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ONDARY TWO

VOLUME 1

MATHEMATICS

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PRELIMINARY EDITION

Educational Services Incorporated
Watertown, Massachusetts, U.S.A.

ENTEBBE MATHEMATICS SERIES

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An Introductory Text for Teachers

NOTE

This book was written at a study and writing workshop held during the summer of 1963 at Entebbe, Uganda. More than fifty mathematicians and mathematics teachers from Africa, the United Kingdom, and the United States attended the Workshop.

The book was edited and produced with the utmost speed so that it might be tried out in African Secondary Schools during 1964. Therefore, there may be errors in editing and production. The African Education Program hopes that students who use this book will appreciate the circumstances under which it was produced.

STUDENT TEXT VOLUME ONE

MATHEMATICS SECONDARY TWO

UNIT 1: RATIONAL NUMBERS

Prepared at the

Entebbe Mathematics Workshops

African Education Program

Educational Services Incorporated

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TO THE STUDENT

In this textbook you have the opportunity to continue in the great experiment which was started in MATHEMATICS SECONDARY ONE. It all began in the summer of 1962 when more than fifty mathematicians and mathematics teachers gathered in Entebbe, Uganda, to begin the preparation of mathematics books more up-to-date than those in use anywhere else in the world. In the summer of 1963 MATHEMATICS SECONDARY TWO was written. The writers came to Entebbe from all the countries in English-speaking Tropical Africa, from the United States, and from the United Kingdom to take part in this project; now you and your teachers must do your share to find out whether this book is as good as it can be made. By using the textbook critically and making the most of the opportunities it offers to make the learning of mathematics an interesting and enriching experience, you will have played a part in the new growth of mathematics education in Africa.

If you studied MATHEMATICS SECONDARY ONE, you know that you can expect to meet many familiar ideas, sometimes expressed in unfamiliar ways, and that there will be more new ideas. You know that careful reading will be important because this book treats mathematics as a language designed to express certain kinds of ideas, and you must learn to read and speak this language.

We shall continue to learn why numbers behave as they do and we shall find that algebra helps us more and more to understand arithmetic ideas. We shall also explore some geometric ideas, observing properties of sets of points. We shall even learn to make algebra and geometry work together, each making the other clearer.

Do not hesitate to ask questions. Try to find the reasons behind things. Talk about your work with your teacher and your fellow students. Above all, try to look upon mathematics as something that can be exciting and rewarding, for if you give it half a chance, that is exactly what it is.

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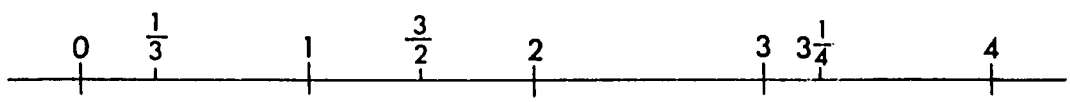
Glossary

CHAPTER 1

ADDITION AND SUBTRACTION OF RATIONAL NUMBERS

1-1 Introduction

In this chapter we shall study all the rational numbers - positive, negative, and zero - and we shall see how to add and subtract them. Although you may have met negative integers in your previous course you are expected here only to know how to add and subtract positive numbers - and that they can be represented on the number line.



In this chapter we shall look at numbers in the following way:

(a) First we shall see why we would like to have more numbers than the positive numbers. (Can you think of reasons why?) Then we shall decide how to represent them on the number line.

(b) Then we shall try to see how we might wish to add them.

(c) Finally we shall see that if the familiar properties of commutativity and associativity are to remain true for addition of any numbers, positive, negative, or zero, then we can find out how we must add them.

For Class Discussion

1. Joseph had 8 shillings and spent 5. How many shillings did he have left? This illustrates the subtraction problem "8 - 5". Can you think of a problem where the answer should be "5 - 8"?

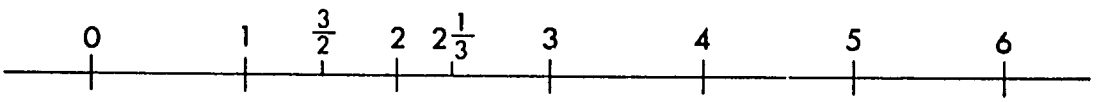
2. If you had negative numbers, how would you want to represent them on the number line? Where would you want the point corresponding to -3? to $-2\frac{1}{2}$?

3. Some business bank accounts are written in black ink if they give amounts of money in hand--while records of money borrowed are kept in red

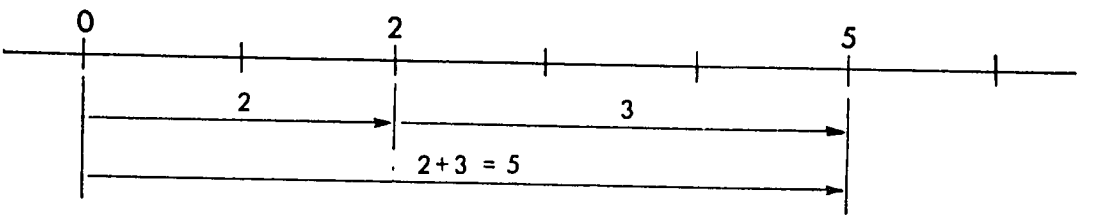
ink. What does this suggest? What meaning can be given to the sum of the numbers represented with black ink and red ink?

1-2 Subtraction and Negative Numbers

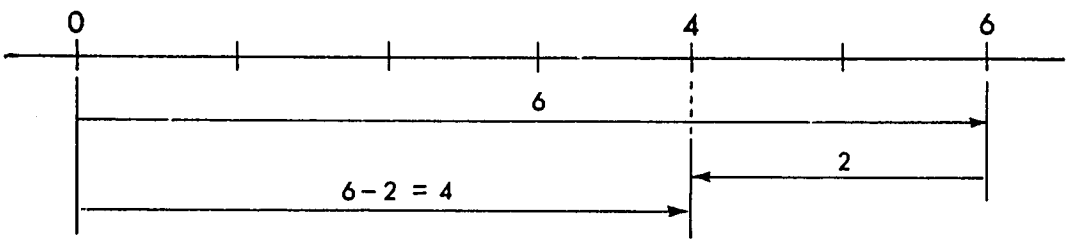
All the numbers which can be represented on the number line to the right of zero we shall now call positive numbers.



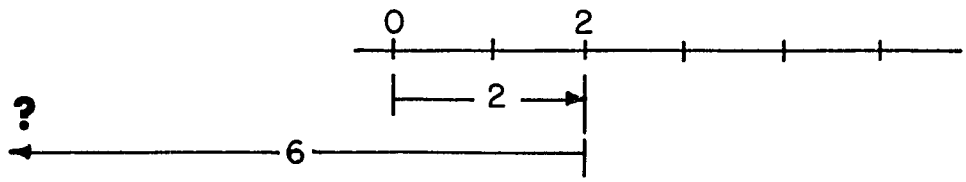
When we add a positive number to a positive number, we move to the right on the number line.



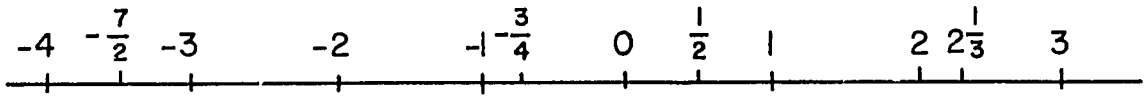
When we subtract a positive number we move to the left on the number line.



We cannot, however, subtract 6 from 2 because there is no place to go on the line.

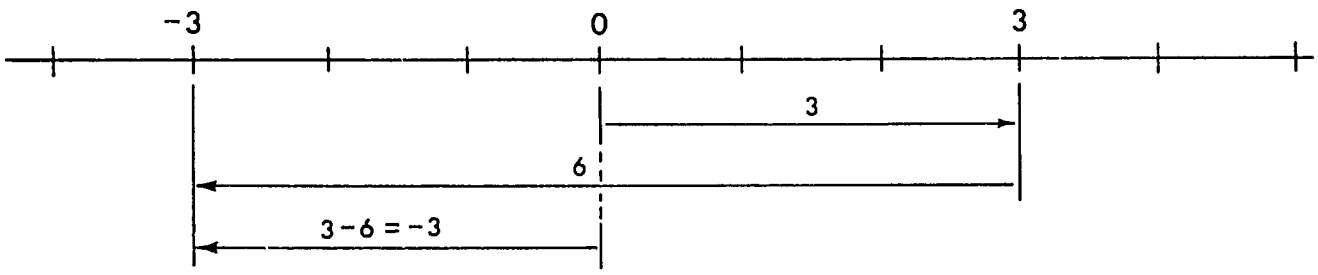


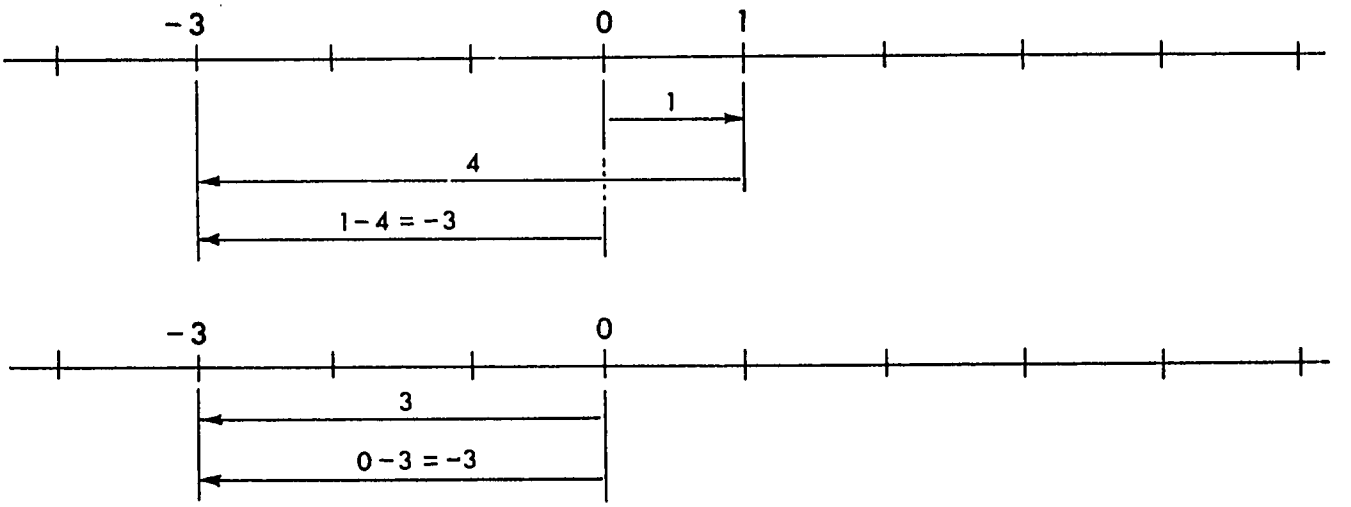
Let us invent new numbers which we shall call <u>negative</u> numbers - and which correspond to points to the left of zero in just the same way that the positive numbers correspond to points to the right of zero. Then the number line can be represented like this:



The positive rational numbers, the corresponding negative rational numbers, and zero together make up the set of all <u>rational numbers</u>.

Now let us agree that subtraction of a positive number should still correspond to movement to the left on the number line, even when it takes us to the left of zero. We always have a place to land on the number line.





Problems 1-2a

1. Draw a picture of the number line and mark with heavy dots the points corresponding to the following numbers:

$$-2,\ -6,\ -\frac{1}{2},\ \frac{9}{2},\ -\frac{9}{2},\ -\frac{2}{4},\ -\frac{18}{5},\ -5\frac{1}{8}$$

2. The numerals $\frac{3}{4}$ and $\frac{6}{8}$ represent the same number, so $\frac{3}{4} = \frac{6}{8}$.

Since to each positive number there is a definite corresponding negative number, we must have $-\frac{3}{4} = -\frac{6}{8}$.

Which of the following negative numbers are equal and correspond to the same point on the line?

$$-\frac{14}{8},\ -\frac{2}{3},\ -\frac{4}{2},\ -\frac{4}{6},\ -\frac{6}{9},\ -\frac{7}{4},\ -1\frac{3}{4},\ -2,\ -\frac{22}{11}$$

3. Do the following subtraction problems on the number line.

(a) $2 - 5$	(d) $1\frac{1}{2} - 3\frac{1}{2}$	(g) $\frac{3}{4} - 2$
(b) $4 - 5$	(e) $\frac{1}{2} - 2\frac{1}{2}$	(h) $\frac{1}{2} - 1\frac{3}{4}$
(c) $0 - 1$	(f) $0 - 2$	(i) $\frac{1}{4} - 1\frac{1}{2}$

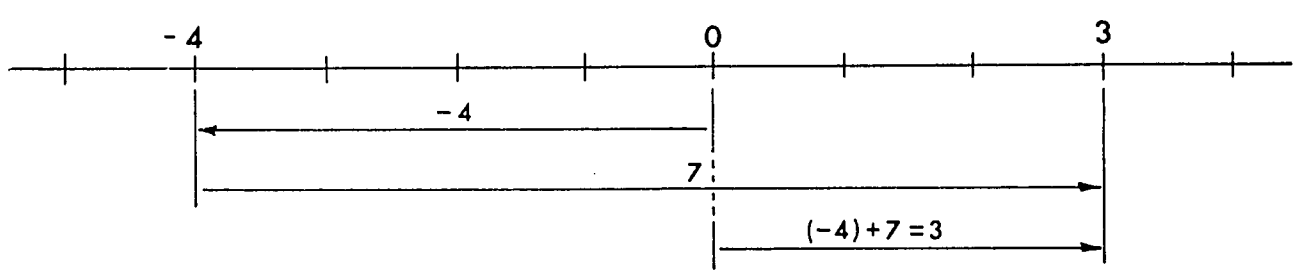
From what we have said about subtraction, we can learn something about addition. Let us recall the meaning of subtraction:

$$\boxed{\begin{array}{c} a - b = c \\ \text{means that} \\ a = c + b \end{array}}$$

If the same meaning is to be used when negative numbers are involved, then

$$3 - 7 = -4 \text{ means that } 3 = (-4) + 7.$$

Since we now see that adding 7 to (-4) must give 3, the addition on the number line must look like this:



In the same way

$$1 - 5 = -4 \text{ means that } (-4) + 5 = 1$$

$$0 - 4 = -4 \text{ means that } (-4) + 4 = 0$$

From examples like these we see that <u>addition of a positive number still corresponds to moving to the right on the line</u>. Then it also must be true that $(-5) + 4 = -1$ because moving 4 to the right from -5 takes us to -1. Similarly $-7\frac{1}{2} + 3 = -4\frac{1}{2}$, etc.

Problems 1-2b

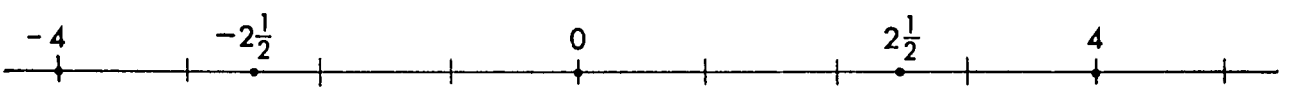
1. Find the following sums and picture each except (k) on the number line.

(a)	$(-1) + 3$	(e)	$(-5\frac{1}{2}) + 3$	(i)	$(-7) + 4\frac{1}{2}$
(b)	$(-2) + 3$	(f)	$(-\frac{11}{3}) + 3$	(j)	$(-5) + 3\frac{1}{3}$
(c)	$(-3) + 3$	(g)	$(-\frac{1}{2}) + 3$	(k)	$(-1000) + 992\frac{1}{4}$
(d)	$(-4) + 3$	(h)	$(-4) + 0$	(l)	$(-0 \cdot 1) + (0 \cdot 1)$

2. Exploratory Problem. How should you add, on the number line, a negative number to a positive number? What Basic Property of addition could help you find out?

1-3 Opposites

To each positive number there is a point to the right of 0 on the number line. To the corresponding negative number, there is a point on the line an equal distance to the left of 0.



> A positive number and the corresponding negative number are called opposites of each other.

Thus

-7 is the opposite of 7

7 is the opposite of -7

$-3\frac{1}{3}$ is the opposite of $3\frac{1}{3}$

$3\frac{1}{3}$ is the opposite of $-3\frac{1}{3}$

We say also that zero is the opposite of zero.

Problems 1-3

1. What are the opposites of -2, $-\frac{3}{2}$, $\frac{7}{2}$, $-\frac{11}{4}$? Plot each number and its opposite on the number line.

2. What is the opposite of the opposite of 1? The opposite of the opposite of the opposite of $2\frac{1}{3}$?

1-4 Addition on the Number Line

We now know how to add a positive number on the number line. How do we add a negative number? We know that

$$(-3) + 5 = 2 \text{ and } (-6) + 3 = -3$$

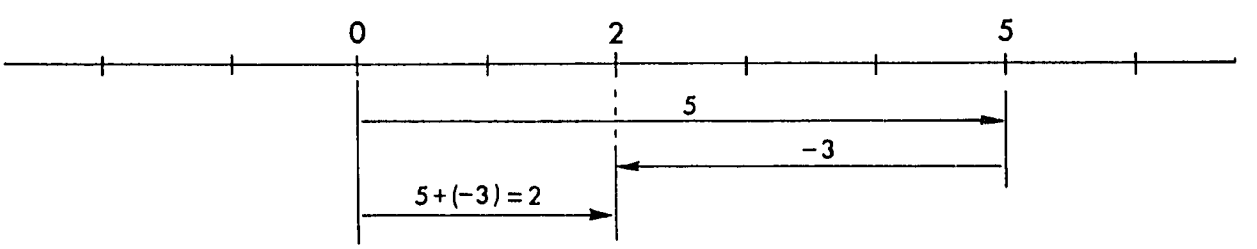
but what are

$$5 + (-3) \text{ and } 3 + (-6)?$$

The fact that we would like to have the commutative property true for addition of all the rational numbers suggests that we agree that

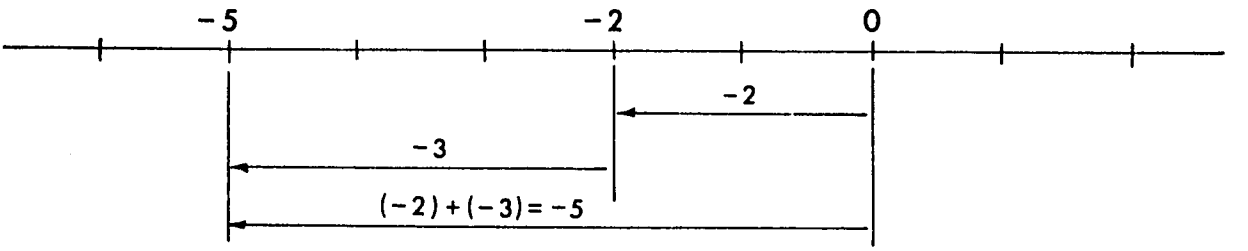
$$5 + (-3) = (-3) + 5 = 2.$$

We can picture the sum $5 + (-3)$ on the line:



In this example, and others like it, addition of a negative number corresponds to moving to the left on the line.

Here we were adding a negative number to a positive number. We shall assume that adding a negative number to a negative number or to zero works the same way. To add a negative number to any given number we move to the left on the line. The picture below shows the sum $(-2) + (-3)$.



We now know how to add, on the line, any two rational numbers, positive, negative, or zero.

Problems 1-4

1. Show the following sums on the number line.

(a) $7 + (-4)$ (e) $4 + (-7)$ (i) $(-5\frac{1}{2}) + (-1\frac{1}{4})$

(b) $4 + (-4)$ (f) $\frac{1}{2} + (-\frac{3}{2})$ (j) $1\frac{1}{4} + 5\frac{1}{2}$

(c) $(-3) + (-4)$ (g) $(-\frac{5}{4}) + 0$ (k) $1\frac{1}{4} + (-5\frac{1}{2})$

(d) $(-3) + 4$ (h) $(-\frac{7}{8}) + \frac{7}{8}$ (l) $5\frac{1}{2} + (-1\frac{1}{4})$

2. What is the sum x + (opposite of x) if

(a) x is positive? (b) x is negative?

In each case picture the sum on the number line.

3. The following sums illustrate some familiar basic properties which were true for positive rational numbers and are still true for all rational numbers. Find each sum by thinking of addition on the number line and name the property which is illustrated.

(a) (i) $(-5) + 11$ (ii) $11 + (-5)$

(b) (i) $(-\frac{3}{4}) + (-\frac{19}{4})$ (ii) $(-\frac{19}{4}) + (-\frac{3}{4})$

(c) (i) $[6 + (-8)] + (-4)$ (ii) $6 + [(-8) + (-4)]$

(d) (i) $(2 \cdot 8) + (-2 \cdot 5)$ (ii) $(-2 \cdot 5) + 2 \cdot 8$

(e) (i) $(-\frac{3}{8}) + [(-\frac{5}{8}) + \frac{1}{4}]$ (ii) $[(-\frac{3}{8}) + (-\frac{5}{8})] + \frac{1}{4}$

(f) (i) $[2\frac{1}{2} + (-6\frac{1}{8})] + 6\frac{1}{8}$ (ii) $2\frac{1}{2} + [(-6\frac{1}{8}) + 6\frac{1}{8}]$

(g) (i) $(-\frac{13}{7}) + \frac{13}{7}$ (ii) $\frac{13}{7} + (-\frac{13}{7})$

(h) (i) $\frac{13}{7} + 0$ (ii) $(-\frac{13}{7}) + 0$

1-5 Addition Using Gains and Losses

Addition of positive and negative numbers can be thought of also in terms of gains and losses of money.

If you gain 5 shillings and lose 8 shillings, you have a net loss of 3 shillings.

$$5 + (-8) = -3$$

If you lose 5 shillings and then lose 8 shillings, you have a net loss of 13 shillings.

$$(-5) + (-8) = -13$$

If you lose $5\frac{1}{2}$ shillings and then gain 8 shillings, you have a net gain of $2\frac{1}{2}$ shillings.

$$(-5\frac{1}{2}) + 8 = 2\frac{1}{2}$$

Problems 1-5

1. Do the following additions thinking of gains and losses.

(a) $17 + (-12)$ (d) $7\frac{1}{2} + 3\frac{1}{2}$ (g) $7\frac{1}{2} + (-3\frac{1}{2})$

(b) $(-32) + (-43)$ (e) $(-7\frac{1}{2}) + (-3\frac{1}{2})$ (h) $(-7\frac{1}{2}) + 7\frac{1}{2}$

(c) $(-14) + 7$ (f) $(-7\frac{1}{2}) + 3\frac{1}{2}$ (i) $(-3\frac{1}{4}) + 0$

2. Find the sums in problem 1 by thinking of the number line. Do you get the same answers?

3. Use addition, thinking of gains and losses, to determine which of the following sentences are true. In the true ones, name the basic property of addition which it illustrates, if there is one.

(a) $(-12) + 17 = (-17) + 12$

(b) $(-300) + (-500) = (-500) + (-300)$

(c) $[8 + (-87)] + (-13) = 8 + [(-87) + (-13)]$

(d) $23 + (-39) = (-23) + (7)$

(e) $(-40\frac{1}{2}) + [39\frac{1}{2} + (-41\frac{1}{2})] = [(-40\frac{1}{2}) + 39\frac{1}{2}] + (-41\frac{1}{2})$

(f) $293 + (-295) = (-295) + 293$

(g) $-29 + 0 = 29 + 0$

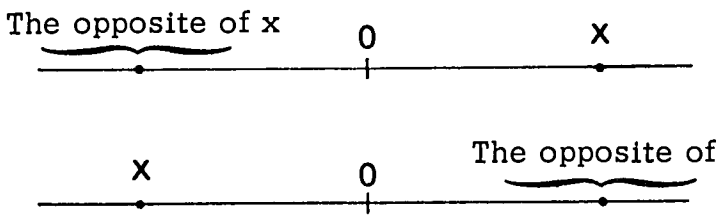
4. A shop has a profit on Monday of 720 shillings. On Tuesday it has a profit of 480 shillings. On Wednesday it has a loss of 1400 shillings (a lorry load of fruit was spoiled). What was the net profit or loss for the three days?

5. A boy earned 8 shillings one day and spent 3 shillings. The next day he earned 5 shillings and spent 9 shillings. How much net gain or loss did he have at the end of the two days?

The number line and thinking of gains and losses have led us to the same sums of rational numbers. Either viewpoint may be used in doing addition of rational numbers.

1-6 Opposites

A number and its opposite are represented on the number line at equal distances from zero like this:



if x is positive

if x is negative

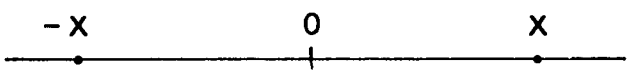
It is a bother always to have to write the words "the opposite of". It will therefore be convenient to have a symbol for these words. Because negative 6 and the opposite of 6 are the same number and we use the symbol "-6" for negative 6, it is natural to use the familiar symbol "-" to denote also "the opposite of". Although the symbol "-" can mean both "negative" and "the opposite of" we shall find there is no confusion. We shall know what to call it from the way it is used. Thus "the opposite of -6" can be written "-(-6)". And

$$-(-5) = \text{the opposite of } -5 = 5$$
$$-(-2\tfrac{1}{2}) = \text{the opposite of } -2\tfrac{1}{2} = 2\tfrac{1}{2}$$

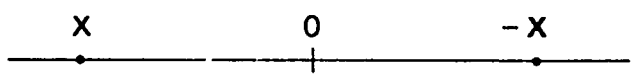
In these the first "-" means "the opposite of" and the second "-" means either "negative" or "the opposite of".

In general, if x is any rational number the opposite of x is written "-x".

If x is positive, -x is negative



If x is negative, -x is positive.



Since -x may sometimes be a positive number, we must be careful to read it "the opposite of x", not "negative x", and not to think of -x as necessarily a negative

number. Thus, if $x = -4$ then $-x = -(-4) = 4$.

We can now state the basic property of opposites which you have observed.

> For any rational number x
>
> $$x + (-x) = 0,$$
>
> and if
>
> $$x + a = 0$$
>
> then a must be the opposite of x, that is $a = -x$.

Finally, we mention another name for $-x$. It is called the <u>additive inverse of x</u>. You will meet this term again in Chapters 6, 7, and 16.

<u>Problems 1-6</u>

1. (a) Locate the opposite of 4 on the number line.
   (b) Locate negative 4 on the number line.
   (c) Are the opposite of 4 and negative 4 the same number?
   (d) Is it correct to use the symbol "-4" for either the opposite of 4 or negative 4?
   (e) In the symbol "-4" is it correct to read "-" <u>either</u> as "the opposite of" or as "negative"?

2. (a) Locate -3 on the number line.
   (b) Locate the opposite of -3 on the number line.
   (c) Are -3 and the opposite of -3 the same number?
   (d) What does -(-3) represent?
   (e) Can -(-3) be read "the opposite of the opposite of 3"?
   (f) Is -(-3) the same number as 3?
   (g) Is the opposite of the opposite of 3 the same as 3?

3. If x is any rational number,
   (a) what is $x + (-x)$? (b) what is a simpler name for $-(-x)$?

4. Give a simpler name for each of the following:
   (a) $-(-17)$ (d) $-(-y)$

(b) $\frac{9}{7} + (-\frac{9}{7})$

(c) the opposite of five

(e) $(-76) + [-(-76)]$

(f) $-[-(a \times b)]$

5. The distance between a positive number and its opposite on the number line is 19. What is the number?

6. CHALLENGE PROBLEM

(a) Let a be a positive rational number and let A be the set of all rational numbers greater than a. Let B be the set of all rational numbers less than $-a$. Let C be the set of all rational numbers between a and $-a$. Picture these sets on the number line. In which of these sets is $-x$ if x is in A? If x is in B? If x is in C?

(b) Now let a be a negative number. Form the sets A, B, and C, such that A is the set of all rational numbers less than a, B is the set of all rational numbers greater than $-a$, C is the set of all rational numbers between a and $-a$, and picture them on the number line. In which of these sets is $-x$ if x is in A? If x is in B? If x is in C?

1-7 Basic Properties of Addition

So far we have seen, by looking at the number line and at gains and losses, how we want to add two rational numbers. We have also seen in each case that: (1) Zero is the number which, added to any number, gives the same number. (2) The sum of a number and its opposite is zero. (3) The commutative property is true. (4) The associative property is true. Therefore, we have the following basic properties.

Basic Properties

1. Addition property of zero.

   If a is any rational number, then $a + 0 = a$.

2. Addition property of opposites.

   If a is any rational number, then $a + (-a) = 0$.

3. Commutative property of addition.

   If x and y are any rational numbers, then

   $$x + y = y + x.$$

4. Associative property of addition.

   If x, y, and z are any rational numbers, then

   $$x + (y + z) = (x + y) + z.$$

Now, in this section we shall look at addition of rational numbers the other way around. We shall use the basic properties to find out how to compute with negative numbers. In other words, in this section we are going to pretend that all you know is

(1) how to add positive numbers and zero.

(2) that the basic properties listed above are true.

We shall see that <u>with this information alone we can do addition with negative numbers</u>.

<u>Example 1</u>. Let us see what $(-7) + 15$ must be.

$(-7) + 15$	$= (-7) + (7 + 8)$	(Why?)	1.
	$= [(-7) + 7] + 8$	(Why?)	2.
	$= 0 + 8$	(Why?)	3.
	$= 8$	(Why?)	4.

In the first step in this example why did we choose to use $(7 + 8)$ as another name for 15? Would $(6 + 9)$ have been an equally good choice?

Example 2. Suppose we know that $(-2) + (-3) = -5$ (see Challenge Problem 4). Let us use this to see what $(-5) + 3$ must be.

$(-5) + 3$	$= [(-2) + (-3)] + 3$	(Why?)	1.
	$= (-2) + [(-3) + 3]$	(Why?)	2.
	$= (-2) + 0$	(Why?)	3.
	$= -2$	(Why?)	4.

Problems 1-7

1. Explain, as in Example 1, why it must be true that $(-2\frac{1}{2}) + 4 = 1\frac{1}{2}$.

2. Explain, as in Example 2, why it must be true that $(-3\frac{1}{2}) + 1\frac{1}{2} = -2$.

3. Sometimes the commutative and associative properties can be used to simplify calculations. Look for opportunities in finding the following sums:

(a) $[(-3\frac{1}{2}) + 17] + 2\frac{1}{2}$

(b) $[\frac{11}{8} + \frac{15}{16}] + (-\frac{3}{16})$

(c) $[79 \cdot 6 + (-168)] + 20 \cdot 4$

(d) $12 + (-20) + 48 + (-30)$

(e) $(-\frac{3}{7}) + \frac{19}{8} + \frac{3}{8} + (-\frac{11}{7})$

4. CHALLENGE PROBLEM

Show, using the basic properties of page 14 that $(-2) + (-3) = -5$. Hint: -5 is the only number which when added to 5 gives 0.

1-8 Subtraction

You are familiar with the idea that subtracting 4 from 10 means finding the number which added to 4 gives 10. In symbols:

$10 - 4$ means that number x for which

$$4 + x = 10.$$

Since $4 + 6 = 10$, we can write

$$10 - 4 = 6.$$

You have often used this idea to check subtractions in arithmetic.

$$93 - 28 = 65 \text{ is correct because}$$
$$28 + 65 = 93.$$

If negative numbers are involved, subtraction can be defined in exactly the same way.

$$(-12) - 8$$

is read

<u>negative 12 minus 8</u>

and it means

that number which added to 8 gives negative 12.

$$(-12) - 8 = x$$
$$8 + x = -12$$

The number x which makes this true is -20. Hence

$$(-12) - 8 = -20.$$

In general, then, we can use what we know about <u>addition</u> of positive and negative numbers to do <u>subtractions</u>.

> If a and b are any two rational numbers, a minus b is the number which added to b gives a. That is $a - b = x$ means that $b + x = a$.

The operation in $a - b$ is called <u>subtraction</u> and the resulting number is called the <u>difference</u>.

For example,

$$12 - (-8) = 20 \text{ because } (-8) + 20 = 12.$$
$$(-6) - 10 = -16 \text{ because } 10 + (-16) = -6.$$
$$(-3) - (-9) = 6 \text{ because } (-9) + 6 = -3.$$

You can think of finding the difference as finding the missing addend in the corresponding sum. Thus to find $12 - (-8)$, you think

$$(-8) + (?) = 12.$$

The missing addend is 20.

Notice that the symbol "-" is read "minus" when it indicates the operation of subtraction and only then. The rest of the time we read it "negative" or "the opposite of", whichever fits the situation.

Problems 1-8

Do the following subtractions by finding the missing addend in the corresponding sum.

1. $23 - 17$
2. $23 - 35$
3. $23 - (-8)$
4. $23 - (-40)$
5. $(-37) - 12$
6. $(-37) - (-5)$
7. $(-37) - (-50)$
8. $(-37) - 37$
9. $(-37) - (-37)$
10. $(-\frac{3}{13}) - \frac{2}{26}$
11. $\frac{7}{24} - (-\frac{7}{12})$
12. $-\frac{4}{35} - (-\frac{5}{14})$
13. Subtract 29 from 12.
14. Subtract (-9) from (-2).
15. From 12 subtract (-15)
16. Find the difference $(-72) - 8$
17. Find 37 minus (-38)
18. What number is 8 less than (-6)?
19. (-7) is less than 5 by how much?
20. By how much is $2 - (-6)$ greater than $-(-3) - (-3)$?
21. What is the opposite of $2 - 5$? of $5 - 2$? of $a - b$?
22. CHALLENGE PROBLEM.

Suppose $c \neq 0$. It could be either positive or negative. When is exactly one of the numbers c, $-c$, $c + (-c)$, $c - (-c)$ positive? When are exactly two positive? Can three of them be positive? Can none of them be positive?

1-9 Subtraction by Adding the Opposite

Problems 1-9a

In the left-hand column perform the subtractions. In the right-hand column perform the corresponding additions. Compare the results.

1.	(a)	$200 - 75$	(b)	$200 + (-75)$
2.	(a)	$\frac{11}{3} - \frac{4}{3}$	(b)	$\frac{11}{3} + (-\frac{4}{3})$
3.	(a)	$5 - 5$	(b)	$5 + (-5)$
4.	(a)	$20 - 30$	(b)	$20 + (-30)$
5.	(a)	$70 - (-65)$	(b)	$70 + 65$
6.	(a)	$(-\frac{2}{5}) - \frac{3}{5}$	(b)	$(-\frac{2}{5}) + (-\frac{3}{5})$
7.	(a)	$(-93) - (-99)$	(b)	$(-93) + 99$
8.	(a)	$(-5\frac{7}{8}) - (-6\frac{1}{8})$	(b)	$(-5\frac{7}{8}) + 6\frac{1}{8}$

For the examples you have just done you observed that subtracting a number gives the same result as adding the opposite of the number. This property makes it possible to express any subtraction directly in terms of addition. Because of this we shall be able to explain properties of subtraction by using known properties of addition.

For any rational numbers x and y, then, whether they be positive, negative, or zero, x minus y is the same as x plus the opposite of y.

> For rational numbers x and y,
>
> $$x - y = x + (-y)$$
>
> To subtract a number from another, add its opposite to the other.

Problems 1-9b

Perform the following subtractions of rational numbers by adding the opposite.

1. $17 - 25$
2. $(-17) - 25$
3. $17 - (-25)$
4. $(-17) - (-25)$
5. $25 - 17$
6. $(-25) - 17$
7. $25 - (-17)$
8. $(-25) - (-17)$
9. $\frac{5}{7} - (-\frac{2}{7})$
10. $(-\frac{9}{8}) - \frac{3}{4}$
11. $(-\frac{13}{5}) - (-\frac{26}{10})$
12. $\frac{4}{15} - \frac{23}{25}$
13. $(-27 \cdot 6) - (13 \cdot 5)$
14. $(-27 \cdot 6) - (-13 \cdot 5)$
15. $8\frac{1}{2} - (-4\frac{1}{4})$
16. $(-\frac{3}{14}) - (-\frac{7}{4})$
17. Subtract $\frac{23}{3}$ from $(-\frac{2}{3})$
18. From $\frac{7}{8}$ subtract $\frac{5}{4}$
19. Subtract $(-\frac{1}{3})$ from $\frac{3}{5}$
20. $(-\frac{7}{5})$ is how much less than $(-\frac{1}{5})$?

1-10 Subtraction on the Number Line

We found that we could add on the number line.

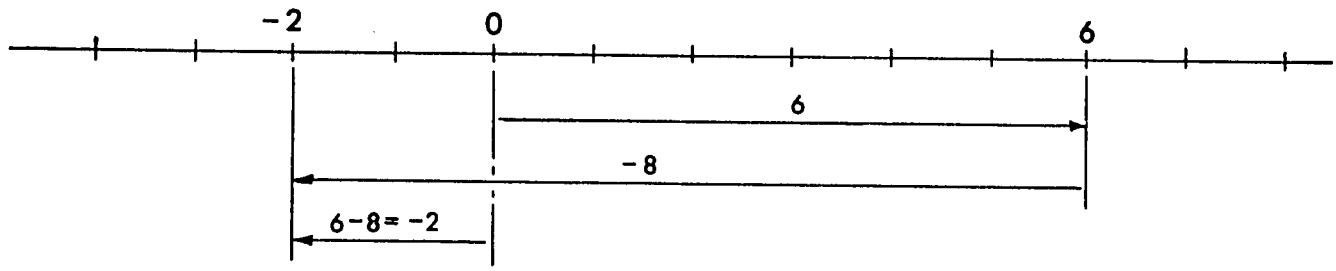
Adding a positive number corresponds to moving to the right.

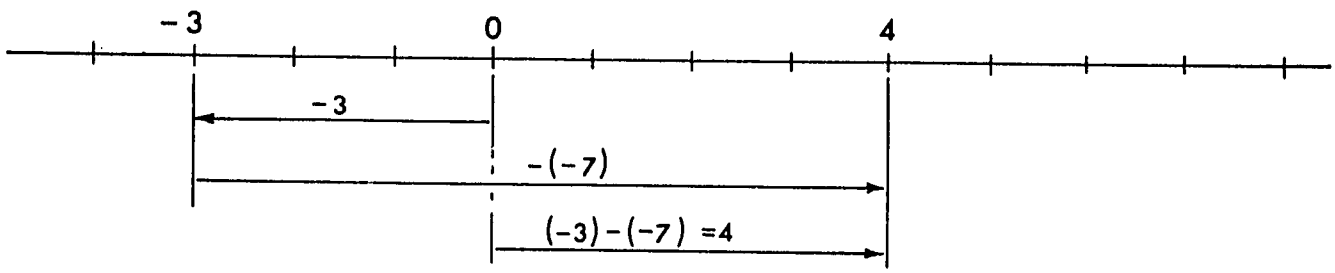
Adding a negative number corresponds to moving to the left.

Since subtracting is adding the opposite, subtracting a positive number corresponds to moving to the left.

Subtracting a negative number corresponds to moving to the right.

For example:





Problems 1-10

Perform the following subtractions on the number line.

1. $2 - 6$
2. $(-3) - (-2)$
3. $(-2) - 4$
4. $(-3) - (-5)$
5. $4 - (-2)$
6. $(-2\frac{1}{3}) - (-3\frac{2}{3})$

CHAPTER 2

MULTIPLICATION AND DIVISION OF RATIONAL NUMBERS

2-1 Introduction

You know how to multiply two numbers if they are positive or zero. In this chapter you will see how to multiply any two rational numbers, positive, negative or zero. Not only will you see how to multiply; you will also see why things are so.

In Chapter 1, you saw that addition of rational numbers is easy to interpret geometrically on the number line. With multiplication, and in particular products like

(negative number) $\times$ (negative number),

there is no naturally satisfying geometric interpretation. What we have to do is rely on the basic properties that we wish our number system to have. We simply assume that the Basic Properties, which we know are true for all positive numbers, still are true for the set of all rational numbers.

From Chapter 1 you know that the commutative and associative properties are true for addition of rational numbers. We agree that we must now have them true for multiplication also. Thus, for example, it is true that

$$(-2)(3) = (3)(-2)$$

$$(-2)[(-3)(4)] = [(-2)(-3)](4)$$

Just as we had 0 as the identity element for addition, giving $(-3) + 0 = (-3)$, now we must have 1 as the identity element for multiplication, giving $(-3) \times 1 = -3$. We want also $(-3) \times 0 = 0$ to be true.

Finally, we want the distributive property to be true. This relates multiplication to addition. For example, it must be true that

$$(-2)[3 + (-3)] = (-2)(3) + (-2)(-3).$$

It is the distributive property that will determine whether a product of two rational numbers is positive or negative.

Basic Properties

For all rational numbers a, b, and c it is true that

$ab = ba$	Commutative property of multiplication
$a(bc) = (ab)c$	Associative property of multiplication
$a(b + c) = ab + ac$	Distributive property
$a \times 1 = a$	Multiplication property of one
$a \times 0 = 0$	Multiplication property of zero

2-2 A Positive Number Times a Negative Number

How can we decide what $4(-2)$ should be? Do you think it should be -8? This is a good guess because $4(-2) = (1 + 1 + 1 + 1)(-2)$ and this should be $(-2) + (-2) + (-2) + (-2)$ which is -8. Let us see how the distributive property shows us that this is so.

Example 1. We suspect that $4(-2) = -8$, that is, that "$4(-2)$" and "-8" are numerals for the same number. This will be true if we can show that

$$4(-2) + 8 = 0$$

because -8 is the only number which when added to 8 gives zero. If $4(-2)$ added to 8 gives 0, then $4(-2)$ must be -8. Here is how it goes. The following is a sequence of true equations.

$4(-2) + 8 = 4(-2) + 4(2)$	(Why?)	1.
$= 4[(-2) + 2]$	(Why?)	2.
$= 4 \times 0$	(Why?)	3.
$= 0$	(Why?)	4.

We have proved that $4(-2) + 8 = 0$ is true, so $4(-2) = -8$.

Under our Basic Properties, then, why is $(-2)4 = -8$ true?

Example 2. Explain why each of the following equations is true.

$$3(-\tfrac{2}{3}) + 3(\tfrac{2}{3}) = 3[(-\tfrac{2}{3}) + \tfrac{2}{3}] \qquad \text{(Why?)} \qquad 1.$$

$$= 3 \times 0 \qquad \text{(Why?)} \qquad 2.$$

$$= 0 \qquad \text{(Why?)} \qquad 3.$$

It follows, as before, that $3(-\frac{2}{3})$ must be the opposite of $3(\frac{2}{3})$, since their sum is 0. Thus

$$3(-\tfrac{2}{3}) = -[3(\tfrac{2}{3})] = -2.$$

Why is $(-\frac{2}{3})3 = -2$ true?

Can you see what the following products must be? Give simpler names for these numbers.

$$(-3)3, \quad 3(-\tfrac{1}{2}), \quad 4(-\tfrac{1}{4}), \quad \tfrac{1}{3}(-2)$$

The examples illustrate the important rule:

The product of a positive number and a negative number is a negative number.

Problems 2-2

1. Write a simple numeral for each of the following products.

(a) $\frac{1}{7}(-2)$ (e) $(-6)\frac{5}{12}$

(b) $\frac{2}{7}(-1)$ (f) $7(-\frac{1}{7})$

(c) $(-\frac{1}{7})2$ (g) $(-1)(\frac{3}{4})$

(d) $6(-\frac{5}{12})$ (h) $(-\frac{3}{4})(\frac{10}{9})$

2. Use the method of Examples 1 and 2 to show that $2(-1) = -2$.

3. Use the method of Examples 1 and 2 to show that $(2\frac{1}{2})(-\frac{3}{2}) = -(\frac{5}{2})(\frac{3}{2}) = -\frac{15}{4}$.

2-3 A Negative Number Times a Negative Number

What do you think $(-4)(-2)$ should be? Do you think it should be 8? This is a good guess. Let us see how the distributive property shows us that this is so.

Example 1. We suspect that $(-4)(-2) = 8$. It will be enough for us to show that $(-4)(-2) + (-8) = 0$. (Why?) Let us try to show this. We have the following sequence of true statements.

$(-4)(-2) + (-8)$	$= (-4)(-2) + (-4)(2)$	(Why?)	1.
	$= (-4)[(-2) + 2]$	(Why?)	2.
	$= (-4) \times 0$	(Why?)	3.
	$= 0$	(Why?)	4.

Since $(-4)(-2) + (-8) = 0$ is true, $(-4)(-2)$ must be the opposite of (-8), that is $(-4)(-2) = 8$.

Why, now, is $(-2)(-4) = 8$?

Example 2. The most important product of two negative numbers is $(-1)(-1)$. Let us see why this product is 1. The product will be 1 if $(-1)(-1) + (-1) = 0$, because $(-1)(-1)$ would then have to be the opposite of (-1).

$(-1)(-1) + (-1)$	$= (-1)(-1) + (-1)(1)$	(Why?)	1.
	$= (-1)[(-1) + 1]$	(Why?)	2.
	$= (-1) \times 0$	(Why?)	3.
	$= 0$	(Why?)	4.

Therefore $(-1)(-1) = 1$.

Example 3. We can now see, using Example 2, that the product of two negative numbers is always positive. The idea is illustrated by $(-4)(-2)$. The following equations are all true.

$(-4)(-2)$	$= [(-1)4][(-1)2]$	(Why?)	1.
	$= [(-1)(-1)][(4)(2)]$	(Why?)	2.
	$= (1)(8)$	(Why?)	3.
	$= 8$	(Why?)	4.

This demonstrates the important rule:

The product of two negative numbers is always a positive number.

One of the things we can see from this rule is that:

If x is not zero, then $x^2 > 0$.

Why is this true?

Problems 2-3

1. Find a simple numeral for each of the following products.

(a) $(-\frac{3}{5})(\frac{2}{3})$

(b) $(-\frac{3}{5})(-\frac{2}{3})$

(c) $(-\frac{16}{3})(-\frac{1}{2})$

(d) $(\frac{14}{15})(-\frac{6}{7})$

(e) $(-\frac{14}{15})(-\frac{6}{7})$

(f) $(-3\frac{1}{3})(-1\frac{2}{7})$

(g) $(-2)(-a)$

(h) $(-3)(-2a)$

(i) $(-3)[(-2)a]$

(j) $(-\frac{3}{2})(-\frac{4}{5})(-\frac{1}{6})$

(k) $(-x)(-y)$

(m) $(-\frac{3}{2})(-x)$

(n) $(-\frac{5}{4})(-\frac{4}{5})$

(o) $(-\frac{a}{b})(-\frac{b}{a})$

(p) $(-\frac{2x}{y})(-\frac{3y}{x})$

2. Explain each of the steps in the following.

$$[(-\tfrac{2}{3})(-\tfrac{3}{4})] + (-\tfrac{1}{2}) = [(-\tfrac{2}{3})(-\tfrac{3}{4})] + [(-\tfrac{2}{3})(\tfrac{3}{4})] \quad \text{(Why?)} \quad 1.$$

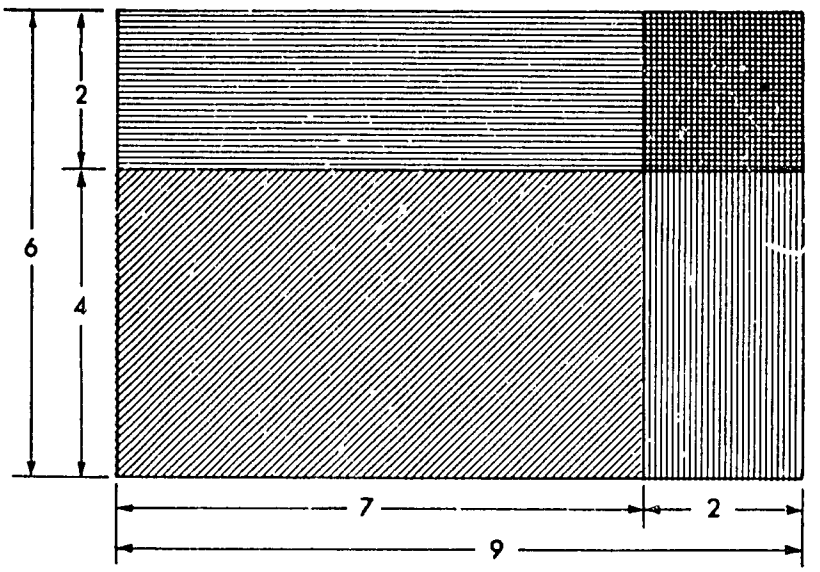
$$= (-\tfrac{2}{3})[(-\tfrac{3}{4}) + \tfrac{3}{4}] \quad \text{(Why?)} \quad 2.$$

$$= (-\tfrac{2}{3})[0] \quad \text{(Why?)} \quad 3.$$

$$= 0 \quad \text{(Why?)} \quad 4.$$

What does this show?

3.



Here is a geometric problem which shows that the product of two negative numbers must be positive if the distributive law is true. The large rectangle is 6 units by 9 units. A two-unit strip is cut off two edges. Explain why the area of the remaining rectangular region is

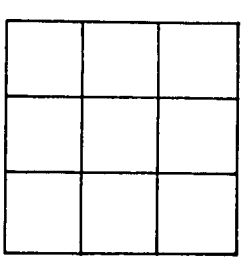
$$\begin{aligned} 28 = (4)(7) = (6-2)(9-2) &= [6+(-2)][9+(-2)] \\ &= [6+(-2)]9 + [6+(-2)](-2) \\ &= 6(9) + (-2)(9) + 6(-2) + (-2)(-2) \\ &= [54 + (-18) + (-12)] + (-2)(-2) \end{aligned}$$

Therefore $28 = 24 + (-2)(-2)$ and $(-2)(-2)$ must be 4.

You can see geometrically what happens. The numbers $(-2)(9)$ and $6(-2)$ correspond to the areas of the two side strips which are cut off the large rectangle. The doubly shaded region in the corner, however, has been subtracted twice, so we have to "add it on again" in the form $(-2)(-2)$.

4. <u>CHALLENGE PROBLEM</u>

A game with two players uses the square array below.



The first player placed a "1" in any of the nine squares. The second player places a "-1" in any of the remaining squares. They continue in turn until all the squares are filled. The first player counts one point for each row and each column for which the product of the three numbers is -1, while the second player counts a point for each row and each column for which the product of the three numbers is 1. What are the possible scores that can result? Is there an advantage for either player? Is there a strategy which will guarantee success for either player?

5. CHALLENGE PROBLEM

Which of the following statements concerning $(a + b)(-a + b)(-a - b)(a - b)$ is true? (a) It is never negative. (b) It is never positive. (c) It is positive for some values of a and b and negative for others.

2-4 Division

Once we know how to multiply positive and negative rational numbers, division is easy. All that is necessary is to remember the definition of division:

$$\boxed{\begin{array}{l} \text{If } ab = c \text{ and } a \neq 0 \\ \text{then } \quad b = c \div a \\ \text{or } \quad b = \frac{c}{a}. \end{array}}$$

Let us see how this works. Consider these examples.

$(-12) \div 4 = -3$	because	$-12 = 4(-3)$
$(-10) \div 5 = -2$	because	$-10 = 5(-2)$
$(10) \div (-5) = -2$	because	$10 = (-5)(-2)$
$(-10) \div \frac{5}{2} = -4$	because	$-10 = \frac{5}{2}(-4)$
$(-\frac{3}{2}) \div (\frac{7}{2}) = -\frac{3}{7}$	because	$-\frac{3}{2} = \frac{7}{2}(-\frac{3}{7})$

$$\frac{-\frac{5}{12}}{-\frac{7}{6}} = \frac{5}{14} \qquad \text{because} \qquad -\frac{5}{12} = (-\frac{7}{6})(\frac{5}{14})$$

Problems 2-4a

Find simple numerals for the following quotients:

1. $(-3) \div (2)$
2. $\frac{-3}{-4}$
3. $(-\frac{1}{4}) \div (\frac{1}{8})$
4. $\frac{-\frac{1}{4}}{-\frac{1}{8}}$
5. $(-\frac{5}{13}) \div (\frac{10}{13})$
6. $\frac{-2\frac{1}{3}}{-1\frac{1}{3}}$
7. $(-\frac{3}{10}) \div (-\frac{6}{5})$
8. $\frac{-2a}{3} \div \frac{-4a}{7}$
9. $(\frac{a}{b}) \div (\frac{c}{d})$

The idea of reciprocal, or multiplicative inverse is convenient in considering division:

> Two numbers a and b are reciprocals of one another if
>
> $$ab = 1.$$

For positive numbers you already know about reciprocals. All that must be done here is to consider negative numbers. Observe that zero has no reciprocal, for no matter what number b is, $0 \times b = 0$, and $0 \times b$ can never be 1.

$-\frac{2}{3}$ and $-\frac{3}{2}$ are reciprocals because $(-\frac{2}{3})(-\frac{3}{2}) = 1$

$(-\frac{7}{5})$ and $(-\frac{5}{7})$ are reciprocals because $(-\frac{7}{5})(-\frac{5}{7}) = 1$

a and $\frac{1}{a}$ are reciprocals because $a\frac{1}{a} = 1$

What are the reciprocals of the following?

$$-3,\ 3,\ -7,\ \frac{4}{3},\ -\frac{4}{3},\ -\frac{12}{13},\ -\frac{13}{12},\ 2\cdot 5,\ -3\frac{1}{4},\ x,\ \frac{1}{x}$$

What can you say about the reciprocal of a positive number? The reciprocal of a negative number?

What can you say about the reciprocal of a number between 0 and 1? The reciprocal of a number between 0 and -1?

What can you say about the reciprocal of a number greater than 1? The reciprocal of a number less than -1?

In the two columns below we have in the first column in each case the <u>quotient</u> of one number by another, in the second column the <u>product</u> of the first number by the <u>reciprocal</u> of the other. Compute each and compare the results.

(a)	$(-18) \div (-9) = ?$	$(-18)(-\frac{1}{9}) = ?$
(b)	$(-12) \div 4 = ?$	$(-12)(\frac{1}{4}) = ?$
(c)	$\frac{28}{-7} = ?$	$28(-\frac{1}{7}) = ?$
(d)	$(-\frac{1}{4}) \div (-\frac{1}{8}) = ?$	$(-\frac{1}{4})(-8) = ?$
(e)	$\frac{-\frac{2}{3}}{\frac{5}{3}} = ?$	$(-\frac{2}{3})(\frac{3}{5}) = ?$

In general, to divide by a number gives the same result as multiplying by its reciprocal. We saw above that if x is not zero, the reciprocal of x is $\frac{1}{x}$. We can therefore say:

$$\boxed{\begin{array}{l}\text{If } x \neq 0, \\ y \div x = y\frac{1}{x}.\end{array}}$$

Problems 2-4b

1. Do the following division problems by multiplying by a reciprocal.

(a) $(-14) \div \frac{7}{2}$

(b) $(-14) \div (-\frac{7}{2})$

(c) $\frac{-14}{-7}$

(d) $(-\frac{5}{3}) \div (-\frac{7}{9})$

(e) $(-\frac{15}{7}) \div (\frac{5}{21})$

(f) $\frac{-1\frac{1}{7}}{-1\frac{3}{5}}$

(g) $\frac{-1}{-\frac{5}{2}}$

(h) $(-1) \div (\frac{a}{b})$

(i) $\frac{-1}{-\frac{a}{b}}$

(j) $\frac{-\frac{a}{b}}{-\frac{2}{3}}$

(k) $(-\frac{a}{b}) \div (\frac{-2a}{b})$

(m) $(-\frac{a}{b}) \div (-\frac{c}{d})$

2. In each of parts (h) to (m) of Problem 1, what restriction, if any, must be put on values of a, b, c, and d ?

3. What is the product of a number and twice its reciprocal?

4. What is the absolute value of the reciprocal of x?

5. What is the reciprocal of the opposite of x?

CHAPTER 3

EQUIVALENT EQUATIONS AND INEQUALITIES

3-1 Introduction

Everyday we use sentences to say what we want, or to explain ideas we have. In your work last year you saw how to write sentences about mathematical ideas. Some examples are:

$$3 + 5 = 8, \qquad 7 > 10, \qquad 4 \neq 2 + 3.$$

The first of these is true, because $3 + 5$ and 8 are both numerals for the same number. Are the second and third sentences true or false? Give reasons for your answer.

Here are more sentences

$$3x + 7 = 13, \qquad x - 9 = 4.$$

These are examples of open sentences.

You will remember that a statement using "=" which says that two numerals are names for the same number is called an equation. Here are some equations.

$$3 + 5 = 8, \qquad 9 = 3 + 10, \qquad x + 7 = 9.$$

The equation $x + 7 = 9$ states that $x + 7$ and 9 are numerals representing the same number. When the number x is 2, the equation is true. When it is not 2 the equation is false. We call $\{2\}$ the <u>truth set</u> or the <u>solution set</u> of the equation. It is the set of all the numbers which make the equation true. When we have found the solution set of an equation we say that we have <u>solved</u> the equation.

In earlier work on equations we have restricted the domain of the variables, x, y, etc. to be the set of whole numbers. By <u>domain of a variable</u> we mean the set of numbers from which the value of the variable may be chosen. In this chapter the domain of the variables will be the set of positive and negative rational numbers and zero, unless, in any special problems we restrict the domain to a smaller set.

3-2 Equivalent Equations

Class Activity

1. What is the truth set of each of these equations?

$3x = 12$	$x = 12(\frac{1}{3})$
$3x + 6 = 18$	$x = (18 - 6) \div 3$
$3x - 7 = 5$	$x + (-7) = -3$

What do you notice about these equations? Do you recall this definition?

> Equations which have the same truth set are called <u>equivalent equations</u>.

How can you know whether two equations are equivalent?

2. Which of the following are pairs of equivalent equations? Find the truth set of each equation.

(a)	$x + 4 = 9$	and	$x = 9 - 4$
(b)	$2x - \frac{3}{8} = \frac{3}{4}$	and	$2x = 1\frac{1}{8}$
(c)	$7x = 19$	and	$x = 3$
(d)	$\frac{3}{5}x = 12$	and	$x = 20$
(e)	$\frac{3}{4}x + \frac{4}{5} = 4\frac{11}{20}$	and	$x = 5$
(f)	$x + (-3) = 7$	and	$x = 7 + 3$
(g)	$x + (-\frac{3}{4}) = -\frac{5}{8}$	and	$x = \frac{3}{4} + (-\frac{5}{8})$
(h)	$\frac{3}{5}x + \frac{2}{3} = \frac{19}{20}$	and	$x = 2$
(i)	$4x + 7 = 15$	and	$2x + 5 = 9$
(j)	$3x - 5 = 16$	and	$4x - 12 = 16$

Were you able to see without first finding their truth set that some of these are equivalent pairs of equations? Can you see any pattern that will help you to recognize equivalent equations? Can you state anything you could do to get an equation equivalent to one you may be given?

3-3 Solving Equations

When we want to solve an equation, we replace it by a succession of simpler equivalent equations until we get an equation whose truth set is easily seen.

Class Activity

Example 1. Solve the equation

$$\frac{1}{2}x - 3 = 1.$$

To solve this equation, we want to find what x must be in order that

$$\frac{1}{2}x - 3 = 1$$

be true. When this equation is true, $\frac{1}{2}x - 3$ and 1 are names for the same number. By adding the number 3 to this number, we see that $(\frac{1}{2}x - 3) + 3$ and $1 + 3$ are both names for the same number. On the other hand when x has a value such that

$$\frac{1}{2}x - 3 = 1$$

is false, then $\frac{1}{2}x - 3$ and 1 are not names for the same number. In this case $(\frac{1}{2}x - 3) + 3$ and $1 + 3$ will not be names for the same number.

Thus the equations

$$\frac{1}{2}x - 3 = 1 \tag{1}$$

and

$$(\frac{1}{2}x - 3) + 3 = 1 + 3 \tag{2}$$

are either both true or both false when x has any specific value, that is, the set of numbers which make the first equation true is the same as the set of numbers which make the second equation true. They are equivalent equations because they have the same truth set. Could we say the same thing had we added 954 instead of 3? Why do you think we chose to add 3?

The equation we have obtained by adding 3 can be written

$$\frac{1}{2}x = 4. \tag{3}$$

You can now see at a glance what the truth set of this equation is.

Suppose, however, you could not. Let us see how we can get an equation where the truth set is even more clearly seen.

When equation (3) is true the number $\frac{1}{2}x$ is the number four. When it is false the number $\frac{1}{2}x$ is <u>not</u> the number four. Explain then why the equation

$$2 \times (\tfrac{1}{2}x) = 2 \times 4 \qquad (4)$$

has the same truth set as equation (3).

We can write equation (4) as

$$x = 8. \qquad (5)$$

(What properties of multiplication have we used here?)
The truth set of equation (5) is $\{8\}$. Hence the truth set of (1) is $\{8\}$.

<u>Example 2</u>. Solve (find the truth set of) the equation

$$3x - \frac{5}{3} = 8 + \frac{5}{3}x + \frac{10}{3}.$$

This equation is equivalent to

$$9x - 5 = 24 + 5x + 10, \qquad \text{(Why?)}$$

And this equation is equivalent to

$$9x = 24 + 5x + 10 + 5, \qquad \text{(Why?)}$$

And this equation is equivalent to

$$9x = 39 + 5x, \qquad \text{(Why?)}$$

And this equation is equivalent to

$$4x = 39, \qquad \text{(Why?)}$$

And this equation is equivalent to

$$x = 9\tfrac{3}{4}. \qquad \text{(Why?)}$$

The truth set of the original equation is $\{9\frac{3}{4}\}$.

Problems 3-3

Find an equivalent equation like equation (5) of Example 1 for each of the following. Write the truth set of each equation.

You should be able to explain why you take each step.

1. $3x - 13 = 26$
2. $2x - 7 = 32 - x$
3. $4x - 9 = -5$
4. $14 = 8 - 2x$
5. $\frac{1}{2}x + \frac{3}{4} = 2$
6. $\frac{3}{2}x + \frac{2}{5} = -\frac{4}{5}$
7. $\frac{3}{4}x + \frac{1}{2} = 1\frac{1}{16}$
8. $\frac{3}{2}y - \frac{2}{5} = \frac{4}{5}$
9. $8 - 3x = 11$
10. $\frac{1}{2}(5x - 4) = x + 5\frac{1}{2}$
11. $18 = \frac{7}{4}y + 5\frac{1}{4}$
12. $18 - 3x = x + 3$
13. $\frac{4}{5}x - \frac{3}{4}x = \frac{7}{15}$
14. $\frac{1}{3}(3x + 9) = x + 3$
15. $\frac{1}{2}(2x + 3) = x + 1$

16. Is there any special name you know for equations with a truth set like that of 14.

17. CHALLENGE PROBLEM

What is the truth set of

$$(1 + x)(2x - 3) = 0?$$

18. CHALLENGE PROBLEM

When the domain of x is the set of all the integers,

$$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\},$$

are the following equations equivalent?

$$(1 + x)(2x - 3) = 0$$
$$x = -1$$

> When one equation has been obtained from another by one or a succession of the following steps the two equations are equivalent.
>
> Adding the same number on both sides.
>
> Multiplying on both sides by the same non-zero number.

3-4 Inequalities

Consider the sentence

$$\frac{3}{4} > 1.$$

It states that the number on the left-hand side is greater than that on the right hand side. You will remember that sentences like this one, and similar ones using the signs $>$ or $<$ are called inequalities. The inequality

$$\frac{3}{4} > 1$$

is false; three-quarters is not greater than one.

A true inequality is

$$\frac{3}{4} < 1.$$

Some open sentences are inequalities; for example

$$x < 2, \qquad x > 3 \qquad \text{and} \qquad 2x - 3 > 7.$$

An inequality is solved when its truth set is known.

We usually have a clear idea of when an inequality about whole numbers is true and when it is false. When an inequality is true the sign $<$ or $>$ points towards the smaller number. We saw in Chapter 1 that the smaller of two numbers needs a positive number added to it to give the other.

When we have the set of all the rational numbers, positive, negative and zero, we still say that <u>the smaller of any two numbers taken from the set is the one that must have a positive number added to it to give the other</u>.

Which number in each of the following pairs is the smaller? Be prepared to give a reason.

(a) $2,\ 3\frac{1}{4}$ (c) $-1,\ 1$ (e) $-2,\ 1$

(b) $-3,\ -1$ (d) $\frac{1}{4},\ \frac{1}{5}$ (f) $-\frac{1}{4},\ -\frac{1}{5}$

How can you tell, using the number line, which of two numbers is the smaller?

Which of the following inequalities are true? which false?

(a) $\frac{9}{11} < \frac{10}{12}$ (e) $-2 < -3$

(b) $3 > -1$ (f) $-1 < 0$

(c) $2 < 3$ (g) $-\frac{1}{4} < -\frac{1}{2}$

(d) $-2 > -3$ (h) $-1000 > -2$

<u>Class Activity</u>

1. Look now at the inequality

$$x > 2.$$

The truth set of this is the set of all numbers greater than two. We write this truth set, $\{\text{all numbers greater than } 2\}$. Are there any negative numbers in the truth set?

The truth set of $x + 1 < 2$ is the set of all numbers less than 1. We can see this because, if x <u>equals</u> 1 then $x + 1 = 2$. If x is <u>less than</u> 1, then the left side becomes less, and $x + 1 < 2$ is true. Does this truth set contain (a) any positive numbers, (b) any negative numbers, (c) zero?

2. What are the truth sets of the following inequalities?

$$x < 3$$
$$2x < 6$$
$$-x > -3 \qquad -x \text{ means the opposite of } x$$
$$x + 2 < 5$$
$$2x + 1 < 7$$
$$3 - x > 0$$

What do you notice about these inequalities?

Inequalities which have the same truth set are called <u>equivalent inequalities</u>.

