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GEOMETRY

Preliminary Edition

MATHEMATICS

Prepared
at the
1964 Entebbe Mathematics Workshop

EDUCATIONAL SERVICES INCORPORATED
Watertown, Massachusetts, U. S. A.

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GEOMETRY

Secondary Three

TEACHERS' GUIDE

Prepared
at the
1964 Entebbe Mathematics Workshop

Preliminary Edition

MATHEMATICS

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PREFACE

This Teachers' Guide, like the text which it accompanies, was prepared at a study and writing workshop held during the summer of 1964 at Entebbe, Uganda, in which mathematicians and teachers from all English-speaking countries of Tropical Africa, the United States, and the United Kingdom participated. This material is intended to follow *Mathematics Secondary Two* and *Mathematics Secondary One* which were prepared at similar workshops during the summers of 1963 and 1962.

I. Preliminary Nature of This Material

Material produced for use in the schools must be considered to be preliminary until it has been tried out in classrooms and modified in accordance with such experience. The Student Text and this Teachers' Guide for MATHEMATICS SECONDARY THREE will be subject to considerable correction and improvement in the light of suggestions from the teachers who use them.

This means that the teacher who uses this material has the responsibility, in addition to helping interpret the new material to the students, of helping identify the areas where improvement is needed. In partial reward for this burden, the teacher will have the satisfaction of knowing that he or she is taking part in an experiment which is of great potential value.

II. Emphasis on Mathematical Ideas

In recent years there has been accumulated much evidence that young students are far more interested in *mathematical ideas* than they have usually been given credit for, and that they are far more competent to deal with such ideas than the current curricula would suggest. A presentation of mathematics which puts its emphasis on concepts rather than the rules of manipulation is likely to lead to far greater satisfaction on the part of the student, and will also lead to greater mathematical competence.

III. Content of the Text

The subjects of geometry and algebra are treated in these two volumes; one is devoted to geometry and the other to algebra. The material is arranged so that there is considerable flexibility in the order in which topics are studied. It is anticipated that any of

the following three procedures could be used in a year's course:

1. SECONDARY THREE GEOMETRY can be studied in its entirety, followed by SECONDARY THREE ALGEBRA. (This order could be reversed, although Chapter 1 of GEOMETRY should be studied before beginning SECONDARY THREE ALGEBRA.)

2. The two volumes can be used simultaneously, devoting two or three periods a week to one volume and the balance of the time to the other.

3. Unit 1 (Chapters 1 and 2) of GEOMETRY can be completed, followed by Unit 1 (Chapters 1 to 7) of ALGEBRA, then Unit 2 (Chapters 3 to 7) of GEOMETRY, Unit 2 (Chapters 8 to 11) of ALGEBRA, and so on.

If the first arrangement is followed, the time should be watched closely, inasmuch as these volumes may include more material than can be covered in a year's course. It would be better to leave part of the GEOMETRY undone and spend approximately equal time on ALGEBRA and GEOMETRY than to complete the GEOMETRY and find only a few weeks left to devote to the ALGEBRA.

The treatment of geometry is formal, adhering to standards of rigour which seem appropriate for students at this level. Many of the ideas will be familiar from the informal geometry of *Mathematics Secondary Two* and some will be new. The emphasis in the classroom should be on the understanding of the properties and their proofs and on developing the ability to solve riders, not on rote memorization. Additional material in geometry will be treated in subsequent textbooks in this series.

The treatment of algebra continues the start made in *Mathematics Secondary One* and *Two* to develop algebra as a unified structure. Techniques and concepts are all related to a few basic ideas with emphasis on *why* the numbers behave as they do. We try to be mathematically sound but at the same time the development is gradual with an effort to involve the students actively. Instead of just telling the ideas to the students outright, the teacher should use problems and careful class discussion to help the students recognize the ideas for themselves.

IV. The General Approach

The best education is the education that the student creates from his own direct efforts. The teacher should resist the temptation to tell the class exactly what to do and how to do it. It is indeed a great temptation, for by such means a class will appear to be proceeding at a rapid pace. If the teacher takes the time and effort to lead the student to think through the idea himself without telling him outright, there is considerably more assurance that the idea will be mastered and retained by the student and will become truly his own.

Here again the demands on the teacher are greater. The teacher, like the student, must be thinking at every moment, for it takes far more insight to lead than to tell. This Guide is intended primarily to assist the teacher in the actual conduct of the class, by methods which are chosen to encourage student imagination and to generate student interest. In addition the Guide contains mathematical background and explanatory material beyond that given in the Text. Answers to the problems are provided.

V. Relationship to Other Materials

The Text is based on the assumption that the student has studied from *Mathemetics Secondary Two*. If a student has not studied all of the earlier text it may be necessary to go back and study portions of that book. However, the student can make good progress in the present Text without having a complete mastery of *Mathematics Secondary Two*.

This Text is designed to prepare the student for further mathematics in either the conventional school curriculum or in curricula which will be prepared in the next few years to follow upon this text. It is also designed to conform to the requirements of the present examinations.

It should be emphasized that this modified curriculum represents a great opportunity for teacher and student alike. In making use of it the school participates in a great experiment to help develop a strong African educational system of which we can be proud. It is a massive joint undertaking, in which students and teachers work side by side with mathematicians of international eminence, from their own country and from a dozen others, to create within Africa something that will be of major significance for Africa itself, and in a large measure for all the world.

SECONDARY _____ REPORT, CHAPTER _____

Would you please fill out one of these forms for each chapter of Secondary material as you complete the chapter. Send it by air mail to Educational Services Incorporated, African Education Program, 108 Water Street, Watertown, Massachusetts 02172, U. S. A. Your comments will help us to improve and to rewrite the material for further use.

Please respond to *all* parts of this report.

Name of Teacher _____

Name of School _____

School Address _____

Date _____

Chapter Number _____ in Secondary Book _____ Volume _____

Chapter Title _____

Number of classes in which you teach the Entebbe Mathematics Workshop Secondary Materials _____

_____

Number of pupils using material _____. Number of periods each class meets per week _____.

Length of class period _____ minutes. Class periods used teaching the chapter _____.

The pupils of my class are of the following ability level (tick one of the following):

fast _____, slow _____, medium _____, mixed ability _____.

My pupils have had, on the average, _____ years of schooling before this present year.

To the best of my judgment this chapter is (tick one of the following): excellent _____,

good _____, fair _____, poor _____, or extremely poor _____.

Why did you rate the chapter in this way? (If you need more space than is provided here, please use an additional sheet.) _____

_____

_____

If you used any supplementary materials in teaching this unit, please describe them. _____

_____

_____

STUDENT TEXT

Please indicate the specific sections which need improvement and how *you* would recommend improving them.

Questions	Comments
1. Should this whole chapter be omitted? Why?	
2. Name specifically any ideas which you think ought to be in this chapter but which are not there. Why should they be included?	
3. Name specifically any topics in this chapter which you think should be omitted. Why should they be omitted?	
4. Give the section and page number of any part of this chapter where you think the explanation is not clear enough. What suggestions do you have on how to make it more clear?	
5. Indicate which sets of problems need more problems to give adequate practice. What kind of problems are needed?	
6. Indicate by page and number any problems which you think are too difficult and problems which you think are too easy.	

STUDENT TEXT

(continued)

Questions	Comments
7. Indicate any problems or illustrations which are ineffective because they use words or ideas which are unfamiliar to your students. For instance, a child who had never been to sea in a schooner would not much appreciate a problem about the area of the jib and the foresail and the mainsail. For the examples which are inappropriate for your children, can you suggest local situations which would be more appropriate in the example.	

TEACHERS' GUIDE

Questions	Comments
1. In what parts of the chapter were the suggestions on how to teach the material not sufficient to help you teach it effectively? What suggestions have you for improving those parts?	
2. In what parts of the chapter were the suggestions on teaching the material trivial and unnecessary?	
3. In some places in the Teachers' Guide there are discussions for the teacher of the mathematical background, going more deeply into the	

Questions	Comments
3. (*continued*) ideas than one would with the students. Indicate by section and page number: (a) which of these discussions were not clear enough to be helpful, (b) which were too brief and need to be expanded, (c) which were unnecessary for you and might be omitted, (d) where no such discussion was given where it was needed.	
4. Indicate any answers to problems which were not sufficiently clear or complete.	

Would you please write any other comments which you believe might be helpful in rewriting this chapter.

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108 Water Street
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Chapter 1
DEDUCTIVE REASONING

This chapter aims at giving the students an introductory and informal approach to the idea of proof and implication, so that they will have some feeling for the developments in ALGEBRA and GEOMETRY. Not more than three or four lessons should be spent on this chapter now, but the students should be encouraged to refer back to it from time to time. For example, when they first consider the converse of a theorem or an indirect proof in geometry, they should look back at this chapter. In ALGEBRA, they will meet the Theorem on Factors of Zero. It will help them then to take another look at this chapter. Later on, in Chapter 8, they will meet the ideas spoken about here, more fully, although strictly speaking there will be no formal logic in this year's work.

1-1 Inductive reasoning

Much of the work that is done in the first one or two years of secondary school appeals to inductive reasoning. Before we formulated the postulates for a field we looked at many special cases showing, for example, the commutativity and the associativity of addition and multiplication, the distributive property, and so on. At this stage, it may be profitable to point out that we have *not proved* the Field Properties for the set of real numbers with the operations of addition and multiplication. In fact, the Field Properties are a set of postulates for the arithmetic of real numbers in the same way that the postulates which will be spoken about in this text are the basis for development of geometry.

Indeed if we look carefully at the way in which we developed the arithmetic of integers and later rational numbers, we realise that, having defined these numbers, we gave meaning to the operations of addition and multiplication by demanding that the Field Properties be satisfied.

Mention is made of Goldbach's conjecture to give the student some little insight into the way in which mathematics can develop. Students who find this interesting might also enjoy reading parts of *Men of Mathematics* by E. J. Bell (Simon and Schuster, New York, 1961 [also available in paperback]). In particular, the chapter on Fermat would be appropriate reading.

1-2 An example of what we mean by a proof

This section aims at bringing out the idea of what is needed to make a proof. The proof is not given here, as we would like those readers who are interested to discover as much as they can for themselves. The proof is given in Section 1-4.

1-3 Definitions

Already in their earlier work and repeatedly in this year's work, the students will meet definitions set out formally. For that reason we have tried here to show them the way in which they will see definitions set out later. It should be stressed that we *do not prove definitions*. As the text says, a definition gives the meaning of a new word or symbol in terms of words or symbols they already know. There are two examples in the text — many more come to mind. For instance, when we understand the word *ray* we can define the word *angle*. When we have the integers, we can give meaning to symbols like 2^3. When we have the real numbers, we can say what we mean by $\sqrt{2}$ or $\sqrt{4}$.

Answers to
PROBLEMS 1-3

Student Text Page 3

1. a. A factor of a positive integer is one of two positive integers which when multiplied give that integer.
   b. A prime number is a number with exactly two different factors.
   c. A pair of lines are parallel when they are coplanar and do not intersect.
   d. An angle is the union of two rays with a common endpoint.
   e. A carnivore is an animal which eats meat.
   f. A quadruped is an animal with four feet.
   g. The empty set is the set which has no members.
   h. A circle is a set of coplanar points all the same distance from a fixed point.

1-4 Deductive reasoning

In this section, a formal proof is given of the statement that the sum of two odd integers is an even integer. The students should find the work easy. They should, however, try to understand the generality of the statements which they are making and that what they prove is true for the sum of *all* pairs of odd integers because the definition holds for *all* odd integers and the Field Properties used are valid for the addition of *all* pairs of numbers. In making the proof, nothing but the definitions and Field Properties are used.

Answers to
PROBLEMS 1-4

Student Text Pages 4-6

These problems are set to show patterns and to encourage their discovery as well as to show deductive methods.

1. This is a simple problem that should offer no difficulty. The answer is 9 years, because 9 is the number that has to be added to 8 to give 17.

2. This problem is set to give those students who were not able to make the proof in Section 1-2, a chance to derive a fairly straightforward proof for themselves.

Since p and q are even integers, they may be taken to be $2m$ and $2n$ where m and n are integers. Their sum is then $2(m + n)$, which is an even integer. Encourage the students to write the reason for the step they have to take so that they can understand the generality of what they are doing. Normally, of course, they will do calculations applying to the Field Properties mentally.

3. The hint should guide the pupils to taking the integers n and m to be of the form $3r$ and $3s$ where r and s stand for integers. They should give the reason why $3r + 3s = 3(r + s)$. The general statement is that the sum of two integers which have 3 as a common factor also has 3 as a factor.

4. If n is the number Tunde thinks of, when he has added 9 he gets $n + 9$, multiplying by 2 gives $2(n + 9)$ and adding 6 gives $2(n + 9) + 6$. Now division by 2 gives $\frac{2(n + 9) + 6}{2} = (n + 9) + 3 = n + 12$. Thus, taking away n always leaves 12.

5. Any even integer can be represented as $2m$, where m is an integer. Its square is $2m \times 2m = 4m^2$ and is divisible by 4.

6. Each of the numbers 3^2, 5^2, 7^2, 9^2 leaves a remainder 1 when divided by 4 or by 8.

The hint makes the problem of proving that the square of any odd number leaves a remainder of 1 when divided by 4 fairly straightforward. The text does not ask about the remainder when the divisor is 8. This takes a little more work, but if any pupil is interested he could try to prove that it is always 1. The proof goes like this:

$$\begin{aligned}(2n + 1)(2n + 1) &= 4n^2 + 4n + 1 \\ &= 4n(n + 1) + 1\end{aligned}$$

Of the numbers n, $n + 1$, one is odd and the other even. Thus $n(n + 1)$ is even and so has a factor 2. Then $4n(n + 1)$ has a factor 8.

7. The numbers $(2 \times 3) + 1$, $(2 \times 3 \times 5) + 1$, $(2 \times 3 \times 5 \times 7) + 1$ are all prime numbers. It is clear that 2, 3, 5, 7, 11 are not factors of $(2 \times 3 \times 5 \times 7 \times 11) + 1$, because each leaves a remainder of 1 when divided into that number.

8. It is clear that $2, 3, 5, 7, \ldots, p$ are not factors of $(2 \times 3 \times 5 \times 7 \times \ldots \times p) + 1$ ($= N$, say), since each leaves a remainder 1 when divided into N. In the cases considered in Problem 7, N was prime. For larger values of p, N may not be prime. However, it follows that if q is a prime factor of N, q must be greater than p, because we defined $\{2, 3, 5, 7, \ldots, p\}$ to be the set of all prime numbers less than or equal to p. Then q may be N itself or it may be a prime number lying between p and N (clearly, N is a number greater than p).

   This argument shows that there is no largest prime number, for if we assumed that p were the largest, we would arrive at a contradiction in finding, as above, a prime number, q, which is larger.

In their work this year, bring to the notice of the students from time to time instances of their deducing general statements or special statements from general statements, as well as instances of inductive reasoning, to give them some over-all idea of how mathematics is structured.

1-5 Implications

This section introduces the words *implication*, *hypothesis*, *conclusion* in an informal way. The ideas will be taken up again in Chapter 8.

In this section when we say *assert* we mean "state as true".

The important ideas are that when an implication is valid, then in any particular instance in which the hypothesis is true, the conclusion must be true; and in any particular case in which the conclusion is false, we know that the hypothesis is also false. If in any particular case we know only that the hypothesis is false, then we

cannot say anything about the conclusion. Likewise, if all we know is that the conclusion is true, then we cannot say anything about the hypothesis.

We have intentionally avoided the terms *if and only if*, and *necessary and sufficient*, as they may produce language difficulties.

The students should, however, learn to use the sign $\Longrightarrow$, although they should be warned against using it where it can have no meaning. It would be unfortunate if students felt they had to pepper the page with $\Longrightarrow$ signs, as sometimes happens with $\therefore$ signs.

In stating theorems, encourage the students to use the sign $\Longrightarrow$. In cases where they meet a theorem and its converse, they could be shown the sign $\Longleftrightarrow$.

For example, they could write:

$$\text{In } \triangle ABC,\ \overline{AB} \equiv \overline{AC} \Longleftrightarrow A\hat{B}C \equiv A\hat{C}B.$$

If p and q stand for statements, and we assert $p \Longleftrightarrow q$, then in any particular instance p and q are either both true or both false. We say p is equivalent to q.

Answers to
PROBLEMS 1-5 A

Student Text Page 7

1.

	Hypothesis	*Conclusion*
a.	I am vaccinated.	I will not get smallpox.
b.	The Earth is flat.	Water falls off its edge.
c.	The number a is greater than the number b.	$(a - b)$ is a positive number.

2. There are a number of suitable conclusions in each case. Here are some suggestions.

a. New Year's Day falls on a Tuesday.
b. I will not reach home before dark.
c. I will go on my bicycle.
d. $p + 1$ and $q + 1$ are even numbers.

3. Again there are a number of suitable answers in each case. Here are some suggestions.

   a. I wear my coat.
   b. x is an even number.
   c. In triangles ABC, DEF, $\overline{AB} \equiv \overline{DE}$, $\overline{BC} \equiv \overline{EF}$, and $A\hat{B}C \equiv D\hat{E}F$.

Answers to
PROBLEMS 1-5 B

Student Text Pages 8-9

1. a. Valid; this is a property of an ordered field.
   b. Valid
   c. Not valid
   d. Not valid without further information. When r is positive, $p > q \Longrightarrow rp > rq$ is valid; otherwise, it is not.
   e. $(p + 3 = 6) \Longrightarrow (p + 5 = 8)$ is valid. The equations $p + 3 = 6$ and $p + 5 = 8$ are equivalent equations. For any particular value of p, $p + 3 = 6$ and $p + 5 = 8$ are either both true or both false statements. This is a case in which we can write $(p + 3 = 6) \Longleftrightarrow (p + 5 = 8)$.

2. a. $(p > -p) \Longrightarrow p > 0$.
   b. $(0 < p < p^2) \Longrightarrow p > 1$.
   c. $(3x + 1 = 7) \Longrightarrow (3x = 6)$.

1-6 Converse

This section points out that the converse of a valid implication is not necessarily valid. It will probably help the students best to understand the meaning of converse if they think of examples for themselves. As the year proceeds, they will have increasing scope for producing examples.

1. a. If pigs fly, then four is six.
   b. If a figure is a triangle, then the figure has three sides.
   c. If the sum is 5, then 2 and 3 are added.
   d. (The lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ define four rays at P) $\Longrightarrow$ (the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at P).
   e. (Jane is younger than John) $\Longrightarrow$ (John is older than Jane).

2. The converse is:

   If $m(\hat{A}) = m(\hat{B})$, then $\hat{A}$ and $\hat{B}$ are right angles. The given implication is valid, but its converse is not.

3. The converse is:

   If the supplement of $\hat{A}$ is an obtuse angle, then $\hat{A}$ is acute.

   Both the implication and its converse are valid.

4. The converse is:

   $$(\overline{AB} \equiv \overline{BP}) \Longrightarrow (P \text{ is the midpoint of } \overline{AB})$$

   The given implication is valid, but its converse is not, since P is not necessarily on the line segment $\overline{AB}$.

Chapter 2
AN INTRODUCTION TO DEDUCTIVE REASONING IN GEOMETRY

2-1 Introduction

The purpose of this chapter is to give the student some experience with deductive reasoning in geometry, done informally, before launching into a full-fledged deductive study of geometry.

There is nothing strange or new about deductive reasoning. As the text points out, it is just clear, common-sense thinking. However, its simplicity should not suggest that it is unimportant. On the other hand, its importance should not suggest that it is difficult to understand. We hope that by doing some informal proofs and then analyzing and criticizing them, the student will begin the more formal treatment with confidence that he is able to understand deductive reasoning.

The aim of the chapter can probably be met even though many students have not mastered all ideas presented. The concepts of this chapter will be seen again (and again). The student should be encouraged to study this chapter a second time after additional work has been done. It would be especially appropriate for him to reread this chapter after Chapter 8.

Some of the problems may be quite difficult for the students. In order to keep morale high, it might be best to use some problems as class activity; that is, work out the solution with suggestions from the class.

In problems which begin with the words "can you do . . . ", we suggest that many students should not be satisfied with a simple yes or no answer. The better students should consider such a question to be followed by the instruction, "If so, then do it; if not,

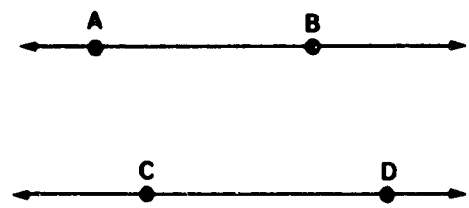
state why not". On the other hand, a student who has the correct answer, but is not sure why his answer is correct is by no means a failure. Intuition, as well as deductive reasoning, must be developed. Thoughtful guesses and conjectures should be encouraged.

You should not attempt to put the proofs in this chapter into the customary form in which "statements" are listed on the left with corresponding "reasons" on the right. Such a form is quite inappropriate, since we have not listed our postulates carefully, and we do not have a backlog of carefully proved theorems available for "reasons".

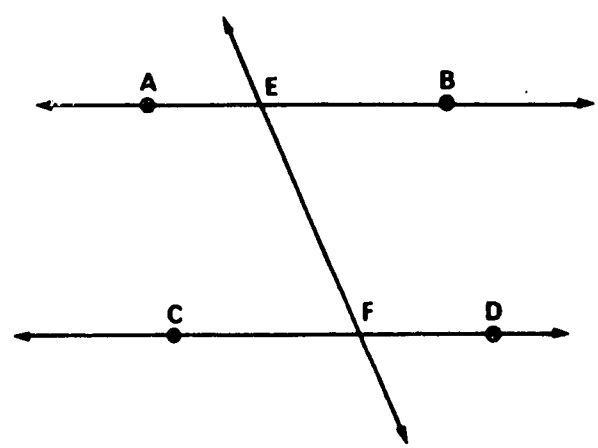
2-2 Some familiar geometric ideas

The ideas of this section are ones which the student has seen in his earlier work. Since he has seen many different geometric ideas, spending a relatively short time on each, some review is necessary.

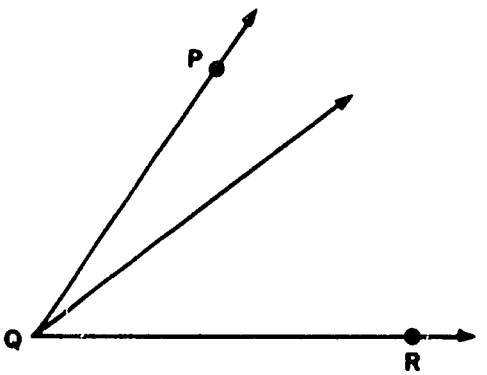
The student should be urged to work actively with paper and pencil while he studies the text. He should be shown how even the simplest figures will be more meaningful if he draws them himself. For example, the figure showing two parallel lines cut by a transversal comes naturally as a sequence of two pictures. First there are simply the two parallel lines, with points A, B, C, D named so lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ can be named.



Then on this picture, the transversal is drawn and points E and F are labelled. As he draws this, the student should think of lines not in the same plane with $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ (such lines will intersect, at most, one of the given lines) or lines parallel to $\overleftrightarrow{AB}$.



For more complicated figures, the finished picture may be overwhelming at first glance. But if the student has learned to find from the text how the picture has been developed, he can draw his own picture. His picture should be more helpful than the one in the text.

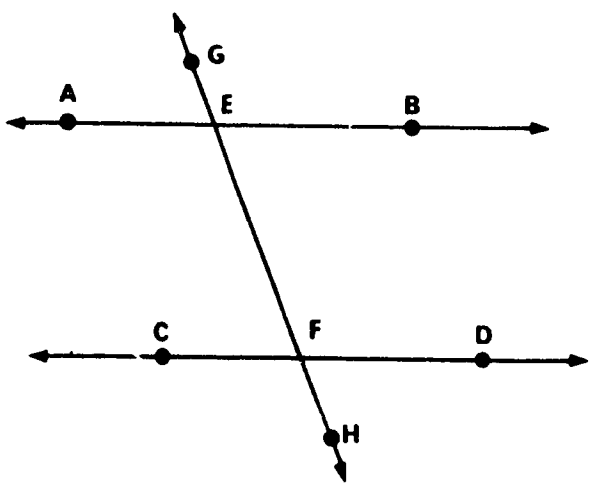


The idea in the last part of this section, concerning the measure of an angle like $P\hat{Q}R$, will appear again in Chapter 6, in connection with Angle Measurement Postulate 2.

Answers to
CLASS ACTIVITY

1. $A\hat{E}F$ and $E\hat{F}D$ form the other pair of alternate interior angles.

2,3. To name the pairs of alternate exterior angles, two more points on $\overleftrightarrow{EF}$ must be named. Referring to the figure, we see that $A\hat{E}G$ and $H\hat{F}D$ form one pair of alternate exterior angles, and $G\hat{E}B$ and $C\hat{F}H$ form the other. $A\hat{E}G$ and $H\hat{F}D$ are called alternate exterior angles, because sides $\overrightarrow{EA}$ and $\overrightarrow{FD}$ lie on alternate sides of the transversal and sides $\overrightarrow{EG}$ and $\overrightarrow{FH}$ are exterior to the region bounded by the parallel lines. Similarly for $G\hat{E}B$ and $C\hat{F}H$.



Perhaps you will want to ask Question 3 before Question 2, since it is easier to explain the name "alternate exterior angles" by referring to a particular pair of angles.

4. From the discussion in the text and from the student's experience with measuring angles, the first figure should clearly suggest the equation $40 + 60 = x$, or $x = 100$.

From the second figure, we obtain

$$108 = 27 + x, \text{ or } x = 81;$$

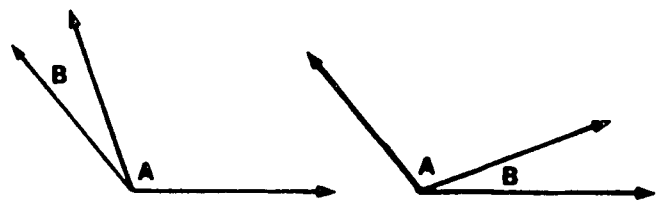
and from the third,

$$85 + 2x = 105, \text{ or } x = 10.$$

Answers to
PROBLEMS 2-2

Student Text Pages 13-14

1. The figure should like either of these. In any case, the measures of $\hat{A}$, $\hat{B}$ and the large angle are about 110. 20 and 130, respectively.



2. $x + 2x = 90$, so $x = 30$.
3. The angles are right angles, so $x = 90$.

2-3 The straight angle

The concept is a simple one and no elaborate discussion should be needed. We will not talk about the interior of a straight angle, so some discussions involving angles will have to exclude straight angles.

2-4 The first deduction

The point of this section is not the two geometric properties, the alternate interior property and the angle-sum property for triangles, but the connection between these two properties. One is deduced from the other. These particular properties were chosen because they are simple and the deductive argument which relates them is simple.

The alternate interior property is one which appears obvious after examining a few well-drawn figures. We derive the angle-sum

property for triangles, a property which is not obvious, from inspection of pictures of triangles.

It should be emphasized that the degree of generality which was assumed for the alternate interior property is carried over to the conclusion. That is, the deduced property states that the angle sum for every triangle, no matter what size or shape, is 180. This sort of generality cannot come from pictures, even if exact measurements were possible, since infinitely many triangles would have to be considered.

Use has been made of the parallel postulate in drawing the line through C parallel to $\overleftrightarrow{AB}$. The parallel postulate is not mentioned at this place in the text since we feel that the point of this informal deduction would be obscured if the existence or uniqueness of this parallel were questioned here. It is discussed in detail in Chapter 11.

Comments on
CLASS ACTIVITY

One would expect that the angle sum of a triangle, obtained by measurement of a figure, would not be exactly 180. It might be in error by as much as 5 or 10 degrees, depending on the size of the sketch and the skill of the student. This would cast no doubt on the derived angle-sum property for triangles, if we still accept the alternate interior property. Measurements of figures may be helpful in discovering what is true, but proofs will not be based on measurements.

Additional
CLASS ACTIVITY

Ask if it was necessary to choose C as the vertex through which the parallel line is drawn. Of course, any of the three vertices could have been chosen. For example, drawing the line through A parallel to BC would give essentially the same proof of the angle-sum property for triangles.

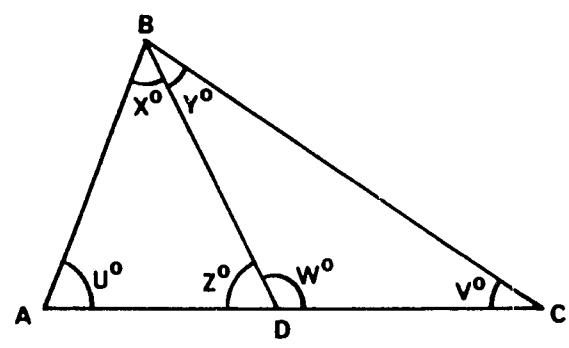
Answers to
PROBLEMS 2-4

1. If all angles of $\triangle ABC$ are congruent, then they each have the same measure, say x. Then the angle-sum property implies that $x + x + x = 180$, or $x = 60$.
2. If, for example, $\hat{A}$ is a right angle, then the measure of $\hat{A}$ is 90. Thus, the sum of the measures of $\hat{B}$ and $\hat{C}$ must be 90. This implies that $\hat{B}$ and $\hat{C}$ are acute angles.
3. It is not possible for a triangle to have two angles, each having measure larger than 90. Thus, the congruent angles in this triangle must be the two unknown angles. We get the equation $102 + x + x = 180$, or $x = 39$, the measure of each of the unknown angles.

Comments on
CLASS ACTIVITY (second part)

Although this is a sketch of an *indirect* proof, that fact need not be mentioned now. We feel that an indirect argument is just another way of thinking carefully and reasonably. The introduction of the term *indirect proof* may make the students unnecessarily apprehensive about it.

The wording used in this problem has been deliberately chosen. If the student can actively put to himself the problem of convincing a mistaken, but reasonable, friend, the argument should be more easily understood.



Referring to the figure, we see that the angle sum for $\triangle ABC$ is $u + (x + y) + v$, which is 182, by assumption. For $\triangle ABD$, we have $u + x + z = 182$ and for $\triangle BCD$, $y + w + v = 182$. Thus, $(u + x + z) + (y + w + v) = 182 + 182$. But the expression on the left is

$$[u + (x + y) + v] + (z + w) = 182 + 180,$$

since $z + w = 180$. Hence, $364 = 362$, a conclusion which must be accepted by our friend who thinks that the angle sum for every triangle is 182. Thus, our friend is forced to discard his erroneous assumption.

Answers to
PROBLEMS 2-4 (continued)

Student Text Page 17

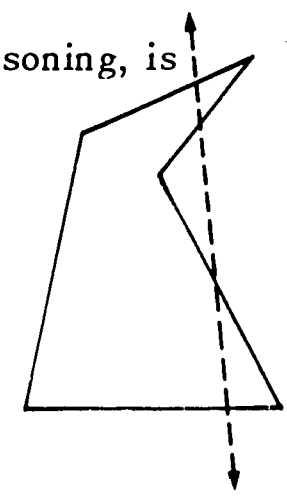
4. The argument here differs from the class activity above only in numbers. The contradiction comes in the equality of $180\frac{1}{2} + 180\frac{1}{2}$ and $180\frac{1}{2} + 180$.
5. Again the argument is esentially the same. The contradiction is $179 + 179 = 179 + 180$.
6. The theorem which is being hinted at is a weaker version of the angle-sum property for triangles. It should read something like this:

   If it is true that *the angle sum for every triangle is the same,* then the angle sum for every triangle must be 180.

   This has not been proved in these exercises, but the three cases considered should give a clear indication of the general argument.

2-5 A second deduction

The chain of properties, linked by deductive reasoning, is lengthened in this and the two following sections.



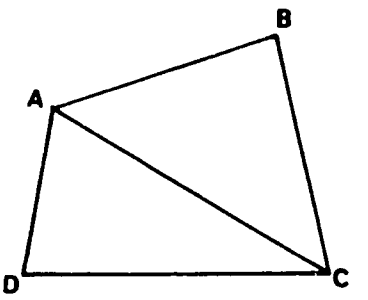
The term *convex polygon* can be defined as follows. A polygon is called a convex polygon in case every line which does not contain a side of the polygon meets the polygon in at most two points. Thus, if a polygon has a vertex "pointing inward", there will be a line which meets the polygon in at least four points, and the polygon is not convex.

Parallelograms (and hence also squares and rectangles) are always convex. So is every triangle. The simplest non-convex polygons are the non-convex quadrilaterals. (This will answer the Class Activity questions.)

In the informal discussion in the text, we prefer not to explain this careful definition to the students. The statement that a convex polygon has all vertices "pointing outward" should convey the idea adequately. Of course, if a very able and curious student should question the definition, there is no reason to withhold the precise definition from him.

The deduction in this section is written in a way which requires the student to make his own figures. As we have mentioned before, the students should develop the habit of drawing their own figures as they study.

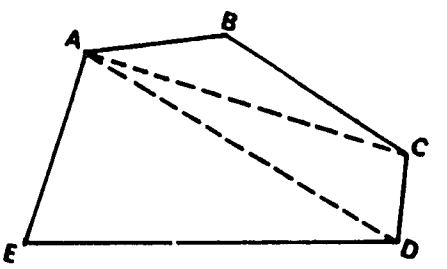
The figure will look something like this. Then the sum of the measures of the four angles of the quadrilateral is the same as the sum of the angle sums for the two triangles, $\triangle ABC$ and $\triangle ACD$. Thus, the angle sum for a convex quadrilateral is 360.

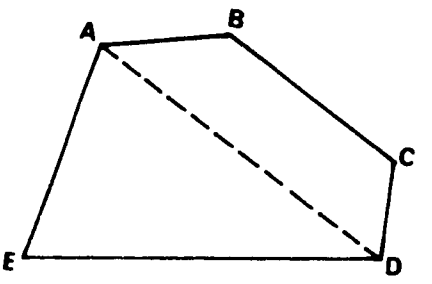


Answers to
PROBLEMS 2-5

Student Text Page 18

2. A convex pentagon can be split into three triangles by drawing two diagonals. A second way is to draw one diagonal, obtaining a convex quadrilateral and a triangle. Thus, the angle sum is 360 + 180, or 540.





3. A convex hexagon can be split into triangles (four of them) or into one triangle and a convex pentagon. The angle sum is 540 + 180, or 720.
4. If Problems 2 and 3 are done by the second method, it should be clear that if the angle sum for all convex $(n - 1)$-gons is known, then the angle sum for convex n-gons is just 180 more. There is really a mathematical induction argument here, but consideration of a few more cases should make the essence of the proof clear. This does not seem like the proper place to begin a discussion of proof by mathematical induction.
5. Regular triangle: each angle has measure

$$\frac{180}{3} = 60.$$

Regular quadrilateral: each angle has measure

$$\frac{2 \times 180}{4} = 90.$$

Regular pentagon: each angle has measure

$$\frac{3 \times 180}{5} = 108.$$

Regular hexagon: each angle has measure

$$\frac{4 \times 180}{6} = 120.$$

Regular n-gon: each angle has measure

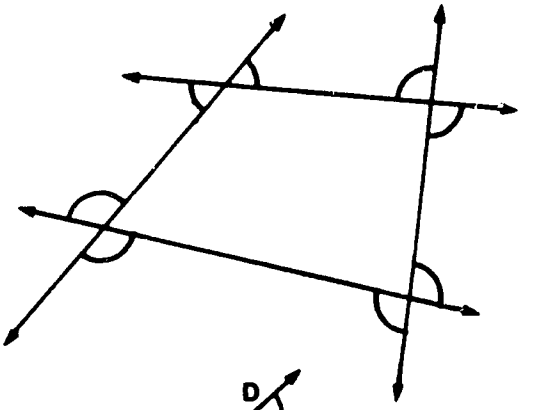
$$\frac{(n - 2) \times 180}{n}.$$

2-6 Additional deductions

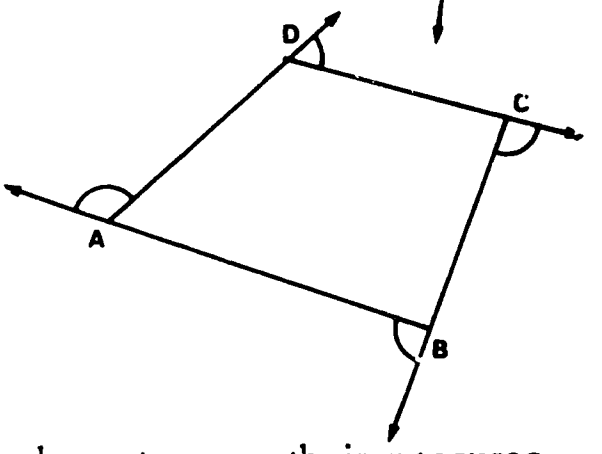
Throughout this section, all polygons considered will be convex polygons.

A polygon of n sides has $2n$ exterior angles, illustrated in the

figure for the case $n = 4$. They come in pairs of congruent angles, so when considering measures of exterior angles it is immaterial which angle of a given pair is examined.



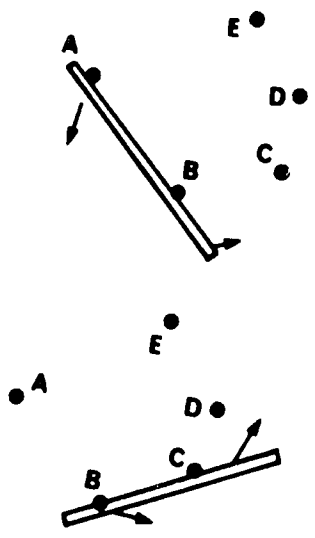
In the deduction, we select, in a systematic way, one exterior angle from each pair. One selection is illustrated in the text. Another suitable choice is shown on the right. Having made such a selection for an n-gon, we have n pairs of adjacent interior and exterior angles.



The angles in each pair are supplementary, so their measures total $n \times 180$. Since the sum of interior angles is $(n - 2) \times 180$, the exterior angles must add to

$$n \times 180 - (n - 2) \times 180 = 2 \times 180 = 360.$$

The result can be made intuitively clear by the following physical device. Locate points at the vertices of a convex polygon; a pentagon will do nicely. Place a straight-edge along AB as shown. First rotate (clockwise) and then slide (to the right) the straight-edge, keeping it in contact with pin B. Soon the straight-edge rests against pins B and C. The angle through which the straight-edge has turned is measured by the exterior angle at B. Continue the process, letting pin C play the role previously played by B. The straight-edge then rests on C and D, after turning an amount measured by the exterior angle at C. Doing this, in succession, a total of five times will return the straight-edge to its original position. It will have turned through one complete revolution (degree measure 360), but it is clear that it has turned through an amount equal to the total of the measures of the exterior angles.

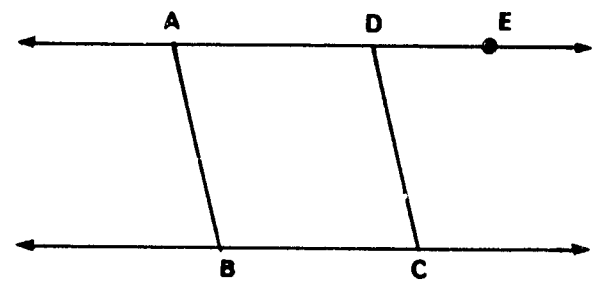


Answers to Problems 2-6 are included in the general discussion of the proof, given at the beginning of this section in the TEACHERS' GUIDE.

2-7 Some related properties of parallelograms

Here are the details of the deduction of this section.

Line $\overleftrightarrow{CD}$ is a transversal of the parallel lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$. Using the alternate interior hypothesis, $C\hat{D}E \equiv B\hat{C}D$. $m(C\hat{D}E) + m(C\hat{D}A) = 180$, so $m(B\hat{C}D) + m(C\hat{D}A) = 180$. This is the first equation asked for in the text. The same argument, applied to parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ and transversal $\overleftrightarrow{AD}$, gives the second equation, $m(B\hat{A}D) + m(C\hat{D}A) = 180$. From the first equation, $m(B\hat{C}D) = 180 - m(C\hat{D}A)$, and, from the second equation, $m(B\hat{A}D) = 180 - m(C\hat{D}A)$. Hence, $m(B\hat{C}D) = m(B\hat{A}D)$, which says that the interior angles at A and C have equal measures.



