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VOLUME 1

Primary Three TEACHERS' GUIDE

Prepared
at the
1964 Entebbe Mathematics Workshop

EDUCATIONAL SERVICES INCORPORATED
Watertown, Massachusetts, U.S.A.

ENTEBBE MATHEMATICS SERIES

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An Introductory Text for Teachers

Volume I - *Structure of Arithmetic* - Chapters 1-21
Volume II - *Structure of Arithmetic* - Chapters 22-46
Volume III - *Foundations of Geometry* - Chapters 47-57

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Preliminary Edition

MATHEMATICS

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PREFACE

The purposes of the Entebbe Mathematics programme for *Primary Three* are:

1. To help children learn the best modern mathematics they can learn
2. To help them discover the ideas of mathematics for themselves

We want children *to learn* good mathematics and *to think* about mathematics. These two purposes are not attained separately but at the same time.

This book was written to help you help your pupils learn and think about mathematics. The book will help you in these ways:

It tells you about the ideas of mathematics that you are teaching; it refers you to sections in your *Teachers' Handbook* where you can read about the ideas.

It tells you how to teach each of the units.

It suggests things you can use in your classroom to help you teach.

It encourages you to let the children think for themselves. (Ask the question "Why?" often and let the children ask questions.)

Testing the Programme

This book and the *Pupil Book* were written to be tested by you and by your pupils. You and other teachers who use the books will be asked to write your opinions of the units you teach. The books will then be revised with the help of your comments. You are asked to do these things:

Teach the material of a unit as best you can, in the way suggested.

Think about the teaching and learning that you and your pupils have done.

Fill out a Unit Evaluation Form for each unit. (See the form at the end of this section.) Give the form to your supervisor.

How to Use the Book

The *first* thing you do before teaching from *Primary Three Entebbe Mathematics* is to *read all the books* carefully (this book, the *Pupil Book* and your *Teachers' Handbook*). They will help you

- ▶Know the whole programme for the year
- ▶Know what things you need to help you teach
- ▶Know what parts of the programme you wish to discuss with other teachers and supervisors
- ▶Know what parts of the programme you want to study about further in your *Teachers' Handbook* or in other books
- ▶Know how this book and the *Pupil Book* fit together
- ▶Know how this book, the *Pupil Book* and your *Teachers' Handbook* are planned to help you

The *second* thing you do is to *study the unit* you are to teach and *make plans* for teaching the unit. This study helps you

- ▶Plan in detail your teaching from day to day
- ▶Know what things to collect to help you teach the unit
- ▶Know what further reading and study you and your pupils should do to teach and learn the ideas in the unit

The *third* thing you do is *teach the unit*. Keep reading this book and your *Teachers' Handbook*. They will help you teach. Remember you teach best when children think things out for themselves. You question and guide. Tell as little as possible.

The *fourth* thing you do is to *think* about the unit and its use by you and the children. Think about how the unit could be improved for you and the pupils.

The *fifth* thing you do is to thoughtfully fill out a Unit Evaluation Form.

The Plan of the Book

This book explains in detail how to teach the fourteen units of the Primary Three programme. A unit is separated into "stages". Each of the stages is organized into several "activities". These activities are the *suggestions for teaching*.

An activity may be a single lesson; two or three activities may become a single lesson; or, you may continue or revise one activity several days. You must decide about how long your pupils need to spend with an activity. Study your children. Listen to them and watch them and decide when they can move to a new activity or stage.

Each unit uses ideas learned in units that come before it. For that reason, it is important that no mathematics be omitted. Some children may need only some of the activities in the book to help them learn an idea. If so, other activities may be omitted.

Helping Children Learn

Children learn best from things they do and by thinking about what they do. You can help by finding worthwhile things for them to do and to think about and by not telling them what to think.

This method means that you too must be doing and thinking as you help the pupils. These are some of the ways you can help children to think about and discover ideas:

Choose good problems and ask pupils to find answers.
Ask questions as pupils work.
Make suggestions as needed.
Suggest the use of things as models for the problem.
Supply a word or words now and then, when a pupil has a good idea he cannot express.
Suggest different methods of work.
Give encouragement at all times.
Help pupils summarize and say what their answers mean.

Outcomes

As you help pupils in mathematics to find their own answers and to think for themselves, you are teaching them mathematics and you are teaching them a way to think about mathematics. You are also helping them to learn a way to solve any problem.

This searching attitude leads to satisfaction and enjoyment for you and your pupils. It is the hope of the writers of the programme that it is not only enjoyed but that much mathematics is learned. Good luck.

Units of Primary Three Mathematics

1. Sets and Whole Numbers
2. Operations on Sets and Whole Numbers
3. Multiplication and Division
4. Geometry I
5. Fractions as Numbers
6. Properties of Operations
7. Techniques of Addition and Subtraction
8. Measurement
9. Techniques of Multiplication and Division I
10. Operations on Fractions I
11. Geometry II
12. Techniques of Multiplication and Division II
13. Operations on Fractions II
14. Fun with Mathematics

PRIMARY UNIT REPORT

Date:_____

Number of Unit_____ in Primary Book_____ Volume_____

Name of Unit:_____

Fill out one of these reports for each unit as you complete it. Send it by air mail to Educational Services Incorporated, 108 Water Street, Watertown, Massachusetts, 02172, U.S.A. Your comments are very much needed. They will be used for the revision of the books.

PLEASE ANSWER ALL THE QUESTIONS IN THIS REPORT

Your Name:_____

Name and Address of Your School:_____

_____

Date you started teaching the Unit:_____

Date you completed the Unit:_____

Number of periods used in teaching the Unit:_____

Length of daily lesson period:_____ minutes.

Answer each of the following questions and *give reasons* for your answers.

Questions	Answers and Reasons for Answers
1. Were the teaching suggestions clear?	
2. Were the teaching suggestions sufficient?	
3. Were the instructions in the Guide helpful in developing the ideas?	
4. Were the suggestions for materials to be used in the classroom sufficient?	
5. Was the background information for teachers in this unit clear, helpful and sufficient?	

What suggestions can you give for the improvement of the *Teachers' Guide* for this Unit?

_____

_____

_____

_____

_____

_____

PUPIL BOOK

Answer each of the following questions and *give reasons* for your answers.

Question	Answers and Reasons for Answers
1. Did your pupils learn the ideas the unit is meant to develop?	
2. Were the pupil pages helpful in developing the ideas of the unit?	
3. Were the pupil pages for this unit sufficient?	
4. Did you use materials other than those suggested in the book?	
5. Were the illustrations in the pupil book suitable?	
6. Did the pupils enjoy this unit?	
7. Did you enjoy teaching this unit?	

What suggestions can you give for the improvement of this Unit?

_____

_____

_____

GENERAL REMARKS and ASSESSMENT of

THE UNIT

To the best of my knowledge, this Unit is _____ (excellent, good, fair, poor) because:

(1) _____

_____

_____

(2) _____

_____

_____

(3) _____

_____

_____

(4) _____

_____

_____

(5) _____

_____

_____

(6) _____

_____

_____

_____
Signature

_____
Date

CONTENTS

		PAGE
UNIT 1 •	Sets and Whole Numbers	1
UNIT 2 •	Operations on Sets and Whole Numbers	15
UNIT 3 •	Multiplication and Division	38
UNIT 4 •	Geometry I	79
UNIT 5 •	Fractions as Numbers	108
UNIT 6 •	Properties of Operations	126
UNIT 7 •	Techniques of Addition and Subtraction	163

UNIT 1

Sets and Whole Numbers

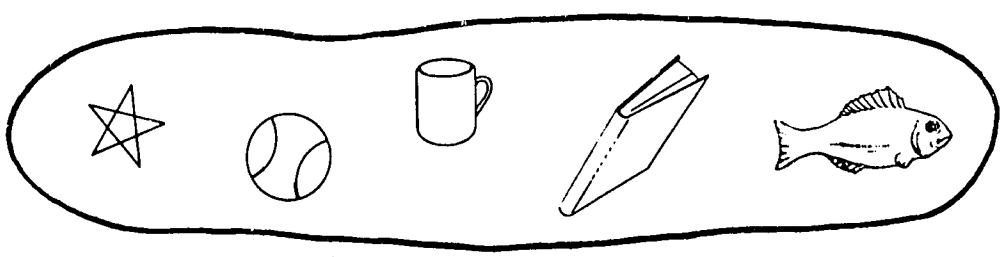
OBJECTIVES

1. To revise the idea of a set and subsets
2. To revise the order of sets and the order of the numbers of the sets
3. To revise place value in decimal numerals and extend place value to thousands
4. To revise renaming of numbers and expanded form of numerals
5. To revise counting by 2's, 3's, . . . , 12's, and extend to counting by 20's, 25's, 50's, 100's, 1000's

Background Information for Teachers

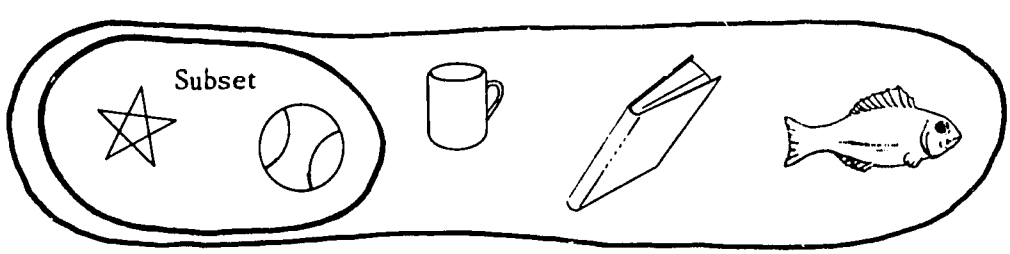
See Part 2, "Arithmetic", pages 9-22, in your *Teachers' Handbook.*

The word *set* is a name used for any collection of objects or things. Each object in a set is called a *member* of the set. Below is a picture of a set:



Sets may be identified by naming their members.

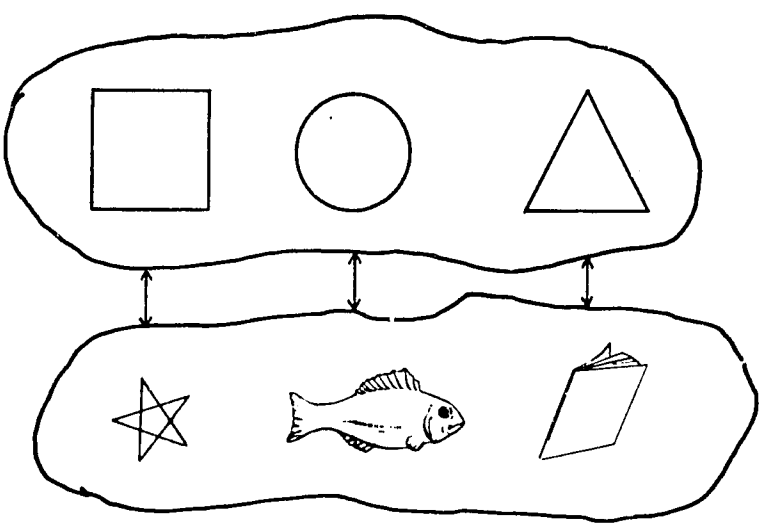
We sometimes need to think about some of the members that belong to a given set. These members form a set also. For example, the set consisting of the star and the ball in the set shown above is called a *subset* of the given set.



There are many possible subsets of the set shown above. In general, given a set called A, then a set called B is a subset of the set A if every member of set B is a member of set A. In the above example, the set B consisting of the star and the ball is a subset of the whole set A because every member of B is also a member of A.

Children will notice many examples of subsets. The set of boys in the classroom is a subset of the set of children in the classroom. The set of even counting numbers (2, 4, 6,. . .) is a subset of the set of counting numbers. The set whose member is 0 is a subset of the set of whole numbers.

Sets are compared with each other by a process called *matching*. If each member of one set can be matched exactly with a member of another set, the two sets are called *equivalent*. The two sets shown below are equivalent.

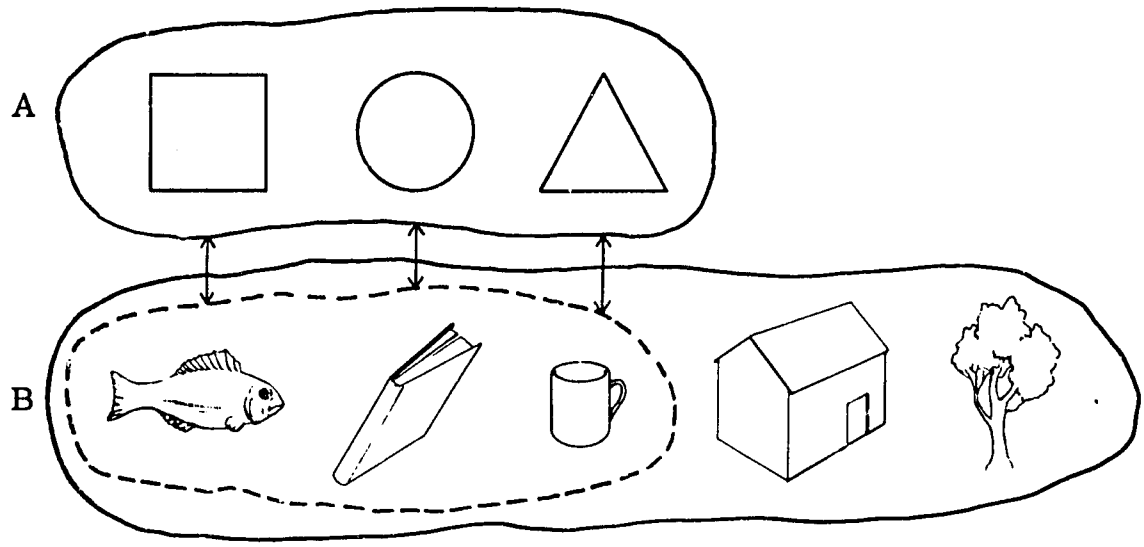


Equivalent sets have something in common. They have the same *number* of members. For example, the set whose members are a star, a fish and a book is described by the number three, and this number also describes every set that is equivalent to the set. Each of the sets equivalent to this set has three members. The symbol 3 is a *numeral* for the number three.

Notice that the number three is an idea which describes all the sets equivalent to {□, ○, △}. The numeral 3 is a *symbol* that we write for the idea of the number.

Some sets are not equivalent to each other. We compare two sets by matching their members. If all the members of the first set are matched with a subset of the second set, with some members of the second set unmatched, then the second set has more members than the first. For example,

in the picture below, all the members of set A are matched with the members of a subset of B, with some members of the set B unmatched.



We say set B has *more members than* set A, and set A has *fewer members than* set B. The number for set A is 3, and the number for set B is 5. Since set A has fewer members than set B, we say that 3 is *less than* 5, and 5 is *greater than* 3. We write:

$$3 < 5 \textit{ and } 5 > 3.$$

In *Primary One* and *Primary Two*, the children separated a set of objects into subsets of hundreds, of tens and of ones and learned to write the numeral for the number of the set. For example, a set of three hundred twenty-four sticks was separated into 3 subsets of a hundred sticks each, 2 subsets of ten sticks each and 4 subsets of one stick each. The children wrote this *decimal numeral* for the number of the set:

3	2	4
hundreds (ten tens)	tens	ones

The 3, the 2 and the 4 are called *digits*. Each digit has a *place value*. The place value of the 3 is hundreds, of the 2 is tens and of the 4 is ones.

Vocabulary

Set, subset, member, match (the members of two sets), more than, fewer than, as many as, number of a set, compare (the numbers of two sets), less than, greater than, equal to

Materials

Several pieces of string to form loops; inch-square symbol cards (< and >) for each child; many small objects, such as seeds and bottle tops, for forming sets and counting; pupil pages 1-2.

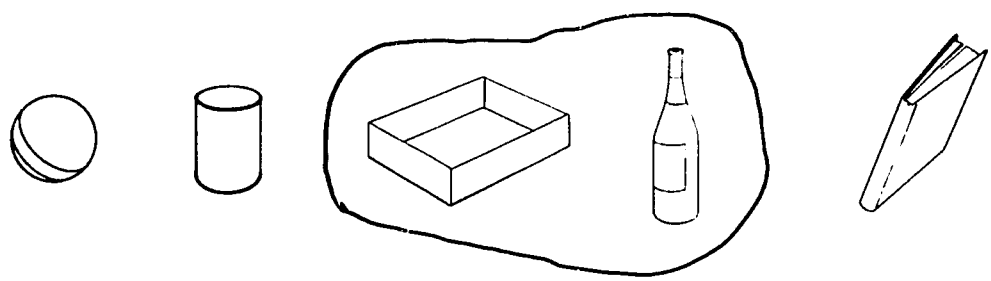
Teaching Procedure

ACTIVITY 1 • Forming Subsets of a Set

Pupil page 1

The purpose of this activity is to revise the idea of a subset of a set.

Put objects on a table to form a set; for example, use objects such as those shown below. Let the children tell the names of the objects in the set. Ask someone to name a *subset* of the *set* on the table. When a child answers, for example, "The subset whose *members* are the box and the bottle", place a loop of string around these members to show the subset.



Remove the string and ask other children to come up and form other subsets of the set on the table. Let the children put the loop of string around any objects. Let them form as many subsets as they can. Some subsets will

have one member, some will have two members, some will have three, some four, and the set itself can be called a subset.

Draw a picture of a set on the blackboard. Use coloured chalk to draw rings around members of subsets. Continue this activity until the idea of a subset is clear. Let the children think of all the objects in the classroom as a set. Ask the children to tell about subsets they can form from this set. Among the many replies may be these:

The set of pencils is a subset of the set of things in the room.

The set of chairs is a subset of the set of things in the room.

The set whose member is the table is a subset of the set of things in the room.

Tell the pupils to open their books to page 1. Get them to describe the picture as showing many sets; for example, sets of birds and sets of people. Ask the pupils to think of the subsets that can be formed from sets shown in the picture. Then ask one pupil at a time to describe a subset. For example,

The set of cows is a subset of the set of animals.

The set of palm trees is a subset of the set of all trees.

ACTIVITY 2 • Comparing Sets and Comparing Numbers

Pupil pages 1-2

The purpose of this activity is to revise the matching of sets.

Form two sets of different objects on your table; for example, a set of three pencils and a set of five books. Ask the children to *match* the members of the set of books with the members of the set of pencils. Ask, "Which set has members left over and not matched?" (The set of books) "Which set has more members?" (The set of books) "Which set has fewer members?" (The set of pencils)

Ask the class to tell the *number* of each set. (The set of pencils has 3 members. The set of books has 5 members.) Ask, "Which number is greater?" (5 is greater than 3.) Let a child write this sentence on the blackboard:

$5 > 3$

Ask, "Which number is less than the other?" (3 is less than 5.) Let a child write on the blackboard:

$3 < 5$

Call three children to the front of the class. Give the first child 2 pencils, the second 6 pencils and the third 9 pencils. Ask, "Has each of the children the same number of pencils?" (No) "Which set has *more* members than each of the other sets? Which set has *fewer* members than each of the other sets?" (Adu's set has more members than John's set, and John's set has more members than Mary's set; Mary's set has fewer members than John's set, and John's set has fewer members than Adu's set.) Let the children write these ideas in symbols. ($9 > 6$, $6 > 2$, $2 < 6$, $6 < 9$)

Repeat this activity with sets having different numbers of members until the pupils can easily compare sets, using the words "more members than", "fewer members than", "as many members as".

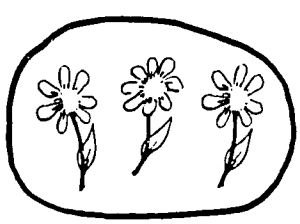
Point out to the children that when they *compare sets* they use one of these terms

more than; fewer than; as many as.

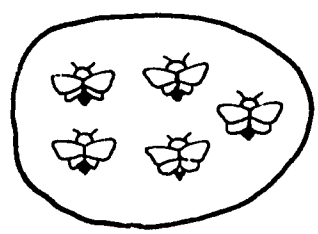
When they *compare the numbers of the sets*, they use the words

greater than;	less than;	equal to.
$>$	$<$	$=$

For example,

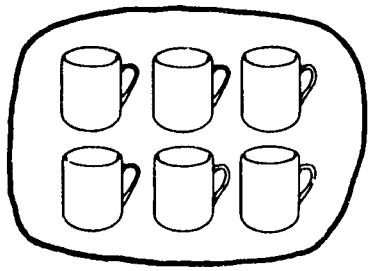


has fewer members than

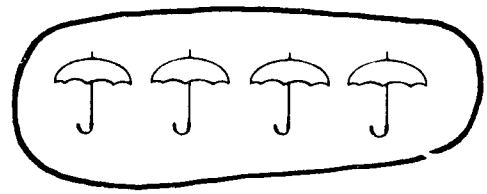


3 is less than 5

$3 < 5$

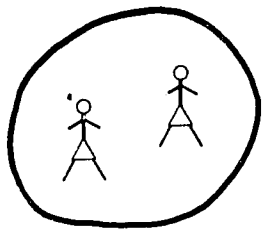


has more members than

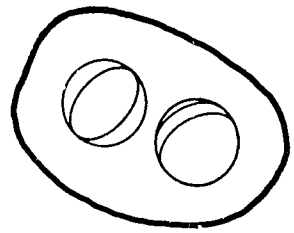


6 is greater than 4

$6 > 4$



has as many members as



2 is equal to 2

$2 = 2$

Revise the symbols for "less than" (<), "greater than" (>), and "equal to" (=) as you do these activities.

Ask the pupils to open their books to page 2. Let them compare the two sets in the left column and then compare the numbers of the sets in the right column. For example, in the first exercise on the page, the children will say, "The set of flowers has more members than the set of bees." Then they will find the numbers for the sets and make the number sentence on the right true by the use of a small card on which is written the "greater than" symbol.

9 $\boxed{>}$ 6

Ask the pupils to turn back to page 1 and look again at the picture. Let them tell about the picture, using the words "fewer than", "more than", "as many as", "less than", "greater than" and "equal to". Answers will be in these forms:

The set of _____ has fewer members than the set of _____.
The set of _____ has more members than the set of _____.
The set of _____ has as many members as the set of _____.
The number of _____ is less than the number of _____.
The number of _____ is greater than the number of _____.
The number of _____ is equal to the number of _____.

Vocabulary

Numeral, tens, hundreds, thousands, place value, rename, expanded form, simplest name for a numeral ·

Materials

More than 200 counting sticks; pieces of string or rubber bands for tying sticks in bundles; pupil pages 3-9

Teaching Procedure

ACTIVITY 1 • Separating Sets into Subsets of Ten Members and One Hundred Members

Place more than 200 sticks on a table. Arrange the class in groups of five or six pupils and give each group some of the sticks.

Tell each group to separate its sticks into subsets of ten sticks and tie each subset in a bundle. Then let each group tell how many sets of ten sticks it has, and how many single sticks are left over. Ask a child in each group to count the sticks of his group while the others in his group watch, listen and check. He might say:

1 ten, 2 tens, 3 tens, 4 tens, 4 tens and 1, 4 tens and 2 ones, ...

Or he might say:

Ten, twenty, thirty, forty, forty-one, forty-two, . . . Encourage the children to give each of these answers.

Let the children decide which group has more sticks, which group has fewer sticks than the others.

Let the children write the *numerals* for the number of their sets of sticks. For example, the numeral for

"4 tens and 5 ones" is written "4 ".

If some of the children are not sure of this, revise the writing of numerals by asking questions such as:

"What is another name for the number '10 ones'? Think of 10 sticks tied in a bundle." (1 ten or 10)

"What is another name for 2 tens?" (20)

"What is another name for 10 tens?" (1 hundred or 100)

Ask a child from each of the groups to put all the sticks together on a table. Choose a child and ask him to separate the sticks on the table into subsets of 100 sticks each. Let other children tie each subset of 100 (10 tens) into a bundle. Get the children to count the sticks by hundreds, tens and ones; for example,

1 hundred; 2 hundreds; 2 hundreds and 1 ten; 2 hundreds and 2 tens; 2 hundreds, 2 tens and 1; 2 hundreds, 2 tens and 2 ones, . . .

Let the class watch and tell if the counting is properly done. Get them to say, for example, "There are 2 hundreds, 2 tens and 7 sticks." Call on another pupil to write the numeral for this number. Let the class decide if the numeral is the correct one. For example, "2 hundreds, 2 tens and 7 ones" is read "two hundred twenty-seven" or "two hundred and twenty-seven". It is written "227".

If needed, remove some bundles of tens and some single sticks or put more sticks on the table and repeat this activity. Each time ask for the number of sticks and for a numeral for the number to be written.

ACTIVITY 2 • Writing Numerals 1-1000

Pupil page 3

Use pupil page 3 to revise the decimal numerals for numbers from 1 to 1000. Let the pupils copy the exercises in their exercise books. Talk with the children about the exercises in which no tens or no ones are given. Ask why the numeral 0 is used to show no tens or no ones. If some children are not sure of their answers, let them show the exercises with sticks in bundles of hundreds, tens and ones.

ACTIVITY 3 • Place Value to Thousands

Pupil pages 4-5

Let the children turn to page 4 in their books. Ask them to look at the set of dots on the page. Get a child to tell the class how many dots are in each small section of dots. (100) Ask, "How many sets of ten dots are in each small section?" (10 sets of ten, or 10 tens)

Ask another child to count aloud the subsets of 100 dots. (1 hundred, 2 hundreds, . . .) When he says "10 hundreds", ask what simpler name we give to a set of 10 hundreds. (1 thousand) Let other children continue.

Let the pupils use loops of string to separate the dots on the page into subsets of a thousand dots. Let them count the dots by subsets of thousands, hundreds, tens and ones; for example,

1 thousand; 2 thousands; 2 thousands and 1 hundred; 2 thousands and 2 hundreds; 2 thousands and 3 hundreds; 2 thousands, 3 hundreds and 1 ten; 2 thousands, 3 hundreds and 2 tens; 2 thousands, 3 hundreds, 2 tens and 1; 2 thousands, 3 hundreds, 2 tens and 2 ones; . . .

Draw a chart like the one below on the blackboard. Let a child write in the chart the number of dots on page 4. You name other numbers and let children write them in the chart; for example, 7284 and 509.

Thousands	Hundreds	Tens	Ones
2	3	2	7

Ask the children to make a chart in their exercise books like the one on the blackboard. Let them turn to page 5 and count the dots by thousands, hundreds, tens and ones. Let them say the number (One thousand, nine hundred, sixty-five) and write the digits of the numeral for the number in their charts.

Thousands	Hundreds	Tens	Ones
1	9	6	5

Ask the pupils to point out the place in the numeral showing thousands,

the place showing hundreds, showing tens, showing ones. Then ask for the *place value* of the "1", the place value of the "9", of the "6" and of the "5."

ACTIVITY 4 • Expanded Numerals

Pupil page 6

On the blackboard write sentences such as:

3 thousands, 4 hundreds, 7 tens and 6 ones is_____.
___thousands,___hundreds,___tens and___one is 4021.

Let the children tell what numerals they can write in the blank spaces to make the sentences true. Continue the use of such sentences as needed by the class. Let the children read the numerals after they are written. Discuss the use of 0 to show there are no hundreds or no tens or no ones.

Ask the children to turn to page 6, copy the exercises and write in the correct numerals to make true sentences.

ACTIVITY 5 • Renaming Numerals

Pupil pages 7-8

Write these numerals on the blackboard:

120	100 + 20	10 × 12	200 − 80

Let the children think about the numerals and what they mean. Guide the children to agree that all of the numerals name the same number, 120. Ask, "Can you write other numerals for 120?" Let them in turn write other names. Some they may write are these:

50 + 50 + 20	100 + 10 + 10	130 − 10
99 + 21	2 × 60	240 ÷ 2

Get the children to decide that each of the numerals written names 120. Say, "When we write one of these numerals for 120, we are *renaming* 120."

Keep all of the numerals for 120 on the blackboard.

Ask the children to rename 124 in many ways and write the numerals for the name on the blackboard. Get the class to decide whether each numeral is a name for 124. Some of the ways in which 124 may be renamed are:

100 + 24	100 + 20 + 4	60 + 64
2 × 62	12 + 12 + 100	125 − 1

Say as you point to "100 + 20 + 4" in the second set of names, "This numeral is the *expanded form* of 124. It tells how many hundreds, how many tens and how many ones are in the number."

Ask, "What is the expanded form for 728?" (700 + 20 + 8) Get the children to say that the expanded form states the name in hundreds, tens and ones. Ask for the expanded form for 7208. (7000 + 200 + 0 + 8) Again get the children to say that the expanded form states the name in thousands, hundreds, tens and ones.

Let the children look at page 7 and read the first sentence written there. It asks them to find all the names on page 7 for 75. Let the children tell all the names they can find for 75. Let them write the names in their exercise books under "75" in a chart such as the following. Ask if one of the names is in expanded form. Tell them to draw a line under that name.

Let them go on with the other exercises on page 7 as needed. (Note that some of the names on page 7 are not names for 75, 24, 144, 520 or 1250.)

Names for 75	Names for 24	Names for 144
60 + 10 + 5	20 + 4	90 + 50 + 4
25 × 3	4 × 6	100 + 44
30 + 5 + 40	12 + 6 + 6	50 + 50 + 44
. . .	. . .	. . .

When the charts are completed, let the children point to the *simplest name* for seventy-five (the numeral 75) and the expanded form of 75. Ask the same question for the other numbers.

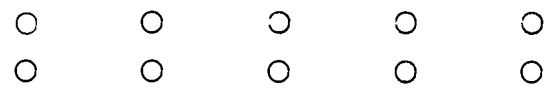
Ask the children to complete the exercises on page 8. In the first group of twelve exercises, they write each numeral in expanded form. In the last twelve exercises, they write the simplest name for a number.

Get the children to use the terms "rename", "expanded form" and "simplest name". For example, they may say, "702 may be renamed as 700 + 2" or "An expanded form of 702 is 700 + 0 + 2 and the simplest name for 700 + 2 is 702."

ACTIVITY 6 • Counting by 2's, 3's, ...

Pupil page 9

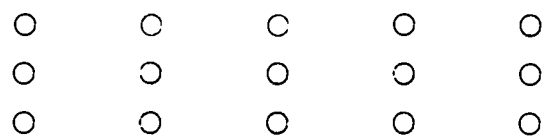
Arrange the class in groups of five or six children. Give each group about 50 small objects to count. Ask the groups to arrange their sets of objects in subsets, each with two members.



and so on.

Let each group count their objects by 2's as one child in the group moves the objects together, two at a time.

Tell the groups to put all the objects back in a pile. Then ask them to arrange their counters in subsets, each with three members.

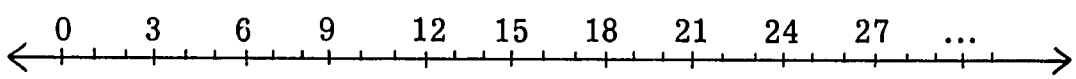


and so on.

Let one child in each group move the subsets together as the rest of the group count by 3's.

Repeat this activity, counting by 4's and then by 5's.

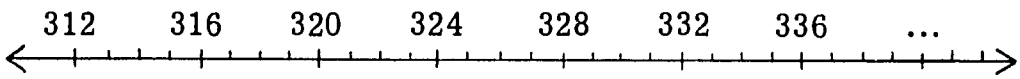
Draw a number line on the blackboard and mark points at equal distances along the line. Label the point on the left "0", and ask a pupil to count the spaces, beginning at 0. When he counts 3 spaces, label that point "3"; when he counts 6 spaces, label that point "6" and so on. Then let him name the numbers he uses as he counts the spaces by 3's: 3, 6, 9, 12, 15 and so on.



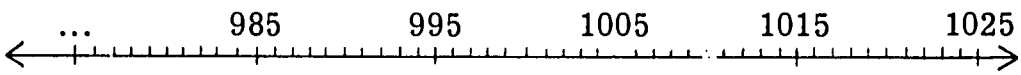
Continue as far as he can go, or until there are no marks remaining on the number line. Let the class decide whether he is right.

Draw other number lines, and let other pupils count the spaces by 4's, by 5's, by 6's, by 7's, by 8's, by 9's and by 10's. In each case, let a child count as far as he can or until there are no more points on the line. He may wish to extend the number line. Do not go beyond about 50 unless a child can do so easily.

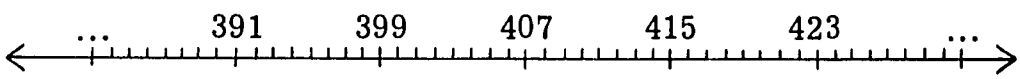
Vary this activity. Sometimes ask the children to say the numbers used in counting without using a number line or objects. At other times, use a number line and begin counting spaces at some number other than 0, and count forward or backward. Also use numbers greater than 50. For example, count spaces on a number line by 4's, beginning at 312.



Or begin at 1025 and count the spaces backward by 10's.



Or begin at 407 and count the spaces in each direction by 8's.



Ask the pupils to turn to page 9. Let them copy the number lines and fill in the missing numerals. If they have difficulty copying the number lines, let them simply write the missing numerals in rows in their exercise books.

UNIT 2

Operations on Sets and Whole Numbers

OBJECTIVES

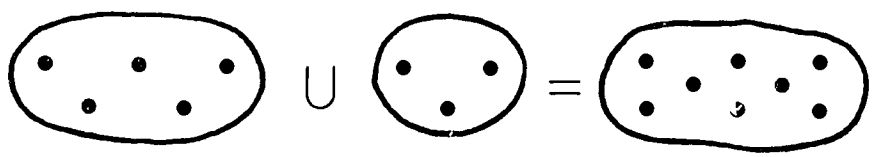
1. To develop the idea of the intersection of two sets
2. To develop the idea of disjoint sets
3. To revise the meaning of union of disjoint sets and the related operation of addition
4. To revise the basic addition facts
5. To revise the meaning of subtraction as finding a missing addend
6. To revise techniques of addition and subtraction of numbers (1-9999)

Background Information for Teachers

See Part 2, "Arithmetic", pages 22-25, 31-32, in your *Teachers' Handbook.*

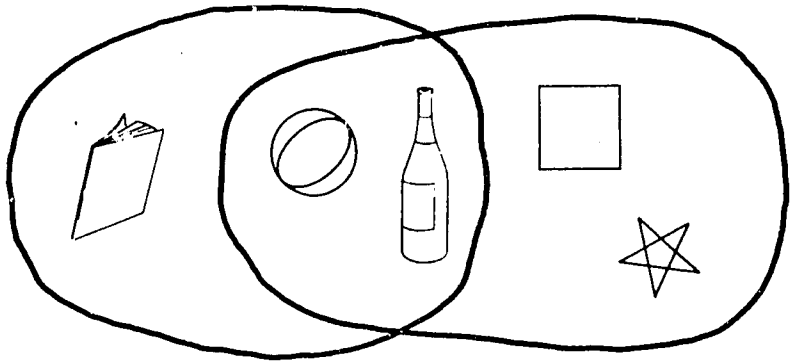
The operations of addition and multiplication on whole numbers are related to certain operations on sets. The children learned in *Primary One* and *Primary Two* to show addition by joining sets of objects together. The operation on sets is called *union.* The corresponding operation on numbers is called *addition.*

Union of sets



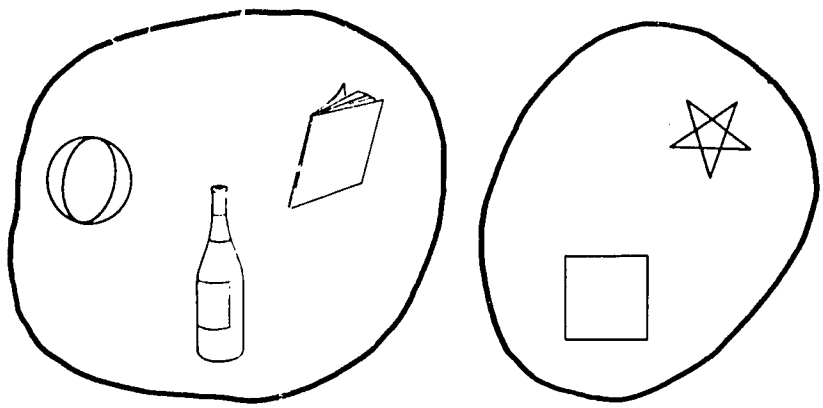
Addition of numbers 5 + 3 = 8

Another operation on sets that we use is called *intersection.* You will notice that some sets intersect and some do not. For example, in the picture below there are two sets shown: a set whose members are the book, the ball and the bottle; and a set whose members are the ball, the bottle, the square and the star.



There are some objects that are members of both sets; the ball and the bottle are members of both sets. We say that these two sets are *intersecting sets* because they have some *common members*. The set whose members are the ball and the bottle is called the *intersection* of the two sets.

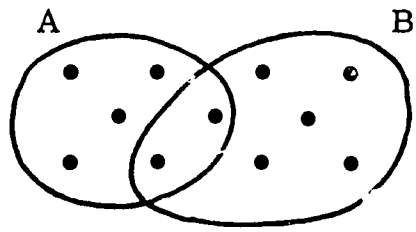
In all the work with sets in *Primary One* and *Primary Two*, the children have joined together sets that do not intersect. For example, the two sets shown below do not have any common members. We say that they do not intersect. These sets are called *disjoint sets*.



Disjoint sets

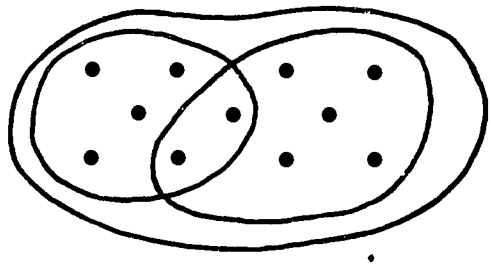
It is very important for *you* to understand that addition of numbers depends on the union of disjoint sets.

We may form the union of two sets even if they are intersecting sets, but this type of union of two sets does not relate to addition. For example, note the union of sets A and B below.



Set A has 6 members and set B has 7 members. The union of set A and set B is the set made by putting together the members of the two sets.

A ∪ B =



The set formed by the union has 11 members. It is not true that 6 + 7 is 11. You see that sets A and B are not disjoint and that their union does not show addition. The union of two sets which are disjoint shows addition.

In this unit, we will not ask the children to form the union of intersecting sets. Do not bring up the problem; but if a child asks about the union of intersecting sets, let him discover that such a union will not show addition. Try to develop the idea that addition is shown by unions of disjoint sets. Always let the children decide that the sets they are joining have no common members.

Also, when a set is separated into two subsets to show subtraction of numbers, we again must be sure that the subsets are disjoint.

STAGE 1 • Intersection of Sets

Vocabulary

Common members, intersecting sets, disjoint sets

Materials

Sets of about five or six small objects for each child; loops of string to form sets; pupil page 10.

Teaching Procedure

ACTIVITY 1 • Making Intersecting Sets

Tell the class, "All the members of the set of girls in this class put their right hands on their heads." Point to individuals and ask, "Is John a member of the set of girls? Is Mary a member of the set of girls?" Let the girls keep their right hands on their heads.

Now say, "All the members of the set of children who are eight years old put their left hands on their heads." Point to children in turn and ask if they are members of this new set. For example, ask, "Is John a member of the set of children who are eight years old?"

Point to a girl who has both hands on her head. Ask, "Is Grace a member of the set of girls?" (Yes) "Is Grace a member of the set of children who are eight years old?" (Yes) "Is Grace a member of both sets?" (Yes) Let the children tell the names of all other pupils who are members of both sets.

Ask, "What can we say about the children with their right hands on their heads?" (They are members of the set of girls.) "What can we say about the children with their left hands on their heads?" (They are members of the set of children who are eight years old.) "What can we say about the children with both hands on their heads?" (They are members of both sets; or, they are members of the set of girls who are eight years old.)

Repeat this activity with two other sets of children that have some members in common. For example,

a. the set of nine-year-old children and the set of boys,
b. the set of girls and the set of children sitting in the front row,
c. the set of children whose names begin with N and the set of eight-year-old children.

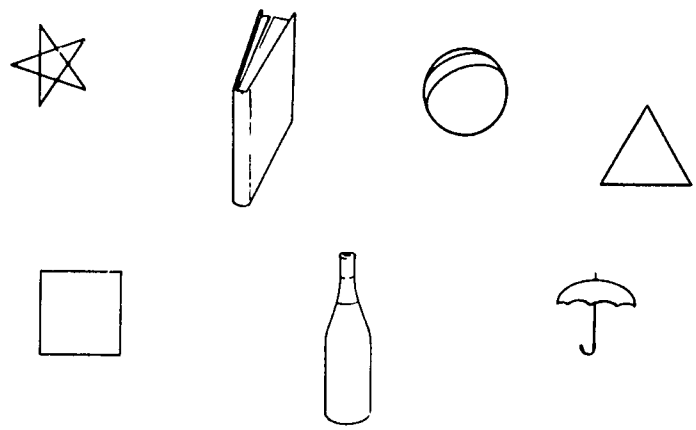
Let the class name the children who are in one set but not in the other; the children who are in both sets; the children who are in neither set.

Now, repeat this activity with two sets that are disjoint. Say, "All the members of the set of girls place their right hands on their heads"; "All the members of the set of boys put their left hands on their heads." Ask,

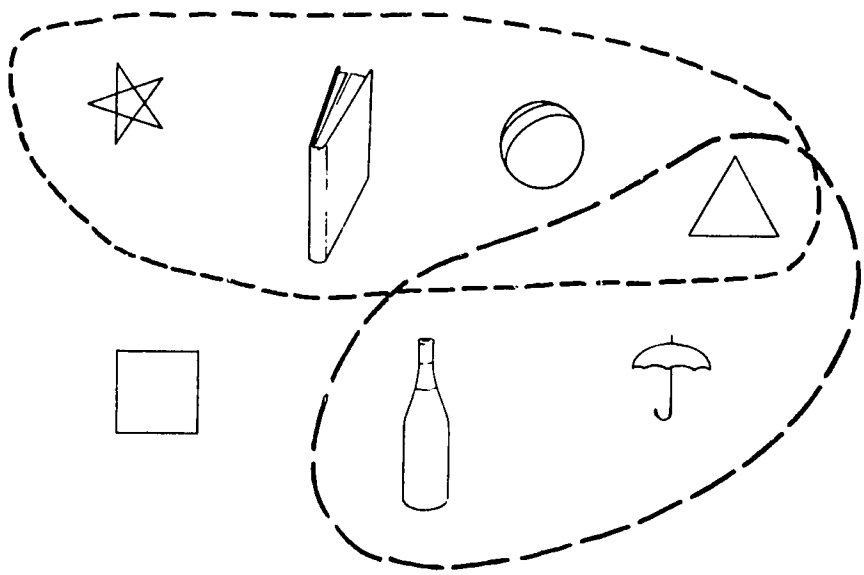
"Which children have both hands on their heads?" (None of the children) "Are there any children who are members of both sets?" (No) "What is the set of children who are in both sets?" (The empty set)

ACTIVITY 2 • Drawing Intersecting Sets

Draw a set of objects on the blackboard.



Let a child draw a ring around some of the objects with a coloured chalk to show a set. Ask him to describe the set he has made. Point to various objects in the picture and ask the children if the objects are members of the set. You draw a ring around another set with chalk of a different colour. Make sure that the two sets have some member or members in common. For example,



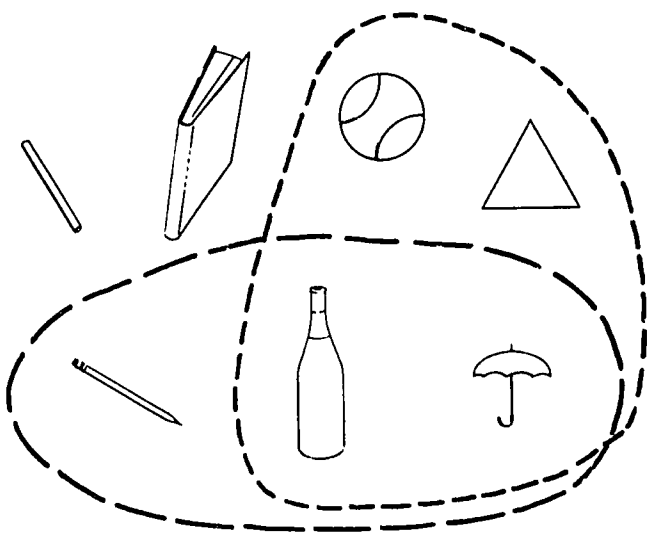
Let the children name, first, the objects that are members of both sets (the objects that are common to both sets), second, the objects that are members of only one set and, third, the objects that are not members of either set.

Ask, "Do the two sets have any *common members*?" (Yes) "Name the common member (or members) of the two sets."

Say, "These two sets are called *intersecting sets* because they have some members in common." Write the words "intersecting sets" on the blackboard and let some of the children say them.

Rub off the rings around the sets on the blackboard. Ask a child to draw rings around two other intersecting sets. Let the class decide if the child chose intersecting sets. Let them then name the common members of the two sets.

Put several objects on a table and let the children put loops of string around intersecting sets. Below is a picture of one of the many possible pairs of intersecting sets that can be formed:



Ask, "What members are in one set and not in the other?" (The pencil is in one set and not in the other. The ball and the triangle are in one set and not in the other.) "What members are not in either set?" (The chalk and the book are not in either set.) "What members are in both sets?" (The umbrella and the bottle are in both sets. They are the common members of the two sets.) Repeat this activity with two other intersecting sets and ask similar questions.

Ask a child to show two sets that are not intersecting sets. For example,

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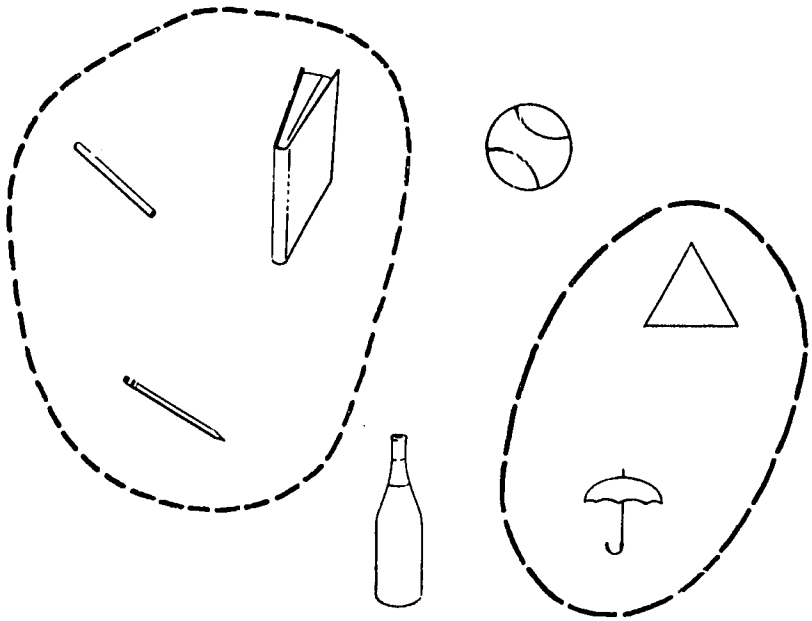


Ask which members are common to both sets. (There are no members common to both sets.) Say that sets which have no members in common are called *disjoint sets*. Write the words "disjoint sets" on the blackboard and let some of the children say them.

Ask other children to show different disjoint sets. Let them tell why the sets are disjoint. (The sets are disjoint because they have no common members.)

Get the children to name two sets of objects in the classroom. Let other children tell whether the sets are intersecting or disjoint.

ACTIVITY 3 • Intersecting Sets and Disjoint Sets

Pupil page 10

Ask the class to turn to page 10 in their books. Say, "On page 10, there are three pictures of the same objects." Let the children name the members of each of the two sets in the first picture. Say, "Name the common members of the two sets." (Triangle and ball) "Are these sets intersecting sets or disjoint sets?" (Intersecting sets, because they have common members.)

Repeat this activity with the second and third pictures. In the third picture, the two sets have no common members and are disjoint sets.

Let the children work in pairs. Let each pair place some small

objects on their table or on the floor. Give them two large loops of string and ask them to make intersecting sets by placing the strings around two sets of objects. Let one child describe to the other child the sets he makes. For example, "I have made two intersecting sets. They have two members in common. The book and the pencil are members of each of the two sets." Or, "I made two intersecting sets. They have three common members. The bottle top, the pencil and the stick are members of both sets."

Ask the children to form two disjoint sets. Let each pupil check that his neighbour has made disjoint sets. Let them say, "I made two disjoint sets. They have no common members."

STAGE 2 • Meaning of Addition

Vocabulary

Union, addition, sum, addend, the union sign (U)

Materials

Many small objects to form sets; bottle tops or beans for counting; pupil pages 11-18

Teaching Procedure

ACTIVITY 1 • Union of Sets and Addition

Pupil page 11

The purpose of this activity is to revise the idea of putting two disjoint sets together to form the union of the sets and to revise the idea of addition of the numbers of the two sets.

Put a collection of many objects on your table and let two pupils select objects to form one set each. Tell them to select disjoint sets.

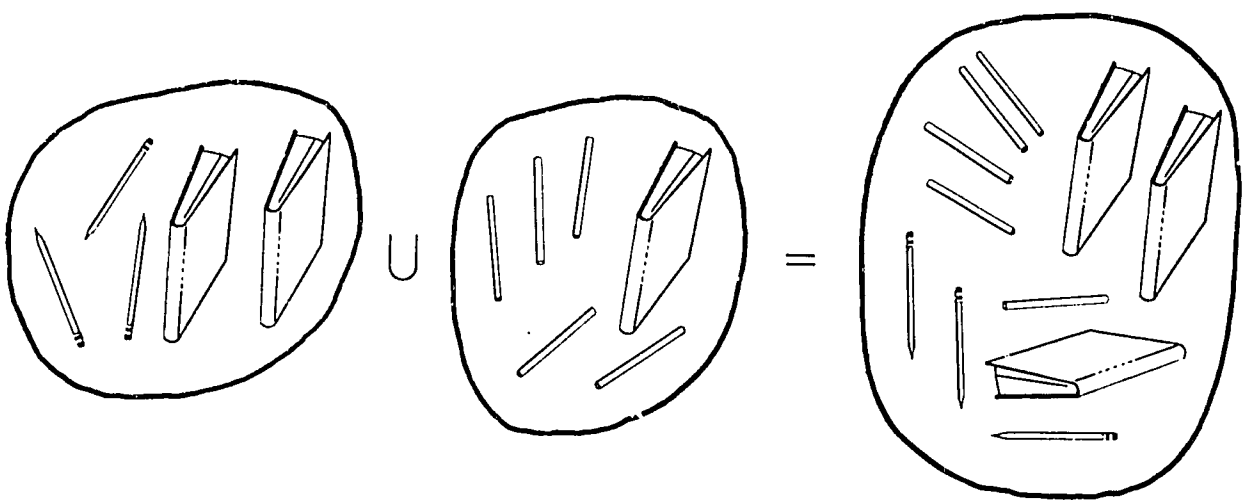
Say, "John has a set and Mary has a set. Are their sets disjoint sets?" (Yes, because there are no common members.) Tell the two children to put

the two sets together. Get them to lay the objects of their two sets together on another table. Ask the children whether a new set is made and to describe the new set. Let them name all the members of the new set.

Say, "The new set was made by putting the two sets together. It is called the *union* of the sets."

Ask another pair of pupils to form two disjoint sets. Again let the class decide that the two sets are disjoint, and again ask the pupils to join their sets to form the union of the sets. Get pupils to tell about this. (The two sets are joined to form the union of the sets.) Continue this activity as other pairs of pupils form the union of two disjoint sets.

Let two pupils hold up their sets. Draw pictures of the sets on the blackboard. Say, "We need to show that we are going to join the sets." Draw the symbol "∪" between the pictures of the sets. Say, "This symbol shows that a union of the sets is made." Ask, "What members are in the union of the sets." Draw a picture of the union as the children name its members. The completed drawing can look like this:



Ask, "How many members are in the first set?" (5) "How many members are in the second set?" (6) "How many members are in the union of the sets?" (11) As these numbers are given, write their numerals under the corresponding sets. Ask, "What operation does this show on the numbers of the sets?" (*Addition.* The sum of 5 and 6 is 11.) Complete the sentence below the pictures of the sets.

$$5 + 6 = 11$$

Let the children talk about the idea that the number of members in the

union of two disjoint sets is the *sum* of the numbers of the sets. Write these sentences on the blackboard as the children talk:

We join two sets to make the union of two sets.
We add two numbers to get the sum of two numbers.

Use several more examples to show the connection between the union of two sets and the sum of the two numbers of the sets. For each addition idea shown, have the children name the *addends* (the two numbers added to get the sum) and the sum. For example, in the sentence 4 + 3 = 7, the addends 4 and 3 are added to get the sum 7.

Let the class turn to page 11. For each exercise, ask them to describe the union of the sets and to answer the questions. For example, in the first exercise:

6 + 3 = 9
The addends are 6 and 3.
The sum is 9.

ACTIVITY 2 • Revision of Addition Facts with Sums to 19

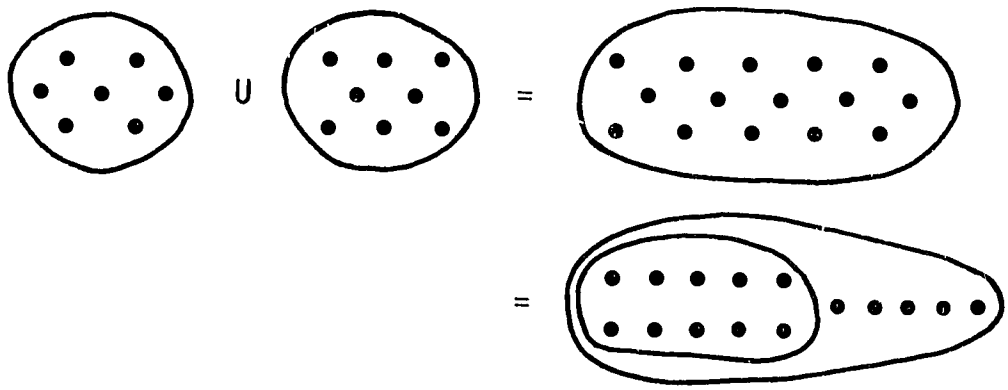
Pupil pages 12-18

If your class needs much revision of these addition facts, make several lessons of this activity. Go over each of the pupil pages 12-18 very carefully. If your class does not need much revision of the addition facts, use only one lesson and only one or two of pupil pages 12-18. The part of the activity that every child should do is that for pupil page 16.

Pupil pages 12, 13 and 14 may be used to revise sums with addends from 0 to 18. Some of the children may still need to join sets of objects and count the members in the union. If a pupil has trouble finding a sum, let him show the addition with sets of objects. Then let him write the numeral for the sum in the addition sentence. For example,

7 + 8 = □

The pupil will form two sets of counters, one with 7 members and the other with 8 members. When the sets are put together, let him arrange the counters in a pattern showing a subset of ten. For example,



7 + 8 = 10 + 5

= 15

If the pupils do not need to form sets, let them give the sums immediately. But if a pupil has any doubt about a sum, do not tell him the sum. Let him discover it for himself by forming unions of two sets.

Pupil page 15 may be used to revise the form for addition sometimes called the "vertical form". Let the pupils copy the exercises and write the correct numerals for the sums. As before, let pupils use sets of objects if they need to, but encourage them to "remember" addition facts.

Pupil page 16 is an addition chart. Let the pupils copy the chart carefully and fill in the correct sums as they occur in their work on other pupil pages. This is an activity in finding sums. It should not be used as a memory device, and it should not be looked at when other sums are being found. When the chart is completed, let the children talk about topics like these: the pairs of addends with the sum of 13; the sums in the "2" row; the sums in the "0" row and column; and the sums in the "1" row and column.

Pupil page 17 provides practice in finding pairs of addends of a given sum. For example, pairs of addends of 15 are 7 and 8, 6 and 9, 5 and 10, 4 and 11 and so on. A game can be made out of this activity by finding as many pairs of addends for a certain number as possible. For example, find the pairs of addends of 15. There are sixteen pairs of addends for 15: 0 + 15, 1 + 14, 2 + 13, 3 + 12,. . ., 14 + 1, 15 + 0. You may wish the children to find the addends of other sums, such as 13, 14 and 16. Let them use their addition chart, page 16, to check their pairs of addends.

Pupil page 18 is a good revision page. It requires thought on the part

of the children. It asks them to find a pair of addends with the same sum as another pair. For example,

$$2 + 7 = \square + 3$$

The children must find the sum 9, think of the fact 6 + 3 = 9, and write "6" in the sentence to make it true. Children may be slow at these exercises. Do not rush them. Encourage them to finish by working in several periods over several days or several weeks.

STAGE 3 • Addition and Subtraction

Vocabulary

Mathematical sentences, addition sentences, missing addend, subtraction sentence, minus, subtract

Materials

Small counters; objects to form sets; pupil pages 19-30

Teaching Procedure

ACTIVITY 1 • Meaning of Subtraction

Pupil pages 19-22

Write these sentences on the blackboard:

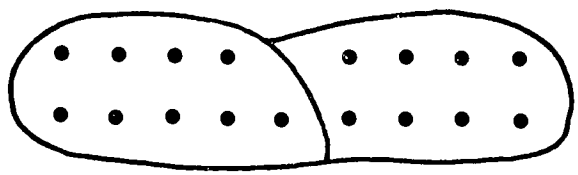
$$9 + \square = 17 \qquad\qquad \square + 9 + 17$$

Say, "These are *mathematical sentences*. We may also call them *addition sentences*." Ask, What is the sum in each sentence?" (17) "What are the addends in the first sentence?" (9, and one is missing. It is 8.) "What are the addends in the second sentence?" (9, and one is missing. It is 8.) Say,

"You are asked to find the *missing addend.*" Let the children tell how they find each missing addend in these exercises:

$5 + \square = 9$ $\qquad$ $8 + \square = 16$ $\qquad$ $12 = 7 + \square$

If a pupil cannot find the missing addend directly in a sentence like $17 = 9 + \square$, let him make a set of 17 objects. Ask him to separate it into two disjoint subsets so that one subset has 9 members. He can now find the number of the other subset by counting its members. This number is the missing addend.



$17 \quad = \quad 9 \quad + \quad \square$

Continue to find the missing addend in sentences until the children can say, "The missing addend is—."

Tell the children, "An addition sentence with a missing addend may be written as a *subtraction sentence.*" Write these sentences:

$17 = 9 + \square$
$17 - 9 = \square$

Get the children to revise addition and subtraction by talking of these things:

The first sentence is an addition sentence.
The second sentence is a subtraction sentence.
The sum in each sentence is 17.
One addend in each is 9.
The other addend is a missing addend. It is 8.

Get them to read the subtraction sentence as "17 *minus* 9 equals 8".

Remind the children that both sentences ask the same question: What is the missing addend?

Let the children revise the idea of the relation between addition sentences and subtraction sentences by turning to page 19 in their books. In each

exercise there are two sentences. Let pupils read these sentences and give a number that makes each one true. Let them decide whether each of the two sentences is true for the same missing number. Get the children to name the sum, the known addend and the missing addend in each sentence of the pair and tell whether these are the same for the two sentences. Let them then tell whether the pair of sentences mean the same thing. Notice that in Exercise 5 the number that make $18 = 11 + \square$ true is *not* the same as the number that makes $\square - 18 = 11$ true.

Ask the children to turn to page 20. On this page there is a sentence on each line. If the sentence is an addition sentence, let the pupils write the corresponding subtraction sentence for it. If the sentence is a subtraction sentence, let them write the corresponding addition sentence. Get the children to find the missing numbers to make the pairs of sentences true.

Pupil pages 21 and 22 can be used to revise subtraction. On page 21, get the pupils to copy each of the five exercises and fill in the boxes with pairs of numerals so that the first number minus the second number is the number whose numeral is shown at the top of the box. For example, the work for the first exercise may begin as follows

7
$\boxed{9} - \boxed{2}$
$\boxed{10} - \boxed{3}$
$\boxed{28} - \square$
and so on

On page 22, there are many subtraction sentences for revision of subtraction. Be sure to keep reminding pupils that when they *subtract* numbers they are finding a missing addend. For example, the second exercise on page 22 is

$$\square - 6 = 3.$$

This sentence asks them to find the missing sum if one addend is 6 and the other is 3. This is an addition exercise.

If some of the children need more revision of this kind, let them work in pairs. Let one write a subtraction sentence and give it to the other child to solve.

Note: The exercises used in this activity will be those that require what is usually called "carrying" and those that do not. However, we will not use the word "carry" (nor "borrow") with children. This programme instead uses the idea of "renaming numbers". The use of these words renaming numbers helps the children develop meaning for addition and subtraction.

Write this mathematical sentence on the blackboard:

$$456 + 213 = n$$

Question the children so they will say things like this about the sentence: "It is an addition sentence. The addends are 456 and 213. The missing sum is n. We add 456 and 213 to find n and make the sentence true."

Ask the children to rename each addend of the sentence and write it in expanded form. Get the children to write the sentence like this:

$$(400 + 50 + 6) + (200 + 10 + 3) = n$$

Let the children talk about how they can find the sum of the six addends. Guide them to agree to add first the ones (6 + 3), then the tens (50 + 10), then the hundreds (400 + 200), and then to rename the sum as 669. You write the work on the blackboard in the form below as they talk.

$$\begin{aligned} 456 + 213 &= n \\ (400 + 50 + 6) + (200 + 10 + 3) &= n \\ 600 + 60 + 9 &= n \\ n \text{ is } 669 \end{aligned}$$

Say, "There is another way to write the numerals in an addition exercise; it makes addition easier for you. We write the numerals one under the other." (See the exercise below.) Let the children tell the steps they used in the sentence form above, and you show how the same steps are used in the new form. The steps are:

1. Rename each addend in its expanded form.
2. Add the ones and then the tens and then the hundreds.
3. Rename the sum 600 + 60 + 9 in its simplest name: 669.

As the children tell about the steps, you complete the vertical form on the blackboard in the same steps, as shown below. (The work on the blackboard at any one time is like that of step 1 or step 2 or step 3. Do not include the sentences at the right; they are for your use.)

$$\begin{array}{r} 456 \\ +\underline{213} \end{array}$$ (Write the exercise.)

1. $$\begin{array}{rcr} 400 + 50 + 6 & = & 456 \\ +\underline{200 + 10 + 3} & = & +\underline{213} \end{array}$$ (Rename each addend in expanded form.)

2. $$\begin{array}{rcr} 400 + 50 + 6 & = & 456 \\ +\underline{200 + 10 + 3} & = & +\underline{213} \\ 600 + 60 + 9 & & \end{array}$$ (Add ones, then tens, then hundreds.)

3. $$\begin{array}{rcr} 400 + 50 + 6 & = & 456 \\ +\underline{200 + 10 + 3} & = & +\underline{213} \\ 600 + 60 + 9 & = & 669 \end{array}$$ (Rename the sum in its simplest name.)

Work through several other exercises with the children. Then let them each add several exercises of this type until they understand the technique. Get them to use only the vertical form above. Their written work will look like that of step 3. Some other exercises you may use are:

$$\begin{array}{cccc} 239 & 341 & 737 & 942 \\ \underline{420} & \underline{537} & \underline{262} & \underline{137} \end{array}$$

Write an addition sentence on the blackboard in which renaming is needed to write the simplest numeral for the sum. For example,

$$38 + 25 = a$$

Let the children tell you the steps they used above.

Write the exercise first in sentence form and then in the vertical form. Notice that step 3 shows the expanded form of the sum a.

First show the steps in sentence form.

$$38 + 25 = a$$

1. $(30 + 8) + (20 + 5) = a$
2. $50 + 13 = a$
3. $50 + 10 + 3 = a$

a is 63

Say, "Now we will use the shorter vertical form." Write the exercise on the blackboard and solve it as the children tell you the steps. (Notice that you will need to leave more space for the renaming in the sum.)

$$\begin{array}{r} 38 \\ +\underline{25} \end{array}$$ (Write the exercise.)

1. $$\begin{array}{rcr} 30 + 8 & = & 38 \\ +\underline{20 + 5} & = & +\underline{25} \end{array}$$ (Rename each addend in expanded form.)

2. $$\begin{array}{rcr} 30 + 8 & = & 38 \\ +\underline{20 + 5} & = & +\underline{25} \\ 50 + 13 & & \end{array}$$ (Add ones, then tens.)

3. $$\begin{array}{rcr} 30 + 8 & = & 38 \\ +\underline{20 + 5} & = & +\underline{25} \\ 50 + 13 = 50 + 10 + 3 & = & 63 \end{array}$$ (Rename the sum in its simplest name.)

Give other examples of addition with renaming of ones and tens until the children have revised this technique. For example,

$$\begin{array}{rcr} 400 + 60 + 3 & & = 463 \\ \underline{200 + 70 + 6} & & = \underline{276} \\ 600 + 130 + 9 = 600 + 100 + 30 + 9 & & = 739 \end{array}$$

Vary the activity by telling a story which asks a question. For example, "Some boys collected shells. James found 638 shells, and Obi found 274. How many shells did they find?" Let the children solve this problem and answer the question. Let them make up other story problems of their own.

Ask the class to turn to page 23. For each of the six exercises, let them tell a story problem which uses the numbers. Then let them solve the problems just as the first one is solved in their book.

ACTIVITY 3 • Finding Missing Addends

Pupil page 24

Write on the blackboard a sentence with a missing addend. For example,

26 + ( ) = 49

Say, "There is a space left within the parentheses for a numeral." Ask the pupils to help you rename 26 and 49 in expanded form. Then write the sentence in this form:

$$20 + 6 + (\quad) = 40 + 9$$

Say, "We want to find the missing addend. We will find it in parts. We will find the number of ones and then the number of tens." Ask, "How many ones must be added to 6 ones to equal 9 ones?" (3) Put "3" in the parentheses. Go on, "How many tens must be added to 2 tens to equal 4 tens?" (2 tens) Put "20" in the parentheses. Then ask, "What must be added to (20 + 6) to equal (40 + 9)?" (20 + 3) You complete "20 + 3" in the parentheses. Now the blackboard looks like this:

$$\begin{aligned} 26 \quad &+ (\qquad) = \quad 49 \\ 20 + 6 &+ (\qquad) = 40 + 9 \\ 20 + 6 &+ (20 + 3) = 40 + 9 \end{aligned}$$

Let the children talk about this work and decide that the sentence

$$26 + 23 = 49$$

is true and that the missing addend is 23.

Say, "What do we call the operation you used to find the missing addend 23?" (Subtraction) Write the sentence 26 + ( ) = 49 as a subtraction sentence: 49 − 26 = ( ). Let the children point out the addition sentence and its corresponding subtraction sentence.

Tell the pupils that the vertical form of writing the subtraction exercise makes the subtraction easier. Say, as you write the "26" under the "49", "We write the numeral for the known addend under the numeral for the sum. (See the exercise below.) Let the children tell the steps they used in the sentence form above, and you show how the same steps are used in the vertical form. The steps are:

1. Rename the sum and the known addend in expanded form.
2. Subtract the ones and then the tens and so on.
3. Rename the missing addend in its simplest name.

As the children give the above, you complete the vertical form on the blackboard in these steps. (Note that the work on the blackboard at any one time is like that of step 1 or step 2 or step 3.)

$$\begin{array}{r} 49 \\ -\underline{26} \end{array}$$

1. $$\begin{array}{rcr} 40 + 9 & = & 49 \\ -\underline{20 + 6} & = & -\underline{26} \end{array}$$

2. $$\begin{array}{rcr} 40 + 9 & = & 49 \\ -\underline{20 + 6} & = & -\underline{26} \\ 20 + 3 & & \end{array}$$

3. $$\begin{array}{rcr} 40 + 9 & = & 49 \\ -\underline{20 + 6} & = & -\underline{26} \\ 20 + 3 & = & 23 \end{array}$$

Let the children work through several other exercises of the same type.

Ask the class to copy and work the exercises on page 24. Their written work for subtraction will look like step 3 above.

Guide the children to rename each number and to begin the subtraction (or addition) with ones, and then go on to tens, to hundreds, and so on. (Notice that there are addition exercises on page 24. These are included not only as revision but so each child will think as he works each exercise.)

ACTIVITY 4 • Subtraction Techniques

Pupil page 25

Revise the technique of subtraction used in Activity 3. Let the children find the missing addend in the sentence $85 - 63 = n$. Get them to review the process of subtraction by making statements like, "We rename each number. We subtracted the ones first and then the tens. We thought of the sum 30 + 2 as 32."

Write this sentence on the blackboard:

$$82 - 54 = n$$

Notice that to find the missing addend in this sentence, the sum will have to

be renamed in a special way. (This is usually called "borrowing". As noted before, this term will not be used in this programme. Other terms which carry more meaning are used.)

Get the children to tell you to write the subtraction exercise in vertical form, and begin the process of subtraction. When this work is on the blackboard, they will find that the process they know cannot be followed.

$$\begin{array}{r} 80 + 2 = 82 \\ \underline{50 + 4} = -\underline{54} \end{array}$$

Ask, "Why can you not go on?" (We cannot subtract the ones and tens as we did before.) Say, "Name 82 in another way so we can subtract." Guide the children to rename 82 in many ways. The children may suggest "78 + 4", "77 + 5", "70 + 12", and so on. Help them to agree that "70 + 12" is a good name to try. Show this on the blackboard. It will now look like this:

$$\begin{array}{rcl} 70 + 12 & = & 80 + 2 = 82 \\ -\underline{50 + 4} & = & -\underline{50 + 4} = -\underline{54} \\ 20 + 8 & & \end{array}$$

The children will now rename 20 + 8 as 28 and say, "The missing addend n is 28" and "The sentence 82 − 54 = 28 is a true sentence."

Let them tell the steps in the process they have used:

1. Rename the sum and the known addend in expanded form.
2. If possible, subtract the ones, then the tens, and so on. If it is not possible, rename the sum so you can subtract. Subtract the ones, then the tens, and so on.
3. Rename the missing addend in its simplest name.

Work through several other exercises with the children until they understand when a second renaming is necessary and they can do the renaming.

Let the children copy and complete the exercises on page 25. Go around the class giving help as needed.

When the children have each finished several of the exercises, let them make up a story problem about each of the exercises. You may get several for each exercise. For each problem, let them answer the question asked by the problem. Here are some stories you might get for the first exercise:

There are 63 children in our town.
Twenty five of them are boys.
How many are girls? (38 are girls.)

Mr. Kanunda has 63 shillings. Mr. Sajin has 25 shillings.
How many more shillings has Mr. Kanunda than Mr. Sajin?
(Mr. Kanunda has 38 shillings more than Mr. Sajin.)

Mr. Akanji has 25 shillings.
He needs 63 shillings to buy clothes for his family.
How many more shillings does he need?
(He needs 38 shillings more.)

ACTIVITY 5 • Addition Techniques Extended to Numbers in the Thousands

Pupil page 26

Write this sentence on the blackboard:

$$785 + 537 = n$$

Quickly go through the following: the renaming of each addend in expanded form; the addition of first the ones, then the tens and then the hundreds. The blackboard will look like this:

$$785 + 537 = n$$
$$(700 + 80 + 5) + (500 + 30 + 7) = n$$
$$1200 + 110 + 12 = n$$

Suggest renaming 1200, 110 and 12 in expanded form and adding ones, then tens, then hundreds and then thousands.

$$(1000 + 200) + (100 + 10) + (10 + 2) = n$$
$$1000 + 300 + 20 + 2 = n$$
$$n \text{ is } 1322$$

Quickly show the same addition exercise in vertical form:

$$
\begin{array}{rcl}
700 + 80 + 5 & = & 785 \\
+\ \underline{500 + 30 + 7} & = + & \underline{537} \\
1200 + 110 + 12 = 1000 + 200 + 100 + 10 + 10 + 2 & = & 1322
\end{array}
$$

Use several more examples like this. Then let the pupils turn to page 26 and copy the exercises. The exercises are to be worked by filling in each sum as was done on the blackboard.

Again let the children make up story problems about a few of the exercises on page 26, and let them answer the questions in their stories. Some story problems they may make about the first exercise are:

There are 756 goats in one town
and 543 goats in another town.
How many goats are in the two towns?
(There are 1299 goats in the two towns.)

There are 756 boys in Addy School.
There are 543 girls in Armar School.
How many children are in the two schools?
(There are 1299 children in the two schools.)

A man travelled 756 miles in one month.
He travelled 543 miles the second month.
How many miles did he travel in the two months?
(He travelled 1299 miles in the two months.)

ACTIVITY 6 • Addition and Subtraction

Pupil pages 27-30

Pupil pages 27 and 28 may be used to give more practice in the techniques of addition and subtraction. If a pupil can solve the exercises on page 28 without writing the numerals in expanded form, let him do so. If not, let him write the expanded forms in his exercise book, rename numerals if necessary, and find the ones, tens and hundreds in the sum or in the missing addend.

Vary this activity by telling stories that ask questions about addition and subtraction, and by letting children ask story problems. For example, "Mary's father caught 328 fish yesterday. He sold 132 of them. How many

did he keep?'' Or, ''A man spent 375 shillings for a bicycle and 187 shillings for clothing. How many shillings did he spend?'' (If shillings are not used in your country, change to the unit of money you use.)

Use pupil pages 29 and 30 to give children experience in solving story problems. Let them read the problems silently and then aloud; then let them give the mathematical sentences for the problems and find the numbers that make the sentences true. Always ask them to answer the questions in the problems after they find the missing numbers in the sentences. They will need guidance from you in each step.

The written work for the first exercise will look like this:

$d = 581 + 407$

$$\begin{array}{r} 500 + 80 + 1 = 581 \\ \underline{400 + \ \ 0 + 7} = \underline{407} \\ 900 + 80 + 8 = 988 \end{array}$$

d is 988

There were 988 letters put in the box today.

UNIT 3

Multiplication and Division

OBJECTIVES

1. To revise the meaning of multiplication
2. To extend multiplication to products of 36 and beyond
3. To introduce division as finding a missing factor
4. To learn basic multiplication and division facts
5. To revise factors and multiples of numbers
6. To find the product of three factors

Background Information for Teachers

See Part II, "Arithmetic", pages 25-28, 34-36, in your *Teachers' Handbook*.

In *Primary Two*, the children learned the meaning of multiplication by mixing sets. They learned to mix two sets to form the set of all pairs of members of the sets. They thought of 4×3 as

4 rows with 3 objects in each row;
Four 3's;
3 added four times.

They also learned to use the words "factors" and "products". They solved problems to find products and missing factors. These problems are related to sentences of two types:

a. $3 \times 6 = n$ is a sentence with a missing product.
b. $5 \times n = 15$ is a sentence with a missing factor.

Sentence a is one in which the product is to be found. Sentence b, in which a factor is missing, is seen so often in arithmetic that we give the process of finding the missing factor a special name and symbol. We call the process *division*, and we use the symbol "÷" for the process. For example, if one of the factors of 15 is 5, we call the missing factor "15 ÷ 5" (15 *divided by* 5). Thus, $15 \div 5 = 3$ because $5 \times 3 = 15$. In this way, we define division by saying that the sentence

$$15 \div 5 = n$$

is true for the same number n as the sentence

$$15 = 5 \times n.$$

In this unit, the meaning of multiplication is revised, and the division symbol is introduced for the process of finding a missing factor. First, the products of factors are extended to 36 and beyond. In doing this, the children will also learn to find missing factors of products to 36.

When multiplying, the children can think of arranging objects in rows. They can even think of objects in one row or with zero rows. Also, they can think of objects arranged with one object in each row or with zero objects in each row. This leads to products with 0 and 1 as factors. In this way, the children understand that when 0 is a factor the product is always 0. That is, for any whole number n,

$$0 \times n = 0.$$

Also, they see that when 1 is a factor the product is always the other factor. In other words, for any whole number n,

$$1 \times n = n.$$

No techniques of multiplication or division are introduced in this unit. Only their meanings are studied. Later, in Unit 9, after some properties of multiplication are discovered in Unit 6, the development of techniques is begun.

Addition and multiplication are operations which are performed on two numbers at a time. When we try to multiply *three* factors, we must first choose two of the factors, find their product, and then multiply this product by the third factor. We need symbols to tell which two factors are to be multiplied first. Such symbols are called *parentheses*. For example, in the sentence

$$3 \times (4 \times 2) = n,$$

the parentheses around 4×2 tell us to multiply these two factors first. Then n is 3×8, or 24. In the sentence

$$(3 \times 4) \times 2 = n,$$

the parentheses tell us to multiply 3×4 first. Then n is 12×2, or 24. The children will see that these two products are the same whenever they start with the same three factors. $3 \times (4 \times 2) = (3 \times 4) \times 2$

In this unit, problems are introduced in which both addition and multiplication are used. When both operations occur in the same sentence, great care must be taken to explain which operation is performed first. The parentheses are needed to show this. For example,

$$3 + (4 \times 2) = n$$

is a sentence in which the multiplication is done first, because the parentheses around 4×2 tell us to add the product of 4×2 to 3. Then n is $3 + 8$, or 11. But in the sentence

$$(3 + 4) \times 2 = n,$$

the parentheses tell us to perform the addition first. In that case, n is 7×2, or 14.

STAGE 1 • Revision of Meaning of Multiplication, Products, Factors

Vocabulary

Mix sets, rows, product, factor, multiply, multiplication sentence, multiplication sign (×), missing factor

Materials

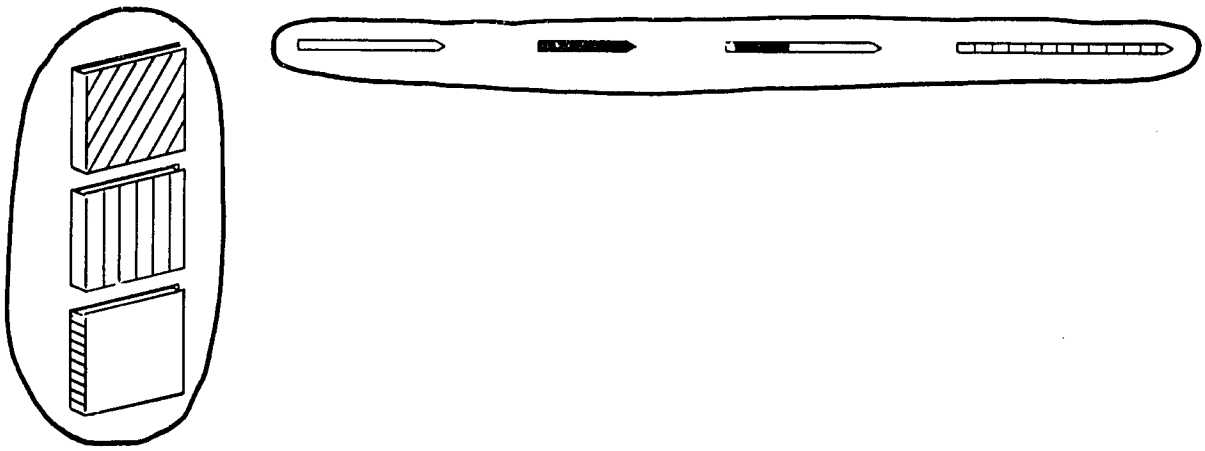
Multiplication sentence cards (5" × 7") for products to 18; pupil pages 31-33

Teaching Procedure

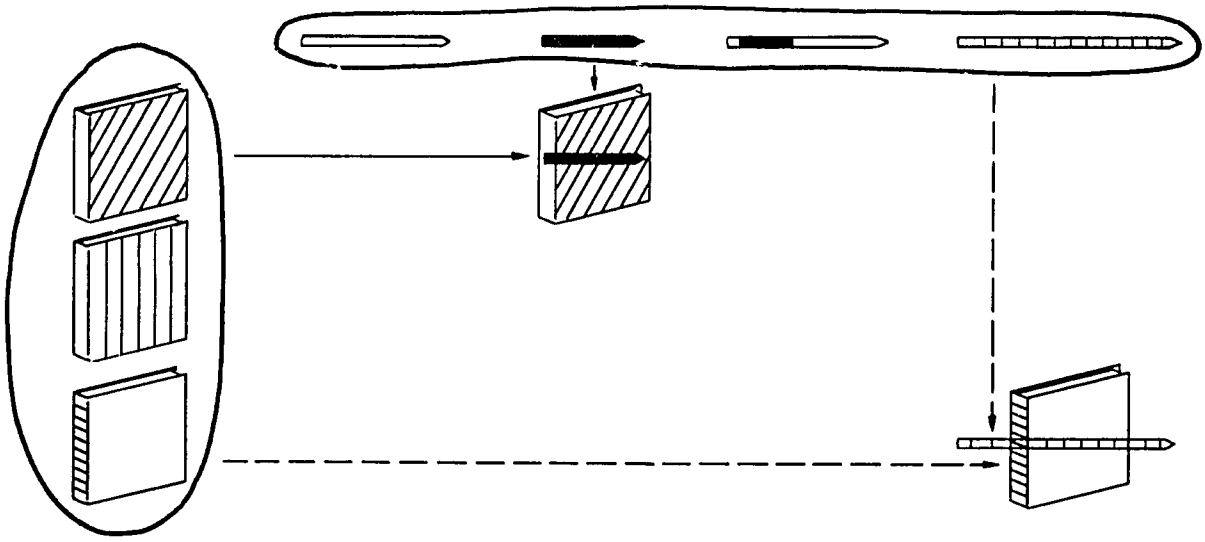
ACTIVITY 1 • Revising the Meaning of Multiplication

Pupil page 31

Show the children two sets of objects. For example, arrange a set of three books (different sizes and colours) and a set of four pencils (different lengths) on your table, as shown below.



Tell the class, "We will *mix* the set of books and the set of pencils to form the set of all pairs of books and pencils." Let pupils in turn come up to the table and put a pencil on a book to show a pair of a book and a pencil. (Let each child move a book to the right and move a pencil down until the pencil is on the book.) Each time a pupil shows a pair, draw a picture of his pair on the blackboard. For example, the first two pairs might look like this in the picture you draw.



Continue until the class decides that every pair of a book and a pencil is made. (There will then be three rows with four pairs in each *row* in the picture.)