3. Which of the following are pairs of equivalent inequalities? What is the truth set of each?

(a)	$x + \frac{1}{2} < 2\frac{1}{2}$	and	$x + 1 < 3$
(b)	$x - 3 > 2$	and	$x - 4 > 1$
(c)	$2x < 3$	and	$x < 1\frac{1}{2}$
(d)	$3x > 5$	and	$x > 1\frac{1}{3}$
(e)	$3x > 5$	and	$x - \frac{2}{3} > 1$
(f)	$x - 4 > 2$	and	$4 - x < -2$
(g)	$x - 5 > 6$	and	$10 - x < 12$
(h)	$3x < 2$	and	$x + \frac{1}{3} < 1$

Can you see any ways of telling that two inequalities are equivalent without first finding their truth set? Are <u>any</u> of the ways of getting equivalent equations also ways of getting equivalent inequalities?

Do you think that <u>all</u> the ways we listed for giving equivalent equations will also give equivalent inequalities? Think very carefully before you give an answer.

3-5 Solving Inequalities

To solve an inequality we can replace it by a succession of equivalent inequalities until one is obtained whose truth set is easily seen.

Class Activity

1. Solve the inequality

$$x + 1 > 4.$$

When this sentence is true, the number represented by $x + 1$ is larger than the number represented by 4. When we subtract 1 from each side, the number on the left will still be larger than the number on the right just as, if Mary is older than Modupe, then one year ago she was older than Modupe. Also when the inequality is false, that is, the number on the left is not larger than that on the right, subtracting one from each side gives another false inequality. Thus the set of numbers which makes the inequality true is not changed if we subtract 1 from each side.

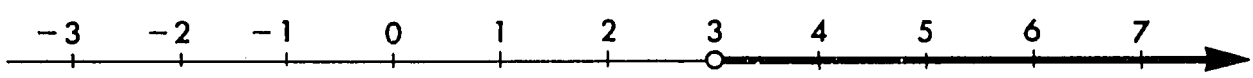
The original inequality is then equivalent to

$$(x + 1) - 1 > 4 - 1$$

or

$$x > 3.$$

The truth set of this sentence is the set of all numbers larger than 3. We can write the truth set, $\{\text{all numbers larger than } 3\}$, and show it on the number line in this way.



2. Solve the inequality

$$\frac{7}{2}x - \frac{4}{3} > \frac{3}{4}x + \frac{2}{3}$$

This inequality is equivalent to

$$\frac{7}{2}x > \frac{3}{4}x + \frac{2}{3} + \frac{4}{3} \qquad \text{(Why?)}$$

and this is equivalent to

$$\frac{7}{2}x - \frac{3}{4}x > 2 \qquad \text{(Why?)}$$

or

$$\frac{11}{4}x > 2.$$

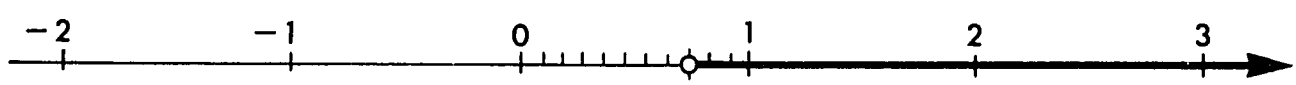
This is equivalent to

$$x > 2 \times \frac{4}{11}. \qquad \text{(Why?)}$$

The given inequality then, is equivalent to

$$x > \frac{8}{11}.$$

The truth set is $\left\{\text{all numbers greater than } \frac{8}{11}\right\}$, and the picture of this on the number line looks like this:



In doing the last example you saw that by multiplying the numbers on both sides of an inequality by $\frac{4}{11}$ you got an equivalent inequality.

Problems 3-5

1. Solve the following inequalities and draw the truth set on the number line.

(a) $x - 4 > 1\frac{1}{2}$

(b) $x - 3 < 2$

(c) $2x - 7 > 3$

(d) $2x + 1 < 1$

(e) $x + \frac{1}{2} > -\frac{3}{2}$

(f) $\frac{1}{3} > x - \frac{4}{5}$

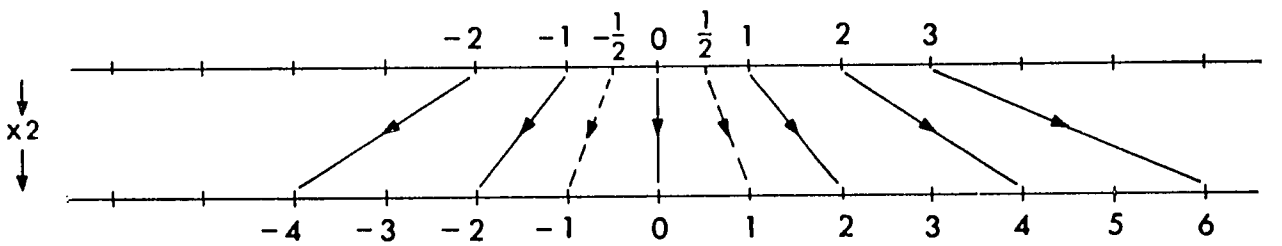
2. CHALLENGE PROBLEM. If $a > b$ is true, show that, no matter what number c is

$$a + c > b + c$$

[Hint. Since $a > b$ is true, there is a positive number p such that $a = b + p$ is true. Now look at $a + c$ and $b + c$.]

3-6 Equivalent Inequalities Obtained by Multiplication

You can show on the number line why an equivalent inequality is obtained by multiplying the numbers on both sides of an inequality by the same positive number. This picture shows multiplication by 2.



We know that "to the left of on the number line" means "less than" and "to the right of on the number line" means greater than.

The picture shows that "less than" and "greater than" relationships are not altered by multiplication by 2.

Can we say the same thing about multiplication by <u>any</u> positive number. Show, using the number line, (as we did above), multiplication by $\frac{1}{2}$, 3, $1\frac{1}{2}$.

Can you say the same thing about multiplication by a negative number? Show, using the number line, (as we did above for showing multiplication by 2) the effect of multiplying by -1. Does the larger of two numbers remain the larger after both have been multiplied by -1?

There is another way we can see the answer to this question. Suppose

$$a > b$$

is true.

Then we can write a true equation

$$a = b + p$$

in which the number p is positive. Multiply the numbers represented on each side of the equal sign by (-1). We obtain

$$(-1)a = (-1)(b + p) \qquad \text{(Why?)}$$

$$(-1)a = (-1)b + (-1)p. \qquad (1) \qquad \text{(Why?)}$$

Is $(-1)p$ a positive number or is it negative? Explain why the equation

$$(-1)a + p = (-1)b$$

is true when equation (1) is true, and how this shows us that

$$(-1)a < (-1)b.$$

This means that multiplying an inequality by (-1) changes the sense of the inequality. That is, to keep a true inequality true we have to interchange $>$ and $<$ when multiplying by (-1). Explain why multiplying by <u>any</u> negative number changes the sense of an inequality.

<u>Problems 3-6a</u>

Which of the following inequalities are true?

1. (a) $3 > 2$ (d) $-3 < -2$
   (b) $3(-1) > 2(-1)$ (e) $(-3)(-1) < (-2)(-1)$
   (c) $3(-1) < 2(-1)$

2. (a) $-2 < 3$ (c) $(-2)(-1) < 3(-1)$
   (b) $(-2)(-1) > 3(-1)$ (d) $2 < -3$

3. (a) $4 < 5$ (d) $4(-\frac{1}{2}) < 5(-\frac{1}{2})$
   (b) $4(2) < 5(2)$ (e) $4(-\frac{1}{2}) > 5(-\frac{1}{2})$
   (c) $4(-2) < 5(-2)$

<u>Class Activity</u>

Find the truth set of the inequality

$$-\frac{2}{3}x + 3 > 0.$$

This is equivalent to

$$-\frac{2}{3}x > -3 \qquad \text{(Why?)}$$

which is equivalent to

$$\frac{2}{3}x < 3 \qquad \text{(What has been done here?)}$$

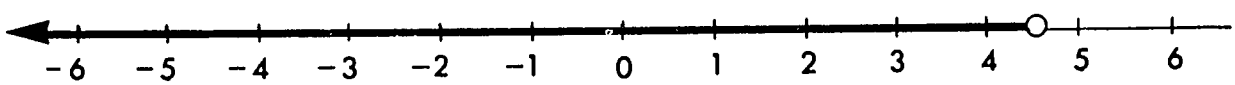
and this is equivalent to

$$x < \frac{3}{2} \times 3 \qquad \text{(Why?)}$$

or

$$x < \frac{9}{2}.$$

The truth set is $\left\{\text{all numbers less than } \frac{9}{2}\right\}$. On the number line it looks like this



> The following operations on an inequality give an equivalent inequality:
>
> 1. Adding the same number on both sides.
> 2. Multiplying on both sides by the same <u>positive</u> number.
> 3. Multiplying on both sides by the same negative number and changing $>$ to $<$ or $<$ to $>$.

<u>Problems 3-6b</u>

1. Find the truth sets of the following inequalities. Show them on the number line.

(a) $2x + 7 > \frac{5}{2}$

(b) $3x + 5 < \frac{1}{17}$

(c) $7 - 2x < 11$

(d) $\frac{4}{5} - \frac{1}{5}x > \frac{2}{7}$

(e) $\frac{2}{3} - \frac{1}{3}x > \frac{4}{3}x + \frac{2}{5}$

2. <u>CHALLENGE PROBLEM</u>. When the inequality

$$a > b$$

is true, and m is a positive number, show that

$$ma > mb.$$

[Hint, begin by expressing a as $b + p$, where p is a positive number.]

3. For what numbers x is $\frac{1}{x} > 1$ true?

4. <u>CHALLENGE PROBLEM</u>. When x is positive, what is the least value that $x + \frac{1}{x}$ can have?

[Hint. Begin by writing $1 + a$ for x. a is greater than -1 when x is positive.]

5. For what numbers x is the following true?

$$-3 < x + 2 < 5$$

Show the truth set on the number line.

[$-3 < x + 2 < 5$ is true when $x + 2 > -3$ <u>and</u> $x + 2 < 5$.]

6. Write the truth set of

$$-5 > x > -3.$$

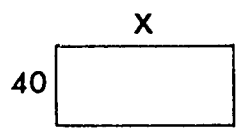
CHAPTER 4

PROBLEMS

4-1 Solving Problems

Frequently problems are stated in words. Sometimes these problems are easily solved using only arithmetic. In other cases, however, it can be easier (because less thought is required) to translate the problem into an equation and then solve the equation. Let us consider two examples.

Example 1. A man wishes to enclose by a fence a rectangular field which is 40 feet wide and has an area of 4800 square feet. How long should the field be?



Let the length of the field be x feet. Then the statement about area gives the equation

$$40x = 4800$$

Solving this equation we get $x = 120$. The field must be 120 feet long.

Observe that in this example there was no need to use any algebra for one could simply write: length is $(4800 \div 40)$ feet. Nevertheless the equation is an exact statement of the conditions of the problem and is easy to write.

Example 2. The sum of two numbers is 24. Twice the first number plus half the second number is equal to 9. Find the numbers.

If the sum of the numbers is 24 and one of the numbers is n, we'll have to subtract n from 24 to find what is left for the other number. We write, then:

If n is the first number then $24 - n$ is the second number. The problem tells us that

$$2n + \frac{1}{2}(24 - n) = 9$$

We solve this equation by obtaining the following equivalent equations:

$$\begin{aligned} 4n + (24 - n) &= 18 \\ 3n + 24 &= 18 \\ 3n &= -6 \\ n &= -2 \end{aligned}$$

If n is -2, then $24 - n$ is 26.

The two numbers therefore are -2 and 26. This checks because their sum is 24, and $2(-2) + \frac{1}{2}(26) = 9$.

Observe that in this problem it would be quite troublesome to use only arithmetic. The equation is a <u>great</u> help.

<u>Problems 4-1</u>

Solve the following problems by first finding an equation and then solving it.

1. A rectangular field is twice as long as it is wide, and is surrounded by a fence which is 370 yards long. How wide is the field?

2. James was 7 shillings in debt. On Monday he earned some money and on Tuesday he earned the same amount. This put him out of debt with 15 shillings left over. How much did he earn on Monday?

3. A boy used a small can to fill a large one with water. The large can holds 4 gallons. After he poured 6 small canfulls into the large can, it was still $\frac{2}{3}$ gallon short of being full. How many gallons did the small can hold?

4. There are 19 animals in a field containing only chickens and sheep. If these animals have a total of 54 feet, how many of them are sheep?

5. A man bought 20 books. Some cost 8 shillings each and some cost 3 shillings and 50 cents each. If the books cost 101 shillings and 50 cents in all, how many of the more expensive books did he buy?

6. I am thinking of a certain number. If I multiply it by 6, add -18 to the product, and then take one-third of this sum, I get -2. What number am I thinking of?

7. Three fourths of a certain number plus three and one-half is equal to five more than two-thirds of the number. What is the number?

4-2 English Phrases and Formulas

In each of the problems of Section 1 you used a letter to represent one of the unknown numbers that you wished to find. The letter is called a variable. Other numbers related to the problem can be expressed in terms of the variable. In this way we get formulas for these numbers.

Example 1. Choose a number, multiply it by 2 and then subtract 5 from the product. Write a formula for the number obtained in terms of the number chosen.

If x is the number chosen then the number obtained is $2x - 5$.

Example 2. John and Modupe have 30 bananas between them. Write a formula for the number that Modupe has, in terms of the number that John has.

Let J be the number of bananas that John has. Then Modupe has $30 - J$ bananas.

Example 3. Manuel rides his bicycle at the uniform speed of 8 miles per hour. How far does he travel in a given time?

If the time he travels is t hours then, the distance he travels is $8t$ miles.

Problems 4-2

1. Given a number, add 10 to it, multiply the sum by 3 and subtract the given number from the resulting product. Write a formula for the number obtained in terms of the given number.

2. There are 50 gallons of water in a tank when a tap is turned on, letting water flow into the tank at the rate of 2 gallons per minute. Write a formula for the amount of water in the tank in terms of the number of minutes since the tap was turned on.

3. A rectangular field has a perimeter of 982 feet. Write a formula for the length of the field in terms of the width.

4. The sum of two numbers is 15. Write a formula for their product in terms of one of the numbers.

5. Write a formula for the sum of one-half of a certain number plus the number itself.

6. Choose a number. Add 7 to this number and multiply the result by 2. Then subtract twice the number chosen. Write a formula for the resulting number in terms of the number chosen. Why do you always get the same result?

7. CHALLENGE PROBLEM. Choose a number. Subtract 2 from it and then divide the result by 2. Multiply this quotient by 5. Add half the chosen number. Add 15. Subtract three times the chosen number. Divide the result by 10. What do you get?

Write a formula for the final result in terms of the number chosen. Explain why the final result is always 1 no matter what number is chosen.

8. CHALLENGE PROBLEM. Make up a number puzzle, similar to that of Problems 6 and 7, and use it on your friends.

4-3 Problems, Equations, and Answers

Problems usually state that two numbers are equal. As you can see from the problems you have solved there are several steps in solving a problem:

(1) Select a variable to represent one of the unknown numbers and show clearly what it represents, drawing a figure where this is possible.

(2) Write formulas for the other numbers in the problem.

(3) Write an equation relating the numbers of the problem.

(4) Solve the equation.

(5) Write the answer to the problem and give the units involved.

(6) Examine your answer to see that it satisfies all the conditions of the original problem. This not only gives you a check on the accuracy of your work but it also sometimes can show that the original problem does not have a meaningful answer.

Example 1. Julius says that the sum of his age and Christopher's age is 24 years. He also says that twice his age plus half of Christopher's age totals 9 years. How old is Julius?

(1) If Julius is x years old,

(2) then Christopher is $24 - x$ years old,

(3) and Julius says that

$$2x + \frac{1}{2}(24 - x) = 9.$$

(4) The equation of (3) is the same as that of Example 2, of Section 1 and has the solution $x = -2$.

(5) Julius is negative 2 years old.

(6) Although an age of -2 years is a ridiculous answer, the number -2 does satisfy the conditions which Julius gave, for Christopher would then be 26 years old $(-2 + 26 = 24)$ and $2(-2) + \frac{1}{2}(26) = -4 + 13 = 9$. Therefore, we may conclude that either Julius made a mistake or was fooling.

Of course most problems have answers which will make sense.

Example 2. Start at -2 on the number line and take a certain number of steps (either right or left) each one unit long, then take twice as many steps in the opposite direction. If you end up at 4 on the line how many steps did you first take and in which direction did you first step?

(1) Let x be the number of steps you first took, where x will be a positive number if you stepped to the right, and a negative number if you stepped to the left.

(2) After x steps you were at $-2+x$. After $2x$ more steps in the opposite direction you were at $(-2 + x) + (-2x)$.

(3) Since you arrived at 4 it is true that

$$(-2 + x) + (-2x) = 4$$

(4) The truth set of this equation is $\{-6\}$, so $x = -6$.

(5) Therefore you first took 6 steps to the left.

(6) The negative answer makes sense because it means movement to the left. It also satisfies the conditions of the problem After 6 steps to the left you are at -8 and after 12 steps to the right you are at 4.

Problems 4-3

Solve the following problems.

1. Sozi wishes to cut a piece of wood $17\frac{1}{2}$ inches long into two pieces such that one piece is $2\frac{3}{4}$ inches longer than the other. What should be the length of each piece?

2. A boy can walk from one town to another at the rate of $3\frac{1}{2}$ mph and cycle back at 8 mph, and thus take $5\frac{3}{4}$ hours over the double journey. How far apart are the two towns?

3. The sum of $\frac{1}{4}$ of a number, $\frac{3}{8}$ of the number, and $\frac{5}{16}$ of the number is $7\frac{1}{2}$. Find the number.

4. A truck went a distance of $67\frac{1}{2}$ miles at a certain speed and returned the same distance at one-half of that speed. The total for the journey was $4\frac{1}{2}$ hours. Find the two speeds.

5. The sum of the ages of a boy and his father is 60 years. Six years ago the boy's age then was one-seventh of his father's age then. What are their ages now?

6. A rectangle is 5 inches longer than it is wide. Its width is three-sixteenths of its perimeter. Find its perimeter.

7. Jacopo ran at a certain speed for one-third of an hour and then walked for half an hour at the rate of 3 miles per hour. At the end of that time he had gone 4 miles. How fast did he run?

8. Find x if $\frac{3x - 2}{5}$ is greater then $\frac{1 - 4x}{10}$ by 5.

9. A rectangular field is two-thirds as wide as it is long. If its perimeter is 320 yards, how wide is the field?

10. A certain angle is $\frac{1}{5}$ as large as its supplementary angle. What is the measure of this angle in degrees?

11. One angle of a triangle is three times as large as one of the other angles. The third angle of the triangle is a 40° angle. What are the measures of the angles of the triangle?

12. Three times the measure of a certain angle is equal to four times the measure of its supplement. What is the angle?

13. At exactly 10 o'clock there are 30 gallons of water in a tank. Water runs steadily into the tank at the rate of $2\frac{1}{2}$ gallons per minute.

(a) Write a formula for the amount of water in the tank t minutes after 10 o'clock. How much water is in the tank 9 minutes after 10 o'clock? At what time was the tank empty?

(b) Suppose there is a drain tap fitted into the tank which is opened at 10 minutes after 10 o'clock, and suppose this drains out all of the water in 10 minutes more. How many gallons per minute go down this drain?

14. Temperatures are usually measured in either degrees Fahrenheit or in degrees Centigrade. If these measures are denoted by F and C, respectively, then they are related by the formula

$$F = \frac{9}{5}C + 32.$$

Find the Centigrade temperatures corresponding to the Fahrenheit temperatures $F = 10, 20, 60, 80$. What Fahrenheit temperatures correspond to $C = 0$? 100? When is $F = C$?

15. Two and one-half plus the reciprocal of a number is equal to one. Find the number.

16. CHALLENGE PROBLEM. A number is chosen, 8 is subtracted from it, and this difference is multiplied by 3 to give a first number. Then 12 is subtracted from the chosen number and the difference is multiplied by 2 to give a second number. If the second number is now subtracted from the first number, the result is the number chosen originally. What was the number chosen originally? Explain.

17. A man sells "widgets" at 3 shillings each. His equipment and the plant he rents cost him 600 shillings per month, and the raw material and labour in each widget cost $1\frac{1}{2}$ shillings. At the end of two months he found that he had a loss of 300 shillings. How many widgets had he sold? How many must he sell per month in order neither to gain nor to lose money? If he said he had lost 200 shillings during those two months, what would you think about his statement?

18. A father is now four times as old as his son. If the sum of their ages ten years ago was sixty, find their present ages.

19. A boy is eight years younger than his sister. If his age three years ago was two-thirds of her age at that time, what are their present ages?

20. A rectangle is 10 feet longer than it is broad. Its perimeter is the same as that of a square whose area is 196 square feet. What is the area of the rectangle?

21. Kwame walked from his room to the top of a hill at the speed of $2\frac{1}{2}$ miles per hour. He returned, without resting at the speed of $4\frac{1}{2}$ miles per hour. If he took 42 minutes altogether, how far is his room from the top of the hill?

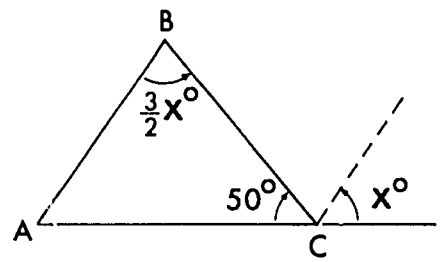
22. One number is greater than another by 15. If five times the larger number minus twice the smaller number is equal to 3, what are the numbers?

23. Musoke bought three plates and four mugs. If each plate cost him three shillings more than each mug and if he paid twenty-three shillings altogether, find the cost of each mug.

24. There are twice as many shilling pieces as there are pound notes in a box. If the total value of the money in the box is 154 pounds, how many shilling pieces are in the box?

25. A boy was told to subtract 7 from a given number and divide the result by 5. Instead he subtracted 5 and divided the result by 7. His answer was 4 less than it should have been. What was the given number?

26. In the figure below the dotted line is parallel to the line through A and B.



If the measures in degrees of the angles are as shown in the figure, what are the measures of the angles of the triangle?

27. <u>CHALLENGE PROBLEM</u>. John was asked to think of a number. First one pupil asked him to add 3 to the number and then multiply the result by 2. A second pupil asked him to subtract 7 from the resulting number and multiply the difference by 5. Finally, a third pupil asked him to subtract 24 from the number just obtained and multiply the result by 3. Suddenly John said, "I now have the same number I started with!" What was John's number?

28. A man has four sons each of whom (except the youngest) is 2 years older than his next younger brother. If the sum of the boys' ages is 52, how old is the oldest son?

29. Divide 102 shillings among three boys A, B, C such that A gets twice as much as B and C gets $1\frac{1}{2}$ times as much as A.

30. "How much did the six eggs cost you?" Kweku asked Ama. "I can't remember," said Ama, "but I bought 5 oranges as well and an orange was 2d.

less than an egg." Since Ama spent 3s. 9d., Kweku was able to find out the price of each egg. How much did each egg cost?

31. CHALLENGE PROBLEM. A bus travelling at the constant speed of 30 miles per hour leaves Accra for Takoradi which is 140 miles away. Another bus travelling at a constant speed leaves Takoradi for Accra 40 minutes after the first bus started. If both buses reach their destinations at the same time, how many miles from Accra were they when they passed each other?

CHAPTER 5

EQUATIONS IN TWO VARIABLES

5-1 An Equation in Two Variables

Suppose that Mr. X and Mr. Y together have eight children. How could we express this fact in the language of algebra?

Since there are two numbers that we do not know, it seems natural to use two variables.

If Mr. X has x children and Mr. Y has y children, then

$$x + y = 8.$$

What are some numbers which will make this equation true? If Mr. X has 2 children, Mr. Y must have 6 children. We show this by saying:

If $x = 2$ and $y = 6$, the equation becomes
$2 + 6 = 8$, which is true.

What are some other pairs of numbers which make the equation true? List them in a table like this.

x	y
2	6
7	1
0	?
6	?
?	?

It takes two numbers, one for x and one for y, to satisfy (or make true) the equation $x + y = 8$. This pair of numbers is written in brackets thus:

$$(2, 6)$$

and we say that

(2, 6) <u>satisfies</u> the equation

or (2, 6) <u>is a solution</u> of the equation

or (2, 6) <u>is a member of the truth set of</u> the equation.

Since it is not the same thing for Mr. X to have 2 children and Mr. Y to have 6 children as it is for Mr. X to have 6 children and Mr. Y to have 2 children, we need a way to show that these are different situations. Your table suggests that (2, 6) and (6, 2) are two different symbols describing these two different situations. Since the order of the two numbers matters, we call (2, 6) <u>an ordered pair of numbers</u> and (6, 2) is a different ordered pair of numbers. How many different ordered pairs of numbers are there which express the number of children of Mr. X and Mr. Y?

In the symbol (6, 2), the first number written is a value of the first variable and the second number written is a value of the second variable. If x and y are the variables, it is customary to consider x the first variable and y the second variable regardless of the order in which they are written in the equation. Thus if we had $3y + x = 18$, the "first variable" would still be x.

Let us look at another example. Find some ordered pairs which satisfy the equation

$$2x + y = 10.$$

For instance (0, 10), (1, 8), (2, 6) are members of the truth set of the equation. Find five more members of the truth set.

Probably you had to suggest an ordered pair such as $(\frac{3}{2}, 7)$ or (8, -6). Is this all right? Since the numbers this time need not represent numbers of children, they need not be whole numbers and we may use any pair of rational numbers which makes the sentence true. Find five more solutions of the equation. How many ordered pairs do you think there are which will satisfy this equation? The pair (2, 6) does satisfy the equation. Does the pair (6, 2) satisfy it?

<u>Problems 5-1</u>

1. Show that the ordered pairs (1, 10), (5, 2), (0, 12), (8, -4) are members of the truth set of the equation $2x + y = 12$.

2. Show that the ordered pairs (6, 0), (2, 8), $(\frac{1}{2}, 11)$ satisfy the equation $2x + y = 12$.

3. Show that the ordered pairs (4, 6), (2, 5), (6, 1) are <u>not</u> members of the truth set of the equation $2x + y = 12$.

4. Find five additional ordered pairs which satisfy the equation $2x + y = 12$.

5. Which of the following ordered pairs satisfy the equation $3x + 4y = 7$?

$$(1, 0), (1, 1), (5, -2), (2, 2), (7, 3), (9, 5)$$

6. Complete the following ordered pairs to give solutions of the equation $x + 3y = 17$.

$$(\quad, 2), (\quad, 5), (\quad, 7), (\quad, 100), (\quad, -2),$$
$$(\quad, \tfrac{4}{3}), (\quad, -\tfrac{7}{3}), (\quad, \tfrac{1}{2}), (\quad, 3\cdot 5)$$

7. If you choose any number whatever for y in the equation $x + 3y = 17$, can you always find the corresponding value of x so that the ordered pair $(x + y)$ is a member of the truth set? Explain how.

8. Complete the following ordered pairs to give solutions of the equation $2x - 3y = 8$.

$$(1, \quad), (7, \quad), (-2, \quad), (5, \quad), (\tfrac{1}{2}, \quad)$$

9. Choose any other value of x and show how the corresponding value of y can always be found to satisfy the equation $2x - 3y = 8$.

10. If x and y are restricted to whole numbers, that is, the domain of x and of y is the set of whole numbers, verify that the truth set of $3x + y = 15$ can be written

$$\{(0, 15), (1, 12), (2, 9), (3, 6), (4, 3), (5, 0)\}$$

11. If the domain of x and of y is the set of whole numbers, how many ordered pairs will there be in the truth set of the equation

$$x - y = 2?$$

Could you write them all as we did in Problem 10?

12. CHALLENGE PROBLEM. Two whole numbers have the property that twice the larger plus three times the smaller is 24. Find the numbers.

5-2 A Problem in Two Variables

I am thinking of two numbers. The first number plus three times the second number is 18. Five times the first number minus three times the second number is also 18. Can you find the numbers?

Here there are two numbers which we do not know and two facts about these numbers are given. This suggests the possibility of using equations in two variables such as those in the preceding section.

If the first number is x and the second number is y, then

$$x + 3y = 18 \text{ and } 5x - 3y = 18$$

If we can find an ordered pair of numbers which satisfies both of these equations, we call this ordered pair a member of the truth set or a solution of the system of simultaneous equations. The word "simultaneous", which means "at the same time", tells us that both equations are to be satisfied by the same pair of numbers. The system is usually written with a bracket to indicate the word "and" between the two equations.

Thus to indicate

Find x and y if

$$x + 3y = 18$$

$$\text{and } 5x - 3y = 18$$

we usually write

$$\begin{cases} x + 3y = 18 \\ 5x - 3y = 18 \end{cases}$$

If $x = 12$ and $y = 2$, the first equation is true: $12 + (3 \times 2) = 18$ is true. The second equation is false: $(5 \times 12) - (3 \times 2) = 18$ is false. Hence $(12, 2)$ satisfies the first equation, but does <u>not</u> satisfy the system of equations.

If $x = 3$ and $y = -1$, the first equation is false: $3 + 3 \times (-1) = 18$ is false. The second equation is true: $(5 \times 3) - 3 \times (-1) = 18$ is true. Hence, $(3, -1)$ satisfies the second equation, but is <u>not</u> a solution of the system of equations.

One way you might find an ordered pair of numbers which satisfies <u>both</u> equations is to make two lists, one of ordered pairs which satisfy the first equation and one of ordered pairs which satisfy the second equation, and see whether the same ordered pair occurs in both lists.

Verify that the following ordered pairs of numbers are members of the truth set of $x + 3y = 18$.

$(0, 6)$	$(-3, 7)$
$(3, 5)$	$(-6, 8)$
$(6, 4)$	$(17, \frac{1}{3})$
$(9, 3)$	$(-2, \frac{20}{3})$
$(12, 2)$	
$(15, 1)$	
$(18, 0)$	
$(21, -1)$	

Verify that the following ordered pairs of numbers are members of the truth set of $5x - 3y = 18$.

$(0, -6)$	$(-3, -11)$
$(3, -1)$	$(-6, -16)$
$(6, 4)$	$(\frac{2}{5}, -\frac{16}{3})$
$(9, 9)$	$(\frac{18}{5}, 0)$
$(12, 14)$	

The ordered pair $(6, 4)$ is in both lists. That is, $6 + (3 \times 4) = 18$ and $(5 \times 6) - (3 \times 4) = 18$ are both true. Hence an answer to our original problem is that the first number is 6 and the second number is 4.

The ordered pair $(6, 4)$ is in the truth set of the <u>system of simultaneous equations</u>

$$\begin{cases} x + 3y = 18 \\ 5x - 3y = 18 \end{cases}$$

Problems 5-2

1. By making two lists as was done above, try to find an ordered pair of numbers in the truth set of the following system:

$$\begin{cases} 4x + 3y = 6 \\ 2x + 5y = -4 \end{cases}$$

2. (a) If the domain of x and of y is restricted to whole numbers, the truth set of the equation $3x + 2y = 18$ is finite. Find it.

   (b) Do the same for the equation $5x + y = 30$.

   (c) Name any ordered pairs which are in both of the truth sets in (a) and (b).

3. Follow the instructions of Problem 2 for the system of equations

$$\begin{cases} 2x + 5y = 35 \\ 6x + y = 23 \end{cases}$$

5-3 Systems of Simultaneous Equations

The method described in the preceding section is not very efficient. It depends on being able to make a good guess. How can we use familiar properties of numbers to find the truth set of a system of equations more easily?

Consider the system

$$\begin{cases} 5x + 3y = 9 \\ -5x + 2y = 1 \end{cases}$$

Our aim will be to find a simpler system which has the same truth set but which has an equation with only one variable. From there on we can use familiar methods.

We shall use the idea that the sum of two numbers is the same no matter what names you use for the numbers. If x and y represent a pair of numbers in the truth set of the system, then $\underline{5x + 3y}$ and $\underline{9}$ are two names for a number and $\underline{-5x + 2y}$ and $\underline{1}$ are names for another number. Since we can use either of the names, the sum of these numbers can be written either as $(5x + 3y) + (-5x + 2y)$ or as $9 + 1$, which means that

$$(5x + 3y) + (-5x + 2y) = 9 + 1$$

is true for the same pair of numbers x and y that we started with.

The advantage of this new equation is that when it is simplified, it has only one variable. We planned it that way.

$$\begin{aligned} (5x + 3y) + (-5x + 2y) &= 9 + 1 \\ 5y &= 10 \\ y &= 2 \end{aligned}$$

Now in the original system of equations we can replace one of the equations (either one) by the simpler equation $y = 2$.

$$\begin{cases} 5x + 3y = 9 \\ y = 2 \end{cases}$$

To make the second equation true y must be 2. Hence y must be 2 in the first equation also. Thus the first equation can be replaced by

$$\begin{aligned} 5x + (3 \times 2) &= 9 \\ \text{or} \qquad 5x &= 3 \\ \text{or} \qquad x &= \frac{3}{5} \end{aligned}$$

Our system has become

$$\begin{cases} x = \frac{3}{5} \\ y = 2 \end{cases}$$

The pair of numbers $(\frac{3}{5}, 2)$ satisfies this last system of equations. Does it satisfy the original system? We must check.

$$(5 \times \tfrac{3}{5}) + (3 \times 2) = 9 \text{ is true}$$

$$(-5) \times \tfrac{3}{5} + (2 \times 2) = 1 \text{ is true.}$$

Hence $\left\{ (\tfrac{3}{5}, 2) \right\}$ is the truth set of the given system.

Suppose we had used the equation $y = 2$ in place of the <u>first</u> equation of the original system instead of the second one, giving the system

$$\begin{cases} y = 2 \\ -5x + 2y = 1 \end{cases}$$

Solve this system and show that you still get the truth set $\left\{ (\tfrac{3}{5}, 2) \right\}$.

Let us consider another example. This time we shall obtain an equation in one variable by using subtraction instead of addition. Do you see why subtraction removes the variable y ? Give a reason for each of the following steps.

$$\begin{cases} 2x + 4y = -12 \\ 5x + 4y = -33 \end{cases}$$

$$(2x + 4y) - (5x + 4y) = (-12) - (-33) \qquad \text{(Why?)}$$

$$-3x = 21 \qquad \text{(Why?)}$$

$$x = -7 \qquad \text{(Why?)}$$

$$\begin{cases} 2x + 4y = -12 \\ x = -7 \end{cases} \qquad \text{(Why?)}$$

$$2(-7) + 4y = -12 \qquad \text{(Why?)}$$

$$4y = 2 \qquad \text{(Why?)}$$

$$y = \tfrac{1}{2} \qquad \text{(Why?)}$$

$$\begin{cases} x = -7 \\ y = \tfrac{1}{2} \end{cases} \qquad \text{(Why?)}$$

Check: $2 \times (-7) + (4 \times \tfrac{1}{2}) = -12$ is true

$5 \times (-7) + (4 \times \tfrac{1}{2}) = -33$ is true.

The truth set is $\left\{ (-7, \tfrac{1}{2}) \right\}$.

<u>Problems 5-3</u>

Find the truth set of each of the following systems of simultaneous equations. Decide carefully when to add and when to subtract.

1. $\begin{cases} 7x - 2y = 29 \\ 7x + y = 38 \end{cases}$

2. $\begin{cases} 10x - 3y = 36 \\ 8x + 3y = 18 \end{cases}$

3. $\begin{cases} 5x - 6y = 16 \\ 7x + 6y = 44 \end{cases}$

4. $\begin{cases} x + 6y = -5 \\ x - 9y = 0 \end{cases}$

5-4 More Systems of Simultaneous Equations

How can we solve the following system?

$$\begin{cases} 3x + 2y = 12 \\ 7x - 4y = 2 \end{cases}$$

Previously we could remove a variable by subtraction or addition because the coefficients of one of the variables in both equations were either the same or opposites. In this example we can bring about a similar situation by replacing the first equation by an equation in which the coefficient of y is 4. Do you see how to do this?

If the equation $3x + 2y = 12$ is true for some ordered pair of numbers, the equation $6x + 4y = 24$ will be true for the same pair of numbers. Why?

If we agree to replace $3x + 2y = 12$ by the equivalent equation $6x + 4y = 24$, we obtain the system

$$\begin{cases} 6x + 4y = 24 \\ 7x - 4y = 2 \end{cases}$$

Can you complete the job of finding a solution? What is the truth set?

Sometimes, to avoid fractions, it is convenient to multiply <u>both</u> equations by a suitably chosen number in order to make the coefficients of one of the variables the same or opposites. For example:

$$\begin{cases} 7x + 4y = 43 \\ 5x + 6y = 26 \end{cases}$$

If we multiply in the first equation by 3 and in the second equation by 2, we obtain an equivalent system in which we can remove y by subtraction.

$$\begin{cases} 21x + 12y = 129 \\ 10x + 12y = 52 \end{cases}$$

Can you now complete the job of finding a solution? What is the truth set?

For the same problem, if you wished to make the coefficients of the x terms the same or opposites, what number would you multiply by in the first equation? What number would you multiply by in the second equation? Show that you obtain the same truth set no matter which variable you choose to eliminate.

Problems 5-4

For problems 1 through 7, find the truth set of each of the systems of simultaneous equations. In problem 1, solve first by eliminating the x variable, and then by eliminating the y variable. After that use whichever of these methods seems simpler.

1. $\begin{cases} 3x - 2y = 7 \\ 4x + 5y = 40 \end{cases}$

2. $\begin{cases} 2x - 7y = 1 \\ 5x + 17y = 37 \end{cases}$

3. $\begin{cases} 8x + 2y = 13 \\ 16x + y = 14 \end{cases}$

4. $\begin{cases} 6x - 5y = 3 \\ 3x + 11y = -39 \end{cases}$

5. $\begin{cases} \frac{2}{5}x + \frac{1}{3}y = 1 \\ \frac{3}{5}x - \frac{1}{9}y = 7 \end{cases}$

6. $\begin{cases} -7x + 12y = -11 \\ 50x + 9y = 47 \end{cases}$

7. $\begin{cases} 12x + 3y = 3 \\ 5 + y = -2 \end{cases}$

8. CHALLENGE PROBLEM. Explain why each of the following steps is justified in finding the truth set of the system.

$$\begin{cases} 3x - 5y = 21 \\ x - 4y = 3 \end{cases}$$

$$\begin{cases} 3x - 5y = 21 \\ x = 3 + 4y \end{cases}$$

$$\begin{cases} 3(3 + 4y) - 5y = 21 \\ x = 3 + 4y \end{cases}$$

Finish finding the truth set of the system.

9. CHALLENGE PROBLEM. Use the method suggested in problem 8 to solve the following systems of simultaneous equations.

(a) $\begin{cases} 2x + 6y = 13 \\ 8x + y = 6 \end{cases}$ (b) $\begin{cases} x + 3y = 5 \\ 7x - 6y = 44 \end{cases}$

10. CHALLENGE PROBLEM. Solve the system of simultaneous equations

$$\begin{cases} 3x + 1 = 0 \\ xy + 2 = 0 \end{cases}$$

5-5 Word Problems Using Two Variables

One person bought 2 pounds of tea and 10 pounds of sugar and paid 24 shillings. At the same prices another person paid 27 shillings for 1 pound of tea and 20 pounds of sugar. What was the price per pound for tea and for sugar?

If tea cost t shillings per pound and sugar cost s shillings per pound, then

$$\begin{cases} 2t + 10s = 24 \\ t + 20s = 27 \end{cases}$$

$$\begin{aligned} (2t + 10s) - 2(t + 20s) &= 24 - (2 \times 27) \\ 2t + 10s - 2t - 40s &= 24 - 54 \\ -30s &= -30 \\ s &= 1 \end{aligned}$$

$$\begin{cases} 2t + 10s = 24 \\ s = 1 \end{cases}$$

$$\begin{aligned} 2t + (10 \times 1) &= 24 \\ 2t &= 14 \\ t &= 7 \end{aligned}$$

$$\begin{cases} t = 7 \\ s = 1 \end{cases}$$

Rather than check in our equations it is safer to check in the words of the problem. If we have made an error in setting up the equations, checking in these incorrect equations will not catch our error.

If tea cost 7 shillings per pound and sugar cost 1 shilling per pound, then

the first person paid 14 shillings for tea and 10 shillings for sugar, or a total of 24 shillings;

the second person paid 7 shillings for tea and 20 shillings for sugar, or a total of 27 shillings.

This is correct. Price of tea: 7 shillings per pound.

Price of sugar: 1 shilling per pound.

Problems 5-5

Solve the following problems, using systems of equations in two variables:

1. I am thinking of two numbers. The first number plus three times the second number is 1. The first minus three times the second is 19. Find the numbers.

2. The sum of two numbers is $\frac{53}{24}$ and the difference of the same two numbers is $\frac{13}{24}$. What are the numbers?

3. In a class there are 10 more boys than there are girls. If there were one more girl, there would be twice as many boys as girls. How many boys and how many girls are in the class?

4. I was able to buy 12 pencils and 10 rulers for 21 shillings. At the same price each I could buy 20 pencils and 4 rulers for 16 shillings. What is the price of one pencil and the price of one ruler?

5. In triangle ABC, $\angle A$ is an angle of 20°. The number of degrees in $\angle B$ is 10 more than 9 times the number of degrees in $\angle C$. Find the size of $\angle B$ and $\angle C$.

6. A rope was cut into two pieces so that one piece was 18 ft. longer than the other. It was also true that one piece was three times as long as the other. How long was each piece and how long was the original rope?

7. An angle is three times its supplement. Find the size of the angle.

8. Twice the length of a rectangle is three times the width. The perimeter is 320 in. Find the dimensions.

9. A boy walked for 4 hours and bicycled for 3 hours covering a total distance distance of 45 miles. Later he walked for 2 hours and bicycled for 4 hours covering 50 miles. What was his average speed of walking and what was his average speed of riding a bicycle if these did not change in the two cases?

10. A carpenter can build a cabinet in 30 hours. If he works for 21 hours at the normal rate of pay, and 9 hours at the overtime rate, his working time costs 56 shillings, while if he works for 27 hours at the normal rate and 3 hours at the overtime rate, his working time costs 52 shillings. Find his normal and overtime rates of pay.

11. <u>CHALLENGE PROBLEM</u>. In decimal numerals a two digit number has the property that it is smaller by 27 than the number with the digits reversed. The sum of the digits of the number is 13. Find the number. (Hint: If the tens' digit is x and the units' digit is y, then the number is $10x + y$. What is the number with the digits reversed?)

12. <u>CHALLENGE PROBLEM</u>. In decimal numerals, if 36 is added to a certain two digit number the result in the number with the digits reversed. The units' digit is the one more than twice the tens' digit. What is the number?

13. <u>CHALLENGE PROBLEM</u>. Two trains leave stations 200 miles apart, the first at noon, the second at 12:15 P.M. They approach each other on parallel tracks and pass at 3 P.M. Each train travels at a constant speed, the one leaving at 12:15 P.M. moving at 10 m.p.h. faster than the other. What are their speeds.

14. <u>CHALLENGE PROBLEM</u>. Two trains leave stations 200 miles apart, the first at noon and the second at 12:15 P.M., each train travelling at a constant speed. They approach each other on parallel tracks and pass at 3 P.M. If

the times of departure had been reversed, they would still have met at 3:00 P.M. What are their speeds?

15. CHALLENGE PROBLEM. The total weight of a mixture of two liquids is 24 lbs. wt. and the total volume is 10 cu. ft. If one cubic foot of one of the liquids weighs 2 lbs. wt. and one cubic foot of the other liquid weighs 3 lbs. wt., what volume of each liquid is present? What weight of each liquid is present?

CHAPTER 6

A STRANGE ARITHMETIC

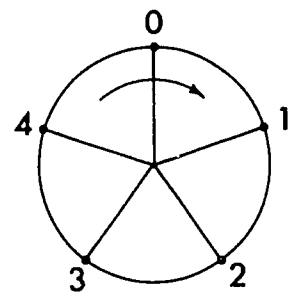
6-1 Introduction

We are going to play a kind of game that will turn out to be quite a bit like arithmetic. This game will help you to understand arithmetic and algebra.

You will remember how we drew a number line to show the whole numbers, and how we could do addition, geometrically, on the line. Now let us see what happens if we arrange numbers on a circle, like a clock. To keep it simple suppose that we decide to use only the five numbers,

0, 1, 2, 3, 4

arranged as in the picture.



We are going to define a strange arithmetic using only these numbers.

6-2 Addition

Let us think of the circle as a kind of clock with five "hours". We shall add the numbers just as we add hours on a clock - when we get past 0 we begin counting all over again. Using this idea we can define an operation, which we call addition. We shall use the symbol $\oplus$ for this operation in order to suggest that it comes from the circle, and also to show that it is not ordinary addition. Examples will show how the game is played.

$1 \oplus 2 = 3$	Because 1 "o'clock" plus 2 "hours" is 3 "o'clock".
$2 \oplus 3 = 0$	Because 2 "o'clock" plus 3 "hours" is 0 o'clock".
$3 \oplus 3 = 1$	Because 3 "o'clock" plus 3 "hours" is 1 "o'clock".
$3 \oplus 4 = 2$	Why ?

We can make an addition table. Part of such a table is shown at the right, where the above "sums" have been filled in.

Draw an addition table for ⊕ and fill in all the blanks.

⊕	0	1	2	3	4
0					
1			3		
2				0	
3				1	2
4					

How is the table different from the addition table for ordinary addition?

How is the table like the table for ordinary addition? Is addition commutative?

Does every number have an additive inverse (or opposite)?

<u>Another rule for ⊕</u>. Here is another way of obtaining the "clock sum" of two numbers:

We shall add them as if they were ordinary numbers with ordinary addition <u>except</u> that if the ordinary sum is 5 or more then we subtract 5 in order to get a number from 0 to 4.

<u>Examples</u>

$1 \oplus 1 = 2$	Because $1 + 1 = 2$
$0 \oplus 3 = 3$	Because $0 + 3 = 3$
$3 \oplus 4 = 2$	Because $3 + 4 = 7$, which is too big, so we subtract 5.
$2 \oplus 3 = 0$	Because $2 + 3 = 5$, which is too big.

Do you see that this rule for addition always gives the same sum as addition on the clock?

Problems 6-2

1. Express the following sums as one of the numbers 0, 1, 2, 3, 4. You may use either the rule for $\oplus$ based on the clock, or the other rule for $\oplus$.

(a) $4 \oplus 1$

(b) $1 \oplus 4$

(c) $3 \oplus 3$

(d) $3 \oplus 4$

(e) $4 \oplus 3$

(f) $0 \oplus 4$

(g) $4 \oplus 0$

(h) $1 \oplus (2 \oplus 3)$

(i) $(1 \oplus 2) \oplus 3$

(j) $2 \oplus (4 \oplus 3)$

(k) $(2 \oplus 4) \oplus 3$

(l) $3 \oplus (4 \oplus 0)$

(m) $(3 \oplus 4) \oplus 0$

(n) $1 \oplus (1 \oplus (1 \oplus (1 \oplus 1)))$

2. You can see from the addition table that addition has the commutative property. Explain why this is so.

3. Addition also has the associative property. What parts of Problem 1 illustrate this?

How might you show that the associative property holds?

4. Find the truth set of the following equations by testing all the numbers, 0, 1, 2, 3, 4. Is there only one number in each truth set? For example, to solve $x \oplus 1 = 2$ we try, in turn, $x = 0, 1, 2, 3, 4$ and find that only $x = 1$ makes the equation true. You may use the addition table.

(a) $x \oplus 1 = 3$

(b) $x \oplus 1 = 4$

(c) $x \oplus 1 = 0$

(d) $x \oplus 1 = 1$

(e) $x \oplus 3 = 0$

(f) $x \oplus 3 = 1$

(g) $x \oplus 3 = 2$

(h) $x \oplus 3 = 3$

(i) $x \oplus 4 = 2$

(j) $x \oplus 4 = 0$

Could you solve these equations in any other way?

5. Perform the following subtractions. Remember that $\underline{a - b = c}$ means that $a = b + c$. Therefore, if $\ominus$ denotes clock subtraction, $a \ominus b = c$ means $a = b \oplus c$.

(a) $3 \ominus 0$

(b) $3 \ominus 1$

(c) $3 \ominus 2$

(d) $3 \ominus 3$

(e) $3 \ominus 4$

(f) $1 \ominus 0$

(g) $1 \ominus 1$

(h) $1 \ominus 2$

(i) $1 \ominus 3$

(j) $1 \ominus 4$

6. Let-us define in our arithmetic, the <u>additive inverse</u>, or <u>opposite</u>, of x to be $\ominus x$, such that $x \oplus (\ominus x) = 0$. Find the additive inverse of each of the numbers 0, 1, 2, 3, 4. Show that each of the subtraction problems of Problem 5 can be done by adding the opposite.

7. How can you see from the addition table that subtraction is always possible?

8. Clock arithmetic is used in telling time. There are twelve numbers on the clock. What is

$2 \oplus 5$?

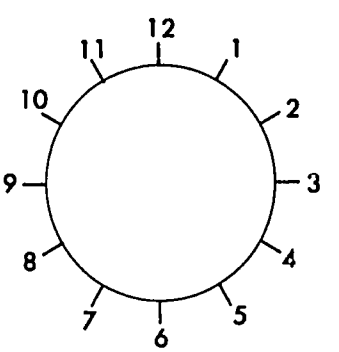
$5 \oplus 7$?

$5 \oplus 8$?

$7 \oplus 8$?

$10 \oplus 4$?

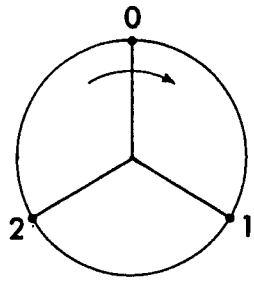
$11 \oplus 11$?



Is there a clock number that behaves like zero?

9. <u>CHALLENGE PROBLEM</u> One can invent other games similar to our game with 0, 1, 2, 3, 4. For example one can define, in a similar way, addition

when the set of numbers consists only of 0, 1, 2.



Make an addition table for this game.

10. CHALLENGE PROBLEM Invent and make addition tables for other games with the set of

(a) four numbers, 0, 1, 2, 3

(b) two numbers, 0, 1.

6-3 Multiplication

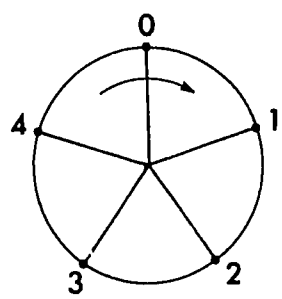
We shall now define, in our system, 0, 1, 2, 3, 4, an operation which we call multiplication and denote by $\otimes$.

Examples

$2 \otimes 3 = 3 \oplus 3 = 1$
(that is two 3's)

$3 \otimes 4 = 4 \oplus 4 \oplus 4 = 2$
(that is three 4's)

$0 \otimes 2 = 0$
(that is no 2's)



Why is $2 \otimes 4 = 3$? $4 \otimes 4 = 1$?

We can now make a multiplication table for $\otimes$. Part of such a table is shown below where the products already obtained have been filled in.

$\otimes$	0	1	2	3	4
0			0		
1			2		
2				1	3
3					2
4					1

Draw a multiplication table for $\otimes$ and fill in all the blanks.

How is the table different from the multiplication table for ordinary multiplication?

How is the table like the table for ordinary multiplication? Is multiplication commutative? Does every number have a reciprocal, or multiplicative inverse?

<u>Another rule for $\otimes$</u>. Here is another way of obtaining the "clock product" of numbers:

We shall multiply them as if they were ordinary numbers with ordinary multiplication <u>except</u> that if the ordinary product is 5 or more then we subtract a multiple of 5 to get a number from 0 to 4.

<u>Examples</u>

$0 \otimes 1 = 0$, because $0 \times 1 = 0$

$1 \otimes 3 = 3$, because $1 \times 3 = 3$

$2 \otimes 3 = 1$, because $2 \times 3 = 6$, which is too big, so we subtract 5.

$3 \otimes 4 = 2$, because $3 \times 4 = 12$, which is too big, so we subtract 10.

Why is $2 \otimes 4 = 3$? $4 \otimes 4 = 1$?

Problems 6-3

1. Express the following products as one of the numbers 0, 1, 2, 3, 4.

(a) $2 \otimes 3$

(b) $3 \otimes 2$

(c) $2 \otimes 4$

(d) $4 \otimes 2$

(e) $4 \otimes (3 \otimes 2)$

(f) $(4 \otimes 3) \otimes 2$

(g) $2 \otimes (3 \otimes 3)$

(h) $(2 \otimes 3) \otimes 3$

(i) $2 \otimes (2 \otimes [2 \otimes 2])$

2. Do you think that multiplication has the associative property? How could you decide?

3. Are the following equations true?

(a) $3 \otimes (4 \oplus 1) = (3 \otimes 4) \oplus (3 \otimes 1)$

(b) $3 \otimes (2 \oplus 3) = (3 \otimes 2) \oplus (3 \otimes 3)$

(c) $4 \otimes (4 \oplus 4) = (4 \otimes 4) \oplus (4 \otimes 4)$

(d) $4 \otimes (3 \oplus 0) = (4 \otimes 3) \oplus (4 \otimes 0)$

(e) $1 \otimes (1 \oplus 1) = (1 \otimes 1) \oplus (1 \otimes 1)$

Do you think that there is a distributive property of multiplication with respect to addition?
How could you decide?

4. Find the truth set of the following equations by testing all numbers, 0, 1, 2, 3, 4. You may use the multiplication table if you choose.

(a) $1 \otimes x = 3$

(b) $2 \otimes x = 3$

(c) $3 \otimes x = 3$

(d) $4 \otimes x = 3$

(e) $0 \otimes x = 3$

(f) $1 \otimes x = 2$

(g) $2 \otimes x = 2$

(h) $3 \otimes x = 2$

(i) $4 \otimes x = 2$

(j) $3 \otimes x = 1$

5. Find the truth sets of the following equations.

(a) $1 \otimes x = 1$ (c) $3 \otimes x = 1$

(b) $2 \otimes x = 1$ (d) $4 \otimes x = 1$

Why does this show that every non-zero number has a reciprocal?

6. Let us define division by saying that $a \div b = c$ means that $a = b \otimes c$. Explain why $1 \div 2 = 3$.

7. How can you see from the multiplication table that division (except by 0) is always possible?

8. <u>CHALLENGE PROBLEM</u> See Challenge Problem 9 of Problems 6-2. What is the rule for multiplication for the clock arithmetic with 0, 1, 2? Is $\otimes$ commutative? Can you solve equations like $2 \otimes x = 1$ in this arithmetic? Does every non-zero number have a reciprocal?

9. <u>CHALLENGE PROBLEM</u> See Challenge Problem 10 of Problems 6-2. Investigate multiplication for each of the parts (a) and (b).

CHAPTER 7

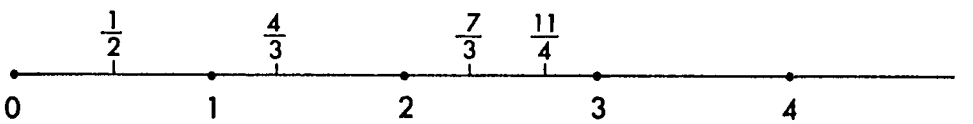
FIELDS

7-1 Number Systems

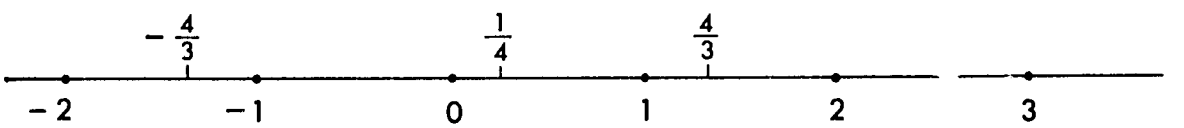
In your study of arithmetic you have met several sets of numbers. You learned first about the whole numbers.

0, 1, 2, 3, ... 99, 100, 101, ... 5833, 5834, ...

Then you learned about the positive rational numbers which you represented on a line.



Finally, you have studied the entire set of rational numbers, positive, negative, and zero. These too you pictured on a line.



Each of these sets of numbers, with their operations of addition and multiplication, can be called a <u>number system.</u> In some of these number systems you cannot perform operations that can be done in others. Thus, for example, with only whole numbers you cannot divide 2 by 3 because $2 \div 3$ is not a whole number. Yet, in the set of positive rational numbers you can divide 2 by 3.

Problems 7-1a

1. In the system of <u>whole numbers</u>, is the sum of two whole numbers a whole number? Is the product of two whole numbers a whole number? Can you always subtract two whole numbers? Can you always divide two whole numbers?

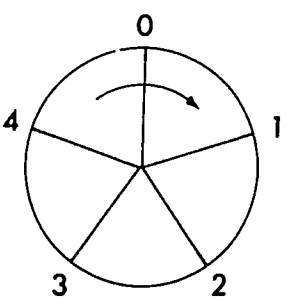
2. In the system of <u>integers</u>,... -2, -1, 0, 1, 2, ..., are the sum and product of two integers always integers? Can you always subtract two integers? Can you always divide two integers?
3. In the system of <u>positive rational numbers</u> are the sum and product of two such numbers always positive rational numbers? Can you always subtract two positive rational numbers? Can you always divide two positive rational numbers?
4. In the system of <u>all rational numbers</u>, which of the operations of addition, subtraction, multiplication, and division are always possible?
5. In which of the number systems do you have the most freedom to add, subtract, multiply, and divide?
6. In each of these number systems is addition commutative? Associative? In each of these number systems is multiplication commutative? Associative? In each of these number systems does the distributive property hold?

In Chapter 6 you studied a strange arithmetic that used the numbers

0, 1, 2, 3, 4

with different addition and multiplication, $\oplus$ and $\otimes$.

The problems which follow will call your attention to some properties of this number system.



Problems 7-1b

1. In the strange arithmetic with 0, 1, 2, 3, 4 and $\oplus$ and $\otimes$ you have a number system. In this number system:

(a) Is the sum of two numbers in the system?

(b) Is the product of two numbers in the system?

(c) Can you always subtract two numbers and get a number of the system?

(d) Can you always divide two numbers of the system and get a number of the system?

2. In the strange arithmetic with 0, 1, 2, 3, 4 and $\oplus$ and $\otimes$ are the commutative, associative, and distributive properties true?

7-2 Fields

In a number system the two fundamental operations are addition and multiplication. Sometimes, in a number system, it is also possible to subtract and divide any two numbers (except for division by zero). Number systems in which, besides adding and multiplying, one can always subtract and divide (except by zero) are especially important. Mathematicians have a name for such number systems. They are called fields.*

You already know about two fields. They are:

(1) The system of rational numbers.

(2) The system with 0, 1, 2, 3, 4, $\oplus$ and $\otimes$.

---

*This is a peculiar name. Do not try to imagine that its meaning can be figured out from the word itself. The word is just a name, just as "John" is a name.

In chapter 16 you will meet another field.

The definition which is given below lists the basic properties that all fields have. You should not have to memorize these properties because they are familiar to you from arithmetic.

Definition of a Field

A field is a set of numbers in which there are two operations, called addition and multiplication, such that the following properties hold.

1. Closure Properties

For any numbers, x and y, in the field their sum $x + y$ is a definite number in the field, and their product xy is a definite number in the field.

2. Commutative Properties

For any numbers, x and y, in the field,

$$x + y = y + x \quad \text{and} \quad xy = yx$$

3. Associative Properties

For any numbers, x, y, and z, in the field,

$$x + (y + z) = (x + y) + z \quad \text{and} \quad x(yz) = (xy)z$$

4. Distributive Property

For any numbers x, y, and z in the field,

$$x(y + z) = xy + xz$$

5. Identity Properties

There are two special numbers in the field, denoted by 0 and 1, such that if x is any number in the field then

$$0 + x = x \quad \text{and} \quad 1x = x$$

6. Inverse Properties

(a) For each number x in the field there is exactly one number, $-x$, in the field, called the additive inverse of x, or the opposite of x, such that

$$x + (-x) = 0$$

(b) For each non-zero number x in the field there is exactly one number, $\frac{1}{x}$, in the field, called the multiplicative inverse of x, or reciprocal of x, such that

$$x \frac{1}{x} = 1$$

Problems 7-2

1. Determine, for each of the following number systems, which of the six field properties hold.
   (a) The whole numbers.
   (b) The odd whole numbers.
   (c) The even whole numbers.
   (d) The integers.
   (e) The negative integers.
   (f) The positive rational numbers and zero.
   (g) The negative rational numbers and zero.
   (h) The set of whole numbers [0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10] with ordinary addition and multiplication.
   (i) The numbers [0, 1, 4] with $\oplus$ and $\otimes$.
   (j) The numbers [0, 1, 2, 3, 4] with $\oplus$ and $\otimes$.
2. Suppose that you are thinking of a set of rational numbers for which all six field properties hold. What must your set of rational numbers be?

3. <u>CHALLENGE PROBLEM</u> In every field it is true that for every number x

$$0 \times x = 0$$

Explain each of the steps, in the argument given below, which proves this fact.

$$\begin{aligned} 0x &= (0 + 0)x \\ &= 0x + 0x \end{aligned}$$

Also, it is true that

$$\begin{aligned} 0 &= (0x) + [-(0x)] \\ &= [(0x) + (0x)] + [-(0x)] \\ &= (0x) + [(0x) + [-(0x)]] \\ &= 0x + 0 \\ &= 0x \end{aligned}$$

GLOSSARY

VOLUME I

Addition, basic properties of (page 14): Fundamental properties of addition from which other properties may be derived.

Additive inverse (page 12): For any number x, the additive inverse is that number a such that $x + a = 0$. See opposite.

Associative property of addition (page 14): $a + (b + c) = (a + b) + c$

Associative property of multiplication (page 22): $a(bc) = (ab)c$

Commutative property of addition (page 14): $a + b = b + a$

Commutative property of multiplication (page 22): $ab = ba$

Difference (page 16): The number resulting from the operation of subtraction.

Distributive property (page 22): $a(b+c) = ab + ac$

Division (page 27): The operation in $a \div b$ or $\frac{a}{b}$. $a \div b = x$ $(b \neq 0)$ means that $bx = a$.

Domain of a variable (page 31): The set of numbers from which the value of the variable may be chosen.

Equivalent equations (page 32): Equations which have the same truth set.

Equivalent inequalities (page 38): Inequalities which have the same truth set.

Field (page 80): A number system having the properties of (1) closure for addition and multiplication, (2) commutativity for addition and multiplication, (3) associativity for addition and multiplication, (4) distributivity, (5) additive and multiplicative identities, (6) additive and multiplicative inverses.

Formula (page 47): A representation of a number in terms of one or more variables.

Minus (page 17): Indicates subtraction of the two numbers mentioned, the second from the first.

Multiplication, basic properties of (page 22): Fundamental properties of multiplication from which other properties may be derived.

Multiplicative inverse (page 28): Two numbers a and b are multiplicative inverses of one another if $ab = 1$. Same as reciprocal.

Negative numbers (page 3): The numbers which correspond to points to the left of zero on the number line.

One (page 22): That number which, multiplied by any number x, gives the same number x as the result.

Open sentence (page 31): Sentence containing a variable.

Opposites (page 6): Two numbers are opposites if they correspond, on the number line, to points on opposite sides of 0 and equally far from 0. Also, the opposite of 0 is 0. The opposite of x is denoted by $-x$. Opposites have the basic property: $x + (-x) = 0$.

Ordered pair of numbers (page 56): Two numbers of which one is designated as first and the other second, written (x, y).

Positive numbers (page 2): The numbers which correspond to points to the right of zero on the number line.

Rational number (page 3): A number which can be expressed as the quotient of two integers.

Reciprocal (page 28): Two numbers a and b are reciprocals of one another if $ab = 1$. Same as multiplicative inverse.

Smaller of two numbers (page 37): The one that must have a positive number added to it to give the other.

Solution of an equation in two variables (page 55): An ordered pair of numbers for which the equation is true.

Solution set (pages 31 and 58): See truth set.

Solve an equation or an inequality (page 31): Find the truth set.

Subtraction (page 16): The operation in $a - b$. $a - b = x$ means that $b + x = a$.

Truth set of an equation in one variable (page 31): The set of all numbers which make the equation true.

Truth set of a system of two equations in two variables (page 58): The set of all ordered pairs of numbers which make both equations true.

Variable (page 47): A symbol, usually a letter, used to represent any one of a given set of numbers.

Zero (page 13): That number which, added to any number x, gives the same number x as the result.

