

3. a. $3 + \square = 8$, $\square = 8 - 3$, $\square = 5$
b. $6 + \square = 15$, $\square = 15 - 6$, $\square = 9$
c. $7 + \square = 12$, $\square = 12 - 7$, $\square = 5$
d. $9 + \square = 27$, $\square = 27 - 9$, $\square = 18$
e. $15 + \square = 32$, $\square = 32 - 15$, $\square = 17$

Exercise 11-3

1. $\square = 13 - 6$, $\square = 13 - 7$
2. $\square = 14 - 5$, $\square = 14 - 9$
3. $\square = 6 - 0$, $\square = 6 - 6$
4. $\square = 12 - 9$, $\square = 12 - 3$
5. $\square = 14 - 7$
6. $\square = 15 - 15$, $\square = 15 - 0$

Exercise 11-4

1. a. $7 - 3 = 4$ **b.** $9 - 4 = 5$ **c.** $8 - 4 = 4$

3. a. $A - B = \{a, c, e, g\}$
The number of elements in $A - B$ is equal to the number of elements in A minus the number of elements in B. This happens when B is a subset of A.

b. $A - B = \{a, c, d, e\}$
The number of elements in $A - B$ is greater than the number of elements in A minus the number of elements in B. This always happens when B contains elements which do not belong to A.

EXERCISE 11-5

1. 0	**2.** 7	**3.** No whole number
4. 1	**5.** No whole number	**6.** 0

EXERCISE 11-6

1. a. $(12 + 4) - 4 = 12$, $(12 - 4) + 4 = 12$
b. $(5 + 0) - 0 = 5$, $(5 - 0) + 0 = 5$
c. $(0 + 0) - 0 = 0$, $(0 - 0) + 0 = 0$

2. a. When B is a subset of A.
b. When A and B are disjoint; that is, when they have no elements in common.

EXERCISE 11-7

2. All whole numbers starting with 1 and ending with 11.
3. 6
4. 2. Eleven points are marked.
5. Subtraction is not commutative.

EXERCISE 11-8

2. Subtract 4 from both sides of $4 < 9$. This gives $0 < 5$.
3. a. Subtract a from both sides of $a < b$. This gives $0 < b - a$.
b. Add a to both sides of $0 < b - a$. This gives $a < b$.

CHAPTER 12

EXERCISE 12-2

1. 27	**2.** 10	**3.** 10	**4.** 7
5. 7	**6.** 20	**7.** 30	**8.** 40

EXERCISE 12-3

1. Let the people be A, B, C. The chart is

	fish	beef	curry
A	(A, fish)	(A, beef)	(A, curry)
B	(B, fish)	(B, beef)	(B, curry)
C	(C, fish)	(C, beef)	(C, curry)

There are 9 combinations.

2. 35

EXERCISE 12-6

2. $1 \times a$ means a added to itself 1 time. Thus $1 \times a = a$.
$b \times 1$ means 1 added to itself b times. Thus

$$b \times 1 = 1 + 1 + \ldots + 1$$

Since there are b ones in the sum the total is b. Thus $b \times 1 = b$.

CHAPTER 13

EXERCISE 13-1

2. 3 3. 12

4. When a whole number is multiplied by 1 the number does not change. When 0 is added to a whole number the number does not change. Thus we call 1 the *identity* for multiplication and 0 the *identity* for addition.

EXERCISE 13-5

1. **a.** $(2 \times 2) + (2 \times 5) = 4 + 10 = 14$
$2 \times (2 + 5) = 2 \times 7 = 14$
b. $(7 \times 6) + (8 \times 6) = 42 + 48 = 90$
$(7 + 8) \times 6 = 15 \times 6 = 90$
c. $(3 \times 4) + (3 \times 5) = 12 + 15 = 27$
$3 \times (4 + 5) = 3 \times 9 = 27$

3. **a.** 4 **b.** 4 **c.** 6 **d.** 2 **e.** 6

4. $(2 + 3) \times (6 + 4) = (2 + 3) \times 6 + (2 + 3) \times 4$
$= (2 \times 6) + (3 \times 6) + (2 \times 4) + (3 \times 4)$

5. $(a + b) \times (c + d) = (a + b) \times c + (a + b) \times d$
$= (a \times c) + (b \times c) + (a \times d) + (b \times d)$

EXERCISE 13-6

1. Statements (*c*), (*e*) and (*f*) are correct. The others are not correct.
3. 4, 5, 6
4. $(1 + 2 + 3) \times 4 = 6 \times 4 = 24$
   Also $(1 \times 4) + (2 \times 4) + (3 \times 4) = 4 + 8 + 12 = 24$

EXERCISE 13-8

1. **a.** $7 \times 6 = 7 \times (5 + 1)$
$$\begin{aligned} &= (7 \times 5) + (7 \times 1) \\ &= (5 + 2) \times 5 + (7 \times 1) \\ &= (5 \times 5) + (2 \times 5) + (7 \times 1) \\ &= 25 + 10 + 7 \\ &= 42 \end{aligned}$$
   The problem can be worked in other ways as well.

   f. $11 \times 11 = (5 + 5 + 1) \times 11$
$$\begin{aligned} &= (5 \times 11) + (5 \times 11) + (1 \times 11) \\ &= 5 \times (5 + 5 + 1) + 5 \times (5 + 5 + 1) + (1 \times 11) \\ &= (5 \times 5) + (5 \times 5) + (5 \times 1) + (5 \times 5) \\ &\qquad + (5 \times 5) + (5 \times 1) + (1 \times 11) \\ &= 25 + 25 + 5 + 25 + 25 + 5 + 11 \\ &= 121 \end{aligned}$$

CHAPTER 14

EXERCISE 14-1

1. **a.** $\square = 45 \div 5$, $\square = 9$ **b.** $\square = 27 \div 3$, $\square = 9$ **c.** $\square = 54 \div 6$, $\square = 9$
   d. $\square = 84 \div 7$, $\square = 12$ **e.** $\square = 36 \div 9$, $\square = 4$
2. Yes, because $0 \times \square = 0$ for all whole numbers.
3. Yes, because $1 \times \square = \square$ for all whole numbers.
4. None, because $0 \times \square = 0$ for all whole numbers.
5. $4 = 28 \div 7$ and $7 = 28 \div 4$
6. The factors are 1, 2, 3, 4, 6, 8, 12, 24.
7. $3 \times \square = 18$, $\square = 18 \div 3$. The number is 6.
8. $(\square \div 5) \times 3 = 6$
   $\square \div 5 = 6 \div 3$
   $\square \div 5 = 2$
   $\square = 2 \times 5$
   $\square = 10$

9. The number is 54. To find it compute the product

$$2 \times 3 \times 3 \times 3$$

EXERCISE 14-2

1. $6 \div 3 = 2$ **2.** $12 \div 4 = 3$

3. $20 \div 4 = 5$ **4.** $5 \div 5 = 1$

EXERCISE 14-3

1. a. $26 - 13 = 13$
$13 - 13 = 0$
Thus $26 \div 13 = 2$

b. $21 - 7 = 14$
$14 - 7 = 7$
$7 - 7 = 0$
Thus $21 \div 7 = 3$

c. 7 **d.** 8 **e.** 5 **f.** 0

EXERCISE 14-4

1. a. 6 **b.** 7, remainder 2 **c.** 7, remainder 5
d. 7 **e.** 7, remainder 7 **f.** 10, remainder 2

2. a. 1, 2, 3, 4, 12 **b.** 1, 23
c. 1, 2, 4, 7, 14, 28 **d.** 1, 2, 3, 4, 6, 9, 12, 18, 36
e. 1, 17 **f.** 1, 2, 4, 5, 10, 20, 25, 50, 100
g. 1 **h.** all counting numbers

3. 7 gallons, 2 shillings left

4. 75

5. a. 2 **b.** 34 **c.** 5
d. 4 **e.** 0 **f.** 6

EXERCISE 14-7

1. $36 = 2 \times 2 \times 3 \times 3$ **2.** 29 is a prime number.

3. $8 = 2 \times 2 \times 2$ **4.** $100 = 2 \times 2 \times 5 \times 5$

5. $91 = 7 \times 13$ **6.** $64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$

7. $60 = 2 \times 2 \times 3 \times 5$ **8.** $500 = 2 \times 2 \times 5 \times 5 \times 5$

9. $3{,}000 = 2 \times 2 \times 2 \times 3 \times 5 \times 5 \times 5$

10. $102 = 2 \times 3 \times 17$

EXERCISE 14-8

1. 1, 3 **2.** 1, 2, 3, 6

3. 1, 2, 4, 5, 8, 10, 20, 40 **4.** 1, 2, 3, 4, 6, 12

5. 1, 3, 9 **6.** 1, 2, 4, 7, 14, 28

7. 1, 7 **8.** 1

9. 1, 11 **10.** 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60

CHAPTER 15

Exercise 15-2

1, 2, 3. No for all three.

CHAPTER 17

Exercise 17-2

1. 2,350 **2.** 10 **3.** 3,700 **4.** 100 **5.** 21
6. 9 **7.** 920 **8.** 100,000 **9.** 40 **10.** 90

11. a. (i) 40_{five} (ii) 120_{five} (iii) $2{,}340_{five}$ (iv) 300_{five}
(v) $3{,}100_{five}$ (vi) $2{,}000_{five}$ (vii) $121{,}000_{five}$
c. Yes

Exercise 17-3

The products are as follows.

1. 608 **2.** 413 **3.** 498 **4.** 442 **5.** 4,891
6. 1,710 **7.** 2,975 **8.** 8,008 **9.** 16,337 **10.** 117,780

11. a. 231_{five} **b.** 224_{five} **c.** $1{,}130_{five}$
d. $2{,}412_{five}$ **e.** $10{,}012_{five}$

Exercise 17-4

The products are as follows.

1. 3,312 **2.** 17,248 **3.** 19,497
4. 7,584 **5.** 360,372 **6.** 101,888
7. $1{,}013_{five}$ **8.** $12{,}234_{five}$ **9.** $33{,}341_{five}$
10. $111{,}243_{five}$

Exercise 17-5

The quotients are as follows.

1. 91 **2.** 89 **3.** 92 **4.** 42 **5.** 1,005
6. 1,030 **7.** 42_{five} **8.** 231_{five} **9.** 121_{five} **10.** 344_{five}

Exercise 17-6

1. 384 **2.** 709 **3.** 1,281 **4.** 208
5. 4,811

CHAPTER 18

Exercise 18-4

2. $1\frac{1}{3}$, $1\frac{4}{5}$, $1\frac{4}{10}$, $4\frac{2}{4}$
4. $\frac{7}{4}$, $\frac{14}{5}$, $\frac{13}{8}$, $\frac{11}{3}$

EXERCISE 18-5

1. $\frac{72}{7}$ or $10\frac{2}{7}$ **2.** $\frac{13}{4}$ or $3\frac{1}{4}$ **3.** $\frac{96}{6}$ or 16
4. $\frac{147}{24}$ or $6\frac{3}{24}$ **5.** $\frac{2176}{322}$ or $6\frac{244}{322}$ **6.** $\frac{625}{10}$ or $62\frac{5}{10}$
7. $\frac{83}{83}$ or 1 **8.** $\frac{1215}{100}$ or $\frac{15}{100}$
12. $3\frac{1}{4}$ oranges **13.** 2

EXERCISE 18-6

1. $A = \frac{4}{3}B$ **2.** $A = \frac{2}{3}B$
3. $A = \frac{3}{2}B$ **4.** $A = \frac{5}{2}B$

EXERCISE 18-9

1. $\frac{5}{6} > \frac{3}{4}$ is true. **2.** $\frac{5}{2} > \frac{5}{3}$
3. $\frac{2}{3} = \frac{8}{12}$
4. **a.** $\frac{1}{2}$ is greater **b.** $\frac{3}{2}$ is greater
c. $\frac{1}{4}$ is greater **d.** $\frac{9}{4}$ is greater
e. $\frac{1}{100}$ is greater **f.** $\frac{300}{100}$ is greater
5. $\frac{a}{b} > \frac{a}{c}$ is true

CHAPTER 19

EXERCISE 19-1

1. **a.** P : $\frac{4}{16}, \frac{2}{8}, \frac{1}{4}$ **b.** P : $\frac{9}{27}, \frac{3}{9}, \frac{1}{3}$
Q : $\frac{12}{16}, \frac{6}{8}, \frac{3}{4}$ Q : $\frac{24}{27}, \frac{8}{9}$
c. P : $11\frac{6}{18}, 11\frac{3}{9}, 11\frac{1}{3}$ **d.** P : $25\frac{6}{16}, 25\frac{3}{8}$
Q : $11\frac{8}{18}, 11\frac{4}{9}$ Q : $25\frac{10}{16}, 25\frac{5}{8}$
5. $\frac{5}{10}, \frac{6}{12}, \frac{7}{14}$. The fractions **a**, **c** and **d** belong to the set.

EXERCISE 19-2

2. $\frac{1}{5} = \frac{2}{10} = \frac{3}{15} = \frac{4}{20} = \frac{5}{25}$
3. $\frac{1}{4} = \frac{2}{8} = \frac{4}{16}$
4. $\frac{3}{2} = \frac{6}{8} = \frac{9}{1} = \frac{12}{16}$
5. **a.** $\frac{6}{24}, \frac{3}{12}, \frac{2}{8}$
b. $\frac{70}{4}$ **c.** 100
7. $\frac{8}{3}, \frac{28}{21}, \frac{44}{33}, \frac{4{,}000{,}000}{3{,}000{,}000}$
8. **a.** 25 orange trees, 75 banana trees
$\frac{25}{100}, \frac{75}{100}$
b. 150 orange trees, 450 banana trees
$\frac{150}{600}, \frac{450}{600}$
c. No, because $\frac{750}{}$ is not a whole number.

EXERCISE 19-3

1. **a.** $\{\frac{1}{5}, \frac{2}{10}, \frac{3}{15}, \frac{4}{20}, \ldots\}$
   b. $\{\frac{2}{7}, \frac{4}{14}, \frac{6}{21}, \frac{8}{28}, \ldots\}$
   c. $\{\frac{5}{4}, \frac{10}{8}, \frac{15}{12}, \frac{20}{16}, \ldots\}$
   d. $\{\frac{4}{9}, \frac{8}{18}, \frac{12}{27}, \frac{16}{36}, \ldots\}$
   e. $\{\frac{19}{6}, \frac{38}{12}, \frac{57}{18}, \frac{76}{24}, \ldots\}$
2. **a.** (i) $\frac{2}{3}$ (ii) $\frac{2}{1}$ (iii) $\frac{3}{2}$
   (iv) $\frac{7}{19}$ (v) $\frac{91}{51}$
   b. $\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \ldots\}$
   $\{\frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \frac{4}{8}, \ldots\}$
   $\{\frac{3}{2}, \frac{6}{4}, \frac{9}{6}, \frac{12}{8}, \ldots\}$
   $\{\frac{7}{12}, \frac{14}{38}, \frac{21}{57}, \frac{28}{76}, \ldots\}$
   $\{\frac{91}{51}, \frac{182}{102}, \frac{273}{153}, \frac{364}{204}, \ldots\}$
3. **a.** $\frac{2}{1}$ **b.** $\{\frac{2}{1}, \frac{4}{2}, \frac{6}{3}, \frac{8}{4}, \ldots\}$ **c.** 2
4. **a.** $\{\frac{0}{1}, \frac{0}{2}, \frac{0}{3}, \frac{0}{4}, \ldots\}$ **b.** 0
5. **a.** 4 **b.** 25 **c.** 84
   d. 96 **e.** 6 **f.** 15
6. **a.** There is no such number.
   b. 10 **c.** 91
   d. There is no such number.

EXERCISE 19-4

2. The fractions in **d** and **e** are equivalent. The others are not.
4. No, because one fraction cannot be equivalent to two different fractions in lowest terms.

CHAPTER 20

EXERCISE 20-1

1. The sums are as follows.
   a. $\frac{3}{5}$ **b.** $\frac{7}{10}$ **c.** $\frac{5}{3}$ or $1\frac{2}{3}$ **d.** $\frac{11}{4}$ or $2\frac{3}{4}$
   e. $2\frac{5}{5}$ or 3 **f.** $3\frac{2}{4}$ or $3\frac{1}{2}$ or $\frac{7}{2}$

EXERCISE 20-2

2. **a.** $\frac{13}{4}$ **b.** $\frac{13}{6}$ **c.** $\frac{8}{6}$ or $\frac{4}{3}$
   d. $\frac{11}{8}$ **e.** $2\frac{7}{8}$ or $\frac{23}{8}$
3. **a.** $\frac{14}{4}$ or $\frac{7}{2}$ **b.** $\frac{53}{8}$ or $6\frac{5}{8}$
   c. $\frac{7}{12}$ **d.** $\frac{17}{15}$ or $1\frac{2}{15}$
   e. $\frac{89}{48}$ or $1\frac{41}{48}$ **f.** $\frac{115}{24}$ or $4\frac{19}{24}$
   g. $\frac{89}{10}$ or $8\frac{9}{10}$ **h.** $\frac{44}{7}$ or $6\frac{2}{7}$
   i. $\frac{117}{20}$ or $5\frac{17}{20}$ **j.** $\frac{117}{90}$ or $12\frac{37}{90}$

4. $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$
5. $\frac{3}{8} + \frac{5}{12} = \frac{76}{96}$ or $\frac{19}{24}$
6. No, because $\frac{5}{8} + \frac{4}{7} = \frac{67}{56}$, which is greater than 1.
7. Yes, he would get the right answer when $a = 0$ and $c = 0$, but no other time. To see this suppose that

$$\frac{(a \times d) + (d \times c)}{b \times d} = \frac{a + c}{b + d}$$

Then we would have $b \times d = b + d$. But this can happen only when $b = 2$ and $d = 2$. In that case $(a \times d) + (b \times c)$ is equal to $2 \times (a + c)$. Thus we would have to have $2 \times (a + c) = a + c$. This can only happen when $a + c = 0$. Thus $a = 0$ and $c = 0$.

8. No, because $\frac{1}{p} + \frac{1}{q} = \frac{p + q}{pq}$ is never equal to $\frac{1}{p + q}$ when p and q are counting numbers.

EXERCISE 20-3

1. **a.** 10, $\frac{7}{10}$ **b.** 24, $\frac{29}{24}$ **c.** 75, $\frac{47}{75}$
   d. 60, $\frac{61}{60}$ **e.** 105, $\frac{118}{105}$ **f.** 120, $\frac{203}{120}$
   g. 630, $\frac{337}{630}$ **h.** 1,386, $\frac{255}{1,386}$ **i.** 2,100, $\frac{10,739}{2,100}$ or $5\frac{239}{2,100}$
   j. 60, $\frac{223}{60}$ or $3\frac{43}{60}$
2. **a.** $\frac{3}{4}$ **b.** $\frac{7}{22}$ **c.** $\frac{13}{4}$ **d.** $\frac{3}{4}$ **e.** $\frac{5}{3}$
3. $\frac{500}{600}$, $\frac{175}{600}$, $\frac{216}{600}$, $\frac{620}{600}$, $\frac{404}{600}$
4. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, $\frac{1}{12}$, $\frac{1}{24}$

EXERCISE 20-4

4. The sums are as follows:
   a. $\frac{15}{10}$ or $\frac{3}{2}$ **b.** $\frac{277}{24}$
   c. $\frac{187}{24}$ **d.** $\frac{3a + 2c}{6}$
   e. $\frac{20 + bc}{5b}$

EXERCISE 20-5

The sums are as follows:

1. $\frac{67}{24}$
2. $2\frac{13}{40}$ or $\frac{93}{40}$
3. $\frac{43}{24}$
4. $\frac{277}{105}$
5. $\frac{6a + 4b + 3c}{48}$

EXERCISE 20-7

3. **a.** $\frac{3}{3}$ or 1 **b.** $\frac{17}{2}$ **c.** $\frac{195}{28}$
   d. $\frac{23}{24}$ **e.** $\frac{79}{80}$

4. a. $\frac{13}{63}$ **b.** $\frac{65}{126}$ **c.** $\frac{34}{35}$

5. $\frac{153}{8}$ inches, or $19\frac{1}{8}$ inches

6. $\frac{5}{24}$

7. $\frac{7}{12}$

8. $\frac{3}{10}$

9. No. For example $\frac{1}{2} - \frac{3}{4}$ cannot be solved with fractions.

CHAPTER 21

Exercise 21-1

3. $\frac{35}{12}$ **4.** 3 **5.** 33 **6.** $\frac{95}{4}$ or $23\frac{3}{4}$

7. $\frac{2}{3}, \frac{2411}{4}$ or $560\frac{1}{4}$ **8.** 20 **9.** $\frac{65}{6}$ or $10\frac{5}{6}$

10. $\frac{132}{5}$ or $26\frac{2}{5}$ **11.** $6\frac{2}{3}$ **12.** 20

Exercise 21-3

1. $\frac{8}{15}$ **2.** $\frac{5}{2}$ **3.** $\frac{5}{8}$ **4.** $\frac{5}{8}$ **5.** $\frac{9}{10}$

6. $\frac{36}{175}$ **7.** $\frac{2,993}{192}$ **8.** $\frac{12,031}{864}$ **9.** $\frac{16}{55}$ **10.** $\frac{117}{476}$

12. $\frac{3}{20}$ **13.** Each gets $\frac{7}{6}$ of a basket

Exercise 21-5

1. The products are as follows:

a. $\frac{560}{117}$ **b.** $\frac{1}{40}$ **c.** $\frac{11,801}{756}$ or $15\frac{551}{756}$

Exercise 21-6

2. The sums and products are as follows:

a. $\frac{355}{27}$ **b.** $\frac{07}{40}$ **c.** $\frac{3,377}{167}$

3.
$$\frac{a}{b} \times \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \times \left(\frac{c \times f + d \times e}{d \times f}\right)$$
$$= \frac{(a \times c \times f) + (a \times d \times e)}{b \times d \times f}$$
$$\left(\frac{a}{b} \times \frac{c}{d}\right) + \left(\frac{a}{b} \times \frac{e}{f}\right) = \frac{a \times c}{b \times d} + \frac{a \times e}{b \times f}$$
$$= \frac{(a \times c \times f) + (a \times d \times e)}{b \times d \times f}$$

Since both expressions are the same we have

$$\frac{a}{b} \times \left(\frac{c}{d} \times \frac{e}{f}\right) = \left(\frac{a}{b} \times \frac{c}{d}\right) + \left(\frac{a}{b} \times \frac{e}{f}\right)$$

EXERCISE 21-8

1. a. $\square = \frac{5}{8} \div \frac{4}{5}$ **b.** $\square = \frac{7}{10} \div \frac{9}{4}$ **c.** $\square = 2\frac{1}{2} \div 1\frac{3}{5}$
d. $\square = 1\frac{7}{8} \div 3\frac{3}{4}$

2. a. $\square \times \frac{3}{5} = \frac{7}{12}$ **b.** $\square \times \frac{8}{3} = \frac{4}{15}$ **c.** $\square \times \frac{12}{11} = \frac{9}{17}$
d. $\square \times \frac{10}{7} = \frac{37}{2}$

3. a. $\frac{7}{5}$ **b.** $\frac{11}{4}$ **c.** $\frac{5}{21}$ **d.** $\frac{32}{91}$ **e.** $\frac{8}{21}$
f. $\frac{13}{86}$ **g.** $\frac{4}{31}$ **h.** $\frac{1}{9}$ **i.** $\frac{1}{5}$ **j.** no inverse

4. a. $\frac{25}{6}$ **b.** $\frac{3}{14}$ **c.** $\frac{1}{5}$ **d.** $\frac{1}{6}$ **e.** $\frac{8}{29}$
f. $\frac{1}{37}$ **g.** $\frac{17}{39}$

5. $\square = 1 \div \frac{0}{b}$ has no solution because $\frac{0}{b} \times \square = 0$.

6. $\square = \frac{a}{b} \div \frac{0}{d}$ never has a solution because $\frac{0}{d} \times \square = 0$.

EXERCISE 21-9

1. a. $\frac{9}{20}$ **b.** $\frac{7}{12}$ **c.** $\frac{5}{2}$ **d.** $\frac{10}{3}$ **e.** $\frac{13}{6}$ **f.** $\frac{68}{215}$
2. a. $\frac{75}{4}$ **b.** 10 **c.** 12 **d.** $\frac{161}{110}$ **e.** $\frac{77}{110}$ **f.** $\frac{153}{13}$
g. $\frac{57}{13}$ **h.** $\frac{17}{35}$ **i.** $\frac{360}{212}$ **j.** $\frac{387}{1,430}$

EXERCISE 21-10

2. $3\frac{1}{3}$ hours
3. $1\frac{1}{2}$ days
4. $\frac{27}{10}$ or $1\frac{11}{10}$ in one hour; $\frac{135}{32}$ or $4\frac{7}{32}$ in $2\frac{1}{2}$ hours.
5. Division is the inverse of multiplication.
6. Multiplication is the inverse of division.

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INDEX OF MAIN SUB-HEADINGS

UNIT I

Introduction, 1

Chapter 1

1–1 Sets and elements, 2; 1–2 Describing sets, 3; 1–3 Describing sets (continued), 5; 1–4 Special sets, 7; 1–5 Equal sets, 9; 1–6 Subsets, 10; 1–7 More about subsets, 13; 1–8 Picturing sets and subsets, 15; 1–9 Equivalent sets, 18; 1–10 More about equivalent sets, 21.

Chapter 2

2–1 Union of sets, 24; 2–2 Picturing the union of sets, 26; 2–3 Intersection of sets, 28; 2–4 Picturing the intersection of sets, 31; 2–5 More about unions and intersections, 32.

UNIT II

Chapter 3

3–1 Comparison of sets by matching, 36; 3–2 Number as a property of sets, 39; 3–3 Sets in natural order, 43.

Chapter 4

4–1 Identifying the whole numbers, 46; 4–2 Order in the set of whole numbers, 47; 4–3 Counting and the counting numbers, 49; 4–4 Finding the number of elements in a set, 50; 4–5 Equivalence using counting sets, 50.

Chapter 5

5–1 Introduction, 52; 5–2 Using counting sets, 54; 5–3 Counting sets used to build the number line, 54; 5–4 Size of the unit piece on the number line, 55; 5–5 Drawing the number line, 56.

Chapter 6

6–1 "Less than" on the number line, 58; 6–2 "Greater than" on the number line, 58; 6–3 "Between" on the number line, 60.

Chapter 7

7–1 Forming the number concept. Abstraction, 62; 7–2 The need for symbols, 63; 7–3 Ways of representing numbers (Egyptian, Babylonian), 64; 7–4 The Hindu-Arabic system of numeration, 66; 7–5 Grouping, 67; 7–6 A symbol for zero, 68; 7–7 Representing numbers on the abacus, 69; 7–8 The empty rod on the abacus, 74; 7–9 The Roman system of numeration, 76.

Chapter 8

8-1 Numeration in base five, 78; 8-2 A base-five abacus, 80; 8-3 Numeration base seven, 82; 8-4 Grouping in twelves, 82; 8-5 Base-two numerals, 84.

Chapter 9

9-1 An example, 86; 9-2 Addition and the union of sets, 87; 9-3 Addition using counting sets, 89; 9-4 Many names for one number, 90; 9-5 Mathematical sentences and the use of the box, 91; 9-6 Missing numbers in addition equations, 92; 9-7 The property of zero in addition, 95; 9-8 Addition table, 96.

Chapter 10

10-1 Commutative property of the union of sets, 98; 10-2 Commutative property of addition, 99; 10-3 The general notion of a variable, 101; 10-4 Associative property of the union of sets, 103; 10-5 Associative property of addition, 105; 10-6 Addition on the number line, 107; 10-7 Commutative and associative properties of addition, 109; 10-8 Generalized properties of addition, 110.

Chapter 11

11-1 Finding the missing addend, 112; 11-2 Subtraction, 114; 11-3 Subtraction facts from addition facts, 115; 11-4 Subtraction as separating a set into subsets, 116; 11-5 Subtraction problems which cannot be solved with whole numbers, 119; 11-6 Subtraction as the inverse of addition, 119; 11-7 Subtraction on the number line, 120; 11-8 Order, and addition and subtraction, 122.

Chapter 12

12-1 Repeated addition, 124; 12-2 Multiplication, 126; 12-3 Multiplication as mixing sets, 127; 12-4 Products of the type $a \times 1$ and $0 \times a$, 129; 12-5 The multiplication table, 131; 12-6 The property of *1*, 132.

Chapter 13

13-1 Multiplication by 0 and by *1*, 133; 13-2 The commutative property of multiplication, 134; 13-3 The commutative property (continued), 137; 13-4 The associative property of multiplication, 138; 13-5 The distributive property, 140; 13-6 The distributive property on the number line, 143; 13-7 Why the properties are important, 144; 13-8 An illustration, 145.

Chapter 14

14-1 The meaning of division, 147; 14-2 Division as separating a set into subsets, 149; 14-3 Division as repeated subtraction, 151; 14-4 Remainders, 153; 14-5 Division by zero, 154; 14-6 Division as the inverse of multiplication, 155; 14-7 Factors and prime numbers, 157; 14-8 Greatest common factors, 157.

Chapter 15

15-1 The whole numbers; natural order, 160; 15-2 Addition and multiplication, 161; 15-3 The properties of 0 and *1*, 163.

Chapter 16

16-1 Basic addition facts and their consequences, 164; 16-2 Addition of multiples of *10*, 167; 16-3 Renaming numbers in the decimal system, 167; 16-4 Addition without regrouping, 169; 16-5 Addition with regrouping, 172; 16-6 Addition in other bases, 176; 16-7 Procedures for subtraction, 180.

Chapter 17

17-1 Introduction, 186; 17-2 Multiplication by multiples of *10*, 186; 17-3 The general procedure for multiplication, 188; 17-4 The short form for multiplication, 190; 17-5 Division by the method of repeated subtraction, 193; 17-6 Division by the method of repeated subtraction (continued), 197.

UNIT III

Chapter 18

18-1 Introduction, 200; 18-2 Names and symbols for fractions, 200; 18-3 Pictures that show fractions, 203; 18-4 Names of fractions (continued), 206; 18-5 Fractions and divisions, 208; 18-6 An example, 213; 18-7 The number line, 215; 18-8 The number line (continued), 216; 18-9 Order properties, 218.

Chapter 19

19-1 Equivalent fractions, 220; 19-2 Equivalent fractions (continued), 223; 19-3 The equivalence property of fractions, 228; 19-4 Tests for equivalent fractions, 231; 19-5 Fractions, 233; 19-6 Whole numbers as fractions, 235; 19-7 The question of zero, 237.

Chapter 20

20-1 Addition of fractions, 240; 20-2 Addition of fractions (continued), 243; 20-3 The least common denominator, 249; 20-4 The commutative property of addition, 253; 20-5 The associative property of addition, 255; 20-6 The property of zero, 256; 20-7 Subtraction of fractions, 258; 20-8 Subtraction as the inverse of addition, 262.

Chapter 21

21-1 Multiplication of fractions, 264; 21-2 Making multiplication simple, 267; 21-3 Multiplication of fractions (continued), 268; 21-4 The commutative property of multiplication, 271; 21-5 The associative property of multiplication, 273; 21-6 The distributive property, 274; 21-7 The property of *1*, 277; 21-8 Division of fractions, 277; 21-9 Division of fractions (continued), 280; 21-10 Division of fractions—the general rule, 282.

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FOREWORD

In this volume of *Basic Concepts of Mathematics*, the Teacher Training Writing Group at the 1964 Entebbe Mathematics Workshop has completed "Structure of Arithmetic", the first part of an experimental text to be used by primary teachers in Training Colleges. The text was planned at the 1963 Entebbe Mathematics Workshop, and four units were written then and published in a preliminary edition. The units now added treat integers, rational numbers, real numbers and approximations. In these units, the number system of arithmetic is further enlarged. The operations of arithmetic are studied with particular attention to problems arising in everyday situations and to the systematic underlying properties of the operations.

As in the earlier units, the exercises have two purposes: to develop and extend the understanding of the mathematical content presented in the text, and to suggest by example kinds of exercises the trainee could create for use in his own classes when he becomes a teacher. Answers for the more difficult exercises in this volume will be found at the end of this book.

The preliminary edition has been produced under pressure of time, and there is still much to be done by way of improving exposition and organization as well as adding to the stock of exercises. To all users, therefore, the Teacher Training Writing Group directs an earnest request for comments and suggestions which can contribute to the work of preparing a more finished text. Reports from experimental use of the preliminary edition are a source of ideas which will make the next edition of greater value to mathematics education.

The succeeding volume of *Basic Concepts of Mathematics*, also prepared by the Teacher Training Writing Group at the 1964 Entebbe Mathematics Workshop, is devoted to "Introduction to Geometry", the second part of the text planned in 1963.

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CONTENTS

	page
Fractions Unit IV	1
CHAPTER 22 *Order Properties of Fractions*	2
CHAPTER 23 *Decimal Fractions*	14
CHAPTER 24 *Operations in Decimal Notation*	32
Integers Unit V	55
CHAPTER 25 *Introduction to Integers*	56
CHAPTER 26 *The Number Line as a Picture for Integers*	62
CHAPTER 27 *Operations on Integers*	70
CHAPTER 28 *Relation of Integers to Whole Numbers*	90
CHAPTER 29 *Operations on Integers* (cont.)	99
The Rational Numbers Unit VI	111
CHAPTER 30 *Negative Fractions*	112
CHAPTER 31 *The Set of Rational Numbers*	120
CHAPTER 32 *Operations on Rational Numbers*	126
CHAPTER 33 *Revision of Numbers*	139
CHAPTER 34 *Revision of Properties of Operations*	144
CHAPTER 35 *Order for Rational Numbers*	152
Real Numbers Unit VII	169
CHAPTER 36 *Real Numbers*	170
CHAPTER 37 *A Geometry Problem*	176
CHAPTER 38 *The View from the Top*	184
Approximations Unit VIII	193
CHAPTER 39 *Approximations as Results of Counting*	194
CHAPTER 40 *Approximations in Measuring*	202
CHAPTER 41 *Rounding Off*	209
CHAPTER 42 *Maximum Error and Relative Error*	216
CHAPTER 43 *Decimal Places and Significant Digits*	220
CHAPTER 44 *Addition and Subtraction of Approximate Numbers*	227
CHAPTER 45 *Multiplication and Division of Approximate Numbers*	236

CHAPTER 46 *Combined Operations on Approximate Numbers* 243
Answers to Selected Problems 247
Index of Main Sub-headings 263

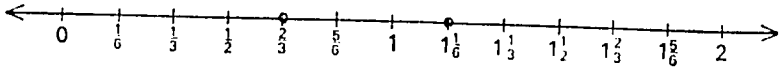
Fractions UNIT IV

Chapter 22

ORDER PROPERTIES OF FRACTIONS

22-1 Revision of order

When you learned about fractions on the number line, you found how to decide whether one fraction was less than another fraction. For example, if you draw a number line showing $\frac{2}{3}$ and $1\frac{1}{6}$,



you see that the point marked $\frac{2}{3}$ is to the left of the point marked $1\frac{1}{6}$. Therefore $\frac{2}{3}$ is less than $1\frac{1}{6}$, and we write $\frac{2}{3} < 1\frac{1}{6}$. Since the point marked $1\frac{1}{6}$ is to the right of the point marked $\frac{2}{3}$, it is also correct to say that $1\frac{1}{6}$ is greater than $\frac{2}{3}$. In symbols, $1\frac{1}{6} > \frac{2}{3}$.

EXERCISE 22-1A

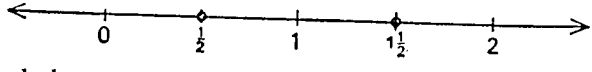
1. Draw a number line and locate each pair of fractions on it. Which fraction in each pair is greater?

   a. $2\frac{1}{3}$ and $\frac{5}{3}$ **b.** $\frac{2}{5}$ and $\frac{1}{3}$

   c. $1\frac{1}{2}$ and $\frac{5}{3}$ **d.** $\frac{5}{4}$ and $1\frac{1}{6}$

2. Write fractions less than each of the following fractions: ^{5}, $\frac{9}{4}$, $\frac{1}{6}$, $\frac{8}{7}$.

Do you remember that if you have three different numbers shown on a number line, one will be between the other two? For example, of the three numbers $\frac{1}{2}$, 1 and $1\frac{1}{2}$, 1 is between $\frac{1}{2}$ and $1\frac{1}{2}$.



In symbols, you can write $\frac{1}{2} < 1 < 1\frac{1}{2}$. This means the same thing as writing

$$\frac{1}{2} < 1 \text{ and } 1 < 1\frac{1}{2}$$

What do you see about $\frac{1}{2}$ and $1\frac{1}{2}$, the two outside points? You see that $\frac{1}{2} < 1\frac{1}{2}$ because $\frac{1}{2}$ is to the left of $1\frac{1}{2}$.

EXERCISE 22-1B

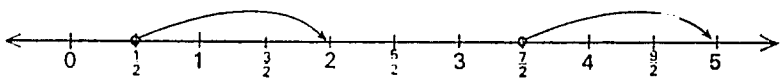
1. Locate the three fractions in each of the following sets on a number line, and say which one is between the other two. Also say which of the outside two fractions is the greater.

   a. $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}$ **b.** $3\frac{1}{2}, 2\frac{1}{2}, 1\frac{1}{2}$
   c. $\frac{2}{3}, \frac{5}{6}, 1$ **d.** $\frac{7}{5}, \frac{1}{2}, \frac{4}{3}$

2. Make up three sets, each containing three different fractions, and show each set on a number line.

22-2 Order and addition and subtraction

Some properties of order which we already know for whole numbers are also true for fractions. Let us start by locating $\frac{1}{2}$ and $\frac{7}{2}$ on a number line. We see that $\frac{1}{2}$ is to the left of $\frac{7}{2}$. That is, $\frac{1}{2} < \frac{7}{2}$. If we add the same fraction $\frac{3}{2}$ to both $\frac{1}{2}$ and $\frac{7}{2}$, we move each to the right a distance of $\frac{3}{2}$.



The sums are 2 and 5, and we notice that $2 < 5$ because 2 is to the left of 5. The sums are in the same order as the original numbers $\frac{1}{2}$ and $\frac{7}{2}$. We have

$$\begin{aligned} \tfrac{1}{2} &< \tfrac{7}{2} \\ \tfrac{1}{2} + \tfrac{3}{2} &< \tfrac{7}{2} + \tfrac{3}{2} \\ 2 &< 5 \end{aligned}$$

Now think of $\frac{a}{b}$ and $\frac{c}{d}$ as any two fractions with $\frac{a}{b} < \frac{c}{d}$. Then $\frac{a}{b}$ will be to the left of $\frac{c}{d}$ on a number line. If we add the fraction $\frac{p}{q}$ to $\frac{a}{b}$ and also to $\frac{c}{d}$, the point for $\frac{a}{b}$ is replaced by a point $\frac{p}{q}$ units

to the right of $\frac{a}{b}$, and the point for $\frac{c}{d}$ is replaced by a point $\frac{p}{q}$ units to the right of $\frac{c}{d}$. The point $\frac{a}{b} + \frac{p}{q}$ is still to the left of $\frac{c}{d} + \frac{p}{q}$. Thus:

If $$\frac{a}{b} < \frac{c}{d}$$

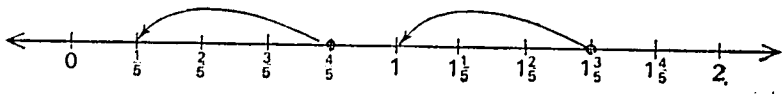
then $$\frac{a}{b} + \frac{p}{q} < \frac{c}{d} + \frac{p}{q}$$

Do you recognize this as one of the properties of order which we knew before only for whole numbers?

Again, let a and b be whole numbers with $a < b$. We have already seen that if c is any whole number which can be subtracted from a to make $a - c$ a whole number, then

$$a - c < b - c$$

The same property is true for fractions.



For example, from both sides of the inequality

$$\tfrac{4}{5} < 1\tfrac{3}{5}$$

we can subtract $\frac{3}{5}$ to obtain

$$\tfrac{4}{5} - \tfrac{3}{5} < 1\tfrac{3}{5} - \tfrac{3}{5}$$

that is, $$\tfrac{1}{5} < 1$$

We still have to be careful that the number we subtract is not too large. For example, we cannot yet subtract $1\frac{1}{5}$ from $\frac{4}{5}$.

The subtraction property can be stated as follows:

If $\frac{a}{b} < \frac{c}{d}$ and if $\frac{p}{q}$ is not greater than $\frac{a}{b}$,

then $$\frac{a}{b} - \frac{p}{q} < \frac{c}{d} - \frac{p}{q}$$

EXERCISE 22-2A

1. Is it true that $\frac{5}{3} < \frac{9}{2}$?
2. By adding $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, 1 to both sides of $\frac{5}{3} < \frac{9}{2}$, find four more true inequalities.
3. By subtracting $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{5}{3}$ from both sides of $\frac{5}{3} < \frac{9}{2}$, find four more true inequalities.
4. Can you subtract $\frac{9}{5}$ from both sides of the inequality $\frac{5}{3} < \frac{9}{2}$?

22-3 Generalized addition property

Let us start once again from the true inequality $\frac{1}{2} < \frac{7}{2}$. In the last section, we added $\frac{3}{2}$ to both sides to obtain $2 < 5$. What will happen if we add a larger number to the right side $\frac{7}{2}$ than to the $\frac{1}{2}$ on the left? For example, since $\frac{3}{2} < \frac{5}{2}$, let us add $\frac{3}{2}$ to the $\frac{1}{2}$ on the left and the larger number $\frac{5}{2}$ to the $\frac{7}{2}$ on the right. Then we obtain the inequality $2 < 6$, which is also true. That is, we start with

$$\tfrac{1}{2} < \tfrac{7}{2} \quad \text{and} \quad \tfrac{3}{2} < \tfrac{5}{2}$$

After adding, we see that the sum of the right-hand sides is greater than the sum of the left-hand sides:

$$\tfrac{1}{2} + \tfrac{3}{2} < \tfrac{7}{2} + \tfrac{5}{2}$$

or $$2 < 6$$

Will this always be the case? Let us start with

$$\tfrac{3}{4} < \tfrac{4}{5} \quad \text{and} \quad \tfrac{2}{5} < \tfrac{1}{2}$$

Is the following true?

$$\tfrac{3}{4} + \tfrac{2}{5} < \tfrac{4}{5} + \tfrac{1}{2}$$

Yes, because $\frac{3}{4} + \frac{2}{5} = \frac{23}{20}$ and $\frac{4}{5} + \frac{1}{2} = \frac{13}{10} = \frac{26}{20}$ and

$$\tfrac{23}{20} < \tfrac{26}{20}$$

The generalized property that these examples illustrate is the *generalized addition property*:

If $$\frac{a}{b} < \frac{c}{d} \quad \text{and} \quad \frac{p}{q} < \frac{r}{s}$$

then $$\frac{a}{b} + \frac{p}{q} < \frac{c}{d} + \frac{r}{s}$$

To see that this is really so, we first add $\frac{p}{q}$ to both sides of

$$\frac{a}{b} < \frac{c}{d}$$

and obtain $$\frac{a}{b} + \frac{p}{q} < \frac{c}{d} + \frac{p}{q}$$

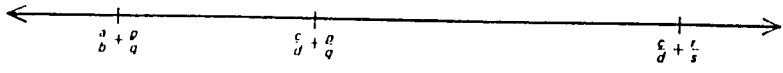
We can get another true inequality by adding $\frac{c}{d}$ to both sides of $\frac{p}{q} < \frac{r}{s}$

This gives us $$\frac{p}{q} + \frac{c}{d} < \frac{r}{s} + \frac{c}{d}$$

or, by using the commutative property of addition,

$$\frac{c}{d} + \frac{p}{q} < \frac{c}{d} + \frac{r}{s}$$

Looking at the second and fourth inequalities in this paragraph, we see that the order of $\frac{a}{b} + \frac{p}{q}$, $\frac{c}{d} + \frac{p}{q}$ and $\frac{c}{d} + \frac{r}{s}$ along the number line must be as shown here.



Of course, you see straightaway that

$$\frac{a}{b} + \frac{p}{q} < \frac{c}{d} + \frac{r}{s}$$

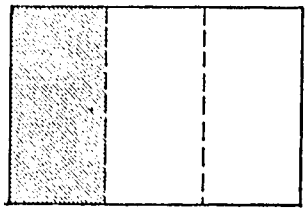
EXERCISE 22-3A

What true inequality do you obtain by using the generalized addition property with each of the following pairs of inequalities?

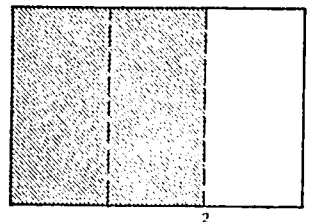
1. $\frac{1}{2} < \frac{3}{4}$ and $1\frac{1}{4} < 1\frac{1}{2}$
2. $\frac{1}{2} < \frac{3}{4}$ and $\frac{2}{3} < \frac{5}{6}$
3. $\frac{3}{5} < \frac{5}{6}$ and $\frac{1}{2} < \frac{2}{3}$
4. $\frac{5}{3} < \frac{9}{4}$ and $\frac{1}{8} < \frac{1}{7}$

22-4 Order and multiplication by $\frac{1}{n}$

You know that $\frac{1}{3} < \frac{2}{3}$. What can you say about $\frac{1}{4} \times \frac{1}{3}$ and $\frac{1}{4} \times \frac{2}{3}$? Which is greater? To find the answer, let us use rectangles to show the original fractions.

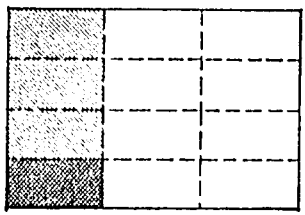


$\frac{1}{3}$

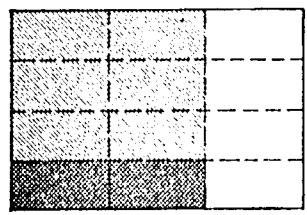


$\frac{2}{3}$

The set of one of three equal parts is smaller than a set of two of the three equal parts. So $\frac{1}{3} < \frac{2}{3}$. To picture $\frac{1}{4} \times \frac{1}{3}$, we divide $\frac{1}{3}$ by lines running from side to side. We can picture $\frac{1}{4} \times \frac{2}{3}$ in the same way.



$\frac{1}{4} \times \frac{1}{3}$



$\frac{1}{4} \times \frac{2}{3}$

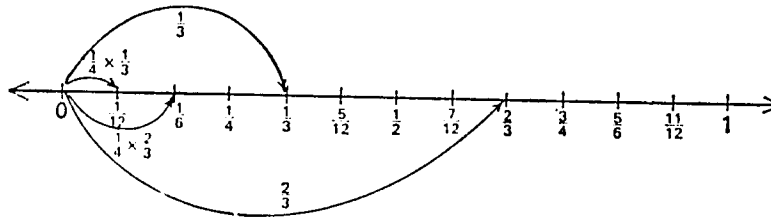
Each rectangle is now cut into 12 equal parts. How many twelfth parts is $\frac{1}{4} \times \frac{1}{3}$? How many twelfth parts is $\frac{1}{4} \times \frac{2}{3}$? You see that from

$$\frac{1}{3} < \frac{2}{3}$$

you can conclude that

$$\tfrac{1}{4} \times \tfrac{1}{3} < \tfrac{1}{4} \times \tfrac{2}{3}$$

We can also show this conclusion on a number line.



Since $\frac{1}{12}$ is less than $\frac{1}{6}$, it follows from the picture that

$$\tfrac{1}{4} \times \tfrac{1}{3} < \tfrac{1}{4} \times \tfrac{2}{3}$$

This illustrates the general rule: Let n be a counting number. If $\frac{a}{b} < \frac{c}{d}$ and both sides are multiplied by $\frac{1}{n}$, then $\frac{1}{n} \times \frac{a}{b} < \frac{1}{n} \times \frac{c}{d}$. It is not difficult to see that this general rule must be true. If $\frac{a}{b}$ is less than $\frac{c}{d}$ and we divide both $\frac{a}{b}$ and $\frac{c}{d}$ into n equal parts, one of the parts of $\frac{a}{b}$ will be smaller than one of the parts of $\frac{c}{d}$. That is to say,

$$\frac{1}{n} \times \frac{a}{b} < \frac{1}{n} \times \frac{c}{d}$$

EXERCISE 22-4A

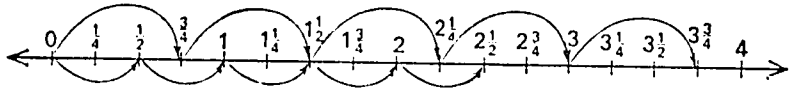
From the inequality $\frac{2}{3} < \frac{3}{4}$, by drawing rectangles and number lines show that the following inequalities are true.

1. $\frac{1}{3} < \frac{3}{8}$ **2.** $\frac{2}{9} < \frac{1}{4}$ **3.** $\frac{1}{6} < \frac{3}{16}$

22-5 Order and multiplication by m

What can we say about multiplying both sides of the inequality $\frac{1}{2} < \frac{3}{4}$ by 5? If we locate $\frac{1}{2}$ and $\frac{3}{4}$ on the number line,

we see that the jump from 0 to $\frac{3}{4}$ is greater than the jump from 0 to $\frac{1}{2}$.



Then five jumps of the size from 0 to $\frac{3}{4}$ will carry you farther to the right on the number line than five jumps of the size 0 to $\frac{1}{2}$. So $5 \times \frac{1}{2} < 5 \times \frac{3}{4}$. From the true inequality

$$\frac{1}{2} < \frac{3}{4}$$

we can conclude that $5 \times \frac{1}{2} < 5 \times \frac{3}{4}$.

You can guess what the general statement must be.

If we know that $$\frac{a}{b} < \frac{c}{d}$$

and m is a counting number, then we can conclude that

$$m \times \frac{a}{b} < m \times \frac{c}{d}$$

Let us see if this guess is true. Imagine jumps of distances $\frac{a}{b}$ and $\frac{c}{d}$ on the number line. The jump for $\frac{c}{d}$ is longer than the jump for $\frac{a}{b}$. Then m jumps of distance $\frac{c}{d}$ will certainly be longer than m jumps of distance $\frac{a}{b}$. That is,

$$m \times \frac{a}{b} < m \times \frac{c}{d}$$

The general statement is true.

EXERCISE 22-5A

1. What new inequalities do you get if you multiply both sides of the inequality $\frac{3}{4} < \frac{4}{3}$ by the following whole numbers?

   2, 3, 4, 5, 6, 12

2. Explain how you can show that $2 \times \frac{2}{3} < 2 \times \frac{5}{4}$ by using rectangles.

22-6 Order and multiplication by $\frac{m}{n}$

Suppose we multiply both sides of the inequality $\frac{2}{3} < \frac{4}{5}$ by $\frac{2}{7}$? What new inequality do you think will be true? Of course, you think $\frac{2}{7} \times \frac{2}{3} < \frac{2}{7} \times \frac{4}{5}$ might be true. But how can you see that it is true? One way, of course, is to multiply the fractions and compare the two sides of the inequality. You will get $\frac{4}{21} < \frac{8}{35}$, which is true.

There is another way if we use our previous results. Suppose you begin by writing $\frac{2}{7} = 2 \times \frac{1}{7}$. Since it is true that $\frac{2}{3} < \frac{4}{5}$, you can conclude that

$$\tfrac{1}{7} \times \tfrac{2}{3} < \tfrac{1}{7} \times \tfrac{4}{5}$$

After further multiplying by the whole number 2, you can conclude that

$$2 \times (\tfrac{1}{7} \times \tfrac{2}{3}) < 2 \times (\tfrac{1}{7} \times \tfrac{4}{5})$$

or

$$(2 \times \tfrac{1}{7}) \times \tfrac{2}{3} < (2 \times \tfrac{1}{7}) \times \tfrac{4}{5}$$

Then

$$\tfrac{2}{7} \times \tfrac{2}{3} < \tfrac{2}{7} \times \tfrac{4}{5}$$

(What property of multiplication did you use to go from the first inequality above to the second inequality?)

The result about order and multiplication that you should now suspect to be true is the following:

If

$$\frac{a}{b} < \frac{c}{d}$$

then we can conclude that

$$\frac{m}{n} \times \frac{a}{b} < \frac{m}{n} \times \frac{c}{d}$$

To see this we write down the following list of inequalities:

$$\frac{a}{b} < \frac{c}{d}$$

$$\frac{1}{n} \times \frac{a}{b} < \frac{1}{n} \times \frac{c}{d}$$

$$m \times \left(\frac{1}{n} \times \frac{a}{b}\right) < m \times \left(\frac{1}{n} \times \frac{c}{d}\right)$$

$$\left(m \times \frac{1}{n}\right) \times \frac{a}{b} < \left(m \times \frac{1}{n}\right) \times \frac{c}{d}$$

$$\frac{m}{n} \times \frac{a}{b} < \frac{m}{n} \times \frac{c}{d}$$

EXERCISE 22-6A

1. Obtain new inequalities by multiplying both sides of the inequality $\frac{5}{4} < \frac{3}{2}$ by $\frac{2}{15}$, $\frac{1}{5}$, $\frac{1}{3}$, $\frac{4}{5}$, 2.
2. From the inequality $\frac{3}{4} < \frac{5}{6}$, how can you conclude that $1 < \frac{10}{9}$? Can you conclude that $9 < 10$?
3. Give the properties of fractions that you use to go from one inequality to the next one in showing the general property in this section that $\frac{m}{n} \times \frac{a}{b} < \frac{m}{n} \times \frac{c}{d}$.
4. Say how you might convince pupils that if $\frac{a}{b} < \frac{c}{d}$ is true, then

$$\frac{a}{b} \div \frac{m}{n} < \frac{c}{d} \div \frac{m}{n}$$

is also true $\left(\text{unless } \frac{m}{n} = 0\right)$.

22-7 Generalized multiplication property

In the last section, we started with the inequality $\frac{2}{3} < \frac{4}{5}$. Let us do so again, but this time multiply the larger number $\frac{4}{5}$ by a greater number than the one we use to multiply $\frac{2}{3}$. Let us multiply $\frac{2}{3}$ by $\frac{3}{5}$, but multiply $\frac{4}{5}$ by $\frac{4}{3}$, which is greater than $\frac{3}{5}$.

From $\qquad \frac{2}{3} < \frac{4}{5}$

and $\qquad \frac{3}{5} < \frac{4}{3}$

we get $\qquad \frac{3}{5} \times \frac{2}{3} < \frac{4}{3} \times \frac{4}{5}$ or $\frac{6}{15} < \frac{16}{15}$

which is a true inequality. The product of the left-hand sides is less than the product of the right-hand sides.

We can do this problem in another way to see that the conclusion must follow. Since $\frac{3}{5} < \frac{4}{3}$, we conclude that

$$\frac{4}{5} \times \frac{3}{5} < \frac{4}{5} \times \frac{4}{3}.$$

That is

$$\frac{12}{25} < \frac{16}{15}$$

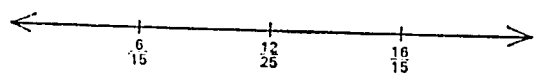
From the true inequality $\frac{2}{3} < \frac{4}{5}$, we can conclude that

$$\frac{3}{5} \times \frac{2}{3} < \frac{3}{5} \times \frac{4}{5}$$

That is

$$\frac{6}{15} < \frac{12}{25}$$

The last inequality tells us that $\frac{12}{25}$ is to the right of $\frac{6}{15}$ on the number line. The inequality $\frac{12}{25} < \frac{16}{15}$ tells us that $\frac{16}{15}$ is to the right of $\frac{12}{25}$. Therefore $\frac{16}{15}$ must be to the right of $\frac{6}{15}$, and $\frac{6}{15} < \frac{16}{15}$.



Thus,

$$\frac{3}{5} \times \frac{2}{3} < \frac{4}{3} \times \frac{4}{5}$$

The general property that we would like to establish is this *generalized multiplication property*:

If

$$\frac{a}{b} < \frac{c}{d} \quad \textit{and} \quad \frac{m}{n} < \frac{p}{q}$$

then

$$\frac{a}{b} \times \frac{m}{n} < \frac{c}{d} \times \frac{p}{q}$$

We will follow the plan we have just used in the example. Since $\frac{m}{n} < \frac{p}{q}$, we can conclude that

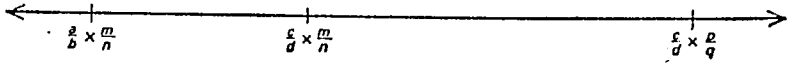
$$\frac{c}{d} \times \frac{m}{n} < \frac{c}{d} \times \frac{p}{q}$$

From the inequality $\frac{a}{b} < \frac{c}{d}$, we can conclude that

$$\frac{m}{n} \times \frac{a}{b} < \frac{m}{n} \times \frac{c}{d} \quad \text{or} \quad \frac{a}{b} \times \frac{m}{n} \times \frac{c}{d} \times \frac{m}{n}$$

(Where did we use the commutative property of multiplication?)

