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SECONDARY TWO

VOLUME 1

TEACHERS' GUIDE

Prepared
at the
Entebbe Mathematics Workshops

PRELIMINARY EDITION

Educational Services Incorporated
Watertown, Massachusetts, U.S.A.

ENTEBBE MATHEMATICS SERIES

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An Introductory Text for Teachers

NOTE

This book was written at a study and writing workshop held during the summer of 1963 at Entebbe, Uganda. More than fifty mathematicians and mathematics teachers from Africa, the United Kingdom, and the United States attended the Workshop.

The book was edited and produced with the utmost speed so that it might be tried out in African Secondary Schools during 1964. Therefore, there may be errors in editing and production. The African Education Program hopes that teachers who use this book will appreciate the circumstances under which it was produced.

TEACHERS' GUIDE VOLUME ONE

MATHEMATICS SECONDARY TWO

UNIT 1: RATIONAL NUMBERS

Prepared at the

Entebbe Mathematics Workshops

African Education Program

Educational Services Incorporated

Watertown, Mass., U.S.A.

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The 1963 Entebbe Mathematics Workshop was directed by Professor Walter Prenowitz of Brooklyn College, Brooklyn, New York, and Professor W. Ted Martin of Massachusetts Institute of Technology, Cambridge, Massachusetts.

Secondary Two manuscripts were written at the 1963 Entebbe Workshop by the following:

Chairman: Professor Kathleen Collard
University of Ibadan
Ibadan, Western Nigeria

Mr. L. T. M. A. Awokoya
Federal Emergency Science School
Onikan, Lagos, Nigeria

Mr. R. D. Baldwin
Munali Secondary School
Lusaka, Northern Rhodesia

Mr. Begeyehou Belai
Ministry of Education
Addis Ababa, Ethiopia

Mr. Hugh Bradley
Kenya Ministry of Education
Nairobi, Kenya

Mr. John Dowsett
Makerere College School
Kampala, Uganda

Miss Joan Holland
Ministry of Education
Nairobi, Kenya

Mr. Julius E. Jonah
Prince of Wales School
Freetown, Sierra Leone

Mr. Herbert J. Kanina
Thika High School
Thika, Kenya

Miss Katherine Kharas
Madison Project, Webster College
Webster Groves, Missouri

Mr. Richard Kimball
Kampala Technical Institute
Kampala, Uganda

Mr. C. M. Linyama
Mongu Secondary School
Mongu, Northern Rhodesia

Mrs. M. W. Maina
Gjakanja Secondary School
Nyeri, Kenya

Dr. Yohannes Menkir
University College, Haile Selassie I University
Addis Ababa, Ethiopia

Mr. Gilbert R. Mwakalukwa
Uhuru Street School
Dar es Salaam, Tanganyika

Mr. Robert L. N. Offurum
King's College
Lagos, Nigeria

Mr. J. J. Okon
Government Secondary School
Afikpo, Eastern Nigeria

Mr. Cyril N. Okosi
Government Secondary School
Owerri, Eastern Nigeria

Mr. J. E. Phythian
Makerere University College
Kampala, Uganda

Mr. Donald E. Richmond
Williams College
Williamstown, Massachusetts

Professor Merrill Shanks
Purdue University
Lafayette, Indiana

Mr. M. A. Soriyan
Government College
Ibadan, Western Nigeria

Mr. Henry Swain
New Trier Township High School
Winnetka, Illinois

Mr. J. H. Swift
Alliance High School
Kikuyu, Kenya

Mr. J. E. Modupe Taylor-Pearce
Secondary Technical School, Technical Institute
Freetown, Sierra Leone

Mr. Husein Yafiy
Secondary Technical School
Zanzibar

The editorial committee at Entebbe consisted of Mr. Henry Swain, Chairman; Mr. Cyril Okosi and Mr. J. Modupe Taylor-Pearce; in the United States of Mr. Henry Swain, Chairman; Miss Katherine Kharas, Mr. Peter Lappan of Lehigh University, Bethlehem, Pennsylvania, Professor Donald Richmond and Professor Merrill Shanks.

The preparation of these books for publication was coordinated at the offices of Educational Services Incorporated by Stanley D. Weinstein. Typing and layout were supervised by Barbara Morse and Sharon E. McGreevey. Editorial services were provided by Edward Webster under the direction of Henry Swain. The text was read critically before publication by Henry Swain. The cover was designed by Nat Bellantone and Edward Webster. Other art work was prepared by Jay Hayner. Production was supervised by Edward Webster and Benjamin T. Richards.

PREFACE

This Teachers' Guide, like the Text which it accompanies, was prepared at a study and writing workshop held during the summer of 1963 at Entebbe, Uganda, in which mathematicians and mathematics teachers from all English-speaking countries of Tropical Africa, the United States, and the United Kingdom participated. This material is intended to follow MATHEMATICS SECONDARY ONE, which was prepared at a similar workshop during the summer of 1962.

I. Preliminary Nature of this Material

Material produced for use in the schools must be considered to be preliminary until it has been tried out in classrooms and modified in accordance with such experience. The Student Text and this Teachers' Guide for MATHEMATICS SECONDARY TWO will be subject to considerable correction and improvement in the light of suggestions from the teachers who use them.

This means that the teacher who uses this material has the responsibility, in addition to helping interpret the new material to students, of helping identify the areas where improvement is needed. In partial reward for this burden the teacher will have the satisfaction of knowing that he or she is taking part in an experiment which is of great potential value.

II. Emphasis on Mathematical Ideas

In recent years there has been accumulated much evidence that young students are far more interested in mathematical ideas than they have usually been given credit for, and that they are far more competent to deal with such ideas than current curricula would suggest. A presentation of mathematics that puts its emphasis on concept rather than on the rules of manipulation is likely to lead to far greater satisfaction on the part of the student, and will also lead to greater mathematical competence.

III. Content of the Text

We build on the simple ideas of the structure of arithmetic which was started in MATHEMATICS SECONDARY ONE. The properties of operations with the rational numbers and later with the real numbers are used as the basis for explaining ideas of algebra. The necessary manipulations take on meaning from the properties instead of being mechanical rules to be learned by rote. Solving equations and inequalities is treated carefully in terms of truth sets.

The portions on geometry are informal and unstructured. Some work on graphing helps a bit to tie together the algebra and geometry. A little work on similar figures makes possible some informal trigonometry.

IV. The General Approach

The best education is the education that the student creates from his own direct efforts. The teacher should resist the temptation to tell the class exactly what to do and how to do it. It is indeed a great temptation, for by such means a class will appear to be proceeding at a rapid pace. If, however the teacher takes the time and effort to lead the student to think through the idea himself without telling him outright, there is considerably more assurance that the idea will be mastered and retained by the student, and will become truly his own.

Here again, the demands upon the teacher are greater. The teacher, like the student, must be thinking at every moment, for it takes far more insight to lead than to tell. This Guide is intended primarily to assist the teacher in the actual conduct of the class, by methods which are chosen to encourage student imagination and to generate student interest. In addition the Guide contains mathematical background and explanatory material beyond that given in the Text. Answers to the problems are provided.

V. Relationship to Other Material

The Text is based on the assumption that the student has studied <u>at least</u> the first two volumes of MATHEMATICS SECONDARY ONE. It is designed to prepare

the student for further mathematics in either the conventional curriculum or in curricula which will be prepared in the next few years to follow upon this Text.

The Text is designed also to conform to the requirements of the present examinations.

It should be repeated that this modified curriculum represents a great opportunity for teacher and student alike. In making use of it the school participates in a great experiment to help develop a strong African educational system of which we can be proud. It is a massive joint undertaking, in which students and teachers work side by side with mathematicians of international eminence, from their own country and from a dozen others, to create within Africa something that will be of major significance for Africa itself, and in a large measure for all the world.

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CHAPTER 1

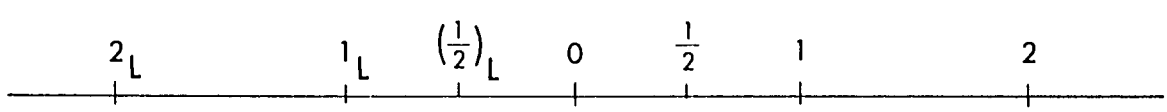
ADDITION AND SUBTRACTION OF RATIONAL NUMBERS

1-1 Introduction

Because this chapter begins a new school year it is especially important that class discussion and exploration be kept lively and interesting. Do not get involved with details and drill. Keep the children's intuition active. What is desired is for the pupil to feel that the invention of negative numbers is a reasonable thing to do, and that addition using these numbers follows naturally from their definition or origin.

There are three principal points of view from which we look at these numbers. Two of these points of view come from physical experience. The third is purely arithmetic. These views are as follows:

1. There certainly are points on the number line to the left of zero. One feels that they should correspond to some kind of numbers. In fact, one could invent the idea of "left-numbers" and represent them by



If this were done, then one would be faced with the problem of deciding how to add using these numbers.

The point of view taken in the text is somewhat different (Section 1-2). The negative numbers are represented on the number line to the left of zero in order to have a place to "land" in subtraction problems. In this case, too, one has to decide how to add using these numbers.

2. The second point of view involves thinking of positive and negative numbers in terms of gains and losses. With this physical interpretation, computation with negative numbers usually is easy and natural.

3. The third point of view is the purely arithmetic one. With only the positive numbers we cannot always subtract. For example, there is no number 3 - 5 if we have only positive numbers to use. Therefore, we are led to invent new numbers in order to make subtraction always possible. If we were to follow such a method of introducing negative numbers, then here too we would have the problem of deciding how addition is to be performed.

The purpose of this section is to get the pupil to ask himself what he wants negative numbers for, and how he wants to add by using them. Presumably the pupils have met negative integers in SECONDARY ONE, so they will be rather familiar with many of the ideas.

You might begin by asking the class why they might want to have negative numbers. Some possible answers might be similar to the following. "In order to subtract 3 from 2, I need negative numbers." "Subtracting (a positive number) is moving to the left so I want numbers to the left of zero." "In dealing with money I want positive numbers for gains and negative numbers for losses."

Should no student suggest reasons for negative numbers, do not do so yourself. Go on to the Class Discussion. You would then return to this question at the end of the discussion. Should there develop a lively discussion of why negative numbers would be desirable, let the discussion go on. You may find that many of the Class Discussion questions will already have been answered.

With the Class Discussion do not press by suggesting any answers. It is likely that previous experience, and natural feeling for the way numbers should behave, will enable the pupils to give a variety of answers to these questions. Let the discussion go on as long as anything new seems to arise, but do not suggest answers if no one volunteers. It could very well happen that the main ideas of the next four sections would be mentioned at this time. Time spent here is made up for by later speed.

Answers to "For Class Discussion" **Student Text Page 1**

1. The purpose of the question is to suggest the gains and losses interpretation of positive and negative numbers. An answer might be, "Joseph had

5 shillings when he broke a window worth 8 shillings which he had to pay for. So he is then 3 shillings in debt." In response to this you might ask, "What is the answer to '5 - 8'?" You would want the answer, "a debt of 3, or negative 3."

2. There probably will be many students who can plot negative numbers on the number line. If so, make sure that they know where they want to graph negative fractions as well as integers. After a pupil has graphed $-2\frac{1}{2}$ you might ask him to graph $-\frac{5}{2}$ and $-\frac{10}{4}$. If he does this correctly, you might ask him why he did so. This anticipates Problem 2 of Problems 1-2a.

3. This bank-account interpretation is essentially the same as gains and losses. It also provides a natural place to discuss addition problems involving negative numbers. You might ask how to add up the total of money in hand and money borrowed, or how to add two negative (red) numbers.

Besides these three items for class discussion, a variety of others could be used which would anticipate the work of the later sections. It is a good policy to keep asking questions as long as the pupils can continue to give good answers and explain (in terms of the line or gains-and-losses) <u>why they think their answers are correct</u>. Here are some possible additional questions.

1. What is $(-3) + 3$? $3 + (-3)$?

Answers such as: "3 to the right of (-3) is 0." "3 to the left of 3 is 0." "A loss of 3 and a gain of 3 is neither a loss nor gain" are acceptable.

2. What is $5 + (-3)$? $(-3) + 5$?

3. On the number line what is addition of a positive number? (For instance, using the number 3, the answer would be: "You move to the right 3 spaces.")

4. On the number line, what is subtraction of a positive number?

5. On the number line, what is addition of a negative number?

Finally, at the end of the discussion mention briefly the three items (a), (b), and (c) on page 1 that are to be considered in this chapter. Parts of (a) and (b) may already have been treated by the class discussion.

1-2 Subtraction and Negative Numbers

This section and the two following are concerned with the number line picture of positive and negative numbers.

In this section the negative numbers are introduced as labels for points to the left of 0 on the number line. We review the fact that subtraction of a positive number corresponds to movement to the left on the line. We observe that some such movements to the left would take us to the left of zero. The negative numbers conveniently name points on the line for us to "land on" when subtraction takes us to the left of zero.

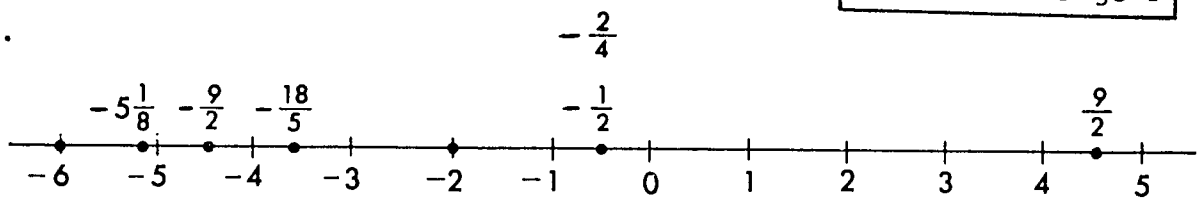
Begin the discussion by drawing the right half of the number line and stating that these numbers will now be called positive numbers. Help the pupils recall how they added on the number line by moving to the right and subtracted by moving to the left. After a few such additions and subtractions, if they seem confident, introduce a subtraction where motion to the left carries us past zero. Help the pupils see the need for new numbers to make such subtractions possible. The association of the new "negative" numbers with their proper points to the left of zero now comes very naturally.

Since the complete set of rational numbers which we now have includes new numbers for which operations such as addition and subtraction are not yet known to us, we must start making some agreements about such operations. From what we just did on the number line, it seems unavoidable that we shall still subtract a positive number on the number line by moving to the left.

Answers to Problems 1-2a

Student Text Page 4

1.

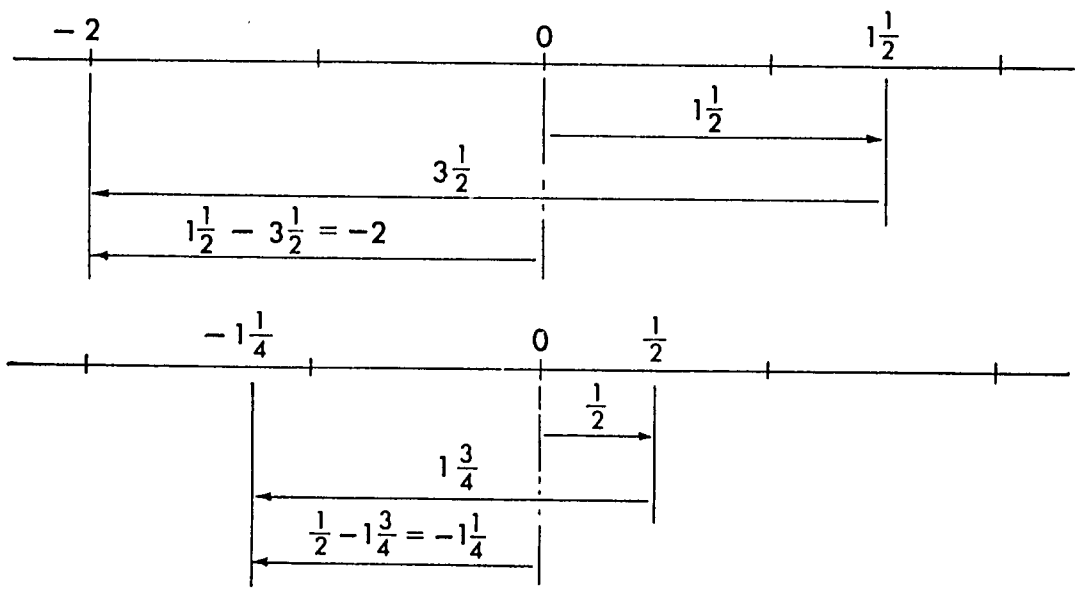


2. $-\frac{14}{8}$, $-\frac{7}{4}$, $-1\frac{3}{4}$ correspond to the same point

$-\frac{2}{3}$, $-\frac{4}{6}$, $-\frac{6}{9}$ correspond to the same point

$-\frac{4}{2}$, -2, $-\frac{22}{11}$ correspond to the same point.

3. Two samples are given of how the work should look. The rest should be similar.



The answers are:

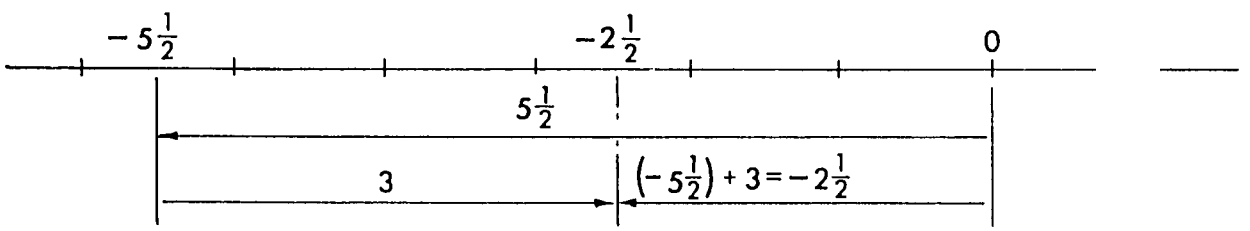
(a) -3	(d) -2	(g) $-1\frac{1}{4}$
(b) -1	(e) -2	(h) $-1\frac{1}{4}$
(c) -1	(f) -2	(i) $-1\frac{1}{4}$

From our number line viewpoint, we now have a way to subtract a positive number from a positive number when the result is negative. From this, by using the basic meaning of subtraction to translate into addition, we obtain a way to add a positive number to a negative number. We are not surprised to find that we can still add a positive number in this new set of numbers by moving to the right on the number line.

Answers to Problems 1-2b

Student Text Page 6

1.

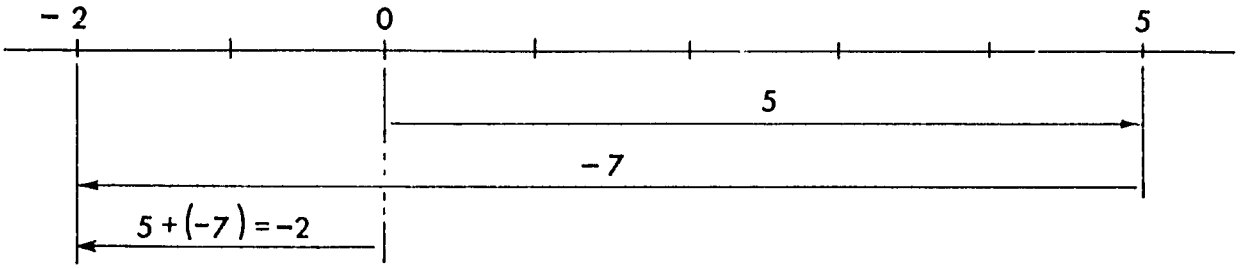


(a) 2	(e) $-2\frac{1}{2}$	(i) $-2\frac{1}{2}$
(b) 1	(f) $-\frac{2}{3}$	(j) $-1\frac{2}{3}$
(c) 0	(g) $2\frac{1}{2}$	(k) $-7\frac{3}{4}$
(d) -1	(h) -4	(l) 0

2. This exploratory question is leading up to one of the ideas presented in Section 4. If the pupils have thought about the idea here, it will be very easy when they reach Section 4.

We have just seen how to add a positive number to a negative on the line. To see how to add a negative number to a positive number all we need to do is to assume that the commutative property still holds in this new set of numbers.

Since $(-7) + 5 = -2$, we expect that $5 + (-7) = -2$. In the first case we moved 7 to the left and then 5 to the right. In the second case, then, we should move 5 to the right and then 7 to the left. We reach the same point. This demonstrates how we can add a negative number by moving to the left.



Do a number of examples like this with the class.

1-3 Opposites

The idea of opposites was mentioned in SECONDARY ONE in connection with integers. It is recalled here for the rational numbers. Opposites are treated more fully a little later in Section 6, but they are introduced now in order that the idea may be present while we are thinking about addition. As you go through the next two sections quietly call attention to the problems where we observe that the sum of a number and its opposite is zero. This is the significant property of opposites which makes them so useful in subtraction.

Answers to Problems 1-3 — Student Text Page 7

1. The opposites of -2, $-\frac{3}{2}$, $\frac{7}{2}$, $-\frac{11}{4}$ are 2, $\frac{3}{2}$, $-\frac{7}{2}$, $\frac{11}{4}$ respectively.

Observe how on the number line a number and its opposite are located at the same distance from zero on opposite sides.

2. The opposite of the opposite of 1 is 1.

The opposite of the opposite of the opposite of $2\frac{1}{3}$ is $-2\frac{1}{3}$.

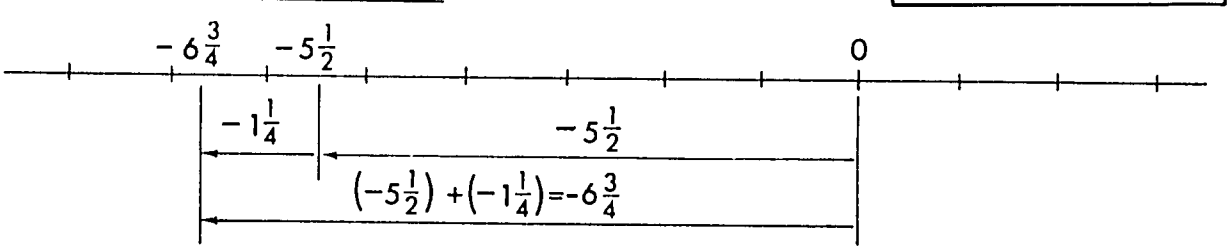
1-4 Addition on the Number Line

Problem 2 in Problems 1-2b has started the pupils thinking about how to add a negative number to a positive number on the number line. The idea is developed here and extended to adding a negative number to a negative number or to zero, the only cases left.

Notice how our rules for addition of any rational numbers are motivated by a desire to continue the validity of properties we already have for the non-negative rational numbers. We want the commutative property of addition still to be true in our larger set of numbers. We want addition of a negative number to be done the same way whether we are adding to a positive number or to a negative number.

Answers to Problems 1-4

Student Text Page 8

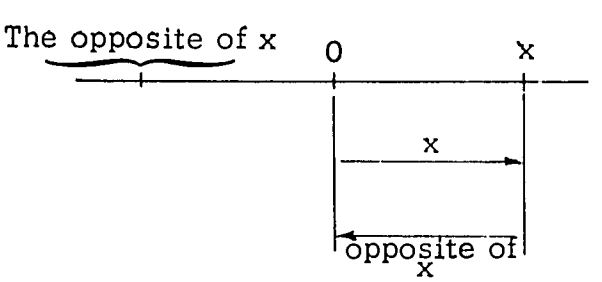


The answers are:

(a) 3	(e) -3	(i) $-6\frac{3}{4}$
(b) 0	(f) -1	(j) $6\frac{3}{4}$
(c) -7	(g) $-\frac{5}{4}$	(k) $-4\frac{1}{4}$
(d) 1	(h) 0	(l) $4\frac{1}{4}$

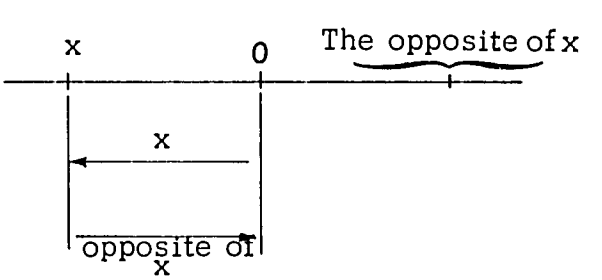
2. The sum of x and the opposite of x is always zero.

(a) x is positive



If x is positive, the opposite of x is negative.

(b) x is negative



If x is negative, the opposite of x is positive.

3.	(a) 6	Commutative property of addition
	(b) $-\frac{11}{2}$	Commutative property of addition
	(c) -6	Associative property of addition
	(d) $0 \cdot 3$	Commutative property of addition
	(e) $-\frac{3}{4}$	Associative property of addition
	(f) $2\frac{1}{2}$	Associative property of addition
	(g) 0	Commutative property of addition
	(h) (i) $\frac{13}{7}$ (ii) $-\frac{13}{7}$	Addition property of zero

---

1-5 Addition Using Gains and Losses

Some pupils understand addition of positive and negative numbers best in terms of the number line, some in terms of gains and losses. We give them experience in both. If any pupils suggest other viewpoints (such as feet above and below normal of the water level in the lake, temperatures on a thermometer, latitudes north and south of the equator) make good use of their suggestions and encourage others.

Answers to Problems 1-5 **Student Text Page 9**

1.	(a) 5	(d) 11	(g) 4
	(b) -75	(e) -11	(h) 0
	(c) -7	(f) -4	(i) $-3\frac{1}{4}$

2. Yes

3. (a) False
   (b) True — Commutative property of addition
   (c) True — Associative property of addition
   (d) True
   (e) True — Associative property of addition
   (f) True — Commutative property of addition
   (g) False

4. $[720 + 480] + (-1400) = 1200 + (-1400) = -200$

   A net loss of 200 shillings.

5. $[8 + (-3)] + [5 + (-9)] = 5 + (-4) = 1$

   A net gain of 1 shilling.

1-6 Opposites

We take a closer look at opposites, introduce the notation "-x" for "the opposite of x", point out the distinction between "the opposite of" and "negative", and state the basic property that the opposite is the unique number which added to a number, gives zero. It is this property which suggests the name "additive inverse" for the opposite of a number.

Try to be careful yourself to use the words "opposite" and "negative" correctly in order to help your pupils learn and use their real meaning. The word "minus" should not be used for either of these meanings because we want to reserve it for the operation of subtraction. It will be easier for your pupils to use these words correctly than it is for you, because they do not have previous habits to change. It is best to tell the pupils frankly that you are having to learn some of these words, too, and ask them to correct you when you forget and misuse a word. When they do correct you, laugh about it and thank them and try not to feel embarrassed. A very good teaching situation is developed if the teacher and pupils can feel they are learning together in an easy friendly way.

In the statement of the addition property of opposites near the end of this section, you see the added sentence

"and if $x + a = 0$, then a must be the opposite of x, that is $a = -x$."

This tells us that the opposite of a number is unique, that is, there is exactly one opposite of any number. This will be a useful property in reaching some further conclusions. For instance, since

$$(a + b) + [-(a + b)] = 0$$

and

$$(a + b) + [(-a) + (-b)] = 0$$

are both true, we conclude that

$$-(a + b) = (-a) + (-b),$$

that is, the opposite of a sum of two numbers is the sum of their opposites.

Answers to Problems 1-6 Student Text Page 12

1. (a) and (b) show that the answer to (c) is "yes".
   (d) Yes. (e) Yes.

2. (a) and (b) show that the answer to (c) is "No".
   (d) The opposite of negative 3, or the opposite of the opposite of 3.
   (e) Yes. (f) Yes. (g) Yes.

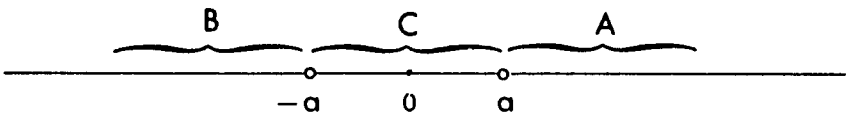
3. (a) Zero (Be careful to read "(-x)" as "the opposite of x".)
   (b) x (Read "-(-x)" as "the opposite of the opposite of x".)

4. (a) 17 (d) y
   (b) 0 (e) 0
   (c) -5 (f) $a \times b$

5. $9\frac{1}{2}$

6. CHALLENGE PROBLEM

(a)

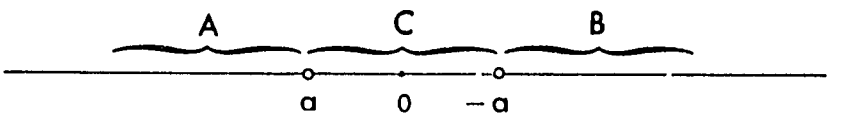


If x is in A, -x is in B.

If x is in B, -x is in A.

If x is in C, -x is in C.

(b)



If x is in A, -x is in B.

If x is in B, -x is in A.

If x is in C, -x is in C.

Note on challenge problems: Some problems throughout the text will be marked CHALLENGE PROBLEM. These are problems which require somewhat more thought and are intended for the more able pupils. The name is intended to give the pupils the feeling that these are not just a lot of hard work but are an opportunity to try their wits. As many as care to or would profit from trying them should be encouraged to do so. Sometimes they will see them quickly, sometimes they may never see them by themselves. In any case do not give them the answers. We want them to develop a spirit of thinking for themselves. After they have tried conscientiously and are still unsuccessful you may help them with some leading questions to try to guide them to the answers.

1-7 Basic Properties of Addition

There are two equally logical ways to introduce an operation such as addition in a new number system and to know how to use it. One method is to decide on a rule for performing the operation and then observe what properties are true if we follow the rule. That is essentially what we did on the number line and with gains and losses. The other method is to start by assuming that the basic properties we want are true and then deriving a rule for the operation from the properties.

It is the second of these viewpoints that we examine briefly in this section. Not all the pupils will fully appreciate this now, but if they get a little feeling for the idea from this brief exposure, that is enough for now.

They will later see this structural approach to algebra developed more fully and will find much satisfaction in the way all the techniques of algebra can be built on and explained by a few basic properties.

Example 1. The reasons are as follows:

Step 1. 7 + 8 is another name for 15. We choose 7 + 8 because of the (-7) which is present and because we look ahead and see that the sum (-7) + 7 will be helpful. We would not choose 6 + 9 for 15 because it is not helpful.

Step 2. Associative property of addition. (Number 4 in the list of basic properties, Student Text Page 14).

Step 3. Addition property of opposites. (Number 2 in the list.)

Step 4. Addition property of zero. (Number 1 in the list.)

Example 2. Since it is a little harder to explain why a negative number can be expressed as the sum of two negative numbers, we leave that explanation for the better students in a challenge problem and assume the idea here in order to be able to demonstrate this case. The reasons are:

Step 1. We assumes that (-2) + (-3) is another name for -5.

Step 2. Associative property of addition.

Step 3. Addition property of opposites.

Step 4. Addition property of zero.

Answers to Problems 1-7

Student Text Page 15

1. $(-2\frac{1}{2}) + 4 = (-2\frac{1}{2}) + [2\frac{1}{2} + 1\frac{1}{2}]$ ($2\frac{1}{2} + 1\frac{1}{2}$ is another name for 4.)

$= [(-2\frac{1}{2}) + 2\frac{1}{2}] + 1\frac{1}{2}$ (Associative property of addition.)

$= 0 + 1\frac{1}{2}$ (Addition property of opposites.)

$= 1\frac{1}{2}$ (Addition property of zero.)

2. $(-3\frac{1}{2}) + 1\frac{1}{2} = (-2) + (-1\frac{1}{2})] + 1\frac{1}{2}$ [$(-2) + (-1\frac{1}{2})$ is another name for $-3\frac{1}{2}$.)

$= (-2) + [(-1\frac{1}{2}) + 1\frac{1}{2}]$ (Associative property of addition)

$= (-2) + 0$ (Addition property of opposites.)

$= -2$ (Addition property of zero.)

3. (a) $[(-3\frac{1}{2}) + 17] + 2\frac{1}{2} = 17 + [(-3\frac{1}{2}) + 2\frac{1}{2}] = 17 + (-1) = 16$

Discuss with the pupils what properties were used in the first step (commutative and associative properties of addition) and why this made the computation easier. Discuss similarly in the remaining parts of this question.

(b) $[\frac{11}{8} + \frac{15}{16}] + (-\frac{3}{16}) = \frac{11}{8} + [\frac{15}{16} + (-\frac{3}{16})] = \frac{11}{8} + \frac{6}{8} = \frac{17}{8}$

(c) $[79 \cdot 6 + (-168)] + 20 \cdot 4 = (79 \cdot 6 + 20 \cdot 4) + (-168)$

$= 100 + (-168) = -68$

(d) $12 + (-20) + 48 + (-30) = (12 + 48) + [(-20) + (-30)]$

$= 60 + (-50) = 10$

(e) $(-\frac{3}{7}) + \frac{19}{8} + \frac{3}{8} + (-\frac{11}{7}) = [(-\frac{3}{7}) + (-\frac{11}{7})] + (\frac{19}{8} + \frac{3}{8})$

$= (-2) + \frac{11}{4} = \frac{3}{4}$

4. CHALLENGE PROBLEM

$$5 + [(-2) + (-3)] = (2 + 3) + [(-2) + (-3)]$$ (2 + 3) is another name for 5)

$$= [2 + (-2)] + [3 + (-3)]$$ (Associative and commutative properties of addition.)

$$= 0 + 0$$ (Addition property of opposites.)

$$= 0$$ (Addition property of zero.)

We now know that (-2) + (-3), when added to 5, gives 0. Since (-5) is the only number which when added to 5 gives 0, it must be true that

$$(-2) + (-3) = -5.$$

1-8 Subtraction

We look at subtraction of rational numbers in three ways. All three are related to addition of rational numbers with which we are now familiar.

1. We define a - b to be the number which added to b gives a.
2. We show that a minus b is the same as a plus the opposite of b.
3. We subtract on the number line by moving in the direction opposite to that used in addition.

After the class studies all three viewpoints, allow each individual pupil to use whichever viewpoint is clearest to him. It seems likely that the second viewpoint will be most popular.

The first method of subtracting is sometimes thought of as finding the missing addend. In a sum the two numbers being added are called addends. When a subtraction is expressed in terms of a sum, we find that one of the addends is unknown or missing.

$$8 - 3$$

means $$3 + (?) = 8$$

To find 8 - 3, then, we must find the missing addend represented by (?) in the the sum. This way of speaking of it is helpful to many pupils.

Answers to Problems 1-8 — Student Text Page 17

1. 6
2. -12
3. 31
4. 63
5. -49
6. -32
7. 13
8. -74
9. 0
10. $-\frac{4}{13}$
11. $\frac{7}{8}$
12. $\frac{17}{70}$
13. -17
14. 7
15. 27
16. -80
17. 75
18. -14
19. 12
20. 2
21. 3, -3, b - a

22. CHALLENGE PROBLEM. Exactly one is positive when c is negative. Two are positive when c is positive. Three cannot be positive, nor can none of them be positive.

1-9 Subtraction by Adding the Opposite

In Problems 1-9a be sure that the pupils see the idea the problems are intended to teach, namely that subtracting a number gives the same result as adding its opposite.

Answers to Problems 1-9a — Student Text Page 18

1. 125
2. $\frac{7}{3}$
3. 0
4. -10
5. 135
6. -1
7. 6
8. $\frac{1}{4}$

The fact that $x - y = x + (-y)$ can be proved from the definition of subtraction.

Let $x - y = d$,		
then $x = y + d$		(Definition of subtraction.)
$x + (-y) = (y + d) + (-y)$		(If $a = b$, then $a + c = b + c$.)
$x + (-y) = d + [y + (-y)]$		(Commutative and associative properties of addition.)
$x + (-y) = d + 0$		(Addition property of opposites.)
$x + (-y) = d$		(Addition property of zero.)

Hence $x - y = x + (-y)$.

Most of the students will not be ready now for a proof like this, but possibly a few of the best students would find it interesting.

An example of how we shall use adding the opposite to explain properties of subtraction is the extension of the distributive property to subtraction. We cannot offer this to the students until we have studied multiplication, but at that time we can say that for any rational numbers a, b, and c,

$$\begin{aligned} a(b - c) &= a[b + (-c)] \\ &= ab + a(-c) \\ &= ab + (-ac) \\ &= ab - ac \end{aligned}$$

Answers to Problems 1-9b

Student Text Page 19

1. -8	6. -42	11. 0	16. $\frac{43}{28}$
2. -42	7. 42	12. $-\frac{49}{75}$	17. $-\frac{25}{3}$
3. 42	8. -8	13. $-41 \cdot 1$	18. $-\frac{3}{8}$
4. 8	9. 1	14. $-14 \cdot 1$	19. $\frac{14}{15}$
5. 8	10. $-\frac{15}{8}$	15. $12\frac{3}{4}$	20. $\frac{6}{5}$

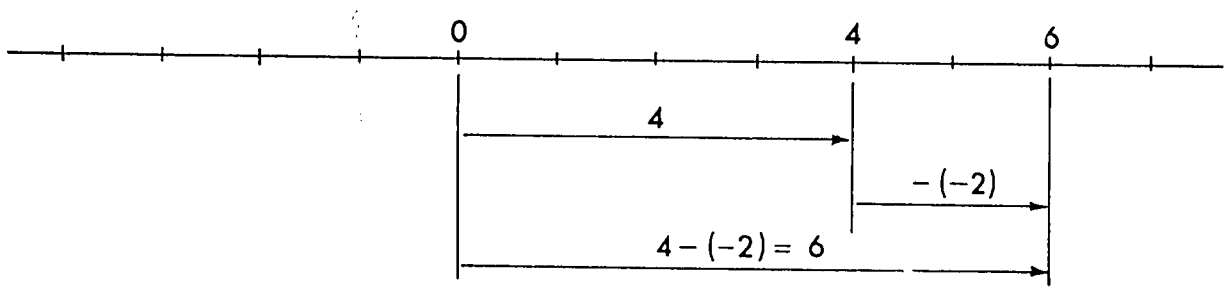
1-10 Subtraction on the Number Line

Observe how directly subtraction on the number line can be explained in terms of adding the opposite.

Answers to Problems 1-10

Student Text Page 20

1. -4
2. -1
3. -6
4. 2
5. 6
6. $1\frac{1}{3}$



CHAPTER 2

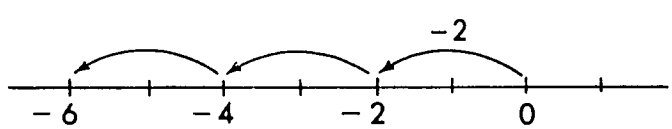
MULTIPLICATION AND DIVISION OF RATIONAL NUMBERS

2-1 Introduction

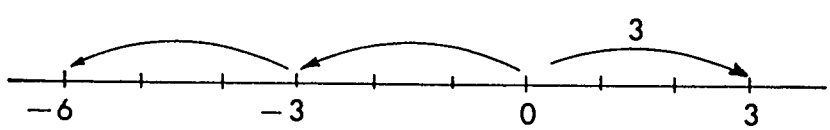
It is the purpose of this chapter to explain the how and why of multiplication and division of positive and negative numbers. Although there is little difficulty in training the pupils to perform the operations correctly, there is a real pedagogical problem in making natural and meaningful the matter of whether the results are positive or negative.

There are several points of view from which this teaching problem could be attacked. One might first suppose that the number line would be a simple and natural device, just as it was for addition and subtraction. Such is not the case, however, because with the line one has to give a meaning to multiplication in a way that may seem quite arbitrary to the pupil. To see how one might have to argue, consider these examples.

Example 1. The product (3)(-2) is quite easy to interpret on the line. One has 3 jumps of -2, giving -6.



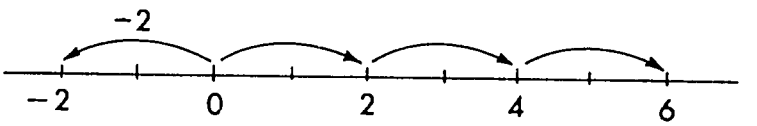
Example 2. The product (-2)(3) requires a different interpretation. The negative factor, (-2) is interpreted as "two jumps backward". We have, then, two jumps of 3 in the opposite direction, and obtain -6.



This explanation may not seem natural to the pupil. Observe that the commutative property will permit us to do Example 2 more simply from Example 1.

Example 3. Now we come to the case that is most likely to cause the student to wonder what multiplication really means. The product (-3)(-2) regarded as

"3 jumps of -2 in the opposite direction" This ends one up at 6.



The objection to this geometric way of justifying the multiplication process is that it may seem quite arbitrary and not "really be" multiplication.

The approach adopted in the text is based on the assumption that the Basic Properties continue to be true for the full set of rational numbers. With this assumption the familiar principles enable us to determine what products are positive and what products are negative.

The essential part of proofs of the rules for multiplying a positive number and a negative number or multiplying two negative numbers is the distributive property. It is through this property alone that a connection between addition and multiplication is established.

Some students may, quite properly, ask why we must assume that the Basic Properties continue to be true. This is a good question. Whenever new numbers are included in a number system, we can define the operations with these numbers in any way we choose. We decide how we would like to have the new, larger system behave. The choice we make is, in a sense, arbitrary but is based on these two considerations: In the first place, if we assume the Basic Properties are true for the new numbers then things work out very nicely. In the second place, computation would be exceedingly troublesome if the rules of the game changed with the values of the variables.

In this section there is nothing for the pupil to do. Just allow the class to recall what the Basic Properties of multiplication are, and make sure that the pupils understand how they are going to proceed.

2-2 A Positive Number Times a Negative Number

Pupils usually agree quickly that a positive number times a negative number is a negative number. One way in which this seems reasonable is through thinking of losses, "If I lose 2 shillings on 4 occasions then I lose, in all, 8 shillings. Therefore $4(-2) = -8$." What is intended in this section is that the student see how the basic principles lead to the same conclusion.

The reasons in Example 1 are as follows:

1. $4(2)$ is another name for 8.
2. The distributive property.
3. The addition property of opposites. (Observe that we must rely on our knowledge of addition.)
4. The multiplication property of 0.

After we have proved that $4(-2) = -8$, it follows that $(-2)4 = -8$ because of the commutative property of multiplication.

The reasons in Example 2 are as follows:

1. The distributive property.
2. The addition property of opposites.
3. The multiplication property of 0.

Since $3(-\frac{2}{3}) = -2$, we know that $(-\frac{2}{3})3 = -2$ because of the commutative property of multiplication.

While "Give simpler names for these numbers," may not be very precise instructions, there is little doubt that the pupil will understand what we want without our having to try spelling out what "simpler names" means. Here the answers expected are -9, $-\frac{3}{2}$, -1, $-\frac{2}{3}$.

Answers to Problems 2-2

Student Text Page 23

1. (a) $-\frac{2}{7}$ (e) $-\frac{5}{2}$

(b) $-\frac{2}{7}$ (f) -1

(c) $-\frac{2}{7}$ (g) $-\frac{3}{4}$

(d) $-\frac{5}{2}$ (h) $-\frac{5}{6}$

2. It will be enough to prove that $2(-1) + 2 = 0$.

$$\begin{aligned} 2(-1) + 2 &= 2(-1) + 2(1) && \text{(Multiplication property of 1.)} \\ &= 2[(-1) + 1] && \text{(Distributive property.)} \\ &= 2 \times 0 && \text{(Addition property of opposites.)} \\ &= 0 && \text{(Multiplication property of 0.)} \end{aligned}$$

Therefore, $2(-1)$ is the opposite of 2, or $2(-1) = -2$.

3. It will be enough to prove that $(2\frac{1}{2})(-\frac{3}{2}) + (2\frac{1}{2})(\frac{3}{2}) = 0$.

$$\begin{aligned} (2\tfrac{1}{2})(-\tfrac{3}{2}) + (2\tfrac{1}{2})(\tfrac{3}{2}) &= \tfrac{5}{2}[(-\tfrac{3}{2}) + \tfrac{3}{2}] && \text{(Distributive property)} \\ &= \tfrac{5}{2} \times 0 && \text{(Addition property of opposites)} \\ &= 0 && \text{(Multiplication property of 0)} \end{aligned}$$

Therefore, $(\frac{5}{2})(-\frac{3}{2}) = -[(\frac{5}{2})(\frac{3}{2})] = -\frac{15}{4}$

2-3 A Negative Number Times a Negative Number

This is the troublesome case. You should not drill the students on the proof. What is important is that pupils understand the proof and agree to accept the fact.

Example 1. To show that $(-4)(-2) = 8$, it will be enough to show that $(-4)(-2) + (-8) = 0$ because, by the addition property of opposites, the only number which added to (-8) gives zero is the opposite of (-8).

The reasons in the sequence of true statements are as follows:

1. $(-4)(2)$ and (-8) are names for the same number as we learned in the preceding section.
2. The distributive property.
3. Addition property of opposites
4. Multiplication property of 0.

Since we now know that $(-4)(-2) = 8$, it follows that $(-2)(-4) = 8$ by the commutative property of multiplication.

Example 2

1. Multiplication property of 1.
2. Distributive property
3. Addition property of opposites.
4. Multiplication property of 0.

Example 3

1. (-4) and $(-1)4$ are names for the same number as we learned in the preceding section. Similarly (-2) and $(-1)2$ are names for the same number.
2. Repeated use of the associative and commutative properties of multiplication.
3. $(-1)(-1) = 1$ by Example 2.
4. Multiplication property of 1.

The statement "If x is not zero, then $x^2 > 0$" is true because

(a) If x is positive, x^2 is the product of two positive numbers, so $x^2 > 0$.

(b) If x is negative, x^2 is the product of two negative numbers, so $x^2 > 0$.

Be sure the class appreciates the fact that if $x \neq 0$, then x^2 is positive.

The development in the last two sections has been in terms of particular examples. While this is appropriate for the pupils we are working with here, it is,

of course, not a general proof. It has suggested a pattern of procedure from which we have taken for granted that a proof for all numbers could be developed.

In case the teacher, or possibly a good pupil, is curious about looking at this more closely, here is the outline of a possible proof.

If a and b are any numbers, positive, negative, or zero,

$$\begin{aligned}(-a)b + ab &= [(-a) + a]b \\ &= 0 \times b \\ &= 0\end{aligned}$$

Hence $(-a)b = -(ab)$.

If a and b are both positive, then $(-a)$ is negative and $-(ab)$ is negative. This proves that the product of a negative number and a positive number is negative.

If a and b are any numbers,

$$\begin{aligned}(-a)(-b) + (-a)b &= (-a)[(-b) + b] \\ &= (-a) \times 0 \\ &= 0\end{aligned}$$

Hence $(-a)(-b) = -[(-a)b] = -[-(ab)] = ab$.

If a and b are both positive, then $(-a)$ and $(-b)$ are both negative and ab is positive. This proves that the product of two negative numbers is a positive number.

Answers to Problems 2-3 — Student Text Page 25

1. (a) $-\frac{2}{5}$ (b) $\frac{2}{5}$ (c) $\frac{8}{3}$ (d) $-\frac{4}{5}$ (e) $\frac{4}{5}$ (f) $\frac{30}{7}$ (g) $2a$ (h) $6a$ (i) $6a$ (j) $-\frac{1}{5}$ (k) xy (m) $\frac{3}{2}x$ (n) 1 (o) 1 (p) 6

In those parts of this problem which involve variables, the pupils will probably get the right answers by assuming that the variables represent positive numbers and that their opposites are negative numbers, and using the principles of positive and negative numbers just studied. If so, accept that viewpoint and don't disturb them.

If we consider that the variables might represent negative numbers, the general properties above [$(-a)b = -(ab)$ and $(-a)(-b) = ab$ for <u>any numbers</u> a and b] can be used and the same answers will be obtained.

In lack of these general properties, a pupil who questions why $(-2)(-a) = 2a$ if a is negative can answer his question by considering cases. If a is negative, $(-a)$ is positive and therefore $(-2)(-a)$ must be negative, but a is negative; so $2a$ is negative, which is what we require.

- In part (k) this viewpoint would call for considering several cases:

  x positive and y positive

  x positive and y negative

  x negative and y positive

  x negative and y negative

  x or y or both zero

In parts (o) and (p) we should be alert to the fact that the variables cannot be zero because they occur in denominators.

2. (1) $(-\frac{2}{3})(\frac{3}{4})$ is another name for $-\frac{1}{2}$.

   (2) Distributive property.

   (3) Addition property of opposites.

   (4) Multiplication property of 0.

   Since $(-\frac{2}{3})(-\frac{3}{4}) + (-\frac{1}{2}) = 0$, we know that $(-\frac{2}{3})(-\frac{3}{4})$ is the opposite of $(-\frac{1}{2})$, or $(-\frac{2}{3})(-\frac{3}{4}) = \frac{1}{2}$.

3. Since $4 = 6 - 2 = 6 + (-2)$ and $7 = 9 - 2 = 9 + (-2)$, the area $(4)(7)$ can be written as shown. By using the distributive property three times, we

obtain a sum of four numbers of which the first three, $6(9)$, $(-2)(9)$, and $6(-2)$, are known. The remaining number, $(-2)(-2)$, can then be found.

This is a good problem for class discussion rather than for outside assignment.

4. CHALLENGE PROBLEM. All pupils will enjoy playing this game. You might divide the class into two teams and let them play several games. If you do this, it would be a good plan to keep a record of how the games come out -- that is, the actual arrangements of 1 and -1 in the squares. From these examples the best students may be able to see a pattern.

You will note that at the end of the game there will be five "1's" and four "-1's". There are 3 rows and 3 columns so the total score is always 6, one for each row and one for each column. Since at least two "-1's" must be in one row or column the second player cannot fail to get a score of at least 2. The possible scores are 4 to 2, 2 to 4, and 0 to 6.

The first player can always win by forcing a 2-1-1 pattern of "-1's" in the rows and columns, such as the pattern shown on the left below. The second player should try for a 2-2 pattern of "-1's" each way, such as that shown on the right.

/–		
	/–	
	/–	/–

–/	–/	
–/	–/	

5. CHALLENGE PROBLEM. $a + b$ and $-a - b$ are opposites, so their product is negative or zero. The same is true of $a - b$ and $-a + b$. The given product $[(a + b)(-a - b)][(a - b)(-a + b)]$ is therefore either zero or the product of two negative numbers in which case it is positive. It can never be negative.

This problem, too, can interest all pupils. It is instructive to examine a number of special cases. For example, one might first try $a = 3$ and $b = 5$. Then try $a = -3$ and $b = 5$, etc.

2-4 Division

The important idea here is the way that the definition of division applies. It probably is wise to put the definition, which is shown in a box in the text, on the blackboard.

Be sure that each pupil sees that the examples are true because of the definition of division. To check, you might ask questions such as:

$-12 = 6(-2)$	What is $(-12) \div 6$?	$(-12) \div (-2)$?
$\frac{3}{2} = (-9)(-\frac{1}{6})$	What is $\frac{3}{2} \div (-9)$?	$\frac{3}{2} \div (-\frac{1}{6})$?

Here is another way of giving insight to the student. Ask questions such as the following. How do you make the following equations true?

$$-14 = (-7)(?)$$

$$-14 = (-4)(?)$$

$$\frac{3}{4} = (\frac{3}{8})(?)$$

$$\frac{3}{4} = (-\frac{3}{2})(?)$$

The student can either do these by division (for example $(-14) \div (-7) = 2$ for the first) -- or better yet just observing <u>what the missing factor should be</u>. It may help to emphasize the analogy between <u>finding the missing addend</u> in subtraction and finding the missing factor in division.

There should be little trouble getting the pupils to answer Problems 2-4a correctly. What is desired is that they understand <u>why</u> their methods work.

It is probably wise to point out clearly how we know, for instance, that $\frac{-2}{3} = -\frac{2}{3}$ or $(-2) \div 3 = -\frac{2}{3}$ because $(-2) = 3(-\frac{2}{3})$.

Answers to Problems 2-4a

Student Text Page 28

1. (a) $-\frac{3}{2}$ (d) 2 (g) $\frac{1}{4}$

(b) $\frac{3}{4}$ (e) $-\frac{1}{2}$ (h) $\frac{7}{6}$

(c) -2 (f) $\frac{7}{4}$ (i) $\frac{ad}{bc}$

It is not necessary to give any rule for obtaining reciprocals. It is best for the student to determine the reciprocal by asking himself the question, "What is the missing factor which gives 1?". It can be helpful to discuss here the analogy with opposites where the question is "What is the missing addend which gives 0?".

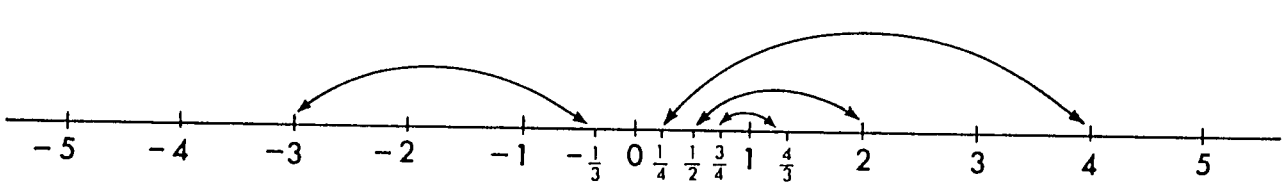
The reciprocals of

$$-3,\ 3,\ -7,\ \frac{4}{3},\ -\frac{4}{3},\ -\frac{12}{13},\ -\frac{13}{12},\ 2\cdot 5,\ -3\frac{1}{4},\ x,\ \frac{1}{x}$$

are $$-\frac{1}{3},\ \frac{1}{3},\ -\frac{1}{7},\ \frac{3}{4},\ -\frac{3}{4},\ -\frac{13}{12},\ -\frac{12}{13},\ 0\cdot 4,\ -\frac{4}{13},\ \frac{1}{x},\ x$$

Some of the properties of reciprocals are worth discussing and are easy to observe. The reciprocal of a positive number is a positive number since their product must be one. Similarly, the reciprocal of a negative number is a negative number. The reciprocal of a number between 0 and 1 is greater than 1; between 0 and -1, is less than -1. The reciprocal of a number greater than 1 is between 0 and 1; less than -1, is between 0 and -1.

These relations can be well demonstrated on the number line.



Just as we found that subtraction can be done by adding the opposite $[a - b = a + (-b)]$, so we find that division can be done by multiplying by the reciprocal $[\frac{a}{b} = a \times \frac{1}{b}]$. Be sure that the pupils see the point in the questions(a) through (e), which are intended to help them discover this idea.

(a) $(-18) \div (-9) = 2$ because $-18 = (-9)2$

$(-18)(-\frac{1}{9}) = 18 \times \frac{1}{9} = 2$

Hence

$(-18) \div (-9) = (-18)(-\frac{1}{9})$

Similarly:

(b) $(-12) \div 4 = -3 = (-12)(\frac{1}{4})$

(c) $\frac{28}{-7} = -4 = 28(-\frac{1}{7})$

(d) $(-\frac{1}{4}) \div (-\frac{1}{8}) = 2 = (-\frac{1}{4})(-8)$

(e) $\dfrac{-\frac{2}{3}}{\frac{5}{3}} = -\frac{2}{5} = (-\frac{2}{3})(\frac{3}{5})$

Encourage the pupils to use the "multiply by the reciprocal" method freely in doing division. It will frequently be most convenient.

Answers to Problems 2-4b

Student Text Page 30

1. (a) -4 (b) 4 (c) 2 (d) $\frac{15}{7}$ (e) -9 (f) $\frac{5}{7}$ (g) $\frac{2}{5}$ (h) $-\frac{b}{a}$ (i) $\frac{b}{a}$ (j) $\frac{3a}{2b}$ (k) $\frac{1}{2}$ (m) $\frac{ad}{bc}$

2. (h) $a \neq 0, b \neq 0$ (i) $a \neq 0, b \neq 0$ (j) $b \neq 0$ (k) $a \neq 0, b \neq 0$ (m) $b \neq 0, c \neq 0, d \neq 0$.

3. $a(2\frac{1}{a}) = 2$

4. $\left|\frac{1}{x}\right| = \frac{1}{|x|}$; the absolute value of the reciprocal of x is the reciprocal of the absolute value of x.

5. $\frac{1}{-x} = -\frac{1}{x}$; the reciprocal of the opposite of x is the opposite of the reciprocal of x.

CHAPTER 3

EQUIVALENT EQUATIONS AND INEQUALITIES

3-1 Introduction

Now that the pupils are familiar with the whole set of rational numbers and their arithmetic properties they can see them in use in solving equations. The problems are kept easy and the emphasis in teaching should be on the meaning of what is being done, rather than on computational details.

This is largely revision. The main ideas are,

(a) What an equation is

(b) What an open sentence is

(c) The truth set of an open sentence.

The children should get practice in recognizing true equations, false equations, and that an equation which is an open sentence becomes true only when the numerals on each side of the equality sign represent the same number. For example, $x = 5$ is an equation which states that x and 5 both name the same number. The truth set is $\{5\}$.

Finally you may find it necessary to define the symbol " $\neq$ " for the class.

For example, "$2 \neq 3$" means "2 is not equal to 3."

3-2 Equivalent Equations

This section is almost entirely class activity. There is no attempt here to teach a technique for solving an equation. We want the pupils to recognize that two equations are equivalent if they have the <u>same truth set</u>, regardless of whether or how one of the equations can be obtained from the other. In the examples the pupil should be able to find the truth sets by inspection.

The answer expected for the question is, "See whether they have the same truth set." Some pupils may want to give you rules, anticipating the results of the next section. If so, encourage them, but try to get them to give valid reasons like those given in the next section.

In Example 2 the two equations of part (j) have the truth set $\{7\}$, so are equivalent. It takes several steps, however, to get one equation from the other by the methods of the next section. Pupils should be encouraged to describe different ways they used to find the truth set of these equations.

All but (c) and (h) are pairs of equivalent equations.

The truth sets are

(a) both $\{5\}$

(b) both $\left\{\frac{9}{16}\right\}$

(c) $\left\{\frac{19}{7}\right\}$ and $\{3\}$

(d) both $\{20\}$

(e) both $\{5\}$

(f) both $\{10\}$

(g) both $\left\{\frac{1}{8}\right\}$

(h) $\left\{\frac{17}{36}\right\}$ and $\{2\}$

(i) both $\{2\}$

(j) both $\{7\}$

3-3 Solving Equations

The sentence at the beginning of this section gives the idea - that by a succession of steps each leading to a new equation that is, in some way, simpler -- we finally arrive at an equation that is solved by inspection. For instance, the truth set of the equation $x = 2$ is $\{2\}$.

The rather careful argument for the solution of Example 1 is worth paying attention to. It may be desirable, however, to do some preliminary work before discussing the example. The idea that the pupil is supposed to grasp is that the numerals on the two sides of the equation represent numbers - and that if the equation is true, the two numerals represent the <u>same</u> number. When we add a number to that number, or multiply that number by another, then only <u>one</u> number is obtained

although we may have two names for that number. This is a fundamental property of numbers.

Thus, if it is true that

$$x = 4$$

then x and 4 are the same number. When we add 2 to that number the sum is a single definite number which we may name either $x + 2$ or $4 + 2$. Thus

$$x + 2 = 4 + 2$$

because both members of the equation represent the number six.

In the same way, when

$$x = 4$$

is false, that is x is not four, then

$$x + 2 = 4 + 2$$

is also false. That is, if x and 4 are names for two different numbers, then $x + 2$ and $4 + 2$ are names for two different numbers.

Therefore the two equations

$$x = 4 \quad \text{and} \quad x + 2 = 4 + 2$$

have the same truth set because any number which makes the first equation true makes the second equation true, and any number which does not make the first equation true does not make the second equation true.

Multiplication by any number likewise leads to equivalent equations. It may be a good idea to consider a very simple example first, using only positive multipliers. For example, if

$$x = 4$$

is true, then x and 4 are the same number. When we multiply this number by the number 2, say, we get one product which may be named either 2x or 8. Thus

$$2x = 8$$

is also true.

Similarly if $x = 4$ is false, that is x is not a numeral for 4, then 2x is not a numeral for eight, so

$$2x = 8$$

is also false.

Now the examples in the text headed "Class Activity" should, with careful class discussion, make good sense to the children.

For Example 1, we chose to add 3 because it gives a simpler equation. The properties used to obtain equation (5) are as follows:

$2 \times (\frac{1}{2}x) = (2 \times \frac{1}{2})x$	Associative property of multiplication
$= 1 \times x$	Definition of reciprocal
$= x$	Multiplication property of one.

For Example 2 the reasons are, in order:

1. Multiplying by 3 or both sides of an equation gives an equivalent equation.

2. Adding 5 on both sides of an equation gives an equivalent equation.

3. 24 + 10 + 5 and 39 are names for the same number. The associative and commutative properties of addition were used also.

4. Adding (-5x) on both sides of an equation gives an equivalent equation.

5. Multiplying by $\frac{1}{4}$ on both sides of equation gives an equivalent equation.

Answers to Problems 3-3 — Student Text Page 35

1. $x = 13$, $\{13\}$
2. $x = 13$, $\{13\}$
3. $x = 1$, $\{1\}$
4. $x = -3$, $\{-3\}$
5. $x = \frac{5}{2}$, $\left\{\frac{5}{2}\right\}$
6. $x = -\frac{4}{5}$, $\left\{-\frac{4}{5}\right\}$
7. $x = \frac{3}{4}$, $\left\{\frac{3}{4}\right\}$
8. $y = \frac{4}{5}$, $\left\{\frac{4}{5}\right\}$
9. $x = -1$, $\{-1\}$
10. $x = 5$, $\{5\}$
11. $y = \frac{51}{7}$, $\left\{\frac{51}{7}\right\}$
12. $x = \frac{15}{4}$, $\left\{\frac{15}{4}\right\}$
13. $x = \frac{28}{3}$, $\left\{\frac{28}{3}\right\}$

14. $x + 3 = x + 3$. The equation is true for <u>all</u> values of x. It is an identity (see 16).

15. $2x + 3 = 2x + 2$. This is true for <u>no</u> value of x.

16. Identity

17. $\left\{-1, \frac{3}{2}\right\}$

18. $2x - 3$ is never zero when x is an integer. Hence the truth set of both equations for the given domain of x is $\{-1\}$. They are equivalent.

Do not give the children the rules to be learnt by rote. The rules should seem obvious and natural from their knowledge of the properties of the rational numbers.

3-4 Inequalities

This section and the next are quite analogous to Sections 3-2 and 3-3 on equations. All these ideas and methods will be met again later. What is important here is that the student develop some feeling for inequalities.

Thus a student might say, "The truth set for $2x < 6$ is the same as the truth set for $x < 3$, because if x is 3 or larger (that is, if x is not less than 3) then $2x$ is 6 or larger." This is quite satisfactory.