At this point you might pose the following question. Is it necessary to prove $m(A\hat{B}C) = m(A\hat{D}C)$ in addition to the work just done? The answer is no, since the pair of interior angles at A and C are an arbitrary pair of opposite angles in a parallelogram. Or another way of looking at this is that we can reletter the figure in such a way that the old labels A, B, C, D are replaced by B, C, D, A, respectively. The argument applies to the relettered figure, word for word, and we see that the pair of interior angles at B and D (old labelling) are congruent.

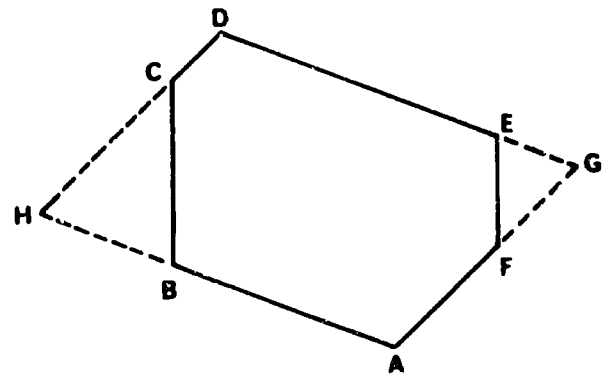
Answers to
PROBLEMS 2-7

Student Text Page 21

1. For most problems of this sort, in which the student is asked to write out the details of a proof, the papers should be read and corrected very carefully. This is especially important at

the beginning, when the students are just beginning to learn what a proof is. It may be helpful to return their corrected papers and ask that rewritten versions be submitted.

2. The figure *ABCDEF* is a hexagon with opposite sides parallel. As the dashed lines indicate, such a figure can be obtained by slicing off opposite corners of a parallelogram (*AHDG*) by parallel slices. From the parallelogram, using the result of this section, $\hat{A} \equiv \hat{D}$. Since *A* and *D* form an arbitrary pair of opposite angles, we have proved that if a hexagon has opposite sides parallel, then opposite angles are congruent.



2-8 A second look at deductive reasoning

This is a section which may not be fully understood by most students on their first exposure to it. It does not seem advisable to insist on mastery of this section at this time. In Chapter 8, these concepts will be treated more thoroughly, at a time when the students' experience has made them better prepared for these concepts. The aim of this section will have been met if the students start thinking about these ideas and if they keep them in mind as they study the following chapters.

The basic terms (*point, line* and so on) on which definitions are based are technically called "undefined terms". This name may sound mysterious to the student and should not be used here, lest the student feel that undefined terms are completely meaningless. We feel that the student should think of lines and points as idealizations of corners of rooms, chalk lines, dots, and so on. His thinking should be based on real things which he can see and understand. The insecurity which would probably be created by the expression "undefined terms" could easily be disastrous at this stage of the students' development.

Illustrations of circular definitions abound in any dictionary. It might be instructive to use a dictionary of a language other than English if one is available.

Answers to
PROBLEMS 2-8

Student Text Page 23

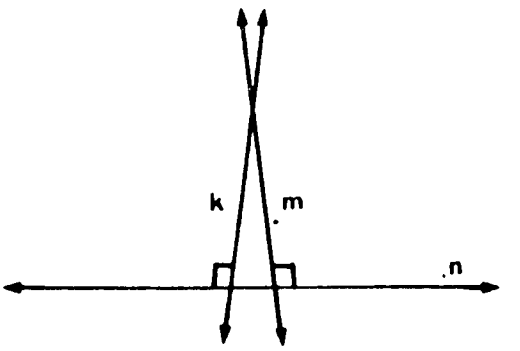
1. Starting with "house" one might find a circular chain of definitions like this:
"house: a building for human use or occupation, abode".
"abode: a dwelling place".
"dwell: abide as a permanent resident, reside".
"reside: dwell permanently, or for a considerable time".
The chain is circular between "dwell" and "reside". Other circular chains may arise from the same initial word.
2. For another circular chain of definitions, it is probably best to start with a noun. Words like "sister", "mother", "commerce" and so on are likely to give easy circular chains.
3. The student is expected to recognize that if *addition* and *natural number* are understood, then the other terms have definitions based on these. *Multiplication* may be defined as successive addition (since we are concerned only with natural numbers in this problem); *division* in terms of multiplication and *divisibility* in terms of division (or multiplication). Then an *even number* is a natural number which is divisible by 2, and all other natural numbers are *odd*. A *prime number* is a natural number other than 1 which is divisible only by 1 and itself.

2-9 Discussion

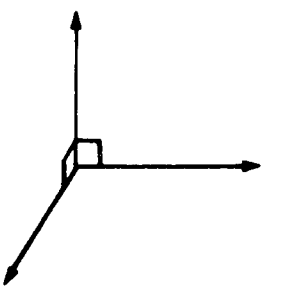
In this section, we are trying to set the stage for the beginning of the careful deductive study of geometry. Weaknesses in the informal arguments are pointed out and hopefully the students will be more critical of proofs and definitions given in subsequent chapters.

2-10 Another informal deduction (optional)

Here is still another deduction, based on the angle-sum property for triangles. If, as developed in the text, there are lines k and m, both perpendicular to n, and if these lines intersect, then a triangle is formed. The base angles of this triangle are both right angles, so the angle sum for this triangle exceeds 180. This is a contradiction, so two lines perpendicular to the same line cannot intersect.



It should be understood that the three lines lie in a plane. Is it possible for two distinct lines to intersect and be perpendicular to the same line, if not in the same plane? This comment could be phrased as a question for class activity.



Section 2-10 could be omitted without loss of continuity. In particular, if the class is anxious to get on with the careful treatment, it might be best to omit this section.

Answers to
PROBLEMS 2-10

Student Text Page 25

1. In the sketch in the text, each of the angles marked at C must be congruent to $\hat{A}$. Thus, these angles are congruent to each other, but the figure clearly indicates that one is larger than the other. Therefore, there cannot be two lines through C parallel to $\overleftrightarrow{AB}$.

Chapter 3
OUR FIRST GEOMETRIC POSTULATES

3-1 Why study geometry?

In this section an effort is made to convey an appreciation of geometry to the students. Your students know that the study of geometry is very practical. What they may not realize is that it is also an intellectually exciting experience if only it is approached in the proper spirit. A study of geometry can call attention to the importance of careful and precise use of words and the beauty of correct and logical reasoning.

3-2 The relation of geometry to the real world

Many of your students will think that lines and points are only the dots and pencil streaks that they mark upon paper. It is not easy to go from the world of physical geometry, where straight lines are taut strings or rays of light (really not *quite* straight), to the imaginary world of mathematical geometry, where lines are so straight that only one line connects each pair of points.

A man can dream of a perfect geometrical world and his dreams of mathematical geometry can transform his knowledge of physical geometry. Pure thought is a powerful tool for obtaining practical knowledge. Many of the secrets of the universe are revealed to the dreamers and thinkers.

The idea that the world of geometrical objects is a world that we can summon into view in our mind's eye may be helpful to students. You might help them in this visualization by pretending to

trace out geometric objects in space with the tip of your finger. For example, you might point with your finger at a spot in space and say, "Imagine a point here". Trace a triangle with your finger in the air and ask a student which is the longest side. Ask a student if he sees a line joining two points in space.

3-3 Deductive reasoning in geometry: the incidence postulates

If your students realize that geometry is the study of an imagined world, then they will have made a long stride toward seeing the need for the formulation of postulates. They have had experience with deductive reasoning. They will know that one cannot reason in a vacuum empty of all ideas. We must accept some statements as true. We must accept some terms without insisting that they be defined in simpler terms.

The Incidence Postulates may seem strange. Students may feel that there ought to be "as many points as they need". Soon we will postulate all the points in space that they could wish for. But we are constructing our geometry little by little. And so for the moment we see only three points in a plane. These are not on any one line. On each of our lines we see two points only. There is not much we can prove about this miniature geometry, but our proofs are very simple and easily understood.

Do not try to amplify the proofs of Theorems 3.1 and 3.2. In particular, do *not* remark that the proof given for Theorem 3.2 is an example of indirect proof.

Answers to
PROBLEMS 3-3A

Student Text Pages 29-30

1. In order to have a few interesting problems here, we ask ourselves what might be the case if we had more points in our plane. Although our postulates only assure us of the existence of three points in each plane, they do not prevent us from thinking about the possibility of more points. If any student complains that we have no right to talk as if there are four points,

X, Y, Z and T, in a plane, point out that we are not asserting that there *are* four points. We are simply wondering what might happen *if* there were.

Since line k contains points Y and Z and line m contains Y and Z, these two lines are the same. Write on the board:

$$k = m,$$

and remind students that every statement of equality is just an assertion that two names are names of the same thing.

2. Some students may say that the student should not make such a drawing because he has not drawn *straight* lines. Our postulates say nothing about "straightness" of line, so we have no basis for a discussion of straightness. Remind the students that our work is based only on the postulates. The comment you want, and surely will get from some student, is that the student is thinking of two different lines, both of which contain P and Q If he understood our postulates, he would know there are not two such lines.

3. Counting problems like these are interesting to students. For the first two questions, answers are, respectively, 4 and 6 lines. For 5 non-collinear points, if 4 are on one line there are 5 lines. If A, B, C are on one line and A, D, E on a different line, there are 6 lines. If A, B, C are on one line and no other line contains 3 of the points, there are 8 lines. If no more than 2 of the 5 points lie on any one line, there are 10 lines.

4. As an introduction to Problem 4 you might mark a point on the board and ask several students to think of names for the point. Then ask for these names. Hopefully, you will get names like M, A, X so that you can write:

$$M = A = X.$$

Students must always be conscious of the fact that different names can name the same thing.

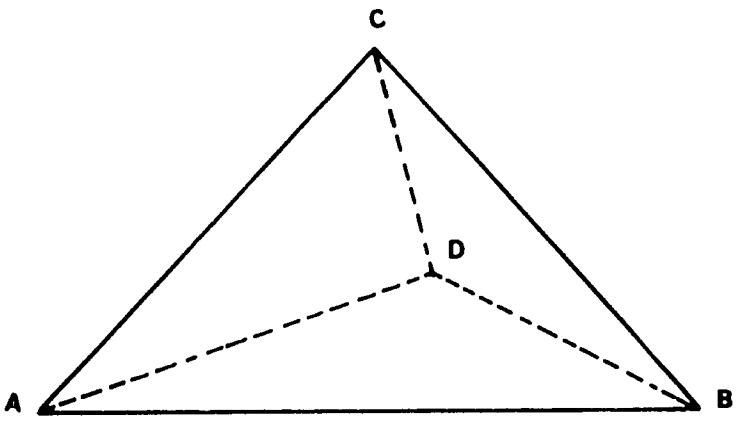
The answer to the problem is that P and Q are two names for the same point. Be sure to write:

$$P = Q.$$

5. The conclusion is:

$$k = m.$$

Our last four postulates enrich our geometry considerably. These postulates lift us off the plane. Now we can see 4 points in in space. To make these problems easier, put a good drawing on the board for reference, *or* make some sort of a model to set on your desk.



Answers to
PROBLEMS 3-3B

Student Text Pages 31-32

1. Four planes: plane *ABC*, plane *BCD*, plane *CDA*, plane *DAB*.
2. Six lines: line *AB*, line *AC*, line *AD*, line *BC*, line *BD*, line *CD*.
3. Lines *AB* and *CD* are skew to each other as are the pair, line *AD* and line *BC*; and the pair, line *AC* and line *BD*.

   If lines *AB* and *CD* did lie in one plane, then that plane would contain all 4 points and we know that no plane has this property.
4. We can prove that 2 planes contain line *AB*, namely, plane *ABC* and plane *ABD*. The easiest way to satisfy oneself that this is all that can be proved is just to list all the planes and match with each plane the point that is *not* in it. This is a proof by looking at all possibilities.

Plane	*Points not in the plane*
ABC	*D*
ABD	*C*
ACD	*B*
BCD	*A*

Of course, our answer would be that the number of planes is the same for every line.

5. We can not prove that there is a pair of parallel planes. In the discussion of Problem 4, we have listed all the planes that we *know* we have. Examining the 6 pairs of planes, we see that each pair intersects in two points.
6. The discussion of Problems 4 and 5 will make this easy.
7. Clever students will realize that they can look simply at the symbol

   "plane *ABC*"

   and state that each of the lines *AB*, *AC* and *BC* has two points in the plane.

   Our Space Postulate 3 tells us that *if* one of these lines contains any more points, then all those points also lie on plane *ABC*.
8. The lines are line *AD*, line *BD* and line *CD*.
9. For example, planes *ABC* and *BDC* both contain points *B* and *C*. Hence, line *BC* is the intersection of these planes. We can prove that the 2 points *B* and *C* lie on both planes, but we can't prove yet that there are any more, since we have not yet postulated that a line contains more than 2 points.

Chapter 4
THE BETWEENNESS POSTULATES

4-1 Introduction

The betweenness postulates will start us on the way to filling space up with points. The seven postulates that we state in this chapter refer to certain simple relationships that seem to be true for the pictures we draw. Since we have announced our intention to list everything that we shall use in making proofs, honesty demands that these postulates be written down. There are many opportunities here for students to strengthen their powers of space visualization. There is much practice in using the postulates to make simple one- and two-step proofs. Without these postulates we would have no precise definitions for segments, rays, angles or triangles.

Make it clear to your pupils that they are not expected to memorize these seven betweenness postulates. For years past students have used the ideas expressed in these postulates by inspecting carefully drawn figures. Your students will be in the stronger position of being able to relate the information they get from the figures they draw to the precise verbal statements that are our postulates.

4-2 The first betweenness postulates

In developing the ideas surrounding these first four postulates the order of presentation should be:

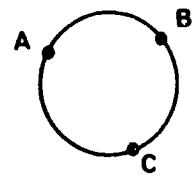
1. Examine pictures.
2. Formulate informally in words the ideas concerning betweenness that the pictures suggest.

3. Examine the postulates carefully to see that they are precise statements of the betweenness properties suggested by our pictures.

It may be possible for you to lead your students to their own formulations of the postulates. At the blackboard mark these points,

and ask, "Would you want to say that this shows B between A and C?" When your pupils dissent, erase B and ask someone to go to the board and mark B where he feels it should be in order that it be *between* A and C. Then get someone to say that B has been marked *on the line* $\overleftrightarrow{AC}$. You yourself might then erase B and mark a new point for B superimposed upon A and ask if they wish to say that B is between A and C if B and A are the same point.

For Postulate 2 you might ask a pupil to mark 3 points on a line so that no one of them is between the other two. You might draw a circle as below and mark 3 points:



Now it is natural to say that each one of these three points is between the other two. Postulate 3 makes it certain that our lines don't behave like circles. We are beginning to get our lines "straightened out", using our postulates to do so.

Postulate 3 brings out clearly how different our imaginary mathematical geometry is from the physical geometry of the real world. In the real world, if we continue making chalk marks between two chalk marks on the blackboard, eventually we cover up all the space between our two "points" and our physical "points" begin to overlap. Only in our imagination can we have a situation like that described in Postulate 3 where we can *think* of infinitely many

points between two given points. It might be helpful to mark two chalk spots on the board a few inches apart and mark chalk spots between, counting them as you do so, until the space between the two original spots is completely covered with chalk spots.

To lead up to Postulate 4, stand 3 pupils in a row and ask the class to name them 1, 2, 3, so that pupil 2 is "in the middle". Then have four pupils stand up and name them 1, 2, 3, 4. Talk about the betweenness relations. Write on the board,

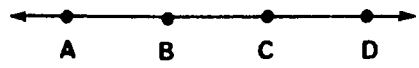
1, 2, 3
1, 2, 4
1, 3, 4
2, 3, 4,

to illustrate your discussion.

Answers to
PROBLEMS 4-2

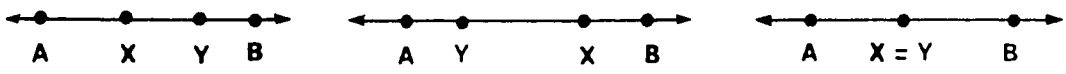
Student Text Page 35

1. There are four betweenness relations for four distinct points on a line. For



they are A, B, C; A, B, D; A, C, D; and B, C, D. There are ten such relations for five distinct collinear points.

2. The points are not collinear. We are using Postulate 2.
3. $A = B$ or $A = C$ or $B = C$. We are using Postulate 2 again.
4. T is between R and S. We use Postulate 2 again.
5. (i) Points X, Y, A, B are on one line (Postulate 1).
   (ii) Points A and B are endpoints for this set of four points. There are three possibilities, shown by these figures:



(We are using Postulate 4.)

(iii) *If* Y is between X and B, *then* X is between A and Y (Postulate 4).

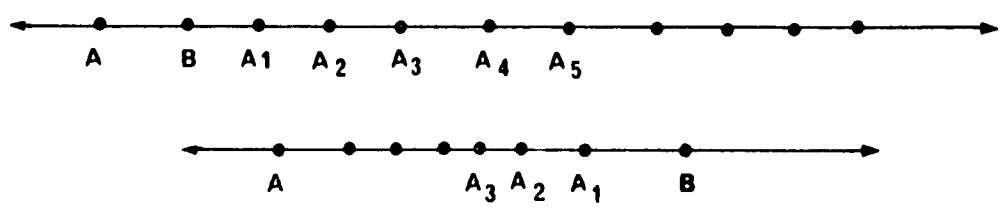
(iv) *If* Y is between A and X, *then* X is between Y and B (Postulate 4).

6. The points fall on the line in the order U, R, T, S *or* S, T, R, U. Besides the two given betweenness relations we see that we also have T between U and S and R between U and S. In order to deduce this from the postulates we must use Postulate 4 which tells us that points U and S are the endpoints so we can label the points either as

$$\begin{array}{cccc} U, & R, & T, & S \\ \downarrow & \downarrow & \downarrow & \downarrow \\ A_1 & A_2 & A_3 & A_4 \end{array} \quad \text{or} \quad \begin{array}{cccc} U, & R, & T, & S \\ \downarrow & \downarrow & \downarrow & \downarrow \\ A_4 & A_3 & A_2 & A_1 \end{array}$$

7. How the student labels these points depends upon how he chooses to mark them on the line.

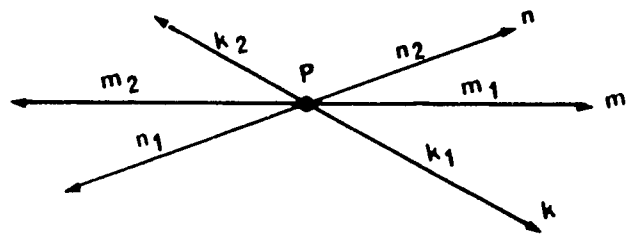
8. To prove that there are infinitely many points on each line, a student might use either figure below in support of his argument.



The student should say that he is repeatedly using Postulate 3. Strictly speaking it is necessary to use Postulate 4 in making any formal proof that these procedures really lead to infinitely many points, but this need not be mentioned. The second figure above suggests a proof that each segment contains infinitely many points.

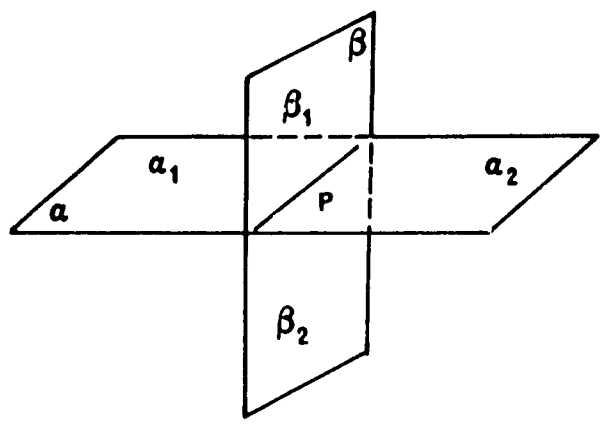
4-3 The separation postulates

Note the uniformity of language used in discussing the three separation situations. The choice of language is deliberate. Language like this helps pupils to see relationships between ideas. Perhaps you should mark a point P on the board and draw several lines through P as below:



Remark that k_1 and k_2 are opposite sides of P on line k; m_1 and m_2 are opposite sides of P on line m. Each point determines two sides on *each* line that contains it.

You could draw the same sort of picture for sides of a plane determined by a line contained in the plane.



Half-planes α_1 and α_2 are opposite sides of line p in plane α; half-planes β_1 and β_2 are opposite sides of line p in plane β. Each line has two sides on *each* plane that contains it.

Answers to
PROBLEMS 4-3

1. P is between A and B.
2. C and A are on opposite sides of P.
3. S may not be in plane α, but if S *is* in plane α, then S is on line k.
4. The three sets into which a point A on a line divides the line are:
   (a) The set containing just point A.
   (b) The set containing all points on one side of A on the line.
   (c) The set containing all points on the other side of A on the line.

   Similarly one can describe three sets of points for each line in a plane and for each plane in space.

4-4 Using the idea of betweenness to make definitions

Your pupils will have met many of these definitions before. The ideas will not be new. What is new, however, is that we can base these precise definitions on our postulates.

Often a student may feel that a definition is clumsily stated and may believe that he can formulate one which is simpler and clearer. Encourage them to try formulating their own definitions. Often through such an attempt a pupil comes to a better understanding of the concept being defined.

We give proofs below for Examples 1 and 2 much as we hope a pupil might give them. Remember that what is important at this stage is that the student see clearly the basic idea of the proof and communicate well enough with his fellow students that they understand his proof.

Example 1. If a point is on $\overline{AB}$, then either that point is A, or it is B or it is between A and B. If it is A or B, then it lies on the line $\overleftrightarrow{AB}$. If it is between A and B, it lies on $\overleftrightarrow{AB}$ because of our first betweenness postulate.

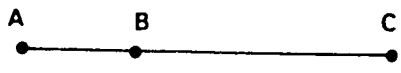
This problem provides you with an opportunity to point out that since we are proving that something is true for *every* point on $\overline{AB}$, we have to think about all of them. A pupil who argued only that points *between* A and B were on line $\overleftrightarrow{AB}$ would be presenting an incomplete proof.

Example 2. Let $\overline{AB}$ be *any* segment on line k. By a betweenness postulate, there is a point C such that B is between A and C. By another betweenness postulate, C is *not* between A and B. Hence, by the definition of a segment, C is not on segment $\overline{AB}$. Moreover, by the first betweenness postulate, C is on line $\overleftrightarrow{AB}$ and so we have shown that if $\overline{AB}$ is any segment on a line, then there is a point on that line which is not a point of the segment.

Answers to
PROBLEMS 4-4

Student Text Page 42

1. If $\overline{AB}$ and $\overline{CD}$ intersected in two points, then the distinct lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ would intersect in two points, and we have proved this false.

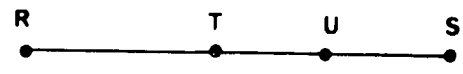
2. a. 

   Intersection is point B.

   b. See Problem 3.

3. Students will see that if two segments intersect in more than one point, then their intersection is a *segment*. Several pictures are possible as shown below.

Intersection is $\overline{TS}$.



Intersection is $\overline{TU}$.

4. Suppose two segments have two distinct points P and Q in common. Then every point between P and Q belongs to both segments, and there are infinitely many such points. Actually

the proof that *if* P and Q are distinct points between A and B, *then* every point X between P and Q is also between A and B depends upon Betweenness Postulate 4. However, this is something that we can let our pictures tell us.

5. There are several possible types of pictures for a, but for b the only picture is this:

($\overrightarrow{AC}$ and $\overrightarrow{AB}$ intersect only at point A.)

6. The proof that if two rays have two distinct points in common, then they have infinitely many points in common is much like the proof for Problem 4. A student should argue first that if P and Q are two points on a ray, then every point between P and Q is on the ray. Formally, this argument would read:
   (a) P and Q are on $\overrightarrow{AB}$.
   (b) X is between P and Q.
   (c) X is on P's side of A on line $\overleftrightarrow{AB}$ (or, if $P = A$, then X is on Q's side).
   (d) X is on ray $\overrightarrow{AB}$ by the definition of ray.

   In order to prove that statement (c) above is true one needs to say that point X is not point A, and if X were not on P's side of A, then A would be between P and X, and so by Betweenness Postulate 4, point A would be between P and Q. This is clearly false.

   This example shows how meticulously one must argue if the betweenness postulates are actually used in making proofs without appealing to drawn figures. Do not hold your students to such standards. Allow them to present arguments in which betweenness properties are obtained from figures.

7 and 8. Note in part a that the line might not lie in the plane of the triangle. This suggest the answer to 8.

9. A and B are on opposite sides of k. A and C are on opposite sides of k. Hence B and C are on the same side and k does not contain a point of segment $\overline{BC}$.

10. Any point X such that B is between A and X is a point of $\hat{A}$ which is not a point of $\triangle ABC$.

11. Three rays determine 3 angles; four rays, 6 angles; five rays, 10 angles. The answers do not depend upon the rays being coplanar.
12. The answers are the same as for Problem 11.
13. Ray $\overrightarrow{AB}$ consists of A and all points on B's side of A. If A were between two points X and Y of the ray, then X and Y would be on opposite sides of A and so would not both be on B's side of A.

4-5 Additional definitions

These definitions are standard. It is not necessary to memorize these definitions, but students should appreciate their clarity and should be able to give definitions equivalent to these. The concept of an *exterior angle* of a triangle is not carefully defined. You might ask your students to supply a precise definition. Perhaps the simplest such is:

> If for $\triangle ABC$ point C is between points B and D, then $D\hat{C}A$ is an *exterior* angle of $\triangle ABC$.

It is intuitively clear that each angle (not a straight angle) divides its plane into three sets: points of the angle, points in the interior of the angle and points in the exterior of the angle. An actual proof of the theorem that *if* point P is in the interior of $\hat{B}$ and R is in the exterior, *then* a point of $\hat{B}$ is between P and R is quite difficult. This is another example of a relationship that seems obvious when we look at pictures, but which cannot be *proved* without the betweenness postulates.

Answers to
PROBLEMS 4-5

Student Text Pages 47-48

1. If P is in the interior of $\triangle ABC$ and Q in the exterior, then there is a point of $\triangle ABC$ between P and Q.
2. c. No such point exists.

3. The interior of $\triangle ABC$ can be defined as the set of all points in the interior of any two of its angles.
4. If a point X is in the interior of $\hat{A}$ it is on B's side of $\overleftrightarrow{AC}$ and on C's side of $\overleftrightarrow{AB}$. If X is in the interior of $\hat{B}$, it is on A's side of $\overleftrightarrow{BC}$ and on C's side of $\overleftrightarrow{AB}$. Hence, if X is in the interior of both $\hat{A}$ and $\hat{B}$, it is on B's side of $\overleftrightarrow{AC}$ and on A's side of $\overleftrightarrow{BC}$. Hence, X is interior to $\hat{C}$.
5. In a and c, $A\hat{B}C$ is not adjacent to $D\hat{B}C$.
6. In b there are ten pairs of adjacent angles.

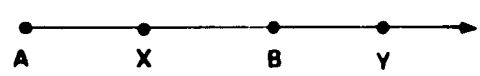
4-6 How we choose symbols and definitions

In mathematics there are infinitely many things for us to think about. We must select certain things upon which we focus our attention. Our definitions reflect these selections.

Often emphasis is misplaced. Students should not expend their energies in the memorization of the particular wording of the definitions lest their attention be drawn away from the essential concepts themselves.

Our choice of symbolism is arbitrary. This symbolism is never essential in itself. It is but an aid in communicating with each other. However, a good choice of symbols certainly aids communication. Whenever possible, the names and symbols should reflect the basic properties of the defined concepts. If you and your pupils decide that some symbolism not introduced in this text is helpful in the study of geometry, feel free to use it.

The fact that definitions can be worded in various ways gives our postulates added importance. The question as to whether or not two differently worded definitions describe the same concept can only be answered by turning to our postulates. The example has already been given of defining the interior of a triangle in two ways. The alternative definition of ray given in Problems 4-6 is a second illustration of these ideas. Consider the two sets of points



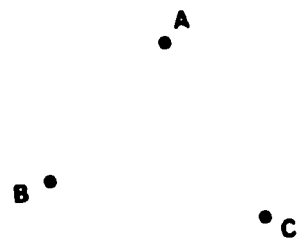
(1) The set consisting of A and all points on B's side of A on $\overleftrightarrow{AB}$.

(2) The set consisting of A, B, all points X between A and B, and all points Y such that B is between A and Y.

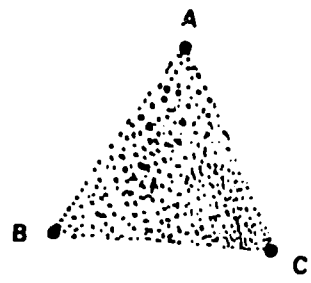
Looking at the figure we see intuitively that the two definitions agree. Note that in the text we do *not* ask pupils to *prove* that the sets are identical. We are content if pupils rely upon the figure for this information. A formal proof can be made using the betweenness postulates, but seems out of place here.

4-7 Filling space with points

Here is an excellent chance to re-emphasize the significance of the incidence and betweenness postulates. Mark three points on the board. Label these points A, B, C as below:



The incidence postulates provide us with these points. The betweenness postulates put points *between* two given points. Have pupils go to the board and mark dots for several such points. First the sides and then the interior of the triangle begin to "fill up" as pictured:



Now use the postulate that if A and B are any points, then there is a point Z such that A is between Z and B. This will extend $\overline{AB}$ to $\overleftrightarrow{AB}$, $\overline{AC}$ to $\overleftrightarrow{AC}$, and so on, and then the students can "see" the regions of the plane filling up with points.

Answers to
PROBLEMS 4-7

Student Text Page 51

1. Let A, B, C, D be four non-coplanar points. Each set of three of these points (there are four such sets) determines a triangle. The union of these triangles and their interiors is the *tetrahedron ABCD*.

   If X is between two points of the tetrahedron and if X is not itself a point of the tetrahedron then X is a point of the *interior* of the tetrahedron.

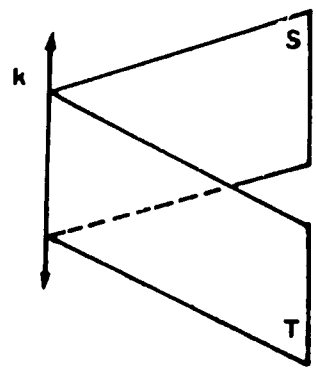
   All points of space which are neither on the tetrahedron nor in the interior of the tetrahedron are points of the *exterior* of the tetrahedron.

2. Think of three planes intersecting as do ceiling, end wall and side wall of a room. These three planes divide space into eight regions. Now imagine a fourth plane which does not pass through the corner of the room and which is parallel to none of the planes already considered. This plane will cut seven of the eight regions. Each of these seven regions has become two regions, so there are 14 + 1 or 15 regions determined by the four planes.

4-8 Another look at the incidence postulates

These incidence theorems could have been proved earlier, but their proofs are just a little more difficult than the first proofs, so they were postponed until now. They will provide a review of the incidence postulates.

1. If k intersects α in two distinct points, A and B, then $k = \overleftrightarrow{AB}$ and the entire line $\overleftrightarrow{AB}$ is contained in α.
2. If there is a point A not on k which is on both α and β, then, by Theorem 4.1, $\alpha = \beta$.
3. The dihedral angle with sides T_1 and T_2 is adjacent to the dihedral angle with sides T_2 and S_1.
4. For the dihedral angle pictured, any point which is between a point of S and a point of T (and which itself is not a point of S or T) is in the interior of the dihedral angle.



Chapter 5
CONGRUENCE, INEQUALITIES AND MEASUREMENT FOR SEGMENTS

5-1 Introduction

The notion of congruence of segments and angles is an abstraction from the concept of *same size* in the physical world. Note that we develop ideas of congruence, and greater than and less than, before speaking of lengths of segments. There is good reason for this. The concepts of larger and smaller and same size are more basic than concepts of measurement. A young child is aware of difference in size long before he learns to measure lengths. In our geometric theorems, we are really not so much concerned with the actual lengths of segments and the measures of angles as we are with comparing sizes of segments and angles. We can decide which of two segments is the greater even if we know nothing about the measurement process.

Students should appreciate the remark that although probably no two sticks in the physical world are *exactly* the same size, yet we can imagine in our geometry segments which are exact copies of each other. This emphasizes again the difference between our imaginary world of geometry and the real world.

5-2 Congruence of segments

This first postulate is an *existence* postulate. Wherever we look in space we see segments congruent to any given segment. This postulate replaces the traditional language: *Cut off from ray*

$\overrightarrow{AB}$ *a segment congruent to* $\overline{CD}$. Such language really conceals the assumption that we make here — namely, that on ray $\overrightarrow{AB}$ *there is* exactly one point X such that $\overline{AX} \equiv \overline{CD}$.

The old language of geometry — *draw, construct, mark, extend, cut off* — is a language appropriate for the physical world but sometimes misleading in mathematics. It tends to obscure the simple mathematical concepts. The real questions are questions of existence. Are there geometric configurations with certain properties? Our postulates tell us what things exist. We don't construct points and lines, we postulate them into existence.

Answers to
PROBLEMS 5-2

Student Text Pages 57-58

These seven problems are designed to stimulate some discussion that will make the congruence postulates seem more significant by pointing out that they are idealizations of certain physical situations.

1. It is unlikely that two carpenters would mark exactly the same spot.
2. $S = T$.
3. Cutting of boards is a clumsy physical operation that gives us approximately the size of board we want. "Cutting off" segments congruent to a given segment in geometry is an act of thought that is completely precise.
4. Two careful measurements of the same rod cannot be expected to give the same result. A slight change of temperature may change the length of a real rod. In geometry, our segments are unaffected by the passage of time, changes in temperature, and so on. Thus, a segment is always congruent to itself.
5. Your friend may decide that the sticks have different lengths. This points out that part (b) of Postulate 2 has no exact counterpart in the physical world.

6, 7. These examples nicely illustrate the difference between part (c) of Postulate 2 and an analogous physical situation. Every

carpenter knows that errors would accumulate and he would soon be cutting boards differing considerably in size from his original one.

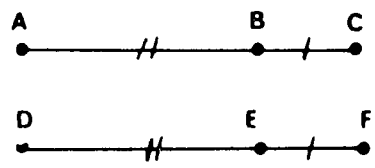
5-3 Inequalities for segments

Note that the definitions of greater than and less than for segments are formulated in terms of congruence and betweenness. If you think about these definitions, you will realize in traditional geometrical work these definitions are accepted but they are not formulated clearly.

The comment following Theorem 5.2, that without additional postulates it is impossible to prove that

$$\overline{AB} > \overline{CD} \Longrightarrow \overline{CD} < \overline{AB},$$

may surprise students. But before such a proof can be made we must state a postulate that tells us something about how segments can be "added". If we stated the *addition postulate* for segments given below we could make this proof.



POSTULATE. *If* B is between A and C, and E is between D and F, and $\overline{AB} \equiv \overline{DE}$ and $\overline{BC} \equiv \overline{EF}$, *then* $\overline{AC} \equiv \overline{DF}$.

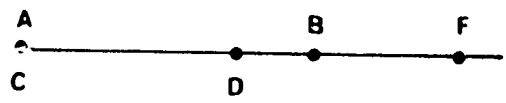
Even with this postulate, the proof that

$$\overline{AB} > \overline{CD} \Longrightarrow \overline{CD} < \overline{AB}$$

is not easy. We avoid the difficulties by postulating instead that segments have lengths and that these lengths add in an appropriate fashion.

Our next two measurement postulates make available ideas of segment length that enable us to give a simple and correct development of inequalities for segments.

1. a. This is true, because every segment is congruent to itself.
   b. This is false, because two different segments can be congruent.
   c. We cannot prove this true or false at the moment, but with the measurement postulates to be stated later, it will be possible to prove it true.
   d. We will be able to prove this false later.
   e. Later we will be able to prove this true.
   f. We can prove this now. We use the fact that $\overline{EF} \equiv \overline{CD} \Longrightarrow \overline{CD} \equiv \overline{EF}$. Because $\overline{AB} \equiv \overline{CD}$ and $\overline{CD} \equiv \overline{EF} \Longrightarrow \overline{AB} \equiv \overline{EF}$, the desired conclusion follows.
   g. We can prove this false now. The figure shows the proof.



2. This problem is difficult in the sense that it requires a fine understanding of the definition of greater than. The student must reason this way. "To decide whether $\overline{AB}$ is greater than $\overline{AC}$, I must look at the point X on ray $\overrightarrow{AC}$ such that $\overline{AX} \equiv \overline{AB}$. This point X *is* the point B. C is between A and B, so C is between A and X. Hence, $\overline{AB} > \overline{AC}$".
3. The argument is as in Problem 2.
4. Here we present a relation between chess players, which we call *greater than* and which does not have the familiar properties of greater than for numbers. This problem emphasizes that we have no right to assume that a relation has special properties merely because we have assigned to it a special name.
5. By asking pupils to prove these two theorems about number inequalities, we are reminding them of these things: The proofs depend upon having available clear definitions of greater than and less than for numbers and also having pos-

tulates for addition. Analogous proofs for segments will have to be made in a manner determined by our definitions of segment inequalities and by our geometric postulates.

Proofs for a and b are given below:

a. $x > y \Longrightarrow x = y + p$ and p is positive. Therefore, $y = x + (-p)$ and $(-p)$ is negative. Hence, $y < x$.

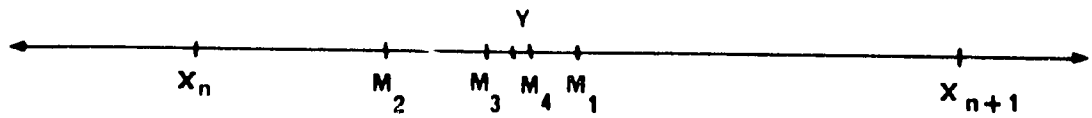
b. $x > y$ and $y > z \Longrightarrow x = y + p$ and $y = z + q$, where p and q are both positive. Then $x = (z + q) + p = z + (q + p)$. But a sum of positive numbers is positive, so $x > z$.

5-4 Measuring the length of a segment

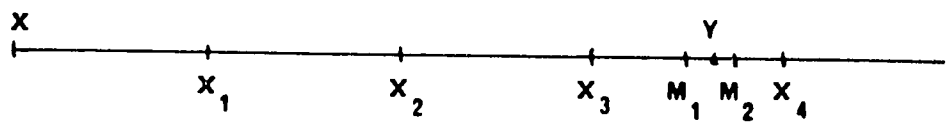
The process of measuring lengths described in the first part of this section is one which should be intuitively acceptable to the students. However, it is not a simple process. In particular, it is necessary to have a firm understanding of the idea of a limit (a concept usually reserved for a university course in calculus) to understand this process completely. By appealing to the decimal representation of real numbers, the limiting process is made to seem reasonable. But the description of this process is not to be regarded as a proof of any measurement properties. Indeed, as is mentioned in the text, we cannot even prove the existence of the midpoint of a segment. There are also other difficulties.

By choosing appropriate postulates we could justify the measurement process and prove the existence of the length of each segment. However, we could not do so without making a careful study of the rather sophisticated notion of a limit of a sequence of real numbers. We have chosen to avoid this, by making a different choice of measurement postulates. Our postulates can be used to justify the described measurement process, but they make it unnecessary to do so. We hope that the students will have some intuitive understanding of the process and that it will serve to make the measurement postulates acceptable.