From $\frac{a}{b} \times \frac{m}{n} < \frac{c}{d} \times \frac{m}{n}$, we know that $\frac{c}{d} \times \frac{m}{n}$ is to the right of $\frac{a}{b} \times \frac{m}{n}$ on the number line. Also $\frac{c}{d} \times \frac{m}{n} < \frac{c}{d} \times \frac{p}{q}$ tells us that $\frac{c}{d} \times \frac{p}{q}$ is to the right of $\frac{c}{d} \times \frac{m}{n}$. Then $\frac{c}{d} \times \frac{p}{q}$ must also be to the right of $\frac{a}{b} \times \frac{m}{n}$.



$$\frac{a}{b} \times \frac{m}{n} \qquad \frac{c}{d} \times \frac{m}{n} \qquad \frac{c}{d} \times \frac{p}{q}$$

That is, $$\frac{a}{b} \times \frac{m}{n} < \frac{c}{d} \times \frac{p}{q}$$

EXERCISE 22-7A

1. Using the same plan as the example, show that $\frac{5}{7} < \frac{7}{8}$ by starting from the inequalities $\frac{3}{7} < \frac{1}{2}$ and $\frac{5}{3} < \frac{7}{4}$.
2. Using the generalized multiplication property, what inequality do you get from each of the following pairs of inequalities?

   $\frac{1}{2} < \frac{3}{4}$ and $\frac{2}{3} < \frac{5}{6}$
   $\frac{3}{5} < \frac{5}{6}$ and $\frac{1}{2} < \frac{2}{3}$
   $\frac{5}{3} < \frac{7}{4}$ and $\frac{1}{8} < \frac{1}{7}$

Chapter 23

DECIMAL FRACTIONS

23-1 Revision of decimal numeration

You will remember that at school you learned another way of writing fractions, that is as decimal fractions. These are the same kind of numbers as the fractions that you have just been studying, but they are written in a different way. Just as whole numbers may be written as Hindu-Arabic numerals, or as Roman numerals or in the way the ancient Egyptians wrote them, so the parts of a whole may be written in two ways. You have already studied ways of introducing your pupils to fractions which are written with a numerator and a denominator, such as $\frac{1}{4}$. We will, in future, refer to fractions written this way as fractions in common form or *common fractions*. We do this to distinguish them from fractions which are written as an extension of the decimal notation for whole numbers, such as ·25. We will refer to fractions written this way as fractions in decimal form or *decimal fractions*. Each common fraction has its decimal fraction equivalent but not every decimal fraction has an equivalent common fraction, as you will see in Unit VII.

You know already that the notation for decimal fractions was invented as an extension of the way we write whole numbers. It uses the ideas of base and place value which are the basis of the Hindu-Arabic notation system. If you will always remember this fact, it will be no harder to understand decimal fractions than it is to understand the decimal notation for whole numbers.

The idea of a base

We will begin by recalling what you know about the decimal notation for whole numbers. You will remember that the notation for whole numbers is based on counting. When large numbers of objects have to be counted it is simpler to group them into equivalent sets, that is, into sets with the same number of members. They can be grouped into sets of 2 or 3 or 4 members, or any number of members you choose. If you group

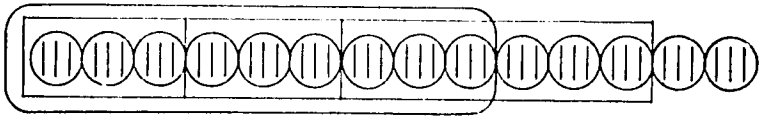
the objects in threes, then you are using *base three*. If you group them in sixes, then you are using *base six*. Traditionally, we group our numbers in tens; that is, we use *base ten*. This is why our system of notation is called the decimal system, from the Latin word *decem* which means "ten". When the number of the set to be counted is large, it is necessary to put groups together to make larger groups, still using the same base number. If 5 ones are grouped together, then 5 fives will make the next larger group. This grouping of groups to make larger groups is continued as far as is necessary. In the decimal system the groups have special names. They are ones, tens, hundreds, thousands, ten-thousands, hundred-thousands, millions and so on.

You will remember that you can help your pupils to understand the idea of grouping by letting them practise grouping a set of sticks, using several different bases. Here is an example you could use with them.

Put out a long row of sticks. Choose a base. We will use base three. First group the sticks in threes, beginning from the left.

Then group the threes into larger groups each of 3 threes.

Continue in this way until you have no more than 2 groups of any kind of group. Here is the last picture. You will see that there is 1 group of 3 groups of 3 threes.



1 group of 3 groups of 3 threes + 1 group of 3 threes + 2 groups of three + 1.

Now you ask your pupils which is the biggest group they

have made? It is 1 group of 3 groups of 3 threes. It can be written as $3 \times 3 \times 3$.

Then ask about the next largest group, the group of 3 threes. There is only 1 of this group also. Then there are 2 groups of three and 1 one. Put these together and you have 1 group of $3 \times 3 \times 3$, 1 group of 3×3, and 2 groups of three and 1 one. Do you remember the short way of writing this? You use the index notation and show $3 \times 3 \times 3$ as 3^3. It is

$$1 \times 3^3 + 1 \times 3^2 + 2 \times 3 + 1 \times 1.$$

EXERCISE 23-1A

1. Set out 43 sticks or draw 43 strokes. Group them in base six and write the result, using index notation showing powers of 6.
2. Using an equivalent set of sticks, group them in base ten and write the result, using index notation.
3. A set of sticks has been grouped in fours and the result written as

$$1 \times 4^3 + 2 \times 4^2 + 3 \times 4 + 0 \times 1$$

Draw a picture to show this grouping.

Place value

What does the numeral $3{,}243_{ten}$ mean? How is it different from $3{,}243_{five}$? It is helpful to your pupils to make a number chart to show the value of each of the digits in a numeral. Here are two number charts, one for base ten and one for base five.

Thousands	*Hundreds*	*Tens*	*Ones*
10^3	10^2	10	1
3	2	4	3

Hundred & twenty-fives	*Twenty-fives*	*Fives*	*Ones*
5^3	5^2	5	1
3	2	4	3

The numeral $3{,}243_{ten}$ can be read from this number chart as three thousand, two hundred and forty-three. Without the number chart, how do you know how to read the number? You

know the value of a digit by its position in the row. The place value of the 4 is ten because the 4 is in the tens place. The place value of the 3 on the left is thousand because it is in the fourth place from the right, the thousands place. Your pupils should be familiar with the value of these places in the decimal system and should be able to read a number easily.

You will remember from your earlier work how to read a number in another base, such as $3{,}243_{five}$. You will remember that we read this as "three two four three base five" and do not attempt to give names to the places. (Why not?)

Change of base

You should remember also how to change a numeral from one base to another. If you have difficulty with the following exercise, look back to Chapter 6.

EXERCISE 23-1B

1. Write the following numerals in the expanded form, using the index notation.

   a. $1{,}213_{four}$ **b.** $23{,}751_{eight}$ **c.** $121{,}304_{five}$

2. Rewrite each of the numbers in Question 1 in the decimal notation.

3. Write the following numbers in base seven.

   a. 65_{ten} **b.** 77_{ten} **c.** $36{,}490_{ten}$

4. Make a number chart for base six and show on it the following numbers.

   a. $35{,}002_{six}$ **b.** $2{,}020_{six}$ **c.** $13{,}452_{six}$

5. What is the value in decimal notation of each 2 in the numerals in the last question?

6. Tell in what base each of the following equations is written.

   a. $2 + 1 = 10$ **b.** $13 - 4 = 4$
   c. $14 \times 4 = 104$ **d.** $26 \div 4 = 5$

7. Write the following numbers in the decimal notation.

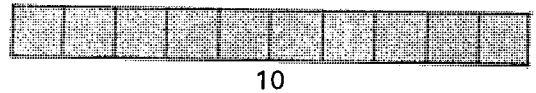
   a. Five hundred thousand, eight hundred and seventeen.
   b. Six million, ninety-two thousand and twenty-three.
   c. Nineteen thousand, nine hundred and nine.

Relationship between the digits in a numeral in the decimal system

Before we introduce the notation for decimal fractions, it is useful to think again about the relationship between one, ten, hundred and so on. Pictures can help your pupils to understand and remember what they learn. One and ten are easy numbers to understand. Ten ones make ten and we can see a set of 10 members by looking at our fingers or toes. One hundred is harder. Where do you see a hundred? A thousand is a big number. Do you ever see a thousand? And so you can think also of a million. How many thousands do you need to make a million? Here is one way to help your pupils to gain some idea of the relative sizes of numbers. If you have graph paper with small squares whose sides are, for example, each $\frac{1}{10}$ inch, use it. If you have not, then use, or make, paper with small squares on it. If you use graph paper, then you will be able to show easily how a thousand is built up from 10 hundreds, how a hundred is built up from 10 tens and how a ten is built up from 10 ones. You are about to make pictures of one, of ten, of a hundred and of a thousand.

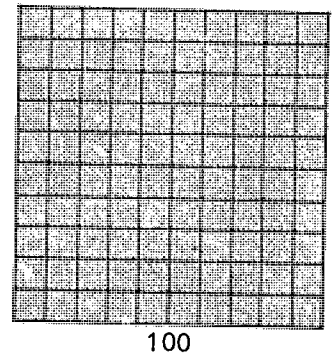
Shade 1 small square. This is 1 one.
Shade 10 small squares. This is a ten-strip.
So 1 ten = 10 ones.

1



10

Next, shade 10 ten-strips.
This is a hundred-square.
So 1 hundred = 10 tens
= 100 ones.



100

Now your pupils can build up a thousand-strip by shading ten of these hundred-squares. What can they write about this new strip? It is a thousand-strip.

So 1 thousand = 10 hundreds
= 100 tens = 1,000 ones.

You could continue further and build up a ten-thousand-square, a hundred-thousand-strip and even a million-square. How many thousand-strips would be needed for a million-square?

A ten-thousand-square needs 10 thousand-strips.
A hundred-thousand-strip needs 100 thousand-strips.
So a million-square needs 1,000 thousand-strips.

Can you write a set of equalities for a million as we did for a thousand? Here it is. Were you right?

1 million = 10 hundred thousands
= 100 ten thousands
= 1,000 thousands
= 10,000 hundreds
= 100,000 tens
= 1,000,000 ones

If you let your pupils draw pictures like those above, it helps them to see the relationships between digits in a numeral. Think of the number 11,111. Each 1 has ten times the value of the digit to the right of it, or 100 times the value of the digit two places to the right of it. You can think of the 3 in 3,641 as

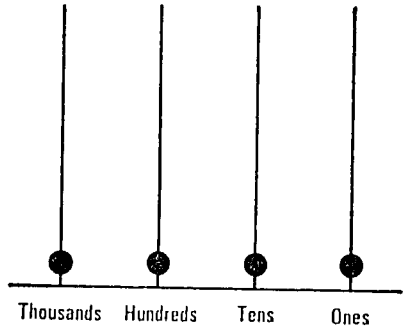
3 thousands or
30 hundreds or
300 tens or
3,000 ones

EXERCISE 23-1C

1. Using squares and the method shown above, draw pictures to show the value of $1,111_{six}$.
2. Give the value of each of the digits underlined in three different ways.
   a. $3\underline{2}$,541 **b.** 5,$\underline{6}$78 **c.** $\underline{1}$0,327
3. **a.** How many tens are there in 362?
   b. How many tens are there in 5,362?
   c. How many hundreds are there in 37,140?

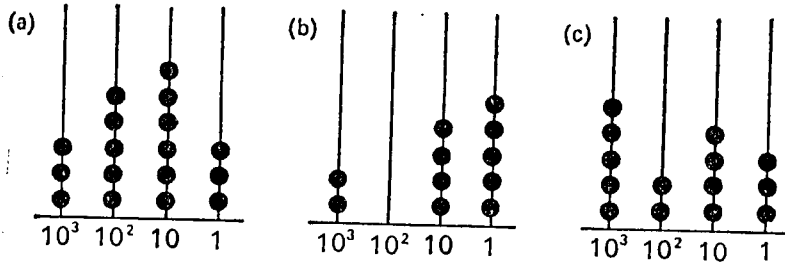
23-2 Decimal fractions

Do you remember how to represent numbers on an abacus? Here is an abacus with four rods. We label the rods as we label the places in a numeral.



The bead on the tens rod is worth 10 of the beads on the ones rod. The bead on the hundreds rod is worth 10 of the beads on the tens rod or 100 of the beads on the ones rod.

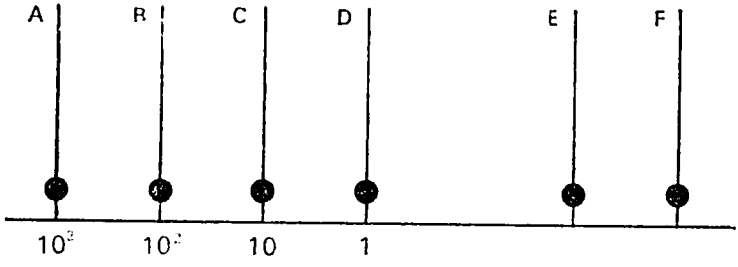
Thus each bead is worth 10 of the beads on the rod next to it on the right.



Shown above are three numbers, represented on the abacus. Can you write them down? Use the index notation first, and then write them in the usual way. Here they are:

(*a*) $3 \times 10^3 + 5 \times 10^2 + 6 \times 10 + 3 \times 1 = 3{,}563$
(*b*) $2 \times 10^3 + 0 \times 10^2 + 4 \times 10 + 5 \times 1 = 2{,}045$
(*c*) $5 \times 10^3 + 2 \times 10^2 + 4 \times 10 + 3 \times 1 = 5{,}243$

It will be useful to us now to think of the relationship between the beads the other way round. To make it easier to write, we will call the rods A, B, C and D.



Ten beads on rod B are worth 1 bead on rod A. So a bead on rod B is worth $\frac{1}{10}$ of a bead on rod A. The same relationship, "$\frac{1}{10}$ of", will be true for each bead and a bead on the rod immediately to the left of it.

A bead on rod C is worth $\frac{1}{10}$ of a bead on rod B.

A bead on rod D is worth $\frac{1}{10}$ of a bead on rod C.

If we place another rod, E, to the right of rod D, we can say that a bead on rod E is worth $\frac{1}{10}$ of a bead on rod D. But a bead on rod D is worth *one*, and so a bead on rod E is worth $\frac{1}{10}$ of one, that is, 1 *tenth*. So we can name rod E the "tenths rod" just as rod D is named the "ones rod" and rod B is named the "hundreds rod". You will note that a space has been left between rod D and rod E. This is to remind us that the whole numbers end with rod D. After rod D we have tenths, which are fractions, *decimal fractions*.

We will now place a rod F to the right of rod E. A bead on rod F must be worth $\frac{1}{10}$ of a bead on rod E. But a bead on rod E is worth $\frac{1}{10}$ of one. So a bead on rod F is worth $\frac{1}{10}$ of $\frac{1}{10}$ of one. In the chapter on multiplication of fractions, you learned how to find that this is $\frac{1}{100}$. But you can help your pupils to understand this better by working it out on the abacus.

How many beads on rod F make 1 bead on rod E? 10.

How many beads on rod E make 1 bead on rod D? 10.

So how many F beads made a D bead? 100. So an F bead is $\frac{1}{100}$ of a D bead. But a D bead is worth one, so an F bead is

worth $\frac{1}{100}$ of $1 = \frac{1}{100}$. So now you can name the F rod the "hundredths rod".

Place value of decimal fractions

You know that the value of a digit in a numeral is shown by its position in the row of digits. Each position has a value of $\frac{1}{10}$ of the position next to it on the left. Using the place-value system, we can write the number shown on the abacus above as 1111 11. The last 1 represents 1 hundredth and the 1 to the left of it represents 1 tenth. You can see that you could easily mistake this number for 111,111, which has no fractional part. You need some way of telling which digits represent whole numbers and which digits represent fractions. That is why we use a dot called the *decimal point*. The number shown on the abacus is then written 1,111·11. This, you remember, is spoken as "one thousand, one hundred and eleven point one one". We do not usually say the value of the decimal fraction when we read it. For another example, we will read 362·54. It is read as "three hundred and sixty-two point five four".

It will be helpful to your pupils to make a number chart which includes decimal fractions. Here is one.

Decimal Point

	Millions	*Hundred-thousands*	*Ten-thousands*	*Thousands*	*Hundreds*	*Tens*	*Ones*	•	*Tenths*	*Hundredths*	*Thousandths*	*Ten-thousandths*	*Hundred-thousandths*	*Millionths*
(*a*)					5	6	9	•	1					
(*b*)						1	2	•	2	3				
(*c*)								•	0	0	0	0	0	1
(*d*)				1	7	8	9	•	3	8	4	0	2	
(*e*)	1	0	0	0	0	0	1	•	0	0	1	0	0	3
(*f*)		1	2	9	6	5	2	•	4	8	5	3	2	4

You will notice that the columns have been extended farther to the right to include thousandths, ten-thousandths, hundred-thousandths and millionths. The place value of each column is $\frac{1}{10}$ of the value of the column next to it on the left. The column on the right of the hundredths column is worth $\frac{1}{10}$ of $\frac{1}{100}$, and this is $\frac{1}{1000}$ or 1 thousandth. Can you show the value of each of the remaining three columns in the same way?

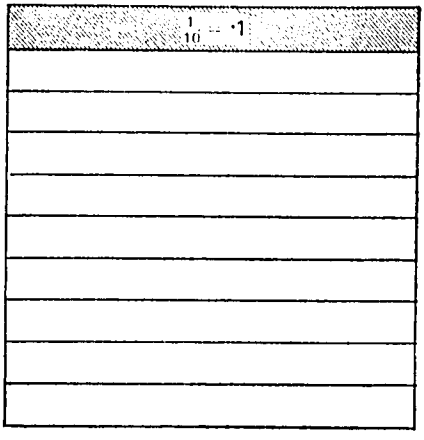
EXERCISE 23-2A

1. Write in full, as was done above for 362·54, the numbers shown on the chart on page 22.
2. Write as common fractions or as whole numbers the digits underlined on the chart.
3. Write each of the numbers (*a*), (*b*), (*c*) and (*d*) shown above in the expanded form. $\left(\text{For example, 23·67 would be } 2 \times 10 + 3 \times 1 + 6 \times \frac{1}{10} + 7 \times \frac{1}{10^2}.\right)$

Pictures of decimal fractions

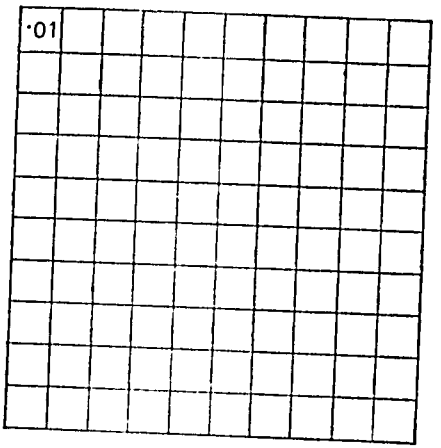
You can show decimal fractions in pictures, using squares as you used them to show whole numbers. Last time you used one small square for 1 one. This time you will need one *large* square to represent 1 *one*. Make a large square with each side $2\frac{1}{2}$ inches. Now divide it into strips each $\frac{1}{4}$ inch wide and $2\frac{1}{2}$ inches long. There will be ten of these strips, so each one represents $\frac{1}{10}$ or ·1. How many strips will represent ·2? ·6?

10 strips, each $\frac{1}{10}$ of 1;
1 one = 10 tenths



Now you want to show ·01. This is $\frac{1}{100}$, and $\frac{1}{100}$ is $\frac{1}{10}$ of $\frac{1}{10}$. So each strip must be divided into 10 small squares. One small square represents ·01, or $\frac{1}{100}$, as there are 100 small squares.

1 one = 10 tenths
= 100 hundredths



If you now divide each small square into 10 small strips, each small strip will be $\frac{1}{10}$ of $\frac{1}{100} = \frac{1}{1000}$ or 1 thousandth. An alternative way of writing a decimal fraction which is less than 1 is to put a 0 in the ones place; for example, 0·2 or 0·35. It is thought that the 0 helps to draw attention to the decimal point.

Regrouping of decimal fractions

Pictures like those opposite can help you to see how many tenths or hundredths there are in a number. You can think of 430 as 43 tens and in the same way you can think of ·43 as 43 hundredths. If you look at picture (*b*), you can see that this is so; 4 is 4 strips and each strip has 10 hundredths in it. So altogether there are $[(4 \times 10) + 3]$ hundredths.

So ·43 = 43 hundredths.

You can also think of this number in tenths. There are 4 tenths and 3 hundredths. One hundredth is $\frac{1}{10}$ of 1 tenth, so you can write ·43 as $[4 + (3 \times \frac{1}{10})]$ tenths.

So ·43 = 4·3 tenths.

In the same way, 430 can be written as 43 tens or as 4·3 hundreds or as 4,300 tenths.

You will have found this way of writing numbers in newspapers. Instead of 3,650,000, you will see 3·65 millions. It is

shorter and easier to read. Or you may see 6·5 thousands. This, in full, would be 6,500. You will read more about this way of writing numbers when you come to the unit on approximations.

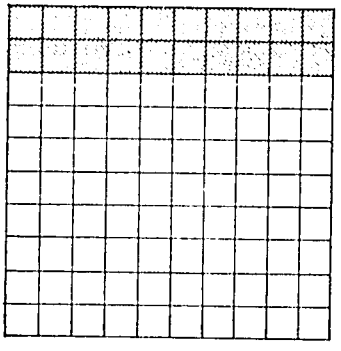
EXERCISE 23-2B

1. Draw pictures to show these decimal fractions.

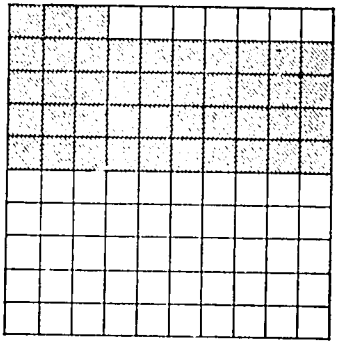
   a. ·3 **b.** ·7 **c.** ·02 **d.** ·07 **e.** ·72 **f.** ·88

2. What decimal fractions are represented by the shaded parts of these pictures if the big square represents 1 one?

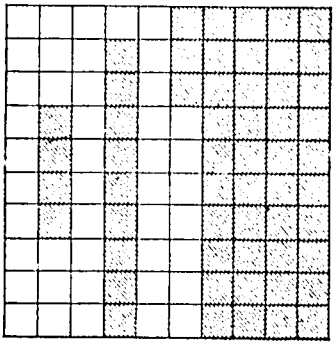
(a)



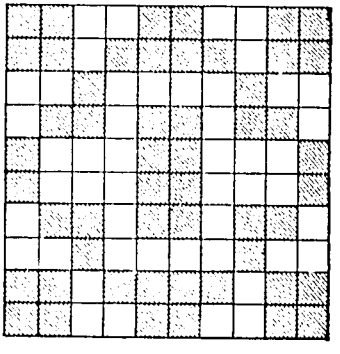
(b)



(c)



(d)



3. Write each of the numbers in Question 2 as tenths and then as hundredths.

4. Write each of these numbers as a decimal fraction of the unit mentioned.

   a. 325 as tens
   b. 325 as tenths
   c. 27·56 as tenths
   d. 394·61 as hundredths
   e. 3,620,000 as millions
   f. 7,200 as thousands
   g. ·037 as thousandths
   h. 2·37 as tens

5. Make a number chart stretching from hundreds to thousandths. Then write these numbers as decimal fractions on your chart.

a. $\frac{2}{10}$ **b.** $\frac{3}{100}$ **c.** $\frac{15}{100}$ **d.** $\frac{15}{10}$
e. $\frac{269}{100}$ **f.** $\frac{269}{100}$ **g.** $\frac{7}{1000}$ **h.** $\frac{29}{1000}$
i. Forty-two tenths **j.** 2·9 hundreds
k. 327 thousandths
l. Twenty-five, five tenths and three thousandths

No improper decimal fractions

There are proper fractions, such as ·36, and also mixed numbers, such as 2·3, in decimal fractions, but there are no improper fractions. Try to write an improper fraction such as $\frac{15}{10}$ as a decimal fraction and you find that you get a mixed number 1·5. This is because the principle of place value makes it impossible to write more than one digit in one column.

Common fractions as decimal fractions

You already know how to write some fractions as decimal fractions. A fraction whose denominator is 10 or 100 or some power of 10 is easily written as a decimal fraction. Examples are $\frac{23}{100} = ·23$, $\frac{5718}{1000} = 5·718$. Some other fractions have denominators which are easily converted into powers of 10 by using the procedure you learned for finding fractions equal to a given fraction.

Do you remember how to rewrite $\frac{2}{3}$ in sixths? Why is it true that $\frac{2}{3} = \frac{4}{6}$? Look back to Chapter 20 if you have forgotten, because you will need this procedure in the next paragraph.

Here are some fractions which have been rewritten with denominators as powers of 10. What can you say about all the denominators of the first fractions of all the sets?

$$\frac{4}{5} = \frac{8}{10} = ·8; \quad \frac{1}{4} = \frac{25}{100} = ·25; \quad \frac{1}{8} = \frac{125}{1000} = ·125$$

The description is very simple. The denominators are 5, 4 and 8, and these numbers are all products of powers of 2 or 5. Numbers which are products of powers of 2 or 5 are factors of a power of 10 and so can be multiplied by a whole number to give a power of 10.

EXERCISE 23-2C

1. Write each of these fractions as an equal fraction with its denominator a power of 10. Then write the corresponding decimal fraction.

a. $\frac{3}{5}$	**b.** $\frac{1}{2}$	**c.** $\frac{3}{4}$	**d.** $\frac{5}{8}$	**e.** $\frac{3}{20}$	**f.** $\frac{21}{25}$
g. $\frac{31}{50}$	**h.** $\frac{31}{500}$	**i.** $\frac{17}{5}$	**j.** $\frac{5}{2}$	**k.** $\frac{27}{4}$	**l.** $\frac{9}{8}$
m. $\frac{45}{20}$	**n.** $\frac{201}{25}$	**o.** $\frac{921}{50}$	**p.** $\frac{17}{16}$		

There is another way to find the decimal fraction equivalent of a common fraction. This is by division. Think of $\frac{1}{2}$. You first met $\frac{1}{2}$ as the missing factor in the multiplication equation

$$2 \times \square = 1$$

This equation corresponds to the division equation

$$1 \div 2 = \square$$

So we can divide 1 by 2 and know that the quotient will be one-half. Let us divide 1 by 2 and see what quotient we can find. You will remember how to set this down as a division exercise.

To find $1 \div 2$:

Ones: There is no whole-number answer to $1 \div 2$, so we regroup 1 as 10 tenths. This we do by simply putting a decimal point after the 1 and adding a 0 in the tenths column.

Ones Tenths
$$\begin{array}{r} \cdot 5 \\ \hline 2\overline{)1\cdot 0} \end{array}$$

Tenths: $10 \div 2 = 5$. We write 5 in the tenths column.

So we have shown that $\frac{1}{2} = \cdot 5$

It is useful also to work this division in common fractions.

$$\begin{aligned} 1 \div 2 &= (10 \times \tfrac{1}{10}) \div 2 \\ &= \tfrac{10}{10} \div 2 = \tfrac{5}{10} = \cdot 5 \end{aligned}$$

Let us next see how we can write $\frac{1}{4}$ as a decimal fraction. We know that $\frac{1}{4} = 1 \div 4$.

To find $1 \div 4$:

Ones: $1 \div 4$ is not a whole number

Regroup 1 one as 10 tenths.

Tenths: $10 \div 4 = (8 + 2) \div 4$
$= 2 + (2 \div 4)$

Regroup the second 2 tenths as 20 hundredths.

Then $1 \div 4 = 2$ tenths $+ 5$ hundredths.

So $\frac{1}{4} = 0{\cdot}25$

Short form

```
Ones Tenths Hundredths
     ·2 5
     10 20
4)1·0 0
```

The numbers written above the digits show how the regrouping is done at each step, beginning with regrouping one as 10 tenths.

Using common fractions, this same problem can be worked as follows:

$$\begin{aligned}\tfrac{1}{4} = 1 \div 4 &= \tfrac{10}{10} \div 4 = \left(\tfrac{8}{10} + \tfrac{2}{10}\right) \div 4 \\ &= \left(\tfrac{8}{10} \div 4\right) + \left(\tfrac{2}{10} \div 4\right) \\ &= \tfrac{2}{10} + \left(\tfrac{2}{10} \times \tfrac{10}{10} \div 4\right) \\ &= \tfrac{2}{10} + \left(\tfrac{20}{100} \div 4\right) \\ &= \tfrac{2}{10} + \tfrac{5}{100} = {\cdot}25\end{aligned}$$

This method with common fractions can be used to explain the short form written above on the right.

Below is another example using only the short form.

We know $\frac{7}{8} = 7 \div 8$. But $7 \div 8$ can be found as follows:

```
Ones Tenths Hundredths Thousandths
     ·8 7 5
     70 60 40
8)7·0 0 0
```

So $\frac{7}{8} = 7 \div 8 = {\cdot}875$

Now try $\frac{1}{3}$. Since $\frac{1}{3} = 1 \div 3$, the short form is as follows:

```
Ones Tenths Hundredths Thousandths Ten-Thousandths
     ·3 3 3 3 . . .
     10 10 10 10
3)1·0 0 0 0 . . .
```

At each step, regrouping and division gives a quotient of 3 and a remainder of 1, which indicates that the process can go on forever. There is no whole number which equals $\frac{10}{3}$, since $\frac{10}{3} = \frac{9}{3} + \frac{1}{3} = 3 + \frac{1}{3}$.

We cannot write all these 3's, so we agree for the present to write only digits to the place value of thousandths and put three dots after this thousandths digit to show that the decimal fraction is unending. That is, we write $\frac{1}{3} = 0{\cdot}333\ldots$, the three dots showing that there are other digits which have not been written down.

Decimal fractions which are found this way from common fractions often have a pattern of recurring or repeating digits after the decimal point. You will meet several of these in this chapter. They are called *recurring decimal fractions*. There is an alternative way of writing a recurring decimal. Place a dot over the digits which recur, but if more than two digits recur, the dots are placed over the first and last recurring digits only. Thus $\frac{1}{3} = {\cdot}\dot{3}$, and ${\cdot}\dot{3}4\dot{8} = {\cdot}348348348\ldots$

Here is another example. We know $\frac{2}{9} = 2 \div 9$

	Ones	Tenths	Hundredths	Thousandths	
		·2	2	2	...
9)	2·	0	0	0	...
		20	20	20	

So $\frac{2}{9} = {\cdot}222\ldots = {\cdot}\dot{2}$

EXERCISE 23-2D

1. Find the decimal fraction equivalent of these common fractions by division.

   a. $\frac{1}{2}$ **b.** $\frac{1}{4}$ **c.** $\frac{3}{4}$ **d.** $\frac{3}{5}$ **e.** $\frac{7}{10}$ **f.** $\frac{5}{8}$ **g.** $\frac{5}{16}$

2. These common fractions that follow all have recurring decimal fraction equivalents. Work each division far enough to be sure you have found the pattern of recurring digits. Put dots over the digits to show the recurring digits.

   a. $\frac{1}{6}$ **b.** $\frac{1}{7}$ **c.** $\frac{1}{9}$ **d.** $\frac{1}{11}$ **e.** $\frac{2}{3}$ **f.** $\frac{5}{6}$ **g.** $\frac{7}{13}$

You will often need to change a common fraction to a decimal fraction and vice versa. It is a good thing to remember these equivalent forms in pairs. Your pupils should also remember them, but be sure that they understand first how to work them out and why they are equivalent. It will help your pupils to draw pictures using the big square of a hundred small squares so that they can see that the *number* represented by a common fraction and by its decimal equivalent is the same number.

EXERCISE 23-2E

1. Fill in the gaps in this table. Do not use more than three places in decimal fractions.

Common fraction	$\frac{1}{2}$	$\frac{1}{5}$		$\frac{3}{4}$	$\frac{6}{8}$		$\frac{2}{5}$	$\frac{1}{10}$		$\frac{37}{100}$
Decimal fraction			·25			·2			·07	

2. Draw pictures to show that each of the pairs of numerals above represents the same number.
3. Give three common fraction names and three decimal fraction names for each of these numbers.

   a. $\frac{2}{3}$ **b.** 31 tens **c.** 31 tenths **d.** 5·7 **e.** $\frac{16}{32}$ **f.** $\frac{3}{8}$

23-3 Fractions in other bases

You know how to write whole numbers in bases other than ten, and you know how to write fractions in base ten. In exactly the same way, you can write fractions in bases other than ten. For example, the base-six notations can be extended to include fractions formed as a result of dividing by 6.

$$243{\cdot}12_{\text{ten}} = [(2 \times 10^2) + (4 \times 10) + (3 \times 1) + (1 \times \tfrac{1}{10}) + (2 \times \tfrac{1}{10^2})]_{\text{ten}}$$

And so: $$243{\cdot}12_{\text{six}} = [(2 \times 6^2) + (4 \times 6) + (3 \times 1) + (1 \times \tfrac{1}{6}) + (2 \times \tfrac{1}{6^2})]_{\text{ten}}$$

EXERCISE 23-3A

1. Write each of the following numbers in the expanded form in base ten:

a. $162{\cdot}5_{\text{seven}}$ **b.** $325{\cdot}321_{\text{six}}$ **c.** $110{\cdot}111_{\text{two}}$

2. Write each of these numbers as a common fraction (base ten) with the lowest possible numerator and the lowest denominator.

a. ${\cdot}375_{\text{ten}}$ **b.** ${\cdot}4_{\text{eight}}$ **c.** ${\cdot}02_{\text{six}}$ **d.** ${\cdot}1_{\text{two}}$
e. $1{\cdot}22_{\text{three}}$ **f.** $2{\cdot}4_{\text{twelve}}$ **g.** $3{\cdot}2_{\text{five}}$ **h.** $1{\cdot}02_{\text{four}}$

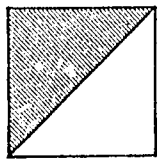
3. Find the numeral equivalent to $\frac{1}{2}$ in each of the following bases. Use three decimal places only. Use three dots to show a recurring decimal fraction.

a. base two **b.** base three **c.** base four
d. base five **e.** base six **f.** base seven
g. base eight

4. Make a third row to the table in Exercise 23-2E, Question 1. Fill this row with the base-six numerals for these fractions.

5. Can a recurring decimal in base ten be an ending numeral in a different base? Give six examples to explain your answer to this question, using the results of previous exercises.

Now that you have worked through this chapter on decimal fractions and fractions in other bases, you will see that there is nothing mysterious about them. They are simply other names for the common fractions. You have seen, too, that the fraction shown shaded in the picture can be called by many different names. Here are some of them.



$\frac{1}{2}$, $\frac{2}{4}$, $\frac{10}{20}$, ${\cdot}5_{\text{ten}}$, ${\cdot}111\ldots_{\text{three}}$, ${\cdot}1_{\text{two}}$, ${\cdot}4_{\text{eight}}$, ${\cdot}2_{\text{four}}$, ${\cdot}222\ldots_{\text{five}}$, ${\cdot}3_{\text{six}}$, ${\cdot}8_{\text{sixteen}}$.

One-half may be written either as an ending numeral (bases 2, 4, 6, 8, . . .) or as a non-ending numeral (bases 3, 5, 7, 9, . . .). So there is nothing difficult about a non-ending or recurring decimal fraction. It is the notation which makes it appear more complicated. What could be simpler than the idea of one-half? Yet in base three it is ·111 . . . . While one-third in base ten is the recurring numeral ·333 . . . , in base three it is ·1.

Chapter 24

OPERATIONS IN DECIMAL NOTATION

24-1 Addition and subtraction

When your pupils thoroughly understand decimal fractions, they should be able to do problems in addition and subtraction with very little difficulty. The methods are exactly the same as for addition and subtraction of whole numbers.

Addition

You will remember that when you add two whole numbers you take each column in turn and add first the ones, then the tens, then the hundreds and so on. If any total is greater than nine, it has to be regrouped. In exactly the same way, you regroup 10 hundredths to give 1 tenth or 10 tenths to give 1 one. Here are two examples of addition; one is addition of whole numbers, and the other is addition of decimal fractions.

Hundreds	Tens	Ones
1	6	7
2	9	8
4	6	5

Ones: $7 + 8 = 15 = 1$ ten $+ 5$ ones.
Write 5 ones.

Tens: $6 + 9 + 1 = 16 = 1$ hundred $+ 6$ tens.
Write 6 tens.

Hundreds: $1 + 2 + 1 = 4$.
Write 4 hundreds.

Tens	Ones	Tenths	Hundredths
1	6	7	6
2	1	8	9
3	8	6	5

Hundredths: 6 + 9 = 15 = 1 tenth + 5 hundredths.
Write 5 hundredths.

Tenths: 7 + 8 + 1 = 16 = 1 one + 6 tenths.
Write 6 tenths.

Ones: 6 + 1 + 1 = 8. Write 8 ones.

Tens: 1 + 2 = 3. Write 3 tens.

If your pupils are to work a problem in which the addends are written horizontally, you will need to remind them to think of the value of each digit and then rearrange the addends vertically. For example, suppose you give them this problem: ·3 + 1·4 + ·0016 + 29 = □. The numbers should be rearranged with the decimal points vertically in line. The first number is 3 tenths and so must be written in the tenths column. Here is the setting out of this calculation and the thinking which should go with it.

		Tens	Ones	Tenth	Hundredths	Thousandths	Ten-Thousandths
(·3)	3 tenths			3			
(1·4)	1 one and 4 tenths		1	4			
(·0016)	0 tenths, 0 hundredths, 1 thousandth, 6 ten-thousandths			0	0	1	6
(29)	2 tens and 9 ones	2	9				
		3	0	7	0	1	6

Pupils who have practised writing decimal fractions on a number chart should not have any difficulty in doing this kind of problem correctly.

EXERCISE 24-1A

1. Find the sums of these sets of numbers, setting out the explanation at the side as you would do for your pupils.

a.	**b.**	**c.**	**d.**
294	21·7	29·87	1·815
519	13·5	96·75	12·504
	9·6		9·139

2. Make up four examples of addition written horizontally and show how you would expect your pupils to work these problems.

Subtraction

Subtraction without regrouping is very simple. Here are two examples. One is using whole numbers and the other is using decimal fractions.

Hundreds	Tens	Ones
3	6	5
2	2	1
1	4	4

Ones	Tenths	Hundredths
3·	2	9
1·	0	7
2·	2	2

Ones: $5 - 1 = 4$
Tens: $6 - 2 = 4$
Hundreds: $3 - 2 = 1$

Hundredths: $9 - 7 = 2$
Tenths: $2 - 0 = 2$
Ones: $3 - 1 = 2$

If you have the problem $86 - 18 = \square$, you see that there are not enough ones in the ones place of 86 in order to subtract 8. So you have to regroup 86 as 7 tens and 16 ones. Then

$$86 = 70 + 16$$
$$18 = 10 + 8$$
$$60 + 8 \quad \text{so} \quad 86 - 18 = 68$$

The procedure is similar for decimal fractions. Suppose you have to find 3·21 − ·08. Here is the thinking.

Hundredths: There are not enough ones in the hundredths place of 3·21 to subtract 8. Regroup 3·21 as 3 ones plus 1 tenth plus 11 hundredths. Then 11 hundredths − 8 hundredths = 3 hundredths.

Tenths: 1 tenth − 0 tenths = 1 tenth.

Ones 3 ones − 0 ones = 3 ones.
So 3·21 − ·08 = 3·13

Ones	Tenths	Hundredths
3·	2	1
·0	8	
3·	1	3

Here is another example. Find 6·5 − 3·77.

Hundredths: The hundredths digit does not appear in 6·5. So regroup 5 tenths as 4 tenths

		Ones	Tenths	Hundredths
	plus 10 hundredths. Then 10 − 7 = 3 hundredths.			
Tenths:	There are not enough tenths in the tenths place of 6·4 to subtract 7, so regroup 6·4 as 5 ones and 14 tenths. Then 14 − 7 = 7 tenths.	6	5	
		3	7	7
Ones:	5 − 3 = 2 ones. So 6·5 − 3·77 = 2·73	2	7	3

EXERCISE 24-1B

1. Explain how to work these problems as you would explain them to your pupils.

   a. 231 − 98 **b.** 87·65 − 64·32
   c. 2·35 − 1·79 **d.** 76·84 − 18·92

2. Which number in each of the following pairs of numbers is the greater and by how much is it greater?

   a. 2·3 and 1·59 **b.** 87·32 and 24·118
   c. ·0017 and ·12 **d.** ·3 and ·168

3. The rainfall in Freetown on a certain week was recorded in inches as follows: Sunday, 1·40 inches; Monday, 3·20 inches; Tuesday, 3·70 inches; Wednesday, 3·21 inches; Thursday, 0·80 inches; Friday, 0·10 inches; Saturday, 0·01 inches.

   a. How many inches of rain fell in Freetown that week?
   b. How many more inches of rainfall were there on Wednesday than on Thursday?

4. Decimal fractions are used in many ways but chiefly to show the results of measurement. Look at a daily newspaper and make a list of the ways in which decimal fractions are used there. Use this list to make up some story problems for your pupils in addition and subtraction. Question 3 shows two kinds of questions you can ask them.

24-2 Multiplication and division by powers of 10

Whole numbers

Perhaps you remember that multiplication of a number by 10 or 100 is quite easy. For example,

$$23 \times 10 = 230 \quad \text{and} \quad 375 \times 100 = 37{,}500$$

The first of these products can be pictured as follows:

Hundreds	Tens	Ones				Hundreds	Tens	Ones
	2	3	×	10	=	2	3	0

The digits of the number 23 have each moved *one* place to the left, so that each digit has a value 10 times as great as it had.

A similar diagram for the second product is as follows:

Ten-thousands	Thousands	Hundreds	Tens	Ones				Ten-thousands	Thousands	Hundreds	Tens	Ones
		3	7	5	×	100	=	3	7,5	0	0	

Each digit in 375 has moved *two* places to the left. It then has a value 100 times as great as it had originally.

What task is performed by the zeros in these products? They show zero ones or zero tens. They are needed so that the other digits are shown in their correct places and get their correct values.

Now think about division by 10, as in 50 ÷ 10 or 560 ÷ 10. You know already that division is the opposite or inverse process to multiplication. Multiplication by 10 moves each digit one place to the left. Division by 10 moves each digit one place to the right. Each digit then becomes $\frac{1}{10}$ as large in value as it was originally. What happens if we divide by 100? Here are charts showing this for several examples.

Hundreds	Tens	Ones					Ones	Tens
	5	0	÷	10	=		5	
5	6	0	÷	10	=	5	6	
5	0	0	÷	100	=		5	

EXERCISE 24-2A

1. Make number charts and use them to show what happens in these problems.

a. 7 × 10 **b.** 56 × 100 **c.** 30 × 10
d. 70 ÷ 10 **e.** 700 ÷ 100 **f.** 560 ÷ 10

2. Make up six more examples like those above and show them on a pair of number charts.

Decimal fractions

Now that you think of multiplication by a power of 10 as moving digits to the left, you will very quickly see how to multiply decimal fractions by powers of 10. It works exactly the same way, because decimal fractions are written in the same way as the whole numbers. Each place in a decimal fraction is worth 10 times as much as the place immediately to the right. Here are some examples written on number charts.

Hundreds	Tens	Ones	Tenths	Hundredths					Hundreds	Tens	Ones	Tenths	Hundredths
			·5		×	10	=				5·		
		7·	5		×	10	=			7	5·		
		·0	6		×	10	=				0·	6	
6	3·	1	7		×	10	=	6	3	1·	7		

You can also work these out by using the expanded form of a decimal fraction. For example,

$$\cdot 5 \times 10 = (5 \times \tfrac{1}{10}) \times 10 = 5 \times 1 = 5$$

$$\begin{aligned} 63{\cdot}17 \times 10 &= [(6 \times 10) + (3 \times 1) + (1 \times \tfrac{1}{10}) \\ &\quad + (7 \times \tfrac{1}{10^2})] \times 10 \\ &= [(6 \times 10) \times 10 + (3 \times 1) \times 10 \\ &\quad + (1 \times \tfrac{1}{10}) \times 10 + (7 \times \tfrac{1}{10^2}) \times 10] \\ &\quad \text{(Note use of distributive property.)} \\ &= (6 \times 10^2) + (3 \times 10) + (1 \times 1) + (7 \times \tfrac{1}{10}) \\ &= 631{\cdot}7 \end{aligned}$$

Did you notice on the last line how every digit in the number 63·17 became worth 10 times as much and so was moved one place to the left?

If you multiply by 100, the digits are moved two places to the left, and so $2{\cdot}3 \times 100 = 230$. You can show this in two stages.

$$\begin{aligned} 2{\cdot}3 \times 100 &= 2{\cdot}3 \times (10 \times 10) = (2{\cdot}3 \times 10) \times 10 \\ &= 23 \times 10 = 230 \end{aligned}$$

Now you will see that division by a power of 10 presents no

difficulty whatever. Just move the digits one place to the right for every 10 by which you divide.

$$\begin{aligned} 73 \div 10 &= (7 \times 10 + 3 \times 1) \div 10 \\ &= (7 \times 10) \div 10 + (3 \times 1) \div 10 \\ &= 7 + (3 \times \tfrac{1}{10}) = 7{\cdot}3 \\ 73 \div 100 &= 73 \div (10 \times 10) = (73 \div 10) \div 10 \\ &= 7{\cdot}3 \div 10 \\ &= {\cdot}73 \end{aligned}$$

You can do the last problem in one step by moving each digit two places to the right straightaway.

$$73 \div 100 = {\cdot}73$$

Your pupils should work these problems in two stages at first and should use a pair of number charts until they really understand what they are doing. Here are some division problems.

Hundreds	Tens	Ones	Tenths	Hundredths				Ones	Tenths	Hundredths	Thousandths	Ten-thousandths
7	3	·			÷	10	=	7	·	3		
9	8	·			÷	100	=		·	9	8	
1	6	·	5		÷	10	=	1	·	6	5	
2	3	7	·	7 1	÷	100	=	2	·	3	7	7 1

EXERCISE 24-2B

1. Find the following by moving the digits:

a. ·7 × 10 **b.** ·09 × 10 **c.** ·32 × 10
d. 5·72 × 10 **e.** ·7 ÷ 10 **f.** 3·09 ÷ 10
g. 97·32 ÷ 10 **h.** 105·72 ÷ 10

2. Find the following by moving the digits:

a. 32 × 100 **b.** 7·5 × 100 **c.** ·012 × 1000
d. 19·401 × 100 **e.** 763 ÷ 100 **f.** 76·2 ÷ 100
g. 79,321 ÷ 1000 **h.** 1 ÷ 100

3. Choose four different types of problems from Questions 1 and 2 and tell how you would help pupils to understand how to find the answer to them.

You will of course have noticed that instead of moving the

digits of a number one place to the left when multiplying by 10, you can instead move the decimal point one place to the right. You can see this in this equation:

$$24{\cdot}81 \times 10 = 248{\cdot}1$$

The point has "jumped" over one digit, the 8. Similarly, when you divide by 100, you can move the decimal point two places to the left instead of moving the digits of the number two places to the right. In the equation

$$52{\cdot}643 \div 100 = {\cdot}52643$$

you see that the point has jumped over two digits, the 2 and the 5. This is sometimes thought to be a quicker method.

24-3 Multiplication and division by whole numbers less than 10

Multiplication

You will remember that multiplication is very similar to addition. Whenever you have more than 9 in any column, you regroup to make 1 or more for the next column. You multiply 17×8 as follows:

Ones: $7 \times 8 = 56 = 5$ tens and 6 ones.
Write 6 ones.
Tens: $1 \times 8 = 8$, $8 + 5 = 13$
$= 1$ hundred and 3 tens.
Write 3 tens and 1 hundred.

	Hundreds	Tens	Ones
		1	7
×			8
	1	3	6

You will see that tenths may be multiplied in a similar way.

$$\cdot 7 = \tfrac{7}{10}$$

$$\tfrac{7}{10} \times 8 = \tfrac{56}{10} = \tfrac{50}{10} + \tfrac{6}{10} = (5 \times 1) + (6 \times \tfrac{1}{10})$$
$$= 5{\cdot}6$$

You need not think this out in this long way, because in the last chapter you practised writing fractions such as $\frac{56}{10}$ as decimal fractions. You can say that $\frac{56}{10} = 5{\cdot}6$ straightaway. Your pupils, however, will need to think about what they are doing more carefully and will need to write their multiplications in a chart at first. Here is $\cdot 76 \times 4$ worked out.

Hundredths: $6 \times 4 = 24$. Regroup as 2 tenths and 4 hundredths.
Write 4 hundredths.

Tenths: $7 \times 4 = 28$, $28 + 2 = 30$. Regroup as 3 ones and 0 tenths.
Write 0 tenths and 3 ones.
So $\cdot 76 \times 4 = 3 \cdot 04$.

	Ones	Tenths	Hundredths
		·7	6
×			4
	3	·0	4

Division

Division by a whole number may give a whole-number answer or it may give a fractional answer. If the answer is not a whole number, then it is often very useful to write the fractional part as a decimal fraction. Decimal fractions can be compared for size more easily than common fractions can be compared and so are more often used in a practical situation such as measuring.

You remember how to do division to obtain a decimal fraction from a common fraction. You worked many examples in Chapter 23. First, we will work two examples without remainders.

First example: 357 ÷ 3

Regroup 357 as 3 hundreds + 3 tens + 27 ones.

Then $357 \div 3 = (300 + 30 + 27) \div 3$
$= (300 \div 3) + (30 \div 3) + (27 \div 3)$
$= 100 + 10 + 9$
$= 119$

Short form

$$\begin{array}{rl} 100 & + 10 + 9 = 119 \\ 3\overline{)357} & \\ 300 & (100 \times 3 = 300) \\ \hline 57 & \\ 30 & (10 \times 3 = 30) \\ \hline 27 & \\ 27 & (9 \times 3 = 27) \\ \hline \end{array}$$

Second example: 27·95 ÷ 5

The regrouping is described in steps.

Tens: $2 \div 5$ is not a whole number. Regroup 2 tens as 20 ones. $20 + 7 = 27$ ones.

Ones: $27 = (\underline{5} \times 5) + \underline{2}$. Write 5 in the ones place. Regroup the 2 as 20 tenths. $20 + 9 = 29$ tenths.

Tenths: $29 = (\underline{5} \times 5) + \underline{4}$. Write 5 in the tenths place. Regroup the 4 as 40 hundredths. $40 + 5 = 45$ hundredths.

Hundredths: $45 = \underline{9} \times 5$. Write 9 in the hundredths place.

So $27{\cdot}95 \div 5 = 5{\cdot}59$

Short form

```
      5·5 9
   5)2 7·9 5
     2 5        (5 × 5 = 25)
       2·9
       2·5      (·5 × 5 = 2·5)
        ·4 5
        ·4 5    (·09 × 5 = ·45)
```

Notice that the result comes from regrouping 27·95 as 25 ones + 25 tenths + 45 hundredths. Division by 5 gives 5 ones + 5 tenths + 9 hundredths = 5·59.

In each example there was no remainder, but in the second example there were decimal fractions involved. The method there was really no different, but the regrouping was a bit harder to write down. So the step-by-step procedure was easier to use. This is often the case.

EXERCISE 24-3A

1. Work these problems.

a. $7{\cdot}4 \times 6$	**b.** $91{\cdot}23 \times 8$	**c.** $1{\cdot}005 \times 2$
d. $11{\cdot}642 \times 5$	**e.** $1{\cdot}274 \times 10$	**f.** $\cdot 125 \times 8$

2. Find the numbers to put into the boxes to make these statements true.

a. $\cdot 5 \times \square = 1{\cdot}5$	**b.** $\cdot 1 \times \square = \cdot 1$
c. $\cdot 1 \times \square = \cdot 8$	**d.** $\cdot 7 \times \square = 7$
e. $\cdot 5 \times \square = 25\cdot$	**f.** $\cdot 3 \times \square = 2{\cdot}4$
g. $\cdot 8 \times \square = 0$	**h.** $1{\cdot}2 \times \square = 3{\cdot}6$
i. $1{\cdot}1 \times \square = 44\cdot$	

3. Work these problems.

a. 336 ÷ 8	**b.** 26·35 ÷ 5	**c.** 2900 ÷ 100
d. 39·83 ÷ 7	**e.** 1·01208 ÷ 4	**f.** 20·712 ÷ 3

4. Explain how you would help your pupils understand how to find the answer to Question 3(b), 26·35 ÷ 5.

5. Make up four division problems which have no remainders.

The two division problems which were worked before the last exercise were shown with the subtraction set down. Many pupils will, of course, be able to find these differences without writing them down, and this shorter method of working is shown in the division problems which follow.

Division with remainders

In a problem such as 25 ÷ 2 = □, there is no whole number answer, but there is a fraction which will make the equation true. It is $\frac{25}{2}$ or $12\frac{1}{2}$. Here is another way to work the problem. Since 25 = (12 × 2) + 1, regroup the 1 as 10 tenths. Division by 2 gives 25 ÷ 2 = 12·5.

Let us look at another example, say 67 ÷ 5 = □.

Tens, Ones: 67 = 50 + 15 + 2
= (10 × 5) + (3 × 5) + 2
= (13 × 5) + 2

Tenths: Regroup 2 as 20 tenths.
Then 67 ÷ 5 = 13·4.

```
      13·4
5)67·0
   20
```

If there are decimal fractions in the number to be divided, there is no difference in the method. The regrouping is carried out in the same way as in our previous examples. Here is another example, 27·1 ÷ 4 = □.

Tens: 2 ÷ 4 is not a whole number.
Regroup 2 tens as 20 ones.
20 + 7 = 27 ones.

Ones: 27 = (6 × 4) + 3
Write 6 in the ones place. Regroup 3 ones as 30 tenths.

$$\begin{array}{r} 6{\cdot}775 \\ {\scriptstyle 20\ 30\ 30\ 20} \\ 4\overline{)27{\cdot}100} \end{array}$$

Tenths: $30 + 1 = 31$ tenths. $31 = (\underline{7} \times 4) + \underline{3}$. Write 7 in the tenths place. Regroup 3 tenths as 30 hundredths.

Hundredths: $30 = (\underline{7} \times 4) + \underline{2}$. Write 7 in the hundredths place. Regroup 2 hundredths as 20 thousandths.

Thousandths: $20 = \underline{5} \times 4$. Write 5 in the thousandths place.

So $27{\cdot}1 \div 4 = 6{\cdot}775$.

Sometimes you will find that the regrouping procedure does not end. This means that the quotient is a non-ending decimal. You have met these non-ending decimals earlier. In a case such as this we could agree to stop after one decimal place or we could work to any number of decimal places. There is no special reason to choose any particular decimal place as the stopping point. When your pupils meet non-ending decimals as quotients in division problems, you should always tell them how many decimal places to use.

For example, let us find $3 \div 7$, working to three decimal places. Using only the short form, we get

$$\begin{array}{r} {\cdot}428\ldots \\ {\scriptstyle 30\ 20\ 60} \\ 7\overline{)3{\cdot}000\ldots} \end{array}$$

So we write $3 \div 7 = 0{\cdot}428\ldots$, the three dots showing that the procedure continues and there are other digits that have not been calculated.

We note that the remainder from the thousandths place is 4 thousandths. This means that

$$3 \div 7 = 0{\cdot}428 + \tfrac{4}{7000} \quad \text{(why?)}$$

The fraction $\frac{4}{7000}$ is the difference between 0·428 and the exact answer to $3 \div 7$. It is the error introduced by cutting off the division procedure. In a later chapter we will discuss such errors in detail.

EXERCISE 24-3B

1. Make up three problems of division of a decimal fraction by a whole number less than 10 and describe how you would help your pupils to think out the working of each problem.

2. Work each problem given below to three decimal places.
   a. $34{\cdot}621 \div 4$ **b.** $5{\cdot}317 \div 3$ **c.** $6 \div 7$
3. Work each problem to the number of places given after it.
   a. $5{\cdot}276 \div 8$ (to four decimal places)
   b. $37{\cdot}01 \div 3$ (to three decimal places)
   c. $\frac{7}{9}$ (to two decimal places)
4. In each problem in Question 3, what is the fraction omitted from the quotient?

24-4 Multiplication in decimal notation

You know that $2 \times 3 = 6$, but what is $2 \times {\cdot}3$? You can work this out in two ways.

1. Since fractions have the commutative property of multiplication, $2 \times {\cdot}3 = {\cdot}3 \times 2$, and you already know that this is ${\cdot}6$. (3 tenths $\times$ 2 = 6 tenths.)
2. Rewrite ${\cdot}3$ in the expanded form as $3 \times \frac{1}{10}$. Then:

$$2 \times {\cdot}3 = 2 \times (3 \times \tfrac{1}{10}) = (2 \times 3) \times \tfrac{1}{10}$$

(Note use of associative property)

$$= 6 \times \tfrac{1}{10} = {\cdot}6$$

This second method is more useful, because it can be used to show how to multiply any two decimal fractions. Let us see whether we can find a rule which works for any two decimal fractions we choose. Here are some examples. Can you see what is happening each time?

1.
$$\begin{aligned} {\cdot}3 \times {\cdot}2 &= (3 \times \tfrac{1}{10}) \times (2 \times \tfrac{1}{10}) \\ &= (3 \times 2) \times (\tfrac{1}{10} \times \tfrac{1}{10}) \end{aligned}$$

(Note use of commutative property.)

$$= 6 \times \tfrac{1}{10^2} = {\cdot}06 \qquad {\cdot}3 \times {\cdot}2 = {\cdot}06$$

2.
$$\begin{aligned} {\cdot}5 \times {\cdot}7 &= (5 \times \tfrac{1}{10}) \times (7 \times \tfrac{1}{10}) \\ &= (5 \times 7) \times (\tfrac{1}{10} \times \tfrac{1}{10}) \\ &= 35 \times \tfrac{1}{10^2} = {\cdot}35 \qquad {\cdot}5 \times {\cdot}7 = {\cdot}35 \end{aligned}$$

3.
$$\begin{aligned} {\cdot}5 \times {\cdot}02 &= (5 \times \tfrac{1}{10}) \times (2 \times \tfrac{1}{10^2}) \\ &= (5 \times 2) \times (\tfrac{1}{10} \times \tfrac{1}{10^2}) \\ &= 10 \times \tfrac{1}{10^3} = {\cdot}010 \qquad \underline{{\cdot}5 \times {\cdot}02 = {\cdot}010} \end{aligned}$$

4. $$\begin{aligned} \cdot 02 \times \cdot 00 &= (2 \times \tfrac{1}{10^2}) \times (3 \times \tfrac{1}{10^3}) \\ &= (2 \times 3) \times (\tfrac{1}{10^2} \times \tfrac{1}{10^3}) \\ &= 6 \times \tfrac{1}{10^5} = \cdot 00006 \end{aligned}$$

$$\cdot 02 \times \cdot 003 = \cdot 00006$$

Did you notice that in every problem you separated the digits in the two numbers from the powers of 10 which show their value?