It is highly desirable at this time to draw pictures of the truth sets on the number line.

In "2" of the Class Activity, use discussion such as the following:

To find the truth set of $2x + 1 < 7$, we notice that when $x = 3$, then $2x + 1 = 7$. If x is less than 3, then $2x + 1$ becomes smaller so $2x + 1 < 7$ is true. The truth set is the set of all numbers less than 3.

For $-x > -3$, we observe that when $x = 3$, then $-x = -3$ is true. If x becomes less than 3, its opposite will become greater. Hence $-x > -3$ is true. The truth set is the set of all numbers less than 3.

In "3", parts (a), (b), (c), (e), (f), and (h) are equivalent inequalities.

The truth sets are

(a) {all numbers less than 2} for both.

(b) {all numbers greater than 5} for both.

(c) {all numbers less than $\frac{3}{2}$} for both.

(d) {all numbers greater than $\frac{5}{3}$} and

{all numbers greater than $\frac{4}{3}$} .

(e) {all numbers greater than $\frac{5}{3}$} for both.

(f) {all numbers greater than 6} for both.

(g) {all numbers greater than 11} and

{all numbers greater than -2} .

(h) {all numbers less than $\frac{2}{3}$} for both.

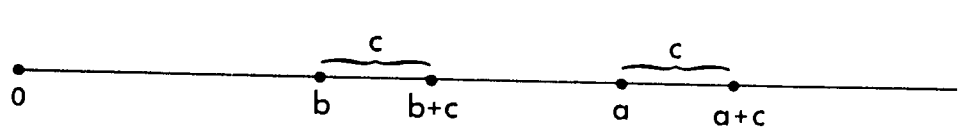
3-5 Solving Inequalities

The examples and problems are intended to lead the pupil to appreciate that:

(1) $\begin{cases} \text{If } a > b \text{ is true then } a + c > b + c \text{ is true.} \\ \text{If } a > b \text{ is true then } a - c > b - c \text{ is true.} \\ \text{If } a > b \text{ is true then } ca > cb \text{ is true } \underline{\text{if } c \text{ is greater than zero}}. \end{cases}$

In this last case observe that c must be a positive number. In the next section the case where c is negative will be discussed.

Do not give the pupils the rules to be learnt by rote. These rules should seem obvious from their knowledge of arithmetic. For example, "If $a > b$ then the number a is bigger than the number b and adding the same number to both a and b won't change the order." It is useful to show this on the number line.



The teacher should be aware of the fact that the pupils at present are thinking of rational numbers only. In the graph on the number line, therefore, only some of the points indicated correspond to the numbers they are thinking of. The rest of the points on the heavy line graph correspond to irrational numbers, which also satisfy the inequality, but which the pupils have not yet met. Together the rational and the irrational numbers on the graph give the complete set of all real numbers in the truth set.

If you have before this mentioned irrational numbers to the class, it would be good to discuss now their presence in the truth set. If not, it is probably better not to mention them now, but later when your pupils do learn about irrational numbers, you could refer back to these pages and notice how the irrational numbers are included in the truth set, even though they were not aware of their presence.

In "2" of the Class Activity the reasons are in order:

(1) Adding $\frac{4}{3}$ on both sides of an inequality gives an equivalent inequality.

(2) Subtracting $\frac{3}{4}x$ on both sides of an inequality gives an equivalent inequality.

(3) Multiplying by $\frac{4}{11}$ on both sides of an inequality gives an equivalent inequality.

Answers to Problems 3-5

Student Text Page 40

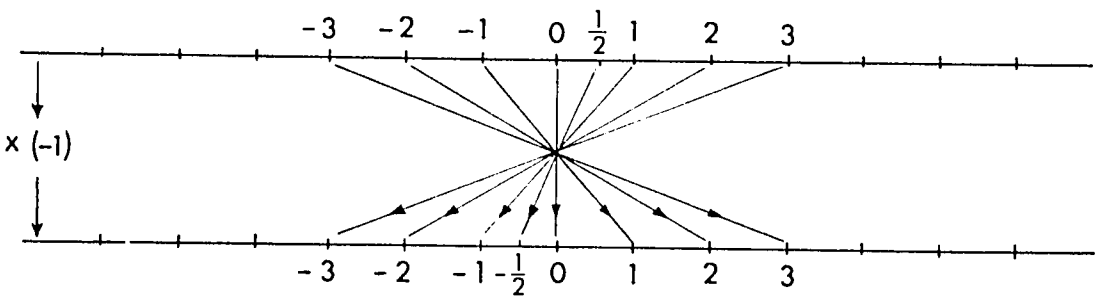
1. (a) $\{$numbers greater than $5\frac{1}{2}\}$
   (b) $\{$numbers less than $5\}$
   (c) $\{$numbers greater than $5\}$
   (d) $\{$all negative numbers$\}$
   (e) $\{$numbers greater than $-2\}$
   (f) $\{$numbers less than $\frac{17}{15}\}$

2. The hint should be sufficient to help some of the pupils. This depends on the definition of greater than. If $a = b + p$ is true, (i.e. numbers a and $b + p$ are the same), than $a + c$ and $(b + p) + c$ are the same, because of the uniqueness of addition. Hence $a + c = (b + c) + p$ is true. This means that $a + c > b + c$, because p is a positive number.

3-6 Equivalent Inequalities Obtained by Multiplication

In this section it is shown that two numbers each multiplied by the same positive number have the same order on the number line as the numbers themselves, whereas two numbers each multiplied by the same negative number come in the reverse order to the numbers themselves. This means that to get an equivalent inequality by multiplying both sides by a negative number, the sign of the inequality has to be changed, $>$ to $<$, and $<$ to $>$.

The picture on the number line of multiplication by -1 would be like this.



For some pupils this visual approach and experimentation with specific numbers will be sufficient introduction to the behavior of inequalities under multiplication by a negative number. The more able pupils will appreciate the formal proof in which the inequality $a > b$ is translated into the equality $a = b + p$ where p is positive.

In stating the methods which lead both to equivalent inequalities and equivalent equations, subtraction is not stated separately, since it is considered included in addition; subtraction of any number is equivalent to adding its opposite.

Similarly division is considered as included in multiplication.

Answers to Problems 3-6a

Student Text Page 42

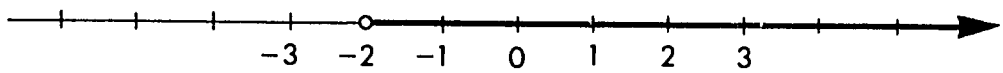
1. (a), (c), (d)
2. (a), (b)
3. (a), (b), (e)

Answers to Problems 3-6b

Student Text Page 43

1. (a) $x > -\frac{9}{4}$
   (b) $x < -\frac{28}{17}$
   (c) $x > -2$
   (d) $x < \frac{18}{7}$
   (e) $x < \frac{4}{25}$

(c)



2. Procedure is like that in the text for multiplication by -1. Here $a > b$ so $a = b + p$, where p is positive. Then $ma = mb + mp$ and mp is positive. Therefore, $ma > mb$ from the definition of "is greater than".

3. Positive values of x less than 1.

4. With $1 + a$ for x.

$$x + \frac{1}{x} = \frac{(1 + a)(1 + a) + 1}{(1 + a)}$$

$$= \frac{2 + 2a + a^2}{1 + a}$$

(Point out the application of the distributive property twice.)

$$= 2 + \frac{a^2}{1 + a}$$

Now consider $\frac{a^2}{1 + a}$. It is always non-negative because the numerator is always positive or zero, and we are given that the denominator (which is x) is positive. Therefore the least value of $x + \frac{1}{x}$ will occur when $a = 0$ and so 2 is the least value of $x + \frac{1}{x}$. This value is obtained when x is 1, or $a = 0$.

5. $\{$Numbers larger than -5 and less than 3.$\}$

6. Since $-3 > -5$, the truth set is the empty set, $\{\ \}$. A number cannot be less than -5 <u>and</u> at the same time greater than -3.

CHAPTER 4

PROBLEMS

4-1 Solving Problems

One of the most important skills a student may acquire is the ability to solve simple problems stated in words. Not only is this ability a useful end in itself, but it also leads the pupil deeper into mathematics. The ability to formulate ideas in symbolic language is a necessity for future study. Success here gives the pupil a feeling of confidence and is a strong motivating factor for further study of mathematics and science.

There are two main questions that the pupil should ask. They are:

(1) "What does x mean?" or "What shall the letter represent this time?"

(2) "How do I write an equation expressing the conditions of the problem?"

Ways of guiding him to resolve these questions are suggested here and in the following sections.

This section is introductory in nature. From the reactions of the pupils you can find those who need more attention. It might be desirable to begin the chapter by having a discussion of the following problem:

A pile of 170 oranges is separated into two piles, such that three-fourths of one pile contains as many oranges as two-thirds of the other pile. How many oranges are in each pile?

Ask for suggestions on how to solve it. Some pupils may find the answer by making several guesses. If they do you can use their method to help them write an equation. Suppose a child guesses that there are 100 in one pile. He would find out if this is correct by seeing if the following is true.

$$\frac{2}{3} \times 100 = \frac{3}{4} \times 70$$

(He found the 70 from 170 - 100.)

Since this is false, his guess was wrong. At this point you can suggest that the pupil replace his guess by x or some other letter. Then he would obtain

$$\frac{2}{3}x = \frac{3}{4}(170 - x)$$

Previous Page Blank

Should he still not see what to do, have him make a second guess. In problems of this kind most children can at least check to see if a guess is correct. In this way they learn naturally to select a variable (the unknown) and to write an equation.

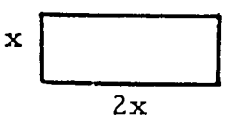
Whenever the answer to a problem involves units of some kind be sure that the pupil makes a statement about the units. In the example above the final answer is:

There are 90 oranges in one pile and 80 oranges in the other.

The examples in the text could very well be explained by pupils at the blackboard.

Answers to Problems 4-1 | Student Text Page 46

1.



If the field is x yards wide, then it is $2x$ yards long.

$$x + 2x + x + 2x = 370$$
$$6x = 370$$
$$x = 61\frac{2}{3}$$

The field is $61\frac{2}{3}$ yards wide.

(This first problem has been written out in full in the form in which the pupil's work should appear. The variable should always be described clearly indicating the unit of measure. The first equation should be a direct translation of the words. It should be possible to read back the words of the problem by looking at the first equation. The answer to the problem should be clearly stated at the end. Only abbreviated solutions are given here for the remaining problems in this section.)

2. If he earned x shillings on Monday then $-7 + x + x = 15$. He earned 11 shillings on Monday.

3. If the small can holds x gallons then $4 = 6x + \frac{2}{3}$. The can holds $\frac{5}{9}$ gallons.

4. If there are s sheep in the field then $4s + 2(19 - s) = 54$. There are 8 sheep.

5. If he bought x expensive books then $8x + (3\frac{1}{2})(20 - x) = 101\frac{1}{2}$. He bought 7 expensive books.

6. If n is the number I am thinking of, then $\frac{1}{3}[6n + (-18)] = -2$. The number is 2.

7. If a is the number then $\frac{3}{4}a + \frac{7}{2} = \frac{2}{3}a + 5$. The number is 18.

4-2 English Phrases and Formulas

In this section we try to help the pupil get started on a problem. In order to write an equation the pupil must first be able to write the formula for any number that appears in the problem. Just as with the problems of the last section, it often helps for the pupil to do the arithmetic in special cases. Then, to obtain his formula all he has to do is replace the special number by the variable x.

In Example 1 you might proceed as follows. Ask a pupil to take twice a number and subtract 5. Then ask him to write on the board what he did. Suppose his number was 8. Then you would want him to write on the board:

$$2 \times 8 - 5$$

Perhaps then you could have other students choose other numbers and write their work on the board. Then ask the class for a general formula that will always work if the number selected is x. It may help to go back to using a box to represent the number chosen, in which case the formula would first be written as:

$$2 \times \square - 5$$

and then, finally, as

$$2x - 5.$$

In Examples 2 and 3 be sure to mention that the formula gives a number of <u>bananas</u>, or of <u>miles</u>.

Answers to Problems 4-2

Student Text Page 47

1. If x is the given number then the number obtained is $3(x + 10) - x$. You need not ask them to simplify this to $2x + 30$. If students wish to make this simplification they should be encouraged to do so but the purpose now is just to obtain a formula.

2. Let t be the number of minutes since the tap was turned on. Then the number of gallons of water in the tank after t minutes is $50 + 2t$.

3. If the field is w feet wide, then its length is $(491 - w)$ feet.

4. If x is one of the numbers then their product is $x(15 - x)$.

5. If n is the certain number then the desired sum is $\frac{1}{2}n + n$.

6. If x is the chosen number then the resulting number is $2(x + 7) - 2x$, which is 14, of course.

7. CHALLENGE PROBLEM. If x is the number chosen the final result is:

$$[5(\frac{x - 2}{2}) + \frac{1}{2}x + 15 - 3x] \div 10$$

which simplifies to give 1.

8. CHALLENGE PROBLEM. This is not difficult but it does take some imagination. Usually pupils enjoy this kind of game. What you want them to get is a seemingly complicated formula which gives a constant answer.

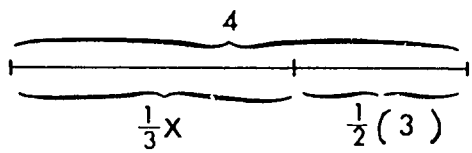
4-3 Problems, Equations and Answers

In teaching this section do not have the pupils memorize the six steps listed in the text. This list is meant as a guide to the pupil and a help to you in telling him how to solve the problems more effectively.

In item (1) it is extremely important that the pupil draw a figure when one is appropriate. This physical picture can do much to simplify the problem. In geometric

problems it will be quite clear what to draw but even in other cases a figure can help.

For example, in Problem 7 an appropriate figure is shown below. We are assuming that Jacopo runs at x miles per hour.



In general, any sort of picture that will help the pupil to understand the problem should be encouraged. Quite often this picture will enable him to understand the parts of the problem. Drawing a picture forces the pupil to think about the different parts of the statement of the problem.

The last two items also should be stressed. Be sure that the pupil knows, and says, what <u>units</u> are involved in his answer. He should have a feeling for the physical magnitude of his result. It is also vital to check to see that the solution to his equation actually satisfies the conditions of the original problem - not only that the equation is satisfied. In this way he can detect mistakes in his solution, or even in the equation he found. It can even happen that the original problem does not have a meaningful answer, in which case step number (6) becomes most important.

Example 1 illustrates this last situation. A lively discussion may develop when you reach the conclusion that $x = -2$. Encourage such conversation. Some children will be disturbed by such a problem, and so it is important for you to convince them that part of the solution is to decide whether or not the statement of the problem makes sense.

Here is another problem which at first sounds as if it must have an answer: A man planned to drive to a city 30 miles distant and return. He had alloted 45 minutes for the entire trip. If he drove to the city at the uniform speed of 40 miles per hour, how fast must he drive on the return trip to arrive on time?

Suppose he returns at x miles per hour. Since 45 minutes is $\frac{3}{4}$ of an hour, we have, <u>if it is possible to fulfill the conditions</u>,

$$\frac{30}{40} + \frac{30}{x} = \frac{3}{4}$$

so $\frac{30}{x} = 0$. But this is not possible for any number x. We conclude that he cannot make the entire trip in the time allotted. Of course, it is easy to see what is wrong. He used up all his allotted time going to the city.

In Example 2 we again have a negative solution to the equation but this time it makes sense in the problem.

As in Section 1, for pupils who have difficulty, it is suggested that you have them:

(a) First guess at the answer, perhaps making several guesses.

(b) Check to see if their guess was correct.

(c) Introduce a variable into their check by replacing the numeral for their guessed number by a box, $\square$, or letter.

At this point they should have an equation.

If a pupil manages to work any problem by straight arithmetic, without first writing an equation, tell him "very good", but insist that he go back and re-do the problem by finding an appropriate equation and solving it, in order that he may get the practice in algebraic methods that he will need for harder problems.

<u>Answers to Problems 4-3</u>

Student Text Page 50

1. $7\frac{3}{8}$ inches and $10\frac{1}{8}$ inches
2. 14 miles
3. 8
4. 45 mph. and $22\frac{1}{2}$ mph
5. 12 years and 48 years
6. 40 inches
7. $7\frac{1}{2}$ mph
8. $5\frac{1}{2}$

9. 64 yards

10. 30°

11. 35°, 40°, and 105°

12. $102\frac{6}{7}^{\circ}$

13. (a) t minutes after 10 o'clock there are $(30 + \frac{5}{2}t)$ gallons in the tank. In 9 minutes, $52\frac{1}{2}$ gallons. Empty at 12 minutes before 10 o'clock $(t = -12)$.
(b) 8 gallons per minute.

14. $C = -12\frac{2}{9}, -6\frac{2}{3}, 15\frac{5}{9}, 26\frac{2}{3}$. $F = 32, 212$. $F = C$ at -40°.

15. $-\frac{2}{3}$

16. <u>CHALLENGE PROBLEM</u> If x is the number chosen then $3(x - 8) - 2(x - 12) = x$. This is true for all x. Therefore any number works.

17. 600 widgets. 400 widgets. His statement and the information about cost imply that he sold $666\frac{2}{3}$ widgets. Since, presumably he does not sell fractions of widgets there is an error somewhere, or, more likely, his statement is an approximation and not intended to be exact.

18. 16 years and 64 years.

19. 19 years and 27 years.

20. 171 square feet.

21. $\frac{9}{8}$ of a mile

22. -24 and -9.

23. 2 shillings.

24. 280 shilling pieces.

25. 82

26. 50°, 52° and 78°.

27. <u>CHALLENGE PROBLEM</u> If x is John's number then:

$$3\left\{5[2(x + 3) - 7] - 24\right\} = x$$

and $x = 3$.

28. 16 years old.

29. A gets 34 shillings, B 17 shillings and C 51 shillings.

30. 5d.

31. <u>CHALLENGE PROBLEM</u>. The time of the trip for the first bus was $4\frac{2}{3}$ hours. That of the second bus was 4 hours. The speed of the second bus was 35 mph. If they were x miles from Accra when they met, then

$$\frac{x}{30} = \frac{140 - x}{35} + \frac{2}{3},$$

$$\text{and } x = 75\frac{5}{13} \text{ miles.}$$

CHAPTER 5

EQUATIONS IN TWO VARIABLES

In this chapter we first examine a single equation in two variables and observe that members of its truth set are ordered pairs of numbers, one for the first variable and one for the second variable. Usually there are many such ordered pairs of numbers in the truth set.

By considering a problem which leads to two equations we find that there is usually only one ordered pair which will satisfy both of two equations. This means that if we have a method of finding such an ordered pair, we have a means of answering a great variety of problems. We proceed to develop a method of solving systems of two linear (first degree) equations in two variables. Later (in Chapter 18) a graphical method of solving such systems is presented. Questions of higher degree equations or of equations in three or more variables will be postponed to later years. It will be helpful to the teacher, however, if he has some understanding of these extensions of the idea as background for the present work.

The chapter is concluded with some problems illustrating the convenience of using two variables.

5-1 An Equation in Two Variables

The main objective of this section is to help the students get used to thinking of two numbers at once, a first one and a second one. In particular we are interested in such ordered pairs of numbers as will make a given sentence true.

The table of pairs of numbers satisfying the equation $x + y = 8$ might be extended as shown. Since x and y stand for numbers of children, they must be whole numbers and these are the only possibilities.

x	y
2	6
7	1
0	8
6	2
1	7
3	5
4	4
5	3
8	0

If possible, lead the students to observe for themselves that 2 in the first column and 6 in the second column does not describe the same situation as 6 in the first column and 2 in the second. Observe it for other pairs of numbers. Then as they grow familiar with the notation for ordered pairs observe the difference in meaning between (7, 1) and (1, 7), etc.

There are 9 different ordered pairs of numbers which express the number of children of Mr. X and Mr. Y, as can be seen from the table.

We shall usually use x and y as the two variables. If other letters, such as p and q, are used, either an arbitary choice must be made as to which is the "first" variable, or frequently they are taken in the order in which they occur in the alphabet.

In the example $2x + y = 10$, nothing has been said to restrict the variables to whole numbers. If possible, let the students discover this themselves when they are asked to find more than six ordered pairs satisfying the equation. Once they start using other rational numbers, they will be able to produce many solutions. Help them conclude that there is an unlimited number of solutions in this case. Also help them realize that (2, 6) is a solution but (6, 2) is not.

Answers to Problems 5-1

Student Text Page 56

1. Most of the present problems are simple enough so that the following form is sufficient for the student to write in verifying a solution of an equation.

   $(2 \times 1) + 10 = 12$ is true.

   $(2 \times 5) + 2 = 12$ is true.

   $(2 \times 0) + 12 = 12$ is true.

   $(2 \times 8) + (-4) = 12$ is true.

2. $(2 \times 6) + 0 = 12$ is true.

   $(2 \times 2) + 8 = 12$ is true.

   $(2 \times \frac{1}{2}) + 11 = 12$ is true.

3. $(2 \times 4) + 6 = 12$ is false.

   $(2 \times 2) + 5 = 12$ is false.

   $(2 \times 6) + 1 = 12$ is false.

4. If some students find fancy solutions, such as $(\frac{57}{8}, -\frac{9}{4})$, they will need a more complete form to show their verification. The following is recommended.

   $(2 \times \frac{57}{8}) + (-\frac{9}{4}) = 12$ is true because

   $(2 \times \frac{57}{8}) + (-\frac{9}{4}) = \frac{57}{4} + (-\frac{9}{4}) = \frac{48}{4} = 12$

   This form will later make clear the verifications of solutions of more involved equations. For instance to show that $(\frac{1}{24}, \frac{5}{9})$ is a solution of the equation $x + 7y = 3 + 9x + y$ we would write:

$$\frac{1}{24} + (7 \times \frac{5}{9}) = 3 + (9 \times \frac{1}{24}) + \frac{5}{9} \text{ is true because}$$

$$\begin{aligned}
\frac{1}{24} + (7 \times \frac{5}{9}) &= \frac{1}{24} + \frac{35}{9} \\
&= (\frac{1}{24} \times \frac{3}{3}) + (\frac{35}{9} \times \frac{8}{8}) \\
&= \frac{3}{72} + \frac{280}{72} \\
&= \frac{283}{72}
\end{aligned}$$

and

$$\begin{aligned}
3 + (9 \times \frac{1}{24}) + \frac{5}{9} &= 3 + \frac{3}{8} + \frac{5}{9} \\
&= (3 \times \frac{72}{72}) + (\frac{3}{8} \times \frac{9}{9}) + (\frac{5}{9} \times \frac{8}{8}) \\
&= \frac{216}{72} + \frac{27}{72} + \frac{40}{72} \\
&= \frac{283}{72}
\end{aligned}$$

5. $(1, 1), (5, -2)$

6. $(11, 2), (2, 5), (-4, 7), (-283, 100), (23, -2), (13, \frac{4}{3}), (24, -\frac{7}{3}), (\frac{31}{2}, \frac{1}{2}), (6 \cdot 5, 3 \cdot 5)$

7. Previous to this we have not suggested any systematic way to find pairs of numbers which will satisfy an equation. The student may just be making good guesses or they may be working out methods of their own. This is fine. It is time, though, to help them see that for any given value of y, they can methodically find the corresponding value of x. At first they may substitute the value of y and solve the resulting equation for x each time.

For example, if $y = -\frac{37}{18}$, then

$$x + [3 \times (-\frac{37}{18})] = 17$$

$$x + (-\frac{37}{6}) = 17$$

$$x = 17 + \frac{37}{6}$$

$$x = \frac{102}{6} + \frac{37}{6}$$

$$x = \frac{139}{6}$$

Hence, $(\frac{139}{6}, -\frac{37}{18})$ is a solution.

If however, they are trying to find several solutions, help them to discover that they can save themselves work by solving the equation for x in terms of y before they substitute any of the given values of y.

$$x + 3y = 17$$

$$x = 17 - 3y$$

Then they have a formula which will give them more directly the value of x for any value of y.

8. It is hoped that the students will use the method suggested in Problem 7. Since values of x are given, the equation should be solved for y in terms of x.

$$2x - 3y = 8$$

$$2x - 8 = 3y$$

$$\frac{1}{3}(2x - 8) = y$$

$(1, -2)$, $(7, 2)$, $(-2, -4)$, $(5, \frac{2}{3})$, $(\frac{1}{2}, -\frac{7}{3})$

9. See Problem 8. If x is $\frac{57}{23}$,

$$y = \frac{1}{3}\left[\left(2 \times \frac{57}{23}\right) - 8\right]$$
$$= \frac{1}{3}\left(\frac{114}{23} - \frac{184}{23}\right)$$
$$= \frac{1}{3}\left(-\frac{70}{23}\right)$$
$$= -\frac{70}{69}$$

Hence, $\left(\frac{57}{23}, -\frac{70}{69}\right)$ is a solution.

Some of the solutions should be verified by direct substitution into the original equation.

10. Each of the given ordered pairs satisfies the equation. Any value of x less than zero would not be a whole number so it is not allowed.

Any value of x greater than 5 makes y negative so is not allowed. Hence this is the complete truth set.

11. There is an unlimited set of solutions of $x-y = 2$. Some of them are $(2,0)$, $(3,1)$, $(4,2)$, $(5,3)$. It is not possible to write all of them, but any one of them is of the form $(x, x-2)$, where x is a whole number greater than 1.

12. <u>CHALLENGE PROBLEM</u>. If the larger whole number is x and the smaller whole number is y, then the equation is $2x + 3y = 24$ and the truth set is $\{(12, 0), (9, 2), (6, 4)\}$. Notice that $(3, 6)$ cannot be used because we chose the first number to be the larger number. If a pupil chooses x as the <u>smaller</u> whole number, the equation is $2y + 3x = 24$ and the truth set is $\{(0, 12), (2, 9), (4, 6)\}$. In either case the required numbers are 12 and 0, 9 and 2, or 6 and 4.

5-2 A Problem in Two Variables

A number problem quite naturally leads us into considering two equations connected by the conjunction "and", for which we are seeking a pair of numbers which will make both equations true. It is not enough to make only one of them true.

Out first method of trying to find a solution is not very efficient but hopefully it helps the pupil to understand exactly what is meant by a solution of the system of simultaneous equations. Separate lists are made of members of the truth sets of the two equations. A solution of the system must be in both lists. With good fortune and a little cleverness one may find a common solution this way. Even if a solution is found, there is nothing about the method which assures us that there are not more solutions of the system. Nevertheless, a little experimenting with this viewpoint is of value for understanding.

Answers to Problems 5-2

Student Text Page 60

1. $(3,-2)$

2. (a) $\{(0,9), (2,6), (4,3), (6,0)\}$
   (b) $\{(0,30), (1,25), (2,20), (3,15), (4,10), (5,5), (6,0)\}$
   (c) $(6,0)$

3. (a) $\{(0,7), (5,5), (10,3), (15,1)\}$
   (b) $\{(0,23), (1,17), (2,11), (3,5)\}$
   (c) There are no pairs in both sets, so the truth set in the domain of whole numbers of the system of two equations is the empty set.

5-3 Systems of Simultaneous Equations

The systematic method which we use here for finding the truth set of a system of two simultaneous equations in two variables is built on the idea of removing one of the variables to give simpler equations with only one variable which we know how to solve. Of course, the new systems of equations must be equivalent to (that is, have the same truth set as) the original system.

Instead of trying to develop a complete theory of equivalent systems of equations, it seems better at this stage -- simpler for the students to understand -- to work in one direction from the original equation and then check. Thus we reason as follows:

If there is a pair of numbers (x, y) which satisfies the system

$$\begin{cases} 5x + 3y = 9 \\ -5x + 2y = 1 \end{cases}$$

then we can show that this same pair of numbers satisfies the system

$$\begin{cases} 5x + 3y = 9 \\ y = 2 \end{cases}$$

and in turn satisfies the system

$$\begin{cases} x = \frac{3}{5} \\ y = 2 \end{cases}$$

In other words, the truth set of the first system is a subset of the truth set of the third system.

In order to know that the truth sets are the same we must show that the truth set of the third system is a subset of the truth set of the first one. We can show that by checking the only solution of the third system, $(\frac{3}{5}, 2)$, in the first system. We do not say all this to the students, but the teacher should be aware of this logical basis.

The addition idea which is used to remove a variable could be formally stated:

If a, b, c, d are numbers, and
if $a = b$ and $c = d$, then
$a + c = b + d$.

The reasoning which explains this, however, is little more than the reasoning given in the text in an informal way. It depends on the closure property of addition. If two numbers are added, the sum is a unique number regardless of what names have been used for the numbers. If a and b are names for a number and c and d are names for another number (or the same one), then the unique sum of the two numbers can be named in various ways, one of which is $a + c$, another is $b + d$. Of course one could also choose the names $a + d$ and $b + c$, but we prefer to choose the ones which serve our purpose.

In this case we have planned ahead. Since the sum of $5x$ and $-5x$ is zero, we can see that addition will give an equation without the variable x.

Since any numbers which make both equations in the original system true will also make the simple equation $y = 2$ true, we can use the new system.

$$\begin{cases} 5x + 3y = 9 \\ y = 2 \end{cases}$$

or equally well the system

$$\begin{cases} y = 2 \\ -5x + 2y = 1 \end{cases}$$

Encourage flexibility in replacing either the first or the second equation by the equation in one variable. Having found y, we substitute this value in whichever of the original equations is then easier to solve for x.

When we reach the system

$$\begin{cases} x = \frac{3}{5} \\ y = 2 \end{cases}$$

we have showed that if an ordered pair satisfies the original system, <u>then</u> it satisfies this final very simple system whose truth set we can see by inspection. It remains to be proved that if an ordered pair of numbers satisfies the final system, <u>then</u> it satisfies the original system. This is why we must always check the ordered pair we obtain by substituting into both of the equations we started with.

In the second example we observe $4y$ in both equations. It is this which suggests using subtraction because $4y - 4y = 0$. Encourage the students to watch the terms in the given equations and to be able to predict whether addition or subtraction will give an equation with only one variable. Write several systems of equations on the board and have them tell which variable can easily be eliminated and whether to use addition or subtraction. Then have them make up some systems and tell the same thing. Possibly they will give some systems where neither addition nor subtraction will work as they stand. This will show nicely the need for an extension of the method, which comes in the next section.

We follow our policy of trying to keep the reasoning alive by asking occasionally for the reasons which justify the detailed steps. The reasons in the second example are as follows:

$(2x + 4y)-(5x + 4y) = (-12)-(-33)$	(Subtraction using different names for the same numbers)
$-3x = 21$	(Simplification using definition of subtraction and the associative, commutative and distributive properties)
$x = -7$	(Multiplication by $-\frac{1}{3}$)

$$\begin{cases} 2x + 4y = -12 \\ x = -7 \end{cases}$$

(Can replace $5x - 4y = -33$ by $x = -7$ because any pairs which make $5x - 4y = -33$ true also make $x = -7$ true)

$2(-7) + 4y = -12$ (x must be -7 in both equations)

$4y = 2$ (Addition of 14)

$y = \frac{1}{2}$ (Multiplication by $\frac{1}{4}$)

$$\begin{cases} x = -7 \\ y = \frac{1}{2} \end{cases}$$

(Can replace $2x + 4y = -12$ by $y = \frac{1}{2}$ because any pairs which make $2x + 4y = -12$ true also make $y = \frac{1}{2}$ true)

Notice that we have rearranged the equations, writing $x = -7$ first. This is because of our habit of calling x the first variable. In writing the ordered pairs $(-7, \frac{1}{2})$ we are careful to put the value of x first.

Answers to Problems 5-3

Student Text Page 62

1. $\{(5,3)\}$
2. $\{(3,-2)\}$
3. $\{(5, \frac{3}{2})\}$
4. $\{(-3, -\frac{1}{3})\}$

5-4 More Systems of Simultaneous Equations

We use the word "coefficient" in this discussion. The pupils have met it before but they may need to be reminded. In $3x + 2y = 12$, the coefficient of y is 2. In $7x - 4y = 2$, the coefficent of y is -4.

We hope that the students will discover that (1) it would be convenient to multiply $2y$ by 2 in the first equation to make it $4y$, (2) we are

permitted to multiply $2y$ by 2 if we also multiply $3x$ and 12 by 2 in the first equation, because the result is an equivalent equation. Since the coefficient of y is now 4 in one equation, and -4 in the other, addition will eliminate the variable y.

Completion of the first example:

By addition,

$$(6x + 4y) + (7x - 4y) = 24 + 2$$
$$13x = 26$$
$$x = 2$$

$$\begin{cases} x = 2 \\ 7x - 4y = 2 \end{cases}$$

$$(7 \times 2) - 4y = 2$$
$$-4y = -12$$
$$y = 3$$

$$\begin{cases} x = 2 \\ y = 3 \end{cases}$$

Check: $(3 \times 2) + (2 \times 3) = 12$ is true.

$(7 \times 2) - (4 \times 3) = 2$ is true.

The truth set is $\{(2,3)\}$.

Completion of the second example:

By subtraction,

$$(21x + 12y) - (10x + 12y) = 129 - 52$$
$$11x = 77$$
$$x = 7$$

$$\begin{cases} x = 7 \\ 7x + 4y = 43 \end{cases}$$

$$(7 \times 7) + 4y = 43$$
$$4y = -6$$
$$y = -\frac{3}{2}$$

$$\begin{cases} x = 7 \\ y = -\frac{3}{2} \end{cases}$$

Check: $(7 \times 7) + [4 \times (-\frac{3}{2})] = 43$ is true.

$(5 \times 7) + [6 \times (-\frac{3}{2})] = 26$ is true.

The truth set is $\{(7, -\frac{3}{2})\}$.

To find the solution of the second example by eliminating the variable x, the children should multiply in the first equation by 5, in the second equation by 7, and subtract.

If in a system such as

$$\begin{cases} 13x + 18y = 5 \\ 17x + 24y = 7 \end{cases}$$

the pupils have any difficulty in deciding what numbers to multiply by in the two equations to make the coefficients of y the same ask them how they would get equivalent fractions with the same denominator in the sum $\frac{13}{18} + \frac{17}{24}$.

Both problems involve finding the least common multiple of 18 and 24.

Answers to Problems 5-4

Student Text Page 64

1. $\{(5,4)\}$
2. $\{(4,1)\}$
3. $\{(\frac{5}{8}, 4)\}$
4. $\{(-2, -3)\}$
5. $\{(10,-9)\}$
6. $\{(1, -\frac{1}{3})\}$
7. $\{(2, -7)\}$
8. CHALLENGE PROBLEM The equations

$$x - 4y = 3$$

and

$$x = 3 + 4y$$

are equivalent ($4y$ has been added to each side of the first equation to obtain the second), so one can be replaced by the other in the system without changing the truth set.

Since for any ordered pair in the truth set, x and (3 + 4y) are names for the same number, we may replace x by (3 + 4y) in the first equation without changing the truth set. The resulting equation has only one variable.

$$3(3 + 4y) - 5y = 21$$
$$9 + 12y - 5y = 21$$
$$7y = 12$$
$$y = \frac{12}{7}$$
$$\begin{cases} y = \frac{12}{7} \\ x = 3 + 4y \end{cases}$$
$$x = 3 + (4 \times \frac{12}{7})$$
$$x = \frac{21}{7} + \frac{48}{7}$$
$$x = \frac{69}{7}$$
$$\begin{cases} x = \frac{69}{7} \\ y = \frac{12}{7} \end{cases}$$

Check: $(3 \times \frac{69}{7}) - (5 \times \frac{12}{7}) = 21$ is true because

$(3 \times \frac{69}{7}) - (5 \times \frac{12}{7}) = \frac{207}{7} - \frac{60}{7} = \frac{147}{7} = 21.$

$\frac{69}{7} - (4 \times \frac{12}{7}) = 3$ is true because

$\frac{69}{7} - (4 \times \frac{12}{7}) = \frac{69}{7} - \frac{48}{7} = \frac{21}{7} = 3.$

The truth set is $\left\{(\frac{69}{7}, \frac{12}{7})\right\}$.

The above method, sometimes called the <u>method</u> of <u>substitution</u>, is often convenient when it is easy to solve one of the equations for one variable in terms of the other. We make the method available to those students who are interested, but we do not consider it a necessity at present.

9. CHALLENGE PROBLEM (a) Solve the second equation for y in terms of x, giving $y = 6 - 8x$. Then use $(6 - 8x)$ for y in the first equation. The truth set is $\left\{\left(\frac{1}{2}, 2\right)\right\}$.

(b) Solve the first equation for x in terms of y, giving $x = 5 - 3y$. Then use $(5 - 3y)$ for x in the second equation. The truth set is $\left\{\left(6, -\frac{1}{3}\right)\right\}$.

10. CHALLENGE PROBLEM From the first equation, $x = -\frac{1}{3}$. Use this value for x in the second equation to show that $y = 6$. The truth set is $\left\{\left(-\frac{1}{3}, 6\right)\right\}$.

5-5 Word Problems Using Two Variables

In solving word problems we have an opportunity to practice clear self expression and good form in written work. Help the students to develop good habits in such things as

1. Describing the variables clearly and showing the units of measure at the start. Not "Let x = butter", which is nonsense, but "If he bought x pounds of butter".

2. Showing sufficient steps in solving the system of equations so the method is clear.

3. Checking in the words of the original problem.

4. Showing the answers to the questions clearly. Not "$x = 3$", but "He bought 3 pounds of butter". Steps 3 and 4 should also help the student eliminate nonsense answers to problems where the domains of the variables are restricted. For example, in a problem involving "He bought x pounds of butter", a solution of "$s = -3$" might be correct for the system of equations, but is not for the word problem.

Answers to Problems 5-5

Student Text Page 66

1. The numbers are 10 and -3.

2. The numbers are $\frac{11}{8}$ and $\frac{5}{6}$.

3. If there are b boys and g girls,

then

$$\begin{cases} b = g + 10 \\ b = 2(g + 1). \end{cases}$$

There are 18 boys and 8 girls.

4. If one pencil costs p shillings and one ruler costs r shillings, then

$$\begin{cases} 12p + 10r = 21 \\ 20p + 4r = 16. \end{cases}$$

Pencils cost $\frac{1}{2}$ shilling each and rullers cost $1\frac{1}{2}$ shillings each.

5. If the size of $\hat{B}$ is x° and the size of $\hat{C}$ is y°, then

$$\begin{cases} x + y + 20 = 180 \\ x = 9y + 10 \end{cases}$$

$\hat{B}$ is an angle of 145° and $\hat{C}$ is an angle of 15°.

6. If the longer piece was x feet long and the other piece was y feet long, then

$$\begin{cases} x = y + 18 \\ x = 3y \end{cases}$$

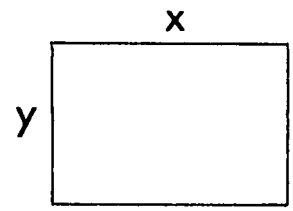
One piece was 27 ft. long and the other was 9 ft. long. The original rope was 36 ft. long.

7. If the size of the angle is x° and the size of its supplement is y°, then

$$\begin{cases} x + y = 180 \\ x = 3y \end{cases}$$

The size of the angle is 135°.

8. If the rectangle is x inches long and y inches wide, then



$$\begin{cases} 2x = 3y \\ 2x + 2y = 320 \end{cases}$$

The rectangle is 96 inches long and 64 inches wide.

9. If he walked at w miles per hour and bicycled at b miles per hour, then

$$\begin{cases} 4w + 3b = 45 \\ 2w + 4b = 50 \end{cases}$$

He walked at 3 m.p.h. and bicycled at 11 m.p.h.

10. If his normal rate of pay is x shillings per hour and his overtime rate is y shillings per hour, then

$$\begin{cases} 21x + 9y = 56 \\ 27x + 3y = 52 \end{cases}$$

His normal rate of pay is $1\frac{2}{3}$ shillings per hour and his overtime rate is $2\frac{1}{3}$ shillings per hour.

11. <u>CHALLENGE PROBLEM</u> If the tens' digit is x and the units digit is y, the number is $10x + y$ and the number with the digits reversed is $10y + x$.

$$\begin{cases} 10x + y = (10y + x) - 27 \\ x + y = 13 \end{cases}$$

$$\begin{cases} 9x - 9y = -27 \\ x + y = 13 \end{cases}$$

The number is 58.

12. <u>CHALLENGE PROBLEM</u> If the tens' digit is x and the units' digit is y, then

$$\begin{cases} (10x + y) + 36 = 10y + x \\ y = 2x + 1 \end{cases}$$

The number is 37.

13. <u>CHALLENGE PROBLEM</u> If the speed of the first train is x m.p.h. and the speed of the second train is y m.p.h. then

$$\begin{cases} 3x + \frac{11}{4}y = 200 \\ y = x + 10 \end{cases}$$

The speed of the first train is 30 m.p.h. and the speed of the second train is 40 m.p.h.

14. <u>CHALLENGE PROBLEM</u> If the speed of the first train is x m.p.h. and the speed of the second train is y m.p.h. then

$$\begin{cases} 3x + \frac{11}{4}y = 200 \\ \frac{11}{4}x + 3y = 200 \end{cases}$$

Both trains travel at the speed of $34\frac{18}{23}$ m.p.h. or about 34·8 m.p.h.

15. <u>CHALLENGE PROBLEM</u> If there are x cu. ft. of one liquid and y cu. ft. of the other, then

$$\begin{cases} x + y = 10 \\ 2x + 3y = 24 \end{cases}$$

There are 6 cu. ft. on one liquid and 4 cu. ft. of the other.

The weights are both 12 lbs. wt.

CHAPTER 6

A STRANGE ARITHMETIC

6-1 Introduction

The main purpose of this short chapter is to show the pupil that the Basic Properties which he has learned for rational numbers hold for other systems. This "game" with numbers provides an enjoyable and valuable experience. The pupil has to follow a few rules step by step and gradually he will see patterns.

Do not over-teach this chapter. About three lessons should be enough. There are no skills to be learned, and nothing to be committed to memory. The pupil is simply to enjoy the experience and figure out in his own way how to do the problems.

A second objective of this chapter is to provide an example of a field (See Chapter 7) different from the field of rational numbers. To be brief, and not quite accurate, in a field one can always add, subtract, multiply, and divide (except by 0). Therefore in teaching this chapter try to get the pupils to see that the four operations can always be performed with these strangely behaving "numbers". Try to bring them to realize this without specific directions from you.

The number system of this chapter is an example of what mathematicians call a finite field - finite because it contains only a finite number of things, in this case five. There are many such fields and some hints to this effect are contained in the problems. Some students may wish to explore further in this direction and should be encouraged to do so. Such algebraic play can be highly stimulating to the imagination.

You might begin by reminding the pupil that in a clock he has an example of numbers arranged around a circle. You should make it quite clear that the only numbers we have are 0, 1, 2, 3, 4. These numbers are not the same as in ordinary arithmetic because the rules for addition and multiplication are different.

6-2 Addition

Before the pupils have looked at the text, draw a picture of the five "hour" clock on the chalkboard and ask them to find a few sums. They should see what is happening

very quickly. Then draw the addition table on the board and fill in the squares as the pupils calculate the sums. The text asks the student to make his own table but this is not absolutely necessary. You will need the table on the board for all to see.

⊕	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

Now ask the question in the text. "How is the table different from the table for ordinary addition"? You may get answers such as:

$4 \oplus 4$ is 3 and not 8

This is true of course, but a more imaginative answer would be:

The sum of two numbers is one of the five numbers.

This states what mathematicians call <u>closure</u>. The system is <u>closed under addition</u>. Of course the set of <u>all</u> whole numbers is also closed under ordinary addition. But the set, $\{0, 1, 2, 3, 4\}$ of whole numbers is not closed under ordinary addition. The sum $2 + 4$ is 6 which is not in the set.

Now ask the second text question. "How is the table like ordinary addition?" You might get correct answers such as:

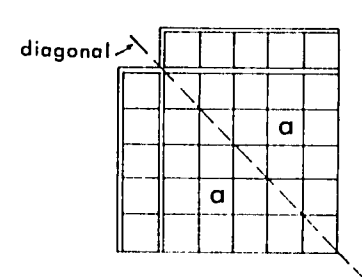
$0 \oplus 1 = 1$

$0 \oplus$ anything = anything

$2 \oplus 3 = 3 \oplus 2$

addition is commutative.

That addition is commutative can be seen by observing that the table is symmetric with respect to the diagonal, as shown in the figure at the right.



That every number has an additive inverse, or opposite, can be seen by observing that every row of the table contains 0.

Another rule for $\oplus$. This arithmetic way of defining addition will appeal to some students more than the geometric clock way. Let a pupil find sums either way he chooses. This arithmetic way can be described as follows: "Add as usual, and then take the sum to be the remainder when you divide by 5". For example:

$2 \oplus 2 = 4$, because the remainder on dividing by 5 is just 4.

$4 \oplus 4 = 3$, because $4 + 4 = 8$ and the remainder on dividing by 5 is 3.

That the two rules for $\oplus$ give the same sums may be checked by direct calculation, or can be understood by careful thought about clock addition. Every time we pass 0 we subtract 5. The addition tables for the two rules are the same.

Answers to Problems 6-2 Student Text Page 71

1. Use this problem in class mainly to see that all pupils can figure out sums from one of the definitions of $\oplus$. The pupil can check his answers from the addition table.

2. The desired answer here is that the table is symmetric with respect to the diagonal. A pupil might express this by saying, "If I fold the table along the diagonal the two halves are the same."

3. Be sure that all understand that addition has the associative property. The only proof at their command is an examination of all cases. This is too tedious to do. Do only enough cases to convince the class. An interesting question to ask here would be: "How many cases would you have to examine?" The answer is: At most 125 because in $a \oplus (b \oplus c)$ there are five possibilities for each of a, b, and c.

4. (a) $\{2\}$ (b) $\{3\}$ (c) $\{4\}$ (d) $\{0\}$ (e) $\{2\}$ (f) $\{3\}$
(g) $\{4\}$ (h) $\{0\}$ (i) $\{3\}$ (j) $\{1\}$

Yes, by looking at the addition table and in the column for the second number find the correct first number. For example, to solve (i) $x \oplus 4 = 2$, we find in the column for 4 that 2 occurs in the row for 3.

5. (a) 3 (b) 2 (c) 1 (d) 0 (e) 4
(f) 1 (g) 0 (h) 4 (i) 3 (j) 2

6. $\ominus 0 = 0$, $\ominus 1 = 4$, $\ominus 2 = 3$, $\ominus 3 = 2$, $\ominus 4 = 1$.

7. Each row contains all the numbers.

8. The number 12 behaves like 0.

The challenge problems are intended mainly for exploration by the better pupils, but there is nothing really hard about them.

9. <u>CHALLENGE PROBLEM.</u> The addition table is shown below:

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

10. <u>CHALLENGE PROBLEM.</u> The addition tables are:

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\oplus$	0	1
0	0	1
1	1	0

Addition tables can be made for clock arithmetics with any number of numbers. The ordinary clock is an example with twelve numbers. Some pupils may wish to examine these other games and if so they should be encouraged. With addition we have complete freedom to choose as many numbers "on the clock" as we wish. With respect to multiplication it will be seen, in the challenge problems of the next section, that we are limited.

6-3 Multiplication

As with addition, explain briefly the rule for multiplication and have the pupils complete the entries in the table, which you will put on the blackboard.

⊗	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

The table is different from that for ordinary multiplication because, for example:

$3 \otimes 3 = 4$ while $3 \times 3 = 9$

Only the numbers 0, 1, 2, 3, 4 occur in the table. If this were ordinary multiplication we would need more numbers than 0, 1, 2, 3, 4. The table shows another example of closure. The system is closed under multiplication.

The table is like that for ordinary multiplication because, for example:

0 ⊗ any number = 0

1 ⊗ any number = the same number

Multiplication is commutative

That multiplication is commutative can be seen by observing that the table is symmetric with respect to the diagonal, in the same way as it was for addition.

Clearly, 0 has no reciprocal. The other numbers do, as can be seen from the table by observing that each of the rows for 1, 2, 3, 4 contains the number 1.

Another rule for ⊗ . Pupils can use either the clock rule or this one. This rule can be described as follows: "Multiply as usual. Then take the remainder when you divide by 5." For example:

$2 \otimes 2 = 4$, because the remainder is 4

$4 \otimes 4 = 1$, because $4 \times 4 = 16$ and the remainder is 1.

That the two rules give the same products can be seen by observing that the multiplication tables are the same.

Answers to Problems 6-3 Student Text Page 75

1. Use this problem mainly to see that all pupils can compute products using one of the two rules. The pupil can check his answers from the multiplication table. Help him observe the examples here of the commutative and the associative property.

2. The associative property is true. To decide the pupil would probably have to examine all cases. A few examples probably will be enough to convince him.

3. The distributive property

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$$

holds once again. A proof would consist of an examination of all cases.

4. (a) $\{3\}$ (b) $\{4\}$ (c) $\{1\}$ (d) $\{2\}$ (e) $\{\ \}$
(f) $\{2\}$ (g) $\{1\}$ (h) $\{4\}$ (i) $\{3\}$ (j) $\{2\}$

5. (a) $\{1\}$ (b) $\{3\}$ (c) $\{2\}$ (d) $\{4\}$

Because the reciprocal of 1 is 1; the reciprocal of 2 is 3; the reciprocal of 3 is 2; the reciprocal of 4 is 4.

6. $1 \div 2 = 3$ because $1 = 2 \otimes 3$

7. The rows for 1, 2, 3, 4 each contain the number 1.

8. CHALLENGE PROBLEM. The multiplication table is shown below:

$\otimes$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

Multiplication is commutative, and associative, and distributive. Every non-zero number has a reciprocal. This system, consisting of the elements 0, 1, and 2, and the operations $\oplus$ and $\otimes$, is another example of a field. See Chapter 7.

9. <u>CHALLENGE PROBLEM</u>. The multiplication tables are:

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

$\otimes$	0	1
0	0	0
1	0	1

In both (a) and (b) multiplication has the commutative, associative, and distributive properties.

However, in part (a) there is a distinct difference from ordinary multiplication namely that $2 \otimes 2 = 0$. Thus we have the product of non-zero numbers being 0. The number 2 in part (a) has no reciprocal. This number system is <u>not</u> a field.

Part (b) is the simplest possible example of a field (See Chapter 7).

The pupil may wonder when things come out "nicely" as in the text, or in Challenge Problem 8, and in part (b) of 9. The answer is this: If the clock arithmetic uses a <u>prime number</u> of numbers then we get a field. Thus in the text the number is 5, which is a prime number. In Challenge Problem 8 the number is 3, which is a prime number.

In part (b) of 9 the number is 2, which is a prime number.

In part (a) of 9 the number is 4 or 2×2, which is composite.

This being the case (and no proof is given here) the next "nice" case would involve seven numbers 0, 1, 2, 3, 4, 5, 6.

One final word about this chapter. Be sure that pupils do not confuse clock arithmetic with the representation of whole numbers using other bases. Thus, for example, in the text we have clock arithmetic which uses only the numbers 0, 1, 2, 3, 4.

In Secondary I the pupil represented ordinary whole numbers to the base five. To do this he used only the numerals "0", "1", "2", "3", "4".

There were however infinitely many numbers:

$0, 1, 2, 3, 4, 10_{five}, 11_{five}, 12_{five}, \ldots$

CHAPTER 7

FIELDS

7-1 Number Systems

The main purpose of this chapter is to acquaint the pupil with the idea of a field. He will meet this idea again in Chapter 16 and again in much more detail, in later years. There is not a great deal to be taught here. All the separate ideas have been met before. This chapter simply puts them all together in a single concept. At most three days should be spent here.

In this section we first call to the pupils' attention that some systems of numbers allow more operations than others. By looking at a few sub-systems of the rational numbers he will realize that none is as convenient as the set of all rational numbers.

Answers to Problems 7-1(a) Student Text Page 77

1. Yes. Yes. No; for example $2 - 3$ is not possible. No; for example $3 \div 5$ is not possible.

2. Yes. Yes. No; for example $(-3) \div (-4)$ is not possible.

3. Yes. No; for example $\frac{1}{2} - \frac{1}{2}$ is not a positive rational number. Yes.

4. One can always add, subtract, and multiply any two rational numbers. One can always divide if the divisor is not zero.

5. The system, or set, of all rational numbers.

6. Yes, to all questions.

Now we consider the strange arithimetic of Chapter 6 and call the pupils' attention to the fact that in this system three of the four operations are freely performable.

Answers to Problems 7-1b

Student Text Page 79

1. (a) Yes. (b) Yes. (c) Yes. (d) Except by zero, yes.

2. Yes to all.

7-2 Fields

You should point out to the pupil that we simply collect together, under one word, field, ideas that he already knows. A way of proceeding is to go directly to the definition of a field in Section 7-2. Have pupils give examples of each of the properties.

If pupils wonder what all the fuss is about explain to them that they will use these properties of a field next year when they continue their study of algebra.

Answers to Problems 7-2

Student Text Page 81

1. (a) Properties 1, 2, 3, 4, 5 hold. Property 6 does not.

   (b) Property 1 does not hold so that 2, 3, 4 do not. Property 5 does not hold because there is no zero. Property 6 fails completely. There are no additive or multiplicative inverses.

   (c) Properties 1, 2, 3, 4 hold. Property 5 does not hold because 1 is not in the set. Property 6 fails to hold also.

(d) Properties 1, 2, 3, 4, 5 hold. Property 6 holds in part. Each number has an opposite, or additive inverse, but, except for 1 and -1, there are no multiplicative inverses.

(e) Property 1 does not hold. Observe that we have closure under addition, but not under multiplication. The other properties also do not hold.

(f) Properties 1, 2, 3, 4, 5 hold. Property 6 holds in part. There are no additive inverses, except for 0, but there are multiplicative inverses, that is reciprocals.

(g) Property 1 does not hold. The system is not closed under multiplication. The other properties also do not hold.

(h) Property 1 does not hold. The system is not closed under either addition or multiplication. Property 5 holds.

(i) Property 1 fails. The system is not closed under addition although it is closed under multiplication. Property 5 holds. Property 6, surprisingly also holds because $1 \oplus 4 = 0$ so 1 and 4 have opposites. Also $4 \otimes 4 = 1$ and $1 \otimes 1 = 1$ so both 1 and 4 have reciprocals.

(j) All properties hold.

2. It must be the set of all rational numbers.

3. <u>CHALLENGE PROBLEM</u>. The reasoning is as follows

$$0x = (0 + 0)x \quad \text{because } 0 = 0 + 0$$

$$(1) \qquad = 0x + 0x \quad \text{because of the distributive property}$$

then

$0 = (0x) + [-(0x)]$, because each number has an opposite and the sum of a number and its opposite is zero.

$= \left[(0x) + (0x)\right] + [-(0x)]$, by formula (1)

$= (0x) + \left[(0x) + [-(0x)]\right]$, by the associative property of addition

$= 0x + 0$, because the sum of a number and its opposite is 0.

$= 0x$, by the identity property of addition.

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Cole, S., Arua Teacher Training College, Kampala, Uganda

Collard, Mrs. K., University of Ibadan, Ibadan, Western Nigeria

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Deakin, J., Registrar, West African Examinations Council, Accra, Ghana

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Fakoya, D. A., Ministry of Education, Lagos, Western Nigeria

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Kazembe, F. N., Ministry of Education, Blantyre, Nyasaland

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Khonje, S. G., Blantyre Secondary School, Blantyre, Nyasaland

Kimball, R., Kampala Technical Institute, Kampala, Uganda

Krouse, H. M., Center for Programmed Instruction, New York City, New York

Lappan, P., Lehigh University, Bethlehem, Pennsylvania

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McLane, Miss L., Weston School System, Weston, Connecticut

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Modu, C. C., West African Examinations Council, Yaba, Nigeria

Mughogho, M. H., Ministry of Education, Lilongwe, Nyasaland

Mwajombe, R., Peramiho Teacher Training College, Peramiho, Songea, Tanganyika

Mwakalukwa, G. R., Uhuru Street School, Dar es Salaam, Tanganyika

Myers, S. S., Educational Testing Service, Princeton, New Jersey

Neville, Mrs. M. C., U.S.A.I.D., Lusaka, Northern Rhodesia

Ngalombi, E. M. K., Kampala Technical Institute, Kampala, Uganda

Oblitey, A., University of Ghana, Legon, Accra, Ghana

Obodai, G. T., Accra City Council Educational Unit, Accra, Ghana

Odunukwe, C., Federal Emergency Science School, Lagos, Western Nigeria

Offurum, R., Kings College, Lagos, Western Nigeria

Okello, L., Ministry of Education, Kampala, Uganda

Okine, J. K., Accra Academy, Accra, Ghana

Okon, J. J., Government Secondary School, Afikpo, Eastern Nigeria

Okosi, C. N., Government Secondary School, Owerri, Eastern Nigeria

Oyelese, J. O., University of Ibadan, Ibadan, Western Nigeria

Pettis, B. J., University of North Carolina, Chapel Hill, North Carolina

Polome, E. G. C., University of Texas, Austin, Texas

Prenowitz, W., Brooklyn College, Brooklyn, New York

Putnam, A. L., University of Chicago, Chicago, Illinois

Phythian, J. E., Makerere University College, Kampala, Uganda

Richmond, D. E., Williams College, Williamstown, Massachusetts

Riske, M., New Zealand House, London, England

Sambolah, E. S., College of West Africa, Monrovia, Liberia

Setidisho, N. O. H., Ministry of Education, Ndola, Northern Rhodesia

Shanks, M., Purdue University, Lafayette, Indiana

Smith, E. M. R., Fourah Bay College, Freetown, Sierra Leone

Soriyan, M. S., Government College, Ibadan, Western Nigeria

Sowden, Miss K. M., Whitelands College, University of London, Putney, London, England

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Swain, H., New Trier Township High School, Winnetka, Illinois

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Williamson, Miss I. B., Ministry of Education, Dar es Salaam, Tanganyika

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STAFF

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TEACHERS' GUIDE

VOLUME 2

MATHEMATICS SECONDARY TWO

UNIT 2: GEOMETRY

Prepared at the

Entebbe Mathematics Workshops

African Education Program
Educational Services Incorporated
Watertown, Mass., U. S. A.

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CHAPTER 8

USE OF INSTRUMENTS

8-1 Introduction

This chapter is primarily intended to: (a) Improve the pupil's facility in using the drawing instruments; (b) Refresh the pupil's mind about angles and triangles; (c) Prepare the way for work on congruency of triangles.

There is no attempt here to teach deductive geometry. The pupil is expected to develop his intuition and recall ideas of Secondary One.

The chapter should be covered rather quickly.

8-2 Ruler and Compasses

Drawing a segment of a desired length by direct use of a ruler is sometimes not as accurate as we wish. A good draughtsman will always take the length from the ruler with a pair of compasses, or dividers. The discussion in the text is intended to make the pupil aware of this fact and so to encourage accurate drawing.

The Pencil Point: Pupils often do not keep their pencil points sharp. The teacher should always stress the sharpening of pencil points in order to avoid getting thick lines.

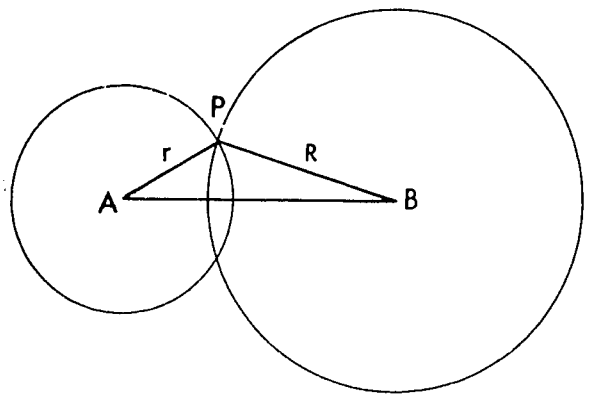
Answers to Problems 8-2 | Student Text Page 84

2. The pupils should observe that $\overline{OP}$ and $\overline{OQ}$ are of the same length (about $1\frac{1}{8}$ inch) and that $\widehat{POA}$ is a right angle.

3. The number of points of intersection is

(a) 2	(d) 0	(g) 2	(j) 0
(b) 2	(e) 0	(h) 2	
(c) 1	(f) 1	(i) 1	

4. Encourage the pupils to try to reach a conclusion without any help. If they are not soon successful, give them the following figure and let them try some more.



Finally, help them see intuitively that $R + r$ must be greater than AB, which gives

$$AB < R + r$$

Also, since $r + AB$ must be greater than R, we have

$$r + AB > R$$

and $$AB > R - r.$$

The required condition for two points of intersection can be stated

$$R - r < AB < R + r.$$

8-3 The Ruler and Set Squares

There are two matters of concern in this section:

(1) Drawing skill in the use of set squares.

(2) Some intuitive insight into the properties of parallel lines and the angles formed with them by a transversal.

In Figure 2 $\overleftrightarrow{OL}$ is parallel to $\overleftrightarrow{PQ}$. Angle A is congruent to angle 1 because they were both determined by the same angle of the set square. The lines $\overleftrightarrow{OL}$ and $\overleftrightarrow{PQ}$ are parallel because of the property

> "If two lines are cut by a transversal so that the corresponding angles are congruent, the lines are parallel."

Do not give this formal statement to the pupils to memorize, but help them talk about the property and see intuitively that it is true. They should use this idea to see <u>why</u>, when the set square is moved along the ruler, the side $\overleftrightarrow{AC}$ will lie along a series of parallel lines. The edge of the ruler is the transversal.

In Figure 3 $\overleftrightarrow{OX}$ is perpendicular to $\overleftrightarrow{PQ}$. Since angle B remains unchanged in size as the set square slides along the ruler, side $\overleftrightarrow{CB}$ is always parallel to its original position because of the property mentioned above. In Figure 1 $\overleftrightarrow{CB}$ is perpendicular to $\overleftrightarrow{PQ}$ because angle C is the right angle of the set square. As $\overleftrightarrow{CB}$ moves parallel to itself it continues to be perpendicular to $\overleftrightarrow{PQ}$ because of the property

> "If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also."

Again, discuss this with the pupils for insight and understanding but do not make it a memory task.

<u>Answers to Problems 8-3</u> Student Text Page 87

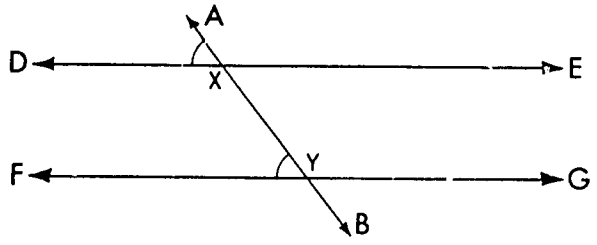
3. Length of the diagonal is approximately 2·8 in.