Make sure that pupils understand the description of segment measurements by duplicating much of the text discussion at the board. The picture of the segment $\overline{X_n X_{n+1}}$ can be "magnified" to show more clearly the successive midpoints:



For a segment XY whose length is between $3 \cdot 5$ and $3 \cdot 75$ the measurement process would give a figure like this:

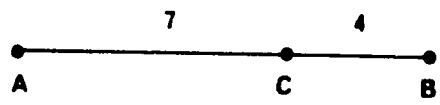


The proof presented for Theorem 5.3 points out that our last three postulates are definitely needed in order to prove that greater than for segments has the expected properties of greater than for numbers. Study this proof carefully. Theorem 5.5 will be used, although the proof is not given in the text. The proof is not difficult; as an example we present a proof of part (ii) here:

From the definition of the inequality for segments, either $\overline{AB} < \overline{CD}$, $\overline{AB} \equiv \overline{CD}$ or $\overline{AB} > \overline{CD}$. We will show that the last two relations lead to contradictions of the hypothesis $AB < CD$. Then it will follow that $\overline{AB} < \overline{CD}$. First, if $\overline{AB} \equiv \overline{CD}$, then $AB = CD$, which contradicts $AB < CD$. (Remember that AB and CD are numbers.) Second, if $\overline{AB} > \overline{CD}$, then by Theorem 5.3, $AB > CD$ must hold, and this contradicts the hypothesis. This completes the proof of (ii).

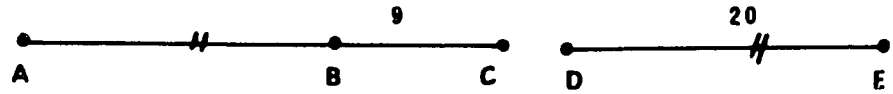
Give the students some problems of the following sort that will give them practice in using these last postulates and Theorems 5.3 and 5.5.

(1)

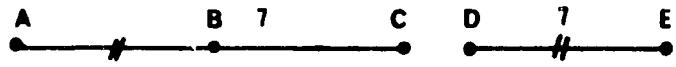


How long is $\overline{AB}$?

Prove that $\overline{AC} > \overline{CB}$.

(2) 

How long is $\overline{AC}$?
Prove that $\overline{BC} < \overline{AB}$.

(3) 

Prove that $\overline{BC} \equiv \overline{AB}$.

Answers to
PROBLEMS 5-4

Student Text Pages 67-68

1. $\overline{AB}$ and $\overline{A'B'}$ have equal lengths since they are congruent. Call their lengths a. $\overline{BC}$ and $\overline{B'C'}$ also have equal lengths for the same reason. Call their lengths b. The length of $\overline{AC}$ is $a + b$, by Segment Measurement Postulate 2. In the same way, $A'C' = a + b$. By the congruence-measurement postulate, $\overline{A'C'} \equiv \overline{AC}$.

This proof written in statement-reason form might look like this.

Statements	*Reasons*
1. $\overline{AB}$ has length a.	Measurement Postulate 1
2. $\overline{BC}$ has length b.	Measurement Postulate 1
3. $\overline{AC}$ has length $a + b$.	Measurement Postulate 2
4. $\overline{A'B'}$ has length a.	Measurement Congruence Postulate
5. $\overline{B'C'}$ has length b.	Measurement Congruence Postulate
6. $\overline{A'C'}$ has length $a + b$.	Measurement Postulate 2
7. $\overline{AC} \equiv \overline{A'C'}$.	Statements 3 and 6 and Measurement Congruence Postulates

2. Call the lengths of $\overline{AB}$, $\overline{BC}$, $\overline{A'B'}$ and $\overline{B'C'}$ the letters r, s, r' and s' respectively. Then we know that $r > r'$ and $s = s'$. By what we know about inequalities for numbers, $r + s > r' + s'$. But $r + s$ is the length of $\overline{AC}$, and $r' + s'$ is the length of $\overline{A'C'}$. Thus, by Theorems 5.3 and 5.5, $\overline{AC} > \overline{A'C'}$.

3. Let x represent the common length of $\overline{AC}$ and $\overline{A'C'}$, and let y represent the common length of $\overline{AB}$ and $\overline{A'B'}$. Let the lengths of $\overline{BC}$ and $\overline{B'C'}$ be respectively r and s. Then

$$y + r = x$$
$$\text{and } y + s = x.$$
$$\text{Therefore } y + r = y + s,$$

which implies $r = s$. Hence, by the congruence-measurement postulate, $\overline{BC} = \overline{B'C'}$.

4. We could define midpoint of segment in two ways, either in terms of congruence or in terms of length:

(1) X is the *midpoint* of $\overline{AB}$ if X is between A and B and $\overline{AX} \equiv \overline{XB}$.

(2) X is the midpoint of $\overline{AB}$ if X is between A and B and $AX = XB$. Our postulates tell us that these definitions are equivalent.

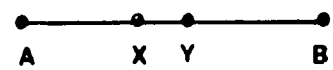
We use our measurement postulates to prove that every segment has a midpoint.

Let $\overline{AB}$ be any segment.

Statements	*Reasons*
1. $\overline{AB}$ has length x.	Measurement postulate
2. There is a point X on $\overrightarrow{AB}$ such that $AX = \frac{x}{2}$.	Measurement postulate
3. X is between A and B.	$\overline{AB} > \overline{AX}$ and X and B are on the same side of A on $\overleftrightarrow{AB}$.

4. $\frac{x}{2} + XB = x$, so $XB = \frac{x}{2}$.	Measurement postulate and algebra
5. $AX \equiv XB$.	Statements 2, 4 and congruence measurement postulate
6. X is the midpoint of $\overline{AB}$.	Statements 3 and 5

5. If $\overline{AB}$ has two midpoints, X and Y, then there are two points on ray $\overrightarrow{AB}$ such that $\frac{1}{2}AB = AX$ and $\frac{1}{2}AB = AY$. This contradicts our postulates.



6.
$$\begin{aligned} AB &= CD \\ AB + BC &= BC + CD \\ AC &= BD \\ \overline{AC} &\equiv \overline{BD} \end{aligned}$$

7.
$$\begin{aligned} \overline{XZ} &\equiv \overline{YT} \\ XZ &\equiv YT \\ XZ - YZ &= YT - YZ \\ XY &= ZT \\ \overline{XY} &\equiv \overline{ZT} \end{aligned}$$

8. The length of $\overline{AB}$ relative to unit segment $\overline{EF}$ is $3 \times 7 = 21$.

9. The length of $\overline{CD}$ using unit segment $\overline{EF}$ is $\frac{1}{2}$. Thus, the length of $\overline{AB}$ using unit segment $\overline{EF}$ is $7 \cdot 4 \times \frac{1}{2} = 3 \cdot 7$. The length of $\overline{EF}$ in terms of $\overline{AB}$ is $\frac{1}{3 \cdot 7} = \frac{10}{37}$.

10. a. xy; b. $\frac{1}{y}$.

Chapter 6
CONGRUENCE, INEQUALITIES AND MEASUREMENT FOR ANGLES

As you work through this chapter, return again and again to Chapter 5 and observe how the ideas are developed in almost exactly the same way. We have deliberately used language identical to that employed in Chapter 5 wherever possible. Be sure that students realize this and so appreciate the symmetry of the development. It should be easy for students to remember with understanding the ten postulates of Chapters 5 and 6.

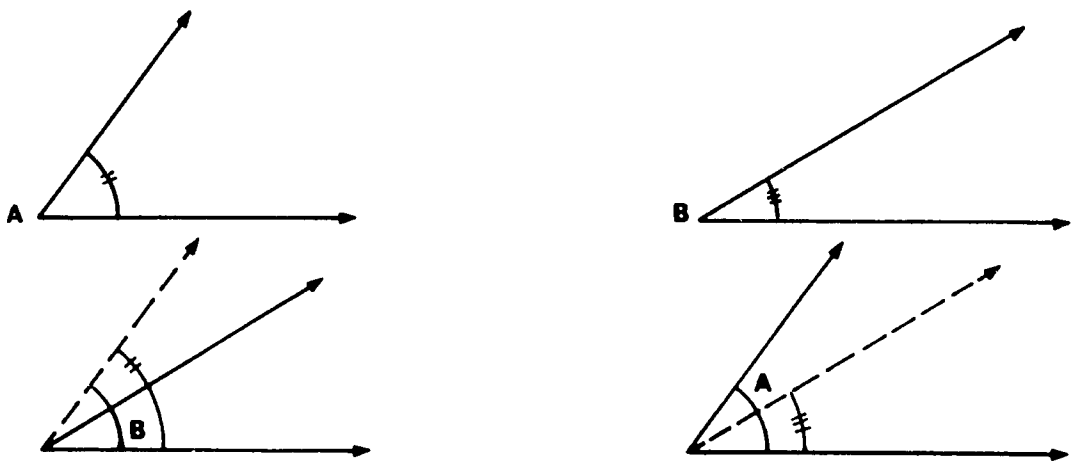
Since this chapter is so much like Chapter 5, our comments will be much less detailed than those for Chapter 5. In Section 6-4, we discuss the two key differences between angle measurement and segment measurement, but emphasize the even more important similarities between the two measurement processes.

In particular, you will want to think again about the comments on Section 5-4 in the TEACHERS' GUIDE as you read Section 6-4 of the STUDENT TEXT.

Answers to PROBLEMS 6-3

Student Text Pages 72-73

1.



2. a. $\hat{A} = \hat{C} \Longrightarrow \hat{A} \equiv \hat{C}$. This is true because every angle is congruent to itself. Like a, each of the other statements is an exact analogue of the statements in Question 1, Problems 5-3. Likewise, the truth, falsity and provability (and reasons) are the same (or analogous). See the answers in Section 5-3 of the TEACHERS' GUIDE.

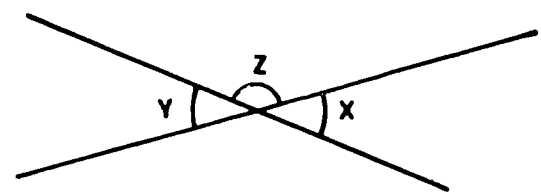
3. This is an exact analogue of Problem 3 of Problems 5-3.

Answers to
PROBLEMS 6-4

Student Text Pages 79-81

See the answers to Problems 5-4. Except for Problem 11, the problems are numbered in a corresponding way, and the solutions from 5-4 can be carried over to 6-4, almost word for word.

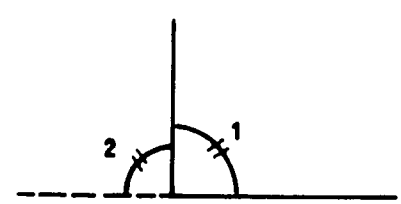
11. If $\hat{X}$ and $\hat{Y}$ are a pair of vertically opposite angles then,
$m(\hat{X}) + m(\hat{Z}) = 180$
and $m(\hat{Y}) + m(\hat{Z}) = 180$,
so $m(\hat{X}) = m(\hat{Y})$.
Hence, $\hat{X} \equiv \hat{Y}$.



6-5 Some definitions

Proofs of Theorems 6.8, 6.9 and 6.12 are quite easy. At least one proof should be written out formally to make it clear what postulates and definitions are being used.

THEOREM 6.8:



Let $\hat{1}$ be any right angle.
Let $\hat{2}$ be the adjacent supplementary angle.
$\hat{1} \equiv \hat{2}$ by definition of right angle,

$m(\hat{1}) = m(\hat{2})$ by congruence-measurement postulate,
$m(\hat{1}) + m(\hat{2}) = 180$ by our choice of degree as the unit measurement,
$m(\hat{1}) = 90$ by arithmetic.

THEOREM 6.9: By Theorem 6.8 any two right angles have the same degree measure, namely 90, and so are congruent.

THEOREM 6.12: By Theorems 6.3 and 6.5 and the definition of acute angle and obtuse angle, an acute angle has a degree measure less than that of a right angle, *that is*, less than 90. Similarly, an obtuse angle has degree measure more than that of a right angle, that is, more than 90.

Answers to
PROBLEMS 6-5

Student Text Pages 84-85

1. a. The perimeter might be $12 + 12 + 7 = 31$ or $12 + 7 + 7 = 26$.
   b. The perimeter is $12 + 12 + 6 = 30$. There is no triangle of sides 6, 6 and 12, although we are not yet able to prove this.
   c. The perimeter is $12 + 12 + 5 = 29$.
2. 53.
3. $\hat{3}$ and $\hat{4}$ are right angles since they arc congruent and supplementary. $\hat{3}$ and $\hat{2}$ are supplementary, so $m(\hat{2}) = 180 - m(\hat{3}) = 90$. Similarly $m(\hat{1}) = 180 - m(\hat{4}) = 90$.
4. $m(\hat{1}) = m(\hat{2})$ and $m(\hat{3}) = m(\hat{4})$. Hence, $2m(\hat{2}) + 2m(\hat{3}) = 180$, or $m(\hat{2}) + m(\hat{3}) = 90$. Therefore, $k \perp m$.
5. 19; 109.
6. If their measures are denoted by x and y, we have $x + y = 90$ and $x = y + 32$. Solving these equations, we find $x = 61$ and $y = 29$.
7. If x is the measure of the angles, we have

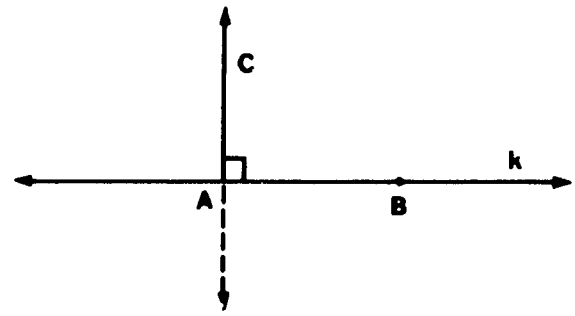
$$x = 20 + \frac{1}{3}(90 - x), \text{ or}$$

a. $x = 37 \cdot 5$.

b. $x = \frac{3}{5}(180 - x) - 10$, so $x = 61 \cdot 25$.

8. The sum of their degree measures is 180, so their non-common sides form a straight angle.

9. Consider just one side of line k (the upper side in the figure). Choose a second point B on k. Then there is exactly one ray $\overrightarrow{AC}$ on the chosen side of k such that $C\hat{A}B$ is a right angle. This unique ray determines a unique line in plane α which is perpendicular to k and contains A.



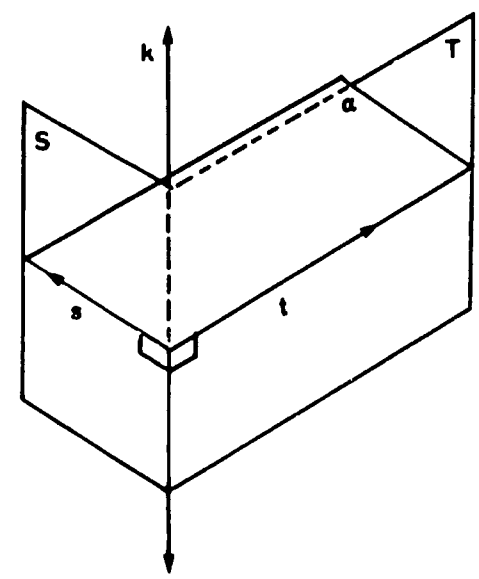
6-6 Dihedral Angles

If you wish you may expand briefly upon this material. We feel that ideas of angle measurement should be casually extended to the measurement of dihedral angles. An informal description of perpendicularity for planes is given. The diagram showing a line piercing a plane but not perpendicular to the plane could be used to convince students that in such a situation there is one and only one line in the plane perpendicular to the line.

The figure shows a dihedral angle formed by half-planes S and T with the line k as common edge. Plane α is perpendicular to k. α intersects S in ray s and α intersects T in ray t.

The degree measure of the dihedral angle is defined to be the degree measure of the angle formed by rays s and t.

Give students some practise estimating degree measures of dihedral angles. Let them do a little drawing to represent 3-dimensional figures.



Chapter 7
CONGRUENCE OF TRIANGLES

7-1 Introduction

A statement of congruence of triangles, like $\triangle ABC \equiv \triangle DEF$, is a very concise way of writing six simpler congruence relations. This is done by giving significance to the order in which the vertices are named. Because of this, the pupils must use great care in writing such congruence statements. Many of the problems in this section are designed just for this purpose. For instance, when you discuss Problem 3 of this section, you could write the six different congruence relations and have the pupils decide why only one seems to be true.

There are six ways of writing each congruence relation. For example,

$$\triangle ABC \equiv \triangle DEF$$

can also be written as

$$\triangle BCA \equiv \triangle EFD,\ \triangle CAB \equiv \triangle FDE, \text{ and so on.}$$

The essential thing is that these relations give the same correspondence between vertices, $A \longleftrightarrow D$, $B \longleftrightarrow E$ and $C \longleftrightarrow F$. To write the six different congruence statements, it is easiest to leave vertices A, B, C in the same order and permute D, E and F. We obtain

$$\begin{array}{ll} \triangle ABC \equiv \triangle DEF, & \triangle ABC \equiv \triangle DFE, \\ \triangle ABC \equiv \triangle EFD, & \triangle ABC \equiv \triangle EDF, \\ \triangle ABC \equiv \triangle FDE, & \triangle ABC \equiv \triangle FED. \end{array}$$

1. $\triangle ABC \equiv \triangle FED$ is true, but the other five congruence statements are false. For example, $\triangle ABC \equiv \triangle DEF$ is false, since $\overline{AB} \equiv \overline{FE}$ and $\overline{FE} \not\equiv \overline{DE}$, so $\overline{AB} \not\equiv \overline{DE}$.

2. The statements $\triangle RST \equiv \triangle LMN$ and $\triangle TRS \equiv \triangle NLM$ are abbreviations for the same set of six statements.

3. The statement $\triangle ABC \equiv \triangle DBC$ asserts that $A\hat{B}C \equiv D\hat{B}C$. The picture shows that this is false. It appears that $\triangle ABC \equiv \triangle DCB$.

4. $\triangle ABC \equiv \triangle EDC$ asserts that $\overline{AC} \equiv \overline{EC}$. The picture shows this to be false. $\triangle ABC \equiv \triangle DEC$ seems to be true.

5. From the first relation, $\overline{RT} \equiv \overline{UW}$, and from the second, $\overline{ST} \equiv \overline{UW}$. Thus, $\overline{ST} \equiv \overline{RT}$ and $\triangle RST$ is isosceles, and of course so is $\triangle UVW$. We can also observe that $\hat{R} \equiv \hat{U}$ and $\hat{S} \equiv \hat{U}$, so $\hat{R} \equiv \hat{S}$.

6. Both triangles are equilateral and equiangular. For example, from the first congruence statement, $\hat{A} \equiv \hat{D}$, $\hat{C} \equiv \hat{F}$ and $\hat{B} \equiv \hat{E}$. From the second, $\hat{D} \equiv \hat{C}$, $\hat{F} \equiv \hat{B}$ and $\hat{E} \equiv \hat{A}$. Thus,

$$\hat{A} \equiv \hat{D} \equiv \hat{C} \equiv \hat{F} \equiv \hat{B} \equiv \hat{E}.$$

Similarly, the six sides are congruent.

7. This follows from the postulates of congruence of segments and congruence of angles which say that each segment (or angle) is congruent to itself.

8. This is false, since $\overline{XY} \equiv \overline{XZ}$ is false.

9. $\triangle ABC$ is isosceles with $\overline{AC} \equiv \overline{BC}$ and $\hat{B} \equiv \hat{A}$. Compare this with Problem 5.

10. $\triangle DEF$ is equilateral and equiangular. Compare this with Problem 6.

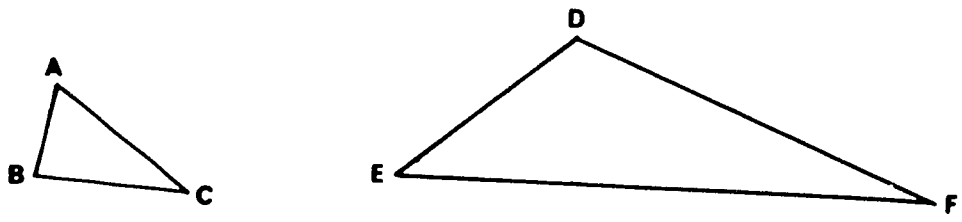
7-2 Proving triangles congruent

Answers to
PROBLEMS 7-2

Student Text Pages 91-92

1. $\hat{D} \Longleftrightarrow \hat{I}$, $\hat{E} \Longleftrightarrow \hat{H}$, $\hat{F} \Longleftrightarrow \hat{J}$, $\overline{DE} \Longleftrightarrow \overline{IH}$, $\overline{DF} \Longleftrightarrow \overline{IJ}$, $\overline{EF} \Longleftrightarrow \overline{HJ}$.
2. a. $\hat{A} \Longleftrightarrow \hat{B}$, $\hat{B} \Longleftrightarrow \hat{A}$, $\hat{C} \Longleftrightarrow \hat{C}$, $\overline{AB} \Longleftrightarrow \overline{BA}$, $\overline{AC} \Longleftrightarrow \overline{BC}$, $\overline{BC} \Longleftrightarrow \overline{AC}$.
   b and c are worked out in the same way.

Do not confuse the idea of setting up a correspondence with that of a congruence. Draw two triangles, clearly not congruent, as below.

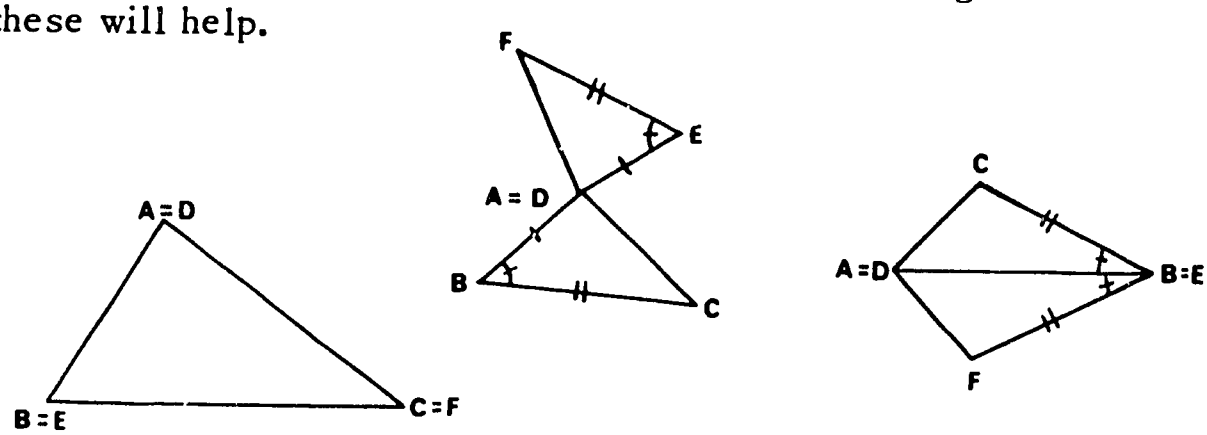


Point out that matching the vertices, sides and angles one to one has nothing to do with the congruence of sides and angles. Of course we are especially interested in correspondences which are also congruences.

7-3 The SAS postulate

This postulate and the ASA postulate are the most intuitively obvious of our congruence postulates. In the physical world there is no question but that two such triangles are copies of each other. In *Secondary Two*, work of the following sort was done, and it may help to do just a little of it now. Give the students measurements for side-angle-side, such as 2·5 inches, 58°, 3·2 inches. Tell the students to construct independently a triangle using this information and then compare their triangles. Except for inaccuracies in construction, the triangles drawn will match.

You should emphasize by drawings that the 6 points A, B, C, D, E, F need not all be different from one another. Figures like these will help.



Answers to
PROBLEMS 7-3

Student Text Pages 94-96

1. $\triangle ABC \equiv \triangle DFE$.
2. No SAS congruence.
3. No SAS congruence.
4. $\triangle TRS \equiv \triangle MNL$.
5. $\triangle ACB \equiv \triangle DCB$.
6. $\triangle MXZ \equiv \triangle MXY$.
7. $\triangle ACB \equiv \triangle ACD$.
8. $\triangle FEG \equiv \triangle HEG$ and also $\triangle FEG \equiv \triangle GEH$.
9. $\triangle AEB \equiv \triangle CED$.
10. $\triangle FJG \equiv \triangle HJI$.
11. $\triangle BAC \equiv \triangle DAE$.
12. The statements are listed above in 8. We can deduce that

$$\hat{F} \equiv E\hat{G}H \equiv E\hat{G}F \equiv \hat{H}.$$

7-4 ASA and SSS

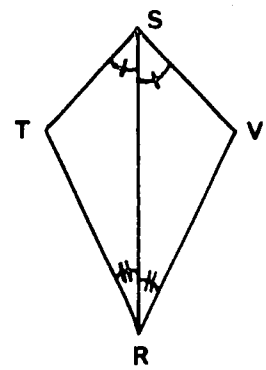
We postpone the proofs of ASA and SSS until the end of the chapter. Both proofs are fairly difficult and it seems best to give students more experience proving simple theorems dealing with congruences of triangles before exposing them to these two proofs.

We choose not to use the AAS theorem for the following reasons. It is not at all intuitively obvious that two triangles which agree in AAS are congruent. There are two ways to prove the AAS theorem. The easiest way is to wait until one has stated the parallel postulate and has proved that the angle sum for every triangle is 180°. It follows that the third angles of the two triangles are congruent and one has the ASA condition satisfied. We will give a proof of AAS congruence, which will depend upon the exterior angle theorem for triangles in Chapter 9. Although we could prove this theorem now and then establish the AAS theorem, we have chosen to postpone the treatment of the exterior angle theorem and related inequalities for triangles to Chapter 9.

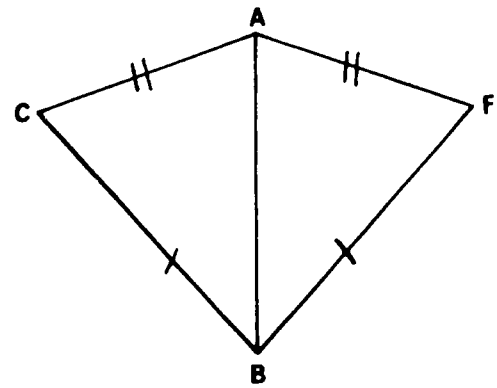
Answers to
PROBLEMS 7-4

Student Text Pages 98-99

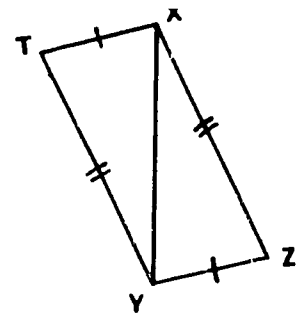
1. $\triangle RTS \equiv \triangle BAC$ by ASA. A picture should be drawn by each student.
2. $\triangle LMN \equiv \triangle FSR$ by SSS.
3. $\triangle RST \equiv \triangle RSV$ by ASA.



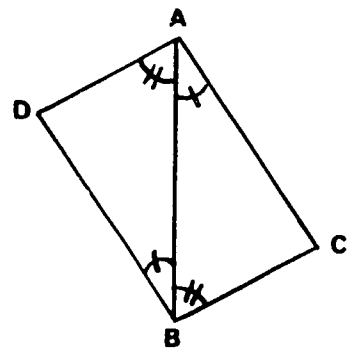
4. $\triangle CAB \equiv \triangle FAB$ by SSS.



5. $\triangle ZXY \equiv \triangle TYX$ by SSS.



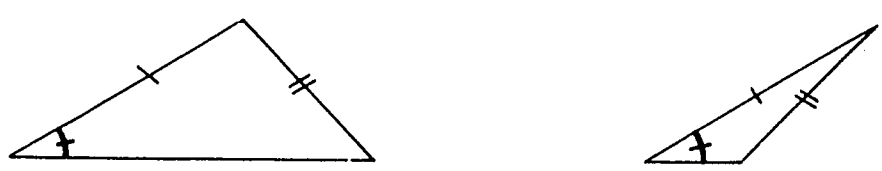
6. $\triangle ABD \equiv \triangle BAC$ by ASA.



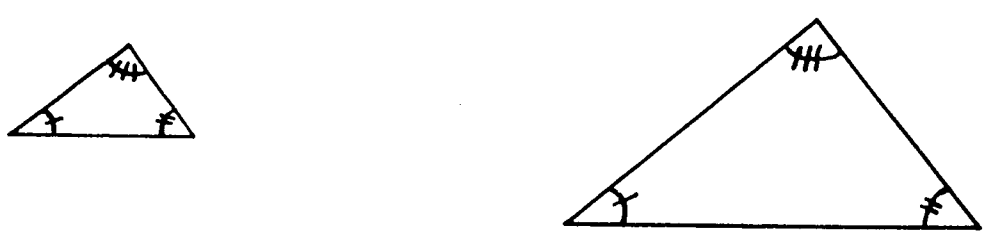
7. $\triangle ABC \equiv \triangle ABC$ is true for any triangle by the second congruence postulates for segments and angles. The second statement is true by SAS, since $\overline{BA} \equiv \overline{CA}$, $\hat{A} \equiv \hat{A}$ and $\overline{AC} \equiv \overline{AB}$. We deduce that $\hat{C} \equiv \hat{B}$. This is a proof that the base angles of an isosceles triangle are congruent.

8. We deduce that $\triangle BCA \equiv \triangle CBA$ by ASA. Therefore, $\overline{AC} \equiv \overline{AB}$. This is the converse of Problem 7.

9.



10.



7-5 Using SAS, ASA, and SSS in proofs

It is in this section that many students should begin to develop the ability to follow a fairly long proof. It is becoming more important now to organize proofs in such a way that it is easier to keep track of the essential ideas. Regardless of what we have said in the STUDENT TEXT, the chief responsibility for helping your students learn how to present lucid proofs falls to the teacher. You can teach them much by the example you set as you write out special proofs at the blackboard. Encourage students to express themselves clearly, but do not insist that proofs be too complete.

When writing a statement of congruence for triangles that is justified by SAS, it helps to write so that the three given congruences can be read off from left to right. That is, arrange the order of your vertices so that

$$\triangle PQT \equiv \triangle MFR \text{ by SAS}$$

follows from the facts

$$\overline{PQ} \equiv \overline{MF};\ \hat{Q} \equiv \hat{F};\ \overline{QT} \equiv \overline{FR}.$$

This notation seems preferable to the notation

$$\triangle PTQ \equiv \triangle MRF \text{ by SAS.}$$

An analogous symbolism for ASA would be to write

$$\triangle ABC \equiv \triangle DEF \text{ by ASA}$$

when one has

$$\hat{A} \equiv \hat{D};\ \overline{AB} \equiv \overline{DE};\ \hat{B} \equiv \hat{E}.$$

This device is a convenient one.

Emphasize throughout the year that there is no single correct form for making proofs. Present some proofs in paragraph style and some proofs written in the statement-reason columnar style. The latter *looks* more formal, but this is only superficial. A proof written out in conversational language can satisfy the demands of mathematical rigour fully as well as a proof written in the other style. Most of our proofs will be incomplete in the sense that as we move along

step by step through our arguments, we will not give all the reasons that justify each of our conclusions.

You have one great advantage in your teaching that is not available to textbook writers. In your teaching, you can develop ideas at the board. Pupils will see the figures as you draw them, presenting the information given in a problem. As you deduce new facts, you can record these facts by suitable markings on the figures, you can draw in new lines, and you can mark new points. Thus, the student sees the flow of ideas. He sees which step in the reasoning comes first, which second, which third and so on. Make the most of this advantage.

With the careful preparation that has been laid in the text, pupils should easily understand the proofs given for Theorems 7.2 and 7.3. We have illustrated in many problems the fact that in a statement of congruence it is not necessary that there be two distinct triangles.

Answers to
PROBLEMS 7-5

Student Text Pages 103-108

1. Accept an informal argument like:
   Because $\overline{AC} \equiv \overline{BC}$, Theorem 7.2 tells us that $\hat{B} \equiv \hat{A}$.
   Since $\overline{BC} \equiv \overline{AB}$, $\hat{A} \equiv \hat{C}$. Thus, $\hat{A} \equiv \hat{B} \equiv \hat{C}$.
2. This is just like Problem 1, except Theorem 7.3 is used.
3. $\triangle XYM \equiv \triangle XZM$ by SSS, so $\hat{Y} \equiv \hat{Z}$. We know by our measurement postulates that $\overline{YZ}$ has a midpoint.
4. $\triangle RTS \equiv \triangle KTS$ by SAS, so $\hat{R} \equiv \hat{K}$. We know $\hat{T}$ has a bisector because of our measurement postulates.
5. $X\hat{M}Y \equiv X\hat{M}Z$, so by definition, $\overline{XM} \perp \overline{YZ}$.
6. This is similar to Problem 5.

7. In case a, A, B and D are on a line. Since $A\hat{B}C$ and $D\hat{B}C$ are congruent by construction they must be right angles. In case b, the triangles are congruent by SAS. Then $A\hat{E}B$ and $D\hat{E}B$ are congruent and $\overleftrightarrow{AD} \perp k$. The difference between this problem and Problem 9 of Problems 6-5 is that in the earlier problem the given point is on the given line. In that case, both the existence and uniqueness of the perpendicular follow from angle congruence postulates. We have established the existence of a perpendicular through a given point not on the given line; the uniqueness of this perpendicular will be proved in Chapter 9 (Corollary 9.3).

8. The reasons are:

   a. SAS b. Step a c. Step a

   d. SSS (Remember that $\overline{BD} \equiv \overline{CE}$ by the figure.)

   e. Step d

   f. Steps b and e and angle subtraction

9. a. By two applications of Theorem 7.2, we have

$$P\hat{A}B \equiv P\hat{B}A \text{ and } C\hat{A}B \equiv C\hat{B}A.$$

   The *figure* tells us that P is in the interior of both $C\hat{A}B$ and $C\hat{B}A$ and so we can subtract angles, getting

$$C\hat{A}P \equiv C\hat{B}P.$$

   b. To prove $\overline{AX} \equiv \overline{XB}$, one needs to look at $\triangle CAP$ and $\triangle CBP$, deducing that they are congruent (SAS). So $A\hat{C}X \equiv B\hat{C}X$. Then $\triangle ACX \equiv \triangle BCX$ (SAS) and the result follows. In presenting this proof at the board students should sketch in line segment $\overline{CP}$, noting that it contains point X. This segment might be sketched in as a dotted line to identify it as something added to the original figure. Of course, segment $\overline{CP}$ *exists*, whether or not we draw a picture of it.

10. a. Theorem 7.3 in $\triangle RST$.

    b. Theorem 7.3 in $\triangle XST$.

    c. $\triangle XSR \equiv \triangle XTR$ by SAS.

11. a. SAS b. SSS c. ASA d. Step c

12. Theorem 7.2 and angle subtraction

13. a. SSS
    b. Theorem 7.3 for $\triangle CKF$
    c. Segment subtraction

14. a. $\triangle BAD \equiv \triangle BCD$ by SSS. Thus, $\triangle ABE \equiv \triangle CBE$ by SAS.
    b. $B\hat{E}A \equiv B\hat{E}C$ from part a.

15. SSS tells us that $\triangle ABC \equiv \triangle DBC$. Hence, $A\hat{B}C \equiv D\hat{B}C$ so $\overrightarrow{BA} = \overrightarrow{BD}$ by an angle congruence postulate.

16. $\hat{B} \equiv \hat{E}$ by Theorem 7.2 in $\triangle ABE$.
    Then $\triangle ABC \equiv \triangle AED$.
    Hence $\overline{AC} \equiv \overline{AD}$ and Theorem 7.3 applies to $\triangle ACD$.

17. M is equidistant from A and B by definition of the midpoint. For any other point P, as sketched, $\triangle PMA \equiv \triangle PMB$ by SAS. Hence $\overline{PA} \equiv \overline{PB}$.

18. a. $\triangle DBC \equiv \triangle ECB$ by SAS.
    Then $D\hat{C}B \equiv E\hat{D}C$ and Theorem 7.3 applies to $\triangle BOC$.
    b. Since $\overline{BE} \equiv \overline{DC}$ and $\overline{BO} \equiv \overline{CO}$, segment subtraction gives the result.
    c. SSS gives $\triangle DOB \equiv \triangle FOC$.

19. The two triangles are congruent in the obvious way by SSS.

20. a. SAS b. SAS c. Steps a and b

21. a. ASA b. ASA c. Steps a and b

22. As the figure shows here, we have the relationships

$$D\hat{B}C < A\hat{B}C \text{ and } \overline{DC} < \overline{AC}.$$

In order to prove that each of the six given statements of congruence is false, we must show that each one implies a statement that contradicts one of the above inequalities. Many of these six parts of Problem 22 are very easily proved, but others require that the student give a short chain of reasoning. This problem provides an excellent opportunity for pupils to organize an argument systematically. We are not sure that all proofs presented below are the shortest possible. Some pupils may improve upon some of them.

a. $\triangle ABC \equiv \triangle DBC$ implies that $A\hat{B}C \equiv D\hat{B}C$, a contradiction.

b. This triangle congruence asserts that $\overline{AC} \equiv \overline{CD}$, and this is false.

c. This congruence implies that $\overline{AB} \equiv \overline{BC}$ and $\overline{BC} \equiv \overline{CD}$. Hence, triangle BCD is isosceles and $C\hat{D}B \equiv C\hat{B}D$. But the congruence also tells us that $B\hat{C}A \equiv C\hat{D}B$, and since $B\hat{C}A \equiv C\hat{B}D$, we get the result that $\triangle DBC$ is equiangular with all three angles congruent to $\hat{C}$. Then $\triangle ABC$ is also equiangular with all angles congruent to $\hat{C}$, and we have deduced the false statement that

$$A\hat{B}C \equiv D\hat{B}C.$$

d. The congruence says that

$$\overline{AC} \equiv \overline{BC} \text{ and } \overline{BC} \equiv \overline{DC}.$$

Hence, we conclude that $\overline{AC} \equiv \overline{DC}$, but this is false.

e. The congruence asserts

$$A\hat{B}C \equiv \hat{C} \text{ and } \hat{C} \equiv C\hat{B}D.$$

Hence, $A\hat{B}C \equiv C\hat{B}D$, and this is false.

f. $\triangle ABC \equiv \triangle CDB \Longrightarrow \hat{A} \equiv \hat{C}$, $\hat{C} \equiv D\hat{B}C$ and $\overline{AC} \equiv \overline{BC}$. But $\hat{A} \equiv \hat{C} \Longrightarrow \overline{AB} \equiv \overline{BC}$. Hence, $\overline{AB} \equiv \overline{AC}$, so $A\hat{B}C \equiv \hat{C}$. Now, $A\hat{B}C \equiv \hat{C}$ and $\hat{C} \equiv D\hat{B}C \Longrightarrow A\hat{B}C \equiv D\hat{B}C$, and this last statement is false.

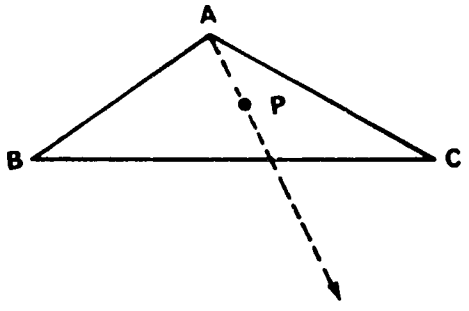
23. a., b., c. $\triangle ABD \equiv \triangle CDB$ by ASA.
d., e. $\triangle ABC \equiv \triangle CDA$ by ASA.
f., g. $\triangle OAD \equiv \triangle OCB$ by ASA.

24. a. $\triangle SRK \equiv \triangle URK$ by SAS.
b. Then $\overline{SK} \equiv \overline{UK}$.
c. $S\hat{K}T \equiv U\hat{K}V$ by congruence of vertical angles.
d. $R\hat{S}K \equiv R\hat{U}K$ by a.
e. $K\hat{S}T \equiv K\hat{U}V$ by d and angle subtraction.
f. $\triangle SKT \equiv \triangle UKV$ by ASA.