Let the children tell about the set of pairs they made. "We mixed a set of 3 books and a set of 4 pencils, and we got a set of pairs. Each pair is made from a book and a pencil. The mixed set has 12 pairs of books and pencils." Ask, "How do you know there are 12 pairs?" Let a child count the pairs in the picture by 4's. (4, 8, 12) Let another child add the pairs in the rows (4 + 4 + 4 = 12) and so on.

Say, "We can make a picture of a set of stars and a set of triangles." You can show the mixed set by drawing pictures of stars in triangles. On the blackboard draw a diagram such as this:

	★	★	★	★	★
△					
△					
△					

Let the class tell you how many members the set of triangles has and how many members the set of stars has. Ask the children in turn to come to the blackboard and help mix the sets by each drawing a star in a triangle (⊿) until they show all the pairs in the mixed set. The diagram will look like this:

	★	★	★	★	★
△	⊿	⊿	⊿	⊿	⊿
△	⊿	⊿	⊿	⊿	⊿
△	⊿	⊿	⊿	⊿	⊿

Say, "We drew every pair of triangles and stars that we get when we mix the set of three triangles with the set of five stars."

Ask, "How many pairs are in the mixed set?" Help the children answer by first asking for the number of pairs in each row and then the number of rows. (There are three rows, with five pairs in each row.) Ask for a

mathematical sentence telling how many pairs are in the mixed set. Write the sentence on the blackboard.

$$3 \times 5 = 15$$

Let the children read the sentence in all the ways they can. You suggest other ways. Here are some ways it can be read.

3 times 5 equals 15.
3 multiplied by 5 equals 15.
The *product* of 3 and 5 is 15.
Multiply 3 and 5 and get 15.
3 and 5 are *factors* of 15.
Three 5's are 15.
5 added three times is 15.
"3×5" and "15" are names for the same number, fifteen.

Say, "The sentence $3 \times 5 = 15$ is a *multiplication sentence*."

Repeat this activity, mixing other sets, such as a set of 4 triangles mixed with a set of 3 stars, a set of 2 triangles mixed with a set of 7 stars and so on. Each time ask for a multiplication sentence which tells the number of pairs in the mixed set. Each time ask for ways of counting the number of pairs and the ways of reading the multiplication sentence. Make sure that the pupils recognize which numbers are the factors and which number is the product. Get them to point in the sentences on the blackboard to the factors, products, *multiplication signs* and equal signs.

3	×	5	=	15
factor	multiplication sign	factor	equal sign	product

Let the pupils turn to page 31 in their books. On this page are exercises in which the children mix sets of squares and sets of dots to form sets of pairs. Let them copy the first exercise and draw the set of pairs. Then ask a pupil to tell what he did. "I mixed a set of 4 squares and a set of 4 dots, and I got a set of 16 pairs of squares and dots." Guide the pupils also to say:

The factors are 4 and 4.
The product is 16.
$4 \times 4 = 16$

Let the class do the same with the other exercises on page 31. If a pupil counts the members of the mixed set one by one, guide him to count by rows, such as 4, 8, 12, 16.

ACTIVITY 2 • Revising Multiplication Sentences

Pupil page 32

Write this number sentence on the blackboard:

$5 \times 3 = \square$

Ask the class to tell you what "5×3" means. Ask for as many meanings as they can give until someone suggests that they can think: "5 rows of things, with 3 in each row". When a child makes this reply, draw on the blackboard 5 rows of dots with 3 dots in each row. Ask the class to suggest things that the dots could stand for and make a story problem. For example, "Each dot could stand for a seed. A boy planted seeds in rows. He made 5 rows and he planted 3 seeds in each row. How many seeds did he plant?" Or, "The dots could stand for sand cakes. A girl bakes sand cakes. She puts the cakes in 5 rows on a tray. There are 3 cakes in each row. How many sand cakes does she bake in the sun?" Ask for the answer to the question in each story problem. Let the class find the answer by counting by threes or by adding threes or by remembering that 5×3 is 15. Ask, "What kind of a problem is this?" (A multiplication problem) "Why?" (We multiply two numbers.)

Vary this activity. Draw on the blackboard 4 rows of dots with 4 dots in each row. Again ask the class to tell story problems about the dots. For example, "The dots can stand for bottles. A boy collected bottles and put them in a basket. Each day he collected four bottles. After 4 days, how many bottles were in the basket?" Ask the class to write in their exercise books a multiplication sentence for the story problem. Tell them to use a box for the missing product. Ask what numeral goes in the box to make the sentence true. ($4 \times 4 = 16$)

Repeat this activity for other sets of dots in rows. For practice, ask

the class to write the following sentences in their exercise books and complete them to make them true:

$5 \times 2 = \square$ $\quad 2 \times 8 = \square$ $\quad 3 \times 5 = \square$
$2 \times 6 = \square$ $\quad 4 \times 3 = \square$

Say, "There is another way to show a missing product in a sentence. We can write a *letter*." For example,

$5 \times 2 = n$ *or* $3 \times 5 = a$

Ask, "What number n makes $5 \times 2 = n$ a true sentence?" Write each of the other sentences with a letter instead of a box. For each, ask, "What number makes the sentence true?"

Write on the blackboard:

$4 \times b = 16$

Ask, "If b is 3, is the sentence true?" (No) "If b is 5, is the sentence true?" (No) "What number is b so that the sentence is true?" (4) For each answer that children give, ask, "Why?" Let a pupil tell you how he got his answer, and let the other children tell whether he is right or wrong.

Ask the class to turn to page 32 in their books. Let them copy the sentences in their exercise books and find the numbers that make the sentences true. For example,

$3 \times 2 = a$, $\quad a$ is 6.

Notice that one of the sentences is an addition sentence, two are multiplication sentences with *missing factors*, one is a substitution sentence and one is a subtraction sentence:

(5) $2 \times a = 16$, $\quad a$ is 8. $\qquad$ (6) $c \times 3 = 12$, $\quad c$ is 4.
(7) $5 + 3 = b$, $\quad b$ is 8. $\qquad$ (9) $12 - e = 5$, $\quad e$ is 7.

Make some multiplication sentence cards as follows. Use cards about 5" × 7". On one side of a card write "10", on the other side write "1 × 10". On another card write "10" on one side and "2 × 5" on the other. Make three cards with "12" on one side. On the other sides write "1 × 12", "2 × 6", "3 × 4". Make two sentence cards for 14, and make three sentence cards for 16. Each card shows a product on one side and factors of the product on the other side.

Arrange all the multiplication sentence cards so that those with the product 10 are together, those with product 12 are together and so on. Hold up all the cards showing "10". Ask the class for the pairs of factors of 10. When someone says "2 and 5", turn the card around showing "2 × 5". When someone says "5 and 2", turn the card around showing "5 × 2" and so on. Do this for the cards showing "12"; for the cards showing "14"; showing "16". If a child gives a pair of factors that are wrong, show him there is no card with those factors. Help him tell why there is no card.

Change this activity. Hold up all the cards showing "12". Say, "One of the factors of 12 is 4. What is the other factor?" When someone answers, let the class decide whether the answer is correct. Then turn the proper card around. (The discussion should include such comments as: "I know the other factor is 3 because 4 times 3 is 12.") Continue this activity, asking for the missing factors of the products 10, 12, 14, and 16.

Ask the pupils to write in their exercise books a mathematical sentence with a letter for the answer to this question: If one factor of 16 is 8, what is the missing factor? Let someone show you what he wrote. If he wrote

$$8 \times n = 16 \quad or \quad n \times 8 = 16,$$

you write it on the blackboard and ask for the number that makes the sentence true. Tell the children they have found a missing factor.

Ask the class to think of a story problem for the sentence with a missing factor: $8 \times n = 16$. You may begin a story about some sweets in a box. Let the factor 8 tell the number of children who share the sweets. Then let the class finish telling the story problem. (How many sweets does each

child get?) In this way, revise problems in sharing equally or separating a set into subsets of the same size. Be sure that the children find the number that makes the sentence true (n is 2) and that they tell the answer to the question in the story problem. (Each pupil gets 2 sweets.)

Repeat this activity, asking for story problems for $3 \times a = 12$, $b \times 6 = 18$ and so on. In each case, let the children find the number that makes the sentence true and tell the answer to the question in the story problem.

Let the class turn to page 33. Ask them which mathematical sentences tell about missing products and which tell about missing factors. Then let them copy the sentences and find the numbers that make the sentences true. (Notice that some of the sentences are addition sentences.)

STAGE 2 • Meaning of Division

Vocabulary

Division, divided by, division sign (÷), division sentence, array

Materials

Pupil pages 34-35

Teaching Procedure

ACTIVITY 1 • Division as Finding a Missing Factor Pupil page 34

Tell a story problem for this sentence:

$b \times 6 = 18$

For example, ' Eighteen people came to a rest house for lunch. They sat down at tables, with 6 people at each table. How many tables did they use?" (b is 3. They used 3 tables.)

After the question in the story problem is answered, ask questions like these: "What kind of a problem is this?" (A problem to find the missing factor) "Why?" (We know one factor of 18 is 6. We found the missing factor to answer the question in the problem.) Say, "We meet problems like this often. We are going to give them a special name. They use a special sign."

Write a "÷" on the blackboard, point to it and say, "When we know one factor of a product, we use this sign to show that we are finding the other factor." Write "18 ÷ 6" and say, "This means that we are finding the missing factor of 18 if we know one factor is 6. What is the missing factor?" (3) "What is another name for 18 ÷ 6?" (3) Point to each symbol in turn and say, "18 *divided by* 6". Let several pupils repeat this. Ask, "What number is 18 divided by 6?" (It is the missing factor, 3.) Tell the class that "÷" is called the *division sign* and that it tells us to find a missing factor.

Write on the blackboard:

18 ÷ 6 = 3.
18 divided by 6 equals 3.

Ask, "Is this sentence true?" (Yes) "Why?" (Because 18 ÷ 6 tells us to find the missing factor of 18. We know that if one factor is 6 the missing factor is 3.)

Write "12 ÷ 4" on the blackboard. Ask children in turn to read the numeral (12 divided by 4) and to tell what it means. (Find the missing factor of 12 if one of the factors is 4.) Ask, "What is the missing factor?" (3) Ask, "Why is it 3?" Encourage answers such as, "The missing factor is 3 because 4 times 3 is 12."

Continue this activity, asking pupils to read the symbols, tell the meaning of the symbols, and find other names for 16 ÷ 8, 15 ÷ 3, 10 ÷ 2, 14 ÷ 7 and so on.

Write this sentence on the blackboard:

$14 \div 7 = n$

Ask, "What number makes the sentence true?" (n is 2.) "Why?" (Because if one factor of 14 is 7, the missing factor is 2.) Ask, "Can you write this sentence another way, using multiplication?" Help the children to remember

that since we have a product and factors we can write the sentence as:

$$7 \times n = 14$$

Ask, "What number makes this sentence true?" (n is 2.) (Make sure the two sentences are written side by side as shown below.)

$$14 \div 7 = n \quad \text{means} \quad 7 \times n = 14$$

Discuss these sentences until the class agrees that the sentences ask for the same number, a missing factor. Get the children to say that these are two different ways of asking the same question about a missing factor. Call one a multiplication sentence and the other a *division sentence*.

Write several sentences on the board, such as:

$$4 \times a = 16 \qquad 10 \div 5 = x \qquad 12 \div 2 = n \qquad 3 \times b = 15$$

Point to each in turn and ask, "What does the sentence ask?" (It asks for a missing factor.) "What kind of a sentence is it?" (The first and last are multiplication sentences, and the second and third are division sentences.) Ask for another way to write each of the sentences. As children tell you a new sentence, write it under its corresponding sentence:

$$\begin{array}{cccc} 4 \times a = 16 & 10 \div 5 = x & 12 \div 2 = n & 3 \times b = 15 \\ 16 \div 4 = a & 5 \times x = 10 & 2 \times n = 12 & 15 \div 3 = b \end{array}$$

The purpose of this activity is to let the children see that every sentence with a missing factor can be written another way: a multiplication sentence can be written as a division sentence, and a division sentence can be written as a multiplication sentence. They should also see, of course, that each of the two sentences is true for the same number.

Ask the class to turn to page 34 in their books. If the sentence involves multiplication, let the pupils write the corresponding division sentence. For example, the first exercise gives this sentence:

$$3 \times n = 15$$

The pupils write the corresponding division sentence. ($15 \div 3 = n$) Then they tell the number that makes this sentence true. (n is 5.) If the given sentence involves division like the second sentence, let the children write a corresponding multiplication sentence. ($6 \times a = 12$) Go over the first few exercises orally.

Let the pupils place a piece of paper beside the sentences on the page. Ask the children to write their new sentences on the paper. Also get them to write in the numerals for the numbers that make the sentence true.

Tell this story problem: "I have 15 beans and lay them in rows. There are 5 in each row. How many rows do I make?" Let a child tell you a multiplication sentence for this problem. ($n \times 5 = 15$) Then let another child write a division sentence for the same problem. ($15 \div 5 = n$) Let still another give a number for n that makes both sentences true. (n is 3.) Finally, get a child to use this number to answer the question. (You make 3 rows of beans.) Tell other story problems of this kind, and ask the class to write in their exercise books two sentences for each problem.

ACTIVITY 2 • Sentences for Problems; Problems for Sentences

Pupil page 35

Tell a story problem about sharing things equally among children or separating things equally in piles or placing the same number of things in several rows. Let the children write a multiplication sentence and a division sentence for the problem.

For example, "16 oranges are put on a tray. They are put in rows, with 4 in each row. How many rows are there?" Ask what kind of a problem this is. (A problem in finding a missing factor) Let the pupils write sentences for the problem:

$$n \times 4 = 16 \qquad 16 \div 4 = n$$

Encourage the children to tell several problems of this kind. Follow the plan outlined above.

Write this sentence on the blackboard:

$$3 \times a = 12$$

Ask the class to tell a story problem for the sentence. Let them try to do this without help or with little help from you. You may suggest sharing or laying objects in rows or in piles. Ask for another sentence for this story problem. ($12 \div 3 = a$)

Continue this activity with other sentences. Each time ask for a story problem, then for another sentence, then for the number that makes the sentences true and finally for the answer to the question in the story problem.

Vary these activities. Sometimes give a story problem and ask for a mathematical sentence. Sometimes give a mathematical sentence and ask for a story problem. Include a story problem with a subtraction sentence or an addition sentence. This helps the children always to think about the kind of problem they are solving.

Let the class open their books to page 35. They will see that each set of dots is arranged in rows. (A set of things arranged in rows, with an equal number of members in each row, is called an *array*.) Ask pupils to make up story problems about the arrays of dots. Some may tell multiplication problems. Some may tell division problems. For example, the first array has 3 rows with 6 in each row. A pupil may tell a multiplication story like this: "The dots are maize seeds. Adu planted 3 rows of seeds and put 6 seeds in each row. How many seeds did he plant?" Let the children write a multiplication sentence for his story,

$$3 \times 6 = n,$$

and tell what number makes this true. (n is 18.)

Another pupil may tell a division story: "The dots are oranges. Obi has 18 oranges and wants to share them equally among three friends. How many oranges does he give each friend?" Let the children write a division sentence for his story,

$$18 \div 3 = t,$$

and tell what number makes the sentence true. (t is 6.) Continue in this way for each array of dots.

Vocabulary

No new vocabulary

Materials

Seeds for counters; 5" × 7" cards to be made into sentence cards for new products and factors; pupil pages 36-41

Teaching Procedure

ACTIVITY 1 • Learning Products to 36

Pupil pages 36-39

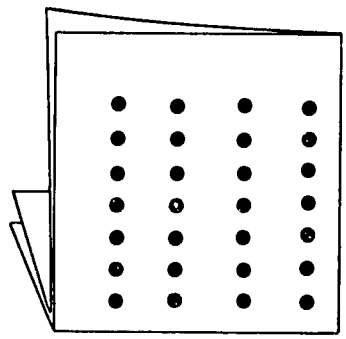
Revise factors of products and products to 20 using the sentence cards. Ask questions such as these: "What is the product of 3 and 6?" and "What other name can we give 3 times 6?" Hold up a sentence card showing "3 × 6". "What is 3 times 6? What is three 6's?" and so on. Vary the language of your questions. Ask some of the pupils how they arrived at the products. (By counting by 6's, by adding 6 + 6 + 6 and so on)

Let the children find products to 36. Write "4 × 6" on the blackboard, and ask what it means. (4 times 6, four 6's, 4 rows of 6 things and so on) Ask, "What is the product of 4 and 6?" If anyone says "24", ask how he decided on 24. (I added 6 + 6 + 6 + 6 or I counted by 6's four times or I know 3 × 6 is 18 and 6 more is 24.) Accept any of these. If an incorrect answer is given, let children decide whether it is correct. If nobody says, "24", let the children put objects in 4 rows of 6 and count the objects in the array.

At this point, let the children make an array of dots that will help them find all the other products to 30. Ask the class to turn to page 36. There they see a 12 by 10 array of dots. Show them how to hold a sheet of paper over the page and trace the array. Say, "We will call this our paper array".

Let the children use their paper arrays to find products in the following way. Write "7 × 4" on the blackboard, and ask for the meaning of the

symbols. Among the replies should be, "7 rows of things with 4 in each row". Ask, "How can you show 7 rows of things with 4 in each row using your paper arrays?" Let pupils suggest folding the paper to show a 7 by 4 array of dots. Ask the children to do this. Let them hold up their 7 by 4 arrays.



After everyone has made a 7 by 4 array, ask, "What is the product of 7×4?" Let one pupil count the dots in his array by 4's. Let another count the dots by adding. $(4 + 4 + 4 + 4 + 4 + 4 + 4)$ Get the children to agree that 7×4 is 28. Let a child make a new sentence card for this product. Guide him to write "28" on one side of the card and "7×4" on the other side.

Continue this activity, finding the products of 5×5, 6×5, 7×5, 5×4, 6×6, 8×4, 9×4, 7×3, 8×3, 9×3, 10×3, 11×3 and 12×3. Let the children fold their paper arrays to show each product. For each product, let a child make a new sentence card.

Ask the class to turn to page 37 in their books. Let the children tell what number makes each of the sentences true. Let them use any help they need, such as their paper arrays or arrays made of seeds.

Use sentence cards to practise finding factors and products up to 36. For example, hold up all the cards with 16 written on one side and ask, "What are factors of 16?" $(1 \times 16, 2 \times 8, 4 \times 4)$ Turn the cards around to show the pairs of factors as a check. If all the factors of 16 are not given, let a pupil fold his paper array in as many ways as he can to show 16 dots. Hold up a card with "7×4" written on it and ask, "What is the product of 7 and 4?" (28) Turn the card over to show the product. No matter what reply a child gives, let him decide whether he is right by looking at an array. Hold up a card showing "24" and say, "One factor of 24 is 4. What is the other factor?" If someone says, "6", ask, "Why?" $(4 \times 6 = 24)$ Let the children show this with their arrays, and then turn the sentence card over to show "4×6". Repeat this activity for other products and other factors.

Ask, "If one factor of 32 is 8, what is the other factor?" Talk about this with the children until the answer, 4, is given. If none of them know, give 32 counters to a pupil and let him put them in eight rows with the same number in each row. Then get him to count the number of counters in each row. Let another child show rows of 8 with his paper array until the array shows 32 dots. Let the children write a number sentence for this. ($8 \times n = 32$) Ask if there is any other way of writing this sentence. ($n \times 8 = 32$ or $32 \div 8 = n$) Repeat this activity, asking for many missing factors of products and the corresponding mathematical sentences.

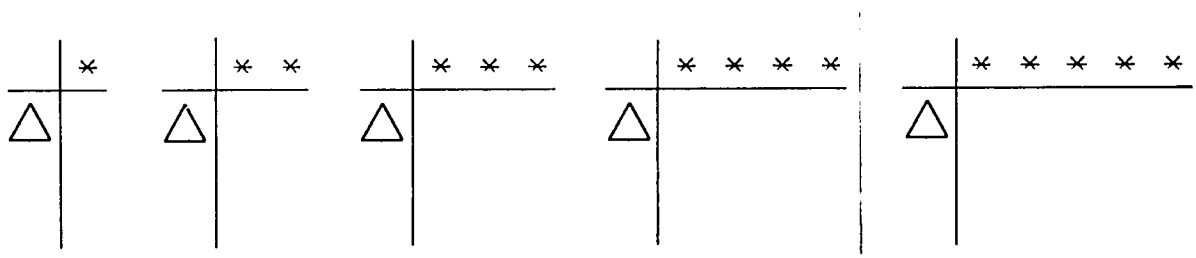
Ask the class to find what numbers make the following sentences true:

$$3 \times y = 9,\ 4 \times c = 12,\ 3 \times a = 24,\ n \times 6 = 30,\ d \times 4 = 32$$

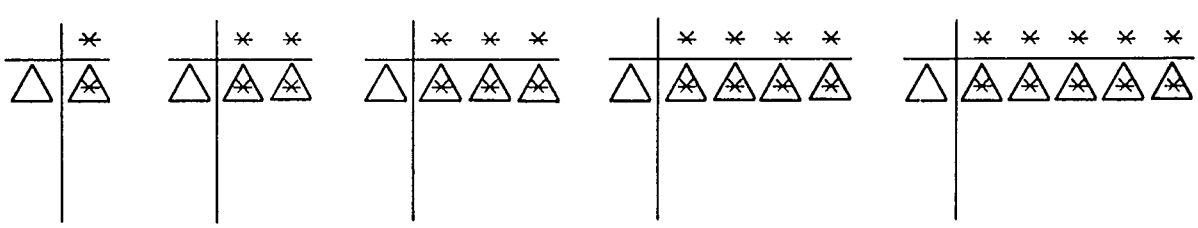
Let the class turn to pupil pages 38 and 39 for practice with products to 36.

ACTIVITY 2 • Revising Multiplication with a Factor 1

Draw these diagrams on the blackboard, and ask the class to mix the sets in each diagram. (Ask them to draw △✳ in each position.)



The completed diagram will look like this:



Ask the class to tell about the multiplication shown by each of the completed diagrams. (One row of one. One row of two. One row of three. One row of four. One row of five.) Let the pupils write a multiplication sentence for

each diagram. ($1 \times 1 = n$, $1 \times 2 = n$, $1 \times 3 = n$, $1 \times 4 = n$, $1 \times 5 = n$) Then ask them to complete the sentences to make them true.

Let the class arrange objects to show: one row of one, two rows of one, three rows of one, six rows of one and so on. Get the children to write a sentence for each array they make. ($1 \times 1 = n$, $2 \times 1 = n$, $3 \times 1 = n$, $5 \times 1 = n$ and so on) Ask, "What numbers make the sentences true?" Let the children copy the true sentences into their exercise books.

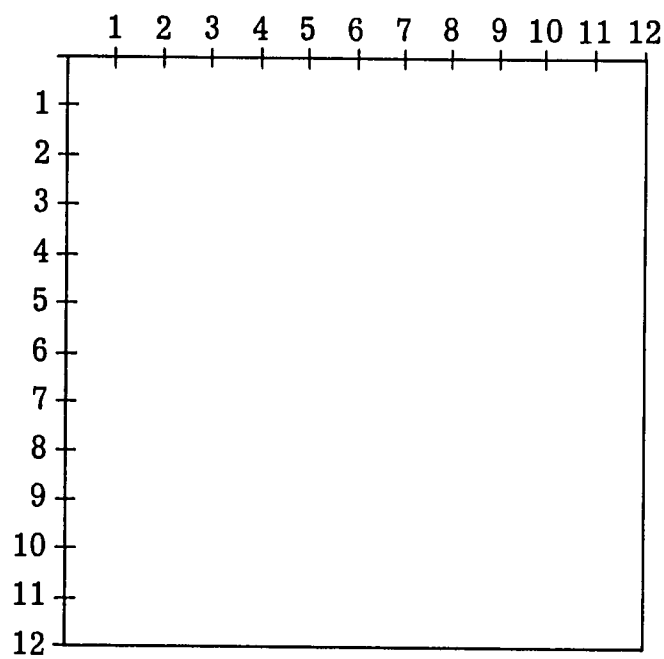
Let the children think about this question: "What is the product of 1 and 63?" Give the children time to think. Some may need to make an array. When the product 63 is given, ask, "Why?" (One row with 63 in the row is 63.) Ask for the product of 1 and several other large numbers. Discuss reasons for the answers. Make new sentence cards showing each factor 1 to 10 with 1:

$$1 \times 1,\ 2 \times 1,\ 3 \times 1,\ \ldots,\ 10 \times 1$$

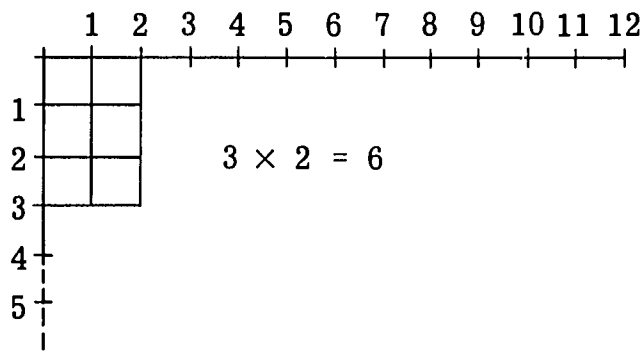
ACTIVITY 3 • Discovering Multiplication Facts with Products up to 36 and Beyond

Pupil page 40

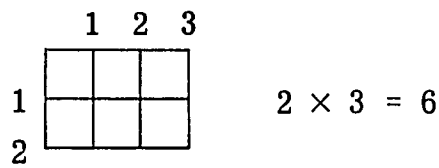
On the blackboard, draw a chart like this, with numerals written as shown:



Say, "We use this chart to help us multiply. We will use it to multiply 3 by 2." Then show the children how to use the chart. Count by 1's down the left column of the chart to "3". Make a mark at the "3". Count by 1's across the top row to "2". Draw a line down from the "2" and a line across from the "3" until the two lines meet, thus completing a box. Draw a line down from "1" and others across from "1" and "2" to form an array of squares, as follows:



Ask the class to count the squares by 2's in rows. (2, 4, 6 squares) Let them add to find the number of squares. Then get the class to draw lines in their exercise books to make a 2 by 3 array of squares. Let them count the squares and say, "The array shows that 2 × 3 = 6."

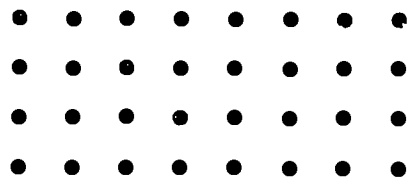


Ask the class to find the following products by drawing arrays of squares: 3 × 6, 4 × 5, 4 × 6, 4 × 7, 4 × 8, 4 × 9, 4 × 10, 5 × 5, 5 × 6, 5 × 7, 5 × 8, 6 × 6 and 6 × 7. Continue to make a new sentence card for each new product discovered.

Let the children make the sentences on page 40 true. If they have difficulty with any sentence, let them draw an array of squares to find the product or missing factor.

ACTIVITY 4 • Revising Meaning of 1 × 0, 2 × 0, . . . , 0 × 1, 0 × 2 . . .

Write "4 × 8" on the blackboard. Ask the class to think about the meaning of 4 × 8. Get them to tell about an array of dots or squares which shows 4 × 8. They may say, "4 rows of 8's", "8 added four times" or "32 dots or squares". Let the pupils show 4 × 8 by placing rows of objects on the table or by drawing rows of marks or squares on the blackboard.



Let the class show that the product of 4 and 8 is 32 by counting by 8's or by adding 8 four times.

Let the pupils talk about the meaning of 4 × 1. Get them to draw an array to show 4 × 1:

Ask what 4 × 0 means. Let the pupils think of an array to show 4 × 0. They will decide that 4 × 0 is shown by an array with 4 rows of objects with no objects in each row. Ask how many objects there are in four rows if there are no objects in any row. (Zero) Let the class decide that 4 × 0 = 0.

Ask the pupils to form rows of objects showing 3 × 4 and 1 × 4. Ask what 0 × 4 means. Let the class talk about this until they decide that 0 × 4 means "no rows of 4's". Ask how many objects there are in the rows if there are no rows. (Zero) Let the class decide that 0 × 4 = 0.

Also, let the class find the following products: 1 × 0, 0 × 1, 2 × 0, 0 × 2, 3 × 0, 0 × 3, 4 × 0, 0 × 4, 5 × 0, 0 × 5, 6 × 0, 0 × 6, 7 × 0, 0 × 7, 8 × 0, 0 × 8, 9 × 0, 0 × 9, 10 × 0, 0 × 10. Let them make sentence cards to be used later in their activities.

Ask, "What is the product of 29 and 0? Why? What is the product of 0 and 51? Why? What is the product of 0 and 0? Why?" Let the children talk

about each question. Ask, "What is the product of any number and zero?" Let the class decide that the answer to all these questions is "zero". Get them to tell why.

Revise these products; 1×1, 2×1, 1×2, 3×1, 1×3, 4×1, 1×4 and so on.

Ask, "What is the product of 19×1? Why? What is the product of 36×1? Why? What is the product of any whole number and one? Why?" Let the class decide that the product is always the same as the number that was multiplied by one. Let them show this with arrays of objects.

ACTIVITY 5 • Story Problems

Tell a story problem such as this:

"A mother has 5 children. She gives each of them 3 bananas. How many bananas does she give in all?"

Let the class tell you what numbers are to be multiplied to get the answer. (If any child does not understand the problem, give each of 5 children 3 objects, let them arrange the objects in rows and count the number of objects.)

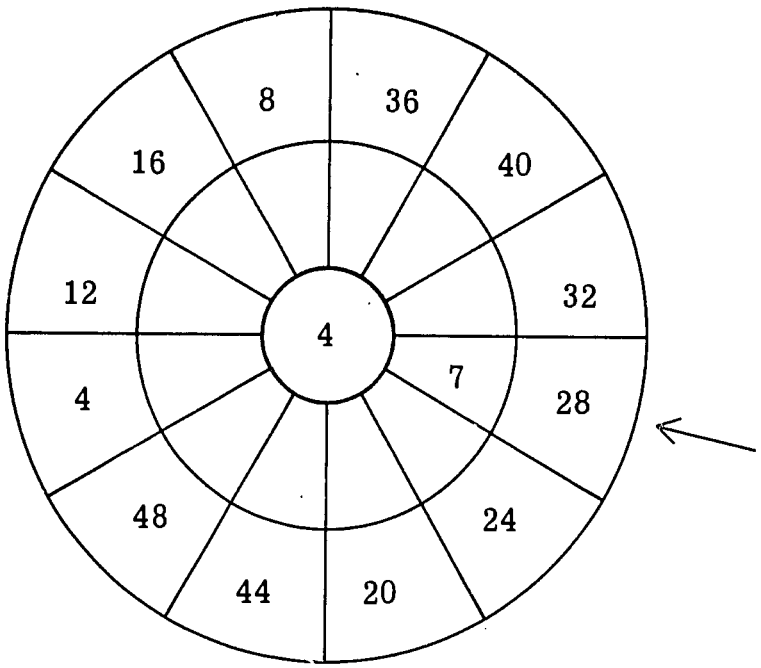
Guide the children to relate the problem to the sentence $5 \times 3 = n$, and ask for the number that makes the sentence true. (n is 15) Ask for the answer to the question in the story problem. (She gives 15 bananas in all.)

Tell other story problems. Let the children relate the problem to a mathematical sentence which needs to be completed; for example:

1. If there are 6 oranges in each of 4 baskets, how many oranges are there in all the baskets?
2. You have 6 rooms in your house. Each room has 5 people in it. How many people are there in your house?

ACTIVITY 6 • A Product-and-Factors Game

Separate the class into two teams. Draw a circular product chart on the blackboard, as shown below:



Write a numeral for a factor in the middle of the product chart. The factor 4 is shown as an example. Write numerals for products in the spaces provided in the outer ring. Leave the spaces in the inner ring for the other factors. With a pointer, touch a numeral for a product; for example, 28. Ask, "If 4 is one factor of 28, what is the other factor?" Let a child give an answer. Write the answer in the inner space as shown, and let the children decide if $4 \times 7 = 28$. A child wins a point for his team if his answer is correct. Continue with other products on the chart in the same way.

Draw another product chart with the factor 6 in the middle. Continue with more charts until every pupil has had at least one turn.

At the end of the game, look at all the completed charts. Relate the finding of a missing factor to division. Say, "If one factor of 28 is 4, what is the other?" (7) "What is $28 \div 4$?" (7)

ACTIVITY 7 • Sentences for Problems

Pupil page 41

Ask the class to turn to page 41 in their books. As they look at each picture, you tell the following story problems. Ask them to write a sentence for the problem and find the number that makes the sentence true. Then ask them to give the answer to the question the problem asks. (Story problem 1 goes with picture 1 and so on.)

1. How many books does a man need for his 4 children if each child is to get 3 books?
2. A mother cooks 2 eggs for each of her 8 children. How many eggs does she cook?
3. I have 14 oranges. How many can I give to each of 7 boys if each is to get the same number of oranges?
4. How many legs have 5 cows?
5. How many children can share 24 pencils if each gets 4 pencils?
6. There were 8 birds on a tree. 7 more birds joined them. How many birds are on the tree now?
7. If I take 7 oranges from a set of 18 oranges, how many oranges do I have left?
8. If you plant 6 rows of flowers with 6 flowers in each row, how many flowers do you plant in all?
9. How many houses can I count in 4 rows if each row has 8 houses?

STAGE 4 • Division of Products to 36 and Beyond

Vocabulary

No new vocabulary

Materials

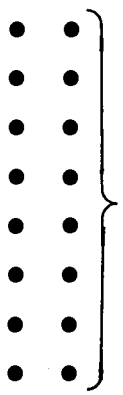
Counters; pupil pages 42-50

Teaching Procedure

ACTIVITY 1 • Revising the Meaning of Division

Pupil page 42

Separate the class into groups of about four and give 16 counters to each group. Tell each group to arrange their counter in rows, so that there are 2 counters in each row. Ask them to count the rows that are formed.



The groups count 8 rows.

Ask one group, "How many rows of counters did you make?" (8) "How many counters are in each row?" (2) "You made an array. What does your array show about multiplying numbers?" (If the product is 16 and one factor is 2, then the other factor is 8.) Let another group decide whether the answers given are correct.

Ask another group to tell a sentence for their array of counters. Get them to use a letter for the missing factor. ($n \times 2 = 16$ *or* $16 \div 2 = n$) Let one of the children write the sentence on the blackboard. Let the class decide whether this is the sentence the array shows. Ask still another group to find a number that makes the sentence true. ($8 \times 2 = 16$ *or* $16 \div 2 = 8$)

Ask another group to tell another sentence for the array they made. If the sentence on the blackboard is $n \times 2 = 16$, expect the new sentence to be $16 \div 2 = n$. (If the group cannot reply, ask whether a division sentence can be made.) If the sentence on the blackboard is $16 \div 2 = n$, expect the new sentence to be $n \times 2 = 16$. Let a child write the new sentence under the other sentence on the blackboard. Finally, ask the class what number makes the second sentence true. (n is 8.)

Let the class talk about the two sentences. Ask, "What number are

you asked for in each sentence?" (One sentence asks us to find the missing factor of 16 if one factor is 2. The other asks us to find the number we get when we divide 16 by 2.) Guide the discussion until the class agrees that these two sentences ask the same thing. (Dividing 16 by 2 is the same as finding the missing factor of 16 if one factor is 2.)

Repeat this activity with another array. Let the groups arrange their 16 counters in rows in as many different ways as they can. Let each group leader tell how many rows his group made and how many counters are in each row. Get a member of each group to write a mathematical sentence the array shows. Use a letter to refer to the number of rows. (The sentences may be: $n \times 16 = 16$ for 1 row of 16; $n \times 8 = 16$ for 2 rows of 8; $n \times 4 = 16$ for 4 rows of 4; $n \times 2 = 16$ for 8 rows of 2; $n \times 1 = 16$ for 16 rows of 1.) Each time a child writes a sentence, let the other groups decide how many rows he is reporting. (They will be deciding what number makes the sentence true.)

Write on the blackboard:

$$12 \div 4 = \square$$

Get the class to talk about the meaning of the sentence in two ways:

1. Let them describe the array which shows the sentence. (We put 12 counters in an array, with 4 counters in each row. We must find the number of rows in the array.)
2. Let them talk about the missing factor in the sentence. (One of the factors of 12 is 4. The sentence asks us to find the missing factor.)

Ask what number makes the sentence true. (3)

Let the pupils check the answer in two ways:

1. Place 12 counters in rows of 4 and count the rows.
2. Multiply 3×4.

Continue this activity with other division sentences, such as $15 \div 3 = a$ and $18 \div 9 = b$.

Ask the children to turn to page 42 in their books. Let them find the missing number in each sentence. Ask them to make the sentences on the page true.

Ask, "Which of the sentences on the page are shown by the picture of

the fishes?" Let the children tell about the sentences and the fishes. (Sentence 1, $5 \times 3 = a$, is pictured by the 5 rows of 3 fishes each. Sentence 2, $3 \times 5 = a$, is shown by turning the book sideways; the fishes are in 3 rows of 5 each. Sentence 6, $15 \div 5 = b$, with the book still turned, is shown by the array of 15 fishes in 3 rows. Sentence 7, $15 \div 5 = c$, with the page in the proper position, is shown by the array of 15 fishes in 5 rows.)

ACTIVITY 2 • Extending Division to Products of 36 and Beyond

Pupil page 43

Separate part of the class into groups of exactly 6 children. (Let others watch.) Give each group 18 seeds. Ask each member of a group in turn to take one seed until all 18 seeds are gone. Let each child count the number of seeds he has. (3)

Let the class discuss what each group did. For example, "18 seeds were shared by 6 children. Each child got 3 seeds." Or "6 of us shared 18 seeds, and each of us got 3." Let them write in their exercise books the mathematical sentence that they showed with seeds. ($18 \div 6 = 3$) Ask a child to read his sentence. (18 divided by 6 equals 3) Write the sentence on the blackboard, and ask another child to read what it says, using the words "product" and "factors". (If one factor of 18 is 6, the missing factor is 3; *or* the product of 6 and 3 is 18.)

Rub out the "3" and replace it with "n". The sentence on the blackboard is now:

$$18 \div 6 = n$$

Ask the class to tell you this division sentence as a multiplication sentence. ($6 \times n = 18$) Write the sentence on the blackboard. Now the blackboard looks like this:

$$18 \div 6 = n$$
$$6 \times n = 18$$

Ask, "What number makes the division sentence true? What number makes the multiplication sentence true? Does the same number make both these sentences true?" (Yes) "What *is* n?" (3) Let the children talk about why n

is the same number in both sentences. (The sentences ask the same question.)

On another part of the blackboard, write this multiplication sentence:

$$\square \times 3 = 24$$

Ask the class to complete the sentence to make it true. If a child completes the sentence, ask how he can show his answer is true. (Make an array of 8 rows of 3 counters; *or* I know $8 \times 3 = 24$.) Ask a child to tell you a division sentence for this multiplication sentence. Let him write it on the blackboard below the multiplication sentence. ($24 \div 3 = \square$) Get the children to talk about these two sentences until they agree that the same number (8) makes both of the sentences true.

Write $30 \div 6 = a$ on the blackboard. Ask, "What number makes this sentence true?" Let the pupils put 30 counters in 6 rows with the same number in each row. Get them to count the number in each row. Ask them to write the sentence as a multiplication sentence. ($6 \times a = 30$) Again let the class agree that the same number (5) makes both sentences true.

Continue this activity with the sentences $b \times 7 = 28$, $30 \div 10 = m$, $27 \div 9 = t$, $3 \times w = 21$ and so on until these ideas are clear:

a. for each division sentence there is a multiplication sentence;
b. for each multiplication sentence there is a division sentence;
c. the two sentences are always true for the same number.

Continue to stress the idea that we divide to find a missing factor. Write "32" on the blackboard. Get a child to name a pair of factors of 32. If someone says "4 and 8", ask, "If one factor of 32 is 4, what is the other factor?" (8) "What is 32 divided by 4?" (8) "If one factor of 32 is 8, what is the other factor?" (4) "What is 32 divided by 8?" (4) On one side of the blackboard, write these results:

$$4 \times 8 = 32,\ 32 \div 8 = 4,\ 32 \div 4 = 8$$

Ask for another pair of factors of 32. When someone replies, for example, "2 and 16", repeat the activity above with these factors. Go on until all the factors are given: 1×32, 2×16, 4×8, 8×4, 16×2, 32×1. If all of these pairs of factors are not given, let the children put 32 counters in rows

in every possible way with the same number of counters in each row.

Do this acitivity for factors of 20, 21, 22, 24, 25, 28, 30, 36.

Ask the class to turn to page 43 in their books and make the sentences true. Make sure each pupil has 36 counters to use to find answers for himself. He may share the counters.

ACTIVITY 3 • Story Problems in Division

Tell the class some story problems. For each story, ask the children to write a division sentence and find the answer to the question asked in the problem. Here are some story problems you may use:

1. I have 18 books to give to 9 children. How many books will one child get if each child gets the same number of books? ($18 \div 9 = n$, n is 2.)
2. My mother has 20 oranges. She gives 4 oranges to each of her children and has no oranges left. How many children does she have? ($20 \div 4 = t$, t is 5.)
3. Four boys want to share 32 pencils. Each gets the same number of pencils. How many will each boy get? ($32 \div 4 = m$, m is 8.)
4. Seat 24 children at desks so that there are 6 children in each row. How many rows are used? ($24 \div 6 = b$, b is 4.)
5. Put 27 bananas in 3 baskets so that there are an equal number in each basket. How many bananas in each basket? ($27 \div 3 = v$, v is 9.)
6. Alice can make 32 dresses in 8 days. How many dresses can she make in 1 day? ($32 \div 8 = y$, y is 4.)
7. I can write 4 words on a strip of paper. How many strips will I need to write 36 words? ($36 \div 4 = c$, c is 9.)
8. What number multiplied by 3 equals 36? ($t \times 3 = 36$, $36 \div 3 = t$, t is 12.)

After the class has written sentences for these story problems, write some mathematical sentences and ask them to tell story problems for the sentences. For example, write this sentence on the blackboard:

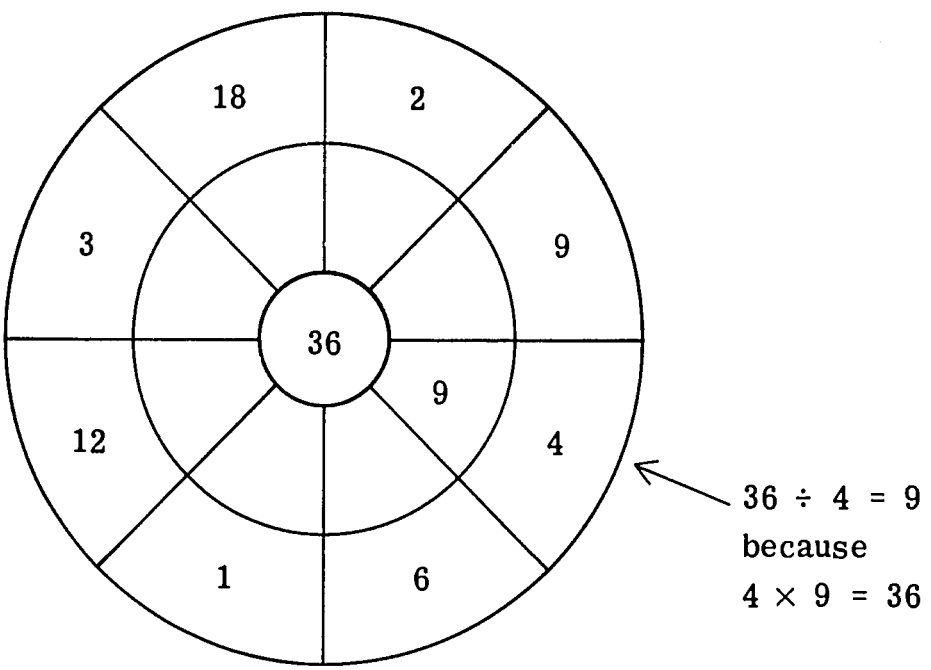
$$24 \div 3 = n$$

Ask the children to think of a story problem for the sentence. Let one or more children tell their stories, and let the class answer the questions asked in the stories. Repeat the activity with $22 \div 11 = a$, $20 \div 2 = c$ and so on.

ACTIVITY 4 • A Division Game

Use a circular product chart to play this division game. The game helps children revise division facts with products to 36. It also helps them to relate division to multiplication.

Draw the chart as shown below. Put a numeral for a product in the middle, numerals for factors in the outer ring and leave blank spaces in the inner ring.



Point to a factor in the outer ring, for example, 4, and say, "4 is a factor of 36. What is 36 divided by 4?" If a child says "9", write "9" in the inner ring and ask, "Is this correct? Why?" ($4 \times 9 = 36$) Go on with the other factors.

Draw other product charts. Use other products in the middle and their factors in the outer ring. Go on as before.

The game may be played between teams, with members of each team in turn finding the missing factors.

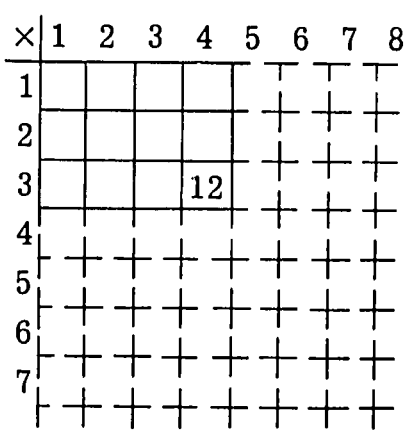
ACTIVITY 5 • Building a Multiplication Chart

Pupil pages 44-45

If needed, use pages 44-45 to give the class more practice in finding products and missing factors. Let them find the numbers that make the sentences true on page 44 with the help of page 45.

In all but five of the sentences on page 44 there is exactly one number that makes each sentence true. In Sentences 5, 6, 10, 11 and 13, two numbers must be found to make each sentence true, and there are several choices. For example, in Sentence 5, the children may say that a is 4 and b is 5, or that a is 2 and b is 10 and so on. Be sure to ask for all pairs of numbers. Let the children refer to page 45 for help by counting rows of trees or cups or tins.

Draw an array of 12 squares by 12 squares on the blackboard, as follows, and let pupils show that 3×4 is 12.



Let them do this by beginning at the top left corner and outlining a 3 by 4 array of squares. Let them count the number of squares that are found by the array and write "12" in the lower right corner where counting ended.

Draw a similar chart on a large piece of cardboard. Each time a pupil multiplies two numbers by outlining an array on the blackboard chart, write the product in the proper space on the cardboard chart. Do this for 1×4, 2×4, 3×4, 4×4, 5×4, 6×4, 7×4, 8×4 and 9×4, 1×5, 2×5,. . .and so on.

Let pupils go on as far as they like with this project, but fill in the cardboard chart at least for the products shown in the chart below:

×	1	2	3	4	5	6	7	8	9	10	11	12
1	1	2	3	4	5	6	7	8	9	10	11	12
2	2	4	6	8	10	12	14	16	18	20	22	24
3	3	6	9	12	15	18	21	24	27	30	33	36
4	4	8	12	16	20	24	28	32	36			
5	5	10	15	20	25	30	35					
6	6	12	18	24	30	36						
7	7	14	21	28	35							
8	8	16	24	32								
9	9	18	27	36								
10	10	20	30	40								
11	11	22	33									
12	12	24	36									

Then let pupils practise reading the chart. For example, ask a child to show the product of 5 and 3 on the chart. He should:

a. point to the "5" in the left column with his left hand,
b. point to the "3" in the top row with his right hand,
c. let his left hand move to the right along the row and his right hand move down the column until his fingers meet on a square marked "15",
d. say, "3 × 5 = 15" as he shows the 5 by 3 array and points to "15".

Continue with other products and factors.

Some pupils will notice in the chart that the numbers in a row (or in a column) show skip counting. Ask a child to tell what number he is counting by as he says the numbers whose numerals are in a row (or a column).

ACTIVITY 6 • Finding Products and Factors

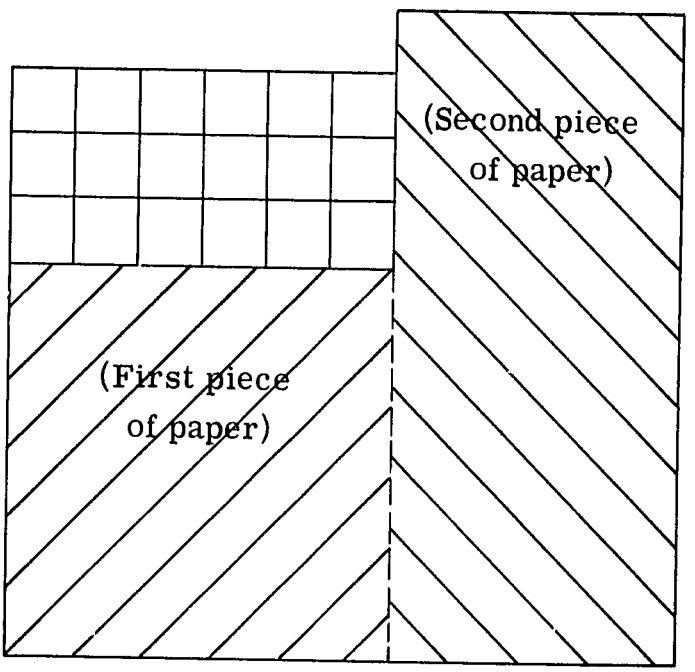
Pupil page 46

Ask the children to look at the chart on page 46 in their books. Give each child two pieces of paper. Show them how to use their papers to make

an array of squares. They cover part of the chart with their papers. Tell them to form an array which shows this sentence:

$$3 \times 6 = n$$

The children should hold their papers on the chart as is shown in this figure:



Ask them to show the 3 by 6 array by moving their hand over the array. Ask them to count the squares in the array. (18) Let them make the sentence true. ($3 \times 6 = 18$)

Continue with other sentences in the same way. For each sentence, let the children use their papers to make an array that helps them complete the sentence.

To revise products and factors, give the pupils a product, for example, 12. Let the pupils decide on an array to show this product. Let them write a multiplication sentence for each array. Continue until all arrays of 12 squares are shown. ($3 \times 4 = 12$, $4 \times 3 = 12$, $2 \times 6 = 12$, $6 \times 2 = 12$) Go on with the products 15, 18, 24, 30 and 36. Continue the activity as needed.

ACTIVITY 7 • Mathematical Sentences

Pupil pages 47-49

Tell some story problems and let the children talk about them. For example,

1. I put stones in a basket, two stones at a time. If I put two stones in the basket ten times, how many stones do I put in the basket? ($2 \times 10 = n$)
2. A teacher bought 15 books for a class of 20. How many more books does he need so that each pupil will have 2 books? ($(20 \times 2) - 15 = n$, *or* $15 + n = 20 \times 2$.)

Let pupils make sentences for the problems. Guide them with your questions. Then let them find the numbers that make the sentences true, and let them answer the questions asked by the problems.

Discuss other kinds of mathematical sentences. Let the children tell what kind of sentence each one is: addition, subtraction, multiplication or division. Find the numbers that make them true. For example,

$17 + 3 = s$	$12 \div 6 = x$	$15 + y = 19$	$5 - a = 1$
$12 - 3 = n$	$16 \div 8 = b$	$b + 9 = 17$	$3 \times q = 12$
$4 \times 7 = f$	$20 \div 10 = r$	$45 + 5 = s$	$12 + n = 20$

See if pupils remember all ways of writing these sentences. For example,

$12 + n = 20$ means $20 - 12 = n$.
$12 + n = 20$ means 12 is one addend of the sum 20.

What is the other?

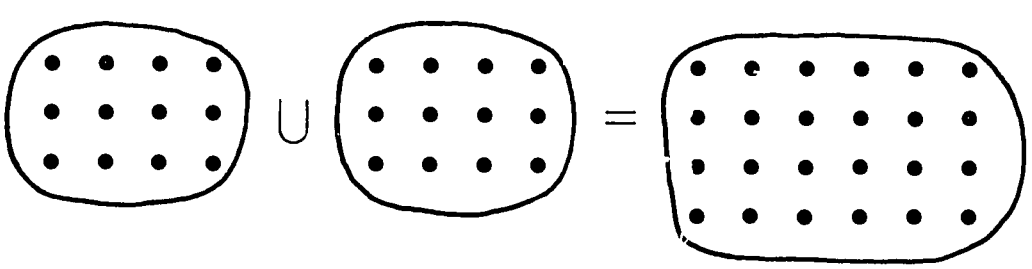
Let pupils turn to page 47 in their books. In each exercise, let them find the number that makes the sentence true. Let them continue with pages 48 and 49 as needed. You may wish to use these pages later as revision.

ACTIVITY 8 • Story Problems

Tell the following story problems. Let pupils write in their exercise books the sentences for the stories. Help them to find the numbers that make

the sentences true. Finally, let a child tell the answer to the question asked by each problem.

1. Three boys have 6 sweets each. How many sweets have all the boys together? ($3 \times 6 = a$; a is 18; the boys have 18 sweets.)
2. Jim has 4 oranges; James has 12. How many more oranges has James than Jim? ($12 - 4 = c$; c is 8; James has 8 more oranges than Jim.)
3. On Monday I bought 3 books; on Tuesday I bought 7 books. I gave 5 of them to my friend. How many had I left? ($(3 + 7) - 5 = t$; t is 5; I had 5 books left.)
4. Twelve dresses are bought for 4 girls. How many dresses does each girl get if each of them gets the same number? ($12 \div 4 = d$; d is 3; each girl gets 3 dresses.)
5. I have 5 rows of orange trees. There are 6 trees in each row. How many trees are there altogether? ($5 \times 6 = r$; r is 30; there are 30 trees altogether.)
6. Thirty-six eggs are placed in 4 rows in a box. How many eggs are in each row? ($36 \div 4 = b$; b is 9; there are 9 eggs in each row.)
7. How many tables will be needed for 30 pupils if 2 pupils sit at each table? ($30 \div 2 = p$; p is 15; 15 tables are needed.)
8. Four boys ate 3 bananas each on Friday. On Saturday two of them ate 2 bananas each. And on Sunday they ate no bananas. How many bananas in all did they eat? ($(4 \times 3) + (2 \times 2) + 0 = b$; $12 + 4 + 0 = b$; b is 16; they ate 16 bananas.)
9. Two boys each had 3 rows of 4 seedlings. If they plant them in one plot, how many rows of 6 plants each will they have?



$((3 \times 4) + (3 \times 4) = a \times 6$

$24 = a \times 6$; a is 4;

(there are 4 rows of 6 seedlings.)

Vocabulary

True, false, parentheses

Materials

Multiplication sentence cards; 30 blocks of sugar or match boxes; pupil pages 50-55

Teaching Procedure

ACTIVITY 1 • True or False Sentences

Pupil pages 50-51

Begin the lesson with a short oral activity, using multiplication sentence cards. Ask for products of given factors, and then for factors of given products. Use different ways of stating the questions.

Ask the class to turn to page 50 in their books. Let the children read the sentences. They will say that some of the sentences are *true* and some are *false*. Ask whether Sentence 1 is true. When a pupil gives an answer, ask, "Why?" Guide him to think about the numerals on each side of the equal sign (=). Ask, "Do 3×5 and 5×4 name the same number?" (No) Let the children agree that $3 \times 5 = 5 \times 4$ is a false sentence.

Ask the same questions about Sentence 2. Guide the pupils to say, "5×3 and 3×5 both name the same number 15; $5 \times 3 = 3 \times 5$ is a true sentence."

Continue asking the children in turn to decide whether sentences are true or false. Let the other children decide whether the answers are correct, and ask them to tell why.

When talking about Sentence 9, ask the class whether they need to multiply 6×4 and 4×6 to decide whether the products are the same. Let the children talk about this. They should say that a 6 by 4 array shows the same number of members as a 4 by 6 array.

Let the pupils turn to page 51 and tell what numbers make the sentences true. If a pupil give a number, say 4, for n in the first sentence, let

the class decide whether the resulting sentence "$2 \times 5 = 5 \times 4$" is true and why. Since it is not true, ask another pupil for a number for n. If he says "2", ask whether the sentence "$2 \times 5 = 5 \times 2$" is true and why.

Tell the pupils that some of the sentences have addition and multiplication in the same sentence. Let the children check these very carefully when pupils suggest numbers to make the sentences true.

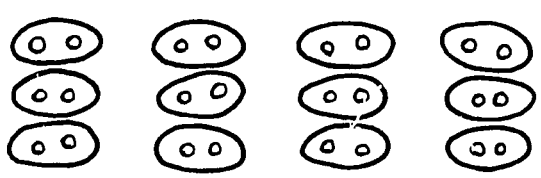
Let the class discuss Sentence 12 until they decide that there is no whole number that makes the sentence true. Let them show this by leaving the space empty or by writing "no number".

ACTIVITY 2 • Problems with Three Factors

Pupil page 52

In this activity, the class will learn to multiply three numbers. They must remember that numbers are multiplied two at a time.

Ask the class to tell you some kinds of crops that are planted in rows. Possible answers are: cotton, maize, beans and groundnuts. Discuss ways of planting the crops; for example, making holes, the number of seeds in each hole, and the number of rows. Then draw a picture of seeds in rows on the blackboard; for example, groundnuts may be planted like this:



4 holes in each row,
2 seeds in each hole,
3 rows.

Ask questions to lead the children to tell you the ways in which the total number of seeds in rows can be found. Some of the answers given by the children should include at least b and c of the following:

a. Add the number of seeds: $2+2+2+2+2+2+2+2+2+2+2+2$.
b. Find the total number of holes, 3×4, then multiply by the number of seeds in each hole, $3 \times 4 \times 2$.
c. Find the number of seeds in one row, 4×2, then multiply by the number of rows, $3 \times 4 \times 2$.
d. Add the number of seeds in each row, $4 \times 2 + 4 \times 2 + 4 \times 2$, $8 + 8 + 8$ or 3×8.

Ask, "How many groundnuts were planted?" Point to answers b and c: $3 \times 4 \times 2$. Ask, "How many factors do we use to get the number of seeds?" (Three) Say, "There are three factors to be multiplied. We can multiply only two numbers at a time. We can decide to multiply 3 and 4." Ask a pupil to give another name for this number. (12) Draw parentheses around these factors, (3×4), and explain that the symbols ( ) are called *parentheses* and mean that we are thinking of the number named by the 3×4. Erase the (3×4) and write 12 in its place. Ask, "What do we do with the number 12 to get the number of seeds?" (Multiply it by two; $12 \times 2 = 24$.) Write:

$$\begin{aligned}(3 \times 4) \times 2 &= 12 \times 2 \\ &= 24\end{aligned}$$

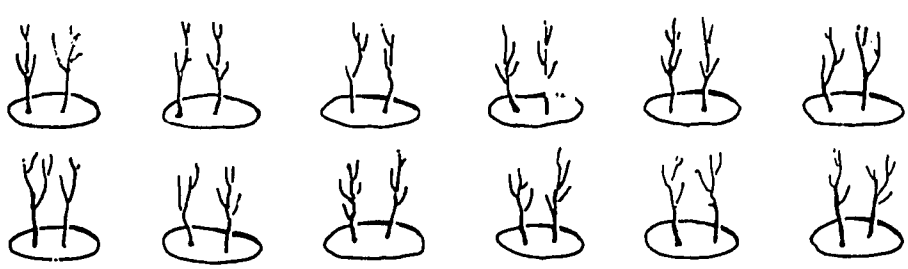
Say, "We could decide first to multiply 4 and 2." Guide the class in a similar way to tell you that the product could be written as:

$$\begin{aligned}3 \times (4 \times 2) &= 3 \times 8 \\ &= 24\end{aligned}$$

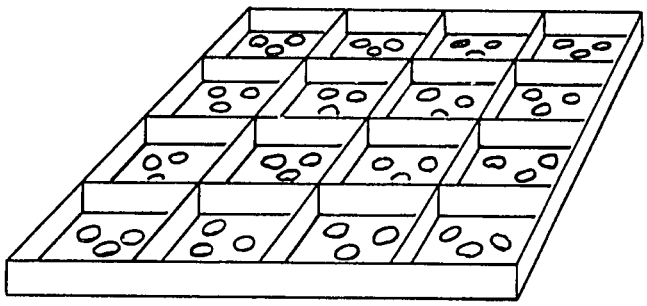
Ask, "What are the factors in each case?" (3, 4, 2) "Is the product the same in each case?" (Yes)

Do several more examples of the same kind.

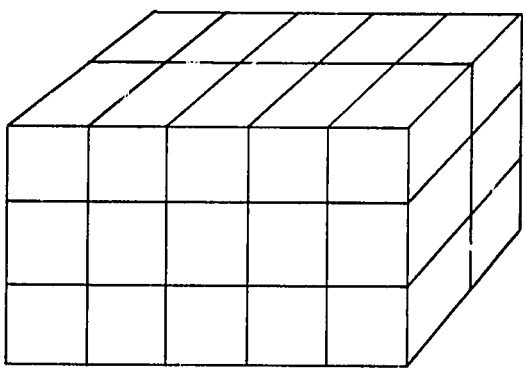
a. Use other crops that are planted in rows, perhaps as shown below:



b. Use examples of games that are played by using rows of holes and counters, in the same way seeds are planted in rows:

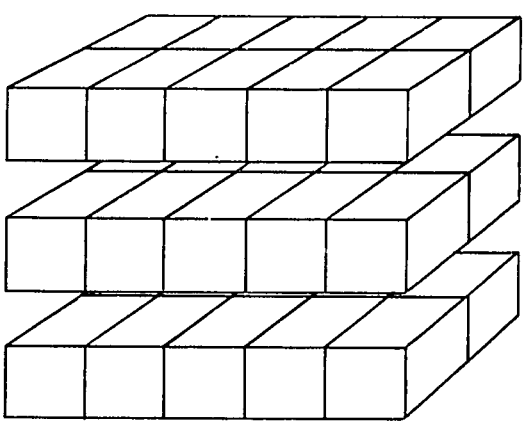


c. Use sugar blocks, bricks, match boxes or any other similar objects. Pile them as shown:



Guide the children to discover different ways of finding the total number of blocks in the pile.

For example, (1) Start from the top.

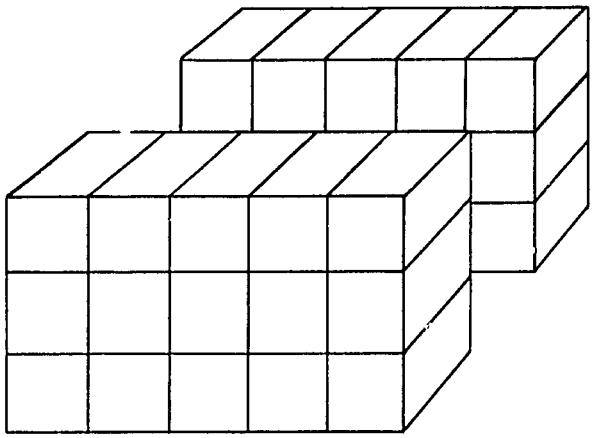


The top layer has two rows with 5 blocks in each row. There are 3 layers. Thus:

$3 \times (2 \times 5) = 3 \times 10$

$= 30$

x (2) Start from the long side. x



The long side has 5 rows with 3 blocks in each row. There are 2 columns. Thus:

$2 \times (3 \times 5) = 2 \times 15$

$= 30$

Ask, "What are the factors in each sentence? What are the products?"

Let the pupils turn to page 52 in their books. Let them tell what they see in Exercise 1. Ask, "How many rows of seeds? (2) How many holes in each row? (4) How many seeds in each hole? (3) What is the total number of seeds?" Let the children suggest ways of finding the answer. Help them to write this sentence:

$$(2 \times 4) \times 3 = n$$
$$8 \times 3 = n$$
$$n \text{ is } 24.$$

Get them to find the number of plants in Exercise 2, and tomatoes in Exercise 3. Let the children discuss Exercises 4 and 5, and ask them to write sentences for the number of blocks.

Help the class to think of multiplication as an operation on two numbers. When they multiply three numbers, they first multiply two of the numbers and then multiply the product by the third number. The pupils will see that the final product is the same number no matter which two factors are chosen first.

ACTIVITY 3 • Grouping Factors

Pupil-page 53

Let the pupils explain how to multiply three factors by multiplying two numbers at a time. Ask the children to turn to page 53 in their books. Get them to decide whether the sentences are true or false and to tell why they

are true or false. (Sentences 2, 4, 7 and 9 are true.) In sentence 2, the factors 3, 2 and 4 are grouped as $3 \times (2 \times 4)$ on the left, and $(3 \times 2) \times 4$ on the right. Let the class say that Sentence 5 is true, although there are different factors on the two sides of the equal sign. Ask the class to find the numbers that make Sentences 10, 11, 12, 13 and 14 true.

ACTIVITY 4 • Solving a Problem Using both Addition and Multiplication

Pupil pages 54-55

If needed, revise factors and products by using multiplication sentence cards. Also tell a few story problems which may be solved using multiplication and a few which may be solved using addition. Let the children give you sentences for the story problems. Also, give some sentences, such as $3 \times 7 = n$ and $4 + 9 = a$, and let the children tell you story problems for the sentences.

Ask the class to turn to page 54 in their books. (The exercises on this page include both addition and multiplication.) Do the first few exercises as a class activity. Let the class copy the other exercises in their exercise books and complete them to make true sentences. Get the pupils to check their answers. For example, in Exercise 3, ask what number is named within the parentheses. (21)

In Exercise 5, let the pupils decide that they are multiplying 10 and 2. (Here the parentheses around $4 + 6$ tell us to think of the number 10.) Let the pupils decide that the operation shown within the parentheses is to be done first and then the other operation.

Tell the class this story problem and ask a pupil to write a sentence for it:

> "Peter is collecting money for a boys' club. He must collect five cents from each pupil in the class. On Monday he collected from two pupils. On Tuesday he collected from 3 pupils. How much money has he collected so far?"

Note: If your country uses a different unit of money, use that unit in your story.

One pupil may write this sentence:

(2×5)	$+$	(3×5)	$=$	$\square$
Number of cents collected on Monday		Number of cents collected on Tuesday		

Or he might write this sentence:

$(2 + 3)$	$\times$	5	$=$	$\square$
Number of pupils who paid on Monday and Tuesday		Number of cents collected from each pupil		Total number of cents collected

By careful questioning and by suggestions, help the children to write each of these sentences. Do not tell them what the sentences should be. If one or the other of these sentences is not given, you might write it. Let the class notice that each of these sentences tells about the same problem. Let the children decide that each sentence is true for the same number. Thus, this sentence is true:

$$(2 \times 5) + (3 \times 5) = (2 + 3) \times 5$$

Ask the class to turn to page 55. Let them tell which sentences are true and which are false.

UNIT 4

Geometry I

OBJECTIVES

1. To revise line segment, square corner, angle, right angle, rectangle, square, circle, end point, tip of corner, ball, box, sphere
2. To revise the joining of two points, the extending of a line segment, line, region
3. To learn the idea of a point and of a plane region
4. To learn to identify and know some characteristics of cube, vertex, simple closed curve
5. To learn the meaning of "beside", "to the right of", "to the left of", "above", "below", "inside", "outside"
6. To learn the meaning of intersection of line segments, of lines, of line segment and circle and of intersection of sides of box.

Background Information for Teachers

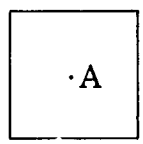
See Part 2, "Geometry and Measurement", pages 67-86, in your *Teachers' Handbook.* (In particular, see "Line segment", page 69; "Congruent line segments", page 70; "Region", page 80; "Points", page 83.) Also see Teachers' Guide, *Primary One*, Unit 13, pages 181-206 and Teachers' Guide, *Primary Two*, Unit 3, pages 63-98 and Unit 13, pages 300-314 and 311-314.

The *Primary One* and *Primary Two* programmes introduce these geometric figures: line segment, line, square corner, triangle, four-sided figure, point and the endpoint of a line segment. They also introduce geometric relations between these figures.

In this unit, these ideas are revised and extended. Relations such as between, beside, above, on and inside are studied. These ideas are called "nearby relations". They are called "nearby" because they are relations between objects which are usually near each other. The relations are discussed with many examples. You will think of others which are known to your pupils.

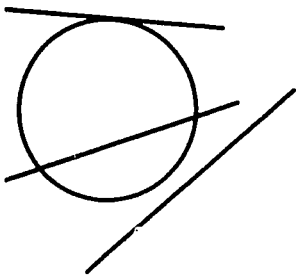
"Nearby relations" exist among objects on a line, on a surface, and in space. Hence, objects in space, such as boxes, bricks, spheres and balls are considered as well as objects in a flat surface, or plane. Some of the objects studied are triangles, four-sided figures, circles, line segments and lines. Geometry is about relations between figures. The plan is to help pupils recognize and name these geometric relations wherever they occur.

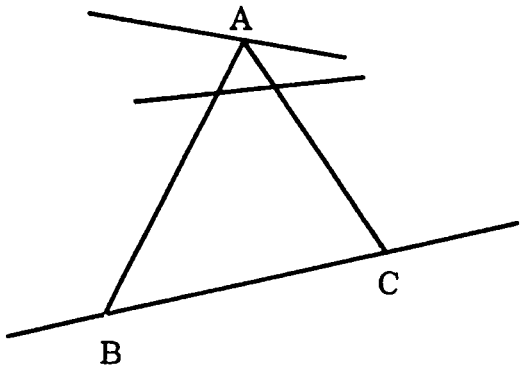
Note on the meaning of "point". In *Primary One* and *Primary Two*, points introduced are those at tips of corners, at ends of line segments and on line segments. In *Primary Three* the idea of a point as *indicating position* is introduced. The earlier examples in *Primary One* and *Primary Two* all relate points to line segments. A point indicating a place is not always on a line. Lines may be drawn through points. A small dot is used to picture a point. Point A in the picture below shows a place near the centre of the square.



Note on the meaning of "curve". A string, whether held loosely or stretched tightly, shows a geometric figure. The name "curve" is given to all the figures shown by a string. Most curves, such as those shown by dropping a string on the floor, do not have special names. But when a string is stretched tightly, a special type of curve is shown that we call a "line segment". Some other special closed curves that can be shown by a string are the circle, triangle and rectangle.

Note on intersection. Intersections of streets, corners of boxes and buildings, knife cuts in cheese and folds and wrinkles in paper are examples of the geometric idea of intersection of figures. In geometry, a place where two figures meet is sometimes a point. The point lies on both figures. Thus, we say the figures have the point in common. In some cases, two figures meet at more than one point. The figures shown by the intersection of the floor and a wall of a room have a line in common; they intersect in a line. A circle and a line may intersect in two points or one point or no points. A triangle and a line may intersect in one point or two points or in a line segment. Pictures of these intersections are shown below.

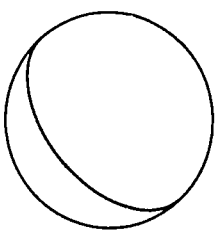




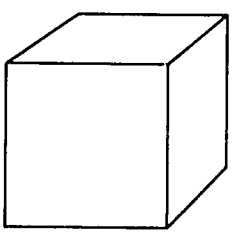
Note on names of curves and regions. In everyday conversation, we often use words like "circle", "triangle" and "rectangle" in two ways. They sometimes are used to mean a plane curve and sometimes are used to mean a plane region inside a curve. For example, people say that the rim of a wheel and the top of a tin both show circles. However, the rim of a wheel shows a circle, while the top of a tin shows the region of a circle.

In this book, "circle", "triangle" and "rectangle" will refer to the curves. We will say "circular region", "triangular region" and "rectangular region" when we refer to the regions. In summary, names of plane figures usually refer to the curves, not the plane regions.

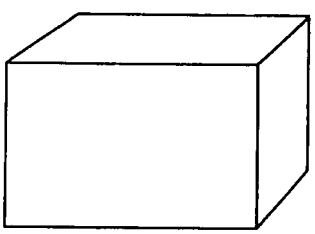
There is a similar relation between solid figures and solid regions. In this book, words like "ball", "brick" and "cube" are the names of solid regions.



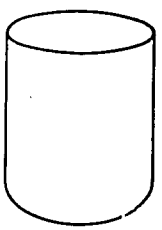
Ball



Cube



Brick



Cylinder

When we name the surface of a ball, we say "sphere". When we name the surface of a cube or a brick, we say "box". We will talk of the "shell", or "skin", of the cylinder. A closed surface is called a solid figure.

Plane as well as solid objects have a region and a boundary. "Figure" is the name used for the boundary. We have plane figures, plane regions, solid figures and solid regions. For example, a "face" of a cube is a plane region and is part of a solid figure.

Note on Materials for Geometry. Make the following materials available in the classroom for you and the pupils to pick up and use or talk about at any time: a wooden straight-edge about 36 inches long, a large block or box, several balls of different sizes, jars and tins to show cylinders, objects in the form of cubes and cones, several cutouts to show different-size circles, three-sided figures, four-sided figures, five-sided figures. The children may help collect these. One of the purposes of Activity 1 of Stage 1 is to collect and prepare these materials.

STAGE 1 • Nearby Relations

Vocabulary

Beside, to the right of, to the left of, between, above, below, points, curve, rectangle, line segments, on, inside, outside, closed curve, plane region, line

Materials

A one-foot piece of string for each child; a 30- or 40-foot piece of string; rubber bands; seeds; pupil pages 56-59. (See also pages 81, 82, 83, 84 and 86.)

Teaching Procedure

ACTIVITY 1 • Collecting Objects that Show Geometric Figures

Let the pupils help you *collect, make* and *identify* different geometric shapes. Many things which show shapes are in the classroom or can be easily obtained. The children may volunteer to bring some things from home. Bring some of the objects yourself, and have materials available for making some others.

Collect objects that show the following figures: points, line segments, lines, corners, square corners, angles, right angles, triangles with two

congruent sides, triangles with no sides congruent, circles, circular regions or disc, rectangles, squares, four-sided figures with no right angles, five-sided figures, regions, spheres, boxes, cubes, pyramids, cylinders, cones and bricks.

Listed below are some objects that show these shapes. You will think of other objects. You and the children collect or make as many of these objects as you can.

1. *Points* are shown by dots on the blackboard, tips of corners of books, ends of sharpened pencils, ends of pins, pin holes in paper, intersections of folds in paper.

2. *Line segments* are shown by edges of books, stretched strings and rubber bands, straight sticks, pencils, grass, straw, some shadows, folds in paper, edges of wooden board, chalk and pencil traces of straight-edges.

3. *Lines* are imagined as the extensions of line segments shown by objects like those listed above. The extensions are shown by drawing arrowheads on the ends of pictures of line segments.

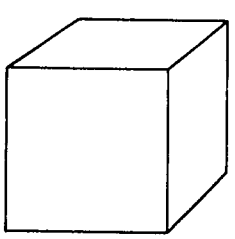
4. *Angles* are shown by corners of objects; *right angles* are shown by square corners.

5. *Triangles, four-sided figures, squares, rectangles,* and *circles* are shown by string, by bent wire, by sticks fastened together, by the edges of cutouts, by drawings on the blackboard.

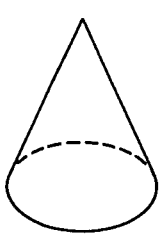
6. *Spheres* are shown by balls; *cylinders* are shown by tins and jars; *boxes* are shown by boxes; *bricks* by blocks of wood or clay and by books; *cubes* are shown by sugar blocks and by models made from paper.

7. *Cones* are shown by roofs on circular buildings and by objects made from paper.

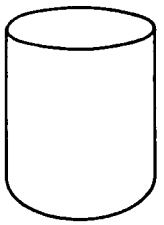
Drawings of solid objects are shown below. Models of these may be made from paper. Directions for making a model of a cone and a model of a cube follow:



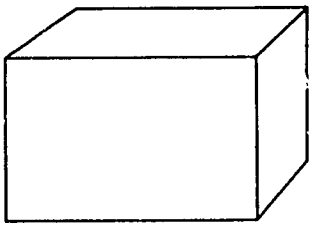
Cube



Cone

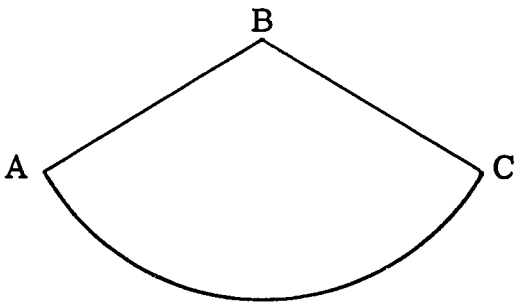


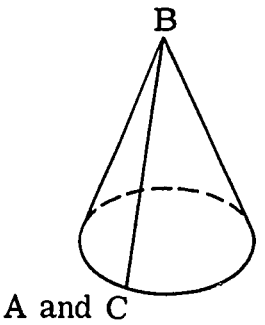
Cylinder



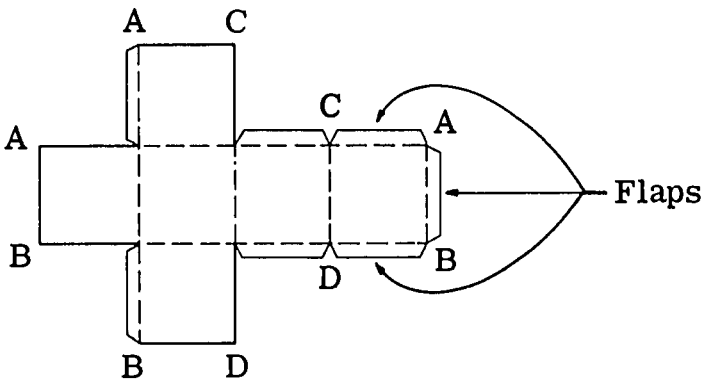
Brick or Block

Making a model of a cone. Let the pupils cut out parts of circles as shown to the left below. The centre of the circle is *B* and *BA* and *BC* are each a radius. The edges *BA* and *BC* are then glued together to make a cone as shown to the right below.





Making a model of a cube. Let the children follow these steps: cut out the pattern below from stiff paper; fold the paper on the dotted lines so the corners with the same letter are together; fasten the edges together to make a cube. (Use glue or plastic tape. Small flaps, as shown in the drawing, are on the edges of the cutout to help glue the model together.)



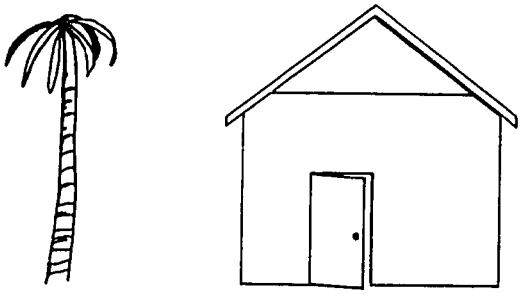
Different sizes of cones and cubes may be made by changing the size of the circles and squares of the cutouts. If line segment *AC* is six inches long, the cube will have edges six inches long.

ACTIVITY 2 • Using the Words "Beside," "To the Right Of," "To the Left Of" and "Between"

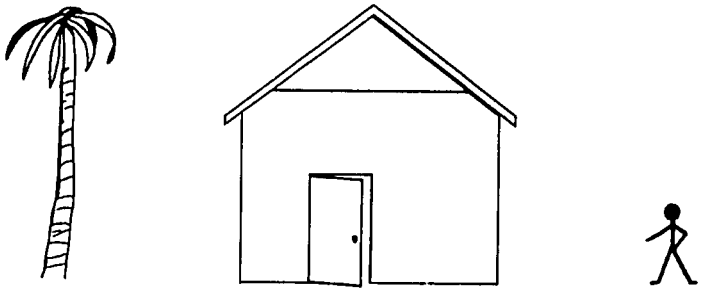
Pupil pages 56-57

Get the pupils to show their right hands and then their left hands. Help those who do not show the proper hands to decide which hands are right hands and which are left hands.

Say, "I am going to tell you about Jonah and Obi. Jonah went looking for his friend Obi. Obi lived in a house *beside* a tree." Draw a house with a tree beside it on the blackboard.



"Jonah called to Obi, "Come out and play." Obi came out of the house and stood beside the house. The house was *between* Obi and the tree." Draw a picture of Obi like this:



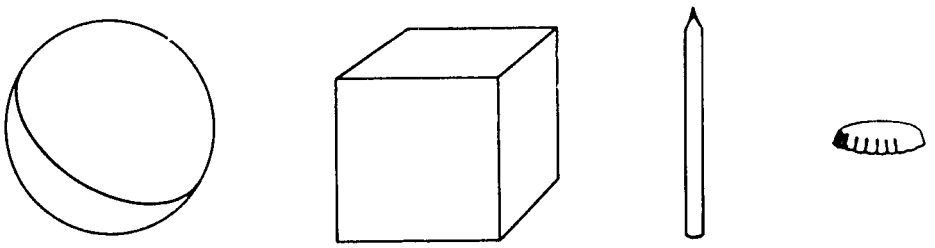
"Jonah said, 'Obi, I can see you *to the right of* your house. The house is *between* you and the tree. The tree is *to the left of* your house'."

Let the children be Jonah and Obi and go through the story. Let Jonah tell about where Obi is standing. Let Obi say things like, "My house is to the right of me. The tree is to the right of me."

Place a box and a ball on a table at the front of the room. Ask a child to place the ball to the left of the box. Say, "The ball is beside the box. The ball is to the left of the box. The box is to the right of the ball." Ask a child, "Where is the ball?" (The ball is beside the box; the ball is to the left of the box.) Ask a child to face the class from the other side of the table. Ask, "Where is the ball?" (To the right of the box)

Let the pupils discuss the fact that the ball is to the right of the box for the child facing the class and to the left of the box for the other children in the room. Let children go to the front of the room, face the class and decide whether this is true.

Give a child a pencil, and ask him to put the pencil to the right of the box. Give another child a bottle top and ask him to put it to the right of the pencil.



Ball Box Pencil Bottle top

Let the pupils tell you the location of the pencil. For example, they may say, "The pencil is between the box and the bottle top. The pencil is beside the box. It is beside the bottle top. The pencil is to the left of the bottle top. The pencil is to the right of the box and to the right of the ball."

Ask a child to face the class as he stands on the other side of the table. Ask him to tell where the pencil is. He will say, among other things, "The pencil is to the right of the bottle top. The pencil is to the left of the box. The pencil is to the left of the ball."