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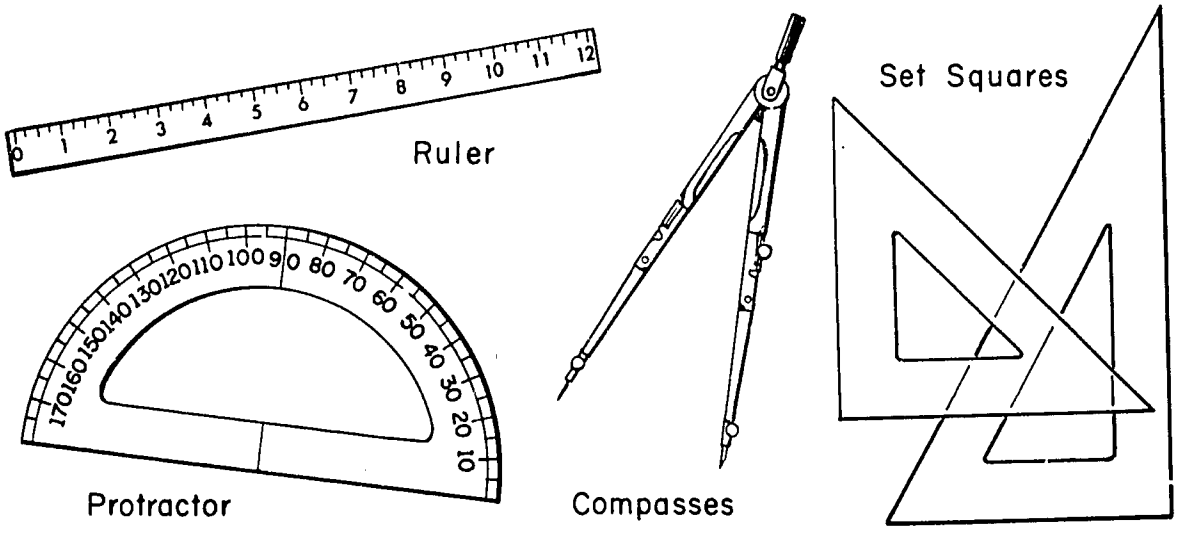
Glossary

CHAPTER 8

USE OF INSTRUMENTS

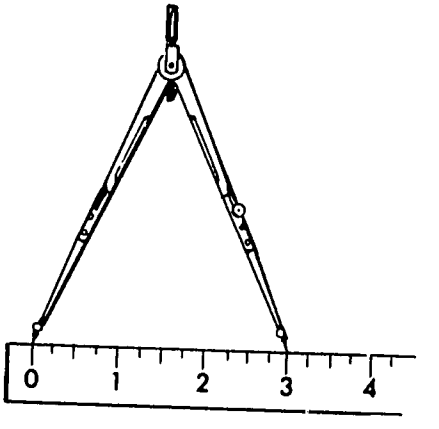
8-1 The Instruments

In this chapter we shall consider how the ruler, the pair of compasses, the protractor and the set squares can be used for drawing parallel and perpendicular lines, for drawing and measuring angles, and for the construction of triangles.

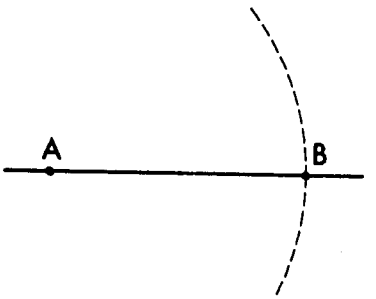


8-2 Ruler and Compasses

Open your compasses on the ruler to give an opening of 3 inches. Be sure the pencil point of the compasses is sharp.



Draw a line segment and mark a point A on the segment. Draw with your compasses a circle with centre at A and radius 3 inches.



If the circle cuts the line at B, how long is the segment $\overline{AB}$? Measure it with your ruler. Why is this method of drawing a segment 3 inches long better than simply measuring with your ruler?

Problems 8-2

1. Draw a line segment and select a point A near one end. With your compasses mark off a segment $\overline{AB}$ 2 inches long. From the point B mark off another segment $\overline{BC}$ 2 inches long. Finally mark off a third segment $\overline{CD}$ 2 inches long. Now use your ruler to measure the total length AD of the three segments. How accurate were you?

2. Using your compasses, draw a segment $\overline{AB}$ 5 inches long. Draw a circle with centre A and a radius of 3 inches, and draw a circle with centre B and a radius of 2·5 inches. Mark the two points of intersection of the circles P and Q. Draw $\overline{PQ}$. The segments $\overline{PQ}$ and $\overline{AB}$ will meet at a point O. Measure $\overline{OP}$ and $\overline{OQ}$. Measure $P\hat{O}A$. Compare your measures with those of others in the class.

3. In each of the following draw two coplanar circles:

a circle with centre at A and radius r,

a circle with centre at B and radius R.

In each case observe in how many points the two circles intersect.

All measurements are in inches.

(a) AB = 3, r = 2, R = $2\frac{1}{2}$

(b) AB = 3, r = 1, R = $2\frac{1}{2}$

(c) AB = 3, r = 1, R = 2

(d) AB = 3, r = 1, R = $1\frac{1}{2}$

(e) $AB = 2, \quad r = \frac{1}{2}, \quad R = 1$

(f) $AB = 2, \quad r = 1, \quad R = 1$

(g) $AB = 2, \quad r = 1, \quad R = 1\frac{1}{2}$

(h) $AB = 2, \quad r = 1, \quad R = 2\frac{1}{2}$

(i) $AB = 2, \quad r = 1, \quad R = 3$

(j) $AB = 2, \quad r = 1, \quad R = 3\frac{1}{2}$

4. On the basis of your experience in Problem 3 try to answer the following question:

If two coplanar circles have centres at A and B and radii of length r and R respectively, what must be true about AB, r, and R if the circles are to intersect in two points?

8-3 The Ruler and Set Squares

Each of your set squares is a right triangle. What are the angles in these triangles?

With either of the set squares we can draw, through a point O, a line perpendicular to a given line $\overleftrightarrow{PQ}$ and if O is not on $\overleftrightarrow{PQ}$ we can draw another line, through O, parallel to $\overleftrightarrow{PQ}$. Here is how to do it.

Draw a line $\overleftrightarrow{PQ}$ and mark a point O not on it. Then place the set square so that one of its shorter sides $\overline{AC}$ lies along $\overleftrightarrow{PQ}$. Place the straight edge along the longest side $\overline{AB}$ of the set square as in Fig. 1. Holding the ruler firmly, slide the set square along the straight edge until the side $\overline{AC}$ passes through O.

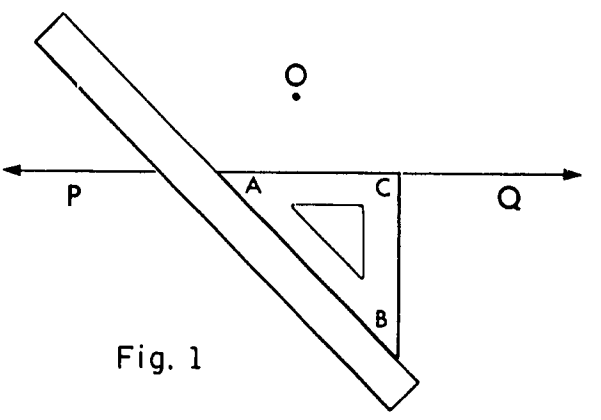


Fig. 1

Draw the line $\overleftrightarrow{OL}$ along side $\overline{AC}$ as in Fig. 2.

What property does the line $\overleftrightarrow{OL}$ have? What do you observe about angle A and angle 1 in Fig. 2 ? Why do these angles have the same measure? Will $\overline{AC}$ always be parallel to $\overleftrightarrow{PQ}$ as long as $\overline{AB}$ stays against the edge of the ruler?

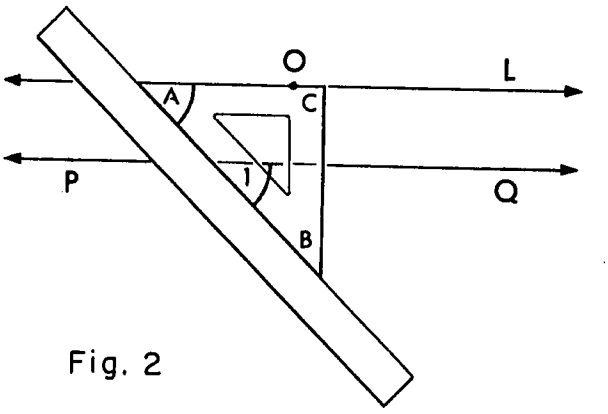


Fig. 2

Now slide the set square along the straightedge until side $\overline{BC}$ passes through O. Draw the line $\overleftrightarrow{OX}$ along side $\overline{BC}$ as in Fig. 3.

What property does $\overleftrightarrow{OX}$ have? Is side $\overline{CB}$ parallel to the original position of $\overline{CB}$ in Fig. 1? Why? Was $\overline{CB}$ perpendicular to $\overleftrightarrow{PQ}$ in Fig. 1? Why? If $\overline{CB}$ moves parallel to itself, will it continue to be perpendicular to $\overleftrightarrow{PQ}$?

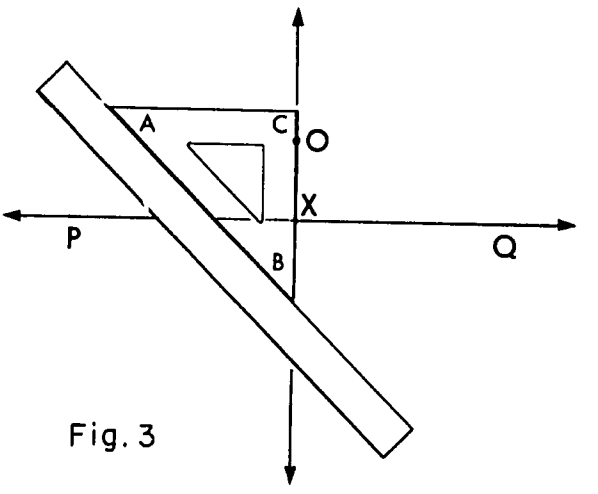


Fig. 3

Figures 1, 2 and 3 are drawn for the case where O is not on the line $\overleftrightarrow{PQ}$. Explain what you would do if O were on $\overleftrightarrow{PQ}$.

Problems 8-3

1. Draw a line $\overleftrightarrow{AB}$. Select a point C not on $\overleftrightarrow{AB}$. Use a set square and straightedge to draw lines through C which are parallel and perpendicular to $\overleftrightarrow{AB}$.

Then choose a point D on $\overleftrightarrow{AB}$ and draw a line through D perpendicular to $\overleftrightarrow{AB}$.

2. Draw a line and mark a point A on it. Use your compasses to mark a point B on the line such that $\overline{AB}$ is 3 inches long.

Use your set square to draw a line through B and perpendicular to $\overleftrightarrow{AB}$. Mark a point C on this perpendicular line such that $\overline{BC}$ is 3 inches long.

3. Use your straightedge and compasses to draw a segment 2 inches long. Then draw a square one of whose sides is the 2 inch segment. Draw a diagonal of the square. How long is it?

4. Draw any triangle. Using a set square, draw, from each vertex a line perpendicular to the opposite side (or to the line containing the opposite side). What do you observe about these three lines?

5. Draw a segment $\overline{AB}$. By measurement find its midpoint. Draw a line, perpendicular to the segment, through the midpoint.

Choose a point C on this perpendicular. Measure $\overline{AC}$ and $\overline{CB}$. What do you find? Would this always be true?

6. Draw any triangle. By measurement find the midpoints of each of the sides. At each midpoint draw a line perpendicular to that side. What do you observe about these lines?

7. Draw two parallel lines $\overleftrightarrow{DE}$ and $\overleftrightarrow{FG}$. Draw a line cutting $\overleftrightarrow{DE}$ at X and $\overleftrightarrow{FG}$ at Y. Name a pair of corresponding angles at X and Y. Are they congruent?

8-4 Construction of Triangles

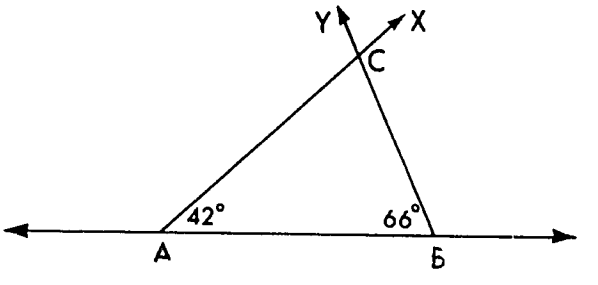
A triangle has three sides and three angles making six elements in all. Of these six elements how many must we know before the triangle can be drawn?

Can we draw a triangle if we know only two of its elements? For example, are two sides, or two angles, or one side and one angle enough to determine a triangle?

Given Two Angles

Suppose two angles of a triangle are angles of 42° and 66°. Could you find the measure of the third angle? How?

If you drew a triangle having an angle of 42° and an angle of 66°, would the third angle of the triangle have to be an angle of 72° in spite of anything you do? Why?



On a line choose two points A and B. With your protractor determine ray $\overrightarrow{AX}$ so that the measure of $\widehat{XAB}$ is 42, and ray $\overrightarrow{BY}$ on the same side of $\overleftrightarrow{AB}$ so that the measure of $\widehat{ABY}$ is 66. Let C be the point where these rays intersect. Measure $\widehat{ACB}$.

If the points A and B had been chosen closer together or farther apart, would the angles have behaved differently? Would the triangle ABC have the same shape? Would it have the same size?

Problems 8-4a

1. Draw a triangle with angles of 50°, 60°, and 70°.
Draw another triangle with the same size angles. Are the two triangles alike in all their parts? Are your triangles like those of other pupils in the class?

2. In triangle ABC the angles at A and B have measures of 40° and 60° respectively and side $\overline{AB}$ is 5 cm. long. Draw the triangle.

Draw a triangle PQR in which the angles at P and Q are 40° and 60° angles respectively and $\overline{PQ}$ is 7 cm. long. In the two triangles measure $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, $\overline{QR}$, $\overline{RP}$, $\overline{PQ}$. If these measures are respectively a, b, c, p, q, r find the ratios $\frac{a}{p}$, $\frac{b}{q}$, $\frac{c}{r}$. What do you notice?

3. What conclusion have you reached about whether knowing two angles is enough information for you to be able to draw a definite triangle?

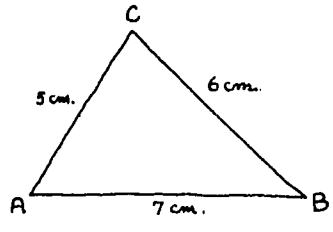
Given Three Sides

We have seen that to know the three angles of a certain triangle will not be enough for us to draw the triangle. If, however, we know the three sides of a triangle can we draw the triangle?

How would you construct a triangle with your ruler and compasses if you knew the three sides?

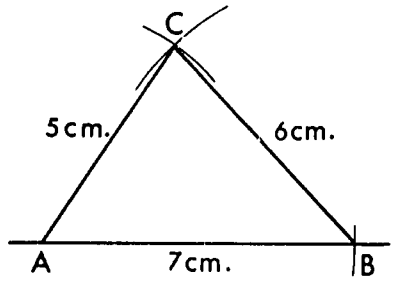
Example.

Construct, a triangle, $\triangle ABC$, such that $\overline{AB}$ is 7 cm. long, $\overline{BC}$ is 6 cm. long, and $\overline{AC}$ is 5 cm. long.



First make a rough sketch and label it correctly.

Next draw a segment, $\overline{AB}$, of length 7 cm. Since $\overline{BC}$ is 6 cm. long, C lies on the circle with centre B and radius 6 cm.



Since $\overline{AC}$ is 5 cm. long, C lies on a circle with centre A and radius 5 cm. With centre B and radius 6 cm. draw an arc about where you expect C to be.

With centre A and radius 5 cm. draw an arc cutting the former arc. Why will the two arcs intersect? The point of intersection of the two arcs is C. Draw $\overline{BC}$ and $\overline{CA}$. Will the method of this example always work?

Problems 8-4b

1. Taking 1 inch as unit of measure, construct a triangle PQR such that PQ = 5, QR = 4 and PR = 3. Measure $\hat{R}$.

2. Taking 1 cm. as unit of measure, construct a triangle ABC such that AB = 2, BC = 6·5 and AC = 5. Measure $\hat{C}$.

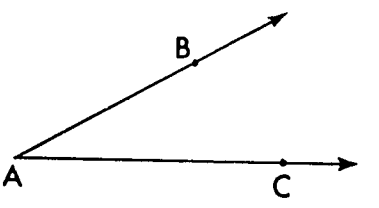
3. Taking 1 cm. as unit of measure, construct a quadrilateral ABCD having AB = 4, BC = 4, CD = 5, DA = 5, AC = 3. Measure $\widehat{DAB}$ and $\widehat{DCB}$. (First draw a rough sketch to help you.)

4. Try to draw triangles whose sides are of lengths
   (a) 7 cm., 4 cm., 2 cm.
   (b) 7 cm., 4 cm., 3 cm.
   (c) 7 cm., 4 cm., 5 cm.

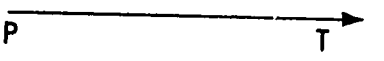
Let x, y, z, be the measures of the sides of a triangle. If x is the measure of the greatest side, what can you say about x and y + z?

5. Draw any three triangles and verify that the sum of the measures of any two sides of a triangle is greater than the measure of the third side.

6. We now can develop a way of copying any given angle without measuring it.

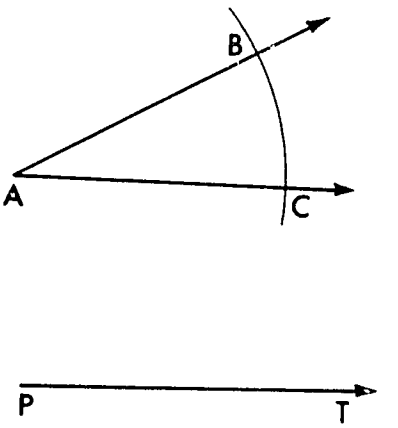


Suppose it is desired to draw an angle of the same measure as angle A and with its vertex at P and one side on $\overrightarrow{PT}$. Choose any points B and C on the rays of A



and draw $\overline{BC}$. Then we can draw a triangle PQR with $\overline{PR}$ on $\overrightarrow{PT}$ and with PQ = AB, QR = BC, and PR = AC. Do this, and then verify with your protractor that angle P and angle A have the same measure.

7. The procedure in Problem 6 can be simplified if the points B and C are chosen by drawing an arc of a circle with centre at A and any convenient radius. Let B and C be the points where this arc intersects the sides of angle A.

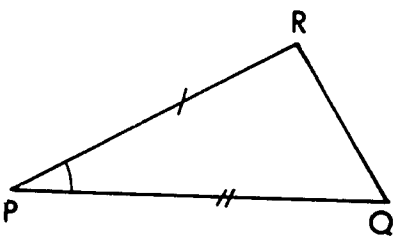


Use this method to copy a given angle A without measuring it.

8-5 Given Two Sides and the Included Angle

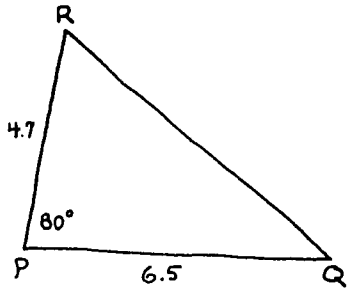
We have seen that the three sides of a triangle determine the triangle. Do two sides and an angle determine a definite triangle?

Let us consider the case where the angle has its vertex at the intersection of the two sides. We call such an angle the included angle. Thus in triangle PQR, $\hat{P}$ is said to be included by the sides $\overline{PR}$ and $\overline{PQ}$ which lie on the rays of $\hat{P}$.



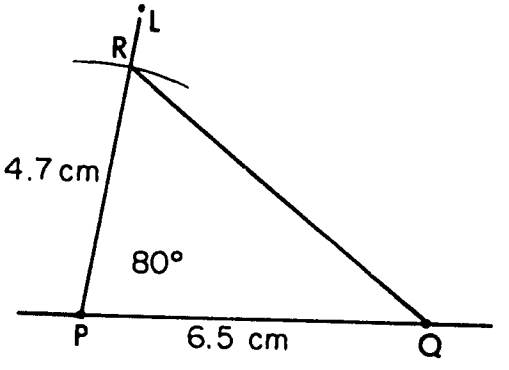
If the angle at P is given, as well as the lengths of $\overline{PQ}$ and $\overline{PR}$, is the length of $\overline{QR}$ determined? If this is the case does this mean that a definite triangle, $\triangle$PQR, is determined?

Let us try to construct a triangle PQR such that $\overline{PQ}$ is 6·5 cm. long, $\overline{PR}$ is 4·7 cm. long, and the size of P is 80°. First draw a rough sketch of the triangle and label it correctly.



Draw $\overline{PQ}$, 6·5 cm. long. With the protractor, draw $\overrightarrow{PL}$ making an angle of 80° with $\overrightarrow{PQ}$.

With P as centre and radius 4·7 cm. describe an arc cutting $\overrightarrow{PL}$. The intersection of $\overrightarrow{PL}$ and this arc is the point R. Draw $\overline{QR}$.



Problems 8-5

1. Draw a triangle $\widehat{PQR}$ in which $\overline{PQ}$ is 8·5 cm. long, $\overline{PR}$ is 7·5 cm. long and the size of $\hat{P}$ is 120°. Measure $\overline{QR}$.

2. Draw a triangle ABC in which $\overline{AB}$ is 3 in. long, $\overline{BC}$ is 3 in. long and the size of $\hat{B}$ is 60°. Measure $\overline{AC}$ and $\hat{C}$.

3. Draw a quadrilateral ABCD in which $\overline{AB}$ is 5 in. long, $\overline{BC}$ is 4 in. long, $\overline{CD}$ is 2 in. long, $\overline{DA}$ is 3 in. long and the size of $\widehat{ABC}$ is 45°. Measure $\overline{AC}$ and $\widehat{ADC}$. (First draw a rough sketch to help you.)

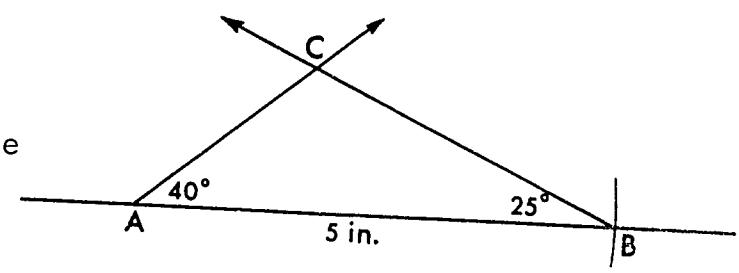
4. Draw a triangle DEF in which the length of $\overline{DE}$ is 8 cm., the size of $\hat{D}$ is 70°, and the length of $\overline{DF}$ is 9 cm. Draw the line through F parallel to $\overleftrightarrow{DE}$ and the line through E perpendicular to $\overleftrightarrow{DE}$. These two lines intersect at G. Measure $\overline{DG}$.

8-6 Given One Side and Two Angles

We saw earlier that if two angles of a triangle are known, the third angle can be found, but the triangle is not determined because the sides can vary in length. If, in addition to the two angles, we know the measure of one side of a triangle, can we draw the triangle?

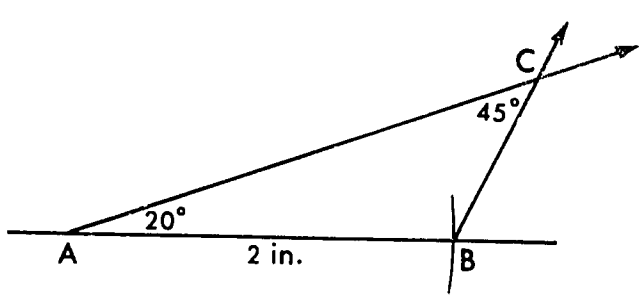
Example 1.

(The side included between the angles.) Draw $\triangle ABC$ in which measure of $\hat{A}$ is 40°, measure of $\hat{B}$ is 25°, and $\overline{AB}$ is 5 inches long.



Example 2.

(The side not included between the angles.) Draw $\triangle ABC$ in which measure of $\hat{A}$ is 20°, measure of $\hat{C}$ is 45°, and $\overline{AB}$ is 2 inches long.



If we find $\hat{B}$, this becomes like Example 1. Why is the measure of $\hat{B}$ equal to 115°?

Problems 8-6

1. Construct a triangle ABC in which the size of $\hat{B}$ is 65°, the size of $\hat{C}$ is 65° and the length of $\overline{BC}$ is 2.9 in. Measure $\overline{AB}$ and $\overline{AC}$.

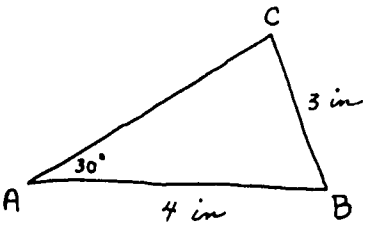
2. Construct a triangle PQR in which the length of $\overline{PQ}$ is 9.4 cm. the size of $\hat{P}$ is 43°, and the size of $\hat{R}$ is 72°. Measure $\overline{QR}$ and $\overline{RP}$.

3. Construct a quadrilateral DEFG such that the size of $\overline{DE}$ is 7·5 cm, the size of $\widehat{GDE}$ is 60°, the size of $\widehat{DEG}$ is 45°, $\overleftrightarrow{GF}$ is parallel to $\overleftrightarrow{DE}$, $\overleftrightarrow{EF}$ is perpendicular to $\overleftrightarrow{DE}$. Measure $\overline{EF}$, $\overline{GF}$, $\widehat{GEF}$, and $\widehat{EGF}$.

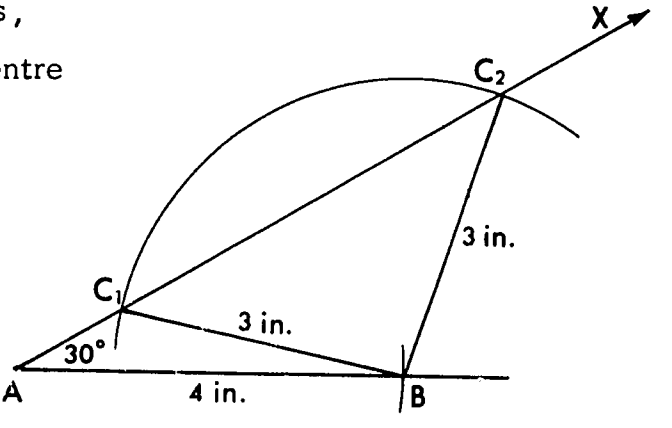
8-7 Given Two Sides and an Angle Not the Included Angle

We saw in Section 8-5 that, if two sides and the included angle of a triangle are given, the triangle is completely determined. If the angle is not included between the given sides, is this still true?

Let us try to draw a triangle, $\triangle ABC$, with $\overline{AB}$ 4 in. long, $\overline{BC}$ 3 in. long, and $\widehat{A}$ an angle of 30°.



After drawing segment $\overline{AB}$ and ray $\overrightarrow{AX}$ making a 30° angle with $\overrightarrow{AB}$, we could find point C by realizing that the distance from B to C is 3 inches, so C is on a circle with B as its centre and a radius of 3 inches. Draw this circle.



Since the circle meets the ray $\overrightarrow{AX}$ in two points, there are evidently two possible positions for C. Let us call them C_1 and C_2.

Consider the two triangles $\triangle ABC_1$ and $\triangle ABC_2$. Do they both have a 30° angle at A? Do they both have $\overline{AB}$ 4 inches long? Are $\overline{BC}_1$ and $\overline{BC}_2$ both 3 inches long? You see that the two triangles both have all the required measurements and yet they are quite different triangles. In this case, then, two different triangles are possible.

Problems 8-7

1. Keeping a 30° angle at A and $\overline{AB}$ still 4 inches long, try to construct $\triangle ABC$ as we did above with:

   (a) $\overline{BC}$ $2\frac{1}{2}$ inches long. Do you still get two triangles?

   (b) $\overline{BC}$ 2 inches long. Do you still get two triangles?

   (c) $\overline{BC}$ 1 inch long. Do you still get two triangles?

   (d) $\overline{BC}$ 4 inches long. Do you still get two triangles?

   (e) $\overline{BC}$ 5 inches long. Do you still get two triangles?

2. If $\hat{A}$ is a right angle and $\overline{AB}$ is 4 inches long, try to construct $\triangle ABC$ with:

   (a) $\overline{BC}$ 5 inches long. How many triangles are possible?

   (b) $\overline{BC}$ 4 inches long. How many triangles are possible?

   (c) $\overline{BC}$ 3 inches long. How many triangles are possible?

3. If $\hat{A}$ is an angle of 120° and $\overline{AB}$ is 4 inches long, try to construct $\triangle ABC$ with:

   (a) $\overline{BC}$ 5 inches long. How many triangles are possible?

   (b) $\overline{BC}$ 4 inches long. How many triangles are possible?

   (c) $\overline{BC}$ 3 inches long. How many triangles are possible?

4. With $\hat{P}$ an angle of 55°, $\overline{PQ}$ 6 cm. long, and $\overline{QR}$ 5 cm. long, two triangles PQR are possible. Draw them and find the two possible measures of $\overline{PR}$ and of $\hat{PRQ}$. What relation do you observe between the two measures of $\hat{PRQ}$?

5. Let a, b, c, respectively, be the measures of the sides opposite $\hat{A}$,

$\hat{B}$, and $\hat{C}$ of $\triangle ABC$. Without making a construction, decide how many triangles are possible in each of the following cases:

(a) $a = 6$, $b = 3$, $c = 10$

(b) $b = 85$, $c = 23$, $\hat{A} = 84^\circ$

(c) $c = 75$, $a = 25$, $\hat{C}$ is 43°

(d) $\hat{A}$ is 50° $\hat{B}$ is 80° $\hat{C}$ is 50°

(e) $\hat{A}$ is 35° $\hat{B}$ is 115° $\hat{C}$ is 40°

(f) $\hat{A}$ is 95° $c = 759$, $\hat{C}$ is 3°

(g) $\hat{B}$ is 167° $a = 50$, $b = 68$

(h) $\hat{C}$ is 29° $c = 380$, $b = 450$

(i) $\hat{A}$ is 79° $b = 2$, $a = 1$

CHAPTER 9

CONGRUENCE

9-1 Size and Shape

For the study of measurement we used the word "congruent" for line segments which have the same length. If we could imagine moving such segments and placing one on the other, we could see that they would fit perfectly.

The idea of congruence was extended to any two geometric figures which have the same size and shape. The possibility of their fitting together exactly, if we could move them, helps to visualize what we mean by congruence.

Problems 9-1

Which of the following are pairs of congruent figures?

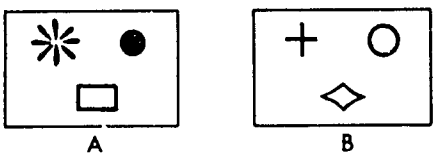
1. Two line segments which have the same length.
2. Two circles which have the same radius.
3. Two rectangles which have equal bases.
4. Two angles which have equal measures.
5. Two rectangles which have equal areas.
6. Two squares whose sides have the same measure.
7. Two squares which have equal areas.
8. Two right circular cones which have congruent bases.

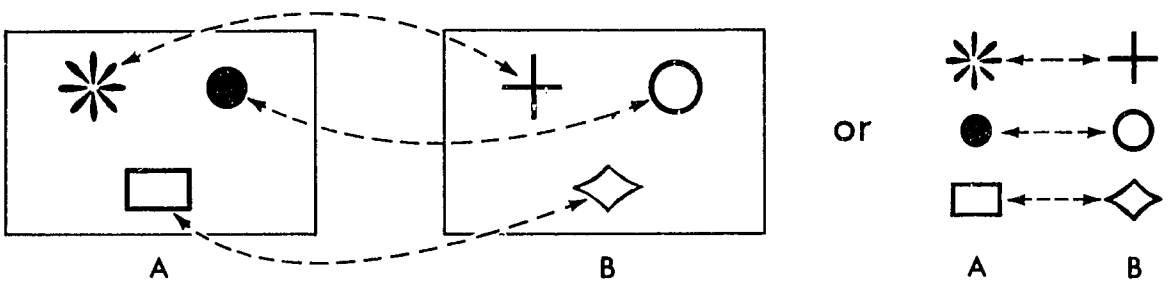
For some geometric figures a single measure determines their congruence.

Angles are congruent if they have the same size.
Line segments are congruent if they have the same length.
Circles are congruent if they have radii of the same length.

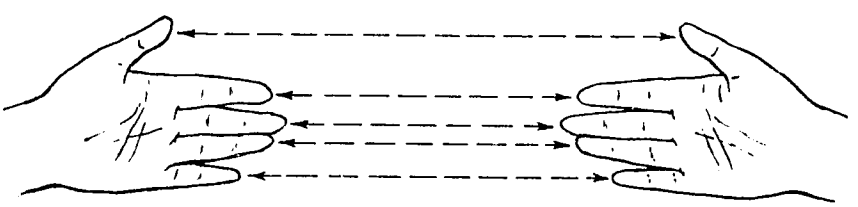
9-2 One-to-one Correspondence

Suppose we have two sets of things, A and B. We can match objects in one set with objects in the other set in various ways. Consider the matching for sets A and B which is shown here.





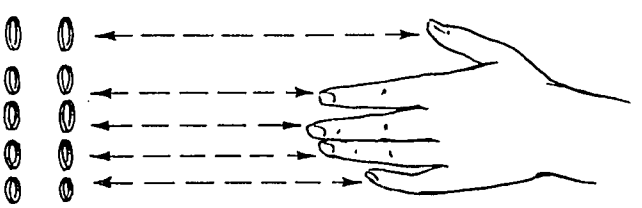
Two sets which are matched like this are said to be in <u>one-to-one correspondence</u>. The fingers of the right hand are in one-to-one correspondence with the fingers of the left hand.



The thumbs correspond.
The fore fingers correspond.
The centre fingers correspond.
The ring fingers correspond.
The little fingers correspond.

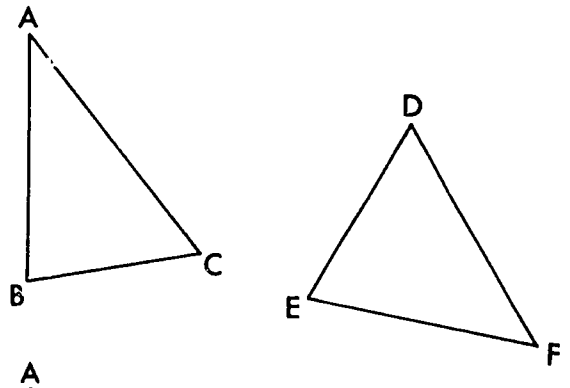
A correspondence is a matching of the members of one set with the members of another set. If each member of either set matches exactly one member of the other set the correspondence is <u>one-to-one</u>.

In contrast with one-to-one correspondence we could imagine a two-to-one correspondence in which two members of one set are matched with one member of the other set. Suppose I want to wear two rings on each finger.



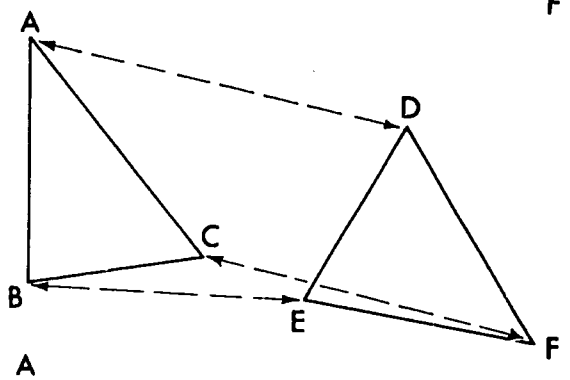
In this case there is a two-to-one correspondence between the set of rings and the set of fingers.

A correspondence can be established between two triangles if we can set up a one-to-one correspondence between the vertices of one triangle and the vertices of the other triangle. A correspondence between their sides follows naturally.

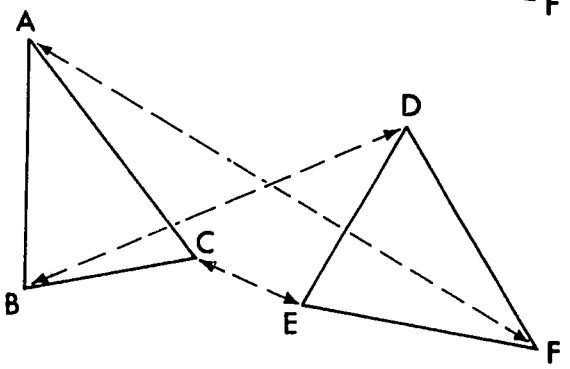


Consider $\triangle ABC$ and $\triangle DEF$.

(a) We can match them in such a way that A corresponds to D, B to E, and C to F. The side $\overline{AB}$ corresponds to $\overline{DE}$ since they join corresponding points. Similarly $\overline{BC}$ corresponds to $\overline{EF}$ and $\overline{CA}$ corresponds to $\overline{FD}$.



(b) Of course, if we match the two triangles so that B corresponds to D and C to E, then A corresponds to F, $\overline{AB}$ to $\overline{FD}$, $\overline{BC}$ to $\overline{DE}$, and $\overline{CA}$ to $\overline{EF}$.



We can observe also a correspondence between angles. If point A corresponds to point F in our triangles, then we can say that $\hat{A}$ corresponds to $\hat{F}$.

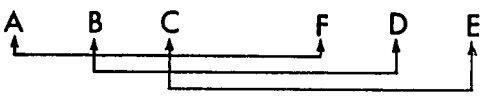
We can write the pairs of corresponding vertices like this:

$$D \longleftrightarrow B$$
$$E \longleftrightarrow C$$
$$F \longleftrightarrow A$$

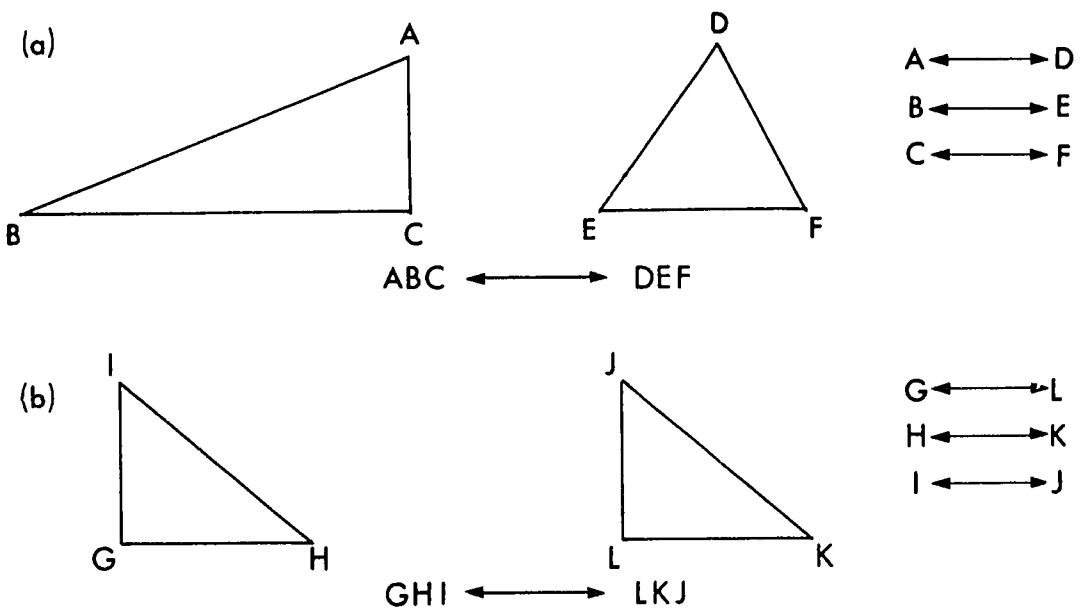
When we match one triangle with another according to this pairing of vertices we show it by writing

ABC ⟷ FDE (⟷ means "is matched with")

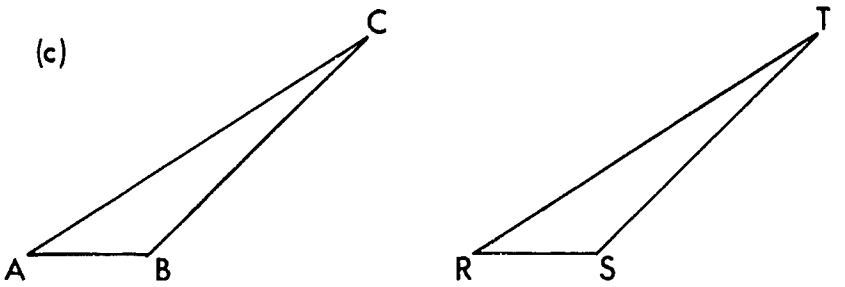
The first letter on the left corresponds to the first letter on the right, the second on the left corresponds to the second on the right, and the third on the left corresponds to the third on the right.



Here are more examples of correspondence between geometric figures indicated by the pairing of corresponding vertices.



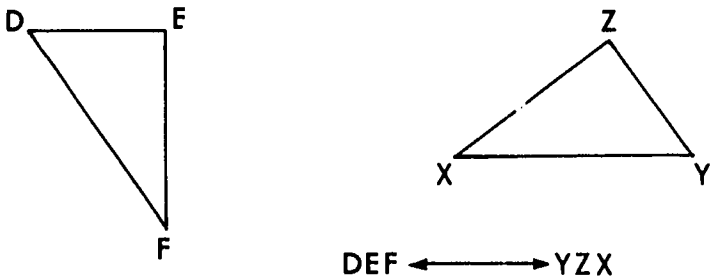
(c)



A ⟷ T
B ⟷ R
C ⟷ S

ABC ⟷ TRS

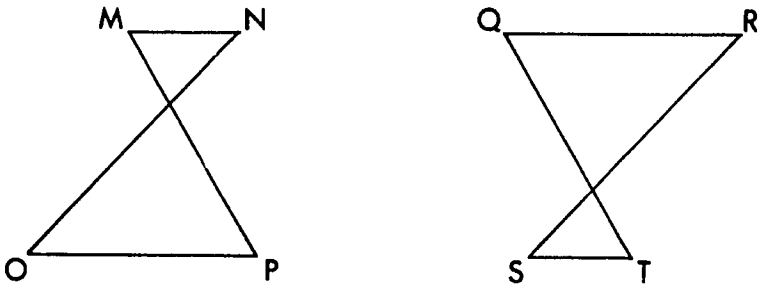
(d)



D ⟷ Y
E ⟷ Z
F ⟷ X

DEF ⟷ YZX

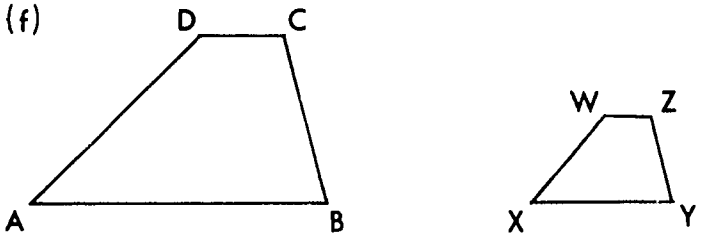
(e)



M ⟷ T
N ⟷ S
O ⟷ R
P ⟷ Q

MNOP ⟷ TSRQ

(f)



A ⟷ X
B ⟷ Y
C ⟷ Z
D ⟷ W

ABCD ⟷ XYZW

Problems 9-2

1. In which of the correspondences (a) - (f) above does it appear that the two figures would fit if they could be moved and placed together according to the correspondence? Use tracings on thin paper, if necessary to test them.

2. In each of the following, can a one-to-one correspondence be made between Set A and Set B?

(a)

(b)

(c)

3. What kind of correspondence can be made between Set A and Set B?

(a)

(b)

(c)

4. If we match two triangles PQR and XYZ so that point P corresponds to point X and Q to Y, what sides and what angles would correspond?

9-3 Testing for Congruence

One way of testing whether two figures are congruent is by placing drawings of them together, one over the other, to see if they can fit exactly in all their corresponding parts. If they do fit exactly, then they are congruent.

Class Activity

Let us now try this method with triangles. We shall need sheets of paper, some ordinary, some thin enough to see through.

1. Draw a scalene triangle LMN with LM = 6, MN = 3, NL = 4, using a cm. as the unit of measure. Now on thin paper draw triangle XYZ with XY = 6, YZ = 3, ZX = 4. Place $\triangle$XYZ over $\triangle$LMN. For which of the following correspondences do they fit?

(a) LMN $\longleftrightarrow$ XYZ

(b) LMN $\longleftrightarrow$ ZXY

(c) LMN $\longleftrightarrow$ ZYX

Do they always fit? In how many ways do they fit?

2. Draw an isosceles triangle PQR with PQ = PR = 4 and QR = 3. Again use a cm. as the unit of measure. Draw another triangle UVW on thin paper with UV = UW = 4 and VW = 3. Place $\triangle$UVW over $\triangle$PQR. For which of the following correspondences do they fit?

(a) PQR $\longleftrightarrow$ VWU

(b) PQR $\longleftrightarrow$ UWV

(c) PQR $\longleftrightarrow$ UVW

Do they always fit?

3. Draw an equilateral triangle ABC of side 3 cm. and on thin paper draw another equilateral triangle DEF of side 3 cm. Place $\triangle$DEF over $\triangle$ABC. For which of the following correspondences do they fit?

(a) ABC $\longleftrightarrow$ DEF

(b) ABC $\longleftrightarrow$ EFD

(c) ABC $\longleftrightarrow$ FDE

(d) ABC $\longleftrightarrow$ DFE

(e) ABC $\longleftrightarrow$ EDF

(f) ABC $\longleftrightarrow$ FED

In how many ways do the triangles fit? In how many ways do they not fit?

Now reviewing the activities 1 - 3 we see that $\triangle ABC$ and $\triangle DEF$ can fit in any of the different positions. $\triangle PQR$ and $\triangle UVW$ can fit in two positions but not in the third. And that $\triangle LMN$ and $\triangle XYZ$ can fit in only one position.

Are $\triangle ABC$ and $\triangle DEF$ congruent?

Are $\triangle PQR$ and $\triangle UVW$ congruent?

Are $\triangle LMN$ and $\triangle XYZ$ congruent?

> If two geometric figures can fit exactly according to at least one correspondence between the figures, then they have the same size and shape and therefore are <u>congruent</u>.

Usually a triangle is congruent to another in only one way such as $\triangle LMN$ to $\triangle XYZ$.

In the case of isosceles triangles, however, we say for example that

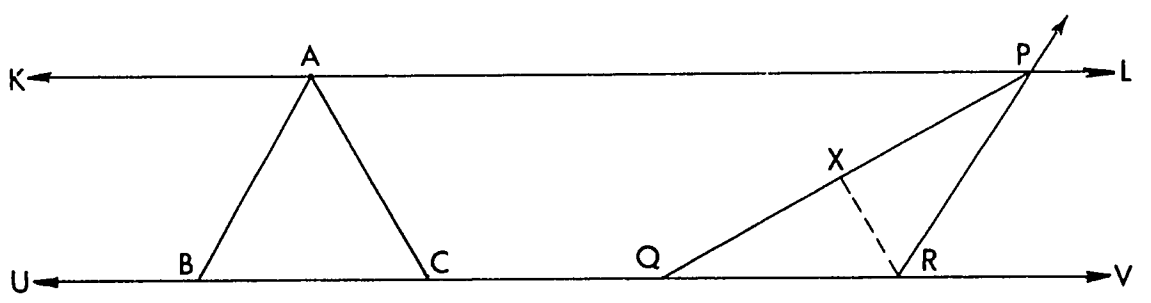
$\triangle PQR$ is congruent to $\triangle UVW$ and also that

$\triangle PQR$ is congruent to $\triangle UWV$.

These are two different statements for two different congruences. The one-to-one correspondence in one statement is not the same as that in the other, so they are not the same congruence.

Similarly with the equilateral triangles there are six different congruences.

4. Draw a line $\overleftrightarrow{UV}$ across a sheet of paper. At the left portion of the paper mark out a line segment $\overline{BC}$ on $\overleftrightarrow{UV}$ of length c cm. and complete an equilateral triangle ABC. Draw $\overleftrightarrow{KL}$ through A parallel to $\overleftrightarrow{UV}$. Some distance to the right of $\triangle ABC$ mark out the line segment $\overline{QR}$ on $\overleftrightarrow{UV}$ of length 3 cm. Draw the ray $\overrightarrow{RP}$ to intersect $\overleftrightarrow{KL}$ at P so that $\widehat{QRP}$ is an angle of 120°. Complete the triangle PQR.



Now let us compare $\triangle ABC$ and $\triangle PQR$.

(a) Can $\triangle PQR$ be fitted exactly on $\triangle ABC$?

(b) Which has a bigger area $\triangle ABC$ or $\triangle PQR$?

(You can compare their areas in two ways: First, the area of a triangle is $\frac{1}{2} \times$ base $\times$ height. The bases of the two triangles are equal [by construction]. Now measure their heights. Are they equal? Are the triangles equal in area? Second, draw $\overline{RX}$ to intersect $\overline{QP}$ at X so that $\widehat{QRX}$ is an angle of 60°. Cut out the triangular region QRP and cut it again along $\overline{RX}$. Now fit $\triangle QRX$ to $\triangle ABC$ with $\overline{QR}$ lying along $\overline{BC}$ and $\overline{RX}$ along $\overline{CA}$. Then fit $\triangle XRP$ with $\overline{PR}$ lying along $\overline{BA}$ and $\overline{RX}$ along $\overline{AC}$. Do the regions fit exactly?)

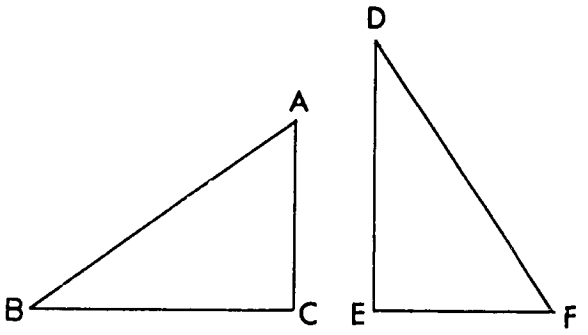
From the comparisons (a) and (b) we find that $\triangle PQR$ does not fit $\triangle ABC$, but that the area of $\triangle ABC$ is equal to the area of $\triangle PQR$. We can say then that $\triangle ABC$ and $\triangle PQR$ have the same size but not the same shape. Are they congruent?

5. Draw an equilateral triangle JKL with sides of length 4 cm. Draw another equilateral triangle MNP with sides of length 6 cm. Do $\triangle JKL$ and $\triangle MNP$ have the same shape? Do they have the same size? Are $\triangle JKL$ and $\triangle MNP$ congruent?

We have seen triangles which have the same size but not the same shape, and triangles which have the same shape but not the same size. In both cases the triangles are not congruent.

For two figures to be congruent they must have the same size and the same shape. If they are flat figures, that is figures in a plane, they must be such that, if we could move them, they could be made to fit exactly. If they are solid figures in space, it is harder to visualize how they may be made to fit together. For example, two balls which have the same size and shape are congruent, but they cannot be fitted together.

6. Here are two triangles ABC and DEF. Are they congruent? (You can use thin paper to trace $\triangle$DEF and fit the tracing on $\triangle$ABC.) If they are congruent, which point of $\triangle$DEF fits on B? Which on C? Which on A?



9-4 Notation for Congruent Figures

In part 6 of the last section we saw that $\triangle$ABC and $\triangle$DEF are congruent and that the correspondence under which they are congruent requires that

$$A \longleftrightarrow F$$
$$B \longleftrightarrow D$$
$$C \longleftrightarrow E$$

In stating the congruence we shall write the letters of the vertices in the order of their correspondence, just as we did with ABC $\longleftrightarrow$ FDE. We shall write $\triangle ABC \equiv \triangle FDE$ ("$\equiv$" means "is congruent to").

As before, the first letter on the left corresponds to the first letter on the right, the second on the left corresponds to the second on the right, and the third on the left corresponds to the third on the right.

This notation tells us not only what vertices correspond, but also what sides and what angles correspond. By the nature of congruence, then, we know that these corresponding parts are congruent. Thus the statement

$$ABCD \equiv PQRS$$

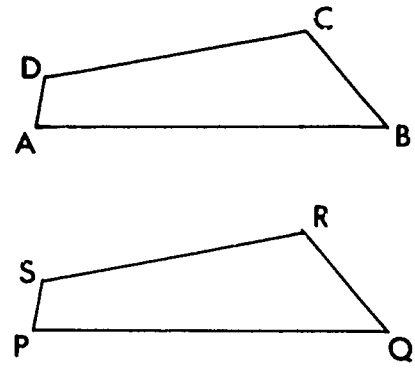
tells us that

$\overline{AB} \equiv \overline{PQ}$ $\hat{A} \equiv \hat{P}$

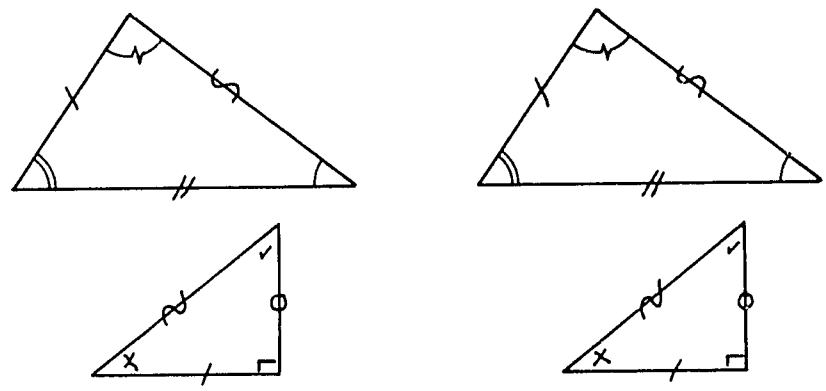
$\overline{BC} \equiv \overline{QR}$ $\hat{B} \equiv \hat{Q}$

$\overline{CD} \equiv \overline{RS}$ $\hat{C} \equiv \hat{R}$

$\overline{DA} \equiv \overline{SP}$ $\hat{D} \equiv \hat{S}$



It is often convenient to indicate a congruence graphically by making identical marks on corresponding sides and angles.



Problems 9-4

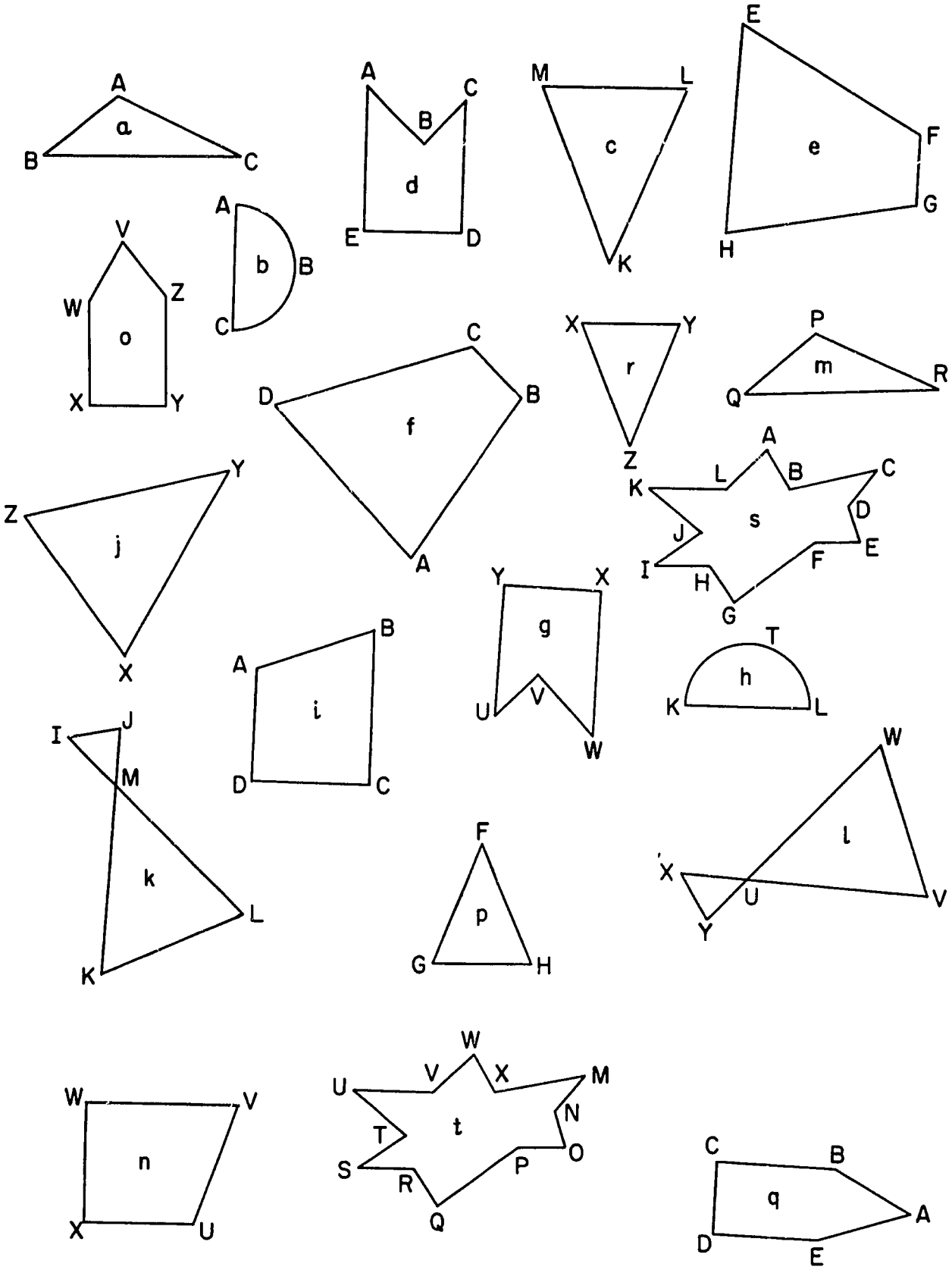
Look at the figures on page 108. Figures a and m appear to be congruent.

(a) Make a list of all the pairs which look as if they might be congruent. List the pairs as follows:

Figure	Its Match
a	m
b	.
c	.
.	.
.	.

(b) Test whether each pair is really congruent by tracing one figure on thin paper and fitting the tracing on the other.

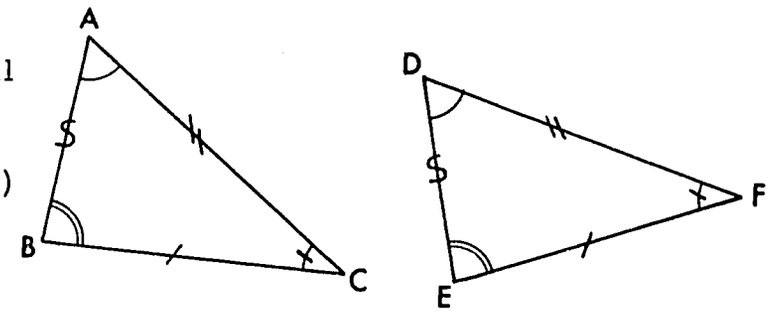
(c) For each pair of congruent figures write the congruence showing the correspondence of vertices. For example, $\triangle ABC \equiv \triangle PQR$.


a
A B C
b
A B C
c
M L K
d
A B C D E
e
E F G H
f
C D B A
g
Y X U V W
h
T K L
i
A B D C
j
Y Z X
k
I J M K L
l
W X U V Y
m
P R Q
n
W V X U
o
V W Z X Y
p
F G H
q
C B A D E
r
X Y Z
s
A L B C K D J E I F H G
t
W V X M U N T O S P R Q

9-5 Congruent Triangles

Hitherto we have decided whether two figures are congruent by trying to fit one to another. Now we want, instead of fitting, to discover some common properties which will show that two figures are congruent. At this stage we shall concern ourselves with triangles only. Triangles are very important in mathematics, science, engineering, and surveying, so we need to know conditions under which they are congruent.

If $\triangle ABC \equiv \triangle DEF$, we know that $\triangle DEF$ must be able to fit exactly in all its parts on $\triangle ABC$. Each triangle has six parts (three sides and three angles) and we know that these parts must be congruent as follows:



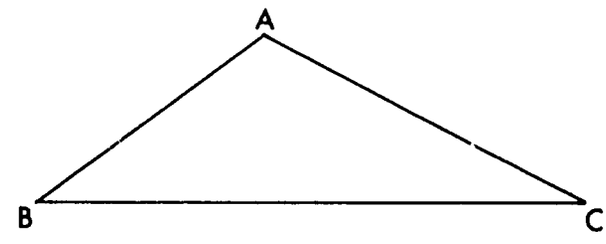
$\overline{BC} \equiv \overline{EF}$	$\hat{A} \equiv \hat{D}$
$\overline{CA} \equiv \overline{FD}$	$\hat{B} \equiv \hat{E}$
$\overline{AB} \equiv \overline{DE}$	$\hat{C} \equiv \hat{F}$

It is true also for triangles that if the six parts of one triangle are congruent to the corresponding six parts of the other triangle, <u>then</u> the triangles are congruent. One way, then, to test congruence of two triangles without any need for moving them would be to verify that all six pairs of corresponding parts are congruent.

You have already learnt to construct triangles using only <u>three</u> given measurements. If sometimes the shape and size of a triangle can be determined without needing to know the measurements of all six parts, perhaps pairs of triangles can sometimes be known to be congruent without needing to know that all six corresponding parts are congruent. Let us consider these cases.

1. <u>Three Sides</u>

   (a) Here is a triangle ABC.



Construct on thin paper a triangle XYZ such that

$$\overline{YZ} \equiv \overline{BC}$$
$$\overline{YX} \equiv \overline{BA}$$
$$\overline{ZX} \equiv \overline{CA}$$

Now compare $\triangle$ ABC and $\triangle$XYZ. Are they congruent?

(b) Using a cm. as the unit of measure, construct on thin paper a triangle ABC with the following measures:

AB = 4 BC = 5 CA = 7

Fit your triangle against other triangles constructed from the same measures by other members of the class. Is there any constructed triangle which does not exactly fit with your own? Do you think it is possible to construct with the given data two triangles which are not congruent?

(c) Here are other sets of data for the construction of $\triangle$ABC

	(i)	(ii)	(iii)	(iv)	(v)
Length of $\overline{AB}$	1·5 in.	4 cm.	4 cm.	2 cm.	2 in.
Length of $\overline{BC}$	2·0 in.	2 cm.	5 cm.	3 cm.	3 in.
Length of $\overline{CA}$	2·5 in.	5 cm.	5 cm.	4 cm.	$3\frac{1}{2}$ in.

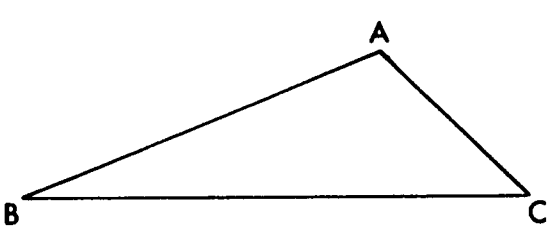
Label each triangle you construct.

Compare your triangles with those drawn by other members of the class, first by fitting triangles drawn from the same set of data, and then by fitting triangles drawn from different sets.

What conclusion can you make from these two constructions 1(a) and 1(b) and from the comparisons in 1(c)?

2. Two Sides and an Angle

(a) Here is a triangle ABC. Draw on thin paper an angle $\widehat{PYQ}$ congruent to $\widehat{ABC}$. Mark off on $\overrightarrow{YP}$ a line segment $\overline{YX}$ congruent to $\overline{BA}$ and on



$\overrightarrow{YQ}$ a line segment $\overline{YZ}$ congruent to $\overline{BC}$.

Compare $\triangle ABC$ and $\triangle XYZ$. Are they congruent?

(b) Construct a triangle ABC with this information.

Length of $\overline{AB}$ is 4 cm., length of $\overline{BC}$ is 6 cm., size of $\widehat{ABC}$ is 30°.

Fit your triangle against other triangles constructed from the same data by other members of the class. Is there any triangle which does not fit exactly with your own? Do you think it is possible to construct with the given data two triangles which are not congruent?

(c) Here are other sets of data for the construction of $\triangle ABC$.

	(i)	(ii)	(iii)	(iv)	(v)
Length of $\overline{AB}$	5 cm.	6 cm.	2 in.	4 cm.	2 in.
Length of $\overline{BC}$	7 cm.	5 cm.	3 in.	7 cm.	5 in.
Size of $\widehat{ABC}$	30°	75°	45°	60°	115°

Label each triangle you construct.

Compare your triangles with those drawn by other members of the class, first by fitting triangles drawn from the same set of data, and then by fitting triangles drawn from different sets.

What conclusion can you make from the two constructions 2(a) and 2(b) and from the comparisons in 2(c)?

(d) Now construct a triangle JKL with this information.

Length of $\overline{JK}$ is 3 in., size of $\widehat{JKL}$ is 30°, length of $\overline{JL}$ is 2 in.

(Note that $\widehat{JKL}$ is not the angle included by the sides $\overline{JK}$ and $\overline{JL}$ whose measures are given.) Compare your triangle with other triangles constructed from the same data by other members of your class. What do you notice? Is it possible to construct with the given data two triangles which are not congruent?

What conclusion would you now make from constructions 2(a), 2(b) and 2(d) and from the comparisons in 2(c)?

(e) Construct a triangle ABC with this information.

Length of $\overline{AC}$ is 1·5 in., size of $A\hat{C}B$ is 90° and $\overline{AB}$ is 4 in.

Note that $A\hat{C}B$ is not the angle included by the sides $\overline{AC}$ and $\overline{AB}$ whose measures are given. Notice also that $A\hat{C}B$ is a right angle. Are all the triangles constructed with the data congruent? Is it possible to construct with the given data two triangles which are not congruent?

Would you now like to modify your previous conclusion made at the end of 2(d)?

3. Two Angles and a Side

(a) Construct a triangle ABC with these data.

Length of $\overline{BC}$ = 3 in., size of $A\hat{B}C$ is 30°, size of $A\hat{C}B$ is 45°.

Compare your triangle with other triangles constructed with the same data. Are all the triangles congruent?

(b) Here are other sets of data for the construction of $\triangle ABC$.

	(i)	(ii)	(iii)	(iv)	(v)
Length of $\overline{BC}$	3 in.				8 cm.
Length of $\overline{AB}$		5 cm.	4 cm.		
Length of $\overline{AC}$				3 in.	
Size of $A\hat{B}C$	40°	50°	30°	45°	55°
Size of $B\hat{C}A$		75°			60°
Size of $B\hat{A}C$	65°		105°	115°	

Make comparisons again as you did in 1(c) and 2(c).

What conclusion would you make from what you observe in 3(a) and 3(b)?

4. Three Angles

Construct a triangle ABC with these data

Size of $\widehat{ABC}$ is 45°

Size of $\widehat{BCA}$ is 70°

Size of $\widehat{CAB}$ is 65°

You can start by drawing any of the three sides and make it any length. Compare your triangle with those made by other pupils. Are all these triangles congruent?