7-6 Making use of figures in proofs

This small section is written to emphasize and clarify the role that our figures play in our proofs. We can use a figure naturally and easily to designate relations of betweenness and incidence. There is no reason why we cannot allow a figure to tell us which one of three points on a line is between the other two. A figure can tell us when three or more points are collinear or three or more lines are concurrent, that is, have a common point of intersection.

It is important that students realize exactly what information they are taking from a figure. Again and again we permit our figures to "think" for us with reference to betweenness and incidence relations. For example, a student may refer to the figure below where P is in the interior of $\triangle ABC$.



The student may say:

Let ray $\overrightarrow{AP}$ intersect $\overrightarrow{BC}$ in point M. The picture tells him that *if* point P is in the interior of $\hat{A}$, then $\overrightarrow{AP}$ does contain a point between B and C.

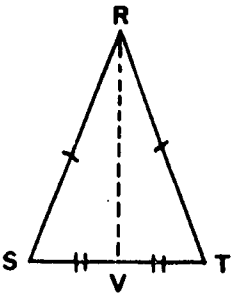
Without our betweenness postulates, we could not deduce this conclusion, which is suggested by our figure. Even with our betweenness postulates the proof is difficult, and we feel that it has no place in the text.

Answers to
PROBLEMS 7-6

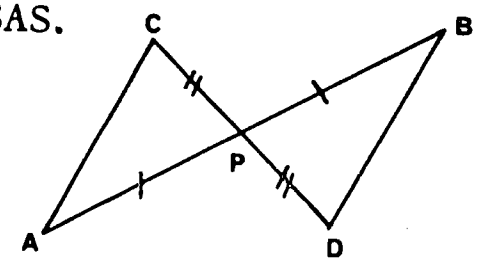
Student Text Page 109

The main task here is one of drawing figures to fit the verbal statements of the problems.

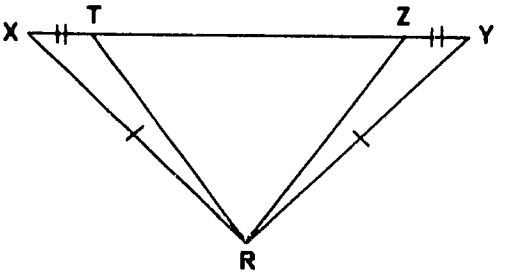
1. $\triangle RSV \equiv \triangle RTV$ by SSS, and the result follows.



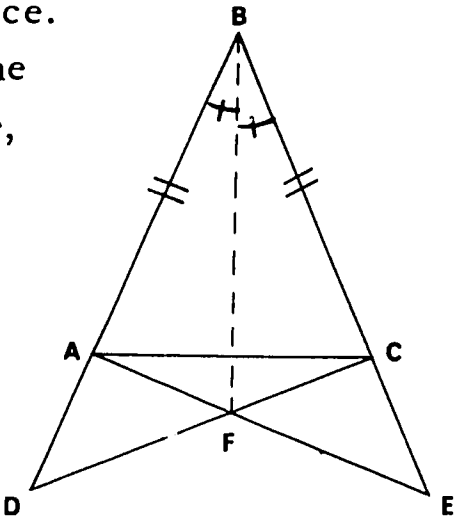
2. Use SAS.



3. $\overline{XR} \equiv \overline{YR} \Longrightarrow \hat{X} \equiv \hat{Y}$.
Then use SAS on $\triangle RXT$ and $\triangle RYZ$.



4. $\overline{AF} \equiv \overline{CF}$ from the ASA congruence. $\triangle AFB \equiv \triangle CFB$. From the same congruence $D\hat{A}F \equiv E\hat{C}F$. Hence, $\triangle DFA \equiv \triangle EFC$ by ASA.



7-7 Congruent triangles in different planes

Answers to PROBLEMS 7-7

Student Text Pages 110-112

These problems are not very difficult. Give much attention to the careful drawing of figures that accurately show the relationships. Show hidden lines as dotted lines.

1. Use ASA.

2. Use SAS.

3. $\triangle APR \equiv \triangle BPR$ by SAS. Then $A\hat{R}P \equiv B\hat{R}P$. Similarly, $A\hat{T}P \equiv B\hat{T}P$, and ASA gives the desired result.

4. In the figure, think of $\triangle ABD$ in the plane of the paper, with point C in front of the paper. Then $\overline{AC}$ intersects the paper in point A only.

   a. Use SSS. b. Step a. c. Use SAS.

5. a. Use SAS.

   b. $\overline{RA} \equiv \overline{TA}$, so use SSS.

   c. $R\hat{S}A \equiv T\hat{S}A$.

   d. Duplicate steps b and c with A replaced by B.

6. $\triangle XYT \equiv \triangle XYZ$ by SAS. Hence, $\overline{XT} \equiv \overline{XZ}$ and $\triangle XTZ$ is isosceles and has congruent base angles.

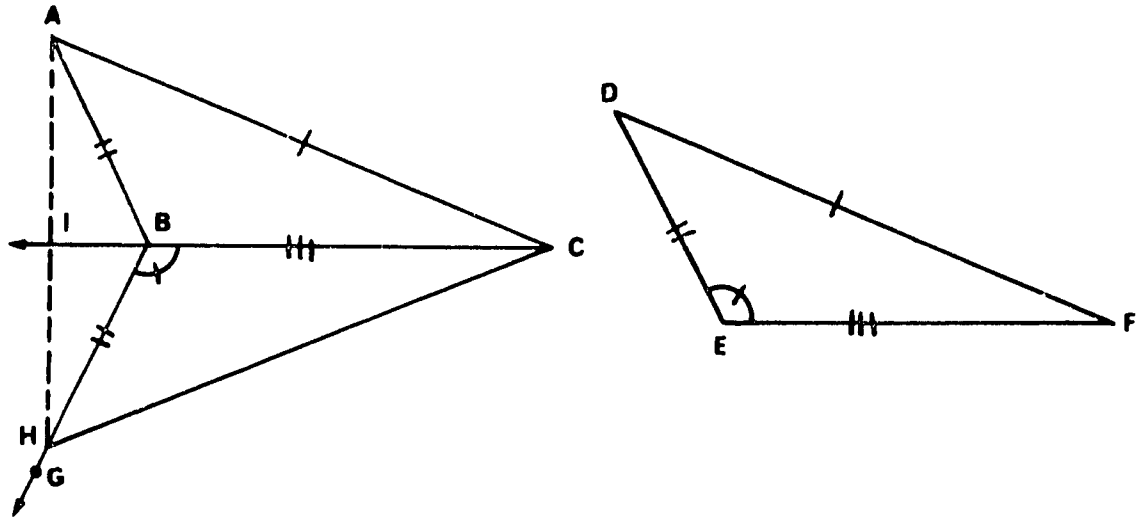
7. $\triangle DBA \equiv \triangle DBC$ by ASA.

7-8 Proofs of ASA and SSS

One could fashion a direct proof for ASA, but the indirect one seems more natural. It is possible that you might decide not to consider these two proofs in detail at this time.

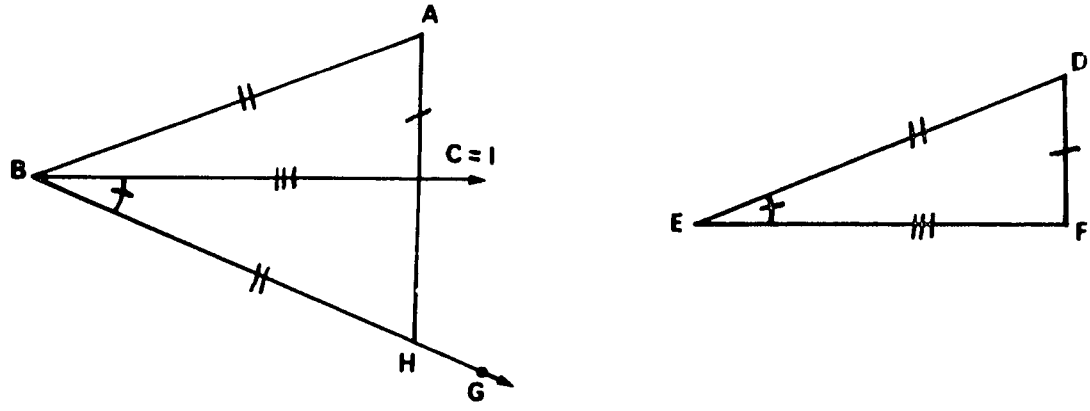
The proof presented for SSS is a little clumsy in that it deals with only one of several cases and other cases must be examined in subsequent problems to complete the discussion. However, the ideas of this proof are the simple ideas related to addition and subtraction of angles and the fact that base angles of an isosceles triangle are congruent. In Problem 10 of this section another proof is presented.

1.



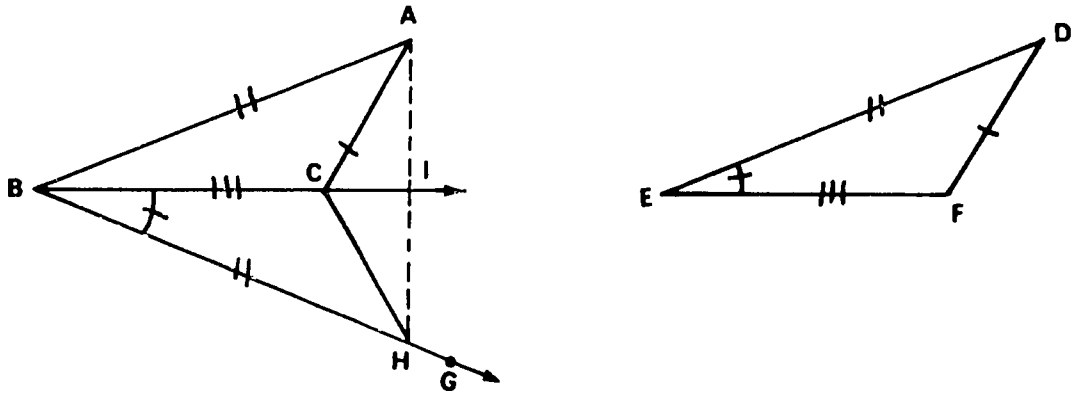
The argument is unchanged, except step 8. The reason for step 8 becomes angle subtraction.

2.



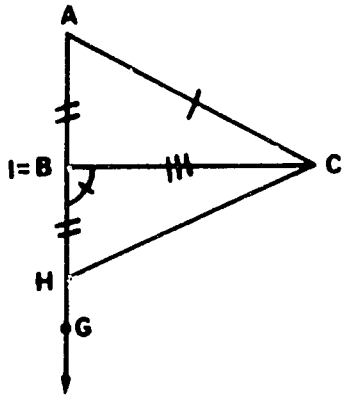
Omit steps 7 and 8.

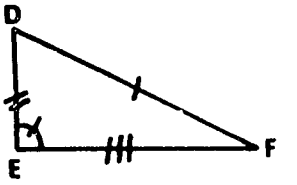
3.



The reason for step 8 becomes angle subtraction.

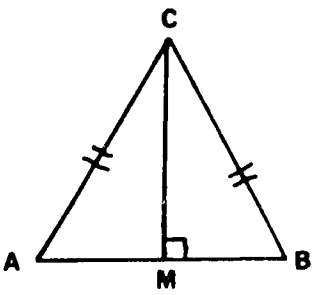
4.





Omit steps 6 and 8.

5. a. Every segment has a midpoint by segment measurement postulates.
   b. $\triangle PMA \equiv \triangle PMB$ by SSS.
   c. $P\hat{M}A \equiv P\hat{M}B$ by step b.
   d. Definition of perpendicularity and step c.
   e. Definition
6. There is a perpendicular bisector of $\overline{AP}$ in each plane which contains $\overline{AB}$.
7. Each is the converse of the other.
8. Apply Theorem 7.7.
9. By Problem 8 the perpendicular bisector of $\overline{AB}$ contains C. Now prove $\triangle AMC \equiv \triangle BMC$ and the result follows.



10. This proof is indirect, but the *essential* difference between this proof and the one in the text is that ray $\overrightarrow{BG}$ is on A's side of $\overleftrightarrow{BC}$.

Reasons

a. Angle congruence postulates.
b. Assumption that $A\hat{B}C \not\equiv \hat{E}$.
c. Segment congruence postulate.
d. Step b. e. SAS f. Step b.

g. $\overline{HC} \equiv \overline{DF}$ by e, and $\overline{AC} \equiv \overline{DF}$ by hypothesis, so $\overline{HC} \equiv \overline{AC}$. If C, H and A were collinear then step d would contradict a segment congruence postulate.

h. $\overline{AB} \equiv \overline{HB}$ and $\overline{AC} \equiv \overline{HC}$ by the hypothesis and step e. Note that steps f and g are used here to insure that H, B, C and C, H, A are vertices of triangles.

i. Problem 8, this section, and step h.

j. Step i is a contradiction. It tells us that a point of $\overline{HA}$, the midpoint, is collinear with B and C. But H was chosen on A's side of $\overleftrightarrow{BC}$.

k. Our assumption that $A\hat{B}C \not\equiv \hat{E}$ has been shown to be impossible, so SAS applies.

Answers to
REVIEW PROBLEMS 7-9

Student Text Pages 117-120

1. By SAS congruence, $\hat{F} \equiv \hat{A}$ and $\overline{AC} \equiv \overline{DF}$. Hence, $m(\hat{F}) = 42$ and $DF = 7 \cdot 2$.

2. Angle addition results tell us that
$m(B\hat{E}C) = 44 - 12 = 48 - 15$, so $12 = 13$.

3,4.

Angle	Measure of Complement	Measure of Supplement
$A\hat{D}B$	48	138
$A\hat{D}C$	4	94
$B\hat{D}C$	46	136

5. $AC = BC = 5 \cdot 21$.

6. $x + 25 = 90$, $x = 75$.

7. $\overline{PE} \equiv \overline{AB}$ so
$PE = 12$.

8. True: a, b, e.
False: c, d.

9. d is the best completion.

10. b and c.

11. False, they may not be collinear.

12. False, their sides may have different lengths.

13. True, a separation postulate.

14. False, a plane separates space.

15. True, incidence postulate.

16. False, a triangle is a union of *segments* joining three non-collinear points.

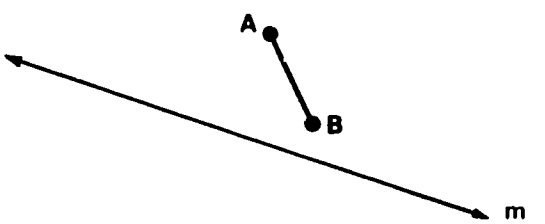
17. False, B is not on either side of k.

18. False, the points may be collinear.

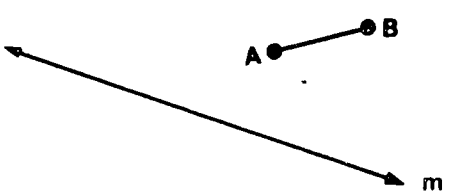
19. H and E are in the interior of $\hat{X}$.
A, B and F are in the exterior of $\hat{X}$.
D and G are on $\hat{X}$.

20.

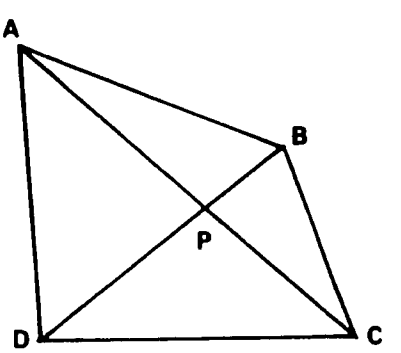
21. A and B are on the same side of m and $\overleftrightarrow{AB}$ is not parallel to m. B is between A and a point of m.



22. A and B are on the same side of m. Either $\overleftrightarrow{AB} \mid\mid m$ or A is between B and a point of m.

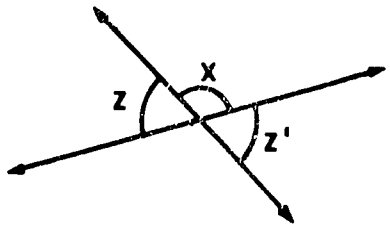


23. Ignoring the intersection point P, there are four triangles. Considering P as a possible vertex, there are four more.

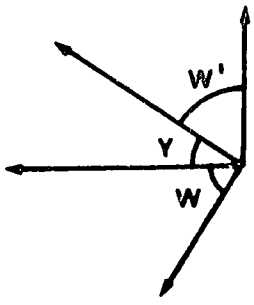


24. Four triangles are formed, the faces of a tetrahedron.

25.



26.



27. If $x + x = 90$, then $x = 45$.

28. If $x = 12 + (180 - x)$, then $x = 96$.

29. $\overline{AB}$ has length r, a positive number. Then sr is positive and our measurement postulates tell us that P can be found on $\overrightarrow{AB}$ such that $AP = sr$. Since $sr < r$ we must have P between A and B.

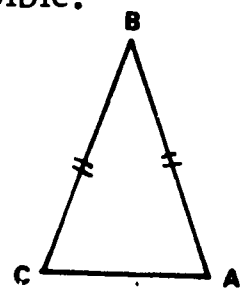
30. Set $s = \dfrac{t}{1 + t}$ and apply the preceding problem. Note that $0 < s < 1$.

If $AQ = \dfrac{t}{1 + t} \times AB = \dfrac{t}{1 + t} \times (AQ + QB)$,

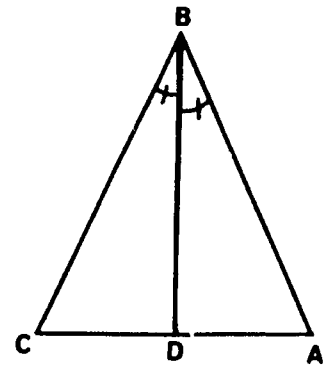
then $\left(\dfrac{1 + t}{t} - 1\right) \times AQ = QB$, or

$$AQ = t \times QB.$$

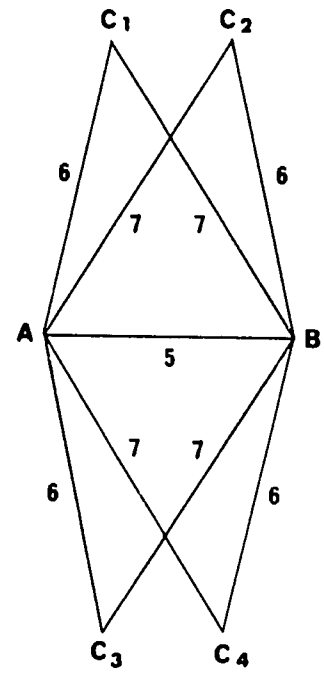
31. $\triangle ABC \equiv \triangle BAC \Longrightarrow \hat{A} \equiv \hat{B}$, which is impossible.
$\triangle ABC \equiv \triangle CBA$ is possible.



32. The angles at D are right angles. Also, $\overline{AB} \equiv \overline{BC}$ and the triangle is isosceles.



33. We can have either $AC = 6$ and $BC = 7$ or $AC = 7$ and $BC = 6$. For each choice we can choose C on either side of $\overleftrightarrow{AB}$. For each of these four possibilities the SSS theorem tells that C is unique.



Chapter 8
LOGIC—A STUDY OF CAREFUL REASONING

8-1 Introduction

In this chapter we consider some concepts which are part of the subject of logic. We do this for two reasons: First, these concepts are of interest and importance in their own right, and they are useful in everyday thinking, and second, these concepts must be understood, at least intuitively, by anyone who would understand a deductive study of geometry. It is this second reason which is of principal importance to us. We feel that many of the difficulties which students have in a course such as this are because of an inadequate understanding of the logical concepts of this chapter. Thus, we feel that a study of these concepts will pay dividends in the form of a better understanding, not only of the deductive study of geometry in this course but in all subsequent mathematical studies.

The principal content of this chapter is in Section 8-2 in which the logical use of the words *and*, *or*, *if . . . then* and *not* is explained. In the last two sections, the concepts of *converse* and *contrapositive* are discussed.

The examples in this first section should motivate the work of the rest of the chapter, principally Section 8-2. They are chosen as examples of logical difficulties which are like some which do frequently arise in the study of geometry. You should devote a generous amount of class time to the discussion of these examples before assigning work in Section 8-2. You should not feel that you have to explain the problems raised by these examples at this time. The explanations depend on an understanding of Section 8-2. The examples

will be explained in that section. At this time, you should encourage open discussion of these examples so that the problems they present will be understood by the students. Some students will be able to see how to resolve these problems using their intuitive knowledge of logic, but certainly a large number of students will be puzzled by some of the examples. We discuss the examples here. Remember that you should not resolve the difficulties in the first classroom discussion. Leave them as questions to be answered as the class progresses through the rest of the chapter.

Example 1: The point of this example is the equivalence of the statements "$a = 0$ or $b = 0$" and "if $a \neq 0$ then $b = 0$". To clear-thinking students this equivalence will be obvious. They might reason (correctly) that in order for "$a = 0$ or $b = 0$" to be true we can either have $a = 0$, or if $a \neq 0$, then we must have $b = 0$. Some students may suggest that the commutative property of multiplication is involved. Actually it is not, and it should not be mentioned unless it is proposed by the students. As we explain later, the statement "p or q" is entirely equivalent to the statement "if not p then q" for any statements p and q. See the discussion of "*if ... then*" in Section 8-2.

Another example of this sort is the equivalence of the statements "three points are collinear or they lie in a unique plane" and "if three points are not collinear, then they lie in a unique plane".

Example 2: Statement (3) is the only one which has the same meaning as "It is not true that all men are honest". Statement (2) says the same thing as (1), but both of them make a stronger statement about the dishonesty of men. Statement (4) says that there definitely are some honest men, while the given statement merely says that there are some dishonest men; perhaps all men are dishonest, perhaps only some. See the discussion of "not" in the next section.

Example 3: Statements (3) and (4) have the same meaning as the given statement; (1) and (2) do not. Statement (4) is just another way of phrasing the given "if ... then" statement. This is discussed in Section 8-4. Statement (3) is the contrapositive of the given implication and is also discussed in 8-4. Statement (2) is the

converse (see Section 8-3) of the given implication. Statement (2) can also be stated as "if a triangle is isosceles, then two angles (called the base angles) are congruent". Although it is true, it has different meaning from the given statement. Statement (1) is the contrapositive of (2) and, therefore, has the same meaning as (2), and, therefore, different meaning from the given statement.

The distinction between an implication, its contrapositive, its converse and the contrapositive of the converse is clearer if one begins with an implication whose converse is false. Examples of this sort are given in 8-3; another is the following: "If it is raining, then the sky is cloudy". Statements analogous to (1) through (4) are listed here:

(1) If it is not raining, then the sky is not cloudy.
(2) If the sky is cloudy, then it is raining.
(3) If the sky is not cloudy, then it is not raining.
(4) In order that it be raining, it must be true that the sky is cloudy.

Example 4: The argument outlined is a valid proof of the fact that there is no rational number whose square is 2. It is a good example of an indirect proof. The students have seen several examples of indirect proofs, so this may be clear to most students. Of course, the idea of an indirect proof is of paramount importance, so all students should eventually understand it. In Section 8-4, this will be discussed.

The proof of the ASA congruence theorem (Section 7-8) is another example of an indirect proof.

Example 5: The student who showed how to construct a line from P perpendicular to k has probably proved that there is *at least one* line perpendicular to k and containing P. To prove that there is *at most one* such line he should probably use an indirect proof. That is, begin by assuming there are two such lines and then show that a contradiction must follow. This is just what is done in Section 9-2, where this is proved.

Segun has confused an "existence theorem" with a "uniqueness theorem". See Section 10-1 where this is discussed in connection with the parallel postulate.

8-2 Agreeing upon the use of words

The title of this section is a good indication of the essence of many logical difficulties. If we all understand exactly the meanings of the words we use, then the capacity for clear, common-sense thinking which most people have will lead us to consistent conclusions.

Of the four terms discussed in this section *and*, *or*, *if ... then*, and *not*, the first three are called logical connectives, because they are used to connect two (or more) simple statements to form a third statement. The last, *not*, is applied to a single statement, which may itself contain some simpler statements, logical connectives and "not". These four terms are discussed in detail, one at a time.

And

The connective, "and" is quite simple. The discussion need not be too lengthy. By carefully chosen simple questions, you can discover when the students understand its use. For example, all students should give correct answers to the following:

Which of these statements are true and which are false?

(a) $3 + 5 = 7$ *and* $6 - 2 = 1$. (False)
(b) $3 + 5 = 8$ *and* $6 - 2 = 1$. (False)
(c) $3 + 5 = 12$ *and* $6 - 2 = 4$. (False)
(d) $3 + 5 = 8$ *and* $6 - 2 = 4$. (True)

Letters p and q (and later also r and s) are used as statement variables in much the same way that x, y, z and so on are used as number variables in algebra. The statement "$x + 3y = 7$" is an open statement. Some of the statements that are obtained from "$x + 3y = 7$" for certain particular numbers x and y are true and others are false. Similarly, the statement "p and q" is a statement which gives rise to true statements when certain particular statements p and q are substituted and which gives rise to false statements for other substitutions. An essential difference is that to know whether "p and q" is true or false, we need only know the

truth or falsity of the statements p and q. It is not necessary to know the particular statements p and q. Of course, if particular statements are given to be substituted for p and q, then we can decide the truth or falsity of "p and q" by deciding the truth or falsity of the particular statements p and q.

Give the students some practice forming particular statements out of ones like "p and q", "(not p) and q", "if p then not q" and so on by replacing p and q by particular statements. For example, p might be replaced by "John is four years old" and q might be replaced by "John is in the third form".

Answers to
PROBLEMS 8-2A

Student Text Page 125

1. The number 2 is the only number which is both even and prime. All other prime numbers are odd and all other even numbers are composite (that is, non-prime). The statement in quotation marks is false for $x = 10$ since 10 is not a prime number, and it is false for $x = 23$ because 23 is odd. It is also false for $x = 27$ since 27 is neither prime nor even.

2. "x is odd and x^2 is odd" is true for all odd natural numbers, since the square of an odd number is odd. (See Problem 6 of Section 1-4.) You might also raise related questions, such as these:

   For what natural numbers x is each of the following true?

   a. "x is odd and x^2 is even". (True for no natural numbers)

   b. "x is even and x^2 is even". (True for all even natural numbers)

   c. "x is even and x^2 is odd". (True for no natural numbers)

3. $\triangle ABC \equiv \triangle ABC$ is true for any triangle ABC, by definition of triangle congruence and by the postulates which say that each segment is congruent to itself and each angle is congruent to itself. Thus, the quoted conjunction is true for just those triangles for which "$\triangle ABC \equiv \triangle ACB$" is true. These are the

triangles in which $\overline{AB} \equiv \overline{AC}$ and $\hat{B} \equiv \hat{C}$, that is, the isosceles triangles ABC with congruent sides $\overline{AB}$ and $\overline{AC}$.

4. When asked to solve the system

$$\begin{cases} x + 2y = 3 \\ 2x + y = 3 \end{cases}$$

you are being asked, "For what ordered pairs of real numbers (x, y) is it true that $x + 2y = 3$ *and* $2x + y = 3$?" When solving you argue as follows:

$$(x + 2y = 3 \text{ and } 2x + y = 3) \Longrightarrow$$

$$\left(x = 3 - 2y \text{ and } x = \frac{3 - y}{2}\right) \Longrightarrow$$

$$\left(x = 3 - 2y \text{ and } 3 - 2y = \frac{3 - y}{2}\right) \Longrightarrow$$

$$(x = 3 - 2y \text{ and } 3y = 3) \Longrightarrow$$

$$(y = 1 \text{ and } x = 1)$$

Thus the ordered pair (1, 1) is the only *possible* solution. The easiest way to see that (1, 1) *is* a solution is to "substitute". That is, verify that if $x = 1$ and $y = 1$, then $x + 2y = 3$ *and* $2x + y = 3$.

Or

The point about the use of "or" which will need emphasizing is that "or" is always used in a *non-exclusive* way in mathematics. That is, in everyday life, if you say "I will go swimming or I will go fishing", you generally imply that you will not do both. The "or" is often used in the sense which *excludes* the possibility that both of the statements linked by "or" are true. In mathematics this is not the case. Always for us "p or q" is true when one *or both* of p, q are true.

Not

The negation of a simple statement is easy to understand. The negation of a more complicated statement, which itself may involve several simple statements with logical connectives, will often involve some very careful reasoning. This result of this reasoning can be displayed clearly by a device called a *truth table*. The truth table relating a statement p to its negation, "not p", is as follows:

Negation:

p	not p
T	F
F	T

The first row of the table tells us that when p is true (T) then (not p) is false (F). The second row tells us that when p is false then (not p) is true.

We can write truth tables for the conjunction and the disjunction:

Conjunction:

p	q	p and q
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction:

p	q	p or q
T	T	T
T	F	T
F	T	T
F	F	F

In these tables, four rows are needed since there are four possible combinations of truth and falsity for statements p and q. The first row of the truth table for the conjunction tells us that (p and q) is true when both of p and q are true. The other three rows

say that (p and q) is false in all other cases. For the disjunction, the truth table tells us that (p or q) is false when both of p and q are false, and in all other cases (p or q) is true.

These two truth tables can be combined:

p	q	p and q	p or q
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Now we can write truth tables for things like [not (p and q)] and [(not p) or (not q)].

p	q	p and q	not (p and q)	not p	not q	(not p) or (not q)
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

In writing the fourth column, for [not (p and q)] in this table we made use of the third column and the truth table for the negation. The latter tells us that if r is any statement, then (not r) is true when r is false and false when r is true. Letting r represent (p and q) we see that the column for [not (p and q)] should be just the opposite of the column for (p and q). In the same way, the column for (not p) is the opposite of the column for p. Likewise for (not q).

To obtain the column for [(not p) or (not q)] we used the columns (not p) and (not q) and the truth table for disjunction. The latter tells us that if r and s are any statements, then (r or s) is false when both r and s are false and true in all other cases. Letting r be (not p) and s be (not q) we can, in a more or less mechanical fashion, make the entries in the column for [(not p) or (not q)]. Thus, when an F appears in both of the columns (not p) and (not q), we enter an F. Otherwise, we enter a T. This column tells us that [(not p) or (not q)] is false when both p and q are true,

and true in all other cases. Of course, the fourth column tells us the same about the statement [not (p and q)]. This shows that the negation of (p and q) has exactly the same meaning as the disjunction [(not p) or (not q)].

For example, the statement [not (2 + 3 = 7 and John is my brother)] has the same meaning as [(not 2 + 3 = 7) or (not John is my brother)]. We would ordinarily write this last statement as (2 + 3 $\neq$ 7 or John is not my brother).

You may wish to test your understanding of truth tables by filling in the entries in the following:

p	q	p or q	not (p or q)	not p	not q	(not p) and (not q)
T	T		F			
T	F		F			
F	T		F			
F	F		T			

You should find that the 4th and 7th columns are identical, thus showing that the negation of (p or q) has the same meaning as the conjunction of (not p) and (not q).

Although truth tables, if fully understood, provide an easy way of finding the meanings of rather complex statements, we suggest that if you use them, you use them as a supplement to (not as a substitute for) careful common-sense thinking. That is, you should be able to see intuitively that the statements

[not (p and q)]

and

[(not p) or (not q)]

have identical meanings. This is the sort of reasoning the students are asked to do in several exercises.

It would undoubtedly be unwise to introduce all students to the use of truth tables, as it would require at least 6 or 8 lessons in order that they acquire facility in their use. A very able student might be interested in studying them on his own. For such a student

(or for your use, as time and energy permit) we suggest either of the following references:

Principles of Mathematics, Second Edition, by C. Allendoerfer and C. Oakley, McGraw-Hill, New York, 1963.

Finite Mathematical Structures, by J. Kemeny, H. Mirkil, J. Snell and G. Thompson, Prentice-Hall, Englewood Cliffs, New Jersey, 1959.

We will return to the use of truth tables in connection with implications and for certain exercises.

Answers to
PROBLEMS 8-2B

Student Text Pages 127-128

1. a. $[x^2 - 2x + 15 = 0] \Longleftrightarrow$
$[(x - 5)(x + 3) = 0] \Longleftrightarrow$
$[(x - 5 = 0) \text{ or } (x + 3 = 0)] \Longleftrightarrow$
$[(x = 5) \text{ or } (x = -3)]$,

so there are two roots, namely 5 and −3. The property we use to justify the second "$\Longleftrightarrow$" is that for real numbers a and b.

$$[ab = 0] \Longleftrightarrow [(a = 0) \text{ or } (b = 0)],$$

the theorem on factors of zero in algebra.

b. $[3x^2 + x - 2 = 0] \Longleftrightarrow [(3x - 2)(x + 1) = 0] \Longleftrightarrow$
$[(3x - 2 = 0) \text{ or } (x + 1 = 0)] \Longleftrightarrow$
$\left[\left(x = \frac{2}{3}\right) \text{ or } (x = -1)\right]$. Solutions: $\frac{2}{3}, -1$.

c. $[x^2 + 10x + 25 = 0] \Longleftrightarrow [(x + 5)(x + 5) = 0] \Longleftrightarrow$
$[(x + 5 = 0) \text{ or } (x + 5 = 0)] \Longleftrightarrow$
$[(x = -5) \text{ or } (x = -5)]$.

Only one solution, namely −5. Notice that it is essential that the "or" be used in the non-exclusive (mathematical) sense here. The truth of $[(x = -5) \text{ or } (x = -5)]$ does not preclude

the possibility that both of the statements in parentheses may be true.

2. "Not p" would be "It is false that yesterday was Saturday" or "Yesterday was not Saturday". The statement "Yesterday was Friday" is false on Monday, Tuesday, Wednesday, Thursday and Friday. On each of these days the statement "Yesterday was not Saturday" is true.

3. a. It is cold and you are wet.
   b. It is cold and you are not wet.
   c. It is not cold and you are not wet.

4. You would usually say, "It is false that it is not cold and you are not wet". However, this is ambiguous. It could mean "(It is false that it is not cold) and you are not wet" instead of the intended meaning, "It is false (that you are not cold and you are not wet)". "It is cold or you are wet" is a simpler statement which has the desired meaning without ambiguity. The students are expected to arrive at this formulation by careful thinking. It is, of course, obtainable as an application of the work done above with truth tables. We showed that [not (r and s)] has the same meaning as [(not r) or (not s)]. If r represents "it is not cold" and if s represents "it is not wet" [that is, if r is (not p) and s is (not q)], then [(not r) or (not s)] is "it is cold or you are wet".

5. a. Tuesday, Wednesday
   b. Sun., Mon., Thurs., Fri., Sat.
   c. Every day except Tuesday
   d. Every day
   e. "p and q" is never true.

6. a, c, f are true; the others are false.
   e. can be written as [(not p) *or* q], which is false.
   f. can be written as [p and (not q)], which is true.
   g. is false since [p and q] is false and so is [(not p) and (not q)].

If . . . then

It will probably be helpful to give the students practice in converting statements from the "if p then q" form to the "(not p) or q" form. Use examples they are familiar with from everyday life, from algebra or from geometry. Possible examples are:

"If I study then I will learn the lesson".
"If $x = y$ then $7 + x - y = 7$".
"If $\overline{AB} < \overline{CD}$ then $\overline{CD} > \overline{AB}$".

Careful thinking about examples like these should help the students to see the meaning of "$p \Longrightarrow q$" and to see that an equivalent statement is "(not p) or q".

Let us find the truth table for "$p \Longrightarrow q$" by using the equivalent statement "(not p) or q".

p	q	not p	(not p) or q	$p \Longrightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

We obtained the second and third columns by using the truth tables for negation and disjunction. We obtained the fifth column by duplicating the fourth column, since we have agreed that [(not p) or q] and [$p \Longrightarrow q$] have identical meanings.

We see that the only case in which "$p \Longrightarrow q$" is false is the case p true and q false. That is, "$p \Longrightarrow q$" clearly says that if p is true then q must be true. Since if p is false, "$p \Longrightarrow q$" says nothing about the truth or falsity of q, we want "$p \Longrightarrow q$" to be true in the cases (1) p false, q false and (2) p false, q true.

In the text we point out how to settle the question raised in Example 1, Section 8-1.

Answers to
PROBLEMS 8-2C

Student Text Page 130

1. a. "(Not p) or q" is "Today is not Wednesday or tomorrow is Thursday". This is true every day.
   b. "If p, then q" is "If today is Wednesday, then tomorrow is Thursday". Of course this is a valid (that is, true) implication, true every day of the week.
2. a. "$4 + 4 \neq 8$" or "$4 + 5 = 10$". False.
   b. "If $4 + 4 = 8$" then "$4 + 5 = 10$". False.
3. a. Yes
   b. Yes. This part is hard to think of in the "if . . . , then" form. The equivalent disjunction in part a. settles the question easily.
4. This is easy to prove. $x + y = 9$ implies that $(x + y) + 1 = 9 + 1$ which implies that $x + (y + 1) = 10$.

Summary

This part should be clear if the rest of Section 8-2 was understood. In Example 4, part (2), let p represent "I go to school" and let q represent "I will pass the test". Then the two statements are "(not p or q)" and "if p then q". For part (3) let p represent "Ruth does not wash the dishes" and let q stand for "Ruth will not go to the party".

Answers to
PROBLEMS 8-2D

Student Text Pages 131-132

1. b. Not $2x + 3 = 11$, or it is false that $2x + 3 = 11$; $2x + 3 \neq 11$. Henceforth, we list only the form of the negation which would ordinarily be used.
   c. There was at least one day this past week when it was not hot.
   d. Everyone in this room is telling the truth.
   e. It is not hot or I am not tired.

f. You will not go with me and I will go.

g. Not all the potatoes are small; or alternately, some of the potatoes are not small.

2. a. p is false.

b. q is true.

c. p is false.

d. p is false. Convert to (not p or q) is true. Hence if q is false, (not p) must be true.

e. [(Not p) or q)] is false; {not [(not p) or q]} is true; [p and (not q)] is true. Hence p is true and q is false.

f. Convert to [(not p) or q] is true. Hence if p is true, q is true.

g. q is true.

h. p is false. Convert to [(not p) or q] is true.

i. No conclusion can be reached.

j. Convert to [(not p) or q] is true; [(not q) or p] is false. The last can be changed to [q and (not p)] is true. Thus, the conclusion is that q is true and p is false.

Truth tables could be used as follows:

p	q	$p \Longrightarrow q$	$q \Longrightarrow p$	not $(q \Longrightarrow p)$
T	T	T	T	F
T	F	F	T	F
F	T	T	F	T
F	F	T	T	F

Comparing the third and fifth columns (remember that we are told that [$p \Longrightarrow q$] is true and that [not $(q \Longrightarrow p)$] is true) we see that they both have the entry T only in the third row. Hence p is false and q is true.

k. We know that if p is true then q is true and if q is true then p is true. Thus, if either is true, so is the other, and hence also if either is false, so is the other. That is {[p and q] or [(not p) and (not q)]} is true.

Alternately, we compare the third and fourth columns of the truth table constructed for part j. Both have entries T in row 1 and in row 4. Thus, either p and q are both true (row 1) or p and q are both false (row 4).

3. a. [p and q] is false, or simply [not (p and q)].
   b. [p and (not q)] is false, or simply $\{$not [p and (not q)]$\}$ or [(not p) *or* q].
   c. [(not p) $\Longrightarrow q$] is false, or $-$[p or q] is false, or [(not p) and (not q)].

4. c. and b. imply that (not q) must be true. But (not q) and ($p \Longrightarrow q$) imply that p is false, that is, (not p) is true.

5. We conclude ($p \Longrightarrow r$). This should be clear from the statements given.

8-3 The converse of an implication

The idea of the converse of an implication is fairly simple, and the students have seen it several times before. The "if" and "only if" language is new. Let us agree to write "if p, then q" as either "$p \Longrightarrow q$" (as we have done already) or "$q \Longleftarrow p$". You can read "$q \Longleftarrow p$" as "q is implied by p". Then we can write "p if q" as "$p \Longleftarrow q$" and we can write "p only if q" as "$p \Longrightarrow q$". Thus, "p if and only if q" can be written "$p \Longleftrightarrow q$". You can read "$p \Longleftrightarrow q$" also as "p implies and is implied by q". The version "p if and only if q" is shorter and more frequently used.