Let the pupils talk about why their statements are different. Get several pupils to make the statement, "The pencil is to the right of the box." Then let them move behind the table and say, "The pencil is to the left of the box."

Repeat this activity, with the pupils talking about the location of the other objects on the table.

Let each pupil place his book and a pencil on his desk. Say things like, "Put your right hand to the right of the book. Put your right hand between the pencil and the book. Put your left hand to the left of the pencil. Put your left hand to the right of the book." Move about the room as you say these things, helping children who need help. Let some of the children give similar directions to the class.

Tell the pupils to look at pupil page 56. Ask a child to read the first sentence in Exercise 1 and tell whether the sentence is a true sentence or a

false sentence. Let the class decide whether his answer is correct. Ask other children to read sentences and tell whether they are true or false sentences. If a sentence is false, ask the child to make a true sentence. After all the sentences are read, say, "Make other statements like these about the rectangle, circle and triangle."

Tell the pupils to read the sentences in Exercise 2. Ask a child, "What word do you put in the blank in the first sentence to make it true?" Let him read his sentence. Ask the other children to decide whether the sentence is true. Ask similar questions for the other sentences.

Tell the pupils to look at the numberline in Exercise 3. You make some statements about the numerals on the number line using the words "beside", "between", "to the right of" and "to the left of". Then ask the children to make other statements about the numerals.

If the pupils already understand the ideas of this activity or if they learn them very quickly, shorten or omit several of the suggested activities. However, be sure to let the pupils use the ideas with the number line of Exercise 3 on page 56.

Say, "Hold your right hand *above* your head. Place your left hand *below* the top of the desk." If all of the pupils do this correctly, they understand "above" and "below", and this activity is then very short. Tell the children to do a few other things, such as the ones listed below. You will think of others.

Hold a pencil below your book; hold your book below your pencil.
Place your right hand below your left hand.
Put your book above your left hand; place your right hand below your book.

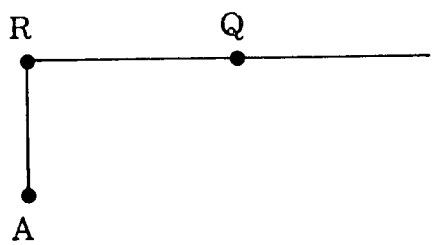
Tell the pupils to look at Exercise 1 on page 57. Ask questions like, "Is the triangle above the square?" (Yes) "Is the circle below the triangle?" (Yes) Let the children say other things about these figures, using "above", "below" and "between".

Ask similar questions about the numerals of the number line in Exercise 2 on pupil page 57. Ask, for example, "Is the numeral 3 above the numeral 1? Is the numeral 4 between the numeral 5 and the numeral 2?" Get the pupils to make complete statements when they answer. Let them ask other questions. (Exercise 3 will be used later, in Activity 4.)

ACTIVITY 3 • Curves and Points

Tell a story like this one: "Once upon a time there was a little ant, Speedy, who wanted to see the world. He said good-bye to all his friends and started out over the flat ground. After a while he came to a rock. He thought, 'I will not climb over this rock. I will turn to the right.' Presently he met a friend, Quickfoot. They touched feelers. 'I am going to see the world,' said Speedy. 'The world is big,' replied Quickfoot. Speedy went on."

Say, "Let us draw a picture of Speedy's path. Where did he start and where did he go?" Let the children tell you what happened. Draw a picture like this on the blackboard:

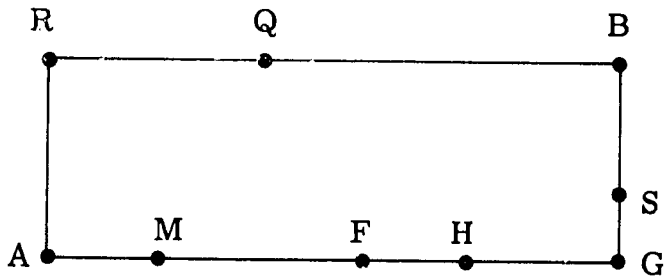


Say, "*A* shows the ant hill where Speedy started. *R* shows the rock. *Q* shows the place where he met his friend Quickfoot. Out picture is not his real path. We call the picture of a path a *curve*. Our picture shows a curve. Pictures of places along the path, like *R* and *Q*, we call *points*. The picture of the path contains *line segments* AR and RQ."

Go on with the story. "Speedy next met a beetle. Speedy was afraid of the beetle, so he turned to the right again. A little farther on he met a scout from the ant hill. 'Speedy, what are you doing so far from home?' asked the scout. 'I am going to see the world,' Speedy replied. 'Find good things,' said the scout, 'and you may be a scout some day.' Speedy soon found a big pile of very tasty young grass. He thought, 'We have not had tasty grass like this to eat for a long, long time. I must tell the other ants.' He took a piece in his mouth and started home. He met three friends, Happy, Fairy and Mund. To each he said, 'Look at the very good grass I found.' Mund ran on ahead. He called to the other ants at the ant hill. 'Here comes Speedy. He has found a pile of fresh new grass.' The other ants ran to Speedy. They asked, 'Where is the grass?' Speedy told them. Soon the ants had eaten all the tasty grass they could eat. The chief of the ants sent for Speedy and said, 'Thank you, Speedy, for finding the delicious grass for us. You are

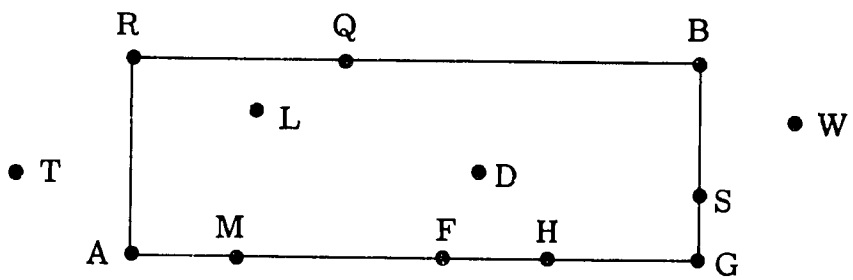
now one of our Chief Scouts.' Speedy was very happy to be a scout."

While telling the story, draw the picture of Speedy's path. The picture will look like this:



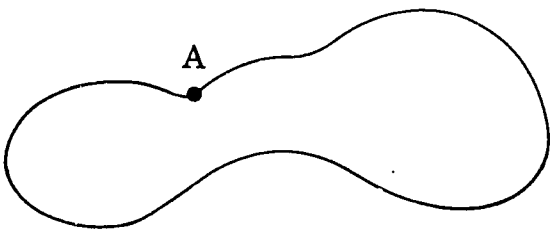
Let the pupils talk about the picture. Get them to put their fingers on the points and say things like, "At point *S*, Speedy met the scout. At point *G*, he found the tasty grass. At point *R*, he came to the rock." Say, "In the picture, points show places."

Go on. "The paths that Scout Speedy took were over a flat field. As he did his scouting, he found many things. One day he found a leaf, a piece of wire, a dog and a tree." Mark points to show the places where Speedy found these things, as shown in the picture below.



Say, "Points *L*, *W*, *D* and *T* show the places where Scout Speedy found the leaf, the piece of wire, the dog and the tree. The picture of Speedy's path is called a curve. We say that point *F* is *on* the curve, point *D* is *inside* the curve and point *W* is *outside* the curve." Ask questions like, "Is point *H* on the curve? Is point *G* inside the curve? Is point *W* inside the curve? Point *L*? Point *A*?" If an answer is "No", ask pupils where the point is. Let them tell where other points are; for example, points *S*, *T* and *R*.

Say, "The curve showing Speedy's path is a *rectangle*. One day Speedy's path was different. It was like this."



Mark some points on, inside and outside this curve. Ask questions like, "Is this point inside the curve? Is this point outside the curve? Is this point on the curve?" Let pupils mark some points and tell where they are.

ACTIVITY 4 • Learning About Regions

Pupil page 57

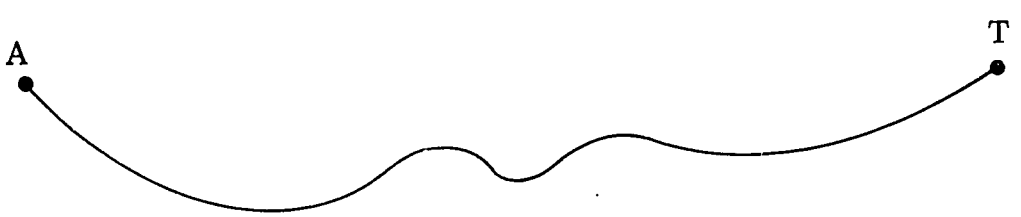
Draw a circle on the blackboard. Mark points inside, outside and on the circle. Ask questions like this about each point: "Is this point inside the circle, outside the circle or on the circle?" Run your finger along the circle and say, "This is a curve." Mark other points and let the pupils tell about their locations. Let pupils mark points and tell where they are.

Say, "A circle is a *closed curve*. A closed curve starts and ends at the same point. An ant like Speedy can start at any point, move along the path shown by the curve and he will return to the point where he started." Ask, "Is a rectangle a closed curve? Can an ant move along the path shown by the rectangle and return to his starting point?" (Yes. A rectangle is a closed curve.)

Ask, "Is a triangle a closed curve?" (Yes) "Is a line segment a picture of a path?" (Yes) "Is a line segment a curve?" (Yes) "Is a line segment a closed curve?" (No)

Note: The children may need to talk about these last two questions. Guide them to call a picture of a path a curve. A path may be straight or it may not be straight. If a path is straight, its picture, a curve, is a line segment. Some people speak of "straight lines" and "curved lines." In later work in mathematics this leads to confusion. In this book, all lines are straight. What some people call "curved lines," we will call "curves."

Draw a curve like this on the blackboard:



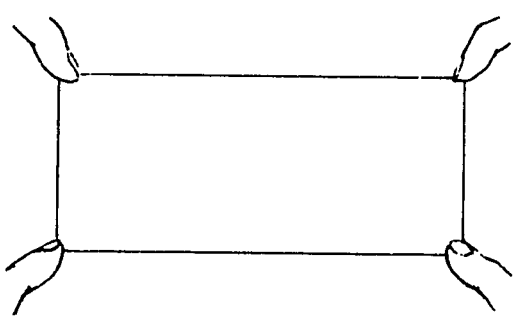
Ask, "Is this a curve?" (Yes) "Is it a closed curve?" (No) "Why?" (We cannot follow the path and come back to the point where we started.)

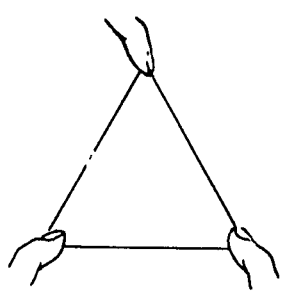
Draw a line segment on the blackboard. Ask "Is this a curve?" (Yes) "Is it a closed curve?" (No) "Why?"

Hold up a piece of string so that it is not straight. Ask, "Does this show a curve? (Yes) Does the curve have endpoints?" (Yes) Get a child to show the endpoints. Tighten the string to show a line segment. Let the children use the string to show many different curves. Guide them to say that the string shows a curve in any position they put it.

Let each pupil put a piece of string about one foot long on his desk to show a curve. Walk about the room to give help as needed. Ask the pupils to stretch their strings to show curves which are straight. Let them say, "The strings show line segments."

Tell the pupils to tie the ends of their strings together. You do the same with a string. Ask, "Does each string with ends tied together show a curve?" (Yes) "What do we call such a curve?" (Closed curve) Let the children show many closed curves with their string loops. Guide them to form rectangles, triangles and circles. Two children may work together when more fingers are needed to hold the string. Let some pupils show the class the figures they made.





Note: In *Primary Three*, only closed curves which do not intersect themselves are discussed with the children. If a child forms a closed curve like the numeral 8, just say, "This is a kind of closed curve." Do not emphasize this type of closed curve.

Say, "Do you remember Speedy, our ant friend? Speedy's paths were in a flat field. The blackboard is flat." Point to a curve on the blackboard. "This curve is on a flat blackboard. A circle is a curve on something flat. A rectangle is a curve on something flat."

Go on. "A rectangle is a closed curve. It has an inside. The inside of a rectangle is flat and is called a *plane region.*" Place your hand on the inside of the rectangle as you say this. "The inside of a circle is also a plane

region." Place your hand on the inside of the circle as you say this.

Hold up a rectangular piece of cardboard. Say, "This shows a plane region. It is the plane region of a rectangle." Pass your fingers over the face of the figure as you say "region". Trace the edge of the cardboard with your finger as you say "rectangle".

Hold up a circular piece of cardboard. Say, "This shows a plane region. It is the plane region of a circle." Run your fingers over the face of the cardboard as you say this.

Hold up a box. Say, "The side of the box is a plane region." Let the children show plane regions on objects in the classroom.

Tell the children to look at pupil page 57, Exercise 3. Let them talk about the closed curves they see and the curves that are not closed. Ask the pupils to name the curves. Note that some curves are named by a single letter. Give each pupil several seeds or other small markers. Say, "Think of these seeds as very small. We will use them to show points. Place a seed on point A. Show point B by placing a seed on point B. Show a point on the line segment BC. Show a point inside the circle D. Show a point on the curve EF." Tell the pupils to show other points on, inside and outside each curve. Say, "Show a point inside curve EF." If there is some hesitation, ask questions like, "Can't you find an inside? Is EF a closed curve?" Guide the children to say such things as, "EF has no inside." Or "EF is not a closed curve."

Tell the pupils to remove the seeds. Ask them to run a hand over the inside of the triangle, over the inside of curve G and along curve G. Get them to show plane regions in the drawings.

ACTIVITY 5 • Extending the Use of "Inside" and "Outside"

Pupil page 58

Say, "Name something you see inside this classroom. Name something outside the classroom. (John is inside the classroom. John's mother is outside the classroom. The tree is outside the classroom. The table is inside the classroom.) Continue until many of the children use the words.

Place a stone inside a box. Put a lid on the box. Put the box on a table beside a pencil. Write on the blackboard, "The stone is inside the box. The pencil is outside the box." Ask a pupil to read the sentences. Make sure the pupils can read and know the meaning of each word.

Tell the pupils to look at page 58, Exercise 1. Ask a child to read the sentences. Ask another child whether they are true sentences.

Tell the pupils to look at Exercise 2. Ask a child to read the sentences and put words in the blanks to make true sentences. (The circle is *inside* the square, and so on.)

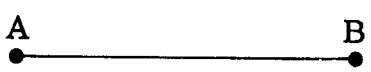
Tell the pupils to look at Exercise 3. Ask them to make statements about the figures, using the words "inside" and "outside". Guide them to include statements like, "The square is inside the circle. The square is outside the triangle."

Let the pupils make similar statements about the figures in Exercise 4. An arrow and a cross and two points, M and L, have been included for variety.

ACTIVITY 6 • Extending a Line Segment; Points on a Line

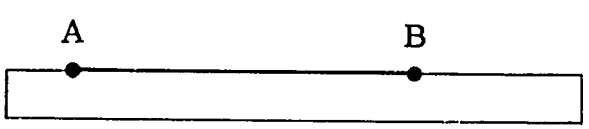
Pupil page 59

Draw a line segment on the blackboard and name the two endpoints. Use a book or similar object to trace the line segment.

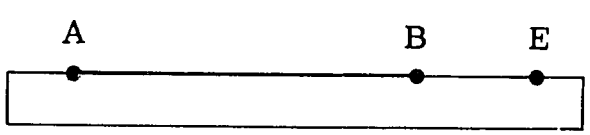


Say, "This is the line segment AB. It is a curve. It has two endpoints, A and B."

Ask, "Can you make a line segment longer than AB?" Help a child place a long straight-edge along the line segment AB like this:



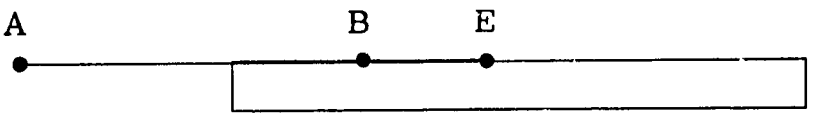
Draw a line segment along the straight-edge to a point marked E.



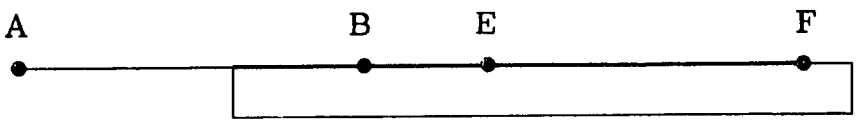
Say, "We had the line segment AB. Now we have a new line segment AE. The line segment AB was extended to make the line segment AE." Ask, "Is

the point B an endpoint of line segment AE?" (No) Ask, "Is point B on line segment AE?" (Yes)

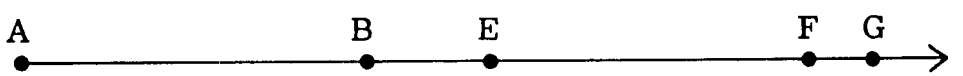
As you hold the straight-edge as shown below, ask, "Can we draw a line segment longer than line segment AE?" (Yes)



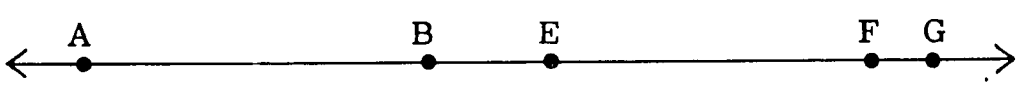
Help a child draw line segment AF.



Extend the line segment several more times, almost to the edge of the blackboard. Let the pupils talk about each extension. Get them to say, "The blackboard is not wide enough for any more extension. If the blackboard were wider, the line segment could be extended on and on." Show this possible extension by putting an arrowhead on the right end of the segment. Guide the children to talk about what the arrowhead means.



Let the pupils discuss whether the line segment can be extended to the left. Get them to decide that it can be extended to the left as far as they please. Let a child show this. (He should draw an arrowhead on the left end of the line.)



Say things like, "We can imagine the line segment is a magic string. It can be stretched and stretched. The magic string is stretched to the tree outside and then to the rock and then on and on."

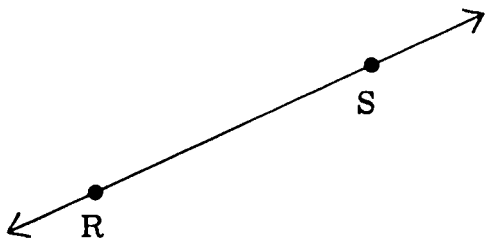
Say, as you point to line segment AB, "This figure is a line segment. We imagine it extends on and on as far as we please in both directions. We drew arrowheads to show this." Point to line AG and say, "We call this figure a *line*."

Ask questions and make statements that get the pupils to say things

like, "Points B and E are on the line. Point F is between points A and G. Point E is to the left of point F."

A line, which extends as far as we please in both directions, is named by giving two points on it. For example, the line AB is also named AE. It is the same line as line AF and line BG. We may also give it other names.

Draw another line on the blackboard as shown below. Mark points R and S.



Ask a pupil, "Can you mark another point on the line RS?" Let him mark a point and name it with his own first initial. Ask, "Can another point be marked on the line RS?" Let another pupil mark his point. Let several pupils mark points. Guide them to say that every pupil in the class can mark a point. Ask, "Could each of you mark two points?" (Yes) "Could all these points be different?" (Yes) "Could each pupil in our school mark a point?" (Yes) "Could all these points be different?" (Yes) (It may be necessary to say that the smaller the chalk mark is made the better the point is shown.) Help the pupils understand that they can mark as many points as they please on the line and then more.

On the blackboard, draw a line segment about a foot long. Take a 30- or 40-foot length of string. Tell the pupils you are going to show the extension of the line segment. Hold a part of the string along the line segment with your hands at the endpoints of the line segment. Then hold a longer part of the string along the line segment. Guide the children to say, "Your arms are not long enough." Ask two pupils to hold the string along the line segment, holding longer and longer parts between them. Let the children decide they could go on and on extending the line segment if the string was longer and longer.

Ask the pupils to imagine a line segment made of magic rubber. Imagine two magic birds each taking an endpoint in his beak and flying away forever in opposite directions. Let the pupils talk about the rubber stretching and stretching to show a line.

Tell the pupils to look at pupil page 59. Tell them to hold a string

along line segment *AB*. Then let them imagine holding longer and longer parts of the string along *AB*. Tell them, "Imagine your string is very long. What will it show? (A line) How far would the line go?" Let the pupils use their imaginations and talk about the line going on and on.

Ask the pupils to place a small seed on the line *LM*. Tell them their seed represents a point. Then let them place another small seed on the line. Then another small seed. Ask, "If the seeds were very small, how many could be put on the line?" The children may begin by saying "5", then "10" and then larger and larger numbers. Some child may finally say something like, "As many as I want and then more and more." This is the idea this activity should develop.

STAGE 2 • Intersection and Lines

Vocabulary

Common, intersect, intersection, intersecting, face (of a box)

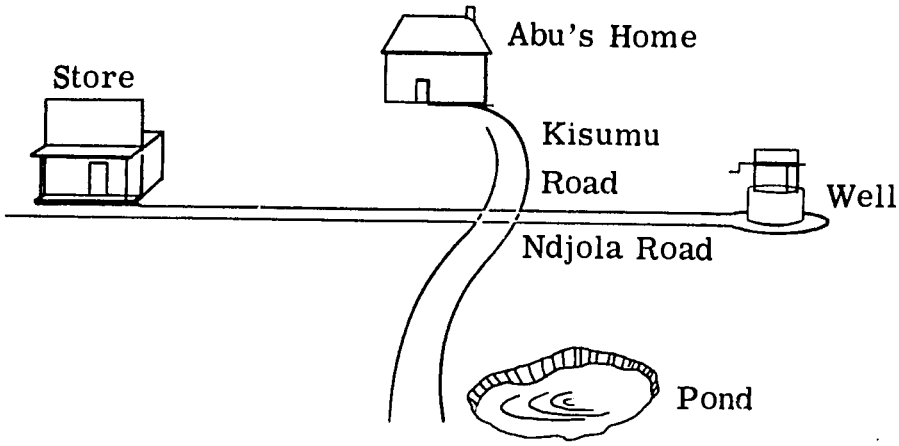
Materials

About 3 sticks, 4-5 inches long, for each child; 3 or 4 small seeds for each child; a piece of string about one foot long for each child; a piece of paper for folding; pupil pages 60-63

Teaching Procedure

ACTIVITY 1 • Intersections of Paths, Curves and Line Segments

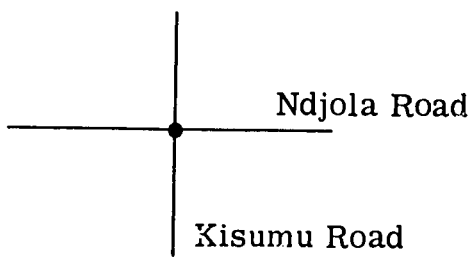
Tell the following story: "Abu was walking along Kisumu Road from his home to a pond. He met his friend Peter, who was walking along Ndjola Road." While you are telling the story, draw the picture below on the blackboard.



" 'What are you doing here on Kisumu Road?' asked Abu. 'You are wrong, Abu,' said Peter. 'We are on Ndjola Road.' They talked and talked. Abu said they were on Kisumu Road and Peter said they were on Ndjola Road."

Ask the class, "Which boy was right?" Let the pupils talk about the question. If no child says, "They both were right," ask a child to come to the blackboard. Get him to put his finger at the point in the picture where the boys met. Ask which road they were on. The pupil will say that the boys were on Kisumu Road and Ndjola Road at the same time. Say, "The place where they met is *common* to both roads. It is the place where the roads *intersect*."

Say, "Let us show the roads with line segments. Where is the place the boys met?" Draw a picture like this:

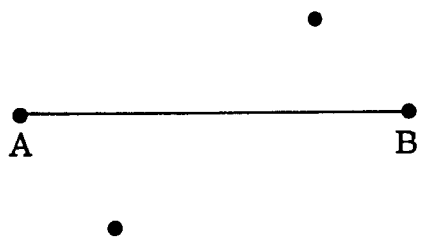


Let a pupil mark the point where the line segments intersect, say Q. Ask, "On which line segment is point Q?" (Both) Say, "The line segments have the point Q in common. Q is the *intersection* of the line segments. If two line segments have a point in common, they intersect at that point." Ask questions and let the pupils talk of these same ideas in their own words.

Mark two points on the blackboard about one foot apart. Ask, "Can you draw a line segment that has these two points as endpoints?" Let the

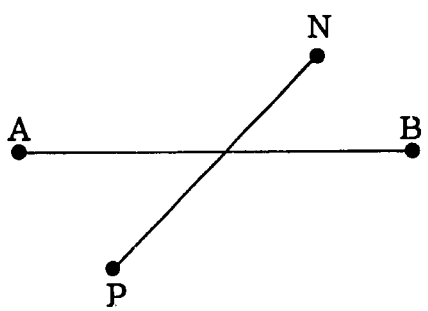
children decide they can. Give a child a straight-edge to use, and let him draw the line. Let another child name the line segment, say *AB*.

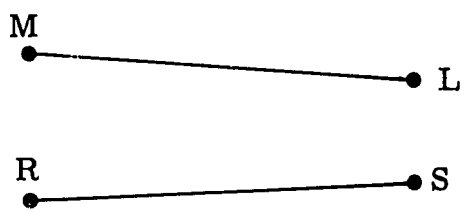
Mark two other points on the blackboard like this:



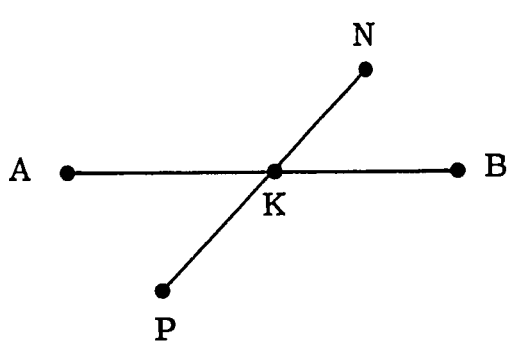
Ask, "Can we draw a line segment with these two points as endpoints?" Give a child the straight-edge and let him draw the line segment and name it, say *PN*. (Note that these two line segments intersect.)

Draw two other line segments which do not intersect. The blackboard will look like this:





Get the pupils to talk about the two pairs of line segments. Get them to say that line segments *ML* and *RS* do not intersect and line segments *AB* and *PN* do intersect. Ask a pupil to point to the place where the line segments *AB* and *PN* intersect. Ask another pupil to mark the point and name the point, say *K*.



Guide the pupils to say, "Point *K* is on line segment *AB*. Point *K* is on line segment *PN*. Line segment *AB* intersects line segment *PN*. Point *K* is

common to both line segments. Point K is the intersection of the line segments."

Let a child draw another line segment on the blackboard. Ask another child to draw a second line segment that intersects the first. Ask, "Do these two line segments intersect?" (Yes) Let one of the children mark and name the point where the two line segments intersect, say J. Say, "These are *intersecting* lines."

Let another child name the endpoints of the intersecting line segments, say MH and XY. Let the children tell all the marked points on each of the two line segments. Guide them to say that J is a point on MH and also on XY.

Tell the pupils to draw in their exercise books two line segments that intersect. (Let them use folded paper for straight-edges.) Ask them to name the endpoints of the two line segments. Tell them to find the point where the line segments intersect, mark it and name it.

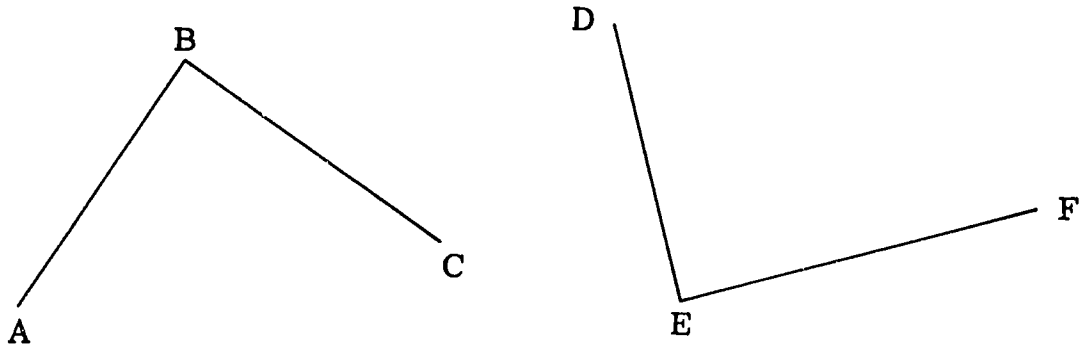
Tell the pupils to turn to page 60 in their books and look at Exercise 1. (Exercises 2 and 3 are used in Activity 2.) Ask a child to give the names of two intersecting line segments. Let the other children talk about the line segments name and decide whether they do intersect. Get them to find other intersecting line segments. Ask for the names of two line segments that do not intersect. Ask a child for the names of the points marked on AB; on CD; on MQ; and so on. Each answer should be talked about; the children should decide whether it is correct and why or why not.

Ask, "If WX is extended, will it intersect line segment SV? How can we decide?" Guide a child to suggest holding a string along line segment WX. Ask the pupils to talk about this and decide whether the line segments intersect. Continue with similar questions.

ACTIVITY 2 • Revision of Square Corners, Angles, Right Angles, Vertex

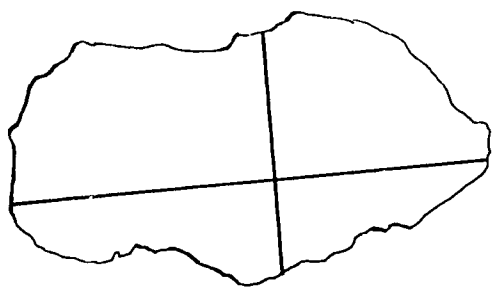
Pupil page 60

Revise the folding of paper to show *square corners*. (See Teachers' Guide, *Primary One*, Unit 13, page 199.) Let each pupil make a square corner and compare his square corner to others by fitting. Let two pupils trace their square corners on the blackboard, naming the *angles* that are formed.



Revise the meaning of *angle* as "a figure formed by two line segments with the same endpoint." Ask, "What kind of angle is formed by the line segments AB and BC?" (*Right angle*) "Do the line segments AB and BC have a point in common?" (They have point B in common. Point B is on both line segments.) "Do the line segments intersect?" (Yes) Ask similar questions about right angle DEF.

Let the pupils unfold their square corners to look like this:

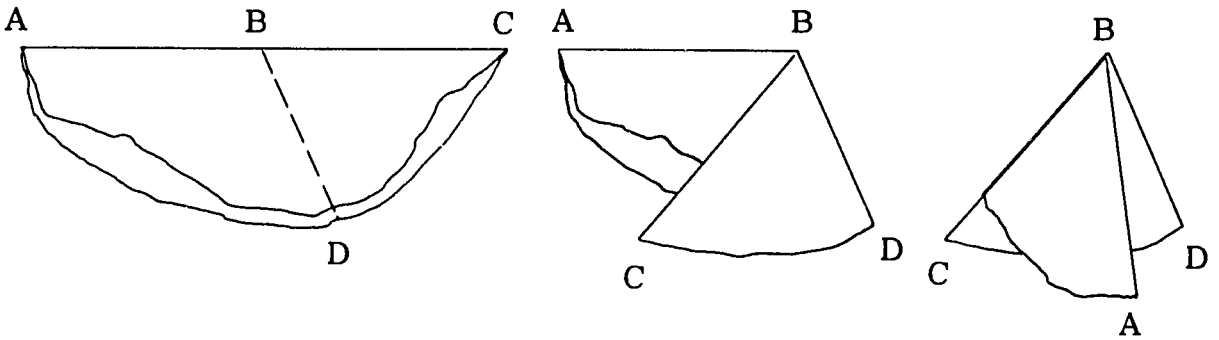


Get them to run a finger along the folds. Ask, "What do the folds show?" (Two line segments that intersect) Ask other children in turn, "Does your paper show two intersecting line segments?" Get them to agree that the folds in the papers do show intersecting line segments. Take a fresh sheet of paper and fold it once and then unfold it. Ask, "What does this single fold show?" (A line segment) Fold the paper again so that the new fold crosses the first fold. Unfold the paper, and ask what the new fold shows. (A line segment) Ask what the two folds show. (Two intersecting line segments)

Ask a pupil to draw on the blackboard a picture of the folds in his paper. Let him use a straight-edge. Get a pupil to name the endpoints in the picture and to say the picture shows two intersecting line segments. Ask, "Why do you say the line segments intersect?" (The line segments have a point in common; a point is on both line segments.)

Fold a sheet of paper to make a corner that is not square. This may

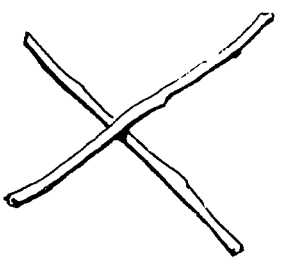
be done as follows: first, fold a piece of paper as shown in the first picture below; second, fold it again along the dotted line *BD*; third, fold it along the line *BC*, as shown in the third picture.



The folded paper forms a corner *CBD*. (*ABD* in the second picture also form a corner.) Trace this corner on the blackboard to form an angle, *CBD*. Ask a pupil to show the line segments and the point where the line segments intersect. Say, "We call this point the *vertex* of the angle."

Let each child fold a corner that is not square. The corners made will not fit each other. Let each pupil trace his corner in his exercise book and label it. Let two pupils trace their corners on the blackboard. Let each pupil label his figure, name the angle and show the vertex.

Hold up two sticks and say, "These sticks show line segments." Hold up the sticks so they cross like this:

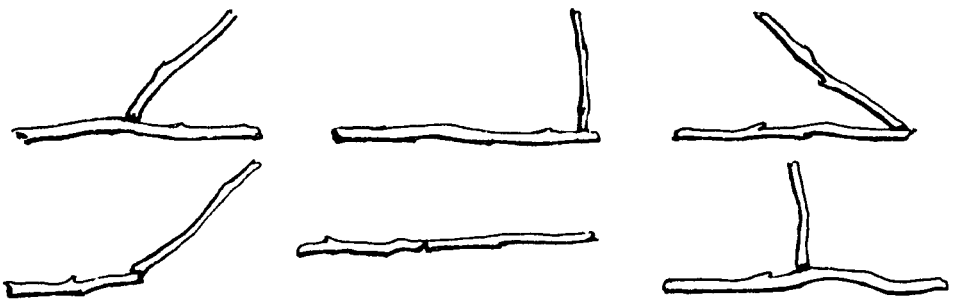


Ask, "What do these sticks show?" (Intersecting line segments) Hold up two sticks so they do not cross. Ask, "Do these show two line segments?" (Yes) "Do the line segments intersect?" (No)

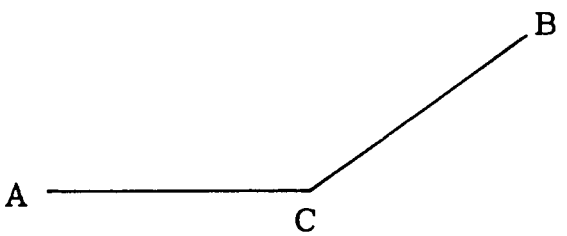
When the sticks are crossed, let a pupil put a finger at the place where the sticks cross. Let him say, "This is the intersection of the two line segments. Here is a point on both line segments."

Give each pupil two sticks. Let the pupils hold up the sticks to show line segments that intersect and then line segments that do not intersect.

Get the pupils to experiment with the sticks to show other kinds of intersection besides crossed sticks. Here are some of the possible patterns:



Let some of the children hold their sticks against the blackboard and draw the intersecting line segments the sticks show. Ask, "Do these line segments have endpoints?" Let the children name the endpoints. If a child has difficulty naming the point of intersection as an endpoint in the figure below, trace your finger from point *A* along the line segment. Ask, "Does this line segment have two end points?" Repeat starting at point *B*. Get the children to see that one point is an endpoint of both line segments.



Let a child run his finger along both line segments to their common endpoint. Guide the children to say, "Line segment *AC* intersects line segment *BC*. *C* is the common point. *C* is the vertex of the angle *ACB*."

Let the pupils look at page 60, Exercise 2. Give the directions listed below. Get several answers to each direction. Let the children talk about each answer and decide whether it is correct and why.

Name two line segments that intersect.
Name the point where two line segments intersect.
Name two line segments that do not intersect. (For example, *AB* and *GE*)
Name a line segment that intersects two other line segments. (*HL*)
Name two line segments that intersect at an endpoint.

Give similar directions for Exercise 3. Continue like this until the children understand:

1. Two lines intersect at a point; the point is on both lines; it is called the point of intersection.
2. There are many types of figures showing two lines that intersect.

Note: It is possible to draw two line segments that intersect in a line segment instead of a point. In the picture below, line segments *AB* and *CD* intersect in line segment *CB*.

Do not include this kind of intersection in the lesson. It will be studied at a later time.

ACTIVITY 3 • Intersection of Line Segments

Pupil page 61

Note: Block letters of the alphabet show geometric figures. All of the letters except "C", "U", "S" and "O" show line segments. Many types of intersection of line segments are shown in these letters. For example, the letter "X" shows two line segments that intersect at a point that is not an endpoint. The letter "L" shows two line segments which intersect at endpoints. The letter "T" shows two line segments intersecting at a point that is an endpoint of one line segment and not an endpoint of the other line segment.

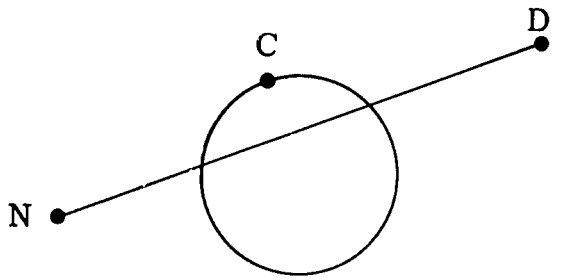
Tell the pupils to turn to page 61 in their books. Ask, "Do you see any line segments on this page?" Remember that the edge of the page shows line segments and that the rectangles in which letters are contained also show line segments. Let the pupils talk about the letters and whether line segments make the letters. Let them also talk about the right angles shown by some of the letters and about the intersections of the line segments to form letters.

Let the pupils draw letters in their exercise books. Tell them to count the number of line segments in each of the letters. Let them tell which letters use one, two three and four line segments. Let them place beans at the points of intersection. Let them talk of the different ways the line segments intersect.

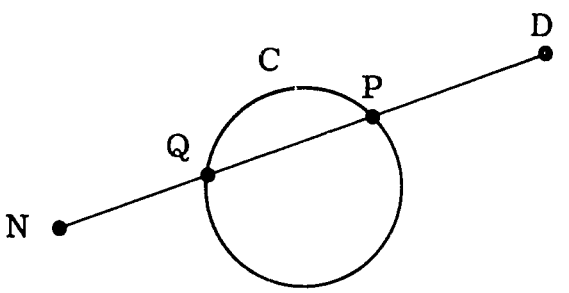
ACTIVITY 4 • Intersection of Curves and Line Segments

Pupil page 62

Draw a circle and a line segment that intersect like this:



Let a pupil name the line segment, say *ND*, and the circle, say *C*. (You may need to say that the circle may be named with a single letter.) Let the pupils talk about the figure. Get them to say that there are two points that are common to both the circle and the line segment. Let a pupil name them, say *Q* and *P*.



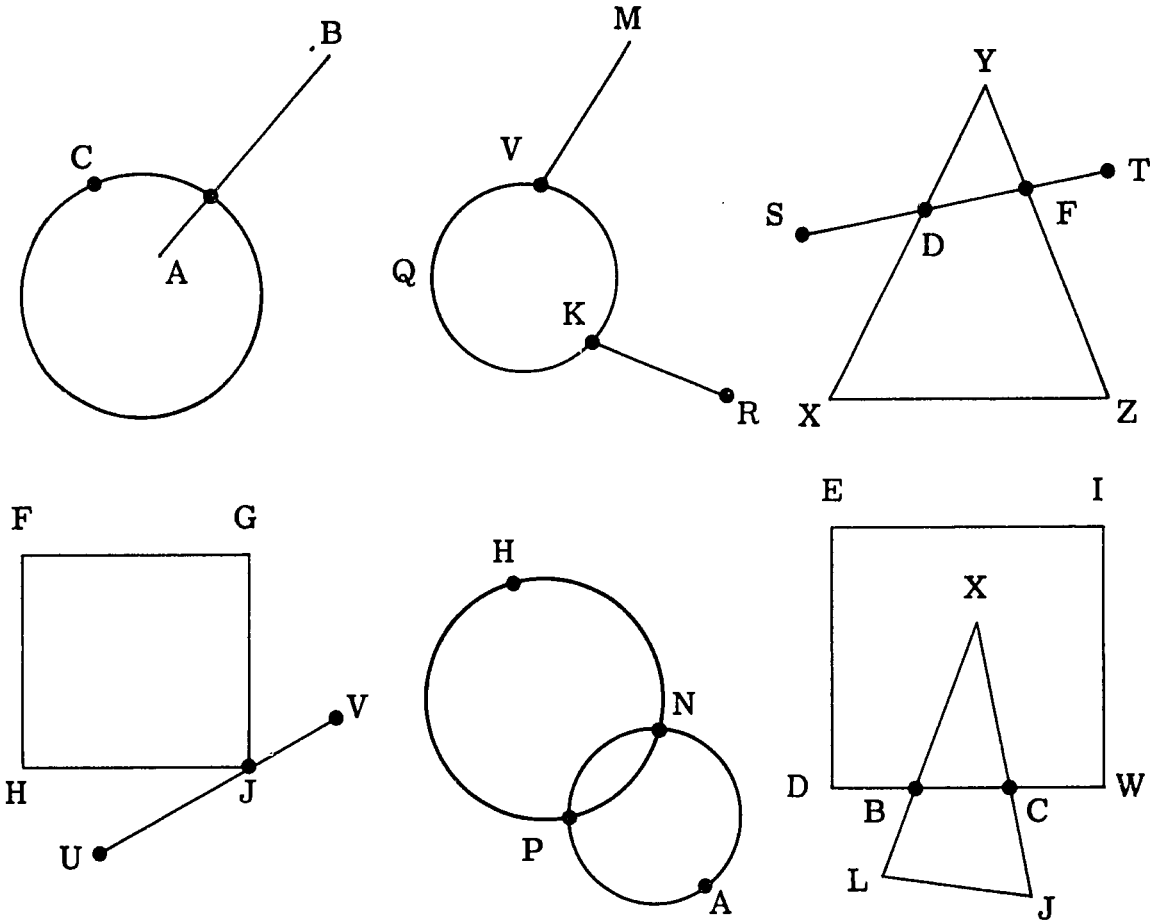
Ask questions like, "Do the circle and the line segment intersect?" (Yes) "Why do you say they intersect?" (The line segment and circle intersect because a point on the line segment is also a point on the circle.) "Where do they intersect?" (*P*) Help them to trace along the circle and line segment and name both point *P* and point *Q*.

Make figures on the blackboard like the ones on the following page, and talk about them in a similar way.

Let the pupils look at page 62. Ask questions like, "Do you see a circle that intersects a line segment?" (Pictures 1, 3, 6 and 8) "In which pictures are there circles that intersect circles?" (5) Let the pupils ask each other questions. Make sure the intersection shown in each picture is discussed.

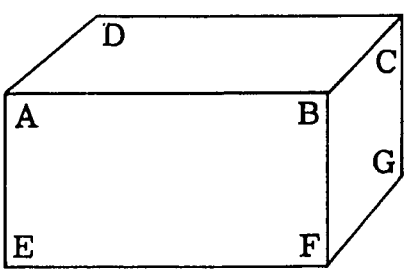
Get the pupils to place seeds on the points where a circle intersects a

triangle. Ask, "What does picture 7 show? Which picture shows a triangle intersecting a square?" (None)



ACTIVITY 5 • Some Intersections in Space

Let the pupils point to the corners of a box and name them. Suppose they name them as they are named in the drawing below. (You can mark the letters with coloured chalk on the box near the corners. H is unseen in the figure below.)



Let a pupil run his fingers along an edge of the box and tell its name. Let other pupils do the same with other edges. Let them say that the edges of the box show line segments.

Say, "Name two of the edges that intersect." (Suppose the answer is *AB* and *BC*.) "Where do these edges intersect?" (*B*) "Why?" (They meet at a common point.) "Is there another line segment that intersects *AB* and *BC*?" (Yes) "Name it." (*BF*) Ask similar questions about other line segments.

Let a pupil run his hand over a *face* of the box. Let him tell the name of the face. (For example, "This is the region *ABCD*.") Let other pupils do the same with the other five faces.

Suppose John and Mombo have named two faces, such as *ABFE* and *CGFB*. Say, "John, point out and name the edges of the face you showed." Then, "Mombo, name the edges of the face you showed." Ask the other children, "Are any of the edges on both faces?" (Yes, *BF*) "Do the faces have an edge in common?" (Yes, *BF*) "Do the two faces intersect?" (Yes) "Why?" (They have an edge in common.) Let the pupils name and run a finger along the edges they talk about.

Note: The idea of two rectangular regions intersecting in a line segment is new. The important idea to be learned is that two intersecting figures have something in common: two line segments intersect in a point; a circle and a line segment may intersect in one or two points; two faces of a box may have an edge in common; two faces of a box may intersect in a line segment.

Let the pupils talk about two intersecting lines and their common point and about two faces of a box that have a common edge. Guide them to agree that the faces intersect.

Repeat with other faces of the box. Be sure to include two faces that do not intersect.

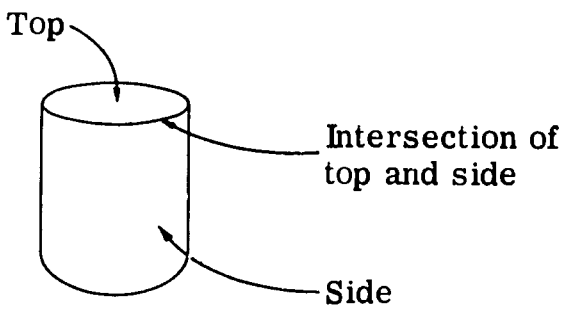
Ask, "Is the classroom a kind of box?" (If your classroom is not like a box, omit this part of the activity.) Let the pupils say that the walls are faces of a box, the floor is the bottom face and the ceiling is the top face.

Ask, "What are the edges of the box shown by the classroom?" Let the pupils point out the edges at the ceiling and floor. Ask questions like, "Do the ceiling and the front wall intersect? Where?" Go on with similar questions.

Let the pupils point out the corners of the room where the edges

intersect. Ask questions like, "Do these two edges intersect?" (Yes) "Do the line segments these two edges show have a point in common?" (Yes) Ask the pupils to show other corners in the room.

Look at the solid figures in your collection, such as the cube and the tin. Ask questions like those above to help the pupils discover the intersections shown. The cube shows intersections like those of the box. The edge at the top of a tin is the intersection of the top and the side. That edge shows a circle. The bottom edge also shows a circle. Let a pupil run his fingers along an edge as he talks about it.



ACTIVITY 6 • Intersections Around Us

Pupil page 63

Objects all around us show intersections. The edges on buildings, such as the edges of roofs and of windows and doors, show line segments that intersect. Some roads intersect other roads. Telegraph poles, wires, fences, paths, shadows and trees show intersections. Ask the pupils to turn to page 63. Get them to tell some places where they see intersections in the picture. Let them be sure to point out edges where faces intersect as well as points where line segments intersect.

Let the pupils point out intersections they can see in the classroom and out the windows. Get them to tell of intersections they know about and cannot see now.

UNIT 5

Fractions as Numbers

OBJECTIVES

1. To revise the meaning of the fractions one-half, one-fourth, one-eighth, one-third and one-sixth
2. To extend the concept of fractions to include one-fifth, one-tenth and one-twelfth and to learn their names and numerals
3. To revise the order of fractions and to compare pairs of fractions that are not equal

Background Information for Teachers

See Part 2, "Arithmetic", pages 52-58 in your *Teachers' Handbook.*

We use the whole numbers 0,1,2,3,4 . . . when we talk about the number of members in a set. But often we want to talk about parts of objects or about separating sets into subsets of the same size. When we talk about the parts of objects or parts of sets or regions, we use another kind of number. These numbers are called *fractions.*

If we cut an orange into three parts of the same size, we say we have separated it into thirds. If we consider two of the three parts, we are thinking of two-thirds of the orange. If we think of a rectangular region separated into three congruent parts, then two of the parts are two-thirds of the rectangular region. If we have a set of three balls, then the number of members in the subset of two balls is two-thirds of the members of the set. The fraction two-thirds is a number that describes all such parts of wholes that consist of two of three parts of the same size.

Fractions have numerals. A numeral for a fraction consists of a pair of numerals. For example, for the fraction two-thirds we may use the symbol $\frac{2}{3}$. The "3" means that we have three parts of the same size; the "2" means that we are thinking of two of the parts.

We associate the whole numbers with points on the number line. The number line helps us show the order of the whole numbers. If a point is at the left of a second point, then the number associated with the first point is less

than the number associated with the second. If a point is at the right of a second point, then the number associated with the first point is greater than the number associated with the second. The numeral 5 is at the left of the numeral 7, and we say $5 < 7$. The numeral 7 is at the right of the numeral 5, and we say $7 > 5$.

We may also associate fractions with points on the number line and set up an order among fractions. Suppose we want to compare the number $\frac{1}{4}$ with the number $\frac{1}{2}$. We locate a point for each of these numbers on the number line. We find the point for $\frac{1}{4}$ is at the left of the point for $\frac{1}{2}$. We say $\frac{1}{4} < \frac{1}{2}$. We find the point for $\frac{1}{2}$ is at the right of the point for $\frac{1}{4}$. We say $\frac{1}{2} > \frac{1}{4}$. If two numerals, such as $\frac{1}{2}$ and $\frac{2}{4}$, name the same point on the line, then we say they are names for the same fraction.

We can relate the comparison of fractions to the comparison of line segments. Let us consider the line segment from the point for 0 to the point for 1. If we compare one-half of that line segment to one-fourth of that line segment, the part described by the number $\frac{1}{2}$ is greater than the part described by the number $\frac{1}{4}$. We say that $\frac{1}{2} > \frac{1}{4}$.

STAGE 1 • Revision of the Meaning of Fractions as Numbers

Vocabulary

Whole, part, one-half, one-third, one-fourth, one-sixth, one-eighth, halves, thirds, fourths, sixths, eighths, fraction, numerator, denominator

Materials

Five 2" × 18" strips of paper for each child; several 1-inch squares of paper for each child; bottle tops; three cutouts of circles, rectangles, squares and triangles for each child; five 5" × 7" numeral $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}\right)$ cards; five boxes (or baskets); pupil pages 64-65

Teaching Procedure

ACTIVITY 1 • Showing Parts of Things

Give each pupil five 2" × 18" strips of paper. (Pupils may work in pairs.) Let them fold one of the strips of paper to make the ends meet, then unfold the paper and tell the number of *parts* into which the *whole* strip has been separated. Ask whether the parts are the same size. Guide the pupils to say, "Each part is *one-half* of the paper." Let a pupil write the numeral for one-half on the blackboard. $\left(\frac{1}{2}\right)$ Ask the pupils to write the numeral for one-half on each part of their folded paper. Ask, "How many *halves* are in the strip?" (Two-halves) Guide the pupils to say, "One-half is one of two parts of the same size."

Let the pupils point to parts of objects in the room that show one-half; for example, one-half of the door, one-half of an exercise book, one-half of a window.

Ask the pupils to fold another 2" × 18" strip of paper into three parts of the same size. Ask someone to show that the three parts are the same size. Let the pupils decide on the fraction that describes each of the parts. (One-third) Ask a pupil to write the numeral for the fraction on the blackboard. $\left(\frac{1}{3}\right)$ Let each pupil write the numeral for one-third on each part of his folded strip. Say, "You have folded your strip into *thirds*. How many thirds are in the whole strip?" (Three-thirds)

Repeat the activity by letting the pupils fold strips into *fourths*, *sixths* and *eighths*. Each time, let them check to see if the parts are the same size, tell the fraction that describes one of the parts and write the numeral for the fraction on each part of the strip. Ask one pupil to write the numeral on the blackboard.

Tell the pupils that each of the numerals on the blackboard names a number. These numbers have a special name. They are called *fractions*.

Point to the numeral for *one-fourth*. Say, "This is the name for the number one-fourth. We call 1 the *numerator* for the fraction. It tells how many parts we are thinking of. For example, we can show *one*-fourth of a

strip of paper. We call 4 the *denominator* for the fraction one-fourth. It tells into how many parts of the same size we have separated the whole. For example, we folded this strip into four parts of the same size.

Say, "Tell the numerator and denominator for the other fractions." Let the pupils tell what each fraction they showed with folded paper means. Guide them to tell about each fraction as you did for one-fourth. Then let them name fractions they know other than the ones named.

ACTIVITY 2 • Fractions for Parts of Regions

Pupil page 64

Ask the pupils to turn to page 64 in their books. Tell them to point to the first picture at the top of the page. Ask, "What is the name of this figure?" (Rectangle) Let the pupils talk about what they know about rectangles. Hold your book up and move your finger along the edges of one of the rectangles pictured. As you do this say, "This is a rectangle." The following questions may be used to guide the discussion. You will think of others.

How many sides does a rectangle have? (4)

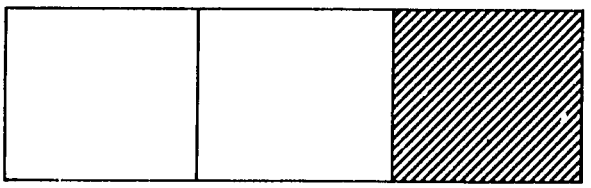
What do you know about line segment *BC* and line segment *AD*? (They are congruent; they fit exactly.)

What do you know about line segment *BA* and line segment *CD*? (They are congruent.)

What do you know about the angles? (They are all right angles.)

Move your hand over the inside of the rectangle. As you do this say, "This is the inside of the rectangle. It is the region of the rectangle."

Point to the second picture. Ask, "What do we call this figure? Into how many parts of the same size has the region been separated?" (2) "How many parts are shaded?" (1) "What number describes the shaded part?" $\left(\frac{1}{2}\right)$ Let the pupils answer similar questions about the shaded parts of the other regions on the page. Give each pupil several one-inch square cards. Ask them to write on a card the numeral for the fraction describing the shaded region of each figure and to place the card beside the figure. Draw the figure below on the blackboard as an example:

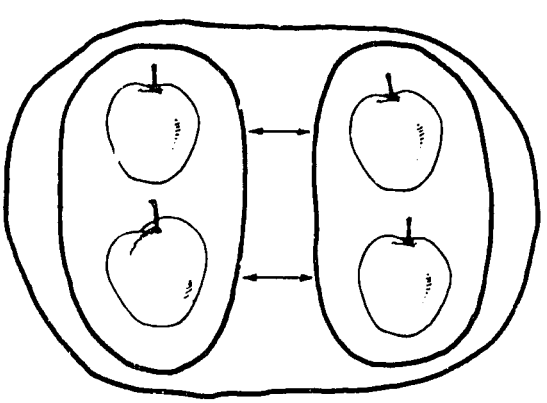


$\frac{1}{3}$

Go around the classroom, giving help as needed. Ask questions so the pupils use the ideas of fraction, numerator and denominator.

ACTIVITY 3 • Fractions for Subsets

Place a set of four objects on your table. Ask a pupil to separate the set into two subsets so that each subset has the same number of members. Check this by asking another pupil to match the members of the subsets. Ask, "How many members are in each subset?" (2) The members may be matched exactly one to one in this way:



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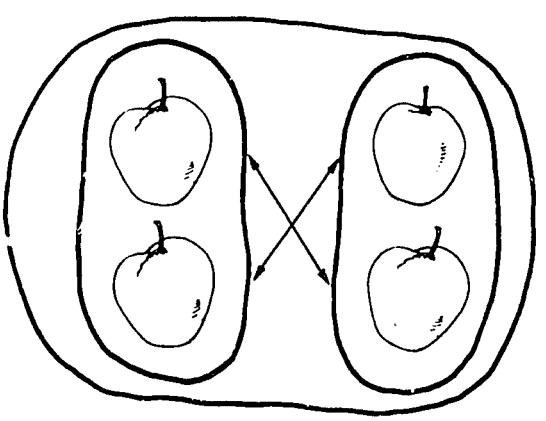
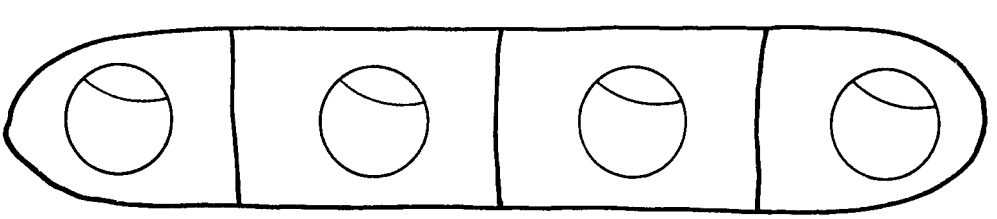


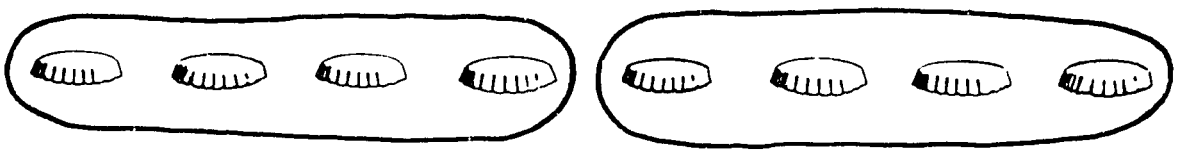
Ask a pupil to point to one subset and tell what fraction describes the part the subset is of the whole set. (The subset has one-half the members of the set.)

Select a pupil to separate a set of four things into four subsets, each with the same number of members. Ask, "What number describes one subset as part of the whole set?" (One-fourth) Let a pupil write the numeral $\left(\frac{1}{4}\right)$ on the blackboard.

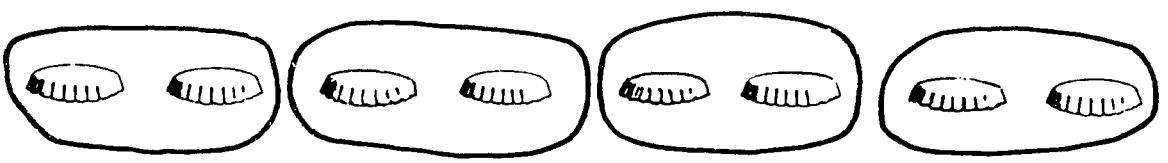


Ask the pupils to put a set of eight bottle tops on their desks. The following directions may be used to guide the pupils' activity and discussion:

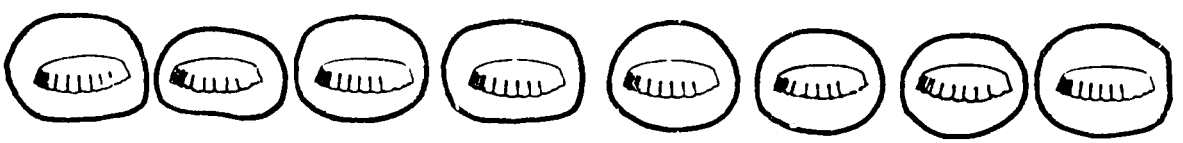
1. Separate your set of bottle tops into two subsets so that each subset has the same number of members. Point to one of the subsets. Tell what fraction describes this part of the whole set. (One-half) Tell whether one-half is a number. (Yes) On a card, write the numeral for the fraction. $\left(\frac{1}{2}\right)$ Place the card by a subset.



2. Separate your set of eight bottle tops into four subsets so that each subset has the same number of members. Point to one subset. Tell what number describes this part of the whole set. (One-fourth) Is this number a fraction? (Yes) On a card, write the numeral for the number. $\left(\frac{1}{4}\right)$ Place the card by a subset.



3. Separate your set of eight bottle tops into eight subsets so that each subset has the same number of members. Tell how many members are in each subset. Tell what number describes each subset. (One-eighth) On a card, write the numeral for the number. $\left(\frac{1}{8}\right)$ Place the card by a subset.



Ask the pupils to remove two bottle tops from their desks so that their set of bottle tops has six members. Repeat activities similar to those above with the sets of six bottle tops. Ask the pupils to separate their sets into two, three and six subsets so that each subset has the same number of members. Each time, let the pupils tell what fraction describes one subset and write the numeral for the fraction.

Ask the pupils to open their books to page 65. Let a pupil read the question at the top of the page. Say, "Each of the sets of beads in the picture has a subset of black beads. There is a fraction that tells us what part the subset is of the whole set." (Point to picture A.) "A subset of set A has black beads. What part of the whole set is the subset of black beads?"

Guide the pupils to say that if they think of the black beads as *one* subset, then that subset is one of two subsets with the same number of members. One subset is one-half of set A.

The following questions may be used to guide the discussion:

How many members has the subset of black beads? (Two)

Is there a subset of white beads in picture A that matches exactly the subset of black beads? (Yes)

Into how many subsets of the same number has the set of beads been separated? (Two)

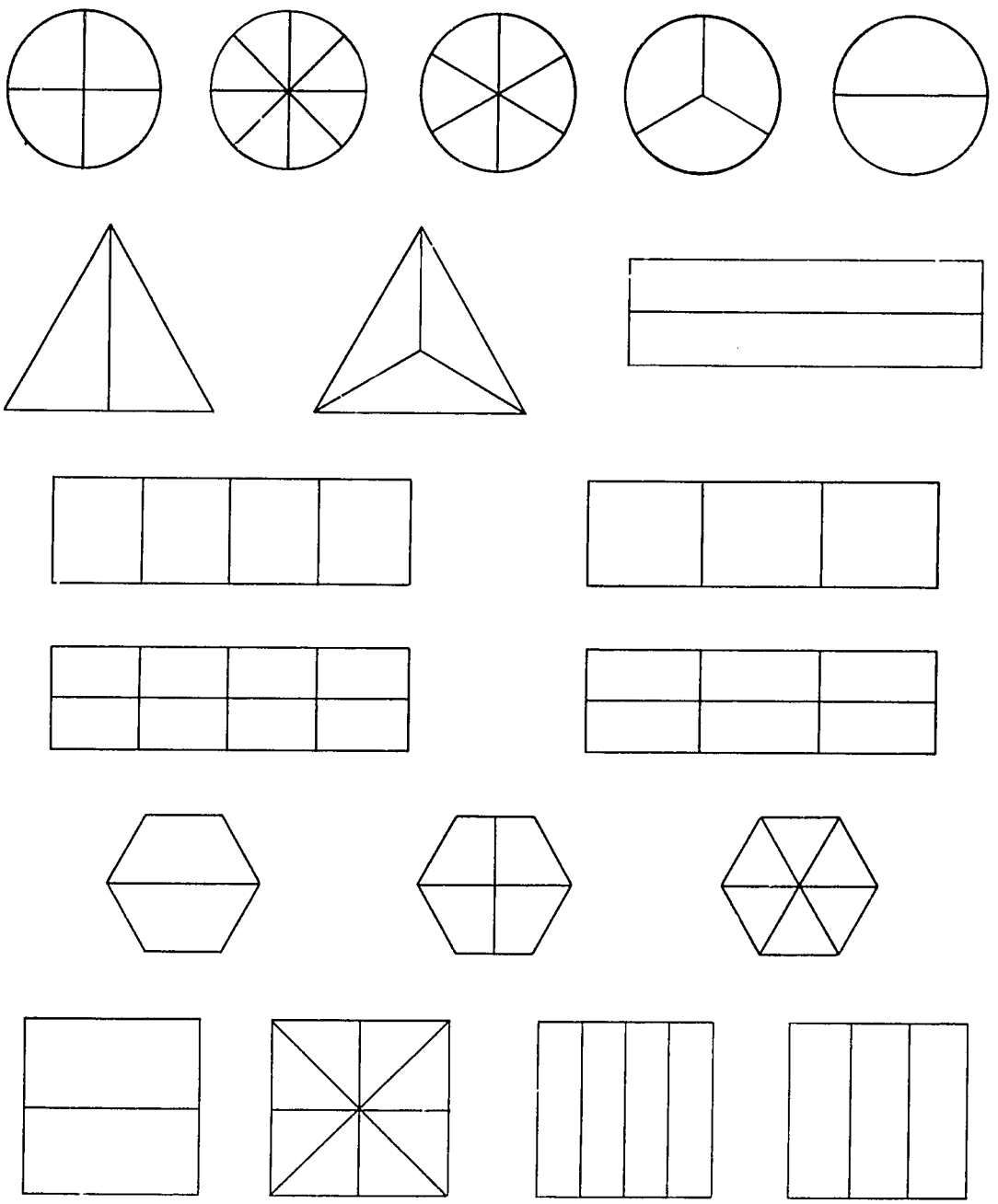
What part is the subset of black beads of the whole set? What fraction describes this? $\left(\frac{1}{2}\right)$

Ask the pupils to write the letters A-M in a column in their exercise books. Ask the pupils to write beside each letter the numeral for the number that describes the subset of black beads in that picture.

A $\frac{1}{2}$ $\left(or\ \frac{2}{4}\right)$	F $\frac{1}{4}$ $\left(or\ \frac{2}{8}\right)$	J $\frac{1}{2}$ $\left(or\ \frac{6}{12}\right)$
B $\frac{1}{4}$	G $\frac{1}{2}$ $\left(or\ \frac{4}{8}\right)$	K $\frac{1}{2}$ $\left(or\ \frac{5}{10}\right)$
C $\frac{1}{3}$	H $\frac{1}{3}$ $\left(or\ \frac{2}{6}\right)$	L $\frac{1}{4}$ $\left(or\ \frac{3}{12}\right)$
D $\frac{1}{6}$	I $\frac{1}{8}$	M $\frac{1}{3}$ $\left(or\ \frac{4}{12}\right)$
E $\frac{1}{2}$ $\left(or\ \frac{3}{6}\right)$		

ACTIVITY 5 • Fractions for Regions

Prepare many cutouts of heavy paper in the shape of circles, triangles, squares, rectangles and hexagons. Draw lines to separate the region of each cutout into parts of the same size, representing halves, fourths, sixths and eighths. Examples are shown below. Give at least three cutouts of different shapes to each pupil.



Let the pupils shade a part of each cutout so that some of the shaded

parts show one-half, some show one-third, some show one-fourth, some show *one-sixth* and others show one-eighth. The shading can be done with crayons, coloured pencils or with lead pencils. At the end of this activity, each pupil should have at least three different shapes shaded to show three different fractions. Let them tell about the shaded parts, tell what fraction describes the shaded part and write the numeral for the fraction on the blackboard.

ACTIVITY 6 • Matching Fractions to Parts of Regions

Place the 5" × 7" cards with the numerals $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$ and $\frac{1}{8}$ written on them, each against an empty box. (Shoe boxes or baskets may be used.) Place the cards so that each pupil in the class can see the numerals.

Give out the cutouts that the pupils shaded in Activity 5. Make sure each pupil has three cutouts with shaded parts that are described by different fractions.

Point to the numeral card for $\frac{1}{2}$ and ask a pupil what number it names. Let the pupils look at their cutouts and decide which cutouts have one-half of their regions shaded. Ask all the pupils who have a cutout with one-half the region shaded to place the cutout in the box marked "$\frac{1}{2}$".

Display the cutouts from the "$\frac{1}{2}$" box for the pupils to see. Ask whether all the half regions are the same size and the same shape. (No) Emphasize the idea that not all halves of regions are the same size and shape but that each part is described as one-half of its own region. The pupils should notice that if two regions are not the same size and shape, then the halves of those regions will not be the same size and shape. Lead the pupils to understand that two-halves of the same region make one whole.

Point to the numeral card for $\frac{1}{3}$. Let a pupil tell the name of the fraction. Ask all who have cutouts with one-third of the region shaded to place them in the box marked "$\frac{1}{3}$". Display all these cutouts and let the pupils talk about the set of all the regions of cutouts that we describe as one-third. Emphasize the fact that the shaded thirds of different regions may be of

different sizes and shapes but that each is one-third of its whole region. Guide the pupils to say, "Three-thirds of the same region make a whole."

Continue in a similar way with the other boxes and cutouts showing fourths, sixths and eighths.

STAGE 2 • Ordering Fractions on the Number Line

Vocabulary

One-fifth, one-tenth, one-twelfth

Materials

Eight 3" × 30" strips of paper; pupil pages 66-67

Teaching Procedure

ACTIVITY 1 • Naming Points on the Number Line

Place two of the 3" × 30" strips of paper on your desk. Hold them up for the class to see. Let the pupils show that the strips are of the same size.

Draw a line segment on the blackboard. It should be more than 60 inches long. Mark a point near, but not on, the left end of the line segment and write the numeral 0 at the point.

Say, "This is the number line. We can mark and name points on the line with numbers. Here is one point. Let us name it 0. Let us mark another point."

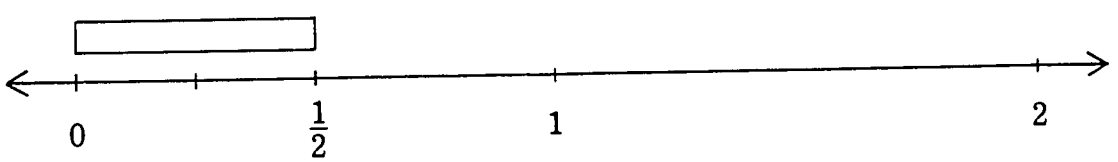
Place one of the strips of paper along the number line, with the left end at the point named 0. Mark a point at the right end of the strip and name it 1. Ask the pupils to tell what this point is named. (1)

Ask a pupil to put his finger where he thinks the point for 2 will be. After he guesses, help him to check his answer. Let him put the strip of paper along the line segment, with the left end of the strip at the point marked 1.

Let him mark a point at the right end of the strip and name it 2. Let the class say whether the pupil's guess about the point for 2 was a good guess. Stress the idea that the line segment from 0 to 1 and the line segment from 1 to 2 are congruent. The edge of the paper strip fits both exactly.

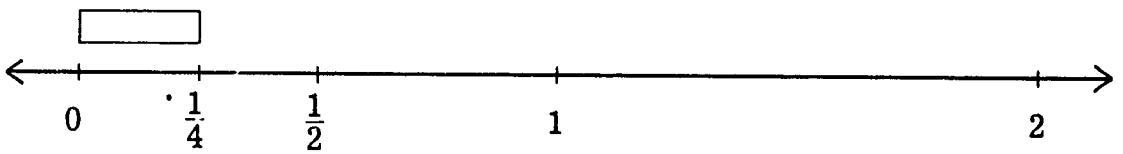
Ask a pupil to fold one paper strip into halves. Ask him to show the class how he folded the strip, to unfold it and to tell into how many parts the strip is separated. (2) Ask, "Are the two parts the same size?" Let the pupils tell the name of the fraction which describes one part. $\left(\frac{1}{2}\right)$ Hold the strip up and say, "We know that the edge of the paper strip fits the line segment from the point for 0 to the point for 1. If we fold the strip, we know that one of the two parts we formed is one-half of the strip." Ask a pupil to use the folded strip to find the point on the number line that we can name $\frac{1}{2}$.

Help a pupil to place the left end of the folded strip at 0 on the number line and mark a point at the right end. Say, "We can name this point $\frac{1}{2}$." Write the numeral for $\frac{1}{2}$ below the marked point. Guide the pupil to make statements such as the following: The $\frac{1}{2}$ point is between the point named 0 and the point named 1; the line segment from 0 to $\frac{1}{2}$ and the line segment from $\frac{1}{2}$ to 1 are congruent; the line segment from 0 to 1 is the whole; the line segment from 0 to $\frac{1}{2}$ is one of two parts of the same size; the two parts make the line segment from 0 to 1.



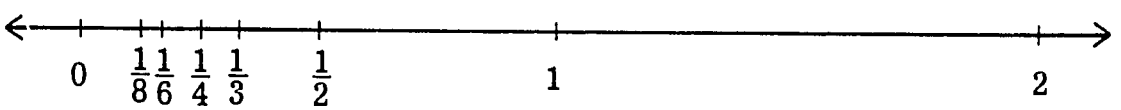
Point to each of the two parts of the strip which $\frac{1}{2}$ describes. Say as you fold each part, "I am separating each of these two parts into two parts of the same size." Unfold the paper and show it to the pupils. Ask, "Into how many parts is the paper separated by the folds?" (4) "Are the four parts all the same size?" (Yes) Let a pupil show one-fourth of the strip. Guide the pupils to say that this part is one of four parts of the same size.

Let another pupil fold the paper so that only one-fourth of the folded paper shows. Let still another pupil use the folded paper to mark the point for $\frac{1}{4}$ on the number line. Let another pupil write the numeral for $\frac{1}{4}$ below the marked point.



Ask, "How do we find the point for $\frac{1}{8}$?" If a pupil suggests that we fold each part of the strip again so that there are eight parts of the same size, let him do this. Let another pupil use the folded paper to mark the $\frac{1}{8}$ point on the line. Ask another pupil to write the numeral for $\frac{1}{8}$ below the mark.

Ask, "Where do you think the point for $\frac{1}{3}$ should be?" Get the pupils to talk about this. Let several pupils put their fingers on the places where they think the point should be. Ask the class how they can decide where the $\frac{1}{3}$ point is. Guide the pupils to say that they can use a paper strip folded into three parts of the same size. Let a pupil fold a strip and use it to mark a point for $\frac{1}{3}$. Write the numeral for the fraction below the mark. Let the pupils tell who made the best guess. Using the same paper strip, follow the plan above to mark and name the point for $\frac{1}{6}$. (*Note:* Save the number line, as it is marked for use in the next activity.)

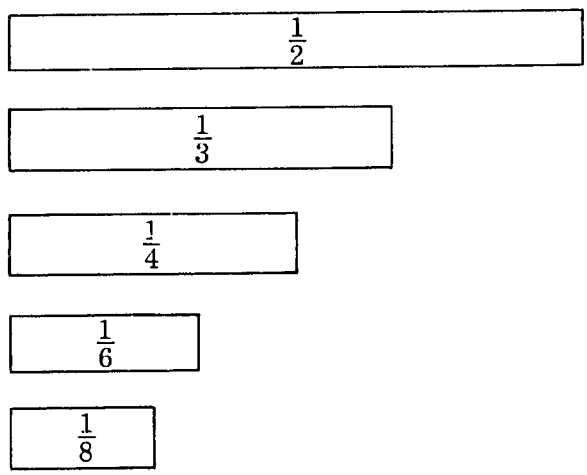


ACTIVITY 2 • Revising the Order of Fractions

Pupil page 66

For this activity, use the number line drawn and marked on the blackboard for Activity 1. Put out five of the 3" × 30" strips of paper.

Let five pupils fold the strips so that one-half of one strip shows, one-fourth of another strip shows, one-eighth of another strip shows, one-third of another strip shows and one-sixth of another strip shows. Ask each pupil to write the numeral for the fraction that describes that part on one part of his strip. Then let the pupils compare the parts they show and decide which part is longest, which is shortest and so on. Ask them to place all the parts of the folded strips on a table in order, from the longest to the shortest.



Ask, "Which part of the strip is longer, the part that shows one-half or the part that shows one-sixth?" (One-half) "We say that the number $\frac{1}{2}$ is greater than the number $\frac{1}{6}$."

Place the part of a strip that shows one-half along the number line to remind the pupils that the edge is congruent to the line segment from the point for 0 to the point for $\frac{1}{2}$. Show in the same way that the one-sixth part of a strip has an edge that is congruent to the line segment from the point for 0 to the point for $\frac{1}{6}$. Ask, "Which of the line segments is longer, the line segment that is one-half the whole line segment from 0 to 1 or the line segment that is one-sixth the line segment from 0 to 1?" (The line segment that is one-half)

Say, "The point for $\frac{1}{2}$ is to the right of the point for $\frac{1}{6}$. One-half is greater than one-sixth." Write the following sentence on the blackboard and ask the children to read it:

$$\frac{1}{2} > \frac{1}{6}$$

Get the pupils to compare the one-third part of a strip and the one-fourth part of a strip. (It should be emphasized that the whole strips are all of the same size.) Ask which of the two parts is greater. (One-third) Say, "One-third is greater than one-fourth." Let two pupils find the points for $\frac{1}{3}$ and $\frac{1}{4}$ on the number line. Ask, "Which is at the right?" $\left(\frac{1}{3}\right)$ Let a pupil write this sentence:

$$\frac{1}{3} > \frac{1}{4}$$

Say, "We can also ask which number is less than another. If $\frac{1}{3}$ is greater than $\frac{1}{4}$, we can say that $\frac{1}{4}$ is less than $\frac{1}{3}$. One-fourth is to the left of $\frac{1}{3}$ on the number line." Let a pupil write the sentence on the blackboard to show which of the numbers is less:

$$\frac{1}{4} < \frac{1}{3}$$

Guide the pupils to use their folded strips and the number line to make statements such as the following: "If $\frac{1}{3}$ is less than $\frac{1}{2}$, then the point for $\frac{1}{3}$ is to the left of the point for $\frac{1}{2}$ on the number line; and if $\frac{1}{2}$ is greater than $\frac{1}{3}$, then the point for $\frac{1}{2}$ is to the right of the point for $\frac{1}{3}$."

Compare several pairs of numbers on the number line. Get the pupils to decide which number of the pair is less and which is greater by noting which point is to the left and which point is to the right. Let them write sentences like these on the blackboard:

$\frac{1}{8} < \frac{1}{4}$ $\frac{1}{4} < \frac{1}{2}$ $\frac{1}{3} > \frac{1}{8}$ $\frac{1}{2} > \frac{1}{3}$ $\frac{1}{6} < \frac{1}{4}$

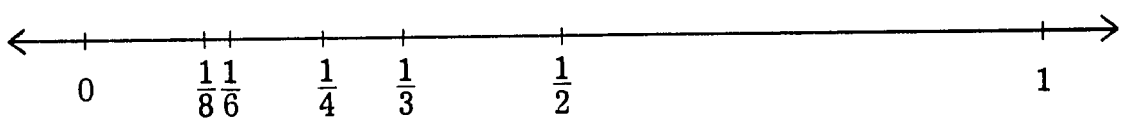
Ask the pupils to turn to page 66 in their books. Let them copy the incomplete sentences in their exercise books and write in the correct symbol, > or < or =, to make each sentence true. Say, "In each sentence, one number is either greater than, less than or equal to the other. Make true sentences by writing the proper symbol." Complete one sentence on the blackboard as an example.

$\frac{1}{3} > \frac{1}{6}$

ACTIVITY 3 • Naming Points on the Number Line

Put out three unused 3" × 30" strips of paper and the strips used in Activities 1 and 2.

Draw the number line on the blackboard, using a strip of paper to mark the segment from 0 to 1. Let the pupils use the folded strips from previous activities to mark and name the $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$ and $\frac{1}{8}$ points.

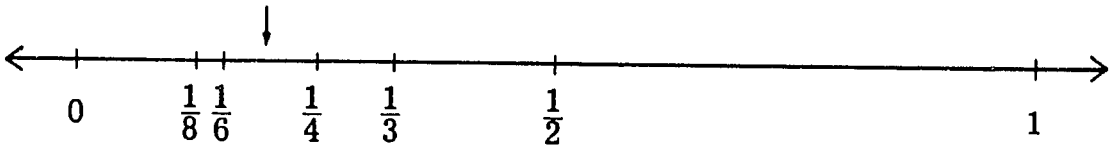


Let the pupils discuss the order of the fractions. Help them discover patterns in the order by noticing that $\frac{1}{8}$ is less than $\frac{1}{6}$, that $\frac{1}{6}$ is less than $\frac{1}{4}$, that $\frac{1}{4}$ is less than $\frac{1}{3}$ and so on. They should notice that the point for $\frac{1}{3}$ is between the points for $\frac{1}{4}$ and $\frac{1}{2}$, that the point for $\frac{1}{6}$ is between the points for $\frac{1}{8}$ and $\frac{1}{4}$ and so on.

Write the numeral $\frac{1}{5}$ on the blackboard in a place apart from the number line. Say, "This is another fraction. Name the fraction." (*One-fifth*) "Look at the number line. Can someone guess where the one-fifth point is?"

Let a pupil put his finger on the place on the number line where he

guesses the point is. Mark it with a small arrow above the line to show that it is a guess.



Ask whether any pupils agree. If some disagree, let them indicate their guesses by marking other arrows. Let them discuss *why* they think a point for $\frac{1}{5}$ should or should not be where suggested. Help them to discover that there are several ways to think about where the $\frac{1}{5}$ point is. They may see a pattern. For example, a pupil may say, "The denominators go 8, 6, 4, 3, 2. I think the denominator 5 goes *between* 6 and 4. I think the point for $\frac{1}{5}$ goes *between* the point for $\frac{1}{6}$ and the point for $\frac{1}{4}$."

They may think of the line segment from 0 to 1 as separated into five congruent line segments. They may think of the paper strip folded into five parts of the same size and try to imagine the size of one of them. They know that if the line segment from 0 to 1 is separated into five congruent line segments, then the endpoint of the line segment beginning at point 0 will be named $\frac{1}{5}$. Say, "These are ways that help us guess where the point is. How can we find and mark the $\frac{1}{5}$ point?"

Use another strip of paper folded into five parts of the same size to locate the one-fifth point and to decide whether the pupils' guesses were good guesses. (Prepare the strip before class. Mark a place on the strip six inches from one end. Since six inches is one-fifth of thirty inches, this will help you to fold the strip quickly into five parts of the same size.)

Let a pupil unfold the strip and count the number of parts of the same size. Get another pupil to use the folded strip to mark the point for $\frac{1}{5}$. Let him write the numeral $\frac{1}{5}$ below the point marked.

Get the pupils to notice that $\frac{1}{5}$ follows the pattern of the numbers on the number line. There is an order in the denominators. The point for $\frac{1}{5}$ is between the points for $\frac{1}{4}$ and $\frac{1}{6}$. Help the pupils to notice also that to locate

$\frac{1}{5}$ we separate the segment from 0 to 1 into more parts of the same size than we did to locate $\frac{1}{4}$ and that each part is shorter.