4. The lines appear to be concurrent. The property that they are concurrent will be proved in a later year. We now simply observe it. If some pupils' lines do not seem to go through one point, discuss how this is caused by inaccuracies in the drawing.

5. We observe that any point on the perpendicular bisector of a segment is equidistant from the endpoints of the segment.

6. We observe inductively that the perpendicular bisectors of the sides of a triangle are concurrent. See Problem 4.

7. You may need to remind the pupils what corresponding angles are. For example $\widehat{AXD}$ and $\widehat{XYF}$ are corresponding angles (see figure on page 82).



We observe the property

> "If two parallel lines are cut by a transversal, the corresponding angles are congruent."

This is the converse of the property used at the beginning of this section.

8-4 Construction of Triangles

In this, and the next three sections the pupil has a chance to discover what must be specified about a triangle in order that it be determined. There is no attempt here at logical deduction. The pupil simply observes that he can actually draw a triangle when certain data are given.

Some pupils will quickly recognize that, if two angles of a triangle are known, the third angle can be found because the sum of the measures of the three angles is 180 . Thus, if the measures of two angles of a triangle are 42 and 66 the measure of the third angle must be 180 - (42 + 66) or 72 .

Many of the pupils may have forgotten the property that the sum of the measures of the angles in a triangle is 180 . If so, let those pupils who remember it tell the others about it. If no one in the class remembers, go on and draw some triangles with two angles given and measure the third angle, as is done in the text. Lead them through this experience to discover the property.

In the example in the text, help the pupils see that, if $\overline{AB}$ were longer or shorter, the angles still could be of measure 42, 66, and 72, but the triangle would look different because its sides would be longer or shorter.

Answers to Problems 8-4a

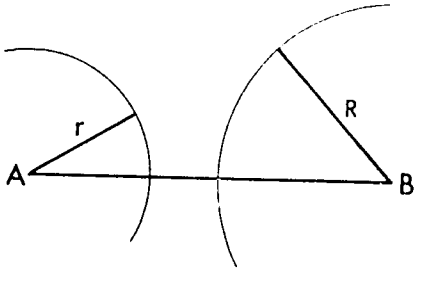
Student Text Page 88

1. Let the pupils compare their triangles and see that, although they are all the same shape, and all have 50°; 60°, and 70° angles as required, the sides of the triangles can be of many different lengths.

2. All the ratios are $\frac{5}{7}$. This problem gives a hint of the property to be studied later that in similar triangles ratios of corresponding sides are equal.

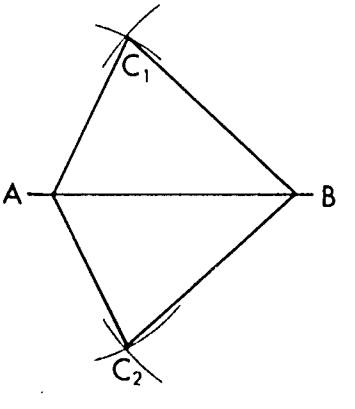
3. Knowing the measures of two angles of a triangle determines the measure of the third angle, but it does not determine a unique triangle.

The case of three sides is perhaps the easiest one to draw. The teacher should be sure that pupils see that the sum of the radii of the two circles must be greater than AB if the circles are to intersect. This requirement is simply a way of saying that the sum of the lengths of two sides of a triangle must be greater than the third. This idea has already been developed in Section 8-2.



It is highly desirable that the pupil first draw a rough sketch of the triangle.

The teacher may let the pupil construct the point C on the other side of $\overline{AB}$. These two positions of C can then be renamed C_1 and C_2. Let the pupil fold the paper along $\overline{AB}$ in order to see that $\triangle ABC_1$ and $\triangle ABC_2$ fit each other exactly.



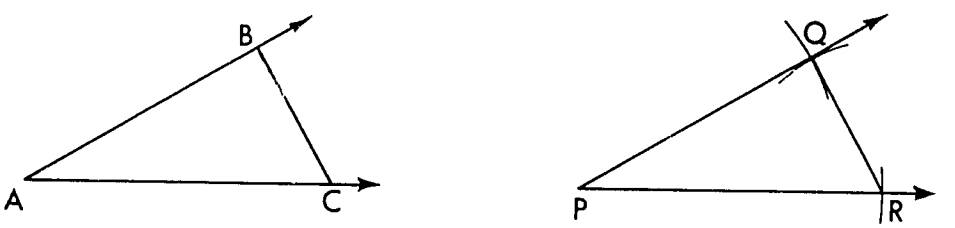
Answers to Problems 8-4b

Student Text Page 90

1. Size of $\hat{R}$ is 90°.
2. Size of $\hat{C}$ is about 14°.
3. Size of $\widehat{DAB}$ is about 141°.
   Size of $\widehat{DCB}$ is about 141°.
4. (a) no triangle is possible; $7 > 4 + 2$.
   (b) no triangle is possible; $7 = 4 + 3$.
   (c) triangle is obtained; $7 < 4 + 5$.

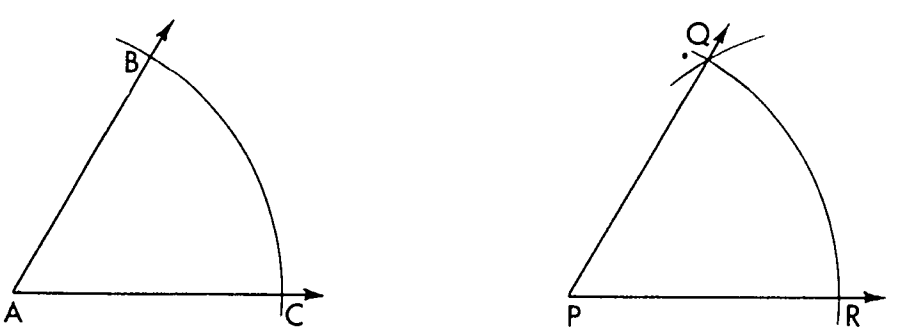
   $x < y + z$.

6.



Since three given sides determine a triangle, $\triangle ABC$ and $\triangle PQR$ must be alike in all their parts. In particular $\hat{A}$ and $\hat{P}$ have the same measure.

7.



In practice it is not necessary to draw the segments $\overline{BC}$ and $\overline{QR}$, since it is only their endpoints which are needed for the construction.

8-5 Given Two Sides and the Included Angle

There is little difficulty here. Once the angle is drawn and the including sides marked off (with compasses) then the triangle is almost complete.

Be careful that the pupils understand that what we do here works only when the known angle is included between the known sides. The case where the angle is not included is considered in Section 8-7.

Answers to Problems 8-5

Student Text Page 92

1. Length of $\overline{QR}$ is about 14·9 cm.

2. Length of $\overline{AC}$ is 3 in.
   Size of $\hat{C}$ is 60°.

3. Length of $\overline{AC}$ is about 3·6 cm.
   Size of $\widehat{ADC}$ is about $89\frac{1}{2}^{\circ}$.

4. Length of $\overline{DG}$ is about 11·7 cm.

8-6 Given One Side and Two Angles

We observe first that the third angle can always be found. The actual construction uses the two angles whose vertices are at the ends of the given side. We say that the side is included by these angles.

Answers to Problems 8-6

Student Text Page 93

1. Length of $\overline{AB}$ is about 3·4 in.
   Length of $\overline{AC}$ is about 3·4 in.

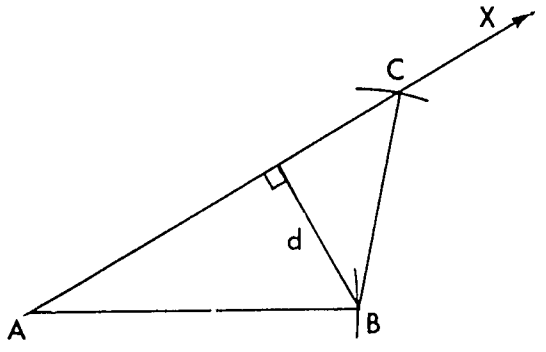
2. Length of $\overline{QR}$ is about 6·7 cm.
   Length of $\overline{RP}$ is about 9·0 cm.

3. Length of $\overline{EF}$ is about 4·8 cm.
   Length of $\overline{GF}$ is about 4·8 cm.
   Size of $\widehat{GEF}$ is 45°.
   Size of $\widehat{EGF}$ is 45°.

8-7 Given Two Sides and an Angle Not the Included Angle

This is the so-called ambiguous case. One can have no triangle, one triangle, or two triangles. The pupil should not try to memorize when these different possibilities occur. In each problem he should draw a figure and decide for himself what happens.

Problems 1, 2, and 3 of Problems 8-7 suggest the various possibilities. Let d be the perpendicular distance from B to $\overrightarrow{AX}$. Then the possibilities can be summarized in the following table.



$\hat{A}$	BC	Number of triangles
acute	BC < d	0
	BC = d	1
	BC > d and BC < AB	2
	BC = AB	1
	BC > AB	1
right or obtuse	BC < AB	0
	BC = AB	0
	BC > AB	1

Do not give this table to the pupils. Let them discover the possibilities themselves. At the end you might possibly let the pupils help you construct such a table on the board, but it is not a real necessity and it certainly should not be memorized.

Answers to Problems 8-7

Student Text Pages 95-96

1. (a) 2 (b) 1 (c) 0 (d) 1 (e) 1

2. (a) 1 (b) 0 (c) 0

3. (a) 1 (b) 0 (c) 0

4. $\overline{PR}$ is about 2·5 in. or 4·4 in. long.
$\widehat{PRQ}$ is about 79° or 101°.
The sum of the two possible measures of $\widehat{PRQ}$ is 180°.
A look at the figure will suggest why this is true.

5. (a) 0 (b) 1 (c) 1 (d) an unlimited number
(e) 0 (f) 1 (g) 1 (h) 2 (i) 0

CHAPTER 9

CONGRUENCE

9-1 Size and Shape

In this chapter we have adopted an informal treatment of congruence. We say that two figures are congruent if they have "the same size and shape". We think of testing whether figures have the same size and shape by supposing that they can be moved about, and observing whether they would fit throughout if we put them together. While this informal approach to congruence is good at the start, it is difficult to give it a sound logical basis, and of course for solid figures in space it is hard to visualize. In Section 9-5 congruence of triangles will be presented without reference to motion, using only congruence of corresponding parts. The general concept of congruence of any two figures without motion will have to wait until a later course. To help the pupils realize that motion is really irrelevant to congruence the teacher should keep using the phrase, "If we could move....".

Answers to Problems 9-1 | Student Text Page 97

The figures in Problems 1, 2, 4, 6, and 7 are congruent.

3. Not congruent if the heights are not equal.
5. Not congruent unless they have equal bases and equal heights.
8. Not congruent unless they have equal heights.

We are here, as we often do, allowing such words as "radius", "base", height", to have two different meanings. We understand which meaning is intended from the context. "Radius" may mean either the segment (which is a set of points) or the length of the segment (which is a number with a unit understood). When we say "circles with the same radius" we are talking about a number. Sometimes we say "radii of the same length" to emphasize what we mean.

Later we shall introduce the notation $\equiv$ to mean "is congruent to". Thus we shall write $\hat{A} \equiv \hat{B}$, $\overline{XY} \equiv \overline{PQ}$, $\triangle ABC \equiv \triangle XYZ$. If you wish to start using this notation now, there is no objection.

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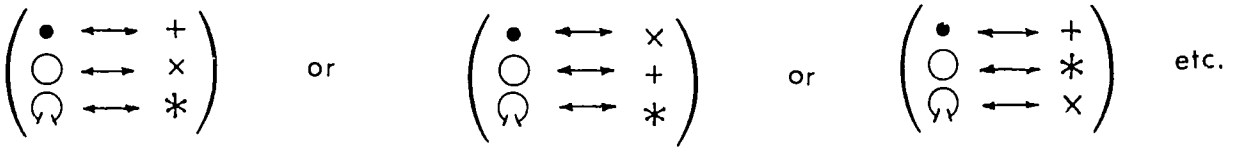
9-2 One-to-one Correspondence

In order to establish congruence between two figures we must have a matching of the parts of the two figures which would fit if the figures were put together. The matching must be a one-to-one correspondence which matches every part of each figure with exactly one part of the other figure. In some circumstances a correspondence can be two-to-one, one-to-three, etc., but for a congruence the correspondence must be one-to-one.

In this section we explain in an elementary way the idea of a correspondence between the members of two sets. The setting up of the correspondence between the elements of two sets is a matter of choice. Take two sets

A one-to-one correspondence can be set up in six ways



For any two triangles a correspondence between their vertices can be set up in any of six ways. Once a correspondence is made for the vertices we can say that angles whose vertices are corresponding points are corresponding angles, and that corresponding sides shall be segments joining pairs of corresponding vertices. Hence a correspondence of vertices determines a correspondence of angles and sides. This is true also of the other examples given which are not triangles.

The notation ABC $\longleftrightarrow$ FDE for indicating a correspondence between vertices will be a great convenience in the study of congruent figures and similar figures. In using this notation one knows what parts are corresponding without looking at the figure.

You will notice that, apart from the informal concept of congruence as having the same size and shape, or being able to fit, no single definition of

"congruent" which will apply to all figures is given here. We say, for instance, that:

Segments are congruent if they have the same length.

Angles are congruent if they have the same size.

Circles are congruent if their radii are congruent.

Triangles are congruent if there is a correspondence between them such that corresponding sides are congruent and corresponding angles are congruent.

We do not try to give conditions for quadrilaterals to be congruent.

While a single precise definition of congruence which covers all these special cases and all others is desirable, it is not easy because it involves the idea of correspondence of points of continuous sets. Such a definition could be:

Two sets of points are <u>congruent</u> provided there is a one-to-one correspondence between the points of the sets which preserves distance.

This is a viewpoint for much later study.

<u>Answers to Problems 9-2</u>

Student Text Page 102

1. (b), (d), (e) would fit. (c) would <u>not</u> fit according to the given correspondence, although it would fit if the correspondence were $ABC \longleftrightarrow RST$.
2. (a) Yes, (b) No, (c) No.
3. (a) Two-to-one (b) Three-to-one (c) One-to-one.
4. $\overline{PQ} \longleftrightarrow \overline{XY}$

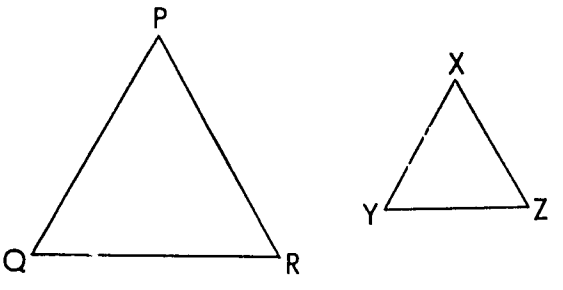
   $\overline{QR} \longleftrightarrow \overline{YZ}$

   $\overline{RP} \longleftrightarrow \overline{ZX}$

   $\hat{P} \longleftrightarrow \hat{X}$

   $\hat{Q} \longleftrightarrow \hat{Y}$

   $\hat{R} \longleftrightarrow \hat{Z}$



9-3 Testing for Congruence

In this section we practice the idea of testing whether two figures are congruent by seeing whether we can make them fit. Besides their regular drawing instruments and paper, the pupils will need a supply of thin semi-transparent paper. After drawing a figure on the thin paper, the pupil can place it over another figure already drawn and see through well enough to know whether the figures fit.

In this section and also in Section 9-5 you probably will run into the problem of failure of drawings to fit because of inaccuracy of the pupil's drawing even though the figures were intended to fit. If this happens, discuss with the class the fact that our drawings are really only representations of the figures we are thinking about. This means that if our drawings are poorly done and are not careful representations of the real thing, they may lead us to wrong conclusions. Then suggest that the poor drawings be done over more carefully to try to reach more dependable conclusions. At the same time point out that no drawing can be a perfect representation of a triangle and a certain degree of error will be expected in every drawing. For instance, just the width of the pencil line introduces some error.

We know that there are six different correspondences between two triangles. Sometimes the triangles will fit for only one of these correspondences; sometimes they fit in more than one way. A correspondence for which the triangles fit is called a congruence. Two triangles are called congruent if there is at least one congruence between them, that is, at least one way they will fit.

Answers to Class Activity

Student Text Page 103

1. The triangles do not always fit. They fit in only one way, i.e. in position (a). This shows there is congruence in only one way.
2. The two triangles fit in positions (b) and (c) but not in (a). Two of the three correspondences are congruences.
3. The triangles are congruent for all six different correspondences between them.

In 1, 2, and 3 the triangles are congruent in each case because in each

case there is at least one correspondence whereby they fit.

4. (a) $\triangle PQR$ cannot fit exactly on $\triangle ABC$

(b) The heights of the two triangles are equal and so are the bases.

The second way of comparison shows also that the areas of $\triangle PQR$ and $\triangle ABC$ are equal because their regions fit.

The two triangles are not congruent.

This construction makes it clear that the figures must have the same shape as well as same size in order to be congruent.

5. $\triangle JKL$ and $\triangle MNP$ have the same shape, but they do not have the same size. They are not congruent. Emphasize the fact that figures must have both same shape and same size to be congruent.

6. $\triangle ABC$ and $\triangle FDE$ are congruent. They fit when $D \longleftrightarrow B$, $E \longleftrightarrow C$, and $F \longleftrightarrow A$.

9-4 Notation for Congruent Figures

Whenever we write a congruence in the form $\triangle ABC \equiv \triangle FDE$, we shall be careful that the letters are in the order of a correspondence for which the figures fit. Make clear to the pupils how the pairs of congruent sides and pairs of congruent angles can be read immediately from this notation.

Answers to Problems 9-4

Student Text Page 107

(a) Figure	Its match	(c) The congruence
a	m	$\triangle ABC \equiv \triangle PQR$
b	h	ABC $\equiv$ KTL or ABC $\equiv$ LTK
c	j	Not congruent
d	g	ABCDE $\equiv$ WVUYX
e	f	HGFE $\equiv$ ABCD
i	n	ABCD $\equiv$ UVWX
k	l	IJKLM $\equiv$ XYWVU
o	q	Not congruent
p	r	FGH $\equiv$ ZYX or FGH $\equiv$ ZXY
s	t	ABCDEFGHIJKL $\equiv$ WXMNOPQRSTUV

(b) All pairs are congruent except (c and j) and (o and q).

9-5 Congruent Triangles

It has already been mentioned that the process of motion and fitting is unnecessary to the idea of congruence. In this section we want to discover some properties which will show when two triangles are congruent without having to use the fitting process. However to discover these properties at this stage we have to use fitting . In a later year the pupils will see the properties developed in a logical deductive system.

Of course two triangles would be congruent if all six pairs of corresponding parts were known to be congruent. We observe, however, that three pairs of congruent corresponding parts are sufficient if they are the right three.

The conditions for congruence of two triangles are stated as congruence of three parts of one triangle to the corresponding parts of the other triangle as follows:

1.	Three sides	SSS
2.	Two sides and the included angle	SAS
3.	Right angle, hypotenuse, and another side	RHS
4.	Two angles and a side	ASA or AAS

In 4. care must be taken that the congruent sides are corresponding sides.

The discovery of each condition is by experiment where generalisation is made from several observations. It is not a proof, of course. The pupils are making conclusions by inductive reasoning. The more observations they make the more likely they are to reach true conclusions. Make this clear to the pupils. As they discover more and more facts they may modify their previous discoveries as in the case of dealing with "two sides and an angle".

See the remarks in Section 9-3 of the Teachers' Guide about possible inaccuracies in the pupils' drawings.

1. Three sides

(a) $\triangle ABC$ and $\triangle XYZ$ are congruent.

(b) All triangles constructed from the same data are congruent

(c) All triangles constructed from the same data are congruent, those from different data are not congruent.

The conclusion is that if three sides of a triangle are congruent to the corresponding three sides of another triangle, then the two triangles are congruent.

2. Two sides and an angle

(a) $\triangle ABC \equiv \triangle XYZ$

(b) All triangles constructed from the data are congruent. It is not possible to construct with the data two triangles which are not congruent.

(c) Comparison shows that triangles constructed from the same data are congruent.

Conclusion they may make: If two sides and an angle of a triangle are congruent to the corresponding two sides and an angle of another triangle they are congruent. (Note: They may not yet discover that the angle must be the included angle.)

(d) The pupils will notice that some triangles fit, some don't. If, by coincidence, all the triangles do fit, give some hints which will lead someone to construct a triangle with these measures which will not fit the others. This shows that when the given angle is the non-included angle, the triangles may not be congruent.

(e) All the triangles constructed with the data are congruent. This shows that if the non-included angle is a right angle congruency holds.

It is true also that if the non-included angle is an obtuse angle, the triangles are congruent.

3. Two angles and a side

(a) All the triangles constructed are congruent.

(b) All the triangles constructed from the same data are congruent.

The conclusion is that if two angles and a side of a triangle are congruent to the corresponding two angles and a side of another triangle, the triangles are congruent.

(Note: Help the pupils to see why it is important that the congruent sides be corresponding sides.)

4. Three angles

The triangles are not necessarily congruent in this case. Later the pupils will learn that this is a condition for similarity of triangles.

Answers to Problems 9-5

Student Text Page 115

1. (a) Congruent ASA

(b) Not congruent (congruent sides are not corresponding)

(c) Congruent SAS

(d) Congruent RHS

(e) Congruent SAS

(f) Congruent SSS

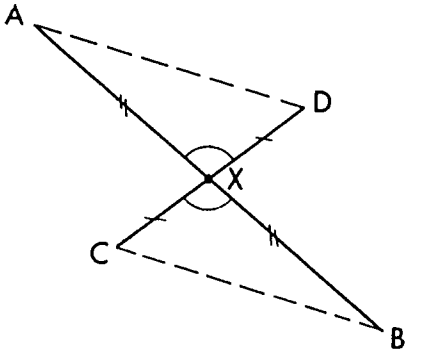
(g) Not congruent (three angles)

(h) Congruent AAS

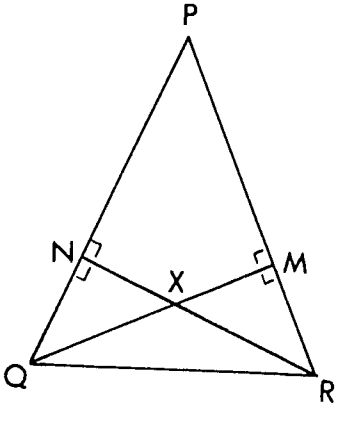
(i) Not congruent (angle not included)

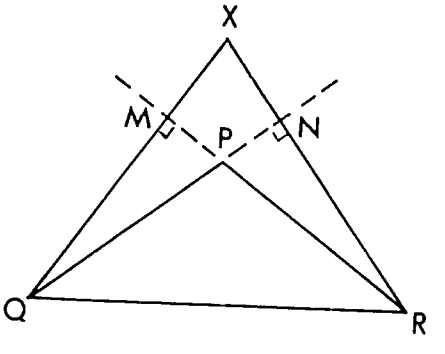
2. $\triangle AXD \equiv \triangle BXC$ by SAS.

We would not write "$\triangle AXD \equiv \triangle CXB$" because the correspondence AXD $\longleftrightarrow$ CXB is not a congruence. They would not fit that way.



3. <u>CHALLENGE PROBLEM</u>





(a) $\triangle PQR$, $\triangle PQM$ $\triangle PRN$, $\triangle QMR$, $\triangle RNQ$, $\triangle QXR$, $\triangle QXN$, $\triangle RXM$

(b) $\triangle PQM \equiv \triangle PRN$; AAS

($\widehat{PMQ} \equiv \widehat{PNR}$, $\hat{P} \equiv \hat{P}$, $\overline{PQ} \equiv \overline{PR}$)

$\triangle QMR \equiv \triangle RNQ$; AAS

($\widehat{QMR} \equiv \widehat{RNQ}$, $\widehat{QRM} \equiv \widehat{RQN}$, $\overline{QR} \equiv \overline{QR}$)

$\triangle QXN \equiv \triangle RXM$: AAS

($\widehat{QXN} \equiv \widehat{RXM}$, $\widehat{QNX} \equiv \widehat{RMX}$, $\overline{QN} \equiv \overline{RM}$)

Conditions for Congruence

This treatment is very informal, but we hope by it to give the pupils a feeling that a general definition for congruence which applies to any two sets of points and which does not depend on motion is possible. The discussion of a correspondence between the points of two segments is particularly informal. If some pupils suggest other ways of setting up such a correspondence, that is fine, but don't expect too much of them. You may want to try applying this general definition of congruence to some other figures such as two angles, two rectangles, or some finite sets of points different from the example given.

Slower students may be expected to omit this section, and even for the better students the teacher should not spend a great deal of time on the section.

CHAPTER 10

BASIC CONSTRUCTIONS

All of the constructions discussed in this chapter are based on congruence. For this reason it would be wise to spend a few minutes looking over again the chapter on congruence before this chapter is begun. Although the pupils will be asked to measure certain segments and certain angles, it should be pointed out to them that the reasons behind the constructions are congruence of the figures.

The pupils should be encouraged to draw each of the figures accurately. The teacher should observe the pupil as he works and make sure that each student uses his drawing instruments correctly. It is very important that the pupil's mistakes be pointed out to him before he has made a habit of them. For this reason, the teacher should check the pupils' work in class and explain the correct procedure again to those individuals who need such an explanation.

10-1 Introduction

The teacher should point out that geometric figures which have certain properties often possess other properties as a result. In the case of an equilateral triangle, the three congruent sides automatically insure us of three congruent angles, each of which is a 60° angle. Thus if we wish to construct a 60° angle, we need only construct an equilateral triangle.

10-2 Bisecting a Line Segment at Right Angles

Observe in the Class Activity that the point P will not necessarily fall between A and B. After the discussion in class, the teacher might raise the following question. In the first construction, if the circles with centres at A and B were congruent, would $\overleftrightarrow{AB}$ still be the perpendicular bisector of $\overline{MN}$? Would $\overline{MN}$ be the perpen-

dicular bisector of $\overleftrightarrow{AB}$? In this case, the students should be able to show that triangles AMN and BMN are congruent and then that triangles AMP and BMP are congruent.

Answers to Problems 10-2 | Student Text Page 119

1. Construct the perpendicular bisector first of the line segment itself, and then that of each of the two segments into which the original segment is divided.

2. Let P be the point where the perpendicular bisector meets AB. Then triangles APX and BPX are congruent.

10-3 Bisecting an Angle

Answers to Problems 10-3 | Student Text Page 119

1. In order that the circles with centres at P and Q may meet, the sum of their radii must be more than PQ.

2. Construct the bisector of the obtuse angle first, and then construct the bisector of each of the acute angles formed.

10-4 Constructing a Perpendicular to a Line at a Point on the Line

Answers to Problems 10-4 | Student Text Page 120

1. The radii must be greater than one-half of XY.

2. Both $\overleftrightarrow{RN}$ and $\overleftrightarrow{LM}$ are perpendicular to $\overleftrightarrow{PQ}$ and in the same plane.

3. $\overline{AC}$ has a length of approximately 2·8 inches. (The exact length is $2\sqrt{2}$ inches.)

10-5 Constructing a Perpendicular to a Line from a Point Not on the Line

Answers to Problems 10-5

Student Text Page 121

1. The radius of the circle with centre at P must be greater than the distance from P to the line $\overleftrightarrow{CD}$. Each of the radii of the circles with centres at C and D must be greater than one-half the distance between C and D.

2. $\overline{AD}$ has length 4 cm., $\overline{BD}$ has length 3 cm., $\overline{CD}$ has length 3 cm.

3. $\overline{AD}$ has length 1·5 inches, $\overline{BD}$ has length 1·4 inches, $\overline{CD}$ has length 2·6 inches.

It is worthy of notice that in the isosceles triangle in Problem 2 the perpendicular line bisects $\overline{BC}$, whereas in Problem 3 where the triangle is not isosceles, $\overline{BC}$ is not bisected.

10-6 Intersecting Lines in Triangles

Encourage the pupils to use a variety of triangles, some acute, some obtuse. Preferably none should be equilateral, isosceles, or right triangles because these are special cases. Discuss informally the conclusions which are suggested inductively in these problems.

Answers to Problems 10-6

Student Text Page 122

1. Yes
2. Yes
3. Yes
4. Yes
5. Yes (If the triangle is obtuse, Z does not lie inside the triangle.)

10-7 Regular Polygons

The figure of the regular hexagon inscribed in a circle reveals several things about a regular hexagon and the circle. The pupil is asked in the latter part of the class activity to recognize that a side of the hexagon is congruent to a radius of the circle.

Encourage the class to state what information they can obtain from the figure. You might ask the question, "What can you say from the figure about the regular hexagon or the circle or both?" Do not suggest any particular information which the figure reveals, but let the class discuss any aspect of the figure it wishes. Some things which may be discussed are the measure of an interior angle of the hexagon and a comparison of the circumference of the circle and the diameter of the circle. (The circumference of the circle is greater than the perimeter of the hexagon, and the perimeter of the hexagon is three times the diameter of the circle.) If the class does not discuss these aspects of the figure, the teacher might try to lead the discussion to them.

CHAPTER 11

SYMMETRY

In this chapter we shall examine various types of symmetry. Symmetry is divided into three main types: symmetry about a line, symmetry about a point, and symmetry about a plane.

One-to-one correspondence has been discussed previously. Since we shall use this property in the study of symmetry, you may find a brief revision of this topic valuable before corresponding points are discussed in the first section.

Much of this subject is intuitive. A great deal of freedom should be allowed in class discussion so that the children can bring out the principles involved.

11-1 Symmetry About a Line

Symmetry about a line is chosen to study first because it is usually the easiest for the children to visualize and grasp.

Class Activity

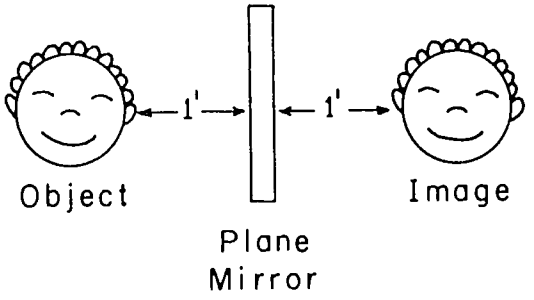
Student Text Page 125

1. (a) Care must be taken that the corners do meet perfectly. Simple cuts such as shown in the student text are best at the beginning; small, more complex designs may discourage the student. Shape C corresponds to shape D, but not to shape O. The corresponding points or sets of points in a symmetry about a line are those which match when the paper is folded.

(b) Yes. Every design of edge PAC has a corresponding design on edge QBD, so curve PAC corresponds to curve QBD. Curves should be discussed as sets of points.

(c) Yes. The regions are images of each other because they are made up of corresponding pairs of points.

2. $\overleftrightarrow{PX}$ can be thought of as a plane mirror, and then A is the image of B in $\overleftrightarrow{PX}$. To reinforce the idea of image, it is worthwhile having a plane mirror on hand. If an object is placed one foot from the plane mirror, the image appears to be one foot "behind" the mirror.



3. The tree diagram may be modified or simplified, or put on the blackboard for class discussion. The children should note that the branches as well as the fruit can be images in the trunk. The trunk is the plane mirror in this example.

(a) Yes. Yes. For every point on the branch on one side of the trunk, there is a corresponding point on the branch on the other side of the trunk.

(b) No. This example is included to point out that corresponding points must be on the line perpendicular to the trunk.

(c) No. The branches are not of the same length, so there is no point on $\overline{YO}$ which corresponds, for example, to J.

(d) If the fruit are thought of as part of the set of points, then the branches are not images of each other; one branch has three fruit, the other only one.

(e) $\overline{GN}$ and $\overline{HN}$.

4. (a) The only axis of symmetry is the vertical line through the middle of the boy.

(b) Left ear, right ear; left hand, right hand; heart, none; left foot, right foot; right eye, left eye; the nose is symmetrical to itself.

(c) The nose and mouth of the boy will probably cause a lively class discussion; they can be divided into two halves which are symmetrical to each other about the axis.

5. (a) Yes

(b) $\overleftrightarrow{AX}$ is the axis of symmetry

(c)

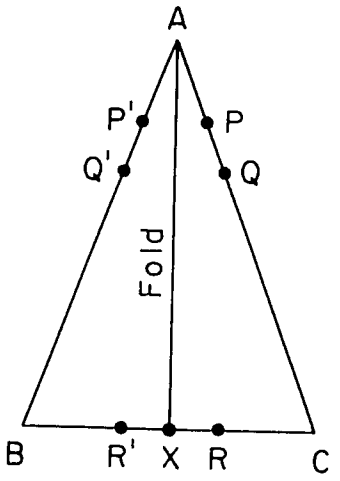


	Image
(1) C	B
(2) $\overline{PQ}$	$\overline{P'Q'}$
(3) $\overline{BR'}$	$\overline{CR}$
(4) $\widehat{ACX}$	$\widehat{ABX}$
(5) $\triangle ABX$	$\triangle ACX$
(6) A	A
(7) $\overline{AX}$	$\overline{AX}$
(8) $\triangle ABC$	$\triangle ACB$

Note that in the table the order of the letters used in naming the image gives the corresponding points; for example, in (3), B corresponds to C, and R' to R.

6. (a) The angles should all be of measure 90 , i.e., each segment is perpendicular to $\overleftrightarrow{AX}$, the axis of symmetry.

(b) (i) Yes (ii) Yes. You may wish to ask if these two conditions would hold for <u>any</u> points made with the compasses in the folded paper in the interior of the triangle. Since the answer is yes, the triangular regions are symmetrical about $\overleftrightarrow{AX}$.

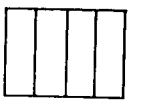
(c) After the definition has been read, you may wish to go back to 5(c) and 6(b) and have the children observe the variety of sets of points which are symmetrical.

7. (a) $\overleftrightarrow{OS}$ is the axis of symmetry.

(b) $\overline{OB}$, $\overline{OB'}$; $\overline{BS}$, $\overline{B'S}$

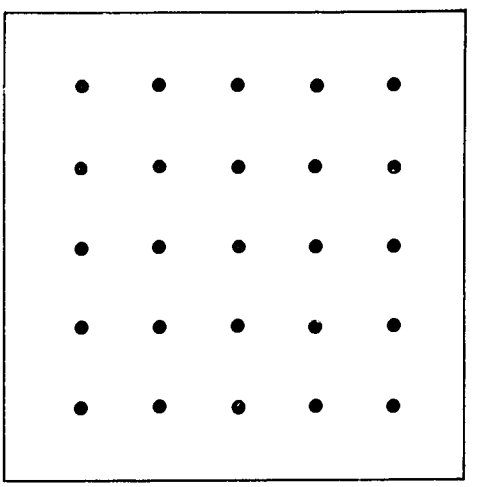
(c) $B'\hat{O}S$ (or $Q\hat{O}S$)

(d) $\triangle OB'S$ (Note that the <u>order</u> gives the point correspondence.)

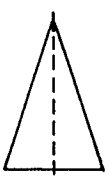
8. Care must be taken in giving directions on folding the paper twice so that the children have a paper that looks like this not this 

when the paper is unfolded. Again, simple cuts should be made. Two axes of symmetry are found -- the two folds in the paper. Other examples of figures with more than one axis of symmetry are: circle, square, some windows, etc.

9. This exercise is a good one to do with rubber band boards if you have them. Rubber band boards are simply flat pieces of wood with nails driven partway in to make a square array. Rubber bands can be stretched over the nails to form a great variety of polygonal figures. It is quicker and more entertaining than drawing the figures on paper.

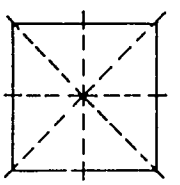


(a)



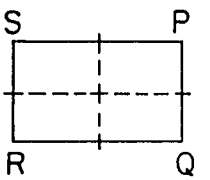
One axis

(b)



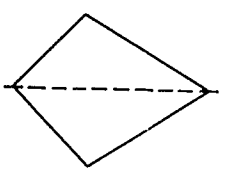
Four axes; the symmetrical regions are rectangles or right triangles or combinations of them.

(c)

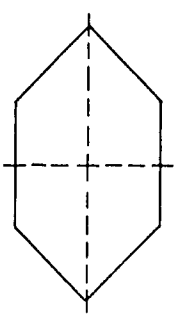


Two axes. (The diagonals are not axes of symmetry since $\overleftrightarrow{PR}$ is not perpendicular to $\overleftrightarrow{SQ}$.)

(d)



One axis; symmetrical regions are triangular.



Two axes; symmetrial regions are either pentagonal or trapezoidal.

(e) There are many possibilities for both categories. You may wish to ask if some of the common geometric shapes, such as parallelograms, trapezoids, etc., have axes of symmetry.

Answers to Problems 11-1

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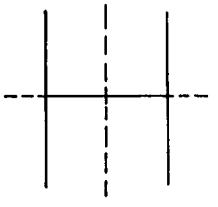
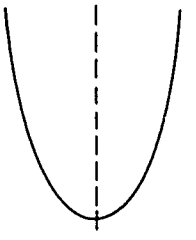
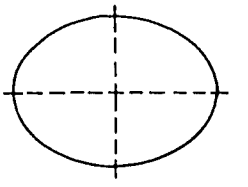
1. (a) Two



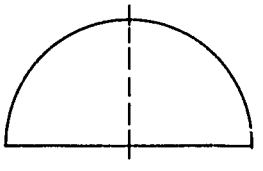
(b) One



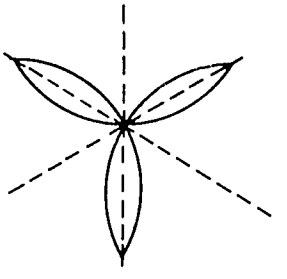
(c) Two



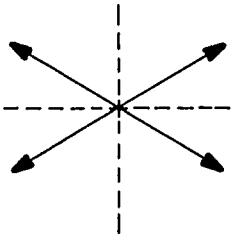
(d) One



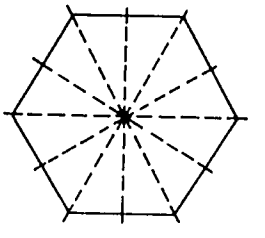
(e) Three



(f) Two



2. Six axes of symmetry.

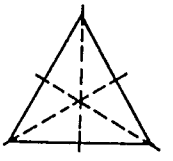


3. (a) Yes. The fold is the axis of symmetry.

(b) The pupil should show that $\overline{AA'}$, $\overline{BB'}$, $\overline{CC'}$, $\overline{DD'}$ are all perpendicular to the fold, and are bisected by the fold.

(c) For example, $\overline{AB}$ is symmetrical to $\overline{A'B'}$, region ABD is symmetrical to region $\overline{A'B'D'}$, etc. An interesting pair of regions is the sets of points on each side of the fold which are in the <u>exterior</u> of the quadrilaterals ABCD and A'B'C'D'.

4. (a) Three axes



(b) None

5. The children can check their answers to this problem by choosing points of their original set of points, connecting each with its corresponding point in the image set, and checking to see if the axis of symmetry is the perpendicular bisector of the line segment formed. Of course, this property should be true of <u>all</u> points in the set, but be satisfied with testing three or four and looking at the rest.

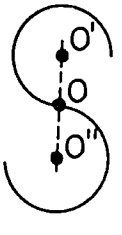
6. (a) The semi-circular regions are symmetrical. The semi-circles are symmetrical curves.

(b) Yes. (i) Yes. (ii) Yes.

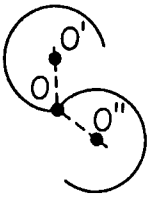
(c) Any diameter lies on an axis of symmetry, so there are as many axes as you wish.

11-2 Symmetry About a Point.

1. (a) You may wish to let the class discuss how to draw the figure shown before they actually draw it with their compasses so that they realize that, if O' and O" are the centres of the circles, the points O, O', and O" need to be collinear if the two circular arcs are to meet smoothly.

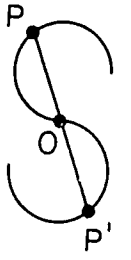


This way



Not this way

(b) The points P and P' are the same distance from O. $\overline{OP} = \overline{OP'}$.



2. This problem is given so that the children can see two kinds of symmetry in the same figure. Note that the point of symmetry may or may not belong to the figure.

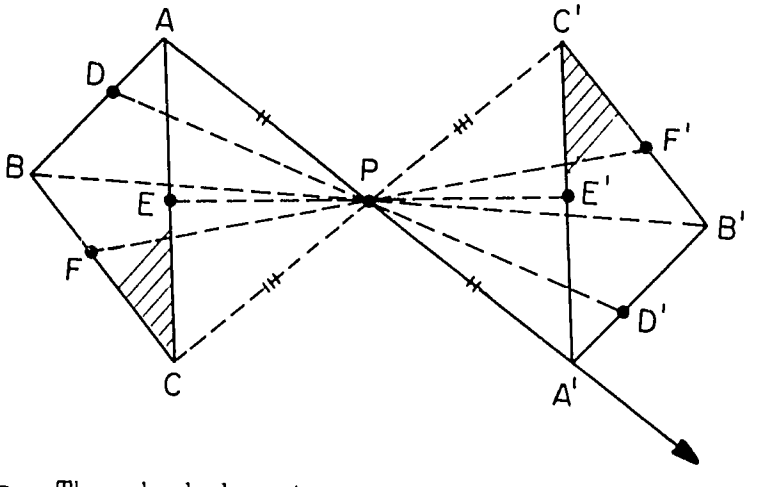
3. (b) The triangles are congruent by ASA.

$S\hat{C}O \equiv R\hat{D}O$ (The right angles formed by the perpendicular lines are congruent.)

$\overline{CO} \equiv \overline{DO}$ (The parts of a bisected segment are congruent.)

$S\hat{O}C \equiv R\hat{O}D$ (Vertical angles are congruent.)

4.



(b) Yes. The shaded regions are symmetrical about P.

Answers to Problems 11-2

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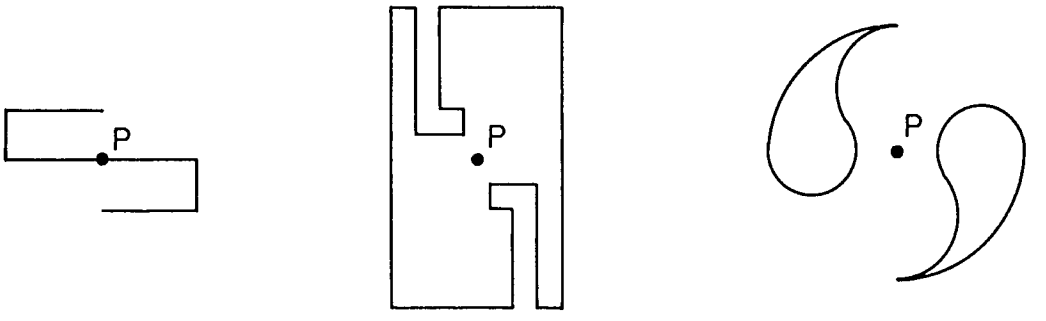
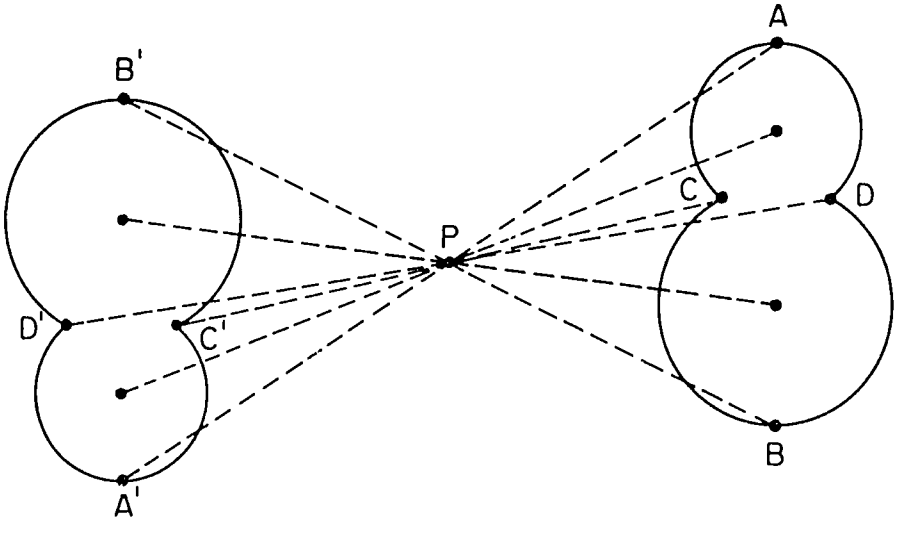
1. (a) (b) (c)



2. (a) $(1, -1)$, $(7, -7)$, $(3\cdot 9, -3\cdot 9)$, $(\frac{7}{8}, -\frac{7}{8})$, etc. In general, $(x, -x)$ where $0 \leq x \leq 10$.

(b) $(-5, -3)$, $(-8, 0)$, $(-4\cdot 76, -3\cdot 24)$, $(-9\frac{1}{2}, 1\frac{1}{2})$, etc. In general, $(-4 - y, -4 + y)$ where $0 \leq y \leq 6$.

3. Children should follow the same process as in 4. of the Class Activity.



Notice that images in point symmetry are upside-down.

11-3 Symmetry About a Plane

Class Activity

1. Each point of one hemisphere has a corresponding point in the other hemisphere. The segment connecting corresponding points is perpendicular to the plane and bisected by it, and hence, the points are points of symmetry about a plane. Since this is true for each pair of points on the surfaces of the hemispheres, the two surfaces are symmetrical about the plane; and since it is also true for the points of the "solid" hemispheres, the two solids are symmetrical about the plane.

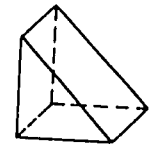
2. Y is at longitude 35° E, latitude 12° S.

3. Other three dimensional figures can be cut from bamboo splinters, twigs, or papyrus with clay, twine, or cellophane tape to bind the corners. Parts of planes cut from paper can be easily inserted in this type of figure.

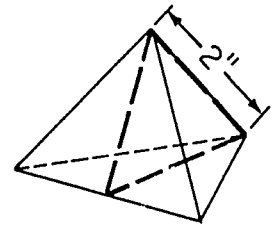
You will have to be careful not to devote too much class time to making models, using up time which you need for the mathematics itself. The pupils will probably enjoy making models and a certain amount of this activity is good for them. After a day of this in class, if the pupils want to do more, encourage them to do it outside class as an extra enrichment activity.

(a) There are three planes of symmetry, each one parallel to opposite faces of the prism and halfway between them.

(b) A cube has nine planes of symmetry. Three are similar to those for the rectangular prism. The other six cut diagonally across opposite faces of the cube, resulting in symmetrical regions like this:

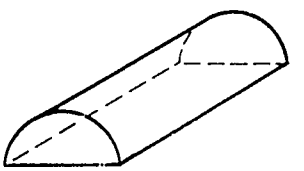


(c) There are six planes of symmetry, one through each edge. An isosceles triangle is formed by the intersection of each plane of symmetry and the tetrahedron. The base of the triangle is 2 inches long. The sides are altitudes of the equilateral triangles.



(d) There are four planes of symmetry in a rectangular pyramid, each one passing through the vertex opposite the base of the pyramid and perpendicular to the base. Two intersect the base in diagonals of the base; the other two divide the base into two congruent rectangles.

(e) Yes, there is an unlimited number of axes of symmetry; one axis is the line through the centre of the two bases of the cylinder; the others contain the diameters of the circle which is parallel to the circular bases of the cylinder and halfway between them. There is also an unlimited number of planes of symmetry; one is the plane containing the circle halfway between the bases (it divides the cylinder into two symmetrical cylinders); the others are planes which contain the axis of symmetry through the centres of the bases Each of these divides the cylinder into two symmetrical sets of points which look like this:



4. The sphere has symmetry about the centre point, about any diameter, and about any plane containing a diameter.

5. Some trees, houses, animals, plants, insects, fruits, etc., have axes and planes of symmetry.

CHAPTER 12

ANGLE PROPERTIES IN CIRCLES

Introduction

The mathematical ideas of this chapter include the following:

(1) In the same circle, or in circles of the same radius, congruent chords subtend congruent central angles and conversely, congruent chords intercept congruent arcs and conversely, congruent central angles are subtended by congruent arcs and conversely.

(2) In a circle if an inscribed angle and a central angle are subtended by the same arc, the measure of the inscribed angle is half the measure of the central angle.

(3) In a circle, inscribed angles which are subtended by the same arc or by congruent arcs are congruent.

Although these topics will be studied again toward the end of a more formal geometry course, they are interesting to study here because:

(1) They provide interesting experience with circles.

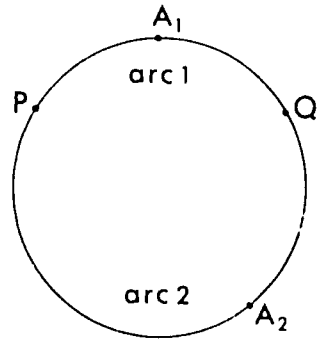
(2) The theorem about inscribed angles shows how a surprising result comes out of other "simpler" propositions.

The emphasis in teaching this chapter should be on pupil discovery and experimentation rather than mastery of any special skills.

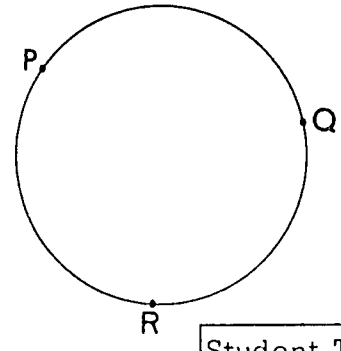
12-1 Arcs

A point on a line separates the line. A point on a circle does not separate the circle. However, two points on a circle do separate the circle into two arcs. Then a point A_1 of one arc (see the figure) and a point A_2 of the other arc are separated by P and Q. This could be described by saying that if "you followed

the circle from A_1 to A_2 you would have to pass through either P or Q".



One cannot speak of the order of three points P, Q, R on a circle. Depending on which arcs one chooses to look at, one can say that Q is between P and R, or P is between R and Q, or R is between P and Q.



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Answers to Problems 12-1

Major arcs	$\overset{\frown}{AB}$	$\overset{\frown}{BC}$	$\overset{\frown}{CD}$	$\overset{\frown}{DA}$	$\overset{\frown}{AC}$	$\overset{\frown}{BD}$
Minor arcs	$\overset{\frown}{AB}$	$\overset{\frown}{BC}$	$\overset{\frown}{CD}$	$\overset{\frown}{DA}$	$\overset{\frown}{AC}$	$\overset{\frown}{BD}$

12-2 Chords

There is an unlimited number of chords of the circle which pass through R.

12-3 Central Angles, Arcs, and Chords

In this section we can reach conclusions about congruent central angles and chords with the help of congruent triangles. To reach conclusions about congruent arcs, we fall back on placing the arcs together to see whether they fit. Our conclusions about congruent arcs are only inductive at this time.

To match arcs you can either cut out regions, as suggested in the text, or if the paper is thin enough, you can place the arcs together and hold them up to the light to see whether they fit. To be able to place them together you can cut

or fold the figure. If you do not want the pupils to cut the paper in their exercise books, give them extra sheets for these experiments.

Answers to Class Activity

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1. $\triangle OAB \equiv \triangle OBC$ by SSS. $\widehat{BOA} \equiv \widehat{COB}$ because they are corresponding angles of the congruent triangles. The arcs $\overset{\frown}{AB}$ and $\overset{\frown}{BC}$ are congruent.

2. We observe that if the chords are not congruent, the central angles will not be congruent and the arcs will not be congruent.

Answers to Problems 12-3a

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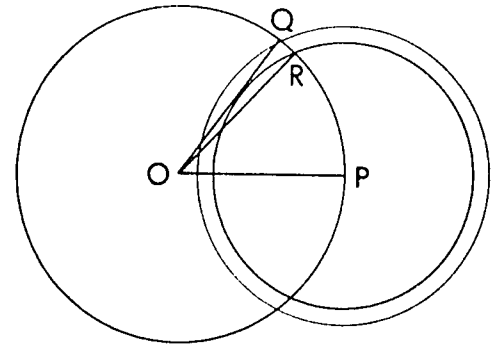
1. In Problem 1 encourage the pupils to vary the lengths of the chords they construct.

2. The figure should look something like this.



Give a variety of examples to illustrate the word "subtend". We can refer to the angle subtended by a flag-staff at the observer's eye.

We generalize from the experiments and say that in a circle, congruent central angles have congruent chords and congruent arcs, congruent chords have congruent central angles and congruent arcs, congruent arcs have congruent chords and congruent central angles.

Answers to Problems 12-3b

Student Text Page 142

1. The central angles are all 60° angles.

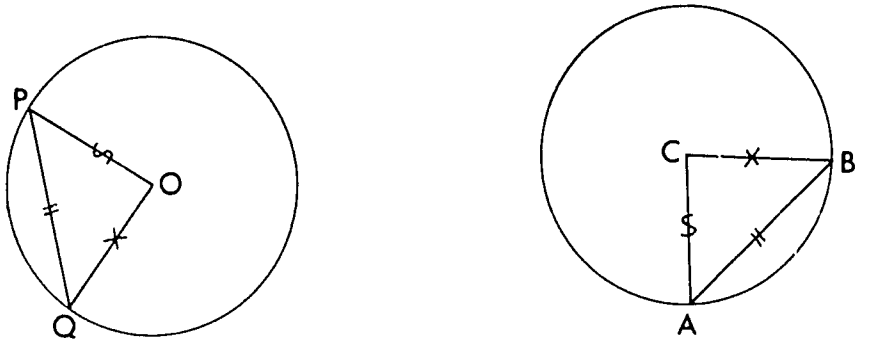
2. The arcs should be congruent.

3. By marking a chord on the bottom of a circular tin and then stretching a string around the tin joining the end points of the chord, we can measure the length of the arc. With congruent chords on the same can, we get arcs of the same length. In this case the arcs are congruent because they are arcs of the same circle. We must be careful, however, not to think that arcs of the same length are always congruent. They might be arcs of circles with quite different radii. Try stretching the same length of string around a much smaller circular tin.

4. The chords are congruent because they are corresponding parts of $\triangle AOB$ and $\triangle COD$, which are congruent by SAS.

5. The arcs should be congruent.

6.



$\triangle OPQ = \triangle CAB$ by SSS. Hence $\widehat{ACB} = \widehat{POQ}$. The arcs are congruent. In this problem we suggest the extension of our ideas about central angles, chords, and arcs to two congruent circles instead of one circle.

12-4 Inscribed Angles

The definition and language are the only topics in this section. You should probably mention that one also uses the less precise language "an angle inscribed in a circle" to mean an angle with its vertex on the circle and points of the circle on the two sides of the angle.

Answers to Problems 12-4 | Student Text Page 144

1. Besides familiarizing the pupil with the definition, this problem anticipates by means of measurement, part of the next section. Don't tell them yet about the next section, but if they guess that all the angles inscribed in the same arc have the same measure, that is fine.

2. This is essentially the same as Problem 1 except that the words "subtend" and "subtended by" are used.

12-5 Central Angles and Inscribed Angles

First let the pupils guess how the measure of $\widehat{APB}$ and measure of $\widehat{AOB}$ compare. Doubtless they will guess that $\widehat{APB}$ is smaller. If nothing more definite than that is suggested turn to the class activity.

Answers to Class Activity | Student Text Page 145

1. The sizes of both $\widehat{APB}$ and $\widehat{AQB}$ are:
   (a) 60° (b) 30° (c) $62\frac{1}{2}^\circ$ (d) 45°

2. $\triangle ORK$ is isosceles because $\overline{OR}$ and $\overline{OK}$ are radii and therefore are congruent. The measure of $\widehat{RKO}$ is 50 because, in an isosceles triangle, the angles opposite the congruent sides are congruent. (If the pupils are not familiar with this fact, let them experiment inductively with it for a while until they are satisfied that it is true.) The measure of $\widehat{KOS}$ is 50 + 50 or 100 . Some pupils may remember the property that the measure of an exterior angle of a triangle is equal to the sum of the

measures of the two remote interior angles. Again let them experiment with it, finding the measure of $\widehat{ROK}$ first and then of $\widehat{KOS}$ from that, until they are satisfied that the property is true.

3. Measure of $\widehat{RKO}$ is x. Measure of $\widehat{KOS}$ is 2x. The reasoning is as in Exercise 2.

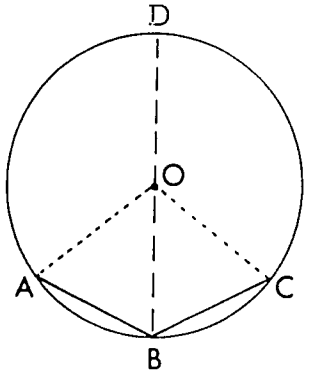
4. (a) 20 (b) 40 (c) 35 (d) 70 (e) 55 (f) 110
   (g) measure of $\widehat{ACB} = \frac{1}{2}$ (measure of $\widehat{AOB}$)

5. (a) x (b) 2x (c) y (d) 2y (e) x + y (f) 2x + 2y
   (g) measure of $\widehat{ACB} = \frac{1}{2}$ (measure of $\widehat{AOB}$)

6. (a) y = 35 (b) z = 110 (c) t = 2x

7. (a) 80 (b) 160 (c) 30 (d) 60 (e) 50 (f) 100
   (g) measure of $\widehat{ACB} = \frac{1}{2}$ (measure of $\widehat{AOB}$)

The conclusion we have been leading to is now formally stated. Observe that it is not mentioned how angles inscribed in the minor arc are related to central angles. This is because the related central angle would have to be an angle of measure greater than 180 and we have not defined any such angles.



If some of the pupils are interested in pursuing this matter, one way to get around the difficulty when $\widehat{ABC}$ is inscribed in a minor arc is to draw a diameter BD through B. Then the measure of $\widehat{ABC}$ is the sum of the measures of $\widehat{ABD}$ and $\widehat{DBC}$, and each of these is inscribed in a major arc so the property can be applied to them.

Answers to Problems 12-5

Student Text Page 147

1. Size of $\widehat{APB}$ is 90

2. m = 40

3. (a) 55 (b) 55 (c) 40 (d) 20 (e) 75 (f) 150
If the pupils need further practice, you can give them more problems similar to this one, or let them make some up for themselves.

4. (a) x (b) 2x (c) y (d) 2y (e) x - y (f) 2x - 2y
(g) measure of $\widehat{ACB} = \frac{1}{2}$ (measure of $\widehat{AOB}$)

12-6 Angles in a Semi-circle

Two viewpoints are given. There is a suggestion of the limit idea. x appears to be approaching 180°. Do this very informally and do not dwell on it.

In the second viewpoint $a = \frac{1}{2}c$ and $b = \frac{1}{2}d$ because of the property in the preceding section. $c + d = 180$, so $a + b = \frac{1}{2}(180) = 90$. The measure of MPN is 90.

Answers to Problems 12-6

Student Text Page 148

1. (a) 32° (b) 48° (c) 20°

2. $\widehat{FBC}$ and $\widehat{EFB}$.

3. $\widehat{XYZ}$ and $\widehat{YZW}$ are right angles because each is inscribed in a semi-circle. Hence $\overline{YX}$ and $\overline{ZW}$ are both perpendicular to $\overline{YZ}$. They are parallel because lines perpendicular to the same line and in the same plane are parallel.

12-7 Angles Inscribed in the Same Arc

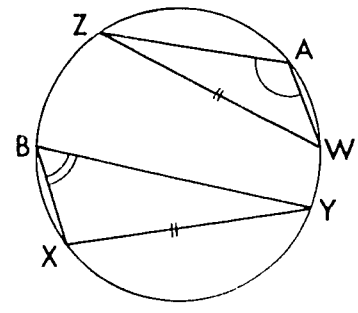
Answers to Class Activity

Student Text Page 149-150

1. The size of each of the inscribed angles is x°. The children can be invited to draw more inscribed angles. Such angles with vertices near the points P and Q should be included. It will be clear that there are infinitely many angles which can be inscribed in any given arc.

2. We have seen that congruent chords in the same circle subtend congruent central angles. We know also that the measure of the inscribed angle is half the measure of the corresponding central angle. It follows that the inscribed angles $\hat{P}$ and $\hat{Q}$ are congruent.

The restriction "if they are both inscribed in major arcs is necessary because without it, we might be tempted to say that, since $\hat{A}$ and $\hat{B}$ intercept congruent chords, they are congruent angles, which would be false for the adjacent figure.



Student Text Page 150

Answers to Problems 12-7

1. (a) $\hat{P} \equiv \hat{T}$, $\widehat{PQT} \equiv \widehat{PRT}$
   (b) $\hat{A} \equiv \hat{D}$, $\widehat{ABD} \equiv \widehat{ACD}$
   There are many pairs of congruent vertical angles here also which you may want to mention, but they are not important at present.

2. (a) $\widehat{AFB} \equiv \widehat{CFD} \equiv \widehat{CGD}$
   (b) $\hat{V} \equiv \hat{U}$

3. $\widehat{BDC}$ is a 40 angle.
   $\widehat{BEC}$ is a 40 angle.
   $\widehat{BOC}$ is an 80 angle.

4. $\widehat{FAH} \equiv \widehat{FBH}$ because they are inscribed in the same major arc.
   $\widehat{BHA} \equiv \widehat{FHB}$ because they intercept congruent chords.
   $\widehat{AKH} \equiv \widehat{BFH}$ because if two angles of a triangle are congruent to two corresponding angles of another triangle, the third angles are congruent.

5. CHALLENGE PROBLEM. Measure of $\widehat{WIU}$ is 124
   The angles can be found in this order:
   $\widehat{WIV}$ is a right angle.
   Measure of $\widehat{WVI}$ is therefore 28.
   Measure of $\widehat{WUI}$ is therefore 28 .

Measure of $\widehat{IWU}$ is therefore 28 .

Measure of $\widehat{WIU}$ is 180 - (28 + 28) or 124 .

6. CHALLENGE PROBLEM. $\widehat{SPT}$ is an angle of 105.

$\widehat{SPR}$ is a right angle.

$\widehat{RPQ} \equiv \widehat{RSQ}$, so measure of $\widehat{RPQ}$ is 15.

Measure of $\widehat{SPT}$ is 90 + 15 or 105.

7. CHALLENGE PROBLEM. Measure of $\widehat{BAE}$ is 22.

$\widehat{ABD}$ is a right angle.