The student should realize that if he is asked to prove an "if and only if" theorem he must make two proofs — one for the "if" and another for the "only if".

Answers to
PROBLEMS 8-3

Student Text Pages 134-135

1. First example: Let p represent "$x = 3$" and let q represent "$x^2 = 9$". Then ($p \Longrightarrow q$) is true but ($q \Longrightarrow p$) is false.
   Second example: Let p represent "$x + 7 = 0$" and let q

represent "$x^2 + 7x = 0$". Since $x^2 + 7x = (x + 7)x$, $(p \Longrightarrow q)$ is true and $(q \Longrightarrow p)$ is false.

These two examples represent common errors in reasoning. In solving $x^2 = 9$, students frequently forget the negative solution, and in solving $x^2 + 7x = 0$, sometimes the common factor x is "cancelled out", and the solution $x = 0$ is lost.

2. a. p: "$\triangle ABC$ is equilateral".
   q: "$\triangle ABC$ is isosceles".
   b. p: "B is between A and C".
   q: "A, B and C are collinear".

3. a. p: "Today is Monday".
   q: "Tomorrow is not Friday".
   b. p: "John is buying some cloth".
   q: "John has some money".

4. Let p represent "For $\triangle ABC$ and $\triangle DEF$, $\overline{AB} \equiv \overline{DE}$, $\overline{BC} \equiv \overline{EF}$ and $\overline{CA} \equiv \overline{FD}$". Let q represent "$\triangle ABC \equiv \triangle DEF$". Then $(p \Longrightarrow q)$ is the SSS congruence theorem. The converse, $(q \Longrightarrow p)$, is true by definition of triangle congruence.

5. $\triangle ABC \equiv \triangle DEF$ *if* $\overline{AB} \equiv \overline{DE}$, $\overline{BC} \equiv \overline{EF}$ and $\overline{CA} \equiv \overline{FD}$.
   In $\triangle ABC$ and $\triangle DEF$, $\overline{AB} \equiv \overline{DE}$, $\overline{BC} \equiv \overline{EF}$ and $\overline{CA} \equiv \overline{FD}$ *only if* $\triangle ABC \equiv \triangle DEF$.
   The converse is true, as we noted in Problem 4, so an "if and only if" statement can be made which includes the SSS congruence theorem. It can be either "p if and only if q" or "q if and only if p", where p and q are as in Problem 4.

6. a. "B is between A and $C \Longrightarrow AB < AC$". This is true. The converse would be "$AB < AC \Longrightarrow B$ is between A and C", or "$AB \not< AC$ or B is between A and C". The converse is false as the figure shows.

8-4 The contrapositive of an implication

The word *contrapositive* looks more complicated than the concept it names. The text explains the idea quite clearly and compares

"$p \Longrightarrow q$" and its contrapositive, "not $q \Longrightarrow$ not p", when both are written as disjunctions. This, together with numerous examples, should convince the students that "$p \Longrightarrow q$" and its contrapositive have identical meanings.

It is good to illustrate this concept with a true implication "$p \Longrightarrow q$" for which the converse "$q \Longrightarrow p$" is false. This will help the students see the distinction between the converse and the contrapositive. The answers to Problems 1, 2 and 3 of the last section are good examples to use. Thus "$(x = 3) \Longrightarrow (x^2 = 9)$" and "$(x^2 \neq 9) \Longrightarrow (x \neq 3)$" are both true, and both have the same meaning. Likewise "(B is between A and C) $\Longrightarrow$ (A, B and C are collinear)" and "(A, B and C are not collinear) $\Longrightarrow$ (B is not between A and C)" say the same thing about the betweenness relation.

Answers to
PROBLEMS 8-4

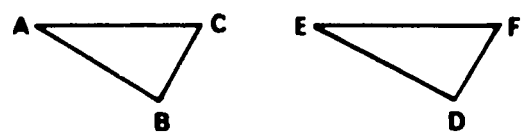
Student Text Page 136

1. $(x + y + 4 \neq 11) \Longrightarrow (x + y \neq 7)$.

2. a. If the sky is cloudy, then it is raining.
   b. If the sky is not cloudy, then it is not raining.
   c. If it is not raining, then the sky is not cloudy.

   The original implication and its contrapositive, b., seem to be true.

3. $\overline{AC} > \overline{DF}$ implies that $\overline{AC} \not\equiv \overline{DF}$. Thus, using the contrapositive of the given implication, we conclude that $\triangle ABC \not\equiv \triangle DEF$. You might ask if it is possible for the two triangles to be congruent. Of course it is, since we have not shown that all of the possible congruences are false. The figure shows $\triangle ABC \not\equiv \triangle DEF$ (because $\overline{AC} > \overline{DF}$) but $\triangle ABC \equiv \triangle EDF$.



4. Problem 2d asks for a conclusion from "$(p \Longrightarrow q)$ is true and q is false". Thus (not $q \Longrightarrow$ not p), the contrapositive, is true. Hence knowing that (not q) is true we conclude that (not p) is true, that is, p is false. Problem 2h is the same.

5. $(a = 5) \Longrightarrow (a^2 = 25)$ is true.
   The contrapositive, $(a^2 \neq 25) \Longrightarrow (a \neq 5)$, is true.
   The converse, $(a^2 = 25) \Longrightarrow (a = 5)$, is false.

Chapter 9
EXTERIOR ANGLE THEOREM AND INEQUALITIES IN TRIANGLES

9-1 Introduction

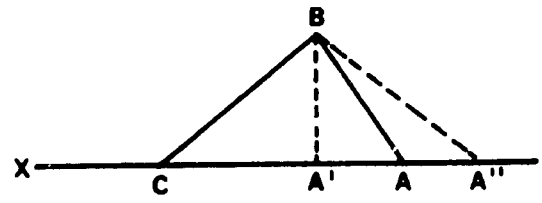
This chapter treats the conventional subject matter of inequalities of angles and of segments as related to triangles. Through questions and answers the students should review the work on angles and segment inequalities before embarking on this chapter.

Numerical questions and exercises motivating some of the theorems have been omitted in some cases. In *Secondary One* and *Two* much drawing and measuring of triangles was done, and the students may find it boring to repeat much of this activity. However, in some cases it may be necessary to do some exploratory measuring before proving the theorems. Such exercises are given in this GUIDE. The usefulness and appropriateness of such exercises must be judged by you.

At this stage of their course, students should recognize that theorems can be proved in a number of ways. They should be encouraged to discover alternative proofs wherever possible, and some of the exercises after the formal proofs of the theorems lead toward such discovery.

The decision as to how detailed a proof should be is a matter between you and the students. In writing our proofs, frequently we have omitted some steps which you and the students will be able to supply. Such omitted steps should be filled in by the students with your help. However, it is important that the students understand and appreciate all the details of the proofs; and the telescoping of several statements into one must be accompanied by this understanding.

1. a. $D\hat{A}B$, $E\hat{B}C$, $F\hat{C}A$
   b. $\hat{C}$, $\hat{B}$
   c. $F\hat{C}A$, $D\hat{A}B$, $E\hat{B}C$
2. a. Four; $\triangle ABC$, $\triangle AFD$, $\triangle CDE$ and $\triangle BEF$
   b. $A\hat{D}E$, $F\hat{E}C$, $B\hat{E}D$
   c. Three; $A\hat{B}E$, $B\hat{E}D$, $C\hat{E}F$
   d. $\triangle ABC$
   e. $C\hat{D}F$ of $\triangle AFD$
3. Yes; $\triangle A'BC$, $\triangle A''BC$



9-2 The exterior angle theorem

If the students cannot recall what they did in *Secondary One*, it may be appropriate to give them the following exercise as a lead to the theorem.

Draw $\triangle ABC$ and let D be a point such that B is between A and D. Measure the exterior angle $C\hat{B}D$ of the triangle and its remote interior angles $\hat{A}$ and $\hat{C}$ and tabulate the result as follows:

Exterior angle	Remote interior angles	
$m(CBD)$	$m(A)$	$m(C)$
.	.	.
.	.	.
.	.	.

Repeat the process by drawing triangles of different sizes and shapes. Compare the exterior angle with its remote interior angles in each case. What general statement is indicated by your results?

Here is an alternative exercise which could be done as class activity, preceding the exterior angle theorem. Draw $\triangle ABC$ using the following data.

AB	$m(\hat{A})$	Measure of exterior angle at vertex B
2 in.	40	60
2 in.	40	50
2 in.	40	42
2 in.	40	40
2 in.	40	30

Has it been possible to obtain a triangle in each case? How do the measures of $\hat{A}$ and of the exterior angle at vertex B compare in the cases in which the data determine a triangle? What about the cases in which the data fail to determine a triangle? Can you draw any conclusions?

In step 4 of the proof of the exterior angle theorem we have used the fact that F is in the interior of $C\hat{B}D$. This can be justified as follows. The problem is equivalent to showing that (1) F and D are on the same side of $\overleftrightarrow{BC}$ and that (2) F and C are on the same side of $\overleftrightarrow{BD}$. Refer to the figure in the text.

(1) B is between A and D, so A and D are on opposite sides of $\overleftrightarrow{BC}$. E, a point of $\overleftrightarrow{BC}$, is between A and F, so A and F are on opposite sides of $\overleftrightarrow{BC}$. Hence, F and D are on the same side of $\overleftrightarrow{BC}$.

(2) Since $\overline{CE}$ contains no points of $\overleftrightarrow{BD}$, C and E are on the same side of $\overleftrightarrow{BD}$. Likewise, $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{BD}$ in point A only; A is not a point of $\overline{EF}$, so E and F are on the same side of $\overleftrightarrow{BD}$. Thus, C, E and F (and in particular, C and F) are on the same side of $\overleftrightarrow{BD}$.

This argument has used some properties of betweenness. Probably most students will not notice the betweenness problem involved in step 4 of the proof, since it is quite easy to see from the diagram that F lies in the interior of $C\hat{B}D$. Unless the students discover the problem themselves, it should not be called to their attention and the formal justification of it may be presented only at

their request and even then perhaps only the more able students should attempt to understand it.

It is not strictly necessary to repeat the proof of Theorem 9.2 for case (b). Instead, we can argue as follows: If the exterior angle theorem is false, then there is some triangle which has an exterior angle not greater than one of its remote interior angles. Label the vertex of this exterior angle B, and label the vertex of the remote interior angle $\hat{C}$. Label the third vertex A. Because the two exterior angles at B are congruent, we can examine either of them. But the proof for case (a) shows that $C\hat{B}D > \hat{C}$, and the theorem cannot be false.

Here are the details of the proof of Corollary 9.3. Suppose there are two perpendiculars $\overline{AB}$ and $\overline{AC}$ through A to the line k. Let D be a point of k such that B is between D and C. Then $A\hat{B}D$ is an exterior angle of $\triangle ABC$, and $A\hat{C}B$ is one of its remote interior angles. But $A\hat{B}D$ and $A\hat{C}B$ are right angles, so the exterior angle theorem provides a contradiction.

Answers to
PROBLEMS 9-2

Student Text Pages 140-141

1. a. $D\hat{A}B > \hat{B}$
   b. $\hat{B} < A\hat{C}F$
   c. $\hat{C} < D\hat{A}B$
   d. $C\hat{B}E > \hat{A}$
   e. $A\hat{C}F > \hat{A}$
   f. $m(C\hat{B}E) + m(\hat{B}) = 180$

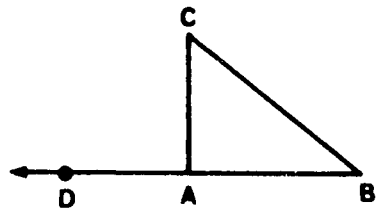
   The results a – e follow from the exterior angle theorem, and f follows from the fact that $C\hat{B}E$ and $\hat{B}$ are supplementary angles.

2. a. $a > 50$; $c > 50$
   b. $b > 60$; $c > 60$
   c. $b > 60$
   d. $c > 60$
   e. $x < 130$; $z < 130$
   f. $y < 120$

3. a. $m(A\hat{D}F) > 30$; $m(C\hat{B}F) > 30$
   b. $m(C\hat{B}F) > 60$
   c. $m(\hat{C}) < 150$; $m(\hat{A}) < m(C\hat{D}F) < m(\hat{E})$, so $m(\hat{A}) < 150$.

4. $\hat{b} < \hat{d} < \hat{c} < \hat{a}$

5. In $\triangle ABC$, $C\hat{A}B$ is a right angle.

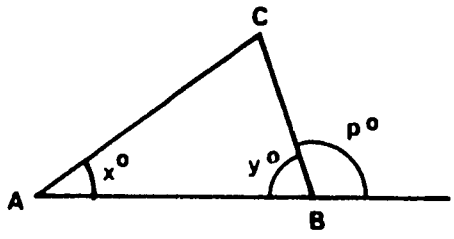


Statements	*Reasons*
a. $m(D\hat{A}C) = 90$.	$D\hat{A}C \equiv C\hat{A}B$
b. $m(\hat{B}) < m(C\hat{A}D)$.	Exterior angle theorem
c. $m(\hat{B}) < 90$.	Statements a and b
Similarly, $m(\hat{C}) < 90$.	

6. *Given*: The figure with the angles as marked.

To prove: $x + y < 180$

Proof:



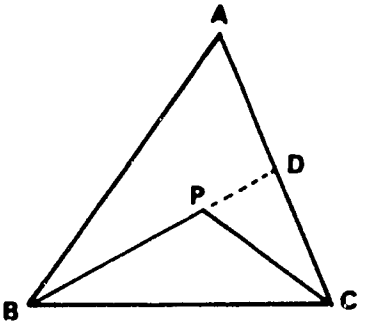
Statements	*Reasons*
a. $x < p$.	Exterior angle theorem
b. $x + y < p + y$.	Statement a and properties of real numbers
c. $p + y = 180$.	Supplementary angles
d. $x + y < 180$.	Statements b and c

7. Suppose in $\triangle ABC$, $\overline{AC} \equiv \overline{BC}$. Then $\hat{A} \equiv \hat{B}$. By Problem 6, $m(\hat{A}) + m(\hat{B}) < 180$, so $2 \times m(\hat{A}) < 180$ or $m(\hat{A}) < 90$.

8. a. From Problem 7, $m(\hat{x}) < 90$. But $m(\hat{x}) + m(\hat{y}) = 180$, $m(\hat{y}) = 180 - m(\hat{x})$. Hence $-m(\hat{x}) > -90$ so $m(\hat{y}) > 180 - 90 = 90$. We have $m(\hat{x}) < 90 < m(\hat{y})$.

   b. Applying the exterior angle theorem to $\triangle BDC$, $\hat{z} < \hat{x}$. Thus, from a, $\hat{z} < \hat{x} < \hat{y}$; so $\hat{z} < \hat{y}$.

9. Let $\overrightarrow{BP}$ intersect $\overline{AC}$ in D.



Statements	*Reasons*
a. $B\hat{P}C > P\hat{D}C$.	Exterior angle theorem in $\triangle BPC$
b. $P\hat{D}C > \hat{A}$.	Exterior angle theorem in $\triangle ABD$
c. $B\hat{P}C > \hat{A}$.	Statements a and b

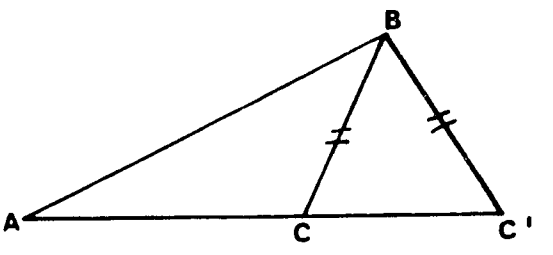
9-3 Hypotenuse-side congruence theorem

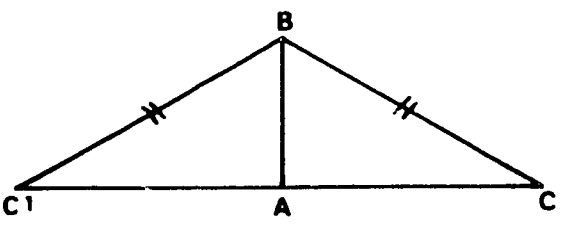
This section contains a fairly important triangle congruence theorem whose proof depends on the exterior angle theorem. The proof is a good example of an indirect proof.

Answers to
PROBLEMS 9-3

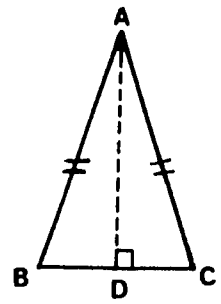
Student Text Page 143

1. In $\triangle ABC$ and $\triangle ABC'$, $\hat{A} \equiv \hat{A}$, $\overline{AB} \equiv \overline{AB}$ and $\overline{BC} \equiv \overline{BC'}$, but the triangles are not congruent. However, if the congruent angles ($\hat{A}$ in the figure above) are right angles, then the figure would look like the figure below. In this case the two triangles are congruent, as we know from Theorem 9.4.





2. In $\triangle ABC$, $\overline{AB} \equiv \overline{AC}$ and $\overline{AD} \perp \overline{BC}$. Then $A\hat{D}C$ and $A\hat{D}B$ are right angles, so by the hypotenuse-side congruence theorem, $\triangle ABD \equiv \triangle ACD$. Thus, $\overline{BD} \equiv \overline{DC}$ (D is the midpoint of the base) and $B\hat{A}D \equiv C\hat{A}D$ ($\overrightarrow{AD}$ bisects the vertex angle).



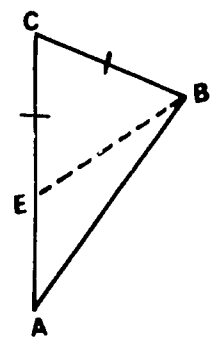
Answers to
PROBLEMS 9-4

Student Text Page 144

1. By the hypothesis, the lengths of two sides of the triangle are unequal and, therefore, one side is longer than the other. Since the lettering of the triangle is immaterial, we can label the longer side as $\overline{AC}$ and the shorter side as $\overline{BC}$.
2. In the construction which leads to $\overline{CD} \equiv \overline{CA}$ with B between C and D.
3. The construction enabled us to apply two results already known to us, namely the base angle property for isosceles triangles and the exterior angle theorem. Then the transitive property of "<" enables us to make the desired comparison.
4. *Given*: In the figure, $\overline{CE} \equiv \overline{CB}$.

   To prove: $C\hat{B}A > \hat{A}$

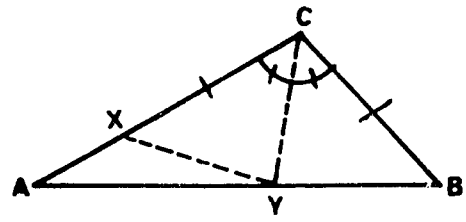
   First proof:



Statements	*Reasons*
a. $C\hat{B}E \equiv B\hat{E}C$.	Base angles of isosceles triangle
b. $B\hat{E}C > \hat{A}$.	Exterior angle theorem
c. $C\hat{B}E > \hat{A}$.	Statements a and b
d. $C\hat{B}A > C\hat{B}E$.	E lies in the interior of $C\hat{B}A$.
e. $C\hat{B}A > \hat{A}$.	Statements c and d

Our construction is justified by the fact that, by the definition of inequalities between segments, $\overline{BC} < \overline{AC}$ implies that there is a point E on $\overline{AC}$ such that $\overline{CE} \equiv \overline{CB}$.

Second proof: Let X be the point between A and C such that $\overline{CX} \equiv \overline{CB}$. Let the bisector of $\hat{C}$ intersect $\overline{AB}$ at Y.



Statements	*Reasons*
a. $\triangle CXY \equiv \triangle CBY$.	SAS
b. $C\hat{X}Y \equiv \hat{B}$.	Statement a
c. $C\hat{X}Y > \hat{A}$.	Exterior angle theorem
d. $\hat{B} > \hat{A}$.	Statements b and c

9-5 Side comparison theorem for a triangle

One proof of Theorem 9.6 presented here is an indirect proof. The method consists of listing all the possible relations between sides $\overline{AC}$ and $\overline{BC}$ and then showing that all but one of these contradict the hypothesis. Then it follows that the one remaining relation must hold.

Notice that Theorems 9.5 and 9.6 are converses of each other. We can restate them as follows:

THEOREM 9.5: For any $\triangle ABC$, $\overline{AC} > \overline{BC} \Longrightarrow \hat{B} > \hat{A}$.

THEOREM 9.6: For any $\triangle ABC$, $\hat{B} > \hat{A} \Longrightarrow \overline{AC} > \overline{CB}$.

These can be combined into one statement:

For any $\triangle ABC$, $\overline{AC} > \overline{CB} \Longleftrightarrow \hat{B} > \hat{A}$.

Answers to
PROBLEMS 9-5

Student Text Pages 145-147

1. $\hat{B}$ is the largest angle; $\hat{C}$ is the smallest angle.
2. $\overline{EF}$ is the longest side; $\overline{DE}$ is the shortest side.
3. $\triangle PQS$: $\overline{PS} < \overline{PQ} < \overline{QS}$; $\triangle RQS$: $\overline{QS} < \overline{RQ} < \overline{RS}$;

$\triangle PQR$: $\overline{PR} < \overline{PQ} < \overline{QR}$.

4. $\hat{C} > \hat{B}$.

5. a. $\overline{FG} < \overline{GH}$, b. $\overline{FH} < \overline{GF}$,
c. $\overline{FG} < \overline{GH} < \overline{HF}$, d. $\overline{HF} < \overline{GF}$ and $\overline{GF} > \overline{GH}$.

6. a. $\hat{Z} > \hat{X}$, b. $\hat{Y} < \hat{X}$,
c. $\hat{Z} > \hat{X} > \hat{Y}$, d. $\hat{X} < \hat{Z}$ and $\hat{Z} > \hat{Y}$.

7.

Statements	*Reasons*
a. $m(A\hat{C}B) > m(A\hat{B}C)$.	Greater angle opposite greater side theorem
b. $\frac{1}{2}m(A\hat{C}B) > \frac{1}{2}m(A\hat{B}C)$.	Step a
c. $m(B\hat{C}D) > m(C\hat{B}D)$.	Step b and hypothesis
d. $\overline{BD} > \overline{CD}$.	Greater angle opposite greater side theorem

8.

Statements	*Reasons*
a. $B\hat{A}C > B\hat{C}A$.	Greater angle opposite greater side theorem
b. $C\hat{A}D > A\hat{C}D$.	Greater angle opposite greater side theorem
c. $m(B\hat{A}C) + m(C\hat{A}D) > m(B\hat{C}A) + m(A\hat{C}D)$	Steps a and b and properties of real numbers
d. $B\hat{A}D > B\hat{C}D$.	Angle addition postulate and statement c
e. $B\hat{A}D > B\hat{C}D$.	Step d

9.

Statements	*Reasons*
a. $E\hat{D}G \equiv D\hat{E}G$.	Base angles of isosceles triangle
b. $E\hat{D}F > E\hat{D}G$.	G lies in the interior of $E\hat{D}F$.
c. $E\hat{D}F > D\hat{E}F$.	Steps a and b
d. $\overline{EF} > \overline{FD}$.	Greater angle opposite greater side theorem

10.

Statements	*Reasons*
a. $C\hat{A}P \equiv P\hat{A}B$.	Right angles are congruent.
b. $A\hat{B}P < C\hat{A}P$.	Exterior angle theorem
c. $A\hat{B}P < P\hat{A}B$.	Steps a and b
d. $\overline{PA} < \overline{PB}$.	Greater angle opposite greater side theorem

11. By Problem 5 of Problems 9-2, the other angles in the triangle are acute. Since the largest side is the one opposite the largest angle, the result is established.

9-6 The triangle inequality

Cases (b), (c), (e) and (f) of the construction exercise do not determine triangles because in each of these cases the sum of two sides is not greater than the third side of the triangle.

In the proof of the triangle inequality in the text we proved that

(i) $AB + BC > AC$. There are two other cases, namely:

(ii) $BC + CA > AB$

(iii) $CA + AB > BC$.

Cases (ii) and (iii) do not need separate proof since they follow from (i) by simply re-labelling the triangle.

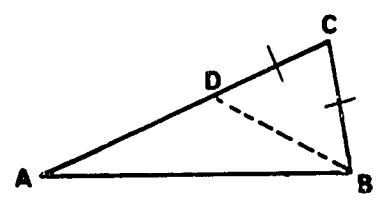
Answers to
PROBLEMS 9-6

Student Text Page 148

1. The construction enables us to obtain a segment equal in length to $AB + BC$ and since that segment and $\overline{AC}$ are two sides of a triangle in which we can compare $C\hat{A}D$ and $A\hat{D}C$ we can apply the side comparison theorem.

2. *Given*: In the figure, $\overline{BC} \equiv \overline{CD}$ and D is between A and C.

   To prove: $AB > AC - BC$, that is, $\overline{AB} > \overline{AD}$.



Proof:

Statements	*Reasons*
a. $D\hat{B}C \equiv C\hat{D}B$.	Base angles of isosceles triangle
b. $C\hat{D}B > A\hat{B}D$.	Exterior angle theorem
c. $B\hat{D}A > D\hat{B}C$.	Exterior angle theorem
d. $B\hat{D}A > C\hat{D}B$.	Steps a and c
e. $B\hat{D}A > A\hat{B}D$.	Steps b and d
f. $\overline{AB} > \overline{AD}$.	Greater angle opposite greater side theorem

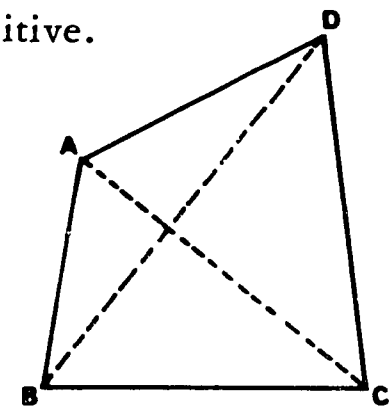
3. Let x be the length of the third side. Then by the triangle inequality we have the following:

   a. $x + 4 > 9$, so
   $x > 5$.

   b. $x + 9 > 4$, or
   $x > -5$,
   which is trival since x must be positive.

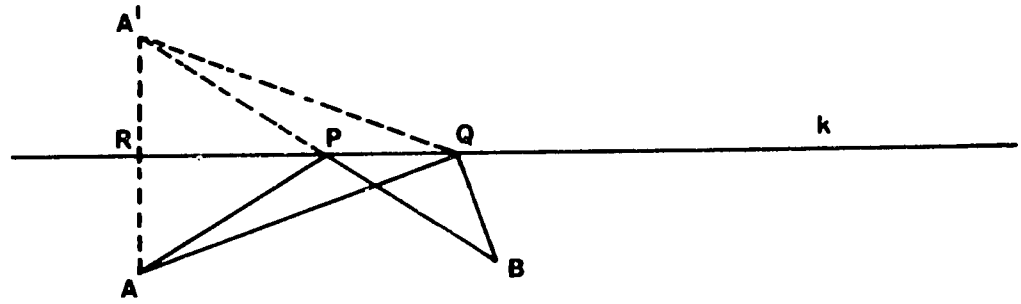
   c. $4 + 9 > x$, or
   $x < 13$.
   Hence, $5 < x < 13$.

4. Consider quadrilateral $ABCD$ with diagonals $\overline{AC}$ and $\overline{BD}$.



Statements	*Reasons*
a. $BD < AB + AD$.	Triangle inequality
b. $BD < BC + CD$.	Triangle inequality
c. $AC < AB + BC$.	Triangle inequality
d. $AC < AD + DC$.	Triangle inequality
e. $2(AC + BD) < 2(AC + BC + CD + DA)$.	Statements a, b, c, and d and properties of real numbers
f. $AC + BD < AB + BC + CD + DA$.	Properties of real numbers

5. When we say that A' is the *mirror image* of A in k, we mean that A' is chosen so that k is the perpendicular bisector of $\overline{AA'}$. The figure is like this:



We are to prove that $AP + PB < AQ + QB$.

We see that $\triangle ARP \equiv \triangle A'RP$ and $\triangle ARQ \equiv \triangle A'RQ$ (by SAS) so it is enough to prove that $A'P + PB < A'Q + QB$. But $A'P + PB = AB$, and $AB < A'Q + QB$ by the triangle inequality.

This problem has an important application in optics, the branch of physics treating properties of light. Imagine a mirror, perpendicular to the paper, with its edge along k. Suppose there is a source of light (for example, a candle) at point A. Some of the light leaving A will reach point B after being reflected in the mirror. The question is, "What is the path of such a light ray?" A physical principle (called D'Alembert's principle) states that the light will be reflected at the point which makes the path travelled by the light as short as possible. (D'Alembert's principle is a sort of "principle of economy".) The result of this problem says that the shortest path is from A to P and from P to B.

9-7 Comparison theorems for two triangles

The proofs of Theorem 9.8 and its converse are rather difficult. If time is running short, then this section could be omitted, or given as an extra assignment for the more able pupils. Problem 11 of Section 9-9 depends on these theorems, so it should not be assigned to students who omit this section.

A drawing and measuring exercise which leads to Theorem 9.8 is the following. Draw $\triangle ABC$ and $\triangle A'B'C'$ using the following data and measure the lengths of BC and $B'C'$.

AB $A'B'$	AC $A'C'$	$m(\hat{A})$	$m(\hat{A}')$	BC	$B'C'$
2	3	60	20	.	.
2	3	60	40	.	.
2	3	60	60	.	.
2	3	60	80	.	.
2	3	60	100	.	.

Compare the lengths of $\overline{BC}$ and $\overline{B'C'}$ for each pair of triangles. How does the measure of $\hat{A}'$ affect the length of $\overline{B'C'}$?

The same exercise suggests that the converse theorem is also true.

Two cases in the proof of THEOREM 9.8 have been left as an exercise for the pupil.

Case (ii): D is on $\overline{BC}$.

Statements	*Reasons*
1. $\overline{BD} \equiv \overline{B'C'}$.	Statement 2 of the proof for case (i)
2. $\overline{BC} > \overline{BD}$.	D is between B and C.
3. $\overline{BC} > \overline{B'C'}$.	Statements 1 and 2

Case (iii): D is in the exterior of $\triangle ABC$.

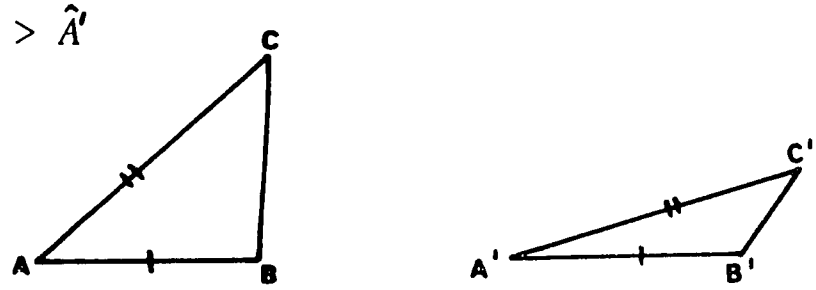
Statements	*Reasons*
1. $\overline{BD} \equiv \overline{B'C'}$.	Statement 2 of the proof in the text
2. $A\hat{C}D \equiv C\hat{D}A$.	Base angles of isosceles triangle
3. $C\hat{D}B > C\hat{D}A$.	A lies in the interior of $C\hat{D}B$.
4. $A\hat{C}D > B\hat{C}D$.	B lies in the interior of $A\hat{C}D$.
5. $C\hat{D}B > B\hat{C}D$.	Statements 2, 3 and 4

6. $\overline{BC} > \overline{BD}$.	Greater angle opposite greater side theorem
7. $BC > \overline{B'C'}$.	Statements 1 and 6

The proof of Theorem 9.9 follows from Theorem 9.8 in the same way that the proof of Theorem 9.6 follows from Theorem 9.5.

Given: $\triangle ABC$ and $\triangle A'B'C'$ with $\overline{AB} \equiv \overline{A'B'}$, $\overline{AC} \equiv \overline{A'C'}$ and $\overline{BC} > \overline{B'C'}$.

To prove: $\hat{A} > \hat{A}'$



Proof: There are three possibilities:

(i) $\hat{A} \equiv \hat{A}'$ (ii) $\hat{A} < \hat{A}'$ (iii) $\hat{A} > \hat{A}'$

If $\hat{A} \equiv \hat{A}'$, then

$\triangle ABC \equiv \triangle A'B'C'$ (by SAS) and hence

$\overline{BC} \equiv \overline{B'C'}$.

This contradicts the hypothesis that $\overline{BC} > \overline{B'C'}$. Hence statement (i) is false.

If $\hat{A} < \hat{A}'$, then by the preceding theorem $\overline{BC} < \overline{B'C'}$, which again contradicts the hypothesis. Hence statement (ii) is false. Therefore (iii) is the only remaining possibility and so

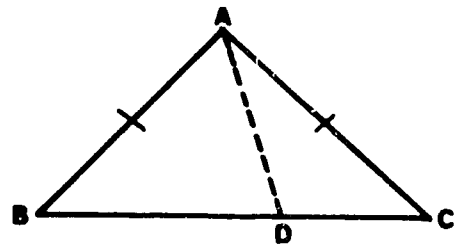
$$\hat{A} > \hat{A}'.$$

Answers to
PROBLEMS 9-7

Student Text Pages 151-152.

1. *Given*: $\triangle ABC$ with $\overline{AB} \equiv \overline{AC}$, $\overline{BD} > \overline{DC}$.

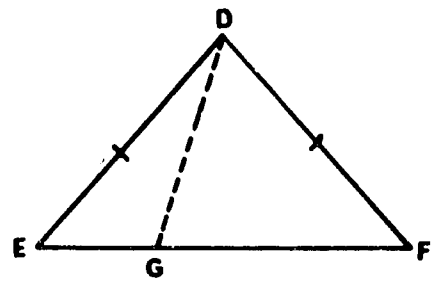
To prove: $B\hat{A}D > C\hat{A}D$



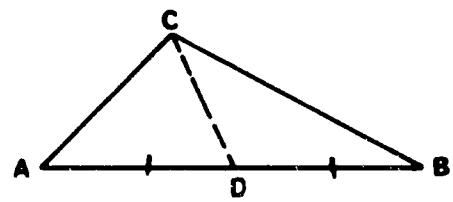
Proof:

Statements	*Reasons*
a. $\overline{AB} \equiv \overline{AC}$.	Hypothesis
b. $\overline{AD} \equiv \overline{AD}$.	Identify
c. $\overline{BD} > \overline{DC}$.	Hypothesis
d. $B\hat{A}D > C\hat{A}D$.	Angle comparison theorem

2. In the figure, $E\hat{D}G < F\hat{D}G$. If we apply the side comparison theorem to $\triangle DEG$ and $\triangle DFG$, the result follows.

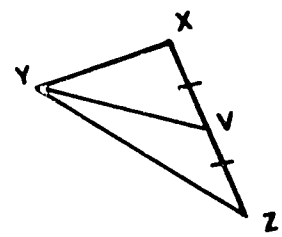


3. In the figure, $B\hat{D}C > A\hat{D}C$. To prove that $\overline{AC} < \overline{BC}$, apply the side comparison theorem to $\triangle ADC$ and $\triangle BDC$.



4. In the figure, $\overline{XY} < \overline{YZ}$. To prove that $m(X\hat{V}Y) < 90$ we reason as follows:

By the angle comparison theorem, $Z\hat{V}Y > X\hat{V}Y$. But $m(Z\hat{V}Y) + m(X\hat{V}Y) = 180$ and it follows that $m(X\hat{V}Y) < 90$ (and $m(Z\hat{V}Y) > 90$).



5. There are only three possible relationships between $\overline{DE}$ and $\overline{EF}$: namely, (i) $\overline{DE} < \overline{EF}$; (ii) $\overline{DE} = \overline{EF}$; (iii) $\overline{DE} > \overline{EF}$.

If $\overline{DE} < \overline{EF}$, then

$D\hat{H}E < F\hat{H}E$, while

$$\text{if } \overline{DE} > \overline{EF}, \text{ then}$$
$$D\hat{H}E > F\hat{H}E.$$

Thus both statements (i) and (iii) lead to contradictions of the hypothesis. Therefore statement (ii) holds and $\overline{DE} = \overline{EF}$. Note that this is another proof of the theorem which states that every point on a perpendicular bisector of a segment is equidistant from the ends of the segment. See Problem 17, Section 7-5.

6. $E\hat{G}F > G\hat{E}H$.

9-8 Some properties of angle bisectors

The results included in this section do not add to the list of comparison theorems, which are the principal business of this chapter. However, consequences of the comparison theorems are used in the proof of the result on angle bisectors.

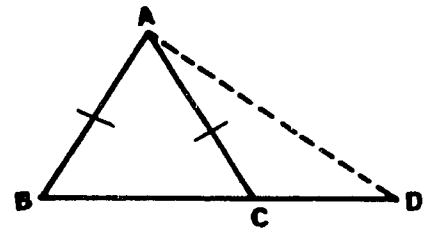
The proofs given in the text are written in prose style. It might be a good exercise to have the students rewrite these proofs in the "statement-reason" form.

Answers to
PROBLEMS 9-8

Student Text Pages 153-155

1. Corollary 9.3 tells us that there is just one segment satisfying the conditions specified in Definition 9.10. If there were more than one such segment, then the definition might give two different numbers as the distance from P to k.
2. The set of points equidistant from A and B is the plane which is perpendicular to $\overline{AB}$ and which contains the midpoint of $\overline{AB}$. It would be reasonable to call this plane the perpendicular bisector of $\overline{AB}$.

Answers to
ADDITIONAL PROBLEMS 9-8

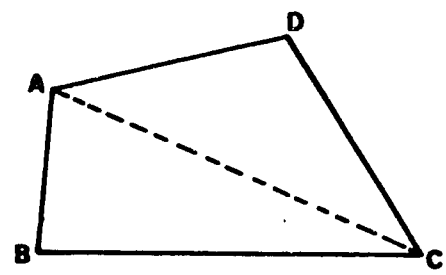


1.

Statements	*Reasons*
a. $\hat{B} \equiv \hat{C}$.	Base angles of isosceles triangle
b. $A\hat{C}B > C\hat{D}A$.	Exterior angle theorem
c. $C\hat{D}A < \hat{B}$.	Steps a and b
d. $\overline{AD} > \overline{AB}$.	Greater angle opposite greater side theorem
e. $\overline{AD} > \overline{AC}$.	Step d and hypothesis

2. Suppose $\overleftrightarrow{AF}$ and $\overleftrightarrow{BE}$ intersect in a point P. Then by the exterior angle theorem applied to $\triangle ABP$, $\hat{y} > \hat{x}$ or $\hat{x} > \hat{y}$ depending on which side of $\overleftrightarrow{AB}$ P is on. This contradicts the hypothesis and hence the supposition is false. Therefore, $\overleftrightarrow{AF}$ and $\overleftrightarrow{BE}$ do not intersect.

3. Let $ABCD$ be the quadrilateral. Consider the diagonal $\overline{AC}$. Then by the triangle inequality,



$$AB + BC > AC \text{ and}$$
$$AC + CD > AD.$$

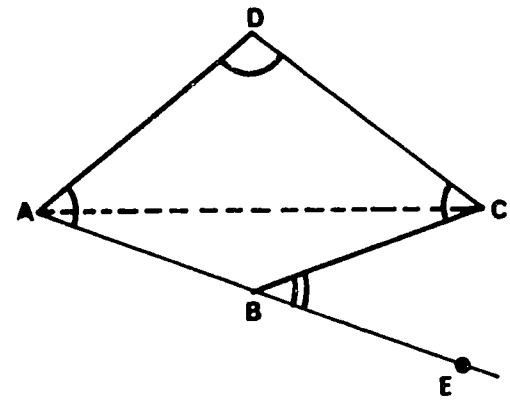
Therefore,

$$AB + BC + CD + AC > AC + AD, \text{ which implies that}$$
$$AB + BC + CD > AD.$$

4. If A, B and C are not collinear then $AC < AB + BC$ by the triangle inequality. If B is between A and C, then $AC = AB + BC$. If $B = A$ or $B = C$, then $AC = AB + BC$ since one of the numbers on the right is zero. If A is between B and C, then $BC > AC$, so certainly $AC < AB + BC$. If C is between A and B, then $AB > AC$ and $AC < AB + BC$. Thus, in any case, AC is less than *or* equal to $AB + BC$.