Ask, "Where do you think we should put the point for one-tenth? Is $\frac{1}{10}$ greater than, or less than $\frac{1}{8}$?" (Less) "Will the point be to the right, or to the left, of the point for $\frac{1}{8}$?" Let the pupils guess the position of the point and mark their guesses with small arrows. Then ask, "How can you find the position of the point for $\frac{1}{10}$?" Get the pupils to suggest that the strip folded into five parts of the same size be folded again to show ten parts of the same size. One of the ten parts of the same size will represent one-tenth of the paper. Let a pupil fold the strip, mark the point on the number line and name the point $\frac{1}{10}$.

Use the same plan to locate the point for *one-twelfth*. The strip folded and used to find the point for $\frac{1}{6}$ can be folded again into twelfths.

ACTIVITY 4 • Naming Points and Comparing Fractions Pupil page 67

Write the following numerals on the blackboard:

$$\frac{1}{4} \qquad \frac{1}{10} \qquad \frac{1}{8} \qquad \frac{1}{3} \qquad \frac{1}{6} \qquad \frac{1}{2} \qquad \frac{1}{5} \qquad \frac{1}{12}$$

Let the pupils talk about what the numerals mean and whether they are in order.

Ask the pupils to open their books to page 67 and read the sentence at the top of the page. Ask them to find the point for 0 on the number line. Then ask them to find the point for 1. Say, "There are other points marked on the number line, but the numeral for the number is not always written. Can you find a numeral for point H?" $\left(\frac{1}{2}\right)$ Write this sentence on the blackboard:

Point H is named $\frac{1}{2}$.

Say, "You complete each of the other sentences by choosing one numeral from the list on the blackboard. Choose the numeral that can be written in the box to make a true sentence. For example, point A has a number name. It is the name of a fraction. Write its numeral in the box. Copy the sentences in your exercise books and make them true sentences." (Some children may need to fold string or strips of paper to decide which fraction to use.)

Ask the pupils to look at the incomplete sentences on the right side of the page. Go over one sentence with the class. Let a pupil read this sentence:

$$\frac{1}{3} < \square$$

Say, "One-third is less than a number. Can you find its name on the list?" Get the pupils to notice that there is more than one number that will make the sentence true. They can choose any number that will make the sentence true. One pupil might write "$\frac{1}{2}$", another might write "1". Say, "Copy the sentences in your exercise books and make them true by writing a numeral in each box." Give help as it is needed.

UNIT 6

Properties of Operations

OBJECTIVES

1. To study mathematical sentences that use several number operations and several variables
2. To learn the meaning of an *operation* on numbers
3. To learn the meaning of a *property* of an operation
4. To discover some properties the basic operations have:
   a. Addition and multiplication have the commutative property.
   b. Addition and multiplication have the associative property.
   c. Multiplication is distributive over addition.
   d. Zero and one have special properties under addition and multiplication.

Background Information for Teachers

See Part II, "Arithmetic", pages 22-24, 36-42, in your *Teachers' Handbook.*

The purpose of this unit is to help the pupils better understand what is meant by an operation on numbers. The pupils have been adding, subtracting, multiplying and dividing numbers for some time. These processes are called *operations.* Now we look at the general meaning of an operation on numbers so that we can discover some "properties" of an operation.

A property of an operation is a special characteristic that the operation has. A property of iron is its hardness, and a property of birds is their ability to fly. The reason we study the properties of the operations is so we can base all the techniques of the operations on their properties. (These techniques are developed in Units 7, 9 and 12.)

The operations that pupils first use are called *binary* operations. For example, addition is a binary operation because it assigns to each *pair* of whole numbers *one* whole number. Addition is an operation that

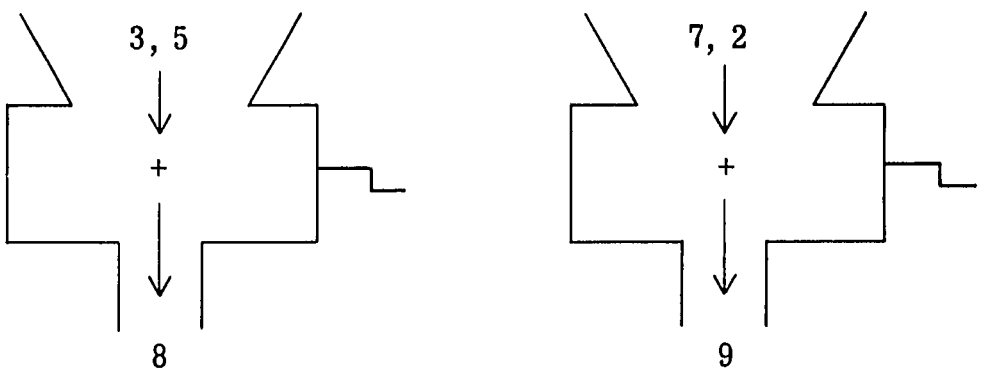
assigns to the pair 3 and 5 the whole number 8,
assigns to the pair 7 and 2 the whole number 9

and so on; for *every pair* of whole numbers, addition assigns one whole number.

We can write this in many ways. For example:

$(3, 5) \rightarrow 8$
$(7, 2) \rightarrow 9$

Or we can think of the addition operation as a machine into which we put a *pair* of numbers and out of which comes *one* number, the sum.



Of course, the usual way to write the fact that addition assigns to the pair 3, 5 the number 8 is as follows:

$3 + 5 = 8$

Later the pupils will use operations on a *single* number. An example of such an operation is the squaring operation, which assigns to each single whole number its square. Squaring assigns to the number 3 the number 9; squaring assigns to the number 5 the number 25; and so on. For example:

$3 \rightarrow 9$
$5 \rightarrow 25$

Only binary operations are studied in *Primary Three*. When we use the word "operation", we will mean binary operation.

We try to help a pupil understand that there are many kinds of operations other than the four they have already studied. By making up new operations, the pupils begin to see that some operations have certain properties and that others do not.

For example, they find that addition and multiplication are commutative operations but that subtraction and division are not.

Commutative property. An operation is commutative if it assigns the same number to the pair a, b as it assigns to the pair b, a, where a and b are any whole numbers. For example,

$$2 + 3 = 3 + 2, \qquad \textit{and} \qquad 5 + 1 = 1 + 5,$$

for any two whole numbers. Also,

$$2 \times 4 = 4 \times 2, \qquad \textit{and} \qquad 3 \times 7 = 7 \times 3,$$

for any two whole numbers. Thus, addition and multiplication are commutative because $a + b = b + a$ and $a \times b = b \times a$ are true sentences for any whole number a and any whole number b.

But some sentences of the following kind are *not* true:

$$6 \div 3 = 3 \div 6 \qquad\qquad 9 - 2 = 2 - 9$$
$$8 \div 2 = 2 \div 8 \qquad\qquad 7 - 4 = 4 - 7$$

This means that subtraction and division do *not* have the commutative property.

Another property of an operation that the pupils study is the associative property. They discover that addition and multiplication are associative operations but that subtraction and division are not.

Associative property. All binary operations assign numbers to pairs of numbers. When three numbers are to be added, we first must add two of them and then add their sum and the third number. This can be done in two ways. For example:

$$\begin{array}{ccc} (3, 5), 7 & & 3, (5, 7) \\ {\scriptstyle +}\downarrow \swarrow & & \downarrow {\scriptstyle +}\downarrow \\ (8, 7) & \xrightarrow{+} 15 & (3, 12) \quad \xrightarrow{+} 15 \end{array}$$

That is:

$$\begin{aligned} (3 + 5) + 7 &= 8 + 7 \\ &= 15 \end{aligned} \qquad \textit{and} \qquad \begin{aligned} 3 + (5 + 7) &= 3 + 12 \\ &= 15 \end{aligned}$$

In both addition and multiplication, the pupils see that the result is the same no matter how the three numbers are grouped when the operation is applied. Thus, we say that addition and multiplication are associative because

$$(a + b) + c = a + (b + c) \quad \text{and} \quad (a \times b) \times c = a \times (b \times c)$$

for any whole number a, any whole number b and any whole number c.

But subtraction and division are *not* associative operations. Some sentences of the following type are *not* true:

$$(8 - 4) - 2 = 8 - (4 - 2); \qquad (8 \div 4) \div 2 = 8 \div (4 \div 2);$$
$$(12 - 6) - 2 = 12 - (6 - 2); \qquad (12 \div 6) \div 2 = 12 \div (6 \div 2)$$

Zero and one. In Unit 3 the pupils found that certain numbers have special properties under addition and multiplication. We call these properties:

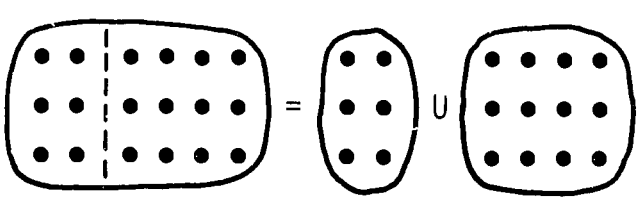
Addition property of 0. For any whole number a, the sentence $0 + a = a$ is true. (For example, $0 + 7 = 7$; $0 + 81 = 81$.)

Multiplication property of 1. For any whole number a, the sentence $1 \times a = a$ is true. (For example, $1 \times 6 = 6$; $1 \times 37 = 37$.)

These properties of 0 and 1 are used over and over again in the development of arithmetic.

Distributive property. The children will discover a property that connects two operations. It is called a *distributive property*. They find that multiplication is *distributive* over addition but that addition is *not* distributive over multiplication.

When a rectangular array is separated into two rectangular arrays, the operations of multiplication and addition are suggested. For example:



$$3 \times (2 + 4) = (3 \times 2) + (3 \times 4)$$

In the same way, the pupils find that both of the following sentences are true:

$$4 \times (1 + 6) = (4 \times 1) + (4 \times 6)$$
$$3 \times (5 + 2) = (3 \times 5) + (3 \times 2)$$

That is, a number multiplied by the sum of two numbers gives the same result as multiplying that number by the first addend and then by the second addend and finding the sum of these problems. We say:

Multiplication is distributive over addition because

$$a \times (b + c) = (a \times b) + (a \times c)$$

is a true sentence for any whole number a, any whole number b and any whole number c.

But when the pupils try to distribute addition over multiplication, they find that some sentences of the following kind are *not* true:

$$3 + (5 \times 2) = (3 + 5) \times (3 + 2)$$
$$4 + (2 \times 3) = (4 + 2) \times (4 + 3)$$

Thus, addition is not distributive over multiplication.

In all the arithmetic and later mathematics that the pupils will study, the property that multiplication is distributive over addition will be called simply the *distributive property*. It will be the most powerful and useful property available to the pupil in all he discovers about the techniques of arithmetic.

In *Primary Three* the pupils should begin to understand what these properties are and how they can be used. But not much emphasis need be placed on the names of the properties. The words "commutative", "associative" and "distributive" should be mentioned, but the pupils should not be drilled on the words. The emphasis should be on the understanding of meanings of the properties rather than on the names.

In this unit there are two minor ideas that are important:

1. The pupils need to learn the use of symbols for grouping. For example, the parentheses in the sentence

$$3 \times (4 + 2) = n$$

tell us that the numerals inside the parentheses name one number.

Then

$$3 \times (4 + 2) \text{ means } 3 \times 6.$$

As another example,

$$(3 \times 4) + 2 \text{ means } 12 + 2.$$

2. The children must understand that when the same letter occurs several times in a sentence, that letter refers to the same number each time it occurs in the sentence. For example, if n is 3, then the sentence $n + (2 + n) = 6$ becomes

$$3 + (2 + 3) = 6 \text{ (a false sentence).}$$

If $n = 2$, then the sentence $n + (2 + n) = 6$ becomes

$$2 + (2 + 2) = 6 \text{ (a true sentence).}$$

STAGE 1 • Mathematical Sentences with Several Operations

Vocabulary

None

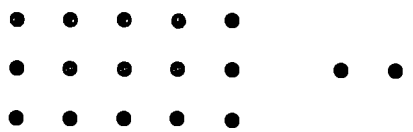
Materials

Many small objects for counters, such as seeds and bottle tops; pupil pages 68-72

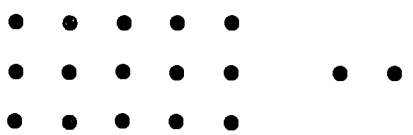
Teaching Procedure

ACTIVITY 1 • Regrouping Sets and Renaming Numbers

Let each pupil place a set of exactly 17 counters on his desk. Ask the pupils to arrange some of their counters in a rectangular array and to place the remaining counters in a group beside it. On the blackboard, draw a picture of one possible arrangement for the pupils to see. For example:



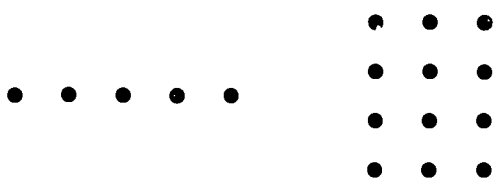
Ask the pupils what product is shown by the dots in the rectangular array. They may say, (3×5). Ask, "How many are left over?" (2) Help the pupils make a mathematical sentence that describes the arrangement of the dots on the blackboard. Write the mathematical sentence below the drawing on the blackboard:



$$17 = (3 \times 5) + 2$$

Ask the pupils to think of other ways of arranging the same number of counters, putting some of the counters in a rectangular array. Let each pupil make a mathematical sentence to describe his arrangement and write the sentence in his exercise book. Let the pupils work in pairs and check each other's work.. If they have difficulty, draw pictures on the blackboard of other possible arrangements of the 17 counters. Ask the pupils in turn to write a mathematical sentence below each of the arrangements.

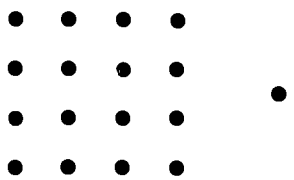
1.



_____ + (____ × ____) = 17

or 17 = _____ + (____ × ____)

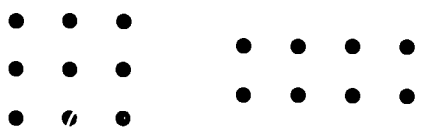
2.



(____ × ____) + ____ = 17

or 17 = (____ × ____) + ____

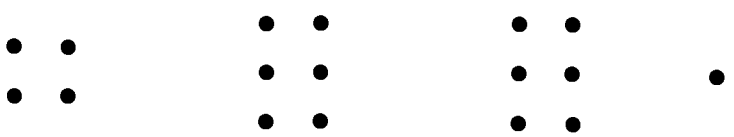
3.



(____ × ____) + (____ × ____) = 17

or 17 = (____ × ____) + (____ × ____)

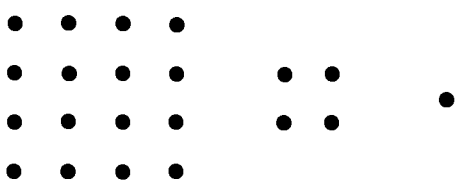
4.



(____ × ____) + (____ × ____) + (____ × ____) + 1 = 17

or 17 = (____ × ____) + (____ × ____) + (____ × ____) + 1

Note: Make sure that the pupils use the parentheses correctly to show which operations are performed first. For example, the arrangement



is properly described by the sentence

$$(4 \times 4) + (2 \times 2) + 1 = 17,$$

but the arrangement is *not* described by the sentence

$$4 \times 4 + 2 \times 2 + 1 = 17.$$

The latter sentence does not tell which operations are done first, and a pupil might think it means

$$4 \times (4 + 2) \times (2 + 1) = 17,$$

which is a false sentence.

Let each pupil get 3 more counters so they each have 20. Let them arrange their 20 counters in different rectangular arrays as they did with the 17 counters. Ask, "Can all of the counters be put in a rectangular array? Can some of the counters be put in an array, with some left over?" Let the pupils arrange the counters in many ways and write a mathematical sentence for each arrangement. Some of the sentences they may write are these:

$10 \times 2 = 20$	*or*	$2 \times 10 = 20$
$20 = 5 \times 4$	*or*	$4 \times 5 = 20$
$(4 \times 4) + 4 = 20$	*or*	$20 = 4 + (4 \times 4)$
$20 = (5 \times 2) + (2 \times 5)$	*or*	$(2 + 5) + (5 \times 2) = 20$
$(3 \times 5) + 5 = 20$	*or*	$20 = 5 + (5 \times 3)$
$20 = (4 \times 2) + 2 + (4 \times 2) + 2$	*or*	$2 + (2 \times 4) + 2 + (4 \times 2) = 20$
$(6 \times 3) + 2 = 20$	*or*	$2 + (3 \times 6) = 20$
$20 = (9 \times 2) + 2$	*or*	$2 + (9 \times 2) = 20$

Encourage each pupil to show at least three different arrangements and write the mathematical sentence for each. In this way, he shows that he can do several different operations on numbers. He also shows that he can make a mathematical sentence and understand what it means.

Allow the pupils to choose any number of counters. For example, one pupil may choose 36. Let him arrange his 36 counters in arrays in different ways and write sentences such as these:

$$12 \times 3 = 36 \quad or \quad (6 \times 2) \times 3 = 36 \quad or \quad 36 = (3 \times 6) + (6 \times 3)$$

ACTIVITY 2 • Solving Mathematical Sentences

Pupil pages 68-69

On pupil page 68, the pupils work with sentences in which there are several operations. Use some or all of the seven examples suggested below to help the pupils before they begin work in their books. Write each example on the blackboard as it is used.

Example one: Ask the pupils to find the number that makes this sentence true:

$$(6 \times 3) + n = 20$$

n is____.

The pupils know that 6×3 is 18, and they know that they must add 2 to 18 to get 20. So if n is 2, the sentence is true.

Example two: Ask, "What number makes this sentence true?"

$$15 = (b \times b) + 6$$

b is_____.

The pupils must know that b is the same number each time it is in the sentence.

Let the pupils talk about the sentence. They might say, "The sentence tells us that we must add $(b \times b)$ to 6 to get 15. That means that $b \times b$ is 9, $b \times b$ is 3×3 and b is 3."

They should decide that if b is 3, then the sentence

$$15 = (b \times b) + 6 \quad \text{becomes} \quad 15 = (3 \times 3) + 6,$$

which is a true sentence. Thus, the number that makes this sentence true is 3.

Example three: Ask the pupils to find the number that makes this sentence true:

$$(n + 5) + n = 21$$

n is_____.

In this sentence, both n's are the same number.

Let the pupils guess what number n is and try it. A pupil might think and say, "If n is 5, then

$$(n + 5) + n = 21 \quad \text{becomes} \quad (5 + 5) + 5 = 21,$$

which is *not* true because $(5 + 5) + 5$ is 15. n is not 5." Another pupil might then try a greater number, 9. He may say, "If n is 9, then

$$(n + 5) + n = 21 \quad \text{becomes} \quad (9 + 5) + 9 = 21,$$

which is *not* true because $(9 + 5) + 9$ is 23. n is not 9."

Another pupil might try a number less than 9. He may say, "If n is 8, then

$$(n + 5) + n = 21 \quad \text{becomes} \quad (8 + 5) + 8 = 21,$$

which is *true*. n is 8."

In this way, the children find that a number, n, which makes that sentence true, is 8.

$$(n + 5) + n = 21$$

n is 8.

Examples four-seven: Write the following examples on the blackboard and let pupils in turn solve the sentences:

$28 = (k \times k) + 3$

k is_____.

$(d + 7) + d = 17$

d is_____.

$(6 \times h) + (3 \times h) = 36$

h is____.

$13 + (n \times 5) = 18$

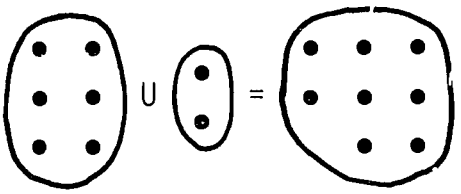
n is_____.

Let the pupils talk about ways of solving sentences.

Ask the pupils to turn to page 68, copy the sentences and find the numbers that make them true. Under each sentence that they copy in their exercise books, let them draw dots in arrays to show the sentence. For example, the first exercise will look like this:

$(3 \times 2) + a = 8$

a is 2.



(Let the pupils use the figure at the top of page 6 as an example.)

Give the pupils help in using parentheses. Ask them to tell which of the symbols $<$ or $>$ or $=$ makes the following sentence true:

$(3 \times 2) \times 2$ ___ $3 \times (2 \times 2)$

In this sentence the symbol $=$ makes a true sentence

because $(3 \times 2) \times 2 = 6 \times 2 = 12$
and $3 \times (2 \times 2) = 3 \times 4 = 12$.

Write this sentence on the blackboard and ask the pupils to make it true:

$4 + (5 \times 3)$ ___ $(3 \times 4) + 6$

The pupils find the number named on the left and the number named on the right. They choose a symbol ($<$ or $>$ or $=$) that makes a true sentence. In this example, they choose the symbol $>$ because

$4 + (5 \times 3)$ is 19,
and $(3 \times 4) \times 6$ is 18
and $19 > 18$, which is true.

Let the pupils turn to page 69, copy the sentences and write a symbol in the blank space to make the sentence true.

ACTIVITY 3 • Solving Problems

Pupil pages 70-72

Note: In this mathematics programme there is a definite plan for teaching the solving of word problems. It has the following steps for pupils to follow:

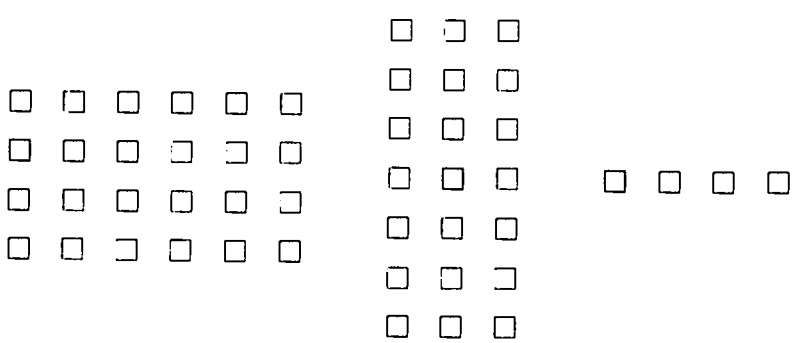
1. Know what the problem is about and the question asked in the problem.
2. Write a mathematical sentence for the problem, with a letter representing the missing number.
3. Find a number that makes the mathematical sentence true.
4. Use the number to answer the question in the problem.

Follow this plan as you help the pupils solve word problems. It is the plan that is used at all levels of mathematics in solving problems.

The purpose of this activity is to help the pupils learn to solve problems stated in words. The problems use several different operations. Before the pupils begin the exercises, help them solve some sample problems. Tell this problem to the pupils and let them talk about it:

Problem 1: Juma collected stamps. Four girls in his class each gave him 6 stamps. Seven boys in his class each gave him 3 stamps. The headmaster gave him 4 stamps. How many stamps did Juma collect?

Let the pupils talk about the problem. Get them to say what question it asks. Write the question on the blackboard. As they talk about the problem they may say Juma got stamps from 4 girls (4 × 6), from 7 boys (7 × 3) and from the headmaster (4). They may suggest drawing a picture like this:



(4 × 6) (7 × 3) 4

Guide the pupils to then make the mathematical sentence:

$$(4 \times 6) + (7 \times 3) + 4 = s$$

Let them talk about what each part (4×6; 7×3; 4; and s) of the mathematical sentence means. Get them to rename the numbers in the mathematical sentence and say

s is 49.

As you write this sentence on the blackboard, ask the pupils whether they can answer the question in the problem. Guide them to restate the question and answer it. Write their answer on the blackboard. The blackboard may now look like this:

How many stamps did Juma collect?

$(4 \times 6) + (7 \times 3) + 4 = s$
s is 49.

Juma collected 49 stamps.

Let the pupils talk about how they answered the question in Problem 1. (We knew the problem and the question it asked; we wrote a mathematical sentence; we found a number to make the mathematical sentence true; and we answered the question asked in the problem.)

Go on in a similar way with Problem 2. The work that you and the pupils may write on the blackboard is given.

Problem 2: One morning a shopkeeper had many eggs in his basket for sale. Five men each bought 8 of the eggs. Four women each bought 7 of the eggs. That night the shopkeeper looked in his basket. He saw 12 eggs left. How many eggs were in his basket in the morning?

How many eggs were in the basket in the morning?

$(5 \times 8) + (4 \times 7) + 12 = e$
e is 80.

There were 80 eggs in the basket.

Let the pupils open their books to pages 70, 71 and 72. There are ten word problems on these pages. Each problem has questions to be answered. Let the pupils tell what numbers to write in the blank spaces to make the sentences true. Let them write the following for each of the exercises: the mathematical sentence; a statement telling what the missing number in the sentence is; and an answer to the question asked in the problem. This is the way their exercise books may look:

1. $3 \times 6 = n$
   n is 18.
   Mary has 18 eggs.

2. $(3 \times 6) + 7 = m$
   m is 25.
   Now Mary has 25 eggs.

3. $a = 5 \times 10$
   a is 50.
   John has 50 bananas left.

4. $d = (5 \times 7) - 8$
   d is 27.
   There are 27 birds left.

5. $(4 \times 3) + (4 \times 2) = g$
   g is 20.
   There are 20 sides.

6. $(4 \times 8) + (2 \times 5) = b$
   b is 42.
   They caught 42 fish.

7. $s = (5 \times 7) + (3 \times 3)$
   s is 44.
   There are 44 small squares.

8. $(7 \times 4) + (4 \times 2) = l$
   l is 36.
   The animals have 36 legs.

9. $(24 \div 2) - 2 = x$
   x is 10.
   John has 10 marbles left.

10. $p = (24 - 6) \div 2$
    p is 9.
    Each boy has 9 fish.

STAGE 2 • What Is an Operation?

Vocabulary

Operation machine, number pairs, operation

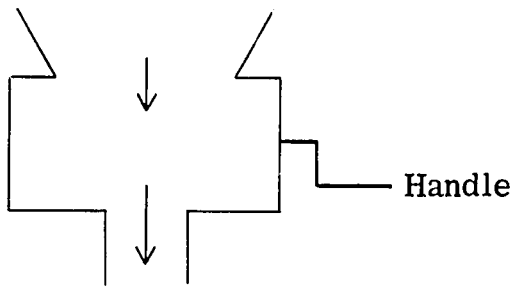
Materials

Cardboard operation machine; 3" × 3" numeral (0-36) cards; 3" × 3" symbol (+, −, ×, ÷) cards; pupil pages 73-76

Teaching Procedure

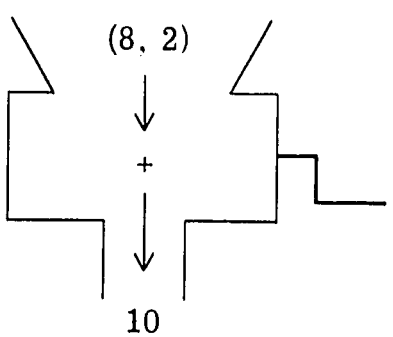
ACTIVITY 1 • Recognizing Operations and Making New Operations

On the blackboard draw a box like this:



Say, "Pretend that this is a machine. It is an *operation machine*. See the handle. We put two numbers into the top of the machine." Write the numerals (8, 2) above the machine. (See the drawing below.) "When we turn the handle, the machine does something to the numbers, and one number pops out of the bottom." Pretend you are turning the handle and write 10 below the box. (See the drawing below.) "What did the machine do to our pair of numbers to get the number 10?" If no one says "add", or even if they do, erase the numerals. Then write (4, 3) above the box. Say, "When we turn the handle, the machine takes the numbers 4 and 3 and out pops the number 7." Write "7" at the bottom. "What did the machine do to our pair of numbers to get the number 7?" Continue putting pairs of numbers at the top and their sum at the bottom.

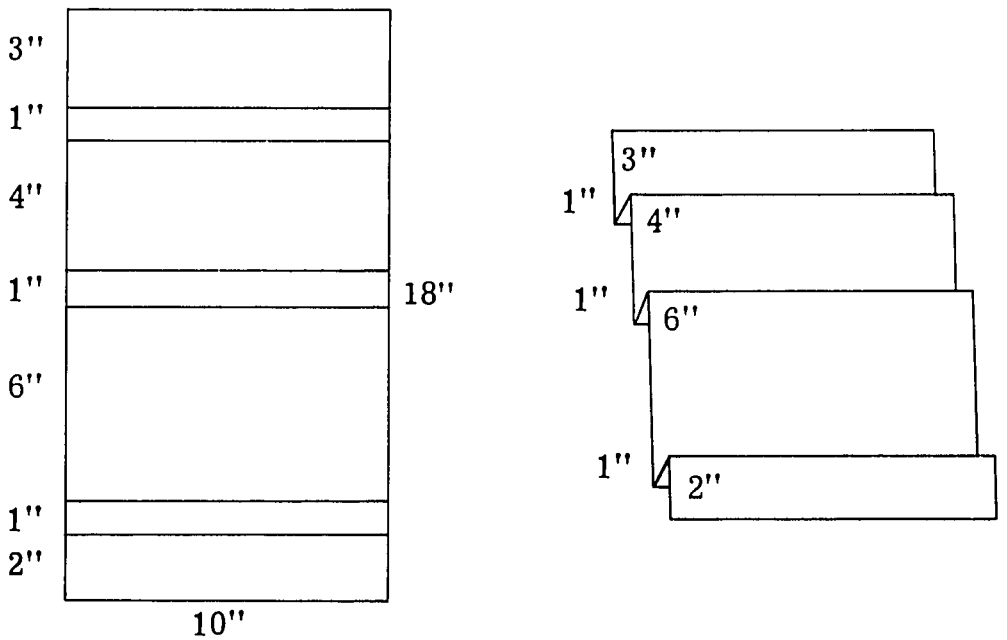
Let the class think about and discuss these questions until they decide that the machine adds the pairs of numbers. Say, "Yes. This is an *addition machine*." Write a plus sign on the middle of the machine. Now the blackboard looks like this for the first example:



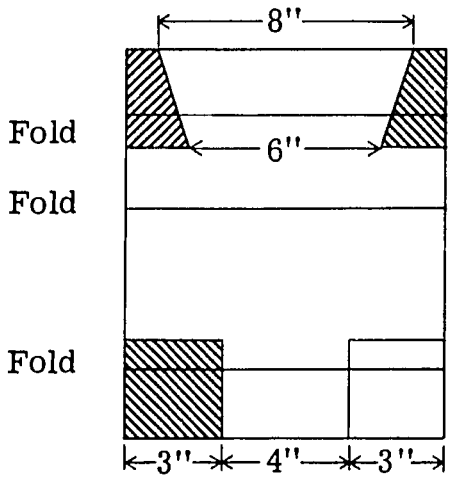
Rub out the numerals and write the pair (6, 3) above the machine. Pretend you are again turning the handle. Ask, "What number will pop out of the addition machine?" (9) Continue this with other *number pairs*. Each time, ask what number will come out of the addition machine. Let the pupils in turn write the numerals for number pairs they want to put into the machine and for the number that pops out.

Bring out a cardboard operation machine, which you made before the lesson from these plans:

Note: To make an operation machine for the pupils, use a 10" × 18" piece of cardboard. Mark these points along the 18" sides: 3", 1", 4", 1", 6", 1", 2", as shown at the left below. Then fold as shown at the right.



Draw a picture of an operation machine on the folded cardboard and cut off the regions that are shaded in the figure below:



Make 3" × 3" numeral and symbol (+, −, ×, ÷) cards to be used with the machine. The cards fit into the folds of the machine.

Fasten the cardboard operation machine to the wall. Place the card marked "×" in the fold for the operation symbol and say, "This is a *multiplication machine.*" Put two numeral cards, for example, 3 and 6, in the top fold and say, "If we put the pair (3, 6) into the multiplication machine and turn the handle, what number comes out?" When the pupils say "18", put a numeral card for 18 in the bottom fold.

Continue this type of activity with the multiplication machine. Let a pupil put a pair of numeral cards at the top of the machine and ask another pupil to pretend he is turning the handle. Let still another pupil tell what number pops out and place the numeral card at the bottom.

After the pupils have used the multiplication machine for a while, remove the "×" card and replace it with the "−" card. Say, "Now we have a *subtraction machine.* We will put these two numbers into the subtraction machine. Put the numeral cards (7, 2) at the top of the machine. (If you use any other pair of numbers, make sure that the first is greater than or equal to the second.) Ask someone to come up and pretend to turn the handle of the machine. Let a pupil tell what number pops out and let him come up and put the numeral for the number at the bottom of the machine. No matter what numeral he puts at the bottom, let the rest of the class decide whether he chose the correct card. Get pupils to tell why the result is correct or incorrect (5 pops out because 7 minus 2 equals 5, or 5 must be added to 2 to get 7). When a pupil puts a number pair like (2, 7) into the machine, the pupils should say, "No number pops out." You say, "The subtraction machine will not work for this number pair."

Continue this activity until the class has some experience in using the subtraction machine.

ACTIVITY 2 • Finding the Operation

Pupil pages 73-76

Let the pupils use the operation machine as they did in Activity 2.

Remove the "−" card from the operation machine. Say, "I put a number pair (8, 2) into the operation machine. "Do we know what *operation* the machine will do?" (No. There is no symbol to tell us.) Say, "We will see

what happens." Get a pupil to turn the handle. Say, "A number is popping out." Put a numeral card for 2 in the bottom of the machine. "The machine gave us 4. What kind of operation machine is this?" Let the pupils discuss this question. If they cannot answer, use other number pairs. Put (8, 4) in the top of the machine and 2 in the bottom. Again ask what kind of machine it is. Continue until the pupils decide that it is a *division machine.*

Place a symbol card for ÷ in the proper place. Go on putting different pairs of numeral cards in the top and asking what number pops out at the bottom. For example, put (6, 2), (9, 3), (12, 3) in the top of the machine. Also try the number pair (2, 6). Guide the pupils to say, "No number pops out." You say, "The division machine will not work for this number pair."

Vary the activity. Sometimes put a pair of numerals in the top and a numeral at the bottom and ask what operation the machine is performing. Other times, put an operation card in the middle and a pair of numeral cards in the top and ask what number pops out at the bottom. Choose the cards so the pupils find all of the four operations. Ask the pupils to turn to pages 73, 74, 75 and 76 in their books and look at the pictures of the operation machines shown there. In some of the pictures of the machines the numeral at the bottom is missing. In some the operation symbol is missing. In others the pair of numerals at the top is missing. Let the pupils talk about these operation machines and decide what the missing numerals or operation symbols are.

The pupils will decide that exactly one numeral is needed at the bottom of each machine. Notice that there is one exception. The last machine on page 75 will not give a whole number because 6 ÷ 4 is not a whole number. If the numeral at the bottom of the machine is given, there may be many different pairs of numbers that can be put at the top to give that numeral at the bottom. For example, look at the fourth row of machines shown on page 73. There are many pairs, such as those shown on the next page, that can be put into the first three machines, but only one pair can be put into the fourth machine.

In the machines with the operation signs missing, the pupils will decide what the operation is. In some cases, there may be two possibilities. For example, on page 73, third row, the last machine can be either a subtraction or a division machine.

Let the pupils draw the machines in their exercise books and put in all

the numerals and symbols. If there are several number pairs, get them to choose one. Let the pupils talk about these exercises.

(14, 7)			
(13, 6)			
(12, 5)	(3, 0)		
(11, 4)	(0, 2)	(4, 4)	
(10, 3)	(2, 0)	(3, 3)	
(9, 2)	(0, 1)	(2, 2)	
(8, 1)	(1, 0)	(1, 1)	
(7, 0)	(0, 0)	(0, 0)	(1, 1)
−	×	−	×
7	0	0	1

Make sure the pupils do all the exercises on pages 73-76 over a period of several days. The exercises help to develop an idea of what an operation is. They also are good revision exercises.

ACTIVITY 3 • Finding a New Operation

Say, "Here is a new operation machine." Put the following pairs of numeral cards at the top of the cardboard operation machine, with corresponding numeral cards at the bottom:

(4, 3)	(5, 2)	(1, 8)	(7, 2)	(3, 4)
4	5	1	7	3

Tell the class that in order to keep a record of the numbers that pop out of the machine for this operation, we write:

(4, 3) → 4
(5, 2) → 5
(1, 8) → 1
(7, 2) → 7
(3, 4) → 3 *and so on.*

Each time you put a number pair at the top and a number at the bottom, write the result on the blackboard as follows:

$$(5, 7) \rightarrow 5$$

The pupils will be puzzled by this operation machine. It uses an operation strange to them. They may doubt that the machine is really performing an operation. Ask, "Does the machine give us a number for each pair of numbers we put into it?" (Yes) Say, "Then the machine is doing an operation. *An operation gives us one number for each pair of numbers.*" Revise the addition and multiplication operations to show that with these operations we get one number for each pair of numbers.

Talk to the pupils again about the new operation. Put more pairs of numbers into the machine:

$$(0, 5) \rightarrow 0$$
$$(4, 1) \rightarrow 4$$

Ask, "If I put the pair (3, 6) into the machine, what number will come out?" If someone says "3", write $(3, 6) \rightarrow 3$. Ask the pupil who answered "3" to keep the operation a secret until the others find it.

If no one says "3", write in "3". Say and write the number pair (6, 3). Ask for a number for this pair. If someone says "6", write this numeral. Continue until the class sees that this operation machine always gives the first of the pair of numbers that is put into the machine.

Since this operation is new to the pupils, let someone give it a name and a symbol. A pupil may suggest that it be called "Firsting" because it always gives the first of the numbers in the pair and give it a symbol as F. Then write:

$$(3, 6) \xrightarrow{F} 3$$

$$(2, 7) \xrightarrow{F} 2$$

$$(6, 3) \xrightarrow{F} 6 \quad \textit{and so on.}$$

Let the pupils use this machine until they see how it works. Ask the pupils to use the name and symbol they made up.

Note: This operation and the others to follow are not important. The pupils do not need to remember them. The operations are used here only to help pupils learn about operation.

Say, "I am thinking of still another operation." This time you do not need to use the cardboard operation machine. Write the pairs and the resulting numbers as follows:

$(0, 2) \rightarrow 3$
$(4, 1) \rightarrow 2$
$(3, 7) \rightarrow 8$
$(5, 0) \rightarrow 1$

Ask a pupil to give a pair of numbers. If he gives (3, 3), write

(3, 3)

and ask what number is given by the operation. If a pupil says "4", write it after the pair. Go on in this way until the class discovers that this new operation always gives *one more than the second number.* Again you can let the class give the operation a name and a symbol. They may call it the "star" operation and use the symbol *. Then

$(5, 2) \xrightarrow{*} 3$
$(6, 6) \xrightarrow{*} 7$
$(1, 3) \xrightarrow{*} 4$ *and so on.*

If the pupils enjoy this activity, continue with some other new operations for them to discover. For example:

1. $(1, 2) \xrightarrow{\triangle} 4$
   $(3, 1) \xrightarrow{\triangle} 4$
   $(4, 2) \xrightarrow{\triangle} 10$
   $(2, 6) \xrightarrow{\triangle} 18$

[The result of this operation is always the product of the two numbers plus the second number: $(4, 3) \xrightarrow{\triangle} (4 \times 3) + 3$.]

2. $(1, 2) \xrightarrow{G} 2$
$(3, 2) \xrightarrow{G} 3$
$(7, 3) \xrightarrow{G} 7$
$(4, 4) \xrightarrow{G} 4$
$(3, 7) \xrightarrow{G} 7$

(The result of this operation is always the greater of the two numbers, if they are different, and the number if they are the same.)

3. $(4, 3) \xrightarrow{O} 11$
$(0, 1) \xrightarrow{O} 1$
$(1, 0) \xrightarrow{O} 2$
$(5, 5) \xrightarrow{O} 15$
$(3, 7) \xrightarrow{O} 13$

[The result of this operation is always 2 times the first number plus the second number. For example, $(3, 4) \xrightarrow{O} (2 \times 3) + 4$.]

Encourage some of the pupils to make up their own operations. Play the game called What Is the Operation? It is played as follows:

Teacher: Tell me a pair of numbers both less than 6.
John: 2 and 4
Teacher writes on the blackboard: $(2, 4) \rightarrow 12$.
Teacher: Tell me another pair of numbers.
Joseph: 0 and 5
Teacher writes: $(0, 5) \rightarrow 10$.
Mary: I think I know the operation. If I say 1, 1, you will write 4.
Teacher: That is right, Mary. We have a secret. You take my place and see if other pupils can find the operation.
Mary: Tell me another pair.
Obi: 3 and 2
Mary writes: $(3, 2) \rightarrow 10$
Obi: I know the operation. If I say 2, 1, you will write 6.
Mary: How do you get 6 from 2 and 1?
Obi: I find 2 times the sum of 2 and 1.

The game can be played from the beginning with a pupil in the place of the teacher. Let a pupil think of an operation and write some examples on the blackboard.

STAGE 3 • Commutative and Associative Properties

Vocabulary

Commutative property, property, associative property

Materials

Six cardboard operation machines; pupil pages 77-79

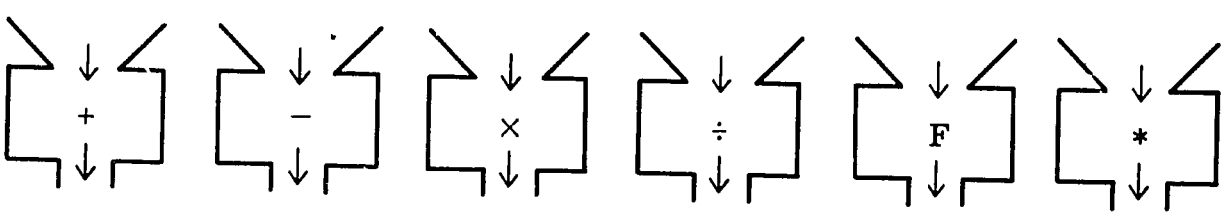
Teaching Procedure

ACTIVITY 1 • The Commutative Property

Pupil page 77

The purpose of the activity is to help the pupils identify a property of operations called the *commutative property*. Let the pupils again think about many operations as they use their operation machines.

Either make several cardboard operation machines or draw them on the blackboard as shown:



Say to the class, "Let us put the same pair of numbers into every one of these operation machines." Get a pupil to suggest a pair, such as (4, 2). Then let the class tell what numbers come out of the machines. Write the results below the machines:

$(4, 2) \xrightarrow{+} 6 \quad (4, 2) \xrightarrow{-} 2 \quad (4, 2) \xrightarrow{\times} 8 \quad (4, 2) \xrightarrow{\div} 2 \quad (4, 2) \xrightarrow{F} 4 \quad (4, 2) \xrightarrow{*} 3$

Ask, "What happens if we put the same two numbers into the operation machines, but this time put 2 in first and 4 in second?" Let the pupils tell what numbers come out of the machines, and write the results below the others:

$(4, 2) \xrightarrow{+} 6$ $(4, 2) \xrightarrow{-} 2$ $(4, 2) \xrightarrow{\times} 8$ $(4, 2) \xrightarrow{\div} 2$ $(4, 2) \xrightarrow{F} 4$ $(4, 2) \xrightarrow{*} 3$

$(2, 4) \xrightarrow{+} 6$ $(2, 4) \xrightarrow{-}$ $(2, 4) \xrightarrow{\times} 8$ $(2, 4) \xrightarrow{\div}$ $(2, 4) \xrightarrow{F} 2$ $(2, 4) \xrightarrow{*} 5$

The pupils will tell you that the subtraction machine and the division machine will not work when (2, 4) is put into the machines. (4 cannot be subtracted from 2 to give a whole number; 2 cannot be divided by 4 to give a whole number.)

Ask, "Which machine gives the same number for (2, 4) as for (4, 2)?" [The addition and multiplication machines give the same number for (2, 4) as for (4, 2).] Try more pairs in the addition and multiplication machines to see whether the numbers that pop out are changed when the order of the numbers of the pair is changed.

$(4, 7) \xrightarrow{+} 11$ $(4, 7) \xrightarrow{\times} 28$

$(7, 4) \xrightarrow{+} 11$ $(7, 4) \xrightarrow{\times} 28$

Note: We usually write $(4, 7) \xrightarrow{+} 11$ as the true sentence $4 + 7 = 11$ and $(4, 7) \xrightarrow{\times} 28$ as the true sentence $4 \times 7 = 28$.

Let the pupils try many number pairs in this way. Guide them to tell about the order of the numbers of a number pair and the addition and multiplication operations. Encourage them to say something like this: "The addition operation gives the same number for a pair no matter which number of a number pair is first." There are many ways to say this. Accept any statement that gives the idea.

Explain to the class that they have found something about addition and multiplication that is always true. They have found a *property* of addition and multiplication. The name of this property is the *commutative property.*

Say, "You found that addition and multiplication have the commutative property. Does subtraction have the commutative property?" (No) Ask a pupil to show this with a subtraction machine. Let him put a pair, for example, (7, 4), into the subtraction machine and get 3. Then he puts (4, 7) into the subtraction machine. He will say that the subtraction machine will not work for (4, 7) because the first number is less than the second. Guide the pupils to say, "We do not get the same number for (4, 7) as we do for

(7, 4) in the subtraction machine. Subtraction does not have the commutative property."

In the same way, guide the class to decide that division does not have the commutative property.

Let them discover that the "firsting" and the "star" operations do not have the commutative property. Do this by showing other examples in which we use pairs written in two different orders:

$$(3, 6) \xrightarrow{F} 3 \qquad (3, 6) \xrightarrow{*} 7$$

$$(6, 3) \xrightarrow{F} 6 \qquad (6, 3) \xrightarrow{*} 4$$

If necessary, look again at some of the other operations that were used in Stage 2 and let the class decide whether they are commutative. For example, the triangle operation does *not* have the commutative property.

$$(5, 2) \xrightarrow{\triangle} 12 \qquad (2, 5) \xrightarrow{\triangle} 15$$

(These sentences may also be written $5 \triangle 2 = 12$ and $2 \triangle 5 = 15$.) The G operation has the commutative property. For example:

$$(2, 1) \xrightarrow{G} 2 \qquad (5, 0) \xrightarrow{G} 5 \qquad (7, 6) \xrightarrow{G} 7$$

$$(1, 2) \xrightarrow{G} 2 \qquad (0, 5) \xrightarrow{G} 5 \qquad (0, 7) \xrightarrow{G} 7$$

But the circle operation does *not* have the commutative property. For example:

$$(3, 4) \xrightarrow{\bigcirc} 10 \qquad (4, 3) \xrightarrow{\bigcirc} 11$$

(These sentences may also be written $3 \bigcirc 4 = 10$ and $4 \bigcirc 3 = 11$.)

Guide the pupils by questions to again say that addition and multiplication have the commutative property. Let them discuss what this means. Ask them to use their operation machines to show what it means for an operation to have the commutative property.

Let the pupils turn to page 77 in their books and talk about the exercises. Get them to find the numbers that make the first few sentences true. Ask a pupil whether he can find one of the missing numbers without adding

or multiplying. Let the pupil explain why. For example, a pupil will say, "I can answer the first exercise without multiplying the numbers; n is 3. We get the same product when we multiply 3 and 4 as when we multiply 4 and 3." Or he may say, "I know n is 3 because $3 \times 4 = 4 \times 3$."

In the same way, let the pupils tell which other sentences can be made true without adding numbers or multiplying numbers. (Sentences 1, 2, 4, 6, 9, 11, 12) Each time, let the pupil explain why he can do this and tell the number that makes the sentences true.

ACTIVITY 2 • Adding and Multiplying Three Numbers

Revise the process of adding three addends and multiplying three factors. (See Primary Two, Unit 4, pages 108-110, and Primary Three, Unit 3, pages 73-76.)

Remind the pupils that addition and multiplication are operations on pairs of numbers. We always put *two* numbers in the operation machine and get *one* number out. Ask, "How can we add three numbers if we can put only two at a time into the addition machine?"

Let the pupils discuss this, using the three numbers 3, 5, 2. Guide them to decide that two of the numbers can be added first, and then the sum can be added to the third. Write this on the blackboard:

$$(3 + 5) + 2$$

Let the pupils tell you that it means $8 + 2$, or 10. Ask whether they could do the addition another way. Guide them to decide that the second and third numbers could be added, and then add the first number to the sum. Write this on the blackboard:

$$3 + (5 + 2)$$

Let the pupils tell you that this means $3 + 7$, or 10. Write this sentence on the blackboard and let the pupils decide whether it is true and *why*:

$$(3 + 5) + 2 = 3 + (5 + 2)$$

Repeat this activity with a set of three other numbers. Let the pupils put the addends in different groups of two numbers to add them. Then get the pupils to follow the same plan in multiplying the same set of three numbers. For example:

$$4, 6, 2: (4 + 6) + 2 = 10 + 2 = 12$$
$$4 + (6 + 2) = 4 + 8 = 12 \quad (4 + 6) + 2 = 4 + (6 + 2)$$
$$3, 4, 2: (3 \times 4) \times 2 = 12 \times 2 = 24$$
$$3 \times (4 \times 2) = 3 \times 8 = 24 \quad 3 \times (4 \times 2) = (3 \times 4) \times 2$$

Repeat for several other sets of three numbers. Use, for example, $2 + 5 + 7$; $2 \times 5 \times 7$; $4 + 6 + 5$; $4 \times 6 \times 5$; $8 + 2 + 4$; $8 \times 2 \times 4$.

Ask, "When we add three numbers, do we get the same sum no matter which two of the numbers we add first?" (Yes) "When we multiply three numbers, do we get the same product no matter which two of the numbers we multiply first?" (Yes) "We have found another property of addition and multiplication. The name we give to this property is the *associative property.*"

Let a pupil tell about this property of adding or multiplying three numbers. Do not require the pupils to use the word "associative".

Ask whether subtraction has the associative property. Check this by trying to subtract three numbers, for example, 8, 4, 2. Let the pupils subtract the first two, then the third:

$$(8 - 4) - 2$$

Then subtract the last two, and subtract this from the first:

$$8 - (4 - 2)$$

Ask these questions: "What is $(8 - 4) - 2$?" ($4 - 2$ or 2) "What is $8 - (4 - 2)$?" ($8 - 2$ or 6). "Do you get the same number when you subtract these three numbers in different groups of the two numbers?" (One way we get 2, but the other way we get 6.) "Does subtraction have the associative property?" (No)

Ask whether division has the associative property. Check with the three numbers 8, 4, 2. "What is $(8 \div 4) \div 2$?" (1) "What is $8 \div (4 \div 2)$?"

(4) "Does $(8 \div 4) \div 2$ equal $8 \div (4 \div 2)$?" (No) "Does division have the associative property?" (No)

ACTIVITY 3 • Which Operations Are Associative? Pupil page 78

Revise the associative properties of addition and multiplication. Let the pupils work several examples, as they did in Activity 2. Ask the questions below, one at a time, and let the pupils discover the answers by using several sets of three numbers. One example is shown for each and also the answer the pupils will give.

Does the "firsting" (F) operation have the associative property?

$(2 \text{ F } 3) \text{ F } 1 = 2 \text{ F } 1 = 2$	$2 \text{ F } (3 \text{ F } 1) = (2 \text{ F } 3) \text{ F } 1$
$2 \text{ F } (3 \text{ F } 1) = 2 \text{ F } 3 = 2$	F has the associative property.

Does the star (*) operation have the associative property?

$(2 * 3) * 1 = 4 * 1 = 2$	$2 * (3 * 1) \neq (2 * 3) * 1$
$2 * (3 * 1) = 2 * 2 = 3$	* does *not* have the associative property.

Does the G operation have the associative property?

$(2 \text{ G } 3) \text{ G } 1 = 3 \text{ G } 1 = 3$	$2 \text{ G } (3 \text{ G } 1) = (2 \text{ G } 3) \text{ G } 1$
$2 \text{ G } (3 \text{ G } 1) = 2 \text{ G } 3 = 3$	G has the associative property.

Does the △ operation have the associative property?

$(2 \triangle 3) \triangle 1 = 9 \triangle 1 = 10$	$(2 \triangle 3) \triangle 1 \neq 2 \triangle (3 \triangle 1)$
$2 \triangle (3 \triangle 1) = 2 \triangle 4 = 12$	△ does *not* have the associative property.

Does the ○ operation have the associative property?

$(2 \bigcirc 3) \bigcirc 1 = 7 \bigcirc 1 = 15$	$(2 \bigcirc 3) \bigcirc 1 \neq 2 \bigcirc (3 \bigcirc 1)$
$2 \bigcirc (3 \bigcirc 1) = 2 \bigcirc 7 = 11$	○ does *not* have the associative property.

Let the pupils turn to page 78 in their books. Ask them to tell which sentences they can solve without adding or multiplying the numbers. For example, a pupil may say, "I know that if n is 6, the first sentence is true. I can tell this without adding because the sum of 2 and 4 and 6 is the same no matter how I group the numbers." Or he may say, "n is 6 because we must add the three numbers 2, 4, 6, and addition has the associative property."

Let the pupils tell which of the other sentences can be solved without doing the operations. (Sentences 1, 3, 5, 6, 8, 10 and 12 can be completed without adding or multiplying because the same three numbers are added or multiplied and these operations have the associative property. Sentences 7 and 11 can be solved by using the commutative property.)

Let the pupils discuss Sentences 2, 4, 9 and 13. Get them to decide that these are not examples of the associative property. In Sentence 2, for example, if the pupils say, "x is 7", they find the sentence is false because subtraction does *not* have the associative property. (x is 5.) In Sentence 4, for example, they cannot say, "b is 16" because division does *not* have the associative property. (b is 4.) Sentences 9 and 13 cannot be solved by use of the commutative property. (z is 8; n is 16.)

ACTIVITY 4 • Properties of 0 and 1

Pupil pages 79-80

Revise these addition facts:

$0 + 1 = 1$; $0 + 2 = 2$; $0 + 3 = 3$; $0 + 4 = 4$; $0 + 5 = 5$; $0 + 6 = 6$; and so on.

Write these sentences on the blackboard and ask what number makes them true.

$0 + 732 = n$ $\quad$ $a + 350 = 350$

The pupils will say that n is 732 and a is 0. You and the pupils write and solve other sentences with 0 as an addend.

Ask, "When a number and 0 are added, what is the sum?" (The sum is the same as the number.) Ask what number makes this sentence true:

$$0 + x = x$$

Many numbers will be named. Let the class decide that the sentence is true for each of the numbers. Guide the pupils to say that $0 + x = x$ is true no matter what number x is. Say, "We have found a property of the number 0. When 0 is an addend, the sum is always the other addend."

Now revise these multiplication facts:

$1 \times 0 = 0$; $1 \times 1 = 1$; $1 \times 2 = 2$; $1 \times 3 = 3$; $1 \times 4 = 4$;
$1 \times 5 = 5$; $1 \times 6 = 6$; and so on.

Let the pupils write other multiplication sentences using the factor 1.

Ask the class to tell you a property of the number 1. If no one can do this, ask for the number that makes these sentences true:

$$1 \times 312 = n \qquad b \times 350 = 350$$

The pupils will say that n is 312 and b is 1. You and the pupils write and solve other sentences with 1 as a factor.

Ask, "When a number and 1 is multiplied what is the product?" (The product is the same as the number.) Ask, "What number makes this sentence true?"

$$1 \times y = y$$

The class will decide that the sentence is true no matter what number y is. Say, "We have found a property of the number 1. If 1 is a factor, the product is always the same as the other factor."

Let the pupils turn to page 79 and decide which sentences are true. Get the pupils to tell what properties are shown by the true sentences. (Sentences 1, 3, 6, 7, 8, 10, 12 are true, and the others are false. Sentence 1 shows the property of 0; Sentence 3 shows the associative property of multiplication; Sentence 6 shows the associative property of addition; Sentence 7 shows the commutative property of multiplication; Sentence 8 shows the property of 1; Sentence 10 shows the property of 0; Sentence 12 shows the property of 1.)

Let the pupils decide what numbers make the sentences true in the exercises on page 80. Let them tell what properties they use to find the numbers. For example, in Sentence 1 we use the commutative property of addition to find that n is 17. Each of the sentences, except 6, 9 and 11, can be solved directly by use of a property. Sentence 12 uses two properties.

STAGE 4 • The Distributive Property

Vocabulary

Distributive property

Materials

Array of dots (See Unit 3, pages 52-53); pupil pages 81-86

Teaching Procedure

ACTIVITY 1 • Adding and Multiplying

Pupil pages 81-82

In this activity the pupils will use addition and multiplication in the same exercise. They will use parentheses to group several numerals together to represent one number.

Write on the blackboard these two numerals:

$$(3 + 5) \times 4 \qquad 3 + (5 \times 4)$$

Ask the pupils to read each. (The first is read "The number 3 + 5 times 4"; the second is read "3 plus the number 5 × 4".) Ask what number is named by the first numeral. (8 × 4, or 32) Say, "32 is the simplest name for (3 + 5) × 4." What number is named by the second numeral? (3 + 20, or 23) Say, "23 is the simplest name for 3 + (5 × 4)."

This revises the meaning of the parentheses as signs of grouping. Remind the pupils that these two numerals written on the blackboard have the same symbols in them, but the numerals are not grouped the same way. Help the pupils understand the importance of the grouping.

Let the class turn to pages 81 and 82, copy the numerals and write a simpler name for each. This will give the pupils practice in using the grouping signs (parentheses).

ACTIVITY 2 • A Pattern in Multiplication and Addition

Pupil pages 83-84

See Unit 3, pages 77-78. Revise the activity in which a story problem is told about the boy Peter who collects money on two days for a boys' club. Let the pupils talk about the two ways in which he can count his total collection:

	(2×5)	$+$	(3×5)	$= \quad n$
	number of cents collected on Monday		number of cents collected on Tuesday	total number of cents collected
or	$(2 + 3)$	$\times$	5	$= \quad n$
	number of pupils who paid on Monday and Tuesday		number of cents collected from each pupil	total number of cents collected

Let the class decide again that these two sentences tell about the same problem, and the same number makes the sentences true.

Say, "We would like to find a property of multiplication and addition together. Let us solve more story problems and try to find such a property."

Let the pupils turn to page 83 and read the first word problem. Help them think about the problem and the two mathematical sentences that tell

about the problem. Let the pupils give the two sentences. Then write them on the blackboard as follows:

(3×6)	$+$	(3×4)	$=$	n
number of sweets in first box		number of sweets in second box		total number of sweets

3	$\times$	$(6 + 4)$	$=$	n
number of rows		number of sweets in each row		total number of sweets

As you point to each part of the sentences, let the pupils tell what it represents. Let the class decide that these two sentences tell the same thing about the problem. Finally, write this sentence:

$$(3 \times 6) + (3 \times 4) = 3 \times (6 + 4)$$

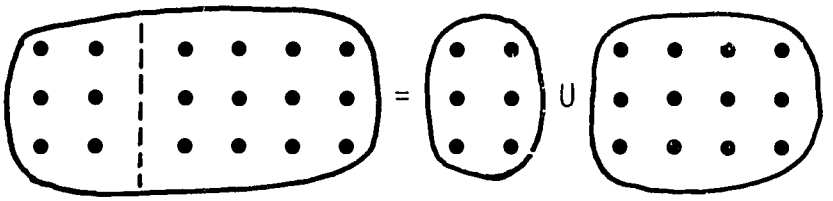
Let the pupils decide that it is a true sentence.

Remind the class that the sentence you wrote has both multiplication and addition in it, in a very special way. Let the class give sentences for the other three problems on page 83. Write them all on the blackboard:

1. $(3 \times 6) + (3 \times 4) = 3 \times (6 + 4)$
2. $(2 \times 5) + (2 \times 5) = 2 \times (5 + 5)$
3. $(5 \times 3) + (5 \times 4) = 5 \times (3 + 4)$
4. $(2 \times 4) + (2 \times 5) = 2 \times (4 + 5)$

In all these true sentences there is a pattern. Let the pupils discover the pattern if they can. (If a number is multiplied by the sum of two addends, the result is the same as multiplying the number by each of the two addends and adding the products.) If the pupils do not see the pattern or cannot explain the pattern, go on to a new way of showing it.

Draw a rectangular array of dots on the blackboard, such as the array shown below, and separate the array into two arrays, as shown:

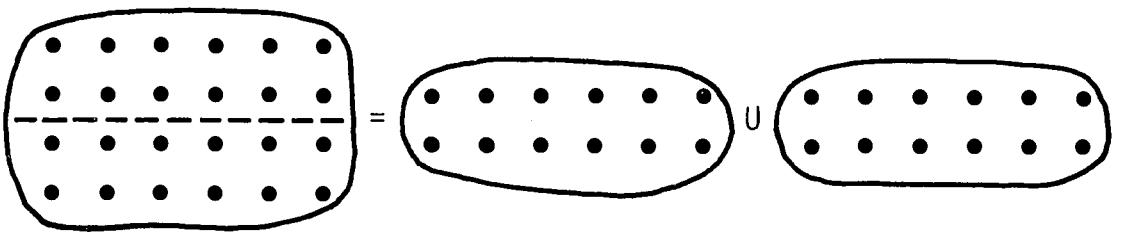


Ask the class for a sentence telling about these arrays. The array on the left is 3 by (2 + 4) and the arrays on the right are 3 by 2 and 3 by 4. They will write:

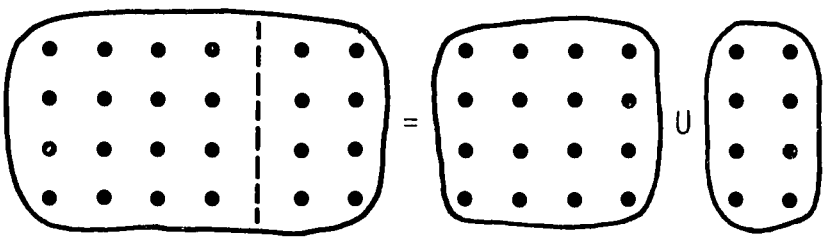
$$3 \times (2 + 4) = (3 \times 2) + (3 \times 4)$$

Let the pupils decide that this sentence is true and that it has the same pattern as the sentences for the story problems on page 83.

Continue with other arrays of dots and separate them in several ways. For example:



$$(2 + 2) \times 6 \quad = \quad (2 \times 6) \quad + \quad (2 \times 6)$$



$$4 \times (4 + 2) \quad = \quad (4 \times 4) \quad + \quad (4 \times 2)$$

Let the pupils fold their own arrays to show the sentences.

Ask the pupils to turn to page 84 and write sentences in their exercise books for each of the separated arrays of dots. The sentences they may write are:

1. $4 \times (3 + 2) = (4 \times 3) + (4 \times 2)$
2. $(3 \times 4) + (5 \times 4) = (3 + 5) \times 4$
3. $(2 + 6) \times 3 = (2 \times 3) + (6 \times 3)$
4. $(2 \times 3) + (2 \times 4) = 2 \times (3 + 4)$

Say, "Let us try to find a pattern in these sentences." Write the numeral "$3 \times (3 + 4)$" on the blackboard. Ask the pupils to show this product by folding their arrays. (They will show three rows of 3 + 4 dots.) Ask, "How many dots are shown in your arrays?" (21) Then let them fold this array to show two arrays: 3×3, and 3×4. Ask how many dots are shown in the two arrays. (9 + 12, or 21) Let a pupil write a sentence to show the relationship of the two sets of arrays he made.

$$3 \times (3 + 4) = (3 \times 3) + (3 \times 4)$$

Continue this activity with the following:

$$4 \times (2 + 3) = (4 \times 2) + (4 \times 3)$$
$$5 \times (1 + 4) = (5 \times 1) + (5 \times 4)$$
$$(4 + 2) \times 3 = (4 \times 3) + (4 \times 3)$$

ACTIVITY 3 • The Distributive Property

Pupil pages 85-86

Write this sentence on the blackboard:

$$3 \times (4 + 2) = (3 \times 4) + (3 \times 2)$$

Say to the class, "This sentence is true. There is a pattern in this sentence between multiplication and addition. We call this pattern the *distributive property.*" Point out that one part of the sentence is a product, $3 \times (4 + 2)$, and the other part is a sum, $(3 \times 4) + (3 \times 2)$. Say, "Let us write some other sentences that use this pattern."

Write the numeral "$4 \times (3 + 5)$" on the blackboard and ask for another numeral that names the same number. The reply to get among others is $(4 \times 3) + (4 \times 5)$. Let pupils show this answer by folding their array of dots.

Write the numeral $(3 \times 4) + (5 \times 4)$ on the blackboard and ask for another numeral that names the same number. Keep questioning the pupils until the numeral $(3 + 5) \times 4$ is given. Write the sentence showing the numbers are equal. Remind the class that they are giving examples of the distributive property. Let them write many sentences that show the use of the distributive property.

Ask the class to turn to pages 85 and 86. Let the pupils copy each exercise on page 85 and use the distributive property to complete the sentence with another numeral naming the same number. For example, Sentence 1 is completed to be

$$2 \times (3 + 5) = (2 \times 3) + (2 \times 5).$$

Sentence 2 is completed to be

$$(3 \times 4) + (2 \times 4) = (3 + 2) \times 4.$$

For page 86, let the pupils try to use the distributive property to find the number that makes the sentence true. For example, in Sentence 1, if t is 2, the sentence is true. In Sentence 2, if m is 4, the sentence is true. In each sentence, ask reasons for the answer given and let the pupils tell how they can find the missing number without adding or multiplying. In Sentences 4 and 10 the pupils will find that they cannot use the distributive property because addition is not distributive over multiplication. In Sentence 4, if they try 10 for b, the sentence will be false. (b is 0.) In Sentence 10, if they try 6 for v, the sentence is false. (v is 2.)

UNIT 7

Techniques of Addition and Subtraction

OBJECTIVES

1. To revise techniques of addition of whole numbers (sum less than 10,000)
2. To extend techniques of addition to addition of three addends
3. To revise techniques of subtraction of whole numbers (sum less than 10,000)
4. To achieve mastery of techniques of addition and subtraction of whole numbers
5. To develop the ability to express the mathematical relationships in a word problem as a mathematical sentence; to find a number that makes the mathematical sentence true; to answer the question asked in the problem

Background Information for Teachers

See Part 2, "Arithmetic", pages 22-25, 31, 32, 38, 45-50 in your *Teachers' Handbook.* See also "Note" about solving word problems, Unit 6, page 138 of this book.

STAGE 1 • Techniques of Addition

Vocabulary

No new vocabulary

Materials

Pupil pages 87-97

Teaching Procedure

ACTIVITY 1 • Revision of Techniques of Addition Pupil pages 87-89

Tell the pupils the following word problem:

> Mr. Mogwe has 62 cows. Mr. Mudenda has 35 cows. How many cows have they altogether?

Ask questions such as the following to help the pupils understand the problem:

> What does the problem tell us? (Mr. Mogwe has 62 cows, and Mr. Mudenda has 35 cows.)
>
> What question is asked in the problem? (How many cows have they altogether?)
>
> How can we find the number of cows they have altogether? (By adding the number of Mr. Mudenda's cows and the number of Mr. Mogwe's cows.)

Guide the pupils to suggest the mathematical sentence below. Let a pupil write it on the blackboard.

$$62 + 35 = c$$

Ask, "What is missing in this sentence, an addend, or the sum?" (Sum) "How can you find the sum?" (By adding 62 and 35) Say, "It will help us to add if we think of these numbers in expanded form." Ask the pupils to rename each number in expanded form. As they tell you the expanded forms, rewrite the mathematical sentence on the blackboard as follows:

$$60 + 2 + 30 + 5 = n$$

Ask the pupils if this mathematical sentence is the same as the first sentence they wrote. (Yes) Let them tell why. (The same numbers are in both mathematical sentences. The numbers just have different names.)

Note: Because of the commutative property for addition, the mathematical sentence may be written $60 + 30 + 2 + 5 = n$. In the following development, do not stress why the 60 and 30 may be grouped together and why the 2 and 5 may be grouped together.

Say, "In this mathematical sentence the expanded notation shows the number of ones and the number of tens. The 2 and 5 tell the number of ones. Can you rewrite the sentence so that the 2 and 5 are separated by a symbol for addition?" (Yes. Change the order of the addends 2 and 30.) Write the following:

$$60 + 30 + 2 + 5 = n$$

Guide the pupils to say, "We can add 2 and 5 and 60 and 30. Two ones plus 5 ones equals 7 ones." (Write "7" as shown below.) "60 plus 30 is the same as 6 tens plus 3 tens, or 9 tens." (Write "90" as shown below.)

$$90 + 7 = n$$

Let the pupils tell why this sentence is the same as the other sentences. (The same numbers are in the sentences. The numbers just have different names.) Ask a pupil to rename $90 + 7$ in its simplest form. Write

n is 97.

Say, "The sum of 62 and 35 is 97."

Write the vertical form for the process of addition as shown below. Say, "Let us show the addition of 62 and 35 another way."

$$\begin{array}{r} 62 \\ \underline{+35} \end{array}$$

Guide the pupils *to think* as described to the centre and right below. Write only what is shown on the left below. (Tell them to first add the number of ones and then the number of tens.)

62	2 ones + 5 ones = 7 ones. Write 7 as shown.
+35	
7	6 tens + 3 tens = 9 tens. Write 9 tens (90) as shown.
+90	
97	7 ones + 9 tens = 97. Write 97 as shown.

Say, "You found the sum, 97. Can you make your mathematical sentence true?" Guide a pupil to write the following sentence:

62 + 35 = 97

Say, "You added numbers to find out something about Mr. Mogwe's and Mr. Mudenda's cows. What question did the problem ask about the cows?" (How many cows do the men have altogether?) "Can you answer the question asked by the problem?" (They have 97 cows altogether.) Write the following incomplete sentence on the blackboard. Let a pupil complete it.

They have 97 cows altogether.

Note: If any of the pupils need more help in addition, let them show the exercise with bundles of sticks in a place-value box.

Write the following sentence on the blackboard:

46 + 12 = p

Use the vertical form for the process of addition. Let the pupils write the exercise as shown on the left below. Guide the pupils to think as described on the right.

46	6 ones + 2 ones = 8 ones. Write 8 as shown.
+12	
8	4 tens + 1 ten = 5 tens. Write 5 tens (50) as shown.
+50	
58	8 ones + 5 tens = 58. Write 58 as shown.

Now help the pupils to see that the form used above may be shortened in the following way. (Emphasize that the number of ones are added first, then the number of tens.)

$$\begin{array}{r} 46 \\ \underline{+12} \\ 58 \end{array}$$

Add the number of ones: 6 + 2 = 8. Write 8 in the ones' place.

Add the number of tens: 4 + 1 = 5. Write 5 in the tens' place.

The sum is 58. *p* is 58.

46 + 12 = 58

Get the pupils to talk about the two forms they have just used to show the addition of 46 and 12. Help them to decide that the last form saves time. Tell them to use it if they understand it.

Ask the pupils to open their books to page 87. Help them work one exercise on the blackboard. Then let them work the rest of the exercises in their exercise books. Give help as needed. As you help, get the pupils to tell you what they think as they add.

Pages 88 and 89 may be used as needed by the pupils now or later. Some will not need to work any of the exercises; some will need to work part of the exercises; and some will need to work all of the exercises.

ACTIVITY 2 • Addition of Three Numbers

Pupil pages 90-92

Tell the pupils this problem:

> John gave me 23 nuts, Joe gave me 34 nuts and Mary gave me 32 nuts. How many nuts did they all give me?

Ask questions like the following to help the pupils understand the problem:

> What does the problem tell us? (John gave me 23 nuts, Joe gave me 34 nuts and Mary gave me 32 nuts.)
>
> What question does the problem ask? (How many nuts did they all give me?)
>
> How do we find the number of nuts they gave me? (We add the numbers.)
>
> How do we show this in a mathematical sentence?

Guide the pupils to suggest the mathematical sentence below. Write the sentence on the blackboard.

$$23 + 34 + 32 = c$$

Say, "This problem is about three sets of nuts. We could join these three sets in many ways and the result would always be the same. We would always have the same number of nuts. How can we add the three numbers?" Guide the pupils to suggest, "We add two numbers at a time. We can add 23 and 34 first. Then to their sum we add 32." As they say this, write $(23 + 34) + 32 = c$. Guide them to say, "We could add 34 and 32 first and then add this sum to 23." As they say this, write $23 + (34 + 32) = c$. Ask, "Are you sure that each of these ways of grouping the numbers gives the same sum?" (Yes) Say, "We can group numbers in either of these ways because of the associative property." If any of the pupils are unsure, let them try both ways. (If needed, let the pupils form sets and join them to show the problem.)

Point to this sentence on the blackboard. (Erase the other sentence.) Say, "We will use this sentence."

$$(23 + 34) + 32 = c$$

Guide the pupils to add the first two addends as shown below to the left. The sum is 57. Then help them to add this sum to the third addend as shown below on the right.

$$\begin{array}{r} 23 \\ \underline{+34} \\ 57 \end{array} \qquad \begin{array}{r} 57 \\ \underline{+32} \\ 89 \end{array}$$

Ask, "What is the sum of the three numbers?" (89) Write this sentence:

c is 89.

Ask, "Could we put these steps together into one step? Let us try." Write the addends as shown:

$$\begin{array}{r} 23 \\ 34 \\ \underline{32} \end{array}$$

Note: The pupils may group the addends together in either an upward or downward direction. The sum is the same. Choose one way (downward is used in this book) and get the pupils to follow it. Later they will learn to group first in one direction (up or down) and then to group in the other direction to check their work.

Guide the pupils to first add the number of ones and then add the number of tens. They will write the exercise as shown on the left below. Their thinking will be somewhat as described on the right.

23	3 ones + 4 ones = 7 ones; 7 ones + 2 ones = 9 ones.
34	Write 9 in ones' place, as shown.
+32	
89	2 tens + 3 tens = 5 tens; 5 tens + 3 tens = 8 tens.
	Write 8 in tens' place, as shown.

Point to the sentence and ask, "Have we found the number that makes our addition sentence true?" (Yes, the number is 89.) Write

$$23 + 34 + 32 = c$$

c is 89.

Say, "Our problem was about nuts. What question did it ask?" (How many nuts did they all give me?) If the pupils are unable to tell the question, tell them the story again and let them tell the question. Then ask for an answer to the question. (They all together gave me 89 nuts.) Write the following incomplete sentence on the blackboard. Ask a pupil to complete the sentence.

They gave me 89 nuts.

Ask the pupils to open their books to page 90. With the help of the pupils, work several of the exercises on the blackboard. Guide the pupils as they work the other exercises in their exercise books. Make sure they understand the process of addition.

You and the pupils may use pages 91 and 92 as needed. You may wish to use them later as revision.

Write this sentence on the blackboard:

$35 + 27 = p$

Write the addends in vertical form, as shown on the left below. Guide the pupils to think and write as described on the right. Suggestions are given to you in parentheses.

$\begin{array}{r} 35 \\ \underline{+27} \\ 62 \end{array}$	5 ones + 7 ones = 12 ones. (Say, "Think of 12 ones in its expanded form as 1 ten + 2 ones. Remember the 1 ten. You will add it with the other tens.") Write 2 in the ones' place. 1 ten + 3 tens = 4 tens; 4 tens + 2 tens = 6 tens. Write 6 in the tens' place.

Let the pupils tell how they add 35 and 27. They will say something like this: "We add 5 ones and 7 ones. The sum is 12. We think of 12 ones as 1 ten and 2 ones. We remember 1 ten and add it to the other tens. We write 2 in the ones' place. We add 1 ten, 3 tens and 2 tens. The sum is 6 tens. We write 6 in the tens' place. The sum of 35 and 27 is 62." Point to the mathematical sentence $35 + 27 = p$ and ask, "Have we found the number that makes this sentence true?" (Yes, 62 is the number.) Write

$35 + 27 = p$
p is 62.

Tell the pupils this word problem:

> There are 513 boys and 429 girls in Chingwele School. How many pupils attend Chingwele School?

It is very important that the pupils learn to express the relationships of the numbers in a problem as a mathematical sentence. The following questions will help the pupils to think of a mathematical sentence. Answers the pupils may give are suggested.

What does the problem tell us? (There are 513 boys and 429 girls in Chingwele School.)

What question does the problem ask? (How many pupils attend Chingwele School?)

How can you find the number of pupils at Chingwele School? (By adding the number of girls and the number of boys. Add 513 and 429.)

What mathematical sentence shows this?

By questions of this type, guide the pupils to suggest the sentence below. Write the sentence on the blackboard.

$$513 + 429 = p$$

Ask, "How do you find p?" (By adding the numbers) Write the addends in vertical form, as shown below at the left. Guide the pupils to think about the addition in the way shown at the right below. The writing on the blackboard will be like that at the left below.

$$\begin{array}{r} 513 \\ +429 \\ \hline 942 \end{array}$$

First add the number of ones. 3 + 9 = 12. Rename 12 ones as 1 ten and 2 ones. Remember 1 ten and write 2 in the ones' place.

Add the number of tens. 1 + 1 = 2; 2 + 2 = 4. Write 4 in the tens' place.

Add the number of hundreds. 5 + 4 = 9. Write 9 in the hundreds' place.

Point to the mathematical sentence $513 + 429 = p$ and ask, "Do you know the number that makes this sentence true?" (Yes, p is 942.) Write

$$513 + 429 = p$$
$$p \text{ is } 942.$$

"We made this sentence for a problem about Chingwele School. What question did the problem ask?" (How many pupils attend Chingwele School?) Ask, "Can you answer the question asked by the problem?" (942 pupils

attend Chingwele School.) Write the following incomplete sentence on the blackboard. Ask a pupil to complete it.

942 pupils attend Chingwele School.

Get the pupils to open their books to page 93. Help them to work one or more of the exercises on the blackboard. Then guide them as they work the rest of the exercises in their exercise books.

ACTIVITY 4 • Revision of Techniques

Pupil page 94

Write the following sentence on the blackboard:

$2771 + 1649 = a$

The pupils will suggest adding the numbers and writing the addends in vertical form. Guide the pupils to think and write in this way:

$$\begin{array}{r} 2771 \\ +1649 \\ \hline 4420 \end{array}$$

First add the number of ones. 1 + 9 = 10. Rename 10 ones as 1 ten. Remember 1 ten and write 0 in the ones' place.

Add the number of tens. 1 + 7 = 8; 8 + 4 = 12. Rename 12 tens as 1 hundred and 2 tens. Remember 1 hundred and write 2 in the tens' place.

Add the number of hundreds. 1 + 7 = 8; 8 + 6 = 14. Rename 14 hundreds as 1 thousand and 4 hundreds. Remember 1 thousand and write 4 in the hundreds' place.

Add the thousands. 1 + 2 = 3; 3 + 1 = 4. Write 4 in the thousands' place.

Point to the mathematical sentence $2771 + 1649 = a$ and ask, "Do you know the number that makes the mathematical sentence true?" (Yes, the number 4420 makes the sentence true.) Write

$2771 + 1649 = a$

a is 4420.

Tell the pupils to open their books to page 94. Help them to work one or more exercises. Ask the pupils to work the rest of the exercises in their exercise books. Go around the class giving help as needed. Ask the pupils to tell you what they think as they add.

ACTIVITY 5 • Addition of Three Numbers

Pupil pages 95-96

Tell the pupils the following word problem:

> From January to April Mr. Phiri, a salesman, travelled 1952 miles. From May to August he travelled 2763 miles, and from August to December he travelled 3124 miles. How far did Mr. Phiri travel in the year?

Ask the pupils to think of a mathematical sentence for the problem. Ask questions to help them think and write this sentence:

$$1952 + 2763 + 3124 = n$$

Ask, "How can you find n?" (By adding the numbers) Let a pupil write the addends as shown on the left below. Help the pupils to add the numbers. Guide them to think about addition as shown on the right.

$\begin{array}{r} 1952 \\ 2763 \\ +3124 \\ \hline 7839 \end{array}$	Add the number of ones. $2 + 3 = 5$; $5 + 4 = 9$. Write 9 in the ones' place. Add the number of tens. $5 + 6 = 11$; $11 + 2 = 13$. Rename 13 tens as 1 hundred and 3 tens. Remember 1 hundred and write 3 in the tens' place. Add the number of hundreds. $1 + 9 = 10$; $10 + 7 = 17$; $17 + 1 = 18$. Rename 18 hundreds as 1 thousand and 8 hundreds. Remember 1 thousand and write 8 in the hundreds' place. Add the number of thousands. $1 + 1 = 2$; $2 + 2 = 4$; $4 + 3 = 7$. Write 7 in the thousands' place.

Point to the mathematical sentence $1952 + 2763 + 3124 = n$ and ask, "Do you know the number that makes the sentence true?" (Yes, the number is 7839.) Write

$1952 + 2763 + 3124 = n$

n is 7839.

Say, "This sentence came from a problem about Mr. Phiri and how far he travelled. What did the problem ask?" (How far did Mr. Phiri travel in the year?) "Can you answer the question?" (Mr. Phiri travelled 7839 miles in the year.) Write the following incomplete sentence on the blackboard. Ask a pupil to complete it.

Mr. Phiri travelled 7839 miles in the year.

Tell the pupils to open their books to page 95. On the blackboard work one or more exercises with the class. Tell the pupils to work the other exercises in their exercise books. Go around the classroom. Guide the pupils as they write mathematical sentences. Listen to them tell what they think as they add. This will help you help them develop good habits of work.

Page 96 may be used by those pupils who need more exercises. It may be used later as revision.

ACTIVITY 6 · Using the Symbols < and >

Pupil page 97

Tell the pupils the following word problem:

> A school ordered 2 gross of exercise books (1 gross = 144). The school received 4 boxes, with 60 exercise books in each box. Did the school receive all the exercise books that were ordered?

Let the pupils talk about the problem. Guide them to say, among other things:

> We need to know how many exercise books were ordered.
> We need to know how many exercise books were received.
> We need to decide if as many books were received as were ordered.

Help the pupils to think in a way similar to this:

> The school ordered some books (144 + 144).
> The school received some of the books ordered (60 + 60 + 60 + 60).

We need to find whether the number of books ordered is the same as the number of books received.
We can write a mathematical sentence that is incomplete:

$$144 + 144 _ _ _ 60 + 60 + 60 + 60$$

We must decide which of the following goes in the sentence to make it true: =, >, <.

Guide them to decide that

$$288 > 240,$$
$$\text{so } 144 + 144 > 60 + 60 + 60 + 60$$

and all the exercise books were not received.