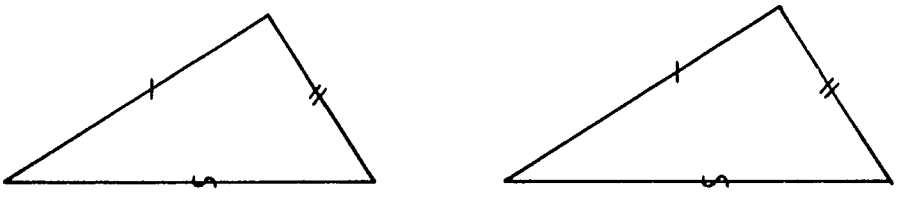
If we are given the measures of two angles of a triangle we can calculate the measure of the third angle. Therefore the data above does not tell us more than giving us only two angles (say, size of $\widehat{ABC}$ is 45°, size of $\widehat{BCA}$ is 70°) and leaving us to calculate the third.

When you did the above construction you used only two of the given angles. The third angle took shape of its own accord because its measure is dependent on the measures of the other two angles. For a triangle to be completely determined, we must have the measures of three independent parts. You chose the measure of a third part yourself when you drew the first side making it any length you liked.

Summary:

We know that a triangle has six parts (three sides and three angles). In order that two triangles be congruent, however, it is enough to know that only three parts of one are congruent to the corresponding parts of the other, provided they are the right three parts. Consider the cases where the parts known to be congruent are (i) three sides, (ii) two sides and one angle (iii) two angles and one side. In each of these cases are the triangles congruent?

(i) Three sides: If three sides of a triangle are congruent to the corresponding three sides of another triangle then the two triangles are congruent. (We may abbreviate this as SSS.)

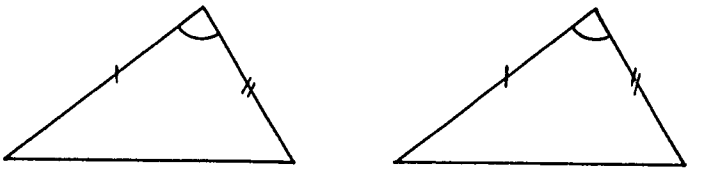


Congruent (SSS)

(ii) <u>Two sides and one angle</u>:

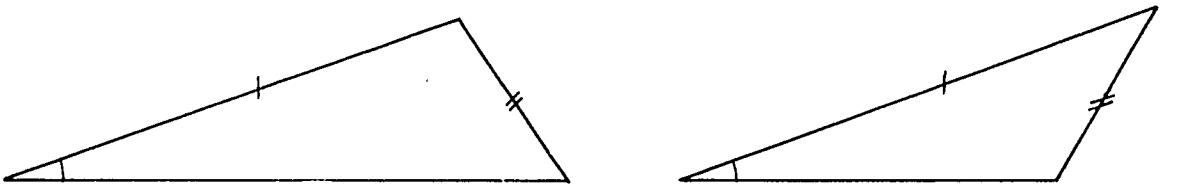
If two sides and an angle of a triangle are congruent to the corresponding two sides and angle of another triangle and

(a) If the angle is <u>included</u> between the sides, the triangles are congruent. (We may abbreviate this as SAS.)



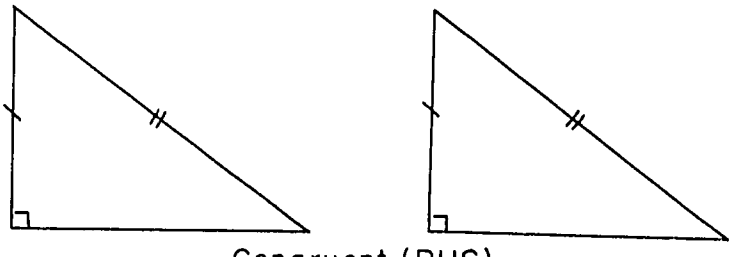
Congruent (SAS)

(b) If the angle is <u>not included</u> between the sides, the triangles may not be congruent.



Not Congruent

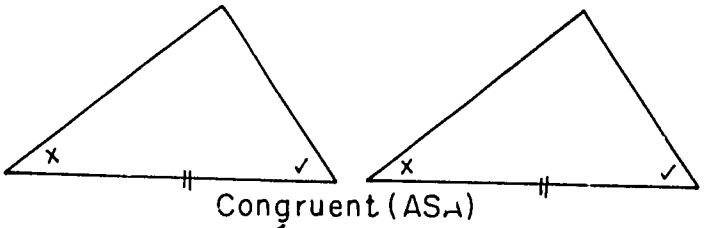
(c) If the angle is <u>not included</u> but is a <u>right angle</u>, the triangles are congruent. (We may abbreviate this as RHS.)



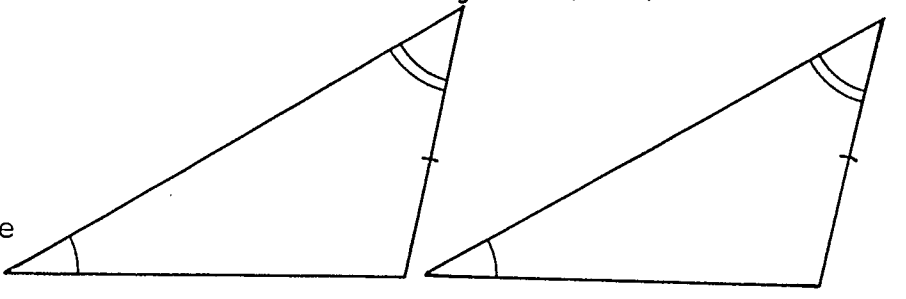
Congruent (RHS)

(iii) <u>Two angles and one side</u>:

If two triangles have two angles of one congruent to two corresponding angles of the other, and a side of one congruent to the <u>corresponding</u> side of the other then the

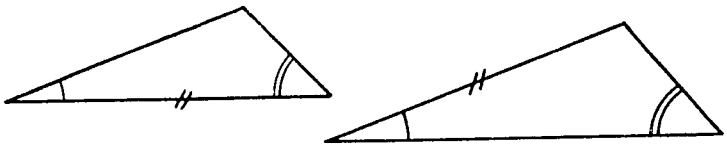


Congruent (ASA)



Congruent (AAS)

two triangles are congruent. (We may abbreviate this as ASA or AAS.)



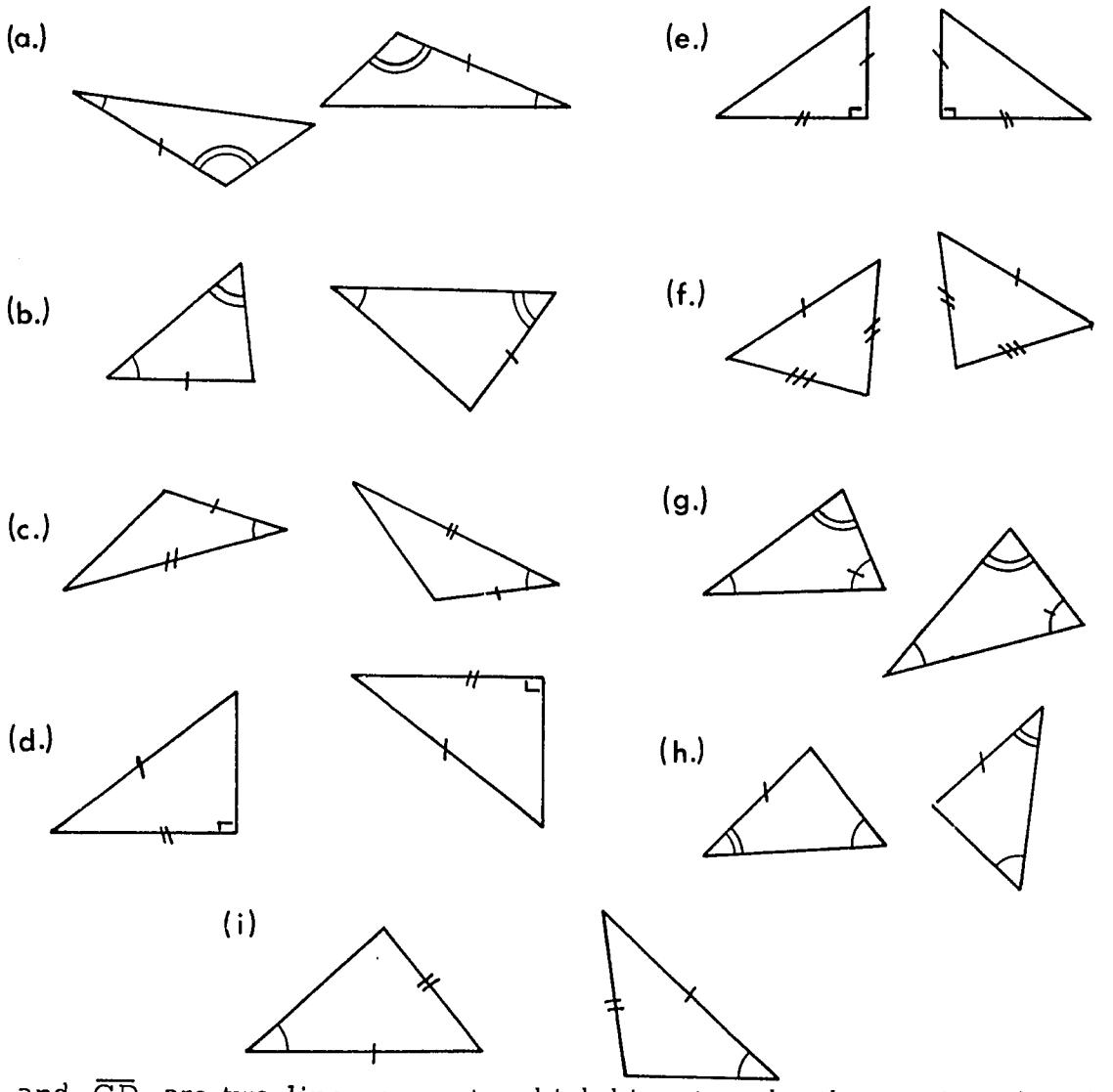
Not Congruent (Congruent sides are not corresponding.)

Here is a list of the four cases of congruency in triangles. The list gives the three parts which must be congruent in two triangles in order to determine that they are congruent.

1.	Three sides	SSS	Side, Side, Side
2.	Two sides and the <u>included</u> angle	SAS	Side, Angle, Side. Notice that in the abbreviation A is between the two S's because the angle must be <u>included</u> by the two sides.
3.	Right angle, hypotenuse, and another side	RHS	Right, Hypotenuse, Side.
4.	Two angles and a side	ASA or AAS	Angle, Side, Angle or Angle, Angle, Side. The congruent sides must be <u>corresponding</u> sides.

<u>Problems 9-5</u>

1. State whether these pairs of triangles are congruent or not; where they are, state the case of congruence, using the abbreviations. (Congruent parts of each pair of triangles are indicated by like symbols on the parts of the drawing.)



2. $\overline{AB}$ and $\overline{CD}$ are two line segments which bisect each other at X. Show by marking congruent parts that $\cong\triangle AXD \cong \triangle BXC$. State the case of congruence. Why would we not write this "$\triangle AXD \cong \triangle CXB$?"

3. <u>CHALLENGE PROBLEM</u>.

An isosceles triangle PQR has $\overline{PQ} = \overline{PR}$. A line segment $\overline{QM}$ is drawn perpendicular to $\overleftrightarrow{PR}$ and intersecting it at M; similarly $\overline{RN}$ is drawn perpendicular to $\overleftrightarrow{PQ}$ and intersecting it at N. Let the intersection of $\overrightarrow{QM}$ and $\overrightarrow{RN}$ be called X.

(a) Name all the triangles formed in the figure.

(b) Pair the congruent triangles, naming them according to their corresponding points. State the case of congruence in each case.

CHAPTER 10

BASIC CONSTRUCTIONS

10-1 Introduction

You have already learnt to make some constructions. For instance, you can draw an equilateral triangle using only straight-edge and compasses. You do this by using your compasses to make the three sides of the triangle congruent. This gives you angles of 60° at each vertex. You realize, then, that you can construct an angle of 60° without using a protractor. (How do you know that they are 60° angles?) Let us now see why certain constructions are possible. We shall call these constructions basic because they form the basis of further construction work.

In this chapter all drawing is to be done with straight-edge and compasses only -- no set squares and no protractors, except to measure angles after they are drawn.

10-2 Bisecting a Line Segment at Right Angles

Class Activity

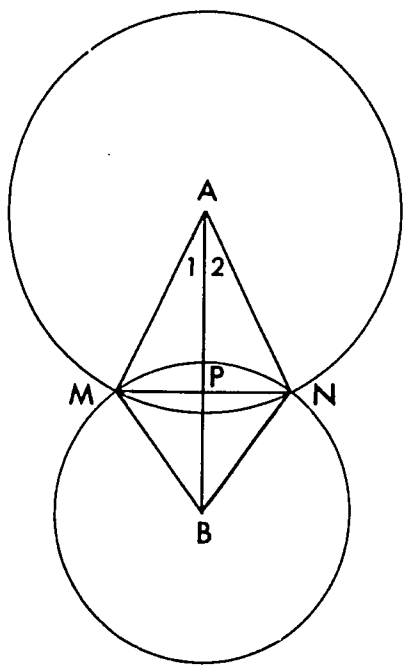
1. (a) Draw any two intersecting circles with centres at A and B.

Name the points of intersection M and N.

Draw $\overline{MN}$ and let it cut $\overleftrightarrow{AB}$ at P.

Measure $\overline{MP}$, $\overline{NP}$. Are they congruent?

Measure the angles at P. Are they congruent?



(b) Draw $\overline{AM}$, $\overline{AN}$, $\overline{BM}$, $\overline{BN}$. From your construction answer the following:

Why are $\triangle AMB$ and $\triangle ANB$ congruent?

Why are the angles marked 1 and 2 congruent?

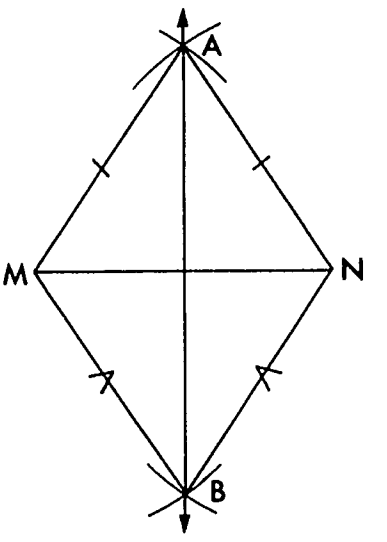
From what you have said above, why are $\triangle AMP$ and $\triangle ANP$ congruent?

Since $\triangle AMP$ and $\triangle ANP$ are congruent, it follows that $\overline{MP}$ is congruent to $\overline{NP}$, and $\widehat{APM}$ is congruent to $\widehat{APN}$.

Since $\widehat{APM}$ and $\widehat{APN}$ are congruent and the sum of their measures is 180 (Why?), it follows that they are right angles. This means that $\overleftrightarrow{AB}$ is perpendicular to $\overline{MN}$.

(c) We have seen that $\overleftrightarrow{AB}$ bisects and is perpendicular to $\overline{MN}$. Therefore $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{MN}$.

2. If we are given any line segment $\overline{MN}$, how can we obtain points A and B so that $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{MN}$?



In part 1, $\overleftrightarrow{AB}$ was the perpendicular bisector of $\overline{MN}$ because $\overline{MA}$ was congruent to $\overline{NA}$ and $\overline{MB}$ was congruent to $\overline{NB}$. There we started with A and B and then located M and N. Here we start with M and N and want to locate A and B. Can we again use circles to find A and B ? If we use M and N as centres of circles how can we find a point A so that $\overline{MA}$ and $\overline{NA}$ are congruent? Can A be found as a point of intersection of two congruent circles? How about the other point where the circles intersect? We can use that point as the point B such that $\overline{MB}$ and $\overline{NB}$ are congruent. Complete the construction and explain why $\overleftrightarrow{AB}$ is the perpendicular bisector of $\overline{MN}$.

Problems 10-2

1. Draw any line segment. Divide it into four congruent parts. Use straight-edge and a pair of compasses only.

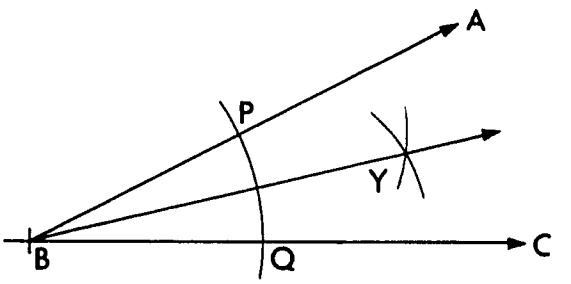
2. Construct the perpendicular bisector of $\overline{AB}$. Take any point X on the perpendicular bisector but not on $\overline{AB}$. Show that $\overline{XA}$ and $\overline{XB}$ are congruent.

10-3 Bisecting an Angle

Class Activity

Draw any angle ABC.

Taking B as centre and using any radius, draw an arc to cut $\overrightarrow{BA}$ at P and $\overrightarrow{BC}$ at Q. With P as centre and any radius greater than half the distance from P to Q, draw an arc in the interior of $\widehat{ABC}$. Taking the same radius and using Q as centre, draw an arc to cut the first one at Y.



Draw $\overrightarrow{BY}$. Draw $\overline{PY}$, $\overline{QY}$ and show that $\triangle BPY \equiv \triangle BQY$.

Since $\triangle BPY \equiv \triangle BQY$, we know that $\widehat{ABY} \equiv \widehat{CBY}$. (Why?) Therefore $\overrightarrow{BY}$ is the bisector of $\widehat{ABC}$.

Problems 10-3

1. In the construction described above why must the circle with centre at P have its radius more than half the distance from P to Q ?

2. Draw an obtuse angle. Construct the rays that divide the angle into four congruent angles.

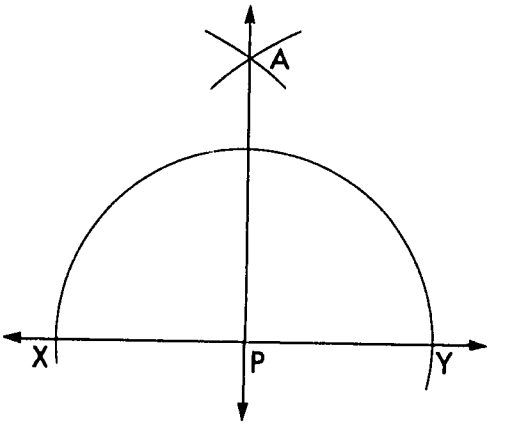
3. Using a pair of compasses and a straight-edge, divide a right-angle into two congruent angles. Check the accuracy of your drawing by measuring the angles with a protractor.

10-4 Constructing a Perpendicular to a Line at a Point on the Line

Class Activity

Let P be the point on a line through which the perpendicular to the line is to be drawn. Take P as centre and any suitable radius, and draw an arc to cut the line at X and Y.

Using X as centre, draw an arc on one side of $\overleftrightarrow{XY}$. Using Y as centre and the same radius, draw another arc to cut the former one at A.



Draw $\overleftrightarrow{AP}$. Draw $\overline{AX}$, $\overline{AY}$ and show that $\widehat{APX}$ and $\widehat{APY}$ are right angles. This shows that $\overleftrightarrow{AP}$ is perpendicular to the given line.

You will note that the construction above has the effect of bisecting an angle of 180°.

Problems 10-4

1. How long must the radii of the circles with centres at X and Y be to ensure that they will intersect?

2. Let Q be any point on the line $\overleftrightarrow{LM}$. Through Q draw with straight-edge and compasses the perpendicular $\overleftrightarrow{PQ}$ to $\overleftrightarrow{LM}$. Through any point R on $\overleftrightarrow{PQ}$, draw $\overleftrightarrow{RN}$ perpendicular to $\overleftrightarrow{PQ}$. $\overleftrightarrow{RN}$ is now parallel to $\overleftrightarrow{LM}$. Why?

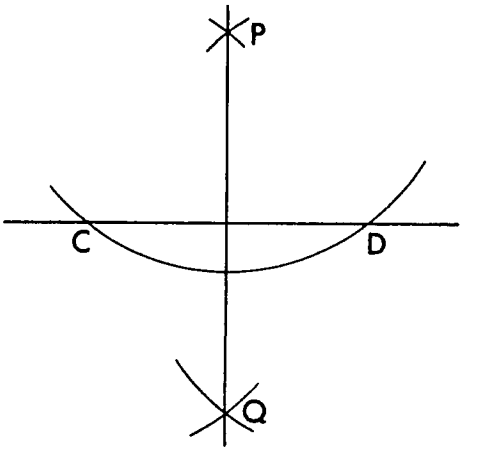
3. Construct a square ABCD with side 2 in. using straight-edge and compasses. Measure $\overline{AC}$.

10-5 Constructing a Perpendicular to a Line from a Point not on the Line

Class Activity

Suppose we have a line and a point P which is not on the line. Using P as centre and a suitable radius, draw an arc to cut the line at C and D.

With C as centre draw an arc. With D as centre draw another arc with the same radius cutting the former arc at Q. Draw $\overleftrightarrow{PQ}$.



Show that $\overleftrightarrow{PQ}$ is the perpendicular required.

Problems 10-5

1. Each of the arcs drawn in this construction must have a "big enough" radius. Explain how big each of these radii must be.

2. Construct an isosceles triangle ABC with $\overline{AB}$ and $\overline{AC}$ each of length 5 cm., and $\overline{BC}$ of length 6 cm.

   From A construct the line perpendicular to $\overline{BC}$, meeting $\overline{BC}$ at D. Measure $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$.

3. Construct triangle ABC with $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ of lengths 2 in., 4 in., and 3 in. respectively.

   From A, construct the line perpendicular to $\overline{BC}$, meeting $\overline{BC}$ at D. Measure $\overline{AD}$, $\overline{BD}$, and $\overline{CD}$.

10-6 Intersecting Lines in Triangles

Problems 10-6

1. Draw a triangle ABC.
   Draw the perpendicular bisectors of $\overline{AB}$, $\overline{AC}$ and let them intersect at P.
   From P draw a perpendicular to $\overline{BC}$ and let it meet $\overline{BC}$ at D.
   Measure $\overline{BD}$, $\overline{CD}$. Are they congruent?

2. Draw a triangle ABC.
   Draw the perpendicular bisectors of $\overline{AC}$, $\overline{BC}$ and let them meet at P.
   Now draw a perpendicular bisector of $\overline{AB}$. Does it pass through P ?

3. Draw any triangle XYZ.
   Draw the bisectors of $\widehat{XYZ}$ and $\widehat{ZXY}$. Let the bisectors intersect at P.
   Draw the bisector of the third angle. Does it pass through P ?

4. Draw any triangle PQR.
   Construct the mid-points L, M, and N of the sides $\overline{PQ}$, $\overline{QR}$, and $\overline{RP}$ respectively.
   Draw $\overline{RL}$, $\overline{PM}$, and $\overline{QN}$. Do these three segments all pass through one point?

5. Draw any triangle JKL.
   From J draw a line perpendicular to $\overleftrightarrow{KL}$.
   From K draw a line perpendicular to $\overleftrightarrow{JL}$.
   Let the point of intersection of these two lines be Z.
   From L draw a line perpendicular to $\overleftrightarrow{JK}$.
   Does this line pass through Z ?

10-7 Regular Polygons

In this section, we shall discuss two kinds of polygons, hexagons and octagons.

A <u>hexagon</u> is a six-sided polygon while an <u>octagon</u> is an eight-sided polygon.

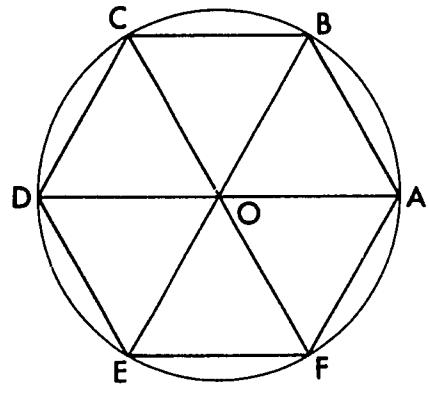
When all the sides of a polygon are congruent and all the angles are congruent then the polygon is said to be a regular polygon.

If all the vertices of a polygon are points in a circle, the polygon is said to be inscribed in the circle.

Class Activity

1. Regular Hexagon

Draw a circle with centre at O. Using your compasses, construct 6 congruent angles of size 60° at the centre of the circle, with O as the vertex of each angle. Let the sides of the angles meet the circle at A, B, C, D, E, F, as shown. Draw $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, and $\overline{FA}$.



You have now drawn a regular hexagon ABCDEF which is inscribed in the circle with centre O. Why is the hexagon regular?

Measure the angles of the hexagon. Are they congruent? Measure the sides too. Are they congruent?

In the hexagon we have six congruent triangles. What kind of triangles are they? Why is each side of the hexagon congruent to the radius of the circle?

If you are given the circle, how could you obtain points A, B, C, D, E, F, without first drawing angles of 60° at the centre of the circle? Now draw any circle and use the radius to construct an inscribed regular hexagon without drawing angles of 60° at the centre of the circle.

2. Regular Octagon

Draw any circle. Draw two diameters, $\overline{AE}$ and $\overline{CG}$, that are perpendicular to one another. Bisect each of the four right-angles, and let the

bisectors meet the circle at B, D, F, H, so that the points A, B, C, D, E, F, G, H are in that order around the circle.

Join the adjacent points of intersection of the circle and the diameters. Then ABCDEFGH is a regular octagon.

Are all the angles congruent?

Are all the sides congruent? Explain.

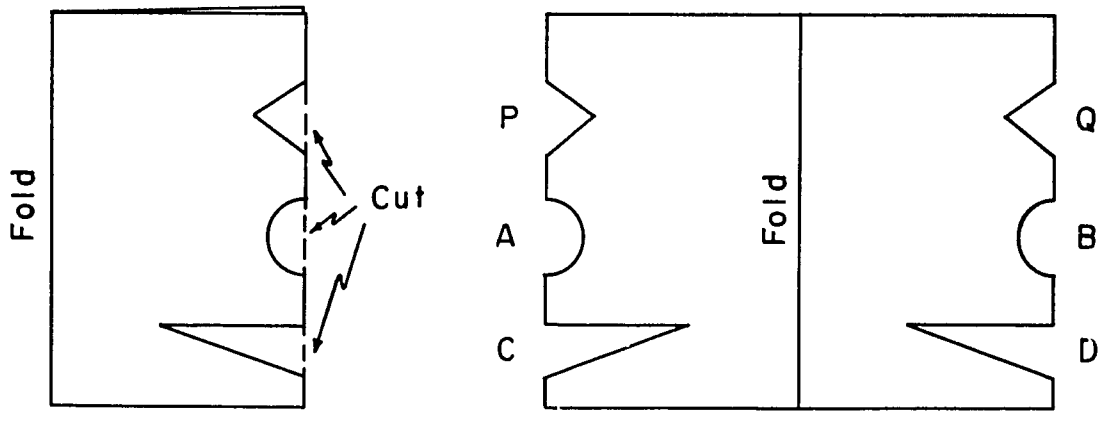
CHAPTER 11

SYMMETRY

11-1 Symmetry about a Line

Class Activity

1. (a) Fold a rectangular piece of paper in half so that the opposite edges fit perfectly. Cut any design in the edge opposite the fold. When you open your paper, do you notice that each design in one edge has a corresponding design on the opposite edge? In the diagram below, shape P corresponds to shape Q, and shape A corresponds to shape B, because they are the same cuts on either side of the fold.



Does shape C correspond to shape D? Does shape C correspond to shape Q?

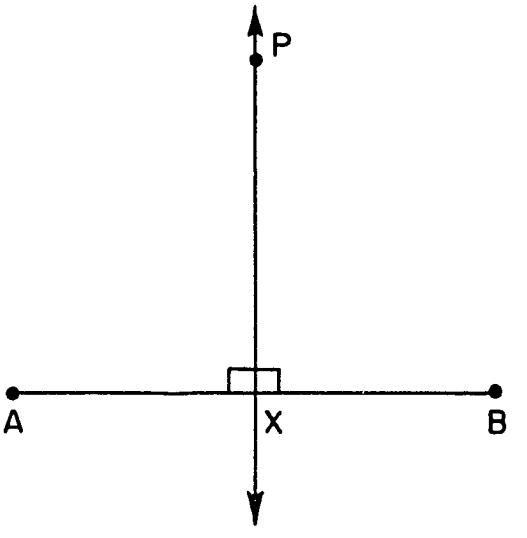
These corresponding parts which are matched when the paper is folded we call <u>images</u> of each other. They are images or reflections in the line of the fold.

(b) Is the cut edge PAC an image of the cut edge QBD ?

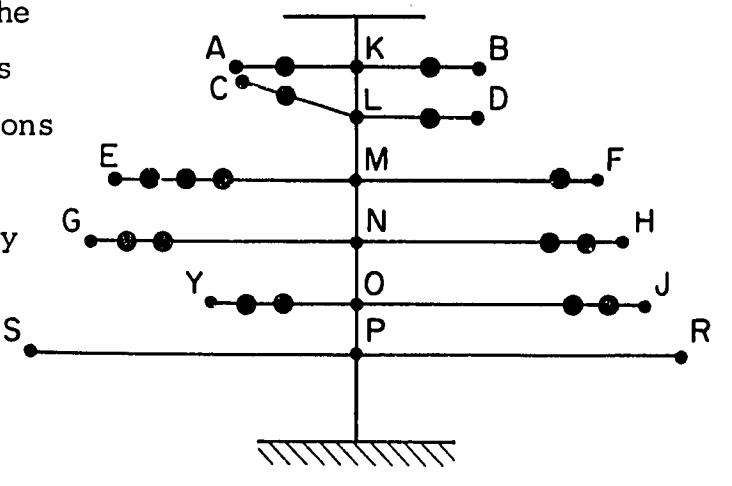
(c) On each side of the fold look at the region enclosed by the edge of the paper. Are the <u>regions</u> images of each other?

2. You have learnt to construct the perpendicular bisector of a line segment. $\overleftrightarrow{PX}$ divides $\overline{AB}$ into two congruent parts at X. You found that $\widehat{PXA} \equiv \widehat{PXB}$ and $\overline{AX} \equiv \overline{BX}$.

Is point A the image of point B in $\overleftrightarrow{PX}$? Mark some other image points on either side of $\overleftrightarrow{PX}$. Note: The order in the statement $\overline{AX} \equiv \overline{BX}$ is chosen to show that A corresponds to B, and X may be thought of as corresponding to itself.



3. The figure to the right represents a tree. The large dots on the branches represent fruit. The tips of the branches and the intersections of the branches with the trunk are labelled as shown. Look carefully at the branches on either side of the trunk.



(a) Does the branch $\overline{AK}$ seem to be the image in the trunk of the branch $\overline{BK}$? Branch $\overline{SP}$ of $\overline{RP}$? Why? Measure to find out.

(b) Does branch $\overline{CL}$ seem to be the image in the trunk of branch $\overline{DL}$? Why? Notice that $\widehat{CLK} \not\equiv \widehat{DLK}$.

(c) Does $\overline{YO}$ seem to be the image in the trunk of branch $\overline{JO}$? Why?

(d) Does branch $\overline{EM}$ seem to be the image in the trunk of branch $\overline{FM}$? If the fruit are thought of as part of the set of points, is your answer still the same? Explain your answer.

(e) Name any other opposite branches that are images of each other in the trunk, considering the fruit as part of the set of points.

> We have been looking at figures some of which contain pairs of images on either side of a central line. The central line is called the <u>axis of symmetry</u> and the pairs of images are said to be <u>symmetrical</u> about the axis.

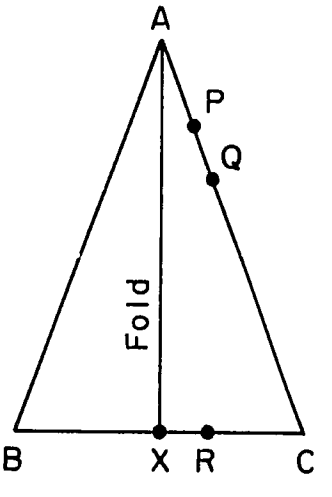
4. (a) Here is a sketch of a boy. Can an axis of symmetry be drawn through this sketch?

	Image
Left Ear	
Left Hand	
Heart	
Left Foot	
Right Eye	
Nose	

(b) Complete the table naming the other part of each symmetrical pair. If there is no symmetrical part to the one mentioned, say "none".

(c) Is any part of the boy on the axis of symmetry?

5. (a) Draw an isosceles triangle having base 3" and congruent sides $\overline{AB}$ and $\overline{AC}$ of length 5". Fold the paper so that $\overline{AC}$ falls on $\overline{AB}$. Will C fall on B ? Let the fold cut $\overline{BC}$ at X.



(b) Has the triangle an axis of symmetry? If so, name it.

(c) Mark any two points P and Q on $\overline{AC}$, and a point R on

$\overline{BC}$. Is there a point on $\overline{AB}$ which is the image of P in $\overleftrightarrow{AX}$? If so, mark it P'. Find and name the image of Q (Q') and of R (R'). Complete the following table:

	Image
(1) Point C	
(2) $\overline{PQ}$	
(3) $\overline{BR'}$	
(4) $\widehat{ACX}$	
(5) $\triangle ABX$	
(6) Point A	
(7) $\overline{AX}$	
(8) $\triangle ABC$	

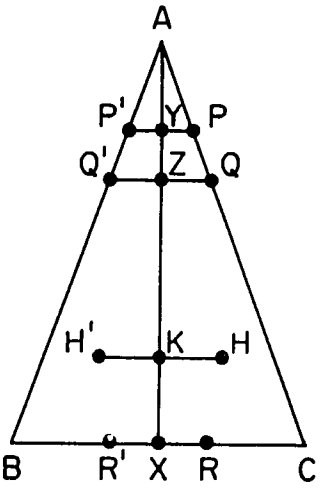
6. (a) On the same $\triangle ABC$ as for problem 5, draw $\overline{PP'}$ and $\overline{QQ'}$ and let them intersect $\overline{AX}$ at Y and Z respectively. Measure $\widehat{AXC}$, $\widehat{AYP}$, and $\widehat{AZQ}$.

(b) Fold the paper again along the same crease. With your compasses make a hole in the folded paper in the interior of the triangle. Open the triangle, and mark the holes H and H'. Draw $\overline{HH'}$, intersecting $\overline{AX}$ at K.

(i) Is $\overline{HH'}$ perpendicular to $\overleftrightarrow{AX}$? Measure to find out.

(ii) Is $\overline{HK} \equiv \overline{H'K}$? Does every point in the region AXC have a symmetrical point in the region AXB ?

(c) Notice again that the segment connecting symmetrical points is perpendicular to the axis of symmetry. The segment connecting symmetrical points is also bisected by the axis of symmetry. Have you seen these two properties before? Look back to other examples to find out.



> Definition: Any set of points in a plane is <u>symmetrical about an axis</u> if for every point in the set not on the axis there is a corresponding point in the set such that the axis is the perpendicular bisector of the segment joining the pair of corresponding points. The set of points on one side of the axis is said to be symmetrical to the set of points on the other side.
>
> Since the nature of the set of points is not otherwise restricted, it may consist of segments, rays, lines, curves, regions, isolated points or any combination of them.

7. Bisect any angle $\widehat{POQ}$, using ruler and compasses. Let $\overrightarrow{OS}$ be the bisector of $\widehat{POQ}$.

(a) Find an axis of symmetry in your figure. On $\overrightarrow{OP}$, choose any point B; on $\overrightarrow{OQ}$ choose B' so that $\overline{OB} = \overline{OB'}$. Draw $\overline{BS}$ and $\overline{B'S}$.

(b) Name two pairs of symmetrical line segments.

(c) What is the image of $\widehat{BOS}$?

(d) What is the image of $\triangle OBS$?

8. Can a figure have more than one axis of symmetry? Fold a rectangular piece of paper in half, matching two opposite edges; then without unfolding, fold it in half again, matching the <u>other</u> pair of opposite edges. Cut a design in two edges which have no folds. Open the paper and tell how many axes of symmetry are found. Name some image regions and image lines about the axes of symmetry.

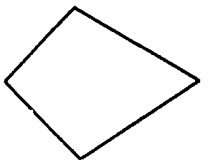
In this exercise you found more than one axis of symmetry. Many important examples of symmetry contain more than one axis. Can you name other figures with more than one axis of symmetry?

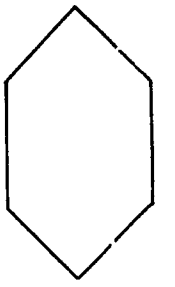
9. (a) Draw an isosceles triangle. Point to an axis of symmetry. How many can you find?

(b) Draw a square. How many axes of symmetry can you find? Point out some symmetrical regions for each of these axes.

(c) How many axes of symmetry has a rectangle? Did you find the same number of axes as you found in a square?

(d) Draw two figures as shown. How many axes of symmetry can you find in each? What symmetrical regions are found for each axis?



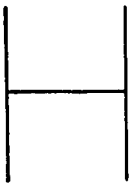


(e) Draw three other figures with axes of symmetry. Find some figure without any axes of symmetry.

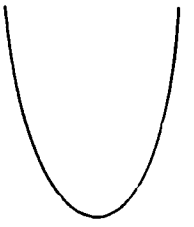
Problems 11-1

1. Copy the following figures. Draw all the axes of symmetry you can find.

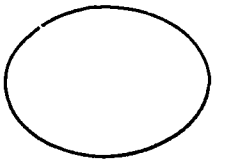
(a)



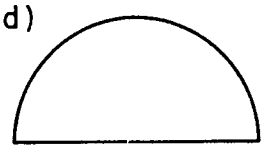
(b)



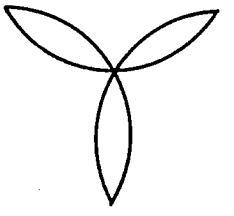
(c)



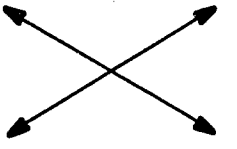
(d)



(e)



(f)



2. Cut a regular hexagon from a piece of paper. Find, by folding, all the axes of symmetry.

3. Fold a piece of paper and with your compasses make any four holes in the folded paper. Open the paper and label the holes on one side of the fold A, B, C, D, and the corresponding holes on the other side, A', B', C', D' respectively.

(a) Does this set of eight points form a symmetric figure about a line? What is the axis of symmetry?

(b) Test by measurement whether the eight points form a symmetric figure.

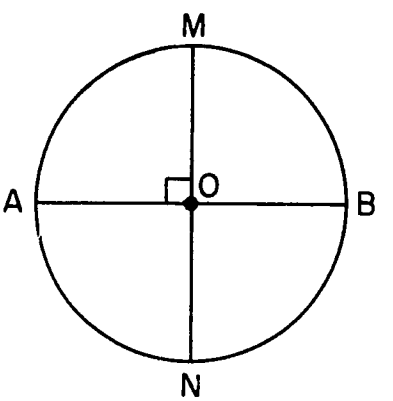
(c) Draw $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, $\overline{BD}$, $\overline{CD}$, and the corresponding segments on the other side. Name some pairs of segments and some pairs of regions which are symmetrical.

4. (a) How many axes of symmetry are found in an equilateral triangle? Show these axes in a figure.

(b) Answer the same question for a scalene triangle.

5. Draw a picture of any set of points. Then draw any line. Draw the image of your picture using the line as axis of symmetry.

6. Draw a circle of radius $1\frac{1}{2}''$ with centre at O. Draw any diameter to meet the circle at A and B. Cut out the circular region and fold along the diameter.



(a) The diameter is on an axis of symmetry. What are the symmetrical regions with respect to this axis? Are there any symmetrical curves about this axis?

(b) Draw a diameter perpendicular to $\overline{AB}$ intersecting the circle at M and N. Is $\overleftrightarrow{MN}$ an axis of symmetry?

(i) Are the points A and B symmetrical about $\overleftrightarrow{MN}$?

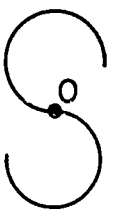
(ii) Are the points M and N symmetrical about $\overleftrightarrow{AB}$?

(c) Can you draw any other axis of symmetry? How many axes of symmetry has a circle?

11-2 Symmetry about a Point

Class Activity

1. (a) With your compasses draw a figure representing the letter S by making two congruent arcs of circles.



Choose the point O so that the curve below O is congruent to the curve above O.

(b) Mark any other point P on the S curve. Let P' be the point where $\overrightarrow{PO}$ again intersects the curve. What can you observe about the points P and P' ? Measure $\overline{PO}$ and $\overline{P'O}$.

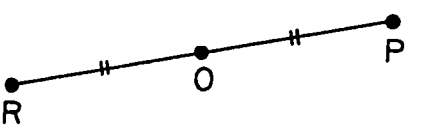
(c) Mark a point Q on the curve. Find Q' corresponding to Q where $\overrightarrow{QO}$ again intersects the curve. Measure $\overline{QO}$ and $\overline{Q'O}$. You find that all pairs of segments, such as $\overline{QO}$ and $\overline{Q'O}$, $\overline{PO}$ and $\overline{P'O}$, are congruent. We say that the S figure is <u>symmetrical about the point O</u>.

2. Draw any circle with centre O. Mark any point P on the circle. Draw $\overline{PO}$ and extend to meet the circle at Q. Q is the point symmetrical to P about centre O. Notice that $\overline{PO} \equiv \overline{QO}$. We see two types of symmetry in a circle:

(a) Symmetry about an axis (a diameter).

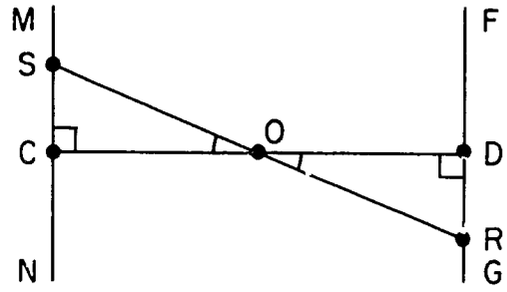
(b) Symmetry about the centre point O.

<u>Definition</u>: Two points P and R are <u>symmetrical to each other about a point O</u> if the segment $\overline{PR}$ is bisected by O. Point O is then a <u>point of symmetry</u>.



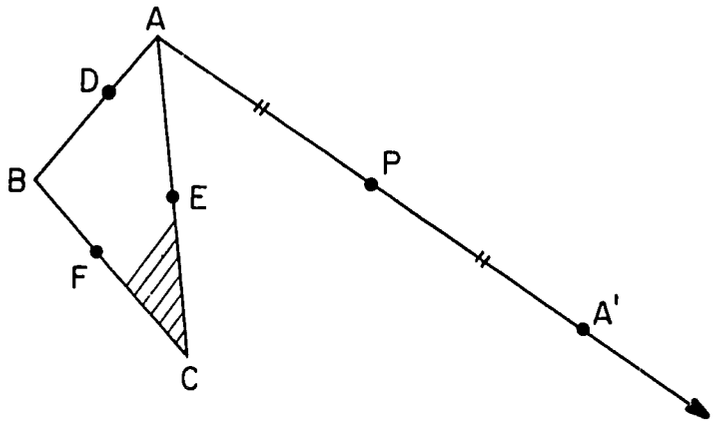
Any set of points is symmetrical about a point O if all points in the set (except O) occur in pairs which are symmetrical about O.

3. (a) Draw a segment $\overline{CD}$ and bisect it at O. At C and D draw $\overleftrightarrow{MN}$ and $\overleftrightarrow{FG}$ perpendicular to $\overline{CD}$.



(b) Choose any point S on $\overleftrightarrow{MN}$. Draw $\overrightarrow{SO}$ to meet $\overleftrightarrow{FG}$ at R. Show without measuring that $\overline{SO} \equiv \overline{RO}$. Which case of triangle congruence did you use? S and R are therefore symmetrical points about O according to our definition. Since each point of $\overleftrightarrow{MN}$ has a symmetrical point in $\overleftrightarrow{FG}$, we say that $\overleftrightarrow{MN}$ is symmetrical to $\overleftrightarrow{FG}$ about the point O.

4. (a) Draw $\triangle ABC$ and mark points D, E, F, on the sides as shown. Choose any point P outside the triangle. Using ruler and compasses, draw $\overrightarrow{AP}$ and extend to A' so that $\overline{AP} \equiv \overline{PA'}$.

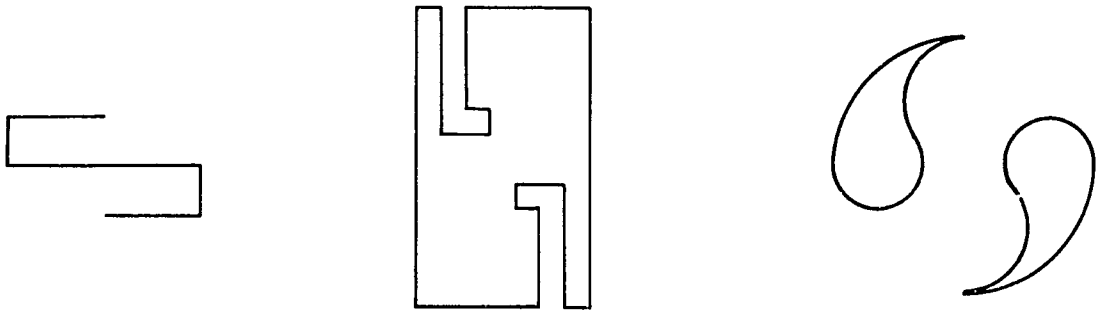


Draw $\overrightarrow{CP}$ and extend to C' so that $\overline{CP} \equiv \overline{PC'}$. Repeat this procedure for points B, D, E, and F which are all on $\triangle ABC$. If you have drawn carefully, C' will be on $\overleftrightarrow{A'E'}$, B' on $\overleftrightarrow{C'F'}$, and A' on $\overleftrightarrow{B'D'}$. $\triangle ABC$ is symmetrical to $\triangle A'B'C'$ about the point P. Each point on $\triangle ABC$ has a corresponding point on $\triangle A'B'C'$ such that the line through P connecting these points is bisected by P.

(b) Shade a region in $\triangle ABC$ as shown. Find a region in $\triangle A'B'C'$ corresponding to your shaded region in $\triangle ABC$. Do these regions appear to be symmetrical about point P ?

Problems 11-2

1. Each of the following figures has a point of symmetry. Trace the figures and mark the point of symmetry.



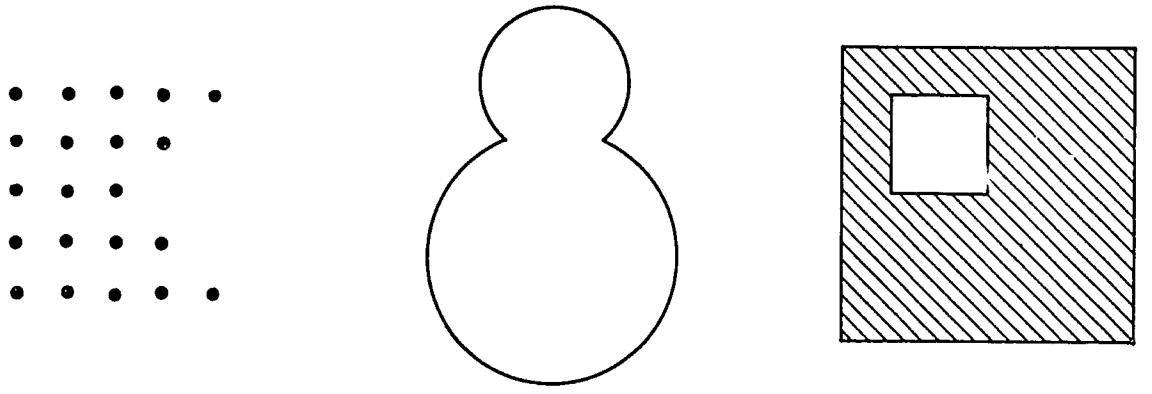
(a) (b) (c)

2. Draw the portion of the number line for the numbers from -10 to 10.

(a) Name five pairs of corresponding points which are symmetrical to each other about zero.

(b) Name five pairs of corresponding points which are symmetrical to each other about -4.

3. Draw any plane set of points. Pick a point P. Draw a symmetrical picture of your original set, making P a point of symmetry. Here are some suggested sets of points to start with. Can you think of an interesting one for yours?



(a) (b) (c)

11-3 Symmetry about a Plane

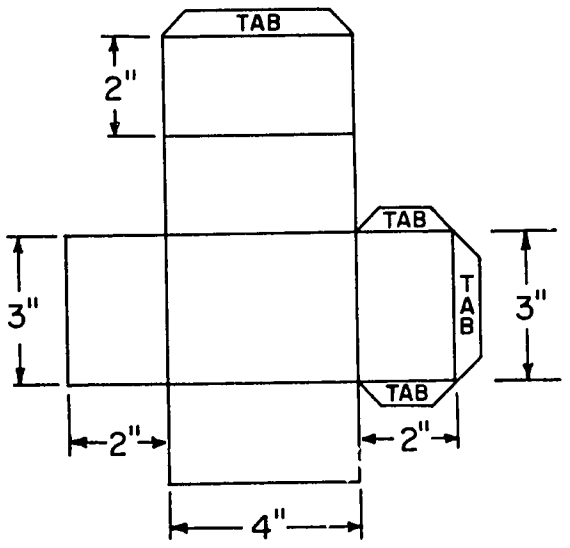
Class Activity

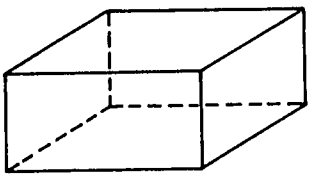
1. Cut an orange (or some other spherical object such as a ball of soft clay) straight through its centre. Put the two parts together again with a piece of cardboard between them. Does there seem to be some kind of symmetry with respect to the cardboard? Can you describe this symmetry? Think of a point M in one half of the orange. If a line from M is perpendicular to the plane P (represented by the cardboard), how would you determine a point N corresponding to M in the other half of the orange? Let $\overline{MN}$ intersect plane P at X. Is $\overline{MX} \equiv \overline{NX}$? Since this correspondence is true for all pairs of points in the orange, we say that the orange is <u>symmetrical about the plane P</u>.

2. If you have a globe of the world find the point X (longitude 35° E, latitude 12° N). Find a point Y symmetrical to X about the plane which contains the equator. What is the latitude and longitude of point Y ?

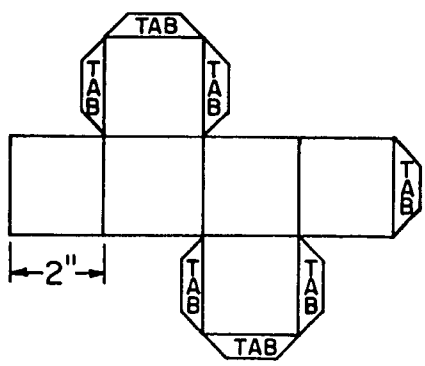
3. The following surfaces may be made by drawing figures like those shown using the suggested measurements on plain or coloured paper. Then cut along the outside boundary, fold along the lines, bend the tabs inward and secure with paste or cellophane tape.

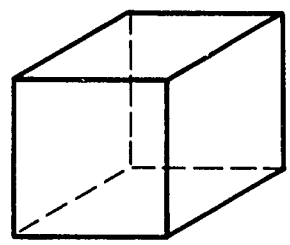
(a) Make a rectangular prism. Are there any planes of symmetry? Indicate some points, segments, rectangular regions, and rectangular prisms which are symmetrical about these planes.



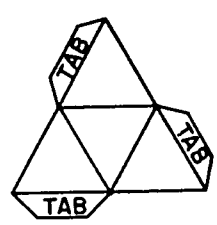


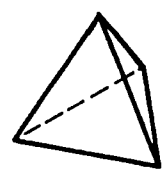
(b) Repeat (a) making a cube instead of a rectangular prism. Make all squares with 2 inch sides.



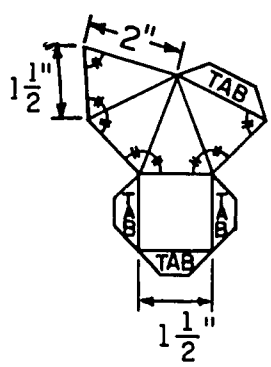


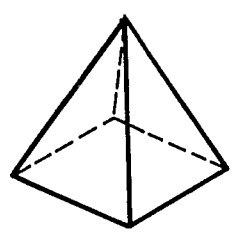
(c) Make a triangular pyramid with all of its faces equilateral triangles with 2" sides. We call such a solid a regular tetrahedron. Find any planes of symmetry in this figure. Show in a picture of a tetrahedron the intersections of one of the planes of symmetry and the faces of the solid. What figure is formed by the lines of intersection of a plane of symmetry and the tetrahedron?



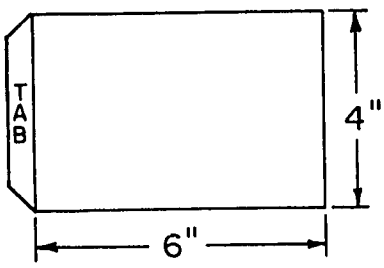


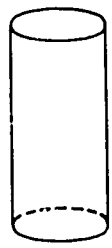
(d) Repeat (c) for a square pyramid. Let the square base have $1\frac{1}{2}$" sides and let the faces be isosceles triangles having $1\frac{1}{2}$" bases and 2" sides.





(e) Make a circular cylinder. Does the cylinder have an axis of symmetry? Does the cylinder have a plane of symmetry? How many planes of symmetry does the cylinder have?





4. Consider a sphere again. What kinds of symmetry does a sphere have?

5. Point out and discuss examples of symmetry of any type both inside and outside the classroom.

Summary: In this chapter you have seen that symmetry may be found with respect to a point, a line, or a plane. In each example of symmetry you made use of corresponding points that were images of each other. To be symmetrical, a pair of corresponding points must have two special properties. The line connecting a pair of symmetrical points is bisected by the point, axis, or plane of symmetry. The axis or plane of symmetry is perpendicular to the line connecting symmetrical points.

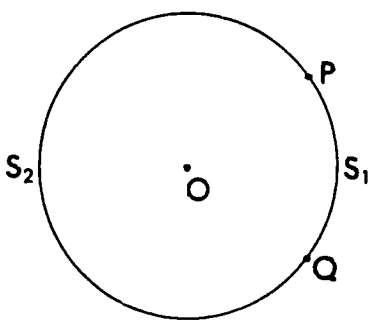
CHAPTER 12

ANGLE PROPERTIES IN CIRCLES

12-1 Arcs

We have seen earlier that any point, X, on a line divides the line into two rays.

You cannot pass along the line from a point of one ray to a point of the other ray without passing through X. Into how many parts does a point on a circle divide a circle? We notice that if we start from a point P on the circle, we can trace the whole circle before we come back to P. The circle is still in one piece. Clearly a point P does not divide a circle into two parts. But if we choose a second point Q, then these two points divide the circle into two parts, S_1 and S_2. We call each part an arc with end points P and Q. We write $\overset{\frown}{PQ}$ (read arc PQ). Do you observe that $\overset{\frown}{PQ}$ can be used for either of the two arcs? How do we know which arc we are talking about? If we choose P and Q so that the segment $\overline{PQ}$ is not a diameter, we call the shorter arc the minor arc (minor $\overset{\frown}{PQ}$) and we call the longer arc the major arc (major $\overset{\frown}{PQ}$).



To distinguish between major and minor arcs, consider the circle with centre O. If $\overline{PQ}$ is not a diameter, the union of P, Q and all the points of the circle in the exterior of $P\hat{O}Q$ is a major arc; the union of P, Q and all points of the circle in the interior of $P\hat{O}Q$ is a minor arc. If $\overline{PQ}$ is a diameter, the union of P, Q and all points of the circle in one of the two half-planes determined by the diameter is a semi-circle.

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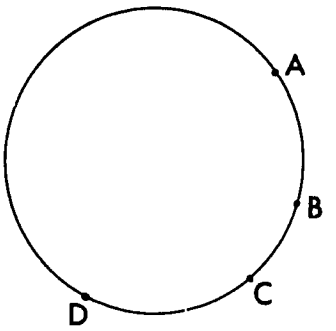
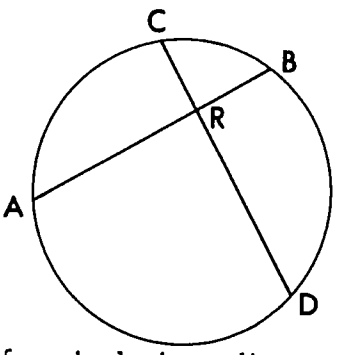
Problems 12-1

Name at least 8 arcs of this circle whose end-points are given. Distinguish between minor and major arcs.



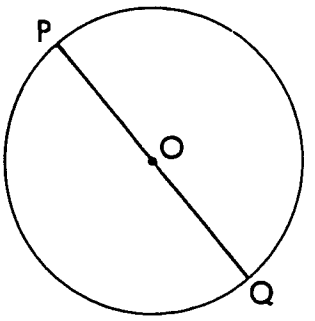
12-2 Chords

A chord of a circle is a line segment whose end points are points of the circle. The figure below shows chords $\overline{AB}$ and $\overline{CD}$. They intersect at R.



Can you show other chords of the circle which pass through R? How many are there?

We recall that a chord which contains the centre of a circle is a diameter. Here is a diameter $\overline{PQ}$.



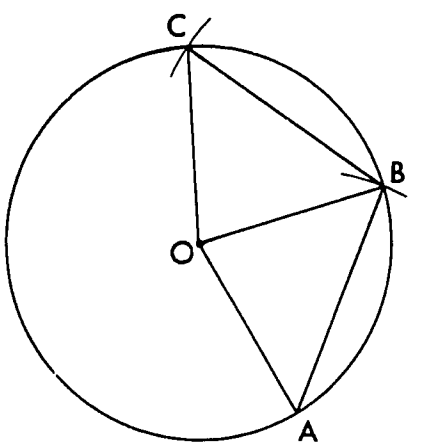
Draw three more diameters. Can you draw more? How many are there?

12-3 Central Angles, Arcs, and Chords

Let us look for some relation between arcs and chords of the same circle.

Class Activity

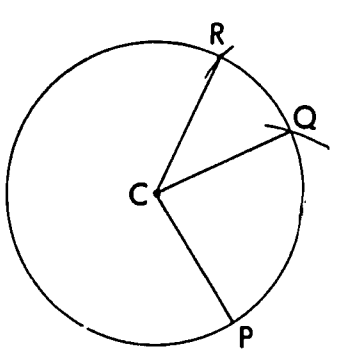
1. Draw a circle with centre O and radius 2 inches long. Choose a point A on the circle. With A as centre and radius 2·5 inches describe an arc to intersect the circle at B. With B as centre and with the same radius describe another arc to intersect the circle at C. Draw the chords $\overline{AB}$, $\overline{BC}$.



Can you show that $\triangle OAB$ and $\triangle OBC$ are congruent? Which case of congruent triangles is used? Is $B\hat{O}A \equiv C\hat{O}B$?

If from your drawing you cut out the region with edges $\overline{OA}$, $\overline{OB}$, and $\widehat{AB}$ and match this with the region with edges $\overline{OB}$, $\overline{OC}$, and $\widehat{BC}$, what do you observe about $\widehat{AB}$ and $\widehat{BC}$?

2. Draw a circle with radius $1\frac{1}{2}$ inches and centre C. Take any point P on the circle and with this point as centre and radius 2 inches draw an arc to cut the circle at Q. With centre Q and radius 1 inch cut the circle at R, not in the interior of $Q\hat{C}P$. Measure $R\hat{C}Q$, $Q\hat{C}P$. We observe that $R\hat{C}Q$ and $Q\hat{C}P$ are not congruent and that $\widehat{PQ}$ and $\widehat{QR}$ could not be made to fit exactly.



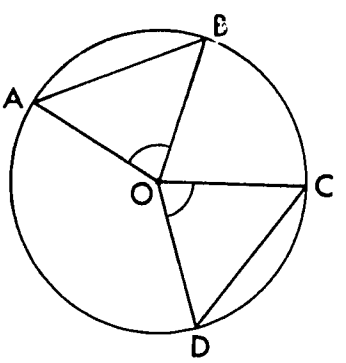
Problems 12-3a

1. Choose a different radius for the circle and suitable pairs of chords and repeat experiments 1. and 2. of the above class activity.

2. Draw a circle with centre O and mark a point P on it. Draw a circle with centre P which intersects the first circle at Q. Draw a third circle with centre

P and radius slightly less than the radius of the second circle. If this circle meets the first circle at R, how do angles POQ and POR compare? How do chords $\overline{PQ}$ and $\overline{PR}$ compare?

Angles such as $B\hat{O}A$ and $C\hat{O}D$ are called central angles of the circle. A central angle of a circle is an angle whose vertex is the centre of the circle.



A central angle of a circle is said to be subtended by a chord or a minor arc of the circle if the sides of the angle pass through the end points of the chord or arc.

Corresponding to any two central angles in a circle there are two chords of the circle which subtend the angles and also two arcs of the circle which subtend the angles.

If the angles are congruent, are the chords congruent?

If the chords are congruent, are the angles congruent?

If the angles are congruent, are the arcs congruent?

If the arcs are congruent, are the angles congruent?

If the chords are congruent, are the arcs congruent?

If the arcs are congruent, are the chords congruent?

Problems 12-3b

1. Draw a circle with radius 2 inches long. Mark off 3 chords $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ each of length 2 inches. Find the size of each central angle.

2. In the figure you drew for Problem 1, use cutting or folding to compare the three minor arcs, $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, $\overset{\frown}{CD}$ by trying to fit them to each other. Are these arcs congruent?

3. Verify by using strings wrapped around the base of a circular tin that congruent chords cut off arcs of the same length in the same circle. Is the fact that two arcs are of the same length sufficient to show that they are congruent?

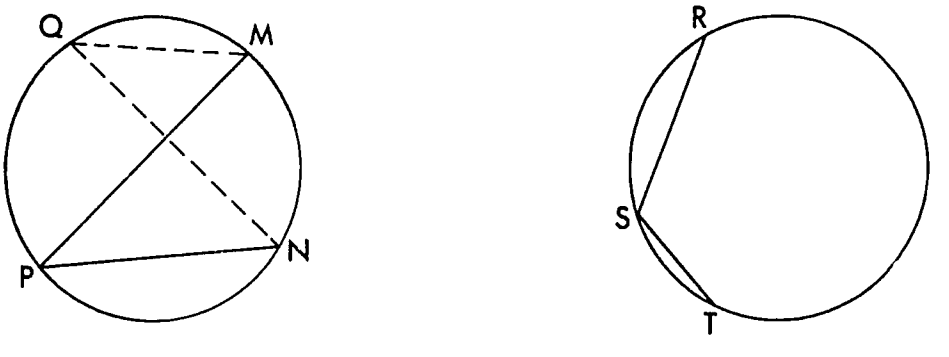
4. In a circle draw two central angles, $\widehat{AOB}$ and $\widehat{COD}$, each of 40°. Why must the chords which subtend these central angles be congruent?

5. In your figure for Problem 4, use cutting or folding to compare the minor arcs $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$. Are these arcs congruent?

6. Draw two circles of the same radius. Draw two congruent chords, one on each circle. Show that the central angles subtended by these chords are congruent. Show by fitting that the arcs intercepted by these chords are congruent.

12-4 Inscribed Angles

The following figure shows several angles inscribed in arcs of circles.



$\widehat{MPN}$ is <u>inscribed</u> in major $\overset{\frown}{MN}$ (and so is $\widehat{MQN}$). Also we say: $\widehat{MPN}$ <u>subtends</u> minor $\overset{\frown}{MN}$.

$\widehat{RST}$ is <u>inscribed</u> in minor $\overset{\frown}{RT}$. Also we say: $\widehat{RST}$ <u>subtends</u> major $\overset{\frown}{RT}$.

<u>Definition.</u> An angle is <u>inscribed</u> in an arc of a circle if the end points of the arc lie on the two sides of the angle and the vertex of the angle is a point, but not an end point of the arc.

We say also that the inscribed angle is <u>subtended</u> by the remaining arc.

<u>Problems 12-4</u>

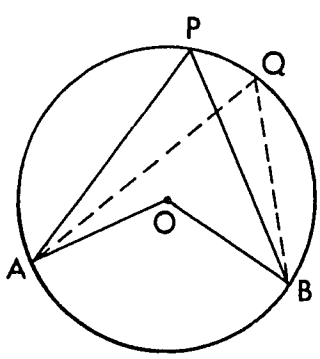
1. Draw a circle and choose an arc of the circle. Inscribe an angle in that arc. Inscribe another angle in that arc. How many angles can be inscribed in the arc? Measure each of the angles you inscribed in the selected arc.

   Repeat your drawing for another circle and an arc of it. Inscribe at least three angles in the arc you select. What do you observe?

2. Draw a circle and select an arc of this circle. Draw an inscribed angle that is subtended by your selected arc. What do you think about the measures of all the inscribed angles which your arc subtends?

12-5 Central Angles and Inscribed Angles

The figure shows two inscribed angles subtended by an arc $\overset{\frown}{AB}$ which also subtends the central angle $\widehat{AOB}$. How do you think the measures of the inscribed angles compare with the measure of the central angle? Is measure of $\widehat{APB}$ more or less than measure of $\widehat{AOB}$? Can you say anything more definite than that? Let us do some experimenting to see what we can learn about these angles.