5. Suppose that A, B and C are not collinear. Then the triangle inequality gives $AC < AB + BC$. Applying Problem 4 to A, C and D, we have $AD \leq AC + CD < (AB + BC) + CD$. Applying Problem 4 to A, D and E, $AE \leq AD + DE < (AB + BC + CD) + DE$ which gives the desired result.

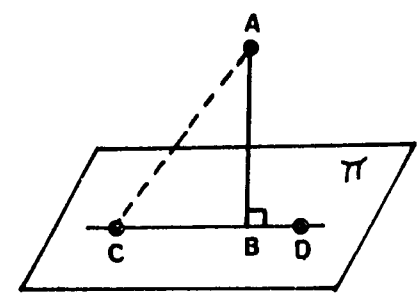
6. Let $ABCD$ be the quadrilateral and let $C\hat{B}E$ be one of its exterior angles. Join $\overline{AC}$. All that can be said about the size of $C\hat{B}E$ is that $C\hat{B}E > B\hat{A}C$ and $C\hat{B}E > A\hat{C}B$. It is impossible to assert that an exterior angle of any convex quadrilateral is greater than each of its remote interior angles. One typical case in which it does not hold is the case of a rectangle where the exterior angle is congruent to each of the remote interior angles.



This solution is a good example of a disproof by *counter-example*. The proposed theorem makes a statement about all convex quadrilaterals. Since we have found one example (the quadrilateral sketched or a rectangle or square) for which the proposed property *does not* hold, then the theorem is false.

7. Suppose there are two lines through P perpendicular to π. Let these lines be $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ where A and B are in π. Then $\overleftrightarrow{AB}$ is in π, so from the definition, $\overleftrightarrow{PA} \perp \overleftrightarrow{AB}$ and $\overleftrightarrow{PB} \perp \overleftrightarrow{AB}$. This is a contradiction of the uniqueness of the line through P perpendicular to a given *line* $\overleftrightarrow{AB}$, Corollary 9.3.

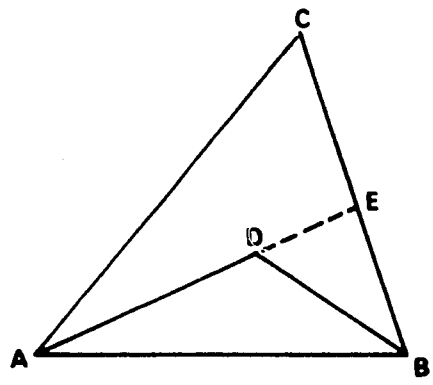
8. In the figure, points B and C are in π and $\overleftrightarrow{AB} \perp \overleftrightarrow{BC}$. We must show that $AC > AB$. We can appeal to Problem 10, Section 9-5, or we can repeat the proof of that problem.



9.

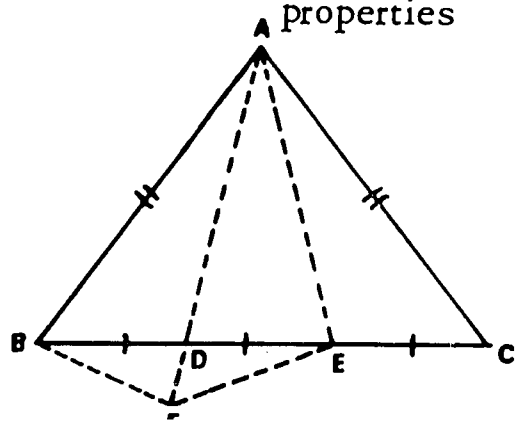
Statements	*Reasons*
a. $\overline{CE} > \overline{AE}$.	Greater angle opposite greater side theorem
b. $\overline{ED} > \overline{EB}$.	Greater angle opposite greater side theorem
c. $CE + ED > AE + EB$.	Statements a and b
d. $CD > AB$.	Statement c

10. *Proof*:



Statements	*Reasons*
a. $AC + CE > AE$.	Triangle inequality
b. $DE + EB > BD$.	Triangle inequality
c. $AC + CE + EB + DE > AE + BD$.	Statements a and b
d. $AD + DE = AE$ and $CE + EB = CB$.	Measurement postulate
e. $AC + CB + DE > AD + DE + BD$.	Statements c and d
f. $AC + BC > AD + BD$.	Statement f and real number properties

11.



Statements	*Reasons*
a. $\hat{B} \equiv \hat{C}$.	Base angles of isosceles triangle
b. $B\hat{D}A > \hat{C}$.	Exterior angle theorem
c. $B\hat{D}A > \hat{B}$.	Statements a and b
d. $\overline{AB} > \overline{AD}$.	Greater angle opposite greater side theorem
e. Let F be on $\overrightarrow{AD}$ such that $\overline{AF} \equiv \overline{AB}$. Join $\overline{BF}$ and $\overline{FE}$.	Segment congruence postulate
f. $B\hat{D}A > A\hat{E}D$.	Exterior angle theorem
g. $\triangle ADB \equiv \triangle AEC$.	Step a, hypothesis, SAS
h. $\overline{AD} \equiv \overline{AE}$.	Step g
i. $A\hat{D}E \equiv A\hat{E}D$.	Step g
j. $B\hat{D}A > A\hat{D}E$.	Steps f, i
k. $B\hat{D}A \equiv E\hat{D}F$.	Vertically opposite angles
l. $A\hat{D}E > D\hat{F}E$.	Exterior angle theorem
m. $E\hat{D}F > D\hat{F}E$.	Steps j, k, l
n. $\overline{EF} > \overline{DE}$.	Greater angle opposite greater side theorem
o. $\overline{EF} > \overline{BD}$.	$\overline{BD} \equiv \overline{DE}$ and statement n
p. $F\hat{A}E > B\hat{A}D$.	Angle comparison theorem for two triangles $\triangle FAE$ and $\triangle BAD$
q. $D\hat{A}E \not\equiv B\hat{A}D$.	Statement p

12. If C_1 is chosen on $\overrightarrow{AC}$ so that $\overline{AC_1} \equiv \overline{A'C'}$, then $\triangle A'B'C' \equiv \triangle ABC_1$ by SAS.

Case (i): If C_1 is between A and C, then the exterior angle theorem implies that $A\hat{C}_1B > A\hat{C}B$, or using the known triangle congruence, $A'\hat{C}'B' > A\hat{C}B$, which contradicts the

hypothesis $\hat{C} \equiv \hat{C}'$. Thus, C_1 cannot be between A and C.

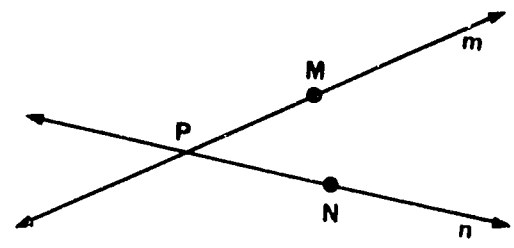
Case (ii): If C is between A and C_1, the exterior angle theorem gives $A\hat{C}B > A\hat{C}_1B$ or $A\hat{C}B > A'\hat{C}'B'$, again a contradiction.

The only remaining possibility, $C = C_1$, must hold. This gives $\overline{AC} \equiv \overline{AC_1} \equiv \overline{A'C'}$ and $\triangle ABC \equiv \triangle A'B'C'$ by SAS.

Chapter 10
ELEMENTARY PROPERTIES OF PARALLEL LINES

10-1 Existence of parallel lines

This chapter contains some theorems about parallel lines whose proofs do not depend on the parallel postulate. In Chapter 11, the parallel postulate will be introduced and further properties of parallel lines will be deduced. For example, the fact that if two parallel lines are cut by a transversal then the alternate interior angles are congruent depends on the parallel postulate for its proof. It will appear in Chapter 11.



The figure shown here should be kept in mind while reading the first paragraph of the text. N is not on the line determined by P and M, for if it were, line $\overleftrightarrow{PN}$, which is n, would coincide with line m. Thus, there is a unique plane α containing M, P and N. If M and P are in α, so is the entire line $\overleftrightarrow{MP}$. In the same way, line $\overleftrightarrow{NP}$ is contained in α.

It is important to emphasize that two lines are parallel if they are *coplanar* and non-intersecting. The reason for introducing the term "skew lines" at this point is to emphasize that two lines which are not coplanar cannot be parallel.

Referring to the box pictured in the text, lines $\overleftrightarrow{AB}$, $\overleftrightarrow{DC}$, $\overleftrightarrow{HG}$ and $\overleftrightarrow{EF}$ are mutually parallel; that is, any two of these four lines form a pair of parallel lines. Similarly, $\overleftrightarrow{AE}$, $\overleftrightarrow{BF}$, $\overleftrightarrow{CG}$ and $\overleftrightarrow{DH}$ are mutually parallel, as are $\overleftrightarrow{AD}$, $\overleftrightarrow{EH}$, $\overleftrightarrow{FG}$ and $\overleftrightarrow{BC}$. The following is a partial list

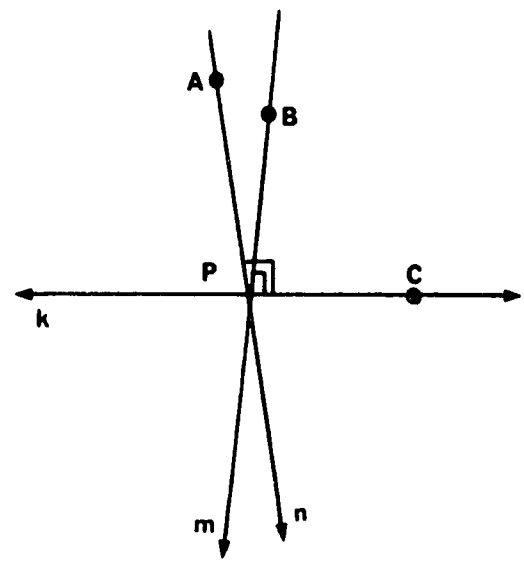
of pairs of skew lines sketched: $\overleftrightarrow{AB}$ and any of $\overleftrightarrow{CG}$, $\overleftrightarrow{DH}$, $\overleftrightarrow{EH}$ and $\overleftrightarrow{FG}$; $\overleftrightarrow{DC}$ and any of $\overleftrightarrow{AE}$, $\overleftrightarrow{BF}$, $\overleftrightarrow{EH}$ and $\overleftrightarrow{FG}$; and so on. In all, there are 24 pairs of skew lines sketched. The vertices of the box determine some pairs of parallel lines (and pairs of skew lines) which are not sketched. For example, $\overleftrightarrow{AF}$ and $\overleftrightarrow{DG}$ are parallel, while $\overleftrightarrow{DE}$ and $\overleftrightarrow{HC}$ are skew.

The proofs in the text are written much as one might explain the proof in an oral discussion. This is a less rigid style than the listing of statements and reasons. A proof written one way is no more, nor less, rigorous than the same proof written the other way. The students should learn to understand proofs in both styles.

If it seems appropriate, the students can be given the exercise of rewriting some of these proofs in the "statement-reason" style used in other parts of the text.

Here are some comments on the proof of Theorem 10.3. Refer to the student text as you read these.

If m and n are not parallel, then they must intersect since, by hypothesis, they are coplanar. Then in $\triangle PQR$ shown in the student text, the exterior angle theorem is contradicted. This shows that m and n cannot intersect at a point not in k. In the second case, the assumption that m and n intersect at a point of k says that rays $\overrightarrow{PA}$ and $\overrightarrow{PB}$ each form a right angle with $\overrightarrow{PC}$. The second angle measurement postulate implies that there is only one such ray in a half-plane with edge k, so if we choose A and B on the same side of k, as shown, we conclude that $\overrightarrow{PA} = \overrightarrow{PB}$. Thus, $m = n$, a contradiction of the hypothesis that m and n are *distinct* lines.



As class activity, the pupils should construct parallels by the two procedures described, both of which involve drawing perpendiculars. In Chapter 8 of *Secondary Two*, construction of perpendiculars was explained. The construction can be done using a protractor

or a set square. Of course, if their drawings are accurate, the indication will be that there is only one line through a given point and parallel to a given line. Thus, it seems that the question of uniqueness of parallels ought to have an affirmative answer, so the parallel postulate of Chapter 11 is motivated.

In the student text, it is stated that in the nineteenth century it was finally shown that the question of uniqueness of parallels could not be answered on the basis of the postulates we have assumed. This was first established independently by J. Bolyai (a Hungarian), N. I. Lobachevsky (a Russian) and K. F. Gauss (a German), and later in another way by B. Riemann (a German). These mathematicians showed that there are consistent geometries much like ours, except that one assumes that through a point there is no line parallel to a given line, or that through a point there are many lines parallel to a given line. Such geometries are called non-Euclidean geometries. Their discovery is an exciting and significant even in the history of mathematics. Some books where this achievement is discussed are:

Basic Concepts of Geometry by W. Prenowitz and M. Jordan (Chapters 2 and 3), Ginn & Company, Boston, 1965;

A Survey of Geometry, Vol. I, (Chapter VII) by Howard Eves, Allyn and Bacon, Boston, 1963;

Elementary Geometry from an Advanced Standpoint by Edwin Moise (Chapter 9), Addison-Wesley Publishing Co., Reading, Massachusetts, 1963;

Non-Euclidean Geometries by Lilian R. Lieber, Galois Institute of Mathematics and Art, Brooklyn, N.Y., 1940.

In order to realize that non-Euclidean geometries are possible, you must think of geometry as an abstract branch of mathematics. That is, although the ideas of geometry have come from consideration of real things, geometry is a study of certain ideas. Thus, when a mathematician decides what postulates he wishes to assume, he is not bound to the behavior of real things. He certainly may be influenced by the way real pictures or real things seem to behave, but his only real concern is that he should not make mutually contradictory assumptions and that his postulates should lead to an interesting body of theorems.

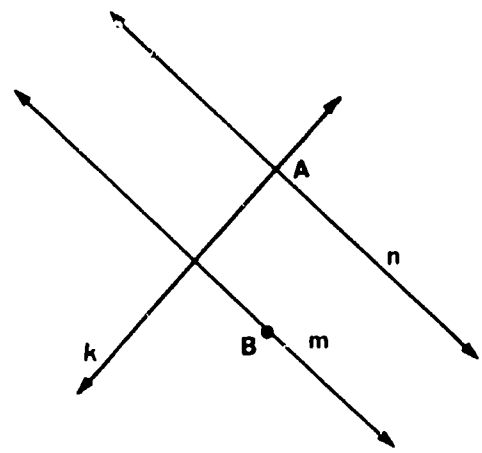
Of course, Euclidean geometry is very useful when applied to real situations by carpenters, surveyors, physicists and others. It is very interesting to know that non-Euclidean geometries are also useful in physics. For example, the work in physics done by Albert Einstein in the first part of this century depends heavily on the non-Euclidean geometry discovered in the middle of the nineteenth century by B. Riemann.

You may or may not want to talk about the subject of non-Euclidean geometries in the classroom. It is certainly not necessary that you do so. However, we do hope that you (and perhaps a few of your students) will be interested in reading more about this in some of the books we mentioned earlier.

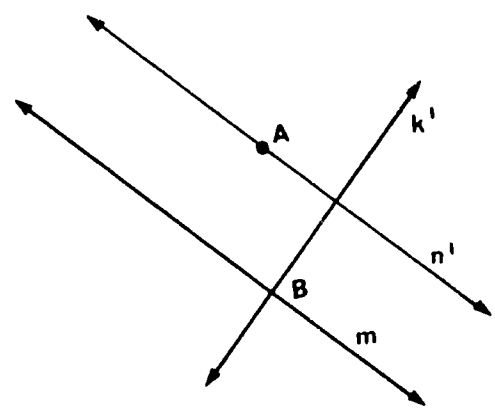
Answers to
PROBLEMS 10-1

Student Text Page 160

1. Following the first method of proof, the constructions would be as follows: First, draw line k through A and perpendicular to m. Second, draw line n through A and perpendicular to k. Then $n \parallel m$, by Theorem 10.3.

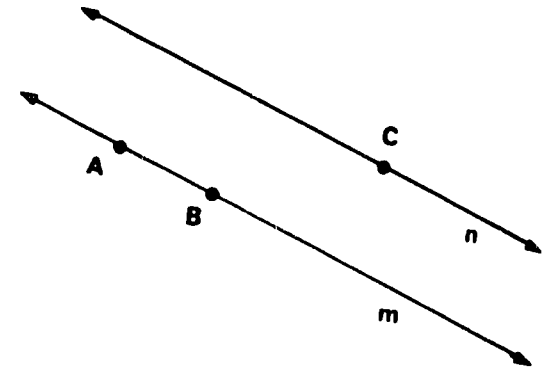


Following the second method, we would construct as follows: Through B draw k' perpendicular to m. Then draw n' through A and perpendicular to k'. Then $n' \parallel m$.

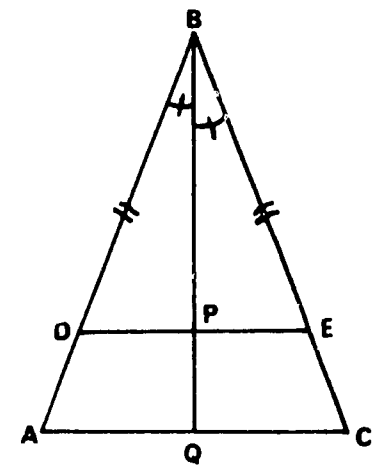


2. Let the parallel lines be m and n; since m and n are parallel, they lie in at least one plane. Let A and B be distinct points on m and let C be any point on n. Then A, B and C are not

collinear, for if so, C would have to be on $\overleftrightarrow{AB}$. But $\overleftrightarrow{AB} = m$ and m and n have no points in common. Hence A, B and C are exactly one plane α. But any plane which contains m and n must contain A, B and C, so α is the only plane which could contain m and n.



3. If the bisector of B is the ray $\overrightarrow{BQ}$, with point P as marked, then we reason as follows: $\overline{BD} \equiv \overline{BE}$ by hypothesis. $\overline{BP} \equiv \overline{BP}$ and, by construction, $P\hat{B}D \equiv P\hat{B}E$. By SAS, $\triangle PBD \equiv \triangle PBE$. Hence, $B\hat{P}E$ and $B\hat{P}D$ are congruent. Since these angles are also supplementary, each is a right angle. That is, is, $\overleftrightarrow{BP} \perp \overleftrightarrow{DE}$.



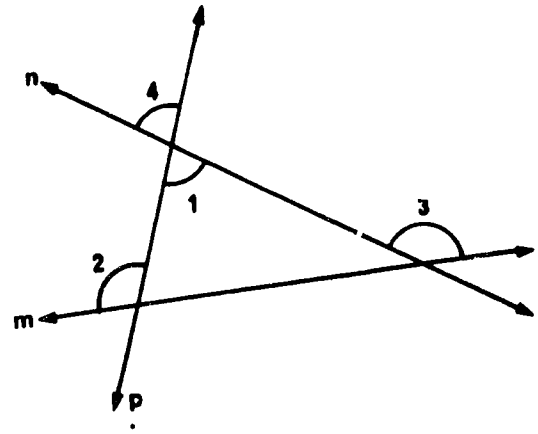
The same argument made on $\triangle BQA$ and $\triangle BQC$ shows that $\overleftrightarrow{BQ} \perp \overleftrightarrow{AC}$. Thus, $\overleftrightarrow{AC}$ and $\overleftrightarrow{DE}$ are perpendicular to the same line (and they are coplanar, since they are in the plane of $\triangle ABC$). So by Theorem 10.3, $\overleftrightarrow{AC} \parallel \overleftrightarrow{DE}$.

If it is necessary to give a hint in addition to the one in the text, you might suggest proving that $\overleftrightarrow{BP} \perp \overleftrightarrow{DE}$ by congruent triangles.

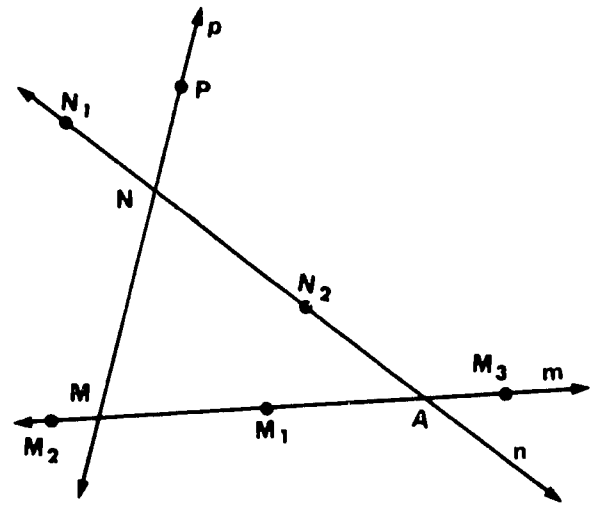
10-2 Transversals of two parallel lines

Definition 10.5 tells you that a transversal of two coplanar lines must intersect these lines in two distinct points. Thus, in the figure in the text, p' is not a transversal of m' and n'. We can regard m as a transversal of p and n, and n can be thought of as a transversal of m and p.

In identifying alternate interior angles, it is important to know which line is the transversal. If p is a transversal of m and n, then $\hat{1}$ and $\hat{2}$ are alternate interior angles. If n is regarded as a transversal of p and m, then $\hat{1}$ and $\hat{3}$ are alternate interior angles. In this case, $\hat{4}$ and $\hat{3}$ are corresponding angles. These distinctions could be brought out in some class activity.



To answer the questions following Definition 10.6, the students will have to name more points. We can name a point N_2 such that N is between N_1 and N_2, and M_2 such that M is between M_1 and M_2. Regarding p as a transversal, the other pair of alternate interior angles is $M\hat{N}N_2$ and $N\hat{M}M_2$. If n is thought of as the transversal, then points P and M_3 need to be named as in the figure. $P\hat{N}N_2$ and $N_2\hat{A}M_1$ are a pair of alternate interior angles, and so are $M\hat{N}N_2$ and $N_2\hat{A}M_3$.

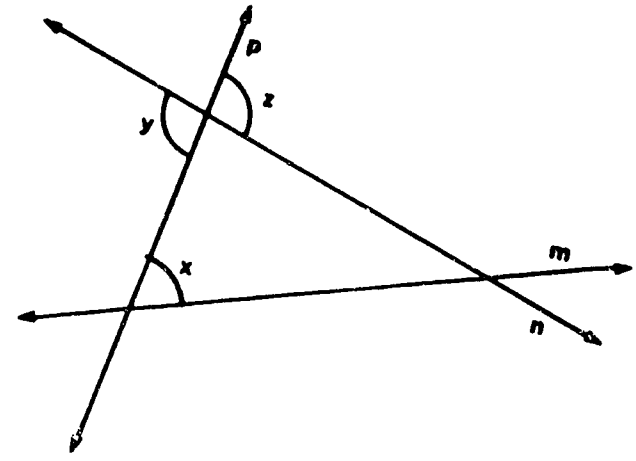


In the proof of Theorem 10.7, the essential point is that $\hat{x}_1$ and $\hat{y}_1$ are supplementary angles, and so are $\hat{x}_2$ and $\hat{y}_2$. Thus, $m(\hat{y}_1) = 180 - m(\hat{x}_1)$ and $m(\hat{y}_2) = 180 - m(\hat{x}_2)$. But $m(\hat{x}_1) = m(\hat{x}_2)$, since $\hat{x}_1 \equiv \hat{x}_2$, so $m(\hat{y}_1) = m(\hat{y}_2)$, and therefore $\hat{y}_1 \equiv \hat{y}_2$.

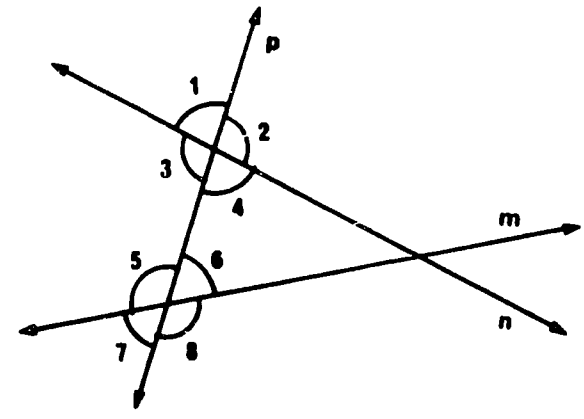
In the proof of Theorem 10.8, we know that the angles in either pair of alternate interior angles are congruent, by Theorem 10.7. Hence, we can consider either pair of alternate interior angles to obtain a contradiction with the exterior angle theorem.

You might ask the class why we did not consider the possibility that m and n intersect at a point of p. The answer is that if that happened, then p would not be a transversal of m and n.

1. In this figure, regard p as a transversal of lines m and n. One pair of alternate interior angles is marked x and y. $\hat{y}$ and $\hat{z}$ are vertically opposite angles, so $\hat{x}$ and $\hat{z}$ are a pair of corresponding angles. Without changing m, n or p, there are four possible ways to mark the angles according to the definition. They are tabulated here, with reference to the numbered figure.

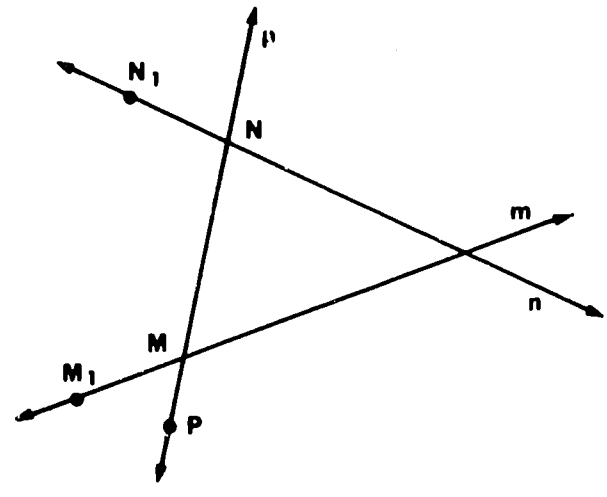


Alternate interior angles $\hat{x}$	$\hat{y}$	$\hat{z}$	Corresponding angles, $\hat{x}$ and $\hat{y}$
$\hat{6}$	$\hat{3}$	$\hat{2}$	$\hat{6}$, $\hat{2}$
$\hat{3}$	$\hat{6}$	$\hat{7}$	$\hat{3}$, $\hat{7}$
$\hat{5}$	$\hat{4}$	$\hat{1}$	$\hat{5}$, $\hat{1}$
$\hat{4}$	$\hat{5}$	$\hat{8}$	$\hat{4}$, $\hat{8}$



The students should be able to name the pairs of corresponding angles when m (or n) is regarded as a transversal of the other two lines.

2. Let the transversal p intersect coplanar lines m and n in points M and N respectively. Let M_1 and N_1 be points of m and n on the same side of p. If P is a point of p which is not on segment $\overline{MN}$, then $P\hat{M}M_1$

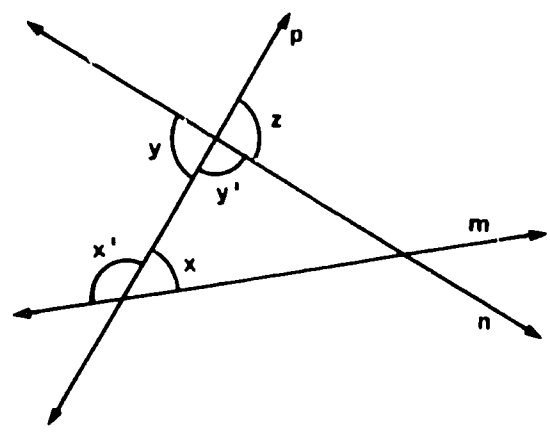


and $P\hat{N}N_1$ are a pair of corresponding angles.

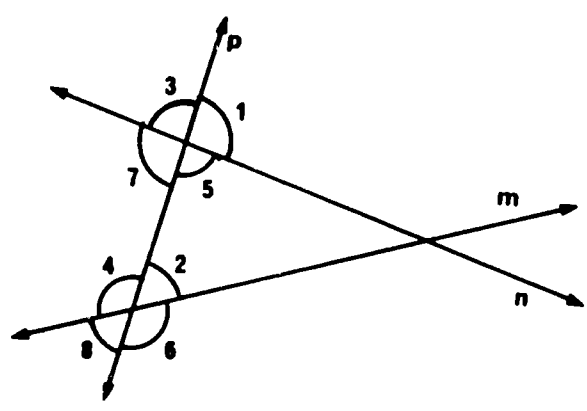
Again, there are four possibilities for the figure. P could have been chosen on the ray opposite to $\overrightarrow{NM}$ as well as on the ray opposite $\overrightarrow{MN}$ which is shown. For each choice of P, we can choose M_1, N_1 on either of the two sides of p.

3. THEOREM. If the angles in one pair of corresponding angles formed by a transversal to two coplanar lines are congruent, then the same is true for the other three pairs of corresponding angles.

Proof: The proof is based on Theorem 10.7, together with the fact that vertically opposite angles are congruent. Specifically, if $\hat{x}$ and $\hat{z}$ are the congruent pair of corresponding angles, then letting $\hat{y}$ be the angle vertically opposite to $\hat{z}$, we see that $\hat{y} \equiv \hat{x}$. Then by Theorem 10.7, $\hat{x}' \equiv \hat{y}'$. Then because of the congruence of vertically opposite angles, we see that the angles in each of the remaining three pairs of corresponding angles are congruent.



This theorem can also be proved without using Theorem 10.7, by basing the proof on consideration of pairs of supplementary angles. Thus, if $\hat{1} \equiv \hat{2}$ then $\hat{3} \equiv \hat{4}$, because $\hat{1}$ and $\hat{3}$ are supplementary and so are $\hat{2}$ and $\hat{4}$. Following the same reasoning, $\hat{5} \equiv \hat{6}$ and then $\hat{7} \equiv \hat{8}$.

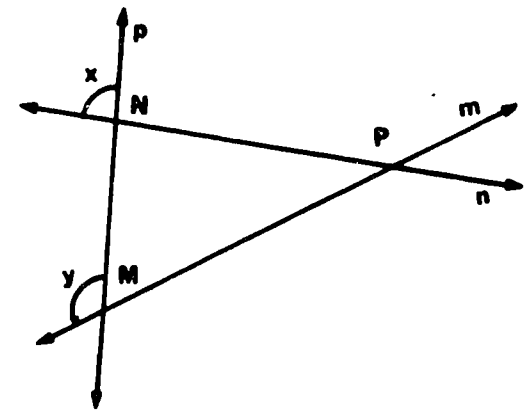


4. THEOREM. Let m and n be coplanar lines with a transversal p. If the angles in one pair of corresponding angles are con-

gruent, then m and n are parallel.

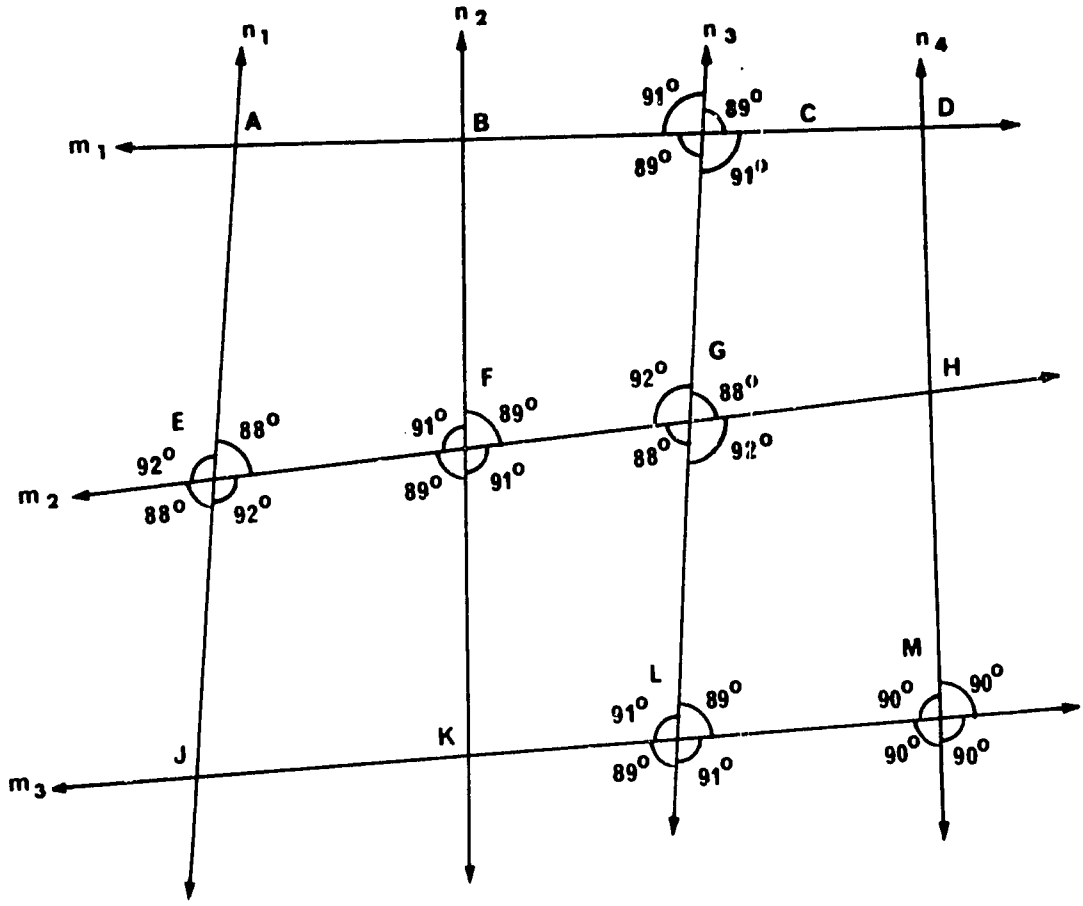
First Proof: In the first proof given for Problem 3, we concluded that the angles in a pair of alternate interior angles must be congruent. Thus, by Theorem 10.8, $m \parallel n$.

Second Proof: Suppose m and n intersect in a point P. The hypothesis is that the angles in one pair of corresponding angles are congruent. By Problem 3, the same is true of every pair of corresponding angles. Thus, $\hat{x} \equiv \hat{y}$. Referring to the figure, $\hat{x} \equiv M\hat{N}P$. But $\hat{y}$ is an exterior angle of $\triangle MNP$, and $M\hat{N}P$ is one of its remote interior angles. Therefore, the exterior angle theorem is contradicted, so m and n cannot intersect.



5. (See figure on next page.) The first step in the solution is to determine the measures of all angles with vertices at C, E, F, G, L and M. Congruence of vertical angles and the definition of supplementary angles will suffice for this. Regarding m_2 as a transversal of n_1 and n_3, we see that alternate interior angles (and also corresponding angles) are congruent. Therefore, $n_1 \parallel n_3$. Regarding n_3 as a transversal of m_1 and m_3, we again find congruence of alternate interior angles and conclude that $m_1 \parallel m_3$.

If we had proved the converse of Theorem 10.8, then from the fact that $n_1 \parallel n_3$ we could find the measures of the angles at A (the same as those at C) and at J (same as those at L). Likewise knowing that $m_1 \parallel m_3$, we would know that measures of angles at D were the same as those at M. This would leave only the angles at B and K unknown. If we also knew the angle-sum theorem for quadrilaterals (which states that the sum of the measures of the interior angles of a convex quadrilateral is 360), then we could find the measures of the angles at B and K. *However*, the proofs of both of these two theorems



(the converse of Theorem 10.8 and the angle-sum theorem) depend upon the uniqueness of parallel lines. Thus, they cannot be proved until the parallel postulate has been added to our list of assumptions.

In Problem 8 of Section 11-3, the students will be referred back to this problem to complete the determination of these angles.

Chapter 11
THE PARALLEL POSTULATE AND SOME CONSEQUENCES

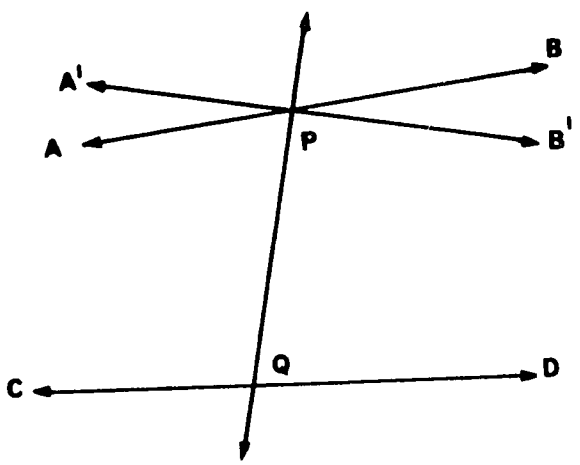
11-1 Introduction

A good introduction could be a discussion on the following lines: We reconsider Theorem 10.4 which states that there is at least one line parallel to a given line and containing a given point not on a given line. The fact that this theorem does not establish uniqueness must be emphasized. It is likely that some pupils may feel that the theorem actually includes uniqueness.

We may also revise Theorem 10.8 which states that if two coplanar lines are cut by a transversal, and if a pair of alternate interior angles are congruent then the lines are parallel. It seems reasonable that the converse of this theorem is true. We then examine what happens when we assume that the parallel through a point not on a given line is not unique. Consider this diagram. Suppose $\overleftrightarrow{AB}$ and $\overleftrightarrow{A'B'}$ are distinct lines through P parallel to $\overleftrightarrow{CD}$. Then, if the converse of Theorem 10.8 were true, we would have $A'\hat{P}Q \equiv P\hat{Q}D$,

and $A\hat{P}Q \equiv P\hat{Q}D$;

that is, $A'\hat{P}Q \equiv A\hat{P}Q$.



However this is impossible by the angle congruence postulates, so it seems that we cannot at the same time have the converse of

Theorem 10.8 and the existence of more than one parallel through the external point P.

11-2 The parallel postulate

With the parallel postulate it is now possible to prove Theorem 11.1. Point out in the proof how the uniqueness of parallels plays a crucial role.

The proofs given in the theorems or in the solutions are not the only ones acceptable. It is quite likely that some students will be able to produce different proofs from the ones given in the text. They should indeed be encouraged to do so provided their proofs are logically correct.

Answers to
PROBLEMS 11-2

Student Text Pages 166-168

1. If k does not intersect n then $k \parallel n$, since they are coplanar. But k and m intersect in a point P, and $m \parallel n$. Thus, through P there are two lines parallel to n, a contradiction of the parallel postulate.

2. Theorem 10.7 states that if angles in one pair of alternate interior angles are congruent the same is true of the other pair.

3. We use the diagram which appears beside Theorem 11.1 in the text.

   Corollary 11.2.

   To Prove: $X\hat{P}B \equiv P\hat{Q}D$.

   $A\hat{P}Q \equiv X\hat{P}B$ since they are vertically opposite angles.

   But from Theorem 11.1, $A\hat{P}Q \equiv P\hat{Q}D$. Therefore, $X\hat{P}B \equiv P\hat{Q}D$.

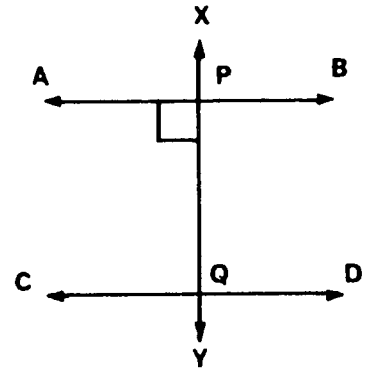
   Corollary 11.3.

   To Prove: $m(Q\hat{P}B) + m(P\hat{Q}D) = 180$.

   $m(A\hat{P}Q) + m(Q\hat{P}B) = m(A\hat{P}B) = 180$. But $A\hat{P}Q \equiv P\hat{Q}B$, so $m(Q\hat{P}B) + m(P\hat{Q}B) = 180$.