Write this incomplete mathematical sentence on the blackboard:

$$37 + 73 _ _ _ 100$$

Ask, "What symbol, =, > or < can you put in this sentence to make it true?" Guide the pupils to say, "We must add 37 and 73." Let them add the numbers decide that 110 is greater than 100 and write

$$110 > 100.$$

Ask the pupils to open their books to page 97. With pupils' help, work one or more of the exercises on the blackboard. Let the pupils work the other exercises in their exercise books.

STAGE 2 • Techniques of Subtraction

Vocabulary

Difference, dozen

Materials

Pupil pages 98-103

Teaching Procedure

ACTIVITY 1 • Revision of Subtraction

Pupil page 98

Tell the pupils the following word problem:

> 289 pupils attend Chisekesi School. 136 of them are girls. How many boys attend Chisekesi School?

Let the pupils talk about the problem. Guide them to express the relationship of the numbers in the problem as a mathematical sentence. The following questions should help them:

> What is the problem about? (There are 289 pupils in Chisekesi School, and 136 of these pupils are girls.)
>
> What does the problem ask? (How many boys attend Chisekesi School?)

Guide the pupils to suggest *either* of the sentences below. When one is given, help them write the other and to say, "The sentences mean the same thing."

$$136 + c = 289$$
$$289 - 136 = c$$

Let the pupils talk about the sentences they made and say, "c is a missing addend." Get them to tell how they can find the missing addend. Write the known addend and the sum as shown below and say, "Let us find the missing addend this way."

$$\begin{array}{r} 136 \\ + \\ \hline 289 \end{array}$$

Ask, "6 ones plus how many ones equals 9 ones?" (3 ones) Write 3 in ones' place. Ask, "3 tens plus how many tens equals 8 tens?" (5 tens) Write 5 in tens' place. Ask, "One hundred plus how many hundreds equals 2 hundreds?" (1 hundred) Write 1 in hundreds' place. The blackboard will look like this:

$$\begin{array}{r} 136 \\ +153 \\ \hline 289 \end{array}$$

Say, "You are subtracting. We will write the numerals in subtraction form and subtract." Guide the pupils to find the missing addend again as they use subtraction language. (Write only what is shown on the left below.)

$$\begin{array}{r} 289 \\ -136 \\ \hline 153 \end{array}$$

First subtract the number of ones. 9 − 6 = 3. Write 3 in ones' place.

Subtract the number of tens. 8 − 3 = 5. Write 5 in the tens' place.

Subtract the number of hundreds. 2 − 1 = 1. Write 1 in hundreds' place.

Let the pupils say that 153 will make their sentences true. Write:

136 + 153 = 289
and 289 − 136 = 153

Guide the pupils to see that if one of the sentences is true, the other sentence is true. Let the pupils add 136 and 153 to make sure the sentence is true.

Say, "We are solving a word problem." Ask, "What question did the problem ask?" (How many boys attend Chisekesi School?) Ask, "Can you answer the question?" (153 boys attend Chisekesi School.) On the blackboard, write the following incomplete sentence. Choose a pupil to help you complete it as shown.

153 boys attend Chisekesi School.

Note: If the pupils need more help in understanding subtraction, let them show the exercises with bundles of sticks in a place-value box.

Note: In some problems that require subtraction, the answer may be called the *difference*. For example, in this problem the difference between 289 pupils and 136 pupils is 153.

Ask the pupils to open their books to page 98. With the pupils' help, work one or more exercises on the blackboard. Then let the pupils work the other exercises in their exercise books. Go around the classroom, giving help as needed.

ACTIVITY 2 • Further Study of Subtraction

Pupil page 99

Tell the pupils the following word problem:

> Mr. Ndzamela had 83 bags of maize. He sold 67 bags. How many bags were not sold?

Ask questions to help the pupils express the relationship of the numbers in the problem as a mathematical sentence. The pupils may suggest either of the following sentences:

$$67 + y = 83$$
$$\textit{or}\ 83 - 67 = y$$

Guide the pupils to say, "y is the missing addend. We subtract numbers to find the missing addend. We subtract the known addend, 67, from the sum, 83." Write the subtraction exercise as shown below:

$$\begin{array}{r} 83 \\ \underline{-67} \end{array}$$

Tell the pupils that the subtraction process depends upon renaming the sum and the known addend. Ask them to give different names for the number 83. They may mention

8 tens and 3 ones	(80 + 3)
7 tens and 13 ones	(70 + 13).

On the blackboard, write the forms shown below:

Subtract	Subtract
8 tens and 3 ones	7 tens and 13 ones
6 tens and 7 ones	6 tens and 7 ones

Ask, "Which of the above forms makes it easier for you to subtract 67 from 83? Which name shall we use for 83?" (7 tens and 13 ones) "Why is it easier to subtract when 83 is renamed as 7 tens and 13 ones?" (We need more than 3 ones to subtract 7 ones from 83.)

Guide the pupils to think and write in the following way:

$$\begin{array}{r} 83 \\ -67 \\ \hline 16 \end{array}$$

First subtract the number of ones. There are not enough ones to subtract 7 ones from 83 when we think of 83 as 8 tens and 3 ones. Rename 83 as 7 tens and 13 ones. 13 − 7 = 6. Write 6 in ones' place.

Subtract the number of tens. 7 − 6 = 1. Write 1 in tens' place.

Point to the sentence 83 − 67 = y. Ask, "Do you know the number that makes this sentence true?" (Yes, 16) Write the following:

67 + 16 = 83
and 83 − 67 = 16

Ask the pupils to add 67 and 16 to make sure the addition sentence is true.

Get the pupils to think again of the word problem. Ask, "What question did the problem ask?" (How many bags of maize were not sold?) Say, "Answer the question asked by the problem." (16 bags of maize were not sold.) Write the following incomplete sentence. Get a pupil to complete it.

16 bags were not sold.

Note: If the pupils need more help in understanding subtraction, let them show the exercises with bundles of sticks in a place-value box.

Ask this question: "What is the *difference* between 526 shillings and 351 shillings?" Get the pupils to write a mathematical sentence for the question. They may suggest either of the following sentences:

$$351 + x = 526$$
$$\textit{or}\ 526 - 351 = x$$

Let the pupils talk about how they can make the sentence true. Guide them to decide to subtract 526 and 351. Write the numerals as shown below:

$$\begin{array}{r} 526 \\ \underline{-351} \end{array}$$

Say, "We first subtract the number of ones, then the number of tens and then the number of hundreds. The subtraction process depends upon renaming the sum." Guide the pupils to study the exercise and to choose the most convenient name for the sum. Ask, "Do you need more ones in the name of the sum?" (No. There are enough ones. 6 ones − 1 one = 5 ones.) "Do you need more tens in the name of the sum?" (Yes. To subtract 5 tens from 526, we need at least 5 tens in the name of 526.) "How can you rename 526 to name more tens?" (4 hundreds, 12 tens and 6 ones)

Write these forms on the blackboard:

Subtract	Subtract
5 hundreds 2 tens 6 ones	5 hundreds 1 ten 16 ones
3 hundreds 5 tens 1 one	3 hundreds 5 tens 1 one

Subtract

4 hundreds 12 tens 6 ones

3 hundreds 5 tens 1 one

Let the pupils examine each of the subtraction exercises and decide that each shows 526 − 351. Get the pupils to try to subtract, using the forms as given. Ask, "Which of these forms makes it easier to subtract?" (The last one) "Which name for 526 is the best one for you to use to subtract?" (4 hundreds, 12 tens and 6 ones)

Help the pupils to think and write as shown below:

$$\begin{array}{r} 526 \\ -351 \\ \hline 175 \end{array}$$

First subtract the number of ones. 6 − 1 = 5. Write 5 in ones' place.

Subtract tens. There are not enough tens to subtract 5 tens from 526 when we think of 526 as 5 hundreds, 2 tens and 6 ones. Rename 526 as 4 hundreds, 12 tens, 6 ones. 12 − 5 = 7. Write 7 in tens' place.

Subtract hundreds. 4 − 3 = 1. Write 1 in hundreds' place.

Guide the pupils to say, "175 is the missing addend." Point to the sentence $351 + x = 526$ and ask, "Have you found the number that will make the sentence true?" (Yes, 175) Write the following:

$$526 - 351 = 175$$

Let the pupils add 351 and 175 to make sure the addition sentence is true.

Ask the pupils to open their books to page 99. You and the pupils work one or more of the exercises on the blackboard. Get the pupils to work the other exercises in their exercise books. Give help as needed. Get the pupils to tell you how they name the sum to subtract. Help them to choose the best name.

ACTIVITY 3 • Using Subtraction

Pupil page 100

Tell the pupils the following word problem:

> Mr. Mulenga had 636 tins in his shop on Monday morning. When he closed his shop on Saturday, he had 379 tins left. He had sold the rest. How many tins had he sold?

Guide the pupils to suggest either of the following mathematical sentences:

$$379 + s = 636$$
$$or\ 636 - 379 = s$$

Write the exercise in vertical form. Guide the pupils to think about the exercise as shown at the right below.

$$\begin{array}{r} 636 \\ -379 \\ \hline 257 \end{array}$$

Subtract the number of ones. There are not enough ones to subtract 9 ones from 636 when we think of 636 as 6 hundreds, 3 tens and 6 ones. Rename 636 as 6 hundreds, 2 tens and 16 ones. 16 − 9 = 7. Write 7 in the ones' place.

Subtract the number of tens. There are not enough tens to subtract 7 tens when we think of 636 as 6 hundreds, 2 tens and 16 ones. Rename 636 as 5 hundreds, 12 tens and 16 ones. 12 − 7 = 5. Write 5 in the tens' place.

Subtract the number of hundreds. 5 − 3 = 2. Write 2 in the hundreds' place.

Point to the mathematical sentence $379 + s = 636$. Get the pupils to say, "257 is the missing addend. 257 makes the sentence true." Write the following:

$$379 + 257 = 636$$
$$\textit{and } 636 - 379 = 257$$

Let the pupils check their work by adding 379 and 257.

Tell the pupils the word problem again about the tins Mr. Mulenga sold. Ask, "What question did the problem ask?" (How many tins had Mr. Mulenga sold?) Say, "Answer the question asked by the problem." (He had sold 257 tins.) Write the following incomplete sentence on the blackboard and get a pupil to complete it:

He had sold <u>257</u> tins.

Write the following on the blackboard:

$$900 - 532 = n$$

Write the exercise in vertical form and guide the pupils to think and write as shown.

$$\begin{array}{r} 900 \\ -532 \\ \hline 368 \end{array}$$

First subtract the number of ones. There are not enough ones to subtract 2 ones from 900 when we think of 900 as 9 hundreds. Rename 900 as 90 tens

and then as 89 tens and 10 ones. 10 − 2 = 8. Write 8 in ones' place.

Subtract the number of tens. 9 − 3 = 6. Write 6 in tens' place.

Subtract the number of hundreds. 8 − 5 = 3. Write 3 in hundreds' place.

Point to the sentence 900 − 532 = n and let the pupils say, "368 makes the sentence true."

Go over the subtraction of this exercise several times with the pupils. Help them to understand the renaming of 900 as 90 tens and 0 ones and then as 89 tens and 10 ones. Write the following on the blackboard:

$$1009 - 781 = n$$

Write the exercise in vertical form and guide the pupils to think as shown below on the right:

$$\begin{array}{r} 1009 \\ -\ 781 \\ \hline 228 \end{array}$$

First subtract the number of ones. 9 − 1 = 8. Write 8 in ones' place.

Subtract the tens. There are not enough tens to subtract 8 tens from 1009 when we think of 1009 as 1 thousand and 9. Rename 1009 as 9 hundred, 10 tens and 9. 10 − 8 = 2. Write 2 in tens' place.

Subtract the hundreds. 9 − 7 = 2. Write 2 in hundreds' place.

Point to the sentence 1009 − 781 = n and ask whether the pupils can now make the sentence true. Get them to write:

$$1009 - 781 = 228$$

Ask the pupils to open their books to page 100. With the pupils' help, work one or more of the exercises on the blackboard. Then get the pupils to work the other exercises in their exercise books. Give help as needed. Go around the class and get the pupils to tell you what they think as they subtract.

ACTIVITY 4 • Using <and>

Pupil page 101

Note: If the pupils are not familiar with the word *dozen*, teach the meaning of the word before beginning this activity.

Tell the pupils the following word problem:

> Mr. Mwenda had 90 mangoes. He sold 3 dozen of them. Did he have enough mangoes left to sell 4 dozen more?

Guide the pupils to think:

> Mr. Mwenda had 90 mangoes. He sold 3 dozen. Did he have enough left to sell 4 dozen more? He sold 3 dozen, or 12 + 12 + 12 = 36.
>
> 90 − 36 _ _ _ 48 (4 dozen is 48.)
>
> Subtract 36 from 90 (90 − 36 = 54.)
>
> Is 54 less than, greater than, or equal to, 48?
>
> 54 > 48

Get the pupils to answer the question asked in the problem. (Mr. Mwenda has enough mangoes left to sell 4 dozen more mangoes.)

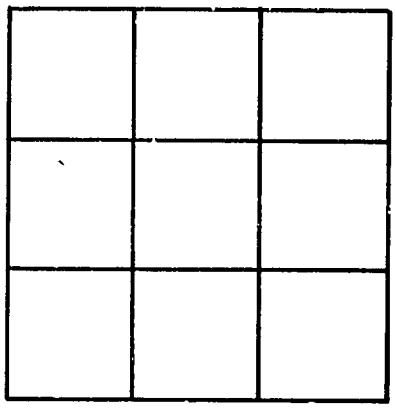
Get the pupils to open their books to page 101. Let them select one exercise and with their help work the exercise on the blackboard. Get the pupils to work the other exercises in their exercise books. Go around the classroom. Observe the pupils and listen to them tell how they subtract. Give help as needed.

ACTIVITY 5 • Magic Squares (Optional)

Pupil page 102

Magic squares are used in this activity to give the pupils practice in adding and subtracting numbers.

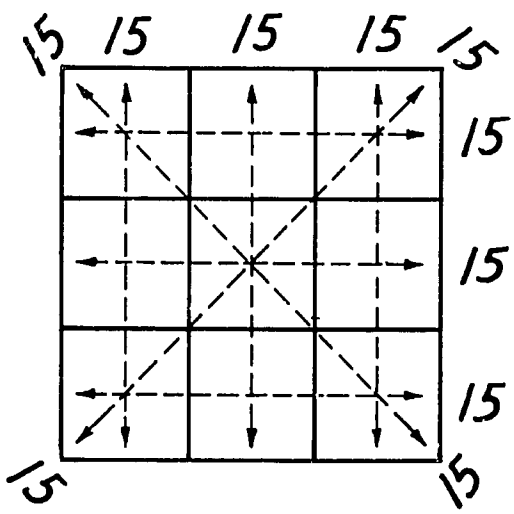
Draw the frame for a 3 by 3 magic square on the blackboard as shown below:



Sum: 15

Say, "This is a magic square. It is not complete. We will finish it in this way." Then give the directions below:

1. Each of the nine numerals 1-9 goes in one of the nine regions of the frame. We will find where they go and write them there. When a numeral is written in one region, we may not write it in any other region.
2. We will write the numerals in such a way that the sum of the three numbers represented by the numerals in any one direction is the same. The three numerals will be in rows, in columns or on the diagonal.
3. In this magic square the sum of the three numbers is 15. The sum of the numbers represented by the numerals in each row is 15. The sum of the numbers represented by the numerals in each column is 15. The sum of the numbers represented by the numerals on each diagonal is 15.



Sum: 15

Say as you write in the numerals shown below, "I will write some of the numerals in the magic square. You find the correct numerals for the other regions."

	3		
9		1	15
	7		
	15		

Sum: 15

Say, "Let us first find the numeral to write in the centre region. What numeral goes in the centre of the frame? How can we find it? Look at the centre row. What addition sentence can you make?

$$9 + n + 1 = 15$$

Get the pupils to look at the centre column, and write

$$3 + n + 7 = 15$$

Ask, "What number will make each of these sentences true?" (5) If pupils have difficulty in discovering the number 5 in the first sentence, guide their thinking in this way:

$$9 + n + 1 = 15$$
$$\textit{or}\ 9 + 1 + n = 15$$

Add 9 and 1:

$$10 + n = 15$$

Write this subtraction sentence:

$n = 15 - 10$

Subtract 10 from 15:

n is 5.

The pupils should think in a similar way about the second sentence:

$3 + n + 7 = 15$
$3 + 7 + n = 15$

Add 3 and 7:

$10 + n = 15$

Write this subtraction sentence:

$n = 15 - 10$

Subtract 10 from 15:

n is 5.

After the pupils understand that the numeral 5 goes in the centre region, they will need to experiment with the other rows and columns. Let them try numerals in regions and decide whether the numerals are in the correct regions. If they follow the rules of the game, they will soon find where to write the other numerals. The answers are shown below but let the pupils discover the answers for themselves.

4	3	8
9	5	1
2	7	6

Ask pupils in turn to write on the blackboard true addition sentences that the magic square shows.

The following sentences are true. (The sentences may be written in other forms.)

Rows	Columns	Diagonals
$(4 + 3) + 8 = 15$	$(4 + 9) + 2 = 15$	$(4 + 5) + 6 = 15$
$(9 + 5) + 1 = 15$	$(3 + 5) + 7 = 15$	$(2 + 5) + 8 = 15$
$(2 + 7) + 6 = 15$	$(8 + 1) + 6 = 15$	

Encourage the pupils to use parentheses to show which two numbers they add first (associative property).

If the pupils enjoy this activity, give them other magic squares to solve. In the one that appears below, the rules are the same as those stated for the magic square above except for these: the sum is 45; and any numerals the pupils know may be used in the magic square.

18		**6**
	15	
24		**12**

Sum: 45

Say to the pupils, "Which region will you fill first?" A pupil may suggest finding the missing numeral in the top row. If so, he should write the sentence and proceed as shown below:

$18 + a + 6 = 45$

$18 + 6 + a = 45$

Add 18 and 6.

$24 + a = 45$

Write this as a subtraction sentence:

$$a = 45 - 24$$

Subtract 24 from 45.

a is 21.

The answers are shown below. Let the pupils discover the answers for themselves.

18	21	6
3	15	27
24	9	12

When pupils understand how magic squares are made, let them open their books to page 102 and fill the regions with the correct numerals. Let them copy the magic squares in their exercise books.

ACTIVITY 6 • Operation Machines

Pupil page 103

Let the pupils open their books to page 103 and discuss the operation machines on the page. They will notice that a part of each exercise is missing. Let them find the missing part.

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VOLUME 2

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PREFACE

The purposes of the Entebbe Mathematics programme for *Primary Three* are:

1. To help children learn the best modern mathematics they can learn
2. To help them discover the ideas of mathematics for themselves

We want children *to learn* good mathematics and *to think* about mathematics. These two purposes are not attained separately but at the same time.

This book was written to help you help your pupils learn and think about mathematics. The book will help you in these ways:

It tells you about the ideas of mathematics that you are teaching; it refers you to sections in your *Teachers' Handbook* where you can read about the ideas.

It tells you how to teach each of the units.

It suggests things you can use in your classroom to help you teach.

It encourages you to let the children think for themselves. (Ask the question "Why?" often and let the children ask questions.)

Testing the Programme

This book and the *Pupil Book* were written to be tested by you and by your pupils. You and other teachers who use the books will be asked to write your opinions of the units you teach. The books will then be revised with the help of your comments. You are asked to do these things:

Teach the material of a unit as best you can, in the way suggested.

Think about the teaching and learning that you and your pupils have done.

Fill out a Unit Evaluation Form for each unit. (See the form at the end of this section.) Give the form to your supervisor.

How to Use the Book

The *first* thing you do before teaching from *Primary Three Entebbe Mathematics* is to *read all the books* carefully (this book, the *Pupil Book* and your *Teachers' Handbook*). They will help you

- ▶Know the whole programme for the year
- ▶Know what things you need to help you teach
- ▶Know what parts of the programme you wish to discuss with other teachers and supervisors
- ▶Know what parts of the programme you want to study about further in your *Teachers' Handbook* or in other books
- ▶Know how this book and the *Pupil Book* fit together
- ▶Know how this book, the *Pupil Book* and your *Teachers' Handbook* are planned to help you

The *second* thing you do is to *study the unit* you are to teach and *make plans* for teaching the unit. This study helps you

- ▶Plan in detail your teaching from day to day
- ▶Know what things to collect to help you teach the unit
- ▶Know what further reading and study you and your pupils should do to teach and learn the ideas in the unit

The *third* thing you do is *teach the unit*. Keep reading this book and your *Teachers' Handbook*. They will help you teach. Remember you teach best when children think things out for themselves. You question and guide. Tell as little as possible.

The *fourth* thing you do is to *think* about the unit and its use by you and the children. Think about how the unit could be improved for you and the pupils.

The *fifth* thing you do is to thoughtfully fill out a Unit Evaluation Form.

The Plan of the Book

This book explains in detail how to teach the fourteen units of the Primary Three programme. A unit is separated into "stages". Each of the stages is organized into several "activities". These activities are the *suggestions for teaching*.

An activity may be a single lesson; two or three activities may become a single lesson; or, you may continue or revise one activity several days. You must decide about how long your pupils need to spend with an activity. Study your children. Listen to them and watch them and decide when they can move to a new activity or stage.

Each unit uses ideas learned in units that come before it. For that reason, it is important that no mathematics be omitted. Some children may need only some of the activities in the book to help them learn an idea. If so, other activities may be omitted.

Helping Children Learn

Children learn best from things they do and by thinking about what they do. You can help by finding worthwhile things for them to do and to think about and by not telling them what to think.

This method means that you too must be doing and thinking as you help the pupils. These are some of the ways you can help children to think about and discover ideas:

Choose good problems and ask pupils to find answers.
Ask questions as pupils work.
Make suggestions as needed.
Suggest the use of things as models for the problem.
Supply a word or words now and then, when a pupil has a good idea he cannot express.
Suggest different methods of work.
Give encouragement at all times.
Help pupils summarize and say what their answers mean.

Outcomes

As you help pupils in mathematics to find their own answers and to think for themselves, you are teaching them mathematics and you are teaching them a way to think about mathematics. You are also helping them to learn a way to solve any problem.

This searching attitude leads to satisfaction and enjoyment for you and your pupils. It is the hope of the writers of the programme that it is not only enjoyed but that much mathematics is learned. Good luck.

Units of Primary Three Mathematics

1. Sets and Whole Numbers
2. Operations on Sets and Whole Numbers
3. Multiplication and Division
4. Geometry I
5. Fractions as Numbers
6. Properties of Operations
7. Techniques of Addition and Subtraction
8. Measurement
9. Techniques of Multiplication and Division I
10. Operations on Fractions I
11. Geometry II
12. Techniques of Multiplication and Division II
13. Operations on Fractions II
14. Fun with Mathematics

PRIMARY UNIT REPORT

Date:_____

Number of Unit_____ in Primary Book_____ Volume_____

Name of Unit:_____ _____

Fill out one of these reports for each unit as you complete it. Send it by air mail to Educational Services Incorporated, 108 Water Street, Watertown, Massachusetts, 02172, U.S.A. Your comments are very much needed. They will be used for the revision of the books.

PLEASE ANSWER ALL THE QUESTIONS IN THIS REPORT

Your Name:_____

Name and Address of Your School:_____

_____

Date you started teaching the Unit:_____

Date you completed the Unit:_____

Number of periods used in teaching the Unit:_____

Length of daily lesson period:_____ minutes.

TEACHERS' GUIDE

Answer each of the following questions and *give reasons* for your answers.

Questions	Answers and Reasons for Answers
1. Were the teaching suggestions clear?	
2. Were the teaching suggestions sufficient?	
3. Were the instructions in the Guide helpful in developing the ideas?	
4. Were the suggestions for materials to be used in the classroom sufficient?	
5. Was the background information for teachers in this unit clear, helpful and sufficient?	

What suggestions can you give for the improvement of the *Teachers' Guide* for this Unit?

_____

_____

_____

_____

_____

_____

PUPIL BOOK

Answer each of the following questions and *give reasons* for your answers.

Question	Answers and Reasons for Answers
1. Did your pupils learn the ideas the unit is meant to develop?	
2. Were the pupil pages helpful in developing the ideas of the unit?	
3. Were the pupil pages for this unit sufficient?	
4. Did you use materials other than those suggested in the book?	
5. Were the illustrations in the pupil book suitable?	
6. Did the pupils enjoy this unit?	
7. Did you enjoy teaching this unit?	

What suggestions can you give for the improvement of this Unit?

_____

_____

_____

GENERAL REMARKS and ASSESSMENT of

THE UNIT

To the best of my knowledge, this Unit is _____ (excellent, good, fair, poor) because:

(1) _____

_____

_____

(2) _____

_____

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(3) _____

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(4) _____

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(5) _____

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(6) _____

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Signature

_____
Date

CONTENTS

		PAGE
UNIT 8	• Measurement	190
UNIT 9	• Techniques of Multiplication and Division I	222
UNIT 10	• Operations on Fractions	257
UNIT 11	• Geometry II	286
UNIT 12	• Techniques of Multiplication and Division II	305
UNIT 13	• Operations on Fractions	324
UNIT 14	• Fun with Mathematics	344

UNIT 8

Measurement

OBJECTIVES

1. To revise the ideas of the measure of length, and to extend the ideas to include measurement in feet and inches
2. To encourage the children to estimate when measuring
3. To develop the idea that measures of length can be added
4. To extend the idea of length to include the perimeter of a figure
5. To extend and refine the techniques of determining the size of regions of rectangles by tiling
6. To revise telling time to the quarter hour, and to introduce telling time to the minute
7. To introduce the idea of capacity, the pint, the quart and the gallon

Background Information for Teachers

See *Teachers' Handbook*, pages 86-91; Teachers' Guide, *Primary One*, Unit 17; Teachers' Guide, *Primary Two*, Unit 7 and Unit 13.

We make comparisons when we measure. For example, when we measure the length of the classroom in feet, we compare this length to a standard unit, the foot. Let us say the length of the room is 20 feet. This measurement consists of a *number* and a *unit*. The number is 20 and the unit is the *foot*. We can use another *unit of length* to make the same measurement. If we use the *inch* as the unit, the length of the room will be approximately 240 inches. Again the measurement consists of a number, 240, and a unit, the inch. There are many different standard units of length in the English System (inch, foot, yard, and so on).

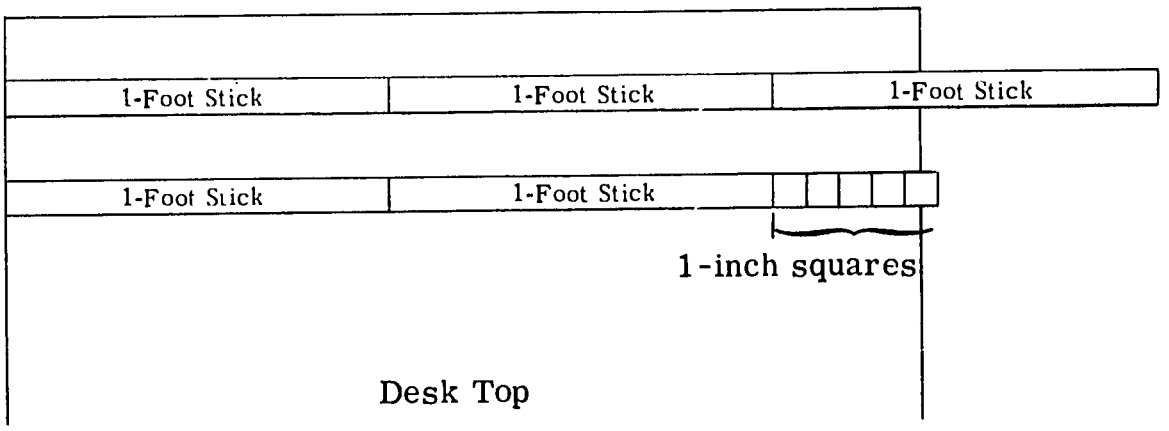
We use a number and a unit when we make measurements. When measuring time, for example, we may say that 15 *minutes* have passed. We use a number, 15, and a unit, the minute, in this measurement. We say that a bag of groundnuts weighs 8 *pounds*. We use a number, 8, and a unit, the pound, in this measurement.

There are two important practical aspects to measurement:

1. The choice of the size of a unit as a standard;
2. Whatever unit we choose, the result of the measurement is rarely a whole number of units.

The choice of the size of a standard unit depends upon what we want to measure and how *precise* we want the measurement to be. There is no fixed rule for making a choice of the size of a unit. Generally, people choose a unit of measure which gives results in small whole numbers. For convenience, the foot is the unit chosen when measuring a distance such as the length of a classroom; for example, 20 feet. Measuring the length of the same classroom in inches gives a result with a larger number of units, 240.

Often it is convenient to use more than one standard unit when making a measurement. Suppose we were measuring the length of a desk top.



If the length of the desk top is measured in 1-foot units, the result is not very precise; the length is reported as "more than 2 feet and less than 3 feet". Using foot units and inch units, the result of the measurement is more precise; we say that the length of the desk is "more than 2 feet 4 inches and less than 2 feet 5 inches". This is obviously more precise than saying that the desk top is "more than 2 feet and less than 3 feet" in length. Whether we make the measurement in feet alone or in feet and inches, the measurement cannot be given exactly in whole units. In each case, there is a length left over which we cannot measure exactly. This "left over" amount causes difficulty in nearly all measurements.

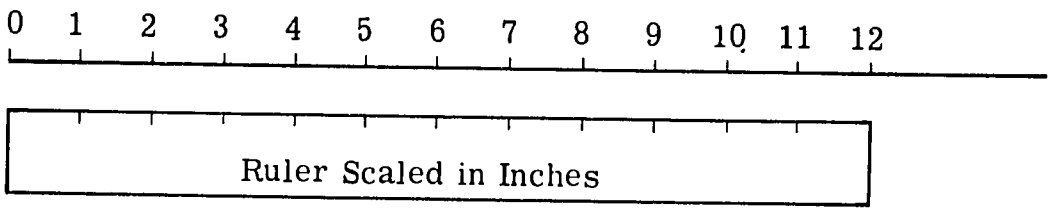
Measuring Distance

In *Primary One* and *Primary Two* the children learned to measure distance with a 1-yard stick, with a 1-foot stick and with 1-inch cardboard squares. It takes a long time to measure a 20-foot distance by laying any of

these sticks end to end to span the distance. A much easier way to make this long measurement is to use a *measuring string*. A measuring string can be made by stretching out a length of string and marking 1-foot *intervals* on it, as shown below.

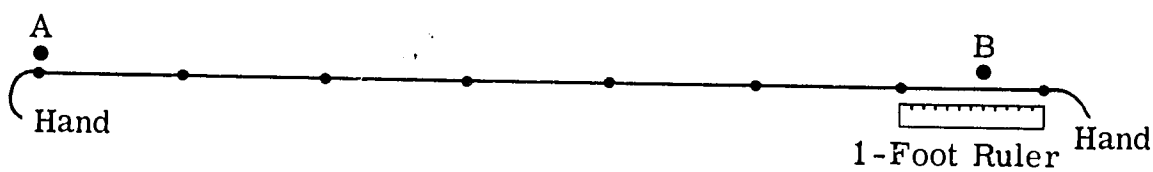
This measuring string can then be used by stretching it out over the distance to be measured. If the distance is greater than the length of the measuring string, the string can be stretched out several times.

A common tool for measuring short distances in inches is the 1-foot *ruler*. A 1-foot ruler is usually made from a piece of wood that has a straight edge. The straight edge is marked off to show twelve 1-inch *intervals*. An object marked off in equal intervals like this is called a *scale*. The common ruler, then, is a 1-foot stick that has one straight edge marked off in a scale of inches. A ruler is a tool that shows a number line. The line segment is shown by the straight edge, and numbers in natural order are shown by the scale of inches. Rulers are easy to make, using a narrow strip of heavy paper or cardboard in the manner shown below.

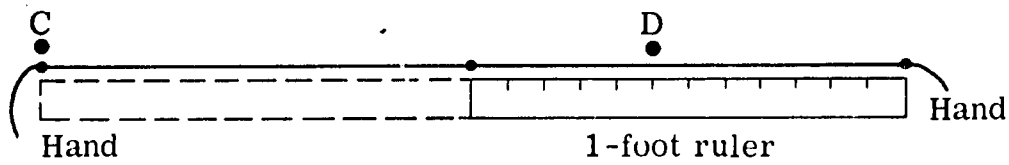


A 1-foot ruler with a scale in inches can be used with a 1-foot stick or with a measuring string to measure distances in feet and inches.

The distance between points A and B, below, is shown as 6 feet on the measuring string and about 7 inches on the ruler (6 feet 7 inches).



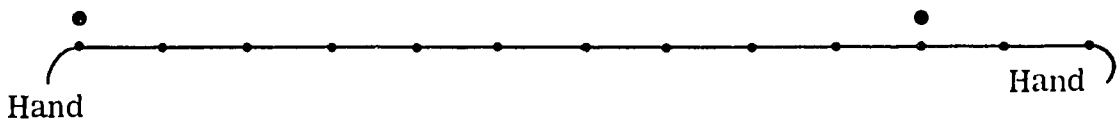
The distance between points C and D is shown as 1 foot 5 inches.



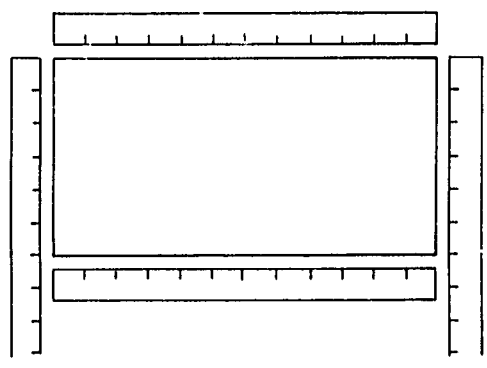
There are two reasons for measuring a distance: first, to find the distance between two points which are already located; second, to help locate the second of two points.

If you are to find the distance between two stakes, stretch the measuring string between the stakes, as shown below. The distance is measured in feet. (The distance between the stakes is more than 12 feet but less than 13 feet.)

Suppose you were asked to place two stones on the ground 10 feet apart. Put one of the stones on the ground. With one end of the measuring string held next to the stone, stretch out the measuring string. Place the second stone at the 10-foot mark of the measuring string.



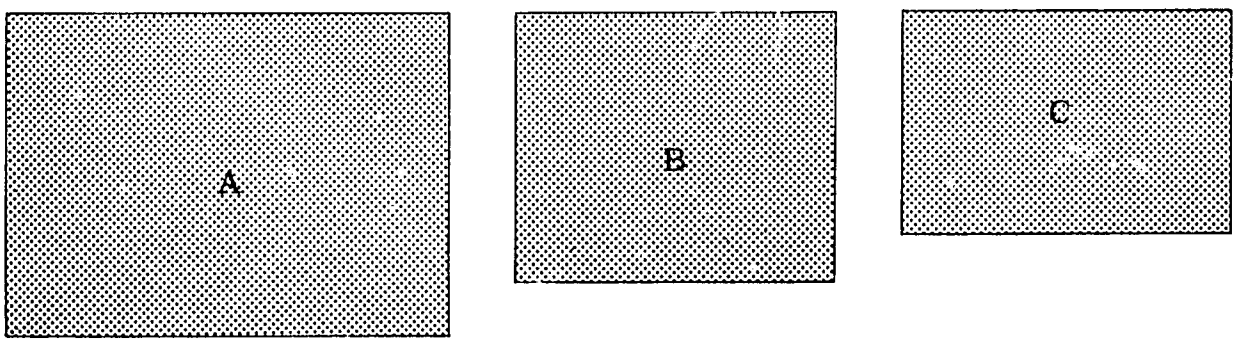
The perimeter of, or distance around, a geometric figure or object can be measured in several different ways. For example, the perimeter of a rectangle can be found by adding the measures of the four line segments. The perimeter of the rectangle in the picture below is found by reading the length of each side from the 1-foot ruler and then finding the sum of 6 + 6 + 12 + 12. (The perimeter is 36 inches.)



Another way to measure perimeter is to stretch a piece of string around the edges of a thick cardboard cutout. Then cut the string when it fits exactly around the edges. This piece of string can then be measured by stretching it along a ruler. Its length is the perimeter of the figure shown by the cutout.

Size of Regions

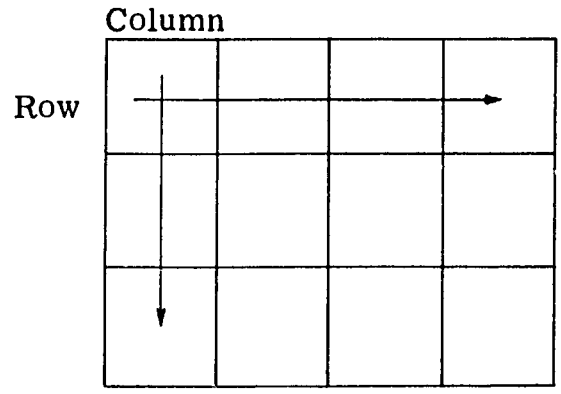
A cardboard cutout that shows a geometric figure also shows a *region*. In the drawings of the three rectangular cutouts below, the faces of the cutouts are shaded to show the *regions*. With our eye we can see that the size of Region A is greater than the size of Region B and greater than the size of Region C. The size of Region C is less than the size of Region A and less than the size of Region B.



The size of a region can be found by several different methods. Each of these methods uses the idea of an *array*.

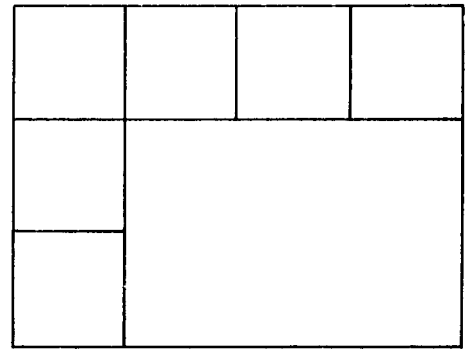
First, the size of a region can be measured by covering the region with square tiles (square cutouts showing congruent regions) and counting the number of tiles. In this method, the unit of measure is the size of the tile. We will call this unit the "tile". (The tile is not a standard unit of measure.) It will be clear from our wording when "tile" means a cutout and when it means a unit of measure.

Second, if the region is rectangular, the tiles will be arranged in rows and columns. Then the number of rows of tiles and the number of tiles per row are counted. By the operation of multiplication, the number of tiles is found. The number of tiles is the size of the region.



Number of rows × number of tiles per row

Third, the number of rows of tiles and the number of tiles per row needed to cover the region of a rectangle can be found without tiling the whole region. The number of tiles per row can be found by tiling the first row. The number of rows of tiles can be found by tiling the first column. The number of tiles needed to cover the entire region can then be found by multiplication.



$3 \times 4 = 12$

Fourth, the number of tiles needed to cover a region can be found without placing any tiles in the region. If the length of the edges of the square tiles is known (1 inch), we can find the number of rows and columns of tiles with a ruler. Then the number of tiles needed can be found by multiplication.

In this unit, the children use 1-inch cardboard squares and $\frac{1}{2}$-inch cardboard squares as tiles when they measure the sizes of regions.

Capacity

It is often convenient to measure liquids and solids in a standard-size container. For example, milk is usually sold in standard-size containers with a *capacity* of 1 quart.

The gallon, the quart and the pint are standard units of capacity. A gallon container has a capacity equal to 4 quarts. A one quart container has a capacity equal to 2 pints.

STAGE 1 • Measurement of Length

Vocabulary

Measure, length, foot, measuring stick, measuring string

Materials

One 1-foot stick for each child; fourteen 1-inch cardboard squares for each child; 20-foot piece of string for every three children; ink; fine paintbrush; cardboard cutouts; stones; pupil page 104

Teaching Procedure

ACTIVITY 1 • Revision of Units of Length (inch, foot and yard)

Pupil page 104

Give each child a 1-foot stick. Say, "We are going to use this stick to *measure*." Let the children talk about this. They may say that they used a stick like this to measure with before. Try to get the children to identify the stick as a 1-*foot* stick. If they do not remember this name, hold up one of the sticks and say, "This is a 1-foot *measuring stick*." Hold the stick between two fingers and say, "It is 1 foot long."

Draw a line segment on the blackboard. Make its length more than 3 feet and less than 4 feet. Ask, "Can you use this stick to find the length of this line segment?" Let one of the children measure the length of the line segment with the 1-foot stick. Let the child tell the length of the line segment.

Ask, "Is the line segment more than 3 feet long?" (Yes) "Is it less than 4 feet long?" (Yes) Get the children to say, "The line segment is more

than 3 feet long and less than 4 feet long." Draw other line segments on the blackboard and let the children measure them with their 1-foot sticks. Get them to use "more than" and "less than" when they tell the *length*.

Let the children measure the edges of objects in the room that are more than 1 foot in length. They should say, for example, "This edge of the desk is more than 2 feet long and less than 3 feet long."

Hold up a Pupil Book. Ask, "How long is this book?" Let the children measure its length with their 1-foot sticks and say how long it is. (It is less than 1 foot long.)

Give each child fourteen 1-inch squares. Hold up one of the squares and run your finger along one of the edges. Ask, "Do you remember the length of each edge of this square?" Let the children talk about this. If they do not remember, say, "Each edge of the cardboard square is 1 inch long." Say, "The Pupil Book is less than 1 foot long. How many inches long is it?" Ask the children to find its length in inches by placing their inch cardboards along an edge. (The book is 11 inches long.) Help the children who need help. Let the children use their inch cardboards to measure other things in the room that are less than 1 foot long. Get the children to use the words "more than" and "less than" as they measure the length of something that is not a whole number of inches in length. (This pencil is more than 4 inches long and less than 5 inches long.)

Hold up a 1-foot stick and ask, "How many inches long is this stick?" Let the children talk about this. Tell the children they can measure the length of a 1-foot stick in inches by fitting several 1-inch cardboard squares along the stick. Let them do this. Get them to say, "A 1-foot stick is 12 inches long."

Tell the children to open their books to page 104. Let them measure the lengths of the line segments on that page with 1-inch cardboard squares. As they work, let them say which line segment is the longest and which is the shortest. (*AB*; *CD* or *GH*) Ask them to find line segments that are the same length. (*CD* and *GH*; *KL* and *OP*) The 9-inch line segments at the left and right margins of the page are also the same length. The $6\frac{1}{2}$-inch line segments at the top and bottom margins are also the same length, but children cannot as yet measure in parts of inches with a ruler. Encourage them to use the words "more than" and "less than" to describe the lengths of line segments that are not a whole number of inches in length. For example, line segment *AB* is more than 10 inches long and less than 11 inches long.

Hold up a Pupil Book opened to page 104. Run your finger along one of the 9-inch line segments that make a border of the page. Ask, "Can you guess the length of this line segment?" Encourage the children to make guesses. Then let them measure the length of the line segment with inch cardboards. Get them to decide whether they made a good guess. Repeat this kind of "guessing-measuring" with the edges of books or other small objects that have straight edges. After each guess, let the children measure and decide whether their guess is a good guess.

ACTIVITY 2 • Measuring Length in Feet

Give each child a 1-foot stick. Arrange the children in groups of three. Give each group a piece of string about 20 feet long.

Take the class outside to a place where the ground is nearly level. Get two children to stand about 15 feet apart and stretch their piece of string between them. Tell each of them to hold the stretched string against the ground with one finger. Ask, "How long is the piece of string stretched between Kesse's finger and Ode's finger?" Let the children make guesses. Then say, "Can you measure the length of the string in feet?" Get the children to talk about this. Let several children work together to measure the length. They can do this by placing 1-foot sticks along the string. Repeat the activity several times, using different children and lengths of stretched string. Let each group of three children repeat the activity. Two children can hold the string while the third child measures the distance with their 1-foot sticks.

Bring the children together again. Say, "We need a special string for measuring lengths. We can make a *measuring string.*" Get two children to stretch a piece of string over a distance of about 20 feet. Ask one of the children to bring a bottle of ink and a small paintbrush from the room. Make an ink mark near one end of the string. Fit a 1-foot stick along the string so one end of it touches the ink mark on the string. With the brush and ink, make another mark on the string at the other end of the 1-foot stick. Continue to make marks on the string at 1-foot intervals. Let the children talk about what you are doing. Make sixteen marks on the string (fifteen 1-foot intervals). Ask the children who are holding the string to hold the string so

that their fingers are on the end marks. Get them to stretch the string again. Ask, "How can we find how long the string is (that Tunde and Ben have stretched)?" Let the children talk about this. Get the children to count the number of 1-foot lengths. (15) Let two other children stretch the string over a shorter distance. (Be sure one end is held with the child's finger on the end mark.) Let the children say how long the stretched string is. Get them to count the number of 1-foot intervals shown by the marks.

Place a stone on the ground so that all of the children can see it. Walk about 10 feet from this stone and place a second stone on the ground. Ask, "Can you find how many feet the first stone is from the second stone?" Get the children to talk about how they can do this. Let two of the children stretch the string between the two stones to find the distance. Get the children to say whether the distance can be measured in a number of feet. Get them to use "more than" and "less than" to describe the distance. Say, "This stone is more than 9 feet but less than 10 feet from the other stone."

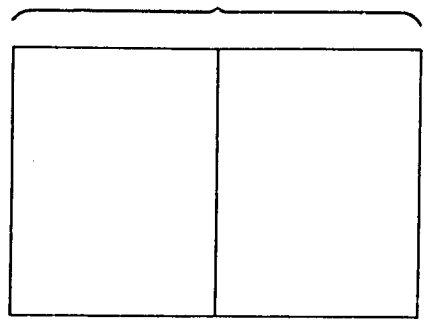
Place another stone on the ground. Ask, "Can you place a third stone 12 feet from the first stone?" Let the children discuss this. If the children do not suggest stretching the string from the first stone and measuring off the 12-foot distance, show them how.

Arrange the children again in groups of three. Let each group make a 15-foot measuring string. Let them use their strings to measure distances between objects and let them place objects at given distances apart.

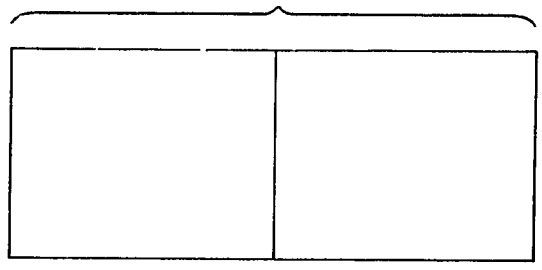
ACTIVITY 3 • Adding the Measures of Lengths

Give each child fourteen 1-inch squares. Get them to measure the long edge of their mathematics books. (11 inches) Ask them to measure the short edge of their books. (It is more than 8 inches but less than 9 inches long.)

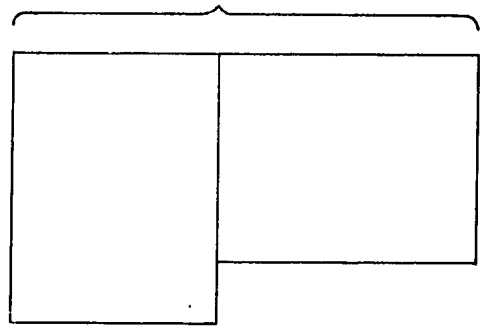
Hold up two mathematics books and fit the long edge of one to the long edge of the other. Ask, "What do you guess is the length of the edge formed when the two books are put together like this?" (Next page.) (The edge is about 17 inches long.) Let them discuss their guesses. Let the children work in groups of two to measure the length of this edge. Ask the children what the measure of this edge is in inches. Let them say, "I made (did not make) a good guess."



Hold up the two mathematics books again. This time fit the short edge of one to the short edge of the other. (See below.) Ask, "What do you guess is the length of the edge formed when the two books are put together like this?" Let the children say what they think the length is. (The edge is 22 inches long.) Let them measure the length of the edge with inch cardboards. Again let the children tell whether they made a good guess.



Ask, "Can you put the books together in a different way to show another edge?" Let the children experiment to find that a long side and a short side can form another edge. Ask them to say what they guess the length of this new edge will be. Let them measure it. (The length of this edge is more than 19 inches and less than 20 inches.)

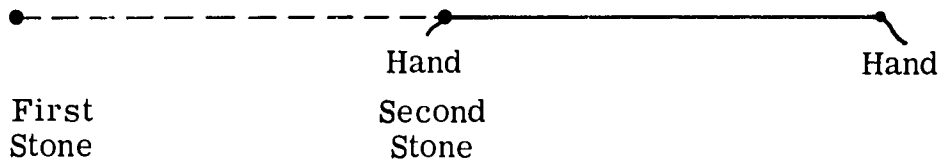


Hold up one of the books and run your finger along the long edge. Say, "This edge is 11 inches long." Hold up the other book and run your finger

along the long edge. Say, "This edge is 11 inches long." Fit the books together to show an edge that is 22 inches in length. Say, "11 plus 11 is 22. When the books are put together like this, they show an edge that is 22 inches in length."

Give the children other rectangular objects or cutouts. Let them measure the lengths of the edges. Let them put different edges together to make new edges. Get them to guess how many inches long the new edges will be. Then tell them to test their guesses by measuring the lengths with inch cardboards.

Take the class outdoors. Give each group of three children the measuring string they made. Place a stone on the ground. Get one of the children to hold the marked end of a measuring string at the point where the stone lies. Get another child to stretch the string out and hold the other end. Place a second stone at the 8-foot mark. Ask a child to tell how far the stones are apart. Tell the children to move the string so that it is stretched from the second stone along the straight line beyond the second stone.



Place a third stone at the 7-foot mark. Ask, "What is the distance between the second and third stones?" Let the children talk about this. Then ask, "What is the distance from the first stone to the third stone?" If the children are puzzled, say, "The distance from the first stone to the second stone is 8 feet. The distance from the second stone to the third stone is 7 feet. What number do we get when we add 7 and 8?" (15) "How can we measure to find the distance between the first stone and the third stone?" Let the children stretch the string to measure the distance between these two stones. (15 feet) *Note:* It may happen that, because of a slight error in the markings, the children will not get exactly 15 feet as the final measurement. Help the children understand why this happens.

Place a stone on the ground. Walk about 6 feet away and place another stone on the ground. Ask, "What is the distance between the two stones?" Encourage the children to guess the distance. Let the children measure this distance with the measuring string. Get them to use "more than" and "less than" when they say what the distance is in feet.

Walk to a spot about 6 feet beyond the second stone and place a third stone on the ground so that it lies along the same straight line as the first two stones. Ask, "What is the distance between the second and third stones?" Let the children guess, and then let several of them stretch the string and measure the distance. Ask, "Is the distance more than 6 feet or less than 6 feet?" Let them say what the distance is, using "more than" and "less than". Ask, "What is the distance between the first and third stones?" Let them guess and then let them measure to see who has made the best guess.

Let the children work in groups. Let them set stones given distances apart. Let them put down two stones and measure the distance between stones with their measuring strings. Encourage them to add the measures of the two distances between three stones lying along a straight line.

STAGE 2 • Using Scaled Rulers

Vocabulary

Ruler, scale, interval, yard, perimeter

Materials

Twelve 1-inch square cardboard for each child; a 1" × 18" paper strip for each child; string; measuring strings; a 12-inch ruler scaled only in inches; one matchbox for every two children; a 5" × 8" rectangular cutout of thick cardboard; rectangular, 4-sided, triangular and circular cardboard cutouts; pupil page 105

Teaching Procedure

ACTIVITY 1 • Making a Ruler

Draw a line segment at least 15 inches long on the blackboard. Mark a point near the left end of the line segment. Make a small mark downward

from the point. Take a 1-inch cardboard square and mark five more points 1 inch apart to the right of the first point. Write a "2" above the third point.

Ask, "Can someone tell what I am drawing on the blackboard?" (A number line) Ask the class to help you finish the number line. Get them to call out the numbers from 0 to 5 as you write the proper numeral. Say, "This is a special kind of number line. It is a number line with 1-inch *intervals* between numerals." Mark more points on the number line. Get the children to call out numbers to complete the natural order from 0 to 15.

Pick up a book that is less than 1 foot long on an edge. Run your finger along the edge of the book and ask, "Do you know a way to measure the length of this edge?" If they suggest using inch cardboards, do it that way. Then ask, "Can you find a way of measuring the length of this edge without using inch cardboards?" Let the children talk about this. Get them to see that the edge can be fitted along the special number line on the blackboard and its length measured in inches. Let one of the children come to the blackboard. Let him fit the book edge along the number line. Let him count the inches and tell the length of the book. Choose an object in the room that cannot be moved easily; for example, the edge of a desk or part of a window. Ask, "Can you think of a way we can measure the length of this edge without using inch cardboards?" Get the children to see that a movable number line with 1-inch intervals between numerals would be useful. Say, "You can make a special number line for measuring length in inches."

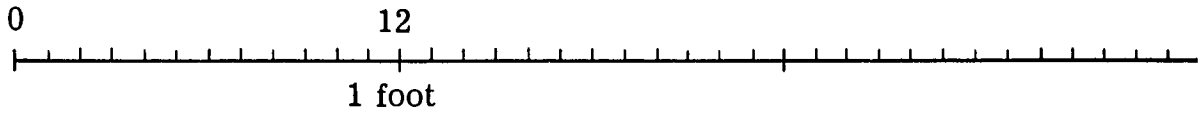
Give each child some 1-inch cardboard squares and a 1" × 18" strip of heavy paper. Help the children to mark a beginning point near the left end of the edge of the strip and to mark points at 1-inch intervals. Give special help to the children who have difficulty. When they finish their special number lines, let the children use them to measure the edges of objects in the room.

Hold up a ruler that is marked off only in inches, not in parts of inches. Say, "This is a *ruler*." Run your finger along the straight edge and say, "This ruler has a straight edge and a *scale* marked in inches." Point to the scale markings on the ruler. Say, "You have just made a ruler. Hold up your ruler. Do you know why it is a ruler?" Get the children to say that

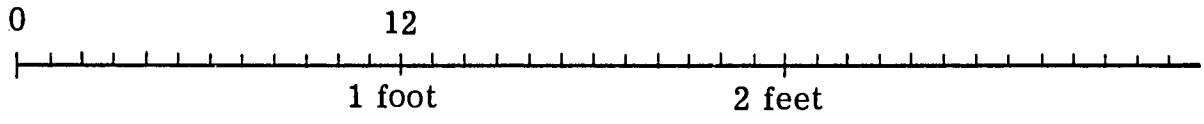
their rulers have straight edges and a scale. If the children do not say this, hold up your ruler and say, "This ruler has a straight edge. Does your ruler have a straight edge? This ruler has a scale. Does your ruler have a scale? The scale on my ruler is marked in inches. How is the scale marked on your ruler?"

Hold up a paper ruler you have made, and fold it back at the 0 mark and at the 12-inch mark. Be sure the children can see what you are doing. Ask, "Can someone tell what I have made?" (A 1-foot ruler marked off in twelve 1-inch intervals) Get the children to see that it is a 1-foot ruler with an inch scale marked on it. Let them fold their rulers at 0 and 12 to make 1-foot rulers.

Extend the line segment on the blackboard. Continue marking off 1-inch intervals until a 36-inch portion of the line segment has been marked. Erase the left end of the line segment up to the 0 mark. Ask, "Can someone show the point that marks off 1 foot on this number line?" Let one of the children come to the blackboard and show the 1-foot point. Mark the 12-inch point as you marked the 0 point. (See below.) Label this point "1 foot".



Run your finger along the number line from the 0 mark to the 12-inch mark. Say, "This segment of the number line is 1 foot long." Ask, "Can you find the point which marks off the second 1-foot line segment on the number line?" Let one of the children show the 2-foot mark. Mark it and label it "2 feet".



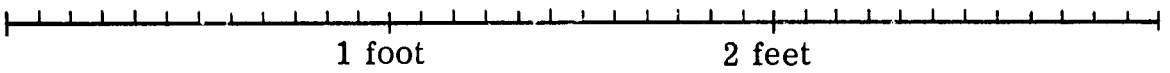
Get the children to count the inch intervals in the second 1-foot segment. Let someone come to the blackboard and show the point which marks 3 feet. Label this point "3 feet".

Get the children to count the 1-inch intervals in the third 1-foot segment. Erase the chalk line to the right of the 3-foot mark.

Ask, "How long is the line segment?" Get the children to count the number of feet. Then let them count the number of inches. Say, "This line

segment is 36 inches long; it is also 3 feet in length." Ask, "Do you know another name for a distance that measures 3 feet?" If no one knows, say "This number line is 36 inches long; it is 3 feet long; it is 1 *yard* long."

Say, "Watch me. I am making the line segment shorter." Rub out two inches from the right end of the line segment.



Ask, "How long is this line segment?" (34 inches, or 2 feet 10 inches) Ask questions so that the children say its length in inches and also in feet and inches.

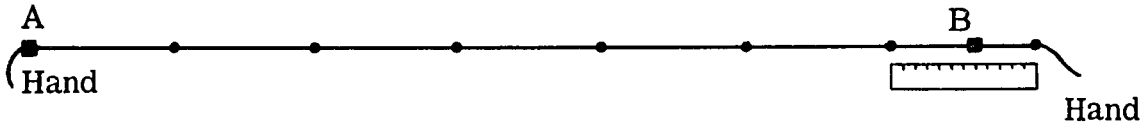
Continue to rub out small segments of the line segment and to ask the length of the line segment in inches and also in feet and inches.

ACTIVITY 2 • Measuring in Feet and Inches

In this activity the children will use 1-foot sticks, 15-foot measuring strings and 12-inch rulers to measure distances in feet and inches. Let them measure many objects in the classroom.

Let the children fit two Pupil Books together and measure the length in feet and inches, and also in inches. Get them to fit the books together in the three different ways. Get them to tell the lengths in inches and also in feet and inches.

Let them use the 15-foot measuring string to measure distances outdoors. Show them how to fit a 12-inch ruler against the last whole foot mark to read the left-over distance in inches.



Get them to say the length is, for example, 6 feet 7 inches. Try to get them to give distances up to 8 feet also in inches.

Note: The children had a brief introduction to the idea of "distance around" a geometric figure in *Primary Two*, Unit 13. In this activity they will learn the meaning of "perimeter" and how to measure it for different geometric figures.

Hold up a 5" × 8" rectangular cutout of thick cardboard. Run your finger along each edge, turning the cutout as you do this. Let the children count the number of edges. (4) Run your finger along one of the 8-inch edges and ask, "What is the length of this edge?" Let the children guess. Get one of the children to use a ruler and measure the length of the edge. Let other children measure the lengths of the other edges. Write the measurement for each edge on the blackboard.

Say, "I am going to stretch a piece of string around the edges of the cardboard cutout." Hold up the cutout and stretch one of the measuring strings around the edges. Ask the children to help you. Let them cut the string where it meets. Stretch out the measuring string, holding it at the points where the string came together around the cutout. Ask, "Does this help us know how many inches it is around the edges of the cardboard cutout?" Let the children talk about this and decide that the string is the same length as the four sides together.

Let the children guess the length of the string. Let someone measure the length of the piece of string. (2 feet 2 inches, or 26 inches) Say, "The length of all of the edges of the cutout together is 2 feet 2 inches; it is 26 inches." Run your finger around the edges of the cutout again and say, "The length of all of the edges of this cutout is called the *perimeter*." Let the children say, "The perimeter of the cutout is 2 feet 2 inches; it is 26 inches."

Ask, "Is there another way we can find the perimeter of the cutout?" If they do not suggest adding the measure of the lengths of the edges together, say, "We measured the length of each edge. The measures of the lengths of these edges are written on the blackboard." Get them to see that the measures of these lengths can be added together to get the number of inches in the perimeter. Write on the blackboard:

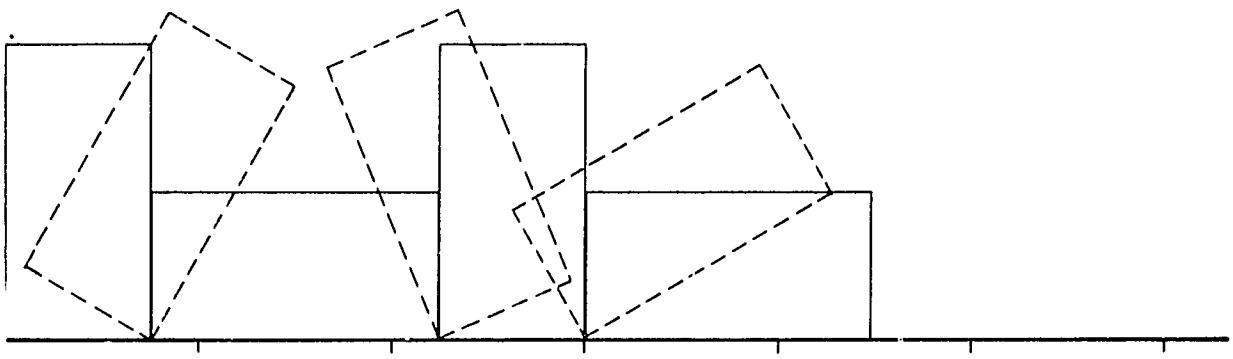
$$5 + 5 + 8 + 8 = 26$$

Let the children help you measure the edges and find the perimeters of other four-sided, triangular and circular cutouts.

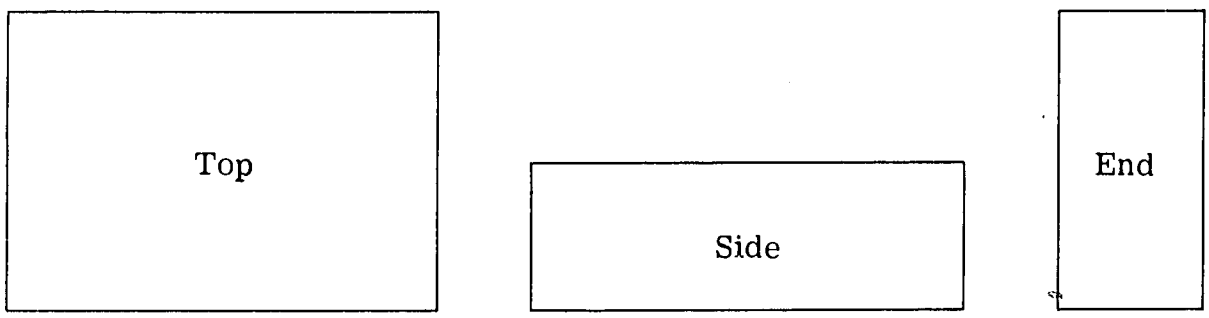
Let the children measure the perimeters of their desk tops. They can use their measuring strings and 12-inch rulers to do this. Let them measure the perimeters of books and other objects in the room.

Tell the children to open their books to page 105. Tell them to measure the edges and find the perimeters of the figures on the page. Get them to say which figure has the longest perimeter (*B*) and which has the shortest perimeter (*D*). Get them to tell which figures have equal perimeters (*A* and *C*).

An inch scale is printed along the left-hand edge of page 105. This scale can be used to measure the perimeter of cutouts or other small objects found in the classroom. For example, a matchbox can be turned edge-over-edge to find the perimeters of the rectangles shown by the faces. (See the figure below.)



The six faces of a matchbox show three different rectangles. Let the children find the perimeter of each face.



One common type of matchbox shows one rectangle with a perimeter of a little more than 7 inches, one with a perimeter of more than 5 inches, and one with a perimeter of a little less than 5 inches. Different types of matchboxes will show rectangles with slightly different perimeters.

Give each two children a matchbox. Point to the inch scale in the book. Say, "Can you measure the lengths of the edges of this matchbox by using the inch scale?" Let them make measurements of the edges. Ask, "How many edges did you find?" (12) Let them hold up their matchboxes and count the number of edges together. Tell them to hold up their boxes and run their fingers along the longest edge. Tell them to run their fingers along the shortest edge. Say, "Does the matchbox show any rectangles?" (Yes, 6) Let the children count the number of rectangles. Get the children to hold up their matchboxes. Say, "Run your fingers around the rectangle shown by the face with the matchbox label on it." Get the children to show other rectangles.

Hold up a matchbox and run your finger around the rectangle with the shortest perimeter. Ask, "What is the perimeter of this rectangle?" Show the children how they can find the perimeter by turning the box edge-over-edge along the inch scale, as shown on page 207. Get the children to say what the perimeter is. Encourage them to use "more than" and "less than" when reporting the perimeter. Ask, "Can you find the perimeter of a rectangle shown by a different face of the matchbox?" Let the children experiment. Watch the children to see if anyone discovers how to measure the perimeter of one of the other rectangles. If they need help, hold up a box and run your fingers around a different rectangle. Let the children measure the length of its perimeter by turning the box edge-over-edge along the inch scale.

Let the children find the perimeters of the other rectangles shown by the box. Let them discover that they can measure the perimeters of six rectangles, but that there are only three different perimeters.

Let them measure the perimeters of other small objects in the room.

STAGE 3 • Size of a Region

Vocabulary

Congruent

Materials

Fourteen 1-inch square cardboard (tiles) for each child; one $\frac{1}{2}$-inch-square tile for each child; a 12-inch ruler; scissors; rectangular cutouts of these dimensions: two 12" × 12", two 12" × 8" and one 8" × 8"; one rectangular cutout for every three children of these dimensions: 8" × 6", 8" × 3", 7" × 4" and 5" × 4"; one triangular cutout; one circular cutout; pupil pages 106-110.

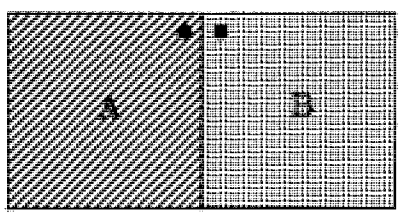
Teaching Procedure

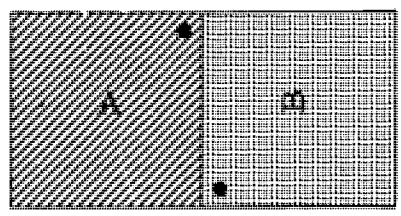
ACTIVITY 1 • Congruence of Geometric Objects

Have ready the cutouts listed in *Materials* above.

Hold up one of the 12" × 12" cutouts and ask, "What kind of figure does this cutout show?" (A square) Hold up the second 12" × 12" cutout and ask what kind of figure it shows. (A square)

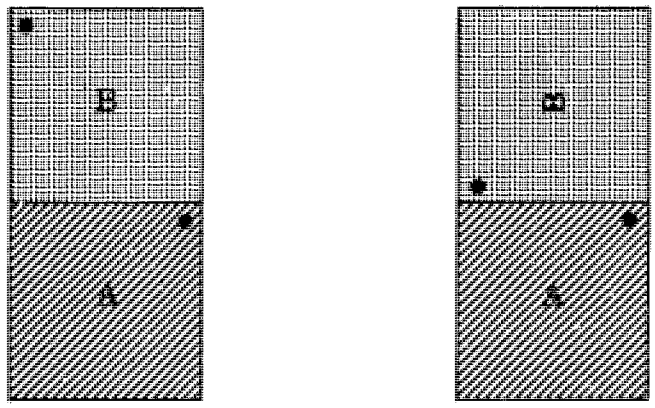
Hold up both of the 12" × 12" squares and fit two of the edges together. Say, "The edge of the cutout in my left hand fits exactly the edge of the cutout in my right hand. The edges are *congruent.*" Keep the cutout in your right hand in the same position, and turn the other cutout so that a different edge is fitted along the edge of the cutout in your right hand. Ask, "Are these two edges congruent?" (Yes) By turning the same cutout each time, fit all four of its edges to one edge of the other cutout. Get the children to say that these edges are also congruent.





The cutout in your left hand can be placed so that one of its edges fits exactly against the top edge of the other cutout. By fitting all four edges of one cutout to each of the four edges of the other cutout, sixteen different

fittings can be made. Do several of these fittings and let the children say that all the edges are congruent.



Give each child two inch square cardboards. Let the children fit the edges of their square cardboards to show congruence.

Hold up one of the 12" × 12" squares. Move your hand over the face of the cardboard and say, "The face of this cutout shows a region." Repeat this for a rectangular cutout which does not show a square, for a circular cutout, and for a triangular cutout. Get the children to say that the face of each of these cutouts shows a region. Let children in turn hold up the cutouts. Let them move their hands over the faces of the cutouts and say that the cutouts show regions. Also get them to say that their inch-square cardboards show regions.

Pick up the two 12" × 12" cutouts and put them together so that one face is on top of the other. Fit the edges of the cutout so that one cutout exactly covers the other cutout. Hold the cutouts up for the children to see. Say, "These cutouts fit together exactly. The squares are *congruent*. They have *congruent edges* and they show *congruent regions*." Ask, "Do your cardboard squares show congruent regions?" Let the children fit their cutouts together and say, "The cutouts show congruent regions."

Repeat this exercise, using the 12" × 8" cutouts. Let the children show that some edges are congruent and some edges are not congruent. Fit the cutouts so that one cutout exactly covers the other and let the children say that the figures are congruent.

Hold up different pairs of cutouts, and let the children try to fit the edges to test for congruence. Then let them test to see whether the cutouts show congruent regions. Revise the exercise until the children understand the idea of congruence or non-congruence of regions.

ACTIVITY 2 • Revision of Regions Inside Geometric Figures

Pupil page 106

Give each of the children twelve 1-inch squares. Ask, "Do your cardboard cutouts show congruent regions?" Let the children stack the cardboards to check for congruence of regions.

Tell the children to turn to page 106 in their books. Let them cover the regions inside the figures with 1-inch squares. Say, "We will call your squares *tiles*." Help the children to lay the tiles on the regions so that the edges of the tiles fit together and do not overlap. (This is called *tiling*.) After the children have had a chance to fit the inch squares inside the figures, ask them questions like the following:

Which figure has the smallest inside region? (*B*)
How many tiles are needed to cover the region inside Figure *B*? (2)
Are there any squares on the page? (Yes, *A* and *D*)
How do you know these figures are squares? (All the edges are congruent, and all the corners are square corners.)
Which figure has the largest inside region? (*F*)
How many tiles are needed to cover the region inside Figure *F*? (10)

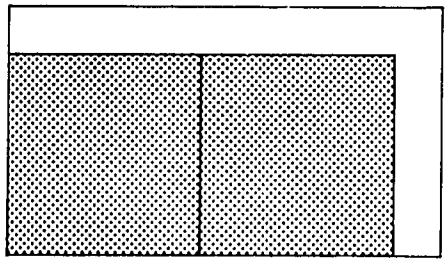
Get the children to tell how many tiles are needed to cover the region inside each figure. Let them tell what they notice about the figures. They may say, for example, "The regions inside Figure *C* and Figure *D* are both covered by 6 tiles", and "We used two times as many tiles to cover the region inside Figure *A* as to cover the region inside Figure *B*."

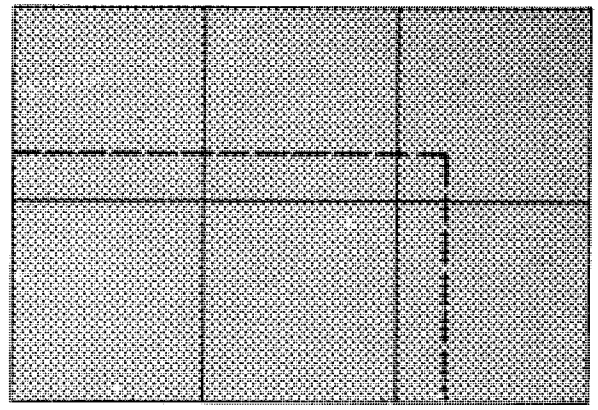
ACTIVITY 3 • Size of the Region, Non-rectangular Figures

Pupil page 107

Note: In the previous activity, the children compared the sizes of the inside regions of rectangles by using 1-inch tiles. The children were never in doubt about which figure had an inside region that was larger, smaller or the same as another figure. The regions could always be covered exactly with inch-square tiles. The children could always make a direct comparison of the region inside one geometric figure and the region inside another figure by counting the number of tiles. In this activity, the children compare the

sizes of the inside regions of figures whose inside regions cannot be covered exactly by 1-inch tiles. The children will cover with 1-inch tiles as much of an inside region as they can. (See the figure at the left below.) Then they will cover all the remaining inside region with as few tiles as possible. (See the figure at the right below.) They will say, "The inside of the region is covered by more than 2 tiles but by fewer than 6 tiles."





Tell the children to open their books to page 107. Give each child twelve 1-inch tiles. Ask, "How many figures do you see on the page?" (6, or 7 if the rectangular border is counted) Let the children fit 1-inch tiles over the inside regions of the figures. Let them say, "The inside regions cannot be covered exactly by the tiles."

Guide the beginning of the activity like this:

Find a figure in whose inside region only 2 whole tiles can be placed. (*F*)
How many tiles can you fit into the inside region of Circle *C*? (1)
Is the size of the inside region of Circle *C* less than the size of the region shown by one tile, or is it more than the size of the region shown by one tile? (More)
Can you find a figure in whose inside region you can fit 6 tiles? (*E*)
How many tiles can you fit in the inside region of Figure *B*? (5)
How many tiles can you fit in the inside region of Figure *A*? (4)

The rest of the activity requires that the children also use their tiles to cover completely the inside regions of the figures. Let the children say, for example, "I used 6 tiles to cover all the inside region of Figure *A*. Its region is covered by more than 4 tiles but fewer than 6 tiles."

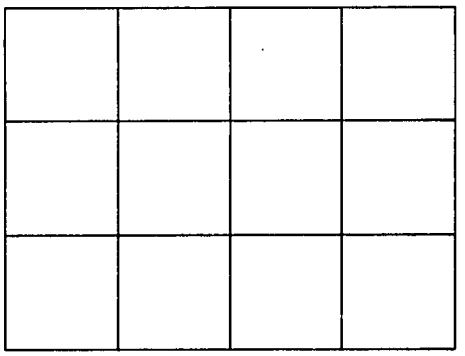
Ask questions like this for each figure:

How many tiles can you fit inside Figure *B*? (5)
How many tiles cover completely the inside of Figure *B*? (10)
How many tiles cover the inside of Figure *B*? (More than 5 and fewer than 10)

ACTIVITY 4 • Arrays and the Sizes of Regions

Pupil page 108

Note: In previous activities, the children found the size of the inside regions of rectangles by tiling regions and by counting the number of tiles that covered the regions. In this activity, they will tile and then use the operation of multiplication to find the number of tiles needed to cover the region inside a rectangle. (See page 195.)



$3 \times 4 = 12$

Give each child twelve tiles. Tell them to turn to page 108 in their books. Let the children tile the inside regions of some of the figures. Hold up a pupil text and point to Figure *A*. Ask, "How many tiles did it take to cover the inside region of Figure *A*?" (6) Ask, "How many rows of tiles are there?" (2) "How many tiles in each row?" (3) Ask, "Can you find the number of tiles without counting them one by one?" Get the children to count the number of tiles by 2's and by 3's. Get them to multiply 2×3 and 3×2. Let them write sentences for what they have said:

$$2 + 2 + 2 = 6$$
$$3 + 3 = 6$$
$$2 \times 3 = 6$$
$$3 \times 2 = 6$$

Get them to say, "The size of Region *A* is the same as the size of 6 tiles. The easiest way to find 6 is to think, '2 × 3 = 6'."

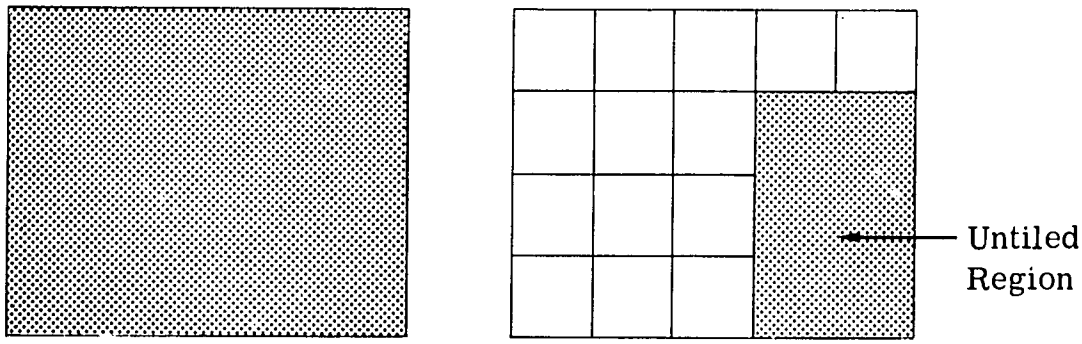
Let them repeat this activity for the other figures on the page.

ACTIVITY 5 • Incomplete Tiling of a Rectangular Region

Pupil page 109

Prepare rectangular cutouts (one of each for every three children) having the following dimensions: 5" × 4"; 3" × 8"; 4" × 7"; 8" × 6".

Give exactly 14 tiles and a 5" × 4" cutout to each group of three children. Get the children to say that the cutouts show rectangles. Tell the children to tile the inside regions shown by the cutouts. They will not have enough tiles. Twenty tiles are needed to cover a 5" × 4" cutout.



No matter how the children tile the region with their 14 tiles, there will remain an untiled part of the region equal to the size of 6 tiles. Get the children to decide how many more tiles are needed to cover the entire region. They may count the number of tiles they have fitted (14) and then move 6 of these to cover the other part of the region. Some children may know that 6 more tiles are needed without actually moving the tiles. Suggest a way to work for those who need help. For example, say to a child who has the region tiled as above, "The top row has 5 tiles, and each of the other rows has only 3. If 2 more tiles are put in each of the rows of 3, these rows would also have 5 tiles."

Give the 3" × 8" and 4" × 7" cutouts to the groups as they are ready for them. Ask the children to fit the 14 tiles in the inside regions shown by the cutouts. Ask them to tell how many tiles are needed to completely cover

the regions. The 3" × 8" cutout requires 24 tiles for the region it shows to be completely covered. The children may move 10 of the 14 tiles to count the number needed to completely cover the region. Twenty-eight tiles are needed to cover the inside region of the 4" × 7" cardboard. The children will fit the 14 tiles in the region. Then they may move all 14 tiles to cover the rest of the region.

Give each child a $\frac{1}{2}$-inch-square tile. (Do not call it a $\frac{1}{2}$-inch tile.) Hold one of the tiles up and say, "These are tiles. Are they the same as the other tiles you used?" Ask the children to say how the tiles differ from the tiles they used before. (These tiles show a region of smaller size than the other tiles.) Get the children to see that these smaller tiles, like the larger tiles, are square.

Tell the children to turn to page 109 in their books. Ask, "How many rectangular figures do you see?" (9) "Do the pictures show tiles in the inside regions?" (Yes) Get the children to fit a small tile over one of the tiles shown in Region *A*. Get them to say that the tiles shown in the regions of the figures on page 109 are square and show the same size regions as their own tiles. Let them count the number of tiles in a row and count the number of rows of tiles needed to cover the region of each rectangular figure. Get the children to count the number of tiles needed to cover each region by 2's, by 4's, 6's, and by 8's, or by whatever number is needed. Encourage the children to find the number of tiles needed to cover a region by *multiplying* the number of tiles in a row by the numbers of rows.

ACTIVITY 6 • Finding the Size of a Rectangular Region

Pupil page 110

Note: The children have found the sizes of rectangular regions by these methods:

1. By counting the number of tiles that covered the region;
2. By multiplying to find the number of tiles in an array that covered the region;
3. By multiplying to find the number of tiles in an array that covered the region when only one row and one column of the array had been tiled.

In this activity, the children find the size of a rectangular region without

using tiles. They will use a ruler to help them think of an array of squares. The array will help them find the size of a rectangular region.

Tell the children to turn to page 110 in their books. Ask, "What kind of figures do you see on this page?" (Rectangles) Ask, "Can you tell the number of 1-inch-square tiles needed to cover the region of Rectangle *A*?" The children will probably say they need some tiles to do this. Give each child a single tile. Ask, "Can you find the number of rows of tiles needed?" (1) "Can you find the number of tiles needed in the row?" (4) Get the children to say that 1 row of tiles is needed, and there are 4 tiles in the row.

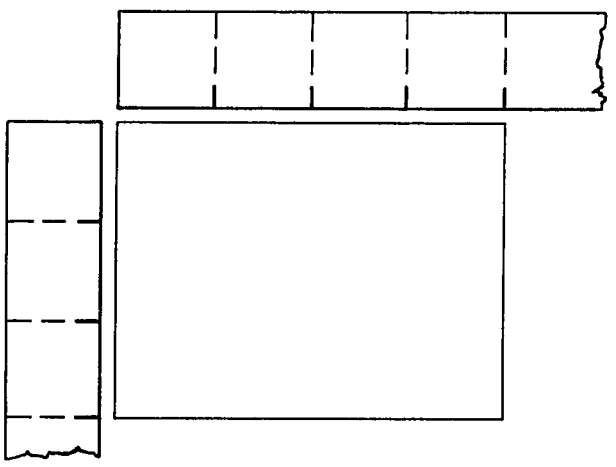
Tell the children to look at Figure *B*. Ask, "How many rows of tiles are needed?" (2) "How many tiles are in each row?" (3) "How many tiles are needed to cover the region?" (6) Write:

$$3 \times 2 = 6$$

Repeat this activity for Figure *C*. (4 rows, 2 in each row; a total of 8 tiles cover the region.) Write:

$$4 \times 2 = 8$$

Give the children their 12-inch rulers. Say, "Use your rulers to help you find out how many tiles are needed to cover the region of Figure *D*?" Get them to fit their rulers along the left edge of Figure *D*. Ask, "How many rows of tiles do you need?" (3) If the children have difficulty, hold up a book and fit a ruler along the left edge of the region. Then fit a 1-inch-square tile against the ruler 3 times and let them count. Ask, "How many



rows of tiles?" (3) If necessary, repeat this process so they can find the number of tiles in each row. (4) Continue until you feel the children can use the ruler to visualise the number of tiles needed to cover the region. Let them use their rulers to help them find the number of tiles that fit in the inside region of Rectangle *E* and Rectangle *F*. Each time, get the children to write the multiplication sentence they use.

Give a 5" × 4" rectangular cutout to every three children. Let them use their rulers to help them find the number of tiles needed to cover the region shown by the cutout. Let them repeat this part of the activity with a 3" × 8" cutout.