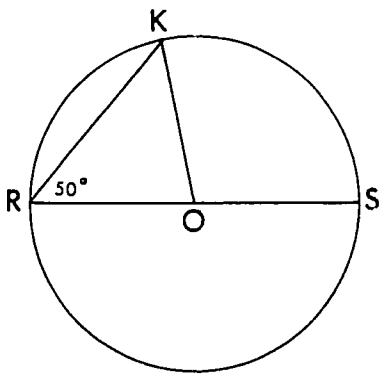
Class Activity

1. (a) Draw a figure like the one above, making $\widehat{AOB}$ an angle of 120°. Find by measurement the size of the inscribed angles $\widehat{APB}$ and $\widehat{AQB}$.

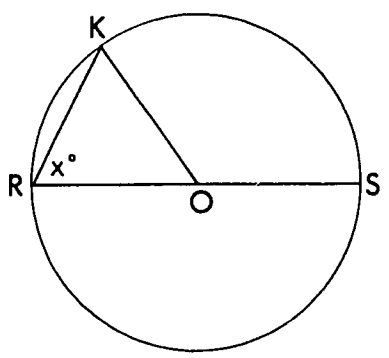
What is the size of $\widehat{APB}$ when $\widehat{AOB}$ is an angle of

(b) 60° (c) 125° (d) 90°

2. If $\overline{RS}$ is a diameter and O is the centre of this circle, why is $\triangle ORK$ an isosceles triangle? If the measure of $\widehat{KRO}$ is 50°, what is the measure of $\widehat{RKO}$? What is the measure of $\widehat{KOS}$? Why?



3. If the measure of $\widehat{KRO}$ is x, what is the measure of $\widehat{RKO}$? What is the measure of $\widehat{KOS}$? Why?

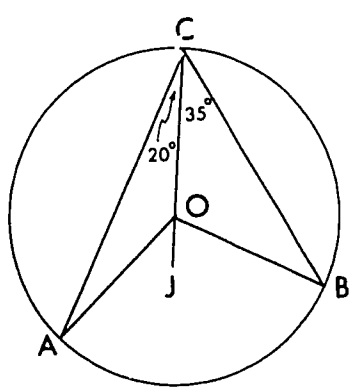


4. Consider central angle $\widehat{AOB}$ and inscribed angle $\widehat{ACB}$. If the measure of $\widehat{ACO}$ is 20° and the measure of $\widehat{BCO}$ is 35°, find the measure of

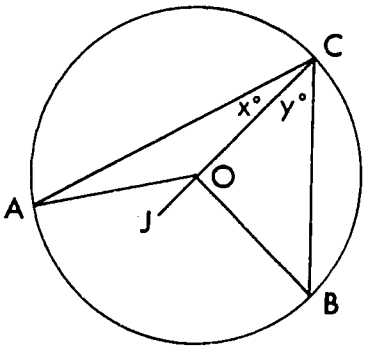
(a) $\widehat{CAO}$ (c) $\widehat{CBO}$ (e) $\widehat{ACB}$

(b) $\widehat{AOJ}$ (d) $\widehat{BOJ}$ (f) $\widehat{AOB}$

(g) What is the relation between $\widehat{ACB}$ and $\widehat{AOB}$?



5. If the measure of $\widehat{ACO}$ is x and the measure of $\widehat{BCO}$ is y, find the measure of

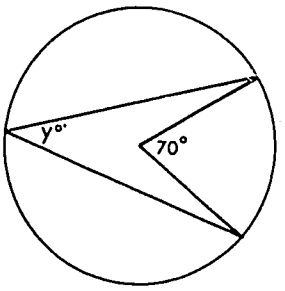


(a) $\widehat{CAO}$ (c) $\widehat{CBO}$ (e) $\widehat{ACB}$

(b) $\widehat{AOJ}$ (d) $\widehat{BOJ}$ (f) $\widehat{AOB}$

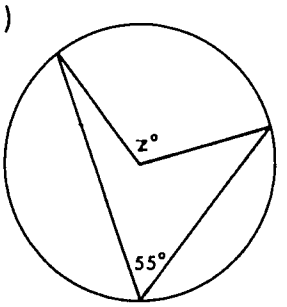
(g) What is the relation between $\widehat{ACB}$ and $\widehat{AOB}$?

6. (a)



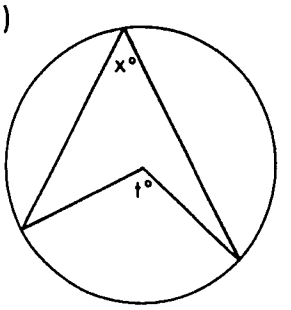
Find y.

(b)



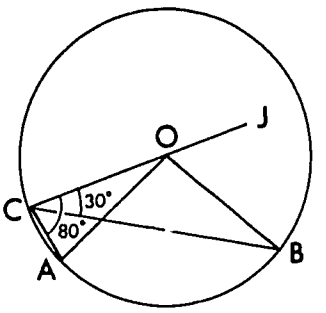
Find z.

(c)



Find t in terms of x.

7. Consider the central angle $\widehat{AOB}$ and inscribed angle $\widehat{ACB}$. If the measure of $\widehat{ACO}$ is 80° and the measure of $\widehat{BCO}$ is 30°, find the measure of



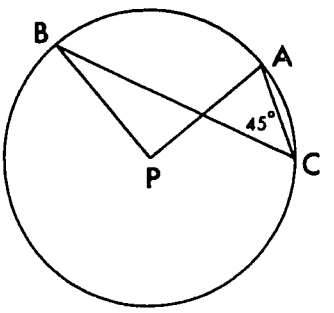
(a) $\widehat{CAO}$ (c) $\widehat{CBO}$ (e) $\widehat{ACB}$

(b) $\widehat{AOJ}$ (d) $\widehat{BOJ}$ (f) $\widehat{AOB}$

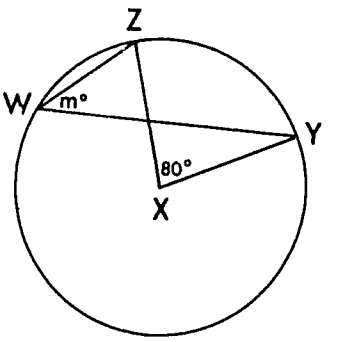
(g) What is the relation between $\widehat{ACB}$ and $\widehat{AOB}$?

In a circle the measure of an inscribed angle in a given major arc is half the measure of the central angle subtended by the remaining arc.

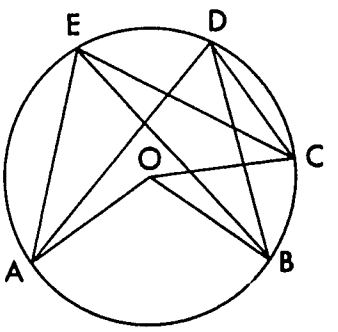
Problems 12-5



1. P is the centre of this circle. The size of $\widehat{BCA}$ is 45°. Find in degrees the size of $\widehat{APB}$.



2. If X is the centre of the circle and the size of $\widehat{ZXY}$ is 80° find m.



3. If the measure of $\widehat{AOB}$ is 110° and the measure of $\widehat{BEC}$ is 20°, find the measure of

(a) $\widehat{ADB}$ (c) $\widehat{BOC}$ (e) $\widehat{AEC}$

(b) $\widehat{AEB}$ (d) $\widehat{BDC}$ (f) $\widehat{AOC}$

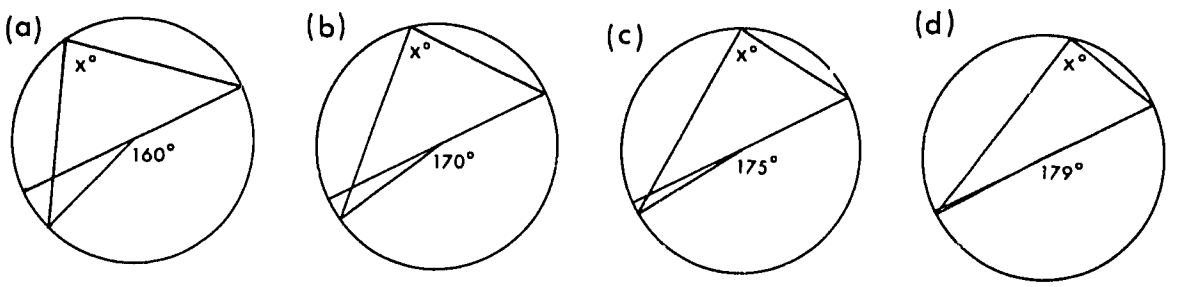
4. Carry through the reasoning of Exercise 7 in the Class Activity for the case where the measure of $\widehat{ACO}$ is x and the measure of $\widehat{BCO}$ is y.

12-6 Angles in a Semi-circle

We hope to discover the measure of angles inscribed in a semi-circle.

Class Activity

Find x in each of the following:



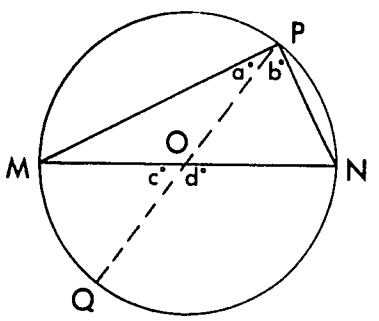
What is x approaching?

Consider $\widehat{MPN}$ inscribed in a semi-circle. Draw diameter $\overline{PQ}$. Why is $a = \frac{1}{2}c$?

Why is $b = \frac{1}{2}d$?

Find $c + d$. Hence find $a + b$.

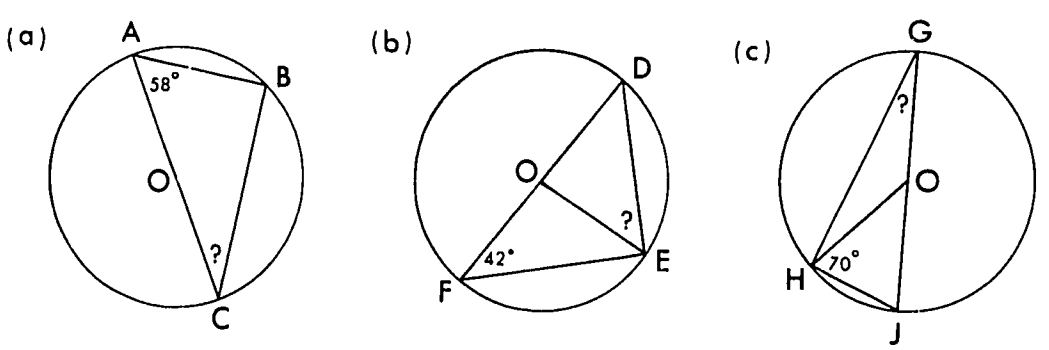
What is the measure of $\widehat{MPN}$?



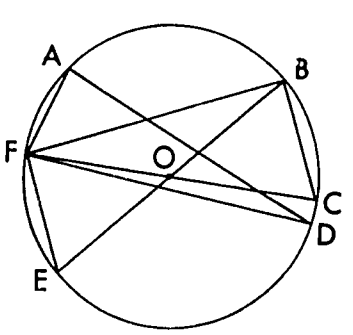
An angle inscribed in a semi-circle is a right angle.

Problems 12-6

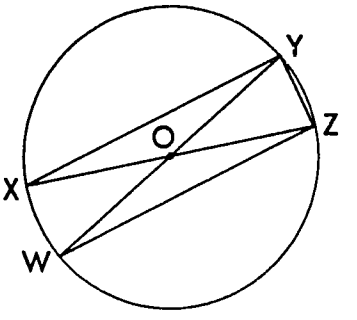
1. If O is the centre, find the size of the angles marked with "?".



2. Name all the right angles in this figure. O is the centre.



3. If O is the centre, why are $\overline{YX}$ and $\overline{ZW}$ both perpendicular to $\overline{YZ}$? Is $\overline{YX}$ parallel to $\overline{ZW}$?

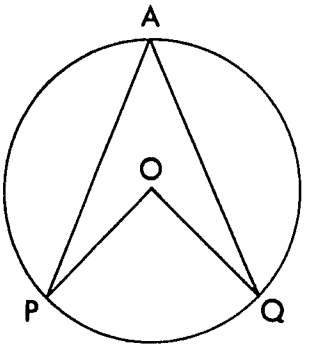


12-7 Angles Inscribed in the Same Arc

Let us look further at inscribed angles.

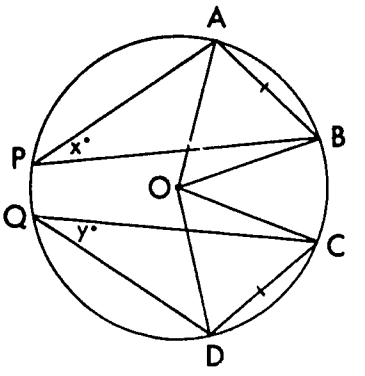
Class Activity

1. $\widehat{PAQ}$ is inscribed in the major $\overset{\frown}{PQ}$. Draw three more angles inscribed in this arc. If the size of the central $\widehat{POQ}$ is $2x^{\circ}$, what is the size of each of the inscribed angles? Why are these inscribed angles all congruent? How many more inscribed angles can we draw in major $\overset{\frown}{PQ}$? Are all these inscribed angles congruent?



Angles inscribed in the same major arc of a circle are congruent.

2. If chords $\overline{AB}$ and $\overline{CD}$ are congruent, are central angles $\widehat{AOB}$ and $\widehat{COD}$ congruent? Why?



If $\hat{P}$ and $\hat{Q}$ are inscribed angles intercepting $\overline{AB}$ and $\overline{CD}$ respectively, why is

$$x = \frac{1}{2} \text{ (measure of } \widehat{AOB}\text{) and}$$

$$y = \frac{1}{2} \text{ (measure of } \widehat{COD}\text{)?}$$

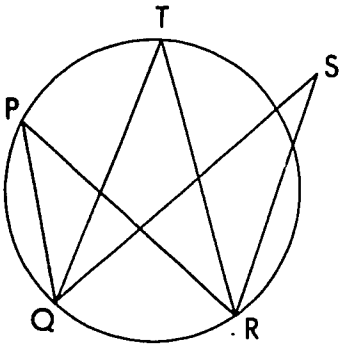
Then why is $x = y$?

For a circle, inscribed angles which intercept congruent chords are congruent angles, if they are both inscribed in major arcs.

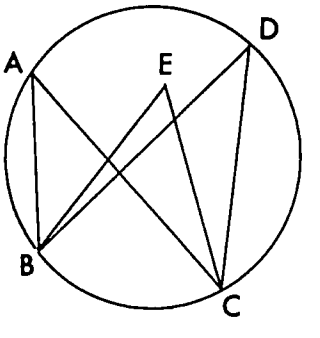
Problems 12-7

1. Name congruent angles in these figures.

(a)

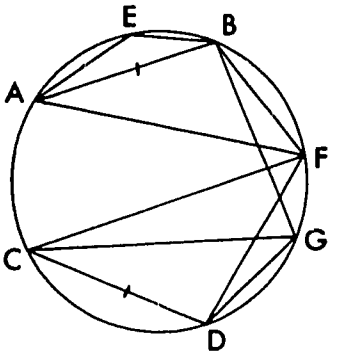


(b)

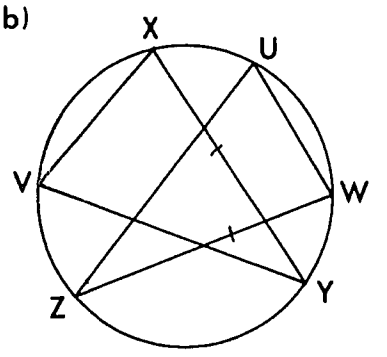


2. If $\overline{AB} \equiv \overline{CD}$ and $\overline{XY} \equiv \overline{ZW}$, name congruent angles in these figures.

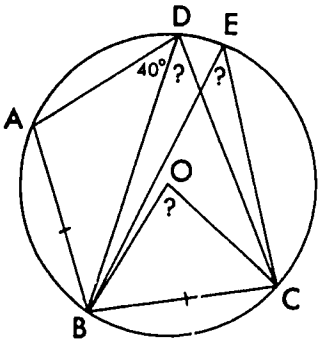
(a)



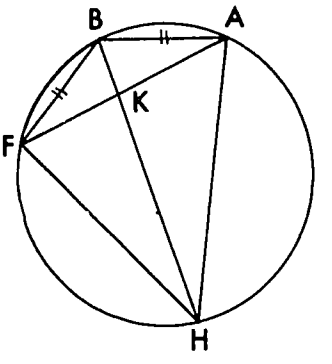
(b)



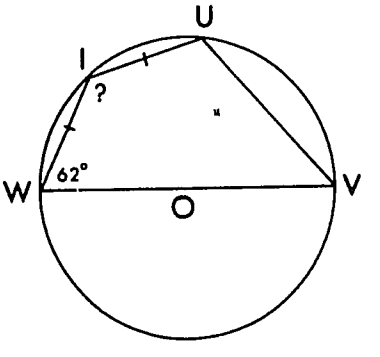
3. If the size of $\widehat{ADB}$ is 40° and $\overline{AB} \equiv \overline{BC}$, find the size of the angles marked with "?".



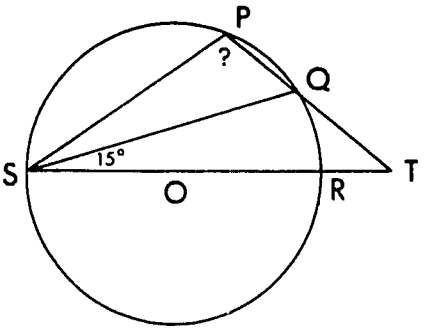
4. If $\overline{AB} \equiv \overline{BF}$, show that the corresponding angles of $\triangle AHK$ and $\triangle BHF$ are congruent.



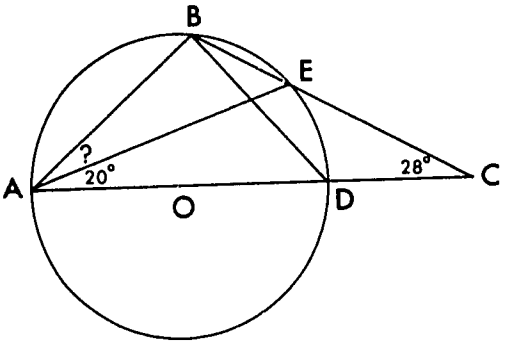
5. <u>CHALLENGE PROBLEM</u>. If $\overline{WI} \equiv \overline{IU}$ and the measure of $\hat{W}$ is 62° and O is the centre, find the measure of $\widehat{WIU}$.
Hint: Draw $\overline{IV}$ and $\overline{UW}$.



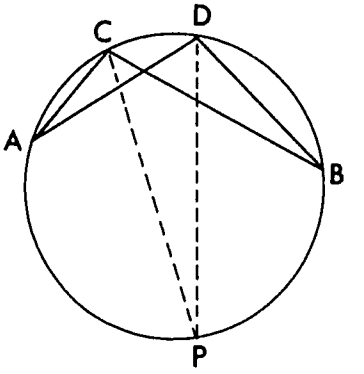
6. CHALLENGE PROBLEM. If O is the centre and the size of $\widehat{QST}$ is 15°, find the size of $\widehat{SPT}$.



7. CHALLENGE PROBLEM. If O is the centre and the measures of $\widehat{EAC}$ and $\widehat{BCA}$ are 20° and 28° respectively, find the measure of $\widehat{BAE}$.

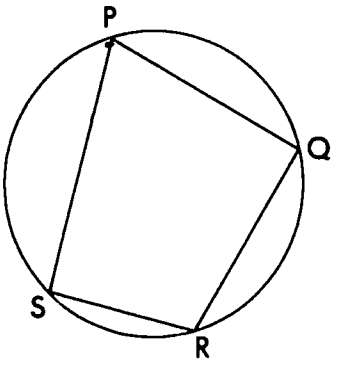


8. CHALLENGE PROBLEM. If $\widehat{ACB}$ and $\widehat{ADB}$ are inscribed in the same minor arc, show that they are congruent.
Hint: On the remaining arc choose a point P which divides it into two minor arcs $\overset{\frown}{AP}$ and $\overset{\frown}{PB}$. Draw $\overline{CP}$ and $\overline{DP}$.



12-8 Inscribed Quadrilaterals

Definition: If the four vertices of a quadrilateral are on a circle, the quadrilateral is inscribed in the circle. PQRS is an inscribed quadrilateral.

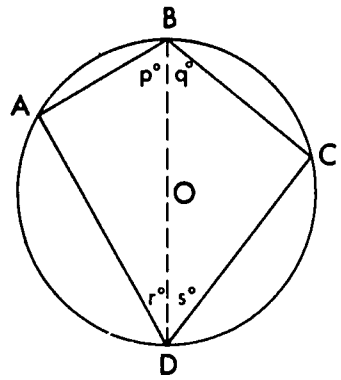


The name cyclic quadrilateral is

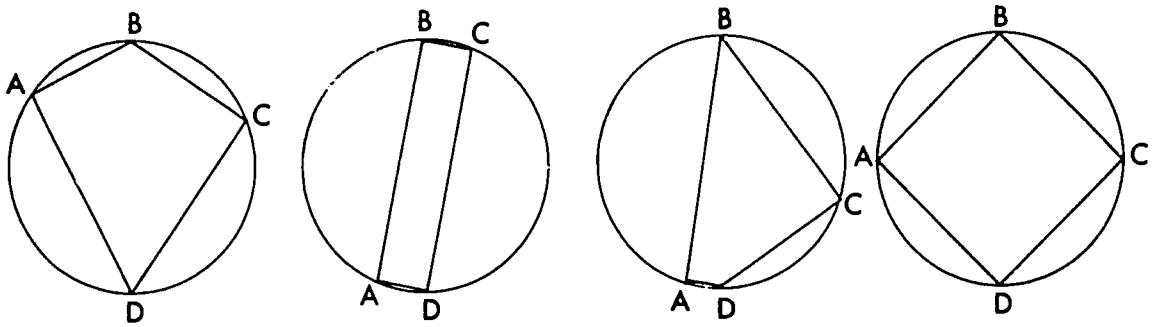
sometimes used for a quadrilateral which can be inscribed in a circle whether or not the circle is drawn.

Class Activity

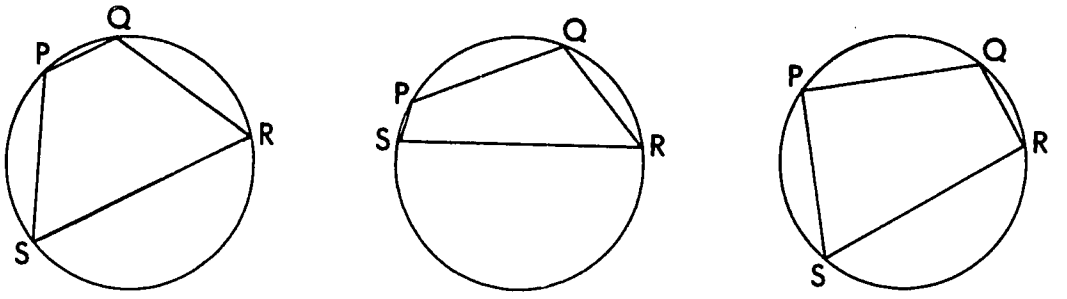
1. Consider the special case of an inscribed quadrilateral ABCD for which the diagonal $\overline{BD}$ is a diameter of the circle. Since $\overline{BD}$ is a diameter, what do you know about the measures of $\hat{A}$ and $\hat{C}$? What is the sum of the measures of angles $\hat{A}$ and $\hat{C}$? Why is it true that $p + r = 90$ and $q + s = 90$. Since, then, $p + r + q + s = 180$, what is the sum of the measures of $\widehat{ABC}$ and $\widehat{ADC}$? Does this discussion suggest a property of an inscribed quadrilateral?



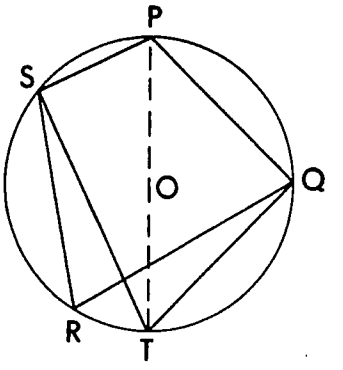
2. Draw four inscribed quadrilaterals similar to (but larger than) the ones shown below. Be sure that $\overline{BD}$ is a diameter. In each case measure angles $\hat{A}$ and $\hat{C}$ and find the sum of their measures; measure angles $\hat{B}$ and $\hat{D}$ and find the sum of their measures. What do you observe?



3. In the inscribed quadrilaterals below no diagonal is a diameter of the circle. Do you think it is still true that the sum of the measures of opposite angles is 180°? Measure them and see.



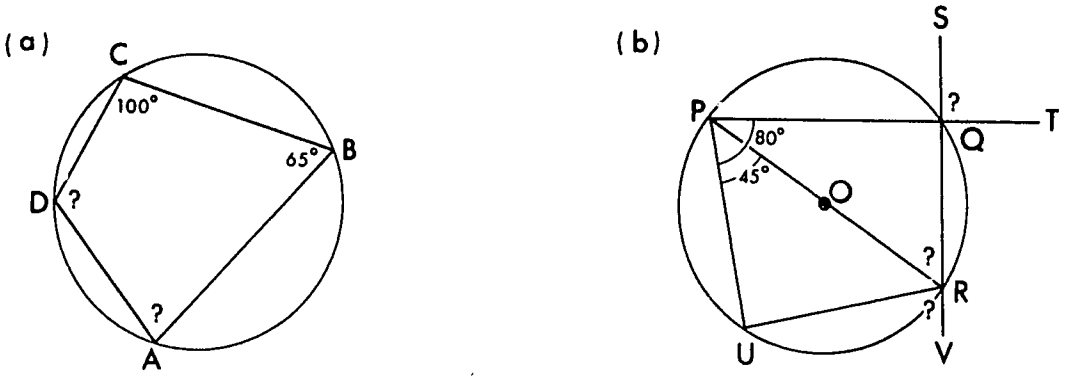
4. Let us try to show that the property we have discovered is always true. If PQRS is the given inscribed quadrilateral, draw diameter $\overline{PT}$, and draw $\overline{ST}$ and $\overline{QT}$. Is $\widehat{SRQ} \equiv \widehat{STQ}$? Why? What is the sum of the measures of angles $\widehat{SPQ}$ and $\widehat{STQ}$? What, then, is the sum of the measures of angles $\widehat{SPQ}$ and $\widehat{SRQ}$? Are these angles supplementary angles?



The sum of the measures of opposite angles of any inscribed quadrilateral is 180 .

Problems 12-8

1. Assume the information given on the figures and find the measures of the angles marked with "?"



2. Complete the following statements:

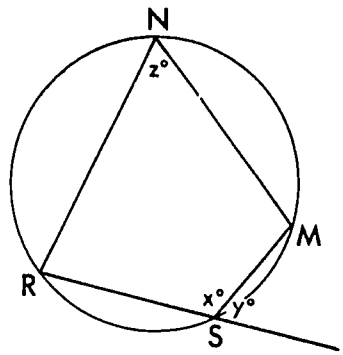
$x + z = ?$ Why?

$x + y = ?$ Why?

$x + y = x + z$ Why?

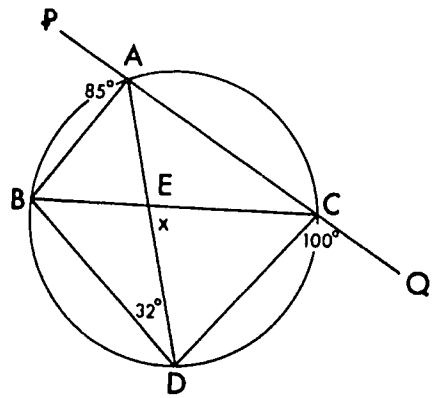
$y = ?$ Why?

What property of inscribed quadrilaterals does this show?

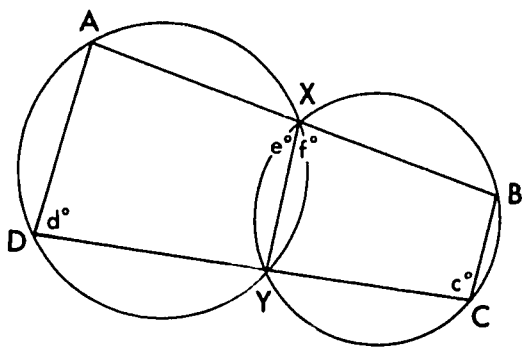


3. <u>CHALLENGE PROBLEM</u>.

Find x.



4. <u>CHALLENGE PROBLEM</u>. $\overline{XY}$ is the common chord of two intersecting circles. $\overline{AB}$ and $\overline{DC}$ intersect the circles as shown in the figure and contain X and Y respectively. Complete the following:



Since XYCB is inscribed,

$e = ?$

Since XYDA is inscribed,

$f = ?$

From these statements it is true that $e + f = ?$

But since A, X, and B are on a line, it is true that $e + f = ?$

What conclusion do you reach?

Can you use this conclusion to show that $\overline{BC}$ is parallel to $\overline{AD}$?

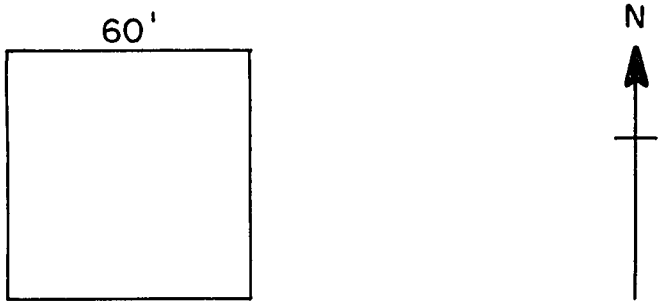
CHAPTER 13

LOCI AND ENVELOPES

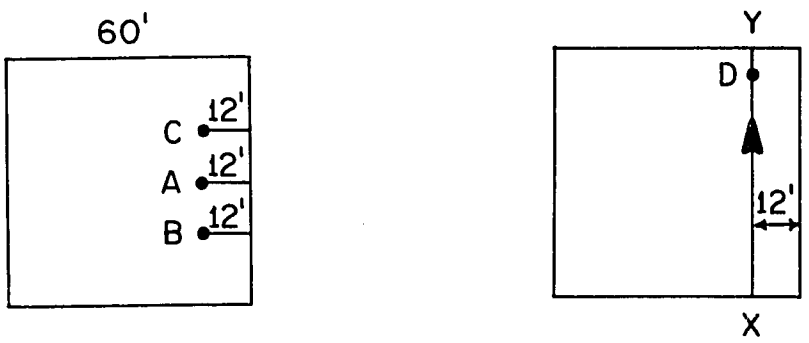
13-1 Examples

Example 1

A man looked for buried treasure in a square courtyard (i.e. walled compound) of side 60 ft. He knew that the treasure had been placed in a hole 3 ft. deep in a position 12 ft. from the east wall.



First he dug a hole at A but found no treasure. Then he tried B and C but again no treasure. Next he decided to dig a trench 3 feet deep from X to Y.



At last he found the treasure at D, very close to the north wall. It was a rich treasure so he was well rewarded for his hard work.

If the man had been given more information he might not have had to dig a trench nearly 60 ft. long. Here is a similar problem to which you should now be able to discover the answer.

Example 2

A square courtyard has sides 40 ft. long. A treasure was buried inside the courtyard so that it was 25 ft. from the north-west corner and 17 ft. from the south wall. How can the exact position be found?

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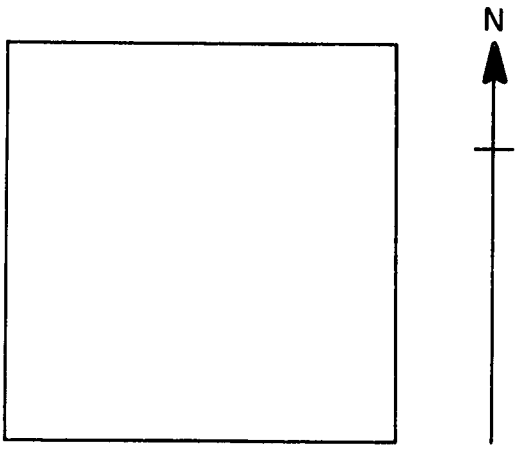


(a) First draw a square with sides 4" long.

(b) On this diagram mark with crosses 5 points which represent approximately positions in the courtyard 25 ft. from the N. W. corner. Does this set of 5 crosses suggest a pattern? If so, let us call it Pattern I. If you marked in 3 more points all 25 ft. from the N. W. corner, would they fit into Pattern I? Could you describe in words the set of all possible points in the courtyard which are 25 ft. from the N. W. corner? What instrument would you use to draw the complete set of points which forms Pattern I in the diagram?

(c) Mark with dots 5 points which represent approximately positions inside the courtyard which are 17 feet from the south wall. Does this set of 5 points suggest a pattern? If so, let us call it Pattern II.

Can you describe in words the set of all possible points inside the courtyard which are 17 ft. from the south wall?

What instruments would you use to draw Pattern II accurately in the diagram?

(d) What is the intersection of the sets forming Pattern I and Pattern II? Where was the treasure buried?

(e) Now draw an accurate diagram of the courtyard using a scale of 1 inch to 10 ft. Draw Patterns I and II accurately. Find by measurement the distance of the treasure from the S. W. corner of the courtyard.

Example 3

A boatman dropped a case into the sea to be picked up by a submarine at night. A radio message was sent to the submarine: "refer to map Q/123. Sixty ft. from pier P, in line with buoys A and B." The submarine commander found map Q/123 and located P, A, and B. (Fig. i.)

(a) He drew the line segment $\overline{AB}$.

(b) He marked 5 dots on the map representing points distant 60 ft. from the pier-end. Now his map looked like Fig. ii.

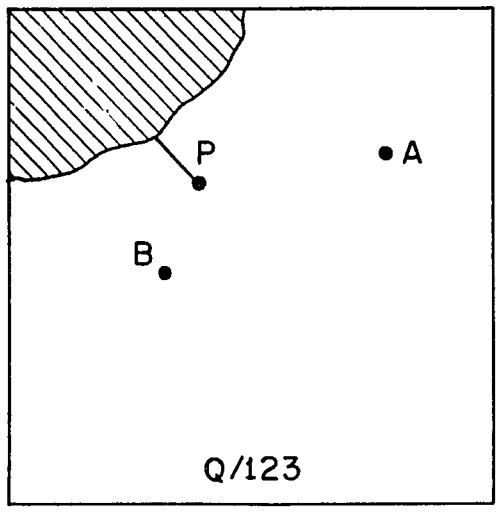


Fig. i

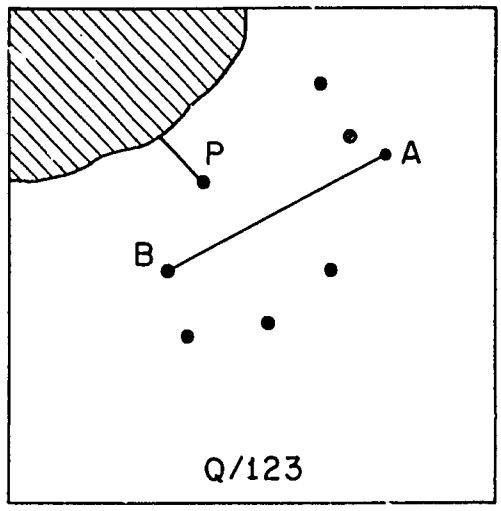


Fig. ii

(c) He drew an arc of a circle to pass through the 5 dots. (What were his centre and radius?) The arc intersected $\overline{AB}$ at C, (Fig. iii).

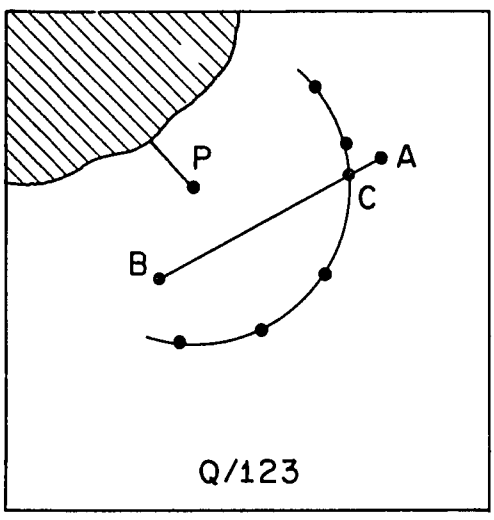


Fig. iii

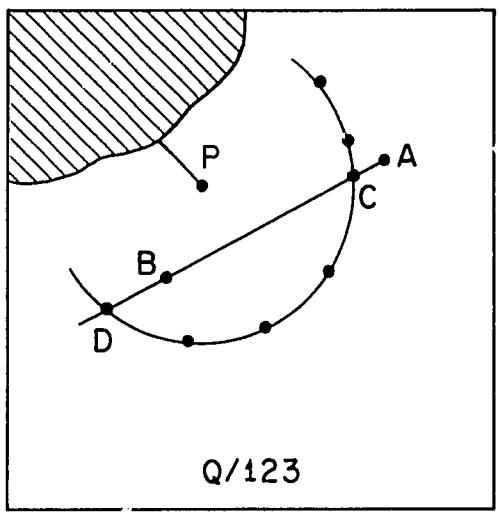


Fig. iv

(d) He sent out a boat from the submarine with instructions to the crew to look for the case at the place represented by C.

(e) The crew made for the pier, tied a sixty foot rope to the end and rowed until the rope was taut. Then they rowed keeping the rope taut until they were directly between A and B. Then one of the crew dived for the case but could not find it.

(f) Next they read the radio message again carefully and looked at the map again. They noticed that the point where the case was dropped was not necessarily between A and B. Then they marked another possible point D on the map (Fig. iv.). To reach D they rowed along the arc of the circle, keeping the rope taut. When they came to a point where buoy B was in line with buoy A they stopped rowing. This time they found the case.

13-2 Loci

In each of these examples we have done we were looking at certain sets of points, sets which are determined by certain given conditions. It is convenient to have a word for such sets of points. The word "locus" serves the purpose rather well. It is a Latin word meaning place. It is used for the set of all points which have a particular property. It is also used for the path or track followed by a moving body, satisfying certain conditions. The plural of locus is loci (pronounced low-sai).

You will understand the use of this word"locus" better if you refer to Examples 1-3.

In Example 1 the locus of all possible positions of the treasure was a line segment 60 ft. long, parallel to the east wall of the courtyard at a distance of 12 feet from it.

In Example 2 there were two loci. One was a part of a circle, with centre at the N.W. corner and radius 25 ft.; the other was a line segment, inside the courtyard, parallel to the south wall, at a distance of 17 feet. The treasure was buried at the intersection of the two loci.

In Example 3 the diagram in Fig. iv showed that the first locus was a line through A and B. The second was a circle with centre P. The two loci intersected at the points C and D. The case was dropped by the boatman at D but the submarine crew began the search at C. How could the boatman have saved them from wasting valuable time?

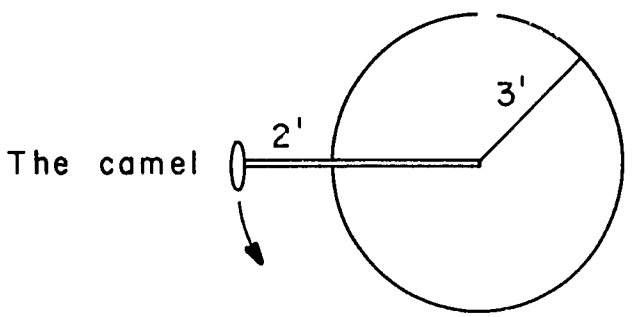
Example 4

Mark two points P and Q 3 inches apart and about halfway down the page. Let us try to answer the question, "What is the locus of all points which are equidistant from P and Q?".

First mark a point A which is 2" from P and also 2" from Q. (Use your compasses to help you do this.) Is A the only point which is 2" from both P and Q? Find a second such point and call it B. Find C and D which are both $2\frac{1}{2}$" from P and Q. Find two more such pairs of points using any radii you like; call the points E, F, G, H. The points A, B, C, . . ., H should form a pattern. This is the locus you are trying to find. Can you draw the locus and describe it in words?

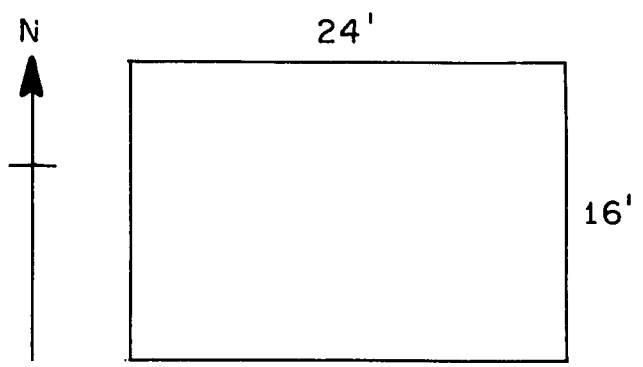
Problems 13-2

1. A camel is attached to a circular oil-mill of radius 3 ft. by a wooden shaft. He turns the mill by walking round it at a distance of 2 feet from the mill. What is the locus in which the camel walks? How far does he walk in completing 1400 turns?



2. A fly walks round a rectangular ceiling 15 ft. by 20 ft. so that it is always 2 ft. from the nearest wall. Draw a rough sketch of the ceiling showing by a dotted line the path (or locus) of the fly.

3. A tree is planted at a distance of 8 ft. from the west wall in a rectangular courtyard 16 ft. by 24 ft. It is also equidistant from the N.W. and the S. W. corners. Copy the diagram and show the position of the tree as the intersection of two loci.



4. A goat is tethered to a post by a rope 10 ft. long. What is the shape of its grazing region? What is its locus if it walks round the post keeping the rope taut? What is its locus if the rope is not taut?

5. A cow is tied by a rope 10 ft. long to a ring which slides on a rail 15 ft. long. Draw a sketch of its grazing region and find its area in square feet. What is the cow's locus if the rope is kept taut?

6. A goat is tethered by a rope 15 ft. long to a bolt in the middle of a wall 40 ft. long. What is the locus of the goat if it walks about, keeping the rope taut?

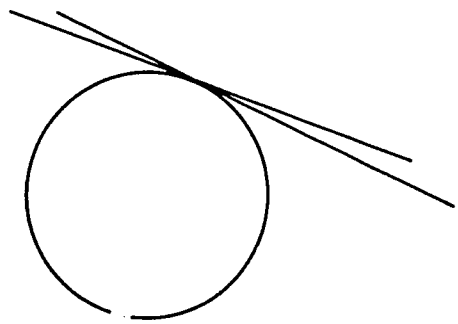
7. What is the locus in Problem 6 if the wall is only 20 ft. long and is the wall of a rectangular house?

13-3 Envelopes

A curve can be suggested by a set of lines as well as by a set of points.

Example 5

Take any circular tin or bottle and get a friend to hold it down firmly on a blank sheet of paper. Hold your ruler flat against the tin and draw a line on the paper following the edge of the ruler. Repeat this after changing the position of the ruler slightly -- keep it flat against the tin but roll it round the tin a small amount. The two lines you have drawn should now look like this.

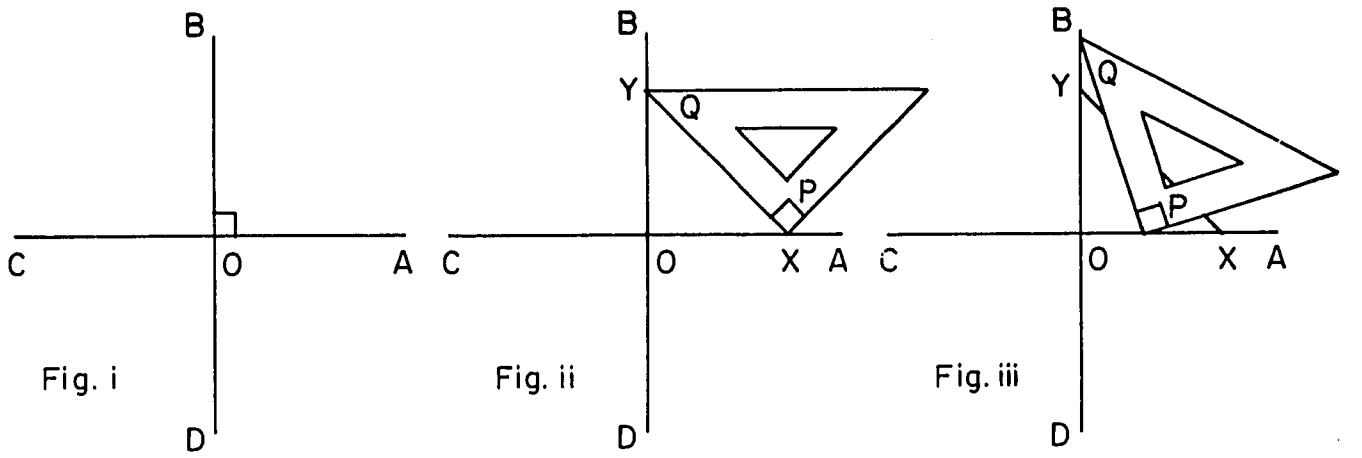


Repeat this 30 or 40 times (the more, the better) until the tin is wrapped in lines. Now take away the tin and look at the pattern of lines. What is the shape of the curve which appears just to touch all the lines?

The curve suggested by the set of lines is a circle. The set of lines is said to envelop the circle. The verb envelop means wrap. The circle is wrapped in lines as a baby is wrapped in a shawl. The noun envelope is used for the curve touched by each line of the set of lines. In Example 5 the envelope was a circle.

Example 6

Draw two perpendicular line segments about 8 inches long near the middle of a sheet of paper, as in Fig. i.



Place your set square so that P, the vertex of the right angle is on $\overrightarrow{OA}$ at X and another vertex, Q, is on $\overrightarrow{OB}$ at Y. Draw $\overline{XY}$ as in Fig. ii.

Repeat this, placing your set square in many different positions but with P always on $\overrightarrow{OA}$ and Q on $\overrightarrow{OB}$. Include some positions where $\overline{PQ}$ is close to $\overline{OA}$ or $\overline{OB}$.

If you draw the line segments carefully and accurately, a curve will appear. The shape will be something like Fig. iv.

So far you have drawn only a quarter of the whole curve. When complete, it looks like Fig. v.

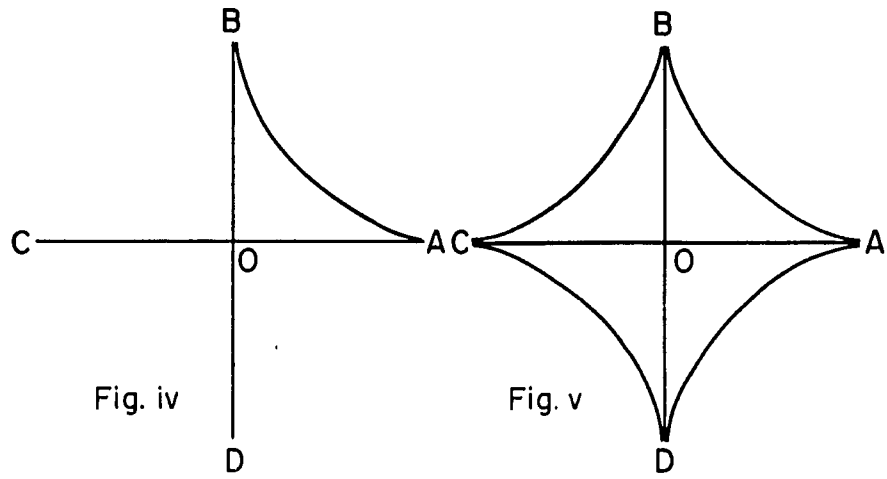


Fig. iv Fig. v

How do you draw the rest of it? You should have no difficulty in using your set square to do this. What are the axes of symmetry for this figure?

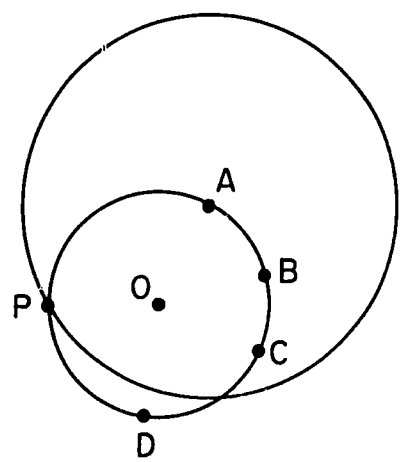
The curve is called the Astroid because it looks like a star. Each line segment of the set you have drawn touches the astroid. The astroid is the envelope of the set of line segments. Each line segment is part of a line and the common property of the set of lines is that the two axes $\overleftrightarrow{OA}$, $\overleftrightarrow{OB}$ intercept (or cut off) a fixed length from each line. The fixed length was provided by the edge of your set square.

Example 7

A curve can be enveloped by a set of other curves.

Draw a base circle with centre O and radius $1\frac{1}{2}$ inches. (O should be placed half-way down the page and distant about $2\frac{1}{2}$ inches from the left-hand edge.)

Let P be the point of the circle furthest to the left. Let A be any other point on the circle. Draw a complete circle, with centre at A and passing through P (with radius $\overline{AP}$). Repeat, taking centres at many other points B, C, ... on the base circle with radii $\overline{BP}$, $\overline{CP}$, ... (It is a good idea to draw the base circle in ink or with coloured pencil.)



As you draw the circles, you will see a very interesting curve called the cardioid appearing at the outer edge. This example of an envelope is a little different because the circles all lie inside the cardioid. They do not appear to wrap around it but mathematically we still say that the circles envelop the cardioid. We also say that the cardioid is the envelope of the circles. In this case you may like to connect the word with the envelope of a letter. The cardioid contains and touches the circles as an envelope contains and touches the pages inside.

Example 8

Look at these pairs of numbers:

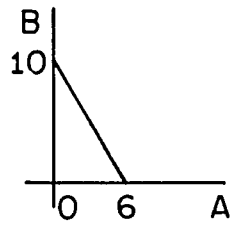
6, 10; 20, 3; 15, 4; 5, 12.

The pairs have a common property. What is it? Other pairs in this set are 1·5, 40; 2, 30; 0·1, 600; 1000, 0·06.

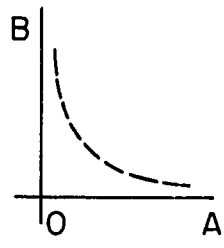
The pairs can be used to draw a set of lines. Take a pair of axes $\overleftrightarrow{OA}$, $\overleftrightarrow{OB}$ at right angles on your graph paper. Mark off 0, 1, 2,..., 12 on each axis using $\frac{1}{4}$ inch as the unit.

To use the first pair 6, 10, join 6 on $\overleftrightarrow{OA}$ to 10 on $\overleftrightarrow{OB}$

Repeat this with any pairs you like whose product is 60.



A curve will appear looking something like this:



If one member of a pair is -6, what is the other? Is there a line segment corresponding to this pair? If so, draw it.

Suggest other pairs of this kind and draw the line segments which match these pairs.

A second curve will now appear. Mathematically the two curves are said to be two branches of the same curve. It is called an hyperbola (pronounced hy<u>per</u>bola).

13-4 Further Examples of Loci and Envelopes

Example 9

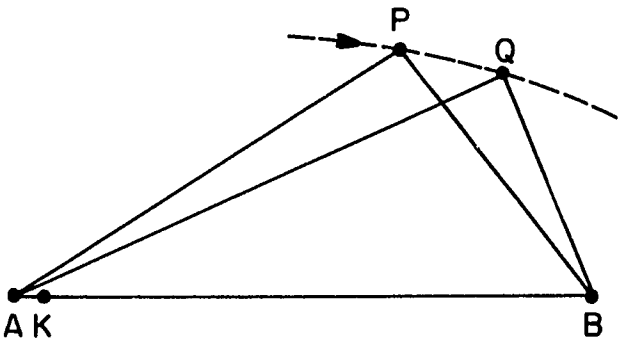
Repeat Example 6 but now, each time you draw a line segment, mark its mid-point with a cross. Do the crosses suggest a pattern? Can you describe the pattern in words? Take any one of the crosses as a centre and draw a circle through O. Repeat with two other crosses. Do you notice anything? Does this suggest a reason for the special pattern into which the crosses fall?

Example 10

There is an easy and interesting way of drawing an oval curve called an ellipse. You need string, drawing-pins or nails, and a flat piece of wood or strong cardboard.

(a) Fix a blank sheet of paper to the wood by means of 2 drawing-pins A and B about 4 inches apart. Make a loop of string about 9 inches long (more than twice AB) and enclose the two drawing-pins. If there is a knot in the string, place it at K near one of the pins.

(b) Put the point of your pencil P into the loop of string and draw a curve, keeping the string taut.



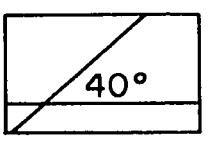
(c) Take away the pins and mark their positions A and B.

(d) A is a <u>focus</u> of the ellipse and so is B. A and B are <u>foci</u>. The curve shows the locus of P, the pencil point, as it moved along the loop of string. If Q is any point of the curve, then AQ + QB is constant. If the whole loop of string is 9 inches long and 4 inches of it is stretched between A and B, then 5 inches of string are left for AQ + QB. This is a special property of an ellipse. It is the locus of points such that the sum of their distances from two fixed points is constant.

(e) Repeat the exercise with AB = 3", then with AB = 2". Notice how the curve changes its shape each time you alter the distance between A and B.

<u>Example 11</u>

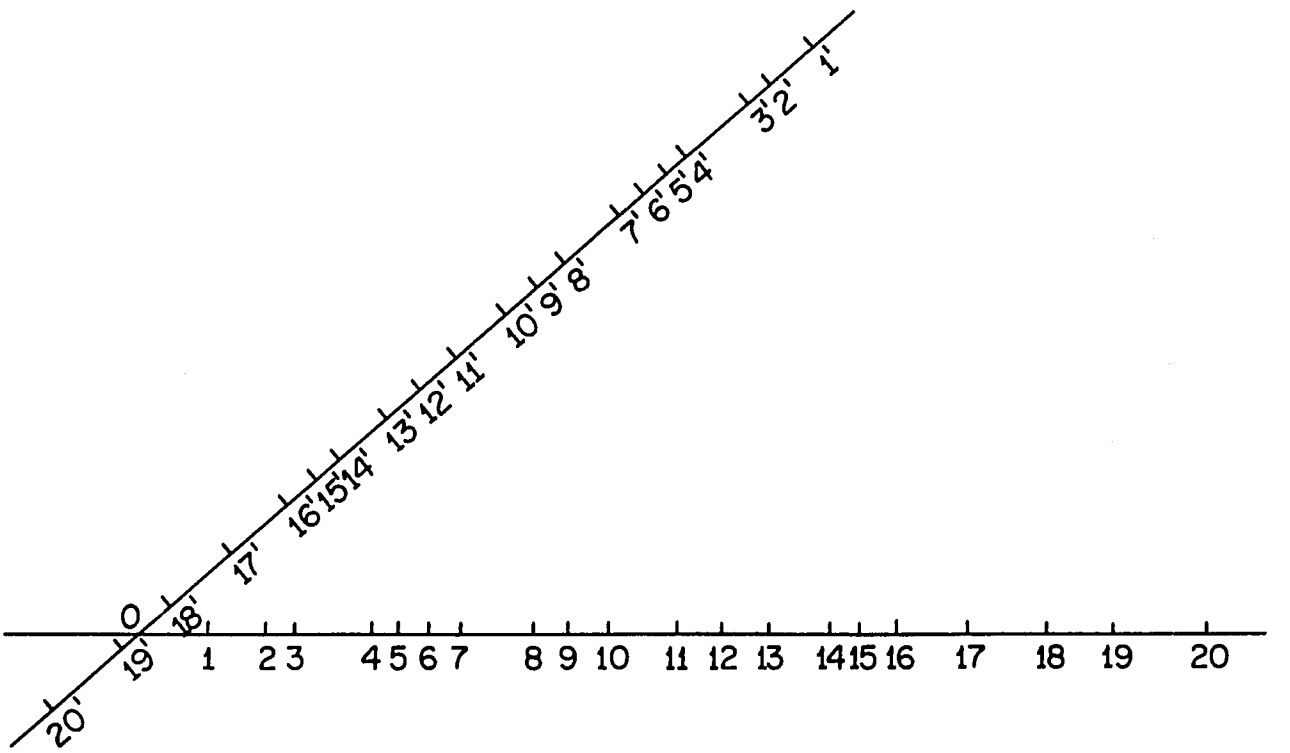
Turn your exercise book sideways. Draw two long line segments at an angle of 40° as shown.



(a) Fold a piece of paper to make a straight edge. Mark 20 points on the edge at irregular intervals. Name them 1 - 20.

(b) Lay the edge against the bottom line and mark the 20 corresponding points naming the points 1 - 20.

(c) Turn the book upside-down and mark 20 corresponding points on the other line. Turn the book back again and name the points 1' - 20'. (Remember to start from the right hand end opposite to the order on the first line.)



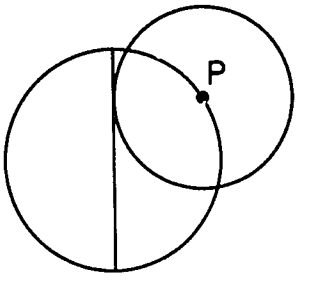
(d) Draw a line through each pair of points 1, 1'; 2, 2'; ...

(e) Observe the pattern of lines. They envelop a curve called a <u>parabola</u>. The path of a ball in the air is approximately a parabola.

The range 1 - 20 is directed away from O, 1' - 20' towards O. This is not necessary. Both ranges could begin near O but in this case the parabola would be so large that you would not have enough of it on the page to be able to see its general shape so well.

Example 12

Draw a base circle with radius $1\frac{1}{2}$ inch, in the centre of a page. Draw a diameter of the circle. Take any point P on the circle as centre and draw a circle which touches the diameter. Repeat this for many points on the base circle.



The circles envelop a curve called the nephroid. Draw the outline of this curve in coloured pencil or ink, when you have finished your drawing.

Summary

In this chapter on loci and envelopes you have been asked to draw a number of well known curves. They have many interesting properties which you may hear about later on. At present you should notice the following:

(a) A curve may be a set of points having a particular property.

(b) A curve may be determined by a set of lines having a particular property.

(c) A curve may be determined by a set of other curves having a particular property.

There is no need to memorize the names or shapes of the curves chosen for examples.

CHAPTER 14

THE PYTHAGOREAN PROPERTY

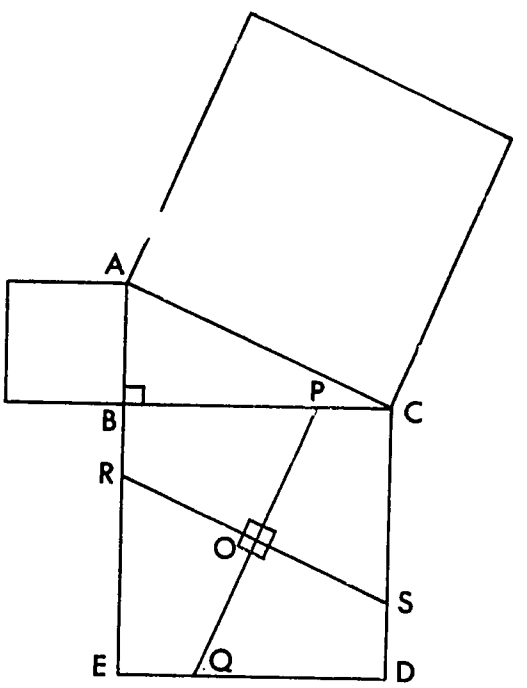
14-1 The Pythagorean Property

In this chapter we wish to investigate a property of the sides of a right-angled triangle.

Class Activity

1. Accurate and careful drawing is needed for the following exercise.

(a) Cut from a piece of stiff paper any triangular region ABC with a right angle at B. Cut also three square regions with sides congruent to the three sides of the triangle respectively. Place the three square regions against their respective sides of the triangular region, as shown in the diagram.



(b) Find O the centre of the square BCDE by lightly drawing its diagonals.

Use your set square to draw:

(1) $\overline{PQ}$ through O, perpendicular to $\overline{AC}$, and

(2) $\overline{RS}$ through O, parallel to $\overline{AC}$.

Mark each of the angles at O with the right angle sign.

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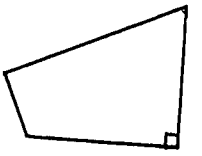
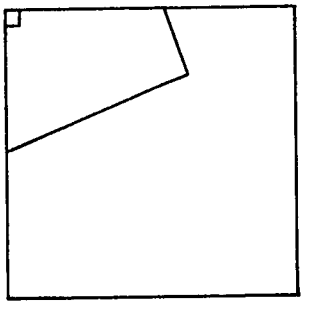
(c) Cut the region BCDE into four pieces by cutting along $\overline{PQ}$ and $\overline{RS}$. Each piece should now look like this.



(d) Place one of the four pieces in the largest of the three original squares with the marked right angle in a corner of the square.



Place the three other pieces with the marked right angles in the other three corners of the largest square so that these pieces all fit together.

(e) What is the shape of the region of the largest square which remains uncovered? Does the remaining small square region fit into this space?

(f) What conclusion do you draw about the areas of the squares?

2. (a) Repeat Exercise 1 using a right-angled triangle having the sides adjacent to the right angle of length 6 cm. and 8 cm. Cut the square region of side 8 cm. into four pieces as before.

(b) Measure the hypotenuse of the triangle and calculate the areas of the following in square centimetres:

(1) Largest square

(2) Smallest square

(3) The square of side 8 cm.

(4) One of the four congruent pieces into which the square of side 8 cm. was cut. Hint: Divide the piece into two right-angled triangular regions to calculate the area.

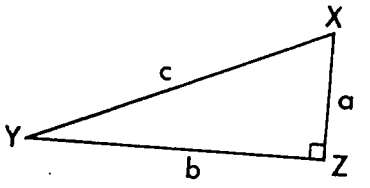
(c) Do these calculations lead you to think that the smallest square region and the four pieces might be expected to fit exactly into the largest square region?

In the preceding exercises you have discovered a relationship involving the three sides of a right angled triangle. A Greek mathematician named Pythagoras discovered the same principle about 2,000 years ago. The property he found still bears his name and is stated as follows:

In any right-angled triangle, the area of the square on the side opposite the right angle is equal to the sum of the areas of the squares on the other two sides.

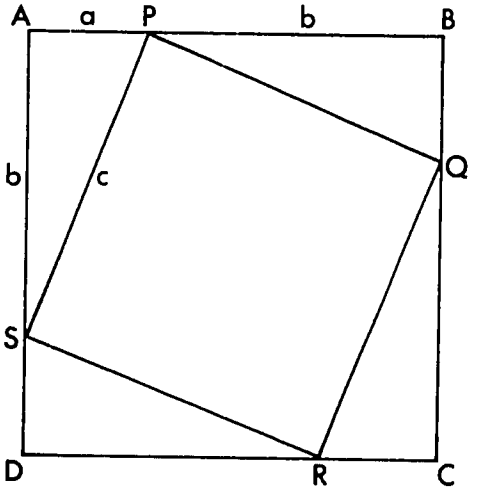
3. Here is another way to demonstrate the Pythagorean property.

(a) Draw any right-angled triangle XYZ with the right angle at Z. Let the measures of $\overline{XZ}$, $\overline{YZ}$, and $\overline{XY}$ be a, b, and c, respectively.



(b) On a line construct $\overline{AP}$ of measure a and $\overline{PB}$ of measure b, so that $\overline{AB}$ is of measure a + b.

Draw carefully the square ABCD on $\overline{AB}$. On the other three sides of this square mark points Q on $\overline{BC}$, R on $\overline{CD}$, S on $\overline{DA}$ with positions corresponding to the position of P, so that BQ = CR = DS = a. Find the centre of ABCD. Draw the segments $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, and $\overline{SP}$.



(c) Trace your diagram on tracing paper. Place the traced drawing over the original. Put a pin through the centre of ABCD so that the tracing is free to turn. Turn the tracing 90°. Does the tracing of PQRS fit perfectly on the original? Turn your tracing 180° and 270° from its original position. Does PQRS still fit? If you have been careful in your drawings you see from symmetry that PQRS is a square.

(d) Observe the following:

(1) The triangle ASP is congruent to the original triangle ZYX (Why?) and hence $SP = c$.

(2) The area of each of the four triangular regions such as ASP is $\frac{1}{2}ab$. Hence the total area of the four triangular regions is $2ab$.

(3) The area of the larger square region is $(a + b)^2$.

(4) The area of the smaller square region is c^2.

(5) The large square region is composed of the smaller square region and the four triangular regions.

(e) It follows that

$$\begin{aligned}(a + b)^2 &= c^2 + 2ab\\ (a + b)(a + b) &= c^2 + 2ab\\ a^2 + 2ab + b^2 &= c^2 + 2ab\\ a^2 + b^2 &= c^2\end{aligned}$$

Thus we see again the Pythagorean property.

We have seen three demonstrations of the Pythagorean property. There are many other ways of showing this property of right-angled triangles, one of which will be given in a later chapter.

The Pythagorean property may be stated either in terms of areas of the squares or in terms of squares of the measures of the sides.

(a) In a right-angled triangle, the area of the square on the hypotenuse is equal to the sum of the areas of the squares on the other two sides.

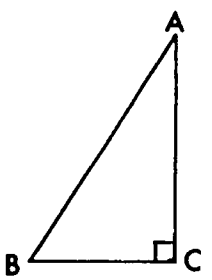
(b) In a right-angled triangle, the square of the measure of the hypotenuse is equal to the sum of the squares of the measures of the other two sides.

Problems 14-1

1. Draw any right-angled triangle ABC with the right angle at C. Measure the sides. Are these equations true for your triangle?