$\cdot 02 \times \cdot 003 = (2 \times 3) \times (\frac{1}{10^2} \times \frac{1}{10^3})$ (omitting one step)

Then you found the products of each pair of factors.

$$= 6 \times \tfrac{1}{10^5}$$

This gave you the product of the digits in the original numbers divided by a power of 10 which tells you the place value of the 6.

$$= \cdot 00006$$

To find where to write the 6, you can think of $6 \times \frac{1}{10^5}$ as $6 \div 10^5$ and you will remember that this can be worked out with a number chart. It means that 6 is moved five places to the right and becomes ·00006. Now where does this five come from? It is the power of 10 which comes from multiplying together the $\frac{1}{10^2}$ and $\frac{1}{10^3}$. But $\frac{1}{10^2}$ tells you that the first number has two places of decimals and the $\frac{1}{10^3}$ tells you that the second number has three places of decimals. So you can add the number of places in the two factors, that is, in the two numbers you multiplied.

This is a very important result and you can help your pupils see this pattern by making a table of numbers and their products. Your pupils can first work out such numbers as $\cdot 3 \times 7$ and $1 \cdot 42 \times 3$, which they already know how to compute. Thus, they can work out by using fractions such numbers as $\cdot 3 \times \cdot 4$, $5 \times \cdot 01$, $\cdot 13 \times \cdot 04$. They then tabulate these results as follows:

Equation	*Number of decimal places in first number*	*Number of decimal places in second number*	*Number of decimal places in product*
$\cdot 3 \times 7 = 2 \cdot 1$	1	0	1
$1 \cdot 42 \times 3 = 4 \cdot 26$	2	0	2
$\cdot 13 \times \cdot 04 = \cdot 0052$	2	2	4

When your pupils have looked at a table like this and thought about the way they find a product using fractions, they will understand the rule. Here it is.

To multiply two decimal fractions: First multiply them without the decimal points. Then make as many decimal places in your product as the sum of the number of decimal places in the two decimal fractions. Here is an example.

$$27{\cdot}312 \times 1{\cdot}25$$

27312	Number of decimal places in the two decimal
125	fractions is $3 + 2 = 5$. Therefore the number
136560	of decimal places in the product must be five.
546240	$27{\cdot}312 \times 1{\cdot}25 = 34{\cdot}14000$
2731200	$= 34{\cdot}14$
3414000	

You will notice that it is very important to keep any zeros in the product until the value of the product is decided. Once the decimal point has been put in, the zeros which are not needed can be omitted.

EXERCISE 24-4A

1. Find the product of each pair of numbers.

a. $2 \times {\cdot}3$ **b.** ${\cdot}7 \times 8$ **c.** ${\cdot}3 \times {\cdot}1$ **d.** ${\cdot}1 \times {\cdot}1$
e. $2 \times {\cdot}1$ **f.** $9 \times 1{\cdot}1$ **g.** $7{\cdot}9 \times 10$ **h.** $1{\cdot}2 \times 1{\cdot}2$
i. ${\cdot}1 \times {\cdot}05$ **j.** ${\cdot}6 \times {\cdot}25$ **k.** $1{\cdot}02 \times {\cdot}7$ **l.** $12{\cdot}5 \times {\cdot}8$

2. Find the product of each pair of numbers.

a. $1{\cdot}142 \times 7{\cdot}3$ **b.** $23{\cdot}121 \times {\cdot}005$
c. $23{\cdot}54 \times 21{\cdot}5$ **d.** ${\cdot}087 \times {\cdot}0014$

3. A pupil writes that ${\cdot}1 \times {\cdot}1 = {\cdot}1$. Explain how you would help him to see his error.

4. Show by working in fractions why

$${\cdot}13 \times {\cdot}04 = {\cdot}0052$$

5. Make a table like the one on page 45 and show on it six equations such as you could use with your pupils to help them discover the rule for multiplication with decimal fractions. How would you make sure that they understood *why* the rule works?

24-5 Division in decimal notation

We have already discussed division of decimal fractions by whole numbers less than 10. Division by whole numbers greater than 10 is similar. Here is an example, $256 \div 16 = \square$.

Regroup 256 as $160 + 80 + 16$

Then

$$\begin{aligned} 256 \div 16 &= (160 + 80 + 16) \div 16 \\ &= (160 \div 16) + (80 \div 16) \\ &\quad + (16 \div 16) \\ &= 10 + 5 + 1 \\ &= 16 \end{aligned}$$

$$\begin{array}{rl} & \underline{10 + 5 + 1} = 16 \\ 16)\overline{256} & \\ \underline{160} & (10 \times 16 = 160) \\ 96 & \\ \underline{80} & (5 \times 16 = 80) \\ 16 & \\ \underline{16} & (1 \times 16 = 16) \end{array}$$

Next think about the problem $25{\cdot}6 \div 16 = \square$

Since $25{\cdot}6 = 256 \div 10$, we expect the answer to our problem to be $16 \div 10 = 1{\cdot}6$. The short form is shown on the right.

$$\begin{array}{r} 1{\cdot}6 \\ 16)\overline{2\,5{\cdot}6} \\ \underline{1\,6} \\ 9{\cdot}6 \\ \underline{9{\cdot}6} \end{array}$$

Division of a decimal fraction by a whole number is quite straightforward, as we have seen. It is possible to turn all division by a decimal fraction into division by a whole number. How is this done? Suppose you want to find $24 \div {\cdot}3$. We would like to work with 3 instead of 0·3. Observe that ${\cdot}3 \times \underline{10} = 3$. If we multiply ·3 by 10 we will divide the answer to our problem by 10. So we must also multiply 24 by 10. In other words, the problem $24 \div {\cdot}3$ is the same as the problem $240 \div 3$. So $24 \div {\cdot}3 = 80$.

Another way to make the divisor a whole number is to think of $24 \div {\cdot}3$ as a fraction.

$$24 \div \cdot 3 = \frac{24}{\cdot 3}$$ (Compare $1 \div 2 = \frac{1}{2}$ and $1 \div 3 = \frac{1}{3}$)

$$\frac{24 \times 10}{\cdot 3 \times 10}$$

$$= \frac{240}{3} = 80$$ (To make an equal fraction)

So $24 \div \cdot 3 = 80$

Here is another example. Find $7 \cdot 2 \div \cdot 09$

$$7 \cdot 2 \div \cdot 09 = \frac{7 \cdot 2}{\cdot 09} = \frac{7 \cdot 2 \times 100}{\cdot 09 \times 100} = \frac{720}{9} = 80$$

So $7 \cdot 2 \div \cdot 09 = 80$

Your pupils will probably need some practice in deciding what they must do to a decimal fraction to make it a whole number. They will need problems such as these:

$$\cdot 31 \times \square = 31$$
$$\cdot 002 \times \square = 2$$
$$73 \cdot 46 \times \square = 7{,}346$$

And these:

What must you do to $\cdot 7$ to make it a whole number?

What must you do to $1 \cdot 32$ to make it a whole number?

This can be a game in which pupils make up questions such as these and ask each other for answers.

EXERCISE 24-5A

1. Make up problems like those above to ask your pupils.
2. Find these quotients:

a. $16 \div 4$	**b.** $21 \div \cdot 3$	**c.** $30 \div \cdot 05$
d. $5 \cdot 6 \div \cdot 7$	**e.** $64 \div \cdot 08$	**f.** $\cdot 032 \div \cdot 4$
g. $55 \div 1 \cdot 1$	**h.** $121 \div \cdot 11$	**i.** $1 \cdot 32 \div \cdot 04$

3. Find these quotients. Do not work more than two decimal places.

a. $33 \cdot 79 \div 23$	**b.** $\cdot 17 \div \cdot 8$
c. $31 \cdot 012 \div \cdot 56$	**d.** $3 \cdot 653 \div 3 \cdot 7$

24-6 Percentage

There is another way of writing a decimal fraction and that is as a percentage. 23%, read 23 per cent, means 23 things out

of every 100 things. If there are 100 boys in a school, then 23% of the boys of the school is 23 boys: 23 out of 100. You will remember when you learned to think of fractions in terms of sets you said that if a set had 5 members, then the fraction of this set represented by 3 of its members is $\frac{3}{5}$. One member is $\frac{1}{5}$ and three members are $\frac{3}{5}$. If the set has 100 members, then 1 member is $\frac{1}{100}$ of the total number of the set and 23 members are $\frac{23}{100}$ of the number of the set. So 23% and $\frac{23}{100}$ are both names for the same number. What decimal fraction can also be used to name the number? $\frac{23}{100}$ is $23 \times \frac{1}{100}$, which is written as ·23. So we have the relationships

$$23\% = \frac{23}{100} = \cdot 23$$

and

$$37\% = \frac{37}{100} = \cdot 37$$

and

$$9\% = \frac{9}{100} = \cdot 09$$

Percentages are frequently used in everyday affairs, in shops, in factories and in government. A 10% discount may be allowed off the prices of articles bought in a certain shop. Money may be invested and earn interest of 2%. A firm of building contractors may decide that it must make a profit of 50% in order to pay its workers. The final profit will be much less. These are some of the uses of percentages, and you should look for more examples and use them to make problems for your pupils.

The percentages mentioned above are ones which can be written as very simple fractions. What fractions name the same numbers as 10%, 2% and 50%? It is easy to find out.

$$10\% = \frac{10}{100} = \frac{1}{10}$$

$$2\% = \frac{2}{100} = \frac{1}{50}$$

$$50\% = \frac{50}{100} = \frac{1}{2}$$

$$3\tfrac{1}{2}\% = \frac{3\frac{1}{2}}{100} = \frac{7}{200} \quad (3\tfrac{1}{2} \text{ out of } 100 = 7 \text{ out of } 200)$$

Percentages must be written as common or decimal fractions before they can be used in calculations. The common fraction is generally more convenient for this purpose, but you should always consider whether using the decimal fraction might reduce the amount of work. Here is an example worked in both ways.

Amodu buys some books and his bill is 40 shillings. If he is allowed 10% discount off his bill, how much must he pay?

By common fractions	*By decimal fractions*
10% of 40 shillings	10% of 40 shillings
$= \frac{1}{10} \times 40$ shillings	$= \cdot 1 \times 40$ shillings
= 4 shillings	= 4 shillings

So Amodu must pay 40 shillings − 4 shillings = 36 shillings.

Your pupils should be able to work simple problems like this in their heads. For this, they should have a sound understanding of percentages and their equivalent fractions.

EXERCISE 24-6A

1. Fill in the gaps in this table.

Percentage	50	25	$12\frac{1}{2}$	75	20	10	5	$2\frac{1}{2}$	$33\frac{1}{3}$	60	35
Common fraction											
Decimal fraction											

2. Work out each of the following percentages in two ways. ("£" denotes "pounds" and $ "dollars".)

a. $5\frac{1}{2}$% of £75 **b.** $3\frac{1}{2}$% of £120

c. 7% of 720 shillings **d.** 4% of $312·00

In all the problems so far, you were given a percentage and asked to find the corresponding fraction. Often we need to know what percentage of a total is represented by a certain amount. For example, what percentage of the whole class of 40 is a group of 35? Now that you know that a percentage is a fraction, you can see that the first step will be to find what fraction the group is of the whole class. The group is 35 out of

40, so this is $\frac{35}{40}$ of the whole class. We want this as a fraction with a denominator of 100; that is, we want it in hundredths. How can we do this? First we can make it simpler. $\frac{35}{40} = \frac{7}{8}$. Now we want to know how many hundredths there are in $\frac{7}{8}$. So we find the quotient $7 \div 8$, and this is ·875. This is 87·5 hundredths. Can you explain why ·875 is 87·5 hundredths? If you have forgotten, you can make a number chart and write ·875 on it and you will see that you have

	8 tenths + 7 hundredths + 5 thousandths
=	87 hundredths + 5 thousandths
=	87 hundredths + ·5 hundredths
=	87·5 hundredths

Do you see what we have done?

87·5 is ·875 × 100 and ·875 is $\frac{7}{8}$.

So 87·5 is $\frac{7}{8} \times 100$. So instead of dividing by 8 and then reading this as hundredths, we can straightaway multiply our fraction by 100. We need not simplify the fraction first. Here is the working to find what percentage of 40 is represented by 35.

$$35 \text{ is } \frac{35}{40} \text{ of } 40$$

$$\frac{35}{\cancel{40}_{2}} \times \cancel{100}^{5} = \frac{175}{2} = 87{\cdot}5$$

So 35 is 87·5% of 40

Here is a problem.

In an examination there were 70 problems. A boy had 55 right. All problems had equal marks. What percentage did the boy have right? Work to one decimal place only.

$$\text{Fraction right} = \frac{55}{70}$$

$$\text{Percentage right} = \frac{55}{70} \times 100 = \frac{550}{7} = 78{\cdot}5\%$$

Another kind of problem is one where we know the percentage and what it represents but we do not know the total.

In a certain town, 78% of the electorate voted in an election.

There were 5,600 ballot papers. How many people were entitled to vote? (That is, how many people were in the electorate?) We can make an equation. The number of the electorate is to be put into the box.

$$78\% \text{ of } \square = 5{,}600$$

$$\frac{78}{100} \times \square = 5{,}600$$

This is an equation with a missing factor and you know what this means. It means division. It can be rewritten as

$$5{,}600 \div \frac{78}{100} = \square$$

This is division by a fraction and to divide by a fraction we multiply by its reciprocal. (Why?) The reciprocal of $\frac{78}{100}$ is $\frac{100}{78}$. $\left(\text{Why? Because } \frac{78}{100} \times \frac{100}{78} = 1.\right)$
So the equation becomes

$$5{,}600 \times \frac{100}{78} = \frac{560{,}000}{78}$$

$$= 7{,}179$$

We did not work this problem further than the ones. (Why not?) So the total electorate is 7,179.

We can fit most problems about percentages to one equation. Sometimes we have to find one part of the equation and sometimes another part. Let us think about an equation where we know everything.

$$25\% \text{ of } 44 = 11$$

This is written as

$$\frac{25}{100} \text{ of } 44 = 11$$

The three problems that can be asked about the situation are:

1. What is 25% of 44?

$$\frac{25}{100} \text{ of } 44 = \square$$

2. What percentage of 44 is 11?

$$\frac{\square}{100} \text{ of } 44 = 11 \quad \text{or} \quad \frac{\square}{100} \times 44 = 11$$

which can be written as division as

$$\frac{\square}{100} = 11 \div 44$$

This we saw was $\square = \frac{11}{44} \times 100$

3. 11 is 25% of what number?

$$\frac{25}{100} \text{ of } \square = 11$$

which can be written as division as

$$11 \div \frac{25}{100} = \square$$

As long as you understand the meaning of percentage, you can work out any problem by using this equation.

EXERCISE 24-6B

1. Write each of these percentages as a common fraction.

a. $3\frac{1}{2}\%$ **b.** $33\frac{1}{3}\%$ **c.** 65%
d. 5·5% **e.** 17% **f.** $16\frac{1}{4}\%$

2. Write each of these fractions as a percentage.

a. $\frac{2}{3}$ **b.** ·4 **c.** 4 **d.** $\frac{4}{5}$
e. ·32 **f.** ·01 **g.** $\frac{7}{100}$ **h.** $\frac{8}{15}$

3. Find each percentage.

a. 5% of 1 hour, in minutes
b. 25% of a year, in days
c. 16% of 200 shillings

4. Find what percentage the first of each pair is of the second of each pair. Work to two decimal places only.

a. 33, 300 **b.** 100, 200
c. 300, 100 **d.** 4, 5
e. 45 shillings, 500 shillings **f.** 55 marks out of 80 marks

5. Find the total amount in each case. Do not work to more than one place of decimals.

a. 4 is 25% of what number?
b. 36 is 20% of what number?
c. 1,500 is 65% of what number?
d. £565 is 32% of how many pounds?

55

Integers UNIT V

Chapter 25

INTRODUCTION TO INTEGERS

25-1 A reminder of subtraction

Think about each of the following examples.

Example 1

Kwame and his sister Araba have 6 bananas. Kwame eats 4 bananas. How many are left for Araba?

Example 2

One classroom has desks for 35 children. A second classroom has desks for 27 children. How many more desks are in the first classroom?

Example 3

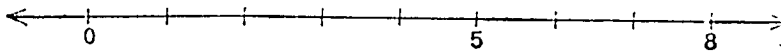
Kofe is 40 inches tall, and Kwame is 40 inches tall. Who is taller? By how many inches?

Example 4

Mary has saved 5 shillings and wants to buy some sandals that cost 9 shillings. How many more shillings must she save?

Example 5

Below is a number line on which the numbers 5 and 8 are shown. How many units to the right must one move to go from 5 to 8?



What do the examples have in common? First, the answer to each is given by a subtraction problem. The answers are:

Example 1: $6 - 4 = \boxed{2}$

Example 2: $35 - 27 = \boxed{8}$

Example 3: $40 - 40 = \boxed{0}$

Example 4: $5 + \boxed{4} = 9$ or $9 - 5 = \boxed{4}$

Example 5: $8 - 5 = \boxed{3}$

Notice that the answer in Example 3 is 0, and that in Example 4 subtraction is thought of as finding a missing addend. (Each example can be thought of in this way.)

There is a second thing which the examples have in common. In each case, the answer is a whole number. Is it always possible to subtract two whole numbers and get an answer which is again a whole number? Unfortunately, it is not. Think about $3 - 5 = \square$, $7 - 10 = \square$, and so on. If one addend is larger than the sum, you are not yet able to handle the problem.

You will remember that you had a similar situation with division. At first you had no answer to a problem such as $8 \div 5 = \square$. By using fractions, you later gave the answer $\frac{8}{5}$. In this unit, you will learn about another kind of number which will make it possible to write an answer to $3 - 5 = \square$.

EXERCISE 25-1A

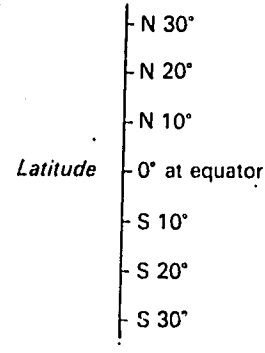
Work the following problems if it is possible to find a whole number answer. Which problems have no whole number answer?

1. $7 + \square = 10$
2. $3 + \square = 1$
3. $11 - 4 = \square$
4. $\square + 2 = 9$
5. $4 - 8 = \square$
6. $2 - 11 = \square$
7. $13 + \square = 13$

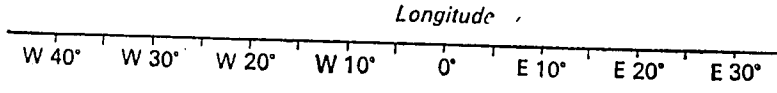
25-2 Physical models

You have learned in geography how the positions of places on the earth are described using latitude and longitude. Latitude is measured in degrees (°) north and south of the equator and longitude in degrees east and west of the meridian of longitude which passes through Greenwich. On a map of the world you will see, along the edges of the page, lines which are marked to show latitude and longitude.

Latitude is marked on a line segment down each side of the map. On the following page is part of one such line segment. You can tell which side is north and which side is south of the equator by the letter N or S marked before the number of degrees.



Longitude is marked on line segments along the top and bottom of the map. Here is part of one of them.

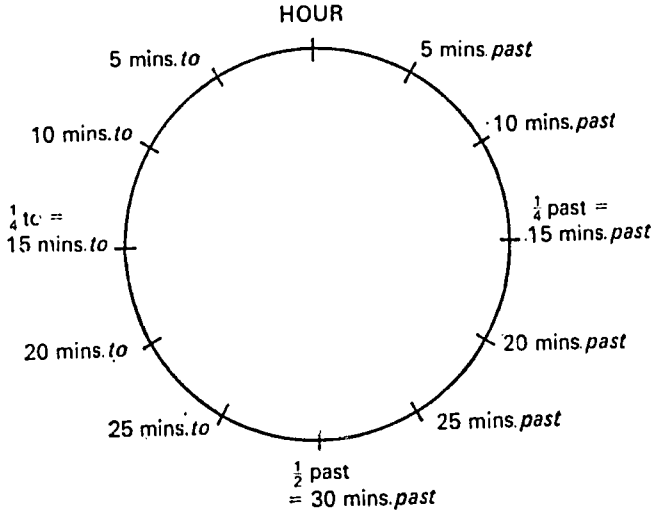


Again, you can tell which is east and which is west by the letters E or W written before the number of degrees.

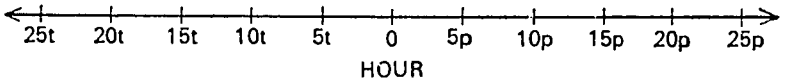
You will have noticed that in each case we measure in two *opposite* directions from a central point which is called zero. Zero degree latitude is on the equator and zero degree longitude is on the meridian through Greenwich.

There are many other things which are measured in two opposite directions. Here are some of them: the height of land above sea level and the depth of the ocean below sea level; the number of years A.D. (*Anno Domini*), and the number of years B.C. (*Before Christ*); time before the hour and past the hour. On the opposite page is a clock-face on which are marked the times before the hour and after the hour. These are the times which are shown by the minute hand.

You will see that these times are measured in opposite directions from the hour. There are the times *past* the hour and the times *to* the hour. Both sets of times are measured from the hour. We will draw a line picture for these times. What time shall we choose as zero time? This must be the hour, because the times are measured from the hour. At the hour there are 0 minutes past or 0 minutes before the hour. On one side of 0 we will mark the times past the hour. On the other side of 0,



we will mark the times before the hour, backward (to the left) from 0. To show which side is past the hour and which side is



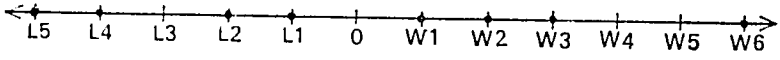
to the hour we have marked each number either p (for *past*) or t (for *to*). Can you see that our line is like the line around the edge of the clock-face? If you cut the line around the edge of the clock-face and straighten it out, where would you make the cut? You want it to look like the number line we have just drawn. You will have to cut the line at half-past.

When the time is 20 past the hour, where will the minute hand be on the number line? At 20p. What will the time be at 10t? 10 to the hour.

The seconds before and after firing off a rocket are also measured in this way. This is called "countdown" and is spoken, "Ten, nine, eight, seven, six, five, four, three, two, one, zero". At zero the rocket is fired. The count goes on after the rocket is fired, "One, two, three" and so on.

You can think of gains and losses in a similar way. If you win 6 shillings at a game of cards, this is not the same thing as losing 6 shillings. If you win 6 shillings, 2 shillings, 3 shillings and 1 shilling, your wins can be shown by dots on a line as

shown in the picture. You must first decide where to put the dot for a result when you neither win nor lose. This will be the zero point. Then you decide on which side of 0 you will mark the line for wins. We usually use the right-hand side for this. Then we can mark losses in the opposite direction.



Wins and Losses

The dots show the wins 1 shilling, 2 shillings, 3 shillings and 6 shillings. What losses are shown? 1 shilling, 2 shillings, 4 shillings and 5 shillings.

In each of these cases you will see that you can make pictures of the measurements by representing them as points on a line. This is how you do it.

1. Draw a line.
2. Mark a point with 0. Call this the zero point.
3. Decide what the zero point represents.
4. Mark the scale for the measurements in one direction.
5. Mark the scale for the measurements in the opposite direction.
6. Give the points on each side of 0 a letter or symbol to enable you to distinguish between them.

EXERCISE 25-2A

Each of the sets of measurements described below can be shown on a line. For each one (*a*) draw the line, (*b*) mark the zero point and say what it represents, and (*c*) mark in the measurements.

1. Longitude east and west of the Greenwich meridian. Mark two points on this line to show the longitude of the most easterly and most westerly parts of the African coast line. Find the longitude of six African towns and mark them also.
2. A hole has been dug in the ground for a mine shaft 100 feet deep. Above it has been built a tower 60 feet high. The tower has platforms at 20 feet and 50 feet. Mark these on the line. Mark also a platform in the mine which is as far below ground as the first platform is above ground.
3. A Centigrade thermometer measures the temperature from 40° below zero to 70° above zero. Mark on the line a temperature of 30° above zero.

4. A shopkeeper has debts of 20 shillings, 50 shillings, 65 shillings and 25 shillings. He has credits of 15 shillings, 25 shillings, 50 shillings, 70 shillings and 35 shillings. Show these amounts on a line.

5. Some boys are to run a race. As some of them are taller and older and some of them are shorter and younger, they are to start from different places behind or ahead of the starting point. Four young boys stand 3 feet, 5 feet, 2 feet and 7 feet in front of the line. Seven boys stand at the starting point and three big boys stand behind the starting point at distances of 3 feet, 6 feet and 8 feet. Show these boys as dots waiting to start their race. (You will need fourteen lines, side by side, for the fourteen boys to stand on.)

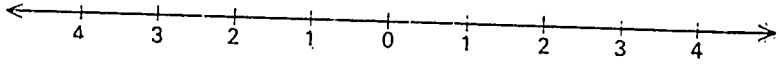
Chapter 26

THE NUMBER LINE AS A PICTURE FOR INTEGERS

26-1 Naming the new numbers

You will have realized while making the lines in the last exercise that it was very like making a number line. When you made a number line before, in Chapter 16, you first chose a point to be zero and marked it 0, and then measured out equal steps to the right. You then used the counting numbers to name the points at the ends of these steps. But when you made the line picture for longitude east and west, you used the counting numbers twice. You used them to the right and to the left of 0. When you made the line picture for the mine shaft and its tower, you drew the line going upwards instead of across the page. But you still used the counting numbers twice, once on each side of the zero point. So you will see that we want a new kind of number line as a picture of any of these things which are measured in two directions.

Look at the number line below.

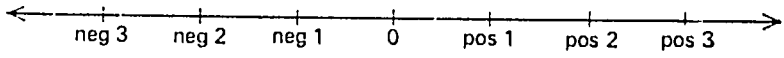


You can see that we must have some way to distinguish between the number 3 to the right of 0 and the number 3 to the left of 0. We want names for the two sets of numbers. We want to make a number line which we can use to show all these situations: right and left, north and south of the equator, east and west of the meridian through Greenwich, ahead of and behind the starting point, above and below sea level, gain and loss, credit and debit, surplus and shortage, after and before and many more. What we need is a way to tell apart numbers on opposite sides of zero. In many of these cases, descriptions such as ahead, above, gain and credit seem to suggest "having something", while their opposites below, behind, loss and debit may seem

to suggest "lacking something". So we distinguish the first as being positive and the second as being negative. On the new number line, we label *positive* numbers on one side of 0 and label *negative* numbers on the other side of 0. The number 0 at the starting point is neither positive nor negative. The positive numbers are conventionally shown to the right of 0 and negative numbers to the left of 0, but we could put the positive numbers above 0 and the negative numbers below 0. This whole set of numbers—positive, negative and zero—we call the SET OF INTEGERS.

The numbers to the left of zero are negative integers and so we will call the 3 on this side *neg 3* for short. The numbers to the right of zero are the positive integers and so we will call the 3 on this side *pos 3* for short.

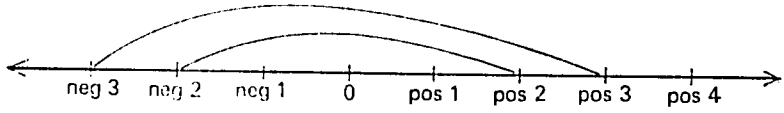
Here is a picture of the number line showing some of the negative integers, zero, and some of the positive integers.



26-2 Zero

The number zero has a special position between these two sets of numbers. You will remember that you first heard of zero, in Chapter 1, as the number of the empty set. You used 0 then to show an empty set. For example, in 202, the number which is equal to $2 \times 10^2 + 0 \times 10 + 2 \times 1$, the set of tens is empty, and in the number which is equal to $3 \times 10^2 + 7 \times 10 + 0 \times 1$, 370, the set of ones is empty. You also used 0 to mark the point on the number line from which you started to measure out unit lengths. So in this way you can once again think of zero as the number of the empty set, the empty set of unit lengths. This is the way we think of zero with the integers. When we think about latitude, 0° is at the equator. For minutes past and to the hour we chose 0 to be the hour and in the problem about the mine, 0 was at ground level. Zero has a very important part to play on this number line. It separates the positive integers from the negative integers and later on we shall see how it helps us in doing addition and subtraction.

26-3 Opposites



You will see on the number line that the integers can be matched in pairs. We can match any integer and its OPPOSITE. Here are some pairs. They are shown on the line also.

neg 3 and pos 3
pos 2 and neg 2
neg 2 and pos 2

Such pairs are called "opposites". You will see that the two members of a pair are the same distance from 0, but they are on opposite sides of it. To find the opposite of an integer, we look for the integer which is on the *opposite* side of 0 and at the *same* distance from 0. We can write this in another way.

The opposite of neg 3 is pos 3.
The opposite of pos 2 is neg 2.
The opposite of pos 15 is neg 15.
The opposite of neg 52 is pos 52.

The opposite of 0 is itself, because 0 is neither positive nor negative. What is the opposite of the opposite of an integer? We can work this out in three stages.

The opp. of the opp. of pos 6 = the opp. of (the opp. of pos 6)
= the opp. of neg 6
= pos 6.

So the opposite of the opposite of pos 6 is pos 6 itself. Is this true if we begin with a negative number? You can see that it must be so. Can you see this for yourself? Begin with the opposite of the opposite of neg 3 and work it out in the same way. You will find that the opposite of the opposite of neg 3 is neg 3 itself. You can see that whether you begin with a positive number or a negative number or 0, the opposite of the opposite will be the number itself, the number you began with.

You can prove this for yourself on the number line also. If you want to find the opposite of any number you begin with that number, you jump to 0 and then you make another jump of the same size and land on the opposite. To find the opposite

of this second number, you simply jump back again the way you came. So to find the opposite of the opposite you jump there and back again. So you need not jump at all; you are at the answer already.

EXERCISE 26-3A

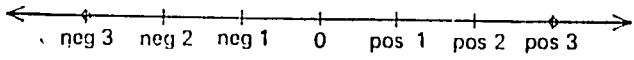
1. What are the opposites of the following measurements?
   a. Latitude N 30°
   b. Longitude W 45°
   c. Temperature 15° above zero
   d. 10 minutes past the hour
   e. A win of 7 shillings
   f. A debt of 50 shillings
2. What are the following?
   a. the opposite of neg 1
   b. the opposite of pos 11
   c. the opposite of pos 17
   d. the opposite of neg 73
   e. the opposite of pos 129
   f. the opp. of the opp. of pos 8
   g. The opp. of the opp. of neg 42
   h. the opp. of the opp. of neg 9
   i. the opp. of the opp. of the opp. of pos 23
   j. the opp. of the opp. of pos 14
   k. the opp. of the opp. of 0

26-4 Order properties

We now have to decide how to compare two integers. You know already that any whole number is less than any other whole number to the right of it on the number line. You will remember that this shows how to decide the order of two whole numbers: the greater is to the right on the number line. Using the same rule for positive integers, here are some examples of inequalities. Recall that the sign "$>$" means *greater than* and the sign "$<$" means *less than*.

pos 3 $<$ pos 6 pos 35 $>$ pos 29
pos 100 $<$ pos 200 pos 1 $>$ 0

You will see that 0 must be less than any positive integer, because all the positive integers are to the right of 0.



Now we want to decide the order of any two integers. Which is greater: neg 2 or neg 3? Let us use the same rule for negative integers that we use for positive integers. Neg 1 is to the left of 0, so by this rule neg 1 is less than 0: neg 1 $<$ 0. Similarly, neg 2 is to the left of neg 1, and so neg 2 $<$ neg 1; neg 3 is to the left of neg 2, and so neg 3 $<$ neg 2, and so on. Any negative integer is therefore less than 0.

So we use the same scheme throughout: on the number line any integer is less than any integer to the right of it. More formally, *If* a *and* b *are any two integers, then* a $<$ b *means that* a *is to the left of* b *on the number line.*

EXERCISE 26–4A

1. Put in the inequality sign, $<$ or $>$, to make each of the following into true statements:

a. pos 6	pos 10	**b.** neg 6	neg 10
c. pos 15	neg 15	**d.** neg 15	pos 15
e. neg 200	neg 1,000	**f.** 0	neg 3
g. 0	pos 8	**h.** neg 11	0

2. Put in an integer to make each of these inequalities true:

a. pos 1 $>$ $\square$	**b.** pos 1 $<$ $\square$
c. neg 5 $>$ $\square$	**d.** neg 5 $<$ $\square$
e. pos 100 $<$ $\square$	**f.** neg 100 $<$ $\square$
g. 0 $>$ $\square$	**h.** 0 $<$ $\square$

Can you attach a meaning to the order of numbers in the physical models of integers?

Can you say that one measurement is *greater* than another? Will this statement always have meaning? There must be such an order because the number line can be used for all such physical models, but are the words "greater than" and "less than" the best ones to use? How would you compare the latitude of a place A at W 27° with the latitude of a place B at E 19°? You would make one of two statements

A is to the west of B
B is to the east of A

You notice that there is no mention of greater than or less than. Here are some more comparisons.

The top of a mountain 1,000 feet high is higher than the top of a house 10 feet high. The top of a house 10 feet high is higher than the bottom of a lake by which it stands.

The Great Pyramid was built about 1800 B.C. and the ancient kingdom of Ghana flourished about A.D. 900. How would you compare these two events? You would say the Great Pyramid was built *before* ancient Ghana flourished, or that ancient Ghana flourished *after* the Great Pyramid was built.

EXERCISE 26-4B

1. Make two comparative statements about each of the following pairs.

a. Longitude N 29° and longitude S 50°.

b. The time 10 past 3 p.m. by your watch and the time 5 to 3 p.m. by your friend's watch.

c. A man on a platform in the mine shaft 53 feet underground and a man who is on the ground.

d. The temperature at noon today, which is 95° Fahrenheit, and the temperature yesterday, which was 102° Fahrenheit.

e. A boy Kofi who starts the race 5 feet ahead of the starting point and a boy Kwesi who starts the race 5 feet behind the starting point.

Now we must think about the order of the opposites of integers. You know that pos 8 > pos 3. What is the order of the opposites of pos 8 and pos 3? We can rewrite the opposites as neg 8 and neg 3 and see that neg 8 < neg 3. You can check this on the number line. So the order relation between two positive integers is reversed between their opposites. You can see this is true for other integers also by working through the next exercise.

EXERCISE 26-4C

Write down the order relationship between each of these pairs followed by the order relationship between their opposites. The first one is done for you.

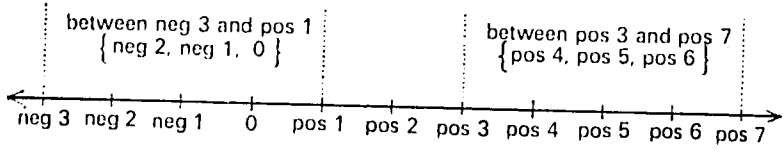
1. neg 3 < pos 2, pos 3 > neg 2 **2.** neg 5 neg 8

3. pos 2 pos 11 **4.** pos 2 neg 11

5. pos 7 0 **6.** 0 neg 2
7. neg 6 neg 1 **8.** pos 10 0

26-5 "Between"

What do we mean when we say that a whole number is *between* two other whole numbers? For example, suppose I say that I am thinking of a whole number between 3 and 7. You know that this whole number must be a member of the set of numbers which are greater than 3 and less than 7. It must be a member of the set {4, 5, 6}. You can see that this way of finding the set of numbers between two numbers will work for the integers also. An integer between pos 3 and pos 7 must be greater than pos 3 and less than pos 7. It must be a member of the set {pos 4, pos 5, pos 6}. Can you write the members of the set of integers between neg 3 and pos 1? Each of these integers must be greater than neg 3 and less than pos 1. The set is {neg 2, neg 1, 0}. You can check each of these examples by looking at the number line.



EXERCISE 26-5A

1. Write the set of integers between these pairs.

a. pos 20 and pos 25 **b.** pos 2 and neg 2
c. neg 3 and 0 **d.** neg 5 and neg 7
e. pos 8 and pos 9 **f.** neg 3 and neg 4
g. pos 2 and 0

2. Write a description of the following sets using the idea of "between":

a. {neg 2, neg 1} **b.** {pos 19, pos 20, pos 21, pos 22}
c. {pos 1, 0, neg 1} **d.** {0} **e.** { }

3. Find how many integers there are in the following sets:

a. {integers between neg 6 and pos 6}
b. {integers between pos 5 and neg 11}

4. Name a set of points which is between each of the pairs of points identified in Exercise 26-4B.

26-6 Summary

You have been introduced to a new set of numbers, the integers. This set is made up of the negative integers, zero and the positive integers. Integers can be represented on a number line and can be given an order of greatness. You will have noticed that the least integer on any number line section you have drawn is always the integer on the extreme left. After this, the integers become greater as you move along the number line from left to right. The greatest integer on any section of the number line you have drawn is always the integer on the extreme right. You are also able to find the set of integers between any two integers by using what you know about order on the line. You have also seen how to find the opposite of an integer and have discovered that to find the opposite of the opposite of a number is to leave the number unchanged.

You will now be wondering whether you can find ways to add, subtract, multiply and divide using these new numbers, and in the next chapters you will see that you can, in fact, do this.

Chapter 27

OPERATIONS ON INTEGERS

27-1 Addition

The positive integers pictured on a number line are so like the whole numbers pictured on a number line that you will wonder whether you can add integers in the same way as you added whole numbers on the number line. Before answering this question, we shall first have to think again about what we mean by these integers. In every case, you measured from 0 a number of units along the line.

To find latitude N 23° you measured 23° from 0° upward.
To find longitude W 19° you measured 19° to the left of 0°.
To find 100 feet below sea level you would measure 100 feet downward.
To find pos 3 you measured 3 steps of 1 unit each to the right of 0.
To find neg 3 you measured 3 steps of 1 unit each to the left of 0.

So you think of the sum of two integers as the result of moving twice along the number line. You can find pos 3 + pos 5 by taking 3 steps from 0 to the right followed by 5 steps from pos 3 to the right. This will bring you to pos 8. You can write

$$\text{pos } 3 + \text{pos } 5 = \text{pos } 8$$

Now you have found the sum of two positive integers. You can see that it was found in just the same way that you found 3 + 5, the sum of two whole numbers, on the number line.

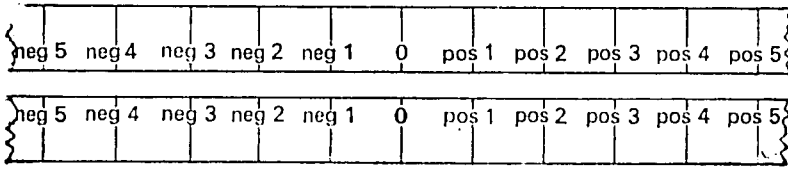
You can check the answer by thinking of gains and losses. A gain of 3 shillings followed by a gain of 5 shillings gives a gain of 8 shillings. As gains may be thought of as positive integers you will see that this is another way of thinking of pos 3 + pos 5 and that this is the same as pos 8.

You can also work out the result of addition and subtraction of integers by using a slide rule. This is how to make it.

You need two strips of ruled paper. Fold each strip into a long strip with the ruled lines across the width like this:

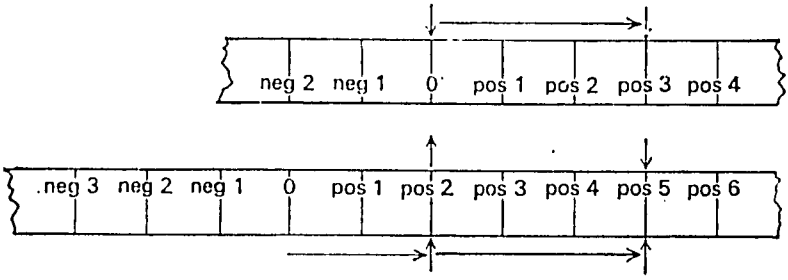
Along the *lower* edge of the first strip mark the integers as if the edge were a number line.

Now mark, in the same way, the *upper* edge of the second strip. The picture shows you how your two strips should look.



This is a slide rule, but yours should have many more numbers than there are in the picture. The more integers you write, the more use you will find for your slide rule.

Now you are ready to find pos 2 + pos 3 on your slide rule. Find pos 2 on the lower strip. Now slide the upper strip along to the right until the zero point of the upper strip is exactly above the point pos 2. Your slide rule should look like this:



From the *lower* zero point, you have moved to the *lower* pos 2 point. Now you want to add pos 3 to the pos 2. To do this, you move 3 more units to the right from pos 2 and you do this with the *upper* strip. The upper zero point is at pos 2 and to add pos 3 you move along the upper strip 3 units to the right, that is, to the pos 3 point. Now you have added pos 3 to pos 2 and the answer will be on the lower strip below the pos 3. What is the

answer? You see that it is pos 5. So you have found by using your slide rule that

$$\text{pos } 2 + \text{pos } 3 = \text{pos } 5$$

EXERCISE 27-1A

1. Without moving your slide rule from the position shown above, find the answers to these problems:

   a. pos 2 + pos 4 **b.** pos 2 + pos 5
   c. pos 2 + pos 2 **d.** pos 2 + pos 1
   e. pos 2 + 0 **f.** pos 2 + pos 6

2. Now use your slide rule to find the answers to these problems:

   a. pos 3 + pos 1 **b.** pos 1 + pos 3
   c. pos 4 + 0 **d.** pos 4 + pos 2
   e. 0 + pos 6 **f.** 0 + 0

3. Make up eight addition problems like these but use greater integers. Check that your slide rule gives you the correct answers.

You know a way to add two positive integers. Can you add two negative integers in the same way? First think of the question in terms of losses and then use your slide rule to see if it gives the same answer. If you have a loss of 3 shillings followed by a loss of 2 shillings, you have lost the same amount as if you had one loss of 5 shillings. Thinking of losses as negative integers, you can write this as neg 3 + neg 2 = neg 5.

What do we mean by addition of negative integers? If it is to have the same meaning as addition of positive integers, it must mean one movement followed by another movement. You can find the sum neg 3 + neg 2 by using your slide rule. Where will you put the 0 of the upper strip? Above the first number, that is, above neg 3. You have now moved 3 steps, from 0 to neg 3. Look along the upper strip from 0 until you find neg 2. You now have added neg 2. What number is in the answer place below neg 2? It will be neg 5. So with the slide rule, also, you have found that

$$\text{neg } 3 + \text{neg } 2 = \text{neg } 5$$

You found this sum by moving 3 steps to the left and then another 2 steps farther to the left.

You now know how to add two positive integers or two nega-

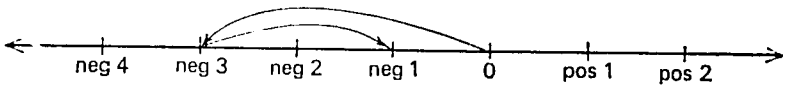
tive integers. Can you work out how to add one negative integer and one positive integer? You should be able to work this out for yourself in the next exercise.

EXERCISE 27-1B

For each of these problems make up a story about gains and losses and then check your answer by using the slide rule.

1. neg 2 + pos 5
2. neg 5 + pos 3
3. neg 3 + 0
4. pos 2 + neg 5
5. pos 5 + neg 2
6. 0 + neg 4
7. pos 3 + neg 3
8. neg 3 + pos 3
9. opposite of neg 3 + neg 3
10. pos 3 + opposite of pos 3

Did you need to use a slide rule all the time? Did you discover that to add a positive integer you move your finger on the number line to the right, and to add a negative integer you move your finger to the left? So that instead of moving your slide rule, you can use one strip only as a number line and count along it. Here is an example. To find neg 3 + pos 2, you will first move to neg 3 on the line and then move 2 steps to the right to add pos 2. This will bring you to neg 1. Here is a picture to show what you have done.



Work the next exercise using a number line only.

EXERCISE 27-1C

1. Make a picture to show how you find each of these sums:

a. pos 2 + pos 4
b. pos 3 + neg 1
c. pos 5 + neg 8
d. pos 1 + neg 1
e. pos 4 + 0
f. 0 + pos 2
g. neg 2 + neg 6
h. neg 3 + pos 2
i. neg 4 + pos 5
j. neg 4 + pos 4
k. neg 1 + 0
l. 0 + neg 1

2. Use a number line to find the following:

a. (pos 3 + neg 5) + pos 4
b. (pos 4 + neg 4) + neg 4

c. neg 5 + (neg 1 + neg 2)
d. (neg 3 + neg 3) + pos 4
e. neg 3 + (neg 3 + pos 4)

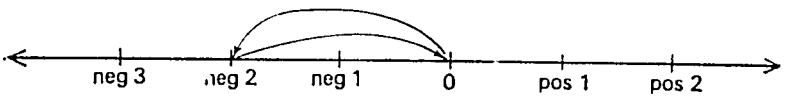
What do you notice about the answers to **d.** and **e.**? Of what property of whole numbers does this remind you?

3. An aeroplane is flying across Africa. The pilot finds that he is at a position whose longitude is E 18°. He then flies 25° to the west. Draw a number line and mark his positions on it. Then make an addition equation to show what he did and where he was finally.

   On another day the pilot starts from a place W 5° and flies east for 14° of longitude. Mark his journey on the number line. Write an addition equation to show what he did. Make up two more problems about the pilot and his aeroplane.

Addition of opposites

When you worked these problems, did you notice something about the result when you added a pair of opposites? What is neg 3 + pos 3? pos 2 + neg 2? pos 100 + neg 100? In each case the answer is 0. You can see that this must be so by looking at a picture of the addition of a pair of opposites. Think of neg 2 + pos 2.



Neg 2 is 2 steps to the left from 0. Pos 2 is 2 steps to the right. 2 steps to the left followed by 2 steps to the right brings you back to where you started. So neg 2 + pos 2 = 0.

You will find this property of opposites very useful later on. It is very important and so we will write it here.

The sum of an integer and its opposite is zero

OR

An integer added to its opposite is zero

EXERCISE 27-1D

For each question, draw the number line and show the addition on it. Write also the addition equation.

1. Arua in Uganda has a latitude of N 3°. Lubushi in Zambia is $10\frac{1}{4}$° of latitude due south of Arua. (Due south means that the two towns are on the same meridian of longitude—they both have longitude E 31°.) What is the latitude of Labushi?
   1° of latitude measures a distance of 70 miles on the surface of the earth. How far is it from Arua to Lubushi?
2. Make up some problems like this about your country and places north and south of it.
3. An aeroplane flying at a height of 3,000 feet above the sea drops a heavy weight which falls through 3,300 feet to the bottom of the sea. How deep is the sea at that point?
4. The minute hand of the clock points to 23 minutes to the hour (4 p.m.). What time will it be in 35 minutes?
5. Kwasi and Kofe were playing a game in which 10 seeds were worth 1 cent and 10 cents were worth 1 shilling. Their wins and losses are given below. How much had each of them won at the end of the game?
   Kwasi: Win 12, Win 15, Lose 13, Lose 6, Win 7.
   Kofe: Win 8, Lose 7, Win 23, Win 2, Lose 15.
6. Make up an addition problem, suitable for your pupils, about each of the situations described in Exercise 25-2A.

27-2 Subtraction

Now you must think about subtraction of integers. You will remember that you have learned to think about subtraction of numbers as finding the missing addend in an addition equation. For example, to find 11 − 5 you would find the missing addend in the equation

$$11 = 5 + \square$$

So once you can add two integers, you should also be able to subtract one from the other. If you have the problem pos 11 − pos 5 = □, you can rewrite the equation as

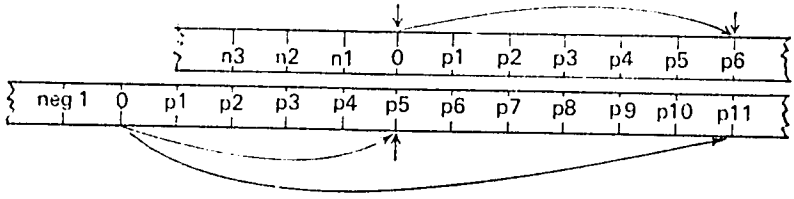
$$\text{pos } 11 = \text{pos } 5 + \square$$

You will probably know at once that the missing addend is pos 6, and so you can write

$$\text{pos } 11 - \text{pos } 5 = \boxed{\text{pos } 6}$$

For harder problems, the slide rule is very useful so let us see how to use it first on the easy problem we have just done. Can you do it yourself? Try first and then read what follows here.

Find the addend you know, pos 5, on the lower strip and move the upper strip so that the zero point is above pos 5. Your slide rule will look like the next picture.



Now you want to know what you must add (on the upper strip) to pos 5 to get pos 11 on the lower strip. So find pos 11 on the lower strip. What number is above it? pos 6. So pos 6 is added to pos 5 to give pos 11.

What is the answer to pos 3 − pos 3? You will know that this is 0 and can check that your slide rule also gives this answer. Can you write the equation showing this problem as finding the missing addend? It will be

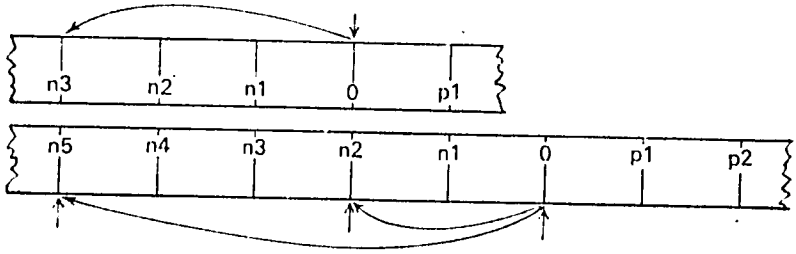
$$\text{pos } 3 + \square = \text{pos } 3$$

You know that 0 is the only number which will make the equation true.

Now we will do subtraction with two negative integers. Think of neg 5 − neg 2. You probably know that neg 5 can be found by taking a step of neg 2 followed by a step of neg 3.

Can you write this as a missing addend problem? What must be added to neg 2 to give neg 5?

Can you work this out with your slide rule? Here is the picture to help you.



The zero point of the upper strip is above neg 2 on the lower strip. What number on the upper strip will give you neg 5 on the lower strip? You see that it is neg 3.

$$\text{neg } 5 = \text{neg } 2 + \boxed{\text{neg } 3}$$

or

$$\text{neg } 5 - \text{neg } 2 = \boxed{\text{neg } 3}$$

You can see that these answers are reasonable by thinking of some real problems. Suppose you have a credit of 5 shillings at a shop and you want to buy a pair of sandals which cost 11 shillings. How much more money do you need? This is a missing addend problem. We can use positive integers for credit and write

$$\text{pos } 11 = \text{pos } 5 + \square$$

You know that you need another 6 shillings and so pos 6 will be put into the box:

$$\text{pos } 11 = \text{pos } 5 + \boxed{\text{pos } 6}$$

The corresponding subtraction equation is

$$\text{pos } 11 - \text{pos } 5 = \boxed{\text{pos } 6}$$

Now suppose instead of a credit of 5 shillings you have a debt of 16 shillings at the shop. The shopkeeper will not allow you to have so large a debt any longer. He says you must reduce it to 6 shillings only. What must you give him?

You can write negative integers for debts and so the missing addend equation will be

$$\text{neg } 16 - \square = \text{neg } 6$$

To reduce your debt from 16 shillings to 6 shillings, you must give the shopkeeper 10 shillings. That is, you add a credit of 10 shillings to your account at the shop. Therefore, we have

$$\text{neg } 16 + \boxed{\text{pos } 10} = \text{neg } 6$$

The corresponding subtraction equation is

$$\text{neg } 6 - \text{neg } 16 = \boxed{\text{pos } 10}$$

When you studied how to help your pupils understand subtraction, you found that there were several different types of problems which subtraction could solve.

One problem was to separate a set into two subsets, to remove one subset, and then to find how many members there were in the remaining subset. You "took away" one subset from the whole set. This way of looking at subtraction is useful to help young children understand what they are doing, but it is not very useful for problems about integers. For instance, it is awkward even to think about a positive subset of a negative set. So we will not use this idea of subtraction. We will think instead of subtraction as *comparing* two sets or two measurements. Think of the two problems about credits and debits. You found the answer by thinking of these as missing addend problems, but they are also comparison problems. In a comparison problem, you find the *difference* between two numbers.

In the first example, you found the difference between two successive credits at the shop:

$$\text{pos } 11 - \text{pos } 5 = \boxed{\text{pos } 6}$$

In the second example, you found the difference between two successive debits at the shop:

$$\text{neg } 6 - \text{neg } 16 = \boxed{\text{pos } 10}$$

In each case you wrote the missing addend equation and then used your slide rule to find the required difference. This is *one* way to find the difference between two integers. Later you will find a quicker way.

EXERCISE 27-2A

1. Rewrite each of the following subtraction equations as a missing addend equation and find the missing integer.

   a. pos 7 − pos 5 = □ **b.** pos 3 − pos 1 = □
   c. pos 3 − pos 3 = □ **d.** pos 2 − 0 = □
   e. neg 7 − neg 7 = □ **f.** neg 4 − neg 2 = □
   g. neg 3 − neg 3 = □ **h.** neg 2 − 0 = □
   i. neg 3 − neg 1 = □

2. A mine shaft 100 feet deep has above it on the ground a tower 60 feet high. A ladder goes from the bottom of the shaft to the top of the tower. Write equations in positive or negative integers which you can use to find the answers to the following problems:

a. A man climbs from the ground to a platform 20 feet high and then climbs another 15 feet. How high is he now?

b. Above his head there is another platform 50 feet from the ground. How much higher is this platform than the first platform?

c. Another man is 25 feet down the mine. How much farther has he to go to reach the bottom?

3. Make up some problems about gains and losses to fit these equations.

a. pos 2 + pos 4 = pos 6 **b.** pos 4 − pos 3 = pos 1

c. neg 4 − neg 1 = neg 3 **d.** neg 2 − neg 2 = 0

Now we come to harder problems. How can we subtract a positive integer from a negative integer, and vice versa? Think of the equation

$$\text{pos } 2 - \text{neg } 1 = \square$$

If you think of this as comparing pos 2 and neg 1 then you can see that you are asking the question "How much greater is pos 2 than neg 1?" This is a missing addend problem again and you can write it

$$\text{pos } 2 = \text{neg } 1 + \square$$

How will you find the answer? You may see at once, by thinking of the position of pos 2 and the position of neg 1 on the number line, that the missing addend is pos 3. (Your pupils will need to work it out with their slide rules and should not be urged to use only the number line just yet. Let them see *why* pos 3 is the only number which will make the equation true.) Here are the equations.

$$\text{pos } 2 = \text{neg } 1 + \boxed{\text{pos } 3}$$

and so

$$\text{pos } 2 - \text{neg } 1 = \boxed{\text{pos } 3}$$

You have now subtracted a negative integer from a positive integer. You could equally well use the same method to subtract a positive integer from a negative integer. Suppose you have

$$\text{neg } 5 - \text{pos } 3 = \square$$

This can be written as

$$\text{neg } 5 = \text{pos } 3 + \square$$

By using your slide rule, you will discover that neg 8 is needed to make this equation true.

$$\text{neg } 5 = \text{pos } 3 + \boxed{\text{neg } 8}$$

so

$$\text{neg } 5 - \text{pos } 3 = \boxed{\text{neg } 8}$$

You can check these answers by thinking about debits and credits. If you have a credit of £2 at the shop and your friend has a debt of £1 at the same shop, then you can compare your credit of £2 with your friend's debt of £1 and say that in that shop you are £3 richer than your friend. You can use positive integers for credits and negative integers for debts and write

$$\text{pos } 2 - \text{neg } 1 = \text{pos } 3$$

Perhaps on another occasion you have a debt of £5 at the shop and your friend has a credit of £3 in the same shop. You can compare your debt of £5 with your friend's credit of £3 and write

$$\text{neg } 5 - \text{pos } 3 = \text{neg } 8$$

The result tells you that in that shop you are £8 poorer than your friend.

You will be realizing now that it seems as though subtraction is always possible with positive and negative integers. From your knowledge of the order of integers, you will have noticed that you can subtract a greater integer from a smaller integer and *have an answer*. You will remember

$$\text{neg } 5 - \text{pos } 3 = \text{neg } 8$$

Neg 5 is less than pos 3 because it is to the left on the number line. Again, you cannot find a whole-number answer to $3 - 5$, but what about pos 3 − pos 5? Try it.

$$\text{pos } 3 - \text{pos } 5 = \square$$

$$\text{pos } 3 = \text{pos } 5 + \square$$

What must be added to pos 5 to get pos 3? It must be neg 2.

$$\text{pos } 3 = \text{pos } 5 + \boxed{\text{neg } 2}$$

so

$$\text{pos } 3 - \text{pos } 5 = \text{neg } 2$$

It looks as if these new numbers will be very useful to us. We

can do any subtraction problem with them. We can find an answer when we subtract a larger number from a smaller number and also when we subtract a positive number from a negative number. You know something important now about the operation of subtraction in the set of integers. No matter what two integers a and b you select, it is always true that there is an integer which is the difference $a - b$. We say that the set of integers is CLOSED under subtraction.