STAGE 4 • The Measurement of Time and Capacity

Vocabulary

Hour, minute, o'clock, pint, quart, gallon, capacity

Materials

Two 3-inch sticks (minute hands) for each child; two $2\frac{1}{2}$-inch sticks (hour hands) for each child; demonstration (or real) clockface; 1-quart jars or bottles; two 1-pint jars; one 1-gallon jar; several jars of other capacities; pupil page 111

Teaching Procedure

ACTIVITY 1 • Reading the Clock

Pupil page 111

Tell the children to open their books to page 111. Ask, "What is pictured on the page?" Let the children talk about the picture. Ask, "Does the picture show an *hour* hand? (No) Does it show a *minute* hand?" (No) Say, "How do you know it is a clockface?" (It shows the numerals 1 to 12 arranged like the numerals on a clockface.) Get the children to talk about the numerals and their arrangement in the picture.

Hold up two sticks of different lengths, one for the hour hand and one for the minute hand. Say, "I have hands for the clock." Get them to tell which is the minute hand and which is the hour hand. Give each child a stick to use as an hour hand. Ask, "Can you put the hour hand on the clockface so that it shows 3 *o'clock*?" Walk around the classroom and check to see that the children have done this. If the children have difficulty, hold up pupil page 111 and an hour-hand stick, and demonstrate the position of the hour hand at 3 o'clock. Ask the children to place the hour hand so that it shows other hours; for example, 6 o'clock and 10 o'clock.

Hold up pupil page 111, and place the hour-hand stick so that it shows a time between 3 o'clock and 4 o'clock. Ask, "What time does this hour hand show?" Let the children talk about this. If they do not say, "A time between 3 o'clock and 4 o'clock", say, "The hour hand is between the numeral 3 and the numeral 4; the hour hand shows a time between 3 o'clock and 4 o'clock." Let the children use their hour-hand sticks to show times between hours. For example, let them place the hour hands to show times between 1 o'clock and 2 o'clock and between 8 o'clock and 9 o'clock.

Give each child a minute-hand stick. Say, "Can you put the hour hand and the minute hand on the clock to show 6 o'clock?" If the children have difficulty doing this, demonstrate the position of the hands at 6 o'clock: Let them use their minute-hand sticks and hour-hand sticks to show times like 1 o'clock, 5 o'clock and so on.

Put the demonstration clock (a real clock if you have one) in a place where the children can see it. Revise the idea of quarter-hour intervals on the clockface. Get the children to understand "quarter past the hour", "half past the hour" and "three quarters past the hour". (See *Primary Two*, Unit 7.)

Say, "Show where to put the hour and minute hands on your clockface so that it 'reads half past 3 o'clock'." Let the children show this time on their clockfaces. Check to see that they can do this. Let them show other times at quarter-hour intervals on their clockfaces. Continue this activity until they understand "quarter past", "half past" and "three quarters past" the hour.

Put the demonstration clock in a place where all the children can see it. Fix the hour hand at the numeral 6 and the minute hand at the numeral 12. Ask, "What time does the clock show?" (6 o'clock) Move the minute hand to three minutes past the hour and move the hour hand very slightly beyond the numeral 6. Ask, "What time does the clock show?" Let the children talk about this. Answers like, "A little past 6 o'clock", or "It is past 6 o'clock", are to be expected. Point to the 3-minute mark on the face and say, "It is 3 minutes past 6 o'clock." Point to the minute marks starting at the numeral 12 and count the 3 one-minute intervals. Say, "The minute hand shows 3 minutes past the hour; it is 3 minutes past 6 o'clock." Move the minute hand to the 5-minute mark. Move the hour hand a little further past the numeral 6. Ask, "What time does the clock show?" Let the children talk about this. Put your finger on the minute marks and get the class to count the number of minutes past the hour. Say, "The minute hand shows 5 minutes past the hour; it is 5 minutes past 6 o'clock." Repeat this activity for 10 minutes past 6 o'clock and 13 minutes past 6 o'clock and so on.

Run your finger around the minute scale on the clockface. Ask, "Can you count the number of one-minute marks on the clockface?" Help the children to count the one-minute intervals by pointing to the marks on the clock as they count. When the children have counted 15 minutes, stop moving your finger and say, "We have counted 15 minutes; we are one quarter of the way around the clockface." Move your finger again and let the children continue to count again, "16, 17, 18, 19" and so on. When the counting reaches 30 minutes, stop moving your finger and say, "We have counted 30 minutes; we are halfway around the clockface." Let the children continue counting until they reach the 45-minute mark. Stop moving your finger and say, "We have reached the 45-minute mark; we are three quarters of the way around the clockface." Let them continue the counting until they reach the 60-minute mark. Say, "We have counted all of the one-minute marks on the clockface; there are 60 minute marks."

Set the hour hand on the demonstration clock a little past the numeral 8. Set the minute hand at the 15-minute mark. Ask, "What time does the clock show?" The children will probably say, "It is quarter past 8 o'clock." Agree with this, and then say, "How many minutes past 8 o'clock does the

minute hand show?" Get the children to count the number of minutes. Say, "It is 15 minutes past 7 o'clock; it is quarter past 7 o'clock."

Continue this activity with the children. Let them count the number of minutes past the hour. Get them to say, "It is 35 minutes past 10 o'clock; it is 43 minutes past 5 o'clock" and so on. Get them to see that half past the hour is 30 minutes past the hour, and three quarters past the hour is 45 minutes past the hour.

ACTIVITY 3 • Measures of Capacity (gallon, quart and pint)

Hold up a 1-*quart* jar or bottle and say, "This is a 1-quart jar; it has a *capacity* of 1 quart." Mark the jar "1 quart". Fill the jar with water and say, "The 1-quart jar now holds 1 quart of water." Pick up a jar that has a capacity of less than 1 quart. Hold up this jar in your left hand and hold the jar containing the quart of water in your right hand. Pour water from the quart jar into the other jar until it is full. Ask, "Does the jar in my left hand have a quart of water in it?" (No) "Does it have less than a quart of water in it?" (Yes) Say, holding up the smaller jar, "This jar has a capacity of less than 1 quart."

Fill the quart jar with water and hold it in your right hand. Pick up the 1-*gallon* jar in your left hand (or put it in place where all the children can see). Pour all of the water from the 1-quart jar into the 1-gallon jar. Ask, "What is the capacity of this large jar?" Let the children say, "The capacity is more than 1 quart." Let a child fill the quart jar with water and pour the second quart of water into the gallon jar. Ask, "How many quarts are in the large jar now?" Get the children to say, "There are 2 quarts of water in the large jar." Ask, "Has the large jar been filled?" (No) "Can more water be put into it?" (Yes) Continue to let the children pour quarts of water into the 1-gallon jar. Let them count the number of quarts that have been poured into the 1-gallon jar. When the jar is full, ask, "What is the capacity of the large jar?" (4 quarts) Say, "The capacity of the large jar is 4 quarts; the capacity is also 1-*gallon*. Let the children say that the big jar has a capacity of 1 gallon. Label the jar "1 gallon". Repeat the activity, using the quart jar filled with water and two 1-*pint* jars. Get them to see that a quart of water is 2 pints of water. Let them tell which jars have a capacity of 1 pint. Label the 1-pint jar.

Remove all of the jars from the top of the desk except those labelled "1 gallon", "1 quart" and "1 pint". Fill a squash bottle with water. Ask, "How much water is in the bottle?" Let the children guess. Ask, "How can we find out who has made the best guess?" If the children do not suggest it, pour the water into the 1-gallon jar. Ask, "Is it more than 1 gallon?" (No) "Is it less than 1 gallon?" (Yes) Pour the water into the quart jar to find if it is more than or less than 1 quart. Pour the water into one of the 1-pint jars to see if it is more than or less than 1 pint.

Repeat this kind of activity with other jars or containers. First let children guess how much water is in a jar or bottle; then let them test to see who has made the best guess. Let children come to the front of the room to pour the water and make the tests.

Revise the idea of gallon, quart and pint. Get the children to see that 2 pints of a liquid make 1 quart, and 4 quarts make 1 gallon.

UNIT 9

Techniques of Multiplication and Division I

OBJECTIVES

1. To revise the multiplication facts for products to 36 and extend the products to 100
2. To relate the idea of multiples of a number to skip counting by that number
3. To develop the techniques of multiplication with one factor less than 10 and one factor greater than 10
4. To revise the division facts for products less than 36 and extend to greater products
5. To develop the techniques of division with the known factor less than 10

Background Information for Teachers

See Part 2, "Arithmetic", pages 47-52, in your *Teachers' Handbook*.

In Unit 3, the children were introduced to the meaning of multiplication and division, and they learned some of the basic multiplication and division facts. In this unit, we develop the techniques of these operations, using the idea of place value, the associative property of multiplication and the distributive property.

We used the idea of renaming numbers in the techniques of addition and subtraction. In a similar way, renaming numbers help make the processes of multiplication and division clear.

First, consider the techniques of multiplication with one factor a multiple of 10 (10, 20, 30, . . . , 100). For example, to find the product of 2 and 30, we think:

$$\begin{aligned} 2 \times 30 &= 2 \times (3 \times 10) \\ &= (2 \times 3) \times 10 \\ &= 6 \times 10 \\ &= 60 \end{aligned}$$

In this example, the associative property is used to express the product as the product of two previously learned facts: $(2 \times 3 = 6)$ and $(6 \times 10 = 60)$.

Next, consider a product of the type 4×36. We find this product by renaming 36 in expanded form and using the distributive property.

$$\begin{aligned} 4 \times 36 &= 4 \times (30 + 6) \\ &= (4 \times 30) + (4 \times 6) \end{aligned}$$

This changes the product to the sum of two previously learned products: $(4 \times 30 = 120)$ and $(4 \times 6 = 24)$. Then:

$$\begin{aligned} 4 \times 36 &= 120 + 24 \\ &= 144 \end{aligned}$$

This may also be written in a vertical form:

$$\begin{array}{r} 30 + 6 \\ \times \quad 4 \\ \hline 120 + 24 = 144 \end{array}$$

In this form, the numerals are arranged in a way that identifies the two products, $4 \times 6 = 24$ and $4 \times 30 = 120$, and shows the use of the distributive property.

The form that the children use in this unit is:

$$\begin{array}{r} 36 \\ \times \quad 4 \\ \hline 24 \\ 120 \\ \hline 144 \end{array}$$

They think, $4 \times 6 = 24$ and $4 \times 30 = 120$. They write the numerals 24 and 120 in column form and find the sum. We stress this arrangement because it is a an important transition to the final, shorter form:

$$\begin{array}{r} 36 \\ \times \quad 4 \\ \hline 144 \end{array}$$

In using the shorter form a child thinks, "$4 \times 6 = 24$; 24 is 2 tens and 4 ones. (He writes '4' in the ones' place and remembers that 2 tens is also a part of the product.) $4 \times 30 = 120$ or 12 tens; 12 tens + 2 tens = 14 tens. (He writes '4' in the tens' place and '1' in the hundreds' place.)" Note that the child is using the distributive property as he finds the two products, $4 \times 6 = 24$ and $4 \times 30 = 120$.

As another example, consider the product of 3 and 153. Again we write the numeral in expanded form and use the distributive property.

$$\begin{aligned} 3 \times 153 &= 3 \times (100 + 50 + 3) \\ &= (3 \times 100) + (3 \times 50) + (3 \times 3) \end{aligned}$$

The product is now the sum of three simpler products: $3 \times 3 = 9$, $3 \times 50 = 150$, $3 \times 100 = 300$. Then:

$$\begin{aligned} 3 \times 153 &= 300 + 150 + 9 \\ &= 459 \end{aligned}$$

This work can be shortened to:

$$\begin{array}{r} 100 + 50 + 3 \\ \times \qquad\quad 3 \\ \hline 300 + 150 + 9 = 459 \end{array}$$

The numerals can also be arranged as follows:

$$\begin{array}{r} 153 \\ \times \quad 3 \\ \hline 9 \\ 150 \\ 300 \\ \hline 459 \end{array}$$

Finally, after the children understand this technique, they may use the shorter form:

$$\begin{array}{r} 153 \\ \times \quad 3 \\ \hline 459 \end{array}$$

Notice that, regardless of the form used, the children always think of the product as the sum of simpler products.

Division is defined in terms of multiplication. When we write "$75 \div 3 = n$", we want the children to think, "What number must be multiplied by 3 to equal 75?" Division is the process of finding a missing factor when the product and one factor are known.

To find n for which "$75 \div 3 = n$" is true, the children look for the missing factor in the sentence

$$3 \times (\ ?\ ) = 75.$$

Since the missing addend is greater than 9, the distributive property suggests that they find two addends each of which is a multiple of 3. (They think of a multiple of 3 that is less than 75.) One convenient addend is 60. They rename 75 as 60 + 15.

$$3 \times (\ ?\ ) = 60 + 15$$

Since 3×20 is 60 and 3×5 is 15, they know that

$$3 \times (20 + 5) = 60 + 15$$

is a true sentence. The missing factor is 20 + 5, or 25.

The missing factor may be found directly by division. By folding arrays of dots or by thinking of the use of the distributive property, the children see that

$$75 \div 3 = (60 + 15) \div 3 = (60 \div 3) + (15 \div 3).$$

In other words, division is *distributive over addition.* This is shown by the children in this way:

$$\begin{array}{r} 20 + 5 = 25 \\ \overline{3)60 + 15} \end{array}$$

This form helps to show that $60 \div 3 = 20$, $15 \div 3 = 5$, and $(60 + 15) \div 3 = 20 + 5$.

Another way the children learn to arrange the written work is:

```
  20 + 5 = 25
3)75
  60        (60 ÷ 3 = 20)
  15        (15 ÷ 3 = 5)
```

In this form, they choose a multiple of 3 (60) and it is subtracted from 75 to find the other addend of 75 (15). Each addend is divided by 3. Any multiple of 3 can be chosen as an addend of 75. For example:

```
  10 + 10 + 5 = 25
3)75
  30        (30 ÷ 3 = 10)
  45
  30        (30 ÷ 3 = 10)
  15        (15 ÷ 3 = 5)
```

Consider another example: $265 \div 5 = n$. To divide 265 by 5, choose as one addend of 265 any multiple of 5 near 265, such as 200. (The children learn by experience that it is convenient to choose a large number for an addend which is also a multiple of 10, of 100 and so on.) Then by subtraction, find that the other addend of 265 is 65. Choose a multiple of 5 near 65, such as 50. Again subtract to find what addend remains (15). The work may be shown like this:

```
   40 + 10 + 3 = 53
5)265
  200       (200 ÷ 5 = 40)
   65
   50       (50 ÷ 5 = 10)
   15       (15 ÷ 5 = 3)
```

The children will become skillful in choosing multiples by practice. In the above example, the children may choose 250 as the multiple of 5 nearest 265. Then the work would look like this:

```
   50 + 3 = 53
5)265
  250       (250 ÷ 5 = 50)
   15       (15 ÷ 5 = 3)
```

STAGE 1 • Revision of the Meaning of Multiplication and Division; Products up to 100

Vocabulary

Multiple

Materials

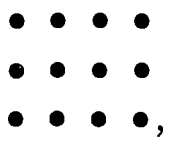
Multiplication sentence cards for products to 36; 10 by 10 array prepared by each pupil in Unit 3; counters; large cardboard (4 feet by 4 feet) for making a multiplication chart; pupil pages 112-116

Teaching Procedure

ACTIVITY 1 • Using Arrays to Show Products

Pupil page 112

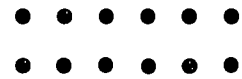
Ask a pupil to arrange 12 seeds or 12 other counters to form an array. After he has formed an array, ask him to tell what product his array shows. If he makes this array,



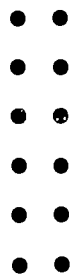
he will say that his array shows that $3 \times 4 = 12$. Draw a picture of his array on the blackboard, and write "$3 \times 4 = 12$" below it. Give a second child 12 other seeds and let him form a different array. Ask him what product his array shows. Draw a picture of his array and write the multiplication sentence below it. Continue this activity with other pupils until every possible array of 12 seeds is shown. The work on the blackboard should then look like this:

$1 \times 12 = 12$

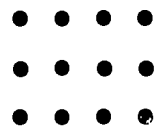
$12 \times 1 = 12$



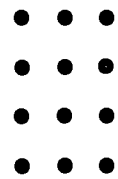
$2 \times 6 = 12$



$6 \times 2 = 12$



$3 \times 4 = 12$

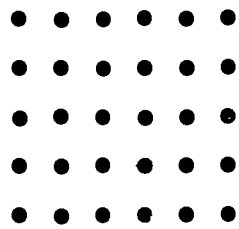


$4 \times 3 = 12$

Be sure that the pupils understand that the first factor of the product shows the number of rows and the second factor shows the number of seeds in each row.

Write another true multiplication sentence on the blackboard, such as $3 \times 5 = 15$. Let a pupil show this product with an array of seeds. Repeat with several other products.

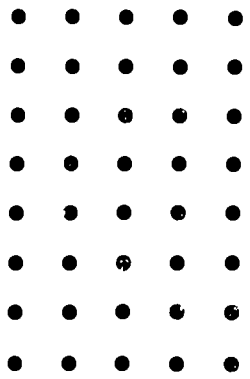
Draw on the blackboard an array of 5 rows with 6 in each row, and ask a pupil to write a true mathematical sentence for the array. ($5 \times 6 = 30$)



$5 \times 6 = 30$

Put other rectangular arrays on the blackboard and ask the same question. Vary the activity by writing mathematical sentences and asking pupils to show with arrays whether the sentences are true.

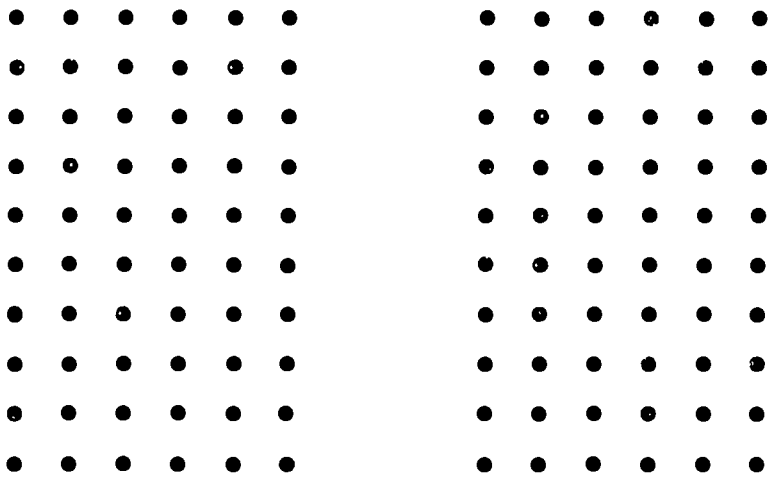
Ask this problem, "In a room, there are 8 rows of chairs with 5 chairs in each row. How many chairs are in the room?" The pupils should be able to solve this problem mentally by thinking that the problem tells about an 8 by 5 array. ($8 \times 5 = 40$) If some children have difficulties, let them show an array for the problem or draw an array on the blackboard.



$8 \times 5 = 40$

Ask other problems similar to the above which lead to other arrays, such as 6 by 6, 7 by 5 and 8 by 4 arrays.

Ask such problems as: "In a room, the seats are arranged in two sections with a corridor at the middle. If in each section there are 10 rows of chairs with 6 chairs in each row, how many chairs are in the room?" If a pupil gives the correct answer (120), let him draw an array for the problem on the blackboard



$10 \times 6 = 60$ $\qquad$ $10 \times 6 = 60$

$60 + 60 = 120$

Guide the children to understand that the number of chairs in the room may be found without counting each chair. Get them to think of the chairs in two arrays of 10 by 6 ($10 \times 6 = 60$; $60 + 60 = 120$) and say, "There are 120 chairs altogether in the room."

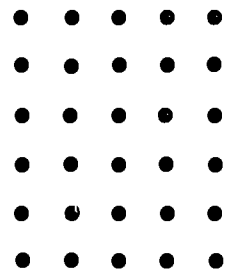
Ask the pupils to open their books to page 112 and work the exercises in their exercise books. Go around the classroom giving help as needed.

ACTIVITY 2 • Revising Products to 36

Pupil page 113

With multiplication sentence cards, revise factors and products to 36. Begin the revision with questions about the sentences on the cards, such as: "What is the product of 6 and 5?" (30) "What is 6 times 5?" (30) "What is six fives?" (30) In some cases, when an answer is given ask the pupils to show how they decided on the product. They should illustrate their answers in any of the following ways:

6×5 is 30; or 6×5 is $5 + 5 + 5 + 5 + 5 + 5$ is 30; or



6 rows of 5 dots is 30 dots; $6 \times 5 = 30$.

Hold up a card with 24 written on one side of it and ask, "What are two factors of 24?" (8×3, 4×6, 12×2, 1×24) Let the children arrange 24 seeds in rectangular arrays in as many ways as possible. This serves as a check as to whether all the factors of 24 have been found. Let the children match sentence cards to the arrays. Repeat this with other products on the cards, such as 36, 32, 27, 16 and 12.

Hold a card with 36 written on it and say, "One factor of 36 is 9, what is the other factor?" When a pupil gives a reply, ask him to show the product with an array. Ask the pupils to express this in a mathematical sentence, indicating the missing factor with the letter n. ($9 \times n = 36$, $n \times 9 = 36$, or

$36 \div 9 = n$.) Be sure that all these forms are given and that the pupils know the sentences have the same meaning. Repeat with other products and factors.

Write on the blackboard the mathematical sentence $3 \times n = 18$. Ask the pupils to find the number that makes the sentence true. (n is 6.) Write some other sentences and ask the same question. After sufficient examples, get the pupils to open their books to page 113. Ask them to find the numbers that make the sentences true.

ACTIVITY 3 • Building a Multiplication Chart

Note: It will be helpful for you to begin to prepare a large chart (about 4 feet by 4 feet) on cardboard of all multiplication facts for use from day to day in the classroom. Draw the chart of this activity in the top left corner. This will leave room for you to extend the chart to a 12 by 12 chart in the activities that follow.

Revise products by helping the children build a multiplication chart to include products up to 36. Draw a chart on the blackboard like the one shown below.

×	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

Ask the children, "What number is 4×5?" (20) Get them to show a 4 by 5 array on the chart by tracing around 4 rows of squares with 5 squares in each row. They do this by counting 4 squares down in the left column and 5 squares across to the column headed 5. They then count the squares in the array. (20) Write 20 in the square forming the intersection of row 4 and column 5. Repeat this with other factors (3×5, 4×9, 6×2, 5×3 and so on) until the chart is completed. The chart will then look like this:

×	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Get the children to check several products again by counting the squares in the arrays formed. Get the pupils to observe that the product written at the intersection of a column and row is the same as the number obtained by skip counting by the number at the head of the row or column. For example, the products along row 3 are 3, 6, 9, 12, 15, 18. These are the numbers used in skip counting by 3 six times. Ask the pupils to skip count by 6, six times. (6, 12, 18, 24, 30, 36) Ask, "What are the products along column 6?" (6, 12, 18, 24, 30, 36) Get the children to notice that 6 × 6 is the number they get by skip counting by 6, six times.

After the class has completed their chart on the blackboard ask each child to make one for himself. Let him make it in an exercise book containing squared paper.

Note: By now the children should know well products of 36 and less and the factors of these numbers.

ACTIVITY 4 • Extending Products to 100

Pupil page 114

Give the children their 10 by 10 array of dots prepared for Unit 3.

Draw a chart on the blackboard as shown on the following page.

Ask a pupil to show a 6 by 9 array of squares and to count the squares by skip counting by 6, nine times. (6, 12, 18, 24, 30, 36, 42, 48, 54) Then ask, "What is 6 × 9?" (54) If the pupils do not understand that the 54 they got by skip counting is the product of 6 and 9, ask them to show the meaning of 6 × 9. They may say any of the following: 6 rows of things with 9 in each row; a rectangular array of 6 rows of objects with 9 objects in each row; 6 nines. Any of these will help them see that 6 × 9 = 54. Let a child show this on the chart by counting 6 squares down along the left and 9 squares

×	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
4										
5										
6									54	
7										
8										
9										
10										

across the top to column 9 and trace around the 6 by 9 array. Let another child count all the squares enclosed. (54) You write 54 in the square that is at the intersection of row 6 and column 9, as shown above.

Write the mathematical sentence $6 \times 9 = n$ on the blackboard and ask for the number n that makes it true. (n is 54.) Ask, "If 9 is a factor of 54, what is the other factor?" (6) Ask a pupil to write this as a mathematical sentence on the blackboard, using n as a missing factor. ($9 \times n = 54$, $n \times 9 = 54$, $54 \div 9 = n$) Accept any of these.

Get the children to show this array by folding their paper arrays. Let them talk about their 6 by 9 array as they did the other arrays. Also relate the process of skip counting to multiplication.

Let the children complete the whole chart. (Complete this section of your 4 foot-by-4 foot cardboard chart also.) Let the children use the chart to answer questions like these:

If 8 is one factor of 48, what is the other?
What is the product of 7 and 9?
There is a 49 on the chart. What are the factors of 49?
I see 48 on the chart twice. Why is it on the chart twice?

Each time a product is talked about, get the children to write a multiplication sentence about the product.

Go on until the children understand how the chart is made and its use in finding products and factors.

Ask the children to put their 10 by 10 array of dots on their tables and to open their books to page 114. Let the children use their paper arrays as

they make the sentences on page 114 true. Ask them to write the sentences in their exercise books.

Do not let the children use the multiplication chart. Ask them to use their rectangular arrays of dots to show, for example, 6×7, and then to skip count to find the product. (The children will need to draw an array for Exercise 6.)

Note: The children should now know well the multiplication facts with products of 36 and less and be able to find products of 100 and less by counting objects in arrays.

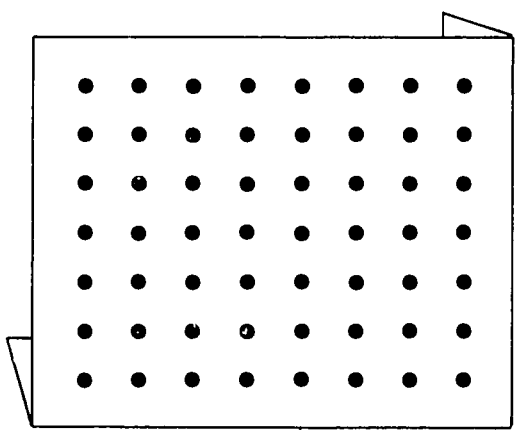
ACTIVITY 5 • Using the Distributive Property to Find New Products

Note: In this activity, the children make use of the distributive property to learn new multiplication facts from facts they already know. For example, the children discover the product of 5 and 9 by renaming 9 as $6 + 3$. Using the distributive property, 5×9 is found by using two known facts, 5×6 and 5×3. This is shown by:

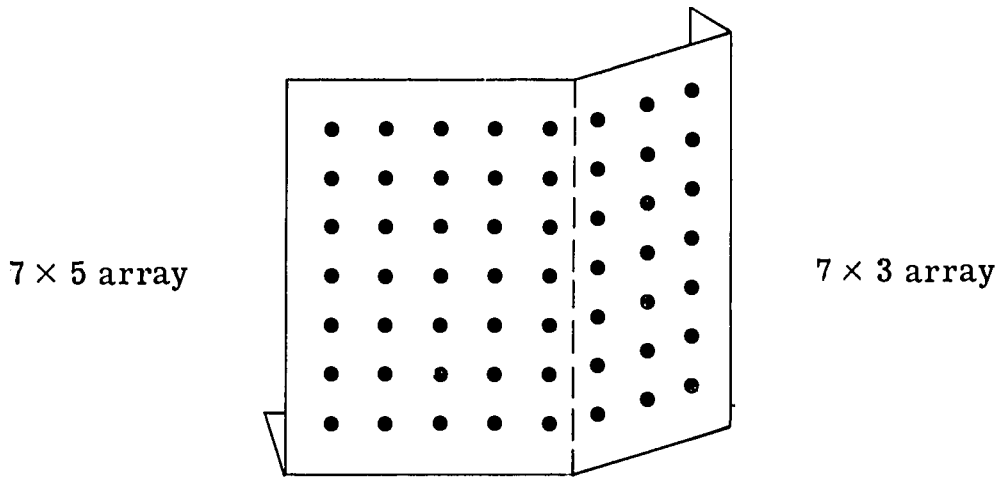
$$\begin{aligned} 5 \times 9 &= 5 \times (6 + 3) \\ &= (5 \times 6) + (5 \times 3) \quad \text{(Using the distributive property)} \\ &= 30 + 15 \\ &= 45 \end{aligned}$$

Let the children put out their 10 by 10 arrays of dots. Get them to fold their arrays to form 7 by 8 arrays.

7 by 8 array



Now say, "We will each separate our array into two arrays by folding." Get them to fold their arrays as shown below. Ask, "What products are shown?" (7×5; 7×3)



$$7 \times 8 = (7 \times 5) + (7 \times 3)$$

Ask, "What is 7×5? What is 7×3? What is $(7 \times 5) + (7 \times 3)$? What is 7×8?" Let the children talk about how they found 7×8. (We made a 7×8 array; we separated it into two arrays of 7×5 and 7×3; we know $7 \times 5 = 35$ and $7 \times 3 = 21$; we know $7 \times 8 = 35 + 21$, or 56.) You say, "You used some products you know to find that $7 \times 8 = 56$."

Get the children to fold their 7 by 8 arrays into two other arrays, for example, 7 by 4 and 7 by 4. Ask, "How many dots in each array?" (28 and 28) "How can we find the number of dots in the large array?" ($28 + 28 = 56$) Let the children say that $7 \times 8 = (7 \times 4) + (7 \times 4)$, or $28 + 28$.

Go on in a similar way with other examples, such as 8×6 and 6×9. Each time let the children make an array and fold it into two arrays so that they use the facts they already know to find a product that is new to them.

They should be able to find 6×8 and perhaps say, "I know $6 \times 8 = 48$ because I know $6 \times 5 = 30$ and $6 \times 3 = 18$ and $30 + 18 = 48$."

ACTIVITY 6 • Introducing Multiples

Pupil page 115

Display your demonstration multiplication chart (10 squares by 10 squares) and let the children talk about how the chart was made. (You may

wish to reproduce the chart on the blackboard.) Ask the pupils to skip count by 6, ten times. (6, 12, 18, 24, 30, 36, 42, 48, 54, 60) Ask, "Do you find the numbers you have just used on the chart?" (Yes; they are the products along row 6 of the chart and down column 6 of the chart.) Get them to understand that the numbers obtained by skip counting by 6, are the products of column 6 and of row 6 on the chart. Help them to say that this is true of all the columns and rows on the chart. For example, the numbers of row 9 are the numbers used in skip counting by 9.

Ask the pupils, "How can you find 8×6 if you do not know it?" They may say any of these: (1) Skip count by 6, eight times; (2) Make an 8 by 6 array of dots and count the dots; (3) Make an array and separate it into smaller arrays. Let them skip count and you write these numbers on the blackboard:

6, 12, 18, 24, 30, 36, 42, 48

Ask, "How do you get each of these numbers by multiplying by 6?" Let them tell you that 6 is 1×6, 12 is 2×6 and so on. Write these products on the blackboard so it looks like this:

6,	12,	18,	24,	30,	36,	42,	48
1×6;	2×6;	3×6;	4×6;	5×6;	6×6;	7×6;	8×6

Say, "The numbers 6, 12, 18 and so on, are called *multiples* of 6. These are numbers we get by multiplying counting numbers by 6."

Ask, "What are some *multiples* of 8?"

8,	16,	24,	32,	40,	48
1×8;	2×8;	3×8;	4×8;	5×8;	6×8

Ask, "Is 40 a multiple of 8? Is 40 the product of some counting number and 8?" (Yes; 40 is 5×8.) Continue with questions about multiples until the children see that when they skip count by a number, they get multiples of that number. Get them to say that a row of the multiplication chart shows the multiples of the number of that row. Also, a column of the chart shows the multiples of the number of that column. For example, every number in column 8 or row 8 (8, 16, 24, 32, 40, 48, 56, 64, 72, 80) is a multiple of 8. In

other words, every number used to skip count by 8 is a multiple of 8. Ask the pupils to give some multiples of 7. (7, 14, 21, 28, 35, 42, 49, . . .) Go on as needed, asking for multiples of 10, 6 and so on.

Let the children again give the multiples of 6 and 8. Write them on the blackboard:

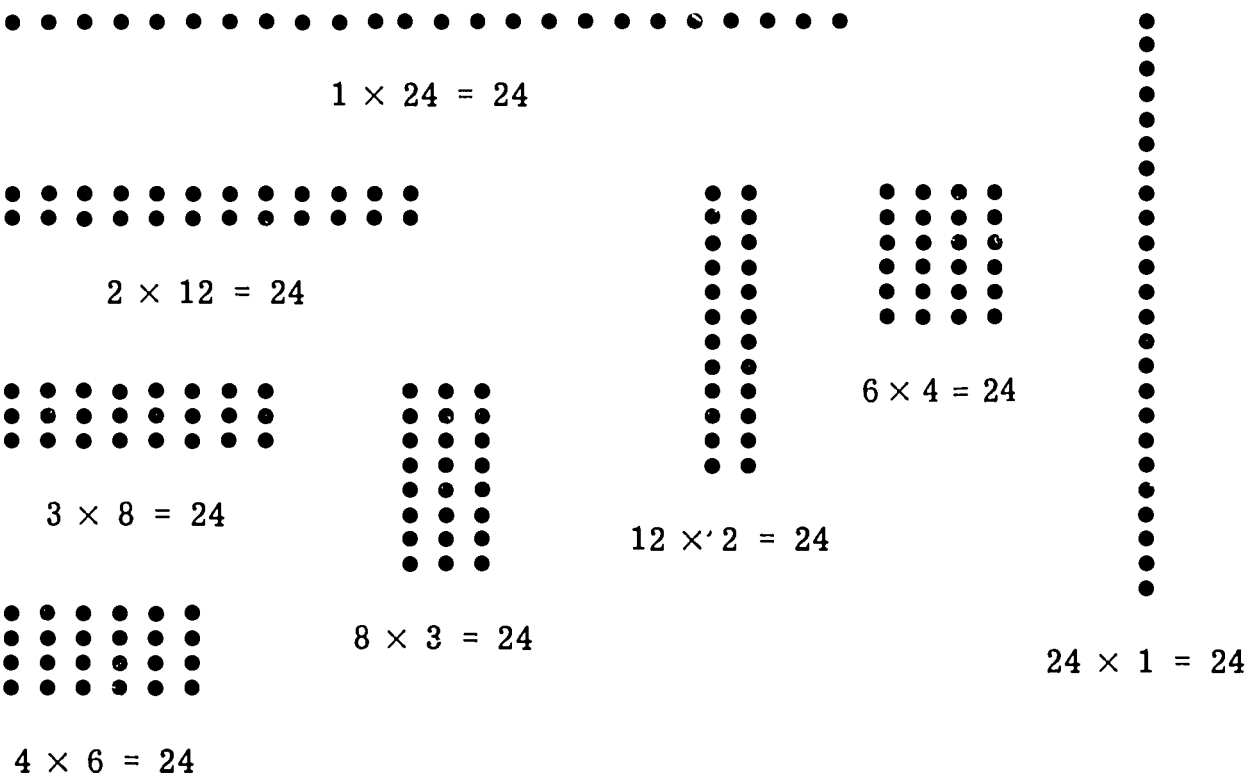
Multiples of 6: 6, 12, 18, 24, 30, 36, 48, . . .
Multiples of 8: 8, 16, 24, 32, 40, 48, 56, . . .

Ask, "What number is a multiple of 6 and is also a multiple of 8?" (24 is a multiple of 6 and of 8. 48 is a multiple of 6 and of 8 and so on.)

Ask the children to name some multiples of 4. Write these on the blackboard under the multiples of 8.

Multiples of 4: 4, 8, 12, 16, 20, 24, 28, . . .

Ask, "What number is a multiple of 4, of 6 and also of 8?" (24) "Is 24 a multiple of any other number?" The children may give several answers. Let them check their answers by looking at the multiplication chart and by arranging 24 counters in rectangular arrays in as many ways as possible. Here are all the possibilities:



With their arrays the children will show that 24 is a multiple of 1, 2, 3, 4, 6 and of 8.

Choose another number, say 30, and ask for all the numbers of which 30 is a multiple. (30 is a multiple of 1 because 1×30 is 30; 30 is a multiple of 2, because 2×15 is 30; 30 is a multiple of 3, because 3×10 is 30; 30 is a multiple of 5, because 5×6 is 30; 30 is a multiple of 6, because 6×5 is 30; 30 is a multiple of 10, because 10×3 is 30; 30 is a multiple of 15, because 15×2 is 30; 30 is a multiple of 30, because 30×1 is 30.) Repeat for other numbers, such as 16, 28, 32. Let the pupils show their answers with arrays, if needed.

Ask the pupils to turn to page 115 in their books. The first two exercises are worked in the book as samples. Read and let the pupils talk about what they mean. Then let the pupils work the other exercises by telling what numerals must be put in the blank spaces to make true sentences.

ACTIVITY 7 • Extending Division Facts to Products Greater Than 36

Pupil page 116

Place 24 counters on the table. Ask three pupils to come up and share the counters equally. Let each pupil tell how many he has. (8) Ask, "What is 24 divided by 3?" (8) Write on the blackboard,

$$24 \div 3 = 8.$$

Ask; "Is this sentence true?" (Yes) "If one factor of 24 is 3, what is the other factor?" (8) Write the sentence again on the blackboard, with 8 replaced with n.

$$24 \div 3 = n$$

Ask the pupils to tell a multiplication sentence that asks the same question. Then write

$$3 \times n = 24.$$

Ask, "What number will make this sentence true?" (8) Get the pupils to

see that the sentences express the same idea and 8 makes both of them true. Guide the pupils to say that 24 is a multiple of 8 and 24 is a multiple of 3.

Repeat this activity with other products and factors. Use sentences such as:

$$36 \div 4 = n; \quad 27 \div 3 = n; \quad 49 \div 7 = n; \quad 70 \div 10 = n$$

In each case, stress the connection between multiplication and division; division is a process of finding a missing factor; and a product is a multiple of each of its factors.

After many examples, ask the pupils to open their books to page 116. Ask them to study and talk about the first exercise. (In the first column, a division sentence is given. In the second column, the corresponding multiplication sentence is given. In the third column, the missing factor is given.) Ask the pupils to copy the sentence from the first column in their exercise books and then fill in the other columns.

STAGE 2 • Development of Techniques of Multiplication

Vocabulary

No new vocabulary

Materials

Two sets of 2" × 3" numeral (10, 20, 30, . . . , 200) cards; 12 by 12 array of dots for use in demonstration; pupil pages 117-125

Teaching Procedure

ACTIVITY 1 • Multiplying a Multiple of 10 by a Number Less Than 10

Pupil page 117

Write on the blackboard the sentence

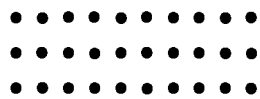
$$3 \times 10 = n.$$

Let the pupils talk about what the sentence means and how they can show its meaning. They will say things like this:

3 × 10 means 10 + 10 + 10 or 30;
An array of 3 rows of 10 counters uses 30 counters;
A 3 by 10 array of counters uses 30 counters;
I can skip count: 10, 20, 30.

Show their ideas on the blackboard as they talk:

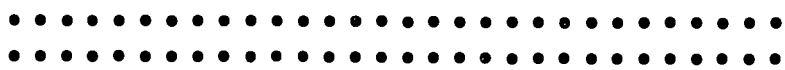
10 + 10 + 10 is 30.



3 × 10 is 30.

Write on the blackboard the sentence $2 \times 30 = n$. Ask the meaning of 2×30. Follow the same plan used with the sentence $3 \times 10 = n$. Ask the children to write their replies on the blackboard:

30 + 30 is 60.



2 × 30 is 60.

Write the sentence $2 \times 30 = n$ on the blackboard again. Say, "We can find n another way." Ask the children to rename 30 as a product with one factor of 10. (3×30) You write on the blackboard:

$$2 \times 30 = 2 \times (3 \times 10)$$

Guide the children to suggest another form of grouping the factors in

$2 \times (3 \times 10)$. The children may recall the associative property of multiplication they learned in Unit 6. Or they may say that 2 times 3 tens is 6 tens. Let them suggest this grouping:

$$2 \times 30 = (2 \times 3) \times 10.$$

Arrange the work on the blackboard as follows:

$$\begin{aligned} 2 \times 30 &= 2 \times (3 \times 10) \\ &= (2 \times 3) \times 10 \\ &= 6 \times 10 \\ &= 60 \end{aligned}$$

Tell the children they will write the work in this way:

$$2 \times 30 = n, \quad \text{or} \quad \begin{array}{r} 30 \\ \underline{\times\ 2} \\ 60 \end{array}$$

Use the same plan for other examples, such as $2 \times 40 = n$; $3 \times 30 = n$; $6 \times 30 = n$. For example, let a child write the product:

$$2 \times 40 = n, \quad \text{or} \quad \begin{array}{r} 40 \\ \underline{\times\ 2} \\ 80 \end{array}$$

Let the pupils explain the work. They may say, for example, "2 times 40 is 2 times 4 tens, or 8 tens."

Give some other examples to be done orally. For example, $5 \times 50 = n$ and $3 \times 80 = n$. Guide the children to say, "5 × 5 tens is 25 tens. 25 tens is 250."

Continue the activity as needed. Sometimes ask the children to give the product orally. Other times let them write the multiplication sentence on the blackboard.

Separate the class into two teams. Prepare two sets of numeral cards with multiples of ten written on each. (10, 20, 30, 40, 50, 60, . . . , 200) Give each team a complete set of cards. There should be two cards bearing each multiple of ten, one for each team. Write on the blackboard a mathematical sentence of either of these types: $5 \times 10 = n$; $3 \times 40 = n$. Ask the pupils having the cards showing the product that makes the sentence true to quickly

come out and place their card over n. Give a point to the team of the first pupil to make the sentence true. You may stop the game after one team has scored ten points, or when most children have skill in finding the product of a multiple of ten and any other factor less than 10.

Let the pupils open their books to page 117 and complete the exercises on the page. The division sentences may first be thought of as multiplication sentences. For example, Sentence 12 may be thought of as $7 \times m = 70$, and m is 10. The sentence is true because 7×10 is 70.

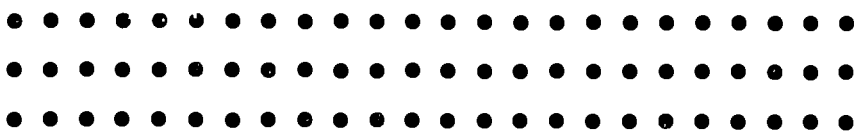
ACTIVITY 2 • Multiplying a Factor Less Than 100 by a Factor Less Than 10

Pupil pages 118–120

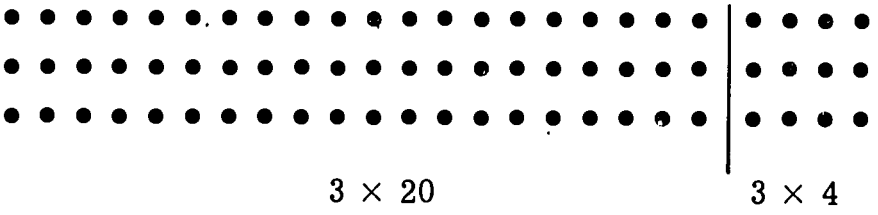
Write on the blackboard the multiplication sentence

$$3 \times 24 = n.$$

Ask one of the pupils to show 3×24 with an array of blocks on your table. Let another child draw a 3 by 24 array on the blackboard:



Guide the children to separate the array into two arrays showing products that they already know. When a pupil suggests a 3 by 20 array and a 3 by 4 array, get all the children to think about the arrangement:



Ask a pupil to find each of the products shown by the parts of the array. ($3 \times 20 = 60$; $3 \times 4 = 12$) Ask, "How did you rename 24 when you separated the array?" ($20 + 4$) Write:

$$
\begin{aligned}
3 \times 24 &= 3 \times (20 + 4) \\
&= (3 \times 20) + (3 \times 4) \\
&= 60 + 12 \\
&= 72
\end{aligned}
$$

Let the pupils help with each step. Get a child to explain that when we write $(3 \times 20) + (3 \times 4)$ as another name for $3 \times (20 + 4)$, we are using the distributive property.

Write "$2 \times 24 = n$" on the blackboard. Ask the pupils to rename 24 in expanded form. (20 + 4) Write:

$$2 \times 24 = 2 \times (20 + 4)$$

Guide the pupils to tell you to change this to the sum of two products:

$$2 \times 24 = (2 \times 20) + (2 \times 4)$$

Arrange the work on the blackboard:

$$
\begin{aligned}
2 \times 24 &= 2 \times (20 + 4) \\
&= (2 \times 20) + (2 \times 4) \\
&= 40 + 8 \\
&= 48
\end{aligned}
$$

Tell the pupils that we will write the exercise in vertical form as shown below. Let the children talk about what each part of the form means.

$$
\begin{array}{r}
20 + 4 \\
\times \quad 2 \\
\hline
40 + 8 = 48
\end{array}
\qquad \text{or} \qquad
\begin{array}{rl}
24 & \\
\times\ 2 & \\
\hline
8 & (2 \times 4 = 8) \\
\underline{40} & (2 \times 20 = 40) \\
48 &
\end{array}
$$

Note: Emphasize the second form because it is the form that will be used by the pupils until the process has been fully mastered.

Give the children another exercise, for example, 2×36, and follow the same steps used for 2×24. Let the children draw an array if needed

and guide them to think and say, "We rename 36 as 30 + 6; we use the distributive property and write (2 × 30) + (2 × 6); we find the products and add." (60 + 12 is 72.) The blackboard looks like this:

$$\begin{aligned} 2 \times 36 &= (2 \times 30) + (2 \times 6) \\ &= 60 + 12 \\ &= 72 \end{aligned}$$

Write the exercise in both forms as shown below. Let the children with your guidance find the products and write numerals to complete these:

$$\begin{array}{r} 30 + 6 \\ \times \quad\quad 2 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 36 \\ \times\ 2 \\ \hline \\ ____ \end{array} \quad \begin{array}{l} \\ \\ (2 \times 6 = \quad\ ) \\ (2 \times 30 = \quad\ ) \end{array}$$

Make sure the pupils understand that to find the product of 2 and 36 they rename 36 as 30 + 6 and find the sum of the two products 2 × 6 and 2 × 30. Let them talk about how the two forms above help them to rename 36 and to find the product.

Continue in a similar way with other exercises, such as 3 × 28. Let the children use this form:

$$\begin{array}{r} 28 \\ \times\ 3 \\ \hline 24 \\ \underline{60} \\ 84 \end{array} \quad \begin{array}{l} \\ \\ (3 \times 8 = 24) \\ (3 \times 20 = 60) \\ \\ \end{array}$$

The children will think, "Rename 28 as 20 + 8; 3 × 8 = 24; 3 × 20 = 60; 24 + 60 = 84."

Always stress the renaming of a factor in expanded form; the use of the distributive property; and the fact that to multiply in these exercises we find the sum of two products.

Ask the children to do the multiplication exercises on page 118. Let them use any of the written forms they choose. Let a child work only as many of the exercises as you think he can do with reasonable success.

On days following, revise the form below and ask the children to use it in finding the products on pages 119 and 120.

$$\begin{array}{r} 32 \\ \times\ 6 \\ \hline 12 \\ 180 \\ \hline 192 \end{array} \quad \begin{array}{l} \\ \\ (6 \times 2 = 12) \\ (6 \times 30 = 180) \\ \\ \end{array}$$

<u>32 × 6 is 192.</u>

Go round the class giving help as needed, talk with the children about what they are doing. They may say things like: To find 6 × 32, I multiply 6 × 2 and then 6 × 30; I use the distributive property; I write down the numerals for the two products and add them.

ACTIVITY 3 • Solving Story Problems

Pupil pages 121-123

Write on the blackboard a word problem like this:

> A bus has 15 rows of seats, with 5 seats in each row. How many people can ride in the bus?

Ask the pupils to read the problem. Let them talk about what the problem tells. (There are 15 rows of seats with 5 seats in each row.) Ask, "What question does the problem ask?" (How many people can ride in the bus?) "Do 15 rows of seats with 5 seats in each row form an array?" (Yes) "What type of an array?" (A 5 by 15 array) Let the pupils give a mathematical sentence the array shows. ($5 \times 15 = n$) Guide the children to see that the number of seats in the bus is the number that makes this sentence true. Let a child find the product of 5 and 15 by writing on the blackboard:

$5 \times 15 = n$

$$\begin{array}{r} 15 \\ \times\ 5 \\ \hline 25 \\ 50 \\ \hline 75 \end{array} \quad \begin{array}{l} \\ \\ (5 \times 5 = 25) \\ (5 \times 10 = 50) \\ \\ \end{array}$$

<u>15 × 5 is 75.</u>

Ask, "What number makes $5 \times 15 = n$ a true sentence?" (n is 75.) "What is the answer to the problem?" (75 people can ride in the bus.)

If needed, work another problem with the children. Then ask the pupils to open their books to pages 121, 122 and 123. For each problem they should

first write a mathematical sentence; then find the number that makes the sentence true; and finally, answer the question asked by the problem. (Let them use the form of multiplication shown just above.)

ACTIVITY 4 • Multiplication (one factor greater than 100)

Pupil pages 124–125

Write on the blackboard the mathematical sentence $3 \times 123 = n$. Ask the children to write 123 in expanded form. (100 + 20 + 3) Then write:

$$3 \times 123 = 3 \times (100 + 20 + 3)$$

Ask the pupils whether the distributive property can be used here. Then write:

$$3 \times (100 + 20 + 3) = (3 \times 100) + (3 \times 20) + (3 \times 3)$$

Say, "We have changed the product to the sum of three simple products: 3×3, 3×20, 3×100." Show the work as follows:

$$\begin{aligned} 3 \times 123 &= 3 \times (100 + 20 + 3) \\ &= (3 \times 100) + (3 \times 20) + (3 \times 3) \\ &= 300 + 60 + 9 \\ &= 369 \end{aligned}$$

Then write the two shorter forms. (Write them one at a time. Let the children talk about them as you write.)

$3 \times 123 = n$

$$\begin{array}{r} 100 + 20 + 3 \\ \times \qquad\qquad 3 \\ \hline 300 + 60 + 9 = 369 \end{array}$$

$$\begin{array}{rl} 123 & \\ \times\ \ 3 & \\ \hline 9 & (3 \times 3 = 9) \\ 60 & (3 \times 20 = 60) \\ \underline{300} & (3 \times 100 = 300) \\ 369 & \end{array}$$

Let a pupil explain each step in each of these forms. (We change the product to the sum of simpler products: 3×3 is 9, 3×20 is 60, 3×100 is 300.

We use the distributive property. We add the three products, 9, 60 and 300. 3×123 is 369.)

Write another example on the blackboard, such as $3 \times 217 = n$. Ask a pupil to write 217 in expanded form. (200 + 10 + 7) Let him arrange the work on the blackboard in this form:

$$3 \times 217 = n \qquad \begin{array}{r} 200 + 10 + 7 \\ \times \qquad\qquad 3 \\ \hline 600 + 30 + 21 = 651 \end{array}$$

Ask another pupil to arrange the work in the shorter form without showing the renaming of 217.

$$3 \times 217 = n \qquad \begin{array}{rl} 217 & \\ \times\ \ 3 & \\ \hline 21 & (3 \times 7 = 21) \\ 30 & (3 \times 10 = 30) \\ \underline{600} & (3 \times 200 = 600) \\ 651 & \end{array}$$

Always ask the pupils to explain each step as the multiplication is done.

Using the same plan, let a pupil work another example on the blackboard, such as $4 \times 162 = n$. His work should look like this:

$$4 \times 162 = n \qquad \begin{array}{r} 100 + 60 + 2 \\ \times \qquad\qquad 4 \\ \hline 400 + 240 + 8 = 648 \end{array} \qquad \text{or} \qquad \begin{array}{rl} 162 & \\ \times\ \ 4 & \\ \hline 8 & (4 \times 2 = 8) \\ 240 & (4 \times 60 = 240) \\ \underline{400} & (4 \times 100 = 400) \\ 648 & \end{array}$$

Write several similar multiplication sentences on the blackboard and let the pupils work them, using only the last form. (The pupils may or may not write the simpler products to the right of the work as shown above.)

The children may discover that they may also write their work as shown on the left below. If they wish let them do several exercises in both forms. Let them talk about why they may use either plan of work. For example:

$$\begin{array}{r} 145 \\ \times\ \ 3 \\ \hline 300 \\ 120 \\ 15 \\ \hline 435 \end{array} \quad \begin{array}{l} \\ \\ (3 \times 100 = 300) \\ (3 \times 40 = 120) \\ (3 \times 5 = 15) \\ \\ \end{array} \qquad \text{or} \qquad \begin{array}{r} 145 \\ \times\ \ 3 \\ \hline 15 \\ 120 \\ 300 \\ \hline 435 \end{array} \quad \begin{array}{l} \\ \\ (3 \times 5 = 15) \\ (3 \times 40 = 120) \\ (3 \times 100 = 300) \\ \\ \end{array}$$

After more practice, ask the children to open their books to page 124 and work the exercises, using only the forms on the right above. Work the first two exercises with them. Their work will look like this:

$$2 \times 143 = n \quad \begin{array}{r} 143 \\ \times\ \ 2 \\ \hline 6 \\ 80 \\ 200 \\ \hline 286 \end{array} \quad \begin{array}{l} \\ \\ (2 \times 3 = 6) \\ (2 \times 40 = 80) \\ (2 \times 100 = 200) \\ \\ \end{array} \qquad 3 \times 102 = n \quad \begin{array}{r} 102 \\ \times\ \ 3 \\ \hline 6 \\ 300 \\ \hline 306 \end{array} \quad \begin{array}{l} \\ \\ (3 \times 2 = 6) \\ (3 \times 100 = 300) \\ \\ \end{array}$$

Let them notice that a sentence like Exercise 11, $78 \times 3 = \square$, may be written as $3 \times 78 = \square$ by use of the commutative property.

Get the children to continue with the exercises on page 125 as needed.

STAGE 3 • Techniques of Division with the Known Factor Not More Than 10

Vocabulary

No new vocabulary

Materials

Pupil pages 126-129

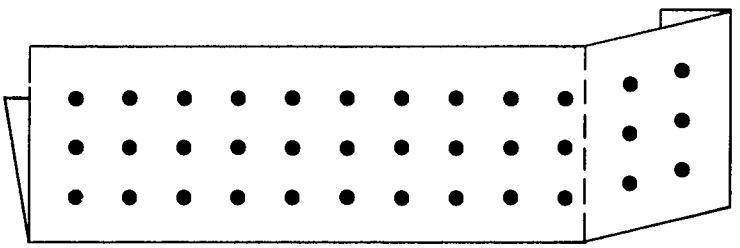
Teaching Procedure

ACTIVITY 1 • Renaming Numbers

pupil page 126

The purpose of this activity is to help pupils rename numbers in the process of dividing.

Write "36" on the blackboard. Ask the pupils to rename this number as the sum of two addends. (33 + 3, 30 + 6, 26 + 10, 28 + 8, 21 + 15, 24 + 12, 18 + 18 and so on) Ask them to rename 36 as the sum of two addends each of which is a multiple of 3. (33 + 3, 30 + 6, 27 + 9, 24 + 12, 21 + 15, 18 + 18) Help the children rename the numbers by folding a demonstration 3 by 12 array in several ways. For example, the renaming of 30 + 6 can be shown as follows:



$3 \times 10 = 30$ $\qquad$ $3 \times 2 = 6$

Ask the pupils to rename 36 as two addends so that one addend is a multiple of 10. (30 + 6, 20 + 16, 10 + 26) Ask, "Can 36 be renamed as the sum of three addends?" (20 + 10 + 6, 6 + 15 + 15 and so on)

Write "256" on the blackboard. Ask the pupils to rename 256 in expanded form. (256 = 200 + 50 + 6) Ask that 256 be renamed as the sum of three addends that are each multiples of 4. (200 + 40 + 16) Write other numerals on the blackboard, some with 2 digits and others with 3 digits. Ask the pupils to rename the numbers in several ways. After some practice, tell the pupils to open their books to page 126 and rename the numbers in the exercises.

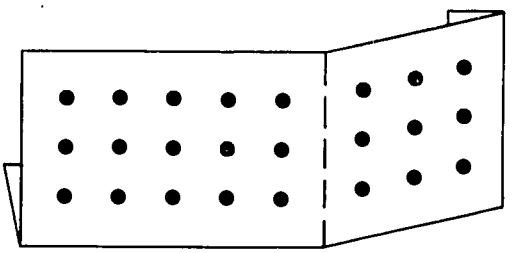
Fold your demonstration array to show a 3 by 8 array. Ask one of the pupils to write a true mathematical sentence that the array shows. ($3 \times 8 = 24$) Ask, "How many dots in the array?" (24) Say, as you rewrite the sentence replacing the 8 with n, "This is a sentence we have when we do not know one factor. What kind of a mathematical sentence do we call it?" (A multiplication sentence with a missing factor)

$$3 \times n = 24$$

Guide a pupil to say that the sentence can be written as a division sentence and write:

$$24 \div 3 = n$$

Say, "Rename 24 as the sum of two multiples of 3." If the children rename 24 as 15 + 9, fold your demonstration array into two arrays of 3 by 15 and 3 by 9 as shown below.



$15 \div 3$ $\qquad$ $9 \div 3$

Say, "We will divide each part of 24 by 3." Ask, "What is $15 \div 3$?" (5) "What is $9 \div 3$?" (3) "What is $(15 \div 3) + (9 \div 3)$?" (5 + 3, or 8) Let a child tell why $24 \div 3$ is another name for $(15 \div 3) + (9 \div 3)$. Write the following on the blackboard as you and the children talk:

$$\begin{aligned} 24 \div 3 &= (15 + 9) \div 3 \\ &= (15 \div 3) + (9 \div 3) \\ &= 5 + 3 \\ &= 8 \end{aligned}$$

Give another example such as:

$$36 \div 4 = (20 + 16) \div 4$$
$$= (20 \div 4) + (16 \div 4)$$

Guide the pupils to say that these examples show a property of division. The property is similar to the distributive property of multiplication over addition. Say, "We will call this the *distributive property of division over addition.*"

Let the pupils show other examples of the distributive property of division. Write these numerals on the blackboard and let pupils give other numerals for the numbers, using the distributive property:

$(18 + 6) \div 3$	$(18 \div 3) + (6 \div 3)$
$(15 + 10) \div 5$	$(15 \div 5) + (10 \div 5)$
$(8 \div 4) + (12 \div 4)$	$(8 + 12) \div 4$
$(12 \div 6) + (24 \div 6)$	$(12 + 24) \div 6$
$(18 + 9) \div 9$	$(18 \div 9) + (9 \div 9)$

ACTIVITY 3 • Dividing with the Product Less Than 100

Pupil pages 127-128

Write on the blackboard the division sentence $36 \div 3 = n$. Make sure the pupils understand these meanings of the division sentence: the sentence means the same as $3 \times n = 36$; the sentence says to find the number which when multiplied by 3 gives 36; the sentence means the product is 36, one factor is 3, and we are to find the missing factor. Write on the blackboard these two sentences:

$$36 \div 3 = n; \qquad 3 \times n = 36$$

Tell the pupils that they are to make the sentences true by finding the missing factor. Say, "You may use either form of the sentence you wish, but think of it as a division sentence." Let the children show a 3 by n array of 36 dots.

Ask the pupils to rename 36 as two addends that are multiples of 3.

(33 + 3, 30 + 6, 27 + 9, 24 + 12, 21 + 15, 18 + 18) Let them choose one of these names to use as the new name for 36. (For example, 24 + 12)

$$36 \div 3 = (24 + 12) \div 3$$

Ask, "How can we use the distributive property of division to rewrite the sentence?" Guide the pupils to suggest this:

$$(24 + 12) \div 3 = (24 \div 3) + (12 \div 3)$$

Get the pupils to say that the division is changed to two simpler divisions which they know. Arrange the work as follows:

$$\begin{aligned} 36 \div 3 &= (24 + 12) \div 3 \\ &= (24 \div 3) + (12 \div 3) \\ &= 8 + 4 \\ &= 12 \end{aligned}$$

Let the class show each stage of the above with an array.

Ask a pupil to rename 36 as the sum of two other addends that are each a multiple of 3. (For example, 30 + 6) Write as you and the children talk about dividing 36 by 3:

$$\begin{aligned} 36 \div 3 &= (30 + 6) \div 3 \\ &= (30 \div 3) + (6 \div 3) \\ &= 10 + 2 \\ &= 12 \end{aligned}$$

Use other pairs of addends of 36 in the same way. Guide the pupils to talk about which pair of addends it is best to choose. (A multiple of ten is easy to use.) Tell the pupils that they get the same result no matter how 36 is renamed.

Let the pupils check whether 12 is the number that makes the sentence $36 \div 3 = n$ true. They will show how to find the missing factor from the multiplication sentence $3 \times n = 36$. Write on the blackboard $3 \times n = 36$. Rename 36 as 30 + 6 and write:

$$3 \times n = 30 + 6$$

Ask, "What multiplied by 10 gives 30? What number multiplied by 3 gives 6?" Write:

$$3 \times (10 + 2) = 30 + 6$$

Ask, "Why is this a true sentence?" (It is true because of the distributive property.)

Write on the blackboard this word problem:

> There are 56 boys in 4 clubs. If each of the clubs has the same number of members, how many boys are in each club?

Ask, "What do you know?" (There are 56 boys in 4 clubs.) "What question does the problem ask?" (How many boys are in each club?) Let the pupils talk about the problem and make a mathematical sentence for the problem. ($56 \div 4 = n$) To find n, ask the pupils to rename 56 as two addends that are each a multiple of 4. Ask them whether one addend should also be a multiple of 10. (It may be. A multiple of 10 is easy to use.)

$$56 \div 4 = (40 + 16) \div 4$$

Ask the pupils to use the distributive property of division. Write the work on the blackboard.

$$\begin{aligned} 56 \div 4 &= (40 + 16) \div 4 \\ &= (40 \div 4) + (16 \div 4) \\ &= 10 + 4 \\ &= 14 \end{aligned}$$

Ask, "What number makes $56 \div 4 = n$ a true sentence?" (n is 14.) "What is the answer to the question in the problem?" (There are 14 boys in each club.)

Say, "We can show the division process like this." Write:

$$56 \div 4 = (40 + 16) \div 4 \qquad \begin{array}{r} 10 + 4 = 14 \\ 4\overline{)40 + 16} \end{array}$$

In this arrangement guide the pupils to think: Forty divided by four is ten; Sixteen divided by 4 is 4; Ten plus four is fourteen.

Also, show the pupils the work arranged like this:

$56 \div 4 = (40 + 16) \div 4$

$$\begin{array}{r|ll} & 10 + 4 = 14 & \\ \hline 4 & 56 & \\ & \underline{40} & (40 \div 4 = 10) \\ & 16 & (16 \div 4 = 4) \end{array}$$

Talk with the pupils about each step in the following way: Select an addend of 56 that is a multiple of 4, for example, 40; subtract 40 from 56 to find the other addend (16); divide each addend by 4; 10 + 4 = 14.

Let a pupil rename 56 as the sum of two other multiples of 4 and divide. For example, a pupil may rename 56 as 32 + 24. Then the work is:

$56 \div 4 = (32 + 24) \div 4$

$$\begin{array}{r|l} & 8 + 6 = 14 \\ \hline 4 & 32 + 24 \end{array} \quad \text{or} \quad \begin{array}{r|ll} & 8 + 6 = 14 & \\ \hline 4 & 56 & \\ & \underline{32} & (32 \div 4 = 8) \\ & 24 & (24 \div 4 = 6) \end{array}$$

Another pupil may rename 56 as the sum of three or more multiples of 4, such as 20 + 12 + 24, 20 + 20 + 16, or 12 + 16 + 12 + 6. Using the name 24 + 20 + 12, the work is as follows:

$56 \div 4 = (24 + 20 + 12) \div 4$

$$\begin{array}{r|ll} & 6 + 5 + 3 = 14 & \\ \hline 4 & 56 & \\ & \underline{24} & (24 \div 4 = 6) \\ & 32 & \\ & \underline{20} & (20 \div 4 = 5) \\ & 12 & (12 \div 4 = 3) \end{array}$$

Let the pupils decide that they can choose any set of multiples of the known factor as addends. They will find, however, that an addend that is a multiple of ten is easy to use.

Let the class work other exercises as needed.

Ask the children to work the exercises on page 127, using this form.

$48 \div 8 = n$

$$\begin{array}{r|l} & 5 + 1 = 6 \\ \hline 8 & 40 + 8 \end{array} \qquad n \text{ is } 6.$$

Let the children work the exercise on page 128, using this form.

$$72 \div 4 = n \qquad \begin{array}{rl} & \underline{10 + 8 = 18} \\ 4)\overline{72} & \\ \underline{40} & (40 \div 4 = 10) \\ 32 & (32 \div 4 = 8) \end{array} \qquad n \text{ is } 18.$$

ACTIVITY 4 • Dividing with Products Less Than 1000 and Given Factor Less Than 10

Pupil page 129

Write on the blackboard the division sentence

$$726 \div 6 = m.$$

Let the pupils rename 726 as a number of hundreds, tens and ones in various ways so that each addend is a multiple of 6. (600 + 120 + 6, 600 + 90 + 30 + 6) Using one name, write:

$$726 \div 6 = (600 + 120 + 6) \div 6.$$

Ask a pupil to tell how to rewrite this using the distributive property of division. Guide them to tell you to write:

$$(600 \div 6) + (120 \div 6) + (6 \div 6)$$

With the help of the pupils complete the division. The written work will be:

$$\begin{aligned} 726 \div 6 &= (600 + 120 + 6) \div 6 \\ &= (600 \div 6) + (120 \div 6) + (6 \div 6) \\ &= 100 + 20 + 1 \\ &= 121 \end{aligned}$$

Let a pupil come up and write the work in a short form. He will write:

```
  100 +  20 + 1 = 121
6)600 + 120 + 6
```

or

```
  100 + 20 + 1 = 121
6)726
  600      (600 ÷ 6 = 100)
  126
  120      (120 ÷ 6 = 20)
    6      (6 ÷ 6 = 1)
```

If the renaming 726 = 600 + 60 + 66 is chosen, the form will look like this:

```
  100 + 10 + 11 = 121
6)726
  600      (600 ÷ 6 = 100)
  126
   60      (60 ÷ 6 = 10)
   66      (66 ÷ 6 = 11)
```

The form you want the pupils to learn to use is shown below. Some may use it now, others will need a longer time to learn the form.

```
  100 + 20 + 1 = 121
6)726
  600      (600 ÷ 6 = 100)
  126
  120      (120 ÷ 6 = 20)
    6      (6 ÷ 6 = 1)
```

In this form, the pupils go through thinking similar to this:

1. I find an addend of 726 that is a multiple of 6, a multiple of 10 or 100, and as large as possible but less than 726. This addend is 600. 600 ÷ 6 = 100.
2. 726 − 600 = 126. 126 is another addend of 726.
3. I find an addend of 126 that is a multiple of 6, a multiple of 10, and as large as possible but less than 126. This addend is 120. 120 ÷ 6 = 20.
4. 126 − 120 = 6. 6 is another addend of 726.
5. Six is a multiple of 6. 6 ÷ 6 = 1.
6. The missing factor is 100 + 20 + 1 or 121. 726 ÷ 6 is 121.

Let the pupils turn to page 129 and work the exercises in their exercise book. Let them use the written form they understand.

UNIT 10

Operations on Fractions

OBJECTIVES

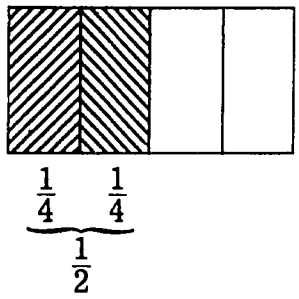
1. To revise the multiples of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$ and to extend the idea to multiples of $\frac{1}{5}$, $\frac{1}{10}$, $\frac{1}{12}$
2. To compare fractions and to understand that one fraction may have many different names
3. To introduce multiplication of a whole number and a fraction through counting unit fractions
4. To introduce addition of fractions with the same denominator
5. To introduce subtraction of fractions by finding the missing addend

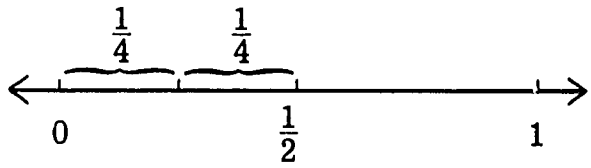
Background Information for Teachers

Also see Background Information for Unit 5 and Part 2, "Arithmetic", pp. 58-66, in your *Teachers' Handbook.*

Fractions are numbers describing parts of wholes: parts of regions, parts of objects, or subsets of sets. A fraction is represented by a pair of numerals which refer to the relationship of part to whole. For example, for the fraction $\frac{1}{6}$, the denominator, 6, refers to the six parts of the same size into which an object is separated, and the numerator, 1, refers to the number of these parts.

Fractions describe parts of a whole, just as whole numbers describe sets of whole objects. Addition of whole numbers is used to describe a set formed by putting two sets together. Addition of fractions is used similarly to describe a part of a whole formed by putting two parts together. For example, $\frac{1}{4} + \frac{1}{4} = \frac{2}{4}$ because two parts that are each $\frac{1}{4}$ of a whole form a part that is $\frac{2}{4}$ of a whole. The fraction $\frac{1}{2}$ also describes this part, so we say, "$\frac{1}{4} + \frac{1}{4} = \frac{1}{2}$". This is shown by the following figures.





When we think of the individual parts of wholes, we can count these parts as units. For example, if we separate a rectangular region into eight congruent regions, we can count the separate regions: one, two, three and so on. We say that three-eighths describes three of the eight parts.

Adding eighths may be done by counting eighths. When we count two parts (one, two) we can show this by the sentence

$$\frac{1}{8} + \frac{1}{8} = \frac{2}{8}.$$

When we count three of the eighths, the sentence becomes

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}.$$

We can also express this as three times one-eighth:

$$3 \times \frac{1}{8} = \frac{3}{8}$$

To take another example, we write

$$\frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} = 4 \times \frac{1}{5} = \frac{4}{5}.$$

Any fraction $\frac{a}{b}$ can be written as $a \times \frac{1}{b}$, where b is a counting number.

The number line is very useful in showing addition and subtraction of fractions with the same denominator. Addition of two numbers is shown on the number line by moving along two successive segments that show the two addends. For example, the sentence "$\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$" is shown by moving

along a segment showing $\frac{3}{6}$ and then along a segment showing $\frac{2}{6}$. As we do this, we move along three spaces or along three segments showing sixths. Then we move along two more of the "sixth" segments and reach the point for five-sixths. We reach this point because we have moved along 3 + 2, or 5, "sixth" segments.

$$\frac{3}{6} + \frac{2}{6} = \frac{3 + 2}{6} = \frac{5}{6}$$

The number line is useful also in finding the missing addend in the subtraction of fractions with the same denominators. In a sentence such as the following, n is the missing addend: $\frac{7}{8} - \frac{3}{8} = n$. This sentence means

$$\frac{3}{8} + n = \frac{7}{8}.$$

We can move along on the number line from 0 to the point for $\frac{3}{8}$, then ask, "How many more of the segments representing eighths do we move along to reach the point for $\frac{7}{8}$?" The answer is "four more segments". Each of the four segments is described as one-eighth of the whole, so the union of the segments is described as four-eighths of the whole. Thus we say that $\frac{4}{8}$ is the missing addend that makes the sentence true. So n is $\frac{4}{8}$.

STAGE 1 • Counting Unit Fractions

Vocabulary

No new vocabulary

Materials

A rectangular piece of paper about 5" × 7" for each child; one square piece of paper about 6" × 6"; three strips of paper about 2" × 18"; three

strips of paper about 3" × 30"; blank numeral cards; string; pupil pages 130-133.

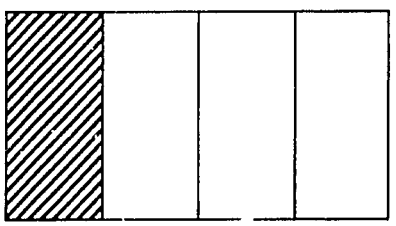
Teaching Procedure

ACTIVITY 1 • Revising Multiples of Fractions

Pupil page 130

Give each child a piece of paper 5" × 7". Ask them to fold the papers into four parts of the same size. (Suggest they fold the papers as shown in the picture below.) Let them unfold the papers and count the fourths. Hold up one paper and point to each part in turn as you count, "One, two, three, four". Say, as you outline the whole, "This shows four fourths." Let the children talk about this.

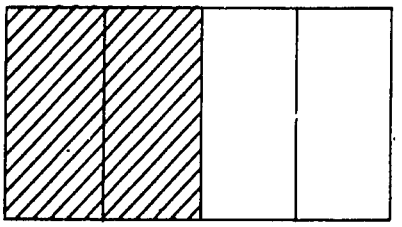
Draw around one of the pieces of paper on the blackboard and draw lines to separate the region shown into four regions of the same size. Shade one of the four regions. Ask, "What part of the whole region is shaded?" (One-fourth) Let the children move their fingers about a region of one-fourth of their papers. Then let a child write the numeral "$\frac{1}{4}$" below the shaded part of the region on the blackboard.



$\frac{1}{4}$

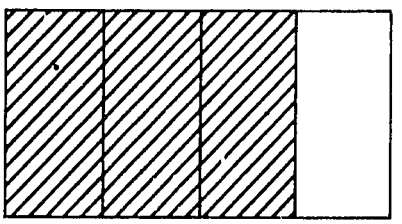
Draw on the blackboard another rectangle exactly like the first. Shade two of the four parts one at a time. Say, "I shaded a part of the whole rectangular region. Count the fourths in the shaded part. How many fourths does the shaded part show?" (Two fourths) Let the children move their fingers about the part of their papers that shows two-fourths. Then write the numeral "$\frac{2}{4}$" just below the middle of the shaded part. Get the children

to fold their papers into halves. Guide them to say, "The part we call two-fourths is the same as the part we call one-half. Two-fourths and one-half are names for the same number."



$\frac{2}{4}$

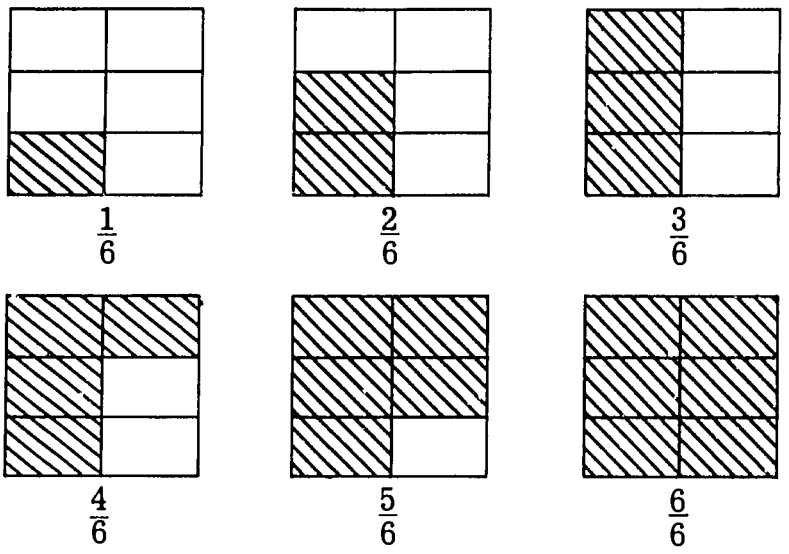
Draw another rectangle like the others on the blackboard and shade three of the four parts. Continue as before. Ask the children to count the fourths as they are shaded. Let them find that the shaded region is three-fourths the whole rectangular region. Ask a child to write the numeral for the fraction below the shaded part.



$\frac{3}{4}$

Draw another rectangle and continue in the same way as you shade all four of the parts. Ask the children the name of the fraction that the region shows. Let a child write a numeral for the fraction. Get the children to say, "All of the region is shaded. We count four fourths. Four fourths is the same as one whole." Write the sentence $\frac{4}{4} = 1$ on the blackboard to express this idea.

Guide a child to fold a 6" × 6" square of paper to show sixths. Help him first fold the paper into three parts of the same size and then to fold the paper again to make six parts of the same size. See the pictures on page 262. Draw around the paper on the blackboard and then draw lines to separate the rectangular region into six parts of the same size. Use the same plan to talk with the children about "sixths" that you used to talk about "fourths". Shade one of the regions at a time and ask, "What part of the region is shaded?" Get the children to tell the name of the fraction that describes the shaded part and write the numeral for the fraction,



As you discuss sixths with the children, emphasize the different names for the same fraction. For example, the children can look at the region shaded to represent three-sixths and see that one-half the rectangular region is shaded. This way they discover that $\frac{3}{6}$ and $\frac{1}{2}$ are names for the same number. Guide the children to write the sentence that expresses this:

$$\frac{3}{6} = \frac{1}{2}$$

Let the children say that in the picture for six-sixths, the whole region is shaded. Thus they see that $\frac{6}{6}$ and 1 are different names for the same number. Let them write:

$$\frac{6}{6} = 1$$

Use the same plan to develop the idea of fractions with other denominators, such as 5, 3, 8 and 10.

Ask the children to point to the Circle *A* on page 130 of their books. Say, "The region of the circle is separated into parts. Are these parts the same size? Into how many parts has the circular region been separated?" (8) "Some of the parts are shaded. What is the name of the fraction that tells how much of the whole region is shaded?" (Three-eighths) Discuss the fact that this number, three-eighths, refers to three of eight parts of

the same size. Let someone write the numeral on the blackboard next to the letter A.

$A, \frac{3}{8}$

Direct the children to copy in their exercise books all the letters that name the circles. Ask them to write next to each letter the numeral for the fraction that describes the shaded region of that figure.

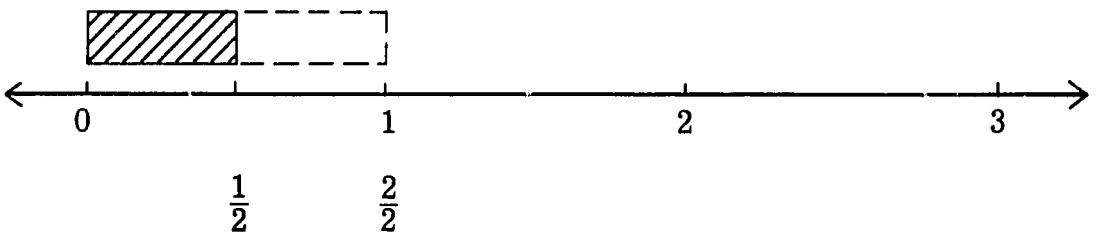
$A, \frac{3}{8}$ $B, \frac{4}{8}$ $C, \frac{7}{8}$ $D, \frac{5}{8}$ $E, \frac{2}{3}$ $F, \frac{3}{4}$ $G, \frac{3}{6}$ $H, \frac{2}{6}$

ACTIVITY 2 • Showing Multiples of Unit Fractions on the Number Line

Pupil page 131

Draw a line segment on the blackboard. Give one child a 2" × 18" strip of paper and help him to mark points for 0, 1, 2 and 3 to make a number line. Let him begin by marking a point for 0 near the left end of the line segment. Then use the strip to mark the points for 1, 2 and 3 by placing the left end of the strip on each point and marking the next point at the right end, as shown below.

Ask another child to fold the strip in half. Let him use the folded strip, holding one end at 0 and marking a point at the other end to show one-half. (See the figure below.) Let him write the numeral "$\frac{1}{2}$" just below the mark. Ask another child to use the same folded strip to count and mark the point for two-halves. They will discover that this is a point that is marked with the label "1". Let the children talk about this and help them to say that they have found another name for the point. It is "two-halves". Ask, "Can you write a numeral for this fraction?" Let a child write "$\frac{2}{2}$" below the point and below the numeral for one.

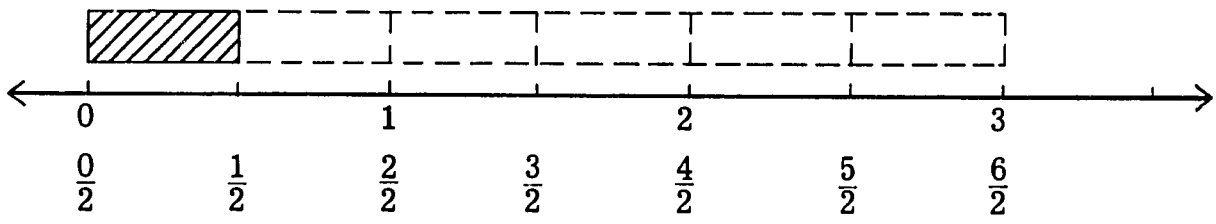


Write the following sentence on the blackboard and say, "This sentence tells us that 'two-halves' and 'one' are names for the same number."

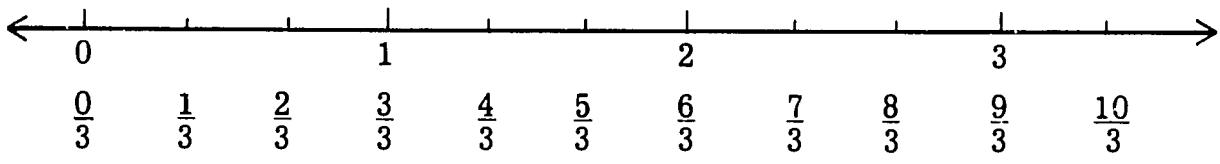
$$\frac{2}{2} = 1$$

Let a child continue marking points, using the strip folded in half, as far as the line is drawn. Let the children count the halves and mark the points: $\frac{1}{2}$, $\frac{2}{2}$, $\frac{3}{2}$ and so on. Explain that some fractions name the same points that the whole numbers on the line name, by writing sentences such as $\frac{4}{2} = 2$ and $\frac{6}{2} = 3$.

Point to the mark labelled "0" and say, "Can you think of another name for this number? How many halves are described by the number 0?" Help the children to conclude that the point is for zero halves. Let a child write the numeral "$\frac{0}{2}$" below the numeral "0".



Draw another number line on the blackboard and use another 2" × 18" strip folded into three parts of the same size to mark thirds. Let the children mark the points and count the thirds. Follow the steps used with the number line for halves.