(a) $(AC)^2 + (BC)^2 = (AB)^2$

(b) $AC + BC = AB$

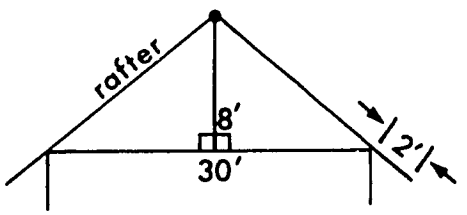


2. Use calculations to find the measure of the third side of the right triangles in the table at the right. In each case the side c is to be taken as the hypotenuse.

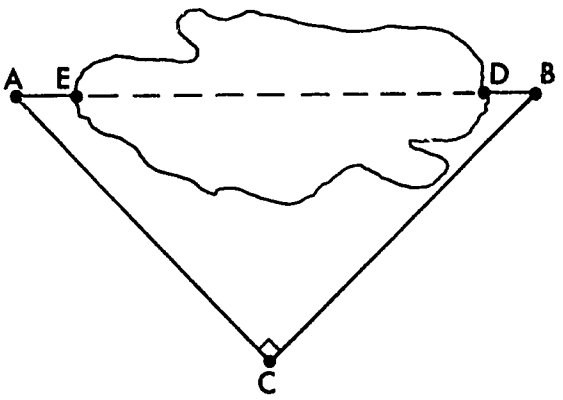
a	b	c
9	12	
2·5	6	
	12	13
1·5		2·5
7	24	
	4	8·5

3. Find the diagonal of a rectangular field 60 yards by 80 yards.

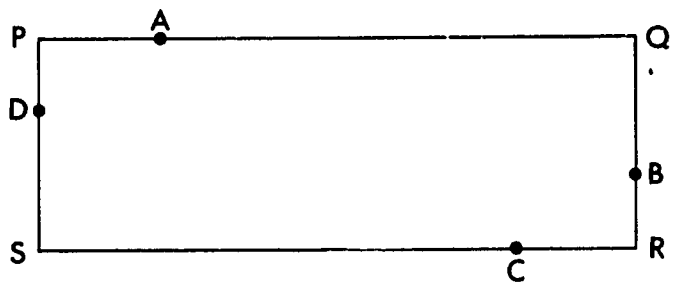
4. The roof of a house is built as shown. How long must each rafter be if it is extended two feet beyond the wall of the house?



5. The figure at the right shows a lake. Segments $\overline{AC}$ and $\overline{CB}$ are laid out with ropes so that there is a right angle at C. The length of $\overline{BC}$ is 100 ft., the length of $\overline{AC}$ is 240 feet, the length of $\overline{BD}$ is 10 feet, and the length of $\overline{AE}$ is 12 feet. Find the length of $\overline{ED}$ across the lake in feet.



6.



Locate A, B, C, D as shown on the sides of rectangle PQRS so that PA = 4, BR = 3, RC = 4, DP = 3, DS = 5, QB = 5, SC = 12, and AQ = 12. Use a centimetre as the unit of measure. Draw $\overline{AD}$, $\overline{DC}$, $\overline{CB}$, and $\overline{BA}$. What figures are formed? Why do you know that $\overline{DA} \equiv \overline{CB}$ and $\overline{DC} \equiv \overline{AB}$ before you measure them? Measure them and verify your prediction.

7. $\overline{AB}$ and $\overline{BC}$ are two chords of a circle at right angles. If AB = 12 and BC = 16, find the diameter of the circle. Give reasons for your answer.

8. On graph paper plot the point (6, 8). How far is this point from the origin?

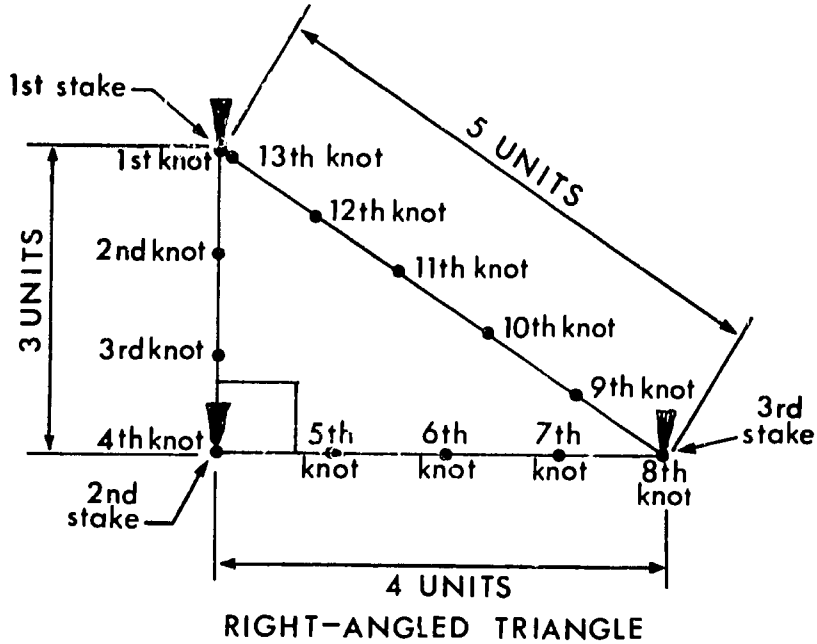
9. On the same paper as used for problem 8 plot two points (4,2) and (10,10). Find the distance between these two points.

14-2 A Related Property

You may have watched workmen marking a football pitch or a tennis court. How do these men make the corners of these fields right angles?

The Egyptians who were fine builders had the same problem as these workmen. Egyptians used a rope with knots tied in it in a similar manner to that used by the workmen. The Egyptian builders who knew this use of a knotted rope were called rope stretchers.

These men knew that, by placing 13 knots at equal intervals along a rope, a right-angled triangle can be formed. Stakes are placed at the first, fourth and eighth knots. The thirteenth knot closes the triangle at the first stake as shown in the diagram.



RIGHT-ANGLED TRIANGLE

The second stake is placed at the desired corner, and the first and third in the directions of the arms of the corner.

You see that the shorter sides of the triangle are three units and four units long, respectively. The longest side of the triangle is five units long.

Why do you suppose this scheme for constructing a right angle worked?

Class Activity

Draw carefully the following triangles.

(a) Sides of length 3 in., 4 in., 5 in.

(b) Sides of length 5 cm., 12 cm., 13 cm.

(c) Sides of length 8 cm., 15 cm., 17 cm.

Copy Table I and Table II. x, y, and z represent the measures of the sides.

Complete Table I showing the squares of the measures of the two shorter sides of the triangles, and the sum of these squares. Complete also Table II showing the squares of the measures of the longest side of the triangles.

Table I

	x^2	y^2	$x^2 + y^2$
(a)	9	16	25
(b)	25		
(c)			

Table II

	z^2
(a)	
(b)	169
(c)	

Compare the numbers in the last column of Table I with the corresponding numbers in Table II. Do the sides of these triangles satisfy the relation which appears in the Pythagorean property? Did you draw these triangles as right-angled triangles? Measure the angles in your triangles with a protractor and see whether they are right angles.

Does this explain why the Egyptians used the knotted rope? What other number of knots at equal intervals could they have used, instead of 13, in order to construct a right angle? At what positions would they have placed the stakes?

Our measurements lead us to suspect that for any triangle with sides of measure a, b, and c, if $a^2 + b^2 = c^2$, then the angle opposite the side

of measure c is a right angle. It can be proved that this is true. Notice that this statement about a triangle is not the same as the Pythagorean theorem. How is it different?

This new property can be stated as follows:

> If the sum of the squares of the measures of two sides of a triangle is equal to the square of the measure of the third side then the triangle is right-angled, and the third side is the hypotenuse.

Let us now compare the two properties we discovered in this chapter.

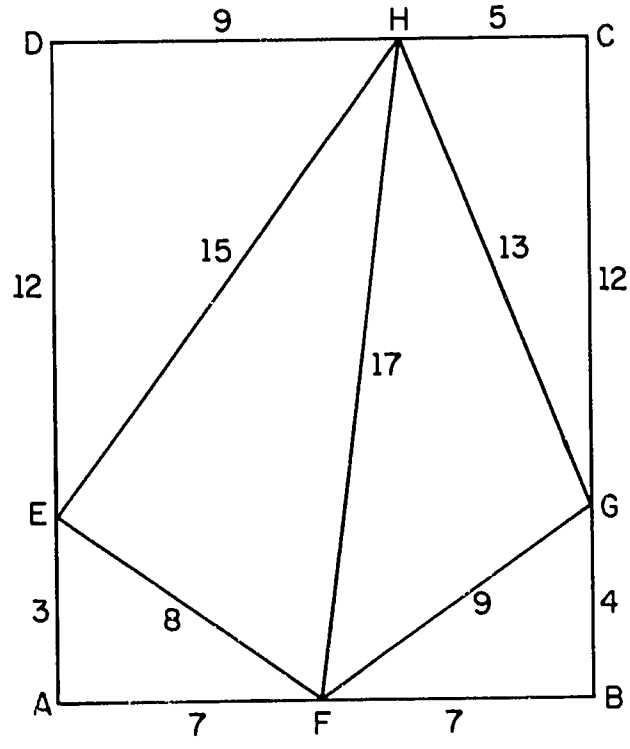
1. If a triangle is a right-angled triangle, then the square of the measure of the hypotenuse is equal to the sum of the squares of the measures of the other two sides.

2. If the square of the measure of one side of a triangle is equal to the sum of the squares of the measures of the other two sides, then the triangle is a right-angled triangle.

The first statement is the Pythagorean property; the second has the two parts of the first statement reversed and is called the <u>converse</u> of the Pythagorean property.

Problems 14-2

1. From the measurements shown on the figure and without measuring angles determine which angles are right angles. Explain your conclusions clearly. (Warning: the figure is imperfectly drawn.)



2. If the following numbers are measures of sides of triangles, which of the triangles are right-angled triangles? Why?

(a) 6, 8, 10

(b) 20, 48, 52

(c) 5, 13, 17

(d) 300, 400, 500

(e) 16, 30, 34

(f) 15, 36, 39

In this chapter we have used the Pythagorean theorem and its converse in triangles where the sum of the squares of the measures of the two sides was a perfect square, so the measure of the hypotenuse was a nice whole number. If the two sides were of measure 2 and 3, for instance, the square of the measure of the hypotenuse would be 13, which is not a perfect square. Such cases call for a further understanding of square roots and skill in approximating such numbers. These topics will be explored in a later chapter.

GLOSSARY

VOLUME 2*

Addition, basic properties of (page 14): Fundamental properties of addition from which other properties may be derived.

Additive inverse (page 12): For any number x, the additive inverse is that number a such that $x + a = 0$. See opposite.

Arcs, major and minor (page 139): The larger and smaller parts of a circle determined by two points on the circle.

Associative property of addition (page 14): $a + (b + c) = (a + b) + c$

Associative property of multiplication (page 22): $a(bc) = (ab)c$

Central angle (page 142): An angle with its vertex at the centre of a circle.

Chord (page 140): A segment with its end points on a circle.

Commutative property of addition (page 14): $a + b = b + a$

Commutative property of multiplication (page 22): $ab = ba$

Congruent angles (page 97): Angles have the same size.

Congruent circles (page 97): Circles which have radii of the same length.

Congruent figures (page 104): Figures which have the same size and shape; figures which can fit exactly according to at least one correspondence.

Congruent line segments (page 97): Line segments which have the same length.

Converse (page 179): A statement in which the hypothesis and conclusion of a previous statement have been interchanged.

Correspondence, one-to-one (page 98): A matching of members of one set with members of another set such that each member of either set matches exactly one member of the other set.

Difference (page 16) The number resulting from the operation of subtraction.

*The glossary for Volume 2 includes the words from the glossary for Volume 1.

Glossary - 2

Distributive property (page 22): $a(b + c) = ab + ac$

Division (page 27): The operation in $a \div b$ or $\frac{a}{b}$. $a \div b = x$ $(b \neq o)$ means that $bx = a$.

Domain of a variable (page 31): The set of numbers from which the value of the variable may be chosen.

Envelope (page 163): The curve determined by a set of lines or curves and touching each line or curve in the set.

Equivalent equations (page 32): Equations which have the same truth set.

Equivalent inequalities (page 38): Inequalities which have the same truth set.

Field (page 80): A number system having the properties of (1) closure for addition and multiplication, (2) commutativity for addition and multiplication, (3) associativity for addition and multiplication, (4) distributivity, (5) additive and multiplicative identities, (6) additive and multiplicative inverses.

Formula (page 47): A representation of a number in terms of one or more variables.

Hexagon (page 122): A six-sided polygon.

Images (page 125): Two sets of points which are symmetrical to each other about a line, point or plane.

Inscribed angle (page 144): An angle with its vertex on a circle and subtended by an arc of the circle.

Inscribed polygon (page 123): A polygon each of whose vertices lies on a circle.

Locus (page 160): The set of all points which have a given property.

Minus (page 17): Indicates subtraction of the two numbers mentioned, the second from the first.

Glossary - 3

Multiplication, basic properties of (page 22): Fundamental properties of multiplication from which other properties may be derived.

Multiplicative inverse (page 28): Two numbers a and b are multiplicative inverses of one another if $ab = 1$. Same as reciprocal.

Negative numbers (page 3): The numbers which correspond to points to the left of zero on the number line.

Octagon (page 122): An eight-sided polygon.

One (page 22): That number which, multiplied by any number x, gives the same number x as the result.

Open sentence (page 6): Sentence containing a variable.

Opposites (page 6): Two numbers are opposites if they correspond, on the number line, to points on opposite sides of 0 and equally far from 0. Also, the opposite of 0 is 0. The opposite of x is denoted by $-x$. Opposites have the basic property: $x + (-x) = 0$.

Ordered pair of numbers (page 56): Two numbers of which one is designated as first and the other second, written (x, y).

Positive numbers (page 2): The numbers which correspond to points to the right of zero on the number line.

Rational number (page 3): A number which can be expressed as the quotient of two integers.

Reciprocal (page 28): Two numbers a and b are reciprocals of one another if $ab = 1$. Same as multiplicative inverse.

Glossary -4

Regular polygon (page 123): A polygon in which all sides are congruent and all angles are congruent.

Smaller of two numbers (page 37): The one that must have a positive number added to it to give the other.

Solution of an equation in two variables (page 55): An ordered pair of numbers for which the equation is true.

Solution set (pages 31 and 58): See truth set.

Solve an equation or an inequality (page 31): Find the truth set.

Subtended by (page 142): An angle is subtended by a chord (or arc) if the ends of the chord (or arc) are on the sides of the angle, but not at the vertex.

Subtraction (page 16): The operation in $a - b$. $a - b = x$ means that $b + x = a$.

Symmetry about an axis (page 129): Two points are symmetrical about an axis if the axis is the perpendicular bisector of the segment joining the points.

Symmetry about a plane (page 135): Two points are symmetrical about a plane if the plane is perpendicular to and bisects the segment joining the points.

Symmetry about a point (page 132): Two points are symmetrical about a point if the point is the midpoint of the segment joining the points.

Truth set of an equation in one variable (page 31): The set of all numbers which make the equation true.

Truth set of a system of two equations in two variables (page 58): The set of all ordered pairs of numbers which make both equations true.

Variable (page 47): A symbol, usually a letter, used to represent any one of a given set of numbers.

Zero (page 13): That number which, added to any number x, gives the same number x as the result.

STUDENT TEXT VOLUME 3

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MATHEMATICS SECONDARY TWO

UNIT 3: REAL NUMBERS
UNIT 4: GRAPHING
UNIT 5: SIMILAR FIGURES AND TRIGONOMETRY

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CHAPTER 15

DECIMAL NUMERALS

15-1 Revision of Base Ten Numeration

Last year we saw that a number like 238 can be written as $2(10^2) + 3(10) + 8$. When we are using the base ten, we say that 238 is written in decimal notation (or decimal form).

When we write it all out, showing the powers of ten, as in the example above, we say that it is written in expanded form.

Problems 15-1

1. Write in expanded form:

   (a) 356 (c) 682,376
   (b) 5,009 (d) 260

2. Write in decimal notation:

   (a) $3(10^2) + 7(10) + 5$ (c) $5(10^4) + 5(10^3) + 3(10^2) + 2(10) + 5$
   (b) $4(10^4) + 3(10^2) + 9$ (d) $8(10^4) + 7$

15-2 Addition and Subtraction with Decimal Numerals

The rules for adding and subtracting integers are easy to understand if we use the expanded form. Notice how the commutative, associative and distributive properties are used.

For example:

$$\begin{aligned} 28 + 37 &= [2(10) + 8] + [3(10) + 7] \\ &= [2(10) + 3(10)] + [8 + 7] \\ &= 5(10) + 15 \\ &= 5(10) + 1(10) + 5 \\ &= 6(10) + 5 = 65 \end{aligned}$$

Show how the steps of the expanded form explain what is happening in our usual short form of writing the addition.

(a)
$$\begin{array}{r} 28 \\ \underline{37} \\ 65 \end{array}$$

(b)
$$\begin{array}{r} 28 \\ \underline{37} \\ 15 \\ \underline{50} \\ 65 \end{array}$$

Similarly the expanded form explains subtraction.

$$38 - 26 = [3(10) + 8] - [2(10) + 6]$$

$$\begin{array}{r} 38 \\ \underline{26} \\ 12 \end{array}$$

Continue to simplify this as in the example above, giving a reason for each step. Can you explain why the normal subtraction method works?

Consider this example where you must regroup. The first two lines are given. Complete the simplification.

$$\begin{aligned} 65 - 47 &= [6(10) + 5] - [4(10) + 7] \\ &= [5(10) + 15] - [4(10) + 7] \end{aligned}$$

Can you pick out the step where the 1(10) was taken from the 6(10) and given to the 5? The associative property of addition permits us to do this. Notice that this step is needed because 7 is larger than 5. We need a number from which we can subtract 7.

Problems 15-2

Use the expanded form to add or subtract. Then check by computing as usual.

1. (a) $23 + 12$ (c) $87 - 35$

   (b) $6304 + 5487$ (d) $1000 - 912$

15-3 Multiplication with Decimal Numerals

Multiplication rules also can be explained using expanded notation.

$$34 \times 7 = [3(10) + 4] \times 7 = [3(10) \times 7] + (4 \times 7) \qquad \text{(Why?)}$$

Continue to simplify this as in the previous examples.

You may be used to writing out your multiplication in one of the ways (a), (b), or (c), given below.

Can you explain each part of (b) and (c)?

(a)	(b)	(c)
34	34	34
7	7	7
238	28	28
	21	210
	238	238

Problems 15-3

1. Use the expanded form to find a decimal numeral for each of the following products and then check by computing in the usual way. Explain what is happening in each part of the usual method.

(a) 37×4 (b) 36×11 (c) 123×49

15-4 Division with Decimal Numerals

To understand the rule for division you must not only know how to multiply and subtract, but you must understand the definition of division.

$a \div b = c$ means that $a = bc$

Example 1. We wish to find a decimal numeral for $736 \div 23$.

You are familiar with the usual long division method shown below.

$$\begin{array}{r} 32 \\ 23\overline{)736} \\ \underline{69} \\ 46 \\ \underline{46} \end{array}$$

We wish to understand why this method works. The object is to find a number q such that

$$736 = 23\,q.$$

We easily see that q must be more than 30 since $23 \times 30 = 690$, and 690 is too small. Also q must be less than 40 since $23 \times 40 = 920$, and 920 is too large.

The desired result is 30 plus a number between 0 and 10. Let us show this by writing the division as shown below.

```
      30 +
23 ) 736
     690
      46
```

23×30 is too small by 46. Since $46 = 23 \times 2$, we see that we must add 2 to 30 to obtain q.

```
      30 + 2
23 ) 736
     690
      46
      46
```

Thus we have found that

$$736 = (23 \times 30) + (23 \times 2)$$

and so $$736 = 23 \times 32$$

or $$736 \div 23 = 32$$

The usual method omits the 0 after 69 and compresses 30 + 2 into 32. Actually we write the 3 and leave the space behind it open until we decide that the next digit is 2.

<u>Example 2</u>. To find $6376 \div 27$ we give below on the right the usual method and on the left a <u>full</u> method of writing which shows more clearly what is going on.

```
      200 + 30 + 6 + 4/27
27 ) 6376
     5400
      976
      810
      166
      162
        4
```

```
      236 4/27
27 ) 6376
     54
      97
      81
      166
      162
        4
```

$$\text{In this case } 6376 = (200 \times 27) + (30 \times 27) + (6 \times 27) + \left(\frac{4}{27} \times 27\right)$$
$$= \left(236\ \frac{4}{27}\right) \times 27$$
$$6376 \div 27 = 236\ \frac{4}{27}$$

Problems 15-4

1. Locate the following quotients between multiples of 10, 100, or 1,000 as directed. For example to locate 547 ÷ 26 between multiples of 10 would be to write "547 ÷ 26 is between 20 and 30".

(a) 736 ÷ 72 between multiples of 10

(b) 4,876 ÷ 13 between multiples of 100

(c) 87,777 ÷ 29 between multiples of 1,000

2. Find the following quotients by the full method. Then find them by the long division method. In one example explain where the numbers of the long division method occur in the full method.

(a) 98 ÷ 7 (b) 612 ÷ 17 (c) 612 ÷ 36 (d) 2,556 ÷ 35

15-5 Rational Numbers in Decimal Notation

Consider 2345, a number written in decimal notation. In expanded form $2345 = 2(10^3) + 3(10^2) + 4(10) + 5(1)$.

We notice that the place value of each digit is 10 times the place value of the digit immediately on the right. Let us now extend the decimal notation to include all rational numbers.

The number twenty-five, three tenths and seven hundredths may be written as

$$2(10) + 5(1) + 3\left(\frac{1}{10}\right) + 7\left(\frac{1}{10^2}\right)$$

We write this in decimal notation as 25·37, using the decimal point "·" to separate the digits representing whole numbers from the digits representing numbers less than one. We say this as "twenty-five point three seven".

Similarly 2345·678 can be written in an expanded form. We say "two thousand three hundred forty-five point six seven eight".

Problems 15-5

1. Write each of the following in decimal notation.

   (a) $4(10^2) + 3\ (10) + 2\ (1) + 5\left(\frac{1}{10}\right) + 6\left(\frac{1}{10^2}\right)$

   (b) $4\left(\frac{1}{10}\right) + 5\left(\frac{1}{10^2}\right) + 6\left(\frac{1}{10^3}\right)$

2. Write each of the following in expanded form.

   (a) 391·345 (b) 600·0102

15-6 Addition and Subtraction of Rational Numbers

When we add two numbers in decimal notation, our work in expanded form would look as follows:

Example 1. 5·64 + 4·62

		tens	ones	tenths	hundredths	
$5\cdot 64 = 5(1) + 6\left(\frac{1}{10}\right) + 4\left(\frac{1}{10^2}\right) =$			5	6	4	= 5·64
$4\cdot 62 = 4(1) + 6\left(\frac{1}{10}\right) + 2\left(\frac{1}{10^2}\right) =$			4	6	2	= 4·62
		1	0	2	6	10·26

Notice how the $12\left(\frac{1}{10}\right)$ becomes $10\left(\frac{1}{10}\right) + 2\left(\frac{1}{10}\right)$ or $1 + 2\left(\frac{1}{10}\right)$ and the 10(1) becomes 1(10).

Subtracting may be performed in a similar way.

Example 2. $9 \cdot 4218 - 6 \cdot 3456$

$$\begin{array}{r} 9 \cdot 4218 \\ -\ \underline{6 \cdot 3456} \\ 3 \cdot 0762 \end{array}$$

How can you justify this result? Use the expanded form to justify the subtraction of hundredths in the example, $4 \cdot 90 - 2 \cdot 43$.

Problems 15-6

1. Add (a) $3 \cdot 12 + 6 \cdot 31$ (b) $6 \cdot 950 + 8 \cdot 432 + 1 \cdot 100$

2. Subtract (a) $7 \cdot 62 - 6 \cdot 51$ (b) $3 \cdot 1200 - 3 \cdot 0104$

15-7 Multiplication and Division by Ten

Multiplication by Ten

You will need to recall that

$$10^2 \times 10 = 100 \times 10 = 10^3$$

and $$\frac{1}{10^2} \times 10 = \frac{1}{100} \times 10 = \frac{1}{10}$$

Consider $8 \cdot 4 \times 10$:

$$\begin{aligned} 8 \cdot 4 \times 10 &= [8(1) + 4\left(\frac{1}{10}\right)] \times 10 \\ &= 80 + 4 \qquad \text{(The distributive property)} \\ &= 84 \end{aligned}$$

Examining this result you will notice that the value of each digit has been multiplied by 10 to give a number 10 times as large and thus the digits in the result have moved one place to the left with reference to the decimal point. Check this observation by finding the product of $1 \cdot 31 \times 10$ with work in the expanded form.

When a number is multiplied by 10^2 or 100 the digits move two places to the left since each digit now has 10^2 times its former value.

Can you make a similar statement for multiplying by 10^3?

Can you make a similar statement for multiplying by 10^n, where n is a positive integer?

Division by Ten

Consider $13 \cdot 15 \div 10$

Find this quotient using the expanded form and the distributive property.

Can you make a statement regarding moving a digit when you are dividing by 10?

Find the quotient $1 \cdot 13 \div 10$ to check your statement.

The result of dividing by 10 is that of moving the digits one place to the right, since each digit now has one tenth its former place value.

Can you make similar statements for dividing by 10^2, 10^3, 10^n where n is a positive integer?

Problems 15-7

1. Multiply each of the following numbers by 10, 100, 1000 and explain one of the answers using expanded notation.

   (a) $0 \cdot 11$ (b) $0 \cdot 012$ (c) $8 \cdot 001$

2. Divide each of the following numbers by 10, 100, 1000 and explain one of your answers using expanded notation.

   (a) 71 (b) $100 \cdot 51$ (c) $0 \cdot 017$

3. If $57 \cdot 6 \times 0 \cdot 675 = 38 \cdot 88$, then $0 \cdot 576 \times 0 \cdot 0675$ is equal to

   (a) $388 \cdot 8$ (b) $38 \cdot 88$ (c) $3 \cdot 888$ (d) $0 \cdot 3888$ (e) $0 \cdot 03888$

4. If $0 \cdot 0107x = 10 \cdot 7$ then x is

   (a) 1 (b) 10 (c) 100 (d) 1000 (e) 10,000

15-8 Decimal Places, Multiplication and Division

Multiplication

The positions of the digits after the decimal point are called decimal places. Thus there are three decimal places in the numeral $41 \cdot 378$ and two decimal places in $0 \cdot 13$.

Consider $1 \cdot 31 \times 2 \cdot 2$.

$$\begin{aligned} 1 \cdot 31 \times 2 \cdot 2 &= \left(131 \times \frac{1}{100}\right) \times \left(22 \times \frac{1}{10}\right) \\ &= (131 \times 22) \times \left(\frac{1}{100} \times \frac{1}{10}\right) \qquad \text{(Why?)} \\ &= 2882 \times \frac{1}{1000} \\ &= 2 \cdot 882 \end{aligned}$$

Notice that the number of decimal places in the answer is the sum of the number of decimal places in the factors. When we multiply numbers in decimal notation, we find the product as with whole numbers and then fix the decimal point according to the sum of the number of decimal places in the factors.

Expand the product $0 \cdot 013 \times 0 \cdot 041$ as in the worked example above.

Division

We first consider division by whole numbers.

Consider $0 \cdot 0555 \div 15$

$$\begin{array}{r} 0 \cdot 0037 \\ 15\overline{)0 \cdot 0555} \\ \underline{0 \cdot 045} \\ 0 \cdot 0105 \\ \underline{0 \cdot 0105} \end{array}$$

It is easy to divide by a whole number. We therefore try to obtain whole number divisors for division.

Example

$$\begin{aligned} 5 \cdot 12 \div 6 \cdot 4 &= \frac{5 \cdot 12}{6 \cdot 4} \\ &= \frac{5 \cdot 12}{6 \cdot 4} \times \frac{10}{10} \qquad \text{(Why?)} \\ &= \frac{51 \cdot 2}{64} \end{aligned}$$

Do you see why we chose to multiply by $\frac{10}{10}$ and not $\frac{100}{100}$ or $\frac{1000}{1000}$ or any other form of the number one? Our main purpose was to make the divisor a whole number.

We can therefore write:

$$\begin{array}{r} 0{\cdot}8 \\ 64\overline{)\,51{\cdot}2} \\ \underline{51{\cdot}2} \end{array}$$

Notice the process used to make the divisor a whole number. Use a similar process to find the quotient $4{\cdot}608 \div 0{\cdot}128$.

In the worked examples in this section the decimal point has been shown in the work. Many people who have to find such quotients frequently have the decimal point showing only in the dividend and the quotient.

Problems 15-8

1. Find these products:

   (a) $1{\cdot}3124 \times 0.02$ (b) $813 \times 0{\cdot}0001$

2. Find these quotients:

   (a) $1{\cdot}3124 \div 0{\cdot}02$ (b) $1{\cdot}69 \div 1{\cdot}3$

3. Find $\dfrac{0{\cdot}3124 \times 0{\cdot}025}{0{\cdot}8 \times 0{\cdot}05}$ as a decimal numeral.

15-9 Rough Approximations in Multiplication and Division

Sometimes it is useful to make a rough approximation of a product or a quotient without multiplying or dividing completely. Often this approximation is a useful check on the accuracy of the completed product or quotient.

Consider the following product:

$$97 \times 5{\cdot}652$$

97 is roughly 100 and $5{\cdot}652$ is roughly 5. A rough approximation of the above product therefore, would be 500. The exact product is $548{\cdot}244$.

Consider again the following division:

$$645 \cdot 575 \div 36 \cdot 89$$

A rough approximation of 645·575 is 600. A rough approximation of 36·89 is 40. Thus (600 ÷ 40) or 15 would be a rough approximation of the above quotient. The exact quotient is 17·5.

Notice that, in these two examples, our rough approximations are not too far from the exact result. In the first case, if we multiplied out 97 × 5·652 and got 54·8244 for the product we would compare this with our rough approximation and suspect that we had put the decimal point at the wrong place. Or suppose our result for the product was 748·244. Again, we would mentally compare this result with our rough approximation and suspect that something was wrong with our multiplication.

Notice also that, in both examples, we have approximated one of the numbers by a larger number and the other by a smaller number. In multiplication this is usually advisable. In division however, it is frequently better to approximate both numbers the same way - either both larger or both smaller. Can you think why this should be so? Consider 635 ÷ 125. Which is the better method of approximating in this case?

Notice also that we did not approximate 97 by 95 or 5·652 by 5·6. We approximate the numbers so that the product or quotient can be done quickly and mentally.

Sometimes we are not interested in the results of a product or a quotient to the last digit and a rough approximation is sufficient for our purpose.

Let us consider the area of the island of Zanzibar. From a map, using the appropriate scale, we might say that the area of the island is approximately the area of a rectangle 55 miles by 15 miles. Using our approximation methods, we find the approximate area of the island in square miles is 60 × 10 or 600.

Here are some rough approximations:

(a) 0·097 × 3·41

0·097 is roughly 0·1 and 3·41 is roughly 3 and so the product should be roughly 0·3.

The exact product is 0·33077.

(b) 0·27 × 4·816

0·27 is roughly 0·25 or $\frac{1}{4}$; 4·816 is roughly 4·8 and $\frac{1}{4}$ × 4·8 is 1·2.

The exact product is 1·30032.

Notice that if we had approximated 4·816 by 5 a useful approximation would still be obtained.

(c) 6·4 ÷ 0·15

6·4 is roughly 6. $\frac{6}{0{\cdot}15} = \frac{600}{15} = 40$

Thus, the quotient is roughly 40.

Problems 15-9

(Do not use paper and pencil)

1. Without calculating say whether the following quotients and products are approximately right or not:

(a) 4·8 × 3·016 = 144·768

(b) 4·8 × 3·016 = 24·4768

(c) 4·8 × 3·016 = 14·4768

(d) 456·3225 × 0·51 = 89·475

(e) 4·433 × 0·97 = 4·3

(f) 4·433 × 0·97 = 43

2. It costs five dollars per day to live in a Y.M.C.A. in New York. 20 shillings is equal to about 2·80 dollars. Estimate roughly how many shillings it would cost for one week in the hotel.

3. Estimate from a map the distance from Accra to Lagos. How many hours would it take to travel this distance at an average speed of 45 miles per hour?

4. Look at a map of East Africa and estimate the area of Lake Victoria.

5. Look at a map of Africa and estimate the area of Ghana. Compare this area with the area of Kenya.

6. The truth set of the equation $\frac{x}{\cdot 004} = 2\cdot 84$ is (a) 12.1 (b) 0·00121 (c) 0·01136 (d) 113·6 (e) 2·836. Find, without multiplying, which answer is likely to be correct.

15-10 Rounding Off

The East African Statistical Bulletin gives the population of Uganda, at mid-year 1961, as 6,845,000 correct to the nearest thousand persons. What does this information tell us? It tells us, for instance, that the population figure was more nearly 6,845,000 persons, than 6,844,000 persons or 6,846,000 persons. The figure may have been 6,844,989 or 6,845,252, but it has been rounded off to 6,845,000, and this gives us an approximation of the population.

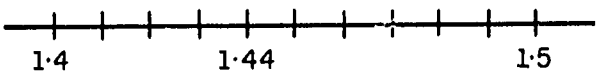
Figures are rounded off to the nearest million or thousand or tenth or thousandth etc., according to the degree of accuracy required. If a comparison is being made between the population of Uganda and that of another country whose population is given as 2,134,000, correct to the nearest thousand persons, it may be sufficient to say Uganda has 7 million people, correct to the nearest million persons. If, however, the population of Uganda is compared with that of a country whose population is given as 6,841,000, correct to the nearest thousand persons, it may not be sufficient to round off in millions, since rounding off in millions will not show the difference. On the other hand, for some purposes that difference may not be significant, and it may be sufficient to say that the two countries are the same size to the nearest million, or even to the nearest ten thousand.

If you are asked to find the length of a given line segment, you will probably use your ruler. This may give you its length to the nearest tenth of an inch. By using more refined methods, it may be possible to find the length to the nearest hundredth

of an inch. These results are approximations of the length of the line segment. It is impossible to measure the length of a line segment exactly. What is important is that the measurement should be reliable as far as it goes.

Rounding Off Decimal Numerals

Suppose we wish to round off 1·44 to the nearest tenth.



In the figure we have divided the interval of 0·1 from 1·4 to 1·5 into ten equal parts. Each part is a tenth of 0·1, or 0·01.

Hence the division marks are at 1·41, 1·42, 1·43, 1·44, 1·45, 1·46, 1·47, 1·48, 1·49.

It can easily be seen that 1·44 is nearer 1·4 than 1·5. Hence we say that 1·44 is 1·4 to the nearest tenth, or correct to one decimal place.

We can make a table of approximations as follows:

1·41	is	nearer	1·4	than	1·5.	Hence	1·41	is	1·4	correct	to	1	decimal	place.
1·42	"	"	1·4	"	1·5.	"	1·42	"	1·4	"	"	"	"	"
1·43	"	"	1·4	"	1·5.	"	1·43	"	1·4	"	"	"	"	"
1·44	"	"	1·4	"	1·5.	"	1·44	"	1·4	"	"	"	"	"
1·46	"	"	1·5	"	1·4.	"	1·46	"	1·5	"	"	"	"	"
1·47	"	"	1·5	"	1·4.	"	1·47	"	1·5	"	"	"	"	"
1·48	"	"	1·5	"	1·4	"	1·48	"	1·5	"	"	"	"	"
1·49	"	"	1·5	"	1·4.	"	1·49	"	1·5	"	"	"	"	"

1·45 is exactly halfway between 1·4 and 1·5. By convention we associate it with the higher of the two numbers. Hence 1·45 is 1·5 correct to 1 decimal place.

Example 1.

Write the following, correct to (a) 2 decimal places, (b) 3 decimal places.

(i) 5·8761 (ii) 68·2459 (iii) 3·1415 (iv) 146·9992.

For a result correct to 2 decimal places, we consider the digit in the third place and round off.

(i) 5·88 (ii) 68·25 (iii) 3·14 (iv) 147·00

For a result correct to 3 decimal places, we consider the digit in the fourth place and round off.

(i) 5·876 (ii) 68·246 (iii) 3·142 (iv) 146·999.

Example 2.

Write $\frac{1}{13}$ as a decimal numeral, correct to 3 decimal places. To do this we need to find $\frac{1}{13}$ as a decimal numeral to 4 places.

$$\begin{array}{r} 0\cdot0769 \\ 13\overline{)\,1\cdot0000} \end{array}$$

Here, considering the fourth figure, we have $\frac{1}{13}$ = 0·077 correct to 3 decimal places.

Problems 15-10

1. Write the following correct to the nearest thousand.

   (a) 6,123,456 (b) 5,432,109 (c) 1,235,678

2. Write the following correct to the nearest whole number.

   (a) 625·1 (b) 96·5 (c) 7·9 (d) 8·45131782

3. Write the following correct to 1 decimal place.

   (a) 25·234 (b) 6·519 (c) 619·991 (d) 8·512

4. Write the following correct to:

   (a) 2 decimal places (b) 3 decimal places

   (i) 1·23456 (ii) 9652·85491

   (iii) 2615·2112 (iv) 8·651397

5. Write the following numbers in decimal notation correct to:

(a) 2 decimal places (b) 3 decimal places

(i) $\frac{1}{3}$ (ii) $\frac{1}{6}$ (iii) $\frac{1}{7}$ (iv) $\frac{3}{11}$ (v) $\frac{5}{9}$

(vi) $\frac{13}{65}$ (vii) $\frac{17}{9}$ (viii) $\frac{81}{7}$ (ix) $\frac{98}{13}$ (x) $\frac{101}{19}$

15-11 Percentages

When you take an examination, the teacher often reports your grades in per cent. He may, for instance, say that Ngozi got 75 per cent. Do you know what this means? The expression "per cent" comes from a Latin phrase "per centum", meaning "for each hundred".

Suppose that in a particular examination in mathematics the teacher gave 10 problems and one pupil had 6 of the problems correct. If all the problems have equal marks, can you write this score in terms of per cent? You will notice that the ratio of the number of correct problems to the total number of problems is $\frac{6}{10}$. From our knowledge of equivalent fractions we know that this ratio can be written as $\frac{12}{20}$, $\frac{18}{30}$, $\frac{30}{50}$, $\frac{60}{100}$, etc. The fraction which indicates the ratio in terms of one hundred is $\frac{60}{100}$. Therefore, we say that the student had 60 percent of the problems right. He had 4 problems wrong. The ratio of the number 4 to the total number of problems given is $\frac{4}{10}$. This again can be written as $\frac{8}{20}$, $\frac{12}{30}$, $\frac{20}{50}$, $\frac{40}{100}$, etc. Similarly we can say that the pupil had 40 per cent of the problems wrong. Thus if a fraction is expressed with 100 as its denominator, it is said to be expressed as a <u>percentage</u>. The numerator of the fraction is called the <u>rate</u>.

For example, $\frac{35}{100}$ is a percentage, denoting 35 out of a hundred. This may be written as 35 per cent or 35%. The symbol % is a short way of writing per cent. It is convenient to have a symbol for $\frac{1}{100}$ and we use % to represent it.

$$\% = \frac{1}{100}$$

Thus we have the following relations:

$$35 \text{ per cent} = \frac{35}{100} = 35 \times \frac{1}{100} = 35\%$$

$$50 \text{ per cent} = \frac{50}{100} = 50 \times \frac{1}{100} = 50\%$$

$$6 \text{ per cent} = \frac{6}{100} = 6 \times \frac{1}{100} = 6\%$$

Problems 15-11a

Express the following in three equivalent forms:

1. 90%
2. 30 per cent
3. $\frac{5}{100}$
4. $7 \times \frac{1}{100}$
5. 55%
6. 100%
7. 45 per cent
8. $\frac{67}{100}$
9. $\frac{15}{100}$
10. $37 \times \frac{1}{100}$

A class of 30 pupils is made up of 18 boys and 12 girls. The ratio of the number of boys to the number of pupils in the class is $\frac{18}{30}$. This ratio can be expressed in various ways as $\frac{18}{30}, \frac{6}{10}, \frac{12}{20}, \frac{30}{50}, \frac{60}{100}, \frac{72}{120}$. Here, boys constitute 60 per cent of the class. Why? We can now state the following:

Any number $\frac{x}{y}$ can be expressed as a percentage by finding a number a such that $\frac{x}{y} = \frac{a}{100} = a \times \frac{1}{100} = a\%$ Here a is called the <u>rate</u>.

Example

Express $\frac{7}{8}$ as a percentage.

Let a denote the rate. Then we have,

$$\frac{7}{8} = \frac{a}{100}$$

$$a = \frac{7}{8} \times 100 = 87\frac{1}{2}$$

$$\frac{7}{8} = 87\frac{1}{2}\%$$

Problems 15-11b

1. Write each of the following numbers as a percentage.

(a) $\frac{2}{3}$	(f) 4	(k) $\frac{3}{5}$
(b) $\frac{1}{2}$	(g) $\frac{1}{3}$	(l) $\frac{4}{5}$
(c) $\frac{3}{4}$	(h) $\frac{5}{3}$	(m) $\frac{5}{5}$
(d) $\frac{4}{3}$	(i) $\frac{1}{5}$	(n) $\frac{8}{15}$
(e) $\frac{7}{5}$	(j) $\frac{2}{5}$	(o) $\frac{6}{5}$

2. Express the following as fractions in simplest form:

(a) $3\frac{1}{2}\%$	(c) 65 per cent
(b) $33\frac{1}{3}\%$	(d) 5.5%

3. Find the value of

(a) $5\frac{1}{2}\%$ of £75	(d) 10% of £230 10s
(b) $3\frac{1}{2}\%$ of £120	(e) 25% of $750.28
(c) 7% of £720	(f) 4% of $312.00

4. Express 33 as a per cent of 300.

5. What per cent of $27\frac{1}{2}$ tons is 4 tons 8 cwt.?

6. A room is 10 ft. 6 in. wide and 15 ft. 6 in. long. What per cent of the length is the width?

7. In an examination 70 problems were given and all problems had equal marks. A boy had 55 right. What percentage is this? In order to pass the examination it is necessary to answer at least 40% correctly. What is the lowest number of correct answers needed to pass?

8. In an election there were two candidates. One of them received 65% of the votes cast and secured a majority of 1500 votes, that is, 1500 votes more than his competitor. How many people voted?

9. In the year 1952 the population of a town was 72,000; in 1961 the population had risen to 85,200. Find the per cent increase of the population.

10. The enrollment in a certain secondary school was 450 in 1950. By 1960 the enrollment had increased by 16%. What was the enrollment in 1960?

11. A primary school had an enrollment of 850 pupils in January 1951. In January 1960 the enrollment was 1120. What was the percentage increase in enrollment?

12. During 1950 a family spent £250 on food. In 1960 the same family spent spent £375 on food. What was the percentage increase in the money spent on food?

15-12 Percentages and Decimal Numerals

In the preceding section you learned that $\frac{65}{100}$ can be written as 65% or as $65 \times \frac{1}{100}$. From your study of decimal numerals you know that $\frac{65}{100}$ can also be written as 0·65. Hence we can write $\frac{65}{100} = 65\% = 0{\cdot}65$. Similarly, $\frac{1}{2} = \frac{50}{100} = 50\% = 0{\cdot}50 = 0{\cdot}5$.

Suppose that we wish to express $\frac{1}{8}$ as a percentage. We know that $\frac{1}{8}$ as a decimal numeral is 0·125 and this can be written as $\frac{12{\cdot}5}{100}$. Hence, $\frac{1}{8}$ as a percentage becomes 12·5%. Similarly we have,

$$0{\cdot}01 = \frac{1}{100} = 1\%$$

$$0{\cdot}375 = \frac{37{\cdot}5}{100} = 37{\cdot}5\%$$

$$1{\cdot}75 = \frac{175}{100} = 175\%$$

Problems 15-12

1. Express the following as percentages:

(a) 0·325
(b) 0·05
(c) 2·3
(d) $\frac{1}{125}$
(e) $\frac{6}{7}$

2. Express the following as decimal numerals:

(a) 33%
(b) $\frac{1}{2}\%$
(c) 1·5%
(d) $12\frac{1}{2}\%$
(e) 250%
(f) 75%

15-13 Application of Percentages

Percentages have a very wide application in everyday transactions. One such application is in the comparison of business transactions in terms of per cent profit or loss.

Profit and Loss

When goods are bought for a sum of money and sold for another, there is a gain or loss on the transaction according as the selling price is more or less than the cost price. In business transactions the actual gains or losses do not give enough comparative information about the transactions. For instance, consider the following:

1. A set of books which cost £25 was sold at £30; and
2. Another set of books which cost £75 was sold at £80.

In each case the actual profit is £5, but one of the transactions is better business than the other. To find out which, we have to express the profit as a percentage of the cost.

In the first case this percentage is 20%.

$$\frac{5}{25} = \frac{20}{100} = 20 \times \frac{1}{100}$$

In the second case the percentage is $6\frac{2}{3}$ %.

$$\frac{5}{75} = \frac{5}{75} \times \frac{100}{100} = \frac{500}{75} \times \frac{1}{100} = 6\frac{2}{3} \times \frac{1}{100}$$

Hence the first transaction is better business. Notice that the profit has been calculated with reference to the cost price. This is the general practice. It is also a general and convenient practice to abbreviate the terms <u>cost price</u> and <u>selling price</u> by the letters <u>c.p.</u> and <u>s.p.</u>, respectively.

<u>Example 1</u>.

A trader bought 20 bicycles for £220 and sold them at £13 10s. each. What was his gain or loss per cent?

$$\text{s.p. of 20 bicycles} = 13\tfrac{1}{2} \times 20 = 270$$

$$\text{gain} = 270 - 220 = 50$$

$$\text{gain per cent} = \frac{50}{220} \times \frac{100}{100} = \frac{5{,}000}{220} \times \frac{1}{100} = 22\frac{8}{11} \times \frac{1}{100}$$

$$= 22\frac{8}{11}\%$$

<u>Example 2</u>.

An article which cost £150 was sold at a loss of 5%. What was the selling price?

$$\begin{aligned} \text{s.p.} &= \text{c.p.} - \text{loss} \\ &= \text{c.p.} - 5\% \text{ of c.p.} \\ &= 150 - \left(\frac{5}{100} \times 150\right) \\ &= 150 - 7\tfrac{1}{2} \\ &= 142\tfrac{1}{2} \end{aligned}$$

The selling price is £142 10s.

Example 3.

By selling goods for £168, a profit of 12% was made by a merchant. How much did the goods cost him?

If the cost of the goods was x pounds, then the profit was $168 - x$ pounds. The profit can be expressed also as $\frac{12}{100}x$ or $\frac{3x}{25}$.

$$168 - x = \frac{3x}{25}$$

$$25(168-x) = 25 \times \frac{3x}{25}$$

$$25(168-x) = 3x$$

$$25 \times 168 - 25x = 3x$$

$$25 \times 168 = 28x$$

$$\frac{25 \times 168}{28} = x$$

$$x = 25 \times 6$$

$$x = 150$$

The goods cost £150.

Problems 15-13(a)

1. A man brought a car for £875 and sold it for £750. Find his loss per cent.

2. A trader bought eggs at 4d. each and sold them at $6\frac{1}{2}$ d. each. Find his per cent profit.

3. A shopkeeper bought eggs at 5s. 6d. a dozen. He found that 10% of the eggs were rotten and sold the rest at 7s. 6d. a dozen. Find his per cent profit or loss.

4. An increase in wages of 15% makes the monthly wage bill of a company £5,980. What is the amount of the increase?

5. A woman bought eggs at 4s. 6d. per dozen. At what price per score must she sell them in order to gain 15%?

6. By selling goods for £175 10s., a profit of 17% was made by a merchant. How much did the goods cost him?

7. By selling goods for £250, a merchant lost 25% on his outlay. For how much should he have sold the goods in order to gain 10%?

8. At a clearance sale an article was reduced in price from 5s. to 4s. 9d. If the first price represents 25% profit, how much is the per cent of profit if the article is sold at the reduced price?

9. How much would I have to pay for a book which cost 3s. to produce, if the printing company sold it to a book seller at 20% profit and the book seller sold it to me at a profit of 25%?

10. A reduction of 25% in the price of rice will enable a trader to buy 50 pounds more for £1 5s. than he could have before the price reduction. What was the price per pound before reduction?

Commissions and Discounts

The business of buying and selling may be conducted by an agent on behalf of another person. For his services the agent is paid a fee called commission, which is a sum calculated as a percentage of the sum received or spent by the agent for his employer. For example: A book salesman is paid a commission of 15% of the selling price of a set of books. If he sells a set of books for £240, his commission is 15% of £240, or £36.

Stores often allow some reduction on cash payment for goods purchased. Such a reduction is normally calculated as a percentage of the original price and is called discount. During a sale, goods may be sold at a discount of, say, 10%. In such a case, goods originally marked £25 would then sell at £22 10s. Discount is also a sum of money deducted from a debt in consideration of its being paid before it is legally due.

Problems 15-13(b)

1. A real estate agent receives a commission of $7\frac{1}{2}$% of the selling price for any sale that he makes. What would be his commission on the sale of a house for £7,200?

2. A certain store gives 20% discount for cash. A woman paid shs. 110 cash for a dress. How much was the marked price of the dress?

3. A bookseller receives $12\frac{1}{2}$% commission on the sale of books for a publisher. In a certain month he received a commission of £75 from the company for his services for that month. How much was his sale for that month?

4. If a car dealer allows 5% cash discount and I paid £855 cash for a car, how much was its marked price?

5. How much cash must a man pay to settle an account of shs. 766.70 at 5% discount?

6. A bookseller received £90 commission for selling a set of books for £640. What was his rate of commission?

Simple Interest

When money is lent, particularly for business, the borrower is expected to pay for the use of the money. The amount of money lent is called the principal and the charge made for the use of the money is called interest.

Interest on money borrowed is paid at definite intervals (monthly, quarterly, half-yearly or yearly.) It is usually reckoned as a percentage of the principal for the period stated until the loan is repaid. If the interest is reckoned on the principal only, throughout the whole term of the loan, it is called simple interest.

The principal together with its interest for a stated period is called the amount for that time. Per annum means in one year.

Example

A sum of £750 was lent for 5 years at 3% per annum. Find the simple interest and the amount.

Interest in pounds on £750 for 1 year at 3% is $\frac{3}{100} \times 750$.

Interest in pounds on £750 for 5 years at 3% is $\frac{3}{100} \times 750 \times 5$

or $112\frac{1}{2}$.

$$\begin{aligned} \text{Amount} &= \text{Principal} + \text{Interest} \\ &= 750 + 112\tfrac{1}{2} \\ &= 862\tfrac{1}{2} \end{aligned}$$

The interest is £112 10s. and the amount is £862 10s.

From this example it is easy to see that the simple interest is equal to the principal multiplied by the rate per cent and the number of years. Thus if a principal of P pounds earns interest of I pounds at the rate of r per cent per year for a period of t years, then

$$I = \frac{Prt}{100}$$

Problems 15-13(c)

1. Find the simple interest on

   (a) £300 for 3 years at 5%.

   (b) £525 10s. for 4 years at $3\frac{1}{2}$%.

   (c) £750 for 3 years 4 months at $2\frac{1}{2}$%.

   (d) 750 dollars for $2\frac{1}{2}$ years at 3%.

   (e) 500 francs for 4 years at 5%.

2. If the simple interest on £500 at 4% per annum is £35, find the time.

3. If the simple interest on £250 for 5 years is £43 15s., find the rate per cent per annum.

4. If £350 amounts to £475 at $2\frac{1}{2}$% per annum, find the time.

5. If the simple interest on a sum of money invested at $3\frac{1}{2}\%$ per annum for 4 years is shs. 420, find the principal.

6. Find the principal which amounts, at simple interest, to £ 729 in 4 years at 2% per annum.

7. Find the quarterly interest on sh.16,800 at $2\frac{1}{2}$ per cent per annum.

8. A man borrows £ 500 on the condition that he repays £ 750 at the end of 4 years. What is the rate of simple interest?

9. A man received shs. 110 as one year's interest from a Post Office Savings Bank which pays 3% interest. How much had he in the bank?

Revision Problems

1. Write in decimal form each of the following:

(a) $30 + 5 + \frac{6}{10} + \frac{3}{100}$

(b) $400 + 3 + \frac{1}{10} + \frac{7}{1000}$

(c) $1 + \frac{1}{100} + \frac{3}{10000}$

(d) $\frac{7}{100} + \frac{7}{1000} + \frac{5}{100000}$

(e) $9000 + 90 + \frac{9}{10} + \frac{9}{1000}$

(f) $58\ \frac{87}{1000}$

(g) $105\ \frac{77}{10000}$

(h) $3\ \frac{1003}{1000000}$

2. Write each of the following decimal numerals in the expanded form:

(a) 1·0249 (b) 5·03 (c) 37035·449

(d) 0·0075 (e) 0·00821 (f) 1000·0001

3. In each of the following equations find x.

(a) $1924 = 1924x$

(b) $25{\cdot}01x = 2501$

(c) $0{\cdot}00157 = 1{\cdot}57x$

(d) $63004x = 6300{\cdot}4$

(e) ${\cdot}000051 = 510x$

(f) $78{\cdot}5x = {\cdot}00785$

(g) $0{\cdot}00107x = 10{\cdot}7$

(h) $0{\cdot}50501x = 50{\cdot}501$

4. The following is a record of rainfall in a certain town in West Africa for the first week of June, 1960:

Monday	3.2 inches
Tuesday	1·0 inches
Wednesday	2·3 inches
Thursday	1·5 inches
Friday	3·0 inches
Saturday	2·5 inches
Sunday	2·0 inches

(a) What is the average rainfall per day for that week? (Give the answer to 1 decimal place.)

(b) By how many inches was the total rainfall for Thursday to Sunday heavier than that for Monday to Wednesday?

(c) By how many inches was the rainfall for Monday heavier than the average rainfall for the week?

5. A clerk gets an increase of 2.5 per cent on his salary. What is his present monthly salary if he was receiving £25 a month before his increase?

6. How many yards are there in 0·125 of a mile?

7. Express 400 metres in yards and inches if 1 metre is equal to 39·37 inches.

8. If a car travelled at an average speed of 45·6 miles per hour, what distance did it cover in 12 hours?

9. On June 1st, 1961, my car had travelled 37477·3 miles. On June 1st, 1962, it had travelled 46285·9 miles.

(a) How many miles did the car cover between June 1, 1961 and June 1, 1962?

(b) How many miles will the car cover after June 1, 1962 before it has travelled 48888·8 miles?

(c) How many miles will the car cover after June 1, 1962 before it has travelled 50000·0 miles?

10. What decimal numerals could you put for a in each of the following equations to make it true?

(a) $a \times 485 = 45\%$ of 485 (b) $a \times 1 = 12\frac{1}{2}\%$ of 1

(c) $2a \times 80 = 15\%$ of 80

11. Express the following percentages as decimal numerals:

(a) 25% (b) 50% (c) 85% (d) $2\frac{1}{2}\%$ (e) 305% (f) $112\frac{1}{2}\%$

12. How may cwt. are there in 5·25 per cent of 48 tons?

13. Ama bought fish at 100 for £1 12s., and sold them at 6d. each; how much profit did she make by buying and selling 1000 fish?

14. A country with a population of 7,250,000 had 54·8% literates. How many were illiterate?

CHAPTER 16

REAL NUMBERS

16-1 Decimal Numerals for Rational Numbers

A rational number can be expressed in decimal notation by considering it as a quotient.

Thus: $\frac{1}{5} = 1 \div 5 = 0 \cdot 2$

and $\frac{3}{8} = 3 \div 8 = 0 \cdot 375$

Problems 16-1a

1. Express as decimal numerals:

(a) $\frac{1}{2}$ (c) $\frac{3}{4}$ (e) $\frac{3}{40}$

(b) $\frac{1}{4}$ (d) $\frac{3}{8}$ (f) $\frac{13}{80}$

2. Express the following as fractions in lowest terms:

(a) $0 \cdot 25$ (b) $1 \cdot 75$ (c) $2 \cdot 8$ (d) $1 \cdot 414$

Repeating Decimal Numerals

Consider the rational number $\frac{3}{22}$. This may be expressed in decimal notation by dividing $3 \cdot 000...$ by 22 as follows:

```
      0·13636...
22 )3·00000...
    2 2
      80
      66
      140
      132
        80
        66
        140
        132
          8
```

where the three dots indicate that you can go on in this way forever. It is clear that the set of digits 36 is repeated over and over again. Why? We call such a numeral a <u>repeating decimal</u>. The answer can be abbreviated by placing dots over the first and the last of the set of digits that repeat.

Hence $\frac{3}{22} = 0{\cdot}13636\ldots = 0{\cdot}1\dot{3}\dot{6}.$

Consider also the decimal numeral for $\frac{6}{37}$.

```
      0·162162...
37 ) 6·000000
     3 7
     2 30
     2 22
        80
        74
         60
         37
         230
         222
           80
           74
            6
```

Do you see why the digits in the quotient are repeating?

If the division does not terminate, there is always a remainder. After a few steps of the division we must come to the same remainder or to the original numerator. Why? In the above division 6 appears as a remainder in the third step and again in the sixth step. The digits in the quotient, then, must repeat in sets of three.

Hence $\frac{6}{37} = 0{\cdot}\dot{1}6\dot{2}.$

The set of digits 162 repeats. We indicate this by placing a dot over the first and last in the set of repeating digits.

Problems 16-1b

Express as decimal numerals:

(1) $\frac{1}{9}$ (4) $\frac{2}{3}$ (7) $\frac{1}{11}$ (9) $\frac{1}{37}$

(2) $\frac{1}{99}$ (5) $\frac{1}{7}$ (8) $\frac{3}{11}$ (10) $\frac{3}{41}$

(3) $\frac{1}{999}$ (6) $\frac{5}{7}$

Terminating Decimals and Repeating Decimals

In expressing fractions in decimal notation, you may have noticed that some fractions come out exactly; that is, they end or terminate, e.g. $\frac{1}{2} = 0\cdot5$ or $\frac{3}{8} = 0\cdot375$. Such decimal numerals are called terminating decimals.

If you try some examples, you will notice that, if a rational number can be expressed as a terminating decimal, then when written in simplest fractional form its denominator will have no prime factors other than 2 and 5. Can you connect this statement with the fact that 10 equals 2×5?

All other rational numbers can be expressed as repeating decimals. Can you show why this should be so?

We shall be able to show also that all repeating decimals represent rational numbers. The following set of problems will help prepare us to do this.

Problems 16-1c

1. By what number has each number in the first column been multiplied to obtain the number in the second column?

(a)	$0\cdot\dot{8}$	$8\cdot\dot{8}$
(b)	$0\cdot\dot{7}\dot{6}$	$76\cdot\dot{7}\dot{6}$
(c)	$0\cdot\dot{6}9\dot{7}$	$697\cdot\dot{6}9\dot{7}$
(d)	$8\cdot\dot{9}3476$	$893476\cdot\dot{9}3476$
(e)	$5\cdot5\dot{2}0\dot{1}$	$5520\cdot1\dot{2}0\dot{1}$

2. If $x = 0\cdot\dot{3}4\dot{5}$, find:

(a) $10x$ (b) $100x$ (c) $1000x$

3. Simplify:

(a) $10x - x$ (b) $100x - x$ (c) $1000x - x$

4. Simplify:

(a) $8\cdot\dot{8} - 0\cdot\dot{8}$ (c) $7245\cdot\dot{2}4\dot{5} - 0\cdot\dot{2}4\dot{5}$

(b) $93476\cdot\dot{9}3476\dot{} - 0\cdot\dot{9}3476\dot{}$ (d) $86\cdot4\dot{7}\dot{5} - 0\cdot0\dot{7}\dot{5}$

Let us consider the question:

Does $0\cdot\dot{1}\dot{5}$ represent a rational number?

We can write

(1) $x = 0\cdot\dot{1}\dot{5}$ and

(2) $100x = 15\cdot\dot{1}\dot{5}$

Subtracting (1) from (2) we obtain:

$$(100 - 1)x = 15 \qquad \text{(Distributive Property)}$$
$$99x = 15$$
$$x = \frac{15}{99} \qquad \text{(Why?)}$$

Hence $0\cdot\dot{1}\dot{5} = \frac{15}{99} = \frac{5}{33}$. This can be checked by doing the division $5 \div 33$.

Since $0\cdot\dot{1}\dot{5}$ can be represented as the quotient of two integers, it is a rational number.

Do you see why we chose to multiply by 100 in the first step and not by 10 or 1000?

Consider $3\cdot\dot{2}16\dot{3}$. Does this repeating decimal represent a rational number? We proceed in a similar way:

$$x = 3\cdot\dot{2}16\dot{3}$$
$$10000x = 32163\cdot\dot{2}16\dot{3} \qquad \text{(Why?)}$$
$$9999x = 32160 \qquad \text{(Why?)}$$
$$x = \frac{32160}{9999}$$
$$x = \frac{10720}{3333}$$

Hence $3\cdot\dot{2}16\dot{3} = \frac{10720}{3333}$

Problems 16-1d

1. Find fractional numerals which indicate that the following decimals represent rational numbers: (Check your answer by dividing.)

(a) $4\cdot\dot{6}$

(b) $0\cdot875$

(c) $0\cdot\dot{1}\dot{4}$

(d) $0\cdot\dot{1}4285\dot{7}$

(e) $3\cdot\dot{2}\dot{7}$

(f) $1\cdot\dot{1}\dot{6}$

16-2 Plotting Repeating Decimals on the Number Line

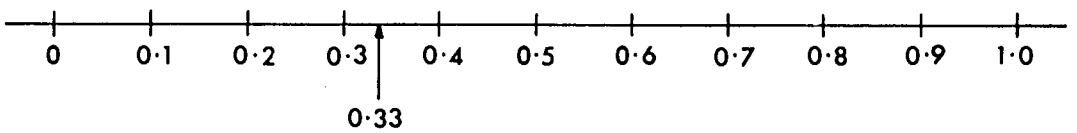
Consider $0\cdot333...$

$0\cdot33$ is another name for $0\cdot3 + 0\cdot03$

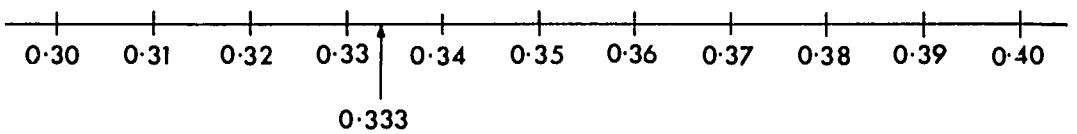
$0\cdot333$ is another name for $0\cdot3 + 0\cdot03 + 0\cdot003$

$0\cdot3333$ is another name for $0\cdot3 + 0\cdot03 + 0\cdot003 + 0\cdot0003$

We divide the segment between 0 and 1 into ten equal parts, each part being of length $0\cdot1$. The point at the end of the third part corresponds to $3 \times 0\cdot1$ or $0\cdot3$.

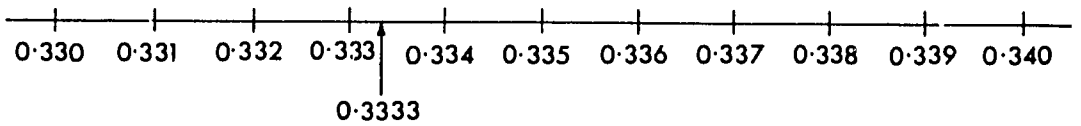


We divide the segment between $0\cdot3$ and $0\cdot4$ into ten equal parts, each part being of length $0\cdot01$. Three of these parts give $3 \times 0\cdot01$ or $0\cdot03$. Thus we find the point that corresponds to $0\cdot3 + 0\cdot03$ or $0\cdot33$.



We continue by dividing the segment between $0\cdot33$ and $0\cdot34$ into ten equal parts and take three of these. The new point will correspond to $0\cdot333$. The process can be continued on and on.

Every time we divide the segment into ten equal parts and take three of them we are getting closer and closer to the point that corresponds to $\frac{1}{3}$ or $0.\dot{3}$.



Problems 16-2

1. Plot each of the following successively on the number line as was done in the example.

(a) 0·6, 0·66, 0·666

(b) 3·6, 3·65, 3·656, 3·6565

16-3 Numbers Which are Not Rational

From the above discussion we can conclude that:

> Every terminating or repeating decimal numeral represents a rational number and every rational number may be represented by a terminating or repeating decimal numeral.

We are, however, left with the question, "Can every number be represented by a terminating or repeating decimal numeral?"

Consider the following decimal numerals:

(1) 1·01

(2) 1·01001

(3) 1·010010001

(4) 1·01001000100001

(5)

Can you find the fifth numeral? Can you find the twentieth numeral? Let us consider the fourth numeral, 1·01001000100001. The digits after the decimal point consist of a 0 followed by a 1, then two 0's followed by a 1, then three 0's followed by a 1, then four 0's followed by a 1. The fifth numeral will have an additional five 0's followed by a 1.

Each of these numerals will represent a rational number, since each is a terminating decimal.

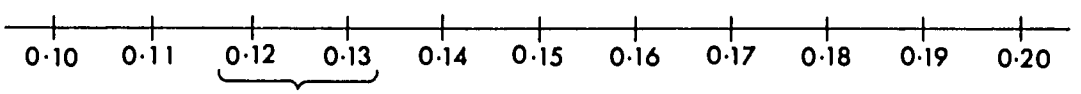
Consider, however, 1·01001000100001000001... where the three dots indicate that the numeral does not end, but continues according to the pattern described above. This is neither a terminating nor a repeating decimal. Hence it does not represent a rational number.

What about 0·1234567891011... ?

How would you continue to write it?
Is it a repeating decimal?
Does it represent a rational number?