What set of numbers is closed under addition? The set of whole numbers is, because there is always a whole number which is the sum $a + b$, whichever whole numbers a and b we select. But the set of integers is also closed under addition because we can always find $a + b$, no matter what integers a and b we select.

(Which set of numbers is closed under division? Is there always a quotient in the set of fractions? If we choose any two fractions $\frac{a}{b}$ and $\frac{c}{d}$, is there always a fraction $\frac{a}{b} \div \frac{c}{d}$? We must be careful here, because we cannot divide by zero.

$$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

This quotient $\frac{ad}{bc}$ can always be found *provided* that neither b nor c is 0. With this condition, the set of fractions is closed under division.)

EXERCISE 27·2B

1. Use your slide rule to find these differences:

a. pos 5 − pos 2	**b.** pos 5 − pos 6
c. pos 2 − pos 2	**d.** pos 5 − neg 1
e. 0 − neg 3	**f.** 0 − pos 2
g. neg 5 − pos 3	**h.** neg 1 − neg 7
i. neg 3 − neg 3	**j.** neg 3 − pos 3

2. Explain how you could use a slide rule to find pos 1 − neg 4.

3. Make up a story problem about wins and losses in a game in which the following equation occurs:

pos 3 − pos 7 = neg 4

Let us think again about comparing two integers. Think about debits and credits. If you have a debt of £3 in a shop and your friend has a credit of £2, then you can compare your financial standing in that shop in two ways. You will be £5 poorer in that shop than your friend, and your friend will be £5 richer than you. Can you write the two equations?

$$\text{neg } 3 - \text{pos } 2 = \text{neg } 5$$

and

$$\text{pos } 2 - \text{neg } 3 = \text{pos } 5$$

These two equations are very closely related. The numerical parts of the differences are the same, but one difference is neg 5 and the other difference is pos 5. Here pos 5 represents "richer by £5" and neg 5 represents "poorer by £5".

These results fit in with the way we usually compare two measurements A and B. We either say that one, A, is greater than the other, B, or that B is less than A. Instead of greater than, we may have several different comparisons, such as higher than, later than, richer than, to the north of and in front of. For less than, the corresponding comparatives are lower than, earlier than, poorer than, to the south of and behind.

In any particular situation, there is no ambiguity. You always know which town of two is to the east by looking at their latitudes. You know which of two wins is greater by comparing the two amounts on a number line. There is a convention—an agreement—about the operation of finding the *difference* between two numbers. The number which is mentioned second is the number which must be subtracted. Find the difference between neg 1 and pos 2 means find neg 1 − pos 2. Find the difference between pos 2 and neg 1 means find pos 2 − neg 1.

We can write this convention as a general rule using a and b to stand for any integers we select.

The difference between a and b is $a - b$

In comparing the two integers we will, of course, say that

a is greater than b if the difference is positive

or a is less than b if the difference is negative

You will notice that the first integer in the difference $a - b$ is written first in the sentence making the comparison.

Here is an example. Find the difference between pos 3 and pos 11. The difference is pos 3 − pos 11.

$$\text{pos } 3 = \text{pos } 11 + \square$$

What must be added to pos 11 to make pos 3? It must be a negative integer, so it will be neg 8.

$$\text{pos } 3 = \text{pos } 11 + \boxed{\text{neg } 8}$$

So we say that pos 3 − pos 11 = neg 8, or the difference between pos 3 and pos 11 is neg 8. Now you know that pos 3 is less than pos 11. If we had asked for the difference between pos 11 and pos 3, we would have

$$\text{pos } 11 - \text{pos } 3 = \text{pos } 8$$

You know that pos 11 is greater than pos 3.

So if the first integer is greater than the second integer, the difference is a positive integer, and if the first integer is less than the second integer, the difference is a negative integer.

This property of the difference of two integers gives you a new way to work out the answer to a subtraction problem. Here it is.

Find the distance on the number line between the two points which correspond to the integers. Then if the first integer is greater than the second, make the answer a *positive* integer. If the first integer is less than the second, make the answer a *negative* integer.

For example, suppose you want to find the difference between neg 4 and neg 7, that is, neg 4 − neg 7. The distance between the points on the number line which correspond to these integers is 3 steps. So the answer will be either pos 3, or neg 3. Look at the numbers. Which is the greater? Which number is to the *right* on the number line? Neg 4 is to the right and so neg 4 is greater than neg 7. So now you can write

$$\text{neg } 4 - \text{neg } 7 = \text{pos } 3$$

Check this by writing it as a missing addend problem:

$$\text{neg } 7 + \boxed{\text{pos } 3} = \text{neg } 4$$

Similarly, the difference between neg 7 and neg 4, that is, neg 7 − neg 4, would be neg 3:

$$\text{neg } 7 - \text{neg } 4 = \boxed{\text{neg } 3}$$

It is useful at first for your pupils to work subtraction problems in this way also, because it helps them to understand subtraction of integers and the meaning of a negative integer result.

EXERCISE 27-2C

1. Use the missing addend method to find the answers to the following problems:

a. pos 3 − pos 8	**b.** neg 3 − neg 8
c. pos 1 − neg 1	**d.** pos 4 − neg 2
e. neg 3 − pos 5	**f.** 0 − neg 4
g. pos 2 − pos 1	**h.** neg 4 − neg 1

Write a sentence to describe and explain each result.

2. Find the difference between the numbers in each of the following pairs by using a number line, *first* finding the distance between the points corresponding to the numbers, and *then* writing pos or neg to show whether the first number is greater than or less than the second number. Write your result first in an equation and secondly in a sentence.

a. pos 2, pos 1	**b.** pos 3, neg 5
c. 0, neg 4	**d.** neg 1, neg 5
e. pos 3, 0	**f.** pos 1, pos 7
g. neg 3, pos 3	**h.** neg 11, neg 2

3. Look at the questions in Exercise 26-4B. You were asked to compare pairs of measurements by saying which position was higher than, or to the east of, the other and so on. Now find the difference between each of the numbers in these pairs. Show the subtraction equation and the corresponding addition equation. Write each answer in a sentence comparing the two measurements.

27-3 Relation between addition and subtraction

Now that you know how to do addition and subtraction with these new numbers, the integers, you see how they are more useful than the whole numbers alone. With the whole numbers, you can find an answer to 5 + 3, and 5 − 3, but not to 3 − 5. With the integers, you can find the sum of any two integers, and you can also find the difference between any

two integers. For example, you can find pos 5 − pos 3 and, also pos 3 − pos 5. With whole numbers you can always do addition, and *sometimes* subtraction. But with the integers you can always do addition and *always* do subtraction.

Do you remember the mathematical way of writing these properties of the set of whole numbers and the set of integers? *The set of whole numbers is closed under addition,* because whatever whole numbers a and b we select, a whole number can always be found which is the sum $a + b$. *The set of integers is closed under addition and subtraction.*

EXERCISE 27-3A

1. Write a sentence similar to the above about:
   a. the set of integers and addition
   b. the set of integers and subtraction
2. What set of numbers is closed under division? How would you explain this to your pupils?

There is another very useful property of addition and subtraction of integers. Exercise 27-3C will help you to discover this property, but first work the short Exercise 27-3B to remind yourself about some very useful pairs of integers.

EXERCISE 27-3B

1. Put an integer into each of these boxes to make the equation true.

a. pos 3 + □ = 0	**b.** neg 3 + □ = 0
c. □ + pos 1 = 0	**d.** □ + neg 5 = 0
e. □ + △ = 0	**f.** □ + 0 = 0

Now answer these questions.

g. What are these pairs of integers called?
h. Draw a number line to show the pair in Question 1b.
i. What is the special property of these pairs of integers?
j. Write three more pairs.

Now that you are sure that you remember opposites and their properties, do the next exercise and see what you can discover about their use in addition and subtraction.

EXERCISE 27-3C

1. Put an integer into each box to make the equations true

a. pos 3 − pos 1 = □
pos 3 + neg 1 = □
pos 3 − pos 1 = pos 3 + □

b. neg 3 + neg 1 = □
neg 3 − pos 1 = □
neg 3 + neg 1 = neg 3 − □

c. pos 2 − pos 6 = □
pos 2 + neg 6 = □
pos 2 − pos 6 = pos 2 + □

d. 0 − pos 7 = □
0 + neg 7 = □
0 − pos 7 = 0 + □

e. 0 − neg 7 = □
0 + □ = pos 7
0 − neg 7 = 0 + □

Do you see here a connection between addition and subtraction? Describe it in a sentence.

What did you discover in the last exercise? Did you find these two properties of addition and subtraction of integers?

Discovery 1

To subtract an integer from another integer you can add its opposite. This is shown in the equations below.

pos 3 − pos 5 = neg 2
pos 3 + (opp. of pos 5) = pos 3 + neg 5
= neg 2

Therefore,

pos 3 − pos 5 = pos 3 + (opp. of pos 5)

Discovery 2

To add an integer to another integer you can subtract its opposite.

neg 3 + pos 4 = pos 1
neg 3 − (opp. of pos 4) = neg 3 − neg 4
= pos 1

Therefore,

neg 3 + pos 4 = neg 3 − (opp. of pos 4)

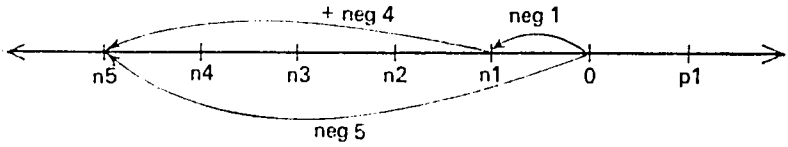
These two properties, linking addition and subtraction of integers, are really only one property: To add an integer to another integer you subtract its opposite, and to subtract an integer from another integer you add its opposite. You will see in the next chapter how this property is used to make addition and subtraction of negative integers much easier.

Now let us think about subtraction on the number line. We have not yet found what movement on the number line corresponds to subtraction. We already have movements for addition on the number line:

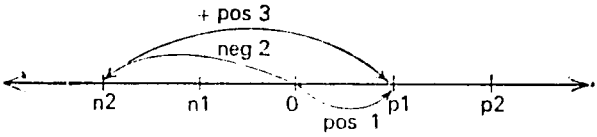
To add a positive integer to another integer, move to the right.

To add a negative integer to another integer, move to the left.

Can you draw the pictures for the two additions neg 1 + neg 4 and neg 2 + pos 3?
Here they are.



neg 1 + neg 4 = neg 5



neg 2 + pos 3 = pos 1

Each of these equations can be rewritten showing subtraction of the opposite. Take the first equation.

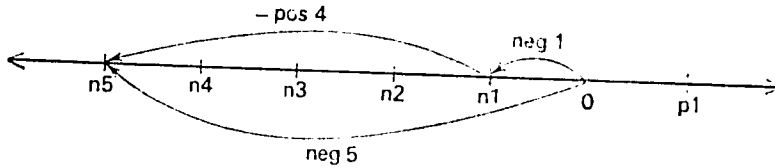
neg 1 + neg 4 = neg 5

becomes neg 1 (opposite of neg 4) = neg 5

or neg 1 − pos 4 = neg 5

So the movement to the left which represents +neg 4 also represents −pos 4. So we have found a meaning for subtraction of a positive integer in terms of a movement on the number line.

To subtract a positive integer move to the left. Here is a picture.



neg 1 – pos 4 = neg 5

You will see that this is the same as the first picture on page 87 except that –pos 4 has replaced +neg 4.

Now consider the second equation,

neg 2 + pos 3 = pos 1

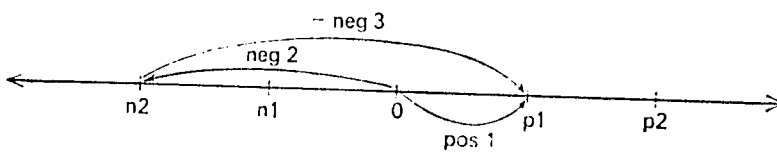
which becomes

neg 2 – (opp. of pos 3) = pos 1

or neg 2 – neg 3 = pos 1

So the movement to the right which represents +pos 3 also represents –neg 3. Now we have a movement for subtraction of a negative integer.

To subtract a negative integer move to the right. Here is a picture.



neg 2 – neg 3 = pos 1

You will see that this is the same as the second picture on page 87 except that –neg 3 has replaced +pos 3.

Put these two discoveries together and you have the movements for subtraction on the number line.

To subtract a positive integer from another integer, move to the left.

To subtract a negative integer from another integer, move to the right.

To summarize what we now know about addition and subtraction as movements we will write:

A move to the right is caused by the

1. addition of a positive integer,
2. subtraction of a negative integer.

A move to the left is caused by the

1. addition of a negative integer,
2. subtraction of a positive integer.

You now have three ways of finding the difference between two integers a and b:

$$a - b = \square$$

Method 1

Rewrite the subtraction equation as an addition equation and find the missing addend c by using a slide rule or the number line:

$$a = b + \boxed{c}$$

Method 2

Find the distance d in units on the number line between the two points. Decide where $a > b$ or $b > a$.

If $a > b$, the difference will be positive:

$$a - b = d > 0$$

If $a < b$, the difference will be negative:

$$a - b = d < 0$$

Method 3

Add the opposite of the second integer in the equation $a - b = \square$:

$$a + (\text{opp. of } b) = \square$$

EXERCISE 27-3D

1. Work the following problems in subtraction of integers by adding their opposites in each case.

a. pos 3 − pos 8
b. pos 12 − pos 7
c. neg 5 − neg 9
d. neg 4 − neg 1
e. neg 3 − pos 7
f. neg 5 − pos 2
g. pos 3 − pos 3
h. neg 5 − pos 5
i. pos 7 − neg 10

Chapter 28

RELATION OF INTEGERS TO WHOLE NUMBERS

In the last three chapters you have learned much about the integers. You know how they are used to show measurement in opposite directions; you know how to find which of two integers is the greater and how to do addition and subtraction with integers. Now you will want to know how they fit in with what you have learned before that. How are the integers related to the whole numbers?

28-1 The positive integers

Think first about the set of positive integers and the set of counting numbers. You will remember that the counting numbers do not include zero. Each set has a least member, pos 1 or 1, and each of these can be matched with the other. All numbers greater than these can be matched in pairs in order of size beginning with pos 2 and 2. Here is a picture of this matching.

Counting numbers	1	2	3	4	5	6 . . .	
	↕	↕	↕	↕	↕	↕	
Positive integers	pos 1	pos 2	pos 3	pos 4	pos 5	pos 6 . . .	

The difference between any two successive counting numbers and the difference between any two successive positive integers is the same, one unit. Each number in either set is one unit greater than the number to the left of it and one unit less than the number to the right of it. If we now consider the set consisting of the counting numbers and zero—that is, the set of whole numbers—and compare it with the set consisting of the positive integers and zero, we can take the comparison further. The number one unit less than pos 1 is 0, and the number one unit

less than 1 is 0. So you can see that there is a very close resemblance between the set of whole numbers and the set made up of zero and the positive integers. It is such a close resemblance that you can show them both on the same number line.

Whole numbers	0	1	2	3	4	5
Zero and pos. integers	0	pos 1	pos 2	pos 3	pos 4	pos 5

Now we must ask, "Do the numbers of both sets behave in the same way when you add or subtract with them?" Do they give corresponding results? Is the answer to 2 + 4 the whole number which corresponds to the positive integer which is the answer to pos 2 + pos 4? The answers are 6 and pos 6 and these *are* corresponding numbers. You can see that this relationship will always be true for addition, but will it be true for subtraction? Work through the next exercise and make a note of any cases where there is not a correspondence between the answer in whole numbers and the answer in positive integers with zero.

EXERCISE 28-1A

1. Find the answers to these problems.

a. 23 + 65 = ☐	pos 23 + pos 65 = ☐
b. 203 − 129 = ☐	pos 203 − pos 129 = ☐
c. 24 − 79 = ☐	pos 24 − pos 79 = ☐
d. 17 − 25 = ☐	pos 17 − pos 25 = ☐
e. 94 − 37 = ☐	pos 94 − pos 37 = ☐

Have you found that the whole numbers and the positive integers with zero give corresponding results for subtraction as long as the first number in the subtraction equation is *not less* than the second number? 25 − 17 has a result 8, and pos 25 − pos 17 has a result pos 8. If you have a subtraction equation where the first number is *less* than the second number, then there *is* a result when you work with integers but this result is a *negative* integer. For example, pos 17 − pos 25 = neg 8. There is no corresponding answer to the whole-number equation. That is, 17 − 25 has no whole-number answer. So if you

work with the *positive integers* and *zero* only, you can see that they behave in exactly the same way as the *whole numbers*. Because of this resemblance, we will not continue to give the positive integers their special label "pos". We say that pos 3 behaves exactly like 3, and so we will write 3 for pos 3.

Instead of pos 1, we write 1.
Instead of pos 2, we write 2.
Instead of pos 3, we write 3.
Instead of pos 100, we write 100.

Here are some equations. They are written first in the old notation and second in the new notation for positive integers.

pos 3 + pos 2 = pos 5 becomes 3 + 2 = 5
pos 5 + neg 3 = pos 2 becomes 5 + neg 3 = 2
pos 5 − pos 2 = pos 3 becomes 5 − 2 = 3
pos 3 + 0 = pos 3 becomes 3 + 0 = 3
0 + pos 1 = pos 1 becomes 0 + 1 = 1
pos 3 − pos 5 = neg 2 becomes 3 − 5 = neg 2

EXERCISE 28-1B

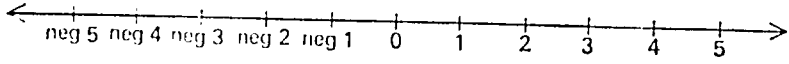
1. Find the answers to the following problems. Then rewrite them, putting whole numbers in place of the corresponding positive integers.

   a. pos 15 + pos 14 **b.** 0 − pos 3
   c. pos 1 − pos 3 **d.** pos 23 − pos 7
   e. pos 17 + neg 9 **f.** pos 3 − neg 2

2. Make up four problems like the ones you have just done.

28-2 The negative integers

Draw a number line for the integers and write the counting numbers where you formerly wrote the positive integers.



You have a number line along which you have measured in two opposite directions. You can still know which 3 is on the right of 0 and which 3 is on the left of 0, because the one on the right is written 3 and the one on the left is written neg 3.

You will ask whether there is a simpler way of writing neg 3. What do you know about neg 3? You know that neg 3 names the point you reach by taking three steps to the left from 0. You can also think of this movement as adding neg 3 to 0. The equation is

$$0 + \text{neg } 3 = \text{neg } 3$$

You have also learned that you can replace + neg 3 by −pos 3, and in the new notation −pos 3 will be −3. Instead of adding neg 3, you can subtract the opposite of neg 3; that is, you can subtract 3. So we can now have two equations

$$0 + \text{neg } 3 = \text{neg } 3$$

and

$$0 - 3 = \text{neg } 3$$

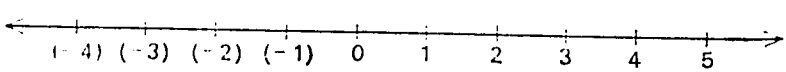
The difference (0 − 3) and the sum (0 + neg 3) are the same number, neg 3:

$$0 + \text{neg } 3 = 0 - \text{neg } 3$$

Because of this equality, we agree to write "− 3" for "neg 3". The minus sign now has two meanings instead of only one. From now on, every minus sign you meet can have two possible meanings and you should know which one is meant. In 0 − 3 the minus sign means the operation of subtraction. In −3 (for neg 3) it means negative. It must mean negative here because no other integer is in front of it from which you can subtract it. It is useful to write − 3, when it is a negative integer, with brackets enclosing it as (− 3) to remind pupils that it does not mean subtract 3 but the integer neg 3.

Instead of neg 1 we now write (− 1)
neg 2 becomes (− 2)
neg 3 becomes (− 3)
neg 251 becomes (− 251)

Each integer except 0 has a new name in the new notation, but the set of integers is the same as before: It consists of the positive integers, zero and the negative integers. The old counting numbers are included in the set of integers as the positive integers. The old whole numbers are now the set of positive integers with zero. Here is the number line labelled with the new notation.



Now you can rewrite equations in the new notation. Here are some examples.

pos 3 + neg 1 = pos 2	becomes	$3 + (-1) = 2$
neg 5 + neg 8 = neg 13	becomes	$(-5) + (-8) = (-13)$
neg 5 − neg 7 = pos 2	becomes	$(-5) - (-7) = 2$
pos 3 − pos 12 = neg 9	becomes	$3 - 12 = (-9)$
pos 5 − pos 2 = pos 3	becomes	$5 - 2 = 3$
0 + neg 5 = neg 5	becomes	$0 + (-5) = (-5)$
0 − pos 2 = neg 2	becomes	$0 - 2 = (-2)$

Each time the minus sign appears in the second column of equations above, it means either subtract or negative. Can you say which it is in every case? Here are three equations. In the first equation the minus means negative:

$$3 + (-1) = 2$$

In the second equation the minus means subtract:

$$5 - 2 = 3$$

In the third equation there are two minus signs. The first means subtract and the second means negative:

$$4 - (-3) = 7$$

EXERCISE 28-2A

1. Find the answer to each of these problems and then write the complete equation in the new notation.

a. neg 13 + neg 71
b. neg 21 − pos 11
c. neg 3 − 0
d. neg 21 − neg 15
e. pos 13 + neg 20
f. 0 + neg 4
g. pos 14 − pos 32
h. pos 3 − neg 7
i. 0 − neg 17

2. In each of the following equations, decide which of the two possible meanings for the minus sign is meant and write the equation in words. Here is an example:
$(-7) - 1 = (-8)$ in words is "neg 7 subtract 1 equals neg 8".

a. $(-3) + 2 = \square$
b. $(-3) - 2 = \square$
c. $4 - (-5) = \square$
d. $4 - 5 = \square$
e. $(-4) - 0 = \square$
f. $(-3) + (-5) = \square$
g. $(-3) - (-5) = \square$
h. $0 - 2 = \square$
i. $0 - (-2) = \square$
j. $(-4) + 2 = \square$

3. Draw a number line showing the integers written in the new notation. Mark on it the points represented by each of the following integers and its opposite. For each pair, make an equation; for example, opp. of 3 = □.

a. 3	**b.** −2	**c.** 7
d. pos 9	**e.** (opposite of 10)	**f.** −5
g. 0	**h.** [opposite of (−6)]	**i.** neg 4

28-3 Opposites

What can we discover about an integer and its opposite in the new notation? Look at these equations.

opp. of 3 = neg 3 becomes opp. of 3 = (−3).
opp. of neg 5 = pos 5 becomes opp. of (−5) = 5.

In each pair of opposites, one integer is positive and the other integer is negative. Can you see that here the minus sign can be thought of in a third way, as saying "the opposite of"?

(−3) is the opposite of 3
−(neg 4) is the opposite of neg 4
−(−4) is the opposite of (−4)

EXERCISE 28-3A

1. For each of the following, first find the integer to make the equation true, then rewrite the equation in the new notation "negative" and for "the opposite of".
Example:

opp. of neg 3 = □
opp. of neg 3 = $\boxed{\text{pos } 3}$
−(−3) = 3

a. opp of pos 2 = □	**b.** opp. of neg 5 = □
c. opp. of 0 = □	**d.** opp. of opp. of pos 1 = □
e. opp. of opp. of neg 3 = □	
f. opp. of opp. of 0 = □	

2. Find the integer which is the simplest way of writing each of the following.

a. (−4)	**b.** [−(−2)]
c. (−0)	**d.** −[−(−6)]

28-4 Addition and subtraction in the new notation

You will be wondering how to perform addition and subtraction in the new notation. Perhaps you already know some rules for adding and subtracting integers, but do you understand how they work? Can you explain them to your pupils?

It will help to think of addition and subtraction on the number line. Think about adding or subtracting a positive integer. The problems are given first in the old notation and then in the new notation.

$$\begin{aligned}
\text{pos } 6 + \text{pos } 2 &= 6 + 2 \text{ steps to right} &&= 8\\
\text{neg } 2 + \text{pos } 4 &= (-2) + 4 \text{ steps to right} &&= 2\\
\text{pos } 6 - \text{pos } 4 &= 6 + 4 \text{ steps to left} &&= 2\\
\text{neg } 2 - \text{pos } 4 &= (-2) + 4 \text{ steps to left} &&= (-6)
\end{aligned}$$

EXERCISE 28-4A

1. Draw a number line to show each of the following and then write the complete equation.

a. $2 + 3$	**b.** $2 - 3$
c. $0 - 2$	**d.** $(-2) + 1$
e. $(-3) + 5$	**f.** $(-4) + 6$

You see that the addition and the subtraction of positive integers from other integers present no difficulty at all. And in the last chapter we learned that to add an integer we can subtract its opposite, and to subtract an integer we can add its opposite. Here are two examples showing these procedures in the new notation.

In the old notation we have

$$\begin{aligned}
\text{pos } 3 + \text{neg } 5 &= \text{pos } 3 - (\text{opp. of neg } 5)\\
&= \text{pos } 3 - \text{pos } 5 = \text{neg } 2
\end{aligned}$$

In the new notation we have

$$\begin{aligned}
3 + (-5) &= 3 - \text{opp. of } (-5)\\
&= 3 - 5\\
3 + (-5) &= (-2)
\end{aligned}$$

Now we will work another problem, first in the old notation:

$$\begin{aligned}
\text{neg } 2 - \text{neg } 3 &= \text{neg } 2 + (\text{opp. of neg } 3)\\
&= \text{neg } 2 + \text{pos } 3\\
&= \text{pos } 1
\end{aligned}$$

In the new notation this becomes

$$(-2) - (-3) = (-2) + [\text{opp. of } (-3)]$$
$$(-2) - (-3) = 1$$

EXERCISE 28-4B

1. Work these problems in the way shown above, first in the old notation and then in the new notation.

a. pos 3 + neg 4	**b.** neg 3 + neg 3
c. 0 + neg 1	**d.** 0 − neg 4
e. pos 3 − neg 2	**f.** neg 5 − neg 2
g. neg 2 − neg 2	**h.** neg 7 + neg 3

2. Work these problems using any method you choose, but explain your method as you would explain it to your pupils.

a. neg 2 + pos 1	**b.** neg 6 + neg 3
c. pos 1 − pos 7	**d.** pos 3 − neg 3

3. Think of the physical models of the integers and make up some problems about them which need addition and subtraction of integers.

28-5 Some properties of integers

Do you remember that when you studied addition of whole numbers and of fractions you found that zero played a particular role in addition? It is an identity element. If 0 is added to a number, that number is unchanged. For example, $9 + 0 = 9$ and $0 + 9 = 9$. When you tested to see if zero was an identity element for subtraction also, you found that it only "half worked". $3 - 0 = 3$, but $0 - 3$ had no answer. *Now* you know the answer to $0 - 3$: it is (-3), neg 3. Try the question again beginning with (-3): $(-3) - 0 = (-3)$, and $0 - (-3) = 3$. So zero does not act as an identity element for subtraction. Do you know why not? It is because of another property which addition possesses and which subtraction does not.

You know that $3 + 5 = 8$ and $5 + 3 = 8$ and also that $5 - 3 = 2$ but $3 - 5 \neq 2$. This shows the commutative property of addition and it shows that subtraction does not possess this property.

In general you know that, for all integers a and b, $a + b$

$= b + a$ but that $a - b \neq b - a$ unless $a = b$. The commutative property is not a property of subtraction.

There is, however, a special relationship between $a - b$ and $b - a$. Can you see what it is? Look at these results.

$$5 - 3 = 2 \qquad 3 - 5 = (-2)$$
$$12 - 4 = 8 \qquad 4 - 12 = (-8)$$
$$16 - 19 = (-3) \qquad 19 - 16 = 3$$

In each pair of equations the differences are opposites, 2 and (-2), 8 and (-8) and (-3) and 3.

So $(a - b)$ and $(b - a)$ are opposites.
So $(a - b) = \text{opp. } (b - a)$
$(a - b) = -(b - a)$.

This is a very useful relationship. If you have, for example, $8 - (3 - 4)$ in an equation, you can replace it by

$$8 + (4 - 3)$$

which is easier to find.

EXERCISE 28-5A

1. Work these problems:

a. (pos 6 + pos 2) − neg 3
b. neg 3 + (neg 2 + pos 1)
c. $[7 + (-2)] + 8$
d. $(-4) + [(-2) + (-1)]$
e. $[(-3) - (-5)] - (-3)$
f. $(-8) - [(-4) + (-2)]$

2. Work these problems in subtraction by "adding the opposite".

a. $(-2) - (3 - 8)$ **b.** $6 - (2 - 1)$
c. $(-4) - [(-2) - 6]$ **d.** $8 - [5 - (-4)]$

Chapter 29

OPERATIONS ON INTEGERS
(continued)

29-1 Addition, using the new notation

You have learned about new numbers, called the integers. You know that they include the counting numbers, zero and the negatives, which are the opposites of the counting numbers. You have learned what it means to add and subtract these numbers, and you have seen that these operations act in the same way as addition and subtraction of counting numbers.

You have also learned an easy way to write these numbers. Instead of writing pos 3, for example, you learned that you could simply write 3. The familiar counting numbers are, in fact, the positive numbers. But when you use them, you must remember, of course, that they are positive, and that they behave as opposites of the negative numbers. And you learned also that instead of neg 3, for example, you could simply write (-3). The symbol (-3) seems to ask you to subtract 3—and there is a very good reason for that, namely, that subtraction of 3 is equivalent to addition of (-3). But the number (-3) is as good a number in its own right as 3. Both are equally important as numbers and deserve your equal respect.

It is useful now to recall some of the ways integers act when you add them, and to use the new way of writing integers in doing so. Thus, you learned to write, for example,

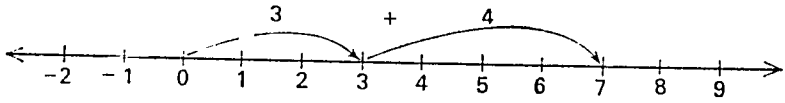
$$\text{pos } 3 + \text{pos } 4 = \text{pos } 7$$

But now you can write simply

$$3 + 4 = 7$$

This new equation means the same thing as the one before it, but it is easier to write. It hides the fact that the numbers are really positive integers, but you should be able to keep that fact in mind by now. You can think of this addition exercise in terms of two successive jumps to the right on the number

line, starting at the zero point. This is shown in the following picture.



You also saw the same pattern for negative integers. Instead of

$$\text{neg } 3 + \text{neg } 4 = \text{neg } 7$$

you can now write

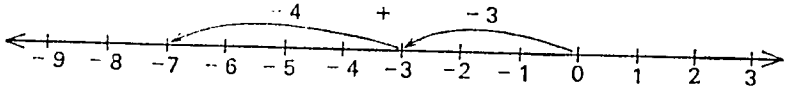
$$(-3) + (-4) = (-7)$$

where the brackets around (-4) remind you that the minus sign, "$-$", in (-4) is not a subtraction sign but rather a marker showing that the number is the opposite of whatever follows the sign. Thus, in this case the "$-$" changes positive 4 to its opposite, neg 4. You can also write this as in the previous chapter

$$(-3) - 4 = (-7)$$

since addition of (-4) is the same as subtraction of 4. In practice, whenever you add a negative number you do it by subtracting its opposite, which is positive. We will write it both ways.

The sum of two negative integers can be represented as the result of two jumps to the left on the number line, starting at the zero point. This is shown in the following picture.



Recall how to add two integers of opposite sign. Addition of a positive integer is represented by a movement to the right and addition of a negative integer by a movement to the left on the number line. Thus the following equations are correct:

$$\text{neg } 3 + \text{pos } 4 = \text{pos } 1$$

and

$$\text{pos } 3 + \text{neg } 4 = \text{neg } 1$$

which can be written more simply as

$$(-3) + 4 = 1$$

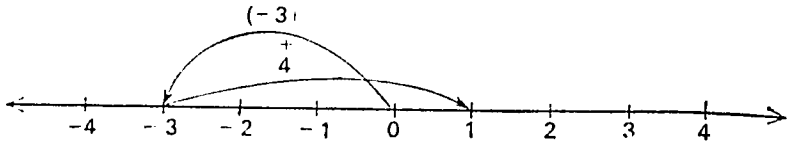
and

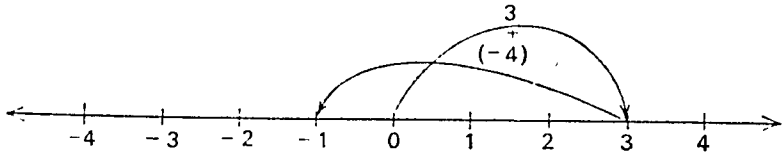
$$3 + (-4) = (-1)$$

which can be written also as

$$3 - 4 = (-1)$$

These can be pictured as follows:





EXERCISE 29-A

Illustrate the following addition problems using the number line:

1. $5 + (-8)$
2. $(-2) + 7$
3. $[3 + (-2)] - 4$
4. $(-7) + \{[(-2) + (-4)] + 8\}$
5. (pos 2 + neg 4) + [(−1) + pos 1]
6. [(pos 3 + neg 2) + pos 5] + (−6)

Remember to do what is required inside the innermost brackets first and then move to the outer brackets.

EXERCISE 29-1A

A pupil complains to you, his teacher, that he does not believe you can add negative integers to anything, because adding makes a number bigger. What would you tell this pupil?

29-2 Subtraction, using the new notation

You saw that subtraction of integers is closely related to subtraction of the counting numbers. Thus the problem

$$\text{pos } 5 - \text{pos } 2 = \text{pos } 3$$

can be rewritten more simply, using counting numbers, as

$$5 - 2 = 3$$

You learned also how to solve the problem, which in a previous section of this book remained unsolved, of subtracting a larger counting number from a smaller counting number, as in this example:

$$5 - 7 = \square$$

This problem can also be considered as a missing addend problem:

$$7 + \square = 5$$

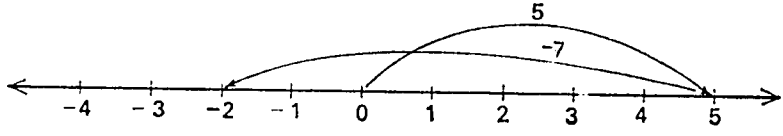
It is impossible to solve this problem using only positive integers, but, using both positive and negative integers, you can say

$$\text{pos } 5 - \text{pos } 7 = \text{neg } 2$$

which can be rewritten

$$5 - 7 = (-2)$$

Such a problem can be pictured on the number line as follows:



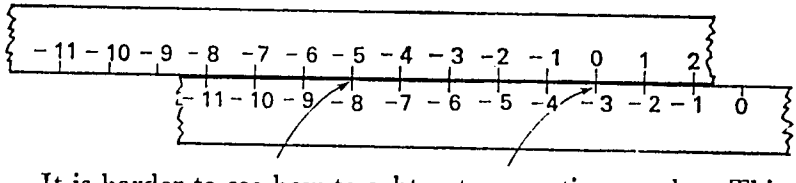
In the same way, it is easy to see that

$$\text{neg } 3 - \text{pos } 5 = \text{neg } 8$$

which can be rewritten

$$(-3) - 5 = (-8)$$

and which can be shown, using your slide rule made from two rulers sliding one on the other, as in the following picture.



It is harder to see how to subtract a negative number. This problem can best be understood by looking for a missing addend. Thus, you can write

$$\text{pos } 3 - \text{neg } 6 = \square$$

so that

$$\text{pos } 3 = \text{neg } 6 + \square$$

Clearly, the result you want to put in the box is pos 9, because

$$\text{pos } 3 = \text{neg } 6 + \text{pos } 9$$

and, thus,

$$\text{pos } 3 - \text{neg } 6 = \text{pos } 9$$

Using the new notation, you can write

$$3 - (-6) = 9$$

$$3 = (-6) + 9$$

The approach remains the same in the case where both numbers in the subtraction problem are negative integers, as in the following example:

$$(-4) - (-7) = 3$$

which is a simplification of the expression

$$\text{neg } 4 - \text{neg } 7 = \text{pos } 3$$

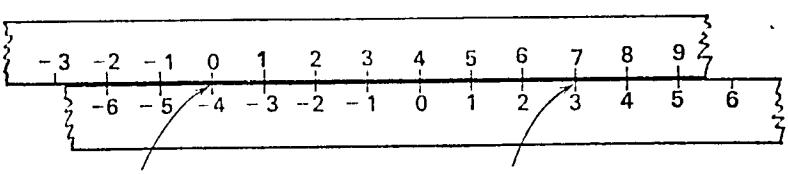
You learned, moreover, that subtraction of one integer from another simply means addition of the opposite integer. This fact made life much easier for you, so that you could write

$$\text{neg } 4 - \text{neg } 7 = \text{neg } 4 + \text{pos } 7 = \text{pos } 3$$

which could be written more simply as

$$(-4) - (-7) = (-4) + 7 = 3$$

This equation can be illustrated as follows, using your slide rule:



Because subtraction and addition are inverse operations, it is possible to subtract any number by adding its opposite, and to add any number by subtracting its opposite.

EXERCISE 29-2A

Simplify each of the following expressions and find the result:

1. $4 - (-7)$
2. $(-5) - (-2)$
3. $(-3) - \{4 + [(-3) - (-4)]\}$
4. $\text{pos } 6 + (\text{neg } 3 - \text{neg } 2)$

5. $[-(\text{neg } 6) + \text{pos } 6] - [(-2) + 3]$
6. $\text{neg } \{[(-2) + (-3)] + [(-2) - \text{neg } 6]\}$

EXERCISE 29-2B

1. A pupil tells you his father has less than nothing in the bank, because the bank put on his account a service charge which was greater than his bank balance. Can you explain this situation for the class?
2. Two men argue as to who is better off: the one who received two gifts of £10 each or the one who had two debts of £10 each cancelled. Each started with the same amount of money. What do you think?
3. Make up similar word problems for the use of your primary school class.

29-3 Multiplication of integers

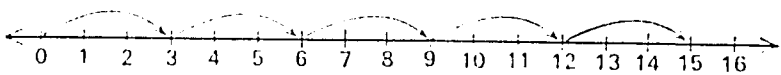
When you studied the addition of counting numbers, you met problems where the same number was added to itself several times. You found that such addition problems could be solved easily by using a new operation, called multiplication. Thus, you learned to replace

$$3 + 3 + 3 + 3 + 3 = 15$$

by the multiplication equation

$$5 \times 3 = 15$$

You saw that such an equation could be pictured by an array of objects or by repeated motions on the number line, as follows:



You can do much the same thing with the integers. Repeated addition of positive integers is the same as the problem given above. Thus,

$$\text{pos } 3 + \text{pos } 3 + \text{pos } 3 + \text{pos } 3 + \text{pos } 3 = \text{pos } 15$$

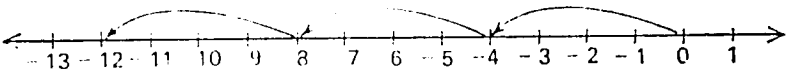
can be written $\quad 5 \times \text{pos } 3 = \text{pos } 15$

In the same way you can do repeated addition of negative

integers. In the following series of equations, the reasons for the successive steps should be clear to you.

$$\text{neg } 4 + \text{neg } 4 + \text{neg } 4 = \text{neg } 12$$
$$3 \times \text{neg } 4 = \text{neg } 12$$
$$3 \times (-4) = (-12)$$

This problem can be pictured on the number line as follows:



In this way, you can see that if you multiply successive integers by a positive integer, the answers form a pattern. Look at the following examples:

$$4 \times 3 = 12$$
$$4 \times 2 = 8$$
$$4 \times 1 = 4$$
$$4 \times 0 = 0$$
$$4 \times (-1) = (-4)$$
$$4 \times (-2) = (-8)$$
$$4 \times (-3) = (-12)$$

One factor (4) is the same in each equation. The other factor is decreased by 1 each time. Each product is 4 *less* than that in the equation above it. It is therefore reasonable to write

$4 \times (-1) = (-4)$, $4 \times (-2) = (-8)$ and $4 \times (-3) = (-12)$

as was done. Again each of these equations can be pictured on the number line as in the problem $3 \times (-4) = (-12)$ above.

Now look at the following examples:

$$4 \times 3 = 12$$
$$3 \times 3 = 9$$
$$2 \times 3 = 6$$
$$1 \times 3 = 3$$
$$0 \times 3 = 0$$
$$(-1) \times 3 = (-3)$$
$$(-2) \times 3 = (-6)$$

Why are the last two products given as (-3) and (-6) respectively? One factor (3) is the same in each equation. The other factor is decreased by 1 each time. Each product is 3 *less* than in the equation above it. So again it is reasonable to write $(-1) \times 3 = (-3)$ and $(-2) \times 3 = (-6)$. However, it does

not make much sense to show these equations on the number line. For example, what would it mean to make (-2) jumps of 3 units each?

With these examples in mind, see if you can find a general rule for multiplying a positive integer by a negative integer. One point of view is the following. You know how to multiply two positive integers or counting numbers. The product is always a positive integer. This operation has certain properties.

It is commutative $(a \times b = b \times a)$

associative $(a \times (b \times c) = (a \times b) \times c)$

and distributive over addition

$$(a \times (b + c) = (a \times b) + (a \times c)).$$

Also, $1 \times a = a$ for any positive integer a. *Assume that these properties continue to hold when you use the set of integers.* What then will be the product of two integers if at least one of the factors is a negative integer?

Result 1

The product of 0 and any integer is 0. That is, $0 \times a = 0$ for any integer a.

Example

Find $0 \times (-2)$.

$0 + 0 = 0$	(Property of 0 in addition.)
$(0 + 0) \times (-2) = 0 \times (-2)$	(Multiply by (-2))
$[0 \times (-2)] + [0 \times (-2)] = [0 \times (-2)]$	(Distributive Law)
$0 \times (-2) = 0$	(Subtract $0 \times (-2)$)

A similar argument works for any integer in place of (-2).

Result 2

$(-1) \times a =$ opposite of $a = -a$, for any integer a.

This can be shown as follows:

$(-1) + 1 = 0$	(The sum of an integer and its opposite is zero)
$[(-1) + 1] \times a = 0 \times a$	(Multiply by a)
$[(-1) \times a] + (1 \times a) = 0$	(Distributive property and Result 1)
$[(-1) \times a] + a = 0$	(Property of 1)

Therefore $(-1) \times a =$ opposite of $a = -a$

In particular, if a is (-1), you see that

$$(-1) \times (-1) = \text{opposite of } (-1) = 1$$

Result 3

$(-a) \times b = -(a \times b)$ for any two integers a and b.

This follows directly from Result 2, for

$$\begin{aligned}(-a) \times b &= [(-1) \times a] \times b && \text{(Result 2)}\\ &= (-1) \times (a \times b) && \text{(Associative property)}\\ &= \text{opposite of } (a \times b) && \text{(Result 2)}\\ &= -(a \times b)\end{aligned}$$

For example, $(-2) \times 4 = -(2 \times 4) = -8$

$(-3) \times 2 = -(3 \times 2) = -6$, and so on.

In particular, if a and b are positive integers, then $(-a)$ is a negative integer, as is $-(x \times b)$. You then see that the product of a negative integer and a positive integer must be a negative integer. The commutative property allows you to change the order of the factors without changing the product, so this last statement can be given as a general rule.

The product of two integers of opposite sign is a negative integer

You already know that the product of two positive integers is a positive integer. What is the product of two negative integers? Look at these examples:

$$\begin{aligned}4 \times (-2) &= -8\\ 3 \times (-2) &= -6\\ 2 \times (-2) &= -4\\ 1 \times (-2) &= -2\\ 0 \times (-2) &= 0\\ (-1) \times (-2) &= ?\\ (-2) \times (-2) &= ?\end{aligned}$$

One factor (-2) is the same in each equation. The other factor is decreased by one each time. Here each product is 2 *more* than that in the equation above it. It is therefore reasonable to expect that

$$(-1) \times (-2) = 2 \quad \text{and} \quad (-2) \times (-2) = 4$$

Result 4

$(-a) \times (-b) = a \times b)$ for any two integers a and b.

This follows directly from Result 2, for

$$\begin{aligned}(-a) \times (-b) &= (-1) \times a \times (-1) \times b && \text{(Result 2)}\\ &= (-1) \times (-1) \times a \times b && \text{(Why?)}\\ &= 1 \times a \times b && \text{(Why?)}\\ &= a \times b && \text{(Why?)}\end{aligned}$$

For example, $(-2) \times (-2) = 2 \times 2 = 4$
$(-3) \times (-4) = 3 \times 4 = 12$, and so on.

In particular, if a and b are positive integers, both $(-a)$ and $(-b)$ are negative integers, while $(a \times b)$ is a positive integer. So you get the following general rule.

The product of two integers of the same sign is a positive integer

EXERCISE 29-3A

1. Perform the indicated operations in each of the following. Remember to do the work inside the innermost brackets first.
   a. $(-5) \times 2$ **b.** $(-6) \times (-4)$ **c.** $3 \times (-7)$
   d. $(-4) \times \{3 - [4 - (-3)]\} - \{(-1) \times [4 + (-2)]\}$
   e. $\{(-2) + [(-3) \times (4 - 1)]\} \times (2 + \{(-5) + [(-3) \times (-4)]\})$
   f. $(8 - \{4 \times [(-2) + 6]\}) \times ((-5) + \{(-3) \times [(-4) + 8]\})$
2. Write an argument for Result 1 using an arbitrary integer a.

29-4 Division of integers

You remember from a previous chapter that division of a whole number by a counting number can be considered in many different ways. You thought of it as sharing a collection of objects among several persons; as finding the number of sets of a given size that can be taken from a given set; as finding a missing factor in a multiplication equation; and so on. In discussing division of integers, you will find the idea of a missing factor most helpful.

It is easy to see how to divide any integer by a positive integer.
For example,

$$4 \div 2 = 2 \quad \text{because} \quad 2 \times 2 = 4$$

Similarly,

$$(-4) \div 2 = (-2) \quad \text{because} \quad (-2) \times 2 = (-4)$$
$$(-12) \div 4 = (-3) \quad \text{because} \quad (-3) \times 4 = (-12)$$

and so on.

Division by a negative integer is handled in the same way. For example,

$$4 \div (-2) = (-2) \quad \text{because} \quad (-2) \times (-2) = 4$$

Similarly, $(-4) \div (-2) = \square$

is the same as $(-2) \times \square = (-4)$

The answer is 2; that is $(-4) \div (-2) = 2$.

Here are more examples:

$$(-16) \div (-4) = 4 \quad \text{because} \quad (-4) \times 4 = (-16)$$
$$(-16) \div 4 = (-4) \quad \text{because} \quad 4 \times (-4) = (-16)$$
$$16 \div (-4) = (-4) \quad \text{because} \quad (-4) \times (-4) = 16$$
$$16 \div 4 = 4 \quad \text{because} \quad 4 \times 4 = 16$$

From these examples and the fact that division and multiplication are inverse operations, you can see that the rules for signs in division of integers are similar to those for multiplication of integers.

The quotient of two integers of the same sign is positive
The quotient of two integers of opposite sign is negative

Finally you see that $0 \div a = 0$ for any *non-zero* integer a, because $0 \times a = 0$. For example, $0 \div (-2) = 0$ because $0 \times (-2) = 0$.

As before for whole numbers, division by 0 is not allowed, since it has no meaning. For example,

$$(-2) \div 0 = \square$$

is the same as $0 \times \square = (-2)$

But there is *no* integer which when multiplied by 0 gives a product of (-2).

EXERCISE 29-4A

1. Perform the indicated operations to get answers for the following problems:

a. $(-6) \div 3$ **b.** $12 \div (-2)$
c. $(-15) \div (-5)$ **d.** $6 \div (-1)$

e. $\{[(-3) + 5] \times [2 + (-8)]\} \div \{[(-2) + [(-2) \times 2)\}$
f. $\{[4 \div [(-1) + 3]\} \times \{[(-3) \times 2] \div [4 + (- \ )]\}$
g. $\{(-3) + [(-2) \times (-6)]\} \div \{[(-4 \times (-1)] - (3 + 4)\}$

EXERCISE 29-4B

1. A pupil tells you that in a multiplication or division problem the easiest way to do it is to forget about the signs and do the problem as he learned it for whole numbers, and then count the number of negative signs in the problem. If the number is odd the answer is negative, and otherwise the answer is positive. Is the pupil correct? What if there were addition and subtraction in the problem also?

EXERCISE 29-4C

1. Do these problems, and see what pattern appears in the answers. Remember what you learned before about inverse operations.

a. $[4 \times (-2)] \div (-2)$ **b.** $[(-2) \times 3] \div 3$
c. $[(-2) \times (-5)] \div (-5)$ **d.** $[8 \times (-1)] \div (-1)$
e. $[(-8) \div 4] \times 4$ **f.** $[15 \div (-3)] \times (-3)$
g. $[(-12) \div (-2)] \times (-2)$ **h.** $[1 \div (-1)] \times (-1)$

It is not necessary to write so many brackets in our equations. For example, if it will not be confusing, we will write -2 instead of (-2), -4 instead of (-4), -259 instead of (-259) and so on.

With brackets	*Without brackets*
$(-4) - 2 = (-2)$	$-4 - 2 = -2$
$4 - (-2) = (-2)$	$4 - (-2) = -2$
$(-5) \times 2 = (-10$	$-5 \times 2 = -10$
$2 \times (-5) = (-10)$	$2 \times (-5) = -10$
$(-4) - (-7) = 3$	$-4 - (-7) = 3$

111

The Rational Numbers UNIT VI

Chapter 30

NEGATIVE FRACTIONS

30-1 Division of integers

The problems we did in the last chapter when we divided one integer by another were easy in one sense. We only worked problems where the answers were also integers. What about problems like the following?

$$(-7) \div 3 = \square$$

We have not done such problems before, and now we must try to find a way of solving them. We know what $(-6) \div 3$ means, and we can give -2 as the answer. What about $(-7) \div 3$?

It is always a good idea in doing mathematical problems to look back to easier and somewhat similar problems which you have solved before. In this case, you should think back to problems of this kind which came up in the discussion of the counting numbers. There we faced the difficulty that the problem

$$7 \div 3 = \square$$

has no answer among the counting numbers. And you remember that we had to find a new kind of number to solve this problem. That number was called a fraction, and was used to name parts of a whole. We found that these new numbers gave us answers to all such otherwise unsolvable division problems.

We will find in the following sections of this chapter that no really new problems arise in dividing one integer by another. If we use what we have already learned about fractions and about integers, we will find that the answer is right in front of us. This is what mathematicians always do when they try to solve a new problem. They look at similar problems they have done before to see if a solution is suggested there.

EXERCISE 30-1A

Which of the following problems can be solved using only integers, and which require new numbers?

1. $18 \div (-3)$
2. $16 \div (-5)$
3. $(-9) \div (-3)$
4. $2 \div (-2)$
5. $17 \div (-3 - 5)$
6. $[-8 \div (-2) \times 3] \div [3 \times (-3) - (-2)]$
7. $[(-8) \div (-2 - 4)] - [3 \div (-3) \times 2]$
8. $[(-7) \times (-5) - 35] \div \{(-16) - [4 \times (-2)]\}$

EXERCISE 30-1B

When you tell your class that there are some numbers they can't yet divide, a pupil tells you that with a sharp enough knife he can divide anything. What do you think you should say to this pupil?

30-2 Division as multiplication with a missing factor

When you tried to solve such problems as $7 \div 3$, you found that it helped to change them into multiplication equations with missing factors. Thus, you wrote

$$3 \times \square = 7$$

and you found the fraction $\frac{7}{3}$ to be the answer. This fraction is, of course, still the answer when you think of the counting numbers as positive integers. But the problem is not so easy when you have negative integers. For example, take the division problem

$$(-7) \div 3 = \square$$

which gives rise to the multiplication equation with missing factor

$$3 \times \square = -7$$

You can guess what the answer to this problem ought to look like if you remember how you solved the two problems

$$3 \times \square = 6$$

and

$$3 \times \square = -6$$

In the one case you can put $6 \div 3 = \frac{6}{3} = 2$ in the box, because

$$3 \times \left(\frac{6}{3}\right) = 6$$

and in the other case you would want to put

$$(-6) \div 3 = \frac{(-6)}{3} = -2$$

in the box, because you would feel that

$$3 \times \left[\frac{(-6)}{3}\right] = -6$$

Thus, if the answer to the problem $7 \div 3 = \square$ is

$$7 \div 3 = \tfrac{7}{3}$$

then the answer to the problem $(-7) \div 3 = \square$ ought to appear in the form

$$(-7) \div 3 = \frac{(-7)}{3}$$

so that

$$3 \times \left[\frac{(-7)}{3}\right] = -7$$

In the same way you can think of the other two cases which might come up, as in these examples:

$$(-3) \times \square = 7$$
$$(-3) \times \square = -7$$

As in the previous case, we can solve the equations

$$(-3) \times \square = 6$$

and

$$(-3) \times \square = -6$$

and obtain

$$6 \div (-3) = \frac{6}{(-3)} = -2$$

$$(-6) \div (-3) = \frac{(-6)}{(-3)} = 2$$

Thus, the answer to our new problems ought to be given in the same way, as

$$7 \div (-3) = \frac{7}{(-3)}$$

and

$$(-7) \div (-3) = \frac{(-7)}{(-3)}$$

We have used the phrases "ought to be" and "ought to appear". You might ask what kind of mathematics *that* is. Mathematics, you might say, should tell you what is, not what

ought to be. But there is a reason for what we have done. All you have seen so far is a juggling of numerals. We showed you some problems you have already done, and then showed you some more problems you have not done. And we asked you to guess only what the answer might look like. In the next section of this chapter, however, you will see that what has been done here does have a real meaning after all, and that these numbers, like $\frac{(-7)}{3}$, do give the solutions to the problems posed.

EXERCISE 30-2A

Look back at those questions in Exercise 30-1A which did not have integers as answers, and tell what the answers "ought to be".

EXERCISE 30-2B

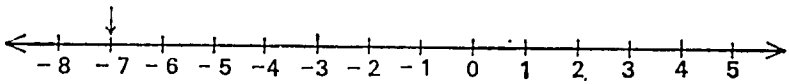
Make up questions like those in Exercise 30-1A which "ought to give" the following answers.

1. $\frac{15}{8}$ **2.** $\frac{(-7)}{2}$ **3.** $\frac{(-8)}{(-4)}$

4. $\frac{12}{(-9)}$ **5.** $\frac{0}{(-1)}$ **6.** $\frac{(-5)}{5}$

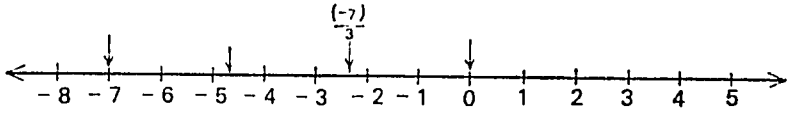
30-3 Interpreting our new numbers on the number line

The problem that faces us now is to decide what a number like $\frac{(-7)}{3}$ might mean. We know what -7 is, and we know how to interpret it. For example, we can place it on the number line as follows:



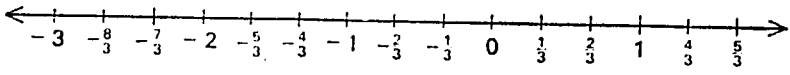
But if we remember that $\frac{7}{3}$ is $\frac{1}{3}$ of the way from 0 to 7, then perhaps we can think of $\frac{(-7)}{3}$ as $\frac{1}{3}$ of the way from 0 to -7. Is there a point which is $\frac{1}{3}$ of the way from 0 to -7? Of course

there is. We just have to take that segment of the number line and break it into three equal parts, and mark the first such point to the left of 0 as $\frac{1}{3}$ of the way from 0 to -7. On the number line it looks like this:



It is just as good a point as $\frac{7}{3}$, and thus $\frac{(-7)}{3}$ deserves its place in our collection of numbers.

Clearly this number $\frac{(-7)}{3}$ is opposite to $\frac{7}{3}$, which is a point to the right of 0 on the number line. In fact, every point to the right of 0 has an opposite point to the left of 0. There is no reason why we should have opposites only for the integers. We can write this opposite as $-\frac{7}{3}$, and we can thus see that $-\frac{7}{3}$ and $\frac{(-7)}{3}$ should mean the same thing. The numbers 7 and -7 are opposites, and thus the numbers $\frac{7}{3}$ and $\frac{(-7)}{3}$, which show $\frac{1}{3}$ part of that which is represented by 7 and -7 respectively, must be opposites. But the opposite of any number x is written $-x$. Thus the opposite of $\frac{7}{3}$ is $-\frac{7}{3}$, and $\frac{(-7)}{3}$ and $-\frac{7}{3}$ are two names for the same number. Thus, we can begin to fill in the number line in the following way:



This picture shows that each fraction to the right of 0 has its opposite to the left of 0, and thus we want to call these new numbers *negative fract' s.*

The only remaining problems concern $\frac{7}{(-3)}$ and $\frac{(-7)}{(-3)}$.

Before trying to give an answer, we should think where these symbols, which "ought to" represent numbers, came from. The problem $(-3) \times \square = 7$ gave rise to the possible answer $\frac{7}{(-3)}$, and thus we should think what the problem $(-3) \times \square = 7$ means. This problem is similar to the problem

$$(-3) \times \square = 6$$

We know the answer to this problem, namely, (-2). To check this, recall that

$$\begin{aligned}(-3) \times (-2) &= (-1) \times 3 \times (-1) \times 2 \\ &= (-1) \times (-1) \times 3 \times 2 \\ &= 3 \times 2 = 6\end{aligned}$$

In the problem $(-3) \times \square = 7$, we get the answer $\left(-\frac{7}{3}\right)$ since

$$(-3) \times \left(-\frac{7}{3}\right) = 7$$

Thus, the result which was suggested by the equation, namely $\frac{7}{(-3)}$, must have the same meaning as $-\frac{7}{3}$. We therefore think of $\frac{7}{(-3)}$ as another name for $-\frac{7}{3}$, which we understand to be the fraction opposite to $\frac{7}{3}$, or $\frac{1}{3}$ part of -7.