Construct a number line for fourths, for fifths, for sixths and for other fractions as needed. Continue to label all points, including the point for zero, with names for the fractions.

Ask the children to open their books to page 131. Give them or let them make blank cards to make into numeral cards. Get the children to

notice that each picture of the number line on the page is marked to show different fractions. Discuss the first picture. Ask, "Into how many segments is the segment from 0 to 1 separated?" (2) Let the children find a point that has a numeral card box beneath it. Say, "What will you label this point?" $\left(\frac{1}{2}\right)$ "Write a numeral on a numeral card and place it in the box." Check to see that the children have written "$\frac{1}{2}$" and have placed the card in the proper box. Let them continue to fill all the boxes with numeral cards.

If needed, the children can use paper strips to help in finding the fractions for the points.

ACTIVITY 3 • Different Names for a Fraction

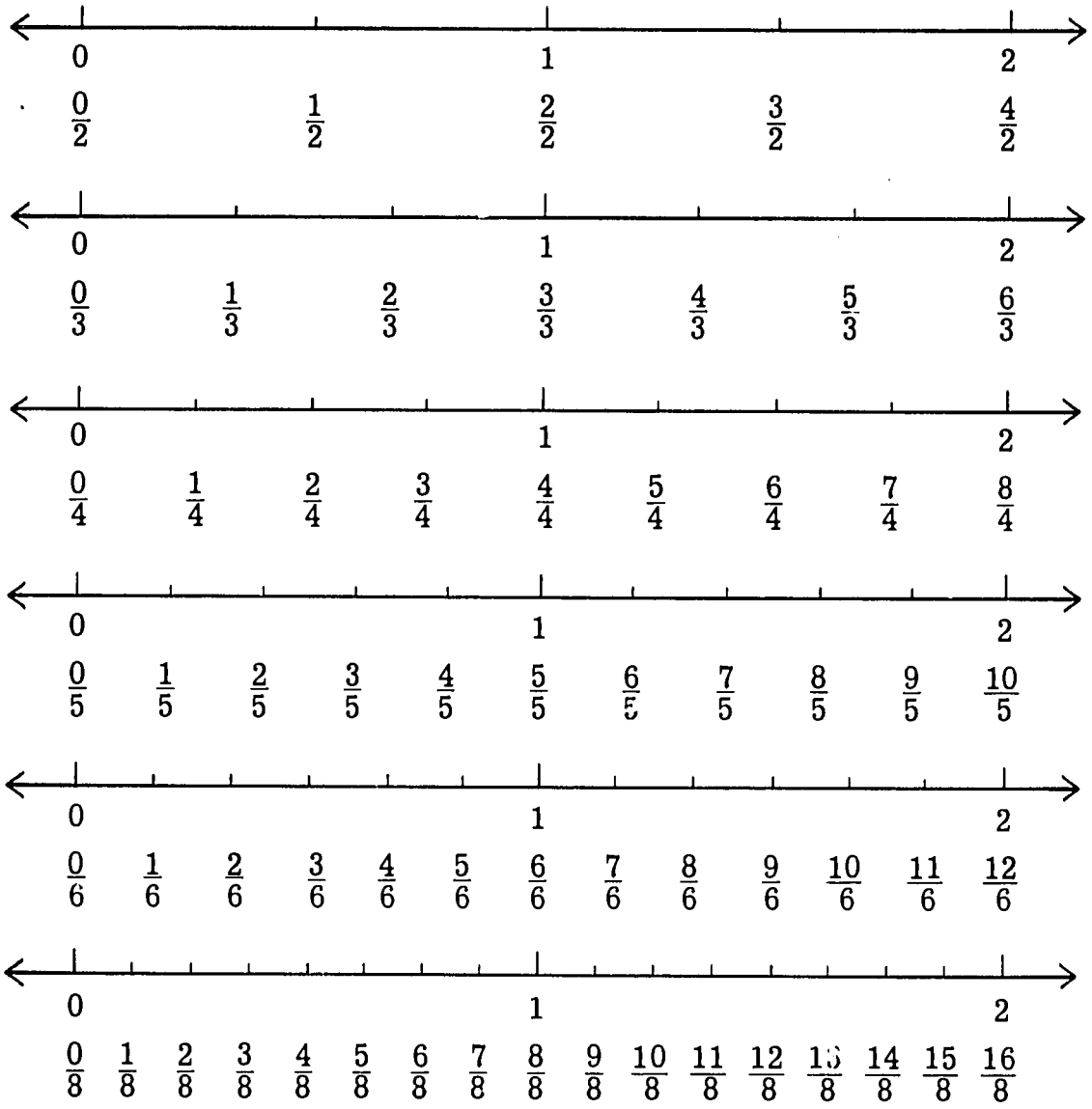
Pupil page 132

Draw six line segments on the blackboard one below the other. Leave enough space between the lines so that numerals for fractions can be written below each line. Use 2" × 30" strips to mark the 0, 1 and 2 points. The points for 0, 1 and 2 on each line should be just below those points on the line above.

Let the children fold the paper strips and mark points for halves, thirds, fourths, fifths, sixths and eighths as shown on the next page. After all the numerals have been written, let the class check whether the segments that are halves are the same size, whether the segments that are thirds are the same size and so on. Let them also check that the numerals are in sequence.

Point to the figure and say, "These are all pictures of the number line, the points for halves are marked on one picture, the points for thirds are marked on another picture and so on." Let the children tell about the idea that 0 in each of the pictures labels the same point on the number line, that 1 in each of the pictures labels the same point on the number line, and 2 in each of the pictures labels the same point on the number line. You can make this clear by stretching a string vertically from the 0 mark of the top picture to the 0 mark of the bottom picture so that the string touches the 0 mark of each picture. Do the same for the point labelled 1. Let them say that this point has many different names. Ask the children to tell some of the names. Write the numerals on the blackboard as they name the fractions. Write the symbol = between each pair to show that the fractions are equal.

$$1 = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} = \frac{5}{5} = \frac{6}{6} = \frac{8}{8}$$



Follow the same plan to show different names for the number 2.

$$2 = \frac{4}{2} = \frac{6}{3} = \frac{8}{4} = \frac{10}{5} = \frac{12}{6} = \frac{16}{8}$$

Ask a child to choose another fraction and find several different names for the fraction. If he chooses $\frac{1}{2}$, for example, help him to stretch the string tightly from the top line to the bottom line to see what points it touches. In this way he can find different names for one-half. Let another child write the numerals on the blackboard as the names are called.

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8}$$

Continue with this procedure, asking children to choose different points and test with the string to find the different names for the points. This plan may be extended to tenths and twelfths by drawing two more number lines.

Leave the number lines on the blackboard so that the children can use them in completing the sentences on page 132. Tell the children to open their books to page 132 and read the instructions at the top. Ask them to copy the incomplete sentences in their exercise books and write numerals in the spaces to make true sentences. Then ask the children to read the instructions for the other exercises. Let them copy these pairs of numerals in their books and make true sentences by using one of the symbols, =, > or <, in each sentence.

ACTIVITY 4 • Discovery of the Idea

Pupil page 133

$$\frac{a}{b} = \frac{a \times c}{b \times c}$$

Write the following sentences on the blackboard and say, "Here are some true sentences you made." Revise briefly the idea that the sentences show different names for the same number.

$\frac{1}{4} = \frac{2}{8}$	$\frac{1}{4} = \frac{3}{12}$	$\frac{1}{5} = \frac{2}{10}$
$\frac{1}{2} = \frac{2}{4}$	$\frac{2}{4} = \frac{6}{12}$	$\frac{2}{3} = \frac{4}{6}$
$\frac{3}{5} = \frac{6}{10}$	$\frac{2}{3} = \frac{6}{9}$	$\frac{1}{3} = \frac{2}{6}$
$\frac{2}{4} = \frac{4}{8}$	$\frac{1}{5} = \frac{3}{15}$	$\frac{3}{4} = \frac{6}{8}$

Discuss these sentences. Guide the children to look for a pattern or to look for similarities in the pairs of fractions. For example, ask them to look at all the pairs of fractions in the first column. Ask, "Is there anything you notice about the pairs of numerators?" Help them to discover that in every pair the second numerator is two times the first numerator. Go on,

"Is there anything you notice about the pairs of denominators?" Help them to discover that in every pair the second denominator is two times the first denominator.

Ask them to look at the pairs of fractions in the second column. Again, guide them to see the relationship between the pairs of numerators and the pairs of denominators. In these examples, they find that the numerator and the denominator of the second fraction are three times the numerator and denominator of the first fraction. Go on to the third column and compare the pairs of fractions in the same way.

In discussing the sentences, help the children to conclude that if both the numerator and denominator of a fraction are multiplied by the same number we get another fraction equal to the first.

Ask, "Do you think this will be true for any fraction we choose?" Let the children suggest other fractions and test their conclusion. In many cases they can use the number lines from Activity 3 to check the result.

Choose some of the sentences from the blackboard and write sentences like these to help the pupils understand the idea.

$$\frac{1}{4} = \frac{1 \times 2}{4 \times 2} = \frac{2}{8}$$

$$\frac{1}{3} = \frac{1 \times 3}{3 \times 3} = \frac{3}{9}$$

$$\frac{2}{3} = \frac{2 \times 2}{3 \times 2} = \frac{4}{6}$$

Ask the children to open their books to page 133 and complete the sentences in their exercise books. The first sentence is completed as an example.

STAGE 2 • Defining $a \times \frac{1}{b}$ as $\frac{a}{b}$

Vocabulary

No new vocabulary

Materials

About ten 2" × 20" strips of paper; pupil pages 134-136.

Teaching Procedure

ACTIVITY 1 • Adding Unit Fractions of the Same Kind

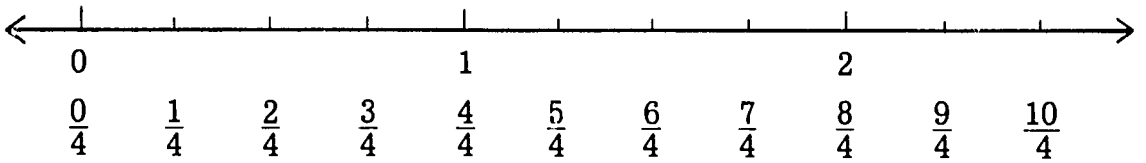
Write the following sentence on the blackboard:

$$\frac{1}{4} + \frac{1}{4} = n$$

Let a child read it. Ask, "What do we mean by $\frac{1}{4} + \frac{1}{4}$?" Let the class talk about this and decide that $\frac{1}{4} + \frac{1}{4}$ describes a part of a whole that is made up of two smaller parts. Each of the smaller parts is one-fourth of the whole. Then $\frac{1}{4} + \frac{1}{4}$ is another name for $\frac{2}{4}$. Guide the children to say that the sentence $\frac{1}{4} + \frac{1}{4} = n$ is true if n is $\frac{2}{4}$.

Say, "We can also count fourths on the number line to show this." Put out two or three 2" × 20" strips of paper. Draw a line segment on the blackboard. It should be more than 40 inches long. Mark a point near the left end of the number line and write the numeral 0 at the point. Help a child use the 2" × 20" strip of paper to mark points on the line segment in this way: Place the strip along the number line with the left end at 0. Mark another point at the right end of the strip and label it 1. Continue marking points on the number line and labelling them with numerals to the right end of the segment.

Ask a child to fold a 2" × 20" strip of paper into fourths. Let another child place the folded strip along the number line and mark the points $\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}$ and so on.



Let the children count, "One fourth, two fourths" as you point to the segments of the line. Ask a child to write the true addition sentence described by the counting:

$$\frac{1}{4} + \frac{1}{4} = \frac{2}{4}$$

Ask the pupils to read another sentence that you write on the blackboard:

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = a$$

Ask, "Can we count fourths on the number line to find a?" (Yes) Let the child who answers point to the line segments and count, "One fourth, two fourths, three fourths". Ask, "What number a makes the sentence true?" $\left(\frac{3}{4}\right)$ Write:

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Ask, "In this sentence, which addends did we add first?" Let the pupils discuss this until they decide that the sentence might be written

$$\left(\frac{1}{4} + \frac{1}{4}\right) + \frac{1}{4} = a \qquad \text{or} \qquad \frac{1}{4} + \left(\frac{1}{4} + \frac{1}{4}\right) = a.$$

Ask, "Will the sum be the same?" Let the children discuss this and count fourths on the number line to help in answering the question. For the first sentence they will count: one fourth and one fourth is two fourths; two fourths and one fourth is three fourths.

$$\left(\frac{1}{4} + \frac{1}{4}\right) + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$$

For the second sentence their counting will show that

$$\frac{1}{4} + \left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4} + \frac{2}{4} = \frac{3}{4}.$$

Follow the same plan used above in adding fourths in sentences expressing addition of thirds, of sixths, of halves and of eighths. Use a number line to show the sentences.

ACTIVITY 2 • Adding Unit Fractions

Pupil page 134

Ask the children to open their books to page 134. Introduce the work on the page by discussing the first two examples with the class. Ask a child to read the first sentence. Let him tell what the sum is. $\left(\frac{2}{3}\right)$ Write the following sentence on the blackboard:

$$\frac{1}{3} + \frac{1}{3} = \frac{2}{3}$$

Ask a child to tell the sum in the second sentence. $\left(\frac{3}{4}\right)$ Say, "The spaces show that we have three addends in the sentence. What numerals should be written in the spaces to make a true sentence?" Let the children tell what the addends are. Write the sentence on the blackboard.

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

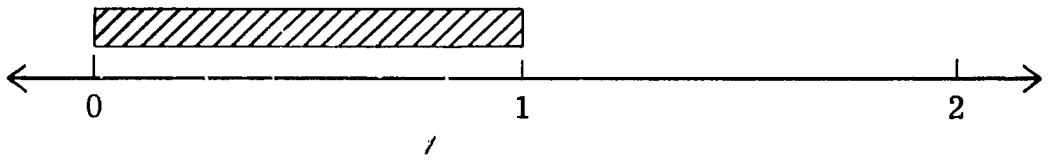
Ask the children to write true sentences for the other exercises in their exercise books.

ACTIVITY 3 • Multiplying a Unit Fraction by a Whole Number

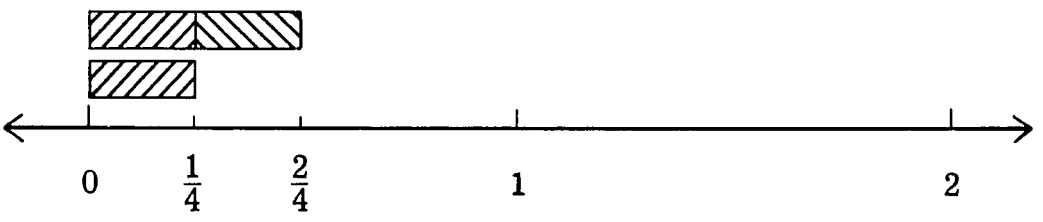
Put out eight 2" × 20" strips of paper. Ask seven children to fold the strips. One strip should be folded into halves, one into fourths, one into

eighths, one into thirds, one into sixths, one into fifths, one into tenths and one strip should remain whole. To help the child fold a strip into fifths, you mark 5 segments on the strip, each of which is 4 inches in length. Similarly, to fold a strip into tenths, mark ten segments, each of which is 2 inches in length. Ask the children to label each part and the whole with a numeral.

Draw a line segment for the number line on the blackboard and mark the point 0 near the left end of the line segment. If convenient, the line segment should be more than 40 inches in length. Select a child to place the whole strip of paper on the line segment with one end on 0 and to write "1" at the point on which the other end of the strip falls. He should continue marking 2, 3 and so on, as needed, along the line segment. See the drawing below.



Ask a child to hold up the strip folded into fourths and to use this strip to mark the *one-fourth* point on the number line. Ask another child to write $\frac{1}{4}$ at the point. Ask another child to unfold the strip to show two fourths. Let him use his strip to mark the two-fourths point. Let another child write $\frac{2}{4}$ at this point.



Fold the strip again into fourths. Ask another child to place this strip on the number line with the left end at the one-fourth point. Let the children think about the point on which the other end of the strip falls. Ask, "How many fourths have we marked?" (Two) Say, "Two of the one-fourth strips reach to the same point as the two-fourth strip." Choose another child to use the strip folded into fourths to mark another one-fourth on the number line. Talk with the children about the line segment from 0 to this point. Count the

fourths. (One, two, three) Match the segment from 0 to this point with the strip unfolded to show three fourths. Write on the blackboard the following sentence:

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \square$$

Say, "By counting we see that we have three fourths. By adding we see that we have three-fourths." Select a child to write the numeral $\frac{3}{4}$ at the point. Then write on the blackboard the following sentence:

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Say, "We have $\frac{1}{4}$ as an addend three times so we counted 3 *one-fourths*. We write this as 3 times *one-fourth*, or $3 \times \frac{1}{4}$." Ask, "What fraction is the product?" $\left(\frac{3}{4}\right)$ Write:

$$3 \times \frac{1}{4} = \frac{3}{4}$$

Say, "By counting or by adding $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$ we see that we use *one-fourth* as an addend 4 times. We write this as 4 times *one-fourth*, or $4 \times \frac{1}{4}$." Ask, "What fraction is the result?" $\left(\frac{4}{4}\right)$ Write:

$$4 \times \frac{1}{4} = \frac{4}{4}$$

Ask, "What is another name for $\frac{4}{4}$?" (1) Write:

$$4 \times \frac{1}{4} = \frac{4}{4} = 1$$

Ask a child to take the strip folded into eights and to use it to mark

segments showing eights on the number line. Follow the procedure just described for fourths. Now ask a child to count the eighths. Write on the blackboard the following sentence:

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \square$$

Say, "By counting or by adding, we see that we have four eights." Select a child to write $\frac{4}{8}$ at the correct point on the number line. Ask, "How many eighths did we count?" (4) "How many eights did we add?" (4) Say, "We have *one-eighth* as an addend 4 times. We write this as 4 times *one-eighth*, or $4 \times \frac{1}{8}$." Ask, "What fraction tells us the result?" $\left(\frac{4}{8}\right)$ Write on the blackboard the following sentence:

$$4 \times \frac{1}{8} = \frac{4}{8}$$

Ask, "What is another name for $\frac{4}{8}$?" $\left(\frac{1}{2}\right)$

Continue the procedure used above for other fractions such as fifths, tenths, thirds, sixths, twelfths, as needed.

ACTIVITY 4 • Multiplying a Unit Fraction by a Whole Number

Pupil page 135

Ask the children to open their books to page 135. Write this sentence on the blackboard.

$$5 \times \frac{1}{12} = \square$$

Ask the pupils to think of the number line marked in twelfths. Let the children tell how many twelfths they count or add to show $5 \times \frac{1}{12}$. (5) Let them also say they counted or used *one-twelfth* as an addend five times. On the blackboard write:

$$5 \times \frac{1}{12} = \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12}$$

Get the children to show the sentence on the number line. Ask, "What fraction describes the line segment shown by the sentence?" $\left(\frac{5}{12}\right)$ Write:

$$5 \times \frac{1}{12} = \square$$

Five-twelfths makes the sentence true. Get the children to look at the first exercise on page 135 and read it. Say, "This sentence has been made true for you. Let us make the other sentences true." Ask a child to read the next sentence and follow the plan used above for $5 \times \frac{1}{12}$. Go on as needed, talking of the sentences with the pupils. Then let them write the true sentences in their exercise books.

If some pupils notice that there are several names for one of the products, encourage them to think about this. For example, in Sentence 4, both $\frac{2}{4}$ and $\frac{1}{2}$ make the sentence true.

ACTIVITY 5 • Relating Addition and Multiplication of Fractions

Pupil page 136

Ask the children to open their books to page 136. Discuss the exercises on the page with the children. Let them say: The first part of each sentence shows the addition of fractions; the second part shows the sum; the third part shows another way of writing the same idea, using multiplication.

Say, "The sentences are not complete. Parts of them are missing." Read the first exercise with the class and talk about it. Let them suggest ways of completing the second sentence. Ask them to make all the sentences true sentences and to write them in their exercise books.

Vocabulary

No new vocabulary

Materials

Pupil pages 137-140.

Teaching Procedure

ACTIVITY 1 • Showing the Addition of Fractions on the Number Line

Pupil page 137

Write the following sentence on the blackboard:

$$\left(\frac{1}{4} + \frac{1}{4}\right) + \frac{1}{4} = n$$

Ask the children to decide what number n makes the sentence true. $\left(\frac{3}{4}\right)$ Let a child write on the blackboard,

$$n \text{ is } \frac{3}{4}.$$

Point to the addends in parentheses in the sentence. Say, "To add, we can count fourths. We count: one, two." Ask, "What is $\frac{1}{4} + \frac{1}{4}$?" $\left(\frac{2}{4}\right)$ "So the number shown in the parentheses is $\frac{2}{4}$. We rewrite the sentence in this way." Write a new sentence below the other.

$$\left(\frac{1}{4} + \frac{1}{4}\right) + \frac{1}{4} = n$$

$$\frac{2}{4} + \frac{1}{4} = n$$

Say, "We found the number n that makes the first sentence true. Will the same number n make the second sentence true?" (Yes, because the addends are the same numbers) "So we can write

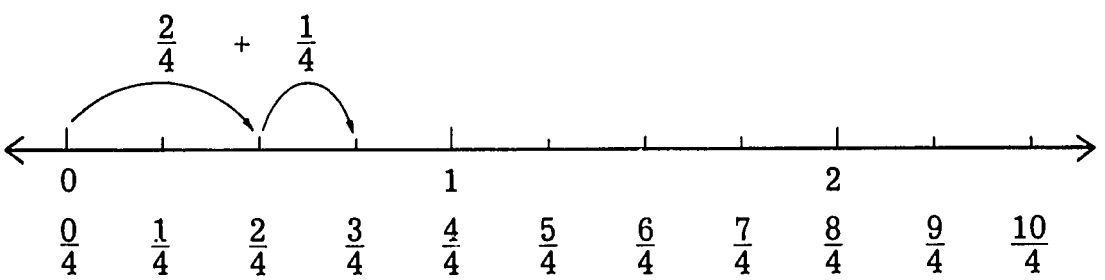
$$\frac{2}{4} + \frac{1}{4} = \frac{3}{4}."$$

Ask, "Could we use the number line to show this?" Get the children to explain how this is done. Help several children make a number line to show the sentence

$$\frac{2}{4} + \frac{1}{4} = \frac{3}{4}.$$

In the discussion, bring out the following:

1. We are adding fourths, so we separate the line into segments that show fourths.
2. We move from 0 along the number line to represent the first addend $\left(2 \text{ spaces or } \frac{2}{4}\right)$ and again to represent the second addend $\left(1 \text{ space or } \frac{1}{4}\right)$.
3. The point at which we end is the point for the sum.



Give many other examples, such as the following. Let the children find the sum on the number line and write true sentences to express the addition.

$\frac{1}{3} + \frac{2}{3} = y$ $\qquad$ $\frac{3}{4} + \frac{3}{4} = m$ $\qquad$ $\frac{2}{6} + \frac{3}{6} = b$

$\frac{3}{5} + \frac{2}{5} = n$ $\qquad$ $\frac{1}{4} + \frac{3}{4} = a$ $\qquad$ $\frac{5}{8} + \frac{1}{8} = c$

After these examples, ask the children to open their books to page 137. Say, "Each picture of the number line shows a sentence about addition. Write the true sentences in your exercise books." Go over the first example with the class.

ACTIVITY 2 • Sentences Showing Addition of Fractions

Pupil page 138

Ask the children to open their books to page 138. Say, "Look at picture 1. Into how many parts of the same size is the region separated?" (4) "The sentence under the picture is about the shaded parts of the region. What does it say?" $\left(\frac{1}{4} + \frac{2}{4} = a\right)$ "What is the first addend?" $\left(\frac{1}{4}\right)$ "What is the second addend?" $\left(\frac{2}{4}\right)$ As they answer, let the children move their fingers over the parts shaded to represent addends.

Say, "The sum is shown by all the shaded part. What is the sum? What number is a if the sentence is true?" $\left(\frac{3}{4}\right)$ Let the children write this sentence in their exercise books:

a is $\frac{3}{4}$.

Let them complete true sentences for each of the other pictures in their exercise books. Give help as needed.

ACTIVITY 3 • Discovery of the Idea

Pupil page 139

$$\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$$

Put many examples of the following kind on the blackboard.

$\frac{3}{4} + \frac{2}{4} = \square$	$\frac{2}{8} + \frac{2}{8} = \square$	$\frac{4}{6} + \frac{1}{6} = \square$
$\frac{2}{5} + \frac{4}{5} = \square$	$\frac{3}{10} + \frac{6}{10} = \square$	$\frac{1}{5} + \frac{3}{5} = \square$

Let children put a numeral for the sum in each of the boxes. If they need help with the addition, let them make and use number lines as in Activity 1.

When the true sentences have been completed, ask the children to look at the denominators. Get them to notice that in each sentence the denominators are the same in the addends and sum. Say, "Look at the numerators in each sentence. Do you notice anything about the numerators in each sentence?"

Help them to discover and say, "If we add the numerators of the addends, we get the numerator of the sum." After the pupils have noticed this pattern, say, "Yes, when the denominators are the same, we add the numerators. Here is an example." Write the following sentence on the blackboard:

$$\frac{3}{4} + \frac{2}{4} = \frac{3 + 2}{4}$$

Ask, "What is 3 + 2?" (5) "So the numerator is five and the denominator is four. What number is this?" $\left(\frac{5}{4}\right)$

Use the same plan and write similar sentences for the other examples on the blackboard.

Let the children discuss each sentence. Let them decide that they can add fractions with the same denominators by adding the whole numbers of the numerators. They will begin to see, for example, that

$$\frac{3}{10} + \frac{6}{10}$$

is found by adding

$$3 + 6.$$

Then

$$\frac{3}{10} + \frac{6}{10} = \frac{9}{10}.$$

If the question of adding fractions with different denominators comes up, let the children try such problems on the number line. Let them discover

that for these fractions the problem is not easy and cannot be done by adding the numerators. Do not discuss this question unless it is asked.

Ask the children to open their books to page 139. Let them copy the sentences in their exercise books and complete them to make true sentences.

ACTIVITY 4 • True and False Sentences

Pupil page 140

Ask the children to turn to page 140 in their books. For each example, let them decide whether the sentence is true.

If the number line is needed, let the children use it. Help them to understand that when the denominators in a sentence are the same, they can add the whole numbers of the numerators. For example, Sentence 3 is true because we know that "2 + 3 = 3 + 2" is true.

STAGE 4 • Finding the Missing Addend

Vocabulary

No new vocabulary

Materials

Pupil pages 141-143

Teaching Procedure

ACTIVITY 1 • Making True Sentences by Shading Regions

Pupil page 141

Ask the children to open their books to page 141. Tell them to draw pictures like those in the book in their exercise books. They may use colour to shade the regions.

Help the pupils as they draw the pictures. It is not necessary that the pictures be copied exactly. But some instructions are helpful. If the children have rulers they can use them to draw the four line segments for the rectangles. If not, they can draw along the edge of some object such as a book. Let them make right angles by tracing corners of books or square corners made by folding paper. Circles may be drawn by tracing round objects such as ink bottles or discs.

The place to draw the lines that cut the regions into parts can be guessed and need not be exact.

Ask a child to read the sentence below the first picture, for example, "One-fourth plus y is equal to three-fourths." Let another child complete the sentence on the blackboard so that it is a true sentence. $\left(\frac{1}{4} + \frac{2}{4} = \frac{3}{4}\right)$ Ask the class what part of the rectangular region is shaded. $\left(\frac{1}{4}\right)$ Point out that the shaded region shows one of the addends in the sentence. Ask, "How many more parts must you shade to show the sum, three fourths?" (2) Let the children use a different colour to shade the second region. Emphasize that each of the coloured regions shows one of the addends in the sentence and the whole shaded region shows the sum. Let each pupil shade the regions in a new colour and write a true sentence beneath each of the figures.

ACTIVITY 2 • Finding the Missing Addend

In this activity let the children try to find missing addends without using shaded regions or folded papers or the number line to help them. For example, write this sentence on the blackboard:

$$\frac{3}{4} + \frac{\square}{4} = \frac{5}{4}$$

Ask what numeral we must put in the box to make the sentence true.

By this time the children will notice that fractions with the same denominator are added by adding their numerators. The children will say, "I add $\frac{2}{4}$ and $\frac{3}{4}$ and get $\frac{5}{4}$." They are sure of this because they know 2 + 3 is 5.

Let the pupils try many exercises of this type, such as these:

$$\frac{\square}{3} + \frac{1}{3} = \frac{2}{3} \qquad \frac{2}{7} + \frac{\square}{7} = \frac{5}{7} \qquad \frac{\square}{9} + \frac{5}{9} = \frac{7}{9}$$

$$\frac{2}{5} + \frac{\square}{5} = \frac{4}{5} \qquad \frac{\square}{6} + \frac{5}{6} = \frac{8}{6} \qquad \frac{5}{8} + \frac{4}{8} = \frac{\square}{8}$$

If necessary, let the children who still have difficulty shade regions or use a number line.

Some children will notice that, for example, the sentence $\frac{\square}{6} + \frac{5}{6} = \frac{8}{6}$ and the sentence $\frac{5}{6} + \frac{\square}{6} = \frac{8}{6}$ are true for the same number. Encourage the children to think about this and to say that $\frac{3}{6} + \frac{5}{6}$ is the same number as $\frac{5}{6} + \frac{3}{6}$ because $3 + 5 = 5 + 3$.

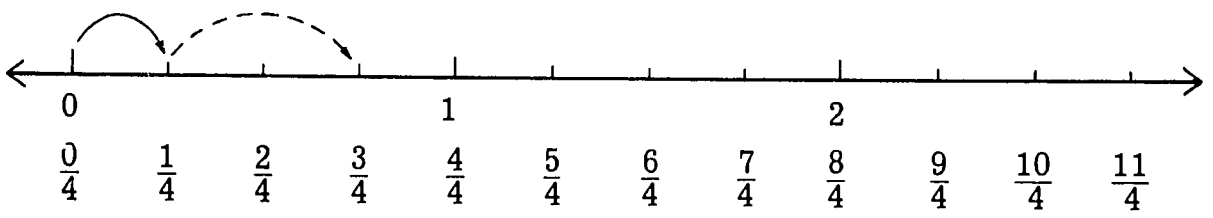
ACTIVITY 3 • Using the Number Line to Find the Missing Addend

Pupil pages 142–143

Write on the blackboard the sentence

$$\frac{1}{4} + n = \frac{3}{4}.$$

Say, "We want to find the missing addend n. We can show the sentence on the number line. One addend is one-fourth. The sum is three-fourths. What points do you think we need to mark on the number line?" Discuss this until the children see that it will be useful to mark points for fourths. Help the class draw the number line and then ask questions such as those suggested below.



At what point do we begin? (0) We can write $\frac{0}{4}$. What number does each space show? (One-fourth) What is the first addend in our sentence? $\left(\frac{1}{4}\right)$ How many spaces do we move to show $\frac{1}{4}$? (One) What is the sum in the sentence? To

show $\frac{3}{4}$, to what mark do we move? If we are at the point for $\frac{1}{4}$, how many spaces must we move to reach the point for $\frac{3}{4}$? (Two) We move two spaces, or two fourths. What number is the missing addend in the sentence? $\left(\frac{2}{4}\right)$ What number is n? $\left(\frac{2}{4}\right)$ Write, "n is $\frac{2}{4}$." The blackboard will look like this:

$$\frac{1}{4} + n = \frac{3}{4}$$

$$n \text{ is } \frac{2}{4}.$$

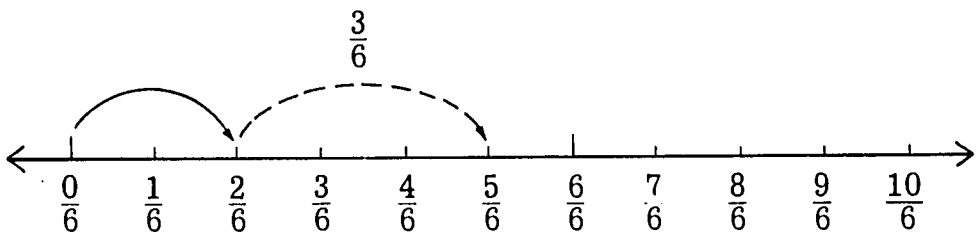
Write another sentence on the blackboard.

$$\frac{2}{6} + y = \frac{5}{6}$$

Draw a line segment to use in drawing the number line. Ask the children how the number line should be marked to show the sentence. (Marked in sixths) Let some of the children mark the number line.

Let a child show the sentence on the number line and ask him to explain his diagram. (See below.) Let him explain it in his own way but he should include these ideas:

1. The first addend is $\frac{2}{6}$.
2. The sum is $\frac{5}{6}$.
3. We move first to the point for $\frac{2}{6}$, then move to the point for $\frac{5}{6}$. The distance from $\frac{2}{6}$ to $\frac{5}{6}$ shows the missing addend.
4. The distance from $\frac{2}{6}$ to $\frac{5}{6}$ is three spaces, or three sixths. The missing addend is $\frac{3}{6}$.



Let another child write

$$y \text{ is } \frac{3}{6}$$

on the blackboard and read the sentence aloud.

Write other sentences with missing addends and let children draw number lines to help them complete true sentences; for example,

$$\frac{3}{5} + a = \frac{4}{5}; \qquad \frac{2}{8} + b = \frac{5}{8}; \qquad \frac{1}{12} + c = \frac{10}{12}.$$

When sentences of the above type are understood, write an example such as

$$n + \frac{2}{6} = \frac{5}{6}.$$

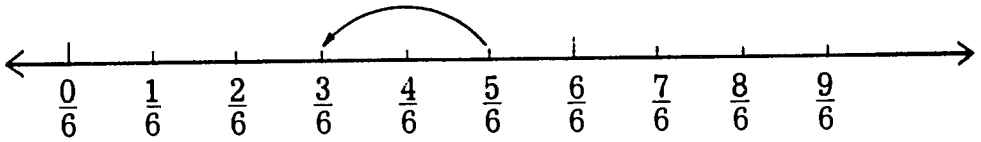
Let the children read and talk about what the sentence means; for example,

1. We are to find the first addend in a sentence. We know the second addend is $\frac{2}{6}$ and the sum is $\frac{5}{6}$; or
2. $\frac{2}{6}$ added to what number gives $\frac{5}{6}$?

Point to the sentence and ask, "How can we use the number line to find the missing addend in this kind of sentence?" Let the children discuss this. You draw the number line for them to use as they talk. In their discussion, help them to bring out what they know about the sentence. For example, they know that:

1. If we make two moves on the number line (each move showing an addend), then we end at the point for $\frac{5}{6}$ (the sum).
2. The second move will be a distance showing two-sixths, or $\frac{2}{6}$.

One suggestion might be that they find the point for $\frac{5}{6}$ and move back the distance for two-sixths. This final point is the point for the first addend. If this is suggested, let the child do this using the number line. Let him tell what number completes the sentence to make it true. $\left(\frac{3}{6}\right)$



Let a child write the completed true sentence and read it aloud.

$$\frac{3}{6} + \frac{2}{6} = \frac{5}{6}$$

n is $\frac{3}{6}$.

Let the pupils make other similar sentences true. Use sentences like these:

$$a + \frac{1}{5} = \frac{3}{5}; \qquad y + \frac{3}{8} = \frac{6}{8}; \qquad m + \frac{1}{10} = \frac{6}{10}.$$

Ask the children to open their books to page 142. Ask someone to read the sentence at the top of the page. Go over one example with the class. Ask, "What number is n?" If none of the children can decide what number n is, let them look at page 143. Tell them to use one of the pictures of the number line to help them. Let them decide which of the line segments would be helpful in completing the first sentence. (The line segment to use is the one marked to show thirds.) Let the children trace along the line with their fingers.

Ask the children to copy all the sentences in their exercise books and complete the true sentences, telling what n is in each case. Remind them that they may use the number lines on page 143 to help in finding the missing addend or to check their answers.

UNIT 11

Geometry II

OBJECTIVES

1. To revise the extension of line segments to lines
2. To learn that two lines intersect in no more than one point, that two points are on only one line, that many points are on a line, and that many lines can intersect at the same point
3. To learn to line up objects and points
4. To revise the construction of figures with line segments
5. To construct networks with line segments and to find and count paths through a network

Background Information for Teachers

See the "Background Information" for Unit 4.

The ideas of line segment and line have been introduced in the *Primary One* and *Primary Two* programmes and revised in Unit 4 of *Primary Three*. These concepts are revised in this unit. The ideas "Two lines intersect in no more than one point" and "Two points are on one and only one line" are introduced and some geometric figures formed from line segments are discussed.

The idea of line is included in the programme of *Primary Three*. Children need to handle and talk about physical objects that show line segments. Children need to recognize straightness in objects, and to be clear that line segments are different from curves that are not straight.

Networks are figures made of line segments. More than two line segments may intersect at the same point. Triangles and rectangles are made of line segments where only two line segments intersect at the same point. These are special cases of networks. Networks are shown by many objects. Some are: edges of boxes, paths in a village, streets in a town, channels in a river system, branches in a tree.

Vocabulary

Line segment, line, intersect, point in common, extend

Materials

Geometry material of Unit 4; seeds; piece of string one foot long for each child; string about 30 feet long; two pieces of string about 5 feet long; pupil page 144

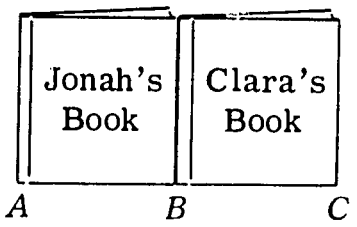
Teaching Procedure

ACTIVITY 1 • Extending Line Segment to Line

Run your finger along the edge of a book and ask, "What does this show?" (Line segment) "What are some other things that show *line segments*?" Let the children point out things in the classroom that show line segments. Put a number of things, such as a string, a box, a cube, a ball and so on, on a table for children to use. Ask questions to get them to show things like the edges of the table, of a window, of a desk and of boards, edges of the room and stretched strings. Try to get each of the children to point out some objects that show line segments.

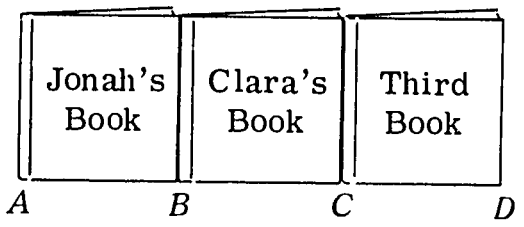
Get a child to draw a picture of a line segment with a straight-edge on the blackboard. Let another child test it with a string. Ask, "Will some one *extend* this line segment?" Choose a child to extend the line segment. Let another child help by holding the straight-edge.

Let two children bring their books to the front of the room and use the edges of the books to show line segments. Ask, "How can we use Clara's line segment to extend Jonah's line segment?" Let the children talk about this. Get them to see that if the books are put side by side, the line segment is extended to form a new line segment.



When books are held as shown in the picture, the line segment AB shown by the edge of Jonah's book is extended to make line segment AC. Get the children to say that the line segment shown by Clara's book is extended. Let a child run his finger along the edges of the two books.

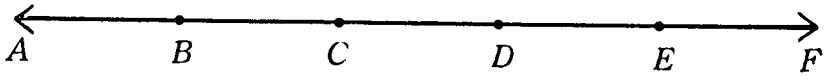
Ask a third child, "Can you extend the line segment shown by Jonah's and Clara's books by using your book?" Let him hold his book beside Jonah's and Clara's as shown below. (Be sure the line segment formed is fairly straight.)



Ask another child to run his finger along the edges showing the line segment AD.

Let other children bring their books and extend the line segment even further. Ask, "How many times can this line segment be extended?" Get the children to say that the line segment can be extended as many times as we please. It can be extended to the edge of the blackboard and if the blackboard were longer, the line segment could be extended more.

Ask, "Can you think of a way we can show that the line segment can be extended on and on?" Let the children talk about this. Say, "Suppose we put arrowheads at the ends of the segment. (Do this.) The arrowheads will remind us that the line goes on and on."



Ask, "What do we call the figure made by a line segment that is extended as many times as we please?" (A *line*)

ACTIVITY 2 • Arranging Objects in a Line

Give pairs of the children three or four small seeds and a piece of string about one foot long. Ask, "Can you arrange your seeds along a line?"

Let the pairs of children arrange their seeds on their table tops. Help those who have difficulty. Ask, "How can you tell if your seeds are arranged along a straight line?" If someone does not suggest testing with a straight-edge or stretched piece of string, ask, "Can you use the string test to see if your seeds lie along a straight line?" Get the children to test their arrangements with stretched strings as shown below.

Seeds in a straight line

Seeds not in a straight line

Let the children practise arranging seeds along a straight line several times and testing the placement of the seeds with a string. Let them use more seeds and place them farther apart along a straight line. They can work in fours, tying their strings together when a longer piece of string is needed.

Ask, "Can you test to see if your seeds are in a straight line without using the string?" Let them talk about this. If a child suggests that he can test with his eye, let him show how to do this. Get the rest of the children to test for straightness by looking along the seeds.

Ask, "Who will draw four points on the blackboard so they are on a line?" Choose a child to try to do this. Let other children check for straightness by eye and by using a piece of string.

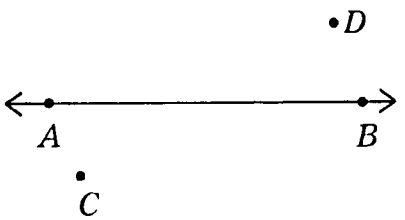
Ask, "Is drawing points in a line on the blackboard like putting seeds in a line?" Let the children talk about this. Get them to say that putting seeds in a line is just like drawing points in a line.

Let each child draw in his exercise book four points that he thinks are in a line. Tell him not to draw the line. Let him check for straightness by eye and then with a stretched string.

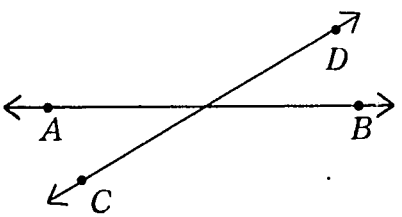
Ask a child to draw a picture of a line on the blackboard. The picture you hope the child will draw is like this:

If he does not draw the arrowheads, ask, "Does the line keep going to the right? How do you show the line keeps going to the right?" Ask similar questions about extending the line to the left. Let a child draw two points on the line and name them *A* and *B*. Say, "The line goes through the points *A* and *B*."

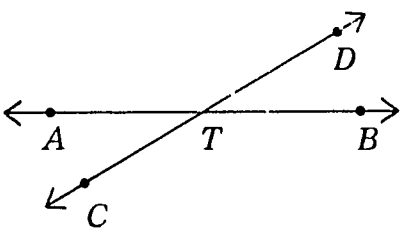
Draw two more points, *C* and *D*, which the line does not go through. The blackboard may look like this:



Ask a child to draw a line through the points *C* and *D*. If needed, ask questions like, "Is there a line segment with *C* and *D* as endpoints?" (Yes) "Can the line segment be extended to a line?" (Yes)



Ask, "What can we say about these two lines?" If no pupil says something like "They *intersect*", or "They have a *point in common*", ask, "Do they intersect? Why?" Let a child show the point where the lines intersect and name it.



Ask a child to show line AB with a string. Get him to hold the string along the line AB. Ask another child if the string shows the line AB. Ask another child if the line stops at B? at the end of the arrow? at the place the string is held? Get the children to say the line goes on and on.

Let another child show the line CD with a string. Ask another child to place his fingers at the point where the lines intersect. Ask what the two strings show. Guide the children to say something like, "The strings show two lines. The lines intersect."

Give each child a piece of string about 1 foot long and some small seeds. Tell the children to turn to pupil page 144. Ask, "What geometric objects are drawn on this page?" (Points) "How many points are drawn?" (16) "Do the points have names?" (Yes, A, B, C, to P)

The list of directions and questions that follows will help the children to think about points, lines and intersections. You will think of others.

Stretch a piece of string to show the line that points B and O are on. Are any of the other points on the line? (No)

Stretch a piece of string to show the line that points B and G are on. Are any of the other points on the line? (No)

Find other pairs of points that are on lines. Let the children say which pairs of points they think lie along lines. (All pairs)

Is point K on the same line as points M and G? (No)

Is point K on the same line as points D and P? (Yes)

Is point I on the same line as points F and O? (No)

Is point O on the same line as points G and K? (No)

Is point B on the same line as points H and D? (Yes)

Can you stretch the piece of string along the line that has five points drawn on it? (A, E, F, K and N)

How many lines can you find on the page that have three of the points on them? (Three: DP; IP; HB)

Find a point that is on the same line that point B is on. (There are 15 such lines.)

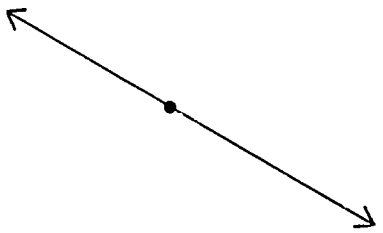
Place a seed so that it will lie along the same line as points M and D.

Find a line that has the points D and M on it. Find the line that has the points F and G on it. Can you place a seed at the point where these two lines intersect? (Give children help if they need it. They must imagine where these two lines intersect. Let them guess where the point is first. Then they can test by stretching two pieces of string

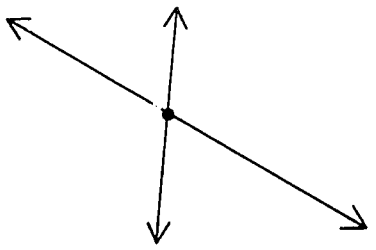
between the proper points.) Let the children practise finding the intersection of lines on the page. Let the children work in pairs, one holding his string along the line *DM*, the other along the line *FG*.

ACTIVITY 4 • Many Lines Through One Point; Only One Line Through Two Points

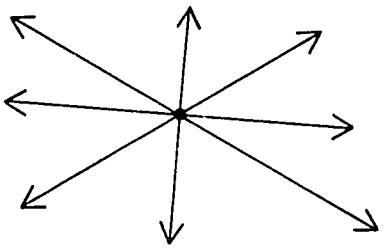
Mark a point on the blackboard. Ask whether a line can be drawn through the point. Let the children talk about this and let one of them come to the blackboard and draw a line. Let another child help him hold the straight-edge up to the blackboard.



Ask, "Did he draw a line through the point?", "Can another line be drawn through the point?" Let the children talk about this. Get another child to draw a second line through the point.

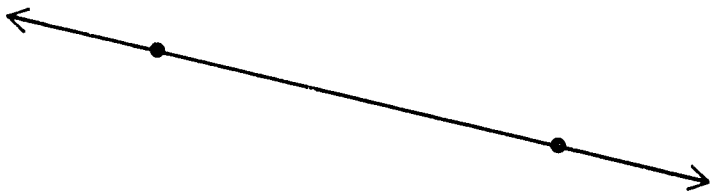


Ask, "Can a third line be drawn through the point? a fourth?" Continue as before, with children deciding that another and still another line can be drawn through the point. Let them draw many lines through the point.

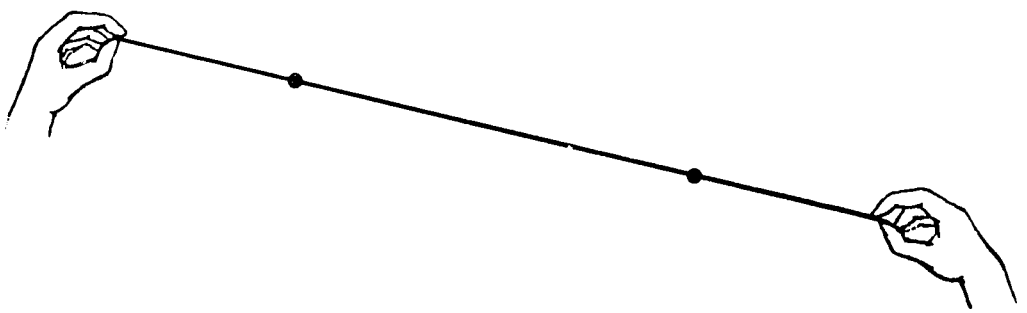


Ask, "How many lines have you drawn through the point? Can more lines be drawn through the point?" Get the children to say that many, many lines can be drawn through the point.

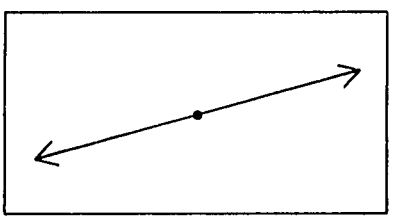
Let a child draw two points on the blackboard. Ask, "Can you draw a line that goes through both of these points?" Let the children talk about this and let one of them draw the line. Let him choose someone to help hold the straight-edge up to the blackboard.



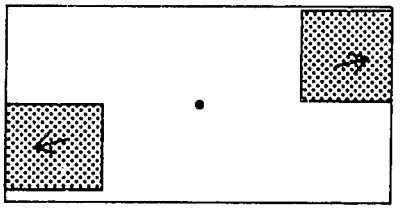
Ask, "Can you draw another line through these two points?" Let the children try. Guide them to discover that only one line can be drawn through two points. Get other children to stretch a string through both of these points, extending it to the edges of the blackboard, then stretching it to the walls. Point again to the two points and ask, "How many lines can we draw through these two points?" (Only one)



Draw a point on the blackboard. Help a child to draw a line through the point extending from one side of the blackboard to the other. (Do not draw a horizontal line.) The diagram below is the picture of an entire section of blackboard.



Quickly rub out the left part of the line. Leave the arrowhead. Rub out the rest of the line except for the original point and the other arrowhead. Cover the two arrowheads with pieces of paper. Leave only the original point for the children to see. Rub out the line well so the children cannot see where it was drawn.



Say, "The line is gone but the point is still there. Will someone put the line back?" Let a child try to redraw the line. Be sure he cannot see the arrowheads under the cardboards. Ask, "Is this new line in the place where the rubbed-out line was?" If they do not agree, let them draw lines where they think the rubbed-out line was.

Remove the cardboards to let the children see whether the rubbed-out line has been replaced. Repeat this exercise several times. Get the children to see that it is difficult to redraw a rubbed-out line when only one point is known. They may say that if they could see the arrowheads they could put the line back.

Change the activity. This time mark two points on the blackboard instead of one. Again let a child draw a line through the points and mark arrowheads at the ends. Again rub out the line except for the arrowheads and the two points. Cover the arrowheads with large pieces of cardboard. Then ask the same question, "Can someone put the line back?" Get the children to see that it is easy to put the line back, because now they see *two* points that were on the line. Only one line can be drawn through two points.

Let a child draw a line on the blackboard. Get another child to draw two points on it, say A, B. Ask still another child to mark two other points on this line, say P and Q, one near one arrow, the other near the other arrow.

Ask a fourth child to take a string and show the line through P and Q.

Ask, "Does the line through P and Q go through A and B? Is the line through P and Q the same as the line through A and B? (Yes) "Is the line through P and B the same as the line through A and Q?" (Yes) Repeat this exercise with other lines and points on the lines.

The children should understand that through any two points of a particular line only that line can be drawn. The children may not be able to say this in words. You try to help them understand the ideas.

STAGE 2 • Geometric Figures Made from Line Segments

Vocabulary

Angle, vertex, vertices, joining two points, congruent angles

Materials

Geometry material of Unit 4; wire model of a triangle; several sticks to make demonstration models of a triangle; three sticks for each two children; several pieces of string; seeds; box; pupil pages 144-147

Teaching Procedure

ACTIVITY 1 • Revision of Angles and of Triangles

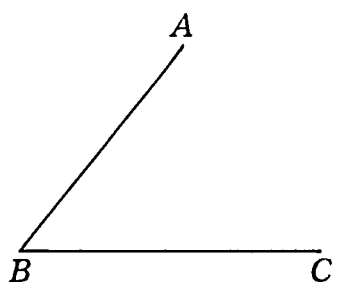
Hold up a book. Run two of your fingers along the two edges that show an angle. Let your fingers meet at the corner. Ask, "What do these two edges show?" You might get several answers, such as two line segments, edges of the book, a corner and an angle. Agree with these answers. If some child says, "An angle", ask a child to draw a picture of the angle on the blackboard. If no child says angle, ask a child to draw a picture of the corner of a book on the blackboard.

Ask questions to get children to name the endpoints of the two line segments that make the figure. The picture will look like this:

Let the children talk about the figure. Get them to say things like: PN is a line segment; NR is a line segment; the line segments intersect; the point N is on both line segments. If no child calls the figure an angle, trace your finger over the angle and say, "This is an *angle*. It is made from two line segments that have the same endpoint. Point N is called the *vertex*." Let them continue to talk about the figure. Guide them to say things like: the two line segments form an angle; the angle is named angle PNR or angle RNP; the point N is the vertex of the angle.

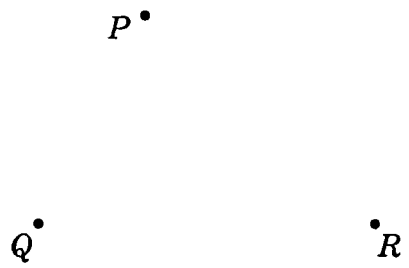
Mark three points on the blackboard. Say, "Who will draw an angle using these three points?" If no child volunteers, choose a child to draw a line segment through two of the points. Let another child name the points. Ask a third child to draw a line segment through two different points. The line segments may form a figure like this.

1.



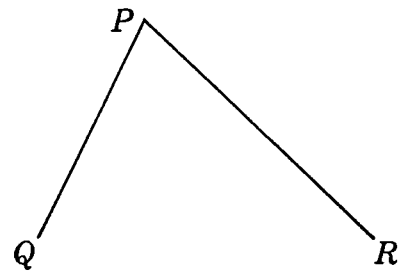
Get the children to agree that: the figure is an angle; it is formed by line segments AB and BC; the line segments intersect at point B; point B is called a vertex.

Mark three points in positions similar to A, B and C at a different part of the blackboard. (Be sure the points are so arranged that the angles formed will be congruent.) Say, "Here are three other points like A, B and C." Ask, "Can we draw an angle using these three points?"



Let a child do so. If he draws angle PQR, say, "John has drawn an angle using P, Q and R. It looks like angle ABC which we just drew. Can a different angle be drawn using P, Q and R?" Then rub out the angle just drawn. With the same three points get the children to draw another angle, say QPR.

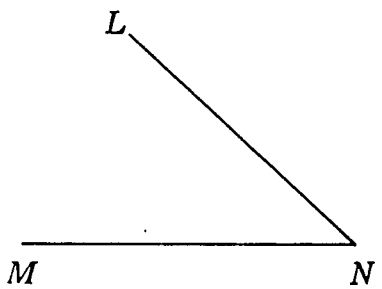
2.



Say, "When we draw a line segment with two points as endpoints, we say we *join the two points* or connect the two points." Let a child say things like, "Points P and Q are joined by the line segment PQ."

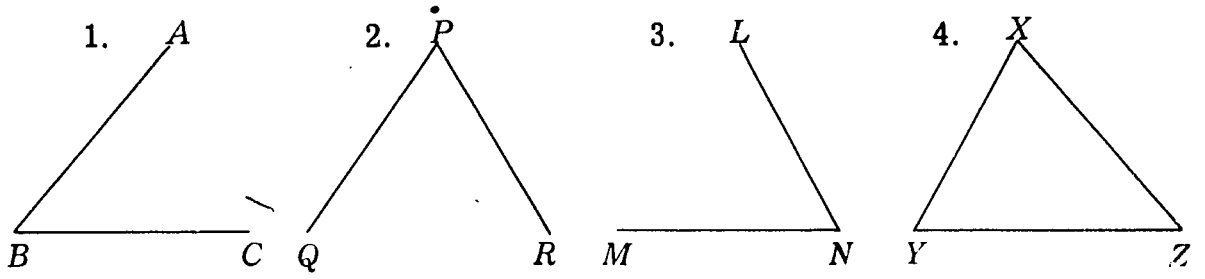
Draw three more points in positions similar to A, B and C at another part of the blackboard. Say, "We have made two angles by joining points; can we draw a third?" Get a child to name the points, say L, M, N.

3.



Let another child draw angle LNM. The blackboard will now have three figures on it.

Draw three more points like A, B and C. Let children draw all three line segments to form a triangle. The blackboard will now look like this.



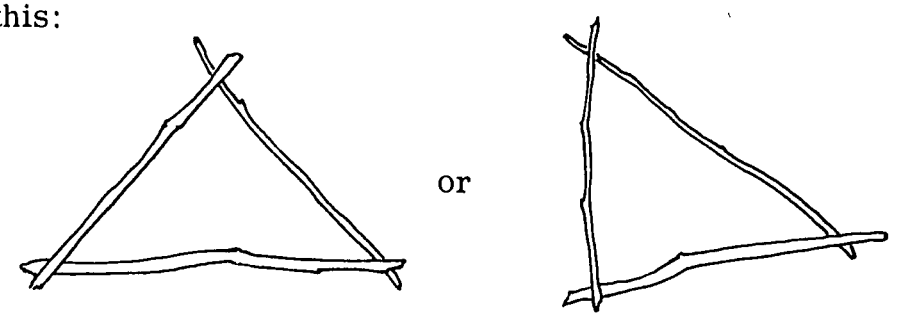
Let the children talk about the triangle. Let them say things like: the triangle has three sides; the sides are line segments; the triangle has three angles; the angles are angle XYZ, angle YXZ, angle XZY; the angles are also named angle ZYX, angle ZXY and angle YZX.

Fold a sheet of paper to form a corner that fits angle ABC. The paper can be folded by following this plan. Fold the paper once to make a straight edge. Lay the edge along the line segment BC. Fold the paper back along line segment AB. The two folds will make a corner that fits angle ABC.

Try this corner on angle XYZ. Ask the children if it fits. If the points X, Y, Z are like the points ABC, the corner will fit. This shows that angle ABC is *congruent* to angle XYZ.

Let pairs of children show angles with sticks and then show triangles. Let them talk about the figures they make.

Draw three points on the blackboard. Let three children come to the blackboard and hold sticks that show the line segments going through pairs of the three points. Choose sticks that are long. The sticks will look something like this:



Let the children say something like, "Part of the sticks show a triangle. We do not use the parts of the sticks that are beyond a vertex." Let a child run his finger along the parts of the sticks that show the triangle. Let another child run his finger along the parts of the sticks that do not show part of a triangle.

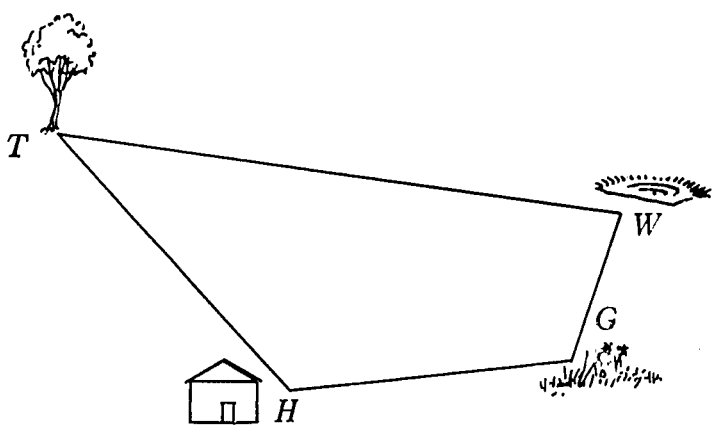
Let the children look at pupil page 144. Let them use sticks to show triangles, determined by 3 points. Tell them to use points O, N and C; then J, F and H; M, G and C; and A, E and N. The last set of three points is

on a line; so the sticks will not show a triangle. Let the children discover this for themselves and tell about it.

ACTIVITY 2 • Drawing of 4-sided Figures

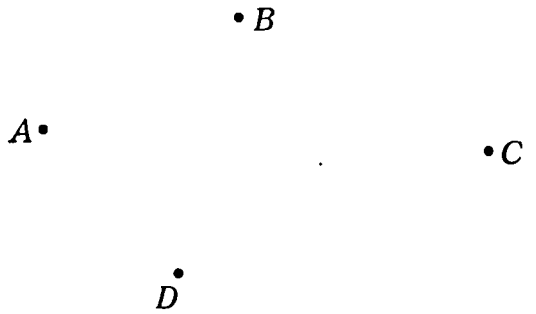
Tell this story: "Babar was a little elephant with big ears who lived in the shade of a tall tree. One morning he awakened early and flapped his ears and flew straight to a waterhole. After drinking, he was hungry. He flew directly to a field of fresh grass. As he was eating, he heard a boy crying. 'What is the matter, little boy?' asked Babar. 'Boo, hoo', cried the little boy. 'I have been walking and walking and cannot find my way home.' 'Never mind', said Babar. 'You hop on my back and we will see what we can do.' The boy hopped on Babar's back. Babar flapped his ears. Before the little boy knew it, they reached his home. The boy's mother was very happy. She gave Babar three bunches of bananas, four pawpaws and five loads of grass. Babar flew happily straight to his home by the tree."

As you tell the story, draw a picture of Babar's path from the tree T, to the waterhole W, to the grassy field G, to the boy's home H, and back to Babar's home by the tree T.



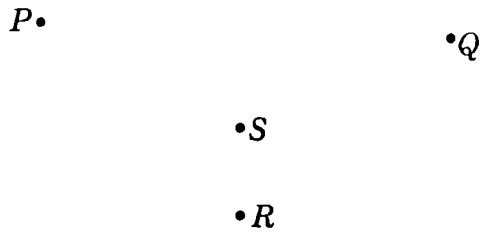
Ask questions like, "What geometric figure does Babar's path show?" (4-sided figure) "How many line segments are there?" (4) "What are the names of the line segments? Name an angle. How many angles are there?" (4) "T is the vertex of what angle?" Say, "T is also called a vertex of the 4-sided figure $TWGH$."

Let a child draw 4 other points on the blackboard. Let another child name them, say A, B, C and D. Ask children to draw a 4-sided figure that has these points as *vertices*. Let a child draw a line segment, another a second line segment and so on.



Let a child trace out the 4-sided figure with his finger. Let other children name the figure, the line segments and the angles.

Repeat the activity with the points P, Q, R and S.



Let the children discover that three different 4-sided figures can be drawn with P, Q, R and S as vertices.

ACTIVITY 3 • Recognizing 3-, 4- and 5-sided Figures

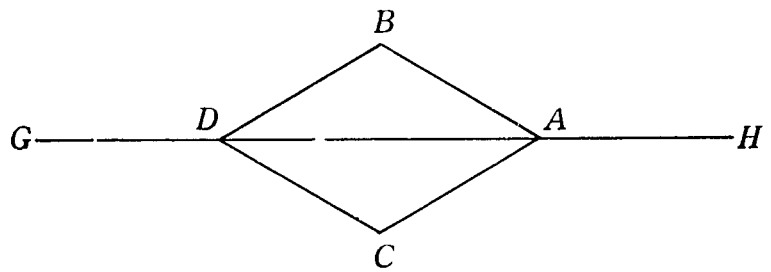
Pupil page 145

Let the children look at pupil page 145. Tell them to put a seed on each 3-sided figure. Ask a child to read the name of a triangle. Ask a second child to read the name of another triangle and so on.

Get the children to do things like: name a triangle; put three seeds on each triangle; name each 4-sided figure; put two seeds on each 4-sided figure; name the angles of each 4-sided figure; and name the sides of each 4-sided figure.

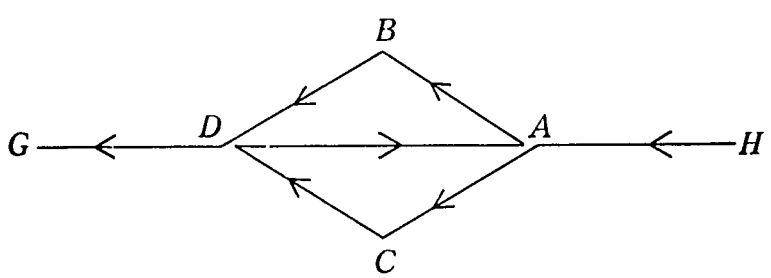
Ask, "Is there a figure on page 145 that is not a triangle and is not a 4-sided figure? Put a marker on each." (There are two.) Let the children name the figures. (One is an angle. The other is a 5-sided figure.)

Tell a story like this as you draw the picture below on the blackboard: "Femi's house has a garden with straight paths going from the back of the house to a gate at the back of the garden. The garden looks like this:"



Say, "*H* shows the back of the house and *G* the gate. The line segments are pictures of the paths. Show a path for Femi to use as he walks from his house *H* to the gate *G*." Let a child show a path by running his finger along the picture. Let him name the points and then the line segments he follows. (For example, *HABDG*)

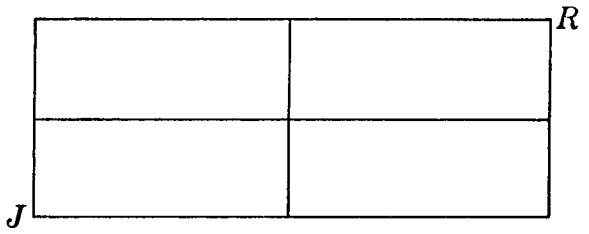
Let a second child show another path Femi may follow. (For example, *HACDG*) Let a third child show still another path. (For example, *HADG*) Ask, "Are there any other ways Femi could go from *H* to *G*?" Some child may suggest an S-shaped path like this:



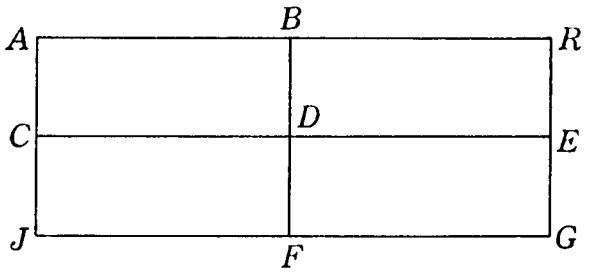
If he does, say, "Yes, this is another path. This path intersects itself. Can you show where it intersects itself?" (At *A* and at *D*)

Ask, "Are there any other paths from *H* to *G* that do not intersect themselves?" (No) "How many paths are there that do not intersect themselves?" (3) Let the children talk about each of these questions and decide on an answer.

Tell this story: "The roads in the town where Joseph and Robert live look like this picture"



Say, "Point J shows Joseph's house and point R shows Robert's house. The line segments are roads." Let the children in turn show each road and name the intersections with letters, for example, as below:



Continue the story, "Joseph wishes to go from his house to Robert's house. What path can he follow to get there?"

Let a child show a path along the roads. Ask the child to name the path Joseph uses by giving the intersections. Ask, "How many intersections does Joseph pass as he takes this path?" One path would be $JCABR$. Joseph passes four intersections in this path. (C, A, B and R)

Let a second child show another path. Ask how many intersections are passed in this path. If no one shows a path passing six intersections, say, "Show me a path passing six intersections."

Ask, "How many paths pass four intersections?"

Similarly, ask for paths passing eight intersections. One such path is $JCABDFGER$. Ask for paths which do not pass the same intersection twice.

Let the children look at page 146 in their books. Ask them to find all the paths from A to B in Exercise 1. Let them list the paths in their exercise books. Let them write the number of intersections passed on each path. Ask, "Which path passes the fewest intersections?" ($ASQB$)

Let the children list in their exercise books paths from point X to point Y in Exercise 2. Ask, "How many paths pass three intersections?" (3)

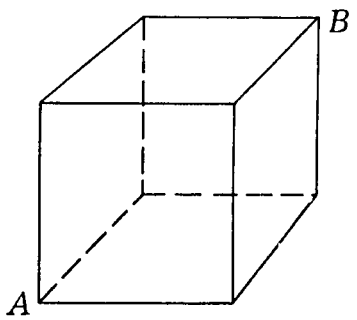
"How many paths pass four intersections?" (None) "Five intersections?" (5) "Can you find a path passing seven intersections?" (Yes. Two. Do not count paths which pass the same intersection twice.)

Use Exercises 3 and 4 in a similar way. Find all the paths from S to T and from W to Z.

ACTIVITY 5 • Networks in Space

Ask the children to show paths between points in the classroom. There are many. Some examples are the edges of the top of a table, the edges and corners of the room, the edges of a book, the edges of pieces of furniture. Note that the line segments forming a network do not need to lie in a plane.

Ask a child to hold up a box. Tell him to run his finger along its twelve edges. Discuss the box, describing and counting edges, corners and faces.



Let the children trace with their fingers the paths that go from point A to point B on the box. Let children name the corners and list paths a spider may follow going from point A to point B.

ACTIVITY 6 • (OPTIONAL) Network Game

Pupil page 147

Many children know this game. It may be used to revise drawing line segments, inside and outside of squares and networks.

Two children play the game. Let them open their books to page 147 and copy the 3 by 3 array of dots. To play the game, the children draw line segments and make squares with the dots as vertices. They draw a line segment between two nearby dots, either up and down or across. In his turn, each player draws one line segment until the line segment completes a square with other line segments already drawn. When this happens he draws another segment. If this completes a second square, he draws a third segment. He continues until the last line segment he draws does not complete a square. His turn is over. He writes his initial in each of the squares he completed. The winner is the player who completes the most squares.

The arrays of dots can be any size. For example, two other arrays are also drawn on page 147. Two children may wish to copy one of these arrays on a sheet of paper or on the blackboard and play the game on it.

UNIT 12

Techniques of Multiplication and Division II

OBJECTIVES

1. To extend the multiplication and division facts to factors of 12
2. To revise the techniques of multiplication with one factor less than 10, and to extend to one factor less than 100
3. To revise the techniques of division with the known factor less than 10, and to extend to known factors of 10, 11, 12 and multiples of 10

Background Information for Teachers

See Background Information for Units 3 and 9. Also see Part 2, "Arithmetic", pages 49-52, in your *Teachers' Handbook.*

STAGE 1 • Extending Multiplication and Division Facts with Factors to 12

Vocabulary

No new vocabulary

Materials

Large multiplication chart (begun in Unit 9); paper array of dots for each pupil (used in Unit 3 and 9); pupil pages 148-157

Teaching Procedure

ACTIVITY 1 • Revising Multiplication and Division Facts with Factors to 10

Pupil pages 148-149

Write this sentence on the blackboard:

$6 \times 8 = n$

Ask the pupils to tell ways they think about the sentence. The replies may include these:

What is the sum when 8 is used as an addend six times?

What is the product of 6 and 8?

How many objects are in 6 rows if there are 8 in each row?

How many objects are in a 6 by 8 array?

Let a pupil tell what number makes the sentence true. (n is 48.) Let another pupil show why n is 48. He may use counters to form a 6 by 8 array and then count the objects in the array. Or he may skip count 6 times by eights. Or he may say, "I know 6×5 is 30 and 6×3 is 18. Then 6×8 is $30 + 18$, or 48." Or he may fold a paper array of dots to show a 6 by 8 array and count the dots.

Let pupils tell how to find several other products with factors of 10 or less. For example, 6×6, 8×7, 9×6, 6×10 and so on.

Vary the practice by naming a number less than 100 and asking for factors of the number. For example, "What are factors of 36?" Consider all replies and let the class decide which are correct. (1×36, 2×18, 3×12, 4×9, 6×6) Let pupils tell how they decided on the factors.

Ask, "If one factor of 36 is 4, what is the missing factor?" (9) Ask a pupil to make up a division sentence for this exercise. ($36 \div 4 = n$) Let another pupil give the number that makes the sentence true. (n is 9.)

Continue practising the multiplication and division facts. Avoid telling. Encourage the children who do not know the multiplication facts to find factors and products for themselves by skip-counting, by folding paper arrays of dots or by laying out rows of counters. The objective, of course, is to get all the pupils to understand multiplication and division and to know the multiplication and division facts so well they can recall them at once.

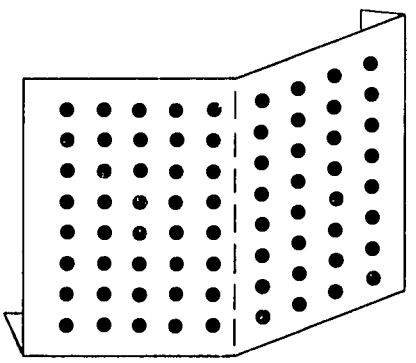
Ask the pupils to turn to pages 148 and 149 in their books. Let them give orally, if they can, the numbers missing in the sentences. Notice that the exercises also include addition and subtraction sentences.

ACTIVITY 2 • Revising Multiplication and Division Facts

Pupil pages 150-151

Revise the meaning of a multiple of a number. Ask, for example, for some multiples of 6. (6, 12, 18, 24, ..., 760) Ask, "Why is 48 a multiple of 6?" (Because 6×8 is 48) Ask for multiples of other numbers. Let the children work exercises of this kind from page 150 in their books.

Continue revising the multiplication facts by asking the children to fold their paper arrays to show products. For example, ask them to show the product 8×9 with their arrays. When the arrays are folded, ask, "How can you fold your array into two arrays that show simpler products?" Let a pupil suggest a way of folding. He may suggest this:



Ask, "What are the two simpler products?" (8×5 and 8×4) Write on the blackboard:

$$8 \times 9 = (8 \times 5) + (8 \times 4)$$

Let a pupil tell what the simpler products are. (40 and 32) "What is 8×9?" ($40 + 32$, or 72)

Let various pupils fold the 8 by 9 array into two arrays in many ways that show two simpler products, and each time write the sentence. [$(8 \times 3) + (8 \times 6) = 8 \times 9$, or $(8 \times 2) + (8 \times 7) = 8 \times 9$] Let pupils discuss the property that these sentences show. (The distributive property) Use this activity with other products, such as 6×7, 7×9 and 6×8.

Let the children turn to page 151 in their books and tell, orally, what products the arrays show, and what two new products are shown for each array. For example, the first exercise shows the product 6×4 and the two simpler products 4×4 and 2×4. Thus,

$$6 \times 4 = (4 \times 4) + (2 \times 4).$$

ACTIVITY 3 • Word Problems

Pupil pages 152-153

Ask a pupil to read aloud the first problem on pupil page 152. Let another pupil tell what is known in the problem. (There are 8 lorries. Each lorry has 6 wheels.) Get another child to tell what question is asked in the problem. (How many wheels did the boy count on all the lorries?) Ask a pupil to give a mathematical sentence for the problem. ($8 \times 6 = n$) Let still another pupil tell what number makes the sentence true. (n is 48.) Finally, ask a pupil to answer the question asked in the problem. (The boy counted 48 wheels.)

Let the pupils read the other problems on pages 152 and 153, write sentences in their exercise books for the problems, find the numbers that make the sentences true, and answer the questions asked in the problems.

ACTIVITY 4 • Multiplication Facts Extended to Factors of 12

Pupil pages 154-156

Use these ways to help the children extend the multiplication facts up to 12×12.

1. Ask pupils to skip-count by a number, say seven, 10 times, and then ask him to continue to 11 and 12 times. For example, to skip-count by sevens, he will say,

   7, 14, 21, ..., 70, <u>77</u>, <u>84</u>.

2. Let the pupils first fold their paper arrays to show products with 11 as a factor or with 12 as a factor. Then ask them to fold that array into two smaller arrays that show simpler products. For example, the 7 by 12 array can be folded to show that:

$$\begin{aligned} 7 \times 12 &= (7 \times 10) + (7 \times 2) \\ &= 70 + 14 \\ &= 84 \end{aligned}$$

3. Let a pupil show a product, using the expanded form of the numeral and the distributive property. For example, to find 8×12, write:

$$
\begin{aligned}
8 \times 12 &= 8 \times (10 + 2) \\
&= (8 \times 10) + (8 \times 2) \\
&= 80 + 16 \\
&= 96
\end{aligned}
$$

Put in view your large multiplication chart, which is filled in with products up to 10×10. Let the pupils tell you what numerals to write in the empty spaces to complete the chart up to 12×12.

While the pupils are learning new multiplication facts, they should at the same time be revising the facts from 0×0 to 10×10.

To help them in this activity, ask the pupils to turn to pages 154 and 155 in their books and find the numbers that make the sentences true. The children will need to know the new multiplication facts as well as the ones they knew before to work these exercises.

Give the children more experience in solving word problems. Let them read aloud the problems on page 156, write mathematical sentences for the problems, complete the sentences, and answer the questions asked in the problems.

ACTIVITY 5 • Division Facts with Known Factors 11 or 12

Pupil pages 157-158

Revise the relationship between multiplication and division. Ask a pupil to tell what is asked by the sentence

$$88 \div 11 = n.$$

He may reply, "What is 88 divided by 11?" or "What number must be multiplied by 11 to give 88?" or "One factor of 88 is 11. What is the missing factor?" or "88 things are put in an array with 11 in each row. How many rows are there?" After he has given one of these, guide the class to give the other interpretations.

Let a pupil give the number that makes the sentence true. (n is 8.) Let another child show the sentence with his array of dots.

Give the children more practice in the multiplication and division facts by letting them work the exercises on pupil page 157. If practice is needed in solving story problems, let them also read and solve the problems on pupil page 158.

Vocabulary

No new vocabulary

Materials

Paper array (12 dots by 12 dots) for each pupil; pupil pages 159-163

Teaching Procedure

ACTIVITY 1 • Revising the Technique of Multiplication with One Factor Less Than 10

Pupil page 159

Tell this story problem: "A man is building a house. He needs 124 tiles for the roof. Each tile costs 6 shillings. How much will all the tiles cost?" (Use the unit of money of your country in the problem.)

Write the problem on the blackboard and help the pupils as they read it. Let the children discuss what is known in the problem (The man needs 124 tiles, and each tile costs 6 shillings.) and what is asked in the problem (How much will all the tiles cost?). Ask a pupil to give a mathematical sentence for the problem. ($6 \times 124 = n$) Write the sentence on the blackboard. Ask how to find the number that makes the sentence true.

Guide the pupils to suggest that they use the techniques they learned for multiplication. Some will remember these techniques from Unit 9. Others will need to revise the techniques.

Say, "How can we find the product of 6 and 124? First, how do we rename 124?" (Write 124 in expanded form.) Then write on the blackboard:

$$6 \times 124 = 6 \times (100 + 20 + 4)$$

"How can we write this as the sum of simpler products?" As the question is answered, write:

$$6 \times 124 = (6 \times 100) + (6 \times 20) + (6 \times 4)$$

Let a pupil give these products as you write:

$$6 \times 124 = 600 + 120 + 24$$

Let a child find the sum 744. Ask a pupil to tell the number that makes the sentence, $6 \times 124 = n$, true. (n is 744.) Finally, let a child answer the question asked by the problem. (All the tiles cost 744 shillings.)

Ask the children to find the same product, 6×124, using the forms they learned before. Remind them that the work can be shown as:

$$\begin{array}{r} 100 + 20 + 4 \\ \times \quad\quad\quad 6 \\ \hline 600 + 120 + 24 = 744 \end{array} \qquad or \qquad \begin{array}{r} 124 \\ \times \quad 6 \\ \hline 24 \\ 120 \\ 600 \\ \hline 744 \end{array}$$

Get the pupils to use the last of these forms in their work if they understand it. In this form, they multiply 6 and 4 (24); 6 and 20 (120); and 6 and 100 (600). Finally, they find 24 + 120 + 600 = 744, and the product of 6 and 124 is 744. As the children use this form, they should be able to tell about each step.

Give the children another example: 7×138. Let them find the product in their exercise books by using the short form:

$$\begin{array}{r} 138 \\ \times \quad 7 \\ \hline 56 \\ 210 \\ 700 \\ \hline 966 \end{array}$$

Let several pupils explain each step in the work.

Let the children turn to page 159 in their books. Call on pupils to read the problems. For each problem, let the children write in their exercise books a mathematical sentence for the problem, find a number that makes the sentence true, and then answer the question in the problem. Let them use the above form for multiplication to find the missing product that makes the sentence true.

ACTIVITY 2 • Practicing the Technique of Multiplication

Pupil pages 160-161

In this activity the children will do many exercises in finding products to help them master the multiplication technique when one of the factors is less than 10.

Let the pupils turn to pages 160 and 161 in their books. Let them find the products, using this form:

$6 \times 243 = n$

$$\begin{array}{r} 243 \\ \times \underline{\quad 6} \\ 18 \\ 240 \\ \underline{1200} \\ 1458 \end{array}$$

As the children work, suggest that they shorten this form even more. Some will be ready to omit the steps in which they write the individual products. They can do the addition without writing the addends. For example, they will write:

$$\begin{array}{r} 243 \\ \times \underline{\quad 6} \\ 1458 \end{array}$$

With this form a child thinks, "6 × 3 is 18; this tells me there are 1 ten and 8 ones in the product. 6 × 40 is 240; this tells me there are 2 hundreds and 4 tens in the product; there are now 2 hundreds, 5 tens and 8 ones in the product; 6 × 200 is 1200; this tells me there are 12 hundreds in the product; there are now 14 hundreds, 5 tens and 8 ones in the product." Each time he has added all the ones, all the tens, and so on, he writes that as a digit in the numeral of the product.

Do not let the pupils use this short form of multiplication until they can understand it. Let them continue with the exercises on pages 160 and 161.

ACTIVITY 3 • Inequalities

Pupil page 162

Ask the pupils to turn to pupil page 162. Let the pupils decide which symbol, <, or > or =, will make each sentence true. In order to decide this, they will have to find some products or sums. Let them use any form they wish for finding the products but when pupils are ready, suggest they use the short form for multiplication.

ACTIVITY 4 • Revising Techniques of Division with a Known Factor Less Than 10

Pupil pages 163-164

Tell this story problem: "A man bought some bricks to build an oven. Each brick cost 2 shillings. He paid 162 shillings for the bricks. How many bricks did he buy?" (Use the unit of money of your country in this problem.)

Let the pupils discuss what is known in the problem and what question is asked in the problem. Let them decide that it is a division problem. Get a child to tell you a mathematical sentence for the problem. Question until both forms of the sentence are given:

$$162 \div 3 = n \quad \text{and} \quad 3 \times n = 162.$$

Let the children discuss these sentences and agree that: both sentences are true for the same number; and to find n they divide 162 by 3.

Revise a technique for dividing 162 by 3. Ask a child to rename 162 as a sum of several addends, each of which is a multiple of 3. Write on the blackboard all the replies. For example,

$162 = 120 + 30 + 12;$
$162 = 150 + 12;$ and so on.