$\widehat{DBE} \equiv \widehat{DAE}$, so measure of $\widehat{DBE}$ is 20.

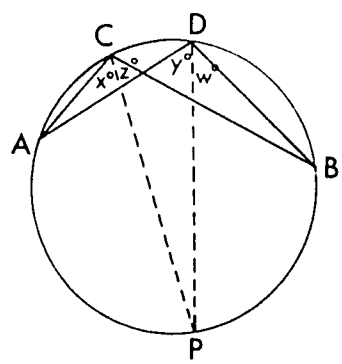
Hence measure of $\widehat{ABE}$ is 90 + 20 or 110.

Measure of $\widehat{BEA}$ is 20 + 28 or 48 . (Exterior angle or triangle)

Finally in $\triangle ABE$, the measure of $\widehat{BAE}$ = 180 - (110 + 48) = 22 .

8. CHALLENGE PROBLEM.

Since $\widehat{ACP}$, $\widehat{PCB}$, $\widehat{ADP}$ and $\widehat{PDB}$ are now all inscribed in major arcs, we know that x = y and z = w. Hence x + z = y + w from which we know that $\widehat{ACB} \equiv \widehat{ADB}$.



12-8 Inscribed Quadrilaterals

The name cyclic quadrilateral may be used if you like it, although it is not necessary for our purposes and it is too bad to emphasize an unfamiliar word if we do not have to.

Answers to Class Activity

Student Text Page 153

1. We study this special case first, and later develop the general case from the special case.

Angles $\hat{A}$ and $\hat{C}$ are right angles because they each are inscribed in a semi-circle. The sum of their measures is 180. p + r = 90 and q + s = 90

because the acute angles of a right triangle are complementary. The sum of the measures of $\widehat{ABC}$ and $\widehat{ADC}$ is 180. At least for this special case, this shows that the opposite angles of an inscribed quadrilateral are supplementary.

2. In each case, the sum of the measures of opposite angles is 180.

3. Yes.

4. $\widehat{SRQ} \equiv \widehat{STQ}$ because they are inscribed in the same major arc. The sum of the measures of $\widehat{SPQ}$ and $\widehat{STQ}$ is 180, as was proved in Exercise 1. This is the special case. We can substitute the measure of $\widehat{SRQ}$ for the measure of $\widehat{STQ}$ to reach the conclusion that the angles $\widehat{SPQ}$ and $\widehat{SRQ}$ are supplementary.

   The question may be raised, "How do we know that arc $\overset{\frown}{SQ}$ in which $\widehat{SRQ}$ and $\widehat{STQ}$ are inscribed is a major arc?" If the pupil has done Challenge Problem 8 in Problems 12-7, the answer is that it does not matter whether the arc is major or minor. If he has not, show him that drawing the diameter through R instead of through P at the start will take care of the matter. If arc $\overset{\frown}{SQ}$ is a semi-circle, there is no problem because $\widehat{SRQ}$ and $\widehat{STQ}$ will both be right angles.

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Answers to Problems 12-8

1. (a) Measure of $\hat{D}$ is 115.
   Measure of $\hat{A}$ is 80.

   (b) Measure of $\widehat{SQT}$ is 90.
   Measure of $\widehat{PRS}$ is 55.
   Measure of $\widehat{URV}$ is 80.

2.

$x + z = 180$	Opposite angles of an inscribed quadrilateral arc supplementary.
$x + y = 180$	Adjacent angles with their exterior sides on a straight line are supplementary.
$x + y = x + z$	Substitution.
$y = z$	Subtraction property of equality.

This shows that an exterior angle of an inscribed quadrilateral is congruent to the opposite interior angle.

3. <u>CHALLENGE PROBLEM</u>. $x = 79$.

Since the measure of $\widehat{PAB}$ is 85, the measure of $\widehat{BDC}$ is 85. (See Problem 2) Hence the measure of $\widehat{ADC}$ is 53. Since the measure of $\widehat{DCQ}$ is 100, the measure of $\widehat{ABD}$ is 100. In $\triangle ABD$, then, the measure of $\widehat{BAD}$ is 48. Hence the measure of $\widehat{BCD}$ is 48. Finally, in $\triangle DEC$, $x = 180-(53 + 48) = 79$.

4. <u>CHALLENGE PROBLEM</u>.

$e = c$

$f = d$

$e + f = c + d$

But $e + f = 180$

Hence $c + d = 180$

$\overline{BC}$ is parallel to $\overline{AD}$ because the interior angles on the same side of the transversal $\overleftrightarrow{DC}$ are supplementary. If the pupils have forgotten this property for proving lines parallel, let them experiment with it until they are satisfied inductively that it is true. Do not dwell on it or try to prove it formally.

CHAPTER 13

LOCI AND ENVELOPES

This chapter has been designed to serve three purposes.

(1) It introduces the pupil to the ideas of locus and envelope. Although the ideas are not easy, the preliminary treatment of them is elementary and well within the grasp of students in the second year of a secondary course.

(2) It also introduces the pupil to a few important well known curves of mathematics. It is not intended to deal with these curves in detail; the question of obtaining each curve from an equation is left for later study.

(3) It makes use of the techniques of construction already learnt, and leads up to the work on graphs in a later chapter.

The emphasis throughout the chapter is on practical activity by the pupils. It is expected that the examples given in the text will be carried out by each pupil individually even though they may be done as class activities under the supervision of the teacher. To avoid time being spent on drawing unsuitable diagrams, suggestions have been made in many examples for a scale and a good starting point on a page.

It is advisable to lay stress on two requirements for successful drawings of loci and envelopes.

(1) A fair degree of accuracy

(2) A sufficient number of points or lines to show very clearly the character of the curve.

If there is not enough time available, some of the problems in Problems 13-2 may be omitted. Similarly, some of the examples of envelopes may be omitted. Since loci and envelopes are closely related ideas, it is not recommended that all the time should be spent on one and none on the other.

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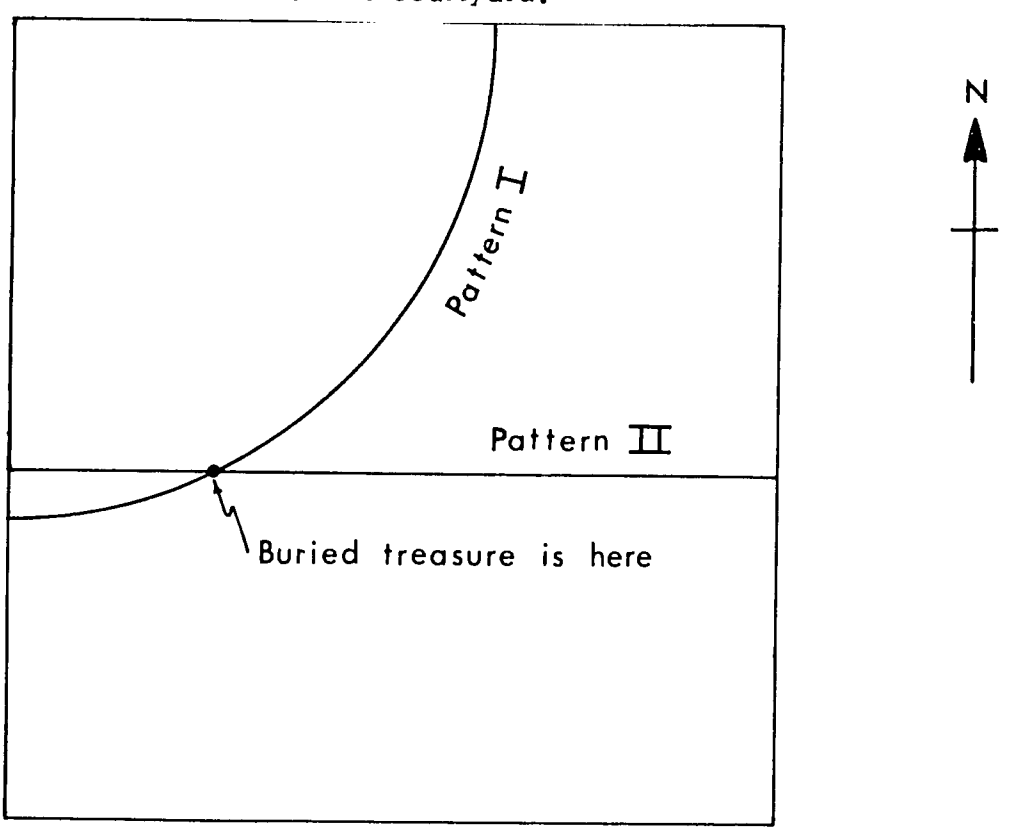
13-1 Examples

These examples are intended to be done as class activities before the word "locus" is introduced in Section 13-2. In this section talk entirely in terms of sets of points.

Pupils often find it difficult to get started on finding the set of points determined by a certain property. A good way to begin is to locate several points which have the particular property of the set. Then those points can be studied to see if they suggest a pattern. In Example 1 three points suggest a line, in Examples 2 and 3, five points suggest a circle.

The diagram for Example 2 is shown below. The treasure is about 20 feet from the southwest corner of the courtyard.



It commonly happens that pupils overlook some of the points of a set satisfying certain conditions. Example 3 illustrates the importance of not overlooking part of the set.

13-2 Loci

There are two slightly different interpretations of the word "locus".

(1) It is the path followed by a point which moves so that it satisfies a particular condition. This concept of a locus is appropriate in physical situations.

(a) The path of the centre of gravity of a ball in the air.

(b) The path of the centre of a wheel travelling in a line along a flat road.

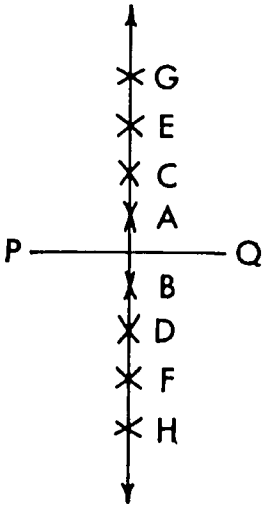
(c) The path of a speck of mud on the rim of the same wheel.

(d) The path of a man walking down the middle of a staircase. This is a very vague condition. It would be necessary to think of something more closely resembling a point, such as the top of the man's head. This path would be a very jumpy, irregular one.

(2) A more precise definition of a locus is that it is the set of all the points which have a particular property. This view is appropriate in studying graphs of equations. This is the view that has been emphasized in the student text and in the summary at the end of the chapter.

At the start of this section use the three examples of Section 1 to help the pupils see that the word "locus" is just a name for something with which they are already familiar.

For Example 3, the boatman could have saved the crew's time by more explicit wording of his message, such as "Sixty feet from Pier P at point X. Buoys A and B. B between A and X."



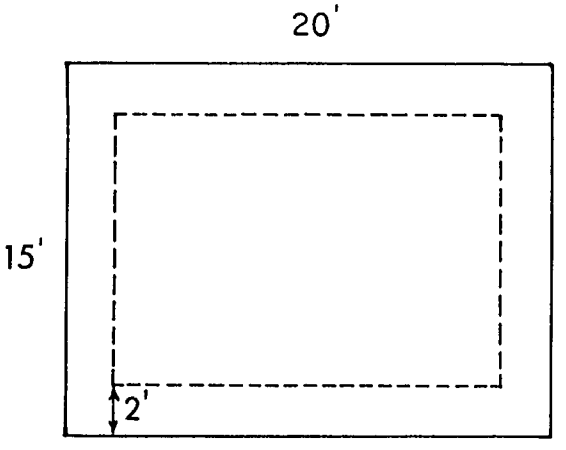
For Example 4, the required locus is the perpendicular bisector of $\overline{PQ}$. PA = QA because the same radius has been used in drawing the arcs which intersect at A.

Answers to Problems 13-2

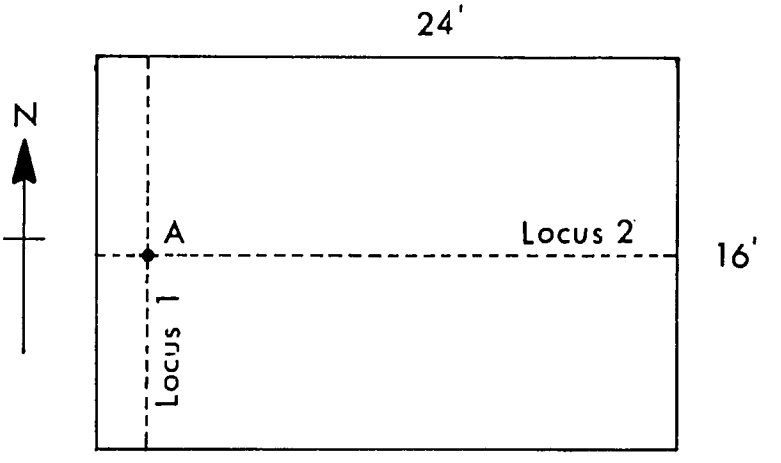
Student Text Page 161

1. The locus is a circle with radius 5 feet and centre at the centre of the oil mill. The distance travelled by the camel is approximately 44,000 feet or $8\frac{1}{3}$ miles.

2.



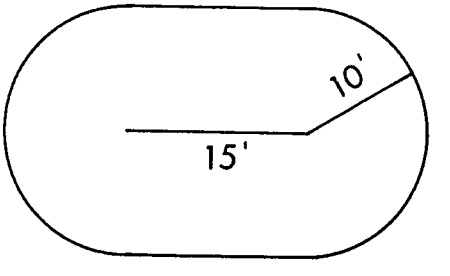
3.



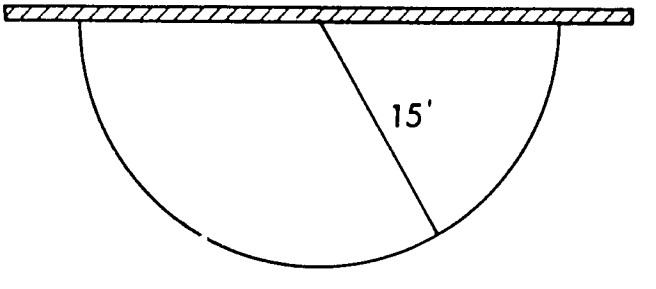
3. The tree is at point A. Since it says that the tree is <u>in</u> the courtyard, there is not a second line 8 ft. from the west wall on the outside.

4. The grazing region is circular with a radius of 10 ft. and its centre at the post. If the goat keeps the rope taut, he walks in a circle. If the rope is not taut, he walks in the interior of a circular region.

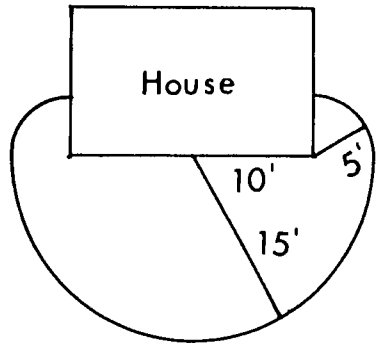
5. The cow's grazing region is indicated. It is the union of a rectangular region and two semi-circular regions. The area is about 614 square feet. If the rope is kept taut, the locus is the boundary of the region shown.



6. The locus is a semi-circle of radius 15 feet.



7.

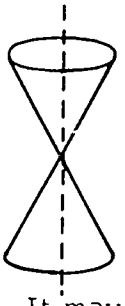


The locus is the union of a semi-circle with radius 15 feet and two quarter circles, each with radius 5 feet as shown.

13-3 Envelopes

While it is not possible at this stage to discuss any of the mathematical theory of envelopes, there is some value in doing some experimenting with sets of lines which envelop certain special curves. The analogy between sets of points and sets of lines is of interest, and some of the resulting curves will be met again later. These sections, however, may be completely omitted if time is needed for more important topics.

The names of curves have been given only for interest. They need not be memorized by the pupils. The astroid is so named from a Latin word "aster" meaning a star; the cardioid from the Greek word for a heart, the nephroid from the Greek word for kidney. The hyperbola, ellipse, parabola, and circle all belong to a very well-known set of curves. They are called collectively conic sections.

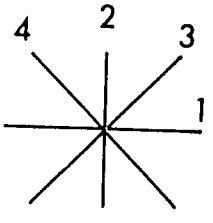


because each may be obtained as a plane section of a double conic surface. A portion of such a surface looks like a pair of funnels placed vertex to vertex with a common axis.

Students are likely to confuse the words envelop and envelope. It may be necessary to teach them the different pronunciations of the two words - en<u>v</u>elop and <u>en</u>velope.

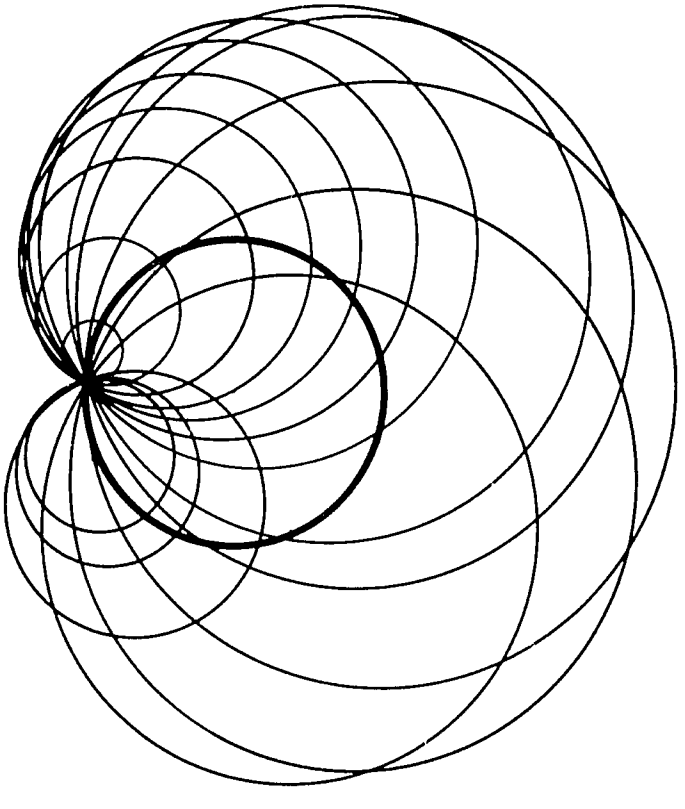
Examples 6, 7, 8 are intended as class-activities leading to discussion on the idea of an envelope. Pupils would enjoy seeing a curve emerge from a carefully drawn set of lines (or curves).

It is advisable for the teacher to draw the envelopes beforehand to compare with results obtained by pupils.

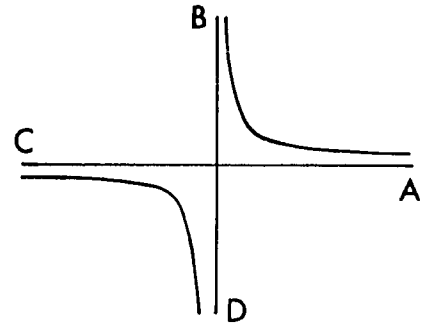


In Example 6 the astroid has four axes of symmetry. The instrument for determining the segments $\overline{XY}$ etc. need not be a set square. The important thing is that the distance corresponding to $\overline{PQ}$ stay constant. You could use a convenient distance, say 4 inches, on a ruler, or even just a piece of cardboard of fixed length.

Example 7.



In Example 8 pairs of negative numbers such as (-6, -10), (-8, -7·5) are necessary to complete the curve by drawing the second branch. The complete curve looks like this.

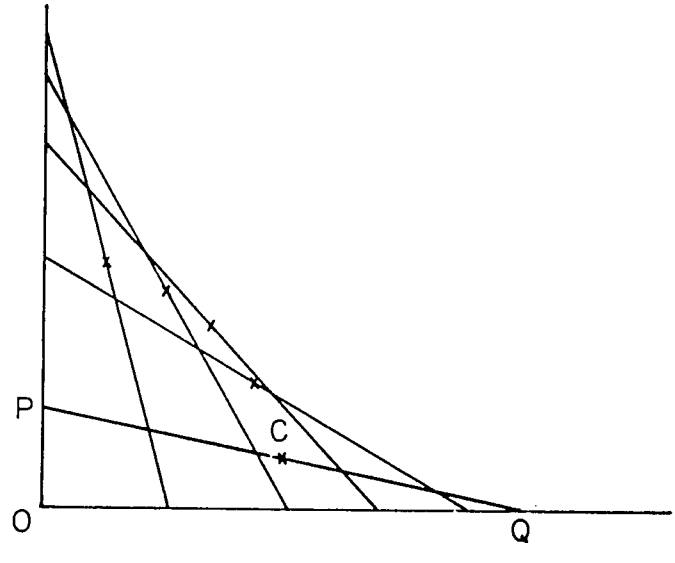


The branches approach but never meet the axes. $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$ are said to be the asymptotes of the curve. This Greek word expresses the fact that the curve and the axes have no points of intersection, but lie very close together.

It should be noted that the pairs of numbers are not pairs of coordinates for points. To avoid any confusion it is advisable not to write the pairs as (6, 10), (20,3), etc.

13-4 Further Examples of Loci and Envelopes

For Example 9 it will be found that the mid-points all lie on a circle with centre O and radius half the fixed length PQ.

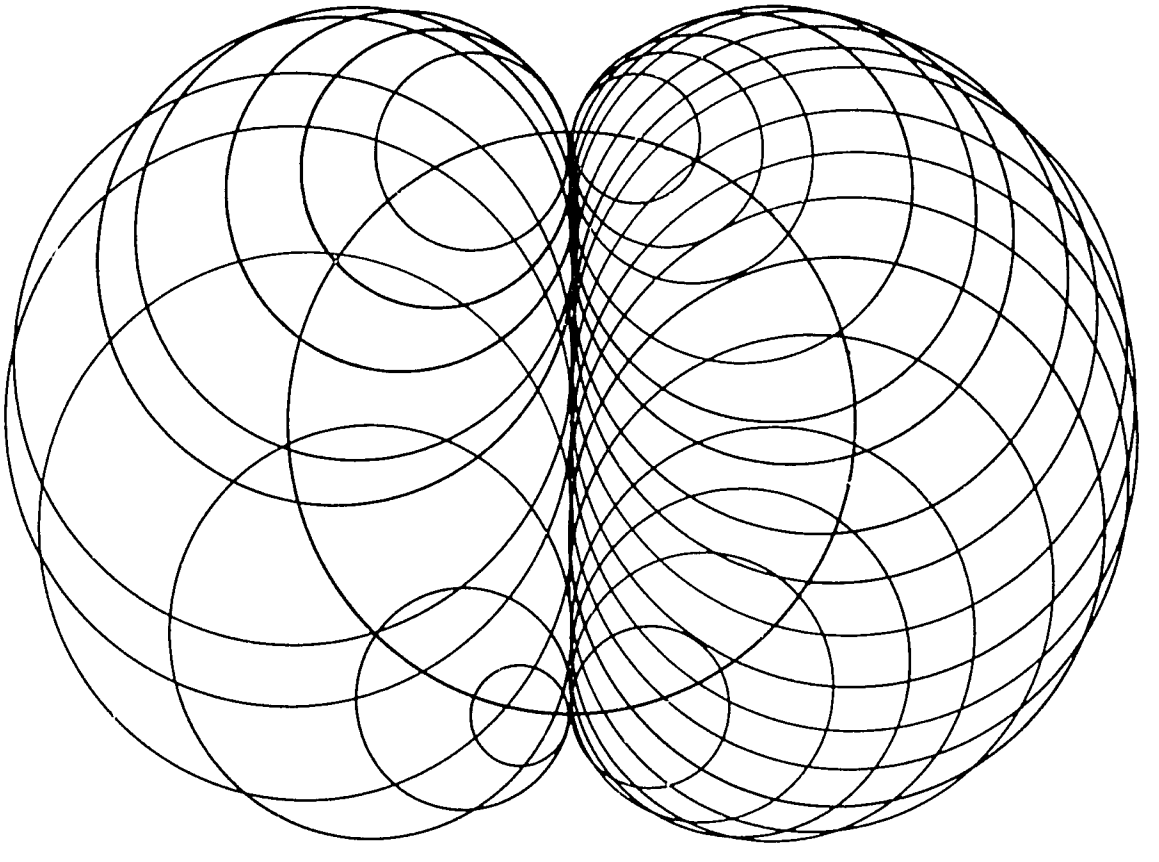


It is known that the mid-point of the hypotenuse of a right triangle is equidistant from the three vertices. Since $\widehat{POQ}$ is a right angle, CO= CP. Since $\overline{PQ}$ is of fixed length, $\overline{CP}$ and hence $\overline{CO}$ must also be of fixed length. It follows that C must be on a circle with its centre at O and radius $\frac{1}{2}$ PQ.

For Example 10 it will be necessary to plan ahead to provide the necessary materials for individual experiments. An alternative but much less satisfactory method would be to do a large-scale drawing on a blackboard. The demonstration should be repeated using different lengths of string and different distances between loci. When the foci are close together the ellipse will look very much like a circle.

Students will meet the parabola of Example 11 again in Unit 5, in connection with squares and square roots, using the equation $y = x^2$. They should be interested to find the same curve appearing in what seems to be a completely different situation.

Note in Example 12 that in the nephroid the diameter is part of the envelope.

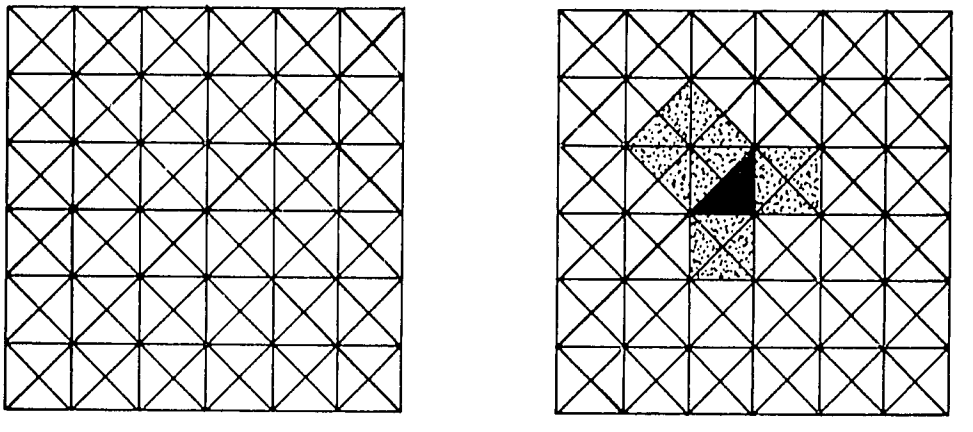


CHAPTER 14

THE PYTHAGOREAN PROPERTY

14-1 The Pythagorean Property

It is said that Pythagoras, the Greek mathematician, on seeing this geometrical pattern, discovered the relationship between the sides of a right-angled triangle.



It is also said that the Egyptians, Sumerians, Babylonians, and Hindus who made use of the 3, 4, 5 right triangle in building, never questioned why this relationship of the lengths of the sides is associated with a right-angled triangle.

Nobody knows whether these two historical notes are true. Today we study the Pythagorean property, named after Pythagoras, because of its general use and importance.

Class Activity

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Great care must be used with the drawing, cutting, and tracing in this section. The following suggestions are made to help facilitate the drawings.

1. (a) The teacher should suggest that the triangular region ABC should be cut with legs about 2 inches and 3 inches respectively. This should allow for a reasonable degree of accuracy. The paper should be thin cardboard so that the pieces can be fitted against each other without slipping.

(b) The centre of the square is found by drawing its diagonals. The centre is the point where the diagonals intersect.

(d) The four cut pieces of region BCDE should fit easily into the square with side $\overline{AC}$ so that the pieces fill up the perimeter of the large square but leave a portion of the middle of the square uncovered.

(e) The uncovered region is a square. The square with side $\overline{AB}$ should fit exactly into this space.

(f) The sum of the areas of the squares with sides $\overline{AB}$ and $\overline{BC}$ is equal to the area of the square with side $\overline{AC}$.

2. (b) (1) 100 cm.^{2}

(2) 36 cm.^{2}

(3) 64 cm.^{2}

(4) 16 cm.^{2}

(c) Yes. $36 + (4 \times 16) = 100$

3. (b) The centre of the square is the point of intersection of its diagonals (see 1 (b) above).

(c) It is very important that the tracing be accurate. Otherwise the rotated tracing will not fit exactly over the original figure. Discuss carefully how PQRS is a square because the sides are all congruent, and the angles, being all congruent, must be right angles.

(e) The final statement, $a^2 + b^2 = c^2$, is an easily remembered skeleton form of the Pythagorean property. However, the students should be encouraged to have a picture of the property in mind and be able to state it in words, as in (a) and (b), rather than as a formula.

Answers to Problems 14-1

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1. (a) Yes

(b) No. Ask the pupils for <u>what</u> triangle ABC the equation AC + BC = AB <u>will</u> be true. They should be able to recognize that it is not true for any triangle.

2. The completed table should look like this:

a	b	c
9	12	15
2·5	6	6·5
5	12	13
1·5	2	2·5
7	24	25
7·5	4	8·5

3. 100 yards

4. 19 feet

5. 238 feet

6. The figures are a parallelogram and four right triangles. DA and CB, DC and AB can be computed by the Pythagorean Property before they are measured. DA = CB = 5. DC = AB = 13.

7. Since inscribed angle $A\hat{B}C$ is a right angle, central angle $A\hat{O}C$ must be an angle of measure 180 and $\overline{AC}$, the hypotenuse of $\triangle ABC$, must be a diameter of the circle. The measure of the diameter is 20.

Problems 8 and 9 on graphs may be done if the class knows something about plotting points.

8. If d is the distance from the origin to the point, then

$$d^2 = 6^2 + 8^2$$
$$d^2 = 100$$
$$d = 10$$

9. If d is the distance between the points, then

$$d^2 = (10 - 4)^2 + (10 - 2)^2$$
$$d^2 = 6^2 + 8^2$$
$$d = 10$$

14-2 A Related Property

The right angles in a tennis court, football pitch, athletics field may have been laid down by the use of a 3, 4, 5 right triangle if not by the knotted rope technique. The teacher may, at this point, obtain some ropes and let the pupils try to lay out a right angle and measure it to verify this practice.

Class Activity

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A paper at least 15 cm. wide will be necessary for this activity. The tables may be put on the blackboard by the teacher and results gathered from the class.

Table I

	x^2	y^2	x^2+y^2
(a)	9	16	25
(b)	25	144	169
(c)	64	225	289

Table II

	z^2
(a)	25
(b)	169
(c)	289

In place of the rope with 13 knots, the following number of knots are some that could be used:

number of knots	31	41	57
1 st stake at knot	1 and 31	1 and 41	1 and 57
2 nd stake at knot	6	9	8
3 rd stake at knot	18	24	32

Help the pupils see the difference between the Pythagorean theorem and its converse. In the first, a right-angled triangle is given and we conclude a relation about its sides. In the second, we are given a relation about the sides of a triangle and conclude that the triangle is right-angled.

Answers to Problems 14-2

Student Text Page 180

1. The right angles are $E\hat{D}H$, $H\hat{E}F$, and $H\hat{C}G$

2. (a) Yes
   (b) Yes
   (c) No
   (d) Yes
   (e) Yes
   (f) Yes

All but (c) are right-angled triangles because of the converse of the Pythagorean Theorem.

PN-AAT-178

TEACHERS' GUIDE VOLUME 3

MATHEMATICS SECONDARY TWO

UNIT 3: REAL NUMBERS
UNIT 4: GRAPHING
UNIT 5: SIMILAR FIGURES AND TRIGONOMETRY

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CHAPTER 15

DECIMAL NUMERALS

This chapter is concerned with a particular way of representing numbers. Much of the work in earlier chapters has been concentrated on properties of the numbers, properties which are not dependent on the way the numbers are written. For example, the commutative property remains valid however we represent numbers. The statements $7 + 9 = 9 + 7$ and $\text{VII} + \text{IX} = \text{IX} + \text{VII}$ are true regardless of what notation is used.

Here we study one important representation of numbers. We study decimal numerals, or more briefly "decimals".

The main feature of the chapter is the clarification, by means of expanded notation, of the four fundamental operations of addition, subtraction, multiplication, and division. With this clarification, the work on percentage follows readily.

15-1 Revision of Base Ten Numeration

This section is a review of what was already done in Secondary One. Hence, do this section in class as rapidly as possible. Begin by asking the class what several decimal numerals represent. Be sure that the index notation is clear to all. Problems 1 and 2 can probably be done orally, or with students putting the work on the board.

15-2 Addition and Subtraction with Decimal Numerals

This material, too, has been discussed in Secondary One.

Addition

Begin the discussion without reference to the text, by asking the pupils to explain simple sums. For example:

$$\begin{array}{r} 53 \\ \underline{42} \\ 95 \end{array} \qquad \begin{array}{r} 57 \\ \underline{64} \\ 121 \end{array} \qquad \begin{array}{r} 637 \\ \underline{795} \\ 1432 \end{array}$$

Keep asking questions until pupils, on their own, make use of expanded notation to explain their work. For example, in $57 + 64 = 121$, you might ask a pupil how he got the "2". If he talks about "carrying one", get him to explain why the "one" is really "ten". Perhaps he may argue that 57 plus 60 is 117 and then 4 more gives 121. Or that 50 plus 60 is 110, 7 plus 4 is 11, and their sum is 121, i.e.,

$$\begin{array}{r} 57 \\ 60 \\ \underline{4} \\ 117 \\ \underline{4} \\ 121 \end{array} \qquad\qquad \begin{array}{r} 50 \\ 7 \\ 60 \\ \underline{4} \\ 110 \\ \underline{11} \\ 121 \end{array}$$

While you would like everyone to discover for himself that

$$\begin{aligned} 57 + 64 &= [5(10) + 7] + [6(10) + 4] \\ &= 11(10) + 11 \\ &= 1(10^2) + 1(10) + 1(10) + 1 \\ &= 121 \end{aligned}$$

not everyone will think of doing it this way. However, when all are aware of how the notation might be used, then you can go to the text. Have pupils explain the steps in the text and how the basic principles are used. Also, show how these steps occur in the usual shorthand computational algorithm.

Subtraction

You may treat as a class activity the completion of the two examples given in the text. Have the pupils explain and discuss the steps in the simplification. Make certain that they really understand the process which is often called "borrowing", but which is really regrouping by the associative property of addition.

Example 1.

$$\begin{aligned} 38 - 26 &= [3(10) + 8] - [2(10) + 6] \\ &= [3(10) - 2(10)] + [8 - 6] \\ &= 1(10) + 2 = 12 \end{aligned}$$

Example 2.

$$\begin{aligned} 65 - 47 &= [6(10) + 5] - [4(10) + 7] \\ &= [5(10) + 15] - [4(10) + 7] \\ &= [5(10) - 4(10)] + [15 - 7] \\ &= 1(10) + 8 = 18 \end{aligned}$$

Answers to Problems 15-2

Student Text Page 182

1. (a) $3(10) + 5 = 35$
   (b) $1(10^4) + 1(10^3) + 7(10^2) + 9(10) + 1 = 11{,}791$
   (c) $5(10) + 2 = 52$
   (d) $8(10) + 8 = 88$

15-3 Multiplication with Decimal Numerals

This material has hardly been mentioned in Secondary One. By this time, however, a pupil should by himself be able to use expanded notation to explain the usual computational algorithm. It is of great importance that the pupil see how the distributive property is applied.

A good way to begin the discussion is to write a problem on the board such as is shown here. Then ask pupils to explain where each digit comes from in the computational algorithm. As soon as the class sees the idea, you can proceed to a study of the text, but do not stop class discussion just to go to the text. It is quite possible that some pupil may suggest something like this:

Algorithm for 42×27

$$\begin{array}{r} 27 \\ \underline{42} \\ 54 \\ \underline{108} \\ 1134 \end{array}$$

$$\begin{aligned} 42 \times 27 &= (40 + 2)(20 + 7) \\ &= [40 \times (20 + 7)] + [2 \times (20 + 7)] \\ &= [(40 \times 20) + (40 \times 7)] + [(2 \times 20) + (2 \times 7)] \\ &= [800 + 280] + [40 + 14] \\ &= 1080 + 54 \end{aligned}$$

Now the problem is reduced to a problem in addition, a problem which has already been discussed in Section 15-2. At this point you can once more ask the class where the different digits in the computational algorithm come from.

Observe that the explanation given by the pupil does not fully use the expanded form. This is quite all right. The purpose of this section is to make perfectly clear to each pupil the reasons underlying the algorithm which he may have learned by rote. Once this has been achieved, there is no necessity for insisting that the pupil write out everything in expanded form.

For example, the multiplication shown above may also be written as follows:

$$\begin{aligned} 42 \times 27 &= [4(10) + 2] \times [2(10) + 7] \\ &= 8(10^2) + 4(10) + 28(10) + 14 \\ &= 8(10^2) + 4(10) + 2(10^2) + 8(10) + 1(10) + 4 \\ &= 10(10^2) + 13(10) + 4 \\ &= 1(10^3) + 1(10^2) + 3(10) + 4 \end{aligned}$$

Remember: <u>Use of expanded notation is not a skill to be learned</u>. Its purpose is to clarify. The example in the text is simplified thus:

$$\begin{aligned} 34 \times 7 = [3(10) + 4] \times 7 &= [3(10) \times 7] + (4 \times 7) \\ &= 21(10) + 28 \\ &= 20(10) + 1(10) + 2(10) + 8 \\ &= 2(10^2) + 1(10) + 2(10) + 8 \\ &= 2(10^2) + 3(10) + 8 \\ &= 238 \end{aligned}$$

Some pupils may need help with the fourth line.

<u>Answers to Problems 15-3</u>

Student Text Page 183

1. (a) $1(10^2) + 4(10) + 8 = 148$

   (b) $3(10^2) + 9(10) + 6 = 396$

   (c) $6(10^3) + 0\ (10^2) + 2(10) + 7 = 6027$

15-4 Division with Decimal Numerals

Begin the discussion by reminding the class of the definition of division in the text. Then start doing a long division problem such as is shown on the right. Ask the class what the 2 means, and what the 94 means. Ask why one selects 2 rather than some other digit. These questions should promote a fair amount of discussion and argument. It is likely that pupils will not keep in mind or tell you the aim of the whole process. That is, they will lose sight of the fact that they are trying to find a decimal numeral for the number q such that

$$\begin{array}{r} 2 \\ 47\,\overline{)1081} \\ 94 \end{array}$$

$$1081 = 47\,q.$$

Keep asking questions until it happens (it may not happen so you will have to tell the class, but try to be patient) that some student remarks that q must be less than 30 because $47 \times 30 = 1410$, and $1410 > 1081$. Also q must be more than 20 because $47 \times 20 = 940$, and $940 < 1081$. Now the class should be able to answer two of your earlier questions. The 2 in the method stands for 20 and the 94 in the method stands for 940.

At this point you can either abandon your example of $1081 \div 47$ and go to the text, or you can continue on with the example. What you should do will depend on how the class feels. Hopefully, the class will want to continue. If so, ask a pupil to explain where the 141 comes from in the long division algorithm shown here. He should be able to tell you that 141 is the difference between 1081 and 940. In other words, he should know that he is looking for a number p such that

$$\begin{array}{r} 23 \\ 47\,\overline{)1081} \\ \underline{94} \\ 141 \\ \underline{141} \end{array}$$

$$1081 = 47(20 + p), \text{ or}$$

$$1081 = (47 \times 20) + (47 \times p), \text{ or}$$

$$141 = 47 \times p$$

The problem now is a simpler one. Obviously, $p = 3$, so

$$1081 = (47 \times 20) + (47 \times 3) = 47 \times 23, \quad \text{and}$$
$$1081 \div 47 = 23$$

Now turn to the text and ask the pupils to follow the two examples and answer the questions in the text.

Answers to Problems 15-4

Student Text Page 185

1. (a) Between 10 and 20
   (b) Between 300 and 400
   (c) Between 3,000 and 4,000

2. (a) $98 = 7x$
   $x = 10 + p$
   where p is between 0 and 10
   $98 = 7(10 + p)$
   $98 = 70 + 7p$
   $28 = 7p$
   $4 = p$
   $x = 10 + 4$

$$\begin{array}{r} 10 + \\ 7\overline{)98} \\ \underline{70} \\ 28 \end{array} \qquad \begin{array}{r} 10 + 4 \\ 7\overline{)98} \\ \underline{70} \\ 28 \\ \underline{28} \end{array} \qquad \begin{array}{r} 14 \\ 7\overline{)98} \\ \underline{7} \\ 28 \\ \underline{28} \end{array}$$

   (b) 36
   (c) 17
   (d) $73\frac{1}{35}$

15-5 Rational Numbers in Decimal Notation

The pupils already know how to write whole numbers in decimal notation. You might take them quickly through a few exercises of writing whole numbers in the expanded form.

You could then start a discussion on the place value of 3 in each of the following whole numbers: (a) 4300, (b) 430, (c) 43, (d) 43,000 and (e) 430,000. Because of the work they have already done on decimal notation they

should have no difficulty in explaining, for example, that the 3 in (a) is 10 times the 3 in (b) and also that the 3 in (b) is 10 times the 3 in (c).

Ask the pupils to write $\frac{43}{10}$ in the expanded form. If they are able to do so after some effort, the exercise may be extended to $\frac{43}{100}$.

They should then be asked to write the number $28\frac{43}{100}$ in the expanded form. If they are able to state that it is

$$2(10) + 8(1) + 4\left(\frac{1}{10}\right) + 3\left(\frac{1}{10^2}\right)$$

ask them whether they can say something about the place values of the digits 2, 8, 4, and 3 in the above expanded number. They should n[illegible]e that the place value of each digit is 10 times the place value of the digit immediately on the right. Now tell them that we write the number $2(10) + 8(1) + 4\left(\frac{1}{10}\right) + 3\left(\frac{1}{10^2}\right)$ in decimal form as 28·43 and use the decimal point to separate the digits representing a whole number from the digits representing a number less than 1. It is read as "twenty-eight point four three".

Answers to Problems 15-5 | Student Text Page 186

1. (a) 432·56

   (b) 0·456

2. (a) $3(10^2) + 9(10) + 1 + 3\left(\frac{1}{10}\right) + 4\left(\frac{1}{10^2}\right) + 5\left(\frac{1}{10^3}\right)$

   (b) $6(10^2) + 0(10) + 0(1) + 0\left(\frac{1}{10}\right) + 1\left(\frac{1}{10^2}\right) + 0\left(\frac{1}{10^3}\right) + 2\left(\frac{1}{10^4}\right)$

   or $6(10^2) + 1\left(\frac{1}{10^2}\right) + 2\left(\frac{1}{10^4}\right)$

15-6 Addition and Subtraction of Rational Numbers

The work which the pupils have already done in addition and subtraction with whole numbers should simplify the work of this section.

Begin by considering a sum such as the following:

$$3 \cdot 64 + 4 \cdot 12$$

Ask the pupils to suggest how this should be done. They may be led to recall what was done with whole numbers and this will suggest the expanded form.

Thus

$$\begin{aligned} 3 \cdot 64 &= 3(1) + 6\left(\frac{1}{10}\right) + 4\left(\frac{1}{10^2}\right) \\ 4 \cdot 12 &= 4(1) + 1\left(\frac{1}{10}\right) + 2\left(\frac{1}{10^2}\right) \\ \hline 3 \cdot 64 + 4 \cdot 12 &= 7(1) + 7\left(\frac{1}{10}\right) + 6\left(\frac{1}{10^2}\right) \end{aligned}$$

Hence, the sum is $7 \cdot 76$. The use of the distributive law should be noted here as the teacher leads the pupils to see how each digit is obtained.

The pupils should soon reach the stage where the process of adding may be understood without using the expanded form.

Subtraction can be studied in a similar manner. Encourage the pupils to check subtraction by addition.

Stress the importance of keeping the decimal points directly under one another when writing decimal numerals to be added or subtracted.

$$\begin{array}{r} 3 \cdot 64 \\ \underline{4 \cdot 12} \\ 7 \cdot 76 \end{array} \qquad \begin{array}{r} 8 \cdot 73 \\ -\ \underline{3 \cdot 45} \\ 5 \cdot 28 \end{array}$$

Answers to Problems 15-6

Student Text Page 187

1. (a) $9 \cdot 43$ (b) $16 \cdot 482$

2. (a) $1 \cdot 11$ (b) $0 \cdot 1096$

15-7 Multiplication and Division by Ten

The pupils have seen the distributive property applied to multiplication with decimal numerals (Section 15-3). A good way to start will, therefore, be to set a problem like $3 \cdot 2 \times 10$ for the children to do on the board in the expanded form.

The problem is done as follows:

$$\begin{aligned} 3 \cdot 2 \times 10 &= \left[3(1) + 2\left(\frac{1}{10}\right)\right] \times 10 \\ &= 3(10) + 2(1) \\ &= 32 \end{aligned}$$

At this stage the teacher could start a discussion with the pupils on place values of the digits in $3 \cdot 2$ and 32. In particular, the teacher could ask them whether they notice what has happened to the place values of the digits in $3 \cdot 2$ after it is multiplied by 10.

They may say that the place values of the digits have changed. Ask them to tell how. Some may be able to say that the value of each digit is now 10 times what it was at first. That is a correct answer, but in order to draw attention to the behavior of the digits in the decimal numeral, the teacher should lead them on to what they may have observed, namely that each digit has moved one place to the left with reference to the decimal point.

Let them see what happens in this example which may be worked on the board:

$72 \cdot 65 \times 10$ There is, as it were,

Result $= 726 \cdot 5$ "promotion" for each digit

Now ask them to tell what happens when a number is multiplied by 100 or 10^2. Lead them to make a general statement about multiplying a number by 10^n where n is a positive integer.

Division by Ten

This should, of course, follow from multiplication by 10. One example may be put on the board for the students:

$$24 \cdot 38 \div 10 \qquad \text{or} \qquad 24 \cdot 38 \times \frac{1}{10}$$

They proceed as follows:

$$24{\cdot}38 \div 10 = [2(10) + 4(1) + 3\left(\frac{1}{10}\right) + 8\left(\frac{1}{10^2}\right)] \times \frac{1}{10}$$

$$= 2(1) + 4\left(\frac{1}{10}\right) + 3\left(\frac{1}{10^2}\right) + 8\left(\frac{1}{10^3}\right) \qquad \text{(The distributive property)}$$

$$= 2{\cdot}438$$

Now ask them to tell what has happened to the digits. It is easy for them to see that the digits have been moved one place to the right with reference to the decimal point.

We may then find out from them whether there has been another "promotion" in this case. This, of course, will lead to their stating that there has been a demotion, i.e., the reverse process.

Then follow up by asking for a statement for dividing by 10^2, 10^3, and 10^n where n is a positive integer.

Answers to Problems 15-7

Student Text Page 188

1. (a) 1·1; 11; 110
   (b) 0·12; 1·2; 12
   (c) 80·01; 800·1; 8001

2. (a) 7·1; 0·71; 0·071
   (b) 10·051; 1·0051; 0·10051
   (c) 0·0017; 0·00017; 0·000017

3. (e) 0·03888

4. (d) 1000

15-8 Decimal Places, Multiplication and Division

The purpose of this section is to show that multiplication and division involving rational numbers in decimal notation are the same as multiplication and division of whole numbers in decimal notation, except that decimal places come into the picture.

Start off by telling the pupils what is meant by decimal places and then have them give orally the number of decimal places of some numerals such as the following: 37·5, 0·24, 40·105, 0·0056, 149·35043.

The next step would be to pose the following problem: What is the truth set for the following equation?

$$2 \cdot 54 = 254 \times a$$

They should be able to give the answer $\left\{ \frac{1}{100} \right\}$. From the above, they are able to see that we can write $2 \cdot 54$ as $254 \times \frac{1}{100}$. Proceed to work $4 \times 2 \cdot 54$ on the board as follows:

$$\begin{aligned} 4 \times 2 \cdot 54 &= 4 \times \left(254 \times \frac{1}{100}\right) \\ &= 1016 \times \frac{1}{100} && \text{(Associative property of multiplication)} \\ &= 10 \cdot 16 \end{aligned}$$

The pupils may now do $1 \cdot 24 \times 3 \cdot 2$ in a similar way. The work will take this form:

$$\begin{aligned} 1 \cdot 24 \times 3 \cdot 2 &= \left(124 \times \frac{1}{100}\right) \times \left(32 \times \frac{1}{10}\right) \\ &= (124 \times 32) \times \left(\frac{1}{100} \times \frac{1}{10}\right) && \text{(Why?)} \\ &= 3968 \times \frac{1}{1000} \\ &= 3 \cdot 968 \end{aligned}$$

Ask the pupils what they notice about the decimal places in the answers of both examples and lead them on through inductive reasoning to the general statement: "When we multiply numbers in decimal notation, we find the product as with whole numbers and then fix the decimal point so that the number of decimal places in the product is the sum of the number of decimal places in the factors."

Division

We go about division first by doing one or two division examples on the blackboard with the pupils. The following will do:

$$\begin{array}{r} 0 \cdot 23 \\ 47 \overline{)10 \cdot 81} \\ \underline{9 \cdot 4} \\ 1 \cdot 41 \\ \underline{1 \cdot 41} \end{array} \qquad\qquad \begin{array}{r} 0 \cdot 0058 \\ 7 \overline{)0 \cdot 0406} \end{array}$$

They see that division is easy with whole number divisors.

The teacher may now ask them: "What should be done in a division like $10 \cdot 81 \div 4 \cdot 7$?" After some effort they might come out with the suggestion that if $4 \cdot 7$ were 47, the problem could be done easily. They may then be asked to put the problem in a form such that the divisor is a whole number.

If $108 \cdot 1 \div 47$ is suggested, ask for the explanation. This may lead to:

$$\frac{10 \cdot 81}{4 \cdot 7} = \frac{10 \cdot 81}{4 \cdot 7} \times \frac{10}{10}$$

We could now generalize by telling them that since it is easy to divide by whole numbers, we try to obtain whole numbers for divisors as was done above.

Answers to Problems 15-8

Student Text Page 190

1. (a) $0 \cdot 026248$ (b) $0 \cdot 0813$

2. (a) $65 \cdot 62$ (b) $1 \cdot 3$

3. $0 \cdot 19525$

15-9 Rough Approximations in Multiplication and Division

This section may be introduced by a game in which the pupils make snap guesses of products. After a number of examples a discussion may be started by having the pupils whose approximations were good describe how they obtained their answers. This could lead to a further discussion of the value of such approximation.

The text does not attempt to give rules for approximations. A few examples are given but each pupil should feel free to use the method easiest for him in the particular example being tackled.

You are aware that almost always when we deal with decimal numerals we are concerned with measurement. For example $2 \cdot 5$ is likely to have appeared as a measure of length, weight, or capacity. In dealing with such measures we know that there is no point in keeping decimal places beyond those which can be

measured in the given situation or beyond those which are justified by the given measurements. The number of decimal places which have meaning depends on the accuracy with which the measurements have been and can be made.

In this section, we have not concerned ourselves with this aspect of decimal numerals. You will find that some of the products have been taken to five or more places. We have assumed that the given numerals are not the result of measurement but that they are exact representations of numbers, from which products can be found exactly. We are concerned now with the computation process with decimal numerals and the "exact products" are given to help demonstrate the usefulness of the rough approximation as a check on the computation.

Discuss with the class the difference between computation with exact numbers which we are studying here, and application to situations involving measurement. Make sure that the pupils understand that in the latter we must frequently throw away some of the last decimal places because they are meaningless.

Answers to Problems 15-9

Student Text Page 192

1. (a) Wrong
   (b) Wrong
   (c) Right
   (d) Wrong
   (e) Right
   (f) Wrong

2. Approximately 250 shillings. If we estimate 5 dollars as about 40 shillings, we get a satisfactory estimate of 280 shillings.

3. 6 hrs., approximately

4. Between 35,000 and 25,000 sq. miles.

5. Ghana has approximately 150,000 sq. miles. Kenya has twice the area of Ghana.

6. (c) 0·01136

15-10 Rounding Off

Pupils may be asked to say whether the population of a country at 8 a.m. on a certain day will be the same as the population of the country at 8 p.m. on the same day. This should lead to the fact that the population may change at every instant of the day, and that it is practically impossible to obtain the exact number of people in a country at any one time, since difficulties may arise by some being in the process of being born and others in the process of dying. Discussion on the practical means of estimating the population of a country should reveal that an approximation to the population is all that is really necessary or practicable.

Records of weights and measures are liable to error because of the nature of the instruments. The measurement that we obtain for the length of a line segment depends on the accuracy of the instrument used, which in turn depends on physical conditions. Thus, the measurement we obtain for the length of a line segment can be only an approximation.

Suppose that we wish to find the area of a rectangle whose dimensions are measured to be 12·5 inches and 9·8 inches, carried to one decimal place, respectively. The statement that the length is 12·5 inches means that it is anything between 12·45 inches and 12·55 inches. In the same way, the breadth is between 9·75 inches and 9·85 inches. The area, then, is between $(12{\cdot}45 \times 9{\cdot}75)$ and $(12{\cdot}55 \times 9{\cdot}85)$ square inches, or between 121·5 square inches and 123·5 square inches. This demonstrates how the approximate nature of the measurements affects the product. We cannot even say what the area will be to the nearest square inch and certainly not to the nearest tenth of a square inch. The best we can say is that the area is about 122 or 123 square inches.

We do not now go into this matter of computation with numbers which are approximations, but it is a matter which should be studied later, and of which the teacher should be aware now.

In talking about mathematics, most of us have acquired language habits which do not always describe accurately the mathematical situation they deal with. An example of this appears in the text in this section when we talk of measuring a line segment. Strictly speaking, a line segment is an abstraction and

cannot be seen and, therefore, cannot be measured. An alternative, rather long-winded, but more accurate, description could be "measure the mark which represents the line segment". Of course, a line segment has associated with it a number which is its measure, but there is no physical measuring process for obtaining it.

A question may arise as to how we know at which level a number like 85,000 has been rounded off. Has it been rounded off at thousands, hundreds, tens or units? In our ordinary decimal numeral, there is no way of saying.

85,000 could be 85,251 or 84,763 rounded off at thousands
or 85,046 or 84,963 rounded off at hundreds
or 85,004 or 84,996 rounded off at tens
or 85,000·4 or 84,999·8 rounded off at units

If, however, we use standard notation to express the number, the accuracy of the number can be clearly indicated. For instance, with standard notation we know that $8 \cdot 50 \times 10^4$ has been rounded off at hundreds.

Not everyone uses the convention we have given for rounding off a number such as 1·45 to one decimal place. Instead the agreement is often to keep the nearest <u>even</u> digit in the last place retained. Thus

1·45 would be rounded off to 1·4
1·75 would be rounded off to 1·8
285,000 would be rounded off to 280,000
0·3615 would be rounded off to 0·362

This procedure is used only when the digit to be dropped is 5 and digits after that digit are zero or not known. An advantage of this convention is that, in a large number of computations with approximations, the errors caused by rounding off tend to balance out instead of accumulating, as they can when the next higher digit is always used.

Answers to Problems 15-10

Student Text Page 195

1. (a) 6,123,000 (b) 5,432,000 (c) 1,236,000

2. (a) 625 (b) 97 (c) 8 (d) 8

3. (a) 25·2 (b) 6·5 (c) 620·0 (c) 8·5

4. Correct to 2 decimal places

(i) 1·23 (ii) 9652·85 (iii) 2615·21 (iv) 8·65

Correct to 3 decimal places

(i) 1·235 (ii) 9652·855 (iii) 2615·211 (iv) 8·651

5. Correct to 2 decimal places

(i) 0·33 (ii) 0·17 (iii) 0·14 (iv) 0·27

(v) 0·56 (vi) 0·20 (vii) 1·89 (viii) 11·57

(ix) 7·54 (x) 5·32

Correct to 3 decimal places

(i) 0·333 (ii) 0·167 (iii) 0·143 (iv) 0·273

(v) 0·556 (vi) 0·200 (vii) 1·889 (viii) 11·571

(ix) 7·538 (x) 5·316

15-11 Percentages

The aim of this section is to introduce that aspect of rational numbers which we call percentages. The definition of the term percentage can be easily understood by the pupils if they recall what they have learnt earlier about fractions. They must have learnt, in particular, that $\frac{7}{10}$ means that a whole thing is divided into ten equal parts and 7 of those parts are represented by the fraction $\frac{7}{10}$. Every part is a tenth part of the whole.

Explain that a "per cent" means a hundredth part. You might at this stage ask a pupil to express "one per cent" as a fraction on the blackboard. He should

write $\frac{1}{100}$. Then ask for some others such as 8 per cent, 30 per cent, 89 per cent, etc. to be written as fractions.

To introduce the use of per cent in expressing a given ratio, you might talk about the boy who got 7 sums out of 10 correct. They know that the ratio of correct sums to total sums is represented by the fraction $\frac{7}{10}$. To show that $\frac{7}{10}$ is the same as $\frac{70}{100}$ they multiply by one in the form $\frac{10}{10}$. Then explain that if a fraction is transformed into one with 100 as its denominator, it is said to be expressed as a <u>percentage;</u> the numerator of the fraction is called the <u>rate</u>. In the percentage $\frac{75}{100}$, 75 denotes the rate. The symbol ($\% = \frac{1}{100}$) can now be introduced. Point out clearly that $\frac{75}{100}$, $75 \times \frac{1}{100}$, 75%, and 75 per cent are four ways of writing the same number.

Some people think of the symbol % as representing $\frac{\quad}{100}$, in other words, part of the fraction symbol. Thus, 35% can be written as $\frac{35}{100}$, the number in front of the symbol % being the numerator of the fraction. This is quite acceptable and may be taught. Whatever concept is taught, however, it is desirable that consistency be maintained throughout. If this second meaning is taught you would want to change the wording in some of the examples, e.g., "$35 \times \frac{1}{100}$" would be omitted in the first example.

<u>Answers to Problems 15-11a</u> Student Text Page 197

1. $90\% = \frac{90}{100} = 90 \times \frac{1}{100} = 90$ per cent. Similarly for the others.

<u>Answers to Problems 15-11b</u> Student Text Page 198

1.

(a)	$66\frac{2}{3}\%$	(f)	400%	(k)	60%
(b)	50%	(g)	$33\frac{1}{3}\%$	(l)	80%
(c)	75%	(h)	$166\frac{2}{3}\%$	(m)	100%
(d)	$133\frac{1}{3}\%$	(i)	20%	(n)	$53\frac{1}{3}\%$
(e)	140%	(j)	40 %	(o)	120%

2. (a) $\frac{7}{200}$ (c) $\frac{13}{20}$

(b) $\frac{1}{3}$ (d) $\frac{11}{200}$

3. (a) £4 2s.6d. (d) £23 1s.0d.

(b) £4 4s.0d. (e) $187.57

(c) £50 8s.0d. (f) $12.48

4. $33 = \frac{11}{100} \times 300 = 11$ per cent of 300

5. 16%

6. $67\frac{23}{31}$%

7. $78\frac{4}{7}$%; 28.

8. 5,000

9. $18\frac{1}{3}$%

10. 522

11. $31\frac{13}{17}$%

12. 50%

15-12 Percentages and Decimal Numerals

The pupils already know how to convert fractions into percentages and vice versa. The aim of this section is to show how we can convert percentages into decimal numerals and vice versa.

The conversion is easily done by using the fraction form as an intermediate step. Thus it is a short step from 45% to $\frac{45}{100}$ and another short step from $\frac{45}{100}$ to 0·45. Point out the following pattern of converting back and forth between the different forms.

Percentage ⟷ Fraction with denominator 100 ⟷ Decimal numeral

Answers to Problems 15-12

Student Text Page 200

1. (a) 32·5%

(b) 5%

(c) 230%

(d) 0·8%

(e) $85\frac{5}{7}$%

2. (a) 0·33 (d) 0·125
(b) 0·005 (e) 2·5
(c) 0·015 (f) 0·75

15-13 Application of Percentages

You might begin by discussing why anyone goes into business. Many reasons may be given for this, but it is generally true that a businessman wishes to make a profit. He buys and sells, and hopes to sell for more than his cost.

You may wish to give some oral exercises in profit and loss:

(a) If I buy a dozen eggs for 1s. 9d. and sell them at 2d. each, how much do I gain? Since I sell the eggs for 12 × 2d. or 2s., I gain the difference between 2s. and 1s. 9d. I gain 3d.

(b) If I buy six exercise books for 3s.6d., how much do I gain or lose if I sell them at 5d. each? I lose 1s. since I sell them for 6 × 5d. or 2s.6d. and 3s.6d. - 2s.6d. is 1s.

Lead the class to the next topic by asking the class which is the better transaction, buying a book for 10s. and selling it for 10s.6d., or buying a book for 5s. and selling it for 5s.6d?

The person who invested in the first book could have used the money to buy two of the second books and in that way gain 1s. Thus, the second transaction is better than the first.

Not every set of transactions may be so clear. Which is the best transaction, buying a book for 9s. and selling it for 12s., or buying a book for 8s. and selling it for 10s., or buying a book for 10s. and selling it for 13s.6d.? We shall be able to find the best transaction if we compare the ratios $\frac{3}{9}$, $\frac{2}{8}$, and $\frac{3\frac{1}{2}}{10}$.

Hence, the ratio of the profit to the cost price gives us an adequate basis of comparison. We express this ratio as a percentage, so that comparison is easy.

Answers to Problems 15-13a

Student Text Page 202

1. $14\frac{2}{7}\%$
2. $62\frac{1}{2}\%$
3. $22\frac{8}{11}\%$
4. £780
5. 8s.$7\frac{1}{2}$d.
6. £150
7. £366 13s.4d.
8. $18\frac{3}{4}\%$
9. 4s.6d.
10. 2d.

Commissions and Discounts

The teacher may prepare for this lesson by obtaining local material on commission and discounts. Shops or newspaper offices may supply this. It will be of value if the students could have a practical notion at first hand of commissions and discounts.

Answers to Problems 15-13b

Student Text Page 204

1. £540
2. shs.137.50
3. £600
4. £900
5. shs.728.38
6. $14\frac{1}{16}\%$

Simple Interest

Many boys and girls use the Post Office bank system. The Post Office uses compound interest to compute amounts. This interest system should be explained to the pupils, and also the reason that simple interest is studied as a prelude to compound interest.