Finally, we have to think about $\frac{(-7)}{(-3)}$. Again we should look at the multiplication equation

$$(-3) \times \square = -7$$

You already know how to do the very similar problem

$$(-3) \times \square = -6$$

which has as its answer

$$2 = (-6) \div (-3)$$

Thus, in the original problem we see that

$$(-3) \times \frac{7}{3} = -7$$

so that $$(-7) \div (-3) = \frac{(-7)}{(-3)} = \frac{7}{3}$$

Thus, we can see that the answer suggested, namely $\frac{(-7)}{(-3)}$, has the same meaning as $\frac{7}{3}$. Once again we think of $\frac{(-7)}{(-3)}$ as another name for $\frac{7}{3}$, with which we are already familiar.

To summarize,

$$\frac{(-7)}{(-3)} = \frac{7}{3} \quad \text{and} \quad \frac{(-7)}{3} = \frac{7}{(-3)} = -\frac{7}{3}$$

EXERCISE 30-3A

In this section, you found three ways of writing a given negative fraction and two ways of writing a given positive fraction. Locate each of the fractions given in Exercise 30-2B on the number line. Then write these fractions in each of the ways possible for it. What do you notice in Question 5 of that exercise?

EXERCISE 30-3B

In Chapter 25, some situations were given in which numbers on both sides of 0 were given physical meanings. Read that chapter over again, and state for each of those situations what negative fractions mean.

EXERCISE 30-3C

Draw up a series of test questions designed to find out whether your pupils understand positive and negative fractions and the relations between them.

EXERCISE 30-3D

Find the opposite of each of the following fractions. Put your answer in "simplest" terms.

1. $\frac{7}{3}$ **2.** $-\frac{5}{4}$ **3.** $\frac{(-5)}{4}$ **4.** $-\frac{(-5)}{4}$

5. $\frac{(-2)}{3}$ **6.** $\frac{2}{(-3)}$ **7.** $-\frac{1}{(-4)}$ **8.** $-\frac{(-6)}{(-5)}$

Chapter 31

THE SET OF RATIONAL NUMBERS

31-1 Fractions, positive and negative

We should stop briefly now, and look at what we have done. You learned about fractions in an earlier chapter, and now you have found that these fractions have opposites. And if you look at the number line, you can see now that it is, or at least seems to be, as crowded as you can make it. Actually, you will find later that there are more numbers to be put on the number line. But let us be content for now with what we have, and try to understand these numbers fully.

We call the fractions we originally had *positive fractions*, and we call our new numbers *negative fractions*. There are many ways of writing any given fraction, of course, just as there are many ways of writing every number. Take the positive fraction $\frac{3}{5}$ and its opposite, the negative fraction $-\frac{3}{5}$, for example. Here are some other ways of writing each of these fractions:

$$\frac{3}{5} = \frac{6}{10} = \frac{150}{250} = \frac{(-3)}{(-5)} = \frac{(-72)}{(-120)}$$

$$-\frac{3}{5} = \frac{(-3)}{5} = \frac{3}{(-5)} = \frac{(-90)}{150} = \frac{33}{(-55)} = -\frac{3000}{5000}$$

Do you remember how you could prove that all these fractions are names for the same number? If not, check back to the chapter on fractions and revise. And remember also that one minus sign in the fraction makes the whole fraction a negative fraction, and two make it a positive fraction. In every case, the simplest way to write a fraction is the first way given in each of the series of equal fractions above. Write it as a positive fraction, without minus signs, or as the opposite of such a positive fraction. We will call this the *standard form*; for example, $\frac{2}{3}$ and $-\frac{2}{3}$. Of course, the fraction may arise in a problem in one of the other forms. If that happens, it is useful to reduce numerator and denominator to the smallest whole numbers possible,

using results obtained earlier. Then put the minus sign, if there remains one, in front.

If you write a fraction in standard form, it is easy to see where it belongs on the number line. All positive fractions are, of course, to the right of 0. Negative fractions, on the other hand, are to the left of 0. Every fraction thus has a place on the number line, and although not every point on the number line indicates a fraction (We will say more about that later!), there is always a fraction as close as you would like to every point on the number line.

You may be thinking at this stage that there are points on the number line which are not fractions, points you already know about. You may say that we have not mentioned the counting numbers and their opposites and 0. After a whole series of chapters in which you studied those numbers, which you learned to call the integers, we seem to have forgotten them again. That is a good question—but if you think about it, you will see that the integers are still here and that they can be written as fractions also. If you look back to Chapter 20, you will see that the counting numbers were written as special fractions, as in the following example:

$$\tfrac{6}{3} = \tfrac{2}{1} = 2$$

The fraction which has 1 as its denominator or whose numerator is equal to the product of the denominator with some counting number was shown to be simply another name for one of the counting numbers. (That is true, of course, because to divide a thing into one part or into some number of equal parts, each of which contains a whole number of members, is the same as division of whole numbers where there is no remainder.)

The same fact is true for negative fractions as well. It is easy to see, for instance, that

$$\frac{(-6)}{3} = -\frac{6}{3} = -2$$

If -6 is broken into 3 equal parts, each will contain -2. For a practical example, think of a debt of £6 shared among 3 people. Clearly, each will pay £2. More complicated examples can be worked in the same way, always remembering first to

put the fraction in standard form. Thus, we have

$$-\frac{(-10)}{(-2)} = -\frac{10}{2} = -5$$

So the fractions, positive and negative, include all the numbers we have used up to this point and can even be extended to include 0, which can be written, for instance, as

$$0 = \tfrac{0}{7}$$

These numbers are an interesting set, and we will think much more about their properties in the following sections of this chapter, as well as in the next few chapters.

EXERCISE 31-1A

Find each fraction in standard form, and four other fractions equal to each of the following. Locate each on the number line.

1. $\dfrac{(-32)}{14}$
2. $7\frac{3}{5}$
3. $-\dfrac{(-20)}{17}$
4. $-\left[-\dfrac{13}{(-13)}\right]$
5. $\dfrac{0}{[-(-11)]}$
6. $-15{\cdot}3$

Remember to do the work within brackets first. Thus, for example,

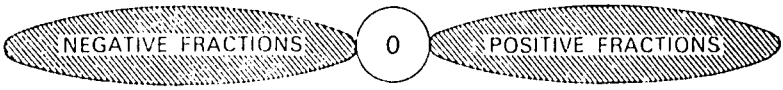
$$-\frac{(-13)}{[-(-17)]} = -\frac{(-13)}{17} = \frac{13}{17} = \frac{26}{34} = \frac{(-26)}{(-34)}$$

EXERCISE 31-1B

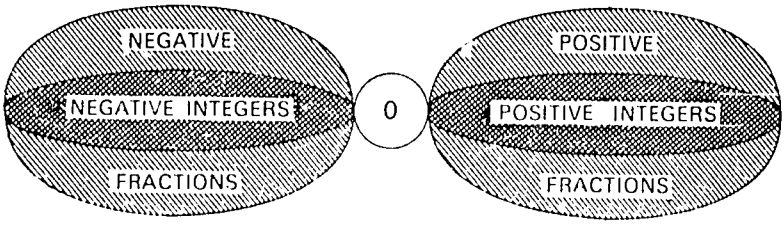
1. Outline a classroom procedure for teaching that it is often important to reduce fractions to standard form and lowest terms when working with them. Your procedure should show them how to make this reduction.
2. Prepare word problems which require students to make use of the fact that a given fraction can be named in several ways. Include both positive and negative fractions.
3. Prepare a classroom demonstration designed to show that some fractions, for example, $\frac{4}{2}$ or $-\frac{6}{2}$, are simply other names for positive or negative integers.

31-2 Definition of the rational numbers

The set of numbers which is made up of the positive fractions, zero and the negative fractions has a special name. It is called the set of RATIONAL NUMBERS. It can be pictured as follows:



The two subsets of fractions—the positive fractions and the negative fractions—contain as subsets the positive integers and the negative integers, respectively. The picture is as follows:

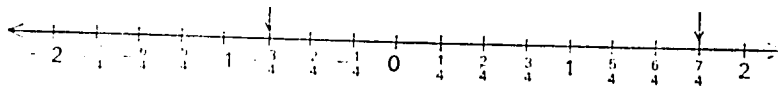


To summarize, the rational numbers contain the positive fractions, zero and the negative fractions. The positive and negative fractions and zero, together, contain the integers. And the integers contain the counting numbers, which is where we began this course in basic concepts of mathematics.

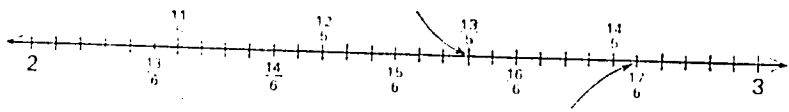
The rational numbers are not only a set of numbers—they are an *ordered set*. The most obvious fact about their order is shown in the diagrams above. The negative fractions are all less than the positive fractions, and the rational numbers to the left of any number on the number line are all less than that number. In a later chapter, you will study the idea of order more closely, just as you will study all the other properties of the rational numbers. For now it is enough simply to state that it is possible to determine of any two fractions which is greater and which is lesser.

For example, consider the two fractions $-\frac{3}{4}$ and $\frac{7}{4}$. This case is obvious. The first fraction is less than 0 and the second is greater than 0. In fact, any negative number x is less than

0, not only because it is to the left of 0 on the number line, but because the difference $0 - x$ is positive. Look at this example on the number line.

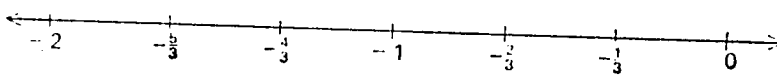


Take another example, $\frac{13}{6}$ and $\frac{17}{6}$. Which is lesser and which is greater? Here is one way in which you can tell. You can draw the number line, place both fractions there and compare them, as follows:



Obviously, $\frac{17}{6}$ is to the right of $\frac{13}{6}$ and is, thus, the greater of the two.

Take a final example: $-\frac{4}{3}$ and $-\frac{2}{3}$. You can show these on the number line as follows. Remember that you can locate any fraction on the number line easily by changing it into a mixed number, if you need to; thus $-\frac{4}{3} = -1\frac{1}{3}$, which is located $1\frac{1}{3}$ units to the left of 0.



Obviously $-\frac{4}{3}$ is to the left of $-\frac{2}{3}$, and, thus,

$$-\frac{4}{3} < -\frac{2}{3}$$

You remember, of course, the meaning of the two symbols "<" and ">". The first means "less than" and the second means "greater than".

EXERCISE 31-2A

Put the following set of fractions in order, from least to greatest, and show their positions on the number line. First simplify each and put it in standard form. Then compare them by pairs, and place each on the number line.

$$\frac{4}{7}, \frac{(-3)}{13}, -2{\cdot}7, 6\tfrac{7}{8}, \frac{(-20)}{(-2)}, \frac{[-(-3)]}{(-4)}, \frac{0}{(-4)}$$

EXERCISE 31-2B

Draw a large poster useful for picturing the set of rational numbers and all the subsets which we have discussed, as on the previous page.

NOTE. We have used the word *fraction* to denote sometimes a number and sometimes the numeral which is its name. Perhaps, strictly speaking, we should not do this. However, it is hard to maintain the distinction in a consistent way, and it seems better not to insist on distinctions which we cannot keep up in practice. It should be clear from the context whether the word *fraction* means a number or a numeral.

The term *rational number*, of course, always denotes a number and not a numeral.

Chapter 32

OPERATIONS ON RATIONAL NUMBERS

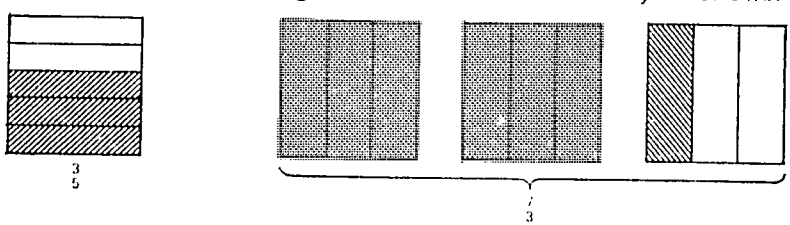
32-1 Addition

It is not really necessary at this point to say anything new about the meanings of the operations. You should be quite familiar not only with their meanings but with how to teach them to young children. Very briefly, addition can be thought of in at least two ways: successive motions on the number line, to the right for positive numbers and to the left for negative numbers; and successive changes of the number of objects in a set, increasing it for addition of positive numbers and decreasing it for addition of negative numbers. You have carried out a thorough and detailed study of this operation for counting numbers, for fractions and for integers. It clearly does not change the picture to include the whole set of rational numbers in this discussion. The only difficult thing at this point—both for you and for the pupils you teach—is to become quick at finding the correct answer when you are faced with a problem.

Let us look at an example, using only positive fractions, to remind ourselves of the methods we learned before. Take the problem

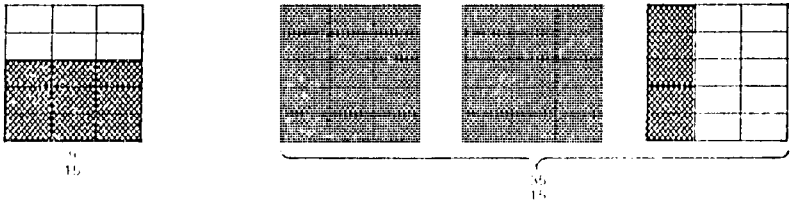
$$\tfrac{3}{5} + \tfrac{7}{3}$$

The way you did such a problem before was to draw a picture showing parts of rectangles for each of the fractions, as follows:



Then you divided each part into smaller pieces so that the resulting small pieces were of the same size. By counting the number of small pieces for each fraction, you could rename both fractions so that they had the same denominator. This was

shown by dividing up each rectangle in the second direction, using the denominator of the other fraction to tell the number of pieces. For the example given above, the picture becomes as follows:



You count the number of small pieces and find the answer:

$$\frac{9}{15} + \frac{35}{15} = \frac{44}{15}$$

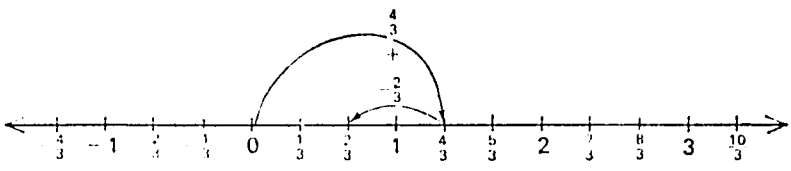
You learned in the study of fractions that there is a short way to carry through this process that saves you having to draw pictures and cut up rectangles. You discovered a rule which did the same thing for the general pair of fractions $\frac{a}{b}$ and $\frac{c}{d}$. The rule was stated as follows:

$$\frac{a}{b} + \frac{c}{d} = \frac{(ad + bc)}{bd}$$

The question now is whether this rule works for all rational numbers, including negative fractions. Let us look at a couple of examples and then come back to the rule. But, as you think about the examples, keep the rule in mind. First take this problem:

$$\frac{4}{3} + \left(-\frac{2}{3}\right) = \frac{4}{3} + \frac{(-2)}{3}$$

This addition can be shown on the number line as follows:



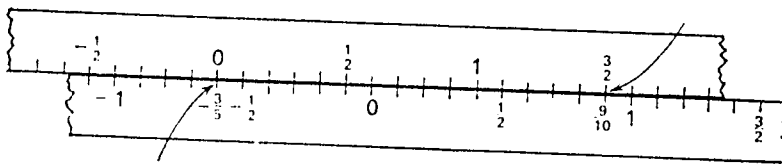
Clearly the answer is $\frac{2}{3}$. If you think back to the problem of cutting bananas into parts, this problem speaks of 4 one-third

pieces and tells you that 2 of them are removed, leaving 2 one-third pieces, or $\frac{2}{3}$ of a banana.

This was an easy problem. Take a somewhat more difficult one; for example,

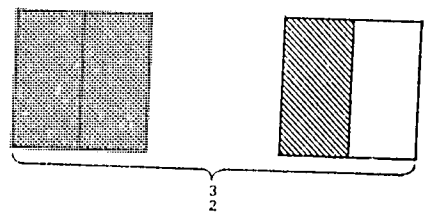
$$-\frac{3}{5}+\frac{3}{2}=\frac{(-3)}{5}+\frac{3}{2}$$

You can look at it on your slide rule in this way:

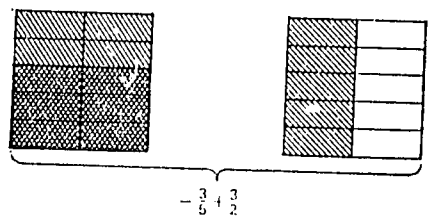


You set your 0 mark on the upper strip to the $-\frac{3}{5}$ mark on the lower. And you read your answer on the lower strip below the $\frac{3}{2}$ mark on the upper strip. That answer is, of course, $\frac{9}{10}$.

You can also think of this problem in terms of rectangles. The fraction $\frac{3}{2}$ can be drawn as follows:



To add $-\frac{3}{5}$ means to remove $\frac{3}{5}$ of a rectangle. Thus, you can redivide each rectangle into 5 parts in the usual way, and remove 3 of those parts. You then count the remaining pieces. You can picture it as follows:

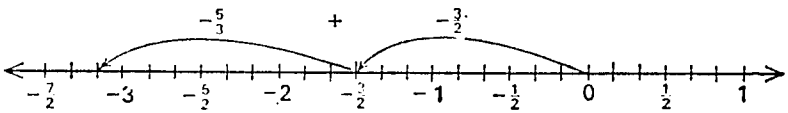


You can see that 9 one-tenth pieces remain, so that the answer is $\frac{9}{10}$.

Where the answer is likely to be a negative fraction, it is not useful to think of dividing rectangles, but you can show such a problem on the number line or the slide rule. Think of the example

$$-\frac{3}{2}+\left(-\frac{5}{3}\right)=\frac{(-3)}{2}+\frac{(-5)}{3}$$

Look at it on the number line.



You can see that the result is $-\frac{19}{6}$ by counting to the left of 0, so that

$$-\frac{3}{2}+\left(-\frac{5}{3}\right)=\frac{(-3)}{2}+\frac{(-5)}{3}=-\frac{19}{6}$$

Let us look for the pattern shown by these examples. In the first case, $\frac{4}{3}+\frac{(-2)}{3}=\frac{2}{3}$, the denominator 3 remained the same, and the resulting numerator was the sum of the two original numerators. In the second case, $\frac{(-3)}{5}+\frac{3}{2}=\frac{9}{10}$, the denominator of the result was 10. If you rewrite each of $\frac{(-3)}{5}$ and $\frac{3}{2}$ with denominator 10, you get $\frac{(-6)}{10}$ and $\frac{15}{10}$. Their sum is, of course, $\frac{9}{10}$. But $-6=(-3)\times 2$ and $15=5\times 3$. Thus, you get

$$\frac{(-3)}{5}+\frac{3}{2}=\frac{[(-3)\times 2]+[5\times 3]}{5\times 2}=\frac{9}{10}$$

In the same way, you get

$$\begin{aligned}\frac{(-3)}{2}+\frac{(-5)}{3}&=\frac{(-9)}{6}+\frac{(-10)}{6}\\&=\frac{[(-3)\times 3]+[2\times(-5)]}{2\times 3}\\&=\frac{(-19)}{6}\end{aligned}$$

But these are obviously examples of the rule for adding positive fractions which is extended to include negative fractions as well. If you think about it, you will see that this must always be true, so that the rule $\frac{a}{b} + \frac{c}{d} = \frac{(ad + bc)}{bd}$ must be true for all rational numbers.

Thus, the same rule you used for positive fractions works for all fractions, both positive and negative.

EXERCISE 32-1A

Find the sums of the following pairs of fractions, illustrating at least one of them by using the number line, the slide rule and rectangles.

1. $\frac{4}{3} + \frac{(-3)}{5}$
2. $\frac{2}{(-3)} + \frac{(-6)}{(-9)}$
3. $-\frac{8}{5} + 1{\cdot}3$
4. $\frac{(-7)}{6} + \frac{0}{(-7)}$
5. $-\frac{3}{5} + \frac{6}{10}$
6. $\frac{(-8)}{(-4)} + \frac{21}{(-7)}$

EXERCISE 32-1B

What fractions suggest the use of money as examples in addition problems? What about fractions like $\frac{1}{3}$? What about fractions like $\frac{1}{6}$?

EXERCISE 32-1C

In Exercise 32-1A, Question 2 presents an important special case. You should, of course, have found the answer 0. This means that the two numbers are opposites, and they cancel each other out. Such pairs of numbers, which have the sum 0, are called additive inverses. Outline a procedure for teaching the meaning of additive inverses to your pupils.

EXERCISE 32-1D

In Exercise 32-1A, Question 4 presents another important special case. The answer is $-\frac{7}{6}$, the same as the first number in the sum. When you add 0 to any number, you do not change that number. The number 0 is called the additive identity element. Outline a procedure for teaching the meaning of the additive identity to your pupils.

32-2 Subtraction

You remember that subtraction is simply the inverse of addition. To subtract a positive number from some other number, you move to the left on the number line; and to subtract a negative number from some other number, you move to the right. Subtracting a positive number decreases the total, while subtracting a negative number increases your total. You can think of subtraction in terms of finding the missing addend in an addition equation. And you can finally think of subtraction as addition of the opposite. You have done many exercises and read many pages on these interpretations of subtraction. Here you need only see that you can understand subtraction of any rational number from any rational number in the same way.

Not only can we understand the meaning of subtraction of rational numbers in terms of the fractions and the integers, but also we can see how to perform such subtraction problems by remembering what we did with integers. There we learned that subtraction of integers meant the addition of the opposite. Thus when you subtract -5 from 3, it is the same as adding 5 to 3. You write this as follows:

$$3 - (-5) = 3 + 5 = 8$$

Can you think of rational numbers in the same way? Look at this problem:

$$\frac{3}{5} - \left(-\frac{7}{3}\right) = \square$$

The answer is the missing addend in this equation:

$$\square + \left(-\frac{7}{3}\right) = \frac{3}{5}$$

Clearly, if you put $\frac{3}{5} + \frac{7}{3}$ in the box, the equation will be true. Thus, you get

$$\begin{aligned} \frac{3}{5} - \left(-\frac{7}{3}\right) &= \frac{3}{5} + \frac{7}{3} \\ &= \frac{(3 \times 3) + (5 \times 7)}{5 \times 3} \qquad \text{(Why?)} \\ &= \frac{9 + 35}{15} = \frac{44}{15} \end{aligned}$$

For another example, take the problem

$$\frac{4}{3} - \frac{6}{5} = \square$$

The answer must be the solution to the equation $\square + \frac{6}{5} = \frac{4}{3}$, which is clearly

$$\begin{aligned}\frac{4}{3} + \left(-\frac{6}{5}\right) &= \frac{4}{3} + \frac{(-6)}{5} \\ &= \frac{[4 \times 5] + [3 \times (-6)]}{3 \times 5} \\ &= \frac{20 - 18}{15} = \frac{2}{15}\end{aligned}$$

Once again you added the opposite to find the result, and you used the rule for addition of rational numbers to do so.

The procedure is always the same. Reduce the rational numbers to the simplest form, change the second of the pair into its opposite, and add.

EXERCISE 32-2A

Perform the following subtractions, and for one of the problems show the meaning with the number line and the slide rule.

1. $\frac{3}{(-7)} - \left(-\frac{5}{3}\right)$
2. $\frac{(-6)}{(-3)} - \frac{2}{(-3)}$
3. $\frac{0}{9} - \frac{(-4)}{(-6)}$
4. $1{\cdot}83 - \frac{47}{50}$

EXERCISE 32-2B

A pupil tells you that 0 is the identity element for subtraction too. Is he correct? Why? How would you explain it to your class?

32-3 Multiplication

You should have no difficulty at this point understanding multiplication of rational numbers, since it follows the same pattern as did the other operations. You remember both how to multiply integers and how to multiply fractions. And if you put this knowledge together, you will see how to multiply rational numbers.

In the first place, if you multiply two numbers of the same sign, your result will be positive, and if you multiply two numbers of opposite sign, your result will be negative. This is as true for fractions as it is for integers.

Think back to our previous work. The rules for signs depended on the result that $(-1) \times a = -a$ for any integer a. Following exactly the same argument as was used there, you can show that

$$(-1) \times \frac{a}{b} = -\frac{a}{b} \quad \text{for any fraction } \frac{a}{b}$$

So you get the same rules for signs in working with fractions.

The actual result obtained for any product of rational numbers can be found by using the procedure you previously learned for fractions, and then using the rule about signs to find the correct sign for your answer. Take the two problems

$$\tfrac{1}{2} \times \tfrac{4}{3} \quad \text{and} \quad \tfrac{1}{2} \times (-\tfrac{4}{3})$$

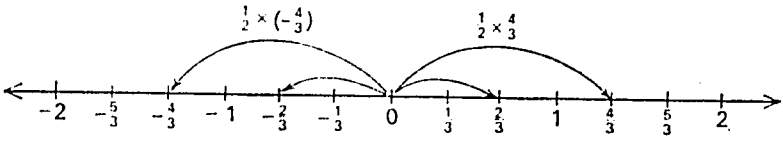
The results are clearly

$$\tfrac{1}{2} \times \tfrac{4}{3} = \tfrac{4}{6} = \tfrac{2}{3}$$

and

$$\tfrac{1}{2} \times (-\tfrac{4}{3}) = -\tfrac{4}{6} = -\tfrac{2}{3}$$

These two results can be pictured on the number line as follows:



In the same way, it is possible to obtain answers for the two problems which use the opposites to these fractions.

$$(-\tfrac{1}{2}) \times \tfrac{4}{3} = -\tfrac{4}{6} = -\tfrac{2}{3}$$

$$(-\tfrac{1}{2}) \times (-\tfrac{4}{3}) = \tfrac{4}{6} = \tfrac{2}{3}$$

There are several important and interesting facts worth noting about multiplication of rational numbers. The first is that the rule for multiplication of fractions is applied in exactly the same form to rational numbers. You remember that rule:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

If you will look at the examples just given, you will see that each one of them fits this rule perfectly, both in terms of the sign and in terms of the numbers. Thus, for example, take the last case and apply the rule:

$$\left(-\frac{1}{2}\right) \times \left(-\frac{4}{3}\right) = \frac{(-1)}{2} \times \frac{(-4)}{3}$$
$$= \frac{[(-1) \times (-4)]}{2 \times 3}$$
$$= \frac{4}{6} = \frac{2}{3}$$

You need to change each of the rational numbers into an equivalent form and then perform the multiplication, but you get the same answer.

The second important fact concerns the number 0. You learned before that if you multiply any fraction by 0 the result is again 0. The same thing is true for rational numbers, as you can easily see. You know already that you can write 0 as a rational number, for instance, $\frac{0}{1}$. Thus, you can write

$$0 \times \frac{(-3)}{2} = \frac{0}{1} \times \frac{(-3)}{2}$$
$$= \frac{0 \times (-3)}{1 \times 2} \times \frac{0}{2} = 0$$

The number 0 has the property that its product with any rational number is 0.

The third important fact concerns the number 1. Again recall what you learned before. If you multiply any fraction by 1, the number remains the same. Clearly you can get the same result for rational numbers, since you can write $1 = \frac{1}{1}$. Thus, for example,

$$1 \times \frac{(-3)}{2} = \frac{1}{1} \times \frac{(-3)}{2}$$
$$= \frac{1 \times (-3)}{1 \times 2} = \left(\frac{-3}{2}\right)$$

A fourth important fact concerns the product of a number with itself. This product is called the *square* of that number, and can be written as follows:

$$\frac{(-2)}{3} \times \frac{(-2)}{3} = \left[\frac{(-2)}{3}\right]^2$$

with a 2 as a right-hand superscript indicating that the number is multiplied by itself. The square of any number a is $a \times a$ and can be written a^2. In the example, the result is clearly

$$\left[\frac{(-2)}{3}\right]^2 = \frac{4}{9}$$

which is positive. In fact, the square of any non-zero rational number is positive. (Why?)

The fifth important point concerns what was previously called the reciprocal of a fraction. If you don't remember what a reciprocal is, look back to Section 21-19 in the chapter on fractions. Thus the reciprocal of the fraction $\frac{2}{3}$ is the fraction $\frac{3}{2}$, where the numerator of the first becomes the denominator of the second and vice versa. (Question: Does the fraction $\frac{0}{3}$ have a reciprocal? Further question: Is there such a fraction as $\frac{3}{0}$? The first question should be enough if you remember your earlier work, but the second question is put in to help your memory!) Consider once again a fraction and its reciprocal and look at their product. For example, for $\frac{2}{3}$ and $\frac{3}{2}$ you have

$$\tfrac{2}{3} \times \tfrac{3}{2} = \tfrac{6}{6} = 1$$

You found earlier that it is always true that the product of a fraction and its reciprocal is the number 1. Is it always true for any rational number also (except, of course, 0)? Look at this example:

$$\frac{(-2)}{3} \times \frac{3}{(-2)} = \frac{(-6)}{(-6)} = 1$$

In general, you can write, where neither a nor b is 0,

$$\frac{a}{b} \times \frac{b}{a} = \frac{ab}{ba} = 1$$

Thus, it is always true that the product of a non-zero fraction and its reciprocal is the number 1. This should remind you of addition, where the sum of a number and its opposite is the number 0.

EXERCISE 32-3A

Find the following products of rational numbers and show at least one of them using the number line.

1. $\frac{3}{2} \times \left(-\frac{1}{4}\right)$
2. $\frac{2}{(-3)} \times 1{\cdot}6$
3. $\left[\frac{2}{5} + \frac{(-6)}{8}\right] \times \left[\frac{7}{4} + \frac{(-3)}{(-4)}\right]$
4. $\left[\frac{(-5)}{7+4} - \frac{3}{(-8)}\right] \times \left[\frac{4}{3} + \frac{1}{(-2)}\right]$
5. $\left[\frac{3}{4} - \frac{(-6)}{(-8)}\right] \times \frac{2}{5}$
6. $\frac{(-6)}{5} \times \frac{(-5)}{6}$

EXERCISE 32-3B

Give three examples of each of the following:

1. A number and its reciprocal
2. A rational number and its square
3. A number and its opposite

EXERCISE 32-3C

Find the squares of the following numbers, and verify that these squares are positive.

1. $\frac{3}{5}$
2. $-\frac{4}{9}$
3. $\frac{(-1)}{2}$
4. $\frac{3}{(-2)}$

32-4 Division

The final topic of this chapter is division, which you remember is the inverse of multiplication. If you think of division in these terms, you should have no difficulty understanding how to divide numbers. In the first place, you learned that the same rule of signs applies as for multiplication, since division can be understood in terms of multiplication problems with missing factors. Thus, if you divide two rational numbers of the same sign, the result is positive, and if you divide two rational numbers of opposite sign, the result is negative.

In the second place, you learned to divide fractions by multi-

plying the first number by the reciprocal of the second, assuming of course, that the second is not 0. This same rule also applies to rational numbers. Thus, for example, to divide $\frac{3}{4}$ by $\frac{1}{2}$ is to solve the problem

$$\tfrac{3}{4} = \square \times \tfrac{1}{2}$$

and the result is clearly

$$\tfrac{3}{4} \div \tfrac{1}{2} = \tfrac{3}{4} \times \tfrac{2}{1} = \tfrac{6}{4} = \tfrac{3}{2}$$

Do you see that $\frac{3}{4} = \frac{3}{2} \times \frac{1}{2}$?

The same reasoning applies to the other possible variations in sign.

$$\frac{3}{4} = \square \times \frac{(-1)}{2}$$

$$\frac{(-3)}{4} = \square \times \frac{1}{2}$$

$$\frac{(-3)}{4} = \square \times \frac{(-1)}{2}$$

The results are clearly obtained in the same way:

$$\frac{3}{4} \div \frac{(-1)}{2} = \frac{3}{4} \times \frac{2}{(-1)} = \frac{6}{(-4)} = -\frac{3}{2}$$

$$\frac{(-3)}{4} \div \frac{1}{2} = \frac{(-3)}{4} \times \frac{2}{1} = \frac{(-6)}{4} = -\frac{3}{2}$$

$$\frac{(-3)}{4} \div \frac{(-1)}{2} = \frac{(-3)}{4} \times \frac{2}{(-1)} = \frac{(-6)}{(-4)} = \frac{3}{2}$$

Only in the case where the second rational number is 0 is this procedure impossible.

EXERCISE 32-4A

Work out the following division problems:

1. $\dfrac{(-3)}{4} \div \left[\dfrac{2}{3} + \dfrac{(-3)}{10}\right]$
2. $\dfrac{5}{8} \div \left(-\dfrac{5}{2}\right)$
3. $\left[\dfrac{1}{2} - \dfrac{2}{3}\right] \div \left[\dfrac{3}{4} + \dfrac{(-1)}{2}\right]$
4. $1{\cdot}8 \div \left(-\dfrac{3}{4}\right)$
5. $2{\cdot}4 \div \left(\dfrac{7}{5} - \dfrac{3}{2}\right)$
6. $\left[\dfrac{(-5)}{3} \times \dfrac{6}{(-10)}\right] \div \left[\dfrac{5}{4} - \dfrac{2}{8}\right]$

EXERCISE 32-4B

Prepare a set of revision exercises which will test your pupils' understanding of the material in this chapter. Include both word problems and strictly numerical problems.

EXERCISE 32-4C

Prepare an examination which covers the material in this unit on the rational numbers and which would enable you to see how much time you should spend on revision.

EXERCISE 32-4D

If one of your fellow students told you that this chapter didn't really teach him anything new but that he had learned it all before in earlier parts of the book, would you agree with him? Why?

EXERCISE 32-4E

Find answers to the following problems, and discuss the relation between multiplication and division which these answers show.

1. $\left[\frac{2}{3} \div \frac{(-4)}{5}\right] \times \frac{(-4)}{5}$

2. $\left[\left(-\frac{1}{5}\right) \div \frac{2 \times 4}{1 - 6}\right] \times \frac{8}{(-5)}$

3. $\left[1{\cdot}7 \times \left(-\frac{4}{9}\right)\right] \div \left[-\left(\frac{2}{3}\right)^2\right]$

4. $[(3 - 5) \times 1{\cdot}1] \div \frac{11}{10}$

Chapter 33

REVISION OF NUMBERS

33-1 Introduction

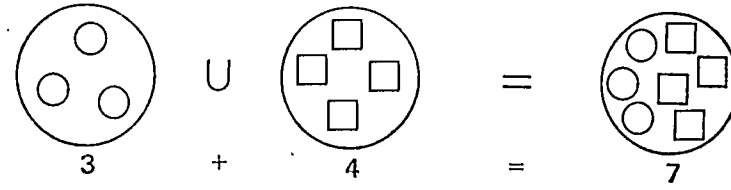
During this course, we have studied numbers. We have learned about many different kinds of numbers. And we are not finished yet. Let us look back over the path we have travelled, and see what the important milestones we have passed have been.

33-2 Sets and counting numbers

We started our study by talking about sets of things. We observed that sometimes the members of two sets could be placed in one-to-one correspondence with each other. We called such sets equivalent. Sometimes it was not possible to place the members of one set in one-to-one correspondence with the members of another set. Such sets are not equivalent. We agreed to say that any two sets which were equivalent to each other had the same number of members, and that any two sets which were not equivalent had different numbers of members. We agreed that if a set A was a subset of a set B and not equivalent to B, then A had a smaller number of members than B and B had a larger number of members than A. So every set has a number of members which is equal to the number of members of every set equivalent to the given set.

These numbers associated with sets were called the counting numbers. Because they were ordered, we were able to picture them on a line by marking equally spaced points, and we agreed that if a number a was less than a number b, the point marked a was to the left of the point marked b.

We learned to add numbers by forming the union of two disjoint sets, whose members we had previously counted, and counting the members of this union.



And we saw that our picture of the number line could be useful in this addition, since we could get the sum of two numbers by stepping along to the right on the number line.

33-3 The empty set and zero

The counting numbers were sufficient to take care of our number needs as long as the answers to our problems could be represented by sets of things. But if a man has a set of 3 shillings and pays 3 shillings for a basket, the money he has left can hardly be described as a set of shillings. We found it convenient to describe the money he has left as the empty set. And then we invented the number 0 to describe the number of members of the empty set. We then had two sets of numbers: the set of counting numbers, and the set whose only member is zero. The union of these two sets we gave the name of whole numbers.

On the number line, the point labelled 0 was our starting point and was to the left of all the points for the counting numbers. And we found that we could still get the sum of two numbers by stepping to the right on the number line, with the understanding that if we were adding 0, we would step 0 units to the right; that is, we would not take a step.

The set of counting numbers is a subset of the set of whole numbers, and we found that all the problems involving counting numbers could still be done by working with the set of whole numbers; also, we could work problems involving the empty set. The extension from counting numbers to whole numbers was not a large one—the whole numbers include only one new member—but it is an important one.

33-4 Fractions

We learned to multiply two whole numbers, and interpreted multiplication as repeated addition. We then defined division

in terms of multiplication. We said that if $3 \times \square = 6$, $\square = 6 \div 3 = 2$, and more generally, if $a \times \square = b$, $\square = b \div a$. But sometimes there is no whole-number answer. For example, in the problem $3 \times \square = 7$, there is no whole-number solution. In such cases, we agreed to invent new numbers, called fractions, which would have the required properties. In the example $3 \times \square = 7$, $\square = 7 \div 3$, and we agreed to call this answer the fraction $\frac{7}{3}$. So $3 \times \frac{7}{3} = 7$. More generally, if $a \times \square = b$, $\square = b \div a = \frac{b}{a}$, and $a \times \frac{b}{a} = b$. These new numbers, the fractions, included the whole numbers. For if a is a whole number, $1 \times \square = a$ has the solution $\square = \frac{a}{1}$. It also has the solution a. So $\frac{a}{1} = a$, and every whole number is a fraction. The set of whole numbers is thus a subset of the set of fractions. Since some fractions are not whole numbers, the set of fractions is an extension of the set of whole numbers.

We saw that we could not assign a meaning to division by 0, and so we were not able to assign a meaning to the fraction $\frac{a}{b}$ if $b = 0$. In the fraction $\frac{a}{b}$, a can represent any member of the set of whole numbers and b any member of the set of counting numbers.

But we have also seen that not all such fractions are different. In fact, $\frac{a}{b} = \frac{ka}{kb}$ for any counting number k.

We learned to add any two fractions, obtaining

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$$

We were able to assign an order to the fractions, and we were able to picture the fractions on the number line, with points between the whole numbers along with the whole numbers themselves. Again, $\frac{a}{b} < \frac{c}{d}$ if $\frac{a}{b}$ is to the left of $\frac{c}{d}$.

We learned to multiply any two fractions, and saw that $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. And we learned that $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

33-5 Integers

Just as the process of division led to the extension of the whole numbers to the fractions, so the process of subtraction also led to an extension of the whole numbers to the integers. If a represents a whole number, we defined a new number, called neg a, and later introduced the notation $-a$ to represent it, which had the property that $a + (-a) = 0$. Since $a + \square = 0$ has, by the definition of subtraction, the solution $\square = 0 - a$, $-a$ is the number obtained by subtracting a from 0.

These new numbers were called the negative integers. The counting numbers were relabelled the positive integers. The complete set of integers is the union of the negative integers, the positive integers and zero.

We were able to establish an order for the integers. We were able to picture the integers on the number line by extending the number line to the left of 0. On the extended line, the smaller of two numbers was still pictured to the left of the larger. We were able to use the line for adding positive integers, as before, by stepping to the right. But we found that stepping to the left on the number line was required for adding negative integers. Since this was the same as subtracting positive integers, we concluded that

$$a + (-b) = a - b$$

Since negative integers may be added on the number line by stepping to the left of 0 in the same manner that positive integers may be added on the right of 0, we concluded that

$$(-a) + (-b) = -(a + b)$$

We saw that the opposite of a negative integer was a positive integer.

The statement $a + (-b) = a - b$ assures us that every subtraction problem of the form $a - b$ can be changed to an addition problem of the form $a + (-b)$.

We learned to multiply two integers and discovered that $(-1) \times a = -a$, $(-a) \times b = -(a \times b)$ and $(-a) \times (-b) = (a \times b)$, for a and b any two integers.

33-6 Rational numbers

Just as the fractions permitted the naming of some of the

points between the whole numbers to the right of 0 on the number line, so we needed to name points between the negative integers to the left of 0. These new numbers, the opposites of the non-zero fractions, we called the negative fractions, and, together with the fractions, they make the set of rational numbers.

33-7 Summary

We now have a very large set of numbers to work with—the set of rational numbers. In the next chapter, we shall revise the properties of operations on these numbers. We have seen that the set of rational numbers is composed of several important subsets. Sometimes we do not need this huge set of numbers when we are solving our problems—sometimes one of the subsets is big enough to take care of what we need to do. But it is comforting to know that with this large set of rational numbers, we can now perform any of the four basic operations on any numbers with just one exception: we are not permitted to divide by 0.

In Unit VII we are going to expand our number system again and for the last time in this course. But we shall still be unable to give meaning to division by 0.

EXERCISE 33-7A

List as many subsets of the set of rational numbers which we have studied and named as you can.

Chapter 34

REVISION OF PROPERTIES OF OPERATIONS

In Chapter 13 we gave a summary of the properties of operations when the operations were performed with whole numbers. Since that chapter, we have studied operations on numbers other than whole numbers; for instance, on integers and rational numbers. We are now, therefore, in a position to expand our summary to include properties of operations on integers and rational numbers.

34-1 Closure under addition and multiplication

A set of numbers is closed under addition if the sum of any two of the members is also a member of the set. It is closed under multiplication if the product of any two of the members is a member of the set.

1. Since $a + b$ is a whole number if a and b are whole numbers, the set of whole numbers is closed under addition. This we know already.

Since $a + b$ is an integer if a and b are integers, the set of integers is closed under addition. For example, $(-8) + 12$ is an integer. Are the rational numbers closed under addition? Is $\frac{p}{q} + \frac{m}{n}$ a rational number whenever $\frac{p}{q}$ and $\frac{m}{n}$ are rational numbers? You see this is so, because

$$\frac{p}{q} + \frac{m}{n} = \frac{pn + qm}{qn} = \frac{\text{an integer}}{\text{another integer}} = \text{a rational number}$$

(Of course, we must be sure that the denominator qn cannot be zero. How do we know that $q \neq 0$ and $n \neq 0$? Could qn be 0?)

2. Since $a \times b$ is a whole number whenever a and b are whole numbers, the set of whole numbers is closed under multiplication.

Since $a \times b$ is an integer whenever a and b are integers, the set of integers is closed under multiplication.

For example, $(-8) \times 12 = -96$, which is an integer.

Is the set of rational numbers closed under multiplication? Is $\frac{p}{q} \times \frac{m}{n}$ a rational number if $\frac{p}{q}$ and $\frac{m}{n}$ are? Yes, because

$$\frac{p}{q} \times \frac{m}{n} = \frac{pm}{qn} = \frac{\text{an integer}}{\text{another integer}} = \text{a rational number}$$

since $qn \neq 0$.

3. The set of whole numbers is *not* closed under subtraction, since, for example, $2 - 7$ is not a whole number.

The set of integers *is* closed under subtraction. So is the set of rational numbers. Can you give examples illustrating this?

4. The set of whole numbers is not closed under division. For example, $2 \div 7$ is not a whole number.

Is the set of integers closed under division? Is $(-2) \div 7$ an integer, for example?

If we divide any rationla number $\frac{p}{q}$ by any non-zero rational number $\frac{m}{n}$, will we always get a rational number? Let us see.

$$\frac{p}{q} \div \frac{m}{n} = \frac{pn}{qm} = \frac{\text{an integer}}{\text{an integer}}$$

Could $qm = 0$? Only if $q = 0$ or $m = 0$. But $q = 0$ is not allowed and if $\frac{m}{n} \neq 0$, m cannot be 0 either. We conclude, then, that the set of all rational numbers *except zero* is closed under division.

EXERCISE 34-1A

1. State under which operations (addition, subtraction, multiplication, division), if any, each of the following sets is closed.
   a. $\{2, 4, 6, 8\}$
   b. The set of all rational numbers
   c. $\{12, 14, 16, 18, \ldots\}$
   d. $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$
   e. The set of all integers
   f. The set of all odd numbers

34-2 Commutative property of addition and multiplication

1. The order in which two whole numbers are added will not affect their sum. That is, $a + b = b + a$ is true when a and b are whole numbers. Is it true when a and b are integers? Yes, it is. For example, $(-8) + 3 = 3 + (-8)$, each side being equal to -5. This is not new. Is this true for rational numbers? Is it true that

$$\frac{p}{q} + \frac{m}{n} = \frac{m}{n} + \frac{p}{q}$$

that is, that

$$\frac{pn + qm}{qn} = \frac{mq + np}{nq}$$

Remember that p, q, m and n are integers. (Which of them can not be 0?) Then $qn = nq$. Why? And $pn + qm = mq + np$. Why?

So we see that the commutative property of addition holds for rational numbers.

2. The order in which two whole numbers are multiplied does not affect the product. That is, $a \times b = b \times a$ when a and b are whole numbers. It is true also for integers, as it is for rational numbers. For example, $(-8) \times 3 = 3 \times (-8)$ since each side is equal to -24, and $(-\frac{1}{8}) \times \frac{2}{3} = \frac{2}{3} \times (-\frac{1}{8})$ since each side is equal to $-\frac{1}{12}$. Can you show that

$$\frac{p}{q} \times \frac{m}{n} = \frac{m}{n} \times \frac{p}{q}$$

in all cases?

We found that subtraction and division do not have the commutative property. Check these statements of subtraction and division to see that neither operation has this property.

Is $(-\frac{4}{5}) - \frac{2}{5} = \frac{2}{5} - (-\frac{4}{5})$?

Is $(-\frac{4}{5}) \div \frac{2}{5} = \frac{2}{5} \div (-\frac{4}{5})$?

34-3 Associative property of addition and multiplication

1. $(a + b) + c = a + (b + c)$ is true for whole numbers. Is it true for integers? Suppose you want to add -8, 7 and -6.

How would you proceed? You could add -8 and 7 and then add -6 to the sum; or you could add to -8 the sum of 7 and -6. But you know you would get the same sum for each: $(-8 + 7) + (-6) = -8 + [7 + (-6)]$. Of course, you would do the same if the given numbers had been rational numbers. So $(a + b) + c = a + (b + c)$ if a, b and c are rational numbers.

2. $a \times (b \times c) = (a \times b) \times c$ is true for whole numbers. Is it true for integers? Is

$$(-8) \times (5 \times 11) = [(-8) \times 5] \times 11?$$

Yes, since each side is equal to -440.

Is it true for rational numbers? Is $\frac{5}{8} \times (\frac{4}{3} \times \frac{2}{3})$ the same as $(\frac{5}{8} \times \frac{4}{3}) \times \frac{2}{3}$? Yes, for each is equal to $\frac{5}{9}$.

It is not difficult to show that for rational numbers,

$$\left(\frac{p}{q} \times \frac{m}{n}\right) \times \frac{r}{s} = \frac{p}{q} \times \left(\frac{m}{n} \times \frac{r}{s}\right)$$

In fact, each is equal to $\frac{pmr}{qns}$.

Subtraction and division do not have this property as you can see by answering the questions below:

Is $(-8) - [3 - (-2)] = [(-8) - 3] - (-2)$?

Is $\frac{1}{4} - (3 - \frac{1}{2}) = (\frac{1}{4} - 3) - \frac{1}{2}$?

Is $[\frac{9}{8} \div (-\frac{1}{2})] \div 4 = \frac{9}{8} \div [(-\frac{1}{2}) \div 4]$?

Is $2 \div [(-\frac{1}{2}) \div \frac{1}{8}] = [2 \div (-\frac{1}{2})] \div \frac{1}{8}$?

34-4 Distributive property

1. $a \times (b + c) = (a \times b) + (a \times c)$ is true when a, b and c are whole numbers. Is this same distributive property true for integers and rational numbers? Check these statements, which illustrate the distributive property of multiplication over addition:

$$(-8) \times [5 + (-2)] = [(-8) \times 5] + [(-8) \times (-2]$$

$$\frac{2}{3} \times [\frac{5}{8} + (-\frac{1}{2})] = (\frac{2}{3} \times \frac{5}{8}) + [\frac{2}{3} \times (-\frac{1}{2})]$$

In the first, each side equals -24, so the statement is true. In the second, each side equals $\frac{1}{12}$, so the statement is true.

2. $a \times (b - c) = (a \times b) - (a \times c)$ is known to be true for

whole numbers. Is it true for integers and rational numbers? Check these statements:

$$(-8) \times [5 - (-2)] = [(-8) \times 5] - [(-8) \times (-2)]$$
$$\tfrac{2}{3} \times [\tfrac{5}{8} - (-\tfrac{1}{2})] = (\tfrac{2}{3} \times \tfrac{5}{8}) - [\tfrac{2}{3} \times (-\tfrac{1}{2})]$$

In the first, each side equals -56, so the statement is true. In the second, each side equals $\frac{3}{4}$, so this statement is also true.

Suppose the statement about multiplication and subtraction had been written in general form. We would like to show that

$$\frac{p}{q} \times \left(\frac{m}{n} - \frac{r}{s}\right) = \left(\frac{p}{q} \times \frac{m}{n}\right) - \left(\frac{p}{q} \times \frac{r}{s}\right)$$

If we work out both sides, we find that the left-hand and right-hand sides are, respectively,

$$\frac{pms - pnr}{qns} \quad \text{and} \quad \frac{pmqs - qnpr}{qnqs}$$

Do you see that these expressions represent the same rational number? Thus the statement is true. In a quite similar way, we could prove that the statement of the distributive property is true.

34-5 Properties of zero and one

We have learned that 0 has the property that

$$0 + a = a + 0$$

when a is any whole number, and that 1 has the property that $1 \times a = a \times 1 = a$, when a is any whole number.

Do 0 and 1 have these properties when a is any integer or any rational number? Yes, they do. We have met these operations when working on integers and rational numbers. For example,

$$0 + (-8) = (-8) + 0 = -8$$
$$0 + \tfrac{1}{2} = \tfrac{1}{2} + 0 = \tfrac{1}{2}$$
$$1 \times (-8) = (-8) \times 1 = -8$$
$$1 \times \tfrac{2}{3} = \tfrac{2}{3} \times 1 = \tfrac{2}{3}$$

In general,

$$\frac{p}{q} + 0 = 0 + \frac{p}{q} = \frac{0}{1} + \frac{p}{q} = \frac{(0 \times q) + (1 \times p)}{1 \times q} = \frac{p}{q}$$

and $\frac{p}{q} \times 1 = 1 \times \frac{p}{q} = \frac{1}{1} \times \frac{p}{q} = \frac{1 \times p}{1 \times q} = \frac{p}{q}$

So 0 is called the *identity element for addition*, and 1 is called the *identity element for multiplication*. (Recall that we have seen that 0 is not an identity element for subtraction, nor is 1 an identity element for division.)

There is another important property of 0. We know that the product of any whole number and 0 is 0. Is this true when the other factor is an integer or a rational number? Are

$$0 \times (-8) = 0 \quad \text{and} \quad 0 \times (-\tfrac{1}{4}) = 0$$

true? From our study of integers and rational numbers, this statement is true: Any number multiplied by 0 will give 0 as answer.

This property of 0 may be written as follows: If a is any rational number, then

$$a \times 0 = 0 \times a = 0$$

34-6 Opposites and reciprocals

Much work has been done with integers and rational numbers on the number line, and you should by now be familiar with such pairs of numbers on the number line as -3 and 3, $-\frac{1}{2}$ and $\frac{1}{2}$, $-a$ and a. One number in each pair is to the right and the other to the left of the zero point—unless they are both 0—and both are at equal distances from it. The sum of the numbers in each pair is zero. We have called the numbers in such a pair *opposites*.

Given a rational number which is not 0, we can find its *reciprocal*. For example, the reciprocal of 7 is $\frac{1}{7}$, of $-\frac{8}{9}$ is $-\frac{9}{8}$, of $\frac{1}{9}$ is 9. The most important property which we discovered about a number and its reciprocal is that the product of the two is always 1. Always keep in mind that the number 0 has no reciprocal.

34-7 Inverse operations

In Chapters 9 and 12 it was mentioned that addition and subtraction are inverses of each other and that multiplication and division have a similar relation to each other. This can be

shown to be true for integers and rational numbers as it was found to be true for whole numbers. Here are the statements expressing these relations:

$$(a - b) + b = a \quad \text{and} \quad (a + b) - b = a$$

if a and b are any rational numbers, and

$$(a \times b) \div b = a \quad \text{and} \quad (a \div b) \times b = a$$

if a and b are any rational numbers (with $b \neq 0$).

34-8 Properties summarized

In this summary, the letters a, b and c represent any rational numbers. Since an integer is a particular kind of rational number, the properties listed will also apply to integers.

(*a*) *Closure properties*

$a + b$ is a rational number.
$a \times b$ is a rational number.

(*b*) *Commutative properties of addition and multiplication*

$a + b = b + a$
$a \times b = b \times a$

(*c*) *Associative properties of addition and multiplication*

$a + (b + c) = (a + b) + c$
$a \times (b \times c) = (a \times b) \times c$

(*d*) *Distributive properties*

$a \times (b + c) = (a \times b) + (a \times c)$
$a \times (b - c) = (a \times b) - (a \times c)$

(*e*) *Properties of zero*

$a + 0 = a$
$a \times 0 = 0$

(*f*) *Property of one*

$a \times 1 = a$

(*g*) *Opposites and reciprocals*

$a + (-a) = 0$
$a \times \frac{1}{a} = 1 \quad (\text{if } a \neq 0)$

(*h*) *Inverse operations*

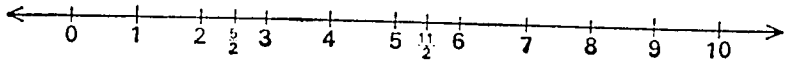
$(a - b) + b = a$
$(a + b) - b = a$
$(a \times b) \div b = a \quad (\text{if } b \neq 0)$
$(a \div b) \times b = a \quad (\text{if } b \neq 0)$

Chapter 35

ORDER FOR RATIONAL NUMBERS

35-1 Order on the number line for positive rational numbers

Do you remember how we described "order" for the set of whole numbers in Chapter 17 and "order" for fractions in Chapter 22? Let us see. Here is a number line.



You will remember that we said that a number a is "greater than" a number b if a is to the *right* of b on the number line. For example, 5 is *greater than* 3 and we see on the number line that 5 is to the *right* of 3. Similarly, $\frac{11}{2}$ is to the *right* of 4 and so $\frac{11}{2}$ is *greater than* 4. In symbols, we write

$$a > b, \quad 5 > 3, \quad \tfrac{11}{2} > 4$$

In a similar way we described the idea of "less than" by saying that a number b is *less than* a number a, if b is to the *left* of a on the number line. Thus 3 is *less than* 5, because it is to the *left* of 5 on the number line. Similarly 4 is to the *left* of $\frac{11}{2}$ on the number line and so 4 is *less than* $\frac{11}{2}$. In symbols, we write

$$b < a, \quad 3 < 5, \quad 4 < \tfrac{11}{2}$$

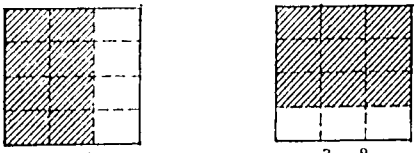
You see that $a > b$ and $b < a$ really mean exactly the same thing. Both inequalities say that a is to the *right* of b on the number line, and this is the same as saying that b is to the *left* of a on the number line.

Exercise 35-1A

1. Draw a number line and locate on it by dots the following:
   a. six consecutive whole numbers greater than 8
   b. four even whole numbers less than 16
   c. the three smallest fractions greater than 3 and having 3 for a denominator

2. a. Find the smallest whole number greater than 6
b. Find the greatest whole number less than 10
c. Find the three largest and the three smallest whole numbers less than $\frac{22}{3}$

In Chapters 19 and 22, you learned to compare two fractions by putting them on the number line. It is not always easy to locate two fractions accurately on the number line when they are very close together. We therefore have to learn another way of finding out which of two fractions is the greater. Suppose we wish to find out which of $\frac{2}{3}$ and $\frac{3}{4}$ is the greater fraction without locating them on the number line. How shall we do it? One way would be to draw a diagram to represent each fraction and then compare the two diagrams. The two fractions $\frac{2}{3}$ and $\frac{3}{4}$ are shown in the diagrams below. Clearly the second picture represents the larger fraction, and so we say that $\frac{3}{4}$ is greater than $\frac{2}{3}$; that is, $\frac{3}{4} > \frac{2}{3}$.



$\frac{2}{3} = \frac{8}{12}$ $\qquad$ $\frac{3}{4} = \frac{9}{12}$

Another method of comparing two fractions is to express both of them with the same denominator and then compare their numerators. Consider the above example where we compared $\frac{2}{3}$ and $\frac{3}{4}$. We may write $\frac{2}{3} = \frac{8}{12}$ and $\frac{3}{4} = \frac{9}{12}$. Hence, by comparing the numerators we see that $9 > 8$ and, therefore, that $\frac{3}{4} > \frac{2}{3}$. Of course we could also have arrived at the same result by taking the difference between $\frac{3}{4}$ and $\frac{2}{3}$. That is $\frac{3}{4} - \frac{2}{3} = \frac{9}{12} - \frac{8}{12} = \frac{1}{12}$, which is positive.