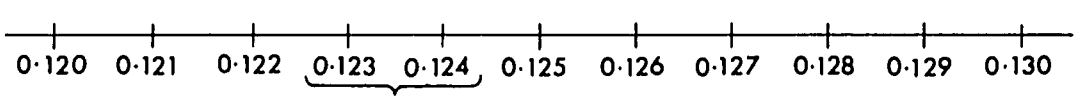
Let us locate on the number line the point which corresponds to 0·1234567891011...

Here is a segment of the line between 0·1 and 0·2 subdivided into hundredths.



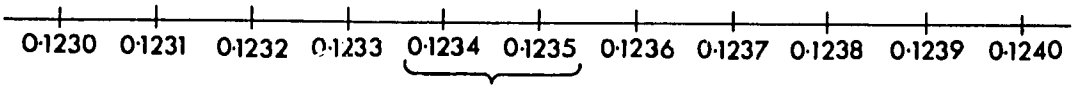
Our required point lies within the segment 0·1 to 0·2, in fact, between 0·12 and 0·13.

We can divide the segment between 0·12 and 0·13 into ten equal parts and plot 0·121, 0·122, 0·123, etc. On an enlarged scale, it will look like this:



Our required point lies between the points marked 0·123 and 0·124.

If we divide this interval (0·123 to 0·124) into ten equal parts and enlarge the scale again we obtain a figure like this:



Our required point lies in the interval from 0·1234 to 0·1235.

We can continue subdividing each segment into ten parts each one-tenth as long as before and find points that correspond to 0·12345 and 0·12346, to 0·123456 and 0·123457, and so on.

In this way we can locate the required point within intervals of length 0·1, 0·01, 0·001, 0·0001, and so on. Since the given decimal does not terminate we shall never locate it exactly in this way. However, we can locate it within an interval as short as we please.

It should be clear that there is a definite point on the number line which corresponds to each decimal.

We conclude

> There are decimal numerals which neither terminate nor repeat. They do not correspond to rational numbers. Every decimal numeral locates a definite point on the number line

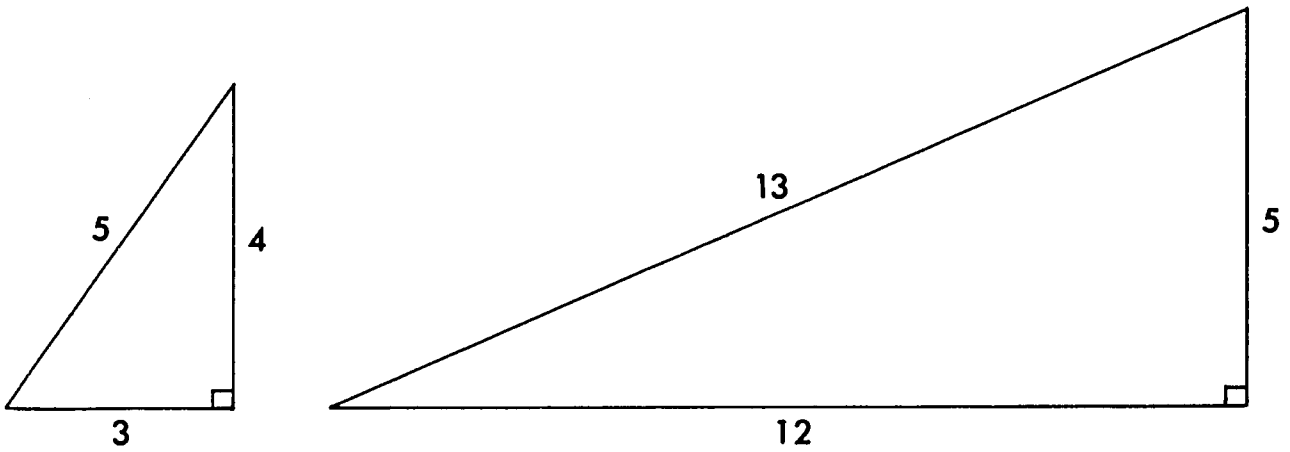
Problems 16-3

1. Use the method of this section to locate within an interval of length 0·0001, the point which corresponds to each of the following decimals:

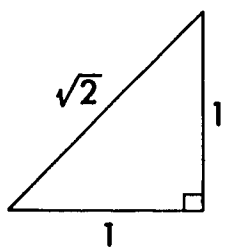
(a) 0·101001... (b) 2·124816

16-4 The Square Root of 2

You remember the Pythagorean Theorem about right-angled triangles. The following figures illustrate this property.

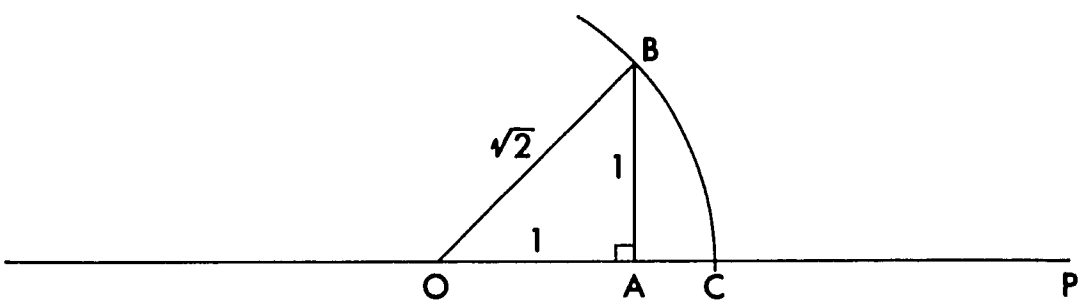


$3^2 + 4^2 = 5^2$ $\qquad$ $5^2 + 12^2 = 13^2$



$1^2 + 1^2 = (\sqrt{2})^2$

Let us find a point that corresponds to $\sqrt{2}$ on the number line.



Draw a segment $\overline{OA}$ that is 1 unit long. At A we construct another segment $\overline{AB}$ that is perpendicular to $\overline{OA}$ and is 1 unit long. Join B to O. The segment $\overline{OB}$, the hypotenuse of $\triangle OAB$, is of length $\sqrt{2}$. (Why?) With O as centre and $\overline{OB}$ as radius draw part of a circle cutting the number line at the point C. OC $= \sqrt{2}$.

$\sqrt{2}$ corresponds to a point on the number line.

How can we find a rational number which is close to $\sqrt{2}$?

Since $(\sqrt{2})^2$ is the number 2, by the definition of square root, we can test some numbers which we think might be close to $\sqrt{2}$ by squaring them. Only if the square of a number is exactly 2 can we say that the number is $\sqrt{2}$.

We might try $\frac{7}{5}$ as our first approximation to $\sqrt{2}$.

Since $\left(\frac{7}{5}\right)^2 = \frac{49}{25}$ and $\frac{49}{25} < 2$ we know that $\frac{7}{5}$ is too small.

Compare the following sentences, both of which are true.

$$(\sqrt{2})(\sqrt{2}) = 2, \qquad \left(\frac{7}{5}\right)\left(\frac{10}{7}\right) = 2.$$

How was the number $\frac{10}{7}$ determined?

Is $\frac{7}{5} \times \frac{10}{7} = (\sqrt{2})(\sqrt{2})$?

If $\frac{7}{5}$ is less than $\sqrt{2}$, what is true of the other factor, $\frac{10}{7}$?

Since $\frac{7}{5}$ is less than $\sqrt{2}$ and $\frac{10}{7}$ is greater than $\sqrt{2}$, the number $\sqrt{2}$ must be between $\frac{7}{5}$ and $\frac{10}{7}$. Therefore, the <u>average</u> of $\frac{7}{5}$ and $\frac{10}{7}$ might be a good next approximation.

$$\frac{1}{2}\left(\frac{7}{5} + \frac{10}{7}\right) = \frac{99}{70}.$$

Now use this second approximation, $\frac{99}{70}$, to write the following equation.

$$\frac{99}{70} \times \frac{140}{99} = 2.$$

How was the second factor $\frac{140}{99}$ determined?

If we reason as before, we see that of the two factors $\frac{99}{70}$ and $\frac{140}{99}$, if one of them is less than $\sqrt{2}$, the other must be greater than $\sqrt{2}$, since their product is $(\sqrt{2})^2$. A third approximation is then found by taking the average of $\frac{99}{70}$ and $\frac{140}{99}$.

$$\frac{1}{2}\left(\frac{99}{70} + \frac{140}{99}\right) = \frac{19601}{13860}.$$

Hence, $\frac{19601}{13860}$ or $1\frac{5741}{13860}$ or about $1 \cdot 4142$ is a third approximation.

If still better approximations are desired, this process of finding the other factor of 2 and averaging the two factors can be continued.

As a test of how close $\frac{19601}{13860}$ is to $\sqrt{2}$ we can square it.

$$\left(\frac{19601}{13860}\right)^2 = \frac{384199201}{192099600} = 2\frac{1}{192099600}$$

Hence, $\frac{19601}{13860}$ is a little larger than $\sqrt{2}$ but not much.

Actually, the square root of 2 lies somewhere between 1·414214 1·414215, or

$$1 \cdot 414214 < \sqrt{2} < 1 \cdot 414215$$

We say that 1·414214 is an approximation to $\sqrt{2}$ to six decimal places.

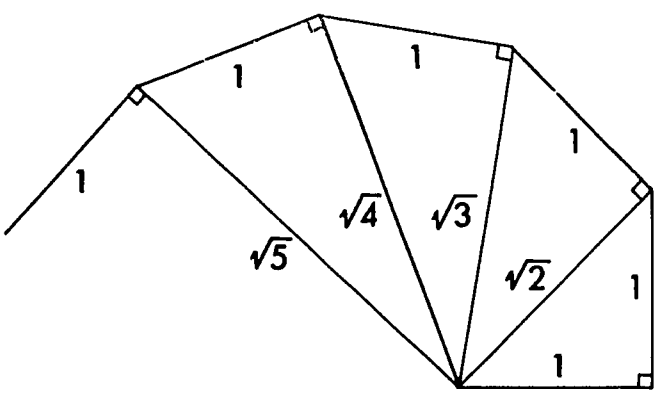
It will be proved in the next section that $\sqrt{2}$ is not a rational number.

We call numbers like $\sqrt{2}$ irrational numbers. They are irrational because they cannot be written as ratios of integers.

We can also find approximate values of square roots by geometry. The following problem illustrates this fact.

Problem 16-4

1. Continue drawing the following diagram as accurately as you can, and by measurement get approximate values of the square roots of the first few whole numbers.



2. Using the arithmetic method described above, find rational approximations to the following square roots: (Make a good first approximation to start with.)

(a) $\sqrt{7}$ (b) $\sqrt{11}$ (c) $\sqrt{13}$ (d) $\sqrt{17}$

16-5 Proof that $\sqrt{2}$ is an Irrational Number

Is $\sqrt{2}$ a rational number? That is, can $\sqrt{2}$ be written as a quotient of two integers?

We have approximated $\sqrt{2}$ by the rational numbers $\frac{7}{5}$, $\frac{10}{7}$, $\frac{99}{70}$, $\frac{140}{99}$. The squares of these numbers are very close to 2, but none of them is exactly equal to 2.

We could go on, trying other possible rational numbers, but no matter how many times we might try without succeeding, we would never be sure that some other choice of rational number might not work. We might begin to suspect that we could never succeed in finding a rational number which is exactly the square root of 2. But how could we be sure?

Let us look at our approximations more closely. We might choose $\frac{99}{70}$. We can see of course that $\left(\frac{99}{70}\right)^2 = \frac{99 \times 99}{70 \times 70} = \frac{9801}{4900}$ is a little more than 2. Can we see without squaring and dividing why $\left(\frac{99}{70}\right)^2$ cannot be exactly equal to 2.

Suppose that $\left(\frac{99}{70}\right)^2 = 2.$

Then $\frac{99 \times 99}{70 \times 70} = 2$

and $99 \times 99 = 2 \times 70 \times 70.$

If we multiply out we get

$$9801 = 9800$$

which is not true.

Can we see, without multiplying, that 99×99 and $2 \times 70 \times 70$ cannot possibly be equal?

Yes, we can. You remember that a whole number can be written as the product of prime factors in only one way.

Suppose that 99×99 were equal to $2 \times 70 \times 70$.

Let us look at the factor 2. This factor does not appear at all on the left-hand side, because 99 is not divisible by 2. But 2 does occur as a factor on the right-hand side. Since $2 \times 70 \times 70 = 2 \times 2 \times 35 \times 2 \times 35$, the factor 2 occurs three times on the right-hand side. If 99×99 and $2 \times 70 \times 70$ did represent the same number, 2 would have to occur the same number of times as a factor in both. We see, then, without multiplying, that since 99×99 and $2 \times 70 \times 70$ cannot be the same number, $\frac{99}{70}$ can not possibly be exactly equal to the square root of 2.

Let us generalize the above argument.

Let $\frac{a}{b}$ be any fraction where a and b are <u>whole numbers</u> and $b \neq 0$.

Suppose that $\frac{a}{b} = \sqrt{2}$.

Then $\left(\frac{a}{b}\right)^2 = 2$

or $\frac{a^2}{b^2} = 2$.

This means that

$$\frac{a \times a}{b \times b} = 2$$

and finally

$$a \times a = 2 \times b \times b.$$

Is this possible? Can $a \times a$ and $2 \times b \times b$ be two numerals for the same number? To answer this question, we ask whether the number of factors 2 on the left and right-hand sides could conceivably be the same.

On the left-hand side $(a \times a)$ there might be no 2's at all. On the right-hand side, $(2 \times b \times b)$, 2 occurs at least once as a factor. In this case, we surely could not have an agreement in the number of 2's.

Suppose then, that 2 does occur as a factor on the left-hand side. Any factor 2 which occurs in the first a occurs also in the second a. Hence there must be an <u>even</u> number of 2's on the left-hand side, (two, four, six,...).

Let us turn to the right-hand side, $(2 \times b \times b)$. What are the possibilities? There is at least one factor 2. There may be more. Every factor 2 that occurs in the first b occurs also in the second b. Hence there must be an <u>odd</u> number of 2's on the right-hand side, (one, three, five,...).

No matter what whole numbers we choose for a and b, the sentence $a \times a = 2 \times b \times b$ cannot be true. (Why?) This means that $\left(\frac{a}{b}\right)^2 = 2$ cannot be true and therefore there is no rational number exactly equal to $\sqrt{2}$. Then $\sqrt{2}$ must be an <u>irrational</u> number.

<u>Problems 16-5</u>

1. Prove that the following are irrational.

   (a) $\sqrt{3}$ (b) $\sqrt{5}$

2. What happens if you attempt to prove that $\sqrt{4}$ is irrational by the procedure described above?

<u>16-6</u> <u>The Real Numbers</u>

We have learned that decimals which terminate or repeat correspond to rational numbers, and that decimals which do not terminate and do not repeat correspond to irrational numbers.

The set of decimals of all kinds corresponds to the union of the set of rational numbers and the set of irrational numbers.

The union of these two sets is called the set of <u>real numbers</u>.

Then every decimal corresponds to a real number.

Every real number locates a definite point on the number line. We have illustrated this for both rational and irrational numbers.

Are there any points on the number line which have not been accounted for? The answer is "No". We can see this as follows:

Imagine any point P on the number line. To be definite, let us choose P between 0 and 1. When we mark the points for $0\cdot1$, $0\cdot2$,...$0\cdot9$, it could happen that P is one of these points. In this case, P corresponds to a terminating decimal, with one decimal place. Otherwise P lies between points marked with successive tenths. Then we know the first place of its decimal representation and we must continue. Now we divide this interval into ten equal parts. If our point P falls on one of the new points of division, we can locate it by a terminating decimal with two decimal places. If not, we have found the first two decimal places and must continue. Going on in this way we see that our given point P corresponds either to a terminating decimal or a non-terminating one. In any case, P corresponds to some decimal and therefore to some real number.

> To every real number there corresponds a single point on the number line, and to every point on the number line there corresponds a real number.

Problems 16-6

Fill in the table with "Yes" or "No".

	Real	Rational	Irrational	Positive	Negative
$\sqrt{3}$					
3					
-3					
$\frac{4}{7}$					
$-0\cdot\dot{8}$					
$0\cdot\dot{1}\dot{7}$					

16-7 Negative Real Numbers

The number line extends indefinitely to the right and to the left. The points to the right of 0 correspond to numbers that are positive.

$$1,\ 2,\ 3,\ \frac{3}{2},\ \frac{19}{3},\ \sqrt{17},\ 1\cdot 67,\ 1\cdot 010010001\ldots$$

(Whenever we write $\sqrt{\ }$ we mean the positive square root.)

The points to the left of 0 correspond to numbers that are negative.

$$-1,\ -2,\ -3,\ -\frac{3}{2},\ -\frac{19}{3},\ -\sqrt{17},\ -1\cdot 67,\ -1\cdot 010010001\ldots$$

(By $-\sqrt{17}$ we mean the negative number which squared gives 17.)

Just as the points to the right of 0 correspond to positive rational and irrational numbers, the points to the left of 0 will correspond to negative rational and irrational numbers.

We remember that if the sum of two numbers a and b is zero, one is called the additive inverse of the other or the opposite of the other. That is:

if $a + b = 0$

a is the opposite of b

and b is the opposite of a.

Thus, since $2 + (-2) = 0$

-2 is the opposite of 2

and 2 is the opposite of -2.

What is the opposite of $\frac{19}{3}$? of $-\frac{13}{5}$?

On the number line, opposites are at the same distance from 0 and on opposite sides.

What should be the opposite of $\sqrt{17}$? In other words, what must we add to $\sqrt{17}$ to obtain 0? Clearly $-\sqrt{17}$.

What real number must be added to $1\cdot 67$ to give 0? What real number must be added to $-1\cdot 010010001$ to give 0?

Problems 16-7

1. Find the opposites of:

(a) $3 \cdot 205$ (b) $-1 \cdot 23$ (c) $\sqrt{3}$ (d) $0 \cdot 2020020002\ldots$

2. Find the opposites of:

(a) $\frac{3}{2}$ (b) $-0 \cdot 1111\ldots$ (c) $\sqrt{6}$ (d) $-0 \cdot 12345\ldots$

16-8 Addition of Real Numbers

As you know, repeating and terminating decimals represent rational numbers. You also know that we can add these numbers.

Problems 16-8a

1. Find the following sums:

(a) $4 \cdot 13 + 5 \cdot 42$ (c) $3 \cdot 86 + 4 \cdot 57 + 0 \cdot 05$

(b) $4 \cdot 13 + 5 \cdot 423$ (d) $4 \cdot 37 + 7 \cdot 321 + 2 \cdot 2$

2. Verify that $0 \cdot \dot{2}7\dot{5} + 0 \cdot \dot{7}1\dot{4}$ equals $0 \cdot \dot{9}8\dot{9}$ by converting the numerals to fractional form (quotients of whole numbers) and adding in that form.

Can irrational numbers be added?

(a) Is $4 + 1 \cdot 030030003\ldots$ a real number?

(b) Is $3 \cdot 1010010001\ldots + 1 \cdot 3030030003\ldots$ a real number?

(c) What can we say about $(\sqrt{2} + \sqrt{3})$?

The answer to (a) is clearly $5 \cdot 030030003\ldots$, since $4 + 1 \cdot 030030003\ldots$ means moving on the number line four more units to the right of $1 \cdot 030030003\ldots$

To perform the addition in (b), we can form successive sums:

$$3\cdot1 + 1\cdot3 = 4\cdot4$$
$$3\cdot10 + 1\cdot30 = 4\cdot40$$
$$3\cdot101 + 1\cdot303 = 4\cdot404 \qquad \text{and so on.}$$

Until we have the sum:

$$\begin{array}{r} 3\cdot1010010001\ldots \\ +\ \underline{1\cdot3030030003\ldots} \\ 4\cdot4040040004\ldots \end{array}$$

We see that in both (a) and (b), the sums are represented by non-terminating and non-repeating decimals. They correspond to irrational numbers. Like all real numbers they can be represented on the number line.

Let us use the number line to help us understand these additions.

Let the point P on the number line correspond to $1\cdot3030030003\ldots$, and let Q correspond to $3\cdot1010010001\ldots$

The point R, such that OR = OP + OQ, represents $4\cdot4040040004\ldots$ There clearly is a definite point R corresponding to a definite real number which is the sum of the given numbers.

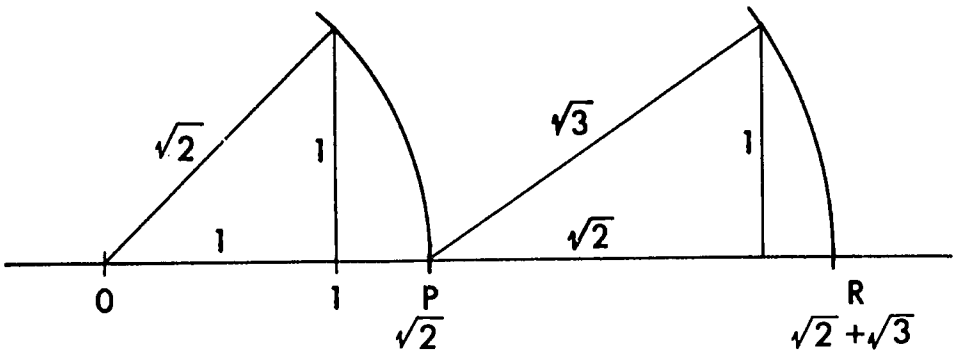
Problems 16-8b

Continue each of the decimals below to eight decimal places assuming that the unwritten digits follow some pattern suggested by the first few and perform the indicated addition:

1. $4\cdot12345\ldots + 5\cdot891011\ldots$
2. $3\cdot101001\ \ldots + 6\cdot152535\ldots$
3. $3\cdot010110111\ldots + 4\cdot102003\ldots$

Addition of Square Roots on the Number Line

As a special case of addition on the number line, we can construct the sum $\sqrt{2} + \sqrt{3}$. We already know how to construct $\sqrt{2}$ and $\sqrt{3}$, as in the following figures:



On the number line, mark P to the right of O such that OP = $\sqrt{2}$ and mark R to the right of P such that PR = $\sqrt{3}$. The coordinate of R, then is $\sqrt{2}+\sqrt{3}$. There clearly is a real number corresponding to $\sqrt{2}+\sqrt{3}$.

Problems 16-8c

Write the first four decimal places for each of the following numbers:

1. $\sqrt{2}$
2. $\sqrt{3}$
3. $\sqrt{2}+\sqrt{3}$
4. $\sqrt{5}$
5. Does $\sqrt{(2+3)} = \sqrt{2}+\sqrt{3}$?
6. Locate $\sqrt{5}$ on the number line by using a geometrical construction.

In question 5 above, notice that

$$\sqrt{(2+3)} = \sqrt{5} \qquad \text{but} \qquad \sqrt{2}+\sqrt{3} \neq \sqrt{5}$$

The Sum of a Rational Number and an Irrational Number is an Irrational Number

Consider $3+\sqrt{2}$. Can this be a rational number?

Let us assume that $3+\sqrt{2}$ is a rational number.

Then $3+\sqrt{2} = \frac{a}{b}$ where a and b are integers and $b \neq 0$

$$\sqrt{2} = \frac{a}{b} - 3$$

$$\sqrt{2} = \frac{a-3b}{b}$$

Since $a-3b$ and b are integers (Why?) $\frac{a-3b}{b}$ is a rational number.

The original assumption leads us to say that $\sqrt{2}$ is a rational number. We know, however, that $\sqrt{2}$ is not rational. Therefore our assumption that $3 + \sqrt{2}$ is rational is false.

In the same way we can prove that the sum of any rational number and any irrational number is an irrational number.

What about the sum of two irrational numbers?

Consider the following addition of two irrational numbers:

$$\begin{array}{r} 0{\cdot}01001000100001\ldots \\ +\ \underline{0{\cdot}10110111011110\ldots} \\ 0{\cdot}11111111111111\ldots \end{array}$$

The resulting decimal represents $\frac{1}{9}$ and therefore the sum is rational.

The sum of two irrational numbers <u>may</u> be a rational number; that is, the set of irrational numbers is not closed under addition.

<u>Problems 16-8d</u>

1. What can you add to each of the following to make each a rational number? (There are many possible answers. In each case give two answers.)

(a) $5 - \sqrt{3}$ (d) $6 + \sqrt{5}$

(b) $4{\cdot}000100001000001\ldots$ (e) $6{\cdot}123456\ldots$

(c) $7{\cdot}030030003\ldots$ (f) $7{\cdot}789101112\ldots$

2. Add:

(a) $3{\cdot}6 + 4{\cdot}21$ (d) $3{\cdot}\dot{4}5\dot{1} + 2{\cdot}\dot{3}2\dot{7}$

(b) $4{\cdot}71 + 0{\cdot}06$ (e) $(7 + \sqrt{3}) + (3 - \sqrt{5})$

(c) $6{\cdot}\dot{2} + 3{\cdot}\dot{5}$ (f) $1{\cdot}04004\ldots + 4{\cdot}40440444\ldots$

3. Compare the following using the symbols $=$ or $\neq$

(a) $\sqrt{(3 + 5)}$, $\sqrt{3} + \sqrt{5}$ and $\sqrt{8}$

(b) $\sqrt{(2 + 5)}$, $\sqrt{2} + \sqrt{5}$ and $\sqrt{7}$

4. Show that $\sqrt{\sqrt{2} + 1}$ is irrational.

16-9 Order of Real Numbers

Suppose that we have two different positive real numbers written as decimals. For example,

$$a = 0 \cdot 32842...$$
$$b = 0 \cdot 32719...$$

How can we compare them so that we can say that one is less than the other? In this example, the two numbers agree in the tenth's place and in the hundredth's place. In the thousandth's place they differ. Let us locate a and b on the number line.

The point that corresponds to a lies to the right of the point that corresponds to $0 \cdot 328$, and the point that corresponds to b is to the left of the point that corresponds to $0 \cdot 328$. Therefore if we go along the number line from left to right we would meet b first. We say, then, that b is less than a and write $b < a$.

In the same way if:

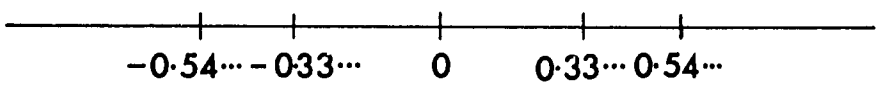
$$a = 0 \cdot 5321...$$
$$b = 0 \cdot 5324...$$

we would come to a before we come to b, regardless of what the digits are after the fourth decimal place.

If we wish to compare two negative real numbers, say:

$$-0 \cdot 54... \qquad \text{and} \quad -0 \cdot 33...$$

we notice that since $0 \cdot 33...$ is to the left of $0 \cdot 54...$



their opposites are in the reverse order. $-0 \cdot 54...$ is to the left of $-0 \cdot 33...$ That is, $-0 \cdot 54... < -0 \cdot 33...$

Of course, any negative number is located to the left of 0 and any positive number is to the right of 0. Therefore, any negative real number is less than any positive real number.

The order of real numbers has the following properties:

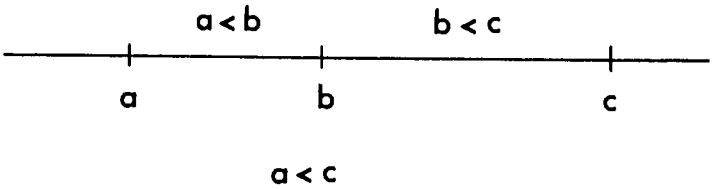
Property 1. If a and b are any real numbers, exactly one of the following is true.

$$a = b \quad \text{or} \quad a < b \quad \text{or} \quad b < a$$

Property 2. If a, b and c are any real numbers

and if $a < b$ and $b < c$

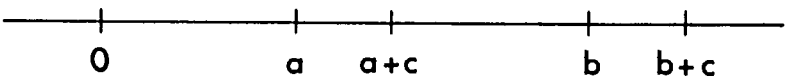
then $a < c$



Property 3. If a, b and c are any real numbers

and if $a < b$

then $a + c < b + c$



Problems 16-9

1. Compare

(a) 0·432... and 0·437...

(b) 3·824... and 3·816...

(c) -0·128... and -0·123...

(d) -0·128... and 0·001...

2. Think of $0 \cdot 432...$ and $0 \cdot 437...$ as

$$0 \cdot 43 + 0 \cdot 002...$$
$$\text{and } 0 \cdot 43 + 0 \cdot 007...$$

Then use Property 3 to conclude that

$$0 \cdot 432... < 0 \cdot 437...$$

3. Use the idea in problem 2 above to compare the numbers in Problem 1(b).

4. Indicate on the number line the truth set of the inequalities.

(a) $x < \sqrt{2}$ (b) $x > -\sqrt{3}$

16-10 Multiplying Real Numbers

As we know, a real number can be written in decimal form. Suppose that we have two real numbers,

For example

$$a = 0 \cdot 1213...$$
$$b = 0 \cdot 2145...$$

How shall we multiply them?

Since $a > 0 \cdot 1$ and $b > 0 \cdot 2$

$$a \times b > (0 \cdot 1)(0 \cdot 2) = 0 \cdot 02$$

Again $a > 0 \cdot 12$ and $b > 0 \cdot 21$

so that $a \times b > (0 \cdot 12)(0 \cdot 21) = 0 \cdot 0252$

Similarly $a \times b > (0 \cdot 121)(0 \cdot 214) = 0 \cdot 025894$

Each time we are closer to the value of the desired product. In each case we have left something off, so that the products we found are always too small.

We can get approximations which are too large by noticing that

$$a < 0\cdot 2 \quad \text{and} \quad b < 0\cdot 3$$
$$a < 0\cdot 13 \quad \text{and} \quad b < 0\cdot 22$$
$$a < 0\cdot 122 \quad \text{and} \quad b < 0\cdot 215 \quad \text{and so on.}$$

From the last of these statements we see that

$$a \times b < (0\cdot 122)(0\cdot 215) = 0\cdot 026230$$

By going on in this way we come closer and closer to the value $a \times b$, but each time the result will be too large.

We know definitely that $a \times b$ is between

$$0\cdot 025894 \quad \text{and} \quad 0\cdot 026230$$

If we continue we can obtain as close an approximation as we please.

In the case of real numbers which are square roots of whole numbers, we can write the products in another form.

For example,

$$\sqrt{2} \times \sqrt{2} = 2$$

since $\sqrt{2}$ is a number whose square is 2.

What is $\sqrt{2}(\sqrt{2} - 1)$?

Using the distributive property we obtain the result

$$\sqrt{2}(\sqrt{2} - 1) = \sqrt{2} \times \sqrt{2} - \sqrt{2} \times 1 = 2 - \sqrt{2}.$$

What is $\sqrt{2} \times \sqrt{3}$?

If we consider the result of squaring the product, we obtain

$$(\sqrt{2} \times \sqrt{3})^2 = (\sqrt{2})^2 \times (\sqrt{3})^2$$ (Use the commutative and associative properties to show that this is true.)

$$(\sqrt{2})^2 \times (\sqrt{3})^2 = 2 \times 3$$ (Definition of square root)

Since $(\sqrt{2} \times \sqrt{3})^2 = 6$

$\sqrt{2} \times \sqrt{3} = \sqrt{6}$ (Why?)

Show in the same way that if a and b are any two positive real numbers,

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$$

What is $\sqrt{2} \times \sqrt{8}$?

So far we have considered only multiplication of two positive real numbers. What happens when we multiply a positive real number by a negative real number?

To find $\sqrt{2} \times (-\sqrt{2})$ consider the following questions.

Why is $\sqrt{2}\left[\sqrt{2} + (-\sqrt{2})\right] = 0$?

Why, then, is $2 + \left[\sqrt{2} \times (-\sqrt{2})\right] = 0$?

And so, why is $\sqrt{2} \times (-\sqrt{2}) = -2$?

What is the result of multiplying two negative real numbers? Is it a positive number or a negative number?

What, for example, is $(-\sqrt{2}) \times (-\sqrt{2})$?

The answer is 2. Can you prove this?

Problems 16-10

1. We wish to multiply a by b where

$$a = 0 \cdot 111\ldots$$
$$b = 0 \cdot 333\ldots$$

Multiply $0 \cdot 1$ by $0 \cdot 3$, $0 \cdot 11$ by $0 \cdot 33$ and so on, to get better approximations to $a \times b$. These results are all too small. Get approximations which are too large by multiplying $0 \cdot 2$ by $0 \cdot 4$, $0 \cdot 12$ by $0 \cdot 34$ and so on.

2. Check your result in Problem 1 from the fact that $a = \frac{1}{9}$ and $b = \frac{1}{3}$, so $a \times b = \frac{1}{27}$.

3. $\sqrt{2} = 1\cdot414...$

$\sqrt{3} = 1\cdot732...$

$\sqrt{6} = 2\cdot449...$

Use the process of Problem 1 to make it plausible that

$$\sqrt{2} \times \sqrt{3} = \sqrt{6}$$

4. Using the distributive property, find $\sqrt{3}\ (\sqrt{2} + 1)$ in terms of square roots.

5. Show that $(\sqrt{2} - 1)\ (\sqrt{2} + 1)$ is a whole number. (Use the distributive property.)

6. Show that $(\sqrt{5} + \sqrt{2}) \times (\sqrt{5} - \sqrt{2})$ is a whole number.

7. <u>CHALLENGE PROBLEM</u>

Show by squaring that

$$\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}} \text{ is rational.}$$

8. <u>CHALLENGE PROBLEM</u>

Show by squaring that

$$\sqrt{2 + \sqrt{3}} + \sqrt{2 - \sqrt{3}} = \sqrt{6}$$

9. <u>CHALLENGE PROBLEM</u>

Find a simple numeral for

$$\sqrt{2 + \sqrt{3}} - \sqrt{2 - \sqrt{3}}$$

10. Find two irrational numbers whose sum is irrational and whose product is rational.

You have learnt that the set of rational numbers under addition and multiplication forms a <u>field</u>.

The set of real numbers under addition and multiplication also forms a field.

That is,

The set of real numbers is closed under addition and multiplication. (The sum and product of two real numbers are real numbers.)

Addition and multiplication of real numbers are commutative and associative.

$x + y = y + x$ $\qquad$ $x \times y = y \times x$

$(x + y) + z = x + (y + z)$ $\qquad$ $(x \times y) \times z = x \times (y \times z)$

There are additive and multiplicative identities (0 and 1) such that

$x + 0 = x$ $\qquad$ $x \times 1 = x$

Every real number x has an additive inverse (opposite) $-x$ such that

$$x + (-x) = 0$$

Every real number x $(x \neq 0)$ has a multiplicative inverse (reciprocal) $\frac{1}{x}$ such that

$$x \times \frac{1}{x} = 1$$

16-11 Absolute Value

We can measure distance along the number line using the segment from 0 to 1 as a unit of length. Thus, the point that corresponds to 2 is at a distance of 2 units from 0. The point that corresponds to -2 is also at a distance 2 from 0. We say that both 2 and -2 have the <u>absolute value</u> 2.

The absolute value of a number a will be written $|a|$.

Thus $|2| = 2$ and $|-2| = 2$.

Similarly, $|0 \cdot 132...| = 0 \cdot 132...$

and $|-0 \cdot 132...| = 0 \cdot 132...$

Any two opposites, a and $-a$, have the same absolute value:

$$|a| = |-a|$$

What is $|0|$? How far is 0 from 0?

Do you see that $|0| = 0$?

We have explained absolute value in terms of the distance from 0 along the number line. This distance is always non-negative. That is, distance can be only positive or 0.

We can say that $|a|$ is that one of the pair of numbers, a and $-a$, which is non-negative.

If a is positive, then $|a| = a$.
For example, since 2 is positive, $|2| = 2$.

If a is negative, then $|a| = -a$. (Notice that when a is negative, its opposite, $-a$, is positive.)

For example, since -2 is negative $|-2| = -(-2) = 2$.

<u>Problems</u> <u>16-11</u>

1. Find the absolute value of each of the following.

   (a) $-0\cdot1234...$ (c) $(-0\cdot111...) - 0\cdot01001...$
   (b) $\frac{4}{3} - \frac{3}{2}$ (d) $-3\cdot68 + 3\cdot89$

2. What is the truth set of $|x| = 2$?

3. Indicate on the number line the truth set of:

   (a) $|x| < 2$ (b) $|x| > 2$

4. Indicate on the number line the truth set of:

   (a) $|x| \leq 2$ (b) $|x| \geq 2$

Note: $\leq$ means "is less than or equal to".

5. Find the truth set of $|6 - x| = 1$.

6. We know that on the number line the distance between the points with coordinates 9 and 12 is 3. We found the distance by taking the difference, 12-9. If we use 9-12, we would have a negative number, but by using absolute value, we can write $|9-12|$ for the distance between the points. Either $|12-9|$ or $|9-12|$

gives the distance. In general $|a-b|$ is the distance between the points with coordinates a and b.

Using this viewpoint, we can say that $|6-x|$ is the distance between the points with coordinates 6 and x. Use this viewpoint to find the truth sets of the following sentences:

(a) $|6-x| = 1$ (c) $|2 - x| = 1$

(b) $|6-x| = 3$ (d) $|2 - x| = 5$

7. Show the truth sets of the following sentences on the number line.

(a) $|5 - x| = 3$ (c) $|(-3) - x| = 2$ (e) $|(-2) - x| = 4$

(b) $|3 - x| = 5$ (d) $|x - 4| = 5$ (f) $|x - (-2)| = 2$

8. Find the truth set and show it on the number line.

(a) $|x - 5| \leq 4$ (b) $|x - 5| \geq 4$

16-12 More About Absolute Value

The Absolute Value of a Product

How is $|a \times b|$ related to $|a|$ and $|b|$?

Let us take a few examples:

If a = 3 and b = 2, then

$|a| = 3, \quad |b| = 2, \quad |a \times b| = |6| = 6$

In this case $|a| \times |b| = |a \times b|$

If a = -3 and b = -2, then $|a| = 3$, $|b| = 2$, $|a \times b| = |6| = 6$. In this case, $|a| \times |b| = |a \times b|$.

If a = 3 and b = -2, then $|a| = 3$, $|b| = 2$, $|a \times b| = |-6| = 6$. In this case, $|a| \times |b| = |a \times b|$.

If a = 3 and b = 0, then $|a| = 3$, $|b| = 0$, $|a \times b| = |0| = 0$. In this case, $|a| \times |b| = |a \times b|$.

We suspect that $|a| \times |b| = |a \times b|$ for all real numbers a and b. This is true because in every case, $|a| \times |b|$ and $|a \times b|$ are both equal to the <u>non-negative</u> number of the two numbers $(a \times b)$ and $-(a \times b)$.

It is true, then, that the absolute value of the product of two real numbers is equal to the product of their absolute values.

The Absolute Value of a Sum

How is $|a + b|$ related to $|a|$ and $|b|$?

You might guess that $|a + b|$ is the sum of $|a|$ and $|b|$.

A few examples should convince you that this is not always true.

Example 1. If $a = 3$ and $b = -1$, $a + b = 2$.

Then $|a| = 3$, $|b| = 1$ and $|a + b| = 2$.

In this case, $|a + b|$ is less than $|a| + |b|$.

Example 2. If $a = -3$ and $b = -1$, $a + b = -4$

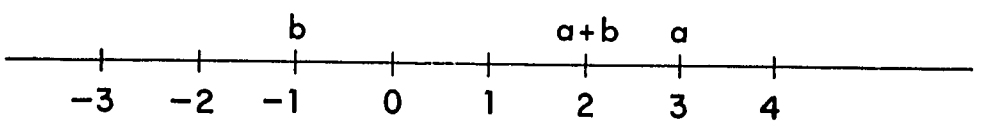
Then $|a| = 3$, $|b| = 1$ and $|a + b| = 4$.

In this case, $|a + b| = |a| + |b|$.

You might try $a = 3$ and $b = 1$, also $a = -3$ and $b = 1$, to see if you can draw a general conclusion.

It will be helpful if we show the results of Examples 1 and 2 on the number line.

In Example 1, a and b are on opposite sides of 0, and a is the farthest away from 0.



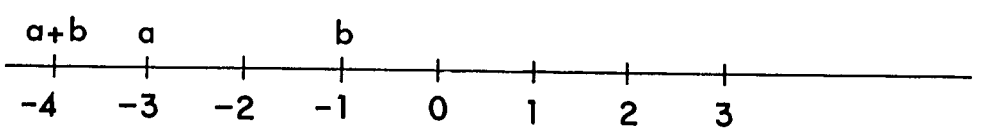
Therefore, $a + b$ must be nearer to 0 than a is.

That is,

$$|a + b| < |a|$$

and surely $$|a + b| < |a| + |b|.$$

In Example 2, a and b are on the same side of 0 at distances 3 and 1.



Their sum is on the same side at distance 4.

Here $$|a + b| = |a| + |b|$$

If we consider the general situation, we can see that one of two things can occur:

(1) If a and b are on the same side of 0 or if one of them is 0

$$|a + b| = |a| + |b|$$

(2) If a and b are on opposite sides of 0,

$$|a + b| < |a| + |b|$$

We can put these two results together in a single open sentence which applies to both cases.

$$|a + b| \leq |a| + |b|$$

We can summarize our results as follows:

For all a and b $\lvert a \times b\rvert = \lvert a\rvert \times \lvert b\rvert$ $\lvert a + b\rvert \leq \lvert a\rvert + \lvert b\rvert$

<u>Problems 16-12</u>

1. Verify that $|a \times b| = |a| \times |b|$ when:
   (a) $a = -\frac{1}{2}$, $b = -\frac{1}{4}$
   (b) $a = \sqrt{3}$, $b = -\sqrt{2}$
   (c) $a = 1 + \sqrt{3}$, $b = 1 - \sqrt{3}$

2. Verify that $|a + b| \leq |a| + |b|$ when:
   (a) $a = -2$, $b = -4$
   (b) $a = 2$, $b = -2$
   (c) $a = \sqrt{2} - 1$, $b = \sqrt{2} + 1$

3. What is a simple name for $|2|^2$? For $|-2|^2$?
   Show that, $|a|^2 = a^2$ whether a is positive, negative or 0.
   Then show that $|a| = \sqrt{a^2}$.

(Remember that when we write $\sqrt{\ }$, it is understood that we take the non-negative square root. If we wanted the negative square root we would write $-\sqrt{\ }$.)

4. Verify that $|a| = \sqrt{a^2}$
for $a = -5$, $a = \frac{1}{4}$, $a = -\frac{1}{9}$, $a = -\sqrt{3}$

5. What is the truth set of $\sqrt{x^2} = |x|$? of $\sqrt{x^2} = x$? of $\sqrt{x^2} = -x$?

6. <u>CHALLENGE PROBLEM.</u> Use $|x| = \sqrt{x^2}$ to prove that $|a \times b| = |a| \times |b|$.
Hint: Is $\sqrt{(a \times b)^2} = \sqrt{a^2} \times \sqrt{b^2}$?

7. <u>CHALLENGE PROBLEM.</u>
   (a) Show that $|a - b| \leq |a| + |b|$
   (b) Use this result to show that $|a - b| \geq |b| - |a|$

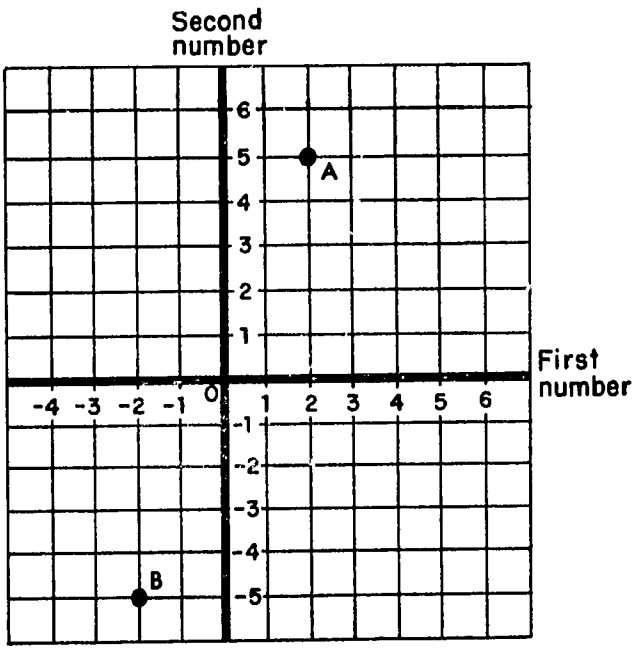
   Hint: Start (b) with $|(a - b) - a| \leq |a - b| + |a|$

CHAPTER 17

INTRODUCTION TO GRAPHING IN THE NUMBER PLANE

17-1 The Number Plane

To the right is a picture of two number lines intersecting at right angles. What first number goes with A ? If you look at the picture, you should see that it is 2. What second number goes with A ? Do you see why the pair of numbers that corresponds to B is (-2, -5)?

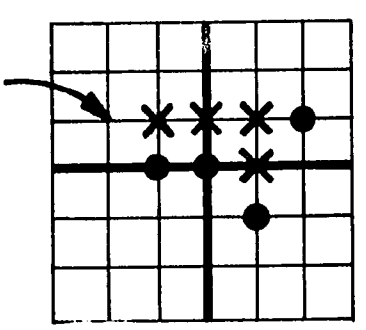


Problems 17-1

1. On a piece of graph paper, draw the diagram that appears above; then mark and label the points that go with each of the following pairs of numbers:

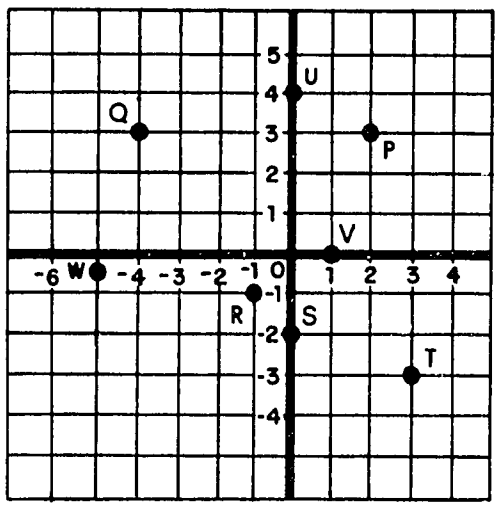
C: (0, 3)	E: (-2, 2)	G: (-2, 5)	I: (10, -4)
D: (3, 0)	F: (6, 2)	H: (0, 0)	J: $(2\frac{1}{2}, 3)$

2. John is on the X-team and wishes to win for his team by placing an X where the arrow is pointing. (Four in a line wins.) He says, "One, negative two." Does he win for his team?



As you have probably already discovered, and as John just learnt, the point that corresponds to (1, -2) is different from the point that corresponds to (-2, 1). The numbers are the same, but the <u>order</u> is different. We call the two numbers an <u>ordered pair</u>, and use brackets when we write them to show that the order is important.

3. Find the ordered pair of numbers that corresponds to each point marked in the diagram to the right. The first one is correctly answered.



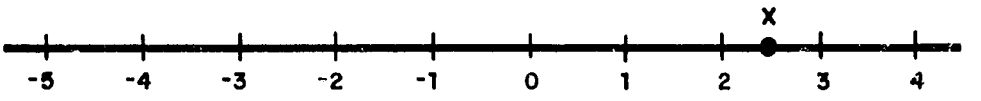
Point	Ordered Pair	Point	Ordered Pair
P	(2, 3)	T	
Q		U	
R		V	
S		W	

4. (a) For any ordered pair of numbers, how many corresponding points in the number plane are there?

(b) For any point in the number plane, how many corresponding ordered pairs of numbers are there?

Coordinates

When you learnt about the number line, you spoke of the <u>coordinate of a point</u>. For example, in the following picture, we call $2\frac{1}{2}$ the coordinate of the point X.



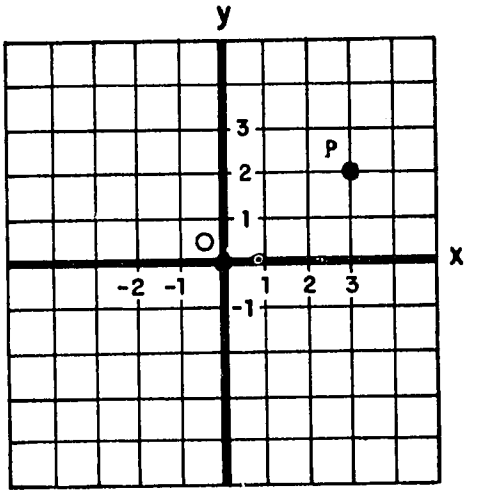
When we use two intersecting number lines, we speak of the coordinates of a point, since we now have a pair of numbers that goes with each point: A first coordinate, and a second coordinate. The special point whose coordinates are both zero, i.e. the point that corresponds to (0, 0), is the point where we start counting when we are marking, or plotting, a point. This point has the special name origin. The two intersecting number-lines are called axes (plural); if you are speaking of one of the axes, for example, the horizontal one, it is called the horizontal axis (singular).

1. The axis marked with x is called the horizontal axis, the first coordinate axis, or the x-axis.

2. The axis marked with y is called the vertical axis, the second coordinate axis, or the y-axis.

3. The x and y axes intersect at the point O, called the origin.

4. The point P has first coordinate 3 and second coordinate 2.



17-2 Graphs as Pictures of Truth Sets

Class Activity

1. What is the truth set of the sentence $-3 < x < 5$ if the domain of x is the set of integers? Draw a picture of the truth set on the number line.

2. What is the truth set of $-3 < x < 5$ if the domain of x is the set of real numbers? Draw a picture of the truth set on the number line.

3. Let us see if we can find the truth set of an open sentence in <u>two</u> variables, say the equation $y = x + 3$. If x is 2, what should y be to make the equation true? If we agree that the first coordinate is the value of x, and the second coordinate the value of y, which of these ordered pairs belong to the truth set of $y = x + 3$?

(a) (5, 8) (c) (4, 1) (e) (-1, 4) (g) $(\frac{1}{2}, 4)$
(b) (0, 3) (d) (2, 5) (f) (-2, 1)

4. Fill in the table to the right with any of the ordered pairs from above that belong to the truth set of $y = x + 3$. Then find two more ordered pairs that belong to the truth set.

x	y
5	8

How many ordered pairs are there in the truth set?

5. For $-3 < x < 5$, you drew a picture of the truth set on the number line. For $y = x + 3$, the truth set contains ordered pairs of numbers, so we shall need a number plane to draw a picture of the truth set. Draw a set of axes at right angles on your graph paper; let us agree to call the horizontal axis the x-axis, and to use the same scale on both axes. Plot the points that correspond to the ordered pairs in your table.

6. Pretend for a moment that you do not know any arithmetic, or how to count, but you do have good eyesight. Mark another point that looks as though it should belong to the truth set of $y = x + 3$. What pair of numbers goes with this last point that you marked? Check to see whether this ordered pair really does make the equation true. If it does not, erase the point and guess again. Mark 3 or 4 more points on the graph paper that look as though they belong to the truth set. Find the ordered pair of numbers that goes with each point after you have marked it, and check to see whether it really does satisfy $y = x + 3$.

7. If you allow the domain of each of the variables in $y = x + 3$ to be the set of real numbers, what picture do you get for the truth set? Can you draw a picture which shows all the ordered pairs in the truth set?

8. For the equation $y = 2x + 3$, what ordered pair with x-coordinate 1 belongs to the truth set? What ordered pair with x-coordinate 2 belongs to the truth set?

x	y
1	?
2	?

On the same graph paper used for exercise 5, plot the points that go with these ordered pairs. (If you marked the points representing the truth set of $y = x + 3$ with $\cdot$, mark these points with $\times$ so that you will be able to tell them apart.) Without doing any more arithmetic or counting, mark another point that you think belongs to the truth set of $y = 2x + 3$. What ordered pair corresponds to the point you marked? Check to see whether this pair of numbers really does make $y = 2x + 3$ true. If it does not, erase it and guess again. Mark 3 or 4 more points on the graph paper that look as though they belong to the truth set of $y = 2x + 3$. Find the ordered pair that goes with each point after you have marked it, and check to see whether it really does satisfy $y = 2x + 3$. Let the domain of both x and y be the set of real numbers, and draw a picture of as much of the truth set of $y = 2x + 3$ as will fit on the paper. The set of points representing the truth set of an open sentence is called a graph of the open sentence.

9. On the same picture of the number plane, graph $y = 3x + 3$.

x	y
1	?

Mark the point that goes with the ordered pair in your table above. Whithout doint any more arithmetic or counting, mark another point that you think belongs to the graph of $y = 3x + 3$. (You may wish to mark the point with $\square$ to tell it apart from points on the other two graphs.) Find the ordered pair of numbers that goes with the point you marked. Check to see whether it really satisfies $y = 3x + 3$. Mark other points that look as though they belong to the graph. Check each one after you have marked it. Complete as much of the graph as you can on the paper.

Problems 17-2a

Assume that the domain of any variable in the rest of this section is the set of real numbers.

1. Graph these equations on the same set of axes:

(a) $y = 5x + 3$ (c) $y = -2x + 3$

(b) $y = 10x + 3$ (d) $y = \frac{1}{2}x + 3$

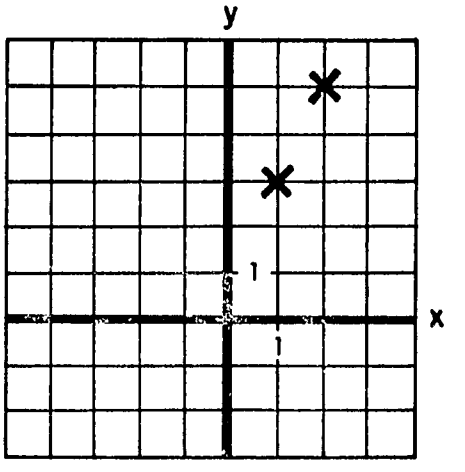
2. Graph these equations on the same set of axes:

(a) $y = 2x + 1$ (c) $y = 2x + (-2)$

(b) $y = 2x + 5$ (d) $y = 2x$

Class Activity

1. If we want an equation whose graph contains the two points marked in the picture, what value should you give to a in the equation $y = ax + 1$?



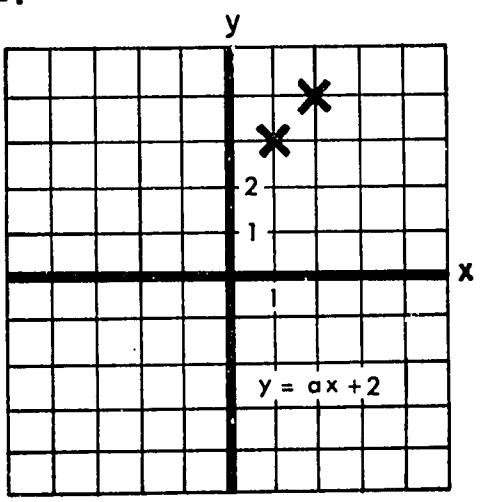
Find the coordinates of the two points marked and check to see whether they both make the equation true.

2. From your previous work in geometry, you know that two points are contained in exactly one line. If you select any of the other points on the line, will they have coordinates that satisfy your equation in Exercise 1?

Problems 17-2b

Find the values of a and b in each of the following equations so that the points marked are on the graph of the equation. For each, check your equation by finding the coordinates of the points and seeing whether they belong to the truth set of your equation.

1.



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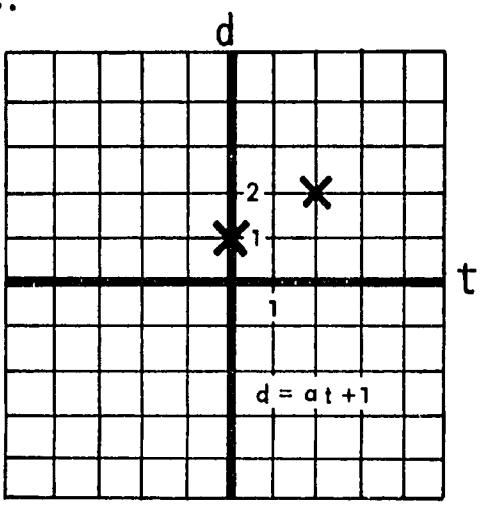

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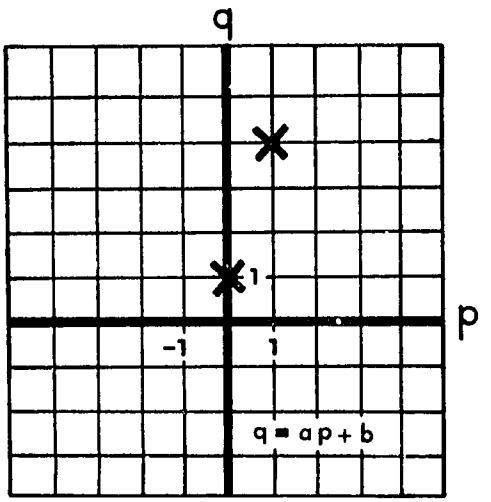

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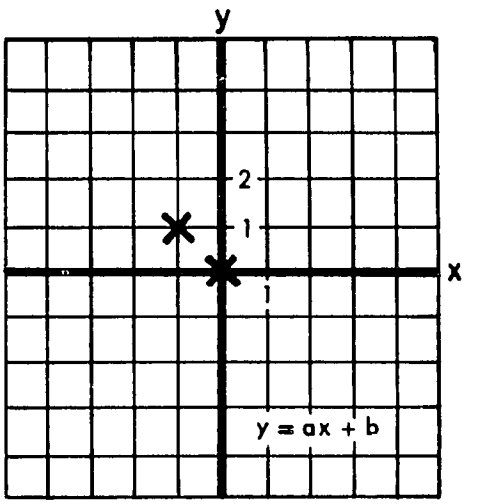

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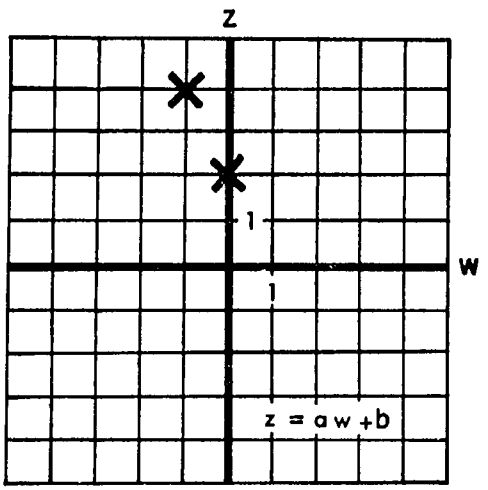

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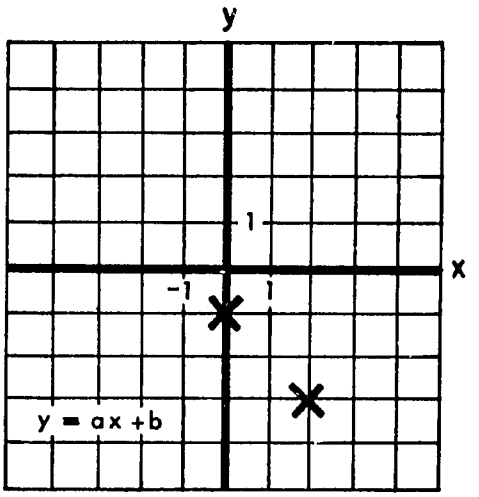

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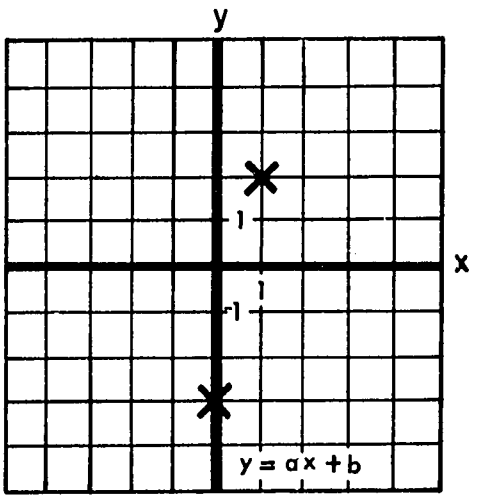

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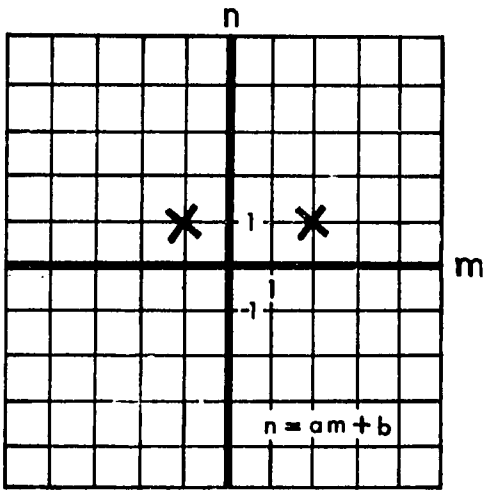


17-3 Guessing Rules

Class Activity

1. John has a rule that he is using on numbers. If you tell him "5" he uses his rule and says "26". If you tell him "3", he says "16". If you tell him "1", he says "6". If you tell him "2", he says "11". If we put this information in a table with the numbers you tell John in the first column, and the numbers John tells you after he has used his rule in the second column, it looks like this:

0	?
1	6
2	11
3	16
4	?
5	26
10	?

What would John say if you told him "0"? "4"? "10"?

Mary says that John's rule is "adding five", because when he used it on "1", he got "6". But Sonko says that "adding five" does not work for the other numbers, so John's rule must be something else. Can you guess John's rule?

2. Here is a table showing how Tunde's rule works for some numbers.

0	0
1	?
2	14
3	?
4	?
7	49
100	700

Try to fill in the question marks and guess Tunde's rule.

3. Here is a table showing how Edison's rule works for some numbers. Try to fill in the question marks and guess Edison's rule.

-1	$-\frac{1}{4}$
0	?
1	$\frac{1}{4}$
2	?
3	?
4	1
8	2
100	25

John says he thinks Edison's rule is "Divide by Four". Does this rule work? Edison says this is not the rule he was using. Can you guess what his rule is?

4. John says he has a short way of writing his rule using two variables, like this: $y = 5x + 1$. Which variable has as its domain all the numbers you might tell John? Which variable has as its domain all the numbers John would tell you after he has used his rule?

Tunde says he has written his rule using p and q. What do you think he wrote?

Edison is not sure how to write his rule using two variables. Can you do it for him?

Problems 17-3

Fill in the question marks and write a rule for each of the following:

1.

x	y
0	7
1	8
2	9
5	?
?	17

2.

w	z
-4	-8
0	?
5	1
?	3
8	4
9	5

3.

s	t
-1	?
0	?
1	2
2	5
3	8
?	14
6	17
7	?

4.

p	q
0	-1
1	?
20	9
100	49
101	?
1000	499

5. Sonko and Edward had nothing to do one afternoon because it was raining, so they decided to play a game of guessing rules. To make the game easier, they agreed that the rules they made up had to be: Multiply by some real number, and then add some real number. The winner was the person who could determine the other's rule from the <u>fewest</u> pairs of numbers. After a while Sonko was winning each time because he needed only to know how Edward's rule worked on two numbers, and he usually asked about 0 and 1. Here are four of the rules Sonko guessed correctly.

(a)

x	y
0	3
1	5

(b)

x	y
0	-1
1	0

(c)

x	y
0	0
1	$\frac{1}{3}$

(d)

x	y
0	5
1	9

What rules did Sonko guess for each of these? Can you write each one as an equation? Why did he ask how the rule worked on zero and one? (You may want to look at the tables and rules you have already guessed to answer this.)

6. Edward got tired of always losing (He had not yet figured out how to guess the rules knowing only how they worked on two numbers.), so he tried to make up a hard rule that Sonko would not be able to guess so easily. Sonko asked about 0 and 1, and the table looked like this:

x	y
0	1
1	2

Sonko said, "That's easy -- it's multiplying by one and adding one," and he wrote it like this: $y = 1x + 1$, but Edward said, "No, that is <u>not</u> my rule!" Sonko looked puzzled, and finally asked about some more numbers. The table looked like this before Sonko finally guessed the right rule:

x	y
0	1
1	2
2	5
8	65
10	101
12	145

Can you guess Edward's rule? Sonko finally decided that his method for guessing the rule knowing about only two numbers did not work this time because Edward's rule was not of the form they had agreed upon: Multiply by a number and then add a number.

7. For each of the rules in this section, decide whether or not its graph would be a straight line.

17-4 Equivalent Equations in Two Variables

You will recall that equations with the same truth set are called equivalent equations.

Look at this pair of equations:

$$y = \frac{1}{2}x - 1$$

$$2y - x + 2 = 0$$

Find three pairs of numbers that make the first equation true. Do they also make the second equation true? Let us try to prove that the two equations are equivalent.

$y = \frac{1}{2}x - 1$	
is equivalent to	If the number represented on each side of an equation is multiplied by the same non-zero number, the truth set remains unchanged.
$2y = 2(\frac{1}{2}x - 1)$	
which is equivalent to	The numeral on the right has been changed by the distributive property to another name for the same number.
$2y = x - 2$	
which is equivalent to	If the same number is added to (or subtracted from) the number represented on each side of an equation, the truth set remains unchanged.
$2y - x + 2 = x - 2 - x + 2$	
which is equivalent to	The numeral on the right has been changed by basic properties of addition and subtraction to another name for the same number.
$2y - x + 2 = 0$	

In practice, while we must think of the reasons why the equations are equivalent, we usually write more briefly as follows:

$$y = \frac{1}{2}x - 1$$

$$2y = 2(\frac{1}{2}x - 1) \qquad \text{Multiplied by 2.}$$

$$2y = x - 2$$

$$2y - x + 2 = x - 2 - x + 2 \qquad \text{Added 2 and subtracted } x.$$

$$2y - x + 2 = 0$$

At each step either we use one of the four rules for equivalent equations or we change the name of a number by using properties of real numbers.