Answers to Problems 15-13c

Student Text Page 205

1. (a) £45
   (b) £73 11s.$4\frac{4}{5}$d.
   (c) £62 10s.
   (d) 56 dollars 25 cents = \$56.25
   (e) 100 francs

2. 1 year 9 months
3. $3\frac{1}{2}$%
4. $14\frac{2}{7}$ years
5. shs.3,000
6. £675
7. shs.105
8. $12\frac{1}{2}$%
9. shs.3,666·67

Answers to Revision Problems

Student Text Page 206

1. (a) 35·63
   (b) 403·107
   (c) 1·0103
   (d) 0·07705
   (e) 9090·909
   (f) 58·087
   (g) 105·0077
   (h) 3·001003

2. (a) $1 + 2\left(\frac{1}{10^2}\right) + 4\left(\frac{1}{10^3}\right) + 9\left(\frac{1}{10^4}\right)$
   (b) $5 + 3\left(\frac{1}{10^2}\right)$
   (c) $3(10^4) + 7(10^3) + 3(10) + 5(1) + 4\left(\frac{1}{10}\right) + 4\left(\frac{1}{10^2}\right) + 9\left(\frac{1}{10^3}\right)$
   (d) $7\left(\frac{1}{10^3}\right) + 5\left(\frac{1}{10^4}\right)$
   (e) $8\left(\frac{1}{10^3}\right) + 2\left(\frac{1}{10^4}\right) + 1\left(\frac{1}{10^5}\right)$
   (f) $1(10^3) + 1\left(\frac{1}{10^4}\right)$

3. (a) $x = 1$

(b) $x = 10^2 = 100$

(c) $x = \frac{1}{10^3} = \frac{1}{1000}$

(d) $x = \frac{1}{10}$

(e) $x = \frac{1}{10^7} = \frac{1}{10000000}$

(f) $x = \frac{1}{10^4} = \frac{1}{10000}$

(g) $x = 10^4 = 10000$

(h) $x = 10^2 = 100$

4. (a) 2·2 inches (b) 2·5 inches (c) 1·0 inch

5. £25 12s.6d.

6. 220 yards

7. 437 yards 16 inches

8. 547 miles

9. (a) 8808·6 miles (b) 2602·9 miles (c) 3714·1 miles

10. (a) 0·45 (b) 0·125 (c) 0·075

11. (a) 0·25 (c) 0·85 (e) 3·05
(b) 0·50 (d) 0·025 (f) 1·125

12. 50·4 cwt.

13. £9

14. 3,280,000

CHAPTER 16

REAL NUMBERS

This chapter has a rather different emphasis from the others in the text. There are fewer problems than usual and emphasis is more on ideas. The problems are chosen to illustrate the ideas rather than to teach procedures which the pupils will often use.

The pupil knows about the rational numbers. He has learned that these correspond to points on the number line. But there are points on the number line which do not correspond to any rational number. A non-terminating, non-repeating decimal numeral like 0·1010010001... locates a point but it does not represent a rational number. The numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, ... locate points on the number line but they are no rational numbers. Ideas like these increase the pupils' understanding. They are particularly important if one goes on in mathematics because they form a good background for the idea of limit.

16-1 Decimal Numerals for Rational Numbers

In this section, the pupil will learn that a rational number, $\frac{a}{b}$ (a, b integers and $b \neq 0$) corresponds to a decimal numeral (or decimal, for short) which either terminates (for example $\frac{1}{4} = 0{\cdot}25$) or repeats without end (for example $\frac{1}{3} = 0{\cdot}333...$).

Conversely, he will learn that a decimal that terminates or repeats corresponds to a rational number. He will enjoy the method by which the rational fraction actually can be found.

You should remind the pupils that the numeral 1 may be written as 1·000... etc., and in general a fraction can be represented as a decimal numeral by dividing the numerator by the denominator.

Answers to Problems 16-1a

Student Text Page 209

1. (a) 0·5
   (b) 0·25
   (c) 0·75
   (d) 0·375
   (e) 0·075
   (f) 0·1625

2. (a) $\frac{1}{4}$
   (b) $\frac{7}{4}$
   (c) $\frac{14}{5}$
   (d) $\frac{707}{500}$

Repeating Decimal Numerals

Ask the class to find the decimal numerals for $\frac{1}{3}$ and $\frac{1}{9}$. Ask the pupils to give the digit in the 10th or the 20th position. Point out to the class that a decimal is not completely known until it is possible to state the digit in any position.

Now ask the class how they could indicate that a digit, or a set of digits, repeats indefinitely, that is, how one might abbreviate the following:

0·333...	(all the digits are 3's)
0·111...	(all the digits are 1's)
0·232323...	(all digits are alternately 2's and 3's)

The pupils may wish to invent their own ways of doing this. You may then use the pupils' notation for a time before leading the class to the standard notation:

$$0{\cdot}333\ldots = 0{\cdot}\dot{3}$$
$$0{\cdot}111\ldots = 0{\cdot}\dot{1}$$
$$0{\cdot}2323\ldots = 0{\cdot}\dot{2}\dot{3}$$

In the division for $\frac{6}{37}$ we must come to the same remainder sooner or later, because the remainder must be less than 37. There are only 36 possible different remainders.

Of course it is possible that there will be some digits at the start of the decimal numeral which are not part of the repeating pattern. For example $\frac{478}{130} = 3.6\dot{7}6923\dot{0}$.

Answers to Problems 16-1b

Student Text Page 210

1. $0.\dot{1}$
2. $0.\dot{0}\dot{1}$
3. $0.\dot{0}0\dot{1}$
4. $0.\dot{6}$
5. $0.\dot{1}4285\dot{7}$
6. $0.\dot{7}1428\dot{5}$
7. $0.\dot{0}\dot{9}$
8. $0.\dot{2}\dot{7}$
9. $0.\dot{0}2\dot{7}$
10. $0.\dot{0}731\dot{7}$

Terminating Decimals and Repeating Decimals

Lead the pupils to see, by means of some examples, that terminating decimals are formed only from rational numbers whose denominators have no prime factors other than 2 or 5. Give them examples of terminating decimal numerals like 0·58, 0·96, and 0·43217. If these decimal numerals are expressed as fractions, their denominators will be powers of 10 and hence have only 2 and 5 as prime factors. When reduced to lowest terms, one or the other of these factors may be missing. For example, $\frac{25}{100} = \frac{5 \times 5}{2 \times 5 \times 2 \times 5} = \frac{1}{2 \times 2}$. In any case there will be no factors in the denominator other than 2 and 5.

The converse also is true. If a fraction has a denominator with no prime factors other than 2 or 5, this denominator can easily be converted into a power of 10 so that the fraction is represented as a terminating decimal.

For example, $\frac{17}{2^2 \times 5^4} = \frac{2^2 \times 17}{2^4 \times 5^4} = \frac{4 \times 17}{10^4} = 0{\cdot}0068$

Incidentally, if a fraction has a denominator in which neither 2 nor 5 occurs as a factor, and numerator 1, the decimal repeats from the beginning.

For example, $\frac{1}{3} = 0\cdot\dot{3}$ and $\frac{1}{7} = 0\cdot\dot{1}4285\dot{7}$

If the denominator has 2 or 5 as a factor and at least one factor other than 2 or 5, the decimal repeats but not from the beginning.

For example, $\frac{1}{6} = \frac{1}{2 \times 3} = 0\cdot1\dot{6}$

The fact that all rational numbers which cannot be expressed as <u>terminating</u> decimal numerals can be expressed as <u>repeating</u> decimal numerals has been explained briefly above for $\frac{6}{37}$. Each remainder in the division process must be less than the divisor, and since the divisor is finite, there is a finite number of different remainders. For a non-terminating decimal the remainder cannot be zero. If the divisor is n, there can be at most (n - 1) different remainders.

Illustrate the discussion by expressing $\frac{1}{7}$ as a decimal numeral. The remainders, in order, are 3, 2, 6, 4, 5, 1 and the next remainder must be one of these numbers; it turns out to be 3.

Discuss carefully the ideas in Problems 16-1c, so the pupils will be ready to go on to the following material easily.

<u>Answers to Problems 16-1c</u> Student Text Page 211

1. (a) 10 (d) 100000
   (b) 100 (e) 1000
   (c) 1000

2. (a) $3\cdot45\dot{3}4\dot{5}$ (b) $34\cdot5\dot{3}4\dot{5}$ (c) $345\cdot\dot{3}4\dot{5}$

3. (a) 9x (b) 99x (c) 999x

4. (a) 8 (b) 93476 (c) 7245 (d) 86.4

The process for finding the rational fraction which corresponds to a repeating decimal, strictly speaking, requires some justification.

Let us illustrate by an example.

In setting $x = 0{\cdot}11111\ldots$ and multiplying by 10 to obtain

$$10x = 1{\cdot}1111\ldots$$

we assume that even though the 1's go on forever, we can multiply by 10 by moving the decimal point one place to the right. This is surely true when the decimal terminates as we saw in the previous chapter. The justification can be given in the case of a non-terminating decimal, but this requires some discussion of infinite series. The pupil is unlikely to worry about this. If he does he can be told that if he goes on with mathematics this matter will be treated fully; or you can observe that

$$\begin{array}{ll} \text{if} & 0{\cdot}11111 < x < 0{\cdot}111111 \\ \text{then} & 1{\cdot}1111 < 10x < 1{\cdot}11111 \end{array}$$

which should convince him.

Any rational fraction found by the method of the text can be divided out to give the original repeating decimal so that in any particular case the pupil can verify the correctness of his result.

The example $0{\cdot}\dot{9} = 0{\cdot}99999\ldots$ is often puzzling to the pupil. He may have difficulty in believing that this decimal represents 1. Of course, the method of the text gives this result.

Thus

$$\begin{aligned} x &= 0{\cdot}\dot{9} \\ 10x &= 9{\cdot}\dot{9} \\ 9x &= 9 \\ x &= 1 \end{aligned}$$

It is worth pointing out that if

$$\frac{1}{3} = 0{\cdot}333\ldots$$

$$\frac{1}{3} \times 3 = 1 = 0{\cdot}999\ldots$$

After studying Section 16-2, the pupil should be able to see that $0{\cdot}\dot{9}$ would not correspond to any point on the number line to the left of the point 1.

Answers to Problems 16-1d

Student Text Page 213

1. (a) $\frac{14}{3}$ (d) $\frac{1}{7}$

(b) $\frac{7}{8}$ (e) $\frac{36}{11}$

(c) $\frac{14}{99}$ (f) $\frac{115}{99}$

16-2 Plotting Repeating Decimals on the Number Line

This section shows how a decimal numeral, even if unending, specifies a definite point on the number line. It should stretch the pupils' imaginations.

16-3 Numbers which are not Rational

In this section, the pupil will learn that there are unending decimals which do not repeat. Since they do not repeat, they cannot represent rational numbers. They represent irrational numbers. They are called irrational because they cannot be written as ratios of integers. These irrational numbers also correspond to points on the number line.

16-4 The Square Root of 2

There is a definite point on the number line that corresponds to $\sqrt{2}$. If there were not, the circle of radius $\sqrt{2}$ and centre O would not intersect the line at C.

We could draw the entire circle. However, we would then have to explain the other point of intersection at $-\sqrt{2}$. It seems best to postpone negative irrational numbers to a later section.

The method explained here of obtaining fractions which approximate $\sqrt{2}$ is widely used in programming computing machines. It rests, first of all, on the fact that if a and b are positive numbers such that

$$a \times b = 2 = \sqrt{2} \times \sqrt{2}$$

and a is less than $\sqrt{2}$, then b must be greater than $\sqrt{2}$. Then with a too small and b too large, when we take $\frac{a+b}{2}$ we find a number halfway between a and b which is our next approximation to $\sqrt{2}$. Repeated use of this procedure narrows down on $\sqrt{2}$ very rapidly.

This method will be used in the next section.

You may be familiar with another method of extracting square roots which has been used in the schools. Very few pupils understand why this method works. The present method is very effective and at the same time is easy to understand.

Answers to Problems 16-4

Student Text Page 219

1. The pupils will probably be unable to obtain the following degree of accuracy, unless they use a very large scale. Accept the nearest tenth as a satisfactory approximation.

$\sqrt{2} \approx 1\cdot41$	$\sqrt{3} \approx 1\cdot73$	$\sqrt{4} = 2$	$\sqrt{5} \approx 2\cdot24$
$\sqrt{6} \approx 2\cdot45$	$\sqrt{7} \approx 2\cdot65$	$\sqrt{8} \approx 2\cdot83$	$\sqrt{9} = 3$

This is a rather amusing geometric construction of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$... in succession. Notice that $\sqrt{4}$, $\sqrt{9}$, $\sqrt{16}$... are whole numbers. As we shall see in Section 16-5, the others are irrational.

2. The particular fractions found will vary with choice of a first approximation. In any case, the third approximation, when written as a decimal numeral, may be correct as far as the fourth digit

(a)	$\sqrt{7}$:	$\frac{13}{5}, \frac{35}{13}, \frac{172}{65}$	2·646
(b)	$\sqrt{11}$:	$\frac{33}{10}, \frac{110}{33}, \frac{2189}{600}$	3·317
(c)	$\sqrt{13}$:	$\frac{7}{2}, \frac{26}{7}, \frac{101}{28}$	3·606
(d)	$\sqrt{17}$:	$\frac{41}{10}, \frac{170}{41}, \frac{3381}{820}$	4·123

16-5 Proof that $\sqrt{2}$ is an Irrational Number

We have given a famous proof. This is real mathematics. The good pupil should get a stimulus from this proof. It is worth spending some time on it.

The actual proof begins with $\frac{a}{b} = \sqrt{2}$ and is fairly short. A good pupil could perhaps begin there. It seems desirable however to begin with numerical illustrations. If necessary the teacher can illustrate the argument by using other fractions, for example if $\sqrt{2}$ were $\frac{10}{7}$ we would have

$$\frac{10 \times 10}{7 \times 7} = 2$$

$$\text{and} \quad 10 \times 10 = 2 \times 7 \times 7$$

This is impossible since there are two factors 2 on the left-hand side and only one factor 2 on the right-hand side.

The proof is based on the Fundamental Theorem of Arithmetic (see Section 4-3 in Secondary One), which says that a given whole number which is not prime can be written as the product of prime factors in only one way, apart from the order in which the factors are written.

Example:

$$75 = 5 \times 15 = 5 \times 5 \times 3$$
$$75 = 3 \times 25 = 3 \times 5 \times 5$$

Answers to Problems 16-5

Student Text Page 222

1. (a) If $\frac{a}{b} = \sqrt{3}$, $3 = \frac{a^2}{b^2}$, $a^2 = 3b^2$, $a \times a = 3 \times b \times b$.

On the left-hand side there would be an even number of factors 3 (zero, two, four, six, ...). On the right-hand side, there would be an odd number of factors 3 (one, three, five,...). Thus $a \times a$ has an even number of 3's and $3 \times b \times b$ has an odd number of 3's.

Since, by the Fundamental Theorem of Arithmetic, a whole number can be expressed as a product of prime factors in only one way, $a \times a = 3 \times b \times b$ cannot be true for any whole numbers a and b. The equation $\frac{a}{b} = \sqrt{3}$ therefore cannot be satisfied by any whole numbers a and b.

(b) If $\frac{a}{b} = \sqrt{5}$, $\frac{a^2}{b^2} = 5$, $a^2 = 5b^2$ and $a \times a = 5 \times b \times b$.

Consider how many factors 5 there could be on the left and right-hand sides. On the left-hand side the possibilities are: zero, two, four...; on the right-hand side: one, three, five... The equation $\frac{a}{b} = \sqrt{5}$ cannot be true for any whole numbers a and b.

2. As before, if we assume that $\frac{a}{b} = \sqrt{4}$ we get

$$a \times a = 4 \times b \times b$$

We do not count the number of factors 4 because 4 is not a prime number. We write

$$a \times a = 2 \times 2 \times b \times b$$

and consider the number of factors 2. This number must be even on both sides. If we take $a = 2$ and $b = 1$ everything works out. Hence we run into no contradiction and cannot prove that $\sqrt{4}$ is irrational.

Another Example.

Prove that $\sqrt{6}$ is irrational.

From $\frac{a}{b} = \sqrt{6}$ we are led to

$$a \times a = 6 \times b \times b = 2 \times 3 \times b \times b.$$

We can consider either the number of factors 2 on both sides of the equation, or the number of factors 3, and carry through the usual argument.

16-6 The Real Numbers

Every decimal corresponds to a definite point on the number line.

The terminating and repeating decimals represent the rational numbers. The other decimals (non-repeating and non-terminating) represent irrational numbers. The union of the set of rational numbers and the set of irrational numbers is the set of real numbers.

Does every point on the number line correspond to a single decimal? There is a slight difficulty here. As we have seen $0{\cdot}999\ldots$ (or $0{\cdot}\dot{9}$) and 1 locate exactly the same point. Similarly $0{\cdot}499\ldots$ (or $0{\cdot}4\dot{9}$) and $0{\cdot}5$ locate the same point. If we decide not to use unending decimals which ultimately have only 9's, then every point on the number line has exactly one corresponding decimal. Decimals of this type will not arise if we follow the procedure described in the text.

Answers to Problems 16-6

Student Text Page 223

	Real	Rational	Irrational	Positive	Negative
$\sqrt{3}$	yes	no	yes	yes	no
3	yes	yes	no	yes	no
-3	yes	yes	no	no	yes
$\frac{4}{7}$	yes	yes	no	yes	no
$-0{\cdot}\dot{8}$	yes	yes	no	no	yes
$0{\cdot}\dot{1}\dot{7}$	yes	yes	no	yes	no

16-7 Negative Real Numbers

We suggest that not much time be spent on this section. The problems can be done orally in class.

We have used "opposite" in preference to "additive inverse" but you should gradually accustom the pupils to the term "additive inverse" which they will meet in later work.

Answers to Problems 16-7 Student Text Page 225

1. (a) $-3\cdot205$ (c) $-\sqrt{3}$
   (b) $1\cdot23$ (d) $-0\cdot2020020002\ldots$
2. (a) $-\frac{3}{2}$ (c) $-\sqrt{6}$
   (b) $0\cdot111\ldots$ (d) $0\cdot12345\ldots$

Note: When we write a number a with ..., it is understood that in writing its opposite $-a$, the dots stand for the same digits as before.

16-8 Addition of Real Numbers

Addition of real numbers is looked at from the point of view of:

1. Adding two rational numbers in decimal notation,
2. Adding numbers in decimal notation when one or both of them are irrational,
3. Interpreting these sums on the number line,
4. Adding square roots on the number line,
5. Considering whether a sum of real numbers will be rational or irrational.

Answers to Problems 16-8a

Student Text Page 225

1. (a) 9·55 (c) 8·48

(b) 9·553 (d) 13·891

2. $\frac{275}{999} + \frac{714}{999} = \frac{989}{999} = 0\cdot\dot{9}\dot{8}\dot{9}$