EXERCISE 35-1B

1. For each pair of rational numbers, determine their order by
(i) locating them on the number line
(ii) writing each pair with the same denominator
(iii) subtracting one from the other
In each case, say which is larger.
a. $\frac{4}{5}$ and $\frac{8}{10}$ **b.** $\frac{3}{5}$ and $\frac{3}{4}$ **c.** $\frac{9}{2}$ and 5 **d.** $\frac{5}{3}$ and $\frac{4}{3}$

2. Arrange the following rational numbers in order, starting with the least and ending with the greatest in each case.

a. $\frac{2}{3}, \frac{1}{2}, \frac{3}{4}, \frac{2}{5}, \frac{5}{6}$
b. $\frac{5}{3}, \frac{3}{2}, 2\frac{1}{2}, 3, \frac{11}{4}$
c. $3\frac{1}{2} + 2\frac{3}{4}; 2\frac{1}{2} + 3\frac{2}{3}; 1\frac{1}{6} + 4\frac{1}{2}; 5\frac{3}{4} + \frac{1}{8}$

Order for decimal fractions

When we wanted to find out which of two fractions was greater, we expressed both fractions as new fractions with the same denominator and then compared their numerators. A similar method is found useful with decimals. For example, 0·5 may be written as $\frac{5}{10}$ and 0·05 as $\frac{5}{100}$. In order to write $\frac{5}{10}$ as a fraction with 100 as a denominator, we have to multiply both numerator and denominator by 10. This gives

$$\frac{5}{10} = \frac{5 \times 10}{10 \times 10} = \frac{50}{100}$$

This means that $\frac{50}{100} > \frac{5}{100}$; that is, $0{\cdot}5 > 0{\cdot}05$.

Here is another example. Which is greater, 0·6 or 0·55? We write $0{\cdot}6 = \frac{6}{10}$ and 0·55 as $\frac{55}{100}$. Since $\frac{6}{10}$ has 10 for denominator, we write $\frac{6}{10} = \frac{6 \times 10}{10 \times 10} = \frac{60}{100}$. Hence, $\frac{60}{100} > \frac{55}{100}$; that is $0{\cdot}6 > 0{\cdot}55$.

EXERCISE 35-1C

1. Arrange the following numbers in order, starting with the least and ending with the greatest:

0·35, 1·35, 3·5, 1·035, 10·35, 0·035, 17·5, 2·25, 2

2. Insert the correct inequality sign in each box below.

a. $\frac{5}{2}$ □ 3·5	**b.** 1·7 □ 1·75	**c.** 0 □ 2·5
d. 3·8 □ 0·38	**e.** 0·75 □ 0·7	**f.** 0·75 □ 0·7
g. 2·5 □ 4	**h.** 5 □ 4·8	**i.** $\frac{22}{5}$ □ 4·2

35-2 Order on the number line for the set of rational numbers

Just as we did for whole numbers and for positive fractions, we use the term *is greater than* with rational numbers to mean "is to the right of" on the number line. If a and b are rational numbers, "a is greater than b", written $a > b$, is interpreted to mean that a is "to the right of" b on the number line.

In a similar way, we use *is less than* for the set of all the rational numbers to mean "is to the left of" on the number line. If a and b are rational numbers, "b is less than a" is written $b < a$.

We may restate the above point by saying that if $a > b$, then on the number line we have to move to the *right* from the point which represents b to get to the point which represents a. This represents the addition of a positive number p to b to get a. We may write,

if $\quad a > b \quad$ then $\quad a = b + p \quad$ and $\quad a - b = p$

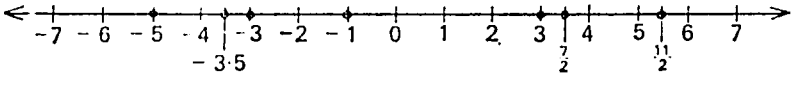
where p is a positive number.

In a similar way, if $a < b$, then we have to move to the *left* from the point which represents b to get to the point which represents a. That is,

if $\quad a < b \quad$ then $\quad a = b + q \quad$ and $\quad a - b = q$

where q is a negative number.

Here is a number line with some rational numbers represented on it.



We easily see that 3 is to the right of -1; $3 > -1$ and $3 = -1 + 4$. Similarly, $-3{\cdot}5 > -5$ and $-3{\cdot}5 = -5 + 1{\cdot}5$, but $\frac{7}{2} < \frac{11}{2}$ and $\frac{7}{2} = \frac{11}{2} + (-2)$.

On a number line, all negative rational numbers are to the left of 0, and so for any negative number k we may write $k < 0$. Also, all positive rational numbers are to the right of 0, and so for any positive number m we may write $m > 0$.

EXERCISE 35-2A

1. Determine which of the following statements are true and which are false. For the true statements of the form $a > b$, find a positive rational number p such that $a = b + p$. For the true statements of the form $a < b$, find a negative rational number q such that $a = b \times q$.

a. $16 < -32$ **b.** $-16 < 32$
c. $-1 < -3$ **d.** $1{\cdot}6 < -3{\cdot}2$
e. $-1{\cdot}6 > 3{\cdot}2$ **f.** $0 > 1$
g. $25 < -26$ **h.** $-25 < 26$
i. $36 > 0$ **j.** $-36 > 0$
k. $-3 > -6$ **l.** $0 < -\frac{3}{2}$
m. $-\frac{3}{2} < 0$ **n.** $-4 + (-2) < -2 + 2$
o. $-5 + 1 > 2 + 1$

Insert the correct inequality sign in the following statements:
a. $4 + (-3) \square -4 + 8$
b. $4 + 8 \square -4 + (-8)$
c. $6 + (-1) \square 6 + 2$
d. $-\frac{3}{2} + (-\frac{5}{2}) \square -\frac{3}{2} + \frac{5}{2}$
e. $-6 + (-1) \square -6 + 2$
f. $-3{\cdot}5 + 3{\cdot}5 \square 3{\cdot}5 + 3{\cdot}5$
g. $-\frac{1}{4} \square (-\frac{1}{3})$

2. For each pair of numbers, determine their order. Write a statement involving the sign $>$ for each pair. Then for each statement $a > b$, find a positive rational number p which makes the statement $a = b + p$ true.

3. **a.** $\frac{4}{5}$ and $-\frac{8}{10}$ **b.** $2{\cdot}5$ and $-5{\cdot}5$ **c.** $-\frac{3}{5}$ and $(-\frac{3}{4})$
d. $-\frac{5}{3}$ and $-\frac{4}{3}$ **e.** $-\frac{9}{2}$ and -5 **f.** $2{\cdot}25$ and $3{\cdot}75$

4. Locate on different number lines
a. four negative integers greater than -6
b. five negative integers less than -5,
c. the six greatest integers less than 4,
d. four negative integers less than 0.

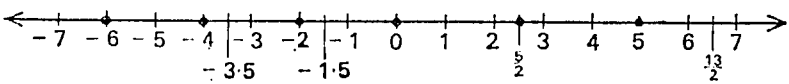
5. From each of the following statements about equality, deduce the corresponding statements about order, using first the sign $<$ and then the sign $>$. For example, from $-3 = -5 + 2$, we see first that $-5 < -3$, and then that $-3 > -5$.

a. $6 = -3 + 9$ **b.** $-2 = -8 + 6$
c. $0 = -4 + 4$ **d.** $12 = 9 + 3$
e. $3{\cdot}25 = -3 + 6{\cdot}25$ **f.** $\frac{3}{4} = \frac{2}{3} + \frac{1}{12}$
g. $-9 = -12 + 3$ **h.** $2 = -8 + 10$

35-3 "Between" for rational numbers

In Chapter 17 we saw that given any three whole numbers a, b and c, we can say which one lies "between" the other two numbers. For example, on a number line we see that 4 is to the left of 7 and so $4 < 7$; also, 10 is to the right of 7 and so $10 > 7$. We can write $4 < 7 < 10$.

What do we mean by "between" for rational numbers? A number line will help us. This number line shows that



$-6 < -4$ and $-4 < -2$. We therefore say that -4 is *between* -6 and -2, and we write $-6 < -4 < -2$. In a similar way, we see from the number line that $\frac{5}{2}$ is between 0 and 5, and we write $0 < \frac{5}{2} < 5$.

EXERCISE 35-3A

1. Arrange the rational numbers a, b, c and d on the number line, on the assumption that a is between b and c, and b is between a and d. Note that there are two possible arrangements.

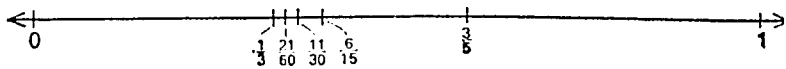
2. Write the set whose members are
   a. the integers between -5 and 5,
   b. the negative integers greater than -4,
   c. the positive integers less than 7,
   d. the integers between 0 and -4,
   e. all the fractions between 1 and 3 with a denominator of 5.

3. Find a rational number between each of the following pairs of rational numbers:
   a. $\frac{1}{3}$ and 1 **b.** $-\frac{1}{3}$ and -1 **c.** $\frac{1}{5}$ and $-\frac{1}{7}$
   d. $-\frac{2}{5}$ and $-\frac{1}{2}$ **e.** $5\frac{1}{2}$ and $5\frac{3}{4}$ **f.** $-\frac{9}{2}$ and $-\frac{8}{3}$

From your answers to Question 3, it will be clear to you that you can find more than one rational number between any pair of rational numbers. For example, what was the rational number you found between $\frac{1}{3}$ and 1? One possibility is $\frac{2}{3}$. What about $\frac{3}{4}$ or $\frac{1}{2}$ or $\frac{3}{5}$? Can you find any more rational numbers between $\frac{1}{3}$ and 1?

We have seen that given any integer we can always tell which integer precedes it or comes after it. Given any three integers we can always tell which is between the other two. The example above seems to suggest that we cannot talk of *the* rational number between $\frac{1}{3}$ and 1, because there are many such rational numbers.

Can we talk of the rational number which follows a rational number? For example, can we find the next rational number after $\frac{1}{3}$? Suppose we take $\frac{3}{5}$, one of the answers we suggested above. We represent both these points on the number line.



We observe that $\frac{1}{3} = \frac{5}{15}$ and $\frac{3}{5} = \frac{9}{15}$ so that $\frac{6}{15}$, $\frac{7}{15}$, $\frac{8}{15}$ are all closer to $\frac{1}{3}$ than $\frac{3}{5}$. If we choose $\frac{6}{15}$ as the next rational number after $\frac{1}{3}$, we note that $\frac{1}{3} = \frac{10}{30}$ and $\frac{6}{15} = \frac{12}{30}$ and so $\frac{11}{30}$ is nearer to $\frac{1}{3}$ than $\frac{6}{15}$. Once more we observe that $\frac{1}{3} = \frac{20}{60}$ and $\frac{11}{30} = \frac{22}{60}$, so $\frac{21}{60}$ is nearer $\frac{1}{3}$ than $\frac{11}{30}$. We may continue this process as long as we wish, so that there is no rational number "next" after a given rational number. A similar argument would show that we cannot identify the rational number "just before" a given rational number. All this suggests to us that between any two rational numbers, no matter how close together they are, there is always a third rational number.

EXERCISE 35-3B

Find four rational numbers between each pair of rational numbers in Question 3 of Exercise 35-3A.

35-4 Two basic properties of order for rational numbers

1. *The Comparison (or Trichotomy) Property*

If a and b are any two rational numbers, then one and only one of the following is true:

$$a > b \quad a = b \quad a < b$$

To see this, find $(a - b)$. The rational number $(a - b)$ is either positive, negative or zero. If $(a - b)$ is positive, then $a > b$. If $(a - b)$ is negative, then $a < b$. If $(a - b)$ is zero, then $a = b$. In the case $a = b$, a and b name the same number.

EXERCISE 35-4A

Put the correct inequality or equality sign into the boxes to make each statement true.

1. $-\frac{6}{5} \square \frac{7}{2}$

2. $-\dfrac{30}{10} \square \dfrac{(-9)}{3}$

3. $5 + \frac{1}{2} \square 5 - \frac{1}{2}$

4. $\frac{2}{3} - \frac{1}{4} \square \frac{1}{12} + \frac{1}{6}$

5. $-7 \square -\frac{14}{4}$

6. $0 \square \dfrac{-1}{-1}$

7. $8 \square 2 - (-6)$

8. $-6 \square -\frac{19}{3}$

2. *The Transitive Property*

Let us look again at the relation $4 < 7$ and $7 < 10$, which we discussed earlier. What conclusion can we draw as to the relation between 4 and 10? Do you see that 4 *is less than* 10? You can verify easily that $4 < 10$ by looking at the number line.

Here is another example. We know that $-6 < 2$ and $2 < 6$. Of course, you see straight away that $-6 < 6$, and again you can check the conclusion by looking at the number line.

As another example, we have $-\frac{1}{2} < \frac{1}{4}$ and $\frac{1}{4} < \frac{1}{2}$. We thus see that $-\frac{1}{2} < \frac{1}{2}$.

Now let a, b and c be any three rational numbers such that $a < b$ and $b < c$. What can be said about the relation between a and c? You can see that $a < c$. This property is called the *transitive property*. A proof of it is sketched following Exercise 35-4C, Question 2. Formally, it says, if a, b and c are any three rational numbers and if $a < b$ and $b < c$, then $a < c$.

EXERCISE 35-4B

1. In each of the following groups of rational numbers, determine their order.

a. 2·5, −5·2, 0 **b.** $-\frac{1}{4}$, $-\frac{1}{2}$, $-\frac{1}{3}$
c. $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{3}$ **d.** 25, 2·05, 2·25
e. 5, −6, 4 **f.** $\frac{4}{3}$, $-\frac{3}{4}$, $\frac{3}{4}$
g. $-\frac{9}{5}$, −2, $-\frac{12}{5}$

The transitive property often makes it easy for us to compare some pairs of fractions. Suppose we wish to find out which of the two fractions $\frac{79}{17}$ and $\frac{97}{19}$ is the greater. By the method described earlier in this chapter, we first write the fractions with the same denominator and compare the numerators. Thus, $\frac{79}{17} = \frac{1501}{323}$, $\frac{97}{19} = \frac{1649}{323}$. By comparing the numerators 1649 and 1501 we see that $\frac{97}{19} > \frac{79}{17}$. You of course know how we arrived at this result. It has involved the finding of the three products 19 × 17, 97 × 17 and 79 × 19, and you will agree that this is a lot of work.

Can the transitive property help us? If we can find a rational number which is between $\frac{79}{17}$ and $\frac{97}{19}$, then we can easily compare the two fractions. We note that $\frac{79}{17} < \frac{85}{17}$; that is, $\frac{79}{17} < 5$. Also $\frac{97}{19} > \frac{95}{19}$; that is, $\frac{97}{19} > 5$. If in the transitive property we take $a = \frac{79}{17}$, $b = 5$ and $c = \frac{97}{19}$, we see that $a < c$; that is, $\frac{79}{17} < \frac{97}{19}$.

EXERCISE 35-4C

1. Determine the order of the following fractions.

a. $\frac{287}{29}$ and $\frac{340}{31}$ **b.** $-\frac{79}{17}$ and $-\frac{97}{19}$
c. $\frac{16}{3}$ and $\frac{19}{5}$ **d.** $\frac{111}{7}$ and $\frac{100}{11}$

2. Sometimes it is not so easy to find the order of two fractions by using the transitive property. Try to find the order of $\frac{3}{7}$ and $\frac{8}{19}$ by using the transitive property.

We shall now sketch a proof of the transitive property, if a, b and c are rational numbers and if $a < b$ and $b < c$, then $a < c$.

It was shown in Section 35-2 that

$$a < b \text{ means } a = b + q_1 \text{ where } q_1 < 0;$$

also,

$$b < c \text{ means } b = c + q_2, \text{ where } q_2 < 0.$$

Hence, $a = b + q_1$

$= (c + q_2) + q_1$ by writing $b = c + q_2$

$= c + (q_2 + q_1)$ by the associative law of addition

$= c + q$ where $q = q_2 + q_1$

But we know that the sum of two negative numbers is another negative number, so q is a negative number.

Hence, $a = c + q$ where q is a negative number

which means that

$$a < c$$

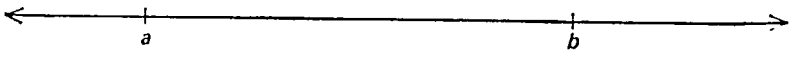
EXERCISE 35-4D

If a, b and c are rational numbers and if $a > b$ and $b > c$, then $a > c$. Prove this property as above, giving a reason for each step in your proof.

35-5 Addition property of order

We have already considered the addition property of order for whole numbers and for fractions in Chapters 17 and 22. Let us now see whether the property is true for the set of all rational numbers.

It will help us to picture addition and order on the number line. Let us first choose two rational numbers a and b on the number line with $a < b$. We remember that the addition of a



positive rational number to both a and b means moving to the right, while the addition of a negative rational number means moving to the left.

If we now add the same *positive* rational number c to a and to b, we see that the point for $a + c$ is c units to the right of the point for a and the point for $b + c$ is also c units to the right of the point for b. The number line indicates that $a + c < b + c$ (see Fig. 1 on p. 162). If we add a *negative* rational number c to a and to b, the points for $a + c$ and $b + c$ will each be the same distance to the left of the points a and b respectively. The number line still suggests that $a + c < b + c$ (Fig. 2).

Fig. 1

Fig. 2

Let us see if we can prove the addition property of order. Starting from $a < b$, we know that $a = b + q$, where q is some negative rational number. We add c to both sides of the equality obtaining $a + c = (b + q) + c$, or $a + c = (b + c) + q$. (Why?) But the last equality tells us that

$$a + c < b + c$$

The order relation between the numbers is preserved. Hence, we have the *addition property of order*:

If a, b and c are rational numbers and if $a < b$, then $a + c < b + c$

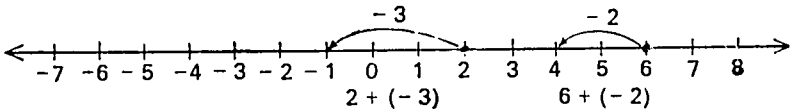
EXERCISE 35-5A

1. Formulate an addition property of order for the relation "$>$" and write out a proof of it.
2. Illustrate the truth of the addition property of order by taking $b = -\frac{3}{2}$, $a = -\frac{5}{2}$, and with c having successively the values $-\frac{3}{2}$, 6, 0, $-\frac{5}{2}$, -4.

35-6 Generalized addition property of order

In the addition property of order considered above, we saw that the addition of the same rational number to both sides of an inequality preserves the order relation. Naturally we may want to know whether an order relation is still preserved when the two rational numbers added to both sides are not equal but have the same order relation between them that the original rational numbers have.

Let us illustrate this with an example on the number line. We know that $2 < 6$ and that $-3 < -2$. What relation holds between $2 + (-3)$ and $6 + (-2)$? Add 2 to (-3); this gives -1. The addition of 6 to -2 gives 4.



What relation exists between -1 and 4? We see that $-1 < 4$. From this we may write,

if $\quad 2 < 6$ and $-3 < -2$

then $\quad 2 + (-3) < 6 + (-2)$

That is, $\quad -1 < 4$

which is true.

Now choose any four rational numbers a, b, c and d with $a < b$ and $c < d$. Add a to c and b to d. Do this with two or more sets of rational numbers. Write in each case the relation which exists between $a + c$ and $b + d$. Your answers will lead you to the generalized addition property of order which states,

If a, b, c and d are rational numbers such that $a < b$ and $c < d$, then $a + c < b + d$

Let us now give a proof of this property.

If $a < b$, then $a + c < b + c$ (by the addition property of order).

If $c < d$, then $b + c < b + d$. (Why?)

Hence, using the transitive property of order, we have

$$a + c < b + d$$

EXERCISE 35-6A

Write the generalized addition property of order for the relation "$>$". Try to make a proof for it.

35-7 Numbers and their opposites

The addition property of order often helps us to see the truth of some properties of numbers which are not at once apparent. One such property is the relation between two rational numbers and their opposites.

By considering a few numerical examples we can easily see that if a and b are rational numbers and if $a < b$, then $-b < -a$. Try to convince yourself of the truth of this property

by drawing a number line and by locating several pairs of rational numbers and their opposites on it.

Let us see now whether the addition property of order can help us to give a proof of the relation which exists between the opposites of a pair of rational numbers. Let a and b be rational numbers with $a < b$. Then adding $-a$ to both sides of the inequality, we have

$a + (-a) < b + (-a)$ (by the addition property of order)

That is, $0 < b + (-a)$ (by the property of opposites)

Again, by adding $-b$ to both sides of the inequality, we get

$$0 + (-b) < b + (-a) + (-b) \quad \text{(Why?)}$$

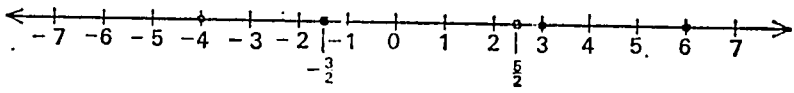
So $$-b < -a \quad \text{(Why?)}$$

Thus, the relation is proved.

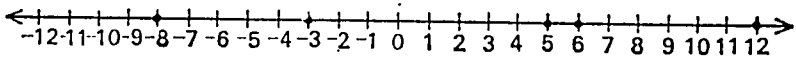
35-8 Multiplication property of order

You will remember from your earlier study of the rational numbers that if a and b are positive rational numbers, then $a \times b$ is a positive rational number and $a \times (-b)$ is a negative rational number. (In fact, it is the opposite of $a \times b$.) That is, the product of two positive rational numbers is always a positive rational number, while the product of a negative rational number and a positive rational number is always a negative rational number.

Consider the set of rational numbers $\{-\frac{3}{2}, -4, 3, \frac{5}{2}, 6\}$ represented on the number line below. Now multiply each element of the set by 2, a positive number. We obtain the set



$\{-3, -8, 6, 5, 12\}$. These numbers are represented on the number line below. On the new number line the points which



correspond to -4 and $-\frac{3}{2}$ are -8 and -3. The relation between -4 and $-\frac{3}{2}$ is $-4 < -\frac{3}{2}$, while the relation between

-8 and -3 is also $-8 < -3$. That is, the order relation is preserved.

Similarly,
since $\frac{5}{2} < 3$, then $2 \times \frac{5}{2} < 2 \times 3$
since $3 < 6$, then $2 \times 3 < 2 \times 6$
since $-\frac{3}{2} < 3$, then $2 \times (-\frac{3}{2}) < 2 \times 3$

These examples suggest that if a and b are rational numbers with $a < b$, then if c is a counting number, $a \times c < b \times c$.

Now let $c = \frac{1}{2}$. Multiplying the set of rational numbers $\{-\frac{3}{2}, -4, 3, \frac{5}{2}, 6\}$ by $\frac{1}{2}$, which is a positive rational number, we get the set $\{-\frac{3}{4}, -2, \frac{3}{2}, \frac{5}{4}, 3\}$. From the number line we see that $-4 < -\frac{3}{2}$ and that $-2 < -\frac{3}{4}$. That is, since $-4 < -\frac{3}{2}$, then $\frac{1}{2} \times (-4) < \frac{1}{2} \times (-\frac{3}{2})$.

Similarly, since $-4 < 6$, then $\frac{1}{2} \times (-4) < \frac{1}{2} \times 6$. This all suggests the *multiplication property of order*:

If a, b and c are rational numbers and if $a < b$ and c is positive, then $a \times c < b \times c$

A proof of this property will be given in the next section.

EXERCISE 35-8A

1. Complete the following statements using one of the symbols $>$ or $<$:

a. If $9 > 7$, then $9 \times 3 \square 7 \times 3$
b. If $-5 < -3$, then $-5 \times 3 \square -3 \times 3$
c. If $a > b$ and $c > 0$, then $a \times c \square b \times c$
d. If $7 > -9$, then $7 \times 4 \square -9 \times 4$
e. If $3 > 0$, then $3 \times 6 \square 0 \times 6$
f. If $a < b$ and $c > 0$, then $a \times c \square b \times c$
g. If $-5 < 5$, then $-5 \times 4 \square 5 \times 4$
h. If $0 < 8$, then $0 \times 2 \square 8 \times 2$

35-9 Proof of the multiplication property of order

The property states that if a, b and c are rational numbers and if $a < b$ and $0 < c$, then $a \times c < b \times c$. If $a < b$, then $a = b + q$ where $q < 0$. Multiplying by c we get

$$\begin{aligned} a \times c &= (b + q) \times c \\ &= (b \times c) + (q \times c) \end{aligned}$$

Now $q \times c$ is a negative rational number, because it is the product of a negative number and a positive number. Therefore $a \times c < b \times c$.

EXERCISE 35-9A

1. Prove that if $a > b$ and $c > 0$, then $a \times c > b \times c$.
2. Choose any three rational numbers a, b and c with $a < b$ and c negative. Find the products $d \times c$ and $b \times c$. What is the relation between $a \times c$ and $b \times c$?
3. Repeat Question 2 with three different sets of numbers a, b and c: (i) choose a and b to be positive and c negative, (ii) choose b to be positive and a and c negative, (iii) choose all three numbers negative. Write down the product $a \times c$ and $b \times c$ in each case and state the order relation between them.
4. Can you deduce a new multiplication property of order from the answers to Questions 2 and 3?

Your answers to Questions 2 and 3 will have shown you that if a, b and c are rational numbers and if $a < b$ and $c < 0$, then $a \times c > b \times c$.

35-10 Generalized multiplication property of order

It was established in Chapter 22 for positive fractions that if

$$\frac{a}{b} < \frac{c}{d} \quad \text{and} \quad \frac{m}{n} < \frac{p}{q}$$

then

$$\frac{a}{b} \times \frac{m}{n} < \frac{c}{d} \times \frac{p}{q}$$

This was proved by using the number line. This property is not true for rational numbers if we allow some of the four numbers to be negative. However, we do have the following property:

If a, b, c and d are rational numbers, with $a < b$ and $c < d$, and a, b, c and d are positive, then $a \times c < b \times d$.

We shall give a proof which uses the multiplication property of order.

If $0 < a < b$ and $0 < c < d$

then $a \times c < b \times c$ and $b \times c < b \times d$ (Why?)

Hence, $a \times c < b \times d$

EXERCISE 35-10A

1. Find three sets, each consisting of four positive rational numbers a, b, c and d satisfying $a < b$ and $c < d$. Verify in each case that the generalized multiplication property of order is satisfied.
2. Rewrite the generalized multiplication property of order given above using $a > b$ and $c > d$. What is the relation between $a \times c$ and $b \times d$?

35-11 Summary of properties of order

In the properties of order given below the sign $>$ may replace the sign $<$ as appropriate.

1. *Comparison Property of Order*

For any rational numbers a and b, one and only one of the following is true: $a < b$, $a = b$, $a > b$.

2. *Transitive Property of Order*

For any rational numbers a, b and c, if $a < b$ and $b < c$, then $a < c$.

3. *Addition Property of Order*

For any rational numbers a, b and c, if $a < b$, then $a + c < b + c$. (NOTE: c may be positive, negative or zero.)

4. *Order Property of Opposites*

For any rational numbers a and b, if $a < b$, then $-b < -a$.

5. *Multiplication Property of Order*

   a. For any rational numbers a, b and c, if $a < b$ and c is *positive*, then $a \times c < b \times c$.
   b. If $a < b$ and c is *negative*, then $a \times c > b \times c$.
   c. For any *positive* rational numbers a, b, c and d, if $a < b$ and $c < d$, then $a \times c < b \times d$.

469

Real Numbers UNIT VII

Previous Page Blank

Chapter 36

REAL NUMBERS

36-1 Introduction

We have used the decimal fraction ·3 as a different way of writing $\frac{3}{10}$. In the same way, ·2 is equal to $\frac{2}{10}$. We can write an equal fraction with the smallest possible numerator and denominator. We then say that the fraction is written in lowest terms. In lowest terms, $\frac{2}{10} = \frac{1}{5}$, so that $\cdot 2 = \frac{1}{5}$.

These are examples of one-place decimal fractions. The same idea applies if there are two, three or even more decimal places. Thus,

$$\cdot 25 = \frac{25}{100} = \frac{1}{4}$$

$$\cdot 05 = \frac{5}{100} = \frac{1}{20}$$

$$\cdot 125 = \frac{125}{1000} = \frac{1}{8}$$

In each case, we have written the fraction in lowest terms.

If we begin with fractions like $\frac{1}{5}$, $\frac{1}{4}$, $\frac{7}{20}$ and $\frac{3}{8}$, it is easy to go in the other direction and express them in decimal form. For these examples, we write

$$\frac{1}{5} = \frac{2}{10} = \cdot 2$$

$$\frac{1}{4} = \frac{25 \times 1}{25 \times 4} = \frac{25}{100} = \cdot 25$$

$$\frac{7}{20} = \frac{5 \times 7}{5 \times 20} = \frac{35}{100} = \cdot 35$$

$$\frac{3}{8} = \frac{125 \times 3}{125 \times 8} = \frac{375}{1000} = \cdot 375$$

36-2 Recurring decimal fractions

Can we find a finite decimal form for any given fraction? It is not hard to see that the answer is "No".

If a fraction $\frac{a}{b}$ is to be written in finite decimal form, it must be possible to "fatten it up" so that b becomes 10 or 100 or 1,000 or some higher power of 10.

Except for 1, the only numbers which divide into 10 are 10, 5 and 2. Any fraction which is equal to a one-place decimal must, therefore, have a denominator which is 10 or 5 or 2. In fact:

$\cdot 1 = \frac{1}{10}$	$\cdot 2 = \frac{2}{10} = \frac{1}{5}$	$\cdot 3 = \frac{3}{10}$
$\cdot 4 = \frac{2}{5}$	$\cdot 5 = \frac{5}{10} = \frac{1}{2}$	$\cdot 6 = \frac{3}{5}$
$\cdot 7 = \frac{7}{10}$	$\cdot 8 = \frac{4}{5}$	$\cdot 9 = \frac{9}{10}$

Any fraction which is equal to a two-place decimal fraction must have a denominator which divides into 100. What are the possibilities? The only ones are 100, 50, 25, 20, 10, 5, 4 and 2. What are the possible denominators for fractions that are equal to a three-place decimal fraction?

Suppose that we have a fraction whose denominator does *not* divide into any power of 10. The fraction $\frac{1}{3}$ is the simplest example. Clearly 3 does not divide into 10 or 100 or 1,000 or any other power of 10. What can we do?

If we try to write $\frac{1}{3}$ as a one-place decimal fraction, we find that $\cdot 3$ is too small and $\cdot 4$ too large, because $3(\cdot 3) = \cdot 9$ and $3(\cdot 4) = 1\cdot 2$. We say that $\frac{1}{3}$ is between $\cdot 3$ and $\cdot 4$ and write

$$\cdot 3 < \tfrac{1}{3} < \cdot 4$$

If we try to write $\frac{1}{3}$ as a two-place decimal fraction, we soon find that $\cdot 33$ is too small and $\cdot 34$ is too large. In fact, $3(\cdot 33) = \cdot 99$ and $3(\cdot 34) = 1\cdot 02$. So

$$\cdot 33 < \tfrac{1}{3} < \cdot 34$$

In the same way, it turns out that

$$\cdot 333 < \tfrac{1}{3} < \cdot 334$$

$$\cdot 3333 < \tfrac{1}{3} < \cdot 3334$$

and so on *forever*. The best that we can do is to write

$$\tfrac{1}{3} = \cdot 333\ldots$$

where the three dots are meant to show that the 3's go on without end. We call the right side of the equal sign a *recurring decimal fraction.*

Let us take another example, $\frac{2}{11}$. Since 11 does not divide into any power of 10, we know that $\frac{2}{11}$ cannot be written as a decimal fraction. We expect to find that it can be written as a recurring decimal fraction. In fact,

$$\tfrac{2}{11} = \cdot 181818\ldots$$

where the digits 1 and 8 recur endlessly.

Let us verify this by doing a long division.

$$\frac{2}{11} = 11\,\overline{\big)\,2\cdot 0000}\;\cdot 1818\ldots$$

$$\begin{array}{r} 1\ \ 1 \\ \hline 90 \\ 88 \\ \hline 20 \\ 1\ \ 1 \\ \hline 9 \end{array}$$

We actually went further with the work than we needed to. When we got the remainder 2 after two divisions (shown by a circle), we were in the same situation as we were when we started. So we know that the later results of division will repeat the earlier ones.

Let us look at one more example. What decimal fraction is $\frac{5}{6}$ equal to? Does 6 divide into any power of 10? Since it does not, we must get a recurring decimal fraction. Let us see what it is:

$$6\,\overline{\big)\,5\cdot 00}\;\cdot 83\ldots$$

$$\begin{array}{r} 4\ 8 \\ \hline 20 \\ 1\ 8 \\ \hline 2 \end{array}$$

Do you see that because the remainders in the two circles are the same, the 3's in the answer must go on endlessly? We write

$$\tfrac{5}{6} = \cdot 8333\ldots$$

EXERCISE 36·2A

1. Find recurring decimal fractions for each of the following fractions:

a. $\frac{1}{9}$ **b.** $\frac{1}{99}$ **c.** $\frac{1}{999}$ **d.** $\frac{5}{11}$
e. $\frac{1}{7}$ **f.** $\frac{2}{7}$ **g.** $\frac{1}{37}$

36·3 Changing unending decimals to common fractions

You see from our work that some fractions are equal to decimal fractions which end and others to decimal fractions which do not end. How did you learn to tell without dividing out whether a fraction is of the first kind or the second?

We discovered something else. When the decimal is unending, *the digits repeat*, at least after a while. For example,

$\frac{2}{11} = \cdot 1818\ldots$ (repeating 18)

$\frac{5}{6} = \cdot 833\ldots$ (repeating 3 after passing the 8)

It is fairly easy to see why the digits must repeat. When dividing by 6, for example, there can be no more than 5 remainders different from 0. (What would happen if the remainder were 0?) Then if we keep on dividing, we must eventually get a remainder that appeared before, and then the digits in the answer start repeating.

Can we go the other way? That is, if we have an unending decimal fraction that repeats, can we find the fraction that it is equal to? Let us see.

What is a common fraction which is equal to $\cdot 3939\ldots$? Let us write

$$\square = \cdot 3939\ldots$$

Notice that two digits repeat. Now if we multiply by 100, we must move the decimal point two places to the right. This gives

$$100 \times \square = 100 \times (\cdot 3939\ldots) = 39\cdot 3939\ldots$$

The unending digits on the right of the decimal point are exactly the same in both cases. (Remember that the 39's go on forever!) So if we subtract these digits from both sides, we get

$$99 \times \square = 39 \quad \text{(exactly)}$$

and therefore the fraction which goes in the box is

$$\frac{39}{99} = \frac{13}{33}$$

You can easily verify that $13 \div 33 = \cdot 3939\ldots$

Let us take another example. We would like to know a common fraction for $\cdot 027027\ldots$ We write

$$\square = \cdot 027027\ldots$$

Multiply by 1000 (why not 100?). We get

$$1000 \times \square = 27\cdot 027027\ldots$$
$$999 \times \square = 27$$
$$\square = \tfrac{27}{999} = \tfrac{3}{111} = \tfrac{1}{37}$$

where the fraction on the right has been written in lowest terms. Again, you should check that $\frac{1}{37} = \cdot 027027\ldots$

As a final example, let us take an unending decimal for which the digits do not repeat from the beginning.

If $\square = \cdot 1333\ldots$ (repeating 3's), what fraction goes into the box? We multiply by 10. Then

$$10 \times \square = 1\cdot 333\ldots$$

and subtracting

$$9 \times \square = 1\cdot 2$$
$$\square = \tfrac{1}{9} \times \tfrac{12}{10} = \tfrac{12}{90} = \tfrac{2}{15}$$

EXERCISE 36-3A

1. Write each of the following unending decimal fractions as a common fraction:

a. $\cdot 222\ldots$	**b.** $\cdot 2323\ldots$
c. $\cdot 234234\ldots$	**d.** $\cdot 1111\ldots$
e. $\cdot 0101\ldots$	**f.** $\cdot 001001\ldots$
g. $\cdot 1666\ldots$	**h.** $\cdot 11010101\ldots$

2. Show how you can use the results of parts **d, e** and **f** of Question 1 to find the answers to parts **a, b** and **c** without working with the box $\square$.

36-4 Irrational numbers and real numbers

We have learned that common fractions can be written as decimal fractions which either end or repeat. We also learned that any decimal fraction which ends or repeats is another name for a common fraction. We can show this in a diagram:

common fractions	$\Longleftrightarrow$	decimal fractions which end or repeat

Have we left out any possibilities? Could we have a decimal fraction that is unending but not repeating? Yes, we could.

An example is

$$\cdot 101001000^{1} \ldots$$

where each time we move along from a 1 we put in an extra 0 before the next 1. In this example, there is no block of digits which repeats.

Another example is

$$\cdot 1234567891011 \ldots$$

where the scheme of writing digits should be clear. Again there is no repetition in the unending digits.

These decimal fractions cannot represent common fractions. They must correspond to a new kind of number, which is called an IRRATIONAL NUMBER.

So far we have talked only about positive decimals. For every positive decimal, ending or unending, there is an opposite written with a minus sign in front of it. Thus, the opposite of $\cdot 101001 \ldots$ is $-\cdot 101001 \ldots$. The numbers that we represent in this way are called negative numbers. When we include them and the number 0 we have the following scheme:

rational numbers	$\Longleftrightarrow$	decimal fractions that end or repeat (whether positive, negative or 0)
irrational numbers	$\Longleftrightarrow$	decimal fractions that do not end and do not repeat (whether positive or negative)

If we put the rational numbers and the irrational numbers together, we get the numbers which correspond to *all* decimal fractions. These numbers are called REAL NUMBERS.

real numbers — { rational numbers / irrational numbers }

EXERCISE 36-4A

Invent some more unending, non-repeating decimal fractions.

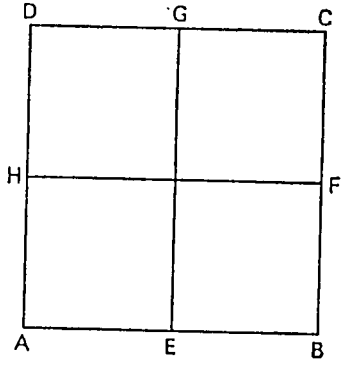
Chapter 37

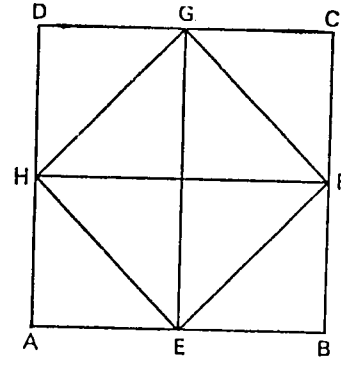
A GEOMETRY PROBLEM

37-1 Introduction

Let ABCD be a square 2 inches on a side. Suppose that E, F, G and H are the midpoints of its sides. If E is joined to G and F to H, ABCD is divided into four squares, all alike, each 1 inch on a side. We therefore see that the area of ABCD is 4 square inches.

We notice that $4 = 2 \times 2$. In general, the number of square inches in a square will be the number of inches on its side multiplied by itself. Show that this is true for squares having 3, 4 and 5 inches for their sides.





Let us now join E, F, G and H. Each of the one-inch squares is cut into two congruent triangles. We see that EFGH is a square consisting of four of these triangles. The area of EFGH is therefore 2 square inches.

How long is a side of EFGH, for example $\overline{EF}$? If S is the length of side $\overline{EF}$, it must be true that S times S is 2.

Could S be a whole number? Certainly not, because $1 \times 1 = 1$ is too small and $2 \times 2 = 4$ is too large. Is S equal to some fraction? If so, the fraction must be between 1 and 2. Remember that we want

$$S \times S = 2$$

A good guess is

$$S = \tfrac{7}{5} (= 1\tfrac{2}{5}).$$

Now $$\tfrac{7}{5} \times \tfrac{7}{5} = \tfrac{49}{25}$$

Because $2 = \tfrac{50}{25}$, we see that $\tfrac{7}{5}$ is a little too small.

Can we do better? Let us divide 2 by $\tfrac{7}{5}$. We get

$$2 \div \tfrac{7}{5} = \tfrac{10}{7}$$

From the meaning of division, this means that

$$\tfrac{7}{5} \times \tfrac{10}{7} = 2$$

Remember that $\tfrac{7}{5}$ was too small.

How about $\tfrac{10}{7}$? Of course

$$\tfrac{10}{7} \times \tfrac{10}{7} = \tfrac{100}{49}$$

We want $$S \times S = 2 = \tfrac{98}{49}$$

so $\tfrac{10}{7}$ is too large. Could we have seen this without multiplying $\tfrac{10}{7}$ by $\tfrac{10}{7}$? Clearly we could. We know that

$$\tfrac{7}{5} \times \tfrac{10}{7} = 2$$

If $\tfrac{7}{5}$ is too small, $\tfrac{10}{7}$ *must* be too large.

What do we know? We know that S is between $\tfrac{7}{5}$ and $\tfrac{10}{7}$. How can we do better? We can average these fractions; that is add them and divide by 2. In this way we get a new fraction, which is between $\tfrac{7}{5}$ and $\tfrac{10}{7}$.

This fraction is $$\frac{\frac{7}{5} + \frac{10}{7}}{2} = \frac{49 + 50}{35} = \frac{99}{70}$$

Let us see if S could be $\tfrac{99}{70}$.

$$\tfrac{99}{70} \times \tfrac{99}{70} = \tfrac{9801}{4900}$$

We want $S \times S = 2 = \tfrac{9800}{4900}$.

We haven't quite succeeded, but we are close. $\tfrac{99}{70}$ is too large but not much too large. Suppose that we divide 2 by $\tfrac{99}{70}$. Will the result be too large or too small? Can you say without doing the arithmetic?

Let us do the arithmetic anyway.

$$2 \div \tfrac{99}{70} = \tfrac{140}{99}$$

$$\tfrac{140}{99} \times \tfrac{140}{99} = \tfrac{19600}{9}$$

If $\tfrac{140}{99}$ had been the required value of S, we should have got $\tfrac{19602}{9801}$ when we multiplied $\tfrac{140}{99}$ by itself. We got a number a

trifle smaller. So now we know that S must be between $\frac{140}{99}$ and $\frac{99}{70}$.

We could average these two results. You should do this and find out whether we have finally succeeded in finding a fractional value for S. You will discover that the answer is "No".

A new question comes to mind. Suppose that we kept on in this way, would we *ever* get an absolutely correct answer? It is very surprising that the answer is "No". We shall show this in the next section.

Meanwhile it will be interesting to change our fractions to decimals. In this way, we can see how close we are getting to the desired result.

First we found S between

$$\frac{7}{5} = 1{\cdot}4 \quad \text{and} \quad \frac{10}{7} = 1\frac{3}{7} = 1{\cdot}428571\ldots$$

Next we located S between

$$\frac{140}{99} = 1{\cdot}414141\ldots \quad \text{and} \quad \frac{99}{70} = 1{\cdot}4142857\ldots$$

What better result were *you* able to find?

EXERCISE 37-1A

Suppose that S is the side of a square with an area 3 square inches so that $S \times S = 3$. Use the scheme of this section to get better fractions which approximate S.

37-2 $S \times S = 2$ has no solution in common fractions

We have tried some fractions $\frac{a}{b}$ to see if we could find the side of a square with area 2 square inches. We did not succeed. Were we unlucky? Or were we lacking in patience? The answer to both questions is "No". *There is no fraction* $\frac{a}{b}$ *which makes*

$$\frac{a}{b} \times \frac{a}{b} \quad 2!$$

This is one of the most famous discoveries in mathematics. The discovery was made by a Greek, a follower of Pythagoras, who lived approximately 600 years before Christ. It produced a crisis in the history of mathematics. Later we shall explain why this discovery was true.

But first let us prove this very surprising result. Suppose that there *is* a fraction $\frac{a}{b}$ for which $\frac{a}{b} \times \frac{a}{b} = 2$, so that

$$S \times S = 2$$

with $S = \frac{a}{b}$.

Now if $S \times S = 2$

$$S = \frac{2}{S}$$

This means that if $S = \frac{a}{b}$, then $\frac{a}{b} = 2 \div \frac{a}{b}$

or

$$\frac{a}{b} = \frac{2b}{a}$$

We must show that this is impossible *no matter how we choose the whole numbers a* and *b*.

We may surely assume that the fraction which solves our problem (if there *is* one) is written in lowest terms, because if it were not in lowest terms we could replace it by a fraction that *was*. For example, if $\frac{14}{10}$ were an answer (it isn't, of course), then $\frac{7}{5}$ would have to be an answer also.

Now we suppose that $\frac{a}{b} = \frac{2b}{a}$ with $\frac{a}{b}$ in the lowest terms. $\frac{2b}{a}$ is supposed to be equal to $\frac{a}{b}$. When are two fractions equal?

Take a definite fraction in lowest terms, say $\frac{10}{7}$. What fractions are equal to $\frac{10}{7}$? The possibilities are

$$\frac{10}{7}, \quad \frac{2 \times 10}{2 \times 7} = \frac{20}{14}, \quad \frac{3 \times 10}{3 \times 7} = \frac{30}{21}$$

and so on. It is enough to look at the possible denominators. They are 7, 14, 21 and so on.

Now if $\frac{a}{b}$ is *any* fraction written in lowest terms, the fractions which are equal to it must have one of the denominators b, $2b$, $3b$, $4b$ and so on. This means that if $\frac{2b}{a} = \frac{a}{b}$, a must be one of

the numbers b, $2b$, $3b$, $4b$ and so on; so that $\frac{a}{b}$ is one of the numbers 1, 2, 3, 4 and so on. That is, $\frac{a}{b}$ must be a whole number.

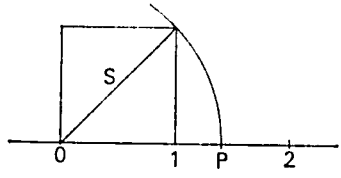
But $S \times S = 2$ has no whole-number solution. So no matter *what* fraction $\frac{a}{b}$ we choose, $\frac{a}{b} \times \frac{a}{b}$ cannot be equal to 2.

EXERCISE 37-2A

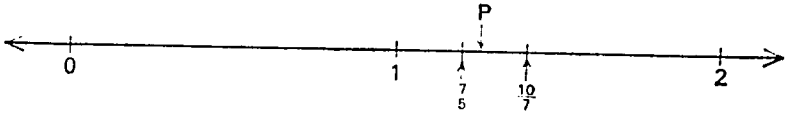
Show that it is not possible to find a fraction $\frac{a}{b}$ so that $\frac{a}{b} \times \frac{a}{b} = 3$.

37-3 More about the number line: irrational numbers

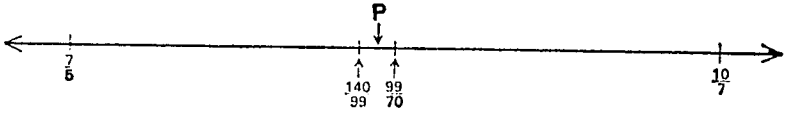
We know how to locate fractions on the number line. For example, we can easily find points to show $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{5}{4}$. Can we locate the number S, the side of a square of area 2? We can certainly do this geometrically. We showed at the beginning of this chapter that S is the length of the diagonal of a square of side 1. Let a number line be drawn along the base of the square with 0 at the left end and 1 at the right end. You can take 0 to be the centre of a circle of radius S. The circle will intersect the number line at a point P between 1 and 2. So we know on the picture how to locate the point P which corresponds to the number S.



But we know that S is not a rational number. Of course S is between $\frac{7}{5}$ and $\frac{10}{7}$. We can locate both of these numbers by points on the number line, and P will lie between these points.



We can do better. P must lie between the points which show $\frac{140}{99}$ and $\frac{99}{70}$.



These points are much closer together than the previous pair. The interval between them is shorter. Also $\frac{140}{99}$ is to the right of $\frac{7}{5}$ and $\frac{99}{70}$ to the left of $\frac{10}{7}$. So our new interval is inside the earlier one.

The important thing to notice is this. If we continue to get new fractions by the same scheme, we can locate P within intervals as short as we please but we never reach P itself. There are points like P on the number line which do not correspond to common fractions. We may call them *irrational points* because they show irrational numbers.

Let us say this in a different way.

Between 1 and 2 there is a point which divides the interval in 2 equal pieces. There are 2 points which separate it into 3 equal pieces, 4 points that separate it into 5 equal pieces and so on. There are 99 points which divide the interval into 100 equal pieces, 999 points which divide it into 1,000 equal pieces. But no matter what measuring stick we choose that divides the interval from 1 to 2 into a number of equal pieces, we cannot use this measuring stick to measure exactly the number whose square is 2. We can say that S and 1 are *incommensurable.* This means that S and 1 have no common measure. This was a discovery that shocked the Greeks when it was first discovered. It showed that the number line as we have met it up to now is full of holes. To say it another way, it shows the need of new kinds of numbers, which we call irrational numbers.

Let us talk about these numbers in terms of decimal fractions. As we learned, rational numbers correspond to decimal fractions which either end or repeat, while irrational numbers correspond to decimal fractions which neither end nor repeat. We saw an example of a decimal of this second sort,

$$\cdot 1010010001 \ldots$$

Our number S must be of this non-ending, non-repeating kind. We found that S was between

$$1{\cdot}414141\ldots \quad \text{and} \quad 1{\cdot}4142857\ldots$$

We can write $1{\cdot}4141 < S < 1{\cdot}4143$.

We have located S within ·0002. Of course with patience we could do still better. We know that S must be represented by an unending decimal that never endlessly repeats.

Can these unending, non-repeating decimals be treated like the rational numbers? Can they be ordered? Can they be added, subtracted, multiplied and divided? If so, do addition, subtraction, multiplication and division have the properties with which we are familiar? The answers to all of these questions are "Yes".

It would take a long discussion to prove in detail that this is true. For our purposes it is sufficient to give some idea of how it could be done.

Which of the decimal fractions ·13275 . . . or ·13268 . . . represents the larger number? The decimal fractions agree in the first three digits 1, 3 and 2. They differ in the next decimal place. Since 7 is greater than 6, the first number is greater than the second. Do you see that ·13275 . . . locates a point on the number line to the right of that for ·13268 . . . ? Is it also clear to you that if a and b are *any* real numbers, there are just three possibilities: $a = b$, $a < b$ and $a > b$? Could you explain why this must be true?

Do you think that $a + b = b + a$ is true for all real numbers a and b? Suppose not. Then for some a and b, it must be true that

$$a + b > b + a \quad \text{or else} \quad a + b < b + a$$

Each of these possibilities will now be shown to lead to a contradiction.

If $a + b > b + a$ is true, then $a + b = (b + a) + p$ where p is a positive number. Now p itself must be representable as a decimal fraction, maybe a small one, say ·0000012 . . . Let us imagine that a and b are written as unending decimal fractions. Let us break off each of them after n decimal places. Suppose that we add the corresponding rational numbers. The result does not depend on the order in which we add. Why? It is because the commutative property holds for rational numbers.

Now look at the equation

$$a + b = (b + a) + p$$

The first n decimal places of $a + b$ agree with the first n decimal places of $b + a$. Then the first n decimal places of p must be 0. Since this must be true no matter how large n is—that is, for as many decimal places as we like—all of the decimal places of p must be 0. This contradicts the assumption that p is positive. So it is impossible that $a + b > b + a$ be true.

In the same way, it can be seen that it is impossible for $a + b < b + a$ to be true. Consequently we know that $a + b = b + a$ is true.

In a quite similar manner we could show that all the properties of addition and multiplication for rational numbers hold for real numbers as well. Also, the order properties of the rational numbers are properties of the real numbers.

Chapter 33

THE VIEW FROM THE TOP

38-1 Looking backward

We have come a long way from our starting point. We began with numbers which could be used to count a set of objects like a herd of cattle. These numbers were known to early man. They appear in the oldest records. Since then man has travelled a long, long road. The idea of number has grown and grown. It is one of the most important ideas that mankind has ever had and one of the most successful. In this book we have tried to show how new kinds of numbers have been invented. We learned about the importance of zero, about the uses of fractions so that numbers could be used not merely to count but to measure. We learned about negative numbers, which help us to include not merely the idea of how many or how much but also the idea of direction, right or left, up or down. We have just extended the idea of number once again to include unending decimals which do not repeat. We have seen that if we want to measure the diagonal of a square of side 1 we need a number of this new kind. Here we have reached the end of our journey. (If you go further in mathematics, you will find that this is not really the end but that there are still new kinds of numbers which man has invented later.) The time has come to look back over the road which we have followed.

We have travelled slowly and patiently. The road has sometimes been dusty and the journey may have been tiring. But we have come to the top of a mountain. We should stop and enjoy the view.

At each stage of our journey we have learned to arrange the numbers in order of lesser or greater. And we have learned to add them, to subtract them, to multiply them and divide them. Let us forget for the moment just how we did this at each stage. These are details—important details but still details. Let us ask ourselves what has been accomplished by bringing in new kinds of numbers and learning to work with them. Man has invented

zero, the fractions, the negative numbers, and the irrational numbers. What for?

At each stage, man has found himself stopped by a difficulty. He wanted to be able to do something which he could not do with the numbers that he already had. There was a roadblock which stood in the way of going ahead. When old ideas fail or do not help, we seek to invent something new. "Necessity is the mother of invention".

For example, we cannot divide 4 by 5 if we have only counting numbers to work with. After fractions were invented, we *could* divide 4 by 5. We cannot subtract 5 from 3 if we have only counting numbers to work with. Negative numbers allow us to do so. After the new kinds of numbers have been invented, we have more freedom. We can remove restrictions.

But a very remarkable thing happens. It *could* be true that the new numbers behave in quite a different way from the old ones. If this were true we should always be having to remember what kinds of numbers we were working with, so that we could know what properties of addition, subtraction, multiplication and division to apply. By good luck it turns out that the properties are the same for counting numbers, for integers, for rational numbers or for real numbers. We do not have to keep learning new principles. This makes things much easier.

At this point it will be useful to reread the introduction, in particular the latter part about the patterns which it was hoped would be discovered. It will be remembered that we thought of the whole numbers as belonging to a club with certain rules. The new kinds of numbers could be admitted to this club because they were able to obey the rules.

38-2 The "club rules" for addition and multiplication

What are the club rules that all of our members are required to obey? First there are the properties of addition and multiplication. These properties were summarized very briefly at the end of Chapter 13, where a, b and c stood for any *whole numbers*. These same properties appeared in Chapter 34 for *rational numbers*. Now we shall use a, b and c to stand for any numbers at all, that is, for any *real numbers* whether rational or irrational. Remember that the set of real numbers includes all the

numbers that we have talked about. Here then are the club rules. First are three for addition:

The Commutative Property of Addition (CA)

$$a + b = b + a$$

The Associative Property of Addition (AA)

$$a + (b + c) = (a + b) + c$$

The Addition Property of Zero (A0)

$$a + 0 = a$$

Then there are three corresponding rules for multiplication:

The Commutative Property of Multiplication (CM)

$$a \times b = b \times a$$

The Associative Property of Multiplication (AM)

$$a \times (b \times c) = (a \times b) \times c$$

The Multiplication Property of One (M1)

$$a \times 1 = a$$

Notice that these three rules can be found from those for addition simply by changing $+$ to $\times$ and 0 to 1. Can you see that 1 behaves as a factor the same way that 0 does as an addend?

There is another rule that connects multiplication and addition:

The Distributive Property (D)

$$a \times (b + c) = (a \times b) + (a \times c)$$

Finally, we had

The Multiplication Property of Zero (M0)

$$a \times 0 = 0$$

In all, we have eight properties which we can think of as club rules for numbers.

Let us look at these rules as requirements that any proposed new members of the number club must obey. For example, suppose we propose the negative integers for membership in the club consisting of 0, 1, 2, 3, ... How must -1 behave if we are going to admit it?

What must $(-1) \times 1$ be equal to? Rule M1 says that

$$a \times 1 = a$$

If -1 is to be a good club member, it must therefore be true that

$$(-1) \times 1 = -1$$

That is, we must be able to use -1 as a particular value of a.

What must $1 \times (-1)$ be equal to? CM says that any members a and b must obey the rule

$$a \times b = b \times a$$

Then $1 \times (-1)$ must be equal to

$$(-1) \times 1$$

which we know is -1. So we must require that

$$1 \times (-1) = -1$$

A harder question is to find what $(-1) \times (-1)$ must be. When we introduced -1, we thought of it as the opposite of 1, so that

$$1 + (-1) = 0$$

So $1 + (-1)$ and 0 are two names for the same number. Then

$$(-1) \times [1 + (-1)] = (-1) \times 0$$

Rule D says that the left side is

$$[(-1) \times 1] + [(-1) \times (-1)]$$

and rule M0 says that the right side $(-1) \times 0$ is 0. So we must require that

$$[(-1) \times 1] + [(-1) \times (-1)] = 0$$

But we know that $(-1) \times 1 = -1$, so it must be true that

$$-1 + [(-1) \times (-1)] = 0$$

Then $(-1) \times (-1)$ must be the opposite of -1, that is 1. So finally we have the requirement

$$(-1) \times (-1) = 1$$

if -1 is to be allowed in the club.

We know of course that -1 does indeed pass all these tests. In fact,

$$1 \times (-1) = -1$$

$$(-1) \times 1 = -1$$

and

$$(-1) \times (-1) = 1$$

as we saw earlier in the book.

EXERCISE 38-2A

1. Show from the rules that

$$(-3) \times a = -(3 \times a)$$

must be true. (HINT: Write $[(-3) + 3] \times a$ in two ways.)