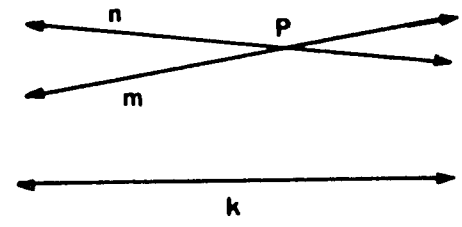
Corollary 11.4.

If $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ are the two parallel lines, the coplanar line $\overleftrightarrow{XY}$ will intersect them at P, Q respectively. (Here we are using the result of Problem 1.) If $\overleftrightarrow{XY}$ is perpendicular to $\overleftrightarrow{AB}$, then the interior angles at P are right angles. By Theorem 11.1 the interior angles at Q are also right angles, and $\overleftrightarrow{XY} \perp \overleftrightarrow{CD}$.

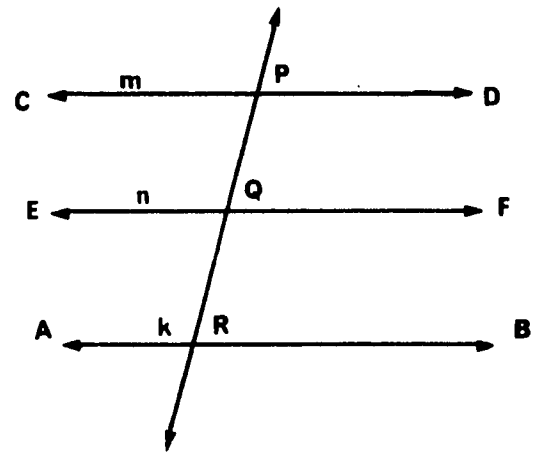


4. $\hat{1} \equiv \hat{3} \equiv \hat{5} \equiv \hat{7}$, so each of these has measure 120.
$\hat{2} \equiv \hat{4} \equiv \hat{6} \equiv \hat{8}$, and each of these has measure 60.

5. We assume m and n not parallel and so they will intersect at some point P. Then through P we have two lines, m and n, each parallel to k. This is impossible, so $m \parallel n$.



A second solution:
Consider the accompanying diagram. $\overleftrightarrow{XY}$ is a transversal to the three lines intersecting them at P, Q and R, respectively.

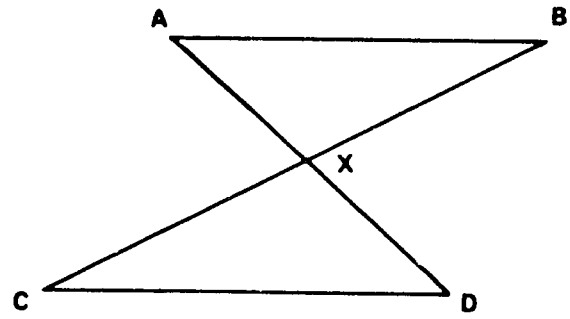


$$C\hat{P}R \equiv P\hat{R}B,$$
$$E\hat{Q}R \equiv P\hat{R}B,$$
$$E\hat{Q}R \equiv P\hat{Q}F.$$

Thus, $C\hat{P}Q \equiv P\hat{Q}F$,

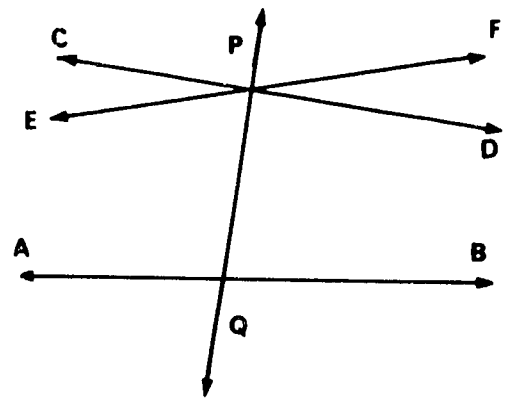
so m is parallel to n by the alternate interior angle theorem. This result is true if the three lines are not coplanar, but the proof in this case is not simple since it involves results connected with intersections of planes.

6. We conclude that $m = n$, for otherwise the parallel postulate is contradicted.



7. $B\hat{A}X \equiv X\hat{D}C$ and
$A\hat{B}X \equiv D\hat{C}X$.
So $\triangle AXB \equiv \triangle DXC$ by ASA,
and it follows that
$\overline{AX} \equiv \overline{DX}$ and $\overline{BX} \equiv \overline{CX}$.

8. We want to show that if we accept the alternate interior angle theorem, then we can prove the uniqueness of parallels.



Given: Line $\overleftrightarrow{AB}$ and point P not on $\overleftrightarrow{AB}$.

To prove that there is at most one line through P parallel to $\overleftrightarrow{AB}$.

We make an indirect proof.

We assume that there are two lines $\overleftrightarrow{CD}$, $\overleftrightarrow{EF}$ through P, both parallel to $\overleftrightarrow{AB}$. Then $C\hat{P}Q \equiv P\hat{Q}B$ and $E\hat{P}Q \equiv P\hat{Q}B$ by our assumed alternate interior angle property. Hence $C\hat{P}Q \equiv E\hat{P}Q$ which, by an angle congruence postulate, implies that $\overrightarrow{PC} = \overrightarrow{PE}$. This proves that $\overleftrightarrow{CD} = \overleftrightarrow{EF}$.

9. We must have $(3x - 10) + (4x - 20) = 180$, so $7x = 210$ or $x = 30$. Thus, $m(P\hat{Q}D) = 100$ and $m(A\hat{P}Q) = 100$.
Hence $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

10. We have $\hat{1} = \hat{5}$, $\hat{1} = \hat{9}$, and $\hat{5} = \widehat{13}$. By considering supplements and vertically opposite angles at the points of intersection we get

$m(\hat{1}) = m(\hat{3}) = m(\hat{5}) = m(\hat{7}) = m(\hat{9}) = m(\widehat{11}) = m(\widehat{13}) = m(\widehat{15}) = 80$,

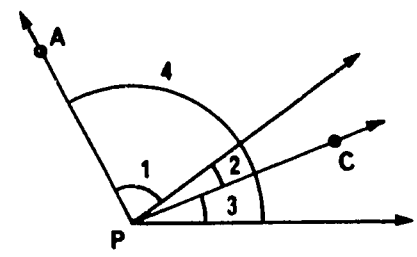
and

$m(\hat{2}) = m(\hat{4}) = m(\hat{6}) = m(\hat{8}) = m(\widehat{10}) = m(\widehat{12}) = m(\widehat{14}) = m(\widehat{16}) = 100$.

11. The four triangles are congruent. Consider $\triangle ADF$ and $\triangle FEC$. We see that $D\hat{A}F \equiv E\hat{F}C$, $A\hat{F}D \equiv F\hat{C}E$, and $\overline{AF} \equiv \overline{FC}$. Hence, $\triangle ADF \equiv \triangle FEC$ by ASA. Consider $\triangle DFE$ and $\triangle DBE$. We see that $B\hat{D}E \equiv F\hat{E}D$ and $B\hat{E}D \equiv F\hat{D}E$. Thus, $\triangle DBE \equiv \triangle EFD$, again by ASA. By similar reasoning, $\triangle EFD \equiv \triangle FEC$. Hence, $\triangle ADF \equiv \triangle DBE \equiv \triangle EDF = \triangle FEC$.

12. $m(\hat{1}) + m(\hat{2}) = m(\hat{3})$ by the second angle measurement postulate, which appears in Section 6-4. This problem and the next problem should serve as a revision of ideas needed in the proof of the angle-sum property for triangles.

13. $m(\hat{1}) + m(\hat{2}) = m(A\hat{P}C)$ and $m(A\hat{P}C) + m(\hat{3}) = m(\hat{4})$. Hence, $m(\hat{1}) + m(\hat{2}) + m(\hat{3}) = m(\hat{4})$. We need a result like this in the proof of the angle-sum property for triangles.



11-3 Angle-sum for triangles

The proof given here for Theorem 11.5 is essentially the same as given informally in Section 2-4. The principal difference is that now we have carefully stated postulates and carefully proved theorems which serve as a firm basis for the proof. It might be well at this point for the students to reread Sections 2-4, 2-8 and 2-9. By now they should have some appreciation for the logical structure of geometry and how this careful development makes many complicated properties depend upon a few simple ones.

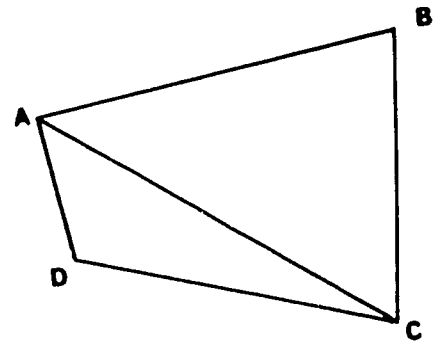
Answers to PROBLEMS 11-3

Student Text Pages 169-170

1. $\hat{B} \equiv D\hat{C}E$ and $\hat{A} \equiv A\hat{C}D$ because $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Thus, $m(\hat{A}) + m(\hat{B}) + m(A\hat{C}B) = m(A\hat{C}D) + m(D\hat{C}E) + m(A\hat{C}B) = 180$.

We also have shown that
$m(\hat{A}) + m(\hat{B}) = m(A\hat{C}D) + m(D\hat{C}E) = m(A\hat{C}E)$. This proves Corollary 11.6.

2. Corollary 11.7 can be proved in the same way this property was proved in Chapter 2. For the convex quadrilateral $ABCD$, draw diagonal $\overline{AC}$. The sum of the measures of the angles of $ABCD$ is the sum of measures of the angles in $\triangle ABC$ plus the angle-sum for $\triangle ACD$.



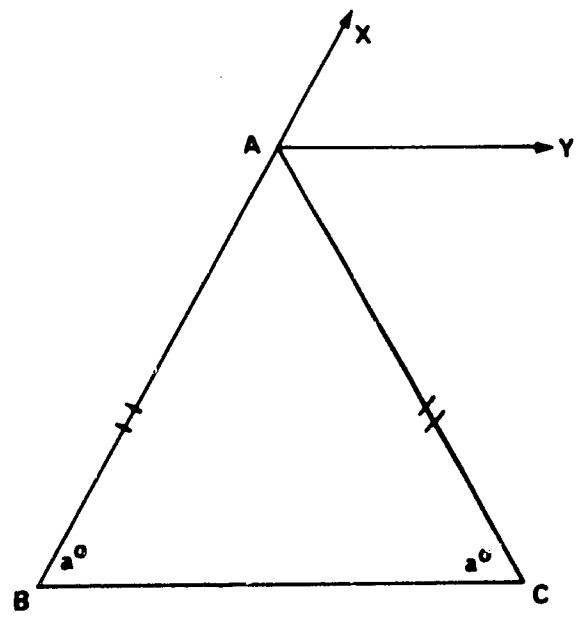
The students could be referred back to Section 2-5.

3. $2x + 3x + x = 180$, so $x = 30$. Thus, $m(\hat{B}) = 60$ and $m(\hat{C}) = 30$.

4. $(2x) + (4x - 18) + (6x - 37) + (3x - 20) = 360$ or $15x = 435$, so $x = 29$. Hence, $2x = 58$; $(4x - 18) = 98$; $(3x - 20) = 67$; and $(6x - 37) = 137$.

5. $m(A\hat{C}D) = 180 - [x + (3x - 21)] = 201 - 4x$.
$\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ provided $B\hat{A}C \equiv A\hat{C}D$, that is $(2x - 9) = 201 - 4x$.
Hence, $x = 35$.

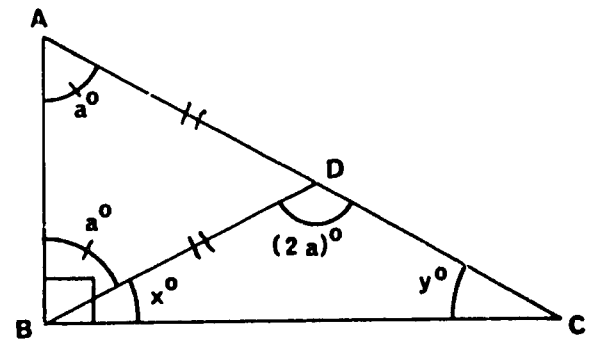
6. If $m(\hat{B}) = a$ then $m(\hat{C}) = a$ and $m(C\hat{A}X) = 2a$.
If $\overrightarrow{AY}$ is the bisector of $C\hat{A}X$ then $m(X\hat{A}Y) = a$. So $\overrightarrow{AY}$ is parallel to $\overline{BC}$ because corresponding angles are congruent.



7. Since $\overline{AD} \equiv \overline{BD}$, $B\hat{A}D \equiv D\hat{B}A$. If the common measure of these angles is a then $m(B\hat{D}C)$, an exterior angle of $\triangle ABD$, is $2a$. Let $m(D\hat{B}C) = x$ and $m(\hat{C}) = y$. Then $x + y + 2a = 180$ and $a + x = 90$. Hence $x = 90 - a$ or $(90 - a) = y + 2a = 180$, or $y = 90 - a$. Now we know that $D\hat{B}C \equiv \hat{C}$, so $\triangle BDC$ is isosceles.

Let us use this problem as a means of illustrating again how a figure, drawn by the student, can help immensely in the solution of a geometric problem.

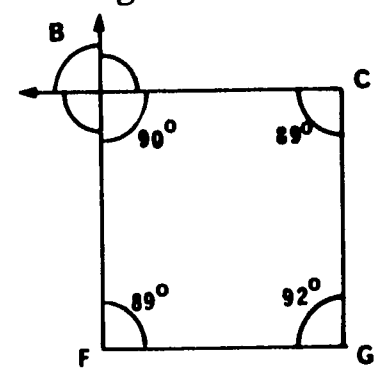
The figure in the text contains all the given information: $A\hat{B}C$ is a right angle and $\overline{AD} \equiv \overline{BD}$. Then mark $D\hat{A}C$ as congruent to $A\hat{B}D$ and label measures a, x and y as specified above. Now mark $B\hat{D}C$ as $(2a)^\circ$. Then the two equations are apparent:



$x + y + 2a = 180$ is the angle-sum for $\triangle BDC$ and $a + x = 90$ is the relation shown at vertex B. After we solve the equations to find $x = 90 - a = y$ we can mark $D\hat{B}C$ and $\hat{C}$ as congruent, and then we know that $\overline{BD} \equiv \overline{DC}$.

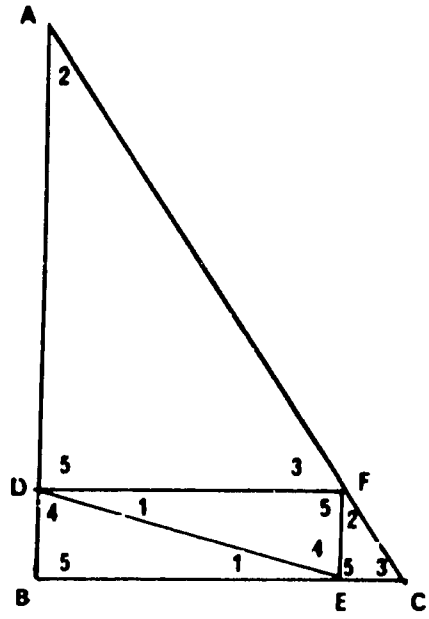
8. Referring to the sketch and solution shown for Problem 5, Section 10-2 of the TEACHERS' GUIDE, we know that $n_1 \parallel n_3$ by the alternate interior angle theorem. Thus, the angles at A are congruent to the corresponding angles at C and the angles at J are congruent to the corresponding angles at L. Similarly, m_1 is now known to be parallel to m_3, so the angles at D are congruent to those at M.

By considering quadrilateral $BCGE$ we find $m(C\hat{B}F) = 90$ since the angle-sum for a convex quadrilateral is 360. Thus, all four angles at B are right angles, and since $m_1 \parallel m_3$, the same is true for the four angles at K.



9. In the figure, the angles labelled 1 and 3 come from $\overline{DF} \parallel \overline{BC}$. From either $\overline{DF} \parallel \overline{BC}$ or $\overline{FE} \parallel \overline{AB}$ we get angles labelled 5. From $\overline{FE} \parallel \overline{AB}$ we get angles labelled 2 and 4.

We have no way of showing that $\overline{DE} \parallel \overline{AC}$, and, in fact, the figure shown here makes it clear that they are not parallel



10. The argument given above shows that $\hat{A} \equiv E\hat{F}C$, $B \equiv D\hat{F}E$ and $\hat{C} \equiv A\hat{F}D$. But clearly the sum of the measures of the three angles marked at F is 180.

11. Three possible figures are shown. In Figure 1, $m(\hat{P}) + 90 + m(\hat{B}) + 90 = 360$, because of the angle-sum property for quadrilateral $QPRB$. Hence, $\hat{B}$ and $\hat{P}$ are supplementary angles.

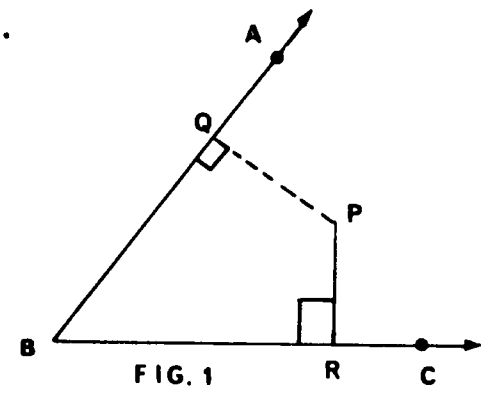


FIG. 1

In Figure 2, $m(\hat{P}) + m(P\hat{S}R) = 90$, $P\hat{S}R \equiv B\hat{S}Q$ and $m(S\hat{B}Q) + m(B\hat{S}Q) = 90$. Hence, $S\hat{B}Q \equiv \hat{P}$. But $A\hat{B}C$ and $S\hat{B}Q$ are supplementary, so $A\hat{B}C$ and $\hat{P}$ are supplementary.

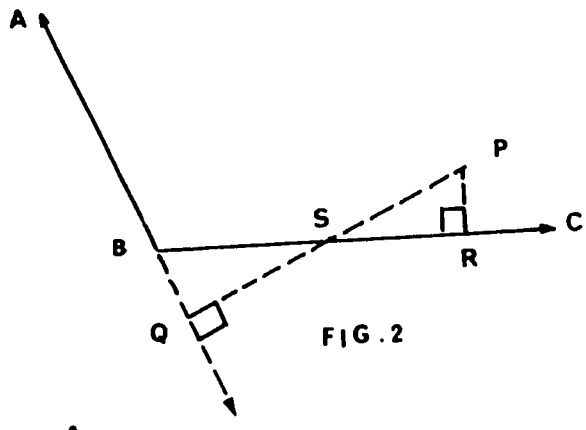


FIG. 2

In Figure 3, $m(\hat{P}) + m(P\hat{B}R) = 90$, so $m(\hat{P}) + m(A\hat{B}C) = 180$ and $\hat{P}$ and $A\hat{B}C$ are again seen to be supplementary.

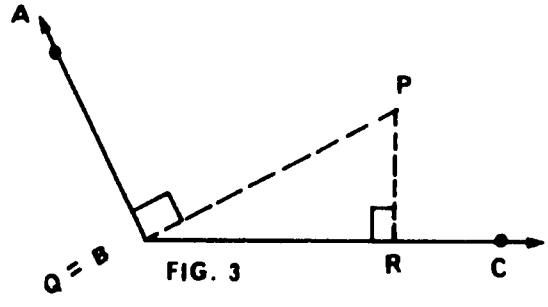


FIG. 3

12. In Figure 1, use the angle-sum for $BQPR$ to find that $m(\hat{P}) + m(R\hat{B}Q) = 180$. But $R\hat{B}Q \equiv A\hat{B}C$, so $\hat{P}$ and $A\hat{B}C$ are supplementary.

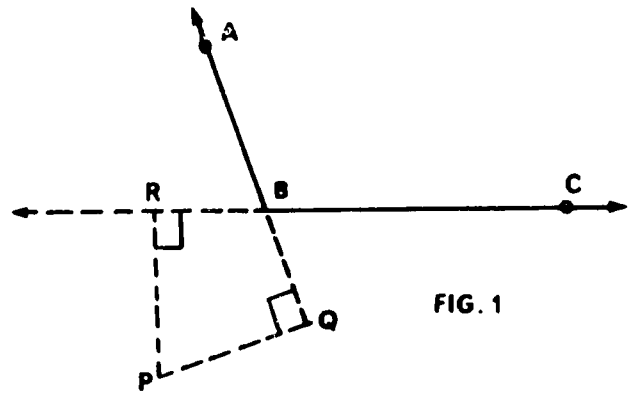


FIG. 1

For the case indicated by Figure 2, $m(\hat{P}) + m(R\hat{B}Q) = 180$ and $m(A\hat{B}C) + m(R\hat{B}Q) = 180$. Hence, in this case, $\hat{P} \equiv A\hat{B}C$.

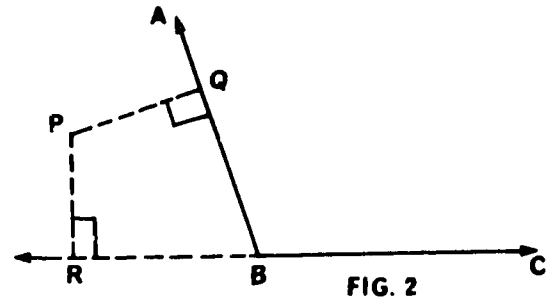


FIG. 2

In Figure 3, $\hat{P} \equiv R\hat{B}A$ since both are complements of $R\hat{B}P$. But, $R\hat{B}A$ and $A\hat{B}C$ are supplementary, so $\hat{P}$ and $A\hat{B}C$ are supplementary. Thus, the relation between $\hat{P}$ and $A\hat{B}C$ depends on where P is located in the exterior of $A\hat{B}C$.

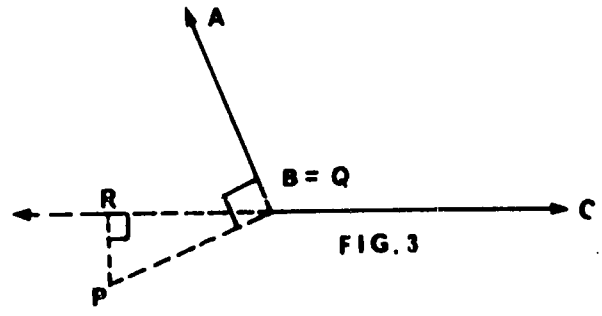
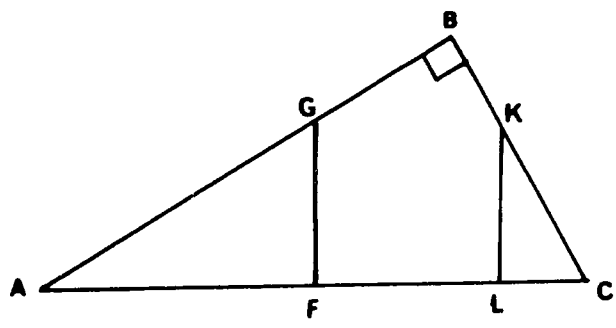


FIG. 3

13. In $\triangle ABC$ and $\triangle A'B'C'$, we have $\overline{AB} \equiv \overline{A'B'}$ and $\hat{A} \equiv \hat{A}'$ and $\hat{C} \equiv \hat{C}'$. Then $m(\hat{B}) = 180 - [m(\hat{A}) + m(\hat{C})] = 180 - [m(\hat{A}') + m(\hat{C}')] = m(\hat{B}')$. Thus, $\hat{B} \equiv \hat{B}'$ and the desired congruence follows from the ASA congruence theorem.

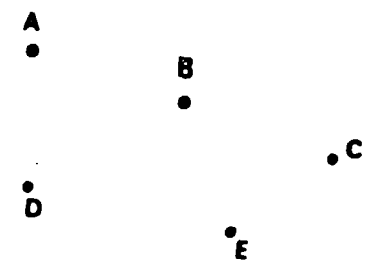
14. Since $\overline{GF} \perp \overline{AC}$ and $\overline{GF} \parallel \overline{KL}$, we know that $\overline{KL} \perp \overline{AC}$. Then $K\hat{L}C$ is a right angle. Hence, from $\triangle KLC$, $L\hat{K}C$ and $\hat{C}$ are complementary. From $\triangle ABC$, $\hat{A}$ and $\hat{C}$ are complementary. Therefore, $L\hat{K}C \equiv \hat{A}$.



11-4 Polygons

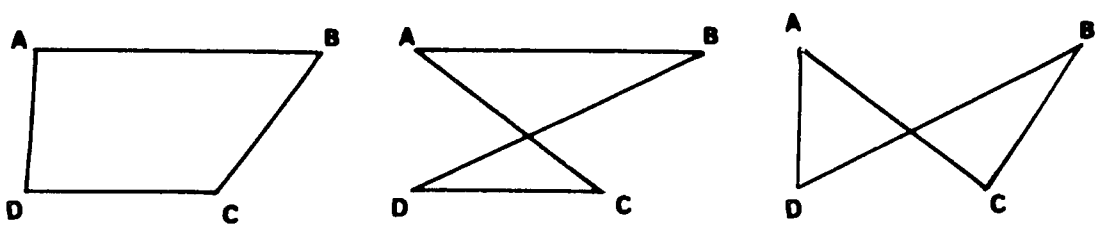
The definition of "polygon" depends on the simple concept "line segment". It might be well to point out that our definition of "line segment" depends on the betweenness properties postulated in Chapter 4. Those postulates are the logical basis which makes this definition more precise than the informal definitions given in *Secondary One* and *Secondary Two*.

Notice that the order in which the vertices appear in the name of a polygon is important. Thus, if points A, B and C are collinear (as in the figure), $ABCDE$ is not the name of a polygon, since three consecutive points are collinear. $ABDCE$ is a polygon (a pentagon). If the vertices are re-arranged in such a way that the cyclic order is maintained, we obtain a new name for the same polygon.



Thus $ABDCE$, $BDCEA$, $DCEAB$, $CEABD$ and $EABDC$ are names for the same pentagon. Also, if the order is reversed, the same polygon is named. That is, there are five additional names for pentagon $ABDCE$. They are $ECDBA$, $CDBAE$, $DBAEC$, $BAECD$ and $AECDB$. The names $ABCDE$, $CDEAB$ and $EDCBA$ are also equivalent. None of these three is the name for a polygon since points A, B, C are collinear. The students should be given plenty of practise with this idea in the form of board work or other class activity.

Mark four points on the board, named A, B, C and D. Ask for different quadrilaterals having those vertices to be drawn and named in all possible ways. If no three of the points are collinear, there will be three different quadrilaterals and each of these can be named in eight ways.



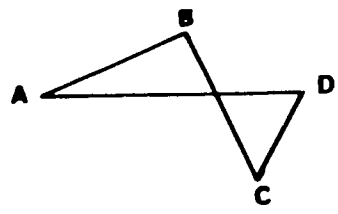
ABCD *ABDC* *ACBD*

If three of the four points are collinear, then there is no quadrilateral having those four points as vertices.

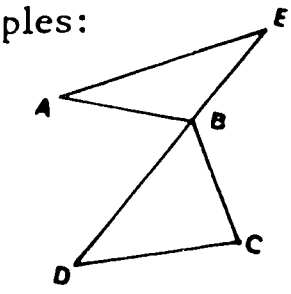
For five points, no three of which are collinear, there will be twelve different pentagons, each of which can be named in ten different ways. If the students seem to be interested in such counting problems, they can devote some time to these with considerable profit. A general analysis of these counting problems depends on some knowledge of combinations and permutations, but for cases to at most six points, some students may be able to find the answers by common-sense reasoning.

It should always be kept in mind that a polygon is a plane figure; that is, the vertices are points lying in the same plane.

The concepts of *simpleness* and *convexity* for polygons can be clarified by class activity. Roughly speaking, a non-simple polygon is one which crosses itself. Some examples:

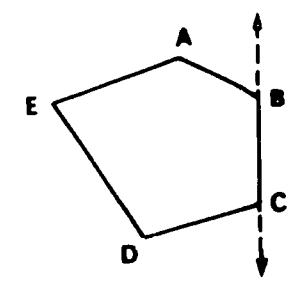


ABCD is not simple.

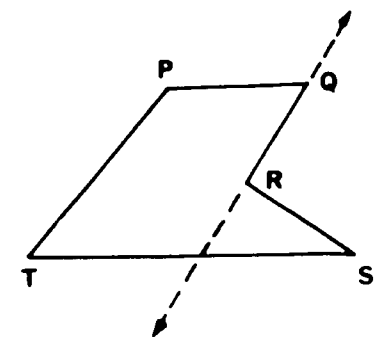


ABCDE is not simple.

There are several equivalent ways of defining convexity for polygons. It is not always easy to prove that these different definitions are equivalent, so only one is given in the text. We list alternate definitions which you may wish to use informally in the classroom as an aid to clarifying the idea of convexity.



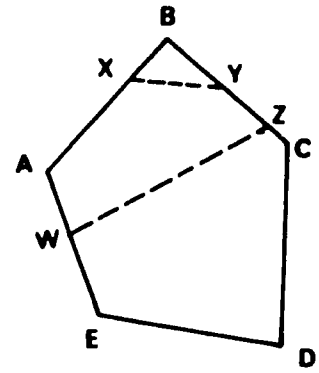
A polygon is convex if for any side $\overline{XY}$ there are vertices of the polygon in only one of the half-planes determined by $\overleftrightarrow{XY}$. Considering side $\overline{BC}$ we see that all vertices of the pentagon are in only one of the half-planes with edge $\overleftrightarrow{BC}$. The same is true for the other four sides.



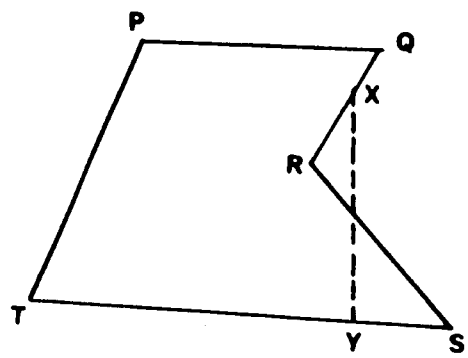
Line $\overleftrightarrow{QR}$ determines two half-planes, and vertex S is in one half-plane and vertices P and T are in the other. This pentagon $PQRST$ is not convex.

A polygon is convex if no segment whose endpoints are on distinct sides of the polygon contains a third point of the polygon.

Pentagon $ABCDE$ is convex. Two segments $\overline{XY}$ and $\overline{WZ}$ are shown. Of course, infinitely many segments of this sort would have to be considered to prove that the pentagon is convex. However, it is clear from the figure that all such segments will have the desired property.

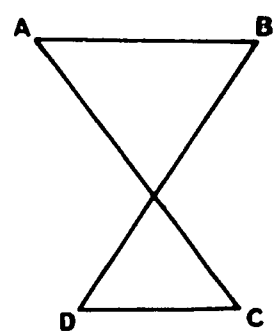


Segment $\overline{XY}$ which contains three points of the pentagon shows that the pentagon $PQRST$ is not convex. Consideration of this one segment $\overline{XY}$ is enough to *prove* that $PQRST$ is not convex.



The adjectives, *adjacent* and *opposite*, applied to quadrilaterals are self-explanatory. The pictures drawn on the board to illustrate these concepts should include some non-convex quadrilaterals.

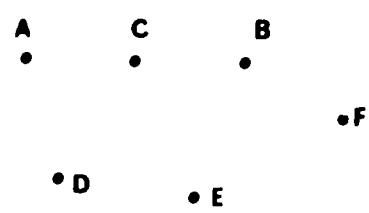
For quadrilateral $ABDC$, vertices A and D are opposite, as are B and C. Pairs of adjacent vertices are A and B, B and D, D and C, C and A. Pairs of opposite sides are $\overline{AB}$ and $\overline{DC}$, $\overline{AC}$ and $\overline{BD}$. Pairs of adjacent sides are $\overline{AB}$ and $\overline{AC}$, $\overline{AB}$ and $\overline{BD}$, $\overline{BD}$ and $\overline{DC}$, $\overline{DC}$ and $\overline{CA}$.



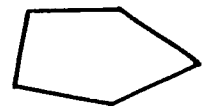
Answers to
PROBLEMS 11-4

Student Text Page 174

1. If A, B and C are collinear, then we must not let A, B and C be consecutive vertices. $ACDBEF$ will do.

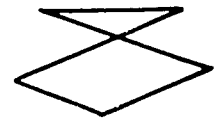


2. Convex and simple:



Non-convex and simple:

Non-convex and non-simple:



It can be proved that every convex polygon is simple, but the proof is by no means simple!

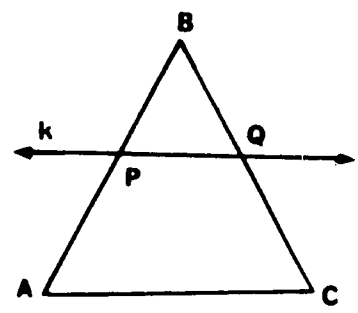
3. Pairs of opposite sides: $\overline{PC}$ and $\overline{XM}$, $\overline{PX}$ and $\overline{MC}$.

Pairs of adjacent sides: $\overline{PC}$ and $\overline{CM}$, $\overline{CM}$ and $\overline{MX}$, $\overline{MX}$ and $\overline{XP}$, $\overline{XP}$ and $\overline{PC}$.

The diagonals join opposite vertices. They are $\overline{PM}$ and $\overline{XC}$.

More problems like Problem 3 should be given as class activity.

4. Suppose k intersects $\overline{AB}$ in P and $\overline{BC}$ in Q where P and Q are not vertices. Then P is between A and B, so A and B are on opposite sides of k. Similarly B and C are on opposite sides of k since Q is between B and C. Thus, A and C must be on the same side of k, and this means that there is no point of k between A and C. Since k does not contain A or C, k does not intersect $\overline{AC}$. This argument applies (with appropriate changes in letters) to any line which intersects two of the sides in points which are not vertices. Hence $\triangle ABC$ is convex.



A student who attempts this problem should be advised to review the separation postulates, Section 4-3.

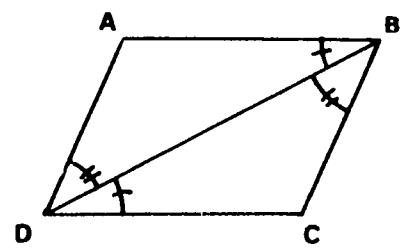
5. There are eight names for quadrilateral $ABCD$. There are two other different quadrilaterals ($ACDB$ and $ABDC$) with the given points as vertices. Each of these has eight names.

11-5 Parallelograms

The agreement that *polygon* used henceforth is to mean "simple, convex polygon" is one of convenience. Occasionally you should propose non-convex or non-simple polygons as a reminder of this agreement.

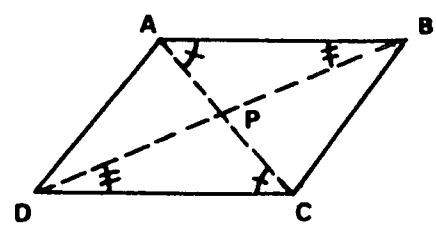
The proofs of the three parts of Theorem 11.12 could be worked out as class activity, with one or two students recording the steps and drawing the figure on the board while the rest of the class make suggestions. Here are the details:

(a) By the alternate interior angle theorem the angles marked are congruent in pairs. Thus, $\triangle ABD \equiv \triangle CDB$ by ASA. Hence $\overline{AB} \equiv \overline{CD}$ and $\overline{AD} \equiv \overline{CB}$.



(b) Also from the congruence of $\triangle ABD$ and $\triangle CDB$, $\hat{A} \equiv \hat{C}$. By the angle measurement postulates, $A\hat{B}C \equiv A\hat{D}C$.

(c) Let P be the intersection of the diagonals. $\overline{AB} \equiv \overline{DC}$ from part (a), and the angles marked are congruent by pairs. Hence, $\triangle ABP \equiv \triangle CDP$ so $\overline{DP} \equiv \overline{BP}$ and P is the midpoint of $\overline{DB}$. Similarly, P is the midpoint of $\overline{AC}$.



Answers to
PROBLEMS 11-5

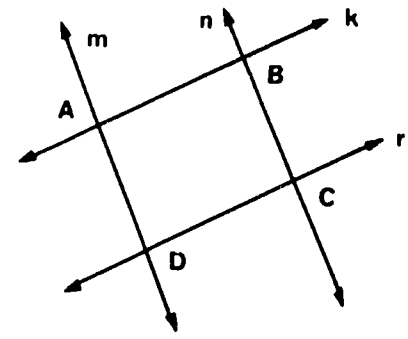
Student Text Page 176

1. See the discussion above.

2. The point of this problem is that the definition of parallelogram

could have been stated without the agreement that *quadrilateral* means "convex and simple quadrilateral".

Let m and n be parallel lines containing two of the sides. Let k be a line containing a third side. Then k must intersect one of m and n, and hence it intersects both. These points of intersection must be vertices, call them A and B. The fourth side is in a line parallel to k, call it r. It is intersected by n. (We are using the fact that if n intersects one of two parallel lines it also intersects the other.) By the same reasoning, m intersects r. Call these points C and D. Since the sides are contained in the four lines m, n, k and r, the quadrilateral must be $ABCD$. Non-adjacent sides cannot intersect since they lie in parallel lines.

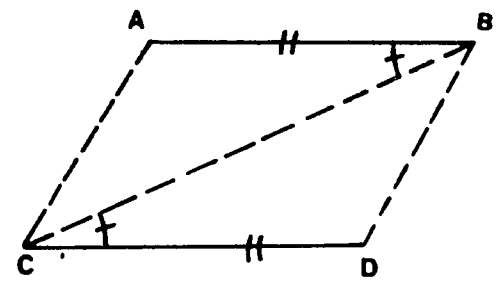


3. *Given*: $\overline{AB} \parallel \overline{CD}$ and $\overline{AB} \equiv \overline{CD}$.

Since the quadrilateral is simple it must be $ABDC$.

To Prove: $\overline{AC} \parallel \overline{BD}$.

Consider diagonal $\overline{BC}$.



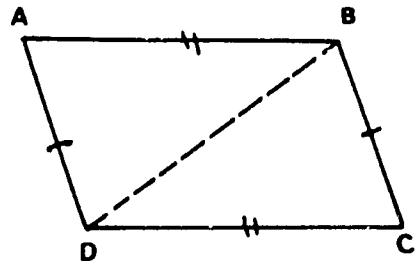
Statements	*Reasons*
a. $A\hat{B}C \equiv B\hat{C}D$.	Alternate interior angle theorem
b. $\triangle ABC \equiv \triangle DCB$.	SAS postulate
c. $A\hat{C}B \equiv D\hat{B}C$.	Step b
d. $\overline{AC} \parallel \overline{BD}$.	Alternate interior angle theorem

The quadrilateral $ABCD$ is non-simple, but it has a pair of sides ($\overline{AB}$ and $\overline{CD}$) parallel and congruent. Thus, the agreement concerning *simpleness* is necessary here.

4. *Given*: $\overline{AB} \equiv \overline{CD}$ and
$\overline{AD} \equiv \overline{BC}$.

To Prove: $\overline{AB} \parallel \overline{CD}$ and
$\overline{AD} \parallel \overline{BC}$.

Consider diagonal $\overline{BD}$.



Statements	*Reasons*
a. $\triangle ABD \equiv \triangle CDB$.	SSS
b. $A\hat{B}D \equiv C\hat{D}B$.	Step a
c. $\overline{AB} \parallel \overline{CD}$.	Alternate interior angle theorem.
d. $A\hat{D}B \equiv C\hat{B}D$.	Step a
e. $\overline{AD} \parallel \overline{BC}$.	Alternate interior angle theorem.