Answers to Problems 16-8b

Student Text Page 226

1.
```
 4·1234567891...
 5·8910111213...
10·0144679104...
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2.
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3·1010010001...
6·1525354555...
9·2535364556...
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3.
```
3·0101101110...
4·1020030004...
7·1121131114...
```

Note: If the pupil finds some other pattern than the one intended here, his results should be accepted.

Addition of Square Roots on the Number Line

We point out incidentally the distinction between $\sqrt{(2+3)}$ and $\sqrt{2} + \sqrt{3}$. In $\sqrt{(2+3)}$ we add first and then take the square root. In $\sqrt{2} + \sqrt{3}$ we take the square roots first and then add. The results are quite different.

Answers to Problems 16-8c

Student Text Page 227

1. 1·4142

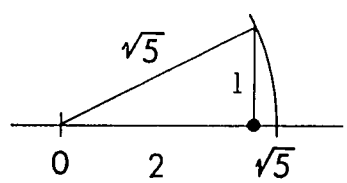
2. 1·7320

3. 3·1462

4. 2·2360

5. No. $2.2360\ldots \neq 3.1462\ldots$

6.



The Sum of a Rational Number and an Irrational Number is an Irrational Number

We write $\sqrt{2} = \frac{a - 3b}{b}$, where a and b are integers and $b \neq 0$. Since the integers are closed under the operation of subtraction, we know $a - 3b$ stands for an integer. Hence, $\frac{a - 3b}{b}$ is a rational number, being the ratio of two integers.

In general the sum of any rational number and any irrational number is irrational.

The formal proof is as follows:

Let R be any rational number and I any irrational number.
Suppose $I + R = R'$
where R' is a rational number.
Then $I = R' - R$.
But, the rational numbers are closed under the operation of subtraction.

That is, the difference of two rational numbers is a rational number. We have a contradiction so $I + R$ cannot be rational. It must be irrational.

Answers to Problems 16-8d

Student Text Page 228

1. (a) $\sqrt{3}$, $5 + \sqrt{3}$
   (b) $-0 \cdot 000100001000001...$, $-1 \cdot 000100001000001...$
   (c) $-0 \cdot 030030003...$, $-1 \cdot 030030003...$
   (d) $-\sqrt{5}$, $1 - \sqrt{5}$
   (e) $-0 \cdot 123456...$, $-1 \cdot 123456...$
   (f) $-0 \cdot 789101112...$, $-1 \cdot 789101112...$

2. (a) $7 \cdot 81$
   (b) $4 \cdot 77$
   (c) $9 \cdot \dot{7}$
   (d) $5 \cdot \dot{7}\dot{7}\dot{8}$
   (e) $10 + \sqrt{3} - \sqrt{5}$
   (f) $5 \cdot 4444... = 5 \cdot \dot{4}$

3. (a) $\sqrt{3 + 5} \neq \sqrt{3} + \sqrt{5}$, $\sqrt{3 + 5} = \sqrt{8}$
   (b) $\sqrt{2 + 5} \neq \sqrt{2} + \sqrt{5}$, $\sqrt{2 + 5} = \sqrt{7}$

In general, $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$ unless $a = 0$ or $b = 0$

For example

$$\sqrt{9} + \sqrt{16} \neq \sqrt{25}$$
$$3 + 4 \neq 5$$

If we assume that

$$\sqrt{a} + \sqrt{b} = \sqrt{a+b}$$

and square both sides of the equation, we obtain

$$a + 2\sqrt{a} \times \sqrt{b} + b = a + b$$

from which it follows that $2 \times \sqrt{a} \times \sqrt{b} = 0$.

This is impossible unless $a = 0$ or $b = 0$ (or both).

4. If $\sqrt{\sqrt{2} + 1}$ were rational, its square $(\sqrt{2} + 1)$ would have to be rational. This is impossible because we know that the sum of a rational number and an irrational number is irrational.

16-9 Order of Real Numbers

The decimal representation of irrational numbers is infinitely long. To compare two irrational numbers, and tell which is larger, we need not ordinarily go far. Certainly $4{\cdot}10100100001\ldots$ is less than $5{\cdot}090090009\ldots$ simply because a number between 4 and 5 is less than a number greater than 5; and $0{\cdot}4321\ldots$ is less than $0{\cdot}435\ldots$ because $0{\cdot}435$ whatever digit comes in the fourth decimal place.

It is suggested in the problems that Property 3 can be used to help compare two real numbers. If you so wish you can incorporate this idea in the text discussion. Thus

$$a = 0{\cdot}32842\ldots = 0{\cdot}32 + 0{\cdot}00842\ldots$$
$$b = 0{\cdot}32719\ldots = 0{\cdot}32 + 0{\cdot}00719\ldots$$

We know $0{\cdot}00719... < 0{\cdot}00842...$

therefore $0{\cdot}32 + 0{\cdot}00719... < 0{\cdot}32 + 0{\cdot}00842...$ by Property 3.

Answers to Problems 16-9

Student Text Page 230

1. (a) $0{\cdot}432... < 0{\cdot}437...$ (c) $-0{\cdot}128... < -0{\cdot}123...$

   (b) $3{\cdot}824... > 3{\cdot}816...$ (d) $-0{\cdot}128... < 0{\cdot}001...$

2. Since $0{\cdot}002... < 0{\cdot}007...$

   $0{\cdot}43 + 0{\cdot}002... < 0{\cdot}43 + 0{\cdot}007...$ by Property 3.

   That is $0{\cdot}432... < 0{\cdot}437...$

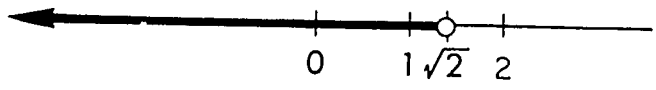
3. $3{\cdot}824... = 3{\cdot}8 + 0{\cdot}024...$

   $3{\cdot}816... = 3{\cdot}8 + 0{\cdot}016...$

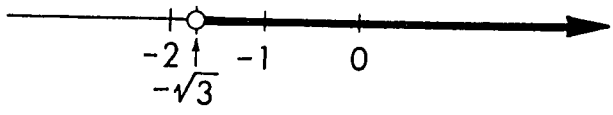
   Since $0{\cdot}024... > 0{\cdot}016$

   $3{\cdot}824... > 3{\cdot}816...$

4. (a)



   (b)



16-10 Multiplying Real Numbers

We observe the basic property of square roots that, for non-negative numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$. We observe also that for real numbers, just as it was for rational numbers, the product of a positive

real number and a negative real number is a negative real number, and the product of two negative real numbers is a positive real number.

Proof that $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$

$$(\sqrt{a} \times \sqrt{b})^2 = (\sqrt{a} \times \sqrt{b}) \times (\sqrt{a} \times \sqrt{b})$$ Definition of Square

$$\left.\begin{aligned} &= [(\sqrt{a} \times \sqrt{b}) \times \sqrt{a}] \times \sqrt{b} \\ &= [\sqrt{a} \times (\sqrt{b} \times \sqrt{a})] \times \sqrt{b} \\ &= [\sqrt{a} \times (\sqrt{a} \times \sqrt{b})] \times \sqrt{b} \\ &= [(\sqrt{a} \times \sqrt{a}) \times \sqrt{b}] \times \sqrt{b} \\ &= (\sqrt{a} \times \sqrt{a}) \times (\sqrt{b} \times \sqrt{b}) \end{aligned}\right\}$$ Associative and commutative properties of multiplication.

$$= (\sqrt{a})^2 \times (\sqrt{b})^2$$

$$= a \times b$$ Definition of square root.

Since $(\sqrt{a} \times \sqrt{b})^2 = a \times b$

$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$

by the definition of square root.

Proof that $\sqrt{2} \times (-\sqrt{2}) = -2$

$$\sqrt{2} \times [\sqrt{2} + (-\sqrt{2})] = \sqrt{2} \times 0 = 0$$ Addition property of opposites and multiplication property of zero.

$$[\sqrt{2} \times \sqrt{2}] + [(\sqrt{2}) \times (-\sqrt{2})] = 0$$ Distributive property.

$$2 + [(\sqrt{2}) \times (-\sqrt{2})] = 0$$ Definition of square root.

Therefore

$$(\sqrt{2}) \times (-\sqrt{2}) = -2$$ Uniqueness of the additive inverse. Any number which, added to 2, gives zero, must be the opposite of 2.

Proof that $(-\sqrt{2}) \times (-\sqrt{2}) = 2$

$$(-\sqrt{2}) \times [\sqrt{2} + (-\sqrt{2})] = 0$$

$$[(-\sqrt{2}) \times (\sqrt{2})] + [(-\sqrt{2}) \times (-\sqrt{2})] = 0$$

$$-2 + [(-\sqrt{2}) \times (-\sqrt{2})] = 0$$

Therefore $(-\sqrt{2}) \times (-\sqrt{2}) = 2$

Answers to Problems 16-10

Student Text Page 233

1. $a \times b > (0\cdot1) \times (0\cdot3) = 0\cdot03$

$> (0\cdot11) \times (0\cdot33) = 0\cdot0363$

$> (0\cdot111) \times (0\cdot333) = 0\cdot036963$

$a \times b < (0\cdot2) \times (0\cdot4) = 0\cdot08$

$< (0\cdot12) \times (0\cdot34) = 0\cdot0408$

$< (0\cdot112) \times 0.334 = 0\cdot037408$

2. $\frac{1}{9} \times \frac{1}{3} = \frac{1}{27} = 0\cdot037037\ldots$

3. $\sqrt{2} \times \sqrt{3} > (1\cdot414) \times (1\cdot732) = 2\cdot449048$

$< (1\cdot415) \times (1\cdot733) = 2\cdot452195$

$\sqrt{6} = 2\cdot44948\ldots$

4. $\sqrt{3}(\sqrt{2} + 1) = [\sqrt{3} \times \sqrt{2}] + \sqrt{3}$

$= \sqrt{6} + \sqrt{3}$

5. $(\sqrt{2} - 1) \times (\sqrt{2} + 1) = [\sqrt{2} \times (\sqrt{2} + 1)] - [1 \times (\sqrt{2} + 1)]$

$= 2 + \sqrt{2} - \sqrt{2} - 1 = 1$

6. $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2}) = [\sqrt{5} \times (\sqrt{5} - \sqrt{2})] + [\sqrt{2} \times (\sqrt{5} - \sqrt{2})]$

$= 5 - \sqrt{10} + \sqrt{10} - 2 = 3$

7. CHALLENGE PROBLEM

$$\left(\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}\right)^2 = 3 + 2\sqrt{2} - 2\left(\sqrt{3 + 2\sqrt{2}}\right)\left(\sqrt{3 - 2\sqrt{2}}\right) + 3 - 2\sqrt{2}$$

$$= 6 - 2\sqrt{9 - 8}$$

$$= 6 - 2\sqrt{1} = 6 - 2 = 4$$

Since $\left(\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}\right)^2 = 4$,

$\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}$ must be 2 or -2.

It must be the positive number because

$3 + 2\sqrt{2} > 3 - 2\sqrt{2}$. Hence $\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}} = 2$.

8. CHALLENGE PROBLEM

$$\left(\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}\right)^2 = 2+\sqrt{3}+2\sqrt{(2+\sqrt{3})\times(2-\sqrt{3})}+2-\sqrt{3}$$
$$= 4+2\sqrt{4-3} = 4+2 = 6$$

Reasoning as in Problem 7 we see that $\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}} = \sqrt{6}.$

9. CHALLENGE PROBLEM

$$\left(\sqrt{2+\sqrt{3}}-\sqrt{2-\sqrt{3}}\right)^2 = 2+\sqrt{3}-2\sqrt{(2+\sqrt{3})\times(2-\sqrt{3})}+2-\sqrt{3}$$
$$= 4-2\sqrt{4-3} = 2$$

Reasoning as in Problem 7 we see that $\sqrt{2+\sqrt{3}}-\sqrt{2-\sqrt{3}} = \sqrt{2}.$

10. $\sqrt{2}$ and $2\sqrt{2}$ for example.

Their sum is $3\sqrt{2}$

Their product is 4.

16-11 Absolute Value

Answers to Problems 16-11

Student Text Page 236

1. (a) $0\cdot1234\ldots$

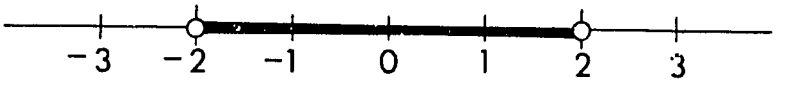
(b) $\left|\frac{4}{3}-\frac{3}{2}\right| = \frac{3}{2}-\frac{4}{3} = \frac{1}{6}$

(c) $\left|(-0\cdot111\ldots)\ -0\cdot01001\ \ldots\right| = \left|-0\cdot12112\ldots\right| = 0\cdot12112\ldots$

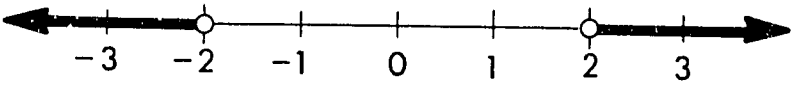
(d) $\left|-3\cdot68+3\cdot89\right| = \left|0\cdot21\right| = 0\cdot21$

2. $\{2,\ -2\}$

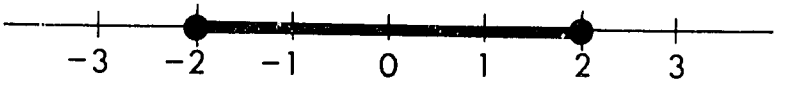
3. (a) $-2 < x < 2$



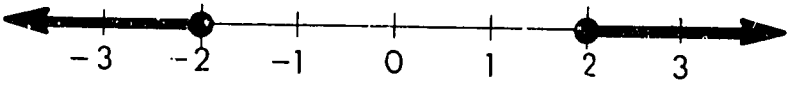
(b) $x < -2$ or $x > 2$



4. (a) $-2 \leq x \leq 2$



(b) $x \leq -2$ or $x \geq 2$



5. For $|6 - x| = 1$ to be true, $6 - x$ has to be 1 or -1.

If $6 - x = 1$, x is 5; if $6 - x = -1$, x is 7.

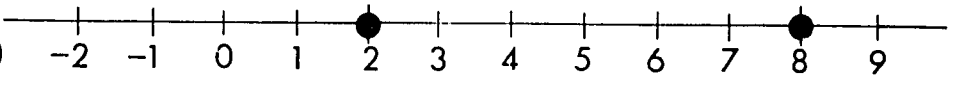
Hence the truth set is $\{5, 7\}$.

6. (a) $\{5, 7\}$ (c) $\{1, 3\}$

(b) $\{3, 9\}$ (d) $\{7, -3\}$

7.

(a)



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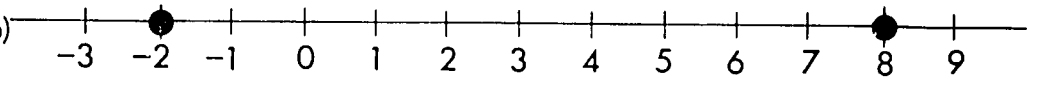

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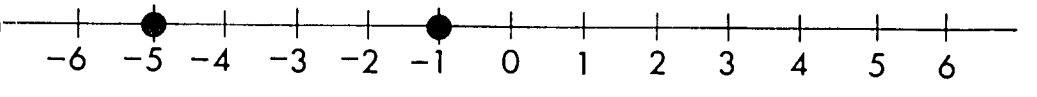

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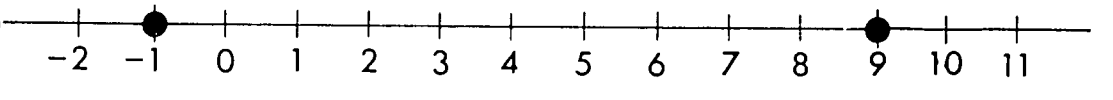

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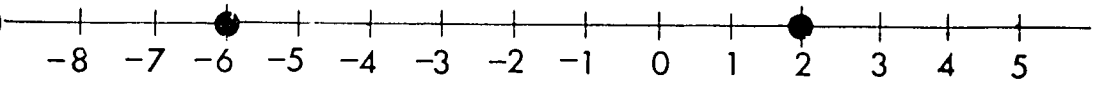

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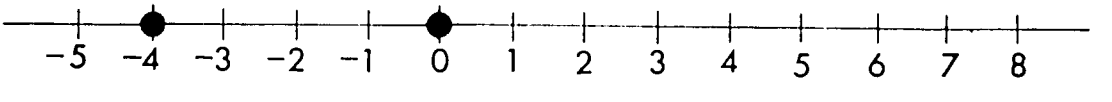
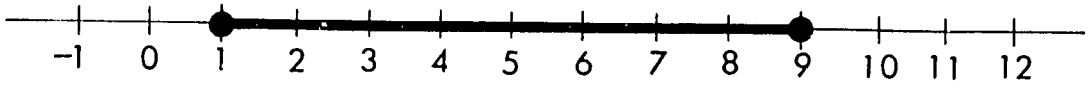
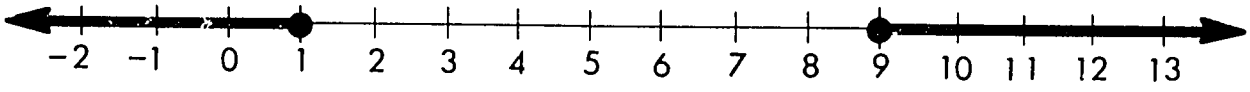


8. (a) $1 \leq x \leq 9$



(b) $x \leq 1$ or $x \geq 9$



16-12 More About Absolute Value

If you are pressed for time, this section can be omitted.

Answers to Problems 16-12

Student Text Page 239

1. (a) $a \times b = \frac{1}{8}$, $|a \times b| = \frac{1}{8}$, $|a| = \frac{1}{2}$, $|b| = \frac{1}{4}$
   (b) $a \times b = -\sqrt{6}$, $|a \times b| = \sqrt{6}$, $|a| = \sqrt{3}$, $|b| = \sqrt{2}$
   (c) $a \times b = -2$, $|a \times b| = 2$, $|a| = 1 + \sqrt{3}$, $|b| = \sqrt{3} - 1$

2. (a) $a = -2$, $b = -4$, $a + b = -6$
   $|a| = 2$, $|b| = 4$, $|a + b| = 6$

   (b) $a = 2$, $b = -2$, $a + b = 0$
   $|a| = 2$, $|b| = 2$, $|a + b| = 0$

   (c) $a = \sqrt{2} - 1$, $b = \sqrt{2} + 1$, $a + b = 2\sqrt{2}$
   $|a| = \sqrt{2} - 1$, $|b| = \sqrt{2} + 1$, $|a + b| = 2\sqrt{2}$
   $|a| + |b| = 2\sqrt{2}$

3. $|2|^2 = 4$ $|-2|^2 = 4$

   $|a|^2 = a^2$ because both of them are the non-negative number $a \times a$.
   If $|a|^2 = a^2$
   then $|a| = \sqrt{a^2}$ by the definition of square root.

4. $a = -5, \quad a^2 = 25, \quad \sqrt{a^2} = \sqrt{25} = 5 = |a|$

$a = \frac{1}{4}, \quad a^2 = \frac{1}{16}, \quad \sqrt{a^2} = \sqrt{\frac{1}{16}} = \frac{1}{4} = |a|$

$a = -\frac{1}{9}, \quad a^2 = \frac{1}{81}, \quad \sqrt{a^2} = \sqrt{\frac{1}{81}} = \frac{1}{9} = |a|$

$a = -\sqrt{3}, \quad a^2 = 3, \quad \sqrt{a^2} = \sqrt{3} = |a|$

5. $\sqrt{x^2} = |x|$ for all values of x.

$\sqrt{x^2} = x$ for $x \geq 0$.

$\sqrt{x^2} = -x$ for $x \leq 0$.

6. <u>CHALLENGE PROBLEM</u>

$$|a \times b| = \sqrt{(a \times b)^2} = \sqrt{a^2 \times b^2} = \sqrt{a^2} \times \sqrt{b^2} = |a| \times |b|$$

7. <u>CHALLENGE PROBLEM</u>

$$|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|$$

$$|(a - b) - a| \leq |a - b| + |a|$$

$$|-b| \leq |a - b| + |a|$$

$$|b| \leq |a - b| + |a|$$

$$|a - b| \geq |b| - |a|$$

CHAPTER 17

INTRODUCTION TO GRAPHING IN THE NUMBER PLANE

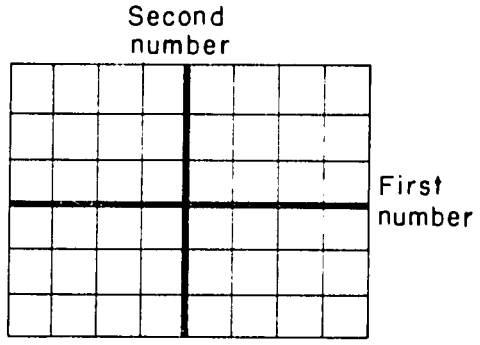
There is provision for much student activity in this chapter. While such activity is time-consuming and you will want to be careful not to spend too long on it, the ideas being presented are significant and deserve careful attention.

17-1 The Number Plane

In the first part of this section the pupils can learn simply and quickly to plot points on the number plane without any formal directions and without learning any ne' language. The language follows later. This part will take only a few minutes of clas time.

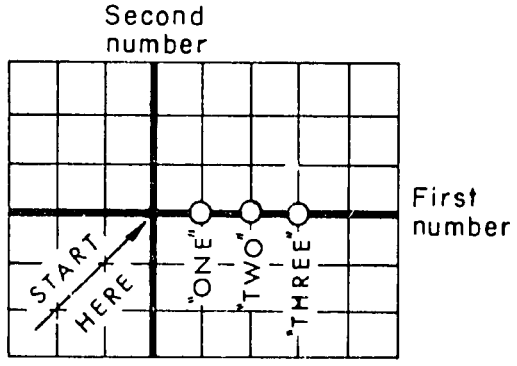
For some classes a brief discussion such as that on the first page of this section in the text will be enough. For classes which need more help you may want to proceed somewhat as follows:

1. Put this diagram on the board (or use a graph board, if you have one).



2. Ask a pupil to name two integers, not too large.

3. Starting at the point where the axes intersect, count over on the first-number axis, saying "One" at the first intersection, "Two" at the second intersection, etc. For example, if the student said "Three and five", the o's in the picture on the right indicate where you put



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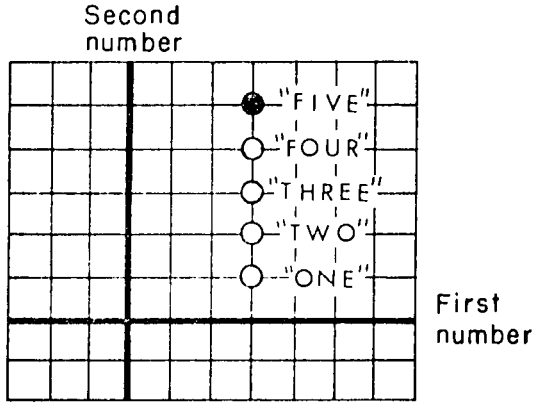
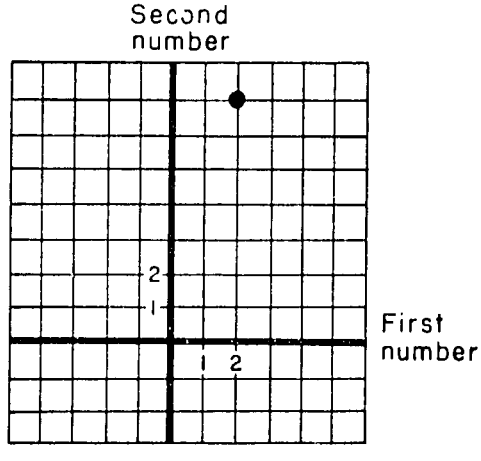
your chalk (but not making a mark on the board), and what you say is written below.

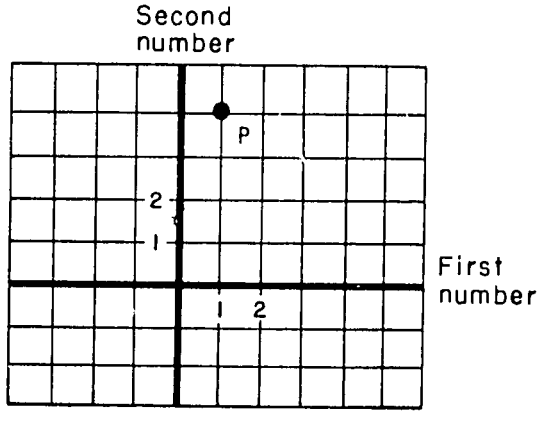
4. Holding your chalk where you said "Three", you might ask the student what the second number he named was. (You may remember, but some of the students may not.) After he has repeated "Five", count aloud up the intersections. When you reach the fifth intersection, mark it with a solid dot. If the student had said "Seven" or "Eight" as one of the numbers, you can add a line or two so that the point can be marked, but if he said "Twenty", you would probably want to ask him for a smaller number.



5. Ask another student to name another pair of integers. Repeat the same process, starting where the axes intersect, counting over for the first number, and up for the second. If the student said "Two and seven", the picture would now look like this:

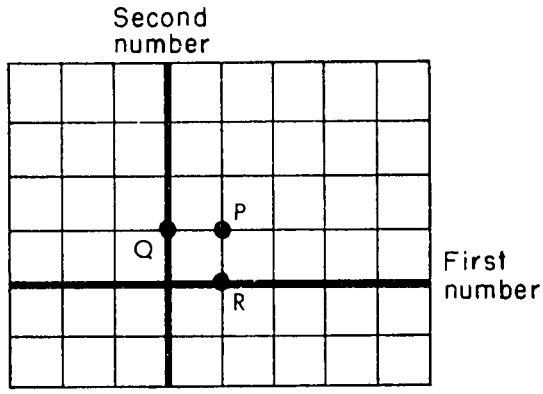


6. Then you can mark a point somewhere at an intersection, say at P, and ask, "What numbers go with this point?" Check the two numbers for the class by counting over and up as you did for the other pairs of numbers.



7. Mark a few more points and have someone in the class give you the pair of numbers that goes with each point, checking each one for them by counting over and up. Have one student name a pair of numbers and ask another to come to the board and mark the corresponding point.

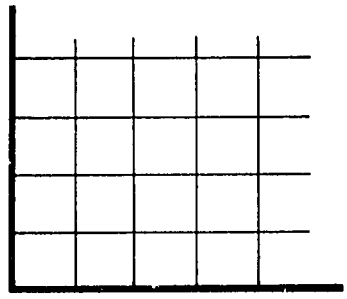
Note: The question of a point being marked on one of the axes, or one with either one or both coordinates negative, probably will not arise. This is all right because it can be introduced later. If it does arise at this time, it is probably best to take care of it here. For example, suppose a student wants to know what numbers correspond to the point Q. Ask someone what numbers he thinks correspond to Q. You may get an answer of "One, one". Mark "One, one" correctly at P, and ask for another guess. Suppose someone says "One, zero". Mark that correctly at R. The next number would probably be the one for Q: "Zero, one."



Dots and Crosses

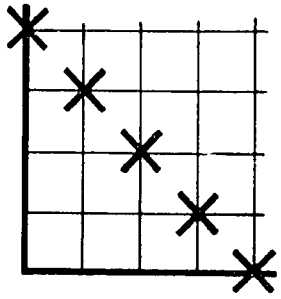
We describe here a game which you may wish to use with your class if they need practice in associating points of the number plane with their ordered pairs of numbers.

Game 1: Put this diagram on the board:

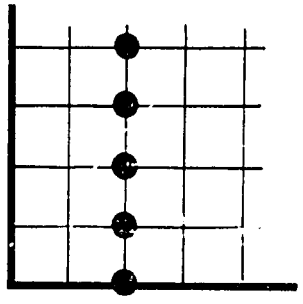


Split the class up into two teams--one can be the cross-team, the other the dot-team. The easiest way is to divide the class down the middle; those on the left are the

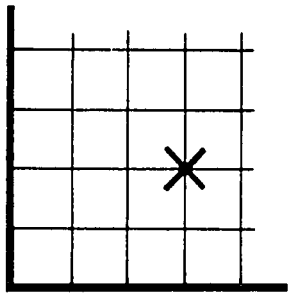
cross-team, on the right the dot-team. Tell the class that a team wins when it gets five of its marks in a line, playing on the intersections in the picture (the picture cannot be made larger). For example, the cross-team would win this game:



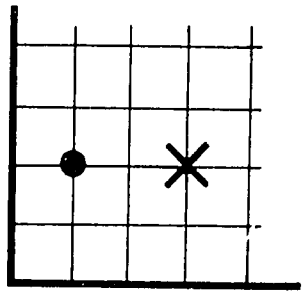
The dot-team wins this one:



Have a person on one team, say the cross-team, name two integers. Suppose he says "Three, two". Count over three, up two, and mark the intersection with ×.



It is now the dot-team's turn. Suppose someone on the team says "One, two". Again, count over and up, and mark this point with a big, heavy dot like this:

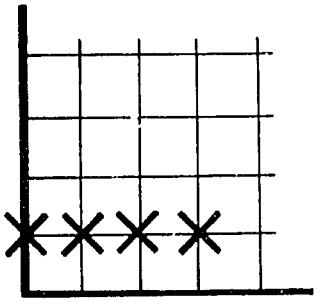


The game continues until one team sets five in a line, or neither side can get five in a line. A team loses its turn if either number named is too large (for example, the point that goes with "Seven, two" will not fit on the picture), <u>or</u> the choice of numbers results in an intersection that already has a mark on it.

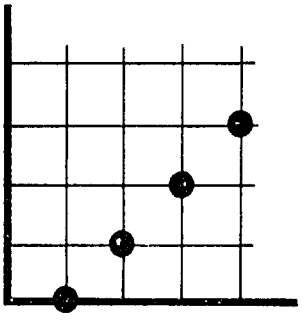
<u>Comments</u>: Having to get five in a line does not make for a very interesting game after the children have had a little practice. Since one team needs four turns to get four in a line, and the other team needs only one turn to stop them, neither team will be able to win. Therefore, after one or two games, you should change the

rule and say that <u>four</u> in an <u>uninterrupted</u> line wins.

Like this:

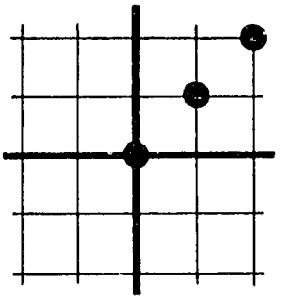


Or perhaps this:



This does make an interesting game, and one team is likely to win if it uses some good strategy. The reason for playing at least one game with five in a line is to force the children to figure out how to get a mark on one of the axes. If they have to figure out for themselves that they need to name zero as one of the numbers, they are likely to remember it.

There is another variation that can be introduced after four or five games, or perhaps earlier. Since the children have learnt about negative numbers previously, the following diagram can be used:



Here, the dot-team has to figure out that "negative one, negative one" is the play that will win for them.

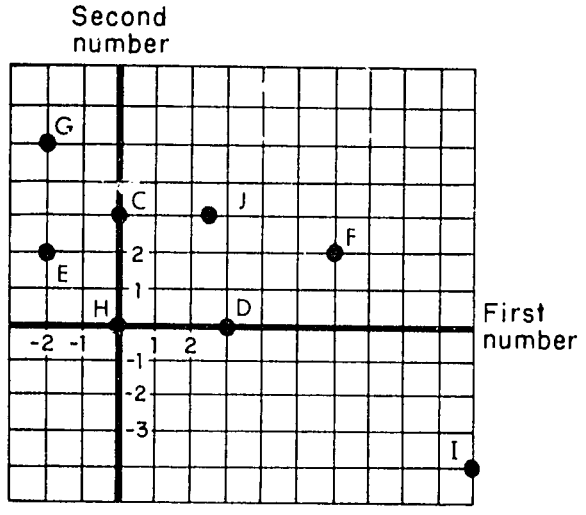
If the teacher does not say anything about negative numbers before the new diagram is put on the board, the class will probably fill up all the intersections which do not use any negative numbers, look a bit puzzled, and perhaps try to tell you "No, I meant you to count to the <u>left</u> one place." You can refuse this by saying, "The only instructions I can accept are the names for two numbers." Eventually, some child will probably announce "negative something, . . ." and get a point in another quadrant. If they start to get discouraged by not being able to do it, you may have to give them a hint.

At this point, the children should be able to handle Section 17-1 in their text, either in class or on their own.

Answers to Problems 17-1

Student Text Page 241

1. The point I, (10, -4) is included to remind the children that the picture of a number plane can be enlarged. The point J, $(2\frac{1}{2}, 3)$, is included to remind them that the numbers are not restricted to the integers.



2. This question emphasizes the importance of the order in the ordered pair of numbers.

3.

P:	(2, 3)	T:	(3, -3)
Q:	(-4, 3)	U:	(0, 4)
R:	(-1, -1)	V:	(1, 0)
S:	(0, -2)	W:	$(-5, -\frac{1}{2})$

4. (a) One

(b) One

You might let the children describe roughly the location of the points that correspond to these ordered pairs:

(a) (-100, -100)

(b) (1000, -2)

(c) $(\sqrt{2}, \sqrt{2})$

(d) $(-\frac{1}{2}, 50)$

Coordinates

You might wish to point out to the children that it is a convention to call the horizontal axis the x-axis; it could be called the y-axis, but it usually is not.

17-2 Graphs as Pictures of Truth Sets

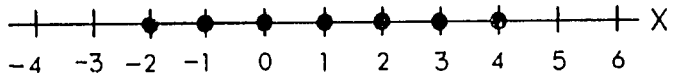
Student Text Page 243

Much of the problem work in this section is referred to and used for further developments in later sections of the chapter. Be sure that the pupils keep their written work available.

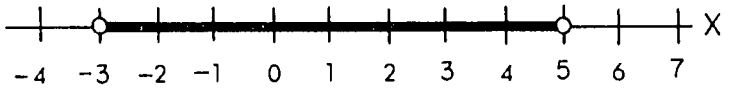
Class Activity

The class activity can be done either on the board with the whole class, or individually on graph paper. Perhaps a combination of the two would be best.

1.



2.



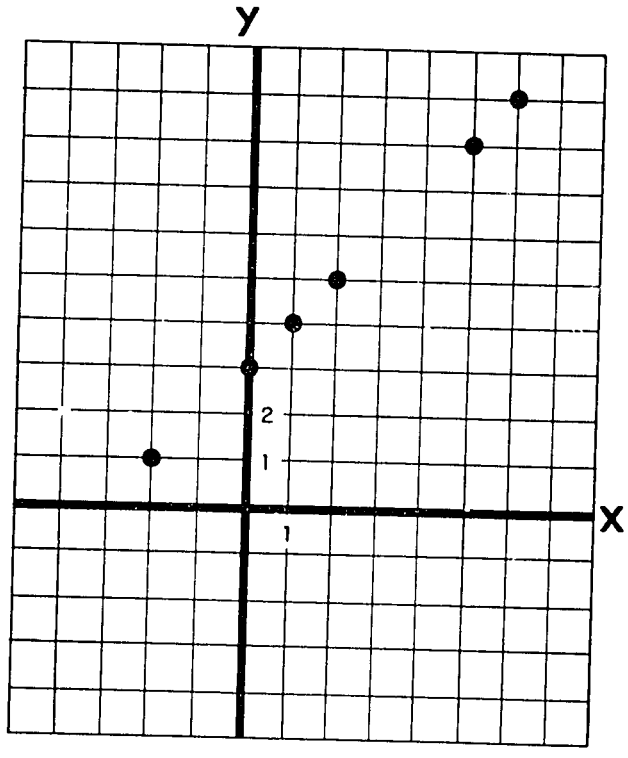
The hollow dots at -3 and 5 indicate that these numbers are not in the truth set. You may need to remind the pupils that the domain of a variable is the set of numbers from which values of the variable may be chosen. For each of these numbers the sentence is either true or false. The truth set of of the open sentence contains those numbers which make it true.

3. For $y = x + 3$, if x is 2, then y is 5. The ordered pairs in (a), (b), (d), and (f) belong to the truth set.

4. The table can be filled in on the blackboard with the pupils filling in two or three more ordered pairs. The number of ordered pairs in the truth set is unlimited.

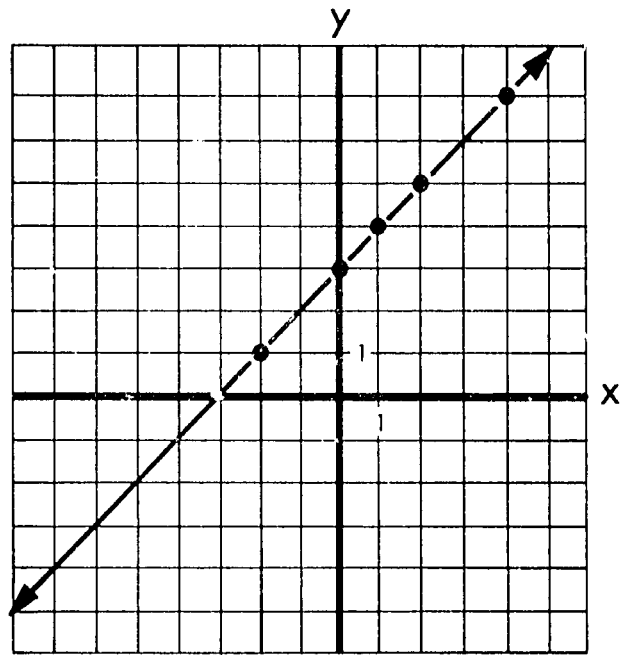
5. For activities 5, 6, 7, and 8, some of the work should probably be done on the blackboard with the whole class before they complete their own graphs individually. For example, the axes can be drawn first on the board, and several children can come up to the board to mark two or three points from

the table, and then the rest of the points can be marked individually on their own papers. Then the remaining points can be marked on the blackboard so that the children can check their graphs for mistakes. The graphs may now look something like this:



6. It is probably a good idea to have several points marked on the board before the children go back to their own graphs so that you can be sure that they understand what they are to do. Ask a pupil to come up and mark a point that he thinks belongs to the truth set <u>without</u> doing any counting. Do <u>not</u> be surprised if he marks one that is <u>not</u> on the line suggested by the other points already marked. Many children do not realize right away that there is a pattern to see. Whatever point is marked, ask for the numbers that go with it, and then go back to the equation $y = x + 3$ and check whether the equation is true for these numbers. This last step is important because a child should realize that he has his own way, independent of the teacher's help, of finding out whether a point belongs to the graph. When several points that belong to the graph have been marked, then the children can mark more on their own graphs. As soon as a child has marked a point on his own graph and checked that it belongs to the graph, he can come up and mark the point on the blackboard graph.

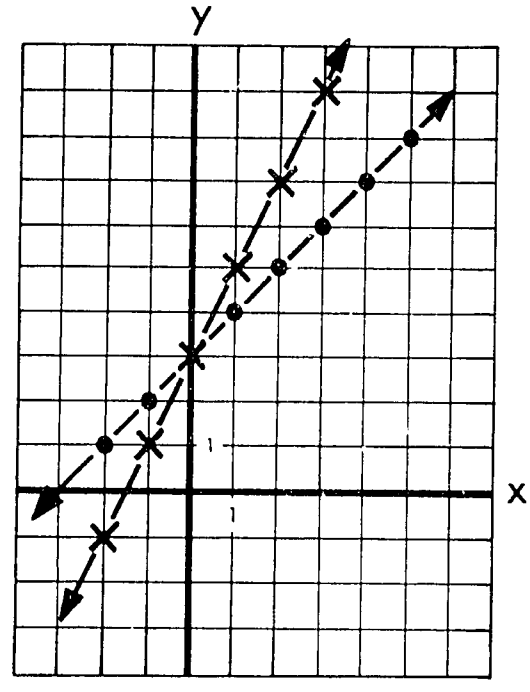
7. Not all the ordered pairs in the truth set can be shown; the line runs off the graph paper.



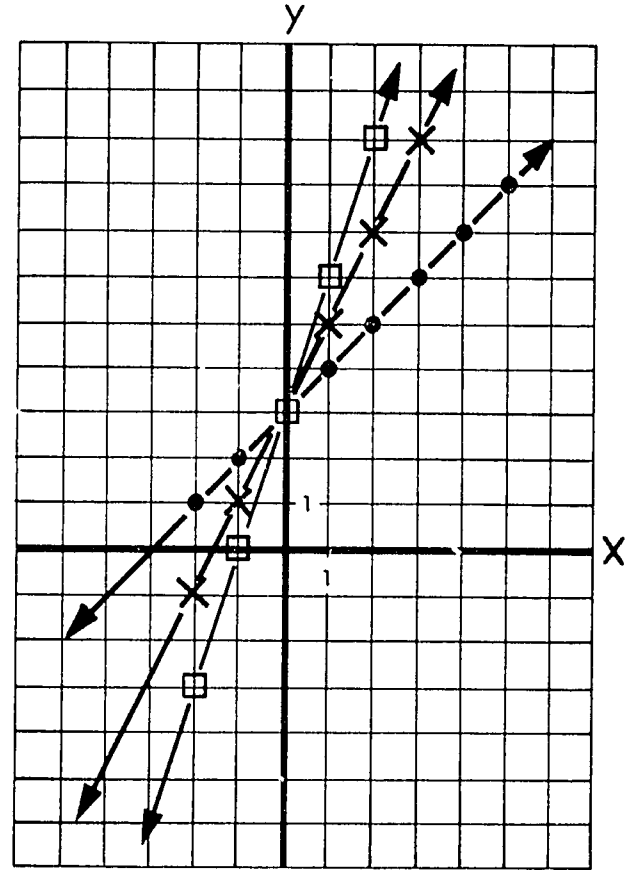
8. There are two things that we wish to give the children an opportunity to discover in the class activity. The first is that the graph of the truth set for some equations is a straight line. This the children found when they finished the graph in 7. The second thing is that the coefficient of the x term for equations of the form $y = ax + b$ determines the "up and over" pattern in going from one point of the graph to another. Many children can discover this for themselves, and for those who do not, it is not necessary to tell them at this point; the subject is dealt with more formally later in this chapter in the section on slope. It is often valuable for children to gain informal experience with a concept before they even know the name attached to the concept.

For the equation $y = 2x + 3$, (1, 5) and (2, 7) belong to the truth set. Have these points marked on the board (perhaps in a different colour if you have coloured chalk). We ask the children to guess where the third point goes even though they may put it in the wrong place. If they do, let them find out for themselves by testing whether the coordinates make the sentence true. Many of them will have discovered by now that the points may be expected to be on a straight line. Some of the children will notice the relation of "up and

over" to the coefficient of x. Again, a few more correct points for $y = 2x + 3$ can be marked, and the children can complete the graph on their individual papers. Their papers should look like this:



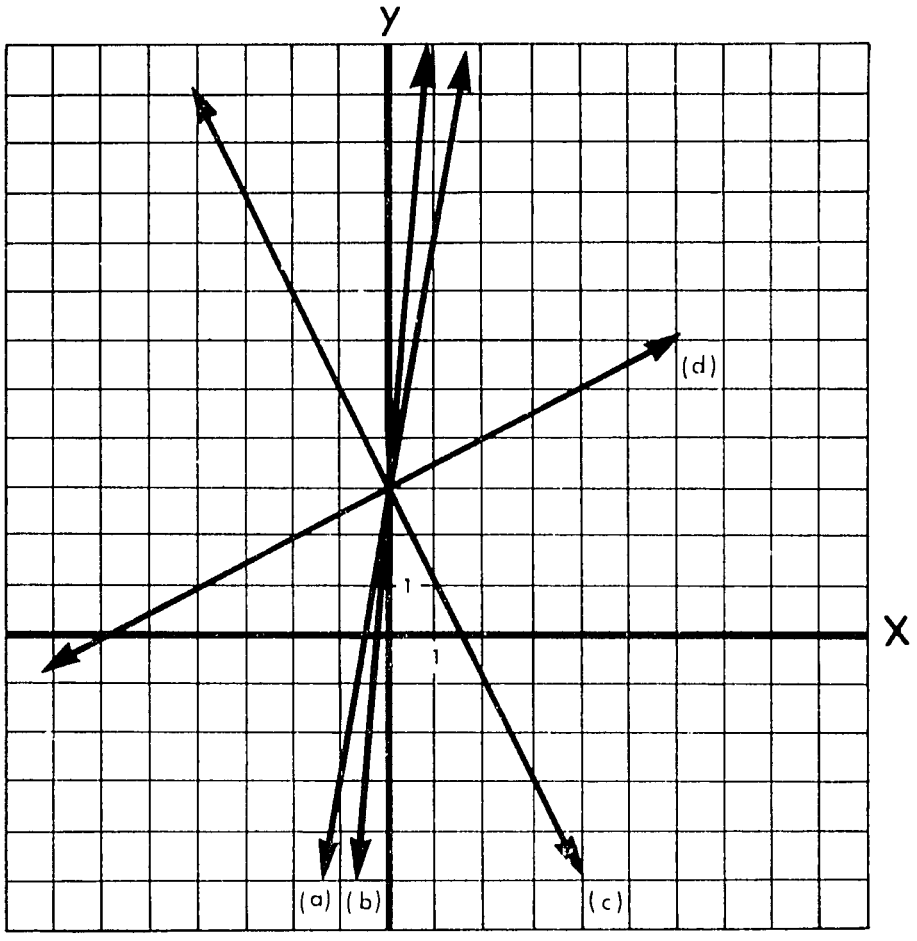
9. This time we let them try to guess a second point after having located only the single point (1,6). Some will recognize the pattern of behaviour quickly and will know that because of the $3x$ in $y = 3x + 3$, the line will climb three units for an increase of one in x. Others may have to do more experimenting before they see the pattern. Their graph paper when $y = 3x + 3$ has been completed will look like this:



Answers to Problems 17-2a

Student Text Page 246

1.

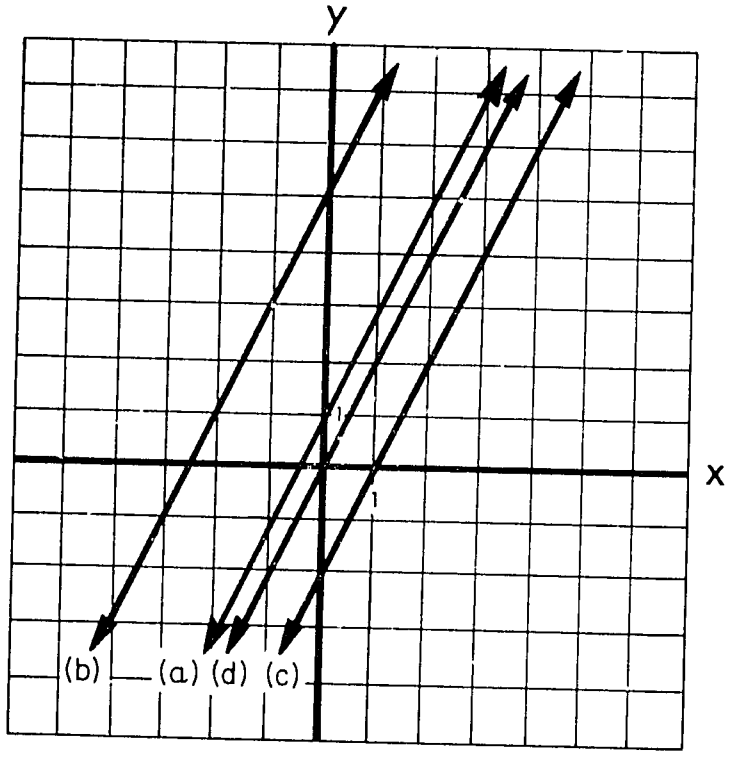


You may want to ask the pupils after they have finished the above graphs to point to where they think the graphs of the following would roughly be:

(a) $y = -10x + 3$ (c) $y = 100x + 3$

(b) $y = -\frac{1}{2}x + 3$ (d) $y = \frac{1}{3}x + 3$

2. Until this point, the children have graphed only straight lines which intersect the y-axis at (0, 3). This exercise is designed so that the children have an opportunity to discover what role the constant term of a linear equation plays in the graph. This idea is introduced formally later in the chapter, so there is no need to use the definition of y-intercept here.



Class Activity

1. This exercise may be a guessing game for the children who have not yet noticed the relationship between the coefficient of x and the "up and over" pattern from one point to another. Accept whatever answers the children give, and have them check the two points to find out if their value of a is correct. They will get the correct value 2 for a sooner or later, and it is good practice for them to test ordered pairs to see whether they make the equation true. If they have a good deal of trouble obtaining the correct value for a, you may wish to do the first two or three of Problems 17-2b in class.

Some pupils may be clever enough to see that, since the point $(1, 3)$ is on the line, the equation $3 = (a \times 1) + 1$ must be true, and this will be true only if a is 2.

2. Yes, all the points on the line have coordinates that make $y = 2x + 1$ true. Have the pupils check two or three of them.

Answers to Problems 17-2b

Student Text Page 247

For the first two problems, the children are asked to find only the coefficient of the x term; for the rest, they are asked to find the constant term, also.

1. $a = 1$
2. $a = \frac{1}{2}$
3. $a = 3;\ b = 1$
4. $a = -1;\ b = 0$
5. $a = -2;\ b = 2$
6. $a = -1;\ b = -1$
7. $a = 5;\ b = -3$
8. $a = 0;\ b = 1$

17-3 Guessing Rules

Class Activity

Student Text Page 249

The purpose of this section is to approach linear equations and straight line graphs from another point of view. The children have already had some experience with graphing a linear equation of the form $y = ax + b$; they have also reversed this process, i.e., starting with two points marked in the number plane, they have written the equation that goes with the line through the points. Two points are determined, however, when you know the ordered pairs that go with them. This section is aimed at having the children find the linear equation determined by two ordered pairs of numbers.

1. John's rule is: Multiply by 5 and add one.

0	1
4	21
10	51

If other rules are suggested first, have the children decide if it works by checking several of the pairs in the table.

2. Tunde's rule is: Multiply by seven.

1	7
3	21
4	28

3. "Divide by four" does work; the rule Edison was thinking of is "multiply by one-fourth" (or perhaps it was "multiply by two and divide by eight").

0	0
2	$\frac{1}{2}$
3	$\frac{3}{4}$
12	3

4. The variable x represents the number which you tell John and y represents the number John tells you after using his rule. Tunde's rule can be written as $q = 7p$, if we assume that p is the number you tell Tunde, and q is the number Tunde tells you. Edison's rule is $y = \frac{1}{4}x$, or $q = \frac{1}{4}p$, etc., depending on what two variables are chosen. John would write the rule as $y = \frac{x}{4}$. The two equations are equivalent, of course.

Before going on to Problems 17-3, the children may enjoy making up and guessing their own rules. For example, you can have two children go outside the room, agree upon a rule, and then let the rest of the children give them numbers on which to work the rule. (You can write the table on the board.) The first child who can guess a rule that works for any number (it may <u>not</u> be the rule the two children were actually using) and write it correctly on the board is allowed to pick an assistant and make up the next rule to be guessed by the class. (The reason for an assistant is that he can act as a checker for the arithmetic involved in using the rule.)

Answers to Problems 17-3

Student Text Page 250

1.

x	y
5	12
10	17

$y = x + 7$

2.

w	z
0	-4
1	-3
7	3

$z = w - 4$

3.

s	t
-1	-4
0	-1
5	14
7	20

$t = 3s - 1$

4.

p	q
1	$-\frac{1}{2}$
101	$49\frac{1}{2}$

$q = \frac{1}{2}p - 1$

(There are other possibilities for the form of the rules that the children may find that might also be correct.)

5. The point of this problem is to have some of the children figure out by themselves the following:

a. If the rule is of the form "multiply by a number and then add a number", the rule is an equation of the form $y = ax + b$.

b. Only two ordered pairs are needed to determine the equation. (They may even associate the equation with its graph and the ordered pairs with the two points needed to determine the line.)

c. Knowing y when x is 0 tells me the constant term in the equation. The coefficient of the x term can be obtained by subtracting the second coordinate for 0 from the second coordinate for 1. (This actually determines algebraically the slope and y-intercept of the linear equation.) Some of the children may discover these two things by examining the tables and rules they have already guessed; some of them perhaps have already realized it when they were guessing rules made up in class.

The four rules Sonko guessed:

(a) $y = 2x + 3$ (c) $y = \frac{1}{3}x$

(b) $y = x - 1$ (d) $y = 4x + 5$

6. The purpose of this question is to indicate to the children that knowing only two ordered pairs determines exactly one equation of the form $y = ax + b$ but there are other equations of different forms which will also have the two ordered pairs in the truth set. In Problem 7, the children may begin to realize that equations of the form $y = ax + b$ have straight line graphs, but equations of different forms may not. Edward's rule is: $y = x^2 + 1$.

7. How the children decide whether the graphs of the rules are straight lines or not does not matter at this point; they will probably decide in different ways. Some may just "know", others may actually draw each graph to decide. All the rules except the last one, $y = x^2 + 1$, have straight line graphs. It is not necessary to complete the graph of $y = x^2 + 1$ (plotting three points will indicate that it is not in a straight line) unless the children are curious

to find out what its graph looks like. It is perhaps helpful to graph in class one or two of the rules from Problem 5 by plotting the two points given so that the children have another opportunity to associate the algebra they did to guess the rule with the geometry they did when they were finding a and b for $y = ax + b$ from the two points marked in the number plane.

17-4 Equivalent Equations in Two Variables

Thus far the children have met linear equations of the form $y = ax + b$. The purpose of this section is to introduce them to linear equations in different forms.

The pupils should find that any ordered pair that makes $y = \frac{1}{2}x - 1$ true also makes $2y - x + 2 = 0$ true and the converse also is true.

Answers to Problems 17-4

Student Text Page 253

1. Equivalent
2. Not equivalent
3. Equivalent

There is a variety of ways in which the equations in 1 and 3 can be proved equivalent. You might ask the children if it matters with which equation you begin the proof.

17-5 Graphs of Linear Equations in Two Variables

We are turning now to consideration of more general equations of the first degree in two variables - equations of the form

$$ax + by + c = 0$$

In this section we observe informally that the graphs are straight lines. We mention the name "linear equations".

Answers for Class Activity

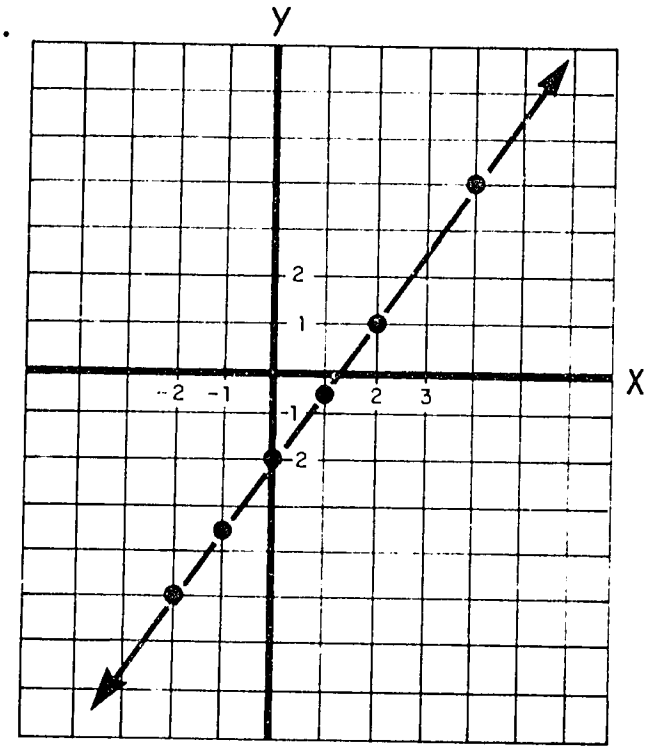
Student Text Page 254

1. (4, 4), (−2, −5), and (0, −2) belong to the truth set of $2y - 3x + 4 = 0$.

2.

x	y
−2	−5
−1	$-3\frac{1}{2}$
0	−2
1	$-\frac{1}{2}$
2	1
4	4

3.

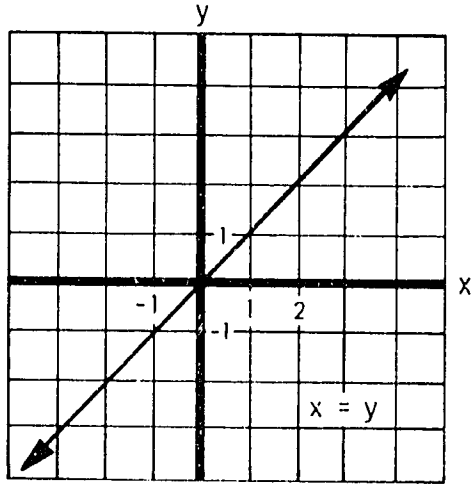


4. $\left(3, 2\frac{1}{2}\right)$, $\left(5, 5\frac{1}{2}\right)$, (−4, −8), (−6, −11)

Answers to Problems 17-5

Student Text Page 255

1.



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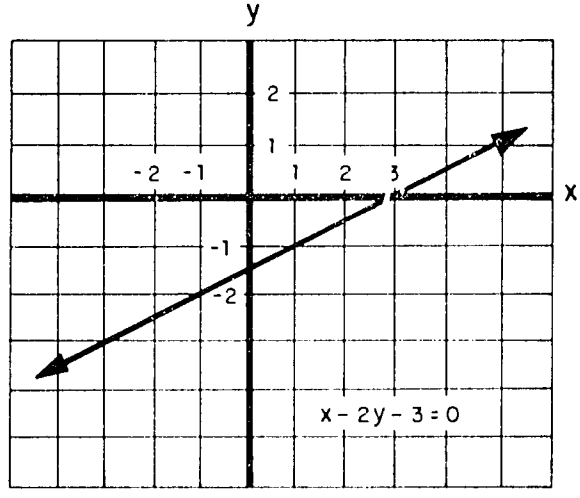

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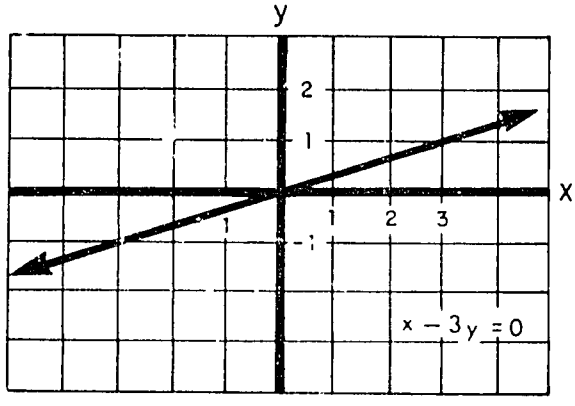
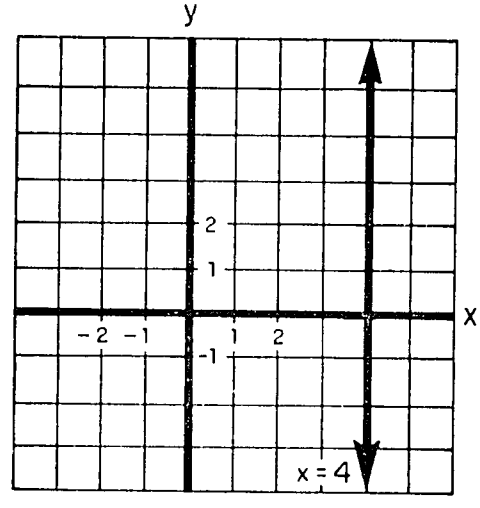
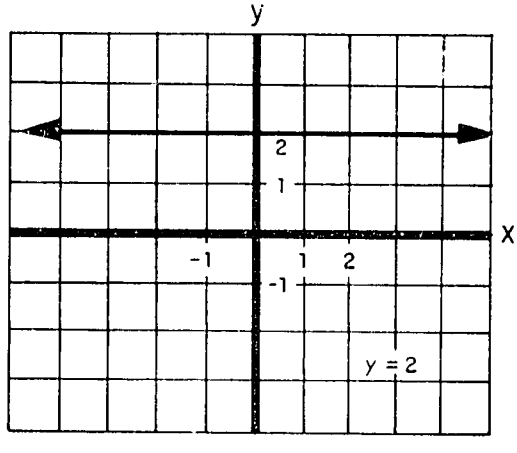


4.



5.



17-6 The Slope of a Straight Line.

Again we are very informal in our approach to the idea of slope. It may be helpful to speak of slope as a convenient measure of the direction of a line. This number is convenient because, as we see later in Section 17-7, it appears in one form of the equation of the line.

The idea that the slope is the same number, no matter what two points on the line are chosen to determine it, is basic. The pupils discover this idea inductively here. No proof is given at this time. Proof of it comes in a later year when the pupils return to the idea of slope from a more formal viewpoint.

Other possible words to use in defining slope are

$$\frac{\text{change in } y}{\text{change in } x} \quad \text{or} \quad \frac{\text{rise}}{\text{run}} .$$

Answers for Class Activity

Student Text Page 255

3. Yes, the children should finish at point F. The ratio is $\frac{6}{4}$ or $\frac{3}{2}$. The ratio $\frac{3}{2}$ should be obtained for all the pairs of points in (a) - (f).

4. Count down 3 units; count to the left 2 units; vertical increase is -3, horizontal increase is -2; ratio is $\frac{-3}{-2}$ or $\frac{3}{2}$.

5. The vertical increase is $-1\frac{1}{2}$; the horizontal increase is -1; the ratio is $\frac{-1\frac{1}{2}}{-1}$ or $\frac{3}{2}$.

6. Vertical increase: $(-3\frac{1}{2}) - (-2) = -1\frac{1}{2}$

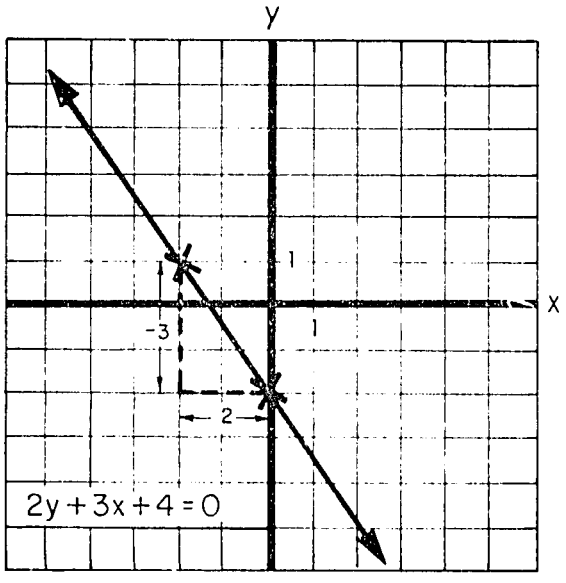
horizontal increase: $(-1) - 0 = -1$

ratio: $\frac{-1\frac{1}{2}}{-1} = \frac{3}{2}$

Answers to Problems 17-6

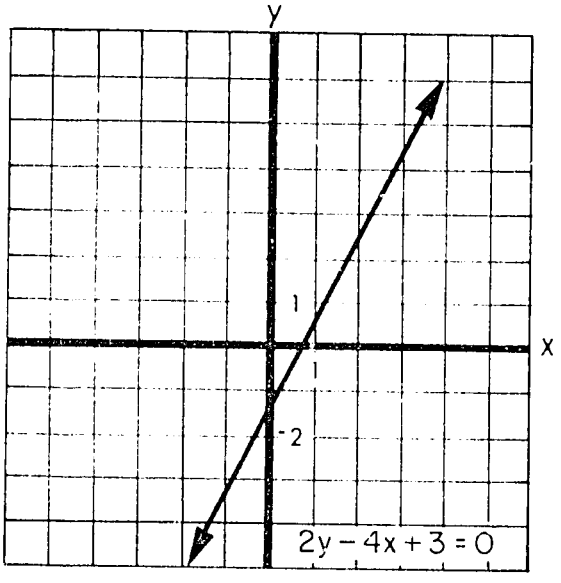
Student Text Page 257

1. (a)



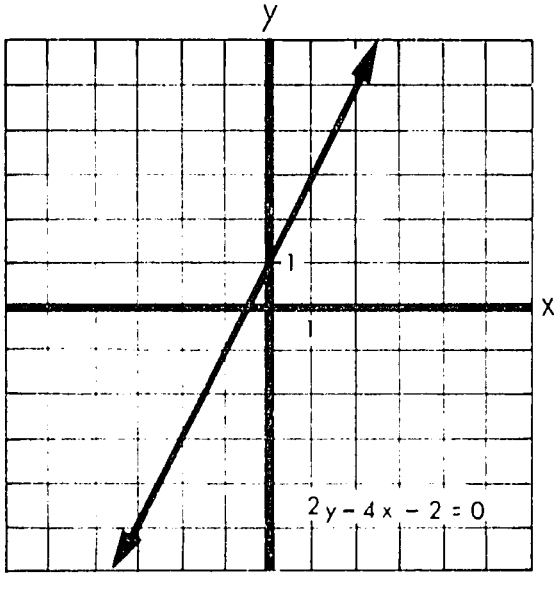
Slope is $-\frac{3}{2}$.

(b)



Slope is 2.

(c)



Slope is 2.

(d)

Same as (c)

2.

Line	Slope
$y = x + 3$	1
$y = 2x + 3$	2
$y = 3x + 3$	3
$y = 5x + 3$	5
$y = 10x + 3$	10
$y = -2x + 3$	-2
$y = \frac{1}{2}x + 3$	$\frac{1}{2}$

3.

1.	1	5.	-2
2.	$\frac{1}{2}$	6.	-1
3.	3	7.	5
4.	-1	8.	0

4. Each of these can be done in two ways, depending on the order in which the points are taken. For example, in Problem 1, one could have gone from (2,4) to (1,3) and obtained these results:

vertical increase: $3 - 4 = -1$

horizontal increase: $1 - 2 = -1$

slope is $\frac{-1}{-1}$, or 1.

17-7 The Slope of a Line from its Equation

Sections 17-7 and 17-8 may be considered optional and omitted if time is limited or if the ability of the class is such that these sections are too difficult at present.

In this section the pupil learns that if the equation of a line is in the form $y = ax + b$, the slope of the graph is a and if the equation is not in this form, it can usually be put in this form with the help of equivalent equations.

Exceptions are lines parallel to the y-axis for which the slope is undefined. Be careful that the pupils do not confuse lines with slope zero (horizontal lines) with lines which have no slope (vertical lines).

The student is led inductively to discover the ideas in this section. The proof that a is the slope in $y = ax + b$ comes later in a more formal treatment.

Proof of equivalence:

(c)	$2y - 4x - 2 = 0$	
	$2y = 4x + 2$	Added $4x + 2$
	$\frac{1}{2}(2y) = \frac{1}{2}(4x + 2)$	Multiplied by $\frac{1}{2}$
(d)	$y = 2x + 1$	

The proof can be reversed if you start with (d).

Answers to Problems 17-7a

Student Text Page 258

1. y increases 5 as x increases 1. (The $5x$ tells us this.)
   y increases 15 as x increases 3. The slope is 5.
   The coefficient of x is 5.

2. The slope and the coefficient of x are both 7.

3. $\frac{3}{2}$ 4. 1 5. -4 6. 1000

You have a chance for a little review of equivalent equations.

$3x + 4y - 5 = 0$ is equivalent to $4y = -3x + 5$ because adding the real numbers $-3x$ and 5 on both sides does not change the truth set.

$4y = -3x + 5$ is equivalent to $y = -\frac{3}{4}x + \frac{5}{4}$ because multiplying on both sides by the non-zero number $\frac{1}{4}$ does not change the truth set.

The slope is $-\frac{3}{4}$.

Answers to Problems 17-7b

Student Text Page 259

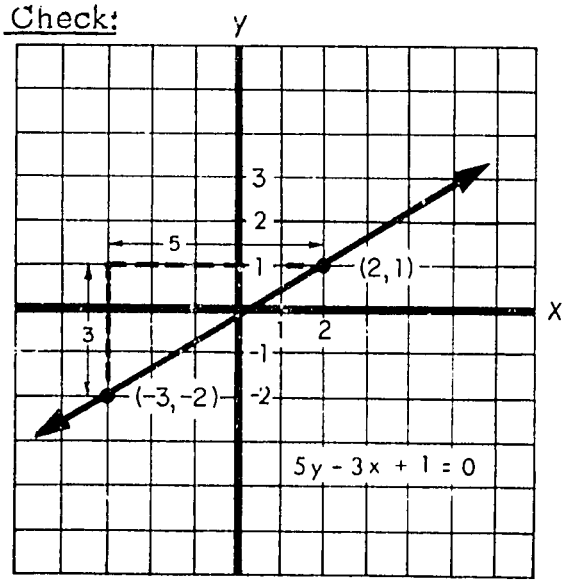
1. $5y - 3x + 1 = 0$

$5y = 3x - 1$

$y = \frac{3}{5}x - \frac{1}{5}$

Slope is $\frac{3}{5}$.

Check:



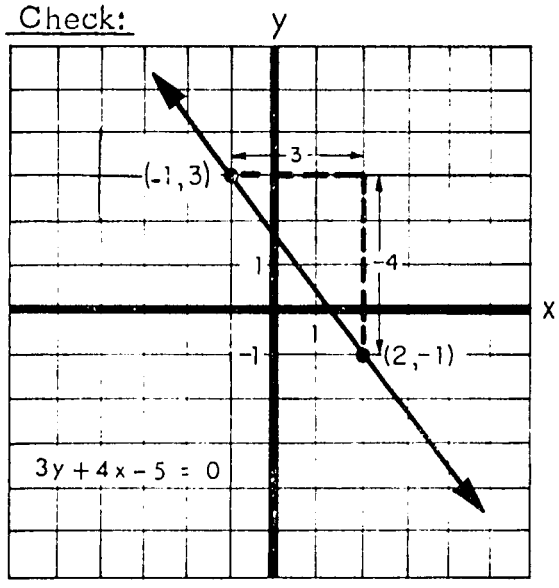
2. $3y + 4x - 5 = 0$

$3y = -4x + 5$

$y = -\frac{4}{3}x + \frac{5}{3}$

Slope is $-\frac{4}{3}$.

Check:



3. $x + 3y - 5 = 0$

$3y = -x + 5$

$y = -\frac{1}{3}x + \frac{5}{3}$

Slope is $-\frac{1}{3}$

Check: Depends on what two points the children choose. Example: From (-1, 2) to (5, 0):

y - increase: $0 - 2 = -2$

x - increase: $5 - (-1) = 6$

Slope is $\frac{-2}{6}$, or $-\frac{1}{3}$

4. $3y - 7 = 0$

$3y = 7$

$y = \frac{7}{3}$

or $y = 0x + \frac{7}{3}$

Slope is 0.

Check: From $\left(0, \frac{7}{3}\right)$ to $\left(1, \frac{7}{3}\right)$:

y - increase: 0

x - increase: 1

Slope is $\frac{0}{1}$, or 0.

5. $2x - 6 = 0$.

This presents a special problem since the line does not have a slope. It is discussed in the text immediately after this problem.

A Special Case

An equation of the form $x = c$, or $ax + b = 0$, has no slope. All lines which are parallel to the y-axis have no slope. Do no confuse these lines with lines which are parallel to the x-axis, which do have slope; their slope is the number zero.

Answers to Problems 17-7c

Student Text Page 261

1. Slope is a negative number.
2. Slope is a positive number.
3. Slope is a negative number.
4. Line has no slope.
5. Slope is zero.
6. Line has no slope.

17-8 The Intercept on the y-axis

If x is 0, y is -2. The coordinates of the point of intersection with the y-axis are $(0, -2)$.

In referring back to the problems in Section 17-2, we find that the second coordinate of the point where the line crosses the y-axis is the same number as the constant term in the equation of the line when the equation is written in the form $y = ax + b$. For example, when the equation of the line is $y = 2x + 5$, the coordinates of the point of intersection with the y-axis are $(0, 5)$.

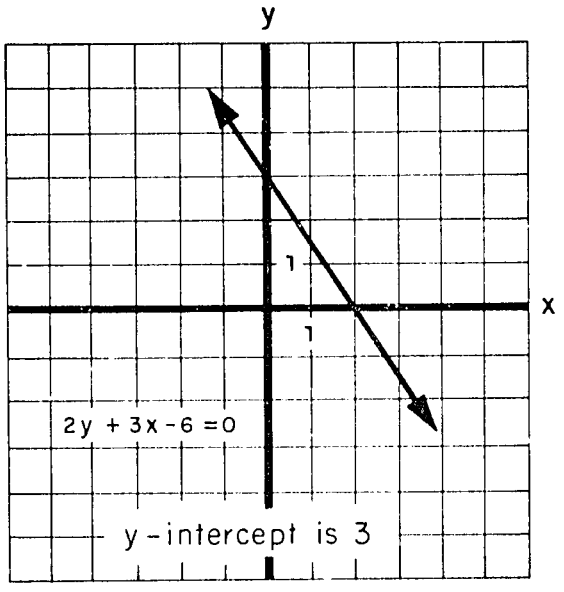
The first coordinate is 0 for every point of intersection with the y-axis.

In the proof, you may wish to discuss the number of points of intersection of a line with the y-axis. If the line is parallel to the y-axis, there is no point of intersection; if the line is the y-axis, then every point of the line is in the intersection; for any other line, there is exactly one point in the intersection.

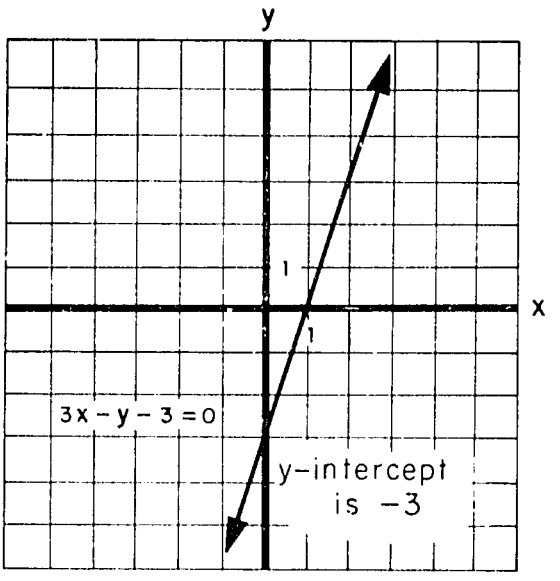
Answers to Problems 17-8

Student Text Page 263

1. (a)



(b)



Check: $2y + 3x - 6 = 0$

$2y = -3x + 6$

$y = -\frac{3}{2}x + 3$

The y-intercept is 3.

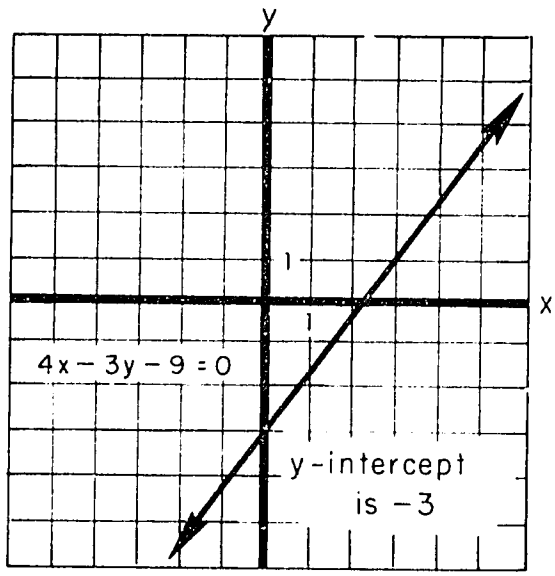
Check: $3x - y - 3 = 0$

$3x - 3 = y$

$y = 3x - 3$

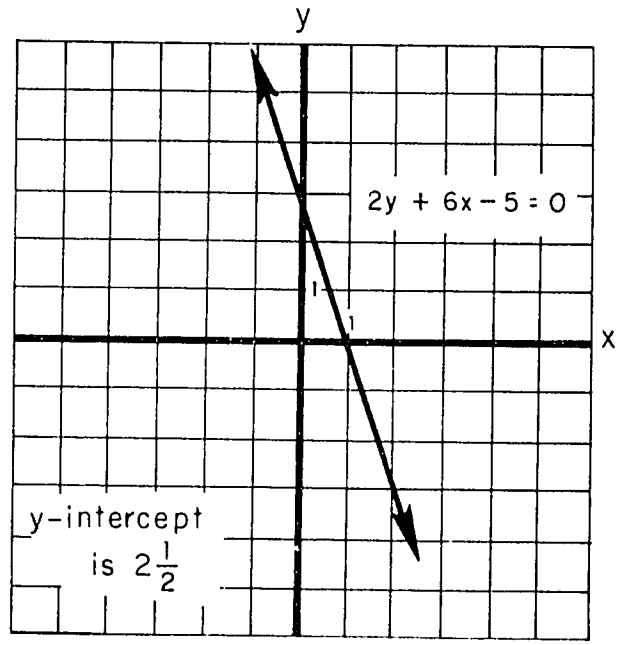
The y-intercept is -3.

2. (a)



Check: If $x = 0$ and $y = -3$,
$4x - 3y - 9 = 0$ becomes
$[4 \times 0] - [3 \times (-3)] - 9 = 0$
which is true.

(b)



Check: If $x = 0$ and $y = \frac{5}{2}$,
$2y + 6x - 5 = 0$ becomes
$\left(2 \times \frac{5}{2}\right) + (6 \times 0) - 5 = 0$
which is true.

3. If a pupil estimates $\left(-\frac{3}{4}\right)$ for the y-intercept, he would try $\left(0, -\frac{3}{4}\right)$ in the equation.

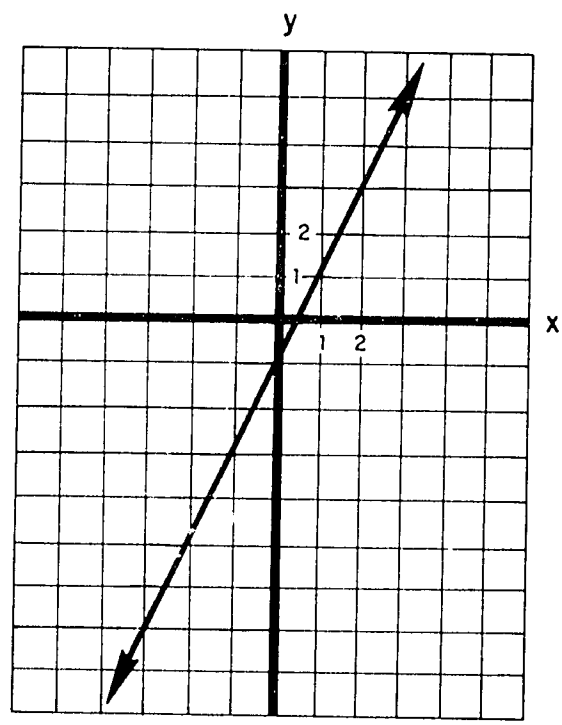
$$[7 \times \left(-\frac{3}{4}\right)] - [14 \times 0] + 6 = -\frac{21}{4} + \frac{24}{4} = \frac{3}{4}$$

$$\frac{3}{4} \neq 0$$

For $(0, -0{\cdot}8)$ we have

$$[7 \times (-0{\cdot}8)] - [14 \times 0] + 6 = -5{\cdot}6 + 6$$
$$= 0{\cdot}4$$

This is a closer estimate.
The exact intercept is $-\frac{6}{7}$, as can be found from the $y = ax + b$ form or by letting x be 0 in the given equation and solving for y.



4. (a) Slope is $-\frac{2}{3}$; y-intercept is 5.

(b) Slope is 100; y-intercept is -3.

(c) $y = 5x + 4$.

(d) $y = -2$.

5. Yes, any equation $x = c$ $(c \neq 0)$ gives a line which is parallel to the y-axis, and therefore has no y-intercept.

6.

	Slope	y-intercept	Equation
(a)	0	2	$y = 2$
(b)	no slope	no intercept	$x = -3$
(c)	2	3	$y = 2x + 3$
(d)	$-\frac{1}{3}$	-1	$y = -\frac{1}{3}x - 1$

7.

	Slope	y-intercept	Equation
(a)	$\frac{6}{1}$ or 6	1	$y = 6x + 1$
(b)	$\frac{0}{3}$ or 0	-3	$y = -3$
(c)	$\frac{-2}{1}$ or -2	0	$y = -2x$
(d)	$\frac{3}{2}$	4	$y = \frac{3}{2}x + 4$

8. Let x be 0 and solve for y.