2. Show that if we follow the rules, $\frac{7}{(-3)}$ must be equal to $\frac{(-7)}{3}$. (HINT: If $\frac{7}{(-3)} = \square$, $7 = (-3) \times \square$ by definition of division. Now use Question 1 and conclude that $3 \times \square = -7$.)

3. Show from A0 that

$$0 + 0 = 0$$

38-3 Simplifying the rules

We have listed some rules that we require numbers to obey to become members of the number club. Can we perhaps simplify these rules? For example, can we make a shorter list that would really say the same thing? The answer is "Yes". In fact, we have already shortened the list from the one that was given in Chapter 13.

There we included $0 + a = a$, $1 \times a = a$ and $0 \times a = 0$. Can you see why it is not necessary to include them in our present list? Can you see for example that $0 + a = a$ follows from $a + 0 = a$ by using CA?

We shall now show that we can also leave out M0, which reads

$$a \times 0 = 0$$

We show that this rule must hold if the other seven rules hold.

According to rule D

$$a \times (b + c) = (a \times b) + (a \times c)$$

If $b = 0$ and $c = 0$, we have

$$a \times (0 + 0) = (a \times 0) + (a \times 0)$$

But we know that $0 + 0 = 0$, so we require that

$$a \times 0 = (a \times 0) + (a \times 0)$$

To simplify the writing, let us call $a \times 0$ by the new name b. Then we must have

$$b = b + b$$

We hope to show that b must be 0.

Rule AA tells us that

$$(-b + b) + b = -b + (b \times b)$$

Now $-b + b = 0$, since $-b$ and b are opposites. Also $b + b = b$. Therefore, we have

$$0 + b = -b + b$$

We know that $0 + b = b$ and that $-b + b = 0$. So finally

$$b = 0$$

That is,

$$a \times 0 = 0$$

which is rule M0.

So in applying tests for new members, it is not necessary to require M0 if we have already satisfied ourselves about the other rules.

38-4 The rules of order

We first met also some properties of order for the counting numbers:

01 if a and b are counting numbers, there are only three possibilities:

$$a = b \quad a < b \quad a > b$$

Again, if a, b and c are any counting numbers:

02 if $a < b$ and $b < c$, then $a < c$
03 if $a < b$, then $a + c < b + c$, and
04 if $a < b$, then $a \times c < b \times c$

These same rules now apply if a, b and c are any real numbers, except that in 04 we must require that $c > 0$ (which is automatically true when c is a counting number). Again, the real numbers are good club members.

We have learned that 04 can be supplemented by:

05 if $a < b$ and $c < 0$, then $a \times c > b \times c$

There is of course no occasion for this rule with counting numbers, because a counting number is *never* less than 0.

These rules too can be simplified. If we say that

$$a < b$$

means that $b = a + p$ where p is a positive number—that is, $p > 0$—we can replace all the rules of order by a new list.

New Order Rules

0′1 If $a > 0$ and $b > 0$, then $a + b > 0$
0′2 If $a > 0$ and $b > 0$, then $a \times b > 0$
0′3 For any real number a, there are exactly three possibilities:

$$a = 0, \quad a > 0, \quad a < 0$$

For example, let us show that 03 follows from our new rules. 03 says that if $a < b$, then $a + c < b + c$. If $a < b$, we can write

$$b = a + p \quad (p > 0)$$

Then

$$\begin{aligned} b + c &\quad (a + p) + c \\ &= a + (p + c) \quad \text{(Why?)} \\ &= a + (c + p) \quad \text{(Why?)} \\ &= (a + c) + p \quad \text{(Why?)} \end{aligned}$$

But then finally $a + c < b + c$.

EXERCISE 38-4A

1. Prove 04 from 0′1, 0′2 and 0′3.

2. Prove 02 from 0′1, 0′2 and 0′3.

3. Prove 01 from 0′1, 0′2 and 0′3.

38-5 Summing up

What we have done in the last two sections is not easy. It is harder than the rest of the book. We have given some examples of the way in which mathematical proofs are constructed. The elementary teacher will not use proofs like this in his own classes. But the teacher should have an idea of what lies ahead for some of his pupils—those who go on in mathematics.

When we continue the study of mathematics we find that more and more simplifications occur. The facts that we know about numbers are connected with each other in surprising ways. The simplifications make mathematics more beautiful and more powerful. But we have to pay a price. The price is

that we have to be prepared to think deeply about our experience. We must not be satisfied with knowing how to get answers in routine ways. We must be willing to ask ourselves "Why?" again and again.

The knowledge that is power is the fruit of our unceasing effort to understand more clearly, more fully and more deeply.

Approximations UNIT VIII

Previous Page Blank

Chapter 39

APPROXIMATIONS AS RESULTS OF COUNTING

39-1 Introduction

Once our pupils held a party and we tried to find out how many attended the party. This was very difficult because when we were counting, several pupils had already left the party, others came after we counted, while many of the people present were moving around. After counting, we got the result 137 pupils. Do you think that this was the exact number of pupils who attended the party? Is counting really always easy and simple?

If you are asked to measure the length of your classroom with a foot ruler and you get the answer 30 feet, can you be sure that this is the exact length? Or could it be 29 feet and some inches, or even 30 feet and some inches? Does measuring give the exact number?

Suppose a tailor needs 3 yards of fabric to make a dress. How many dresses can he make from a piece of 40 yards of fabric? Dividing 40 by 3, you obtain $13\frac{1}{3}$. It is clear that your answer would not be $13\frac{1}{3}$ dresses. You will say that the tailor can make 13 dresses. Such rounding off is often used in everyday life.

These three examples have something in common. What is it? We are now going to consider in detail the use of numbers in instances such as given above. We shall discuss what are commonly called "approximate numbers" or "approximations".

39-2 Approximations in counting

We have all learned how to count and how to make use of the set of counting numbers. We also know the set of whole numbers, which is the set of counting numbers and zero.

Suppose you ask one of your pupils to count the number of pupils in your class or the shillings in his pocket. No doubt his answer will be correct, and he will give you the exact number.

He will count the pupils or shillings one by one or in groups. This will be easy if the number of pupils is rather small. If you asked him to count the number of windows in a very big building, then he can get the exact number if he counts carefully. You could be quite sure about the answer if some other pupils counted the windows and got the same number. But counting the number of members in a set becomes harder as the set becomes larger. Even so, it is possible in many instances to obtain accurate counts of large sets. Sometimes it is quite necessary to obtain accurate counts. For example, a bank teller must count the exact amount of money at the end of the day. There are, however, instances when counting the number of members in a set is extremely hard or even impossible.

Would it be easy or even possible for the government to count the exact number of people in your country? The number of people in your country does not stay the same even for one day. For many purposes, however, the government has to know how many people live in various regions and in the whole country. Of course, it is practically impossible to count all of these people. Besides, does the government really need to know the *exact* number of people?

There is another example in which it is very difficult to count the number of members in a certain set. Suppose you ask a pupil to find the number of trees in a certain park or piece of land. If he counts and gives the answer 563, do you think his answer is exact? It probably is not exact for the following reasons. First, it was rather inconvenient and hard to count such a large number of trees scattered about without recounting some and without missing others. Next, it was probably difficult for the pupil to decide whether the dead trees or some larger bushes should be counted or not. In other words, it was difficult for him to determine exactly what things were members of the set of trees. If some other pupils count the number of trees in the park, they will probably obtain different answers, perhaps 559, or 550 or 571. In fact, it would be interesting to see whether the pupil who first counted would get his original answer if he counted again. In situations like these we are usually quite satisfied with approximate, rather than precise, results. The numbers in the statements below are certainly not exact. We will call them *approximate numbers.*

The population of Uganda is 6,780,000.
There were two thousand people present at the lecture.
Our college library contains 6,700 books.
Ali has 300 chickens at home.

EXERCISE 39-2A

1. Ask your pupils to try to count how many people are in your school in one day. What makes it hard to obtain an exact number? Do you think that the number they count would change from time to time during the day? Suppose instead you asked them to count how many different people in all were in your school on a certain day. Would they still have difficulties?

2. Ask your pupils to count some of the sets in the examples we have given.

3. Find other situations you can use with your pupils to show that the results of counting are not always exact.

39-3 Averages

By using many examples, you can convince your pupils that we can often obtain only approximate numbers in counting certain large sets. Of course, you will want them to obtain the most accurate approximations that they can. We will now see how to make sure that the approximate answers are rather accurate.

If four pupils tried to find out how many chickens Ali has at home, each pupil would probably get a different result by counting. Why is it so? Suppose the first pupil counts 295, the second 305, the third 304 and the fourth 297. Which answer do you think would be the best? You might think that a good answer would be one between 295 and 305. In order to find a good answer between 295 and 305 for the number of chickens, we can proceed as follows: we find what we call the *arithmetic mean* or *average* of all results of counting. We first find the sum of all the results. Then we divide the sum by the number of terms in the sum. The quotient obtained is called the arithmetic

mean or average of the numbers that we started with. For example, the average of the numbers 17 and 25 is

$$(17 + 25) \div 2 = 21$$

Going back to Ali's chickens, the sum of the counting results is

$$295 + 305 + 304 + 297 = 1{,}201$$

After dividing 1,201 by 4 (the number of counts), we get the average 300·25. Taking into account the objects we are dealing with, we may say 300 chickens is the final result. Remember that this result is only approximate. We do not claim that it is precise, but it is certainly more precise and reliable than any one of the four individual counts. Finding the average is a good way of obtaining an approximate answer when repeated counting gives different numbers as results.

EXERCISE 39-3A

1. Find the average (arithmetic mean) of each of the following sets of numbers. If the quotient is an unending decimal, write the answer to one decimal place and then write three dots, . . . , to show that the answer is unending.

a. 18, 22, 23	**b.** 22, 23, 26, 29
c. 101, 102, 105, 108	**d.** 248, 251, 252, 267
e. 61, 63, 64	**f.** 248, 251, 257, 267

2. Using fractions, write the averages that were decimals in Question 1.
3. Make up examples to use with your class of situations in which you would want to find averages.

39-4 Deviation

We have said that the average of several counts of a large set can be taken as a good answer for the number of members in the set. However, this number may look somewhat artificial to your pupils, and they may raise questions such as: How does this number correspond to the reality? How reliable is it as a solution to our problem?

Let us use the following example to try to see how to answer these questions. Suppose you ask your class to determine the number of grains of rice in one ounce of rice. Let five pupils

weigh five separate heaps of one ounce of rice each, and count the number of grains in each heap. Suppose they got the following numbers:

308, 332, 328, 342, 307

The average of these numbers is $1{,}617 \div 5 = 323{\cdot}4$. The digit "3" for the hundreds appears in each of the five counts, and therefore we may conclude that we can rely on the number of hundreds. Thus, we say that the digit 3 is *reliable*. The digit "2" for the tens in the average is *questionable*, because in the five countings we got various digits in the tens place, namely

0, 3, 2, 4, 0

The digit "3" in the ones place in the average is clearly not reliable at all and, thus, *worthless*. Therefore, that digit as well as the digit in the tenths place (·4) ought to be rejected in the final result.

Since the right-hand two digits (3·4) in the average are worthless, the answer 320 would be just as good an answer. Therefore, we will say that the number of grains of rice in a heap of one ounce is approximately 320. We can be quite sure about the first digit of this number, which indicates the hundreds of grains. In the second digit (2), which expresses the number of tens, there may be a small inaccuracy. About the remaining digits, we just cannot say anything.

We may *summarize* our procedure as follows:

1. Find the average.
2. Compare the average with each separate count.
3. The digits which are the same in every count are reliable and are to be kept in the final result.
4. Take the next digit in the average even though it is questionable.
5. Replace all remaining digits by zeros, since they are worthless. (More definite instructions about replacing rejected digits by zeros will be given in Chapter 40.)

EXERCISE 39-4A

1. Suppose five pupils in a class counted the number of books in the school library. The results of their five counts were

275, 274, 278, 279, 271

Find the average of the five counts.

2. Which digits in the average are reliable, which are questionable and which are worthless?

3. What would you say is the final number of books?

We have used the words *reliable*, *questionable* and *worthless*. You may feel that they have not been explained sufficiently; perhaps there are still questions in your mind about the procedure. Let us now discuss more carefully how we can tell whether a given digit is reliable enough to keep.

We have seen that the numbers obtained in the separate counts of grains of rice are different: 308, 332, 328, 342, 307. Each of these numbers is different also from the calculated average 323·4. Suppose we find now how much each count differs from the average. We will call these differences the *deviations from the average*. In our example of rice, they are:

$$323{\cdot}4 - 308 = 15{\cdot}4$$
$$332 - 323{\cdot}4 = 8{\cdot}6$$
$$328 - 323{\cdot}4 = 4{\cdot}6$$
$$342 - 323{\cdot}4 = 18{\cdot}6$$
$$323{\cdot}4 - 307 = 16{\cdot}4$$

(NOTE: To find the deviation from the average we subtract the smaller number from the larger.)

Now we find the average of these deviations by adding them and dividing their sum by 5. $63{\cdot}6 \div 5 = 12{\cdot}72$. This quotient, 12·72, is called the *average deviation*.

In our example the left-most digit (1) in the average deviation 12·72 is in the tens place. Therefore, the digit in the tens place (2) in the average 323·4 we will call questionable. We keep the digit "2" for tens in the average 323·4 as the first questionable digit. We replace all the digits to the right of the tens place by zeros. As the final result, we get 320. In order to avoid misunderstandings, it is sometimes convenient to underline the "2" as the questionable digit in the final result $3\underline{2}0$.

This method can be used in any problem, not keeping any digits beyond the left-most place in the average deviation.

We can set out the whole problem as follows:

One ounce each	*Number of grains*	*Deviation from the average*
First counting	308	15·4
Second counting	332	8·6
Third counting	328	4·6
Fourth counting	342	18·6
Fifth counting	307	16·4
Sum	1,617	Average 63·6
Average	323·4	deviation 12·72

The number of grains in one ounce is approximately 320.

EXERCISE 39-4B

1. Indicate whether the number appearing in each of the following statements is exact, or approximate:
   a. According to the class registers, the school has 387 pupils.
   b. The town has 14,700 inhabitants.
   c. John received 125 shillings for the work done.
   d. During the month, Ali worked 6 days overtime.
   e. The train had been on its way for $3\frac{1}{2}$ days.
   f. The sum of the ages of father, mother and son is 112 years.
   g. The store sold 463 pairs of shoes in a week.
   h. 6,200 people visited the museum in a month.
   i. The theatre sold 527 tickets yesterday.
   j. The dairy farm produces 430 quarts of milk a day.
   k. The machine weighs 1,325 pounds.
   l. The room is 12 yards, 5 inches long.
   m. The flight lasted 1 hour and 17 minutes.
2. You have probably noticed that the average is always between the smallest and the largest of the numbers that you start with. Explain how you might convince a class that this is always so.
3. On five different walks, a pupil counted the number of steps he made in 100 metres, and obtained the following numbers:

   132, 150, 138, 147, 143

   What is his average number of steps in 100 metres?
4. Suppose you count the number of people watching a foot-

ball match. You count six times and get a different result each time. Your counts were

574, 562, 573, 567, 580, 571

a. Why do you think you got different results?

b. Find the average of your six counts. Also find the deviations from the average and the average deviation. Indicate in the average of your counts the reliable, questionable and worthless digits.

c. What answer will you finally give for the number of people watching the football match?

5. Make up more problems of this type for your pupils to work out.

Chapter 40

APPROXIMATIONS IN MEASURING

40-1 Approximate measurements

You saw in the previous chapter that the results of counting the number of members in sets are sometimes exact but often they are only approximate numbers. Let us now consider what happens when we measure lengths and weights of objects or periods of time.

What do you think will be the standard unit for measuring lengths of main roads and railways? If a chart indicates that the distance from Dar es Salaam to Nairobi is 498 miles, does it mean that this is an exact number or perhaps that it may be 498·5 or even 497·5 miles? When measuring such great distances, we usually disregard a difference amounting to less than a mile in the final results. This means that for our purpose we are quite satisfied if we find the approximate number of miles, with a *precision* to one mile. Parts of a mile are in practice neglected.

However, when measuring material for dresses or curtains, we do realize that a difference of even one inch or half an inch is important and has to be taken into account. In such cases, tenths of an inch only can be neglected.

What would you as a teacher say if a pupil was told to draw in his notebook a line segment 2·3 inches long, and his segment was only 2·1 inches long? Would you say that the pupil has done it correctly, because a few tenths of an inch do not matter? In such cases it does matter, because you required him to be precise to the nearest tenth of an inch—the pupil's segment 2·1 inches long is not correct. There are even instances where more precision is important. For example, those who design precision instruments, such as wrist watches, require precision to lengths so small that we cannot observe them with our eyes.

From the several examples above, the following conclusion is easily reached: when we consider measuring lengths in

practical life, we see that in each case there is some desired unit of length used, while smaller units are ignored. Measurement of length always gives us an *approximate number.*

Measurement of time also has different degrees of precision. When an adult is asked to give his age, he will do it in terms of whole years. A mother expresses the age of her small child in terms of years and months, neglecting days. The length of a class lesson or of a football match is usually given in hours and minutes, ignoring seconds. However, in such sports as running or swimming, seconds and even tenths of seconds are counted.

EXERCISE 40-1A

1. In a way similar to our discussion of measuring length and time, explain how approximate numbers are obtained when weighing various objects.
2. What unit of weight is usually used in each of the following cases? What units can be neglected in each case?
   a. A shopkeeper weighing sugar
   b. A postman weighing letters
   c. A nurse weighing a new-born baby
   d. A doctor weighing an adult
3. What is the degree of precision used in railway and airline time-tables?

We have shown how approximate numbers are obtained when we measure quantities—lengths, weights and periods of time—and how in each case the appropriate unit of measurement is chosen according to the need. (In this unit, we will use the term "quantity" rather informally to denote things that can be measured.)

On the other hand, it is also very important to understand that *we can never get an exact number from any of these measurements of quantities.* Among the essential reasons for this impossibility are

(*a*) the inaccuracy of measuring instruments, and
(*b*) the inaccuracy of human senses.

In some cases, repeated measurements of the same quantity

could not give even the same approximate number because of

(*c*) the changing conditions under which the successive measurements are made.

You should discuss reasons (*a*) and (*b*), and give examples. To discuss (*c*), think, for example, of the influence of the temperature on the length of an object, or of the evaporation of a liquid whose weight is to be found.

Conclusions

Every measurement gives only an approximate value of what is measured, and it is carried out with a certain definite precision. When we record the result of measuring, we show which units have been considered and which ignored.

(REMINDER: The results of measuring are always approximate numbers. As we have seen in the previous chapter, the numbers obtained as results of counting are sometimes exact numbers and sometimes approximate numbers.)

EXERCISE 40-1B

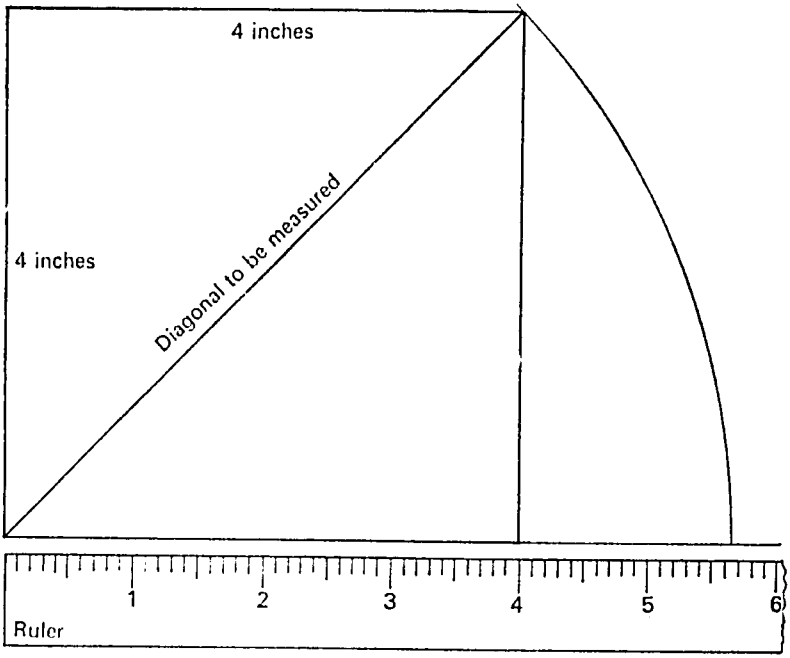
1. What definite standard unit is used in each of the following, in order to get reasonable measurements? What units can be neglected in each case?
   a. An architect designing a house
   b. A surveyor mapping a city
   c. A shoemaker taking the size for a pair of shoes

40-2 Basic agreement for recording approximate numbers

There is a method of recording the results of counting and measuring, showing clearly the precision of these results. This method will be applied in the following example.

Problem

To measure with a foot ruler, having marks of tenths of an inch, the length of the diagonal of a square whose side is 4 inches. We use the symbol $<$ for "is less than", and the symbol $\approx$ "is approximately equal to", to write the result as follows.



1. 5 in. $< x <$ 6 in.
2. $x \approx$ 6 in.
3. 5·6 in. $< x <$ 5·7 in.
4. $x \approx$ 5·65 in.

Explanations

1. x, the length to be found, is between 5 and 6 inches to the nearest inch.
2. Measured to the nearest inch, x appears to be nearer to the 6-inch mark than to the 5-inch mark.
3. Measuring to the nearest tenth of an inch, we see from the figure that x lies between 5·6 inches and 5·7 inches.
4. From the figure, we are unable to decide whether x is nearer to 5·6 inches or 5·7 inches and so we may conclude that $x \approx$ 5·65 inches, taking the average of 5·6 and 5·7.

In our final result, we have two reliable digits: namely, the digit for ones (5) and the digit for tenths (6). The third digit (5) for hundredths is questionable. The final result of our measurement does not allow us to say that length x contains exactly 5 hundredths of an inch beyond 5·6 inches.

(Actually, if we had made a very precise drawing and had used a more accurate and precise ruler, we might have found that the length x, with a precision to the nearest thousandth of an inch, is equal to 5·657 inches. Therefore, taking a precision to one hundredth of an inch, it is correct to write $x \approx 5{\cdot}66$ inches.)

Let us consider next an example of measuring and recording temperature. What does the recording "T $\approx$ 37° C" mean? It says that we measured with a precision to one degree. On the other hand, if with a more precise thermometer we record "T $\approx$ 37·0° C", the "0" indicates that we measured with a precision to one tenth of a degree.

The examples above of recording approximate numbers are based on the following agreement.

BASIC AGREEMENT

An approximate result should be recorded in such a way that its last digit to the right indicates its precision. All digits, except the last, ought to be reliable. Only the last digit is questionable and may be slightly inaccurate.

40-3 Repeated measurements

As we have mentioned before, it often happens that when we measure the same quantity again we get a somewhat different result, even though we use the same instrument each time. This happens frequently in measuring long distances. In such situations, we obtain the most precise result by finding the average of all the results of the repeated measurements. (This we do in the same way as when dealing with several counts of a large set.) The average is then rewritten, preserving all reliable digits and only one questionable digit. In order to know which digits in the average are to be kept, it is useful to find the average deviation. In the previous chapter, we studied how this is done. Let us illustrate the method by an example, measuring length in metres.

Example

We are to measure the length x of a building, using a metric ruler marked for centimetres. The results of six successive measurements are as follows.

Measuring length x	*Result in metres*	*Deviation from average in metres*	
No. 1	51·63	0·352	
No. 2	52·12	0·138	
No. 3	52·20	0·218	
No. 4	51·87	0·112	51·9 m. $< x <$ 52·0 m.
No. 5	51·91	0·072	$x \approx$ 52·0 m.
No. 6	52·16	0·178	
Sum	311·89	1·070	
Average	311·89 ÷ 6 = 51·982	0·178	

The highest order digit of the average deviation is the tenths place. We thus conclude that the tenths digit in the number 51·982 is questionable, and therefore the digits of hundredths and of thousandths are to be rejected as worthless. Moreover,

$$51{\cdot}9 < x < 52{\cdot}0$$

Of these two numbers, 52·0 is closer to the average that we calculated, so we accept it as the final result:

$$x \approx 52{\cdot}0 \text{ m}$$

EXERCISE 40-3A

1. Indicate the reliable, questionable and worthless digits in each of the following approximate numbers. Write down each of these numbers according to the basic agreement.
   a. 25·43 with a precision to the nearest one
   b. ·2502 with a precision to the nearest hundredth
   c. 52·03 with a precision to the nearest tenth
2. Five weighings of the same object gave the following results in (pounds):

   2·834, 2·832, 2·837, 2·833, 2·835

   a. Find the average weight.
   b. Indicate in the average the reliable, questionable and worthless digits.
   c. Write the final result according to the basic agreement.
3. Four measurements of the same distance have given the following results (in yards):

   2,648, 2,656, 2,663, 2,678

a. Find the average of the four numbers.

b. Find the deviations from the average and the average deviation.

c. Indicate in the average the reliable, questionable and worthless digits, and write the final result according to the basic agreement.

Chapter 41

ROUNDING OFF

41-1 Introduction

We have previously discussed two situations in which we obtain approximate numbers: counting and measuring. We will consider now a third way of getting such numbers.

You have already done some problems in arithmetic in which you had to record the resulting answer "to the nearest ten" or "to the nearest unit". What you actually had to do was to replace your answer (a natural number, decimal fraction or unending decimal) with a simpler number close to it. The simpler number was to have fewer non-zero digits. Such replacement is called *rounding off.*

The following examples illustrate the process of rounding off.

Example 1

The census shows that a certain city has 246,143 inhabitants. Suppose a friend of yours asks you how many people live in that city. If you know that he does not need a very precise answer, would you say to him 246,143 people? Of course not. You would probably simply answer 246 thousand.

Example 2

There are certainly cases when results ought to be expressed to the highest degree of precision possible. For example, the assets of a bank must be recorded in the yearly report to the nearest pound. For general information, however, it is sufficient to know that the assets of a bank are 57 million pounds rather than 56,967,146 pounds.

In the two examples, certain numbers were rounded off. The results of rounding off are clearly approximate numbers.

41-2 Rounding up and rounding down

Rounding off can be done in two ways: we can "round off upwards" (*Example 2*) or "round off downwards" (*Example 1*).

To avoid long phrases, we shall call these two ways "rounding up" and "rounding down", respectively.

Rounding off numbers is easy. To *round down* a number to a digit in a certain place, we replace all the digits of the number written to the right of that place by zeros. For example, 274 rounded down to tens is 270, 27·4 rounded down to ones is 27, 27·4 rounded down to tens is 20.

To *round up* a number to a certain place, we add one to the digit in that place and replace all digits to the right of it by zeros. For example,

274 rounded up to tens is 280
27·4 rounded up to ones is 28
27·4 rounded up to tens is 30

Consider this complete example of rounding off the number 217·5073:

217·5073	rounded down to hundreds is	200
	rounded up to hundreds is	300
	rounded down to tens is	210
	rounded up to tens is	220
	rounded down to ones is	217
	rounded up to ones is	218
	rounded down to tenths is	217·5
	rounded up to tenths is	217·6
	rounded down to hundredths is	217·50
	rounded up to hundredths is	217·51
	rounded down to thousandths is	217·507
	rounded up to thousandths is	217·508

It is clear that in all these cases the original number is increased by rounding up and decreased by rounding down.

You may ask when we apply rounding up, or rounding down. The answer to this question is often suggested by the situation we are dealing with or by the conditions of the problem.

Example 3

We want to divide 50 shillings equally among 6 people. How much will each get?

To get the answer, you must divide 50 by 6. But the quotient of $50 \div 6$ results in the unending decimal 8·333. . . . Rounding down to the order of ones gives us 8 shillings and rounding

down to the order of hundredths gives us 8·33 shillings. If everyone gets 8 shillings, we remain with 2 shillings. If everyone gets 8·33 shillings, there will be 2 cents left over because $8{\cdot}33 \times 6 = 49{\cdot}98$ shillings. The latter is no doubt the best we can do, because a cent is the smallest coin. Rounding down is certainly the only appropriate procedure here, because if we round up 8·333 . . . to hundredths we get 8·34 shillings. We cannot give everyone 8·34 shillings, because $8{\cdot}34 \times 6 = 50{\cdot}04$ shillings and there are only 50 shillings to be shared.

Example 4

A group of 14 pupils decided to collect 100 pounds of oranges for the children of an orphanage. How many pounds should each pupil collect?

$100 \div 14 = 7{\cdot}14285714 \ldots$. Rounding down is not applicable here because less would be collected than aimed for. In this case, it is necessary to round up to get *at least* 100 pounds of oranges. Rounding up to ones, we get 8. So if each pupil collects 8 pounds, together they get $8 \times 12 = 112$ pounds, which is substantially more than wanted. Rounding up to tenths gives 7·2 pounds for each pupil, and in all they collect $7{\cdot}2 \times 14 = 100{\cdot}8$ pounds, which is quite close to the desired 100 pounds and reasonable from the point of view of weighing oranges. A practical answer to our problem is that each pupil should collect at least 7·2 pounds of oranges. If 100 shillings (and not 100 pounds of oranges) were to be collected by the students, it would have been proper to round up to hundredths to obtain 7·15 shillings for each student. Altogether they would then collect $7{\cdot}15 \times 14 = 100{\cdot}10$ shillings.

41-3 Fundamental rules for rounding off

The examples in the last section illustrate two cases when the conditions of the problem actually show whether a given number is to be rounded down or rounded up. It was also clear to what place the rounding off should be made. The question naturally arises: What kind of rounding off is to be applied when there is no indication what to do?

As you have seen, rounding down replaces the given number by a second number which is smaller than the given number,

while rounding up replaces the given number by a number which is larger. When a given number is to be rounded off and there is no special indication whether it should be up or down, it is reasonable to round off so that the number obtained differs as little as possible from the original number. For example, the number 17·384 rounded down to ones is 17, which is ·384 less than the given number. On the other hand, 17·384 rounded up to ones gives 18, which is ·616 greater than the original number. Certainly 17 is closer than 18 to the original number. So here it is better to round down.

Suppose now we want to round off to tenths instead of ones. You see that 17·384 rounded down to tenths is 17·3, which is ·084 less than the given number. But rounding up to tenths gives 17·4, which is only ·016 greater than the given number. Therefore, the better result in rounding off to tenths is obtained by rounding up.

We see that if it is permissible either to round up or to round down a given number, it is better to round down when the first rejected digit is less than 5 and to round up if the first rejected digit is greater than 5. In each of these cases, we will obtain a closer approximation; that is, the rounded-off number is closer to the original number.

Suppose you want to round off ·2604 to tenths. It is better to round up to ·3, because that differs from ·2604 by ·0396. Rounding down results in ·2, which differs by ·0604 from ·2604. If, however, we have to round off the same number ·2604 to hundredths, we should round *down* since

$$\cdot 2604 - \cdot 26 = \cdot 0004$$

and

$$\cdot 27 - \cdot 2604 = \cdot 0096$$

You may notice that we have not said how to round off numbers in which the first rejected digit is 5. We consider here the following two cases.

1. The first rejected digit is 5 and it is followed by digits some of which are non-zero digits. For example,

round off 43,503 to thousands

and round off ·257 to tenths

It is easy to see that here we get a closer approximation by rounding up. Show that this is so. Thus, to round off 43,503

to thousands, we round up and obtain 44,000. To round off ·257 to tenths, we round up and get ·3. Therefore, if the first rejected digit is 5 and is followed by digits, some of which are non-zero digits, then we round up.

2. The first rejected digit is 5 which is followed by zeros only, or the first rejected digit is 5 and it is the last digit in our number. For example,

round off 43,500 to thousands

round off 45 to tens

and round off 7·5 to ones

If we round down 43,500 to thousands, we obtain 43,000; if we round up, we get 44,000. Each of these rounded-off numbers differs from the original number by 500. We may say that they are "equally close" approximations. The same remark applies to rounding off the other two numbers. In cases like these, we simply agree to round up. Therefore,

43,500 rounded off to thousands is 44,000
45 rounded off to tens is 50
7·5 rounded off to ones is 8

(In some treatments of approximate numbers, the following *agreement* is made, which we will *not* use in this text.

If the first rejected digit is 5 which is followed by zeros only, or if the first rejected digit is 5 and it is the last digit, then we round *down* if the digit before 5 is *even*, and round *up* if the digit before 5 is *odd*.)

Here is our *fundamental rule for rounding off numbers*. We will always apply it if there are no special reasons to either round down or round up

If it is permissible either to round up or to round down a given number, we round it down when the first rejected digit is 0, 1, 2, 3 or 4, and round it up if the first rejected digit is 5, 6, 7, 8 or 9.

You should have already seen that the result of rounding off is always a number which represents an approximate value of the given number. It is an *approximate number*. The difference between the given number and the rounded-off number depends entirely on the way the rounding off is done. If a given number is to be rounded off to a certain place and if it is known that we have to round down or have to round up, then the

difference between the result and the given number does not exceed but may come close to *one* unit in the last place preserved. If, however, the problem does not show us which way to round off, we will use the fundamental rule. Then the difference between the given number and the result will never be more than *one-half* of the unit in the last place kept. If an approximate value of a quantity differs from its exact value by not more than one-half of a unit in the last place kept, then we say that *all digits of the approximation are accurate.* Therefore, if we obtain an approximation by applying the fundamental rule for rounding off, then all the digits in the approximation are accurate. For example: if we rounded off the number $2\frac{5}{7} = 2{\cdot}7142\ldots$ to hundredths, we would get the approximate number 2·71 with all digits accurate.

EXERCISE 41-3A

1. Round down to tens each of the following numbers and find the error of rounding down (the difference between the number and the rounded-down number).

   503, 817, 4,305, 21,658, 12,814, 17,715

2. Round off to tens each of the numbers in Question 1 and find the error of rounding off (from the larger number, subtract the smaller number).

3. Round up to thousands each of the following numbers and find the error of rounding up.

   23,458, 17,501, 13,709, 60,500, 100,998, 365,651, 1,349,673

4. Round off to thousands each of the numbers in Question 3 and find the error of rounding off.

5. Round off to ones each of the following numbers and find the error of rounding off.

   ·8, 2·55, 3·7, 15·5, 41·4, ·379, ·49, 1·813

6. Round off to tenths each of the following numbers.

   8·512, 11·395, ·403, 6·15, 4·08, 6·17, 10·0098

7. Round off to hundredths each of the following numbers.

   9·647, 12·784, ·231, 1·054, 19·6723, ·455

8. **a.** Indicate the reliable, questionable and worthless digits in each of the following approximate numbers.

b. Round off each number to the place of its questionable digit.

c. Write down each number according to the basic agreement of Chapter 40.

343 with a precision to the nearest ten
6750 with a precision to the nearest hundred
47·0983 with a precision to the nearest hundredth
9·0015 with a precision to the nearest thousandth.

9. Three experiments to find the weight in grams of 1 cubic centimetre of the same piece of iron gave the following results: 7·62, 7·80, 7·64. Find the average. Indicate in it the reliable, questionable and worthless digits. Round off the average to the place of the questionable digit. Write down the final result according to the basic agreement.

Chapter 42

MAXIMUM ERROR AND RELATIVE ERROR

42-1 Maximum error—precision

We have seen that every measurement of length, weight, time and so on can be made only approximately and the result is an approximate number. Even when it is possible to find an exact number (in counting the members of a set, for example), it is sometimes sufficient to know only its approximate value.

Here is an example. A pupil worked after school and saved money for a holiday. The exact amount was 101·30 shillings. When asked how much money he had saved, he answered "about 100 shillings." It is clear that the exact number representing his savings and the approximate number he gave are different. The pupil got the approximate number by rounding off. Similarly, the exact value of a measured quantity and the result of measuring are different.

The difference between the exact value of a measured or counted quantity and its approximate value is called the maximum or absolute error. In the quoted example, the maximum error is equal to 1·30 shillings.

You know already that exact values are known only very rarely, for example, in some cases of counting. This means, of course, that the actual value of the maximum error can very seldom be found exactly. However, in carrying our various measurements we can usually give the *bounds* or *limits* of the maximum error. In other words, we can expect to find out that the maximum error *does not exceed* a definite number.

For example, if you weigh an object on a shop scale, the maximum error will usually not be more than one ounce. But on laboratory scales, you can weigh an object so that the maximum error is no more than one-half of one hundredth of an ounce (that is, $\frac{1}{200}$ of an ounce).

42-2 Relative error—accuracy

We must, however, realize that the maximum error does not give us an idea of the quality or *accuracy* of the measurement. In other words, the maximum error does not indicate how accurately the measuring has been done. *The maximum error tells us only about the precision of the measurement.* For example, if the maximum error in a measurement is $\frac{1}{2}$ yard, then the measurement is made with a *precision* to the nearest yard. Conversely, if a weighing is made with a precision to the nearest pound, then the maximum error in the weighing is $\frac{1}{2}$ pound.

Suppose we made two measurements with a *maximum error of* $\frac{1}{2}$ *inch*: the first was of the length of a bridge, and we obtained 20 yards; the second was of the length of a book, and we obtained 12 inches. We say that each measurement was made with a *precision to the nearest inch.*

It is clear that the first measurement was done very carefully and is of high quality, but the second measurement is quite rough and unsatisfactory.

The same can be said about weighing. An error of one ounce in 50 pounds is usually not important. But an error of one ounce in $\frac{1}{2}$ pound can seldom be allowed. You can now see that to evaluate the quality of a measurement, it is *not* the maximum error that is important. Instead, it is how the maximum error compares with the measured value itself. In other words, we would like to know what part of the measured quantity the maximum error represents. Let us go back to the measurements of the lengths of the bridge and the book.

In measuring 20 yards, an error of $\frac{1}{2}$ inch is only $\frac{1}{1440}$ part of the length. However, in measuring the length of the book an error of $\frac{1}{2}$ inch is $\frac{1}{24}$ part of the measured quantity. *The fraction obtained by dividing the maximum error by the measured value is called the relative error.* As we have seen before, we do not know how good or accurate a measurement is by knowing the maximum error alone. It is the relative error that tells how accurate the measurement is. For example, we can compare the relative errors when measuring the bridge and the length of the book. These are the numbers $\frac{1}{1440}$ and $\frac{1}{24}$, respectively. The first fraction is one-sixtieth ($\frac{1}{60}$) of the second fraction, so the *accuracy* of the first measurement is much higher than the accuracy of the second.

It is usual to express the relative errors as percentages. Then it is easy to compare the accuracy of two different measurements. The relative error in measuring the length of the bridge is $\frac{1}{1440} \times 100 = \cdot 0695\%$ and the relative error in measuring the length of the book is $\frac{1}{24} \times 100 = 4\cdot 17\%$. Certainly, a measurement with a relative error of $\cdot 0695\%$ is much more accurate than a measurement with a relative error of $4\cdot 17\%$.

A special notation is often used to show the precision of a measurement. Suppose we measure a certain length d with a precision to the nearest inch, and obtain the result 132 inches. This result is then written in the form

$$d \approx 132\ (\pm\ \cdot 5) \text{ inches,}$$

since $\cdot 5$ inch is the maximum error here.

EXERCISE 42-2A

1. Find the maximum error of the approximate number $\cdot 66$, if its exact value is $\frac{2}{3}$.
2. Find the maximum error for each of the fractions

$$\tfrac{2}{7},\ \tfrac{5}{13},\ \tfrac{4}{19}$$

expressed by the approximations

$$\cdot 28,\ \cdot 384,\ \cdot 2105$$

respectively.

3. Find the relative error (in percentage) of the approximate number $5\cdot 47$, if its maximum error is $\cdot 005$.
4. Express the number $5\frac{2}{7}$ by an approximate decimal fraction with a precision to the nearest hundredth. Find the maximum error and the relative error (as a percentage) of the approximate number.
5. The width of a narrow street measured with a precision to the nearest ten centimetres is $7\cdot 6$ metres. The length of the street measured with a precision to the nearest metre is 76 metres. Which of these measurements is more precise? Is one of the measurements more accurate than the other?
6. Measuring a segment of length $8\cdot 75$ centimetres, we made an error of $\cdot 25$ centimetres. Measuring another segment of length $10\cdot 5$ metres, we made an error of 25 centimetres. Which of the two measurements is more accurate?

7. Measuring a segment 25 inches long, a student obtained 25·2 inches. Find the relative error of the measurement.
8. The volume of a container is 25 cubic inches. A pupil, however, computed the volume as 24·6 cubic inches. Find his maximum and relative errors.
9. Using the formula on the relationship between an approximate number, the maximum error and the relative error, complete the table.

Approximate number	*Maximum error*	*Relative error*
4·1	·05	
·654	·001	
48·4		·1%
·348		·5%
260	5	
3·40		·5%

Chapter 43

DECIMAL PLACES AND SIGNIFICANT DIGITS

43-1 Decimal places

We have learned how approximate numbers are obtained from *counting*, *measuring* and *rounding off*. We have seen that counting sometimes gives exact values, while measuring and rounding off always give approximate numbers.

In order to be able to discuss and understand operations such as addition and multiplication of approximate numbers, we have to study in more detail the notion of the precision of an approximate number. From one point of view this was done in Chapter 42. We will now consider two new ideas which are also closely related to the notion of precision: *decimal places* and *significant digits*.

Do you remember what the decimal places of a number are? You studied them in Chapter 23. *All digits of a number written to the right of the decimal point are called the decimal places of the number.* For example, the numbers 7·2, 6·03, ·417 have one, two and three decimal places, respectively. The number 46 has no (or zero) decimal places.

43-2 Significant digits

The concept of significant digits is a harder one, and we will have to develop it in several successive stages. In dealing with exact numbers, you may have heard the term significant digit. First of all, any digits from 1 to 9 appearing in a number are significant, because each of these digits shows a definite number of units in the place where that digit appears. For example, in the number 56·71, there are four significant digits: 5, 6, 7 and 1. To see this, $56{\cdot}71 = 5 \times 10 + 6 \times 1 + 7 \times \frac{1}{10} + 1 \times \frac{1}{100}$, because the 5 shows that the number contains 5 tens, the 6 shows that the number contains 6 ones, and similarly the 7 and the 1 show 7 tenths and 1 hundredth.

In the same way each *digit zero* that is *between digits from 1 to 9* is also called significant. In the number 603, the digit 0 is a significant digit because it indicates that there are no, or zero, tens in the number which has 6 hundreds and 3 ones.

On the other hand, the number ·023 has only two significant digits, 2 and 3. You see that this number has 2 hundredths and 3 thousandths, and this completely describes it. The digit 0 here is not a significant digit, because it is only used to locate the decimal point. It performs the role of a place (position) holder only, and we will not regard it as a significant digit. *In a decimal fraction, all 0 digits to the left of the first non-zero digit are not significant digits.* The numbers ·001, ·25, ·0305 have one, two and three significant digits, respectively.

Let us now consider the *digit 0* and its meaning when it is written *at the end of a decimal fraction.* Here it is important to know whether the decimal fraction is an *exact* number, or an *approximate* number.

If the decimal fraction is an exact number, the digits 0 written at the end do not have any significance. The decimal fractions 3·8, 3·80, 3·8000 represent the same number. Therefore, *a digit 0 when written at the end of an exact decimal fraction is not significant,* and it does not make any difference whether we omit the zero or write it.

The situation is completely different when the digit 0 is written at the end of a decimal fraction which represents an approximate number. We will show that the 0 has in this case a definite meaning. Consider, for example, the two approximate numbers 3·8 and 3·80, differing only by the digit 0 at the end. These two decimal fractions represent two different approximate numbers for the following reason.

The approximate number 3·8 could have been obtained from rounding off to tenths such numbers as 3·81, 3·82, 3·83, 3·84, or 3·75, 3·76, 3·77, 3·78, 3·79. This means that originally in our number there might have been hundredths or thousandths, but the number was rounded off to tenths. Suppose the approximate number 3·8 was obtained by measuring. Then the digit on the right (8) is in the tenths place and shows that the measurement was made with a precision to the nearest tenth.

If, however, an approximate number is written as 3·80, it

means that the ones (3), the tenths (8) and the hundredths (0) are known to us. If 3·80 were obtained by rounding off to hundredths, the original number might have had thousandths. Suppose 3·80 is obtained by measuring. Then the digit on the right (0) is in the hundredths place and shows that the measurment was made with a precision to the nearest hundredth.

We see that *the digit 0 appearing at the end of an approximate decimal fraction has a definite meaning*, and is therefore to be considered as a *significant digit*.

We will now give special attention to approximate values written as whole numbers. An approximate whole number may contain zeros at the right-hand end. Such a zero is a significant digit if it shows the absence of units in its place. But often a zero at the end replaces a worthless or unknown digit. Then the zero is not a significant digit.

Let us look at an example. Suppose the approximate value of a weight is 14·7 kilograms. This number has three significant digits. If we express this approximate value in grams, we get the number 14,700 because there are 1,000 grams in a kilogram. This number also has only three significant digits, because the two 0 digits at the end replace unknown digits.

If however the approximate number 14,700 grams was obtained by using a more precise scale, which weighs with a precision to the nearest gram, then this approximate number 14,700 has five significant digits. To express this approximate number in kilograms, we would have to write it as 14·700 kilograms. The last zero at the end is written in the thousandths place. This says that the measurement was made with a precision to the nearest thousandth of a kilogram; that is, to the nearest gram.

Thus, there is a difficulty in reading an approximate whole number ending in zeros. We know that the number of significant digits in an approximate whole number with digits 0 at the end depends on the precision of that number. For example, if we look at the approximate number 2,400, as it is written down, we cannot decide in which of the following three ways it was obtained.

1. The number 2,400 may be the result of rounding off to the nearest hundred or of measuring with a maximum error of 50. Then neither zero is a significant digit.

2. It may be the result of rounding off to the nearest ten or of measuring with a maximum error of 5. This makes the first zero a significant digit, and the second zero not a significant digit.
3. It may be the result of rounding off to the nearest one or of measuring with a maximum error of $\frac{1}{2}$. Then both zeros are significant digits.

We summarize our discussion in the following detailed statement of the meaning of a significant digit in an approximate number.

If an approximate number is written according to the basic agreement in Chapter 40 then:

1. *Any of the digits 1, 2, . . . , 9 are significant.*
2. *Any digit 0 at the right-hand end of an approximate decimal fraction is significant.*
3. *Any digit 0 at the right-hand end of an approximate whole number in the place showing the precision of the approximate number (or of the measurement) is significant.*
4. *Any digit 0 between significant digits is significant.*

43-3 Examples

The approximate decimal fractions 8·2, 7·06, ·1230, ·061 have two, three, four and two significant digits, respectively.

250 precise to the nearest one has three significant digits, because the 0 is in the ones place showing the precision.

2,500 precise to the nearest ten has three significant digits. The first zero from the left is a significant digit, since it is in the place showing the precision. The last zero (in the ones place) is not significant.

2,500 precise to the nearest one has four significant digits, because the last zero on the right is in the place showing the precision; and the zero in the tens place is significant, because it is between two significant digits, the 5 and the 0 at the end.

2,050 precise to the nearest one has four significant digits.

Let us look at some other examples of rounded-off numbers. If we round off the number 2,803 to tens, we obtain the approximate number 2,800 with three significant digits. If 2,803 is rounded off to hundreds, we also obtain the approximate number 2,800, however with only two significant digits.

Consider a rod measured to be 124 millimetres long with a precision to the nearest millimetre (10 millimetres make a centimetre). The number 124 has three significant digits. If we round it off to tens, we obtain the approximate number 120, containing only two significant digits.

In order to avoid any misunderstanding concerning the digits 0 at the end of approximate whole numbers, it is better to leave out the 0 digits which replace worthless or rejected digits (that is, which are not significant) and to change to larger units. For example, when rounding off the number 83,542 to hundreds, it is better to write 83·5 thousands rather than 83,500. If the three zeros at the end of 3,569,000 square metres are not significant, it would be better to write 3·569 square kilometres (1 square kilometre equals 1,000,000 square metres).

Sometimes it is not convenient to write an approximate number in larger units and drop the non-significant 0 digits. Then it would be important to say which of the zeros are worthless. One way of doing this is to underline the questionable digit, as we have done in Chapter 39. For example,

if $x \approx 36$ kilometres, then $x \approx 3\underline{6}{,}000$ metres
(two significant digits);

if $y \approx 8{\cdot}4$, then $y \approx 8\underline{4}0$ centimetres
(two significant digits)

or $y \approx 8\underline{4}00$ millimetres
(two significant digits).

We would be allowed to write 36 kilometres $\approx$ 36,000 metres only if we measured the distance with a precision to the nearest metre.

For a final review of the notions of decimal places and significant digits, look at this list of approximate numbers.

7, one significant digit, no decimal places
·7, one significant digit, one decimal place
·07, one significant digit, two decimal places
·070, two significant digits, three decimal places
·37, two significant digits, two decimal places
2·037, four significant digits, three decimal places
·307, three significant digits, three decimal places
2·0370, five significant digits, four decimal places.

43-4 Comparison of approximate numbers

We should now point out that in order to give an idea of the precision and accuracy of an approximate number, we can tell the *number of its decimal places* or the *number of its significant digits*.

The method of counting the number of decimal places is recommended, when we compare approximate values of the same quantity. For example, the first weighing of an object is 14·7 grams, and the second weighing, using a more precise scale is, 14·684 grams. The second approximate value is clearly more accurate and more precise than the first, because it has three decimal places, while the first has only one decimal place.

In changing from one unit of measurement to another in the metric system, the number of decimal places changes. But the number of significant digits remains unchanged. For example 254 centimetres = 2·54 metres. The number of significant digits in each number is three. But the first number has no decimal places, while the other has two. For this reason, it is a good idea to compare the accuracy of various approximate numbers by counting the number of their significant digits. For example, if measuring a segment resulted in the number 6·3 centimetres, and measuring the length of a field gave the number 254 metres, we must admit that the second approximate number is more accurate than the first, since the second has three significant digits and the first only two.

43-5 Exact whole numbers

The major part of the discussion in this chapter was devoted to the meaning of significant digits in approximate numbers. For the sake of completeness, we give now a rather simple statement on the meaning of significant digits in *exact* numbers.

In exact whole numbers, all the digits are significant.

By the significant digits of an exact decimal fraction, we mean all its digits except zeros written to the left of its first non-zero digit and zeros written at the right-hand end.

The exact whole numbers 45, 305, 27,108, 560,000 have two, three, five and six significant digits, respectively.

The exact decimal fractions 8·2, 7·06, ·1230, ·61 have two, three, three and two significant digits, respectively.

EXERCISE 43-5A

1. How many significant digits has each of the following approximate numbers given with a precision to the nearest ten?

   230, 480, 2,080, 81,050, 70,190, 13,700, 12,000, 201,000

2. How many significant digits has each of the following exact numbers?

   230, 480, 2,080, 81,050, 70,190, 13,700, 12,000, 201,000

3. How many significant digits has each of the following approximate numbers, given with a precision to the nearest hundred?

   32,400, 70,300, 190,100, 149,000, 10,050,000

4. How many decimal places and how many significant digits has each of the following approximate decimal fractions?
   8·5, ·42, ·703, 6·05, 1·003, 201·03, ·03, ·004, ·005, 2·60, 8·240, 8·040, ·070, ·2080, ·300, 2·500, 603·100, 2004·50

5. How many decimal places and how many significant digits has each of the numbers listed in Question 4, if they are given as exact decimal fractions?

6. Explain the difference between the two recordings: "the length of the segment is 12 inches" and "the length of the segment is 12·0 inches".

7. Recalling that there are 1,000 grams in a kilogram, express each of the following in kilograms if the 0 digits at the end are significant.

   2,860 grams, 8,700 grams, 250 grams, 23,400 grams

8. Express in kilograms each of the numbers in Question 7 if the 0 digits at the end are non-significant.

Chapter 44

ADDITION AND SUBTRACTION OF APPROXIMATE NUMBERS

44-1 Introduction

Let us look at a simple problem from everyday life. Find the length of a fence around a rectangular field. To solve this problem, we must first measure the length and the width of the rectangle. Suppose we obtain the approximate numbers 225 yards and 112 yards. To find the answer to our problem, the perimeter of the rectangle, we must add the lengths of the four sides, which are these approximate numbers:

$$225 + 225 + 112 + 112 = 674$$

In this way we are led to perform the operation of addition on approximate numbers. We clearly obtain the approximate number 674 yards.

If we had to find the area of the same field, we would get it by multiplying the two approximate numbers:

$$225 \times 112 = 25{,}200$$

Thus, in the second case we must perform the operation of multiplication on approximate numbers. The result of 25,200 square yards is clearly also an approximate number.

The question naturally arises, what kind of approximate number is this area? In other words, which digits of the approximate number 25,200 are reliable and which are not? We ask the same question about the length we found for the fence.

We must thus discuss computations with approximate numbers. The results of such operations are also approximate numbers.

As we have seen before, approximate numbers are obtained from *counting*, *measuring* and *rounding off*. We see now that besides these three sources, there is still a fourth source for obtaining approximate numbers, namely from computations or *operations*. Whenever we calculate with numbers, one or more of which is

approximate, the result of the calculation is an approximate number.

We will discover some rules which tell us which digits of such a sum or product are reliable, and how to record the answer according to our basic agreement.

We will study addition and subtraction of approximate numbers in two stages:

1. Addition and subtraction of approximate whole numbers.
2. Addition and subtraction of approximate decimal fractions.

44-2 Addition and subtraction of approximate whole numbers

In the previous example of the rectangular field whose length and width are 225 and 112 yards respectively, we saw that the fence all round had to be 674 yards long. However, we need to find out how reliable the digits of 674 are. We note that the length 225 yards and the width 112 yards are approximate numbers with a precision to the nearest yard. Thus, the ones digits 5, 5, 2, 2 of the terms in the sum

$$225 + 225 + 112 + 112$$

are questionable, which leads us to believe that certainly the ones digit 4 in the answer 674 is questionable.

Let us now consider a slightly harder problem. In a certain region there is a town with 720 people (counted with a precision to the nearest ten), two villages with 234 and 88 people and farm land with a population of 4,300 people (counted with a precision to the nearest hundred). Find the total population of the region.

Adding the four numbers in the usual way, we obtain

$$\begin{array}{r} 4{,}300 \\ 720 \\ 234 \\ 88 \\ \hline 5{,}342 \\ \hline \end{array}$$

Since the terms are approximate numbers obtained by counting various large sets, the number 5,342 is clearly also an approxi-

mate number. The question is which digits of this sum are to be kept in the final answer.

In the first number, 4,300, precise to the nearest hundred, the tens digit and the ones digit are unknown to us. We do know the tens digit and the ones digit in the number 88. However, when we add these to unknown digits in the number 4,300, the tens digit and the ones digit in the sum 5,342 remain unknown. We simply have to disregard them in the final result. When we first add the terms, we will take into account the tens digits and the ones digits in those terms in which they are known. But then we will round off the sum obtained to get the final result.

Let us write down the problem as follows. In place of digits unknown to us we will write the letter "U" for "unknown".

$$\begin{array}{r} 43UU \\ 72U \\ 234 \\ 88 \\ \hline 52UU \\ \hline \end{array}$$

Rounding off the sum 5,342 (obtained in the usual way), we reject the worthless digits of tens and of ones and obtain

$$5{,}342 \approx 5{,}300 \text{ for the final answer.}$$

We see that in the final result we rejected—that is, replaced by zeros—the digits in those places in the sum for which the digits in even one of the addends are unknown.

We can write down what we have discussed as the following rule:

In adding approximate whole numbers, we reject in the final result (according to the fundamental rule for rounding off) digits in those places in the sum for which the digits are unknown even in one of the approximate terms. (This rule will be included in a more general rule later.)

We use this rule even if there are one or more exact numbers among the terms. (In the example above, 88 was an exact number.)

Consider now the following simple problem. From a stock of 480 pounds of sugar, 117 pounds were sold in one day. How much sugar remained?

Here we have to subtract approximate numbers.

$$(a)\quad \begin{array}{r} 480 \\ -117 \\ \hline 363 \\ \hline \end{array} \qquad\qquad (b)\quad \begin{array}{r} 48\mathrm{U} \\ -117 \\ \hline 37\mathrm{U} \\ \hline \end{array}$$

Taking into consideration the difference 363 obtained in the usual way (*a*), and the unknown digit in the ones place (*b*), we round off 363 to tens to obtain the final result, 360 pounds.

It is not hard to see that our rule for adding approximate whole numbers will also apply to subtraction.

44-3 Addition and subtraction of approximate decimal fractions

Suppose two or more approximate numbers, written according to our basic agreement, have *the same number of decimal places*. Then all but the last digit of each number is reliable, the last digit of each being questionable. We will see that if we simply add the numbers, then all digits in the sum except the last are reliable and the last is questionable. The sum obtained is thus automatically recorded according to our basic agreement. For example:

$$\begin{array}{r} 18{\cdot}6 \\ +\ 23{\cdot}9 \\ \hline 42{\cdot}5 \\ \hline \end{array}$$

We will verify that this procedure gives the correct results by using some rather extensive reasoning based on the meaning of approximate numbers and on properties of inequalities.

From our work on order properties, we know the following statement to be always true:

If a, b, c, d, e, f are any numbers such that

$$c < a < d$$

and

$$e < b < f$$

then

$$c + e < a + b < d + f$$

You may read, "If a is between c and d, and b is between e and f, then $a + b$ is between $c + e$ and $d + f$." In other words, if two double inequalities hold, then the double inequality obtained by adding the corresponding terms also holds.