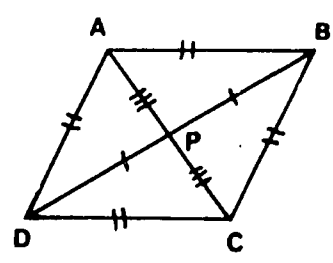
11-6 Some special parallelograms

If in parallelogram $ABCD$ we have $\overline{AB} \equiv \overline{BC}$, then all sides are congruent (and we have a rhombus) since $\overline{AB} \equiv \overline{CD}$ and $\overline{BC} \equiv \overline{AD}$ for all parallelograms $ABCD$. Likewise, if $\hat{A} \equiv \hat{B}$, then all angles are congruent since for all parallelograms $ABCD$, $\hat{A} \equiv \hat{C}$ and $\hat{B} \equiv \hat{D}$. Since the angle sum for a quadrilateral is 360, each of the angles has measure 90, and the figure is a rectangle.

Answers to
PROBLEMS 11-6

Student Text Pages 177-178

1. a. Since the diagonals of any parallelogram bisect each other, we need only prove that the diagonals of a rhombus are perpendicular. We have $\triangle ABP \equiv \triangle ADP$ by SSS. Hence $A\hat{P}B \equiv A\hat{P}D$. But $A\hat{P}B$ and $A\hat{P}D$ are supplementary, so $\overline{AC} \perp \overline{DB}$.



b. From the triangle congruence above, $A\hat{B}P \equiv A\hat{D}P$. But $D\hat{B}C \equiv A\hat{D}P$ by the alternate interior angle theorem so $A\hat{B}D \equiv D\hat{B}C$ and $A\hat{B}C$ is bisected by $\overline{BD}$.

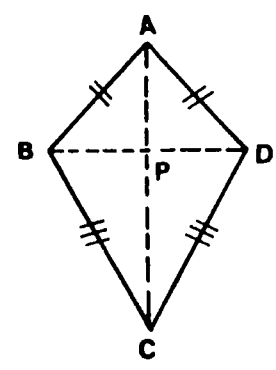
2. a. True, because of Problem 1 and the fact that a square is a rhombus
   b. True, for the same reason

3. a. Yes, by definition
   b. No. Consider the figure:
   c. All sides of a rhombus are congruent.
   d. All angles of a rectangle are right angles.
   e. No
   f. Yes
   g. Yes
   h. No
   i. Yes
   j. Yes
   k. No
   l. No
   m. No
   n. Yes, a square is an example.
   o. Congruent angles
   p. Congruent sides

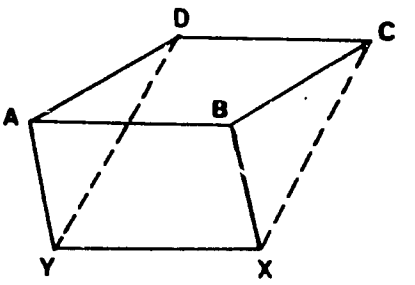
4. A rhombus is a quadrilateral with all sides congruent. A rectangle is a quadrilateral with all angles congruent.

5. $\overline{AC} \perp \overline{BD}$ and $\overline{BP} \equiv \overline{PD}$.
   Also, $\overline{AC}$ bisects $B\hat{A}D$ and $B\hat{C}D$.

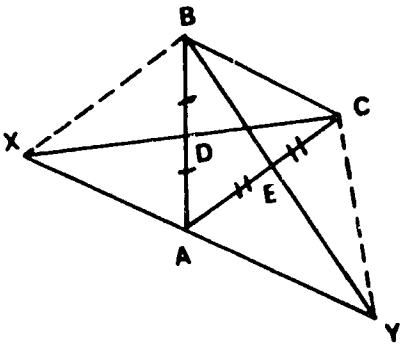
   Proof: Since A is equidistant from B and D, A is on the perpendicular bisector of $\overline{BD}$. The same is true for C. Thus, $\overline{AC} \perp \overline{BD}$ and $\overline{BP} \equiv \overline{PD}$.
   $A\hat{B}P \equiv A\hat{D}P$ since $\triangle ABD$ is isosceles, so by SAS $\triangle ABP \equiv \triangle ADP$. Thus, $D\hat{A}P \equiv B\hat{A}P$ and $\overrightarrow{AC}$ bisects $B\hat{A}D$. Similarly, one can show $B\hat{C}P \equiv D\hat{C}P$.



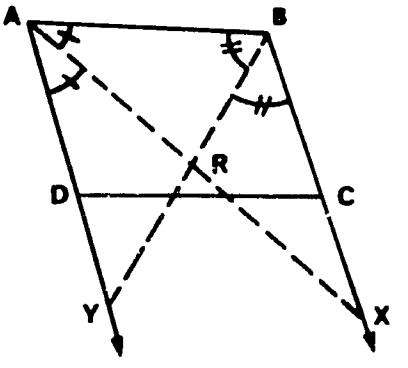
6. Since $\overline{DC}$ and $\overline{YX}$ are each parallel to $\overline{AB}$ they are parallel to each other, unless, of course, C, D, X and Y are collinear. Also, $\overline{DC} \equiv \overline{AB} \equiv \overline{YX}$, so if $DCXY$ is a quadrilateral it has a pair of opposite sides congruent and parallel, and therefore, it is a parallelogram.



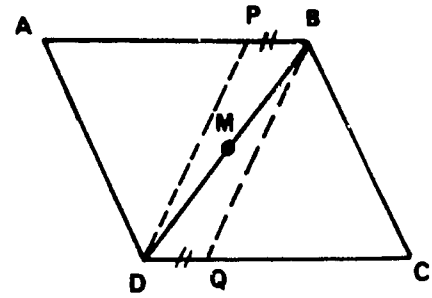
7. $ACBX$ is a quadrilateral whose diagonals bisect each other; therefore, it is a parallelogram. Thus $\overline{AX} \parallel \overline{BC}$ and $\overline{AY} \parallel \overline{BC}$. But through A there is only one line parallel to $\overline{BC}$, so X, Y and A must be collinear.



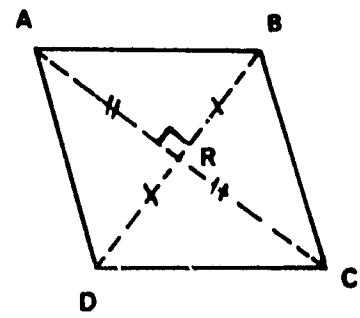
8. Let R be the intersection of $\overline{AX}$ and $\overline{BY}$. Then $m(R\hat{A}B) + m(R\hat{B}A) = \frac{1}{2}[m(D\hat{A}B) + m(C\hat{B}A)] = 90$, since $D\hat{A}B$ and $C\hat{B}A$ are supplementary. Thus, $A\hat{R}B$ must be a right angle, since the sum of the measures of the other two angles in $\Delta\, ABR$ is 90. Thus all angles at R are right angles. Since $\overline{AR}$ bisects $Y\hat{A}B$, we see that $\Delta\, BAR \equiv \Delta\, YAR$ by ASA. Thus, $\overline{AY} \equiv \overline{AB}$. Similarly $\overline{BX} \equiv \overline{AB}$, so $ABXY$ has the sides in one pair of opposite sides congruent and parallel. Thus, $ABXY$ is a parallelogram. Since a pair of adjacent sides are congruent it is also a rhombus.



9. $PBQD$ is a parallelogram since $\overline{PB}$ and $\overline{DQ}$ are both congruent and parallel. Now let M be the midpoint of $\overline{DB}$. M is also the midpoint of $\overline{PQ}$ (since the diagonals of parallelogram $PBQD$ bisect each other). For a similar reason, M is the midpoint of $\overline{AC}$.



10. First, since the diagonals bisect each other the quadrilateral is a parallelogram $ABCD$. $\overline{AC} \perp \overline{BD}$ by hypothesis. Thus, $\triangle ARB \equiv \triangle ARD$ by SAS. This proves that $\overline{AB} \equiv \overline{AD}$, which means that all four sides are congruent.



Chapter 12
TRANSVERSAL TO MANY PARALLEL LINES

12-1 Parallel lines in a triangle

The chapter opens with *midpoint theorems* relating to the triangle.

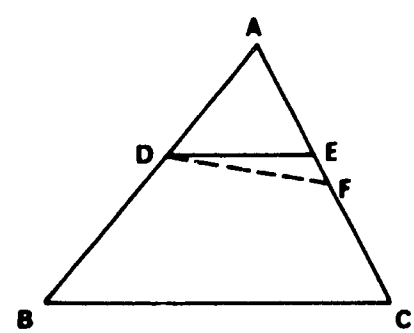
The suggested Class Activity brings out the theorem and its related theorem as clearly as possible before giving the formal proof of the main theorem and a series of leading questions suggesting the proof of the related theorem.

The details of the proof of Theorem 12.2 are as follows:

1. $BDFC$ is a parallelogram, since opposite sides are parallel.
2. $\overline{CF} \equiv \overline{BD}$ from step 1.
3. $\overline{BD} \equiv \overline{DA}$ since D is the midpoint of $\overline{AB}$.
4. $\overline{DA} \equiv \overline{CF}$ from steps 2 and 3.
5. $\triangle ADE \equiv \triangle CFE$ by AAS. We have $A\hat{D}E \equiv C\hat{F}E$ by the alternate interior angle theorem, and $A\hat{E}D \equiv C\hat{E}F$ since they are vertically opposite.
6. $\overline{AE} \equiv \overline{CE}$ from step 5.

There are other ways of looking at Theorem 12.2. We can make direct use of the parallel postulate to obtain another proof.

D is the midpoint of $\overline{AB}$ in triangle ABC. The line through D drawn parallel to $\overline{BC}$ cuts $\overline{AC}$ at E. We *suppose* E is not the midpoint of $\overline{AC}$ and also suppose F is the midpoint of $\overline{AC}$. From Theorem 12.1, $\overline{DF} \parallel \overline{BC}$. By hypothesis, $\overline{DE} \parallel \overline{BC}$. By the parallel postulate, $\overleftrightarrow{DE} = \overleftrightarrow{DF}$ so $E = F$. Thus E is the midpoint.



If the students are urged to look for their own proof of Theorem 12.2 before reading the questions in the text, some of them may discover this easier proof.

12-2 Parallel lines in a trapezium

Here are details for the proof of Theorem 12.3:

1, 2, 3. Applying Theorem 12.1 to $\triangle ADB$ we see that $PK = \frac{1}{2}AB$ and $\overline{PK} \parallel \overline{AB}$.

4, 5. The same theorem used with $\triangle BCD$ tells us that $KQ = \frac{1}{2}CD$ and $\overline{KQ} \parallel \overline{CD}$.

6, 7. $\overline{KQ} \parallel \overline{DC}$ and $\overline{DC} \parallel \overline{AB}$. Since $\overleftrightarrow{KQ}$ is distinct from $\overleftrightarrow{AB}$, $\overline{KQ} \parallel \overline{AB}$. Also $\overline{PK} \parallel \overline{AB}$ from step 3.

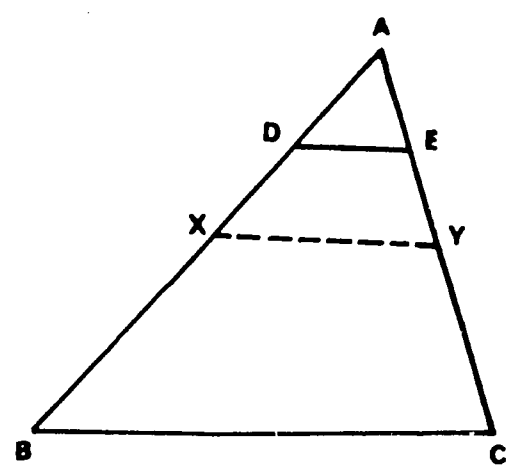
8, 9. Since there is at most one line through K parallel to $\overleftrightarrow{AB}$, it must be that P, K and Q are collinear.

If four parallel lines intercept three congruent segments on some transversal, then they do so on any transversal. This can be proved by two applications of Theorem 12.5.

Answers to
PROBLEMS 12-2

Student Text Pages 184-187

1. This is a simple problem which can be solved by constructing *one* line. If any students have difficulty with it, it may help them in doing other problems if they are directed into finding the construction so that they have in the figure a situation that will enable them to use the theorems just

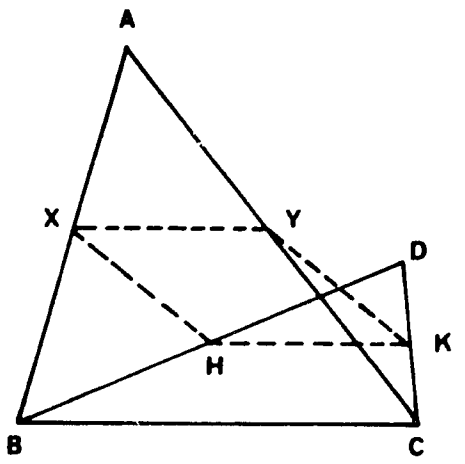


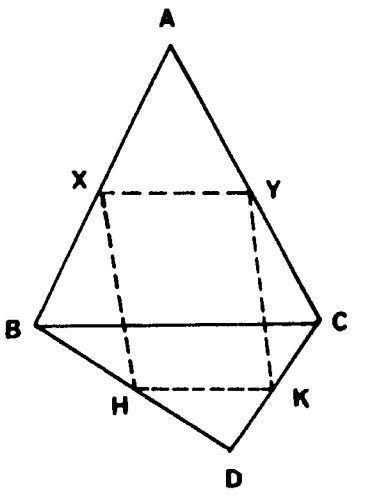
learned. The construction is to join the midpoints X, Y of $\overline{AB}$, $\overline{AC}$, respectively. Then apply the midpoint theorem first to triangle ABC and then to triangle AXY. We have

$$XY = \frac{1}{2}BC \text{ and } DE = \frac{1}{2}XY, \text{ so } DE = \frac{1}{2}(XY) = \frac{1}{2}\left(\frac{1}{2}BC\right),$$

that is, $DE = \frac{1}{4}BC$. Also $\overline{XY} \parallel \overline{BC}$ and $\overline{DE} \parallel \overline{XY}$. Since $\overline{DE}$ is distinct from $\overline{BC}$, it follows that $\overline{DE} \parallel \overline{BC}$.

2. This problem is quite interesting. It is possible to make quite a number of diagrams that will apply. The points A and D may lie on the same side of $\overleftrightarrow{BC}$, as in the first figure. The vertices,





A and D, may also lie on opposite sides of $\overleftrightarrow{BC}$ as in the second figure. In either case the proof is the same. We apply the midpoint theorem to triangle ABC and triangle DBC to get

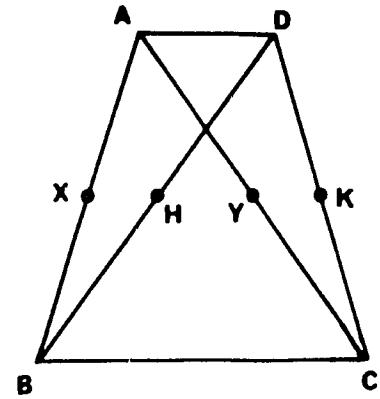
$$XY = \frac{1}{2}BC \text{ and } \overline{XY} \parallel \overline{BC},$$

$$\text{and } HK = \frac{1}{2}BC \text{ and } \overline{HK} \parallel \overline{BC},$$

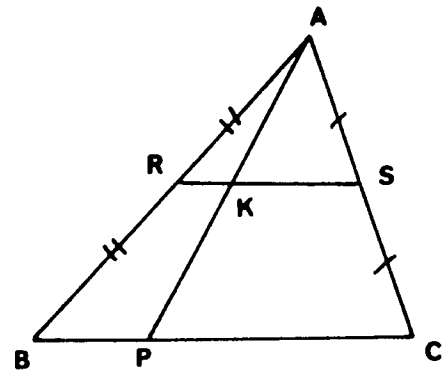
$$\text{so that } XY = HK \text{ and } \overline{XY} \parallel \overline{HK}.$$

It follows that $XYKH$ is a parallelogram.

If $\overline{AD}$ is parallel to $\overline{BC}$, then $ABCD$ is a trapezium. In this case the points X, Y, H, K will be collinear and we do not get a parallelogram. In fact, the four points X, Y, H, K lie on the median of the trapezium.

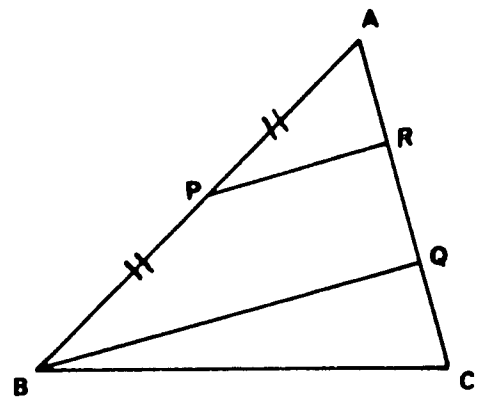


3. There is no construction necessary for this. We apply the midpoint theorem at once to the triangle ABC to get $\overline{RS} \parallel \overline{BC}$. The related theorem is then applied to the triangle ABP to show that K is the midpoint of $\overline{AP}$; that is, $\overline{RS}$ bisects $\overline{AP}$.

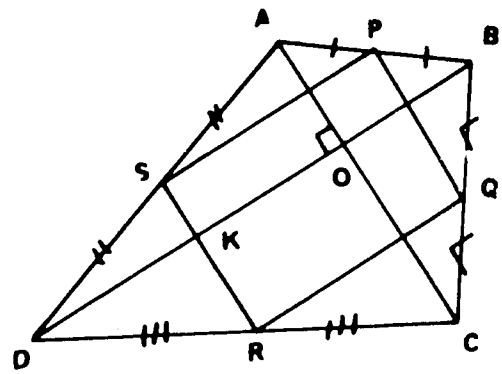


4. We apply the midpoint theorem to triangle ABQ to find $\overline{AR} \equiv \overline{RQ}$. And so from the data given,

$$QC = \frac{1}{2}AQ = AR.$$



5. In this problem we are going to apply the midpoint theorem and the theorems relating to parallel lines cut by a transversal with its associated angles: alternate interior and corresponding angles. We have for the triangle ABC, $\overline{PQ} \parallel \overline{AC}$ and

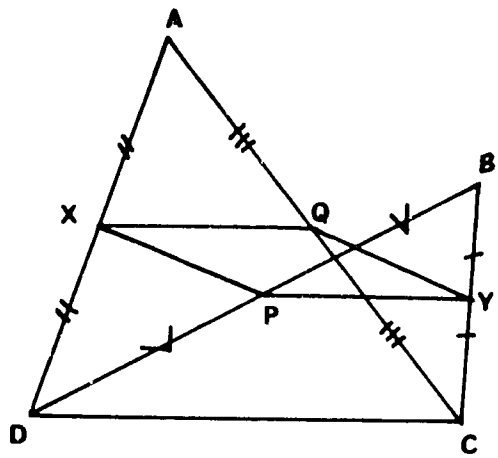


$PQ = \frac{1}{2}AC$. In the triangle DAC, $\overline{SR} \parallel \overline{AC}$ and $SR = \frac{1}{2}AC$. So $\overline{SR} \parallel \overline{PQ}$ and $SR = PQ$. It follows that $PQRS$ is a paralelogram. Now $A\hat{O}K \equiv O\hat{K}R$, since $\overline{AC} \parallel \overline{SR}$. Thus, $O\hat{K}R$ is a

right angle and $O\hat{K}S \equiv O\hat{K}R$. By the midpoint theorem, $\overline{RQ} \parallel \overline{BD}$, so $S\hat{K}B \equiv S\hat{R}Q$. Thus, $PQRS$ is a parallelogram in which one angle is a right angle. Therefore, it is a rectangle.

6. Referring to the figure for Problem 5, we see that the midpoint theorem (used in $\triangle ABC$ and in $\triangle ACD$) gives $\overline{PQ} \parallel \overline{SR}$. Similarly, $\overline{PS} \parallel \overline{QR}$.

7. Problem 7 is very much like Problem 2, for X, P, Y, Q would be collinear if the quadrilateral $ABCD$ were a trapezium. Treating the general case, we find from $\triangle ADC$, that



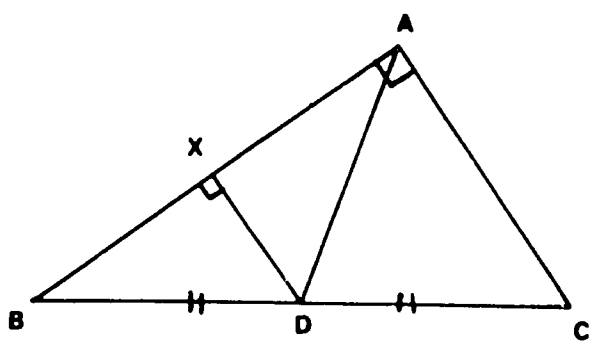
$XQ = \frac{1}{2} CD$ and $\overline{XQ} \parallel \overline{DC}$.

In $\triangle BDC$,

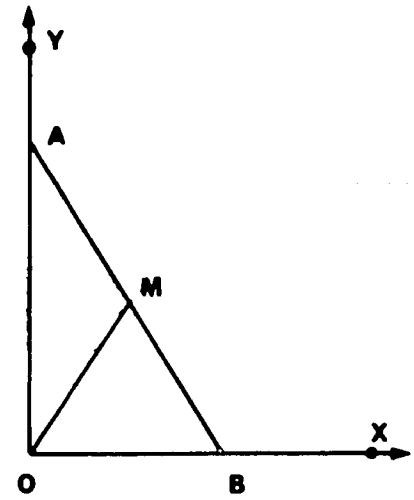
$PY = \frac{1}{2} DC$ and $\overline{PY} \parallel \overline{DC}$.

So $XQ = PY$ and $\overline{XQ} \parallel \overline{PY}$. Thus, $XPYQ$ is a parallelogram.

8. In this problem we apply the theorem related to the midpoint theorem. The construction done here enables to use the theorem we know. In $\triangle ABC$, $\overline{BD} \equiv \overline{DC}$, $\overline{DX} \parallel \overline{CA}$, so $\overline{BX} \equiv \overline{XA}$. We now prove $\triangle DXA \equiv \triangle DXB$ in order to get $\overline{DA} \equiv \overline{DB}$. These triangles are congruent by the SAS postulate, for $\overline{DX} \equiv \overline{DX}$, $D\hat{X}A \equiv D\hat{X}B$, and $\overline{XA} \equiv \overline{XB}$. Having proved the triangles congruent, we know that $\overline{DA} \equiv \overline{DB} \equiv \overline{DC}$.



9. The students may need some explanation of the term *locus*. The question could be rephrased as follows: Describe the collection of points M such that M is the midpoint of a segment of fixed length, with one endpoint on $\overrightarrow{OY}$ and the other on $\overrightarrow{OX}$.



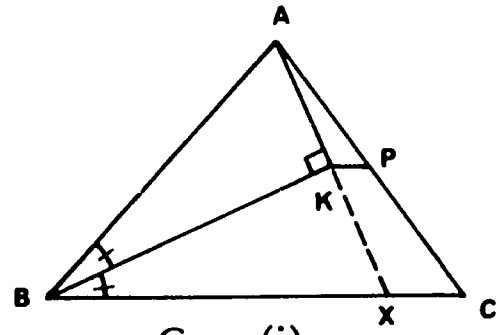
From Problem 8, we see that $OM = \frac{1}{2}AB$; that is, OM is the same for all positions of the ladder. Thus, M is always on a circle of radius $\frac{1}{2}AB$ with its centre at O.

10. Three cases have to be considered.

(i) K within $\triangle ABC$.

(ii) K outside $\triangle ABC$.

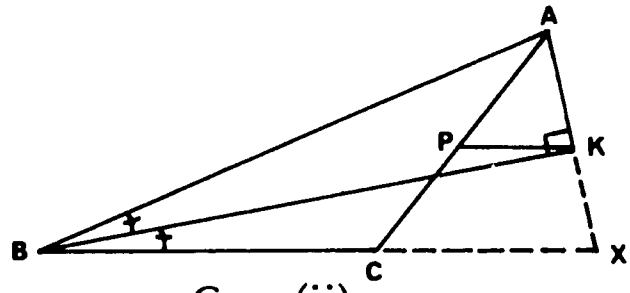
(iii) K on AC.



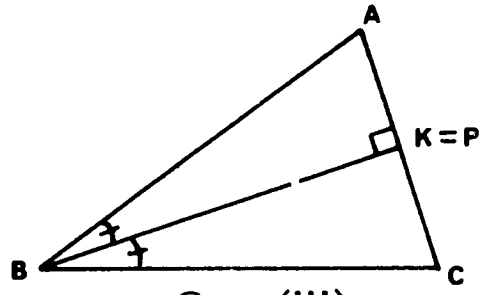
Case (i)

Case (iii) is easy by ASA congruence of $\triangle APB$ and $\triangle CPB$.

Cases (i) and (ii) can be treated by the same argument. In these cases, $\triangle AKB \equiv \triangle XKB$ by ASA. Thus, $\overline{AK} \equiv \overline{KX}$. Since $\overline{PK} \parallel \overline{CX}$, we can apply the theorem related to the midpoint theorem to $\triangle ACX$, to find that $\overline{AP} \equiv \overline{PC}$.

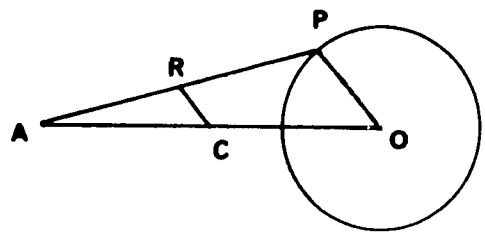


Case (ii)



Case (iii)

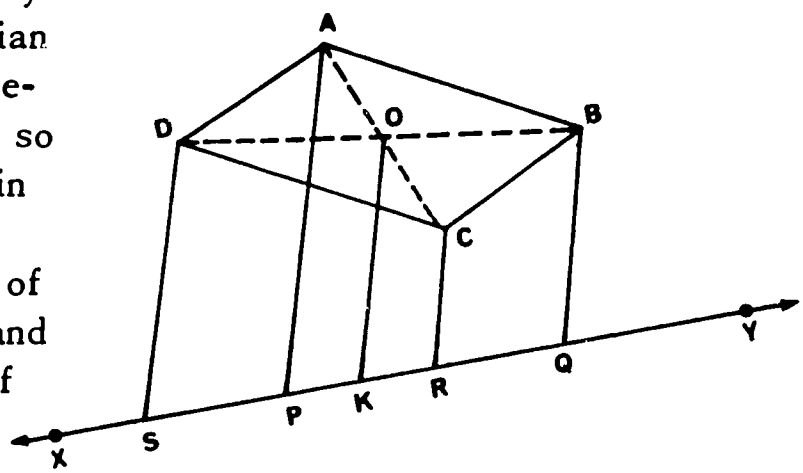
11. This is a case of direct application of the midpoint theorem. Clearly, because A and O are fixed points, the midpoint of $\overline{AO}$ is fixed. In $\triangle AOP$, $CR = \frac{1}{2}OP$ and OP is a fixed real number. So R moves as P moves, but its distance from C is constant. So its locus is a circle with centre C and radius $\frac{1}{2}OP$.



12. This problem relates to the equal intercept theorem. From $\overline{AD} \parallel \overline{BE} \parallel \overline{CF}$ and $\overline{AB} \equiv \overline{BC}$, it follows that $\overline{DE} \equiv \overline{EF}$. And because $\overline{DX} \parallel \overline{EY} \parallel \overline{FZ}$ and $\overline{DE} \equiv \overline{EF}$, we have $\overline{XY} \equiv \overline{YZ}$. $\overleftrightarrow{AC}$, $\overleftrightarrow{DF}$ and $\overleftrightarrow{XZ}$ need not be coplanar.

13. This problem is solved by applying both the equal intercept theorem and the midpoint theorem. The construction introduced makes this possible. From the equal intercept theorem, $\overline{CK} \equiv \overline{KD}$ and $\overline{AX} \equiv \overline{XD}$. In $\triangle ABD$, we obtain $HX = \frac{1}{2}BD$. In $\triangle DAC$, $KX = \frac{1}{2}AC$. $HK = HX - KX \quad \frac{1}{2}BD - \frac{1}{2}AC = \frac{1}{2}(BD - AC)$.

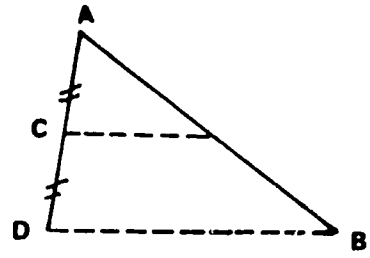
14. The diagram clearly suggests the median theorem for a trapezium, for we have so many trapeziums in it. Furthermore, $\overline{OK}$ is the median of trapezium $BQSD$ and also the median of trapezium $ACRP$. So, applying the median theorem, we



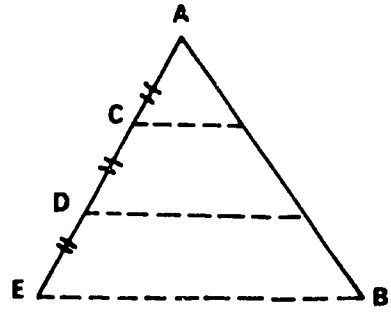
have $2\,OK = AP + CR$ and $2\,OK = BQ + DS$. Therefore, $AP + CR = BQ + DS$.

15. Here the equal intercept theorem is used to make some clasical constructions.

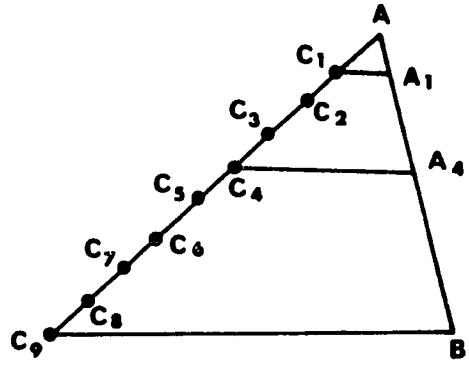
a. If $\overline{AB}$ is the segment to be bisected, lay off $\overline{AC} \equiv \overline{CD}$ on another ray $\overrightarrow{AD}$. The line through C parallel to $\overline{DB}$ bisects $\overline{AB}$.



b. If $\overline{AB}$ is to be trisected, lay off $\overline{AC} \equiv \overline{CD} \equiv \overline{DE}$ as in the figure. Lines through C and D parallel to $\overline{EB}$ cut off three congruent segments, each having length $\frac{1}{3}AB$.

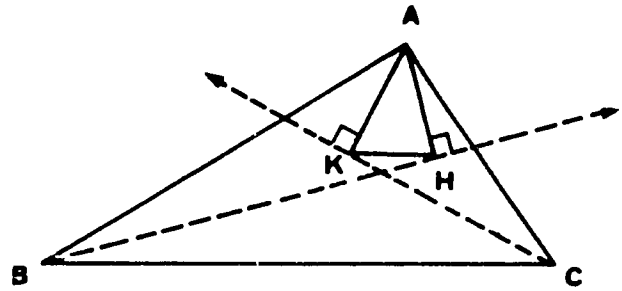


c. By a construction like part b, divide AB into nine congruent segments $\overline{AA_1}$, $\overline{A_1A_2}$, and so on. If $AA_1 = a$, then $AA_4 = 4a$ and $A_4B = 5a$. Thus, $5 \times AA_4 = 20a$ and $4 \times A_4B = 20a$, and A_4 is the desired point P.



The remaining problems are *Challenge Problems*. Probably a good number of students may never try them, but for the few who venture that far, the problems will prove rewarding. They should be assigned sparingly, if at all, to avoid discouraging the students.

16. Again the construction introduced brings out to the bright students what to do. From Problem 10, the line through K parallel to $\overline{BC}$ bisects $\overline{AC}$, and hence also bisects $\overline{AB}$. Also from Problem 10, the line through H parallel to $\overline{BC}$ bisects both $\overline{AC}$ and $\overline{AB}$. Thus, these two lines are identical, and $\overline{HK} \parallel \overline{BC}$.

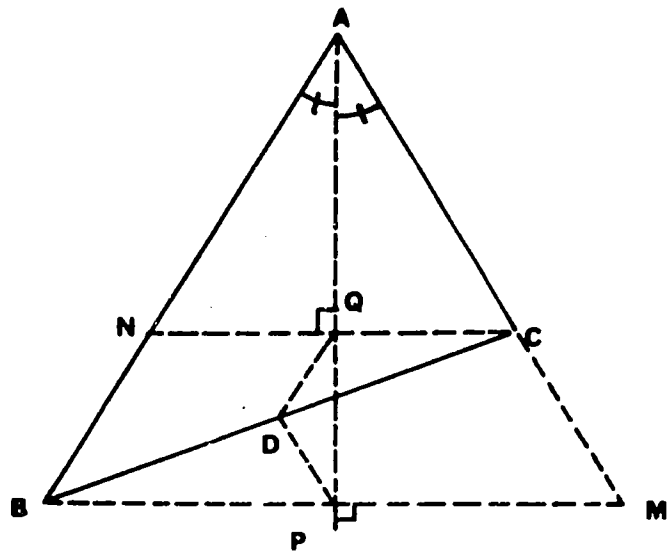


17. Here we prove $\triangle APB \equiv \triangle APM$ by ASA. Therefore, $\overline{BP} \equiv \overline{PM}$ and $\overline{AB} \equiv \overline{AM}$. In $\triangle BCM$, $\overline{BD} \equiv \overline{DC}$ and $BP \equiv PM$,

so $PD = \frac{1}{2} CM$

$= \frac{1}{2}(AM - AC)$

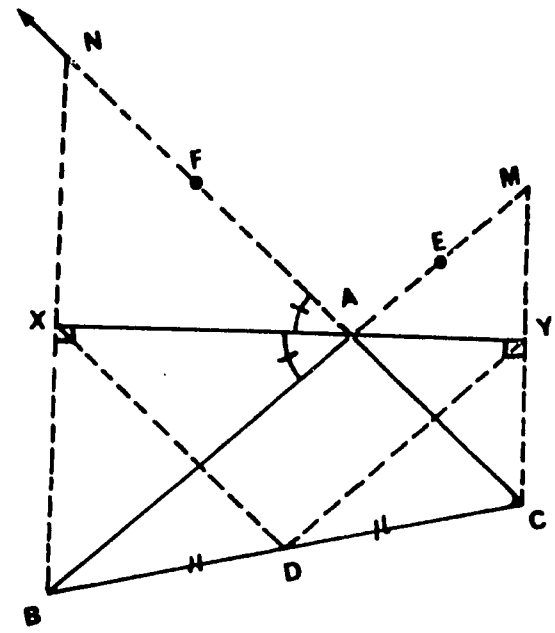
$= \frac{1}{2}(AB - AC)$.



For the second part $\triangle AQC \equiv \triangle AQN$ as before. Thus $\overline{CQ} \equiv \overline{QN}$ and $\overline{AC} \equiv \overline{AN}$. In $\triangle CNB$, $\overline{CD} \equiv \overline{DB}$ and $\overline{CQ} \equiv \overline{QN}$. Hence, $DQ = \frac{1}{2} BN$

$= \frac{1}{2}(AB - AN) = \frac{1}{2}(AB - AC) = PD$, and $\triangle PDQ$ is isosceles.

18. This is similar to Problem 17, so either can be tackled first. Only the outline is given.



$\triangle AXB \equiv \triangle AXN$ by ASA. Hence, $\overline{AB} \equiv \overline{AN}$ and $\overline{BX} \equiv \overline{NX}$. In $\triangle BCN$, $\overline{BD} \equiv \overline{DC}$ and $\overline{BX} \equiv \overline{XN}$,

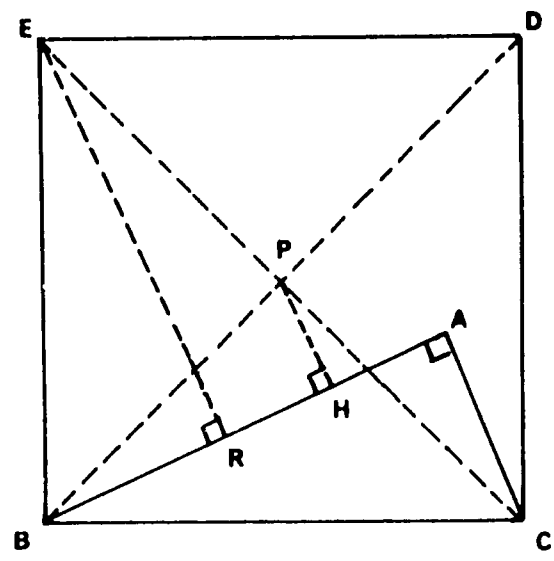
so $DX = \frac{1}{2}CN$

$= \frac{1}{2}(CA + AN)$

$= \frac{1}{2}(CA + AB)$.

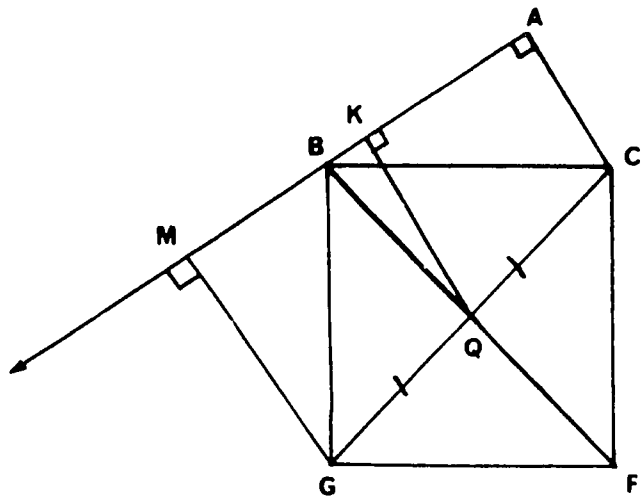
Also $\triangle AYC \equiv \triangle AYM$, so $\overline{CY} \equiv \overline{MY}$ and $\overline{AC} \equiv \overline{AM}$. In $\triangle CBM$, $\overline{CD} \equiv \overline{DB}$ and $\overline{CY} \equiv \overline{YM}$, so $XY = \frac{1}{2}BM$ $= \frac{1}{2}(BA + AM) = \frac{1}{2}(BA + AC)$. Thus, $DX = DY$ and $\triangle DXY$ is isosceles.

19. Go over Problem 13 again before trying this problem, because the result obtained there will be needed here. By what was shown there, we can say that because $\overline{CA}$, $\overline{PH}$ and $\overline{ER}$ are parallel and $\overline{EP} \equiv \overline{PC}$, $ER - AC = 2PH$. We call this equation a. We now want to show that ER is equal to AB.



To do this, we prove $\triangle ABC \equiv \triangle REB$. $B\hat{A}C \equiv E\hat{R}B$, $\overline{BC} \equiv \overline{EB}$, and $A\hat{C}B \equiv R\hat{B}E$, since they are complements of $A\hat{B}C$. Thus, $\triangle ABC \equiv \triangle REB$ by ASA. Then, $\overline{ER} \equiv \overline{BA}$ so a. becomes $AB - AC = 2PH$.

20. This problem is similar to Problem 19. Let M be the foot of the perpendicular from G to $\overrightarrow{AB}$. Then in $\triangle ABC$ and $\triangle MGB$



$B\hat{A}C \equiv G\hat{M}B$, $M\hat{B}G \equiv A\hat{C}B$, and $\overline{BC} \equiv \overline{GB}$. Thus, $\triangle ABC \equiv \triangle MGB$, so $\overline{AB} \equiv \overline{MG}$. We now use the median theorem for the trapezium $ACGM$, giving $2KQ = AC + GM = AC + AB$.

21. As a challenge problem this presents relatively little difficulty, for clearly $\triangle OMP \equiv \triangle OMQ$. Hence $\overline{PM} \equiv \overline{MQ}$. By the equal intercept theorem $\overline{HM} \equiv \overline{MK}$. Therefore, $HM - PM = MK - MQ$, so $HP = KQ$.

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