$$5y + 35 = 0$$
$$5y = -35$$
$$y = -7$$

When $x = 0$, $y = -7$, so the y-intercept is -7.

9. Use the method of Problem 8, but do it mentally.

(a) 4 (d) -15

(b) 8 (e) $\frac{3}{2}$

(c) $\frac{4}{7}$ (f) 4

10. <u>CHALLENGE PROBLEM.</u> The intercept on the x-axis is the x-coordinate of the point where the graph intersects the x-axis. The y-coordinate at this point is 0.

(a) 12

(b) $-\frac{8}{3}$

(c) $-\frac{1}{2}$

(d) $\frac{5}{32}$

(e) $\frac{5}{3}$

(f) $-\frac{8}{3}$

11. <u>CHALLENGE PROBLEM.</u> Since the point with first coordinate 0 is not given, the pupils will be looking for some other way to find the intercept on the y-axis. Some may find it by drawing the graph. Some may see that y goes from 7 to 4 (a change of -3) as x goes from 2 to 1, so y must go from 4 to 1 as x goes from 1 to 0. Some may write $y = 3x + b$ (they have found that the slope is 3) and say that the coordinates (1, 4) must make this true. This gives them an equation which they can solve for b.

The equation is $y = 3x + 1$.

CHAPTER 18

USING STRAIGHT LINE GRAPHS

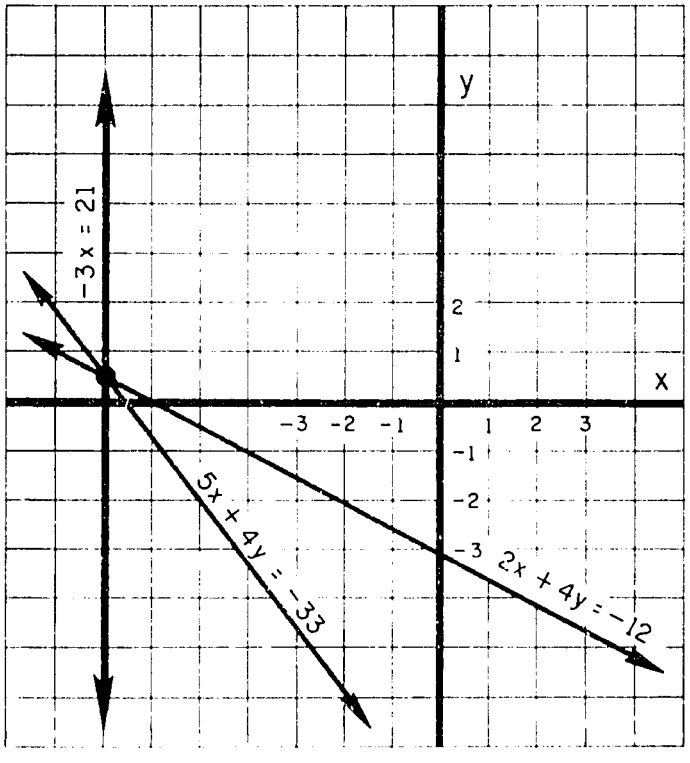
18-1 Systems of Simultaneous Equations

The purpose of this section is to illustrate graphically the solution of systems of simultaneous linear equations, and to give examples of the special cases that arise.

Answers to Class Activity

Student Text Page 267

1.



2. $(-7, \frac{1}{2})$

3. $(-7, \frac{1}{2})$. The three lines intersect at the same point.

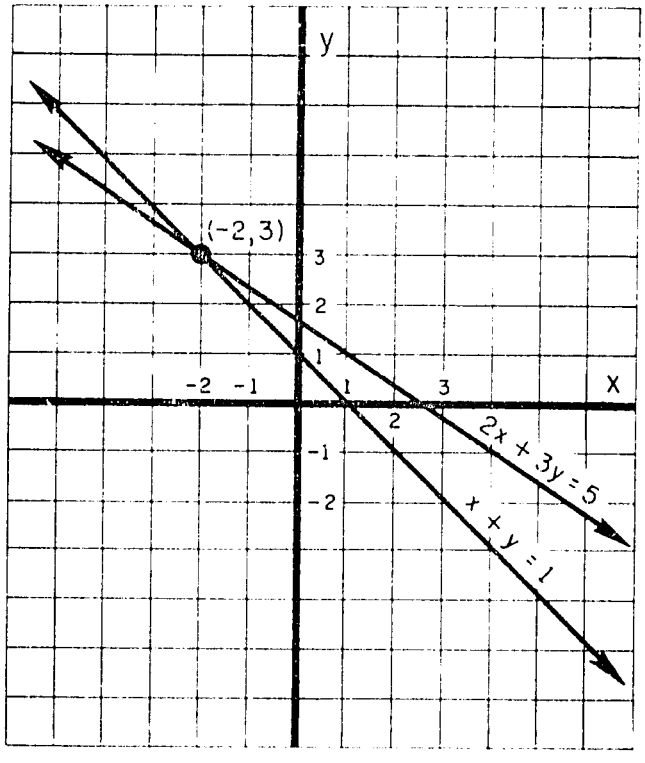
4. Yes, if the straight lines intersect, they will do so at the same point. This should not surprise the children since the coordinates of the point of intersection of two lines is the solution to the corresponding system of simultaneous equations, and if two pairs of simultaneous equations are equivalent, they have the same solution.

5. $(-7, \frac{1}{2})$. The children should realize that the solution of the system of simultaneous equations is the pair of coordinates of intersection of their graphs.

Answers to Problems 18-1a

Student Text Page 268

1.

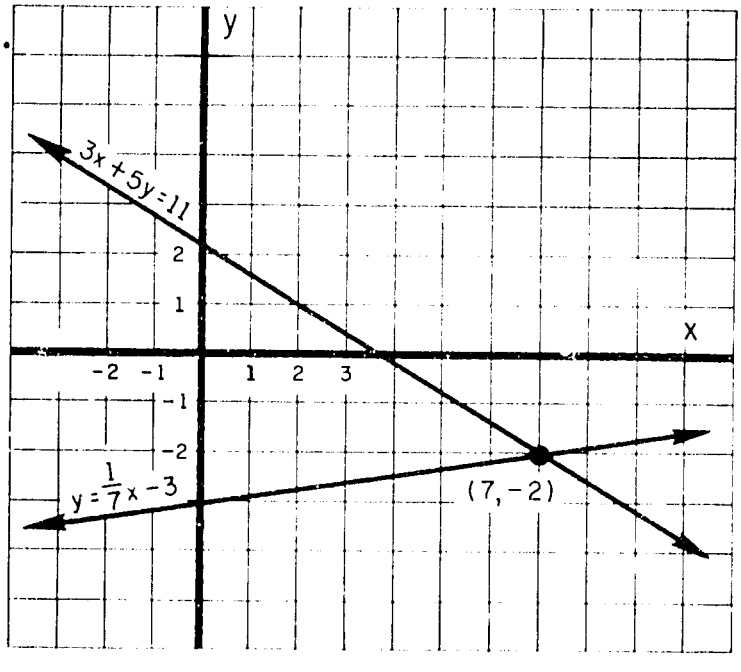


Check:

$2 \times (-2) + 3 \times 3 = 5$ is true.

$-2 + 3 = 1$ is true.

2.



Check:

$3 \times 7 + 5 \times (-2) = 11$ is true.

$-2 = \frac{1}{7} \times 7 - 3$ is true.

3. A problem where the solution involves fractions was included to illustrate to the children the difficulty in obtaining the solution from a graph; if accuracy is desired, the algebraic method should be used.

$$\begin{cases} 7x + 4y = 11 \\ 14x + 3y = 12 \end{cases}$$

$$\begin{cases} 14x + 8y = 22 \\ 14x + 3y = 12 \end{cases}$$

$$5y = 10$$

$$y = 2$$

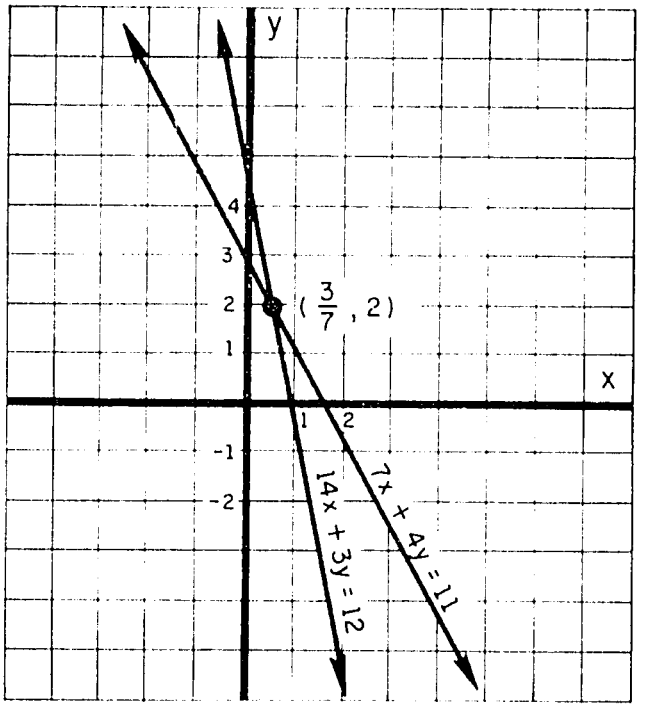
$$\begin{cases} 7x + 4y = 11 \\ y = 2 \end{cases}$$

$$7x + 4 \times 2 = 11$$

$$7x = 3$$

$$x = \frac{3}{7}$$

$$\begin{cases} x = \frac{3}{7} \\ y = 2 \end{cases}$$



There are two cases where a pair of simultaneous equations does not have a unique solution. The truth set of

$$\begin{cases} 3x - y + 1 = 0 \\ 2y - 6x - 7 = 0 \end{cases}$$

is the empty set. The children will discover that their graphs are parallel lines which, of course, have no point of intersection. This explains why the truth set is the empty set.

The system of equations

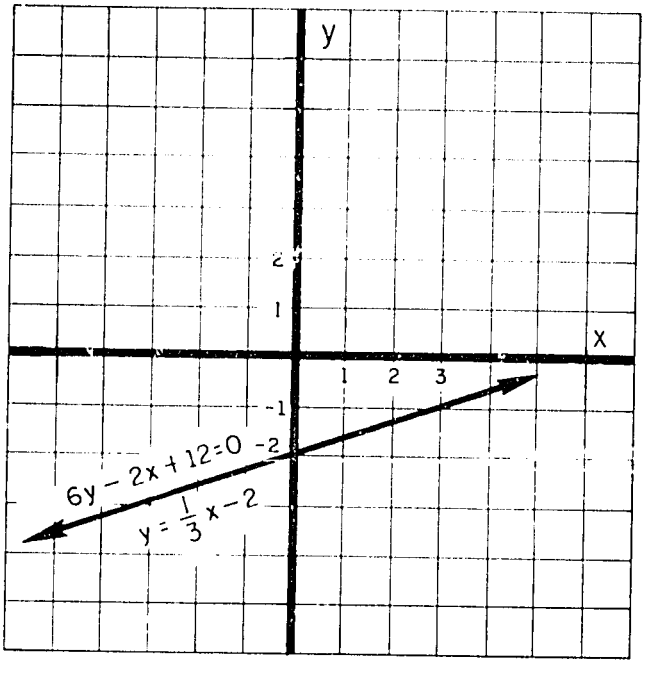
$$\begin{cases} y = \frac{1}{3}x - 2 \\ 6y - 2x + 12 = 0 \end{cases}$$

has many solutions, in particular (3, -1), (0, -2), and (9, 1). Since two distinct lines cannot intersect in more than one point, some of the children may be able to predict that the two equations have the same graph before they actually draw their graph in Exercise 1 of the Class Activity.

Answers to Class Activity

Student Text Page 269

1. You may wish to ask the pupils if they can prove the two equations equivalent without using the graph.



2. An equivalent system can be obtained in which the two equations are identical.

$$\begin{cases} \frac{1}{3}x - 2 = y \\ 6y - 2x + 12 = 0 \end{cases} \qquad \begin{cases} 3y - x + 6 = 0 \\ 6y - 2x + 12 = 0 \end{cases}$$

$$\begin{cases} x - 6 = 3y \\ 6y - 2x + 12 = 0 \end{cases} \qquad \begin{cases} 6y - 2x + 12 = 0 \\ 6y - 2x + 12 = 0 \end{cases}$$

3. The children may need to discuss this question before they decide on an answer. The truth set can be described using the graph as the set of ordered pairs that correspond to the points on the line; or algebraically as

the set of ordered pairs satisfying $y = \frac{1}{3}x - 2$ (or $6y - 2x + 12 = 0$); or as the set of ordered pairs of the form $(x, \frac{1}{3}x - 2)$.

Answers to Problems 18-1b

Student Text Page 269

1. $\{(-3, 5)\}$
2. $\{\quad\}$ (the empty set)
3. $\left\{\left(7, \frac{5}{8}\right)\right\}$
4. $\left\{\left(x, \frac{1}{15}x + \frac{2}{5}\right)\right\}$ or the set of all ordered pairs of numbers which satisfy either one of the given equations.

18-2 Proportionality

Answers to Questions in Text

Student Text Page 270

1. (b) and (c).
2. 7 for (b); $\frac{1}{4}$ for (c).
3. (a) $y = x + 7$ (c) $y = \frac{1}{4}x$

   (b) $y = 7x$ (d) $y = x^2 + 1$

(b) and (c) show proportionality because the rule can be written in the form $y = kx$; (a) and (d) cannot.

4. The pupils should realize that the graph of an equation showing proportionality must be a straight line (which (d) is not) and it must pass through the origin (which (a) does not).

5. Yes, the equation is linear with slope k and intercept 0 on the y-axis. The point (0, 0) will be on the graph of $y = kx$ for any k.

Answers to Problems 18-2

Student Text Page 271

1. x is proportional to y with constant of proportionality $\frac{1}{k}$. The proof of this is simple:

If $y = kx, \quad k \neq 0$

Then $\frac{1}{k} y = \frac{1}{k}(kx)$, Multiply by the non-zero number $\frac{1}{k}$

$\frac{1}{k} y = \left(\frac{1}{k} \times k\right)x,$

and $\frac{1}{k} y = x$

2. (b) $k = \frac{7}{2}$

3. (b) and (d), since they both contain the point (0, 0) and do not have slope zero. (a) and (c) do not contain (0, 0). The assumption that the rule is a linear equation is important, since other wise (d) might be $y = x^2$. For (b), $k = 100$; for (d) $k = -2$.

4. (b) with $k = -1$

(c) with $k = 3$

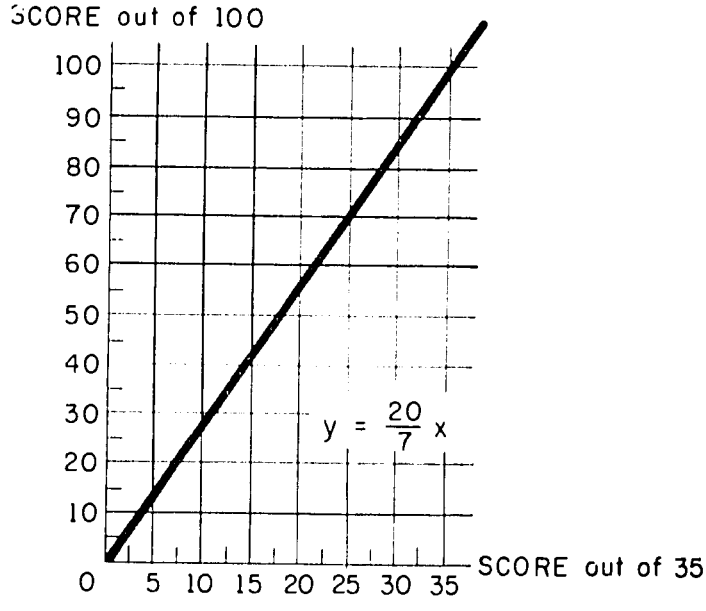
5. (a) For $\frac{28}{35} = \frac{x}{100}$, $x = 80$, and 80 is the grade put on the paper.

For $\frac{23}{35} = \frac{x}{100}$, $x = 65\frac{5}{7}$, and 65 is the paper grade.

(b) For $y = \frac{20}{7} \times 14$, y is 40, and 40 is the paper grade.

For $y = \frac{20}{7} \times 18$, y is $51\frac{3}{7}$, and 50 is the paper grade.

(c)



Number correct	10	34	20	17	30	26	4	12	31	24
Grade	30	95	55	50	85	75	10	35	90	70

18-3 A Practical Application

Introduction

This section is an optional one. Although there is no necessity to take up this material, it nevertheless provides a variety of interesting problems and these problems can provide the pupils with better understanding of linear equations.

We give here a brief introduction to what is known as linear programming. The problems are of a type which manufacturers meet when they try to make or distribute their products in the most economical way. The central mathematical topic is that of linear inequalities and the first objective is to show that a line (such as $x + y = 3$) divides the plane into two half-planes (given by $x + y > 3$ and $x + y < 3$).

Not all pupils should be able to carry out a problem on their own. This does not matter. The purpose is to develop insight into linear equations and inequalities, and to show the usefulness of inequalities.

1. Graphs of Inequalities

As mentioned in the introduction, the aim here is to show that a linear equation gives a line which divides the rest of the plane into two half-planes, each given by a linear inequality.

After completing the discussion in the text of the graphs of $x + y = 3$, $x + y > 3$, and $x + y < 3$, you may wish to review the ideas with the following Class Activity.

1. Have a pupil draw the graph of $2x + y = 10$ on the blackboard.
2. Is the point (4, 2) on the graph?

3. If the point (4, y) is in the half-plane above the graph, what numbers can y be?

4. Show all points (4, y) such that $y > 2$.

5. What values can $2x + y$ have for these points (4, y) above the graph?

6. If $2x + y > 10$, where is the point (4, y)?

7. If $x = 3$, what must y be to give points (x, y) in the upper half-plane?

8. What inequality will describe all points in the upper half-plane?

9. What inequality gives all points in the lower half-plane?

Other questions at this point might be:

10. What inequality describes the half-plane

(a) to the right of the graph of $x + 4y = 12$?

(b) to the left of the graph of $x + 4y = 12$?

(Examine the point (x, 2), for instance, for values of x which put the point on, to the right of, or to the left of the given line.)

11. Draw the graphs of the inequalities (on the same axes) $x + y > 2$ and $x + 3y \leq 6$. Name four points whose coordinates satisfy both inequalities.

2. An Application (Linear Programming)

It is not necessary to make a point of introducing the name "linear programming". Some classes may be put off by the name while others may be stimulated by it.

The first task is to establish the inequalities. This does not differ in any way from the methods used in working problems earlier in the course. It should not present much difficulty if it is not rushed.

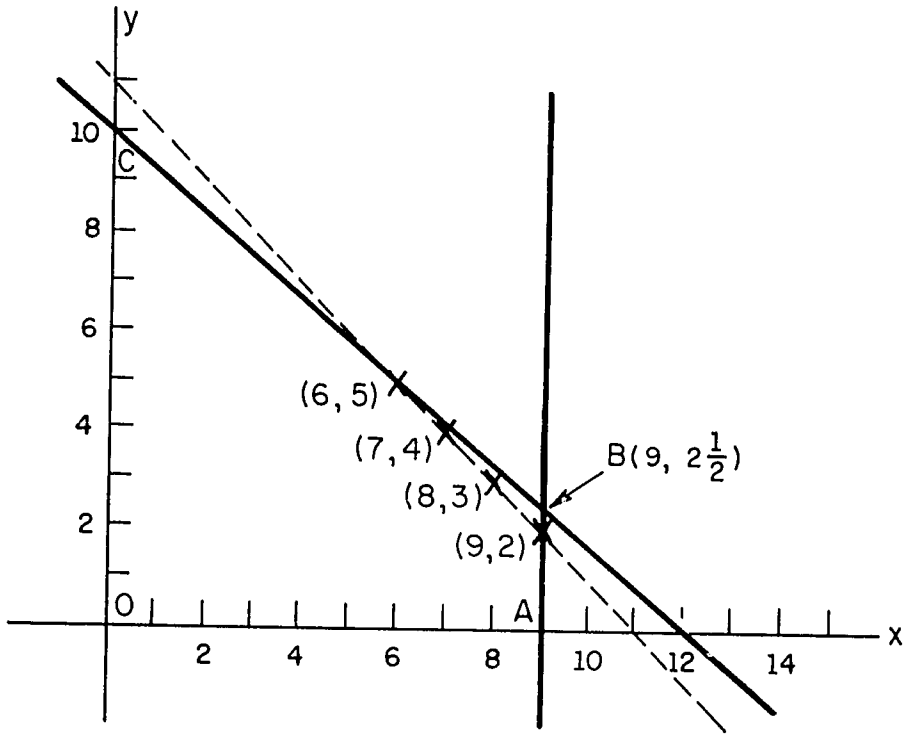
When graphing the lines which give the half-planes, the easiest way will usually be to find where the lines cut the axes. These two points should be all that is necessary to draw the lines. The important thing is to be able to draw the lines quickly.

The polygon containing the points whose coordinates satisfy all the inequalities, will be built up step by step. The order is not important.

To indicate the half-plane given, for instance, by $5x + 6y \leq 60$, we shade the region for which this is <u>not</u> satisfied. This is not the usual practice, but it gives a clearer diagram eventually. You may find that a coloured pencil makes the shading more clear. We eventually have an unshaded region which is the set of points whose coordinates satisfy all the inequalities. This is shown in the figure on page 276 of the Student Text.

Having drawn the polygon, we must now find the point (x, y) within it which makes $(x + y)$, the number of books, as large as possible. That is, we want the line (with equation $x + y = c$) which cuts the polygon and has c as large as possible. This is most easily found by using a set square and ruler. Arrange the instruments so that one edge of the set square gives lines parallel to $x + y = c$. It is then easy to see, by sliding the set square along the ruler, that the required line is the one which passes through B. This is $x + y = 11$. The solution is $x = 6$, $y = 5$, that is, 6 books from Nazarili and 5 books from Shariff.

It will be necessary to point out that x and y must be whole numbers. We obviously cannot buy $7\frac{1}{2}$ books. This problem was made easier by the fact that the point B itself has whole number co-ordinates. This does not always happen. Consider the case when $x \leq 9$. B is now the point $(9, 2\frac{1}{2})$ and the required line through B has the equation $x + y = 11\frac{1}{2}$. Since $x + y$ must be a whole number, we use the line $x + y = 11$. We see from the figure on the next page that there are four points on this line with integral coordinates and which are contained in the region OABC. They are $(9, 2)$, $(8, 3)$, $(7, 4)$ and $(6, 5)$.



Any of these, then would be a solution. There is one more factor to consider. Although it is not so stated in the question, we should expect the cost in this case to be kept as low as possible. That is, we want $5x + 6y$ to be as low as possible. The four points make $5x + 6y$ equal to 57, 58, 59, and 60 respectively, so the best solution is $x = 9$, $y = 2$, that is, 9 books from Nazarili and 2 from Shariff.

If Nazarili had 5 books, one should buy 5 books from Nazarili and 5 from Shariff. If Nazarili had 4 books, one should buy 4 books from Nazarili and 6 from Shariff.

3. Another Application

Very little additional comment is needed on this second example. The method is exactly the same. The unit "ton-miles" will need careful explanation. Reference should also be made to other compound units such as man-hours, machine-hours, etc. The manufacturer should make 10 tons per day of A and 15 tons per day of B. If the profit were £2 per ton for both A and B, he should make $8\frac{1}{2}$ tons per day of A and 17 tons per day of B.

Answers to Problems 18-3 | Student Text Page 280

1. (a) (6, 2) (b) $(\frac{15}{4}, \frac{15}{2})$

(c) (6, 6)

(d) Any point on the line $2x + 3y = 30$ between the points (6, 6) and $(\frac{15}{4}, \frac{15}{2})$

2. The inequalities are

$$5x + 5y \geq 50$$
$$15x + 5y \geq 60$$

You may wish to point out that these are equivalent to

$$x + y \geq 10$$
$$3x + y \geq 12$$

but this is not necessary.

The cost, $2x + y$, is to be a minimum.

The solution is 1 oz. of "X" to 9 oz. of "Y".

3. This table shows the number of units transported.

From \ To	Thompson Falls	Nanyuki
Nairobi	$15 - x$	$20 - y$
Nakuru	x	y

Number of units sent from Nakuru is $x + y$. Therefore $x + y \leq 20$.

Number of units sent from Nairobi is $35 - (x + y)$. Therefore

$$35 - (x + y) \leq 25$$

or $$10 \leq x + y$$ Subtract 25 from both sides and add $x + y$ to both sides.

Also, 15 units go to Thompson Falls and 20 units go to Nanyuki.

Therefore $x \leq 15$

$y \leq 20$

The system of inequalities is

$$\begin{cases} 10 \leq x + y \leq 20 \\ x \leq 15 \\ y \leq 20 \end{cases}$$

The cost of transport will be

$$[36(15 - x) + 45(20 - y) + 30x + 42y] \text{ shillings}$$

or $$(1440 - 6x - 3y) \text{ shillings}$$

We want this to be as small as possible so we want $6x + 3y$ to be as large as possible. The solution is $x = 15$, $y = 5$. So numbers of units sent are:

From \ To	Thompson Falls	Nanyuki
Nairobi	0	15
Nakuru	15	5

4. The completed table is:

From \ To	P_1	P_2	P_3
S_1	x	y	$9 - (x + y)$
S_2	$5 - x$	$6 - y$	$6 - [(5 - x) + (6 - y)]$ or $(x + y) - 5$

The inequalities are

$x \leq 5$ (from $5 - x \geq 0$)

$y \leq 6$ (from $6 - y \geq 0$)

$x + y \leq 9$ (from $9 - (x + y) \geq 0$)

$x + y \geq 5$ (from $(x + y) - 5 \geq 0$)

Cost of transport in shillings is:

$$6x + 3y + 4[9 - (x + y)] + 4(5 - x) + 2(6 - y) + 6[(x + y) - 5]$$

or $$4x + 3y + 38$$

$4x + 3y$ must therefore be as small as possible.

The solution is:

$$x = 0, \quad y = 5$$

The table indicating the number of tons delivered is:

From \ To	P_1	P_2	P_3
S_1	0	5	4
S_2	5	1	0

5. Let the mine at A operate for x days per week.

Let the mine at B operate for y days per week.

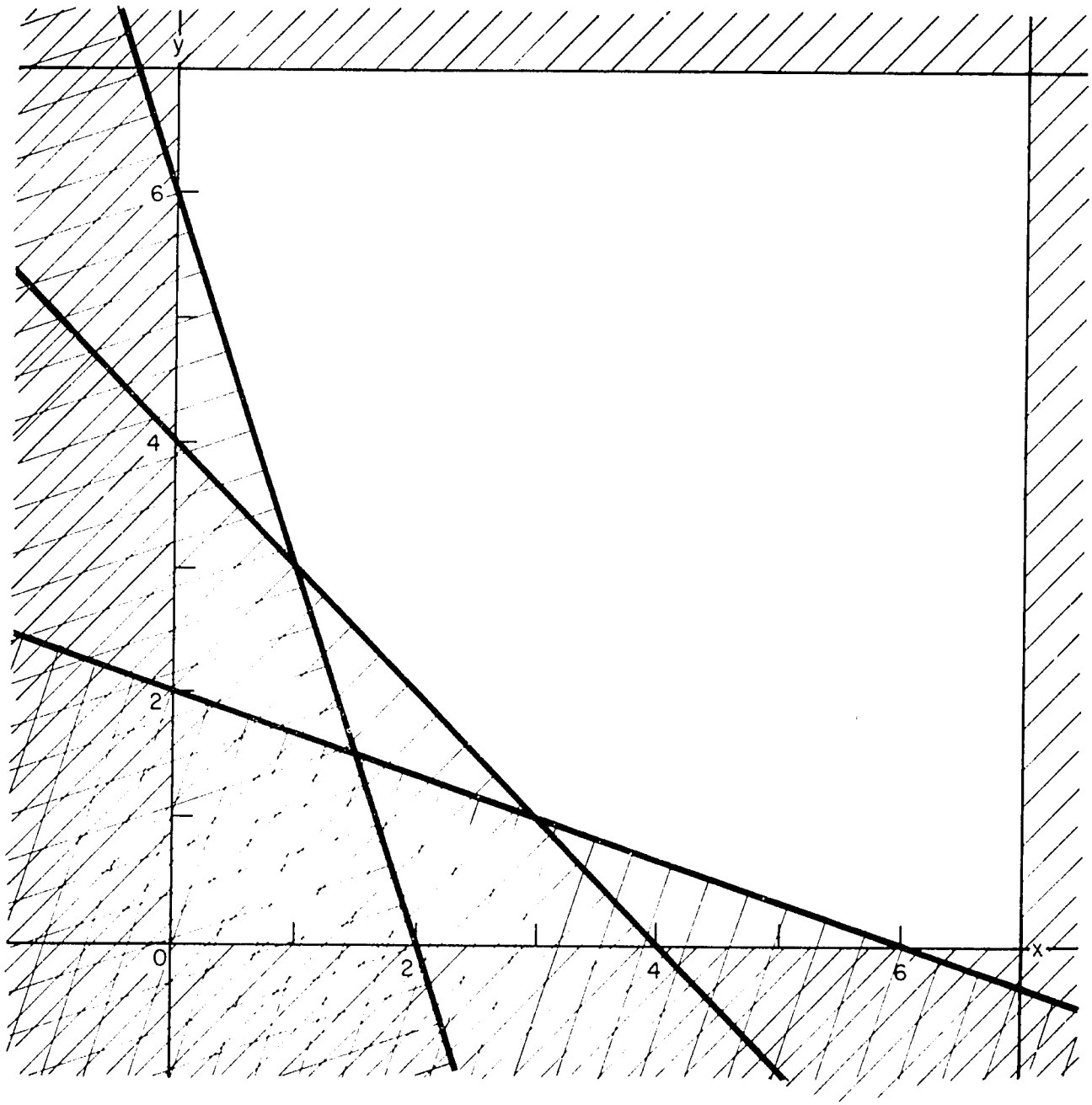
The following table helps to clarify the information:

Mine \ Ore	Grade 1	Grade 2	Grade 3	Cost per day
A	3	1	2	£40
B	1	1	6	£30
Orders for	6	4	12	

The system of inequalities is:

$$\begin{cases} 3x + y \geq 6 \\ x + y \geq 4 \\ 2x + 6y \geq 12 \\ 0 \leq x \leq 7 \\ 0 \leq y \leq 7 \end{cases}$$

The cost per week of running the mines is £(40x + 30y), that is, 10(4x + 3y). We therefore want 4x + 3y to be as small as possible.



The solution is: $x = 1$ $y = 3$

Mine A works for 1 day per week.

Mine B works for 3 days per week.

CHAPTER 19

OTHER GRAPHS

Much of this chapter may be considered optional if it is not appropriate for your pupils at this time. Look ahead through the chapter and plan in advance what parts of the chapter you want to emphasize.

19-1 Non-linear Graphs

The purpose of this section is to acquaint the children with some graphs which are not straight lines. Since all these graphs will be considered at a more advanced level in later years, how much you do with this section should be determined by time and the children's interest. If time is short, you may wish to spend only 15 or 20 minutes in class on the two graphs in the class activity, and either omit the rest or assign the problems as long-range individual projects or perhaps as extra-credit work for interested children. If your class has much difficulty doing the graphs, or finds them uninteresting, then it is probably a good idea to omit most of the problems. You may find, however, that many children get considerable enjoyment from discovering that graphs of open sentences can be circles, squares, parabolas, etc. as well as straight lines. If the class seems to enjoy the section, it is suggested that you spend a part of class time on Problems 19-1 for several days, as you go on into other sections.

Teaching Suggestions for Class Activity

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The graph of $y = |x|$. The graph may be done either on the blackboard with the class working together, or on graph paper with each child making his own graph.

1.

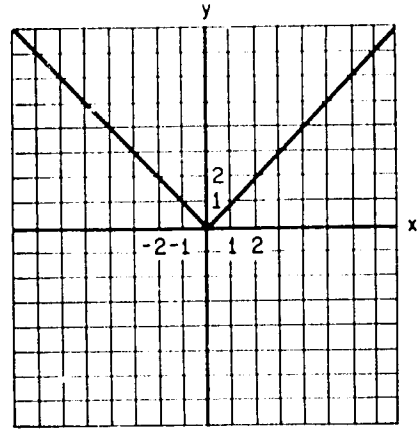
x	0	1	2	3	4	5
y	0	1	2	3	4	5

3.

x	-1	-2	-3	-4	-5
y	1	2	3	4	5

4. $y = -x$

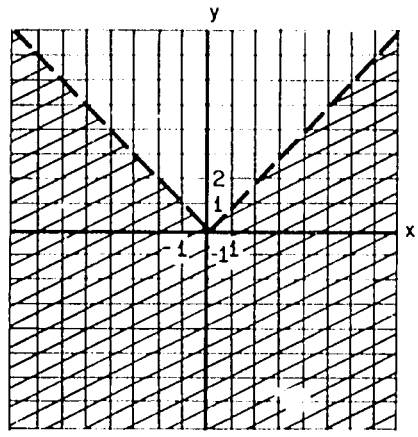
5. Completed graph of $y = |x|$.



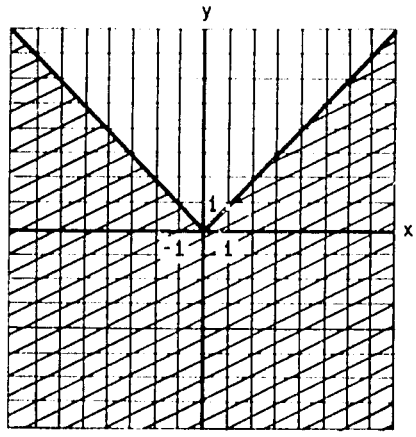
6. The graphs $y = |x|$ and $y = x$ are the same for $x \geq 0$; the graphs of $y = |x|$ and $y = -x$ are the same for $x \leq 0$.

7. This question's purpose is to have the children notice the connection between the graph and the equation. Since $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x \leq 0$, the children should not be surprised that the graph of $y = |x|$ is made up of parts of the graphs of $y = x$ and $y = -x$. Additional questions you can ask: "According to your graph, what are the possible values of y?" Answer: $y \geq 0$, i.e. y cannot be negative. "Could you have predicted this before you drew the graph?" Answer: Yes, y cannot be negative because the absolute value of any number cannot be negative.

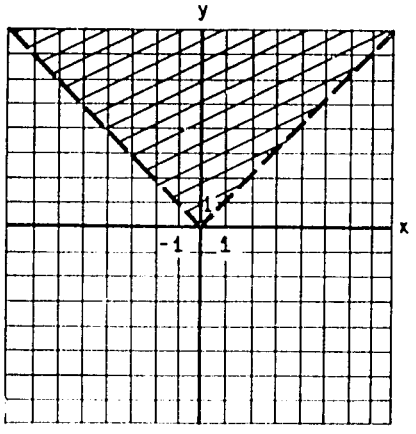
8. (a) The graph of $y < |x|$ is the region below the graph of $y = |x|$. In the picture below, the rays are dashed to show that these points are <u>not</u> part of the graph; the shading is used to show points that do belong to the graph.



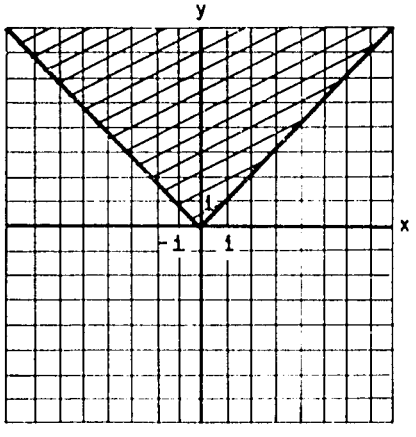
(b) The graph of $y \leq |x|$ is the region below the graph of $y = |x|$ <u>and</u> the graph of $y = |x|$.



(c) The graph of $y > |x|$ is the region above the graph of $y = |x|$.



(d) The graph of $y \geq |x|$ is the region above the graph of $y = |x|$, <u>and</u> the graph of $y = |x|$.



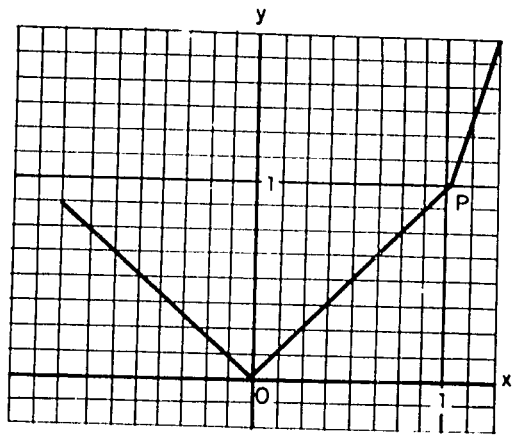
<u>The graph of $y = x^2$</u>

1.

x	-2	-1	0	1	2	3	4	5
y	4	1	0	1	4	9	16	25

3. This section needs to be handled with care. If the class is doing the graph on the blackboard, you can ask someone to come up and complete the graph as he thinks it should be. If the children are making their own graphs, you can have all the <u>different</u> ways they have completed the graph put on the board. They should realize that the graph is symmetric about the y-axis, and therefore, since (3, 9) is in the truth set, (-3, 9) is also. It is likely that many, or even all, of the children will complete the graph by connecting the points by line segments, which is not surprising since their previous experience has not included curves. If no one objects to the line segments,

you can direct their attention to the piece of their graph for x between 0 and 1, perhaps draw an enlarged picture on the blackboard,



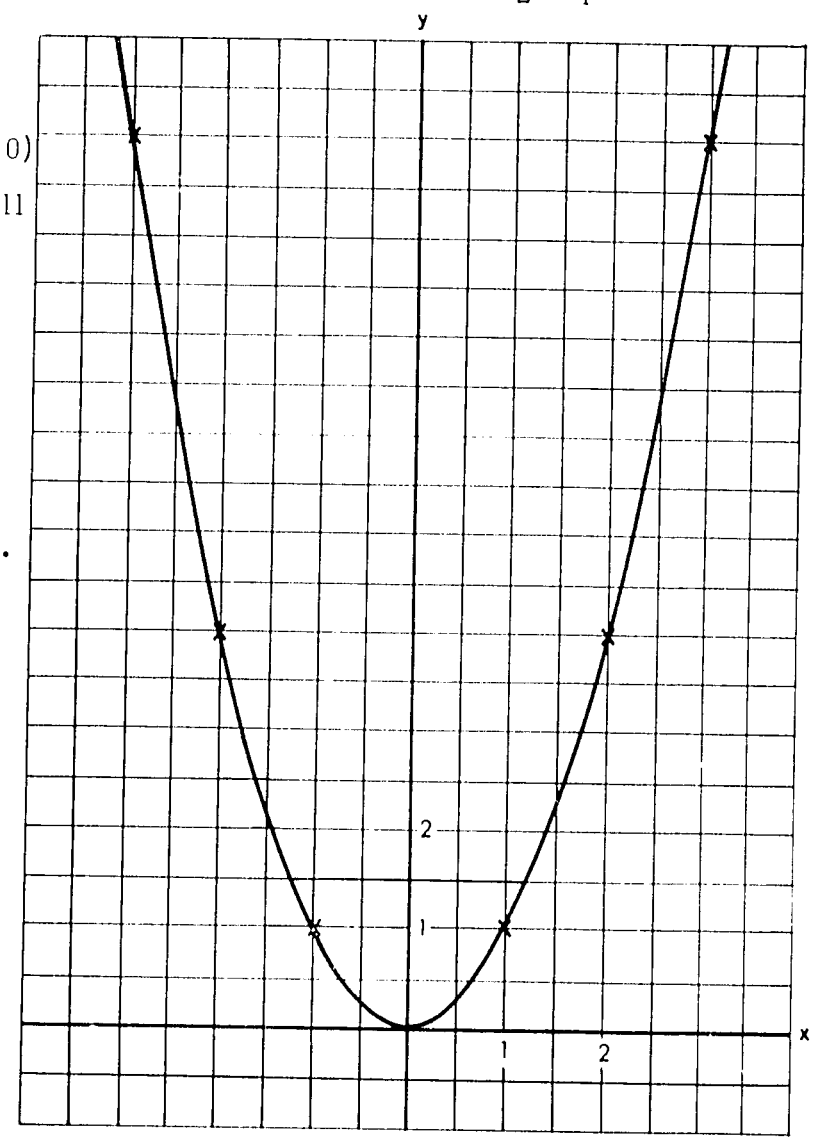
and ask the following questions:

(a) What is the equation of the line that contains $\overline{OP}$? [Ans: $y = x$].

(b) If x is $\frac{1}{2}$, and $y = x$, what is y? [Ans: $\frac{1}{2}$]

(c) Is $(\frac{1}{2}, \frac{1}{2})$ in the truth set of $y = x^2$? [Ans: No, $(\frac{1}{2}, \frac{1}{4})$ is.]

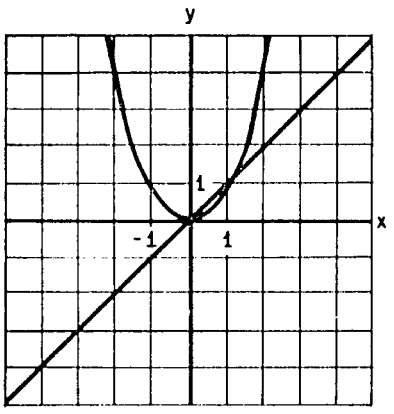
At this point in the discussion, the children should understand that connecting (0, 0) and (1, 1) by a line segment will not give a correct graph of $y = x^2$, and that they need to find more ordered pairs that belong to the truth set. It is probably a good idea to find pairs for such x-values as $\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}$. After the points have been plotted, someone can come up and mark a point that he thinks belongs to the graph, estimate the ordered pair that goes with it, and check it in the equation to see how close it is. Points can then be marked for other parts of the graph, and finally the curve drawn.



Some questions you can ask when the graph of $y = x^2$ is completed are:

1. Looking at the graph, do you find any points with a negative y-coordinate? [No.] Could you have predicted this <u>before</u> you drew the graph? [Yes; y cannot be negative because the square of any real number cannot be negative.]

2. Have the children draw the graph of $y = x$ on the same axes as $y = x^2$,



and ask:

(a) For what values of x are the y-values of $y = x^2$ greater than the y-values of $y = x$? [For $x > 1$ and $x < 0$; from the graphs we see that $y = x^2$ is <u>above</u> $y = x$; from the algebra we know that the square of any number greater than 1 or less than 0 is greater than the number.]

(b) For what values of x are the y-values of $y = x^2$ <u>less</u> than the y-values of $y = x$? [For x between 0 and 1; the graphs show that $y = x^2$ is <u>below</u> $y = x$; also the square of any number between 0 and 1 is <u>less</u> than the number.]

(c) For which values of x are the y-values of $y = x^2$ and $y = x$

equal? [For $x = 0$ and $x = 1$; the graphs intersect at the points (0, 0) and (1, 1).]

3. For $y = x$, if x increases one, how much does y increase? [1]. For $y = x^2$, if x increases one, how much does y increase? [It depends on the value of x.]

4. Is (12, 144) in the truth set of $y = x^2$? [Yes.] What other ordered pair do you then know is in the truth set? [(-12, 144)] Is there a line of symmetry for the graph of $y = x^2$? [Yes, the y-axis.] What is its equation? [$x = 0$.]

Answers to Problems 19-1

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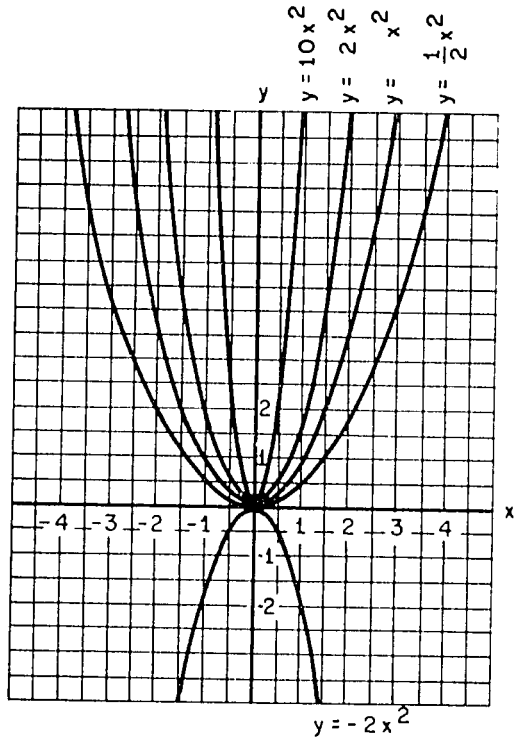
1. This problem you may wish to do in class right after doing the graph of $y = x^2$ if there is time. Additional questions you might ask when the graphs to the right have been completed:

(a) Without drawing the graphs, how could you describe the graphs of $y = -x^2$, $y = -10x^2$, $y = -\frac{1}{2}x^2$? [Several answers are possible; for example, the graph of $y = -10x^2$ could be described as the graph of $y = 10x^2$ "flipped over" the x-axis; or the graphs of all of them can be obtained by turning the paper upside-down and re-labeling the axes.]

(b) Would $y = 100x^2$ be thinner or fatter than $y = x^2$? [Thinner]



How about $y = \frac{1}{10}x^2$? [Fatter.] Can you name an equation of a parabola that would be very fat? [$y = \frac{1}{100}x^2$, $y = \frac{1}{1000}x^2$, etc.]

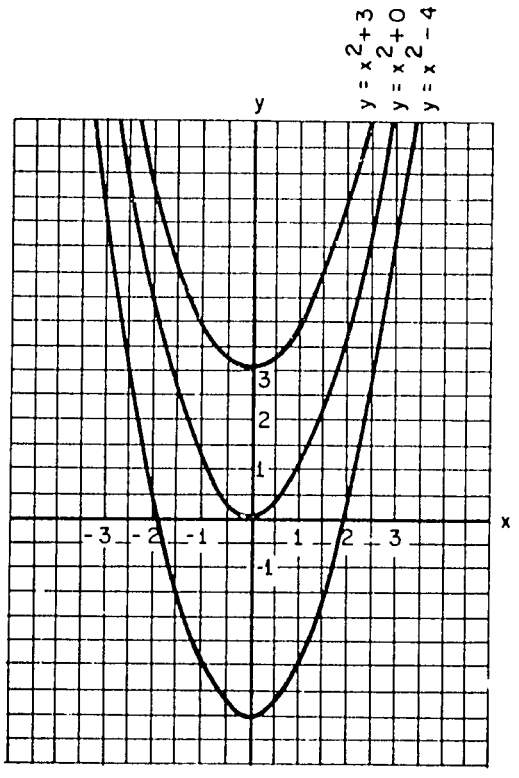
(c) When $x > 0$, for which graph does the y-value increase faster, $y = 100x^2$ or $y = x^2$? [$y = 100x^2$]; $y = \frac{1}{100}x^2$ or $y = x^2$? [$y = x^2$.]

(d) <u>If</u> the children can use variables well, you can describe <u>all</u> the parabolas whose turning points are at the origin by the equation $y = kx^2$, where k is any real number, and then ask the following questions:

(1) For what values of k does the parabola open up? [$k > 0$.] For what values of k does the parabola open down? [$k < 0$.] What happens when $k = 0$? [$y = kx^2$ becomes $y = 0$. The equation for the x-axis.]

(2) For what values of k is $y = kx^2$ thinner than $y = x^2$? [$k > 1$ or $k < -1$.] For what values of k is $y = kx^2$ fatter than $y = x^2$? [$-1 < k < 1$.]

2.



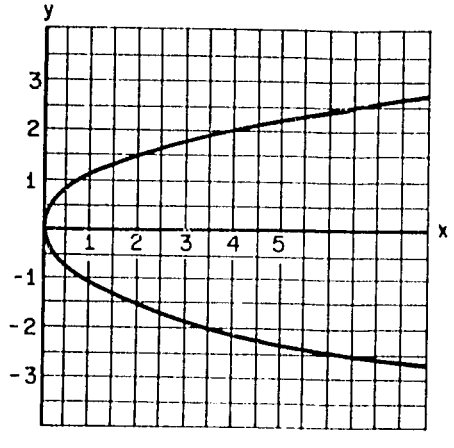
Additional questions you can ask when the above graphs are completed:

(a) Without graphing, can you describe the graph of $y = x^2 + 1$? of $y = x^2 + 100$? of $y = x^2 - 100$? of $y = -x^2 + 100$? [The first three graphs have the same shape as $y = x^2$, but have been moved up or down the y-axis so that the turning points are at 1, 100, and -100 respectively; the last one is the graph of $y = x^2$ "flipped over" the x-axis and moved up the y-axis 100 units.]

(b) Without graphing, can you describe the graph of $y = 2x^2 + 5$? [Same shape as $y = 2x^2$ (thinner than $y = x^2$) but moved up the y-axis 5 units.]

(c) Again, if the children can handle variables well, you can now discuss the graphs of $y = x^2 + p$, for all real numbers p. The ordered pair (0, p) gives the parabola's turning point. In addition, the graphs of $y = kx^2 + p$ can be discussed. The children may be able to give rough descriptions of the parabola for various values of k and p; for example, they might describe the graph of $y = -100x^2 + 10$ as: A parabola much thinner than $y = x^2$, opening down, with turning point at (0, 10). An interesting special case occurs for $k = 0$. The graphs are a family of straight lines $y = p$ parallel to the x-axis.

3.

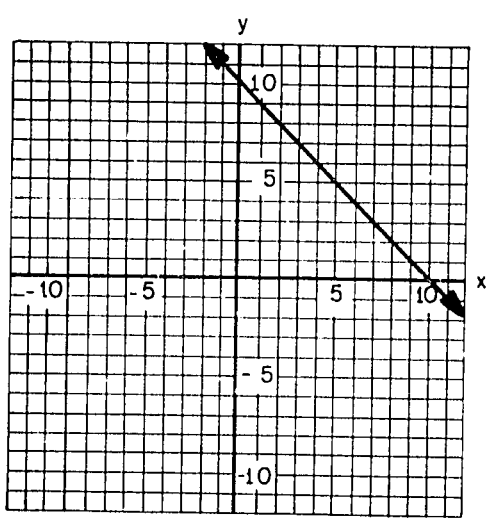


$x = y^2$

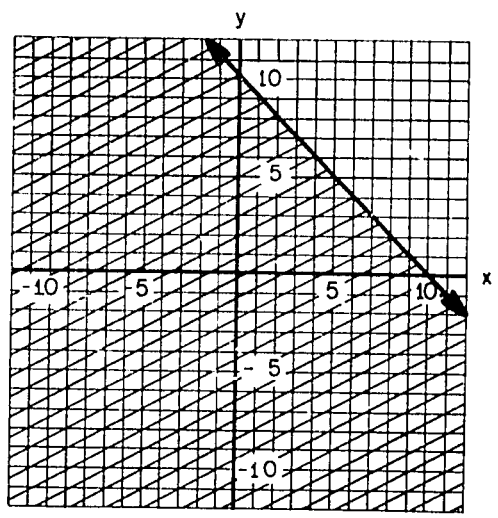
Additional questions similar to those in 1, 2, and 3 can be asked when this graph is completed. For example, describe the graphs of $x = -y^2$, $x = y^2 + 3$, $x = -10y^2 + 2$, etc. If the children happen to ask how you can get a parabola whose graph has its turning point off the axes, you can have them try something like $y = (x + 1)^2 - 2$, whose turning point is at $(-1, -2)$.

4.

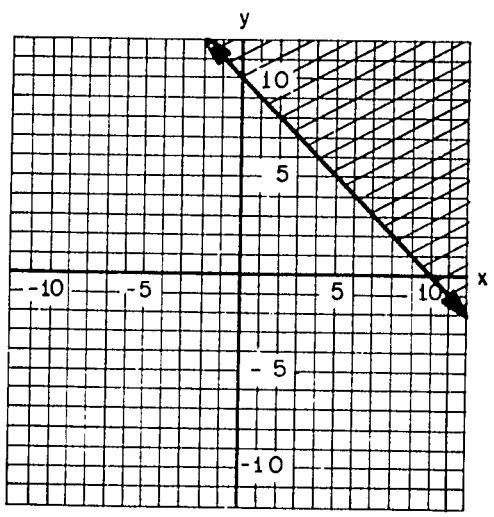
(a)



$x + y = 10$

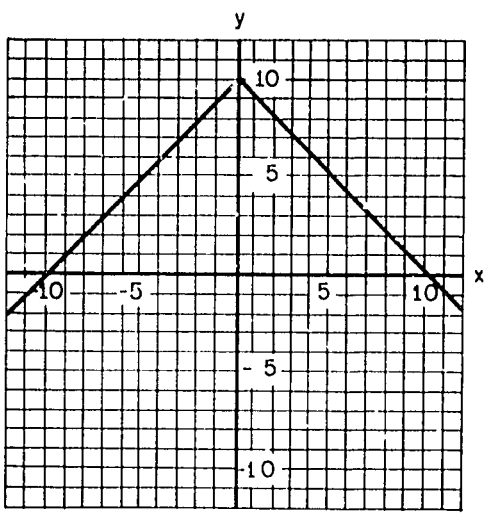


$x + y \leq 10$

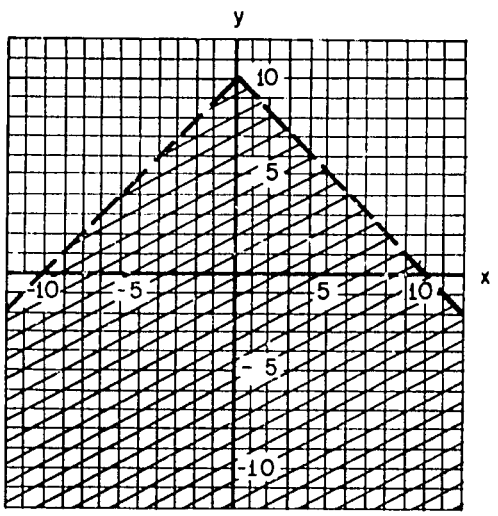


$x + y \geq 10$

(b)

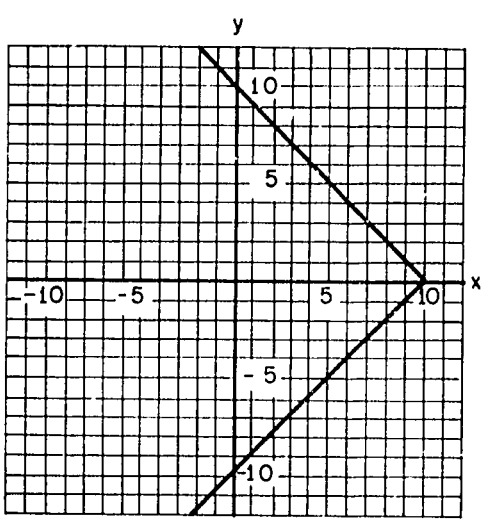


$|x| + y = 10$

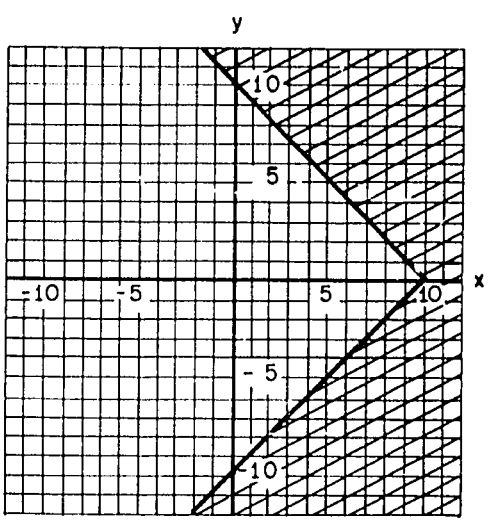


$|x| + y < 10$

(c)

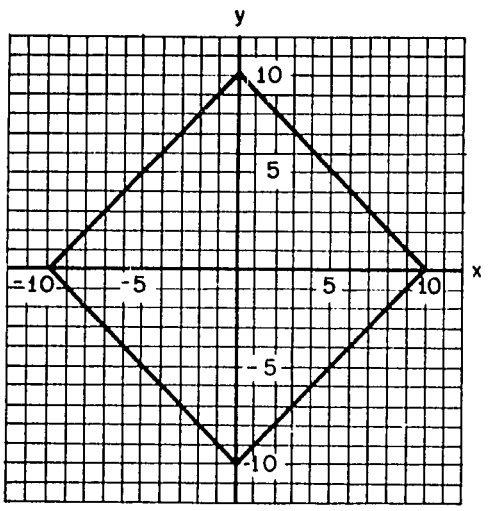


$x + |y| = 10$

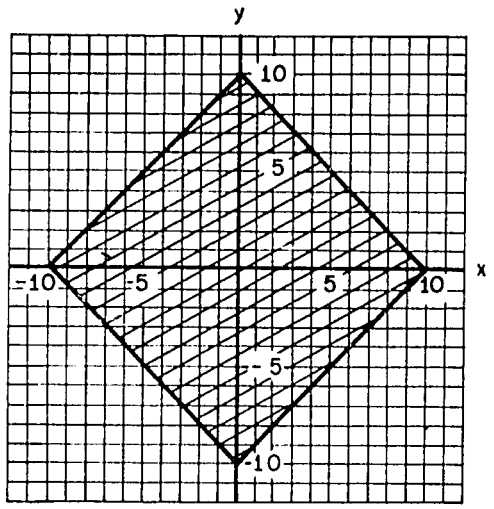


$x + |y| \geq 10$

(d)

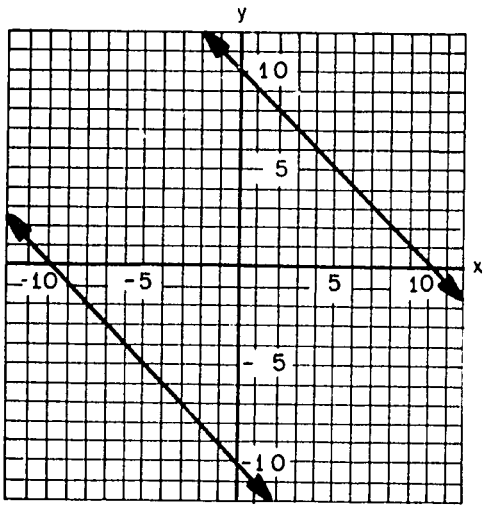


$|x| + |y| = 10$

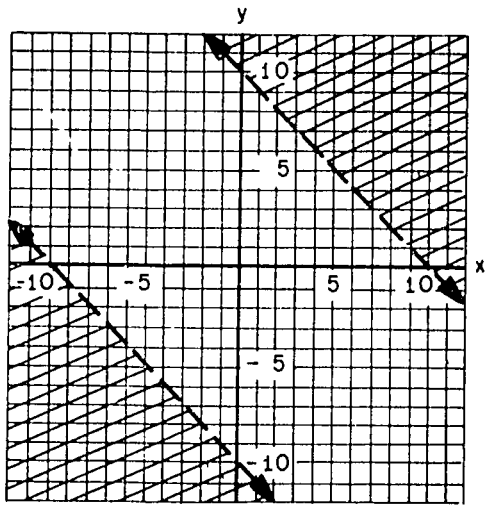


$|x| + |y| \leq 10$

(e)



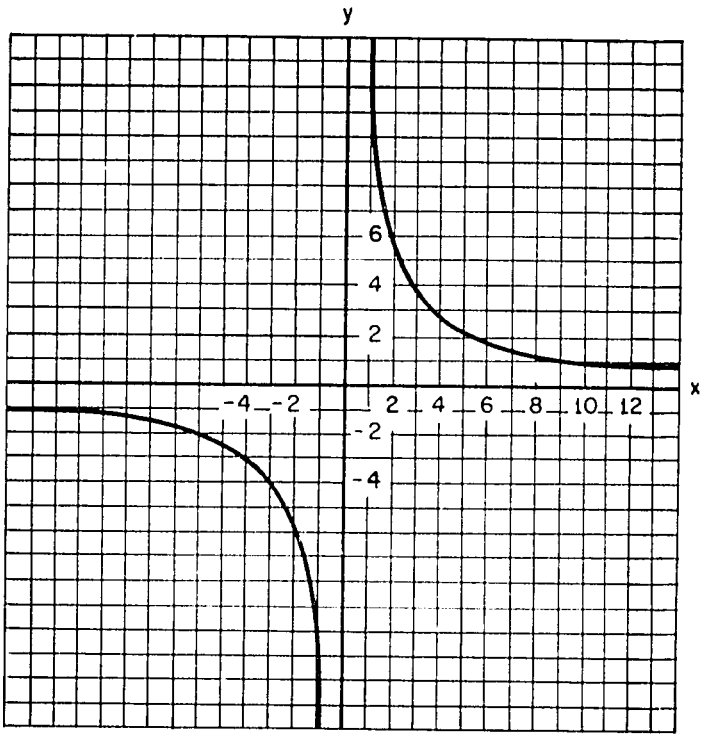
$|x + y| = 10$



$|x + y| > 10$

5. Since this is an identity, the graph is the number-plane; i.e., every ordered pair of numbers is in the truth set.

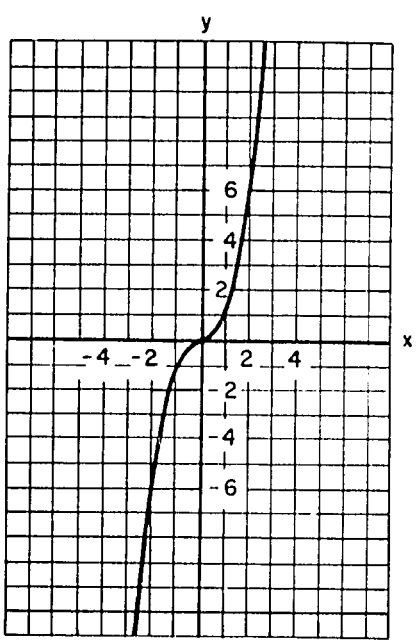
6.



$xy = 12$

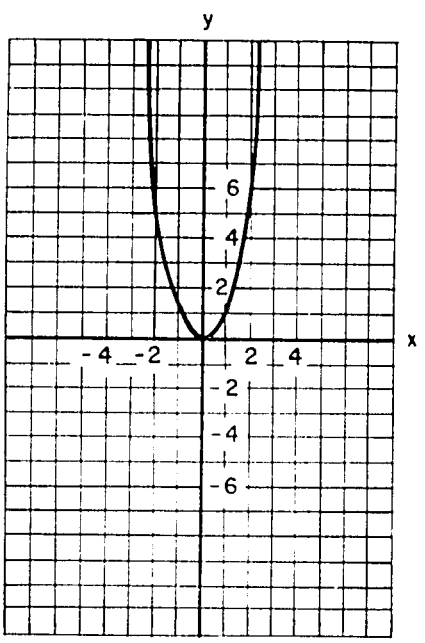
7.

(a)



$y = x^3$

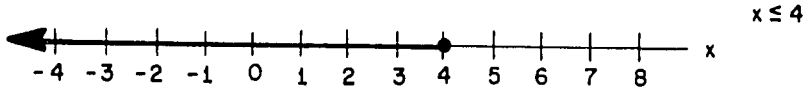
(b)



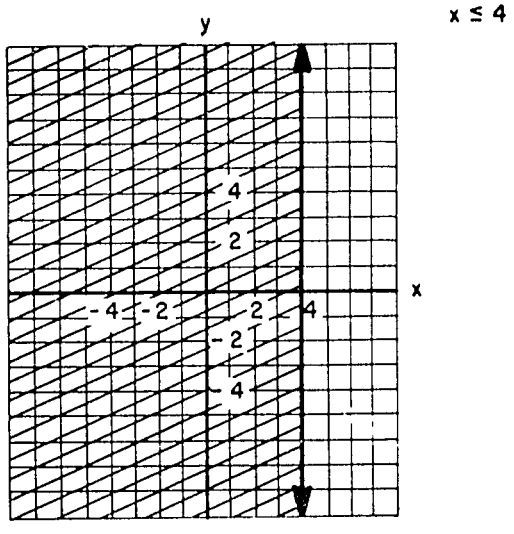
$y = |x|^3$

8. CHALLENGE PROBLEM

(a)



(b)



(c) The children have already gone from the number-line to two intersecting number-lines to get the number-plane; it is hoped that they can figure out that if they consider three mutually perpendicular number-lines intersecting at one point, they can describe any point in space by an ordered triple of numbers. They can have practice visualizing this by using a corner of the room and describing points in the room by an ordered triple of numbers. The first coordinate can be the distance of the point from the left side wall, the second coordinate its distance from the floor, and the third coordinate its distance from the front wall. (The distance can be measured in

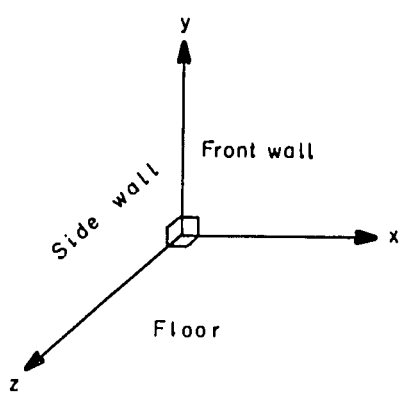
units of feet, yards, etc. as long as the same unit is used.) The children might describe the following points by ordered triples:

(1) One corner of the teacher's desk.

(2) The top of someone's head.

(3) A point marked with chalk on the blackboard. (If the board is at the front of the room, the third coordinate will be zero.)

(4) A point somewhere on the floor. (The second co-ordinate will be zero.) You can ask someone to find a point whose first co-ordinate is zero. (Any point on the left-wall.) You can ask where the origin, (0, 0, 0), is, and then where the three axes are, and which is the x-axis (first co-ordinate), which is the y-axis (second co-ordinate). They will need a third variable, say z, for the third axis. They should now have this picture in mind:



Ask the children how they could describe with an ordered triple of numbers a point which is 2 units in from the left side wall, 3 units <u>underneath</u> the floor, and 5 units in from the front wall. [(2, -3, 5)]. They should realize that the three axes extend to include the negative numbers too.

For the graph of $x \leq 4$, if they have trouble seeing what it would be, start with $x = 4$, i.e. all the points whose first co-ordinate is 4. Since y and z are not limited in any way, this graph will be the whole plane of points 4 units to the right of the plane determined by the y-axis and the z-axis. The graph of $x \leq 4$ in the number-space consists of all points to the <u>left</u> of or on this plane. The children can visualize this by thinking of putting up a partition in their room which is four units from the left side wall. The points on one side of the partition will have the first co-ordinate bigger than four, on the other side less than four.

9. (a) $-2 \leq x \leq 3$
   (b) $x \geq 2$
   (c) $y < 2$
   (d) $y > x + 2$
   (e) $y \geq |x| - 1$
   (f) $y \leq -x^2$

10. <u>CHALLENGE PROBLEM</u>.

		Lines of Symmetry
1.	All parts	$x = 0$ (y-axis)
2.	All parts	$x = 0$ (y-axis)
3.		$y = 0$ (x-axis)
4.	(a)	$y = x + k$ for any k
	(b)	$x = 0$ (y-axis)
	(c)	$y = 0$ (x-axis)
	(d)	$x = 0$ (y-axis) and $y = 0$ (x-axis) and $y = x$ and $y = -x$
	(e)	$y = x + k$ for any k and $y = -x$
5.		Any line
6.		$y = x$, $y = -x$
7.	(a)	None
8.	(b)	$x = 0$ (y-axis)

19-2 The Graph of $y = x^2$ and Tables of Squares and Square Roots

Answers for Class Activity

Student Text Page 287

1.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	25	16	9	4	1	0	1	4	9	16	25

Any value of x and its opposite correspond to the same value of y.

2. The children should realize from looking at their previous sketch of $y = x^2$ that the x-axis should be placed at the bottom of their graph paper; because of symmetry, the y-axis should be down the middle of the paper. The scale to be used on each axis will depend on the particular graph paper you have, and the class should discuss this before they put the scales on the axes. The reason for including negative values of x for the graph is that we wish to have a graphical illustration that a positive number has two square roots when we come to this section later in the chapter.

3. The purpose of these questions is to remind the children that the graph of $y = x^2$ is a smooth curve, not a series of line segments.

x	0·5	1·5	2·5	3·5	4·5
y	0·25	2·25	6·25	12·25	20·25

A Table of Squares

The children will find that reading squares from their graphs does not give great accuracy. They can estimate to only one decimal place, and this may give a relatively large error even though they have drawn their graphs carefully. The children may have a variety of answers for how to obtain greater accuracy. A few of the possibilities are listed here.

1. Use a larger scale. This will mean a larger piece of graph paper will be needed to show the values of x from -5 to 5, or only a portion of this range can be shown on the paper.
2. Use a pencil with a very sharp point.
3. Plot more points before completing the curve.
4. Compute the squares arithmetically. (This, of course, will give the exact square.)

A Printed Table of Squares

The students should be encouraged to decide where the decimal point belongs by estimating the size of the answer. The method illustrated in the examples should give the students some help in locating the position of the decimal point. In all cases, the student should look at the size of his answer to make sure the answer is reasonable.

Answers to Problems 19-2a

Student Text Page 290

(a) 5·76

(b) 57,600

(c) 50,410,000

(d) 0·005041

(e) 86·49

(f) 0·0000008649

A Table of Square Roots

The purpose of the discussion in the first paragraph is to lead the children to see that since $y = x^2$ and $\sqrt{y} = |x|$ are equivalent equations (and therefore have the same graphs), they can use their graph of $y = x^2$ to find square roots of numbers from 0 to 25.

Since $\sqrt{y}$ represents only the non-negative square root of y, $\sqrt{y} = x$ is true only when $x \geq 0$. If $x < 0$, we must take the opposite of x to obtain a positive number, so if $x < 0$, $\sqrt{y} = -x$. Both these cases are included in the statement $\sqrt{y} = |x|$.

Yes, $y = x^2$ and $\sqrt{y} = |x|$ are equivalent equations.

Answers to Problems 19-2b

Student Text Page 291

1.

y	1	2	3	4	5	6	7	8	9	10	11	12	13
$\sqrt{y} = \|x\|$	1	1·4	1·7	2	2·2	2·4	2·6	2·8	3	3·2	3·3	3·5	3·6

y	14	15	16	17	18	19	20	21	22	23	24	25
$\sqrt{y} = \|x\|$	3·7	3·9	4	4·1	4·2	4·4	4·5	4·6	4·7	4·8	4·9	5

2.

y	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
$\sqrt{y} = \|x\|$	0·32	0·45	0·55	0·63	0·71	0·77	0·84	0·89	0·95

A Printed Table of Square Roots

This part may be omitted, especially if the children are not yet ready to handle the necessary work with radicals.

The children should find that the square roots of 400, 4, and 0·04 are 20, 2, and 0·2, all having the same digit 2. They can see from Problem 2 of Problems 19-2b that the square root of 0·4 is about 0·63, which has entirely different digits.

The children should be able to find values for only part of the table.

n	3600	360	36	3·6	0·36	0·036	0·0036
$\sqrt{n}$	60		6		0·6		0·06

When they have found that $\sqrt{360} \approx 18{\cdot}97$, they can find values for the rest of the table.

n	3600	360	36	3·6	0·36	0·036	0·0036
$\sqrt{n}$	60	18·97	6	1·897	0·6	0·1897	0·06

Answers to Problems 19-2c

Student Text Page 293

1. (a) 3·873 (d) 27·57

(b) 12·25 (e) 275·7

(c) 1·225 (f) 0·02757

2. CHALLENGE PROBLEM

A table of cube roots can be constructed either graphically or arithmetically. The graphical method is much easier, the arithmetic more accurate, but very time consuming. Probably a combination of the two would work best - the graph for a first approximation, and then arithmetic to get the accuracy needed. The graphical readings would require a graph of $y = x^3$.

3. CHALLENGE PROBLEM

Any real number has exactly one real cube root. The student should be reminded that this is not the case for square roots, where no negative number has a real square root, zero has one square root, and any positive number has two square roots.

4. CHALLENGE PROBLEM

The cube root table would need three columns, namely $\sqrt[3]{n}$, $\sqrt[3]{10n}$, and $\sqrt[3]{100n}$.

$\sqrt[3]{32}$ can be read in the $\sqrt[3]{n}$ column.

$\sqrt[3]{320}$ is $\sqrt[3]{32 \times 10}$ and can be read in the $\sqrt[3]{10\,n}$ column.

$\sqrt[3]{3200}$ is $\sqrt[3]{32 \times 100}$ and can be read in the $\sqrt[3]{100n}$ column.

$$\sqrt[3]{3{\cdot}2} = \sqrt[3]{32 \times 100 \times \frac{1}{1000}} = \sqrt[3]{32 \times 100} \times \frac{1}{10}$$

$$\sqrt[3]{0{\cdot}32} = \sqrt[3]{32 \times 10 \times \frac{1}{1000}} = \sqrt[3]{32 \times 10} \times \frac{1}{10}$$

$$\sqrt[3]{0{\cdot}032} = \sqrt[3]{32 \times \frac{1}{1000}} = \sqrt[3]{32} \times \frac{1}{10}$$

$$\sqrt[3]{320{,}000} = \sqrt[3]{32 \times 10 \times 1000} = \sqrt[3]{32 \times 10} \times 10$$

$\sqrt[3]{32}$ and $\sqrt[3]{0{\cdot}032}$ have the same digits, namely 317...

$\sqrt[3]{320}$, $\sqrt[3]{0{\cdot}32}$, and $\sqrt[3]{320{,}000}$ have the same digits, namely 684...

$\sqrt[3]{3200}$ and $\sqrt[3]{3{\cdot}2}$ have the same digits, namely 1474...

19-3 Other Graphs

This section, as was 19-1, is optional and should be done only if time permits and the children seem to enjoy graphing.

Teaching Suggestions for Class Activity

Student Text Page 294

One of the purposes of this section is to have the children realize that they can find some information about a graph by first considering the equation. The graph and work can be done on the board or individually as you choose.

1. y^2 becomes smaller.

2. The smallest either x^2 or y^2 can be is zero.

3. The largest x^2 can be is 25. It cannot be 100 since y^2 cannot be negative.

4. The possible x-values are from -5 to 5; same for y.

6.

x	-5	-4	-3	-2	-1	1	2	3	4	5	0
y	0	3 -3	4 -4	4·6 -4·6	4.9 -4.9	4·9 -4·9	4·6 -4·6	4 -4	3 -3	0	5 -5

7. Yes; additional ordered pairs:

x	4·6	-4·6	4·9	-4·9
y	2 -2	2 -2	1 -1	1 -1

The children will probably realize that from one ordered pair in the truth set, they can get others without doing additional arithmetic. It should now be obvious from the points plotted that the graph is a circle.

8. A circle is the set of all points at a given distance from a given point. You might ask the children what the centre and radius are for their graphs.

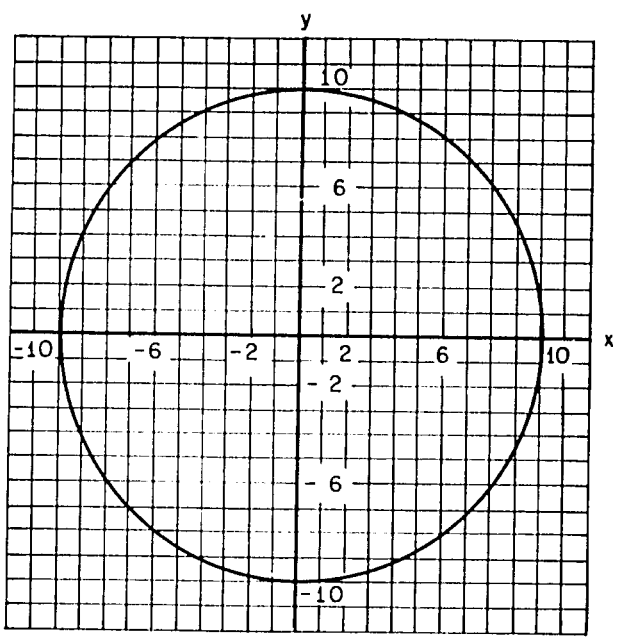
9. (a) The points in the interior of the circle.
   (b) The points in the exterior of the circle.
   (c) The circle <u>and</u> the points in the interior of the circle.
   (d) The circle <u>and</u> the points in the exterior of the circle.

Answers to Problems 19-3

Student Text Page 296

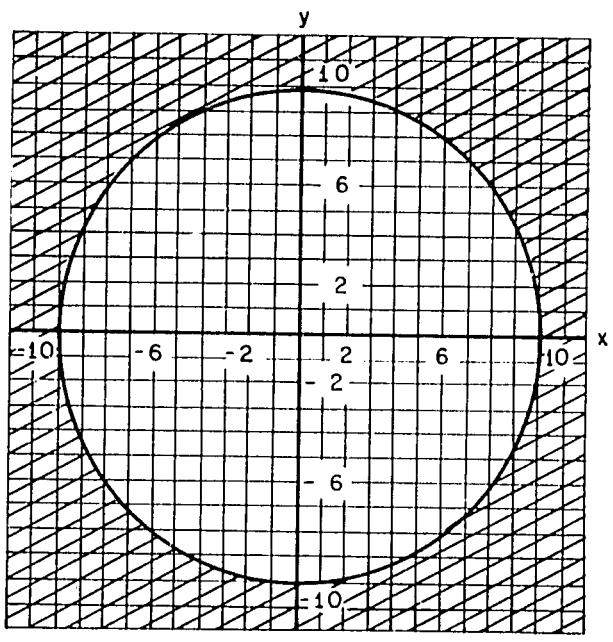
1.

(a)



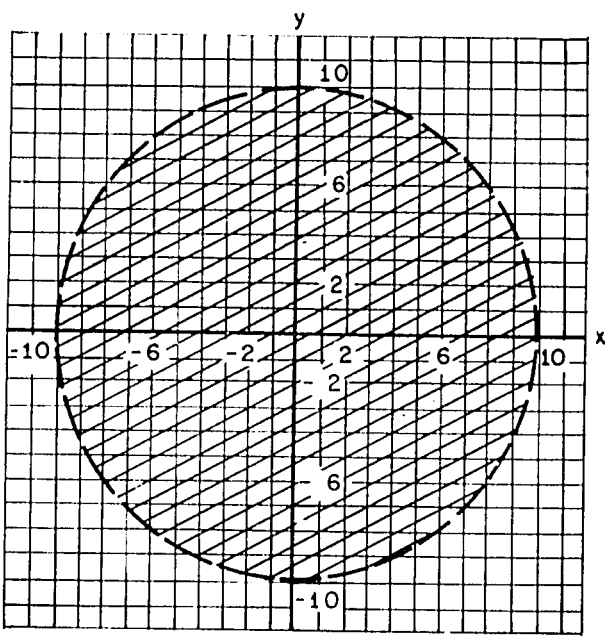
$x^2 + y^2 = 100$

(b)



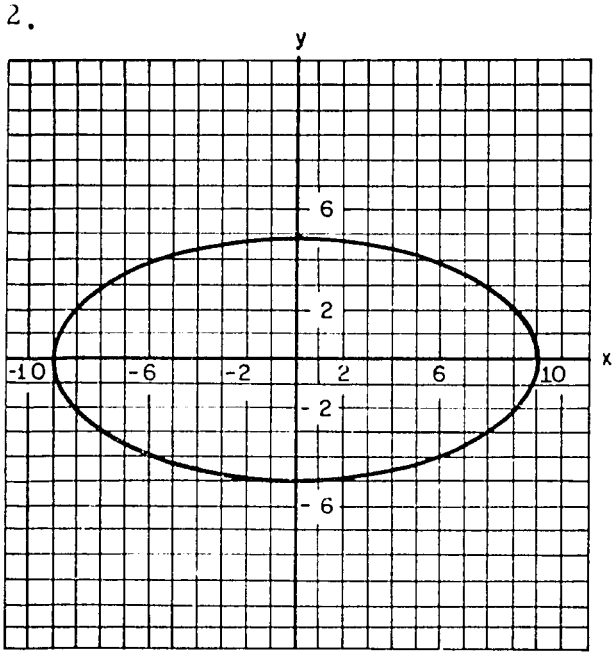
$x^2 + y^2 \geq 100$

(c)



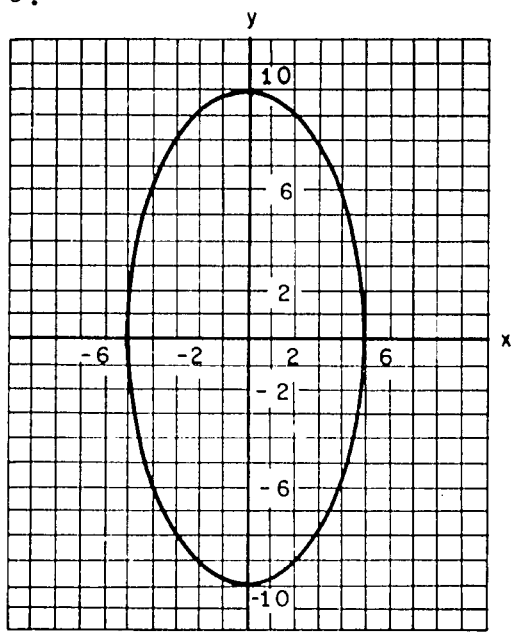
$x^2 + y^2 < 100$

2.



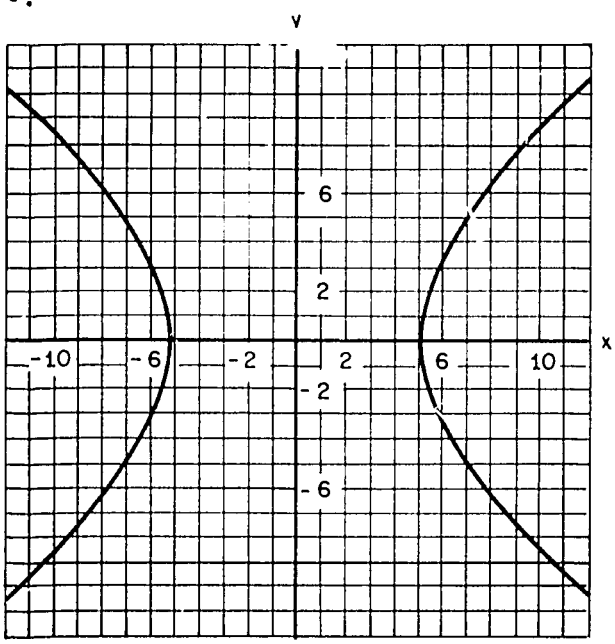
$x^2 + 4y^2 = 100$

3.



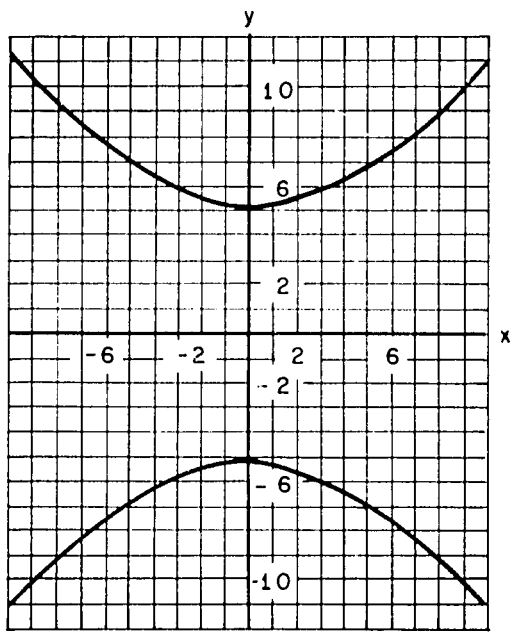
$4x^2 + y^2 = 100$

4.



$x^2 - y^2 = 25$

5.



$y^2 - x^2 = 25$

6. 1. (a) circle

2. ellipse; refer to Example 10 in Section 13-4, string experiment.

3. ellipse.

4. hyperbola; refer to Example 8 in Section 13-3.

5. hyperbola.

Additional questions you may ask after graphs are completed: Suppose you have just finished the graph in Problem 2 and you feel lazy, how could you obtain the graph in Problem 3 with very little effort? [Turn the paper through 90° and re-label the axes.] How could you obtain the graph in Problem 5 with very little work? [Turn the graph in Problem 4 through 90° and relabel the axes.]

7. For the graphs in Problem 1, any line through the center of circle is a line of symmetry. Since the center is the origin, the family of lines of symmetry can be given by $y = ax$, where a is any real number.

For the graphs in Problems 2-5, the x-axis (equation $y = 0$) and the y-axis (equation $x = 0$) are lines of symmetry.

19-4 More Ways of Picturing Numerical Information

The main aim of this section is to show clearly the difference between a discrete and a continuous variable. These terms need not be mentioned to the pupils.

1. Scatter Diagram: As mentioned in the student's text, this helps us see whether there is any relationship between the two sets of data, the heights and spans of boys. The name "scatter diagram" comes from the fact that the points are scattered on the graph paper.

The construction of the scatter diagram should be explained (or discussed) fully. There is no special significance in putting the heights on the horizontal axis. Questions such as, "What is the greatest (smallest) height (span) we have?" should help the pupils see why we start labelling our axes where we do (around 60) and stop

where we do (in the seventies). This should also bring out the reason why the scales we have shown have been used. In response to the question, "What do you think the diagram tells us?" you should expect quite a number of answers. Here you should consider each answer given and in turn try to ask the child who gives the answer why he thinks the diagram tells us what he says it does. The diagram indicates that the span of a boy 66 inches tall may be anything from 63 inches to 67 inches. You might ask the pupils whether they can name some factors which contribute to such a wide scatter. Included in these might be errors in measurement and different arm lengths.

At this stage, the pupils should be familiar with the two scales of temperature from their geography and science lessons. They should know that 32° F and 0° C represent the same temperature.

The difference between this scatter diagram and the one of heights and arm spans is that the former shows much more scatter than the latter. The formula $F = \frac{9}{5} C + 32$ if not already learnt in the science lessons, should be introduced. The pupils should be able to see that it is an equation of a straight line.

The points of the scatter diagram only approximate a straight line, again because of errors in reading our thermometer scales. Let the pupils draw what they think is the "best" straight line connecting the points, and then ask them why they think the line they have drawn is the best.

The points do not all lie exactly on a straight line because of our errors in measurement, in this case errors in reading the thermometer scales. If there were no such errors all the points would lie on a straight line. The pupils may be familiar with experiments in physics such as Hooke's law from which scatter diagrams may be drawn which approximate straight lines. You may want to seek out and refer to such experiments if they exist in your school.

Class Activity

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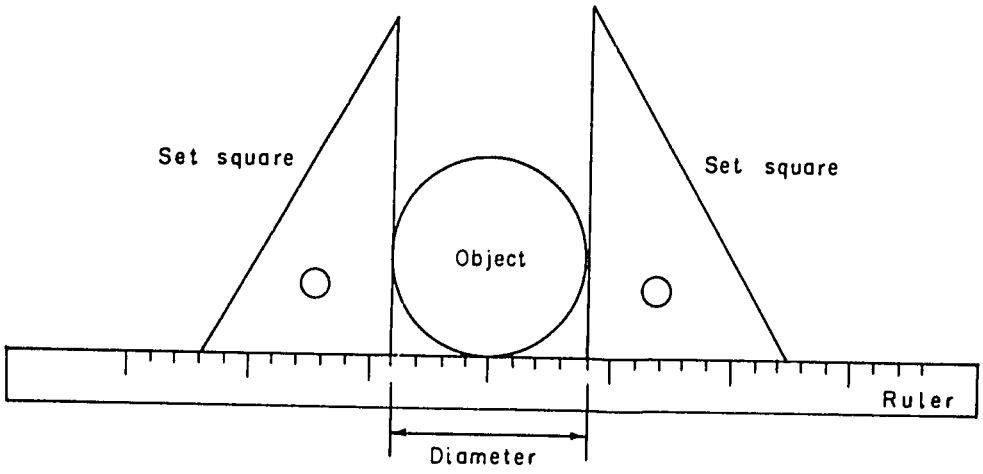
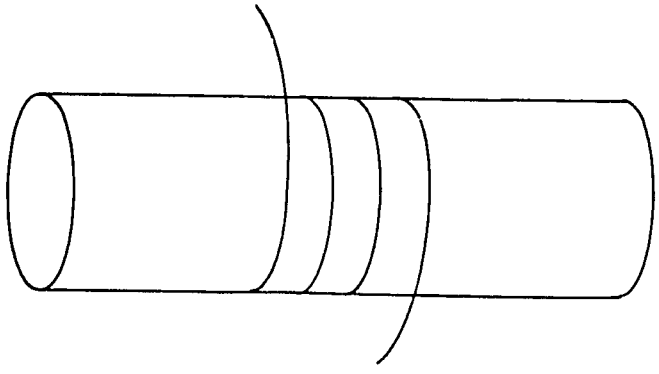
In doing the exercises in measurement, it may be a good idea to divide the class in groups of about four each. Methods of measuring the diameters and circumferences of the circular objects should be explained. One method for measuring the diameter is shown in the figure below.



For the circumference a number of ways may be used. Some of these are illustrated below:

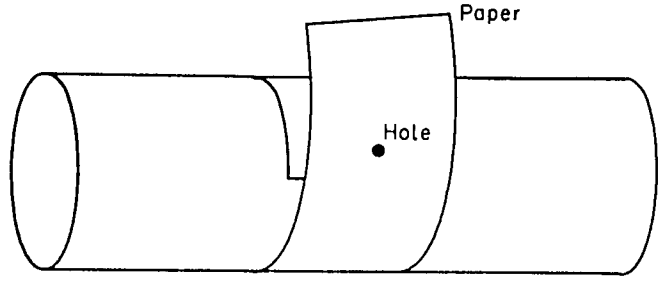
(a)



A piece of string is wrapped tightly round the object. Each complete turn represents the length of the circumference of the object. Wrapping ten

turns and dividing the total length by 10 gives a good approximation to the circumference.

(b)



A piece of paper is wrapped tightly round the object until it overlaps. A pin is then pushed into the paper so that it makes a hole right through the paper. When the paper is unwrapped, the distance between the two pin holes is the measure of the circumference.