Return now to our example. Let a and b be the approximate numbers that we are representing by 18·6 and 23·9, respectively.

$$18{\cdot}55 < a < 18{\cdot}65$$

and, similarly, $23{\cdot}85 < b < 23{\cdot}95$

We add and obtain $42{\cdot}40 < a + b < 42{\cdot}60$

Therefore, $a + b = 18{\cdot}6 + 23{\cdot}9 \approx 42{\cdot}5$

We move now to the case when the approximate terms to be added have *different numbers of decimal places.* We have to be careful here, because the sum obtained in the usual way will contain worthless digits and will have to be rounded off. Consider the following example.

Let a machine weigh 3·507 kilograms, and let a wooden box, in which the machine is placed, weigh 2·8 kilograms. What is the total weight of the box with the machine inside?

To find the answer we proceed as follows.

$$(a)\quad \begin{array}{r} 3{\cdot}507 \\ +\ 2{\cdot}8 \\ \hline 6{\cdot}307 \\ \hline \end{array} \qquad (b)\quad \begin{array}{r} 3{\cdot}5\,0\,7 \\ +\ 2{\cdot}8\mathrm{UU} \\ \hline 6{\cdot}3\mathrm{UU} \\ \hline \end{array} \qquad (c)\quad \begin{array}{r} +\ 3{\cdot}507 \\ +\ 2{\cdot}8 \\ \hline 6{\cdot}307 \\ \hline 6{\cdot}3 \end{array}$$

We will now explain what we have done. In (a) we added the decimal fractions in the usual way, treating the terms as if they were exact, and not approximate, numbers. In (b) we show that such simplified addition of approximate numbers with different numbers of decimal places is inappropriate. In the sum in (a) there are worthless digits. Therefore, the answer must not be written according to our basic agreement. Actually, in the first term we know the ones, tenths, hundredths and thousandths. In the second term we know only ones and tenths, and nothing about the further decimal places. It is, therefore, clear that in the sum the hundredths digit (0) and the thousandths digit (7) do not deserve any confidence at all. They are worthless and ought to be rejected. The addition of the approximate numbers should be done as shown in (c).

Using double inequalities as we did before, we can verify that method (c) gives the correct result. Let a and b be the approximate numbers that we are representing by 3·507 and 2·8, respectively.

$$\begin{aligned} 3{\cdot}5065 &< a < 3{\cdot}5075 \\ 2{\cdot}75 \quad &< b < 2{\cdot}85 \\ \hline 6{\cdot}2565 &< a + b < 6{\cdot}3575 \\ 6{\cdot}2 \quad &< a + b < 6{\cdot}4 \\ & a + b = 3{\cdot}507 + 2{\cdot}8 \approx 6{\cdot}3 \end{aligned}$$

Notice that the left-hand sum, 6·2565, is not rounded off according to the fundamental rule, but is rounded *down*. The reason is this. If $6{\cdot}2 < 6{\cdot}2565$ and $6{\cdot}2565 < a + b$, then it follows, by transitivity of "less than", that $6{\cdot}2 < a + b$. But it does not follow that $6{\cdot}3 < a + b$. Similarly, the right-hand sum is rounded *up*.

Subtraction can be treated in exactly the same way. For example, let us subtract the approximate number 14·2714 from the approximate number 42·7.

$$\begin{array}{r} 42{\cdot}7 \\ -14{\cdot}2714 \\ \hline 28{\cdot}4286 \\ \hline \end{array} \qquad \begin{array}{r} 42{\cdot}7\text{UUU} \\ -14{\cdot}2\ 7\ 1\ 4 \\ 28{\cdot}5\text{UUU} \end{array} \qquad \begin{array}{r} 42{\cdot}7 \\ -14{\cdot}2714 \\ \hline 28{\cdot}4286 \\ \hline 28{\cdot}4 \end{array}$$

Again the correct way of recording the subtraction is the third one.

We will verify that this procedure for subtracting approximate numbers is correct by again using the meaning of approximate numbers and properties of inequalities. To do it, we first have to obtain some more facts about inequalities.

We saw that if $c < a < d$ and $e < b < f$ are true statements, then $c + e < a + b < d + f$. Do you think the corresponding subtraction inequalities are true? Is $c - e < a - b < d - f$? In fact, this is not true. To show that the statement is false, it is enough to give one instance when it is false. Suppose a and b are such that

$$2 < a < 3$$

and

$$1 < b < 3$$

Then the subtraction inequalities would be

$$1 < a - b < 0$$

But the difference $a - b$ cannot at the same time be greater than 1 and less than 0. So our statement is false.

Is there not another way we can use inequalities in subtraction? It is not hard to see that the following statement does hold:

If $c < a$

and $e > b$

then $c - e < a - b$

The proof is very simple. $c < a$ and $e > b$ mean the same thing as $c < a$ and $-e < -b$. We can add the last two inequalities and obtain

$$c + (-e) < a + (-b)$$

that is, $c - e < a - b$

Check this statement by substituting any numbers for a, b, c, e. When the last statement is extended to double inequalities, we obtain the following true statement.

If a, b, c, d, e, f; are any numbers such that

$c < a < d$

and $e > b > f$

then $c - e < a - b < d - f$

On the left, from a number less than a, we subtract a number greater than b.

On the right, from a number greater than a, we subtract a number less than b.

With the help of our new double subtraction inequality, we can see that our method of subtracting approximate numbers is correct. Here is the previous problem.

$$42{\cdot}65 < a < 42{\cdot}75$$
$$14{\cdot}27145 > b > 14{\cdot}27135$$
$$28{\cdot}37855 < a - b < 28{\cdot}5$$
$$a - b \approx 28{\cdot}4$$

From the discussion of the examples on addition and subtraction of approximate numbers, it is clear that even one unknown digit in any place makes the digit in that place in the answer worthless. Therefore, in addition and subtraction of approximate decimal fractions we will use the following rule.

ADDITION AND SUBTRACTION RULE

When adding or subtracting approximate decimal fractions, we preserve only as many decimal places in the result as there are in the approximate term with the least number of decimal places.

We clearly see that our addition and subtraction rule is based on the idea of *decimal places.*

The addition rule for approximate whole numbers that we have already discovered is really contained here. To see this, let us return to the problem in Section 44-2 about the population of a region. This time let us use a larger unit, say hundreds, to represent the counts. We then have:

The population of the farm land	43	hundreds
The population of the town	7·2	hundreds
The population of the first village	2·34	hundreds
The population of the second village	·88	hundreds

The problem is now reduced to adding approximate decimal fractions, and we apply our addition and subtraction rule.

(*a*)	(*b*)	(*c*)
43	43·UU	43
7·2	7·2 U	7·2
2·34	2· 3 4	2·34
·88	· 8 8	·88
53·42	52·UU	53·42
		53 hundreds

We round off the sum in (*c*) to ones because the term 43 has no decimal places.

Thus, we see that to apply our addition and subtraction rule to approximate whole numbers, we only have to avoid zeros which replace unknown or rejected digits and express the approximate terms in larger units.

EXERCISE 44-3A

1. Find the sum of the approximate numbers.

a.	**b.**	**c.**
·53	2130	2·725
·038	420	·6482
	35200	·01686

2. For Question 1a, carry out the complete analysis using double inequalities.

3. Find the differences of the approximate numbers.

a.	**b.**	**c.**
1430	5·10	8·53
− 275	− ·282	− ·0065

4. For Question 3b, carry out the complete analysis using double inequalities.
5. A rectangular field has length 1,240 yards and width 136 yards. Find the perimeter of the field.
6. A park had 7,300 trees. In one year 860 trees were cut down. How many trees remained in the park?
7. A wire was cut up into four parts of lengths 3·54 yards, ·756 yards, 8·49 yards, 1·138 yards. Find the original length of the wire.
8. A bottle of milk weighs 2·42 pounds. The weight of the bottle is ·543 pounds. What is the weight of the milk?

Chapter 45

MULTIPLICATION AND DIVISION OF APPROXIMATE NUMBERS

45-1 Introduction

In this chapter we will discuss the remaining two operations on approximate numbers—multiplication and division. In Chapter 44 we developed a procedure for adding and subtracting approximate numbers in two stages—first for approximate whole numbers, then for approximate decimal fractions. At the end of the chapter, however, we showed that addition and subtraction of approximate whole numbers can easily be reduced to the same operations on approximate decimal fractions. We will find the same situation in multiplication and division. As a matter of fact, we will not even find it important to distinguish between the two types of approximate numbers. Instead, we will simply discuss a single procedure for multiplying and dividing approximate numbers.

45-2 Multiplication

Consider a very simple problem of the same kind as at the beginning of Chapter 44. Suppose the sides of a rectangular field are 254 yards and 194 yards long, measured with a precision to the nearest yard. To find the area of the field, we multiply 254 and 194 and obtain 49,276 square yards. Since the measures of the sides are approximate numbers, it is clear that their product only approximately gives the area and probably has to be rounded off. Again, we must ask which digits of this product should be retained in the final result.

Because of the precision of the measurement, we know that the unknown exact values of the length and of the width of the field are greater than or equal to 253·5 yards and 193·5 yards, respectively. And we know they are less than 254·5 yards and 194·5 yards, respectively. Therefore, the rectangular area is

greater than 49,052·25 square yards (253·5 × 193·5), but less than 49,500·25 square yards (254·5 × 194·5).

As you can see, we have just used the following true statement about inequalities:

If a, b, c, d, e, f are any *non-negative* numbers such that

$$c < a < d$$

and

$$e < b < f$$

then

$$c \times e < a \times b < d \times f$$

The assumption that all the numbers involved are non-negative is essential. Can you give an example that shows that the statement is not necessarily true if you allow some of the numbers to be negative?

We can now set down the solution to our problem as follows:

$$253{\cdot}5 < a < 254{\cdot}5$$
$$193{\cdot}5 < b < 194{\cdot}5$$

Therefore,

$$49{,}052{\cdot}25 < a \times b < 49{,}500{\cdot}25$$
$$49{,}000 < a \times b < 49{,}500$$
$$a \times b \approx 49{,}300$$

This longer procedure shows us that the first two digits of our original product 49,276 for the area are reliable (4 and 9), and the third digit (2) is questionable. According to our basic agreement, only these first three digits ought to be preserved. This means that we have to round off the product 49,276 to hundreds to obtain the final result of 49,300 square yards.

This problem showed us that if we multiply two approximate numbers, each having three significant digits, their product is an approximate number also containing three significant digits.

The use of "U" in place of unknown digits leads to the same result:

```
(a)   254        (b)   2 5 4 U        (c)   254
      194              1 9 4 U              194
      ----             -------              ----
      1016             UUUU                 1016
     2286            1 0 1 6 U            2286
     254           2 2 8 6 U              254
     -----         2 5 4 U                -----
     49276         4 9 2 UUUU             49300
```

Rounding off the product 49,276 to three significant digits as required by (*b*), we obtain the final answer 49,300 (*c*).

In a similar way we can show that if we multiply two approximate numbers, one with three significant digits and the other with two significant digits, the product is an approximate number with two significant digits. In general, the product of two approximate numbers will be an approximate number with as many significant digits as the number of significant digits in the factor with the lesser number of significant digits. As we will illustrate later, exactly the same can be said about division.

MULTIPLICATION AND DIVISION RULE

When multiplying or dividing approximate numbers, in the result we preserve as many significant digits as there are in the original approximate number with the lesser number of significant digits.

We see now clearly that our multiplication and division rule is based on the idea of *significant digits*. On the other hand, our rule for addition and subtraction is based on the idea of *decimal places*.

Here is an additional illustration of the satisfactory results obtained using our multiplication and division rule. We will consider a problem using common fractions represented by decimal fractions.

Find the product of $1\frac{10}{11}$ and $1\frac{5}{6}$, following the instructions below:

a. Represent the common fractions as decimal fractions.

b. Round off the decimal representing $1\frac{10}{11}$ to *four* significant digits.

c. Round off the decimal representing $1\frac{5}{6}$ to *three* significant digits.

$$1\tfrac{10}{11} = 1{\cdot}9090\ldots \approx 1{\cdot}909$$

$$1\tfrac{5}{6} = 1{\cdot}8333\ldots \approx 1{\cdot}83$$

$$1{\cdot}909 \times 1{\cdot}83 = 3{\cdot}49347 \qquad \approx 3{\cdot}49$$

The result is rounded off to *three* significant digits, as many as there are in the factor with the lesser number of significant digits.

Let us now compare the approximate result with the exact value of the product of the two fractions.

$$1\tfrac{10}{11} \times 1\tfrac{5}{6} = \tfrac{21}{11} \times \tfrac{11}{6} = \tfrac{21}{6} = \tfrac{7}{2} = 3{\cdot}50$$

Our approximate result 3·49 differs from the exact product 3·50 by only one unit in the third significant digit. In fact, we expect the last significant digit of an approximate number to be questionable. Therefore, the result we obtained by rounding off the product to three significant digits, according to our rule for multiplication and division, is satisfactory.

45-3 Division

We now consider a problem leading to division of approximate numbers. Measurements show that the weight of a piece of iron is 491 grams and that its volume is 63 cubic centimetres. Find the weight of one cubic centimetre of this iron.

$$491 \div 63 = 7{\cdot}79\ldots \approx 7{\cdot}8$$

Here the dividend is an approximate number with *three* significant digits, the divisor an approximate number with only *two* significant digits. Our rule says that we should preserve *two* significant digits in the quotient.

Let us verify our solution by assuming that the unknown exact weight of the piece of iron is greater than or equal to 490·5 grams and less than 491·5 grams, and that its unknown exact volume is greater than or equal to 62·5 cubic centimetres and less than 63·5 cubic centimetres.

We know that any quotient of positive numbers decreases if we decrease the dividend or increase the divisor. Also, a quotient of positive numbers increases if we increase the dividend or decrease the divisor.

It follows that:

$490{\cdot}5 \div 63{\cdot}5 = 7{\cdot}724\ldots$ is less than the quotient
$491{\cdot}5 \div 62{\cdot}5 = 7{\cdot}864\ldots$ is greater than the quotient

Let us make a general statement about the division of double inequalities.

If a, b, c, d, e, f are any *positive* numbers such that

$$c < a < d$$

and

$$f > b > e$$

then $$c \div f < a \div b < d \div e$$

A number *less* than a is divided by a number *greater* than b.

A number *greater* than a is divided by a number *less* than b.

Do you see a similarity between this statement and the statement involving subtraction of double inequalities?

Can you illustrate by an example that this is not necessarily true if any of the numbers involved are negative?

Here is a solution of our problem using the statement about division of double inequalities.

$$490{\cdot}5 < a < 491{\cdot}5$$
$$63{\cdot}5 > b > 62{\cdot}5$$

Therefore $490{\cdot}5 \div 63{\cdot}5 < a \div b < 491{\cdot}5 \div 62{\cdot}5$

that is, $7{\cdot}724\ldots < a \div b < 7{\cdot}864\ldots$

$$7{\cdot}7 < a \div b < 7{\cdot}9$$
$$a \div b \approx 7{\cdot}8$$

We see that in the quotient $491 \div 63 = 7{\cdot}79\ldots$, obtained at the beginning of our problem, the first digit is reliable, the second digit is questionable and the remaining digits are worthless. We conclude therefore that our rounding off to two significant digits (7·8) is correct, and in accord with our basic agreement for recording approximate numbers.

We obtain the same result, if we write "U" instead of the unknown digits in the division.

```
          7·8 U
63·U)491·UU
     441 U
      50 UU
      50 4 U
         UU
```

45-4 Operations with one approximate number and one exact number

Our multiplication and division rule can also be applied in multiplication or division when one number is an approximate number and the other is an *exact* number. For example, multiply an exact number. A satisfactory answer is obtained by preserv-

ing in the product four significant digits, the same number of significant digits as in the approximate factor. The number of significant digits in the exact number is simply not taken into account.

Multiply the approximate number 24·3 by the exact number 34. In the product 826·2 we keep three significant digits, because the approximate factor 24·3 has three significant digits. Therefore, the answer is

$$24{\cdot}3 \times 34 \approx 826$$

We conclude with the following rule:

When applying our multiplication and division rule to multiplication or division of an approximate number by an exact number, we keep in the final answer as many significant digits as in the approximate number (and disregard the number of significant digits of the exact number).

EXERCISE 45-4A

1. Find the products of the approximate numbers.

a. ·53	**b.** 4800	**c.** 1928
·06	523	·00552

2. For Question 1a, carry out the complete analysis using double inequalities.

3. Find the product of the approximate number ·431 and the exact number 54.

4. A shop received 183 boxes. Each box contained 24 pounds of oranges. Find the weight of the oranges received.

5. Find the quotients of the approximate numbers.
   a. ·06 ÷ 2·3 **b.** 800 ÷ 35 **c.** ·385 ÷ 27

6. Find the quotient of the approximate number 2,600 divided by the exact number 165.

7. Find the product $\frac{85}{16} \times \frac{8}{23}$ in two ways.
   a. Represent the common fractions as decimal fractions and take each of them with a precision to the nearest thousandth.
   b. Multiply the common fractions and represent the result as a decimal fraction. Then take it with a precision to the nearest thousandth.

8. Find the quotient $\frac{16}{3} \div \frac{45}{4}$ with a precision to the nearest hundredth in two ways.

a. Represent each common fraction as a decimal fraction with two decimal places. Divide the decimal fractions.

b. Divide the common fractions and represent the quotient as a decimal fraction with two decimal places.

Chapter 46

COMBINED OPERATIONS ON APPROXIMATE NUMBERS

46-1 Introduction

In the previous two chapters we learned how to add, subtract, multiply and divide approximate numbers. Many simple problems, however, require a combination of these operations.

For example, suppose measurement of the length, width and height of a parallelepiped gives the following approximate numbers:

$$\begin{aligned} &\text{length}, \quad l = 34{\cdot}7 \text{ cm.} \\ &\text{width}, \quad w = 26{\cdot}8 \text{ cm.} \\ &\text{height}, \quad h = 42{\cdot}1 \text{ cm.} \end{aligned}$$

To find the volume, V, of the parallelepiped we use the known formula

$$V = l \times w \times h = 34{\cdot}7 \times 26{\cdot}8 \times 42{\cdot}1 \text{ cubic cm.}$$

We have to perform two multiplications using our rule for multiplying or dividing approximate numbers. First we find $34{\cdot}7 \times 26{\cdot}8$. Then we multiply the product, an intermediate result, by $42{\cdot}1$ to get the final result. Note the distinction we make between the final result, obtained after the last operation is performed, and the intermediate results obtained at earlier stages.

If we are asked to find the total surface area of the same solid we have

$$\begin{aligned} S &= (l \times w) + (l \times w) + (l \times h) + (l \times h) + (w \times h) + (w \times h) \\ &= (2 \times l \times w) + (2 \times l \times h) + (2 \times w \times h) \\ &= (2 \times 34{\cdot}7 \times 26{\cdot}8) + (2 \times 34{\cdot}7 \times 42{\cdot}1) + (2 \times 26{\cdot}8 \times 42{\cdot}1) \end{aligned}$$

Here we have to multiply six times and add twice.

The rules concerning the order of performing combined operations on exact numbers are also valid for approximate numbers. Thus to find the surface area, S, we proceed in the following order. First we perform all multiplications, and then

add the three terms of the sum. How many intermediate results will we have?

Could we solve problems involving more than one operation on approximate numbers by rounding off the intermediate results by our rules? If we did this we would increase the errors of our approximate numbers even more by rounding off. The accumulation of rounding-off errors could substantially influence the final result. It turns out that in many cases of combined operations, this influence is greatly reduced in the final result if in each of the intermediate results we preserve one more digit than our rules for operations on approximate numbers tell us to preserve. Let us underline these extra digits. In the final result, however, we will reject the extra digit. Here is the rule we will use.

INTERMEDIATE RESULT RULE

In solving problems which involve more than one operation on approximate numbers, we preserve in the intermediate results one more digit than recommended by our rules for operations on approximate numbers. In determining the number of significant digits in an intermediate result, the extra digit is not counted, according to our basic agreement for rounding off approximate numbers.

46-2 An example

We will now illustrate this procedure for combined operations. Compute the value of the quotient

$$\frac{2\frac{13}{16} \times 1\frac{7}{18}}{(2\frac{13}{16} - 1\frac{7}{18}) \times \frac{2}{7}}$$

Let us first represent the given common fractions as decimal fractions and then round them off to hundredths. We have

$$2\tfrac{13}{16} = 2{\cdot}8125\ldots \approx 2{\cdot}81$$
$$1\tfrac{7}{18} = 1{\cdot}3888\ldots \approx 1{\cdot}39$$
$$\tfrac{2}{7} = {\cdot}2857\ldots \approx {\cdot}29$$

Now we must compute

$$\frac{2{\cdot}81 \times 1{\cdot}39}{(2{\cdot}81 - 1{\cdot}39) \times {\cdot}29}$$

(1) Intermediate result:

```
  2·81
  1·39
  ----
  2529
  843
 281
 -----
 39059
 -----
 3·906
```

(2) Intermediate result:

```
  2·81
 −1·39
 -----
  1·42
```

(3) Intermediate result:

```
   1·42
    ·29
  12 78
  28 4
  -----
  ·4118
  -----
  ·412
```

(4) Final result:

$3{\cdot}90x \div 41x$
$= 9{\cdot}48 \approx 9{\cdot}5$

Final answer $= 9{\cdot}5$

As the first operation, we multiplied two approximate numbers with three significant digits each. But we rounded off the product to four significant digits, underlining the last digit (6) as the extra digit.

The second operation gave as a difference an approximate number having two decimal places and three significant digits. Note that there was no extra digit to preserve.

As the third operation, we multiplied approximate numbers with three and two significant digits. The product was rounded off to three significant digits, the third digit (2) being marked as an extra digit.

The fourth operation consisted in dividing an approximate number with three significant digits by an approximate number with two significant digits (the extra digits do not enter into this count). This gives the final result and therefore we reject the extra digit, preserving only two significant digits.

Let us check our approximate answer by calculating the exact answer using the original common fractions. We obtain

$$9\tfrac{49}{164} = 9{\cdot}603\ldots$$

We see, therefore, that in the approximate value 9·5 only the last digit was questionable. This is in accordance with our basic agreement for recording approximate numbers.

EXERCISE 46-2A

1. Perform the operations on the approximate numbers.
   a. $2{\cdot}98 - (1{\cdot}4 + {\cdot}387)$
   b. $23{,}000 - (2{,}645 + 15{,}300 - 1{,}639)$
   c. $(562 \div 87) \times 7$
   d. $2{\cdot}75 + (1{\cdot}2 - {\cdot}30103)$
   e. $36{,}408 \div (236 \times 28)$
   f. $5{,}325 \times [(832{,}860 \div 211) \times 37]$
   g. $[4{\cdot}5 - ({\cdot}03 \times 1{\cdot}5)] \div 7{\cdot}8$
2. Find the volume, V, of the parallelepiped with length, $l = 34{\cdot}7$ cm., width, $w = 26{\cdot}8$ cm., and height, $h = 42{\cdot}1$ cm.
3. Find the total surface area of the parallelepiped in Question 2.

ANSWERS TO SELECTED PROBLEMS

CHAPTER 22

Exercise 22-2A

4. You cannot subtract $\frac{9}{5}$ from both sides of $\frac{5}{3} < \frac{9}{2}$, because $\frac{5}{3} - \frac{9}{5}$ is not yet defined.

Exercise 22-6A

4. A somewhat formal way you could use would be this. Since

$$\frac{a}{b} \div \frac{m}{n} = \frac{a}{b} \times \frac{n}{m}\left(\text{if } \frac{m}{n} \neq 0\right) \text{ and } \frac{c}{d} \div \frac{m}{n} = \frac{c}{d} \times \frac{n}{m}$$

the equality

$$\frac{a}{b} \div \frac{m}{n} < \frac{c}{d} \div \frac{m}{n}$$

is the same as the inequality

$$\frac{a}{b} \times \frac{n}{m} < \frac{c}{d} \times \frac{n}{m}$$

CHAPTER 23

Exercise 23-1B

2. a. 103 **b.** 10217 **c.** 4579

3. a. 122_{seven} **b.** 140_{seven} **c.** 211246_{seven}

6. a. base three **b.** base five **c.** base six **d.** base seven

Exercise 23-2C

1. a. ·6	**b.** ·5	**c.** ·75	**d.** ·625
e. ·15	**f.** ·84	**g.** ·62	**h.** ·062
i. 3·4	**j.** 2·5	**k.** 6·75	**l.** 1·125
m. 2·25	**n.** 8·04	**o.** 18·42	**p.** 1·0625

Exercise 23-2D

1. a. ·5	**b.** ·25	**c.** ·75	**d.** ·6
e. ·7	**f.** ·625	**g.** ·3125	
2. a. $\cdot 1\dot{6}$	**b.** $\cdot\dot{1}4285\dot{7}$	**c.** $\cdot\dot{1}$	**d.** $\cdot\dot{0}\dot{9}$
e. $\cdot\dot{6}$	**f.** $\cdot 8\dot{3}$	**g.** $\dot{5}\cdot 3846\dot{1}$	

EXERCISE 23-3A

2. a. $\frac{3}{8}$ **b.** $\frac{1}{2}$ **c.** $\frac{1}{18}$ **d.** $\frac{1}{2}$ **e.** $\frac{17}{9}$ **f.** $\frac{7}{3}$ **g.** $\frac{17}{5}$ **h.** $\frac{9}{8}$

3. a. ·1 **b.** ·111 . . . **c.** ·2 **d.** ·222 . . .
e. ·3 **f.** ·333 . . . **g.** ·4

CHAPTER 24

EXERCISE 24-1A

1. a. 813 **b.** 44·8 **c.** 126·62 **d.** 23·458

EXERCISE 24-1B

2. a. 2·3 is greater by 0·71 **b.** 87·32 is greater by 63·202
c. 0·12 is greater by 0·1183 **d.** 0·3 is greater by 0·132

3. a. 12·42 inches **b.** 2·41 inches

EXERCISE 24-2B

1. a. 7 **b.** 0·9 **c.** 3·2 **d.** 57·2
e. 0·07 **f.** 0·309 **g.** 9·732 **h.** 10·572

2. a. 3200 **b.** 750 **c.** 12 **d.** 1940·1
e. 7·63 **f.** 0·762 **g.** 79·321 **h.** 0·01

EXERCISE 24-3A

1. a. 44·4 **b.** 729·84 **c.** 2·01 **d.** 58·21 **e.** 12·74 **f.** 1

2. a. 3 **b.** 1 **c.** 8 **d.** 10 **e.** 50 **f.** 8
g. 0 **h.** 3 **i.** 40

3. a. 42 **b.** 5·27 **c.** 29 **d.** 5·69 **e.** 0·25302 **f.** 6·904

EXERCISE 24-3B

2. a. 8·655 **b.** 1·772 **c.** 0·857

3. a. 0·6595 **b.** 12·336 **c.** 0·77

EXERCISE 24-4A

1. a. 0·6 **b.** 5·6 **c.** 0·03 **d.** 0·01 **e.** 0·2 **f.** 9·9
g. 79 **h.** 1·44 **i.** 0·005 **j.** 0·15 **k.** 0·714 **l.** 10

2. a. 8·3366 **b.** 0·115605 **c.** 506·11 **d.** 0·0001218

EXERCISE 24-5A

2. a. 40 **b.** 70 **c.** 600 **d.** 8 **e.** 800 **f.** 0·08
g. 50 **h.** 1100 **i.** 33

3. a. 1·46 **b.** 0·21 **c.** 55·37 **d.** 0·98

EXERCISE 24-6A

1.

Percentage	50	25	$12\frac{1}{2}$	75	20	10
Common fraction	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{3}{4}$	$\frac{1}{5}$	$\frac{1}{10}$
Decimal fraction	0·5	0·25	0·125	0·75	0·2	0·1

Percentage	5	$2\frac{1}{2}$	$33\frac{1}{3}$	60	35
Common fraction	$\frac{1}{20}$	$\frac{1}{40}$	$\frac{1}{3}$	$\frac{3}{5}$	$\frac{7}{20}$
Decimal fraction	0·05	0·025	$0{\cdot}3\dot{3}$	0·6	0·35

2. **a.** £$4\frac{1}{8}$ **b.** £$4\frac{1}{5}$ **c.** $50\frac{2}{5}$ shillings **d.** \$12·48

EXERCISE 24-6B

1. **a.** $\frac{7}{200}$ **b.** $\frac{1}{3}$ **c.** $\frac{13}{20}$ **d.** $\frac{11}{200}$ **e.** $\frac{17}{100}$ **f.** $\frac{13}{80}$

2. **a.** $66\frac{2}{3}$% **b.** 40% **c.** 400% **d.** 80% **e.** 32% **f.** 1% **g.** 7% **h.** $53\frac{1}{3}$%

3. **a.** 3 minutes **b.** 91 days **c.** 32 shillings

4. **a.** 11% **b.** 50% **c.** 300% **d.** 80% **e.** 9% **f.** $68\frac{3}{4}$ marks

5. **a.** 16 **b.** 180 **c.** 2,307·6 **d.** £1765·6

CHAPTER 25

EXERCISE 25-1A

1. 3 **2.** No whole-number answer

3. 7 **4.** 7

5. No whole-number answer **6.** No whole-number answer

7. 0

CHAPTER 26

EXERCISE 26-3A

1. **a.** Latitude S 30° **b.** Longitude E 45°
c. Temperature 15° below zero **d.** 10 minutes before the hour
e. A loss of 7 shillings **f.** A gain of 50 shillings

2. **a.** pos 1 **b.** neg 11 **c.** neg 17 **d.** pos 73

e. neg 129 **f.** pos 8 **g.** neg 42 **h.** neg 9
i. neg 23 **j.** pos 14 **k.** zero

EXERCISE 26-4A

1. **a.** pos 6 < pos 10 **b.** neg 6 > neg 10
   c. pos 15 > neg 15 **d.** neg 15 < pos 15
   e. neg 200 > neg 1000 **f.** 0 > neg 3
   g. 0 < pos 8 **h.** neg 11 < 0

EXERCISE 26-4B

1. **a.** The first place is north of the second place.
   The second place is south of the first place.
   b. My watch is ahead of my friend's watch.
   My friend's watch is behind my watch.
   c. The man on the platform is below the man on the ground.
   The man on the ground is above the man on the platform.
   d. Noon today is colder than noon yesterday.
   Noon yesterday was hotter than noon today.
   e. Kofi starts ahead of Kwesi.
   Kwesi starts behind Kofi.

EXERCISE 26-4C

2. neg 5 > neg 8; pos 5 < pos 8
3. pos 2 < pos 11; neg 2 > neg 11
4. pos 2 > neg 11; neg 2 < pos 11
5. pos 7 > 0; neg 7 < 0
6. 0 > neg 2; 0 < pos 2
7. neg 6 < neg 1; pos 6 > pos 1
8. pos 10 > 0; neg 10 < 0

EXERCISE 26-5A

1. **a.** {pos 21, pos 22, pos 23, pos 24} **b.** {pos 1, 0, neg 1}
   c. {neg 2, neg 1} **d.** {neg 6} **e.** { } **f.** { } **g.** {pos 1}
2. **a.** Set of integers between neg 3 and 0
   b. Set of integers between pos 18 and pos 23
   c. Set of integers between pos 2 and neg 2
   d. Set of integers between neg 1 and pos 1
   e. Set of integers between pos 101 and pos 102

CHAPTER 27

EXERCISE 27-1A

1. **a.** pos 6 **b.** pos 7 **c.** pos 4 **d.** pos 3 **e.** pos 2 **f.** pos 8
2. **a.** pos 4 **b.** pos 4 **c.** pos 4 **d.** pos 6 **e.** pos 6 **f.** 0

EXERCISE 27-2B

1. **a.** pos 3 **b.** neg 1 **c.** 0 **d.** pos 6
   e. pos 3 **f.** neg 2 **g.** neg 8 **h.** pos 6
   i. 0 **j.** neg 6

EXERCISE 27-2C

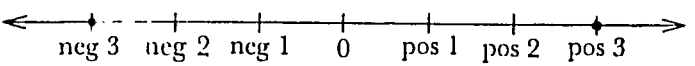
2. **a.** pos 2 − pos 1 = □; pos 2 − pos 1 = pos 1
   b. pos 3 − neg 5 = □; neg 5 + pos 8 = pos 3
   c. 0 − neg 4 = □; neg 4 + pos 4 = 0
   d. neg 1 − neg 5 = □; neg 5 − neg 4 = neg 1
   e. pos 3 − 0 = □; 0 + pos 3 = pos 3
   f. pos 1 − pos 7 = □; pos 7 − pos 6 = pos 1
   g. neg 3 − pos 3 = □; pos 3 − pos 6 = neg 3
   h. neg 11 − neg 2 = □; neg 2 + neg 9 = neg 11
3. **a.** N 29° − S 50° = □; S 50° + □ = N 29°;
   S 50° + N 79° = N 29°
   b. 3.10 p.m. − 2.55 p.m. = □
   2.55 p.m. + □ = 3.10 p.m.
   2.55 p.m. + 15 min. = 3.10 p.m.
   c. −53 ft. − 0 = □; 0 + □ = −53 ft.;
   0 − 53 ft. = −53 ft.
   d. 95° − 102° = □; 102° + □ = 95°;
   102° − 7° = 95°
   e. 5 ft. ahead − 5 ft. behind = □
   5 ft. behind + □ = 5 ft. ahead
   5 ft. behind + 10 ft. = 5 ft. ahead

EXERCISE 27-3A

1. **a.** The set of integers is closed under addition.
   b. The set of integers is closed under subtraction.
2. The set of fractions, excepting 0, is closed under division.

EXERCISE 27-3B

1. **a.** pos 3 + $\boxed{\text{neg 3}}$ = 0 **b.** neg 3 + $\boxed{\text{pos 3}}$ = 0
   c. $\boxed{\text{neg 1}}$ + pos 1 = 0 **d.** $\boxed{\text{pos 5}}$ + neg 5 = 0
   e. $\boxed{\text{pos 9}}$ + $\boxed{\text{neg 9}}$ = 0 **f.** $\boxed{0}$ + 0 = 0 **g.** opposites
   h.



   i. Their sum is 0.

EXERCISE 27-3C

1. a. pos 3 − pos 1 = $\boxed{\text{pos 2}}$

pos 3 + neg 1 = $\boxed{\text{pos 2}}$

pos 3 − pos 1 = pos 3 + $\boxed{\text{neg 1}}$

b. neg 3 + neg 1 = $\boxed{\text{neg 4}}$

neg 3 − pos 1 = $\boxed{\text{neg 4}}$

neg 3 + neg 1 = neg 3 − $\boxed{\text{pos 1}}$

c. pos 2 − pos 6 = $\boxed{\text{neg 4}}$

pos 2 + neg 6 = $\boxed{\text{neg 4}}$

pos 2 − pos 6 = pos 2 + $\boxed{\text{neg 6}}$

d. 0 − pos 7 = $\boxed{\text{neg 7}}$

0 + neg 7 = $\boxed{\text{neg 7}}$

0 − pos 7 = 0 + $\boxed{\text{neg 7}}$

e. 0 − neg 7 = $\boxed{\text{pos 7}}$

0 + $\boxed{\text{pos 7}}$ = pos 7

0 − neg 7 = 0 + $\boxed{\text{pos 7}}$

CHAPTER 28

EXERCISE 28-3A

2. a. −4 **b.** 2 **c.** 0 **d.** (−6)

EXERCISE 28-4B

1. a. pos 3 + neg 4 = pos 3 − (opp. of neg 4)
= pos 3 − pos 4
= neg 1

or 3 + (−4) = 3 − [opp. of (−4)]
= 3 − 4 = (−1)

b. neg 3 + neg 3 = neg 3 − (opp. of neg 3)
= neg 3 − pos 3
= neg 6

or (−3) + (−3) = (−3) − [opp. of (−3)]
= (−3) −3
= (−6)

c. $0 + \text{neg } 1 = 0 - [\text{opp. of neg } 1]$
$= 0 - [\text{pos } 1]$
$= \text{neg } 1$
or $0 + (-1) = 0 - [\text{opp. of } (-1)]$
$= 0 - 1$
$= (-1)$

d. $0 - \text{neg } 4 = 0 + [\text{opp. of neg } 4]$
$= 0 + [\text{pos } 4]$
$= \text{pos } 4$
or $0 - (-4) = 0 + [\text{opp. of } (-4)]$
$= 0 + 4$
$= 4$

e. $\text{pos } 3 - \text{neg } 2 = \text{pos } 3 + [\text{opp. of neg } 2]$
$= \text{pos } 3 + [\text{pos } 2]$
$= \text{pos } 5$
or $3 - (-2) = 3 + [\text{opp. of } (-2)]$
$= 3 + 2$
$= 5$

f. $\text{neg } 5 - \text{neg } 2 = \text{neg } 5 + [\text{opp. of neg } 2]$
$= \text{neg } 5 + \text{pos } 2$
$= \text{neg } 3$
or $-5 - (-2) = (-5) + [\text{opp. of } (-2)]$
$= (-5) + 2$
$= (-3)$

g. $\text{neg } 2 - \text{neg } 2 = \text{neg } 2 + [\text{opp. of neg } 2]$
$= \text{neg } 2 + \text{pos } 2$
$= 0$
or $(-2) - (-2) = (-2) + [\text{opp. of } (-2)]$
$= (-2) + 2$
$= 0$

h. $\text{neg } 7 + \text{neg } 3 = \text{neg } 7 - [\text{opp. of neg } 3]$
$= \text{neg } 7 - [\text{pos } 3]$
$= \text{neg } 7 - \text{pos } 3$
$= \text{neg } 10$
or $(-7) + (-3) = (-7) - [\text{opp. of } (-3)]$
$= (-7) - 3$
$= -10$

EXERCISE 28-5A

1. a. $\text{pos } 8 - \text{neg } 3 = 8 - (-3) = 8 + 3 = 11$
b. $\text{neg } 3 + (\text{neg } 2 + \text{pos } 1) = \text{neg } 3 + \text{neg } 1 = \text{neg } 4 = (-4)$
c. $[7 + (-2)] + 8 = [7 - 2] + 8 = 5 + 8 = 13$
d. $(-4) + [(-2) + (-1)] = (-4) + [(-2) - 1]$
$= (-4) + (-3) = (-7)$

e. $[(-3) - (-5)] - (-3) = [(-3) + 5] + 3 = 2 + 3 = 5$
f. $(-8) - [(-4) + (-2)] = (-8) - [(-4) - 2]$
$= (-8) - (-6) = (-8) + 6 = (-2)$

2. a. $(-2) - (3 - 8) = (-2) + 8 - 3) = (-2) + 5 = 3$
b. $6 - (2 - 1) = 6 + (1 - 2) = 6 - 1 = 5$
c. $(-4) - [(-2) - 6] = (-4) + [6 - (-2)]$
$= (-4) + [6 + 2] = (-4) + 8 = 4$
d. $8 - [5 - (-4)] = 8 + [(-4) - 5] = 8 + (-9) = (-1)$

CHAPTER 29

Exercise 29-2A

1. 11 **2.** (-3) **3.** (-8) **4.** 5 **5.** 11 **6.** 1

Exercise 29-3A

a. (-10) **b.** 24 **c.** (-21) **d.** 18 **e.** (-99) **f.** 136

Exercise 29-4A

a. (-2) **b.** (-6) **c.** 3 **d.** (-6) **e.** 2 **f.** (-6) **g.** -3

CHAPTER 30

Exercise 30-1A

Those using integers: 1, 3, 4, 6, 8.
Those needing new numbers: 2, 5, 7.

CHAPTER 31

Exercise 31-1A

1. $-\frac{16}{7}$ **2.** $\frac{38}{5}$ **3.** $-\frac{20}{17}$ **4.** -1 **5.** 0 **6.** $-\frac{153}{10}$

Exercise 31-2A

$-\frac{27}{10}$, $-\frac{3}{4}$, $-\frac{3}{13}$, 0, $\frac{4}{7}$, $\frac{55}{8}$, 10

CHAPTER 32

Exercise 32-1A

1. $\frac{11}{15}$ **2.** 0 **3.** $\frac{3}{10}$ **4.** $-\frac{7}{6}$ **5.** 0 **6.** -1

Exercise 32-2A

1. $\frac{26}{21}$ **2.** $\frac{8}{3}$ **3.** $-\frac{2}{3}$ **4.** $\frac{89}{100}$

EXERCISE 32-3A

1. $-\frac{3}{8}$ **2.** $-\frac{16}{15}$ **3.** $-\frac{7}{8}$ **4.** $-\frac{35}{528}$ **5.** 0 **6.** 1

EXERCISE 32-3C

1. $\frac{9}{25}$ **2.** $\frac{16}{81}$ **3.** $\frac{1}{4}$ **4.** $\frac{9}{4}$

EXERCISE 32-4A

1. $-\frac{45}{22}$ **2.** $-\frac{1}{4}$ **3.** $-\frac{2}{3}$ **4.** $-\frac{12}{5}$ **5.** -24 **6.** 1

EXERCISE 32-4E

1. $\frac{2}{3}$ **2.** $-\frac{1}{5}$ **3.** 1·7 **4.** -2

CHAPTER 34

EXERCISE 34-1A

1. a. Not closed under any
b. Under addition, subtraction and multiplication
c. Under addition and multiplication
d. Not closed under any
e. Under addition, subtraction and multiplication
f. Closed under multiplication

CHAPTER 35

EXERCISE 35-1B

1. (ii) **a.** $\frac{8}{10}$ and $\frac{8}{10}$, same number
b. $\frac{12}{20}$ and $\frac{15}{20}$, $\frac{3}{4}$ is larger
c. $\frac{9}{2}$ and $\frac{10}{2}$, 5 is larger.
d. $\frac{5}{3}$ and $\frac{4}{3}$, $\frac{5}{3}$ is larger.
(iii) **a.** 0 **b.** $\frac{3}{20}$ **c.** $\frac{1}{2}$ **d.** $\frac{1}{3}$

2. a. $\frac{2}{5}$; $\frac{1}{2}$; $\frac{2}{3}$; $\frac{3}{4}$; $\frac{5}{6}$
b. $\frac{2}{3}$; $\frac{5}{3}$; $2\frac{1}{2}$; $\frac{11}{4}$; 3
c. $1\frac{1}{6} + 4\frac{1}{2}$; $5\frac{3}{4} + \frac{1}{8}$; $2\frac{1}{2} + 3\frac{2}{3}$; $3\frac{1}{2} + 2\frac{3}{4}$

EXERCISE 35-1C

1. 0·035, 0·35, 1·035, 1·35, 2, 2·25, 3·5, 10·35, 17·5

EXERCISE 35-2A

1. a. False **b.** True $-16 = 32 + (-48)$ **c.** False
d. False **e.** False **f.** False **g.** False
h. True $-25 = 26 + (-51)$ **i.** True $36 = 0 + 36$
j. False **k.** True $-3 = -6 + 3$ **l.** False

m. True $-\frac{3}{2} = 0 + (-\frac{3}{2})$

n. True $[-4 + (-2)] = [-2 + 2] + (-6)$ **o.** False

3. a. $\frac{4}{5} > -\frac{8}{10}$; $\frac{4}{5} = -\frac{8}{10} + \frac{8}{5}$

b. $2{\cdot}5 > -5{\cdot}5$; $2{\cdot}5 = -5{\cdot}5 + 8$

c. $-\frac{3}{5} > -\frac{3}{4}$; $-\frac{3}{5} = -\frac{3}{4} + \frac{3}{20}$

d. $-\frac{4}{3} > -\frac{5}{3}$; $-\frac{4}{3} = -\frac{5}{3} + \frac{1}{3}$

e. $-\frac{9}{2} > -5$; $-\frac{9}{2} = -5 + \frac{1}{2}$

f. $3{\cdot}75 > 2{\cdot}25$; $3{\cdot}75 = 2{\cdot}25 + 1{\cdot}5$

5. a. $6 > -3$; $-3 < 6$ **b.** $-2 > -8$; $-8 < -2$

c. $0 > 4$; $-4 < 0$ **d.** $12 > 9$; $9 < 12$

e. $3{\cdot}25 > -3$; $-3 < 3{\cdot}25$ **f.** $\frac{3}{4} > \frac{2}{3}$; $\frac{2}{3} < \frac{3}{4}$

g. $-9 > -12$; $-12 < -9$ **h.** $2 > -8$; $-8 < 2$

EXERCISE 35-4A

1. $<$ **2.** $=$ **3.** $>$ **4.** $>$ **5.** $<$ **6.** $<$ **7.** $=$ **8.** $>$

EXERCISE 35-4B

1. a. $-5{\cdot}2 < 0 < 2{\cdot}5$ **b.** $-\frac{1}{2} < -\frac{1}{3} < -\frac{1}{4}$

c. $\frac{1}{4} < \frac{1}{3} < \frac{1}{2}$ **d.** $2{\cdot}05 < 2{\cdot}25 < 25$

e. $-6 < 4 < 5$ **f.** $-\frac{3}{4} < \frac{3}{4} < \frac{4}{3}$ **g.** $-\frac{12}{5} < -2 < -\frac{9}{5}$

EXERCISE 35-4C

1. a. $\frac{287}{29} < \frac{340}{31}$ **b.** $-\frac{79}{17} > -\frac{97}{19}$ **c.** $\frac{16}{3} > \frac{19}{5}$ **d.** $\frac{111}{7} < \frac{190}{11}$

EXERCISE 35-8A

1. a. $>$ **b.** $<$ **c.** $>$ **d.** $>$ **e.** $>$ **f.** $<$ **g.** $<$ **h.** $<$

CHAPTER 36

EXERCISE 36-2A

1. a. $\cdot 11111\ldots$ **b.** $\cdot 010101\ldots$ **c.** $\cdot 001001\ldots$

d. $\cdot 4545\ldots$ **e.** $\cdot 142857142857\ldots$

f. $\cdot 285714285714\ldots$ **g.** $\cdot 027027\ldots$

EXERCISE 36-3A

1. a. $\frac{2}{9}$ **b.** $\frac{23}{99}$ **c.** $\frac{234}{999} = \frac{26}{111}$ **d.** $\frac{1}{9}$

e. $\frac{1}{99}$ **f.** $\frac{1}{999}$ **g.** $\frac{1}{9}$ **h.** $\frac{109}{999}$

2. a. $\cdot 222\ldots = 2 \times (\cdot 111\ldots)$
$= 2 \times \frac{1}{9} = \frac{2}{9}$

b. $\cdot 2323\ldots = 23 \times (\cdot 0101\ldots)$
$= 23 \times (\frac{1}{99}) = \frac{23}{99}$

c. $\cdot 234234\ldots = 234 \times (\cdot 001001\ldots)$
$= 234 \times \frac{1}{999} = \frac{234}{999} = \frac{26}{111}$

EXERCISE 36-4A

Some possible answers are:

a. ·2020020002 . . .
b. ·01011011101111 . . .
c. ·030330333 . . .
d. ·100100001000000100 . . .
(ones at 1st, 4th, 9th, 16th places and so on)
e. The same as **d.** with 1's replaced by 2, 3, 4, . . . or 9.
f. ·234567891011 . . . **g.** ·34567891011 . . .

CHAPTER 37

EXERCISE 37-1A

Successive trials might be $\frac{7}{4}$, $\frac{12}{7}$, $\dfrac{\frac{12}{7} + \frac{7}{4}}{2} = \frac{97}{56}$, $\frac{168}{97}$.

EXERCISE 37-2A

If $\frac{a}{b} \times \frac{a}{b} = 3$, then $\frac{a}{b} = \frac{3b}{a}$. Also, $1 < \frac{a}{b} < 2$.

Let $\frac{a}{b}$ be in lowest terms. Then a, the denominator on the right, must be one of b, $2b$, $3b$, $4b$ and so on.

That is, $\frac{a}{b}$ must equal one of 1, 2, 3, 4, ... This is impossible since $\frac{a}{b}$ must lie between 1 and 2.

CHAPTER 38

EXERCISE 38-2A

1. $[(-3) + 3] \times a = [(-3) \times a] + (3 \times a)$
$0 \times a = [(-3) \times a] + (3 \times a)$
$0 = [(-3) \times a] + (3 \times a)$

Hence $(-3) \times a$ is the opposite of $3 \times a$, that is,

$$(-3) \times a = -(3 \times a)$$

2. By Question 1,

$$7 = (-3) \times \frac{7}{(-3)} = -\left(3 \times \frac{7}{(-3)}\right)$$

Hence,
$$3 \times \frac{7}{(-3)} = -7$$

and
$$\frac{7}{(-3)} = \frac{(-7)}{3}$$

3. $a + 0 = a$ (AO)
This holds for all values of a, in particular for $a = 0$. Hence, $0 + 0 = 0$

EXERCISE 38-4A

1. If $a < b$ $\quad b = a + p$ where $p > 0$
Then $\quad b \times c = a \times c + p \times c$ [D]
since $p > 0$ and $c > 0$, $p \times c > 0$ [0'2]
Then by definition, $\quad a \times c < b \times c$

2. If $a < b$ $\quad b = a + p, p > 0$
If $b < c$ $\quad c = b + p', p' > 0$
Then $\quad c = (a + p) + p'$
$= a + (p + p')$
Since $p > 0$ and $p' > 0$
$p + p' > 0$ [0'1]
Hence, by definition, $a < c$, as required.

3. Given a and b, any two real numbers, consider the real number $c = a - b$. By [0'3], there are three possibilities:
$c = 0$
$c > 0$
or $\quad c < 0$
That is, $\quad$ (1) $a - b = 0$
(2) $a - b > 0$
or $\quad$ (3) $a - b < 0$
Now $a = (a - b) + b$.
(1) If $a - b = 0$, $a = b$
(2) If $a - b > 0$, $a - b$ is a positive number p so that $a = p + b$. This means by definition that $a > b$
(3) If $a - b < 0$, $b - a > 0$
Since $(b - a) + a = b$
$b = a +$ a positive number.
By definition then, $a < b$

CHAPTER 39

Exercise 39-3A

1. a. 21 **b.** 25 **c.** 104 **d.** 254·5 **e.** 62·6 . . . **f.** 255·7 . . .

2. d. $254\frac{1}{2}$ **e.** $62\frac{2}{3}$ **f.** $255\frac{3}{4}$

Exercise 39-4A

1. 275·4

2. The (2) and (7) are reliable. The (5) is questionable, and the (4) is worthless.

3. 275 books

Exercise 39-4B

E means exact, A means approximate.

1. a. E **b.** A **c.** E **d.** A **e.** A **f.** A **g.** E
h. A **i.** E **j.** A **k.** A **l.** A **m.** A

3. 140 steps

4. b.

Counts	*Deviation from average*
574	2·8
562	9·2
573	1·8
567	4·2
580	8·8
571	·2

Average 571·2 Average deviation 4·5

Reliable digit: 5 Questionable: 7 Worthless: 1·2

CHAPTER 40

Exercise 40-3A

1. a. 254 **b.** ·25 **c.** 52·0

2. a. 2·8342 **c.** 2·834 pounds

3. a. 2661·25 **c.** 2660

CHAPTER 41

Exercise 41-3A

8. c. 340, 6800, 47·10, 9·002 **9.** 7·7

CHAPTER 42

Exercise 42-2A

1. ·007 **2.** ·006, ·0007, ·00003 **3.** ·1% **4.** 5·29, ·005, ·09%
5. The same **6.** The second **7.** ·8% **8.** 1·6%

CHAPTER 43

Exercise 43-5A

1. 2, 2, 3, 4, 4, 4, 4, 5 **2.** 3, 3, 4, 5, 5, 5, 5, 6 **3.** 3, 3, 4, 4, 6

4. Decimal places 1, 2, 3, 2, 3, 2, 2, 3, 4, 2, 3, 3
Significant digits 2, 2, 3, 3, 4, 5, 1, 1, 1, 3, 4, 4
Decimal places 3, 4, 3, 3, 3, 2
Significant digits 2, 4, 3, 4, 6, 6

5. Decimal places 1, 2, 3, 2, 3, 2, 2, 3, 4, 1, 2, 2
Significant digits 2, 2, 3, 3, 4, 5, 1, 1, 1, 2, 3, 3
Decimal places 2, 3, 1, 1, 1, 1
Significant digits 1, 3, 1, 2, 4, 5

CHAPTER 44

Exercise 44-3A

1. a. ·56 **b.** 37800 **c.** 3·390
3. a. 1160 **b.** 4·82 **c.** 8·52
5. 2,750 yards **6.** 6,400 **7.** 13·92 yards **8.** 1·88 pounds

CHAPTER 45

Exercise 45-4A

1. a. ·03 **b.** 2,500,000 **c.** 10·6
3. 23·3 **4.** 4,400 pounds
5. a. ·03 **b.** 20 **c.** ·014
6. 16 **7.** 1·848 **8.** ·47

CHAPTER 46

Exercise 46-2A

1. a. 1·2 **b.** 7,000 **c.** 50 **d.** 3·6
e. 5·5 **f.** 150,000 **g.** ·57

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INDEX OF MAIN SUB-HEADINGS

Unit 4

Chapter 22

22-1 Revision of order, 2; 22-2 Order and addition and subtraction, 3; 22-3 Generalized addition property, 5; 22-4 Order and multiplication by $\frac{1}{n}$, 7; 22-5 Order and multiplication by m, 8; 22-6 Order and multiplication by $\frac{m}{n}$, 10; 22-7 Generalized multiplication property, 11.

Chapter 23

23-1 Revision of decimal numeration, 14; 23-2 Decimal fractions, 20; 23-3 Fractions in other bases, 30.

Chapter 24

24-1 Addition and subtraction, 32; 24-2 Multiplication and division by powers of 10, 35; 24-3 Multiplication and division by whole numbers less than 10, 39; 24-4 Multiplication in decimal notation, 44; 24-5 Division in decimal notation, 47; 24-6 Percentage, 48.

Chapter 25

25-1 A reminder of subtraction, 56; 25-2 Physical models, 57.

Chapter 26

26-1 Naming the new numbers, 62; 26-2 Zero, 63; 26-3 Opposites, 64; 26-4 Order properties, 65; 26-5 "Between", 68; 26-6 Summary, 69.

Chapter 27

27-1 Addition, 70; 27-2 Subtraction, 75; 27-3 Relation between addition and subtraction, 84.

Chapter 28

28-1 The positive integers, 90; 28-2 The negative integers, 92; 28-3 Opposites, 95; 28-4 Addition and subtraction in the new notation, 96; 28-5 Some properties of integers, 97.

Chapter 29

29-1 Addition, using the new notation, 99; 29-2 Subtraction, using the new notation, 101; 29-3 Multiplication of integers, 104; 29-4 Division of integers, 108.

Previous Page Blank

Chapter 30

30-1 Division of integers, 112; 30-2 Division as multiplication with a missing factor, 113; 30-3 Interpreting our new numbers on the number line, 115.

Chapter 31

31-1 Fractions, positive and negative, 120; 31-2 Definition of the rational numbers, 123.

Chapter 32

32-1 Addition, 126; 32-2 Subtraction, 131; 32-3 Multiplication, 132; 32-4 Division, 136.

Chapter 33

33-1 Introduction, 139; 33-2 Sets and counting numbers, 139; 33-3 The empty set and zero, 140; 33-4 Fractions, 140; 33-5 Integers; 33-6 Rational numbers, 142; 33-7 Summary, 143.

Chapter 34

34-1 Closure under addition and multiplication, 144; 34-2 Commutative property of addition and multiplication, 146; 34-3 Associative property of addition and multiplication, 146; 34-4 Distributive property, 147; 34-5 Properties of zero and one, 148; 34-6 Opposites and reciprocals, 149; 34-7 Inverse operations, 149; 34-8 Properties summarized, 150.

Chapter 35

35-1 Order on the number line for positive rational numbers, 152; 35-2 Order on the number line for the set of rational numbers, 155; 35-3 "Between" for rational numbers, 157; 35-4 Two basic properties of order for rational numbers, 159; 35-5 Addition property of order, 161; 35-6 Generalized addition property of order, 162; 35-7 Numbers and their opposites, 163; 35-8 Multiplication property of order, 164; 35-9 Proof of the multiplication property of order, 165; 35-10 Generalized multiplication property of order, 166; 35-11 Summary of properties of order, 167.

Chapter 36

36-1 Introduction, 170; 36-2 Recurring decimal fractions, 170; 36-3 Changing unending decimals to common fractions, 173; 36-4 Irrational numbers and real numbers, 174.

Chapter 37

37-1 Introduction, 176; 37-2 $S \times S = 2$ has no solution in common fractions, 178; 37-3 More about the number line: irrational numbers, 180.

Chapter 38

38-1 Looking backward, 184; 38-2 The "club rules" for addition and multiplication, 185; 38-3 Simplifying the rules, 188; 38-4 The rules of order, 189; 38-5 Summing up, 190.

Chapter 39

39-1 Introduction, 194; 39-2 Approximations in counting, 194; 39-3 Averages, 196; 39-4 Deviation, 197.

Chapter 40

40-1 Approximate measurements, 202; 40-2 Basic agreement for recording approximate numbers, 204; 40-3 Repeated measurements, 206.

Chapter 41

41-1 Introduction, 209; 41-2 Rounding up and rounding down, 209; 41-3 Fundamental rules for rounding off, 211.

Chapter 42

42-1 Maximum error-precision, 216; 42-2 Relative error—accuracy, 217.

Chapter 43

43-1 Decimal places, 220; 43-2 Significant digits, 220; 43-3 Examples, 223; 43-4 Comparison of approximate numbers, 225; 43-5 Exact whole numbers, 225.

Chapter 44

44-1 Introduction, 227; 44-2 Addition and subtraction of approximate whole numbers, 228; 44-3 Addition and subtraction of approximate decimal fractions, 230.

Chapter 45

45-1 Introduction, 236; 45-2 Multiplication, 236; 45-3 Division, 239; 45-4 Operations with one approximate number and one exact number, 240.

Chapter 46

46-1 Introduction, 243; 46-2 An example, 244.