Then, for each of these renamings, write a division statement:

$162 \div 3 = (120 + 30 + 12) \div 3;$
$162 \div 3 = (150 + 12) \div 3;$ and so on.

Let the children tell how each statement can be written as simpler division exercises:

$$\begin{aligned} 162 \div 3 &= (120 \div 3) + (30 \div 3) + (12 \div 3) \\ &= 40 + 10 + 4 \\ &= 54; \\ 162 \div 3 &= (150 \div 3) + (12 \div 3) \\ &= 50 + 4 \\ &= 54; \text{ and so on.} \end{aligned}$$

Get a child to tell about each step in one of these exercises. Ask another pupil to give the number that makes $162 \div 3 = n$ a true sentence. (n is 54.) Let another pupil answer the question asked in the problem. (The man bought 54 bricks.)

Tell the children they can use a shorter form to show division.

$162 = 150 + 12$

$$\begin{array}{r} 50 + 4 = 54 \\ 3\overline{)150 + 12} \end{array}$$

Also, revise this form of division:

$$\begin{array}{rl} \underline{50 + 4} = 54 & \\ 3\overline{)162} & \\ \underline{150} & (150 \div 3 = 50) \\ 12 & (12 \div 3 = 4) \end{array}$$

Let a child explain each step in this form. (I used a large addend of 162 that is a multiple of 3. I used 150 because 150 is 3 times 50. I subtracted 150 from 162 to find the other addend of 162. $162 - 150 = 12$. I know that 12 is a multiple of 3 because $3 \times 4 = 12$. $150 \div 3$ is 50 and $12 \div 3$ is 4. $50 + 4 = 54$.)

Ask, "How can we check this exercise?" (Multiply to decide whether $3 \times 54 = 162$ is a true sentence.) Let a pupil multiply 3 and 54. He will write on the blackboard:

$$\begin{array}{r} 54 \\ \times\ \underline{3} \\ 12 \\ \underline{150} \\ 162 \end{array} \qquad \textit{or} \qquad \begin{array}{r} 54 \\ \times\ \underline{3} \\ 162 \end{array}$$

Guide the pupils to say, "$3 \times 54 = 162$ is true. This tells us that $162 \div 3 = 54$ is true."

Let the pupils turn to page 163 in their books. Let them read the problems; write in their exercise books a mathematical sentence for each problem; and find the number that makes each sentence true. Notice that some of the problems are solved by finding a product and others are solved by finding a missing factor, that is, by dividing. Let them use any form they understand as they multiply and divide. Get them to write the answer to the question asked by the problem. For example, the work for Problem 1 should look like this:

1. $354 \div 3 = n$

$$\begin{array}{rl} \underline{100 + 10 + 8} = 118 & \\ 3\overline{)354} & \\ \underline{300} & (300 \div 3 = 100) \\ 54 & \\ \underline{30} & (30 \div 3 = 10) \\ 24 & (24 \div 3 = 8) \end{array}$$

n *is* 118.

The boy made 118 holes.

In order to practise the techniques of multiplication and division, let the children work the exercises on pages 163 and 164 in their exercise books.

STAGE 3 • Extension of Techniques of Multiplication

Vocabulary

No new vocabulary

Materials

12 by 12 arrays of dots for each pupil; pupil pages 165-168

Teaching Procedure

ACTIVITY 1 • Multiplying Multiples of Ten

Pupil page 165

Quickly revise the multiplication facts to 12×12 both orally and in writing. Vary the type of revision by asking such questions as, "What is 8×9?" and "If a product is 132 and one factor is 11, what is the other factor?" Let pupils fold their arrays of dots to show a fact if they do not remember it. Remember to *let them find the facts for themselves.*

Ask for some multiples of numbers such as 7, 6 and 9. In particular, ask for some multiples of 10. (10, 20, 30, ..., 120)

Get the children to think of a way to find the product of two multiples of 10. For example, write this sentence on the blackboard:

$$40 \times 30 = n$$

Let the children talk about how to find n. Guide them to suggest the sentence and work below:

$$
\begin{aligned}
40 \times 30 &= (4 \times 10) \times (3 \times 10) \\
&= (4 \times 3) \times (10 \times 10) \\
&= 12 \times 100 \\
&= 1200
\end{aligned}
$$

Ask, "Is the product also a multiple of 10?" (Yes) "Is the product a multiple of 100?" (Yes)

Give several other examples, such as:

$$
\begin{aligned}
70 \times 80 &= (7 \times 10) \times (8 \times 10) \\
&= (7 \times 8) \times (10 \times 10) \\
&= 56 \times 100 \\
&= 5600
\end{aligned}
$$

Let the children discuss a way of multiplying two multiples of 10. Help them decide that the product is always a multiple of 10×10, that is, a multiple of 100. Give them some other examples to solve in their exercise books, using the form below. For example:

$$
\begin{array}{r} 80 \\ \times\ \underline{90} \\ 7200 \end{array}
\qquad
\begin{array}{r} 110 \\ \times\ \underline{40} \\ 4400 \end{array}
\qquad
\begin{array}{r} 90 \\ \times\ \underline{60} \\ 5400 \end{array}
$$

Ask the children to turn to page 165 in their books and find the products. Get them to write the products in their exercise books. Allow the children to use their arrays if needed.

ACTIVITY 2 • Using a Multiple of Ten as a Factor

Pupil page 166

Write this sentence on the blackboard:

$$40 \times 38 = n$$

Say, "We can use our short form for multiplication to find this product." Then write:

$$
\begin{array}{r} 38 \\ \times\ \underline{40} \end{array}
$$

Say, "Think of 38 in expanded form. What is 40×8?" (320) "What is 40×30?" (1200) As children reply to these questions, write:

$$\begin{array}{r} 38 \\ \times \ \underline{40} \\ 320 \\ \underline{1200} \\ 1520 \end{array} \quad \begin{array}{l} \\ \\ (40 \times 8 = 320) \\ (40 \times 30 = 1200) \\ \\ \end{array}$$

Ask, "What number makes $40 \times 38 = n$ a true sentence?" (n is 1520.)

Give several such examples for the pupils to work in their exercise books. (Choose a multiple of 10 as one of the factors.) Ask the class to open their books to page 166 and work the exercises. In some of the exercises the pupils must decide which symbol, $<$, $>$ or $=$, makes the sentence true. In the other exercises, they must find the numbers that make the sentences true. Give help as needed.

ACTIVITY 3 • Multiplying Factors That Are Each Less Than 100

Pupil page 167

Write this sentence on the blackboard:

$$n = 39 \times 28$$

Ask, "How can you find n, the product of 39 and 28?" Let the children talk of ways of finding n. (Count objects arranged in 39 rows with 28 in each row; use the distributive property to change to simpler products.) If you do not get the last suggestion, make it yourself. Ask for the expanded form of 28, and write:

$$\begin{aligned} 28 \times 39 &= (20 + 8) \times 39 \\ &= (20 \times 39) + (8 \times 39) \end{aligned}$$

Let the pupils explain each step in the above sentences. (The distributive property is used to change the product to the sum of two simpler products.) Ask a pupil for the next step. (Find the products 20×39 and 8×39.) Let pupils come to the blackboard and show how to get these products.

$$\begin{array}{r} 39 \\ \times \ \underline{20} \\ 180 \\ \underline{600} \\ 780 \end{array} \qquad \begin{array}{r} 39 \\ \times \ \underline{8} \\ 72 \\ \underline{240} \\ 312 \end{array}$$

Let another pupil add the products to get 1092. Then write:

$$\begin{aligned} 28 \times 39 &= (20 \times 39) + (8 \times 39) \\ &= 780 + 312 \\ &= 1092 \end{aligned}$$

Show the pupils a way to put all this together in a shorter form.

$$\begin{array}{rl} & 39 \\ \times & \underline{28} \\ & 72 \quad (8 \times 9 = 72) \\ & 240 \quad (8 \times 30 = 240) \\ & 180 \quad (20 \times 9 = 180) \\ & \underline{600} \quad (20 \times 30 = 600) \\ & 1092 \end{array}$$

Give several more examples. For each example, put the first form on the blackboard with the help of the children and then show the shorter form. For example:

$$\begin{aligned} 36 \times 43 &= (30 + 6) \times 43 \\ &= (30 \times 43) + (6 \times 43) \\ &= 1290 + 258 \\ &= 1548 \end{aligned}$$

$$\begin{array}{rl} & 43 \\ \times & \underline{30} \\ & 90 \\ & \underline{1200} \\ & 1290 \end{array} \qquad \begin{array}{rl} & 43 \\ \times & \underline{6} \\ & 18 \\ & \underline{240} \\ & 258 \end{array} \qquad \begin{array}{r} 1290 \\ \underline{258} \\ 1548 \end{array} \qquad \begin{array}{rl} & 43 \\ \times & \underline{36} \\ & 18 \quad (6 \times 3 = 18) \\ & 240 \quad (6 \times 40 = 240) \\ & 90 \quad (30 \times 3 = 90) \\ & \underline{1200} \quad (30 \times 40 = 1200) \\ & 1548 \end{array}$$

After several examples have been worked, let the pupils turn to page 167 in their books. Let them use any form they choose to find the product as they work in their exercise books. By now some of the pupils are ready to shorten the form to this:

$$\begin{array}{rl} & 52 \\ \times & \underline{37} \\ & 364 \quad (7 \times 52 = 364) \\ & \underline{1560} \quad (30 \times 52 = 1560) \\ & 1924 \end{array}$$

ACTIVITY 4 • Multiplying with Only One Factor Less Than 100

Write this sentence on the blackboard:

$$57 \times 125 = n$$

Guide the children to suggest renaming 57 in expanded form. Write out the steps in horizontal form. With the help of the pupils, show the use of the distributive property.

$$\begin{aligned} 57 \times 125 &= (50 + 7) \times 125 \\ &= (50 \times 125) + (7 \times 125) \end{aligned}$$

Guide the children to say that the distributive property is used to change the product to the sum of two simpler products. Again, let them find each of the products, using any form.

$$\begin{array}{r} 125 \\ \times \quad 50 \\ \hline 250 \\ 1000 \\ 5000 \\ \hline 6250 \end{array} \qquad \begin{array}{l} \\ \\ (50 \times 5 = 250) \\ (50 \times 20 = 1000) \\ (50 \times 100 = 5000) \\ \\ \end{array} \qquad \begin{array}{r} 125 \\ \times \quad 7 \\ \hline 35 \\ 140 \\ 700 \\ \hline 875 \end{array} \qquad \begin{array}{l} \\ \\ (7 \times 5 = 35) \\ (7 \times 20 = 140) \\ (7 \times 100 = 700) \\ \\ \end{array}$$

Then, $57 \times 125 = 6250 + 875,$

$= 7125.$

Let the children tell the steps in the technique they use.

Let those children who can, use this form:

$$\begin{array}{r} 125 \\ \times \quad 57 \\ \hline 875 \\ 6250 \\ \hline 7125 \end{array} \qquad \begin{array}{l} \\ \\ (7 \times 125 = 875) \\ (50 \times 125 = 6250) \\ \\ \end{array}$$

After several more examples, let the pupils work the exercises on pupil page 168. Notice that some of the exercises include sums as well as products, and some are in the form of story problems.

Vocabulary

No new vocabulary

Materials

Pupil pages 169-171.

Teaching Procedure

ACTIVITY 1 • Division with a Multiple of Ten as the Known Factor

Pupil page 169

In Stage 2 the pupils revised the technique of dividing by a number less than 10. In this activity, the technique is extended to division by a multiple of 10. Write this sentence on the blackboard:

$$420 \div 30 = n$$

Guide the pupils to suggest renaming 420 as the sum of addends that are each a multiple of 30. (420 = 300 + 120; 420 = 240 + 180; and so on.) Write:

$$\begin{aligned} 420 \div 30 &= (300 + 120) \div 30 \\ &= (300 \div 30) + (120 \div 30) \\ &= 10 + 4 = 14 \end{aligned}$$

Ask a pupil to work this exercise, using a shorter form for division. Help him to write:

$$\begin{array}{r} 10 + 4 \\ 30\overline{)300 + 120} \end{array} = 14 \qquad \text{or} \qquad \begin{array}{r} 10 + 4 \\ 30\overline{)420} \\ \underline{300} \\ 120 \end{array} = 14 \quad \begin{array}{l} \\ \\ (300 \div 30 = 10) \\ (120 \div 30 = 4) \end{array}$$

Let the pupil tell what number n makes the sentence $420 \div 30 = n$ true. (n is 14.) Let another pupil check this by writing a multiplication sentence

for the same exercise. ($30 \times 14 = 420$) Let him multiply 30 and 14 to decide whether the sentence is a true sentence.

Continue with several other examples at the blackboard, such as:

$$2040 \div 60 = t$$

Let pupils tell how to proceed as you divide:

```
      20 + 10 + 4 = 34                      30 + 4 = 34
60⌈2040                                 60⌈2040
   1200  (1200 ÷ 60 = 20)    or            1800  (1800 ÷ 60 = 30)
    840                                     240  (240 ÷ 60 = 4)
    600  (600 ÷ 60 = 10)
    240  (240 ÷ 60 = 4)
```

t is 34.

Accept any way of renaming 2040, provided each addend is a multiple of 60. Always suggest that the children check the division by multiplication. For example, "$2040 \div 60 = 34$" is true because "$34 \times 60 = 2040$" is true.

Let the pupils work the exercises on page 169 in their books. Notice that some of the exercises have missing factors and some have missing addends. Let the pupils use any forms to do the operations. The story problems are solved by writing a mathematical sentence, finding a number that makes the sentence true, and answering the question asked in the problem.

ACTIVITY 2 • Division with the Known Factor Less Than 13

Pupil page 170

In Stage 1 of this unit the children extended their multiplication facts to 12×12. For each multiplication fact, they learned division facts. For example, since $7 \times 12 = 84$ they know that $84 \div 7 = 12$ and $84 \div 12 = 7$. Use these facts now to extend division techniques to known factors of 11 and 12.

For example, let the class help you to work this exercise:

$$275 \div 11 = n$$

Ask, "What large addend of 275 is a multiple of 11?" (The children may suggest 220, or 20×11.) Write the form for the division on the blackboard and continue the work as the children guide.

$$
\begin{array}{r l}
 & \underline{20 + 5} = 25 \\
11 \overline{)\,275} & \\
 & \underline{220} \quad (220 \div 11 = 20) \\
 & 55 \quad (55 \div 11 = 5)
\end{array}
\qquad n \text{ is } 25.
$$

Let the children check the work by showing that the sentence $11 \times 25 = 275$ is a true sentence.

On pupil page 170 the children will find exercises to be solved. Some ask for products and others require division. Some are addition sentences. Let the pupils use any forms they understand to perform the operations in the exercises.

ACTIVITY 3 • Division Process with the Known Factor Not a Multiple of the Product

Pupil page 171

Note: The children have been solving division exercises of the type

$$369 \div 9 = n$$

by finding the whole number n for which

$$9 \times n = 369$$

is true. That is, 369 is a multiple of 9.

The children will be curious about division sentences when the known factor is *not* a multiple of the product. Use the following problem as an example:

Kuli has 166 nuts. He shares them equally among 6 of his friends. How many nuts does each of his friends get?

Let the children tell ways to think about this problem. Among the ways are these:

1. Put 166 nuts in an array with 6 in each row. (Will the nuts form an array, or will some nuts be left over?)
2. Write the division sentence $166 \div 6 = n$. Ask whether there is a whole number that makes the sentence true.
3. Write the multiplication sentence $6 \times n = 166$. Ask whether there is a whole number that makes the sentence true.

In order to answer the problem, let the children set up and work out the division as they have done before. Do not suggest that the exercise is different than others.

$$
\begin{array}{r|l}
 & \underline{20 + 7} \\
6 & 166 \\
 & \underline{120} \quad (120 \div 6 = 20) \\
 & 46 \\
 & \underline{42} \quad (42 \div 6 = 7) \\
 & 4 \quad (4 \div 6 = ?)
\end{array}
$$

The children will notice from the above form that 166 is thought of as the sum of three addends,

$$166 = 120 + 46 + 4,$$

but not all the addends are multiples of 6. Ask, "If Kuli gives each of his 6 friends 20 nuts and then 7 nuts, will he give away all of his 166 nuts?" Let the class decide that 6×27 is 162. This means that 4 nuts are not given away.

Ask, "Can we find a whole number that makes the sentence $166 \div 6 = n$ true?" (No) "How many of Kuli's nuts can be shared?" (162) "How many did he have left?" (4) Help the children to write this in the form of a true sentence:

$$166 = (6 \times 27) + 4$$

Give other examples that use this division process. For example, ask, "Is $347 \div 11$ a whole number?" Let a pupil begin a division process to answer this question:

$$
\begin{array}{r|l}
 & \underline{30 + 1} = 31 \\
11 & 347 \\
 & \underline{330} \quad (330 \div 11 = 30) \\
 & 17 \\
 & \underline{11} \quad (11 \div 11 = 1) \\
 & 6
\end{array}
\qquad\qquad 347 = (11 \times 31) + 6
$$

He will say that $347 \div 11$ is not a whole number, because 347 is not a multiple of 11. Ask, "What is the greatest number less than 347 that is a multiple of 11?" (341) "What true sentence tells us about this division process?" [$347 = (11 \times 31) + 6$] "How can 347 be renamed as the sum of two addends so that one addend is the largest multiple of 11?" ($347 = 341 + 6$)

Let the pupils turn to page 171 in their books. Let them read the problems with your help and write the answers asked for.

UNIT 13

Operations on Fractions

OBJECTIVES

1. To revise addition of fractions with the same denominator
2. To introduce subtraction of fractions with the same denominator
3. To show that some fractions may be considered as a whole number plus a fraction
4. To relate a fraction to the division of a whole number by a counting number

Background Information for Teachers

Also see "Background Information" for Units 5 and 10, and Part 2, "Arithmetic", Section 8, pages 57, 59, 60, in your *Teachers' Handbook.*

When we add fractions, the sum is a fraction. This fraction may also be a whole number. For example, $\frac{2}{2}$, $\frac{4}{4}$ and $\frac{3}{3}$ are names for the whole num ber 1, and $\frac{4}{2}$, $\frac{6}{3}$ and $\frac{10}{5}$ are names for 2 and so on. Sometimes the sum is a fraction which is not a whole number and which is greater than 1. We can write the names for these fractions in several ways. For example, $\frac{5}{4}$ is $1 + \frac{1}{4}$, or $1\frac{1}{4}$, and $\frac{7}{2}$ is $3 + \frac{1}{2}$, or $3\frac{1}{2}$.

We can think of $\frac{5}{4}$ and $\frac{7}{2}$ in the following way: Since $\frac{4}{4} = 1$, $\frac{5}{4}$ is $\frac{4}{4} + \frac{1}{4}$, or $1 + \frac{1}{4}$; and since $\frac{6}{2} = 3$, $\frac{7}{2}$ is $\frac{6}{2} + \frac{1}{2}$, or $3 + \frac{1}{2}$. This can be shown by the number line, on which the point for $\frac{5}{4}$ is between the points for 1 and 2. Further, we can find the point for 1 and see that $\frac{5}{4}$ corresponds to a point which is $\frac{1}{4}$ of a unit beyond the point for 1; that is, the point for $1 + \frac{1}{4}$.

We can divide any whole number p by any counting number q and the result is a number. This number is the fraction $\frac{p}{q}$. For example, $4 \div 2 = 2$, but another name for 2 is $\frac{4}{2}$. Thus, $4 \div 2 = \frac{4}{2}$. We can show this on the number line. For $4 \div 2 = \frac{4}{2}$, the fraction $\frac{4}{2}$ resulting from the division operation

is also a whole number. But sometimes the result is not a whole number. For example, $3 \div 4 = \frac{3}{4}$. The sentence $3 \div 4 = n$ means $4 \times n = 3$. Since $4 \times \frac{3}{4} = 3$ is a true sentence, we know that $3 \div 4 = \frac{3}{4}$ is a true sentence.

In the same way, we can show that $1 \div 5 = \frac{1}{5}$, $7 \div 2 = \frac{7}{2}$, $2 \div 3 = \frac{2}{3}$ and so on. In other words, $\frac{1}{5}$ means $1 \div 5$, $\frac{7}{2}$ means $7 \div 2$ and $\frac{2}{3}$ means $2 \div 3$. The fraction $\frac{a}{b}$ means $a \div b$, if a is a whole number and b is a counting number.

As was seen before, $\frac{2}{3}$ also means $2 \times \frac{1}{3}$, $\frac{7}{2}$ means $7 \times \frac{1}{2}$ and $\frac{a}{b}$ means $a \times \frac{1}{b}$. Thus, the fraction $\frac{a}{b}$ means $a \div b$ and $a \times \frac{1}{b}$.

Since $\frac{a}{b}$ means $a \div b$, b cannot be zero, because division by zero is not possible. To see why, look at an example: $6 \div 0 = n$. The related multiplication sentence is $0 \times n = 6$. But there is no number n which will make this last sentence true, because 0 times any number is 0. So the sentence $6 \div 0 = n$ cannot be solved and $\frac{6}{0}$ is not a numeral for any fraction.

STAGE 1 • Revision of Addition of Fractions Having the Same Denominators

Vocabulary

No new vocabulary

Materials

Pupil page 172

Teaching Procedure

ACTIVITY 1 • Adding Fractions

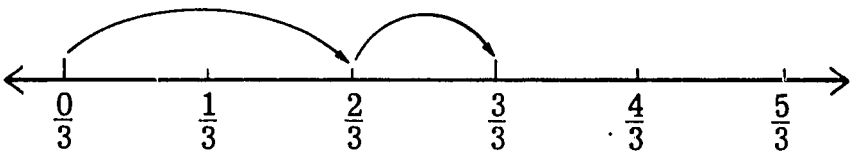
Pupil page 172

Revise addition of fractions by discussing examples such as:

$$\frac{2}{3} + \frac{1}{3} = n$$

Write the sentence on the blackboard. Ask the children how to find the number n that makes the sentence true. Let them suggest various ways to find n. Among the possible ways are the following:

1. Use the number line as shown below.



2. Write $\frac{2}{3} + \frac{1}{3}$ as $\frac{2+1}{3}$ and add the numerators.

3. Count the thirds, "one-third, two-thirds, three-thirds", since

$$\frac{2}{3} + \frac{1}{3} = \left(\frac{1}{3} + \frac{1}{3}\right) + \frac{1}{3}.$$

Carry out the suggestions until the children agree that n is $\frac{3}{3}$ (or 1). Let a child write "n is $\frac{3}{3}$" on the blackboard.

Use the same plan with other examples, such as:

$$\frac{3}{4} + \frac{2}{4} = y \qquad \frac{3}{8} + \frac{2}{8} = a \qquad \frac{1}{6} + \frac{4}{6} = m$$

$$\frac{1}{3} + \frac{1}{3} = b \qquad \frac{4}{8} + \frac{3}{8} = c \qquad \frac{1}{4} + \frac{2}{4} = z$$

For each sentence, let the children show different ways of finding the number that makes the sentence true. Emphasize that to add fractions with the same denominator, we add the numerators to find the numerator of the sum. For example:

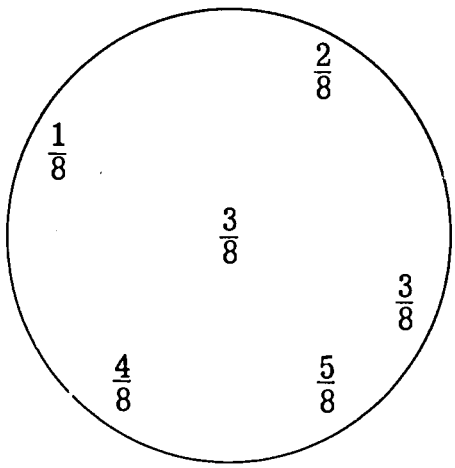
$$\frac{3}{4} + \frac{2}{4} = \frac{3+2}{4} = \frac{5}{4}$$

Ask the children to open their books to page 172. Let a child read aloud the first sentence, and let the children talk about how to make the sentence true. Tell the children to number from 1 through 20 in their exercise books and, for each sentence on page 172, write the numeral for the number n that makes the sentence true.

Note: The aim in this stage is for pupils to master the addition of fractions with the same denominators. It may be helpful to place the children in groups according to their levels of achievement. Some children may need more help and more practice than others.

The games described here are just a few of the many that are suitable. You will think of other activities.

Draw a circle on the blackboard and write the numerals for fractions inside the circle, as shown below. Place one fraction numeral in the centre.



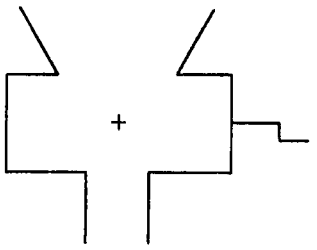
Choose a leader. Give him a stick to use as a pointer. Let him point to any of the numerals around the circle; for example, $\frac{1}{8}$. Let another child add $\frac{1}{8}$ to the fraction that is shown at the centre, $\frac{3}{8}$. The child should respond, "$\frac{3}{8} + \frac{1}{8} = \frac{4}{8}$." The leader continues to point at random to different numerals and to choose other children to give sums. Replace the "$\frac{3}{8}$" in the centre with "$\frac{2}{8}$" and repeat.

Draw other circles for halves, thirds, fourths and so on and use them for the game. The aim is to provide practise with exercises such as these:

$$\frac{1}{2} + \frac{1}{2} = \frac{2}{2} \qquad \frac{1}{4} + \frac{2}{4} = \frac{3}{4} \qquad \frac{1}{8} + \frac{2}{8} = \frac{3}{8}$$

$$\frac{3}{6} + \frac{2}{6} = \frac{5}{6} \qquad \frac{5}{8} + \frac{2}{8} = \frac{7}{8} \qquad \frac{3}{12} + \frac{4}{12} = \frac{7}{12}$$

Another game uses operation machines. Draw an operation machine on the blackboard.



Say, "This is an operation machine. It is an addition machine. When we put two fractions into the machine and turn the handle, a fraction pops out at the bottom." Write "$\left(\frac{3}{4}, \frac{2}{4}\right)$" at the top of the machine. Pretend to turn the handle. Ask, "What fraction pops out?" $\left(\frac{5}{4}\right)$

Use the machine for other pairs of fractions having the same denominator. Then, when a child gives a correct answer, let him put another pair of fractions into the machine. (He writes their numerals at the top of the machine.) This child chooses someone to turn the handle and tell what fraction pops out. If he names the correct fraction, let him put another pair of fractions into the machine. Continue as needed.

STAGE 2 • Subtraction of Fractions Having the Same Denominators

Vocabulary

No new vocabulary

Materials

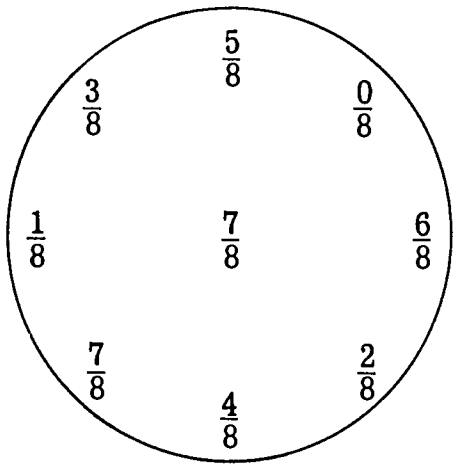
Pupil pages 173-175

Teaching Procedure

ACTIVITY 1 • Games to Find Missing Addends

Pupil page 173

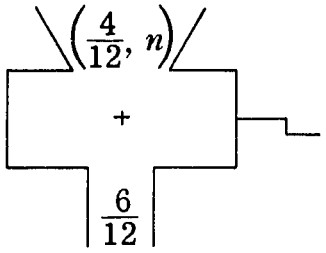
Draw a circle on the blackboard and write numerals for fractions inside the circle, as shown below.



In this game, the fraction $\frac{7}{8}$ is a sum. The other fractions in turn are used as known addends. Select a child to point to any of the other numerals, say $\frac{3}{8}$. Let another child tell what fraction added to $\frac{3}{8}$ gives the sum $\frac{7}{8}$. (The child is asked to make the following sentence true: $\frac{3}{8} + a = \frac{7}{8}$.) He finds the missing addend, $\frac{4}{8}$. Draw other circles for fractions having other denominators and follow the same plan.

Another game can be played in which children work in groups of three at the blackboard or at their desks. Let one child write a numeral for a fraction that is the sum of a pair of fractions. Let a second child write a numeral for one of the fractions of the pair having that sum. Let the third child write the numeral for the other addend.

Another way to give practice in finding missing addends is to use the operation machine. Draw an addition machine and put in numerals, as shown on the next page.



Say, "When the two numbers, $\frac{4}{12}$ and n, are put into the machine and the handle is turned, the number $\frac{6}{12}$ pops out. What number is n?" $\left(\frac{2}{12}\right)$ Go on using the operation machine with other fractions.

Let the children turn to page 173 in their books and tell the missing number for each machine shown there.

ACTIVITY 2 • Subtracting Numbers to Find Missing Addends

Pupil page 174

Note: In the preceding activity, the pupils revised the idea of finding missing addends. For example, if the sum is $\frac{7}{12}$ and one addend is $\frac{3}{12}$, the missing addend is to be found. We use the sentence $\frac{3}{12} + b = \frac{7}{12}$ or the sentence $b + \frac{3}{12} = \frac{7}{12}$ to ask for the missing addend. Many pupils will be able to find the missing addend directly, because they know that $\frac{4}{12}$ is the number that we add to $\frac{3}{12}$ to obtain the required sum $\frac{7}{12}$. (They know this because $4 + 3 = 7$.)

If a pupil cannot find the missing addend directly in this activity, provide revision activities as needed. These children may need more use of the number line.

Briefly revise subtraction of whole numbers, pointing out that in the study of whole numbers we subtract by finding a missing addend. Write these sentences on the blackboard:

$$15 = 8 + n$$
$$15 - 8 = n$$

Say, "In both sentences, n is the missing addend." Let a child read the sentences aloud. (The second sentence may be read as "fifteen minus eight" or as "eight subtracted from fifteen".) Let the children talk about

the way the two sentences are related. (Both sentences have the same meaning. One is an addition sentence and one is a subtraction sentence. The number n in both sentences is the same number. $15 - 8$, or n, is the number added to 8 to give 15.)

Ask, "Do you think subtraction of fractions is like subtraction of whole numbers?" Give an example so that the children may talk about this question. For example:

$$\frac{11}{12} - \frac{5}{12} = n$$

The discussion should emphasize (1) that n is the missing addend that makes the sentence true and (2) that the addition sentence is $\frac{11}{12} = \frac{5}{12} + n$. When the addition sentence is suggested, let a child write it on the blackboard below the subtraction sentence, like this:

$$\frac{11}{12} - \frac{5}{12} = n$$
$$\frac{11}{12} = \frac{5}{12} + n$$

Let a child read both sentences. (Eleven-twelfths minus five-twelfths equals n; Eleven-twelfths equals five-twelfths plus n.) Ask, "What is the sum in both sentences?" $\left(\frac{11}{12}\right)$ "What is the known addend?" $\left(\frac{5}{12}\right)$ "What number n makes both sentences true?" $\left(\frac{6}{12}\right)$ Let a child write the numeral for the addend on the blackboard in a sentence like this:

$$n \text{ is } \frac{6}{12}.$$

Give several similar examples of pairs of sentences. Emphasize these ideas:

1. One is an addition sentence and the other is a subtraction sentence.
2. Both sentences give the sum and one addend.
3. Both sentences ask, "What is the missing addend?"
4. We subtract fractions to find the missing addend.

Some of the children may notice, for example, that since $\frac{5}{6} - \frac{3}{6} = \frac{2}{6}$, then $\frac{5}{6} - \frac{3}{6} = \frac{5-3}{6}$. Encourage these children to use this way of thinking

about the subtraction of fractions. They will begin to understand that to subtract fractions with the same denominator we may subtract the numerators.

Ask the pupils to open their books to page 174. Say, "In each exercise there are two sentences. One is an addition sentence and one is a subtraction sentence. Both have the same meaning. Both sentences are true for the same missing addend."

Go over the first exercise with the class. Let them decide that if a is $\frac{2}{6}$ both sentences are true. In their exercise books they write:

1. a is $\frac{2}{6}$.

Tell them to complete the other exercises in the same way.

STAGE 3 • Different Names for the Same Fraction

Vocabulary

No new vocabulary

Materials

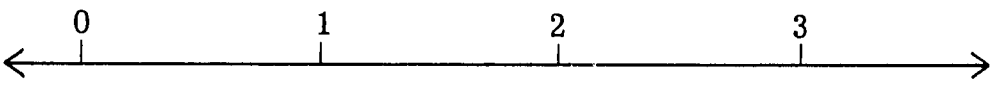
Three 2" × 18" strips of paper; pupil page 175

Teaching Procedure

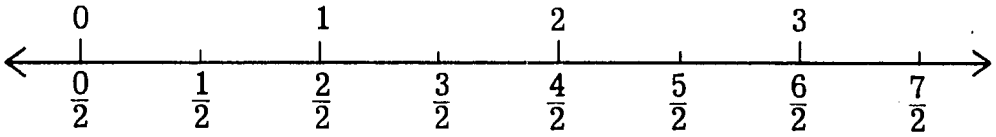
ACTIVITY 1 • Different Names for Fractions, Such as $\frac{5}{2} = 2 + \frac{1}{2}$

Pupil page 175

Draw a line segment on the blackboard. Mark off several 18" unit lengths with one 2" × 18" strip of paper and label the points.



Let a child fold the 2" × 18" strip of paper into halves. Call another child to mark one-half-unit segments. Let other children label these points on the number line.



Let the class count the halves shown on the number line as "one half, two halves, three halves" and so on. Call children in turn to find the points for fractions you name; for example, two halves, six halves.

Ask children to look at the number line and find points that are labelled with numerals for fractions and also with numerals for whole numbers; for example, $\frac{2}{2}$, $\frac{4}{2}$ and $\frac{6}{2}$. Each time ask, "What is another name for the fraction?" For example, the point for $\frac{2}{2}$ is also labelled "1". Ask children to write sentences on the blackboard that show these different names, such as:

$$\frac{2}{2} = 1 \qquad \frac{4}{2} = 2$$

Move your finger along the line segment from the point for 0 to the point for $\frac{5}{2}$ and say, "This line segment is the union of two whole segments and a half segment. We say that $\frac{5}{2}$ is another name for $2 + \frac{1}{2}$."

Ask, "Is $\frac{5}{2}$ greater than 2?" (Yes, its point is to the right of the point for 2 on the number line.) "Can you give another name for 2?" $\left(\frac{4}{2}\right)$ "We see that $\frac{5}{2}$ is $\frac{4}{2} + \frac{1}{2}$ or $2 + \frac{1}{2}$." Let a child write this sentence on the blackboard:

$$\frac{5}{2} = \frac{4}{2} + \frac{1}{2} = 2 + \frac{1}{2}$$

Move your hand along the segment from the point for 0 to the point for $\frac{3}{2}$. Say, "This is the union of the segment described by the number 1 and this segment which we describe by one-half. Can you give another name for $\frac{3}{2}$?" $\left(1 + \frac{1}{2}\right)$

Ask, "Is $\frac{3}{2}$ greater, or less, than 1? (Greater; the point for $\frac{3}{2}$ is to the right of the point for 1.) Find the point for 1 on the number line. Does it have another name?" $\left(\frac{2}{2}\right)$ Say, "Another way to name $\frac{3}{2}$ is $\frac{2}{2} + \frac{1}{2}$ or $1 + \frac{1}{2}$." Let a child write these names on the blackboard:

$$\frac{3}{2} = \frac{2}{2} + \frac{1}{2} = 1 + \frac{1}{2}$$

Write the numeral for $\frac{7}{2}$ on the blackboard. Let children use the number line as they tell and write different ways to name $\frac{7}{2}$.

$$\frac{7}{2} = \frac{6}{2} + \frac{1}{2} = 3 + \frac{1}{2}$$

Note: There are many ways these fractions can be named other than the suggested answers. For example, $\frac{7}{2}$ can also be named $2 + \frac{3}{2}$ or $\frac{4}{2} + \frac{3}{2}$ and so on. If such a name is suggested by a child, accept it and go on to show that $3 + \frac{1}{2}$ is also a name.

Draw other pictures of the number line and mark them in thirds, fourths, eighths, sixths and so on. Use the above plan to help children find different names for other fractions. For example:

$$\frac{5}{4} = 1 + \frac{1}{4} \qquad \frac{8}{6} = 1 + \frac{2}{6} \qquad \frac{5}{3} = 1 + \frac{2}{3} \qquad \frac{13}{4} = 3 + \frac{1}{4}$$

Ask the children to turn to page 175 in their books, and discuss the first example with them. Ask "What other name can we give to the point for $\frac{5}{2}$ on the number line?" Let the children trace along the line segment which has as endpoints the points for 0 and $\frac{5}{2}$. Then help them to see, first, that $\frac{2}{2} = 1$ and $\frac{4}{2} = 2$, and to see, second, that to get to the point for $\frac{5}{2}$ you move another half segment. Therefore $\frac{5}{2} = 2 + \frac{1}{2}$.

Explain the example $1 + \frac{1}{2} = \frac{2}{2} + \frac{1}{2} = \frac{3}{2}$, using the number line at the top of the page. Ask the children to complete all the exercises, using the number lines as needed.

Vocabulary

No new vocabulary

Materials

Paper strips, 2" × 16", 2" × 24", 2" × 32"; pupil pages 176-177

Teaching Procedure

ACTIVITY 1 • Revising Division

Write the following sentences on the blackboard:

$6 \div 2 = n$	$6 \div 3 = a$	$4 \div 2 = m$
$8 \div 2 = y$	$8 \div 4 = b$	$6 \div 6 = k$

Ask the children to discuss the meaning of these sentences. The discussion should bring out such things as:

1. These are all sentences about division of two counting numbers.
2. The letter (n, y and so on) in each sentence names a number.
3. Division is related to multiplication; so we know, for example, $6 \div 2 = n$ means $2 \times n = 6$. That is, "$6 \div 2 = n$" and "$2 \times n = 6$" are both true for the same number n.
4. In each sentence the number that makes the sentence true is a missing factor.

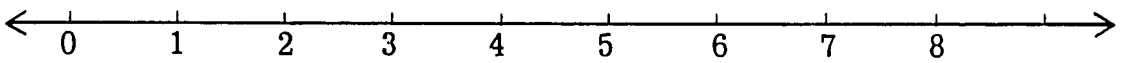
Let different children rewrite the sentences on the blackboard as multiplication sentences, placing each multiplication sentence under its corresponding division sentence.

$2 \times n = 6$	$3 \times a = 6$	$2 \times m = 4$
$2 \times y = 8$	$4 \times b = 8$	$6 \times k = 6$

Ask the children to tell what number the letter must name in each of the sentences to make a true sentence. Write the numbers on the blackboard. The blackboard for the first sentence will look like this:

$6 \div 2 = n$
$2 \times n = 6$
n is 3.

Make a picture of the number line on the blackboard. Make it at least 36 inches in length. Mark a point near the left end for 0. Mark other points for the whole numbers up to and including 8. Make the line segments from one point to the next 4 inches in length; for example, from 0 to 1 will be 4 inches.



Say, "We can use the number line to check our sentences. Look at the sentences $6 \div 2 = n$ and $2 \times n = 6$. How can we show this on our line? We said that if n is 3 these sentences are true. How can we check this?" Help the children to understand that, since two times the number is six, if we begin at the point for 0 and move three spaces two times then we will be at the point for 6. So if the number is 3, as suggested, and if we move three spaces twice (or three spaces and then three spaces more), we will be at the point for 6. Let a child trace along the line with his finger to show this and to check the truth of the completed sentences.

Say, "We find that when we complete the sentences to make them true we can check them on the number line. But if at first we do not know what number to try, can we use the number line to find the correct number?" Encourage the children to see that we can separate the line segment from 0 to 6 into two congruent segments. Each segment shows the number which when multiplied by 2 gives six. Ask, "How can we show this using a paper strip or string?"

Get the children to suggest that a strip the same length as the segment from 0 to 6 may be folded into two parts of the same size. Use a paper strip 2" × 24" or a string 24" long, and hold it along the segment of the line from the point for 0 to the point for 6. Let a child fold it into two parts of the same size. Let him hold the folded strip along the segment, with the left end at the point for 0. Ask the class, "At what point is the right end of the

strip? (3) "In this way we find that $6 \div 2$ is what number?" (3)

Discuss the next exercise, $8 \div 2 = y$, in the same way. Show on the number line that $2 \times 4 = 8$ and show that 4 is the number that makes the sentence true; that is, y is 4. Emphasize again that the pupils can choose a number that they think will make the sentence true and then check it on the number line. But if they do not know what number to try, they can fold a strip of paper into parts and compare it with the segments of the number line. For this example, use a paper strip 32 inches in length (the same length as the segment from the point for 0 to the point for 8). Let a child fold it into two parts of the same size. Ask the children why it should be folded into *two* parts. (Two is the known factor.) Let a child hold the strip along the number line with the left end at the point for 0. He will find that the right end is at the point for 4 and say, "y is 4." Write the sentences on the blackboard in this way:

$8 \div 2 = y$
$2 \times y = 8$
y is 4.

Continue with the same plan for the other sentences. Keep the paper strips and the number line on the blackboard for use in the next activity.

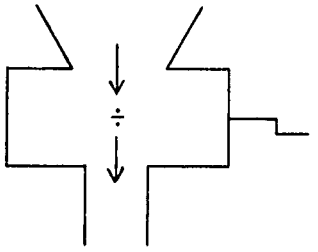
ACTIVITY 2 • Fractions as the Result of Division

Put the following pairs of numerals on the blackboard:

(6,2) (8,2) (6,3) (8,4) (4,2) (6,6)

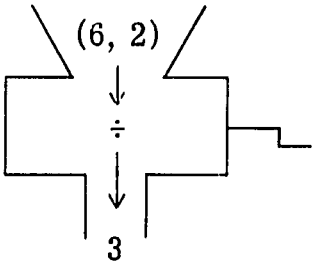
Say, "In the last lesson, we used some division sentences. In each sentence, there was a pair of counting numbers. I have written the numerals for the pairs of numbers on the blackboard. We divided the first number by the second. For each pair, we found that the division operation gave us another number. We found the point for these numbers on the number line."

Draw an operation machine on the blackboard.



Ask, "Do you remember our operation machines? If we put two numbers into the top of the machine and turn the handle, the machine operates on the two numbers and one number pops out at the bottom. Pretend this is a division machine. What happens if we put in the pair (6,2)?"

Help the children to see that with this pair of numbers in the division operation machine the number 3 will pop out. Write the numerals in the diagram.



Say, "We write this as a sentence." Write:

$$6 \div 2 = 3$$

Ask, "What number will pop out of the division machine if we put in the pair of numbers (8,2)?" (4) Let a child write the numerals for this pair and for the result in the diagram. Let another child write the division sentence.

Continue with the other examples in the same way. Get the children to notice that all the resulting numbers are whole numbers.

After completing these examples, ask, "Do you think we can put any pair of counting numbers into the division machine and always get a counting number?" (No) Let the children talk about this. Let them put in 3 and 2 and find that no whole number pops out. Ask, "Can another kind of number pop out of the machine?" (Yes)

Guide the discussion so that the pupils think about the problem of dividing 3 by 2. Let them show this by separating a segment on the number

line. Say, "If the division machine pops out a number from the pair 3 and 2, we should be able to find the number on the number line."

Help the children to see that the segment of the number line from 0 to 3 can be cut into two congruent segments. Show this by folding a paper strip. Let a whole segment represent the first counting number in the pair, 3, and the number of congruent segments, two, represent the second. Say, "We can find a point on the line for the number that pops out of the division machine. That number is the fraction $\frac{3}{2}$."

Next write the following pairs of numerals on the blackboard:

(7,3) (7,4) (7,2) (6,4) (5,2) (5,3)

Say, "Here are numerals for some other pairs of counting numbers. Do you think that the division machine will pop out a number for each pair? (Yes) "Do you think that each number it pops out will be a whole number?" The answer to this is "No", but let the children suggest their own answers. If some children say, "Yes", let them try to guess what whole number will be attached to a particular pair.

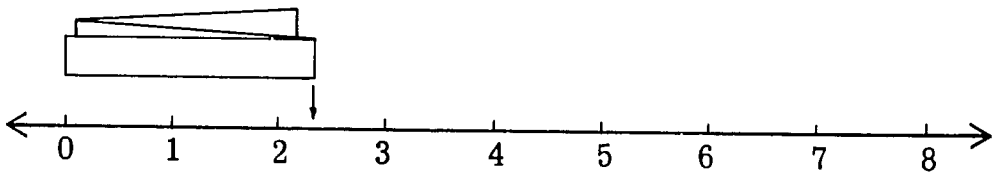
Take the pair (7,3) as an example. Ask, "If we put the pair of numbers (7,3) into the division machine, what number do you think will pop out?"

Get the children to tell how they find the solution to

$$7 \div 3 = n$$

on a number line. For example, they can take a paper strip which is the same length as the segment of the line from 0 to 7 (a 28" strip in this case). They understand this because 7 is the first number of the pair. Ask, "What is the second number of the pair? Into how many parts will we fold the strip?" (3)

Use the number line and the folded strip to find the point for $7 \div 3$.



The point will be between the points for 2 and 3. Ask, "Is this a point for a number?" Let the children discuss their answers until they understand that

this is a point for a fraction, and so it is a point for a number. Guide the children to say, "It is a point for a number, but we see it is not one of the whole numbers. It is a point for a fraction." Let the children tell what fraction they think it is. When someone suggests a number, ask him to fold a strip (4 inches in length, the length of the segment from 2 to 3) and check his answer. If he suggests $\frac{1}{4}$, he folds the strip into four congruent parts and discovers that he is wrong. If he suggests $\frac{1}{3}$, he folds the strip into three congruent parts and discovers that he is probably right. The point is found to be the point for $2 + \frac{1}{3}$ or $\frac{7}{3}$. Emphasize that the point has both names, so we can write:

$$7 \div 3 = 2 + \frac{1}{3} \qquad \text{or} \qquad 7 \div 3 = \frac{7}{3}$$

Emphasize that we use the number line to make guesses about the number we get when we divide two counting numbers.

Use the same procedure with the number line to help the children guess that

$$6 \div 4 = 1 + \frac{2}{4}.$$

Get them to notice also that the point named $1 + \frac{2}{4}$ is also named $\frac{6}{4}$. Write on the blackboard:

$$6 \div 4 = \frac{6}{4}$$

The same plan can be followed with the pair (5,2) to complete these sentences:

$$5 \div 2 = 2 + \frac{1}{2} \qquad \text{and} \qquad 5 \div 2 = \frac{5}{2}$$

ACTIVITY 3 • Relating Division and Fractions

Revise some examples like those of the previous activity. Write the following sentences on the blackboard. Remind the children that they use

the number line to help them guess the number they get when they divide in these sentences:

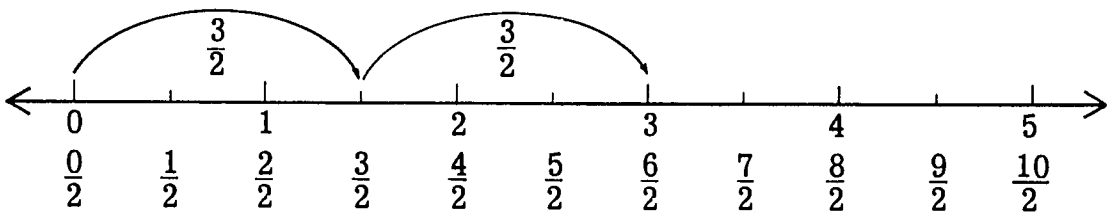
$$7 \div 2 = \frac{7}{2} \qquad 3 \div 2 = \frac{3}{2} \qquad 7 \div 3 = \frac{7}{3}$$

Say, "Check your guesses by thinking of the meaning of these sentences. What are the multiplication sentences that mean the same?" Let children write the related multiplication sentences under the division sentences:

$$7 \div 2 = \frac{7}{2} \qquad 3 \div 2 = \frac{3}{2} \qquad 7 \div 3 = \frac{7}{3}$$

$$2 \times \frac{7}{2} = 7 \qquad 2 \times \frac{3}{2} = 3 \qquad 3 \times \frac{7}{3} = 7$$

Ask, "How can you show on the number line that the multiplication sentences are true?" For example, $2 \times \frac{3}{2}$ means $\frac{3}{2} + \frac{3}{2}$. Guide them to say, "If we move from 0 along two segments, each showing $\frac{3}{2}$, we come to the point for 3."



Guide the children to notice the pattern in the division sentences. (The first number in the pair of numbers is the numerator of the fraction and the second number is the denominator of the fraction.) Say, "Now we have another way of expressing division. This first sentence (point to it) says that $\frac{7}{2}$ means $7 \div 2$; this sentence says that $\frac{3}{2}$ means $3 \div 2$; this sentence says that $\frac{7}{3}$ means $7 \div 3$. This is true for the names of all fractions. A fraction is a number we get when we divide two counting numbers."

The children should use this idea in doing other examples. For instance, write on the blackboard,

$$6 \div 4 = n$$

and ask, "What number n makes this sentence true?" If the children think

about the previous example, they will say, "n is $\frac{6}{4}$" Ask for the related multiplication sentence: $4 \times n = 6$. Say, "If n is $\frac{6}{4}$, then $4 \times n$ is $4 \times \frac{6}{4}$. How can we show this product on the number line?" Let the children tell how to move along four segments, each segment showing $\frac{6}{4}$, until they come to the point for 6. Say, "We find that $4 \times \frac{6}{4} = 6$. This proves that $6 \div 4 = \frac{6}{4}$." Ask, "Is there another name for this number?" Use the same picture of the number line so that the children can see that $\frac{6}{4}$ is also $1 + \frac{2}{4}$. They may also discover that the same fraction can be called $1 + \frac{1}{2}$. If so, say, "We see on the number line that $\frac{6}{4}$ and $1 + \frac{2}{4}$ and $1 + \frac{1}{2}$ name the same point on the line. We can think about these different names in another way. We know that $\frac{4}{4}$ is what whole number? (1) Then $\frac{6}{4}$ is 1 and how many more? $\left(\frac{2}{4}\right)$ Write:

$$\frac{6}{4} = \frac{4}{4} + \frac{2}{4} = 1 + \frac{2}{4}$$

Show that we may write $\frac{6}{4} = 1 + \frac{2}{4}$ or $\frac{6}{4} = 1 + \frac{1}{2}$. Emphasize again that $\frac{6}{4}$ means $6 \div 4$.

Use the same plan for many similar examples, such as:

$$5 \div 2 = \frac{5}{2} = 2 + \frac{1}{2} \qquad 5 \div 3 = \frac{5}{3} = 1 + \frac{2}{3}$$

Go on until the children understand that:

1. Any counting number can be divided by another. The result is a number; it may be a fraction that is a whole number, or it may be a fraction that is not a whole number.
2. Any numeral written in a form such as $\frac{5}{2}$ can refer to division; in this case, to $5 \div 2$. We know that $5 \div 2 = \frac{5}{2}$ because we found that $2 \times \frac{5}{2} = 5$.

ACTIVITY 4 • Relating Division and Fractions

Pupil page 176

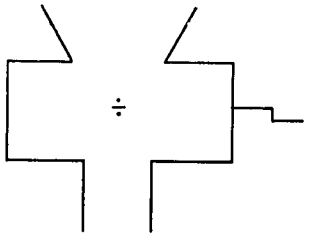
The first line of the table on page 176 has an example worked out.

Ask the children to complete the rest of the table by writing other names for the fractions in the spaces in the same way. They may use the number line at the top of the page to help them find different names for a number. The number line is marked to represent whole numbers, fourths and halves.

ACTIVITY 5 • Division Resulting in Fractions

Pupil page 177

Draw a picture of a division machine on the blackboard.



Say, "We have put many different pairs of counting numbers into the division machine. Has a number always popped out?" (Yes)

Let children take turns writing a pair of numerals above the picture and ask other children to write a numeral for the number that pops out under the picture. Encourage examples such as (5,2) and (3,2) and also examples such as (6,3), (8,2) and so on.

Say, "The division machine always assigns a fraction to a pair of counting numbers. Sometimes the fraction is a counting number; sometimes it is not."

Ask the pupils to open their books to page 177. Ask a child to read the words at the top of the page. Go over the first example with the children. Say, "The pair of numbers we put in our division machine is the pair (7,3). What number does the machine assign to this pair?" $\left(\frac{7}{3}\right)$ Tell the children to do the other exercises, writing the numerals for the resulting fractions in their exercise books.

When the work is completed, discuss it with the class. Emphasize these ideas:

1. For any two counting numbers we choose, division assigns a number.
2. A fraction is a number we get when we divide two counting numbers; for example, $\frac{5}{2}$ is the number we get when we divide 5 by 2: $5 \div 2 = \frac{5}{2}$.

UNIT 14
Fun with Mathematics

OBJECTIVES

1. To apply knowledge of mathematics to problems, puzzles and games
2. To explore mathematical patterns and relations

Background Information for Teachers

The underlying aim of this unit is the enjoyment of mathematics. But the activities also encourage the children to think about mathematical concepts and relations, to search for new ideas in approaching a problem, and to explore new ways of looking at old ideas. In doing this, much learning can take place.

Approach the activities of the unit lightly, in the spirit of fun and recreation. Make no pressures on the students to achieve mastery of a body of concepts or skills. But from solving puzzles and riddles, playing games and looking for patterns, extended insights into important mathematical relations will result. These insights, though not fully formed concepts, are preparation for mathematical ideas to be developed later.

Let the children work on an activity only as long as it is enjoyed. If some activities prove too difficult for some children, let them go on to another. Full solutions are not always necessary in order for learning to take place.

STAGE 1 • Multiples and Factors

Vocabulary

No new vocabulary

Materials

A 2" × 3" card (or paper) for each child; no pupil pages

ACTIVITY 1 • Classifying Counting Numbers

On the 2" × 3" cards write the numerals for counting numbers from 1 up to and including the number of children in the class. Give each child one of the cards. Tell the children that the cards tell their names.

Say, "Each of you has a number name. I will think of a subset of the counting numbers. I will tell you something about the set. If your number is in the set of numbers I am thinking of, stand up."

Use the following as clues to the subsets of counting numbers.

1. Each of the numbers in the set I am thinking of is a factor of six. We can also say that 6 is a multiple of all these counting numbers. (Children with the 1, 2, 3, 6 cards will stand because 6 is a multiple of these numbers.)
2. Each of the numbers I am thinking of is a factor of 18. Eighteen is a multiple of all these numbers. (Children with 1, 2, 3, 6, 9, 18 cards will stand.)
3. Continue with other sets of numbers that have, for example, 10, 12, 24, 25 or 32 as a multiple.
4. I am thinking of the set of numbers that have 10 as a factor. (Children with 10, 20, 30, 40 cards will stand.)
5. I am thinking of the set of numbers that have 8 as a factor. (Children with 8, 16, 24, 32, 40 cards will stand.)
6. Continue with other numbers, such as 2, 3, 4, as factors.
7. I am thinking of the set of numbers that are multiples of 12.
8. I am thinking of the set of numbers that are multiples of 8.
9. I am thinking of the set of even counting numbers. They are all multiples of 2.
10. I am thinking of the set of odd counting numbers. They are not multiples of 2.
11. I am thinking of the set of numbers that are not multiples of 2 and are not multiples of 3.
12. I am thinking of all the multiples of 1. (All the children will stand, since every counting number is a multiple of 1.)

After each clue, check to see that all the children, and only the children, whose numbers are in the set described are standing. To make this a game, points may be given for correct answers.

Use other similar examples with multiples and factors, odd and even numbers. If the children can use more difficult clues, the following may be used.

1. The numbers less than 21 that are multiples of 5
2. The even numbers less than 35
3. The odd numbers greater than 10 and less than 19

ACTIVITY 2 • The Game of Buzz

Choose a number, for example, three. Let the children say the counting numbers in sequence beginning with one. The first child says, "One"; the second says, "Two"; and so on. Whenever the number a child is to say is a multiple of three, tell him to say "Buzz" instead of the number. For example, the third child says, "Buzz", the fourth says, "Four", the fifth says, "Five", and the sixth says, "Buzz", and so on.

If a child says the name of a number that is a multiple of three rather than "Buzz" ask him to stand up. Let him continue to play the game as he stands. The game goes on until only one child remains seated. He is the winner.

When the counting numbers get too large, let the children start counting with "One" again and continue the game. If the children become good at this game, change to multiples of four. Also try a variation which is more difficult. Choose two numbers, say 3 and 4. When a child comes to a number that is a multiple of 3 or of 4, he says, "Buzz". But when he comes to a number that is a multiple of *both* 3 and 4 (for example, 12), he says, "Buzz-Buzz".

After the game, write the numerals for the "Buzz-Buzz" numbers on the blackboard. Let the children say the numbers that are in the set of common multiples of the two numbers.

The children may notice that the game is easier if one of the two numbers is a multiple of the other (for example, 3 and 6). Thus they may discover that if 6 is a multiple of 3, then every multiple of 6 is a multiple of 3 and so on.

ACTIVITY 3 • Readiness for Prime Numbers

This game uses multiples of counting numbers. Before the game, give the children a short revision of multiples. For example, write the numeral for a counting number, say 18, on the blackboard. Ask the children to tell the set of all counting numbers for which 18 is a multiple. Use a similar plan with other numbers.

To begin the game, write on the blackboard the numerals for the counting numbers in order beginning with 2 and ending with 30. Separate the class into two teams. The first team takes the first number, 2, and decides the set of numbers for which 2 is a multiple. Call on a member of the first team to tell how many members the set has. (2) Then ask a team member to list all numbers in the set. (1,2) A point is scored for each correct answer. Write, "Factors of 2:1,2."

Let the second team decide on the numbers for which 3 is a multiple. Get one child to tell the number of members in the set, (2) and another child to write, "Factors of 3:1,3." Continue with the other numbers in order.

Encourage the children to notice:

1. Every set has 1 as a member. One is a factor of every counting number.
2. Every set has the counting number itself as a member.
3. Every set contains at least two members, the counting number itself and one.
4. Several sets have only two members, the counting number itself and one.

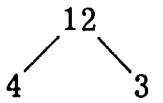
After the game, let the children point to the sets that have only two members. Thus they find that counting numbers with only two factors are special kinds of numbers.

ACTIVITY 4 • Factor Trees

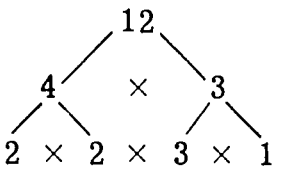
Write the numerals for these numbers on the blackboard:

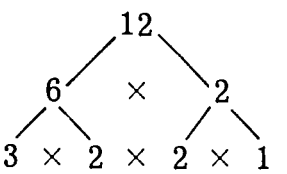
12, 20, 18, 24, 15

Tell the children that for each number we can make a factor tree. Ask, "What are two factors of 12?" Let a child suggest two factors and write their numerals below the "12". Draw lines from the "12" to each of the factor numerals. For example, if 4 and 3 are the factors given, the blackboard looks like this:



Ask for two factors of each of the factors 4 and 3 in turn. Write the numerals for the factors below the factors. See the factor tree to the left below. Continue until no more factors can be given without repetition. (Do not use 1 as a factor if there is another pair of factors. Write 1 in each factor tree only once.) For example, a completed factor tree for 12 is shown below on the left. If the children first say the factors 6 and 2, the factor tree will be as shown on the right below.





Let the children make factor trees in their exercise books for other numbers. They may notice that factor trees of a given number have the same factors at the bottom of the tree although in different order. They may also notice that some numbers have only one factor tree and it is a short one, for example, the numbers 2, 3, 5, 7, 11 and so on. These numbers have only two factors, the number itself and one.

STAGE 2 • Games with Fractions

Vocabulary

No new vocabulary

Materials

2" × 3" fraction numeral cards, one for each child; no pupil pages

ACTIVITY 1 • Classifying Fractions

Give each child in the class a card on which is written a different numeral for a fraction. Among the fractions include those that have several different names. $\left(\text{For example, } \frac{3}{4} \text{ and } \frac{6}{8} \text{ and } \frac{9}{12} \text{ and so on}\right)$ Also include many that are whole numbers. $\left(\frac{6}{2}, \frac{4}{2}, \frac{5}{1}, \frac{8}{4} \text{ and so on}\right)$

Say, "Your card tells you your number name." Give directions such as these to the children:

1. If you have the name of a fraction, stand up. (All children)
2. If your fraction is another name for a whole number, stand up. (After each direction, let the class decide whether the children standing should be standing.)
3. If your fraction is not another name for a whole number, stand up.
4. If your fraction is less than one, stand up.
5. If your fraction name is greater than 1, stand up.
6. If your fraction name is less than 2, stand up.
7. If your fraction name is greater than 1 and less than 2, stand up.

You think of other ways of classifying the fractions and use them.

ACTIVITY 2 • Names for Fractions as a Game

Give one of the numeral cards used in Activity 1 to each child. Tell the children to look at their cards, to remember the fraction and then to place their cards *face down* on their tables.

Choose one child to be "It". Ask him to bring his card to the front of the classroom, to show it to the class and to tell the name of his number, say $\frac{6}{8}$. He asks, "Who has another name for my fraction?" Ask all the children who think they do to raise their hands.

The child who is "It" goes in turn to each of the children with raised hands. He tries to guess the name on the card. He may ask one question. For example, he might ask, "What is the numerator of your fraction?" If

the answer is, "Three", then he has a clue that the name on the card is "$\frac{3}{4}$", because $\frac{3}{4}$ is another name for $\frac{6}{8}$.

If the child who is "It" guesses correctly, he goes on to another child with a raised hand. If he continues without a mistake, he scores 10 points. Then another child is "It". However, if he guesses incorrectly, he loses his turn and scores only points for his total number of correct answers. If a child raises his hand when his card does *not* have a name for the same fraction, he loses one point.

Each time a child is "It", let him and other children who have other names for his fraction display their cards in front of the class. Let the class decide whether the cards all show names for the same number.

This game may be played for a few minutes each day for a week or more. At the end of the time, each child should have been "It" several times.

STAGE 3 • Puzzles with Operations

Vocabulary

No new vocabulary

Materials

Counters; pupil pages 178-179

ACTIVITY 1 • Cryptograms

Pupil page 178

Ask the children to turn to page 178 in their books. Let a child read the sentence at the top of the page. Say, "In each exercise, an operation sign shows either addition, subtraction or multiplication. Some of the numbers are shown in code. A letter of the alphabet is the code name for a number. If you study the exercise carefully, you will be able to tell what number each code letter stands for. See if you can discover each code."

Ask the children to write the numerals 1-16 in their exercise books. After each numeral, tell the number each letter in the problem has replaced. The answers are these:

1. T is 8. M is 3.	2. S is 2. Q is 9.	3. R is 8.	4. C is 8. B is 7. A is 6.
5. X is 8. W is 2.	6. S is 9. R is 6.	7. T is 6. N is 5.	8. Y is 2. X is 8.
9. F is 4. G is 3.	10. K is 8. L is 4.	11. P is 4. M is 5.	12. T is 1. S is 9. R is 2.
13. Y is 3.	14. Z is 2.	15. W is 6.	16. U is 3.

ACTIVITY 2 • Riddles

Pupil page 179

Ask the children to read the riddles on page 179. Let them answer the exercises independently in their exercise books or orally as a class activity.

Discuss the last riddle after the children have finished. Let some of the children tell what number they thought of, then perform the operations. They will discover that no matter what number they choose, they end with that number. They will see that this is true because they first add the number to itself (doubling it), then subtract it, which cancels the first step.

ACTIVITY 3 • Arrays

Give each child from 12 to 18 counters. Vary the number so that some have 12, some have 13, some have 14 and so on. Tell them to count the objects and then use them to make an array that has exactly 3 counters in each row.

Some children will use all their counters, for example, 12 counters make a 4 by 3 array, 15 counters make a 5 by 3 array and so on. Some children will have counters left; for example, a child with 14 counters can make a 4 by 3 array and will have 2 counters left over.

Discuss the different arrays and those that have counters remaining. Write sentences that the arrays show. For example,

$12 = 4 \times 3$ $\quad 13 = (4 \times 3) + 1$ $\quad 14 = (4 \times 3) + 2$

$15 = 5 \times 3$ $\quad 16 = (5 \times 3) + 1$ $\quad 17 = (5 \times 3) + 2$ $\quad 18 = 6 \times 3$

Then let them form arrays with 2 in each row. For each number of counters, let a child write a sentence to express the result. For example, for 9 counters they will make a 2 by 4 array and write "$(2 \times 4) + 1$."

Finally, put away the counters and let the children try to write sentences like these without the use of arrays. Give them a number, say 15, ask them to think of 15 as two addends and to make one addend the largest multiple of 2 they can. $[(2 \times 7) + 1]$ Let them use the arithmetic they know to discover that for any number we can find two addends, one of which is the largest multiple of a second given number.

For some numbers, one addend may be zero as in

$$9 = (3 \times 3) + 0$$

and for some, the second addend is greater than zero as in

$$11 = (3 \times 3) + 2.$$

Let the children use this to think about problems like the following:

1. If you have 8 oranges and you want to share them equally among two friends and yourself, how many will each of the three persons get? (The children may either say that each person gets 2 oranges and there are 2 left over, or they may say that each gets 2 and $\frac{1}{3}$ oranges.)
2. If you have 9 sweets and wish to share them equally between 2 people, how many sweets will each get?
3. If you have 7 marbles and you want to give the same number to each of 3 boys, how many will you give to each? (The children say that each boy gets 2 and there is one marble left, because he cannot cut the marble into parts as he can the orange.)

Give other similar examples.

STAGE 4 • Number Patterns and Numeral Puzzles

Vocabulary

No new vocabulary

Materials

Pupil pages 180-181

ACTIVITY 1 • Adding One to a Number

Name numbers one at a time, and for each let the children take turns in adding one to that number. For example, if you say "twelve", the child says "thirteen".

Each time you name a number make it greater than the last number named. Begin with several in ones and tens, then go to hundreds, then to thousands and so on. Go on as far as possible. In addition to saying the name, write the numeral on the blackboard and let the child write his answer on the blackboard.

Numbers you might use are: 1, 7, 10, 12, 19, 26, 35, 39, 47, 68, 100, 226, 548, 701, 978, 1000, 1001, 7008, 9246, 10000, 100000, 1000000.

Begin again with another series when the numbers become too large to name or to understand. Finally, get the children to answer these questions:

Can you always add one to any number named?

Can you always name a number larger than any number named?

ACTIVITY 2 • Number Series

Pupil page 180

Ask the children to open their books to page 180. Say, "On each line there is a series of numbers named. Find the pattern used and fill in the blank spaces, using the same pattern."

Go over Exercise 1 with the class. Let them talk about it and discover that in this series 2 is added to each number in turn. In other words, this pattern is the series of even counting numbers 2, 4, 6, 8, 10 and so on.

Ask the children to copy the numerals for each exercise in their exercise books and then continue the pattern by replacing the blanks with the correct numerals.

Answers to the exercises:

1. 2, 4, 6, 8, 10, 12, 14, 16
2. 1, 3, 5, 7, 9, 11, 13, 15
3. 5, 10, 15, 20, 25, 30, 35, 40
4. 10, 9, 8, 7, 6, 5, 4, 3
5. 10, 20, 30, 40, 50, 60, 70, 80
6. 1000, 900, 800, 700, 600, 500, 400
7. 60, 58, 56, 54, 52, 50, 48, 46
8. 1, 2, 4, 8, 16, 32, 64, 128, 256
9. 1, 3, 6, 10, 15, 21, 28, 36, 45, 55

After they have completed the page, go over all the exercises, discussing the patterns with the class.

ACTIVITY 3 • "No More Than Three" Game

This is a game for two children to play. A pile of any number of seeds (or other small objects) is placed in front of the players. Each player in turn takes 1, 2 or 3 seeds as he wishes. The object of the game is to be the player who takes the last seed. He is the winner.

A good number of seeds to begin with is about 15 or 20. Neither player should know exactly how many seeds there are, nor can he touch the seeds to count them. If he wishes to know the number, he must estimate or somehow find the number without touching the seeds.

The children will discover a best method for playing the game so they will win. Do not tell the children the best way. Let them find it by playing the game and trying to win.

ACTIVITY 4 • Cross-numeral Puzzle

Pupil page 181

Let the children open their books to page 181. Ask them to copy the puzzle in their exercise books and use the clues beneath the puzzle to complete it. Go over one or two clues to make them clear.

For example, for clue 1-across, ask, "What is the simplest name for $(5 \times 6) + 3$?" (33) Tell the children to place the digits for "33" in boxes 1 and 2 in their puzzles. Let the children tell you that the simplest name for 1-down is 33 and that a "3" goes in box 1 and a "3" goes in box 5. Let the children continue working alone on their puzzles.

STAGE 5 • Co-ordinates

Vocabulary

Finding points

Materials

Small seeds, pupil page 182

ACTIVITY 1 • Points in a Network

Pupil page 182

Introduce the idea of finding points by drawing on the blackboard the picture that is on page 182 of the pupil books. Say, "We can locate points in this region by using these two number lines and pairs of numbers. Suppose we wish to *find a point* where two lines intersect."

Mark a large "X" where the vertical line up from 1 and the horizontal line from 2 intersect. Say, "We can name this point with an ordered pair of numbers. The first number is 1. It tells us to move from 0 across 1 space. The second number is 2. It tells us to move up from the bottom line 2 spaces. This is the point marked "X". Write numerals for the ordered

pair on the blackboard.

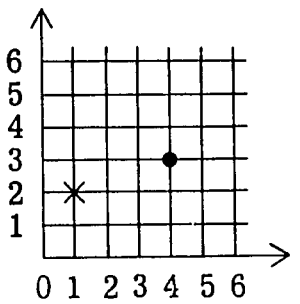
(1, 2)

"The first number tells us how many spaces to move across. The second tells us how many spaces to move up."

Write on the blackboard the ordered pair of numerals

(4, 3).

Ask the children to think about what point these numbers name. Then let a child find the point, marking it with a large dot. The picture on the blackboard with the two points marked should look like this:



Continue letting children suggest pairs of numbers and other children find the points for these pairs. They should notice that on this picture they can only use numbers from 0 to 6. (To use larger numbers, the picture can be enlarged.)

After the children know how to find points, ask the pupils to open their book to page 182. Give each child some small seeds to mark the points named by the ordered pairs given. After all these points are marked, ask them to write in their exercise books the pairs of numerals for the marked points not covered.

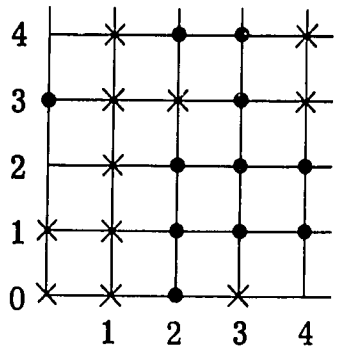
ACTIVITY 2 • Tic-Tac-Toe

Put several large diagrams on the blackboard, like the one on page 182 of the pupil books except mark each line just to 4. See the picture below. Do not mark any points.

Separate the class into two teams with about half the children on each team. Let the teams take turns naming an ordered pair of numbers and you plot the points on the picture on the blackboard. Use a large "X" to mark

the points named by team A and a large dot to mark the points named by team B. The first team to get all points in one line covered by their mark (X or dot) wins the game. The line may be either horizontal or vertical.

For example, the following picture shows that the X team has won.



Each time, let different children from the team name ordered pairs of numbers until the game is won or until all points are marked. If a point is named that has already been marked or cannot be marked, the team loses that turn.

Repeat as needed.

STAGE 6 • Fun with Geometry

Vocabulary

No new vocabulary

Materials

A 1-foot piece of string for each child; pupil pages 183-187

Teaching Procedure

ACTIVITY 1 • Paths in a Maze

Pupil page 183

Ask the children to open their books to page 183. Say, "I will tell you a story about this picture."

Once there was a king who had a very beautiful daughter. She was so beautiful that all the young men in the kingdom and in the neighboring kingdoms wanted to marry her. But the king said, "My daughter will marry only the cleverest man among you." So the king had a house built for his daughter and around it built many many fences with many many gates. He said, "Whichever man goes through the gates to the house in the shortest time will marry my beautiful daughter." Many men tried but a handsome prince from the next kingdom went through in less than one minute. He had found the shortest path and so he married the king's daughter.

Let the children try to find the shortest path to the centre of the maze by tracing with their fingers along the path. Say, "Could you reach the house in one minute?" Keep the time to see if anyone can do better than the prince.

ACTIVITY 2 • Inside and Outside

Pupil page 184

Let the children open their books to page 184. Tell the following story.

In the town of Heebie there is a house of walls shaped like the diagram on page 184. The owner of the house asked one of his friends to stay in the house for one week without going outside the walls. The friend did not want to stay inside the walls. The owner said, "You may choose where you stay in my house of walls. I want you to stay for one week at point A, B, C, D, E, F, G or H." The friend took a look and decided that he would stay for his friend because some of the places where he could stay were not even inside the walls.

Ask the children to find one point inside the closed figure and one point outside. Then ask them to name other points and decide whether they are inside or outside. (C, D, F and G are outside the walls.)

Let them try to discover a rule which can be used to tell whether any point is inside or outside. Give them a clue if needed. The way to decide is related to the number of line segments intersected by a line segment from the named point to a point outside the diagram of the house.

ACTIVITY 3 • Guessing Numbers of Bricks

Pupil page 185

Say, "Robert's uncle is building a new house of bricks. He had several piles of bricks and asked Robert to count the number of bricks in each. The pictures of these piles are on page 185 of your books. Can you guess the number in each pile?".

Discuss each picture with the children. Let them guess the numbers and talk about ways of making a good guess.

(A, 1; B, 2; C, 12; D, 15; E, 15; F, 14; G, 10 or 11; H, 9; I, 12 or 13.)

ACTIVITY 4 • Geometric Figures on Maps

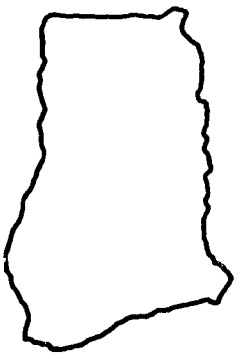
Pupil pages 186-187

Maps show us the location of countries, continents, rivers, mountains, oceans and cities. The map on page 186 of the pupil book shows the continent of Africa. Some of the countries of Africa and their capital cities are also shown.

We identify countries on a map by their shapes. The border of a country shows a geometric figure whose shape is usually different from the shape of the border of other countries. Sometimes a section of the border of a country shows a line segment, but most of the border is an irregular curve that is not a line segment. The border of a country sets it apart from all other countries and defines the region of the country. Cities are usually shown by points on a map.

The map of the African country of Zambia illustrates how geometric ideas are related to maps. Most of the border of Zambia is an irregular curve, but at five places the border shows line segments. At point *A* two line segments intersect forming a right angle.

Maps come in different sizes, but the shape of the border (and region) of a country give it an identity that is easily recognized. The West African country of Ghana has a characteristic shape on any map.



In this activity the children will learn to recognize countries by their shapes and then find the countries on a different-size map. The map of Africa on page 186 shows ten countries. The drawings on page 187 show these countries separately and larger. The capital cities are shown as points.

Tell the children to open their books to page 186. Guide them to say that this page has a drawing of the continent of Africa. Page 187, which is opposite page 186, shows larger drawings of individual countries. Hold up a pupil book and point to the drawing of your own country on page 187. Say, "This is a picture of our country." Run your finger around the border and say, "This is the border of our country." (We will use Malawi as an example.) Point to the region inside the border and say, "This is the region of Malawi." Ask, "Can you find a point inside the border of Malawi?" Let the children find the point. Say, "The point shows the location of Zomba, the capital city of Malawi."

Tell the children to look at the map of Africa on page 186. Ask. "Can you find the region of Malawi on this map?" If the children have difficulty finding the country, help them by asking questions like the following:

(Point to Nigeria on the map) "Is this the region of Malawi?" (No) Get the children to see that the shape of the border is not like that of Malawi. Eliminate several other countries that are obviously not

Malawi. Then point to Malawi on the map and ask, "Do you think this is Malawi?" Get the children to say that it has the same shape as the other drawing of Malawi.

Ask, "Can you find the point that shows the location of Zomba, the capital of Malawi?" Let the children find the point.

Repeat this activity with drawings of other countries. Get the children to identify the individual country and the point showing the capital city. If the border shows any line segments, let the children identify these and test for straightness with a piece of string. Then let them find the country on the map of Africa. Continue this until all ten countries have been identified and located on the large map.

Ask, "Can you find any countries whose borders touch other countries we have located?" (Ethiopia-Kenya, Kenya-Uganda, Kenya-Tanzania, Uganda-Tanzania, Liberia-Sierra Leone, Zambia-Malawi)

Ask, "Can you find any countries whose borders do not touch other countries we have located?" (Ghana and Nigeria)

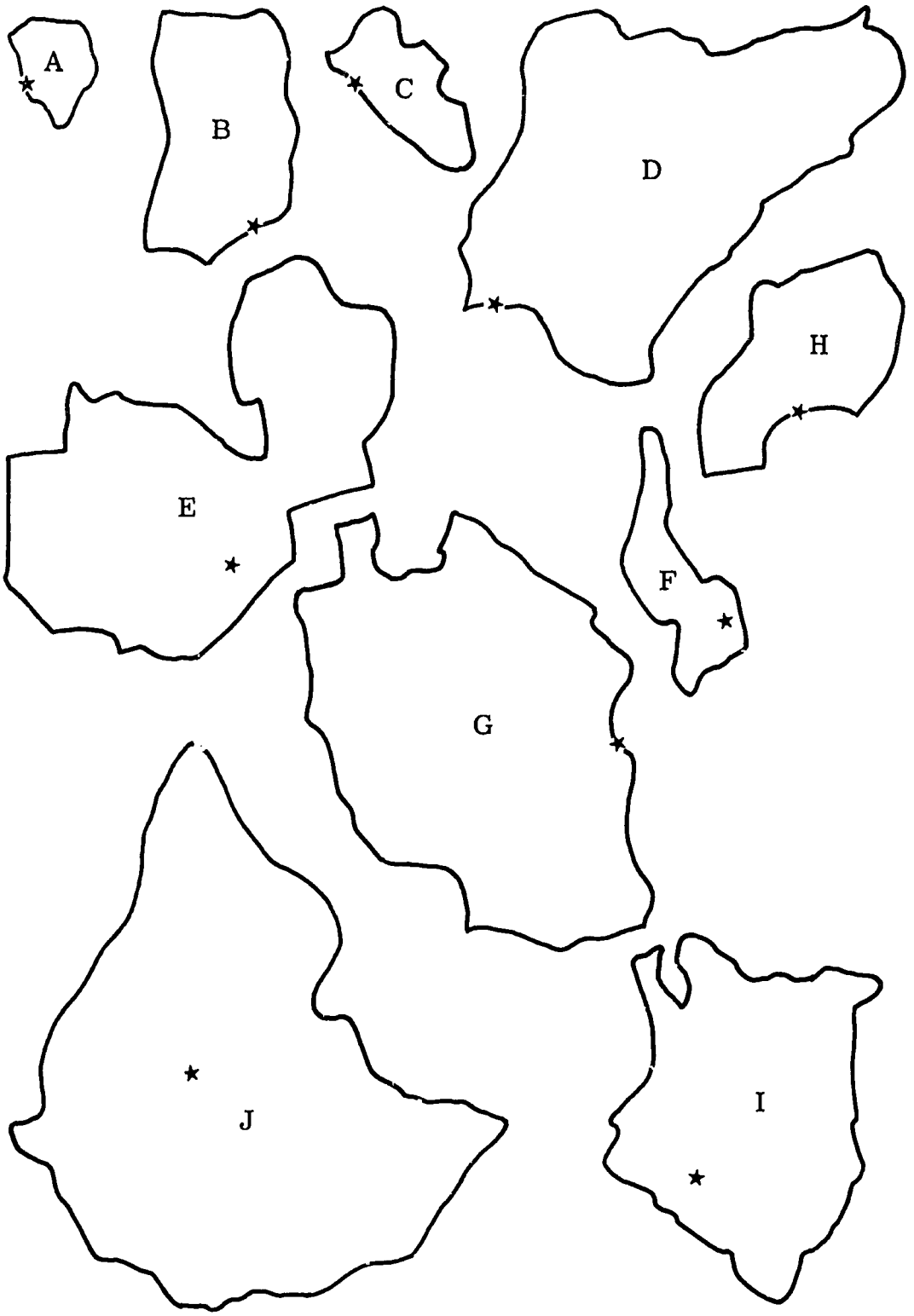
Give each two children a 1-foot piece of string. Get the children to put their fingers on the region of Nigeria. Ask, "Can you find the point that shows Lagos, the capital city of Nigeria?" Let them find this point. Ask, "Can you put your finger on the region of Kenya?" Let them point to the proper region. Say, "Can you find the point that shows Nairobi, the capital city of Kenya?" Let them find this point. Say, "Can you stretch a piece of string to show the line on which both Nairobi and Lagos lie?" Let the children, using the string in turn, do this. Continue this activity, getting the children to show the straight lines along which different pairs of capitals lie.

Let the children work in pairs. Get them to stretch their piece of string between different pairs of capital cities. Let them discover that some of the lines intersect one another. For example, a line shown by a piece of string stretched between Dar es Salaam and Accra will intersect the line shown by a piece of string stretched between Addis Ababa and Lusaka.

Let the children discover which two cities are the greatest distance from one another (longest line segment) and which two cities are closest together (shortest line segment). Let the children discover whether there are any cities that are the same distance apart. (Congruent line segments)

The following diagram identifies the countries and capital cities on the maps on page 187 of the pupil book.

A. Sierra Leone	—	Freetown	F. Malawi	—	Zomba
B. Ghana	—	Accra	G. Tanzania	—	Dar es Salaam
C. Liberia	—	Monrovia	H. Uganda	—	Kampala
D. Nigeria	—	Lagos	I. Kenya	—	Nairobi
E. Zambia	—	Lusaka	J. Ethiopia	—	Addis Ababa



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