(c) Another method is to roll the object on a ruler. A point is marked on the circular edge of the object. The marked point is placed on the ruler and the object is rolled carefully along the ruler (without slipping) until marked point is again on the ruler. The measure of the distance between the two positions is the measure of the circumference.

For the measurement of heights, use a vertical scale graduated in inches on the wall or blackboard. The pupils then stand against the wall (with their backs to the wall). A ruler on their heads helps to read the scale to the nearest inch. For the spans, use a horizontal scale on the blackboard, again graduated at one inch intervals.

2. Continuous Line Chart: The rainfall chart is an example of a quantity which is not varying continuously. The rainfall for the entire month may have fallen entirely in one, two, or ten days of the month. Some days may not have had any rain at all. Therefore we are able to say from the graph only that the accumulated rainfall for the month was so much. We can plot only twelve points on the graph, representing the twelve months of the year. The chart cannot tell us what was happening on various days of the month. With the temperature figures, we could have as many points as we please by reading the thermometer more frequently. The temperature is continuously changing. We therefore join the points we have decided to plot by a continuous free-hand line. There is no equation that we know of which connects the temperature and the time. Therefore we do not need to use a ruler to draw straight lines connecting the points to get the pattern of the changing temperature. From this kind of graph we could estimate the temperature at any given time during the day. For example, if we had read the temperature at 11:00 a.m. it is likely that it would have been about 77° F.

The pupils should be asked to draw the continuous line and then estimate from the graph what they think the temperature might have been at times other than the ones at which it was actually read and recorded.

Other examples of data suitable for this kind of graph are:

(a) the weight of a child at different ages.

(b) the temperature of a cooling body at different times.

(c) the girth of a tree at different ages.

(d) the height of a child at different ages.

The pupils should be encouraged to give a number of examples and the examples should be discussed to see whether they are suitable.

CHAPTER 20

SIMILAR FIGURES

20-1 Shapes of Plane Figures

The children know from previous work that congruent figures have the same shape and size. In this section, polygons which have the same shape but not necessarily the same size are being investigated. The chief aim of the section is to help the pupils to discover through class discussion the conditions under which two polygons have the same shape (i.e., are similar).

The discussion on the pair of circles and pair of squares should not take long. Here you tell them that each pair has the same shape without actually defining shape. They may have an intuitive idea of same shape; however, the idea of compressing or stretching uniformly to get one figure congruent to the other suggests in an indirect manner the idea of equality between the ratios of corresponding sides and the congruency of corresponding angles.

In order to talk clearly about "same shape", we recall the idea of one-to-one correspondence between parts of the two figures. The pupils became quite familiar with this in the chapter on congruence. A brief review of the idea and the notation should be sufficient.

Make sure that the pupils understand that ABCD $\longleftrightarrow$ RSTV and BCDA $\longleftrightarrow$ STVR are two equally good ways to write the <u>same</u> correspondence, but that ABCD $\longleftrightarrow$ RSTV and ABCD $\longleftrightarrow$ STVR are two <u>different</u> correspondences because different points correspond.

We start with squares since they are so obviously of the same shape. Help the pupils observe that the corresponding angles are congruent. They may not see right away how to express the relation between the measures of corresponding sides.

Previous Page Blank

Don't tell them. They will have more opportunity to find out as they go on. Let them experiment some with the measures they find. If someone discovers that the ratio of the measures of corresponding sides is $\frac{4}{3}$, very good. If not, go on.

The square and the rhombus obviously do not have the same shape. The corresponding angles are not congruent.

The two irregular quadrilaterals <u>are</u> of the same shape although they are turned in different positions. An advantage of the clear statement of the correspondence shows up here. The pupils should find that the corresponding angles are congruent and more of them may see the common ratio, $\frac{1}{2}$, of the measures of corresponding sides.

In the Class Activity, help the class to discuss carefully the various examples, to see the importance of stating the correspondence, and hopefully to formulate the two basic requirements for polygons of the same shape, namely

(1) Corresponding angles are congruent

(2) Ratios of measures of corresponding sides are equal.

The word "similar" may be introduced any time now when they seem ready for it.

<u>Answers to Class Activity</u> | Student Text Page 307

1. The following pairs have the same shape:

(a) ABCD $\longleftrightarrow$ PQRS (In this case, other correspondences, such as ABCD $\longleftrightarrow$ RSPQ, would also work.)

(d) ABCDE $\longleftrightarrow$ PQRST

(i) CTSA $\longleftrightarrow$ RMLK

(j) AFKP $\longleftrightarrow$ DQTL

(m) All circles have the same shape.

(b) and (c) are obviously not similar.

In (e), (h) and (k) the corresponding angles are congruent but ratios of measures of corresponding sides are not equal.

In (f) there are some congruent angles, but they are not <u>corresponding</u> angles.

In (g) the ratios of measures of corresponding sides are equal but the corresponding angles are not congruent.

Note: Remember in these that CTSA $\longleftrightarrow$ RMLK and SACT $\longleftrightarrow$ LKRM, for instance, name the same correspondence and either one is correct.

2. The polygons in (a) and (d) are similar because both conditions (1) and (2) above are satisfied. The polygons in (b) are not similar because condition (2) is not satisfied (although (1) is). The polygons in (c) are not similar because condition (1) is not satisfied (although (2) is).

Note: Help the pupils to see that it is not only <u>visually</u> they should test for similarity. They might be tempted to judge just by looks in the case of two rectangles. For example, in part (b) they may feel just by looking that the two rectangles are not similar. If so, change the measures of the sides of the smaller rectangle to 1 and 1·75 and ask them whether the two rectangles are still dissimilar. If they use a visual method only they may say the rectangles are still dissimilar whereas they are similar.

Help them to see also, in the case of two polygons, each with at least four sides, that (1) if only the corresponding angles are congruent, they are not similar and (2) if only the ratios of the measures of corresponding sides are equal, they are not similar. You need both (1) and (2) for the two polygons to be similar.

Congruent polygons are similar because they satisfy both the conditions for similarity of polygons. The ratio of the measures of corresponding sides of two congruent polygons is 1.

Answers to Problems 20-1

Student Text Page 310

1. Not similar. Condition (2) not satisfied.

2. Similar. Both conditions satisfied. Ratio is $\frac{b}{a}$. Since the triangles are equilateral, six correspondences are possible: XYZ $\longleftrightarrow$ PQR, XYZ $\longleftrightarrow$ QRP, XYZ $\longleftrightarrow$ RPQ, XYZ $\longleftrightarrow$ PRQ, XYZ $\longleftrightarrow$ QPR, XYZ $\longleftrightarrow$ RQP.

3. Similar. Both conditions satisfied. Ratio is $\frac{2}{1}$. Symmetry makes two correspondences possible: PORQ $\longleftrightarrow$ WTXS, PORQ $\longleftrightarrow$ XTWS.

4. Not similar. Neither condition satisfied.

5. Similar. Both conditions satisfied. Ratio is $\frac{3}{2}$. PRT $\longleftrightarrow$ FHG.

6. Similar. Both conditions satisfied. Ratio is $\frac{2}{1}$. FEJHG $\longleftrightarrow$ TSRQP.

7. Not similar. Condition (2) not satisfied.

8. Not similar. Condition (1) not satisfied.

In Problems 3, 5, and 6 the angles which are unmarked are determined by the other angles, which are known. For example in Problem 6, the measure of $\hat{E}$ is

$$540 - (p + x + y + z).$$

20-2 Scale Drawing - Plans and Maps

The purpose of this section is to show how similar polygons are used in the drawing of plans and maps. The treatment is not meant to be exhaustive at all and not much time should be spent on it.

You may want to speak of floor plans of rooms in buildings in addition to plans of pieces of land.

In the first example we say "one possible plan" because, by choosing different ratios for corresponding sides, many different plans can be drawn. The rectangle drawn is similar to the piece of land because the corresponding angles are congruent and the ratios of the measures of corresponding sides are equal. Since 30 feet is (30×12) inches, the ratio is $\frac{5}{30 \times 12}$ or $\frac{1}{72}$. If the plan is drawn 10 inches long and 8 inches wide, the ratio of corresponding sides is $\frac{1}{36}$.

These different ratios introduce the idea of scale. You may want to give more examples to strengthen the idea.

The architect will use a scale of $\frac{1}{12}$ to make a plan 30 inches long and 24 inches wide, the largest one he can make on paper 32 inches long and 27 inches wide with a 1 inch minimum margin.

An instructive class project is to let the pupils measure the cricket pitch or football field or even their classroom, decide on a scale, and draw plans. You may encourage them to use different scales and in the end compare their plans.

During this project may be a good time to help them to see what are the fewest measurements they need before the plan of a quadrilateral region can be drawn. See Problem 2 in Problems 20-2.

During the study of maps it would be helpful, of course, if you can have some actual maps from which the pupils can determine the scale and which they can use to find lengths of roads and rivers, distances between specified points, and areas of some simple rectangular regions.

Discuss with the class the difficulties in map construction for large regions of the earth's surface resulting from the fact that the earth is spherical. For small regions such as a town or city, the map on flat paper can be quite similar in all its parts to the region which it represents. The curved surface of a large region, such as the continent of Africa, cannot be similar in all its parts to the map on flat paper. There must be some distortion. Further difficulty is caused by mountains and valleys which certainly are not flat.

For a scale of 1:200, 1 inch on the map represents $16\frac{2}{3}$ feet. If the scale is given by "1 inch represents 1 mile", the ratio will be $\frac{1}{5280 \times 12}$ or 1:63,360.

Answers to Problems 20-2

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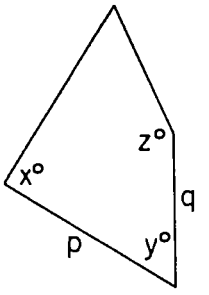
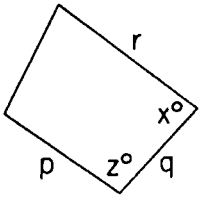
1. (b)	(i)	$a' = 108'$	$r° = 94°$	$s° = 61°$
	(ii)	$b' = 41'$	$c' = 36'$	$t° = 80°$
	(iii)	$d' = 44'$	$u° = 17°$	$v° = 13°$
	(iv)	$x° = 46°$	$y° = 129°$	$w° = 115°$

2. Sufficient measurements are:

(a) Three angles and any two sides. (The fourth angle can be found since the sum of the measures of the angles is known to be 360.)



(b) Three sides and two angles both included between known sides.



In the case of squares one side is sufficient measurement. (The right angle is known.) For rectangles two adjacent sides are sufficient. For parallelograms two adjacent sides and one angle are sufficient.

3. 65 feet

4. 8 feet

5. 56 feet

6. 1:1440

7. (a) 2000 inches or about 167 feet

   (b) 1600 inches or about 133 feet

   (c) Approximately 22,200 square feet.

20-3 Similar Triangles

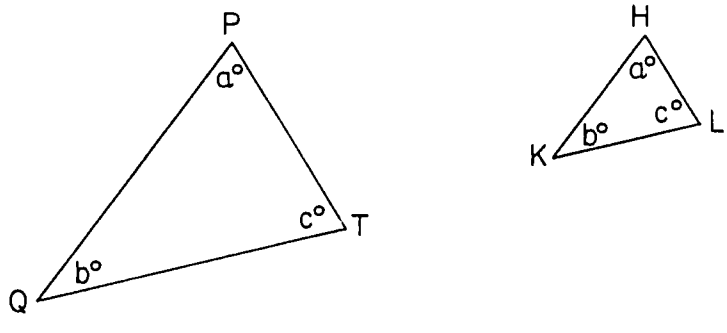
In this section we are dealing with a special case of similar polygons. The main purpose here is to help the pupils recognize that, although in general there are two conditions which must be satisfied if two polygons are to be similar, in the case of triangles, if one of those conditions is satisfied, the other will be automatically satisfied as a consequence of the first one. Thus for triangles it is sufficient, in testing for similarity, to show that either one of the conditions is satisfied - corresponding angles congruent or ratios of measures of corresponding sides equal.

The pupils are led to discover this in the Class Activities through experiment and inductive reasoning. No formal proofs are intended at this time, but help them through the discussion to be satisfied about these two tests for similar triangles.

There are also other conditions under which two triangles can be similar but we do not intend treating them until later in the school course when a more deductive approach will be used.

In order to get satisfactory and meaningful results in the Class Activities and the Problems in this section the pupils will need to make their measurements very carefully.

Make sure that the pupils write the names of similar triangles according to the correspondence under which they are similar. For example, the two triangles shown below are similar.



This should be written:

$\triangle PQT$ is similar to $\triangle HKL$.

Sometimes show just the correspondence as we did earlier:

$PQT \longleftrightarrow HKL$.

In this way, one sees clearly the corresponding angles that are congruent and the corresponding sides whose ratios are equal.

Answers to Problems 20-3 | Student Text Page 317

1. (a) $w' = 10'$ $a^\circ = 48'$ $b^\circ = 92^\circ$
   (b) $x' = 20'$ $y' = 10'$ $c^\circ = 30^\circ$
   (c) $z' = 13'$ $d^\circ = 23^\circ$ $e^\circ = 67^\circ$
   (d) $p' = 9'$ $q' = 7'$ $f^\circ = 65^\circ$

2. Height of pole is $7\frac{1}{2}$ ft.

3. (a) $p = \frac{12}{5}$ $q = \frac{10}{3}$
   (b) $\hat{P} \equiv \hat{Z}$ $\hat{Q} \equiv \hat{X}$ $\hat{R} \equiv \hat{Y}$
   (c) $c = \frac{15}{2}$ $d = \frac{18}{5}$

4. (a) $\triangle XPQ$ and $\triangle XRS$
   (b) $\frac{XP}{XR} = \frac{PQ}{RS} = \frac{QX}{SX}$
   (c) $XR = 4$ $PR = 2$
   (d) $\frac{XQ}{XS} = \frac{1}{2}$ $\frac{XQ}{QS} = 1$
   (e) $\overline{XQ} \equiv \overline{QS}$

5. (a) $\left.\begin{array}{l}\widehat{ABC} \equiv \widehat{ADE} \\ \widehat{ACB} \equiv \widehat{AED}\end{array}\right\}$ If parallel lines are cut by a transversal, the corresponding angles are congruent.
   (b) $\triangle ABC$ is similar to $\triangle ADE$
   (c) $\frac{AB}{AD} = \frac{BC}{DE} = \frac{CA}{EA}$

(d) $DE = \frac{10}{3}$

(e) $AC = 2.7$

6. (a) $\widehat{PQT} \equiv \widehat{PRU} \equiv \widehat{PSV}$

$\widehat{PTQ} \equiv \widehat{PUR} \equiv \widehat{PVS}$

(b) $\triangle PQT$ and $\triangle PRU$ and $\triangle PSV$ are similar.

(c) (i) $\frac{PQ}{PR} = \frac{PT}{PU} = \frac{QT}{RU}$

(ii) $\frac{PR}{PS} = \frac{RU}{SV} = \frac{PU}{PV}$

(iii) $\frac{PQ}{PS} = \frac{PT}{PV} = \frac{QT}{SV}$

20-4 Similar Right-Angled Triangles

This section is a special case of the preceding section. It is treated as a section because of two things:

1. Similar right-angled triangles are used to show the Pythagorean Property, and

2. The use of similar right-angled triangles leads to certain equal ratios which we shall meet soon in trigonometry.

The Class Activity in this section leads the pupils to see the simple condition under which two right angled triangles are similar; that is:

If an acute angle of a right-angled triangle is congruent to an acute angle of another right-angled triangle, then the two triangles are similar.

Remember to watch that the names of similar triangles are written according to the correspondence under which the triangles are similar.

Answers to Class Activity

Student Text Page 320

1. (a) $\hat{A}$ and $\hat{D}$ are 60° angles.

(b) $\triangle ABC$ is similar to $\triangle DEF$ because the corresponding angles are congruent.

(c) $\frac{AB}{BC} = \frac{DE}{EF}$ because ratios of measures of corresponding sides of similar triangles are equal.

2. (a) No, there are many such triangles.

(b) $\triangle PQR$ is similar to $\triangle XYZ$.

3. (a) (i) 90° (ii) 66° (iii) 24° (iv) 66°

(b) Three

(c) $\triangle ABC$, $\triangle ADB$, and $\triangle BDC$ are similar because their corresponding angles are congruent.

4. (a) $\triangle XYH$, $\triangle XZY$, and $\triangle YZH$ are similar.

(b) $\frac{p}{z} = \frac{z}{y}$, $z^2 = py$

$\frac{q}{x} = \frac{x}{y}$, $x^2 = qy$

$z^2 + x^2$	$= py + qy$	Addition property of equality
	$= (p + q)y$	Distributive property
	$= y \times y$	$p + q = y$
	$= y^2$	

The result we have obtained, that $z^2 + x^2 = y^2$, is called the Pythagorean Property of a right-angled triangle. It was studied from a different viewpoint in Chapter 14. No numerical applications of the property are given here because our main interest at present is similar triangles.

Answers to Problems 20-4

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1. $\triangle PAB$, $\triangle QAC$, $\triangle RAD$, $\triangle SAE$.

2. The triangles are similar because they are right-angled triangles and angle A is common to them all.

3. Since the triangles are similar, it follows that the ratios of corresponding sides must be equal. Therefore, from the first two triangles, it

follows that $\frac{a}{b} = \frac{p}{q}$. From the first and third, it follows that $\frac{a}{c} = \frac{p}{r}$. From the first and fourth, it follows that $\frac{a}{d} = \frac{p}{s}$.

4. (a) If both sides of the first of the two equations are multiplied by $\frac{q}{a}$ the second equation is obtained. The two equations are therefore equivalent.

(b) Same, using $\frac{r}{a}$.

(c) Same, using $\frac{s}{a}$.

5. Since $\frac{q}{b}$, $\frac{r}{c}$, $\frac{s}{d}$ are all equal to $\frac{p}{a}$, we can write $\frac{p}{a} = \frac{q}{b} = \frac{r}{c} = \frac{s}{d}$. We shall meet this ratio later as the tangent of $\hat{A}$.

CHAPTER 21

TRIGONOMETRY

This chapter seeks to give students an introduction to the simplest problems of trigonometry. The principles of similar triangles, and especially those of similar right-angled triangles studied in the last chapter, are used here to introduce trigonometric ratios. Problems of determining heights by indirect measurement are expected to lead into the study of similar right-angled triangles, which in turn leads into the definition of the trigonometric ratios.

21-1 Direct and Indirect Measurement

Class discussion should bring out the various methods that the students would use to measure heights of objects. The teacher should also lead them to distinguish direct measurement from indirect measurement. The method of using the Scouts' staff should be discussed as an example of indirect measurement, and the teacher should bring out the basic principles underlying it.

Class Activity

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1. This activity will be performed out of doors. One student should carry out the instructions under the close observation of the other students. The teacher should have a tape measure ready for taking measurements.

2. (a) Take the measurements with the help of the students. If, for example, the length of $\overline{BC}$ is 3 ft., the length of $\overline{AE}$ is 10 ft., the length of $\overline{AC}$ is 2 ft , a scale drawing may be made as below.

Scale: Let 1 cm. represent 1 ft.

3 cm. represents the length of $\overline{BC}$.

2 cm. represents the length of $\overline{AC}$.

10 cm. represents the length of $\overline{AE}$.

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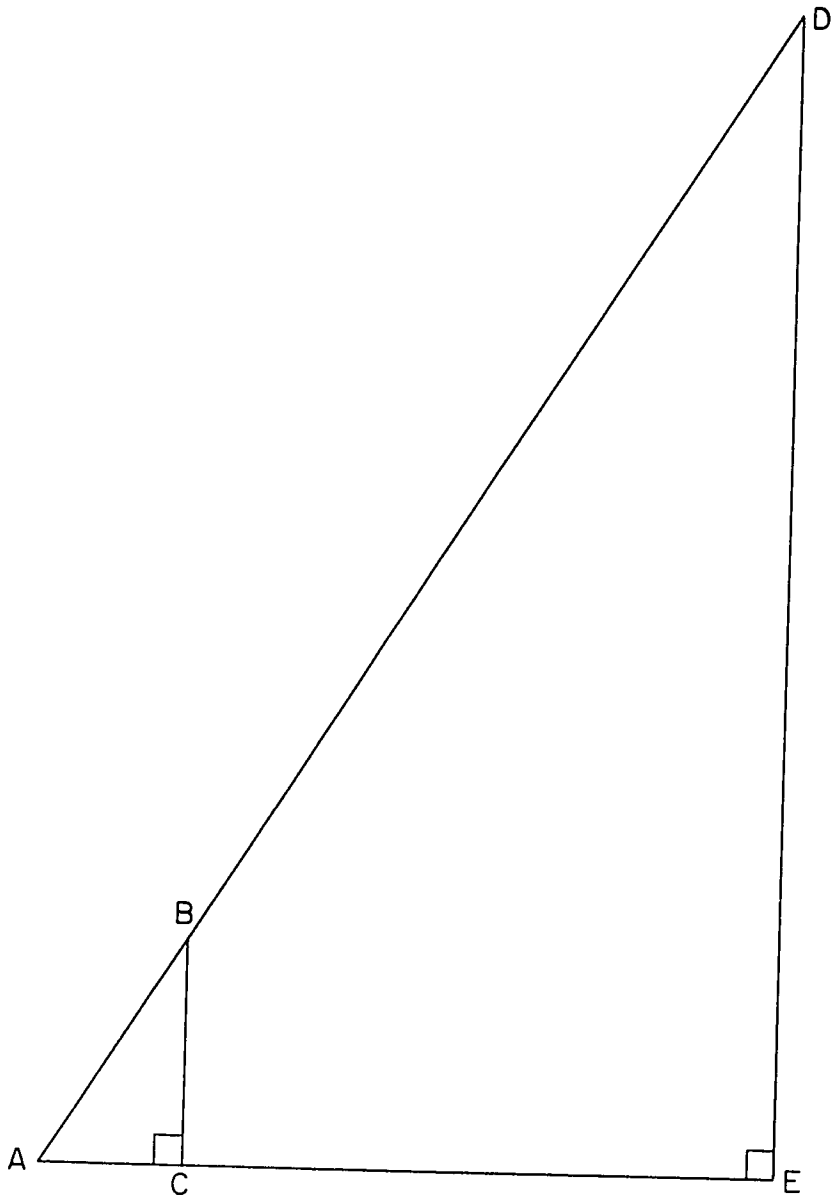

The length of $\overline{DE}$ in the figure is found by measurement to be 15 cm. Hence, the actual length of $\overline{DE}$ is 15 ft. If the eye level of the observer is 5 ft. above the ground, then the height of the tree is 20 ft.

(b) $\triangle ABC$ and $\triangle ADE$ are similar because

(1) both are right-angled triangles, and

(2) $B\hat{A}C$ and $D\hat{A}E$ are the same angle.

(c) (i) $\frac{DE}{BC} = \frac{AE}{AC}$

(2) $\frac{BC}{AC} = \frac{DE}{AE}$

(d) Since $\frac{DE}{BC} = \frac{AE}{AC}$

$$DE = \frac{AE \times BC}{AC} = \frac{10 \times 3}{2} = 15$$

Hence if the height of the observer is 5 ft., the height of the tree is 20 ft.

3. (a) One meaning of "elevate" is "raise". The eyes are "raised" or elevated a certain angle from the horizontal to look at the top of the object.

4. This activity is practically a repeat of 1, 2, and 3 above, and may be done as "Home Activity".

5. Make a rough sketch of the required drawing. Discuss with pupils the choice of a suitable scale. A good choice would be 1 cm. to represent 20 ft. A careful drawing will give 346 ft. for the height of the tower, but 350 ft. is a good approximation.

21-2 Tangents

The sides of a right-angled triangle containing the right angle may be named in terms of one of the acute angles of the triangle. The teacher may ask the pupils to suggest suitable names before telling them the usual names.

Answers to Problems 21-2a

Student Text Page 325

1. (a) $\overline{XY}$ (b) $\overline{YZ}$ (c) $\overline{YZ}$ (d) $\overline{XY}$

Previous work on similar right-angled triangles should be used to lead the pupils to see that for acute angle $\hat{A}$, $\frac{\text{measure of side opposite } \hat{A}}{\text{measure of side adjacent to } \hat{A}}$ is a constant ratio. The name tangent of the angle $\hat{A}$ has been given to this ratio.

Class Activity

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The triangles are similar because they are right-angled triangles and all have a common acute angle at O.

$\frac{X_1 M_1}{O X_1} = \frac{X_2 M_2}{OX_2} = \frac{X_3 M_3}{OX_3} =$ ...because they are ratios of corresponding sides of similar triangles.

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Answers to Problems 21-2b

1. $\tan 10^\circ \approx 0{\cdot}18$

$\tan 20^\circ \approx 0{\cdot}36$

$\tan 30^\circ \approx 0{\cdot}58$

$\tan 40^\circ \approx 0{\cdot}84$

$\tan 50^\circ \approx 1{\cdot}19$

We use here the symbol " $\approx$ " meaning " is approximately " to emphasize the fact that these decimals are only approximations. In the case of $\tan 60^\circ = \sqrt{3}$, an exact value is expressed by $\sqrt{3}$ and " $=$ " can properly be used. Most of the time, however, only approximate values of the trigonometric ratios are being used.

As is customary, we shall soon relax and use " $=$ " where we really mean " $\approx$ ", assuming that the meaning is clear from the context.

In the results which the pupils obtain an error of 2 or 3 in the hundredths' place should be expected from inaccuracies in drawing and measurement. Do not demand anything closer.

2. Note here that an angle of 15° is obtained by bisecting an angle of 30°. If two rays make angles of 60° and 90° respectively with the horizontal, the bisector of these two rays makes an angle of 75° with the horizontal.

(a) $\tan 15^\circ \approx 0{\cdot}27$

(b) $\tan 30^\circ \approx 0{\cdot}58$

(c) $\tan 45^\circ = 1{\cdot}00$

(d) $\tan 60^\circ \approx 1{\cdot}73$

(e) $\tan 75^\circ \approx 3{\cdot}73$

3. (a) 60°

(b) 1 inch. Since $\overline{AD}$ is the altitude of the equilateral triangle, $\overline{AD}$ bisects $\overline{BC}$.

(c) By the Pythagorean Property,

$$(AD)^2 + (BD)^2 = (AB)^2$$
$$(AD)^2 + 1^2 = 2^2$$
$$(AD)^2 = 3$$
$$AD = \sqrt{3}$$

$\overline{AD}$ is $\sqrt{3}$ inches long.

(d) $\tan 60^\circ = \sqrt{3} \approx 1{\cdot}73$

$$\tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \approx \frac{1.73}{3} \approx 0.58$$

4. $\tan 45^\circ = 1$

5.

size of the angle	tangent of the angle
10°	0·18
15°	0·27
20°	0·36
25°	0·47
30°	0·58
40°	0·84
45°	1·00
50°	1·19
60°	1·73
75°	3·73

6. a) $\tan A\hat{B}C = \frac{y}{100} = \tan 30^\circ = 0{\cdot}58$

(b) $\frac{y}{100} = 0{\cdot}58$

$y = 58$

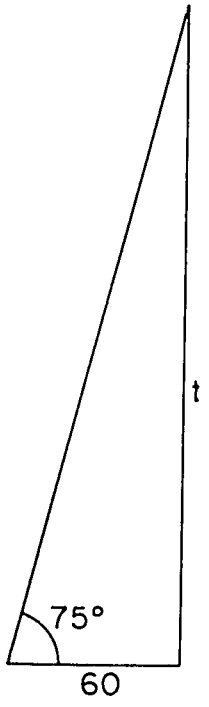
The measure of $\overline{AC}$ is about 58.

7. $\frac{x}{50} = \tan 25^\circ$

$\frac{x}{50} = 0{\cdot}47$

$x = 23{\cdot}5$ or about 24.

8. If the tower is t feet high, then



$\frac{t}{60} = \tan 75^\circ$

$\frac{t}{60} = 3{\cdot}73$

$t = 60 \times 3{\cdot}73$

$= 223{\cdot}8$

The tower is about 224 feet high.

9. If the tent is h feet high, then

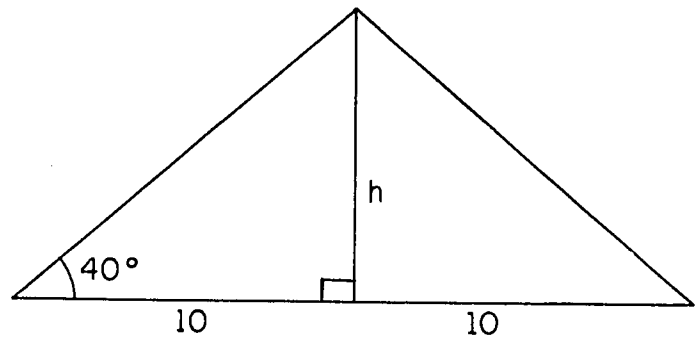
$$\frac{h}{10} = \tan 40^{\circ}$$

$$h = 10 \tan 40^{\circ}$$

$$h = 10 \times 0{\cdot}83$$

$$h = 8{\cdot}3$$

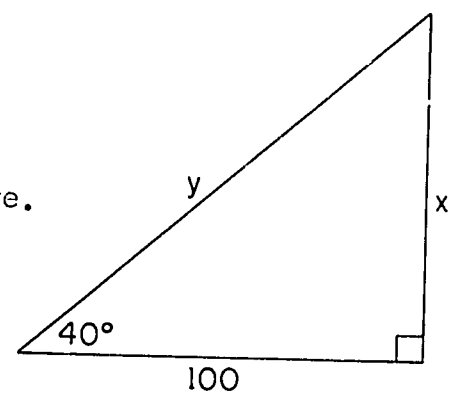
The tent is 8·3 ft. high.



21-3 Sines and Cosines

You may want to bring out tne need for the sine and cosine ratios by considering a problem like this.

Find y in the figure.



One may say $x = 100 \tan 40^\circ$. Then by use of the Pythagorean Property,

$$y^2 = (100 \tan 40^\circ)^2 + 100^2$$

$$= 100^2 (\tan^2 40^\circ + 1)$$

$$y = 100 \sqrt{\tan^2 40^\circ + 1}$$

The value of y may thus be obtained.

Then invite the pupils to seek another, and perhaps simpler way to find y. The figure used to introduce the tangent of an angle may be considered again and the pupils should be encouraged to discover the other constant ratios. The usual names may then be given to these ratios.

Answers to Problems 21-3a

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1. (a) $\sin 30^\circ = \frac{1}{2}$ (d) $\cos 60^\circ = \frac{1}{2}$

(b) $\cos 30^\circ = \frac{\sqrt{3}}{2}$ (e) $\sin 45^\circ = \frac{\sqrt{2}}{2}$

(c) $\sin 60^\circ = \frac{\sqrt{3}}{2}$ (f) $\cos 45^\circ = \frac{\sqrt{2}}{2}$

2. (a) $\sin 20^\circ \approx 0{\cdot}34$ (c) $\sin 40^\circ \approx 0{\cdot}64$

(b) $\cos 20^\circ \approx 0{\cdot}94$ (d) $\cos 40^\circ \approx 0{\cdot}77$

3.

Size of the angle	sine of the angle	cosine of the angle
20°	0·34	0·94
30°	0·50	0·87
40°	0·64	0·77
45°	0·71	0·71
60°	0·87	0·50

4. (a) The sine of the angle.

(b) $\sin A\hat{B}C = \frac{x}{50} = \sin 30^\circ$

$x = 50 \sin 30^\circ = 50 \times 0{\cdot}5 = 25$

5. $\frac{y}{10} = \cos 40^{\circ}$

$y = 10 \cos 40^{\circ} = 10 \times 0{\cdot}77 = 7{\cdot}7$

6. (a) $\sin \hat{B} = \frac{3}{5}$

(b) $\cos \hat{B} = \frac{4}{5}$

(c) $\tan \hat{B} = \frac{3}{4}$

(d) $\sin \hat{A} = \frac{4}{5}$

(e) $\cos \hat{A} = \frac{3}{5}$

(f) $\tan \hat{A} = \frac{4}{3}$

Relations to be observed

(i) $\sin \hat{B} = \cos \hat{A}$

(ii) $\cos \hat{B} = \sin \hat{A}$

(iii) $\frac{\sin \hat{B}}{\cos \hat{B}} = \tan \hat{B}$

(iv) $\frac{\sin \hat{A}}{\cos \hat{A}} = \tan \hat{A}$

7. (a) $\sin \hat{C} = \frac{c}{b}$

(b) $\cos \hat{A} = \frac{c}{b}$

(c) $\sin \hat{A} = \frac{a}{b}$

(d) $\cos \hat{C} = \frac{a}{b}$

(e) $\frac{\sin \hat{A}}{\cos \hat{A}} = \frac{a}{b} \div \frac{c}{b} = \frac{a}{c}$

(f) $\tan \hat{A} = \frac{a}{c}$

(g) $\frac{\sin \hat{C}}{\cos \hat{C}} = \frac{c}{b} \div \frac{a}{b} = \frac{c}{a}$

(h) $\tan \hat{C} = \frac{c}{a}$

Relations to be observed:

(i) $\sin \hat{C} = \cos \hat{A}$

(ii) $\cos \hat{C} = \sin \hat{A}$

(iii) $\frac{\sin \hat{A}}{\cos \hat{A}} = \tan \hat{A}$

(iv) $\frac{\sin \hat{C}}{\cos \hat{C}} = \tan \hat{C}$

The teacher may here point out further that if $\hat{C}$ is an angle of x°, then $\hat{A}$ is an angle of $(90-x)^{\circ}$, and

(i) $\sin x^{\circ} = \cos (90-x)^{\circ}$

(ii) $\cos x^{\circ} = \sin (90-x)^{\circ}$

8. (a) $\sin \hat{A} = \frac{a}{b}$ (c) $\cos \hat{A} = \frac{c}{b}$

(b) $(\sin \hat{A})^2 = \frac{a^2}{b^2}$ (d) $(\cos \hat{A})^2 = \frac{c^2}{b^2}$

$$\sin^2 \hat{A} + \cos^2 \hat{A} = \frac{a^2}{b^2} + \frac{c^2}{b^2} = \frac{a^2 + c^2}{b^2} = 1$$

The conclusions of this section have been developed for acute angles only. They are, of course, true for any angles for which the trigonometric ratios are defined, as will be proved in a later course.

Answers to Problems 21-3b

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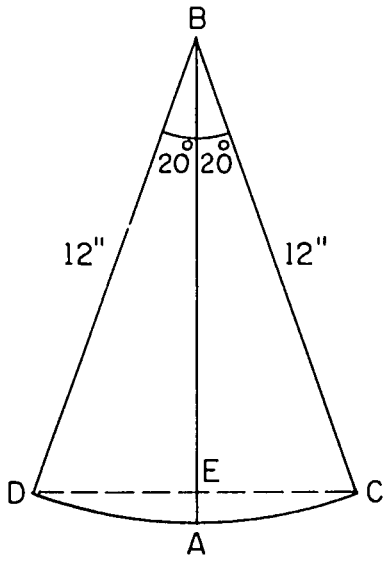
1. 16 ft., 9 ft.
2. 213 yd. north. 213 yd. east.
3. 8·7 cm., 5·0 cm.
4. 35 ft.
5. 45°
6. 60°
7. The electric pole is 21 feet high, and the shadow when the angle of elevation is 45° is 21 feet long.
8. The figure shows the pendulum in three positions $\overline{AB}$, $\overline{CB}$, and $\overline{DB}$, where $A\hat{B}C$ is an angle of 20°. We wish to find the length of $\overline{AE}$. We first observe that $\frac{EB}{CB} = \cos 20^\circ$ so $EB = CB \times \cos 20^\circ$.
$EB \approx 12 \times 0{\cdot}94 \approx 11{\cdot}28$.
Since $\overline{AB}$ is 12 inches long,
$AE = AB - BE \approx 12 - 11{\cdot}28 \approx 0{\cdot}72$.
The pendulum rises approximately 0·72 inches.



21-4 Graphs of the Trigonometric Ratios

As was the case with square roots, the trigonometric ratios are graphed and the graphs are used as an introduction to the use of standard tables. The ratios for 0° and 90° are avoided because our definition of the three ratios in terms of sides of a right-angled triangle does not apply to angles of sizes 0° and 90°. We shall consider the ratios for these angles at a later stage.

Class Activity

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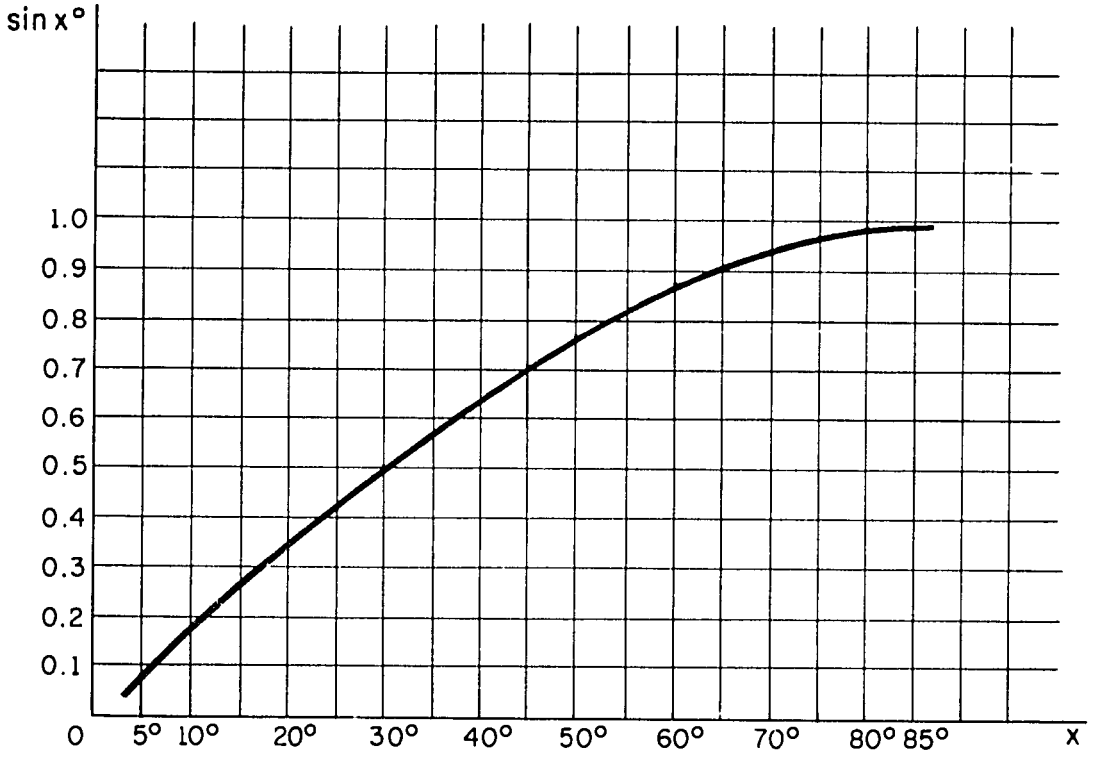
1. The units for this activity should be as large as possible. Since this activity requires much care and time, it may be given as "Home Acitivity". It is helpful to remember that $\sin x^{\circ} = \cos (90-x)^{\circ}$.

1. (e)

size of the angle	sine of the angle	cosine of the angle
5°	0·09	0·99
10°	0·17	0·98
15°	0·26	0·97
20°	0·34	0·94
25°	0·42	0·91
30°	0·50	0·87
35°	0·57	0·82
40°	0·64	0·77
45°	0·71	0·71
50°	0·77	0·64
55°	0·82	0·57
60°	0·87	0·50
65°	0·91	0·42
70°	0·94	0·34
75°	0·97	0·26
80°	0·98	0·17
85°	0·99	0·09

2.



3.

	(a)	0·39	(d)	30
	(b)	0·66	(e)	24
	(c)	0·16	(f)	44

21-5 Tables

Introduce the pupils to standard tables, giving them the opportunity of comparing the results obtained from their graphs with the standard results.

Class Activity

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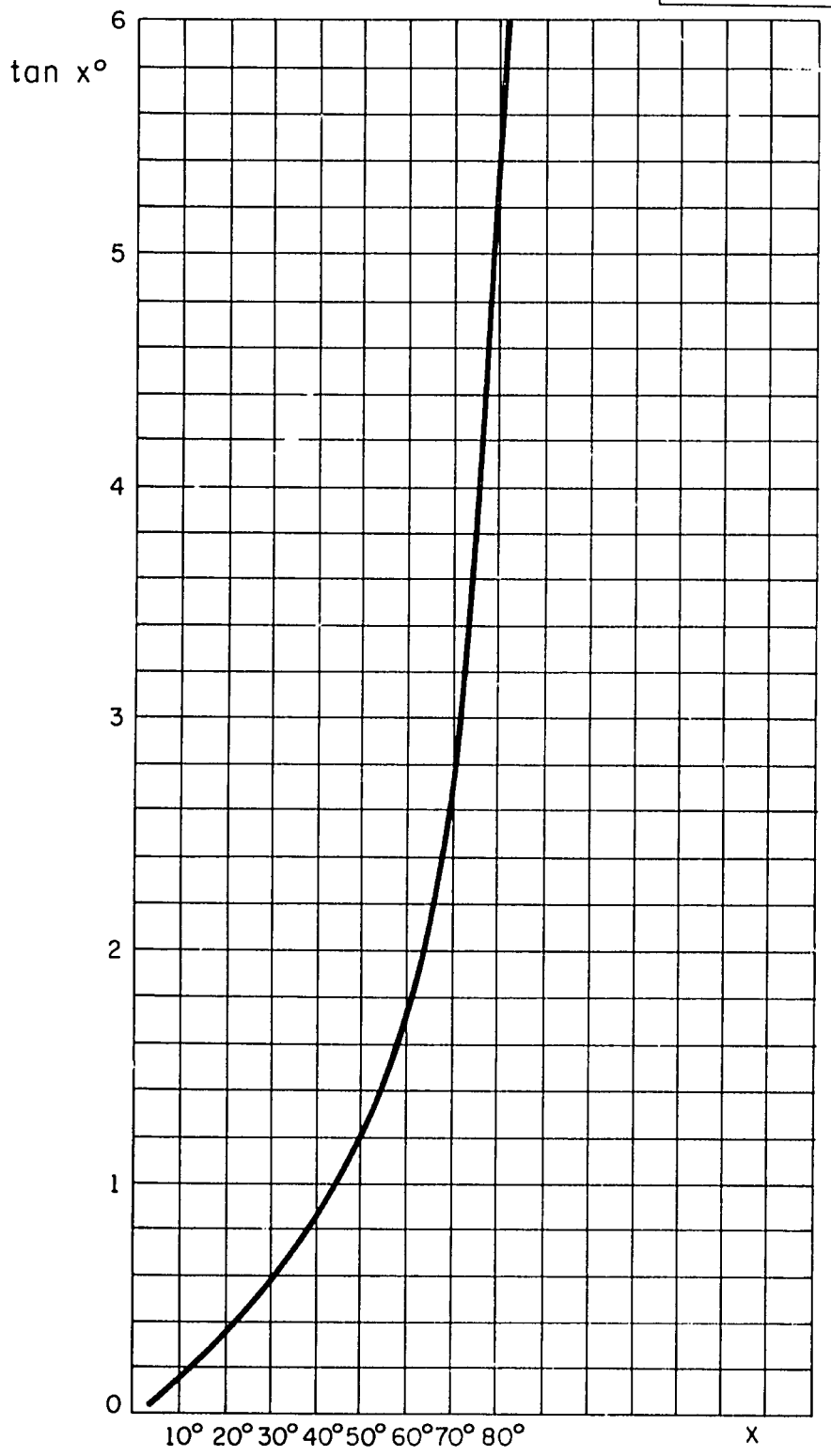

Answers to Problems 21-5

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1. 33·5 ft.
2. 40·5 cm.
3. 63°
4. 57°
5. 24 yd.
6. 93 miles north, 118 miles east
7. 12·4 square units.
8. $x = 9 \cdot 5$, $y = 6 \cdot 9$, $z = 3 \cdot 1$

   The area is 47·5 square units.

   See how many pupils discover that the area could have been found without finding y and z because BC = AC.
9. Follow the same technique as in Problem 8, constructing the altitude through either the 69° angle or the 53° angle. The area is 15·9 square units.

CHAPTER 22

AREAS OF SIMILAR FIGURES

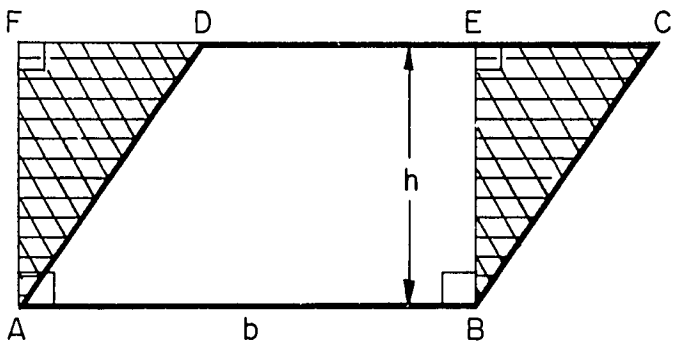
In this chapter we begin with a summary of work on areas. We use the precise phrases "area of a rectangular region", "area of a triangular region", and, in general, "area of a polygonal region" at first, but later we relax and use the simpler phrases "area of a rectangle", "area of a triangle" and "area of a polygon", respectively, knowing that the real meaning will be clear from the context.

22-1 Areas of Basic Figures

1. The section begins with a rectangle 4 units by 3 units divided up into unit squares, one of them shaded, to recall the idea that area can be determined by counting the number of unit squares into which the figure may be divided. Remind the pupils how we can think of multiplying the number of unit squares in a row by the number of rows to find the total number of unit squares. In the examples, the required answers in the units stated will be:

(a) 35 sq. in.
(b) 6 sq. cm.
(c) 8 sq. miles
(d) 12 sq. ft.
(e) ab sq. cm.

2. In the diagram, the triangles AFD and BEC are congruent and hence equal in area. Thus area $\triangle$AFD + area ABED = area $\triangle$BEC + area ABED. The union of the triangular region AFD and the quadrilateral region ABED is the rectangular region ABEF, while the union of the triangular region BEC and the quadrilateral region ABED is the parallelogram region ABCD. Thus the rectangular region ABEF and the parallelogram region ABCD have equal areas. Since the area of the rectangular region is already known to be bh we find that the area of the parallelogram region is also bh.



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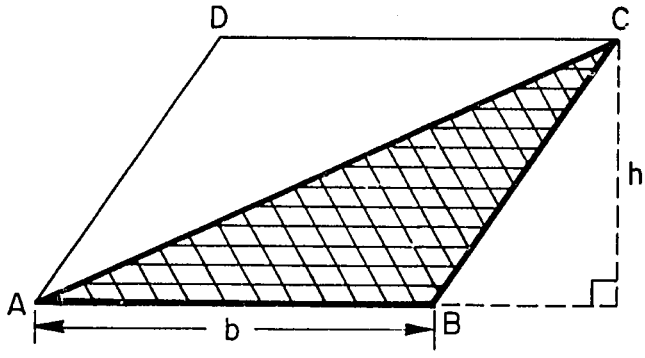
3. The diagonal $\overline{AC}$ divides the parallelogram ABCD into two congruent triangles ABC and ADC. Hence the area of the triangle ABC is one-half the area of the parallelogram ABCD. The area of the parallelogram is bh, so the area of the triangle is $\frac{1}{2}bh$.



4. A trigonometric expression for the area of a triangle is found by observing that,

$$\text{since } \frac{h}{b} = \sin \hat{C}, \qquad h = b \sin \hat{C}.$$

Then, since $A = \frac{1}{2}ah$, we have $A = \frac{1}{2}ab \sin \hat{C}$.

Of course, this result can be written using any side as the base. In any case we see that it is "one half the product of the measures of two sides times the sine of the included angle".

$$\frac{1}{2}ab \sin \hat{C} = \frac{1}{2}bc \sin \hat{A} = \frac{1}{2}ca \sin \hat{B}$$

From these equations the powerful Law of Sines may be quickly derived. You may want to derive it and show some of its usefulness. Divide by $\frac{1}{2}abc$.

$$\frac{\frac{1}{2}ab \sin \hat{C}}{\frac{1}{2}abc} = \frac{\frac{1}{2}bc \sin \hat{A}}{\frac{1}{2}abc} = \frac{\frac{1}{2}ca \sin \hat{B}}{\frac{1}{2}abc}$$

$$\frac{\sin \hat{C}}{c} = \frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b}$$

or, if you prefer,

$$\frac{c}{\sin \hat{C}} = \frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}}$$

Answers to Problems 22-1 | Student Text Page 341

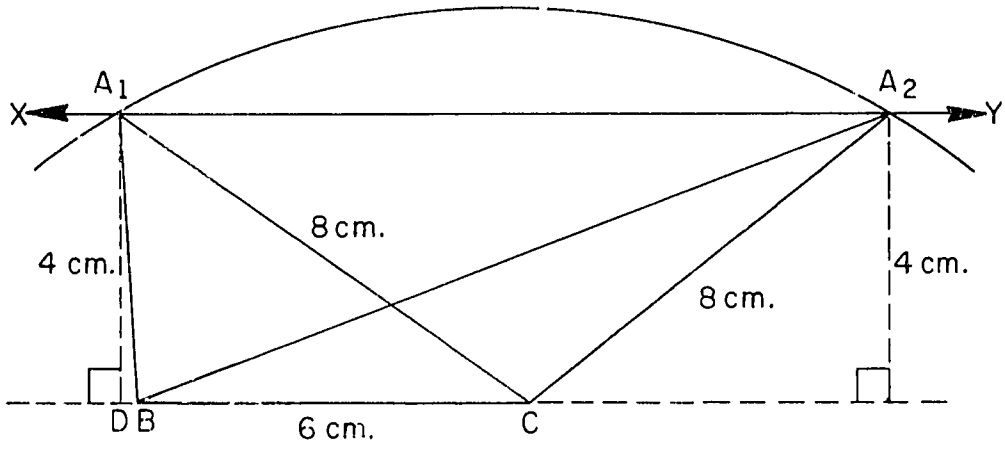
1. Each of the triangles is a right-angled triangle, as we know by the converse of the Pythagorean Theorem. Test them by showing that $c^2 = a^2 + b^2$.

(a) 6 sq. in. (c) 13·5 sq. in.

(b) 30 sq. in. (d) 84 sq. in.

2. Since the altitude from A is 4 cm., the vertex A must be on a line $\overleftrightarrow{XY}$ parallel to $\overline{BC}$ and 4 cm. away. Draw $\overleftrightarrow{XY}$ and with C as centre and a radius of 8 cm. Draw an arc cutting $\overleftrightarrow{XY}$ at A_1 and A_2.

The two triangles possible are $\triangle A_1BC$ and $\triangle A_2BC$.



To find the length of the altitude from B in each case, we use the fact that the area of the triangle is half the product of any one of the sides and the altitude drawn to that side.

Thus $\frac{1}{2} \times 6 \times 4 = \frac{1}{2} \times 8 \times h$ where h represents the required altitude from B. This gives $12 = 4h$, so $h = 3$. The altitude from B is 3 cm. After computing the altitude from B this way, have the pupils draw the altitudes in the two triangles, measure them, and verify that they are both 3 cm.

3. Let the triangle be ABC and the altitudes through A, B, C be 4 in., 5 in., 6 in. respectively, then

$$\frac{1}{2} \times a \times 4 = 24 \qquad a = 12$$

$$\frac{1}{2} \times b \times 5 = 24 \qquad b = 9\cdot6$$

$$\frac{1}{2} \times c \times 6 = 24 \qquad c = 8$$

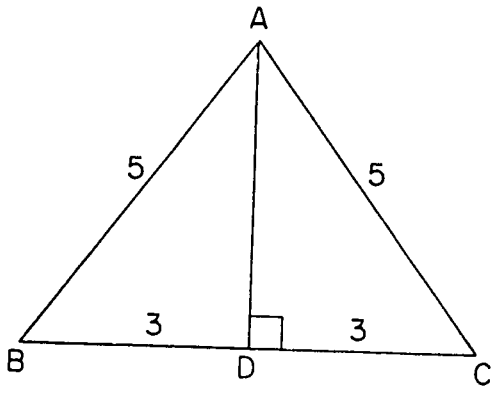
$(a + b + c) = 29\cdot6$, so the perimeter is $29\cdot6$ in.

4. (a) Area of $\triangle ABC = \frac{1}{2} \times 5 \times 6 \times \sin 30^\circ$

$$= \frac{1}{2} \times 5 \times 6 \times \frac{1}{2} = 7\frac{1}{2}$$

The area is $7\frac{1}{2}$ square inches.

(b) Draw the altitude $\overline{AD}$. This bisects the base $\overline{BC}$ of the isosceles triangle ABC.



Obtain the length of $\overline{AD}$ by using the Theorem of Pythagoras.

$$AD = \sqrt{5^2 - 3^2} = 4$$

Area of $\triangle ABC = \frac{1}{2} \times 6 \times 4 = 12$

The area is 12 sq. in.

(c) Area of $\triangle ABC = \frac{1}{2} \times 4 \times 7 \sin 60^\circ$

$$= \frac{1}{2} \times 4 \times 7 \times \frac{\sqrt{3}}{2} = 7\sqrt{3}$$

The area is $7\sqrt{3}$ sq. in. or about 12 sq. in.

5. (a) Area of $\triangle ABC = \frac{1}{2} \times 8 \times 7 \times \sin 40^\circ$

$$= 28 \sin 40^\circ \approx 28 \times 0\cdot64 \approx 18$$

(b) Area of $\triangle ABC = \frac{1}{2} \times 4 \times 5 \sin 62^\circ$
$= 10 \sin 62^\circ \approx 10 \times 0{\cdot}88 \approx 8{\cdot}8$

22-2 Ratios in Similar Triangles

The purpose of this section is to demonstrate experimentally that if two triangles are similar then corresponding segments are proportional. In particular we shall need to know in the following section that corresponding altitudes of similar triangles are proportional to corresponding sides.

Answers to Problems 22-2

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1. (a) $\frac{AE}{A'E'} = \frac{12}{3} = 4$

(b) $\frac{BA}{B'A'} = \frac{7}{1} = 7$

(c) $\frac{C'A'}{CA} = \frac{3}{4}$

(d) $\frac{AC}{A'C'} = \frac{18}{6} = 3$

(e) $\frac{C'D'}{CD} = \frac{\frac{3}{2}}{2} = \frac{3}{4}$

(f) $\frac{C'D'}{CD} = \frac{350}{750} = \frac{7}{15}$

2. (a) $\frac{PR}{12} = \frac{7}{14}$; $PR = 6$

(b) $\frac{RN}{6} = \frac{3}{12}$; $RN = \frac{3}{2}$

(c) $\frac{RT}{4} = \frac{2}{14}$; $RT = \frac{4}{7}$

(d) $\frac{RT}{4} = \frac{10}{5}$; $RT = 8$

(e) $\frac{RQ}{5} = \frac{1}{6}$; $RQ = \frac{5}{6}$

(f) $\frac{TR}{4} = \frac{10}{14}$; $TR = \frac{20}{7} = 2\frac{6}{7}$

22-3 Ratios of Areas of Similar Triangles

The purpose of this section is to demonstrate that the ratio of the areas of two similar triangles is the same as the ratio of the squares of corresponding segments.

The use of the subscripts 1, 2, 3, ... in this context enables the pupil to know the ratio of corresponding sides.

Problems 22-3

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1. $\triangle ADE$ is similar to $\triangle ABC$.

$$\frac{\text{area } \triangle ADE}{\text{area } \triangle ABC} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

2. $\triangle AXY$ is similar to $\triangle ABC$

(a) If $\frac{AX}{AB} = \frac{1}{4}$, then $\frac{\text{area } \triangle AXY}{\text{area } \triangle ABC} = \frac{1}{16}$

(b) If $\frac{AY}{AC} = \frac{2}{3}$, then $\frac{\text{area } \triangle AXY}{\text{area } \triangle ABC} = \frac{4}{9}$

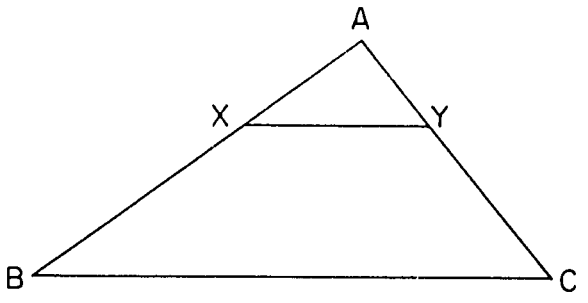
(c) If $\frac{AX}{XB} = \frac{1}{2}$, then $\frac{AX}{AB} = \frac{1}{3}$ and $\frac{\text{area } \triangle AXY}{\text{area } \triangle ABC} = \frac{1}{9}$

(d) If $\frac{XY}{BC} = \frac{3}{5}$, then $\frac{\text{area } \triangle AXY}{\text{area } \triangle ABC} = \frac{9}{25}$

(e) If $\frac{AY}{AC} = \frac{h}{k}$, then $\frac{\text{area } \triangle AXY}{\text{area } \triangle ABC} = \frac{h^2}{k^2}$

(f) If $\frac{AX}{XB} = \frac{k}{2}$, then $\frac{AX}{AB} = \frac{k}{k+2}$ and $\frac{\text{area } \triangle AXY}{\text{area } \triangle ABC} = \frac{k^2}{(k+2)^2}$

(g) If $\frac{AY}{YC} = \frac{h}{k}$, then $\frac{AY}{AC} = \frac{h}{h+k}$ and $\frac{\text{area } \triangle AXY}{\text{area } \triangle ABC} = \frac{h^2}{(h+k)^2}$



3. (a) $\frac{\text{area } \triangle XYZ}{\text{area } \triangle ABC} = \left(\frac{XY}{AB}\right)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$

Hence $\frac{\text{area } \triangle XYZ}{8} = \frac{9}{4}$

area $\triangle XYZ$ is 18 sq. in.

(b) $\left(\frac{BC}{YZ}\right)^2 = \frac{\text{area } \triangle ABC}{\text{area } \triangle XYZ} = \frac{1}{4}$

Let BC = a. Then

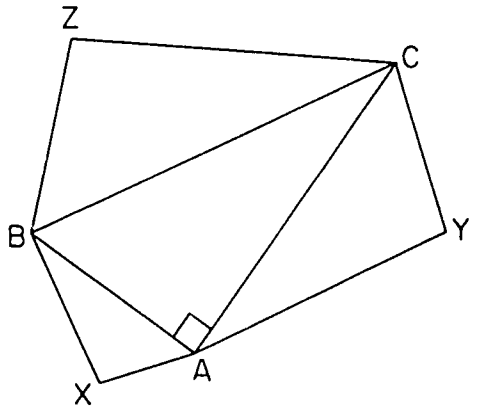
$$\left(\frac{a}{6}\right)^2 = \frac{1}{4}$$

$$a^2 = \frac{1}{4} \times 36 = 9$$

$$a = 3$$

$\overline{BC}$ is 3 inches long.

4. <u>CHALLENGE PROBLEM</u>.



We know that $(AB)^2 + (CA)^2 = (BC)^2$ and also that $\overline{AB}$, $\overline{CA}$, and $\overline{BC}$ are corresponding sides of the similar triangles ABX, CAY, and BCZ. Thus

$$\frac{\text{area } \triangle ABX}{\text{area } \triangle CAY} = \frac{(AB)^2}{(CA)^2} \quad \text{and} \quad \frac{\text{area } \triangle CAY}{\text{area } \triangle BCZ} = \frac{(CA)^2}{(BC)^2}$$

From these we can obtain

$$\frac{(AB)^2}{\text{area } \triangle ABX} = \frac{(CA)^2}{\text{area } \triangle CAY} = \frac{(BC)^2}{\text{area } \triangle BCZ}$$

If we call this common ratio r, we have

$$(AB)^2 = r(\text{area } \triangle ABX)$$
$$(CA)^2 = r(\text{area } \triangle CAY)$$
$$(BC)^2 = r(\text{area } \triangle BCZ)$$

By substituting these values in the Pythagorean relation above and dividing by r, we obtain

$$\text{area } \triangle ABX + \text{area } \triangle CAY = \text{area } \triangle BCZ$$

22-4 Ratios of Areas of Other Similar Polygons

The more general result is illustrated by means of similar quadrilaterals. The basic idea is to divide the two similar figures into corresponding sets of similar triangles, for which we already know how the areas compare. Once the figures are so divided, we can obtain the desired results by addition.

In the class activity, we note that corresponding angles of the quadrilaterals ABCD and EFGH are congruent and that corresponding sides are proportional. The pupils have not yet studied enough properties of similar triangles to be able to prove that $\triangle ABC$ is similar to $\triangle EFG$, but let them measure angles and sides in a carefully drawn figure enough to convince them that the triangles actually are similar. Since $\triangle ABC$ is similar to $\triangle EFG$ and $\triangle ACD$ is similar to $\triangle EGH$, their areas are proportional to the squares of the measures of corresponding sides. Thus we obtain the result that the ratio of the areas of ABCD and EFGH is equal to the ratio of the squares of the measures of any pair of corresponding sides. In fact, we could use any corresponding parts, such as the diagonals $\overline{AC}$ and $\overline{EG}$, in place of corresponding sides.