Explain why the following equations are equivalent:

$$2x + 6y - 7 = 0$$

$$2x + 6y - 7 + 7 = 0 + 7$$

$$2x + 6y = 7$$

$$2x + 6y - 2x = 7 - 2x$$

$$6y = 7 - 2x$$

$$\frac{1}{6}(6y) = \frac{1}{6}(7 - 2x)$$

$$y = -\frac{1}{3}x + \frac{7}{6}$$

Problems 17-4

Which of the following are pairs of equivalent equations? If the equations of a pair are not equivalent, prove it by finding an ordered pair of numbers that belongs to the truth set of one, but does not belong to the truth set of the other. If the equations of a pair are equivalent, prove it by using the appropriate rules for equivalent equations and properties of real numbers.

1. $y = \frac{2}{3}x + 2$

   $3y - 2x - 6 = 0$

2. $4y - 5x + 1 = 0$

   $y = \frac{4}{5}x - 4$

3. $10x + 5y - 3 = 0$

   $y = -2x + \frac{3}{5}$

17-5 Graphs of Linear Equations in Two Variables

You have seen that the graphs of equations such as

$$y = 3x + 2$$
$$y = -\frac{1}{2}x + 8$$
$$y = x - \frac{9}{2}$$

are straight lines and you have observed some relations between the positions of those straight lines and the numbers in the equations.

What can we learn about the following equations?

$$5x + 2y - 3 = 0$$
$$7x - y = 0$$
$$2y - 3x + 4 = 0$$

Let us experiment with some.

Class Activity

1. Consider the equation $2y - 3x + 4 = 0$. Which of these ordered pairs belong to the truth set? (4, 4), (1, 2), (-2, -5), (0, -2), (-2, 0).

2. Find numbers for the question marks in the table so that each pair of numbers makes the equation $2y - 3x + 4 = 0$ true.

x	y
-2	?
-1	?
0	?
1	?
2	?
?	4

3. Plot the points represented in the table. Do they appear to lie on a straight line? How could we test other points of the line to see whether they belong to the graph of $2y - 3x + 4 = 0$?

4. Draw the line through the points you have plotted and from the line find numbers for the question marks.

(3, ?), (5, ?), (?, -8), (?, -11)

Do the resulting ordered pairs of numbers satisfy the equation? Since equations such as $2y - 3x + 4 = 0$ do have straight line graphs, they are often referred to as linear equations.

Problems 17-5

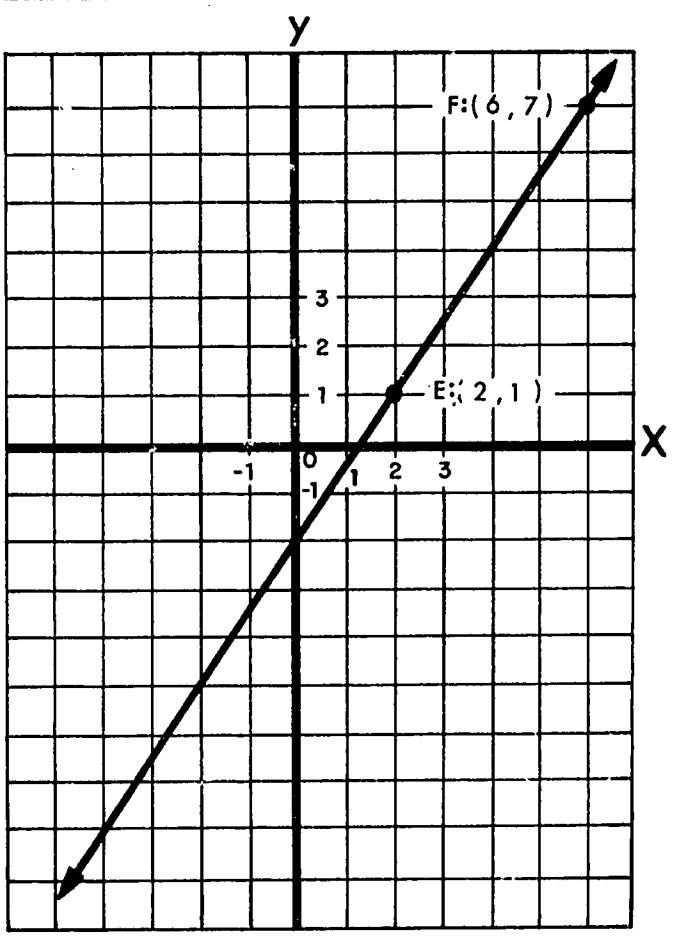
Graph each of the following equations by making a table of five ordered pairs of the truth set, plotting the points, and drawing the straight line through the points.

1. $x = y$
2. $x - 2y - 3 = 0$
3. $x - 3y = 0$
4. $x = 4$ (Suggestion: You may think of this as $x + 0y = 4$ if you wish.)
5. $y = 2$

17-6 The Slope of a Straight Line

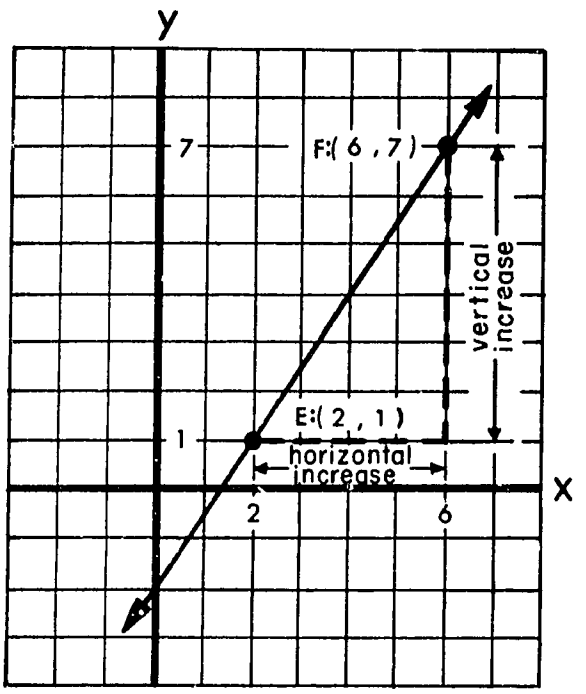
Class Activity

1. On your graph of $2y - 3x + 4 = 0$ that you made for the last section, label the point which corresponds to (2, 1) with E and (6, 7) with F as in the picture.



2. Label the following points on your graph: A: (-4, -8); B: (-2, -5); C: $(-1, -3\frac{1}{2})$; D: (0, -2).

3. Put your pencil point on E. Count over four units to the right, and up six units. Did you finish at F ? By counting over four units, you increased the x-coordinate by four; by counting up six units you increased the y-coordinate by six. This can be shown in a diagram.



What is the ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$?

The same result can be obtained by using arithmetic. In going from E: (2, 1) to F: (6, 7), the vertical (or y-coordinate) increase is: 7 - 1 = 6; the horizontal (or x-coordinate) increase is: 6 - 2 = 4. The ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$ is $\frac{6}{4}$, or $\frac{3}{2}$. From your graph, find the ratio $\frac{\text{vertical increase}}{\text{horizontal increase}}$, for the following pairs of points:

(a) D to E

(b) D to F

(c) C to D

(d) A to B

(e) A to D

(f) A to F

4. Put your pencil point on the point E. If you wanted to reach D, would you count up or down? How many? Would you count over to the left or right? How many? In going from E to D the vertical increase is <u>negative</u>; the horizontal increase is <u>negative</u>. What is the ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$?

5. What is the vertical increase in going from D to C ? What is the horizontal increase? What is the ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$?

6. The ratio $\frac{\text{vertical increase}}{\text{horizontal increase}}$ in going from E: (2, 1) to D: (0, -2) which you obtained in Exercise 4 can be found by arithmetic:

$$\text{vertical increase: } -2 - 1 = -3$$

$$\text{horizontal increase: } 0 - 2 = -2$$

$$\frac{\text{vertical increase}}{\text{horizontal increase}}: \frac{-3}{-2} = \frac{3}{2}$$

Show how you can obtain your answer to Exercise 5 by using arithmetic.

In your work with the equation $2y - 3x + 4 = 0$ and its graph, you found that the ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$ was the same for any two points considered.

The constant ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$ for any two points on a line is called the <u>slope</u> of the line.

How many points of a line do you need to find the slope of that line?

<u>Problems 17-6</u>

1. Draw the graph of each of the following equations. Choose two points on the line in your completed graph and determine the slope of the line.

(a) $2y + 3x + 4 = 0$ (c) $2y - 4x - 2 = 0$

(b) $2y - 4x + 3 = 0$ (d) $y = 2x + 1$

2. In Section 17-2 you graphed $y = x + 3$, $y = 2x + 3$, and $y = 3x + 3$ on the same set of axes. Look at these graphs you made and find the slope of each line. Do the same for the graphs you made for Problem 1 of Problems 17-2a.

3. Refer to Problems 17-2b and for each, find the slope of the line that contains the two points pictured.

4. For each problem in Problems 17-2b, show how the slope can be obtained by computation. For example, for Problem 1, the coordinates of the two points are (1, 3) and (2, 4). If we go from (1, 3) to (2, 4), we have:

$$\text{vertical increase: } 4 - 3 = 1$$
$$\text{horizontal increase: } 2 - 1 = 1$$

Therefore, the slope is $\frac{1}{1}$, or 1.

17-7 The Slope of a Line from its Equation

In Problem 1 of Problems 17-6 you may have noticed that the graphs of (c) $2y - 4x - 2 = 0$ and (d) $y = 2x + 1$ were the same. Can you prove that the two equations are equivalent by the method you used in Section 17-4 ?

You may also have noticed that the coefficient of the x-term in $y = 2x + 1$ is the same as the slope of the line. You should have found this true also of the equations referred to in Problem 2 of Problems 17-6. The coefficient of the x-term in $2y - 4x - 2 = 0$, however, is -4. Thus, in this form of the equation, the coefficient of the x-term is <u>not</u> the slope.

Problems 17-7a

1. In the graph of the equation $y = 5x + 3$, how much does y increase if the x-increase is 1? How much does y increase if the x-increase is 3? What is the slope of the line? What is the coefficient of x ?

2. Answer the same questions for $y = 7x + 9$.
3. Answer the same questions for $y = \frac{3}{2}x - 2$.
4. Answer the same questions for $y = x + 8$.
5. Answer the same questions for $y = -4x + 2$.
6. Answer the same questions for $y = 1000x + 4$.

We have mentioned that an equation whose graph is a straight line is called a linear equation. There can be several forms for the equation of a line. We have observed that a linear equation in the special form $y = ax + b$ tells us the slope of the line. For the equation $y = 5x + 3$ the slope of the graph is 5; for the equation $y = -4x + 2$ the slope is -4. We make the following generalization:

If a linear equation is in the form $y = ax + b$, then a is the slope of the line.

If a linear equation is <u>not</u> in the form $y = ax + b$, we can try to find an <u>equivalent equation</u> which is in that form. Thus, to find the slope of the graph of $3x + 4y - 5 = 0$, we could proceed as follows:

$$3x + 4y - 5 = 0$$

is equivalent to $\quad 4y = -3x + 5 \quad$ (Why?)

which is equivalent to $\quad y = -\frac{3}{4}x + \frac{5}{4} \quad$ (Why?)

What, then, is the slope of the graph?

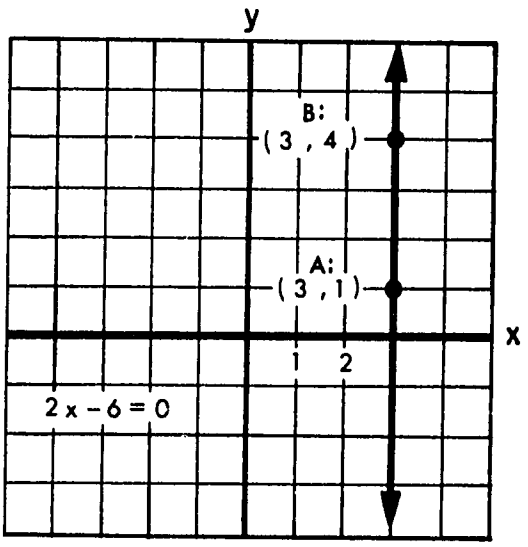
<u>Problems 17-7b</u>

For each of the following equations, obtain the slope, where possible, from an equivalent equation in the $y = ax + b$ form. Check 1 and 2 by drawing a graph and finding the ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$ for two points; check 3 and 4 by finding two ordered pairs in the truth set and computing the slope; use either method for 5.

1. $5y - 3x + 1 = 0$
2. $3y + 4x - 5 = 0$
3. $x + 3y - 5 = 0$
4. $3y - 7 = 0$
5. $2x - 6 = 0$

A Special Case

Were you able to find a slope of the line for $2x - 6 = 0$ in Problem 5? Examine the graph of $2x - 6 = 0$ to the right, and find the change in y and in x from A: (3, 1) to B: (3, 4).



You should find that the vertical increase is 3 and the horizontal increase is 0. When we consider the slope, that is, the ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$ we find that $\frac{3}{0}$ is <u>not</u> a real number. We obtain the same result by computing:

$$y\text{ - increase: } 4 - 1 = 3$$

$$x\text{ - increase: } 3 - 3 = 0$$

Since $\frac{3}{0}$ does not make sense, slope does not exist for this line.

If we attempt to find an equivalent equation in the form $y = ax + b$ for $2x - 6 = 0$, we find that we cannot. Even if we introduce a y-term into the equation obtaining

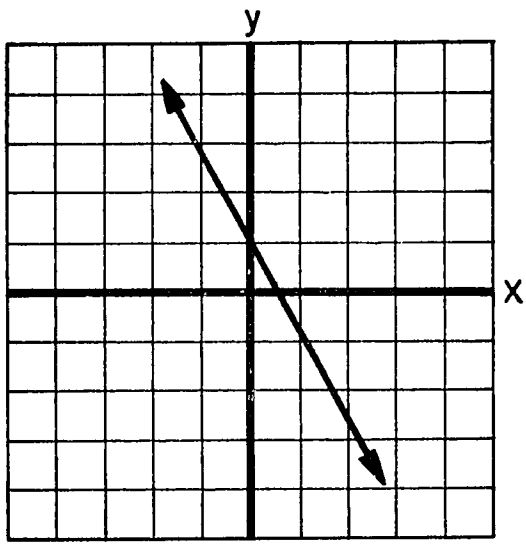
$$0y + 2x - 6 = 0$$

we cannot solve for y because we cannot divide by 0. The linear equation $2x - 6 = 0$ does not have a slope. Can you name another equation whose graph is a straight line that does not have a slope? We might have expected that the slope would never exist for a line in which the horizontal increase between any two points is 0. How can you describe all such lines?

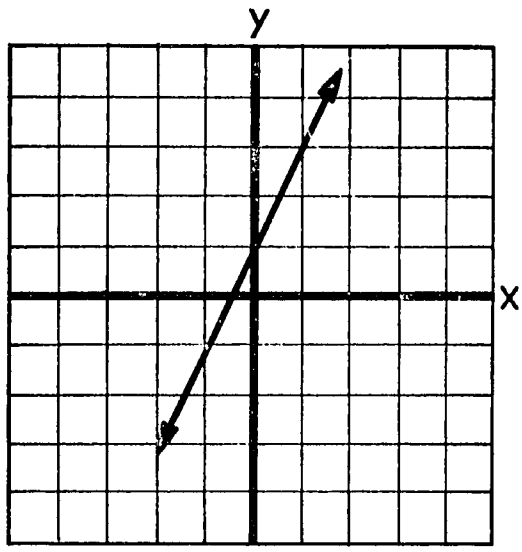
Problems 17-7c

For which of the following lines is the slope positive, for which is the slope negative, for which is the slope zero, for which does the slope not exist?

1.



2.



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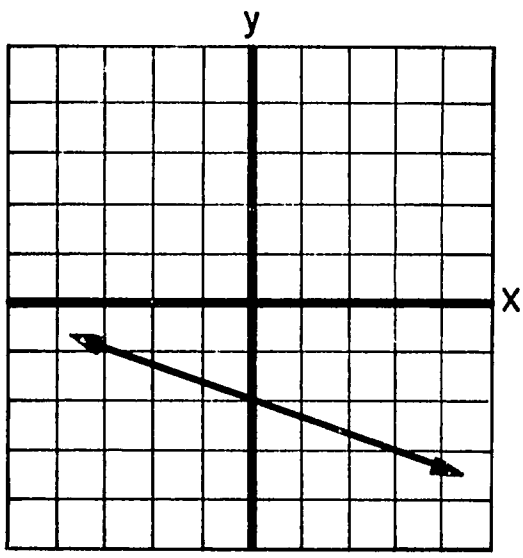

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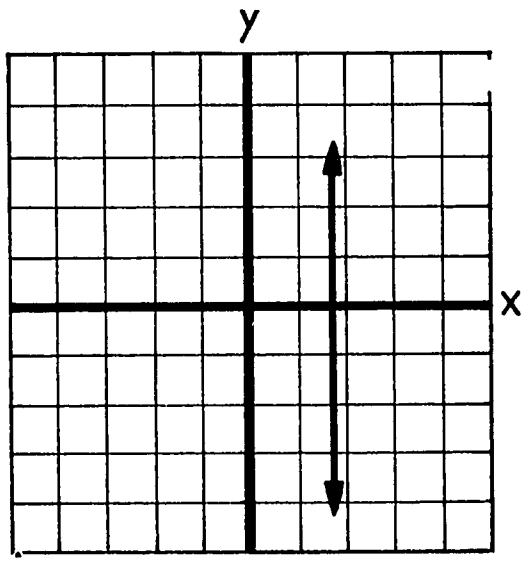

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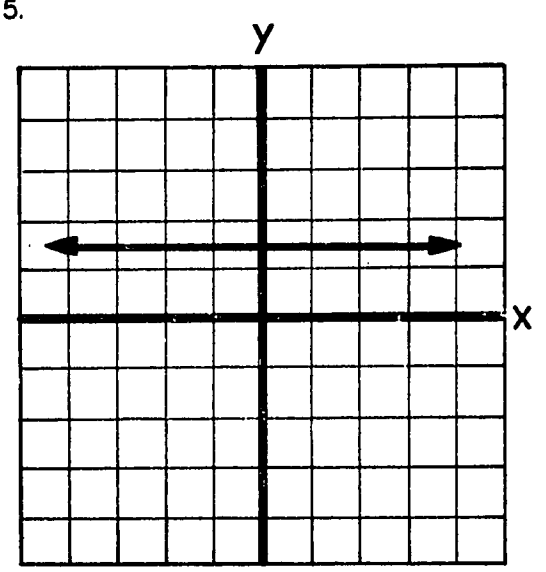
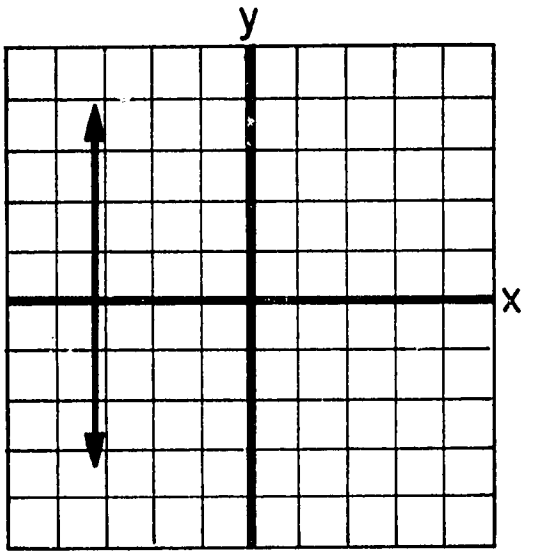
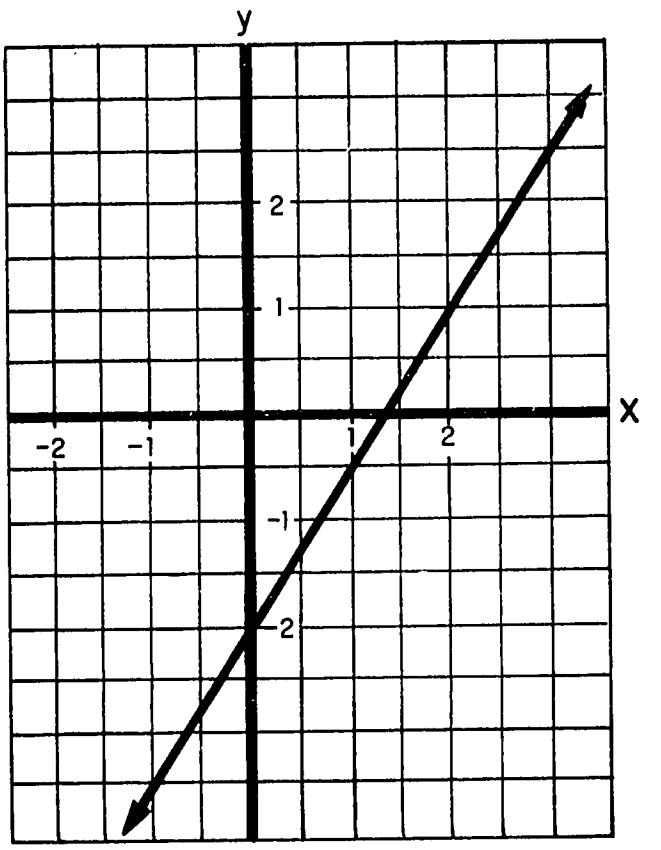


6.



17-8 The Intercept on the y-axis

To the right is an enlarged picture of a portion of the graph of $2y - 3x + 4 = 0$ which we considered in Section 17-5.



If x is 0, what is y ? What are the coordinates of the point where the line intersects the y-axis? Refer back to your graphs for Problem 2 of Problems 17-2a. For each line, find the coordinates of the point where the line intersects the y-axis. Compare the second coordinate of this point with the constant term in the equation of the line. Do the same for the problems in Problems 17-2b.

The second coordinate of the point where a line intersects the y-axis is called the <u>intercept</u> on the y-axis.

What is the first coordinate of the point where a line intersects the y-axis?

You may have noticed that if an equation is in the form $y = ax + b$, then b is the intercept of the line. The proof of this is simple. Since any point on the y-axis has 0 for its x-coordinate, the equation $y = ax + b$ becomes $y = a \times 0 + b$ for a point on the y-axis; therefore, y is b, and $(0, b)$ belongs to the truth set of $y = ax + b$. Since a nonvertical line can intersect the y-axis in at most one point, the intercept is b. For any equation $y = ax + b$, the intercept on the y-axis is b.

<u>Problems 17-8</u>

1. Graph each of the following and read the y-intercept from the graph; check by writing the equation in $y = ax + b$ form and reading the y-intercept from the equation.

   (a) $2y + 3x - 6 = 0$ (b) $3x - y - 3 = 0$

2. Graph each of the following and read the y-intercept n from the graph; check by seeing whether $(0, n)$ makes the equation true.

   (a) $4x - 3y - 9 = 0$ (b) $2y + 6x - 5 = 0$

3. Graph the equation $7y - 14x + 6 = 0$. When you try to read the intercept on the y-axis from the graph, you will find it difficult to read exactly. Estimate it as closely as you can from your graph and test your estimate by seeing whether your coordinates make the equation true. If the two sides of

the equation differ by very little, your estimate is close. You may then want to find the exact intercept by some other method which does not depend on the graph.

4. If we combine what we know about the slope and the intercept of a line, we obtain the following:

For any equation $y = ax + b$,
a is the slope of the line and
b is the intercept on the y-axis.

For the equations in (a) and (b), name the slope and y-intercept; for (c) and (d), find an equation which has the slope and y-intercept as given.

(a) $y = -\frac{2}{3}x + 5$

(b) $y = 100x - 3$

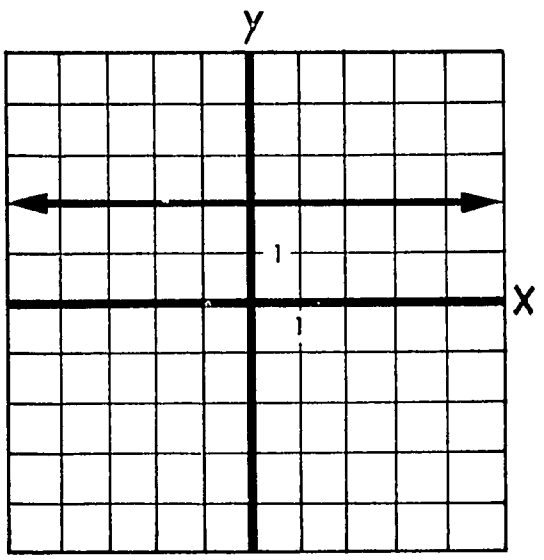
(c) Slope is 5; y-intercept is 4.

(d) Slope is 0; y-intercept is -2.

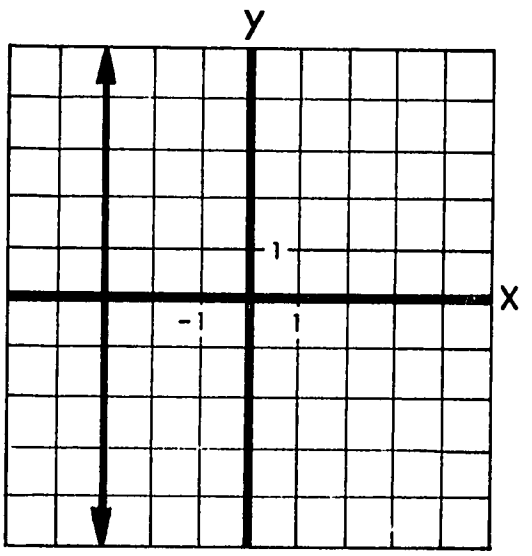
5. In the last section you found that not every linear equation has a slope. Can you find a linear equation that does not have an intercept on the y-axis?

6. For each of the following lines, determine the slope and y-intercept, and write an equation whose truth set is pictured.

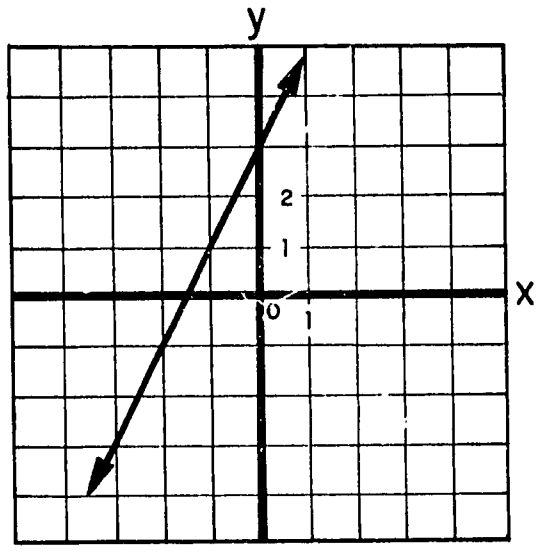
(a)



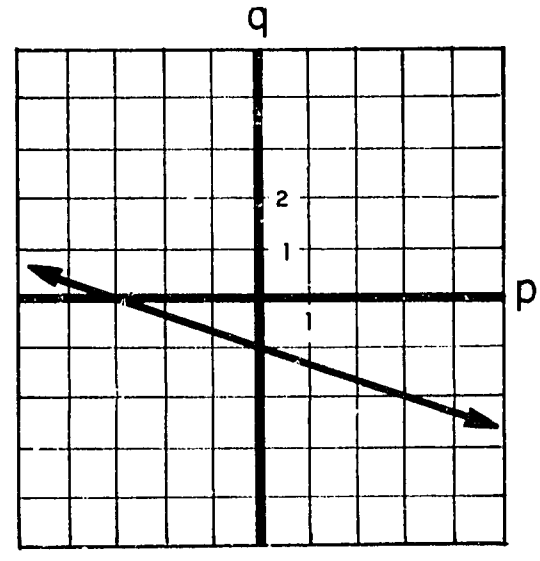
(b)



(c)



(d)



7. For each of the following, determine an equation, the graph of which contains the two points named.

(a) (0, 1) and (1, 7)

(b) (0, -3) and (3, -3)

(c) (0, 0) and (1, -2)

(d) (0, 4) and (2, 7)

8. Can you find the intercept on the y-axis for the equation $5y - 13x + 35 = 0$ without putting the equation in the form $y = ax + b$?

9. Find the intercept on the y-axis without doing any writing:

(a) $x + 3y = 12$

(b) $3x - y + 8 = 0$

(c) $7y - 8x - 4 = 0$

(d) $x + y + 15 = 97x$

(e) $\frac{3}{5}x + \frac{2}{3}y = 1$

(f) $2x + 3y = 8 + 5x + y$

10. <u>CHALLENGE PROBLEM</u>. What meaning might appropriately be given to "the intercept on the x-axis"? Find this number for each part of Problem 9.

11. <u>CHALLENGE PROBLEM</u>. Find an equation of the line which contains the points (1, 4) and (2, 7).

CHAPTER 18

USING STRAIGHT LINE GRAPHS

18-1 Systems of Simultaneous Equations

In Chapter 5 you solved the following pair of simultaneous equations:

$$\begin{cases} 2x + 4y = -12 \\ 5x + 4y = -33 \end{cases}$$

By subtracting, you obtained another equation, $-3x = 21$, and you then considered this pair of equations

$$\begin{cases} 2x + 4y = -12 \\ -3x = 21 \end{cases}$$

which is equivalent to the pair above.

Class Activity

1. On the same set of axes, graph the three equations $2x + 4y = -12$, $5x + 4y = -33$, and $-3x = 21$.

2. Read from the graph the coordinates of the point of intersection of the lines whose equations are $2x + 4y = -12$ and $5x + 4y = -33$.

3. Read from the graph the coordinates of the point of intersection of the lines whose equations are $2x + 4y = -12$ and $-3x = 21$.

4. You should have found that the graphs of the two pairs of equivalent equations intersect at the same point. Do you think this will occur for the graphs of any two pairs of equivalent equations?

5. Without doing any more algebra do you know the solution of

$$\begin{cases} 2x + 4y = -12 \\ 5x + 4y = -33 \end{cases} ?$$

Previous Page Blank

Problems 18-1a

For each of the following pairs of equations, obtain the solution by graphing; for problems 1 and 2, check by seeing whether your solution makes both equations true; for problem 3, check by finding the solution using equivalent systems of equations as you did in Chapter 5.

1. $\begin{cases} 2x + 3y = 5 \\ x + y = 1 \end{cases}$

2. $\begin{cases} 3x + 5y = 11 \\ y = \frac{1}{7}x - 3 \end{cases}$

3. $\begin{cases} 7x + 4y = 11 \\ 14x + 3y = 12 \end{cases}$

Suppose we wish to find the solution for this system of equations:

$$\begin{cases} 3x - y + 1 = 0 \\ 2y - 6x - 7 = 0 \end{cases}$$

We might proceed by using algebra to find equivalent pairs of equations. Give a reason why each of the following pairs is equivalent to the pair above.

$$\begin{cases} 3x - y + 1 = 0 \\ -6x + 2y - 7 = 0 \end{cases}$$

$$\begin{cases} 6x - 2y + 2 = 0 \\ -6x + 2y - 7 = 0 \end{cases}$$

If we add the last pair of equations, we obtain: $0 + 0 - 5 = 0$, or $-5 = 0$. Our attempt to find the solution of this system of equations algebraically resulted in a sentence which is false.

Class Activity

Graph the pair of equations on the same set of axes and see whether you can explain why the attempt to solve them algebraically resulted in a

false statement. What is the truth set for this system of simultaneous equations?

Let us examine another system of equations:

$$\begin{cases} y = \frac{1}{3}x - 2 \\ 6y - 2x + 12 = 0. \end{cases}$$

Is (3, -1) a solution?

Is (0, -2) a solution?

Is (9, 1) a solution?

The graph of each of the above equations is a straight line. Can two lines intersect in more than one point?

Class Activity

1. Graph the pair of equations above and see whether you can explain why there is more than one solution for this pair of equations.

2. Try to solve the same pair of equations algebraically. What happens?

3. What is the truth set of this system of equations?

Problems 18-1b

Using any method you wish, find the truth set for each of the following:

1. $\begin{cases} 3x + 4y - 11 = 0 \\ 2x + 3y = 9 \end{cases}$

2. $\begin{cases} 2x - y + 8 = 0 \\ 4x - 2y + 17 = 0 \end{cases}$

3. $\begin{cases} 2x + 8y = 19 \\ x - 16y + 3 = 0 \end{cases}$

4. $\begin{cases} \frac{1}{3}x - 5y + 2 = 0 \\ 30y - 2x = 12 \end{cases}$

18-2 Proportionality

You may recall that we say <u>y is proportional to x</u> if $y = kx$, and k is any non-zero real number called the <u>constant of proportionality</u>. Let us examine the tables of some of the rules you met in Chapter 17.

(a)

x	y
0	7
1	8
2	9
10	17

(b)

a	b
0	0
2	14
7	49
100	700

(c)

x	y
-1	$-\frac{1}{4}$
0	0
2	$\frac{1}{2}$
4	1
8	2

(d)

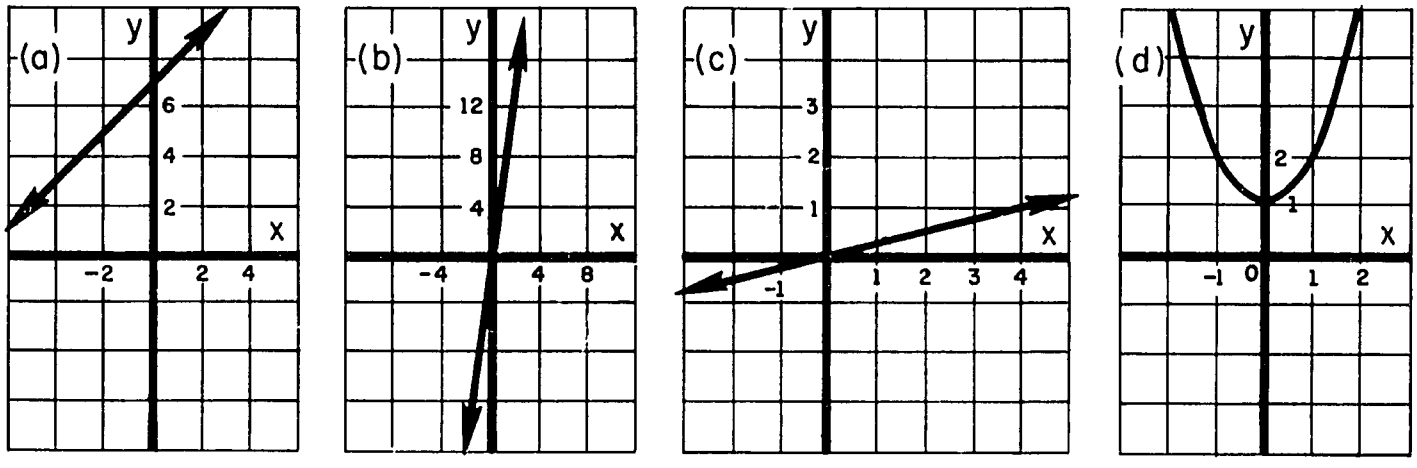
x	y
0	1
1	2
2	5
8	65
10	101
12	145

1. For which of these rules is the second variable proportional to the first variable?

2. For those that are proportional, what is the constant of proportionality?

3. Write an equation for each of the rules (a) - (d). Describe how to tell if y is proportional to x by examining the equation.

4. Here are graphs of the four rules.



From examining the equations you should have been able to know that (a) and (d) do not have proportionality, while (b) and (c) do. Can you describe how to tell whether two variables are proportional from examining the graph?

5. Let us examine the equation $y = kx$ again. Is the equation linear, that is, can it be put in the form of $y = ax + b$? What is the slope of $y = kx$? What is its intercept on the y-axis? What point must the graph of $y = kx$ contain, for <u>any</u> value of k?

Your answers to the previous questions should lead you to the following conclusion:

> y is proportional to x if there is a linear equation relating y and x, the graph of which goes through the origin and has a non-zero slope. The constant of proportionality is the slope of the line.

<u>Problems 18-2</u>

1. If y is proportional to x with a constant of proportionality k, is x proportional to y? If your answer is "yes", then what is the constant of proportionality?

2. For which of the following is y proportional to x ? What is the constant of proportionality?

(a) $y = 3(x + 2)$

(b) $7x - 2y = 0$

(c) $y = 0$

(d) $y = 2x^2$

3. For each of the following, assume that a rule can be written which is a linear equation. Which relations show that y is proportional to x ?

(a)

x	y
-1	0
5	2

(b)

x	y
0	0
1	100

(c)

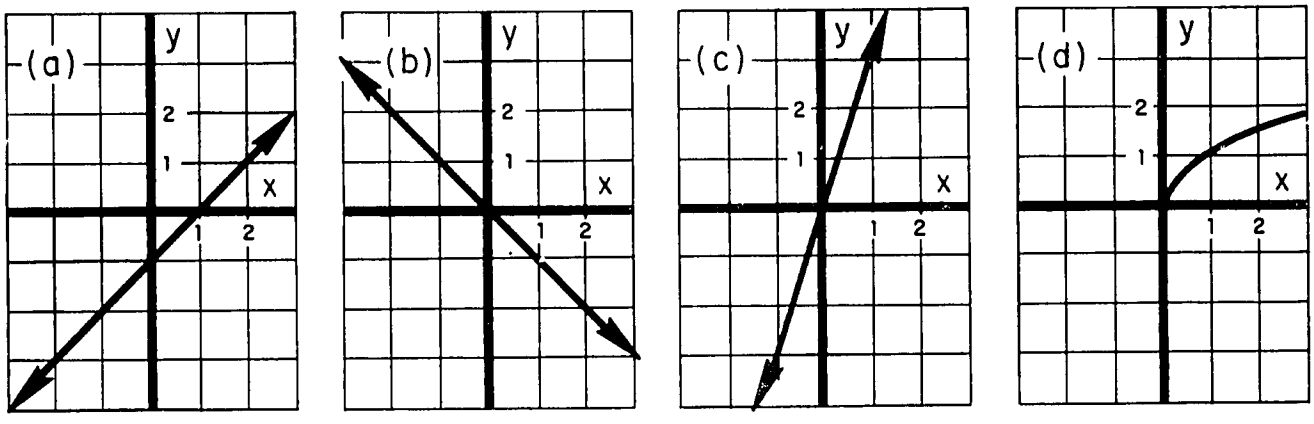
x	y
$\frac{1}{2}$	0
0	3

(d)

x	y
0	0
-2	4

For those that show proportionality, can you tell what the constant of proportionality is?

4. For which of the following is y proportional to x ? What, if any, is the constant of proportionality?



5. (a) A class of 45 children took an examination which had 35 questions. The teacher scored each paper and wrote the number of correct answers at the top of the paper. Then he gave the papers to Sonko, and asked him to figure out what grade out of 100 each paper should have if the allowable grades were numerals ending in either 0 or 5. For example, if Sonko had a grade of 71 out of 100 for a paper, then he was to round it off to 70. The

first paper Sonko looked at had 28 out of 35 correct, so he set up the following proportion: $\frac{28}{35} = \frac{x}{100}$. What value should Sonko get for x? What grade should he put on the paper?

"That was easy," thought Sonko. The next paper had a score of 23. So Sonko set up this proportion: $\frac{23}{35} = \frac{x}{100}$. What value should he get for x? What grade should he put on the paper?

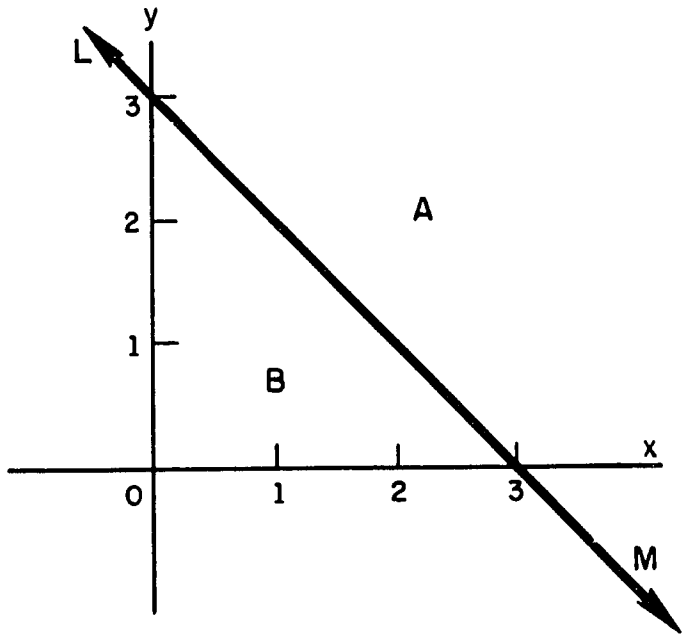
(b) Sonko decided at this point that this was too much work - it would take all day to do 43 more papers. After thinking for a minute, he decided that the grade out of 100 <u>before</u> rounding off was proportional to the number correct out of 35, so he could save time by finding the constant of proportionality. Since 100 had to be $k \times 35$, Sonko found that k must be $\frac{100}{35}$, or $\frac{20}{7}$. The next paper had a score of 14, so Sonko wrote: $y = \frac{20}{7} \times 14$. What value should he obtain for y? What grade should he put on the paper? The next paper had a score of 18. What value for y should Sonko obtain? What grade should he put on the paper?

(c) Sonko still had 41 papers to go when he began to think, "Since the exact number out of 100 is not needed, maybe I could use a graph... Let me see, I need 35 units on the x-axis and 100 units on the y-axis." So he drew the axes on a large piece of graph paper and marked the scales. Then he thought, "(0, 0) and (35, 100) have to be points on the graph," so he marked them, drew the line, and finished putting grades on the remaining 41 papers in 7 minutes. He even had time to check the grades for the first four papers. Use Sonko's last method to find the grades for the next 10 papers with scores of 10, 34, 20, 17, 30, 26, 4, 12, 31, and 24.

18-3 A Practical Application

1. Graphs of Inequalities

Look at the graph shown below of the equation, $x + y = 3$. Let us call the line $\overleftrightarrow{LM}$.



The line divides the rest of the plane into two half-planes. Let us try to find the open sentences whose truth sets give us these half-planes.

Name the coordinates of some points in the half-plane A.

Find the values of $x + y$ at these points.

Compare these values with the values of $x + y$ at points on the line $\overleftrightarrow{LM}$.

What do you notice?

The point (0, 3) lies on the line $\overleftrightarrow{LM}$. What can you say about y when the point $(0, y)$ lies in the half-plane A ? What can you say about the value of $x + y$ at this point. Is it greater or less than 3?

Can you find any point in the half-plane A at which the value of $x + y$ is not greater than 3?

You can now see that any member of the truth set of

$$x + y > 3,$$

gives a point in the half-plane A, and the coordinates of any point in the half-plane A is a member of the truth set of $x + y > 3$.

Make similar statements about points in the half-plane B.

2. An Application

Ideas like those above can be used in the solution of many problems. Let us look at an example:

A teacher has 60/- to spend on text books for his class.

At Nazarali's they cost 5/- each.

At Shariff's they cost 6/- each.

But Nazarali has only 6 of the books. How must the teacher buy the books in order to obtain the greatest number?

This is an easy problem and you will probably be able to give the answer using only your common sense. But let us see how we could solve the problem using graphical methods and then you will be able to use this method to solve harder problems.

Let x be the number of books bought from Nazarali.

Let y be the number of books bought from Shariff.

The cost of the books is then $5x + 6y$ shillings and so $5x + 6y \leq 60$.

Since Nazarali has only 6 books $x \leq 6$.

We know that $x \geq 0$ and $y \geq 0$.

We shall therefore need to draw the lines.

$5x + 6y = 60$, $x = 6$, $x = 0$, $y = 0$ on the same axes.

When we graph an inequality such as $5x + 6y \leq 60$, it is customary to shade the half-plane whose coordinates satisfy $5x + 6y \leq 60$. In this work however this sometimes gives a rather untidy diagram. We shall instead, shade the region for which $5x + 6y > 60$, and then the clear part of the plane is the part in which we are interested.

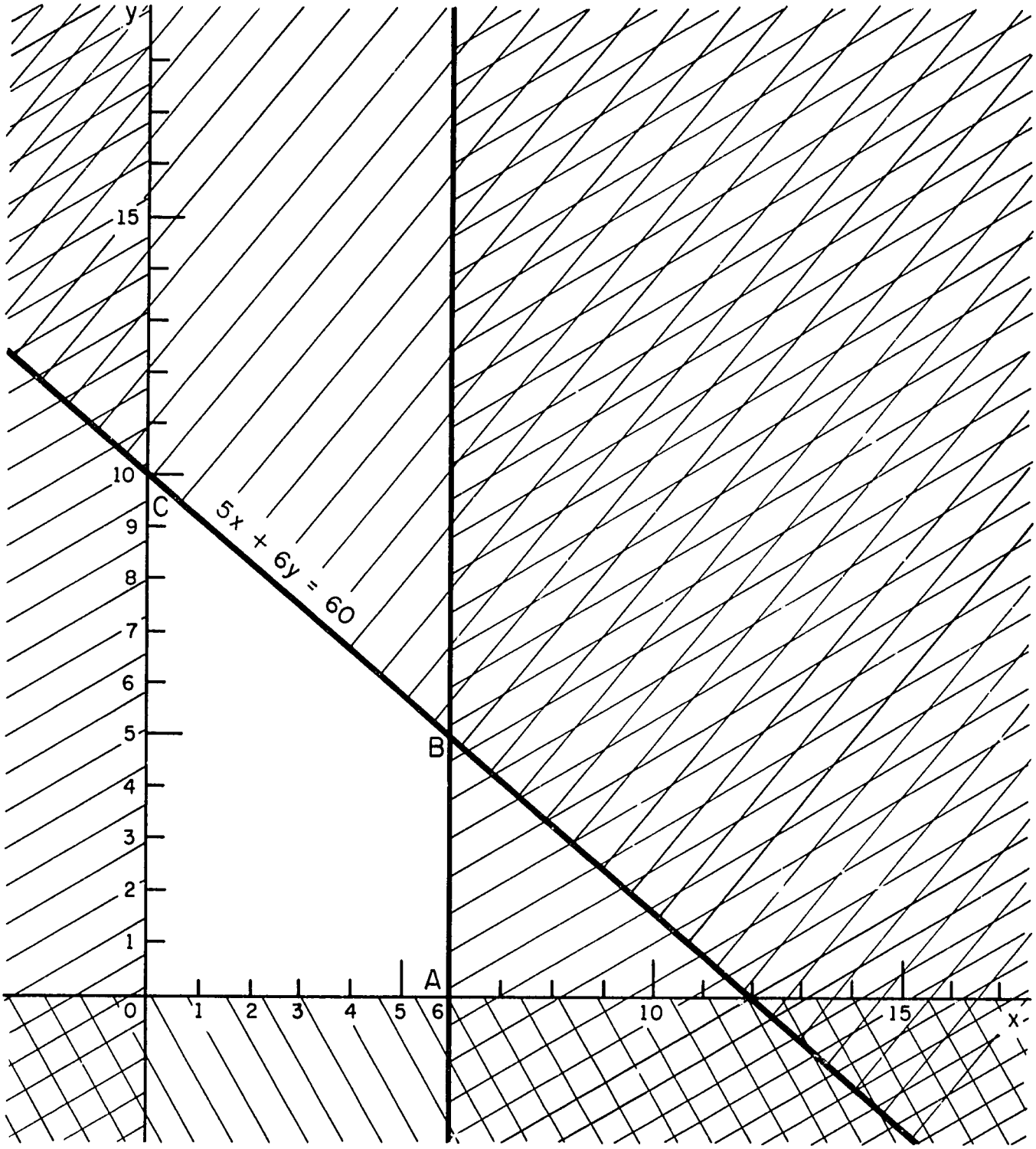


Fig. 1

The truth set of $5x + 6y \leq 60$ gives us the set of points in the region below the line $5x + 6y = 60$, so we shade the region above the line (a coloured pencil might make it more clear). Similarly, we shade the regions:-

to the right of $x = 6$

to the left of $x = 0$

below $y = 0$.

We are now left with an unshaded region within the quadrilateral OABC as seen in Fig. 1. The coordinates of points in this region are members of the truth set of the system of simultaneous inequalities.

$$\left.\begin{array}{r} 5x + 6y \leq 60 \\ 0 \leq x \leq 6 \\ 0 \leq y \end{array}\right\} \quad \text{---------} \ (1)$$

We now want the member of this set with whole number coordinates (we cannot buy $7\frac{1}{2}$ books) which makes $x + y$ as large as possible.

The lines containing the truth sets for $x + y = 5$, 10, and 15 are drawn on the graph. (See Fig. 2) It is clear that there are members of the truth set of (1) above for which $x + y = 5$ and for which $x + y = 10$ but not for which $x + y = 15$.

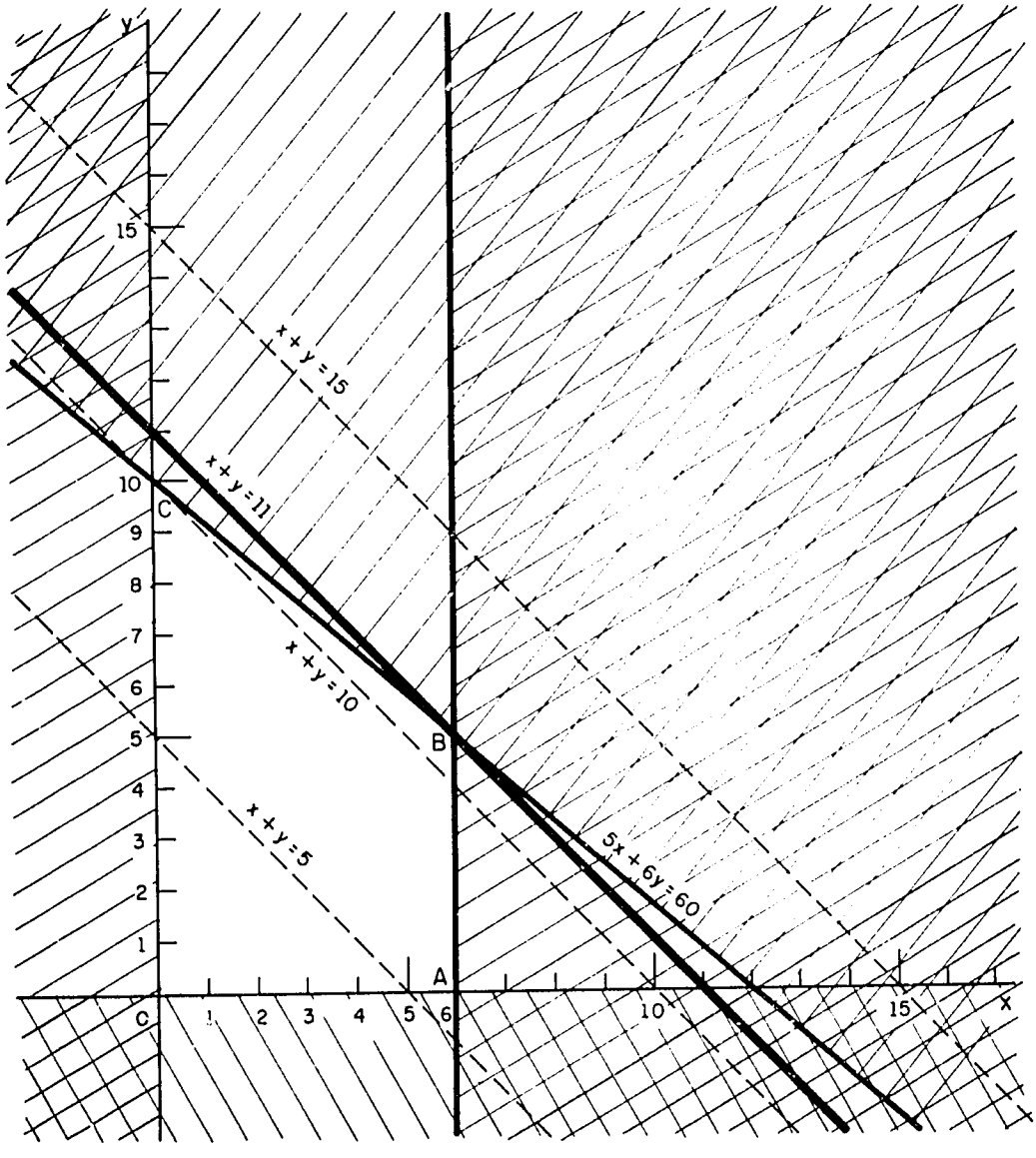


Fig. 2.

The lines $x + y = 5$, $x + y = 10$, and $x + y = 15$ are members of a set of parallel lines. There are other members of the set along which the value of $x + y$ is different from 5, 10, and 15. Which member of the set of parallel lines cuts the unshaded region, and has the largest value of $x + y$?

What is this value of $x + y$?

What are the coordinates of the point where it cuts the quadrilateral OABC ?

You may find it easier to use your ruler and set square to answer these questions.

Where, then, does the teacher buy his books?

What would the solution be if Nazarali had

(1) 9 books?

(2) 5 books?

(3) 4 books?

3. <u>Another Application</u>

Here is a longer example. This is a kind of problem which a manufacturer might meet.

A manufacturer makes 2 products A and B. For every ton of A he makes, he can make from 1 to 2 tons of B.

Not more than 10 tons of A can be made in 1 day.

The product A can be sold at a town 30 miles away at a profit of £ 2 per ton.

The product B can be sold at a town 20 miles away at a profit of £ 1 per ton.

But there is only enough transport for 600 ton-miles per day. (This means that in 1 day 600 tons can be carried 1 mile or 30 tons carried 20 miles or 10 tons carried 60 miles, etc.)

How many tons of each product does the manufacturer make each day to obtain the greatest profit?

If the manufacturer makes x tons per day of A and y tons per day of B, we have

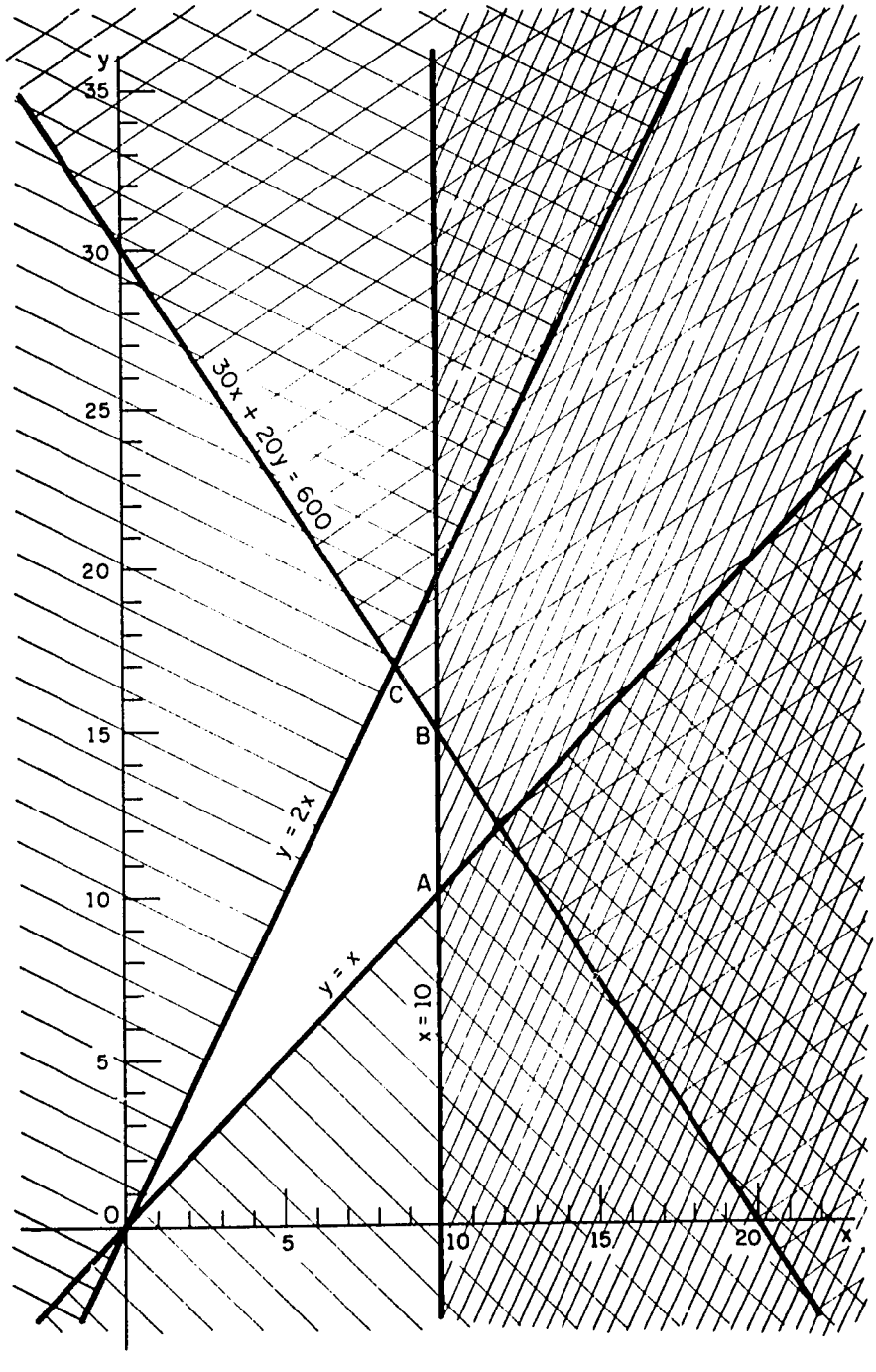
$y \leq 2x$	Why?
$y \geq x$	Why?
$x \leq 10$	Why?

and, from looking at the transport information, we have

$$30x + 20y \leq 600.$$

How do we get this?

Therefore we need to draw the graphs of $y = x$, $y = 2x$, $x = 10$, and $30x + 20y = 600$ on the same axes.



Having done this we now have the unshaded region OABC. This is the set of points whose coordinates satisfy the system

$$\left.\begin{array}{r} y \geq x \\ y \leq 2x \\ x \leq 10 \\ 30x + 20y \leq 600 \end{array}\right\}$$

We want the point in this quadrilateral which makes the profit as large as possible. The profit is £ $(2x + y)$. Why?

We must therefore find the member of the set of parallel lines given by $2x + y = d$, which cuts the quadrilateral OABC and for which d is as large as possible.

Use your ruler and set square to find this line.

Where does it cut the quadrilateral?

How many tons of A and B should the manufacturer make per day?

What would the solution be if the profit on both A and B were £2 per ton?

<u>Problems 18-3</u>

Here are more examples you can try. The first is not a word problem:

1. Construct the polygon which contains the points whose coordinates are in the truth set of the simultaneous inequalities:

$$\left.\begin{array}{r} y \leq 2x \\ x \leq 6 \\ y \geq 2 \\ 2x + 3y \leq 30 \end{array}\right\}$$

Find the members of this truth set which make:

(a) $y - x$ a minimum

(b) $y - x$ a maximum

(c) $5y + 6x$ a minimum

(d) $4x + 6y$ a maximum

2. A manufacturer makes two compounds "X" and "Y" which contain

the following quantities of Vitamins A and B:

	"X"	"Y"
Vitamin A	5 units per oz.	5 units per oz.
Vitamin B	15 units per oz.	5 units per oz.

"X" costs 2/- per oz. and "Y" costs 1/- per oz. The manufacturer wants a mixture of "X" and "Y" to contain at least 50 units of Vitamin A and at least 60 units of Vitamin B. How many ounces of each does he use to make the mixture as cheaply as possible? (Hint: Let the manufacturer mix x oz. of "X" with y oz. of "Y".)

3. A manufacturer has two warehouses in Kenya. One is in Nairobi and contains 25 units of his product. The other is in Nakuru and contains 20 units of his product. He has to supply a shop in Nanyuki with 20 units and a shop in Thompson Falls with 15 units. The costs of transport for each unit are as follows:

From \ To	Thompson Falls	Nanyuki
Nairobi	36/-	45/-
Nakuru	30/-	42/-

How does the manufacturer send his products as cheaply as possible? (Hint: If x units are sent from Nakuru to Thompson Falls, how many are sent from Nairobi to Thompson Falls? If y units are sent from Nakuru to Nanyuki, how many are sent from Nairobi to Nanyuki?)

4. A builder has 2 stores at S_1 and S_2. He is building houses at 3 places, P_1, P_2, P_3. He needs 5 tons of bricks at P_1, 6 tons at P_2, and 4 tons at P_3. The stores contain 9 tons at S_1 and 6 tons at S_2.

The transport costs per ton are as follows:

From \ To	P_1	P_2	P_3
S_1	6/-	3/-	4/-
S_2	4/-	2/-	6/-

How does the builder send the bricks at minimum cost? (Hint: If x tons are sent from S_1 to P_1 and y tons sent from S_1 to P_2, complete the following table:

From \ To	P_1	P_2	P_3
S_1	x	y	
S_2	5 - x		

5. A mining company owns 2 small copper mines, each of which produces 3 grades of copper ore, grade 1, grade 2, and grade 3. The first mine at A costs £ 40 per day to run and produces 3 tons of grade 1, 1 ton of grade 2, and 2 tons of grade 3 ore per day. The second mine costs £ 30 per day to run and produces 1 ton of grade 1, 1 ton of grade 2, and 6 tons of grade 3 ore per day.

The company has orders for 6 tons of grade 1, 4 tons of grade 2, and 12 tons of grade 3 ore per week.

For how many days each week should each mine work, to fill the order as cheaply as possible?

Hint: Arrange the information in a table like that shown below:-

Mine \ Ore	Grade 1	Grade 2	Grade 3	Cost per day
A				
B				
Orders for				

CHAPTER 19

OTHER GRAPHS

19-1 Non-linear Graphs

Thus far we have been concerned with the straight line graphs of equations of the form $y = ax + b$, or $ax + by + c = 0$. Let us investigate the graphs of some equations which are not of these forms.

Class Activity

The graph of $y = |x|$

Recall that the absolute value of x is the same as x when x is positive or zero, and that it is the opposite of x when x is negative. One way to get an idea of what the graph will look like is to find some ordered pairs of numbers that belong to the truth set, plot the points, and see whether there is a pattern.

1. Find the values of y which would complete this table.

x	0	1	2	3	4	5
y						

2. Plot the points that correspond to these ordered pairs.

3. Thus far you should find that the graph is the same as for $y = x$. We have not yet considered any negative values for x. Find the values of y which would complete this table, and then plot the points.

x	-1	-2	-3	-4	-5
y					

4. This part of the graph is not the same as $y = x$, but it is the same as part of the graph of a straight line. Can you name the equation for it?

5. Complete the graph of $y = |x|$.

6. For what values of x is the graph of $y = |x|$ the same as for $y = x$? For what values of x is the graph the same as for $y = -x$?

7. By recalling the definition of $|x|$, could you have predicted what the graph of $y = |x|$ would be <u>without</u> having graphed it?

8. Describe the graph of each of the following open sentences in terms of the graph of $y = |x|$:

(a) $y < |x|$ (c) $y > |x|$

(b) $y \leq |x|$ (d) $y \geq |x|$

<u>The graph of $y = x^2$</u>

Again, we can begin by finding some ordered pairs in the truth set, plotting the points, and seeing whether there is a pattern.

1. For $y = x^2$, find the values of y which would complete this table.

x	-2	-1	0	1	2	3	4	5
y								

2. Plot the points on graph paper or on the blackboard. Use the same scale on both axes; unless you use small units, you may find that there are several points that do not fit on your graph.

3. You should have found that the points you plotted do <u>not</u> lie on a straight line. Decide how you think the graph should be completed, and then discuss your decision with the rest of the class. Graphs such as the one you have constructed are called <u>parabolas</u>.

<u>Problems 19-1</u>

Sketch the graph of each open sentence. Let the x-axis be horizontal, the y-axis vertical, and use the same scales on both.

1. Sketch these on the same set of axes.

(a) $y = x^2$

(b) $y = 2x^2$

(c) $y = -2x^2$

(d) $y = 10x^2$

(e) $y = \frac{1}{2}x^2$

2. Sketch these on the same set of axes.

(a) $y = x^2 + 3$

(b) $y = x^2 + 0$

(c) $y = x^2 - 4$

3. $x = y^2$

4. Use a separate set of axes for each of the following:

(a) $x + y = 10$; $x + y \leq 10$; $x + y \geq 10$

(b) $|x| + y = 10$; $|x| + y < 10$

(c) $x + |y| = 10$; $x + |y| \geq 10$

(d) $|x| + |y| = 10$; $|x| + |y| \leq 10$

(e) $|x + y| = 10$; $|x + y| > 10$

5. $x + y = y + x$

6. $xy = 12$

7. (a) $y = x^3$

(b) $|x|^3$

8. <u>CHALLENGE PROBLEM</u>

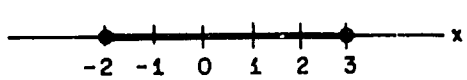
(a) Graph $x \leq 4$ on the number-<u>line</u>.

(b) Graph $x \leq 4$ on the number-<u>plane</u>.

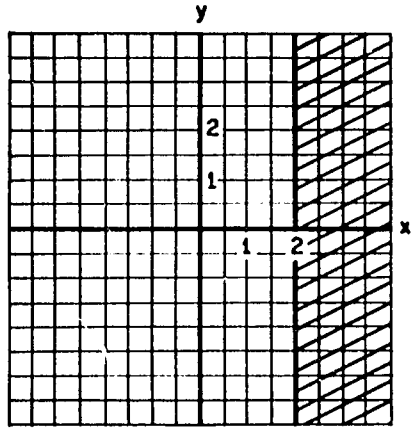
(c) What do you think the graph of $x \leq 4$ in the number-<u>space</u> would be?

9. Consider each of the following as a graph of a truth set, and write an open sentence for each. (The shaded portions indicate the points of the graph. A solid boundary of a region belongs to the graph; a dashed boundary does not.)

(a)



(b)



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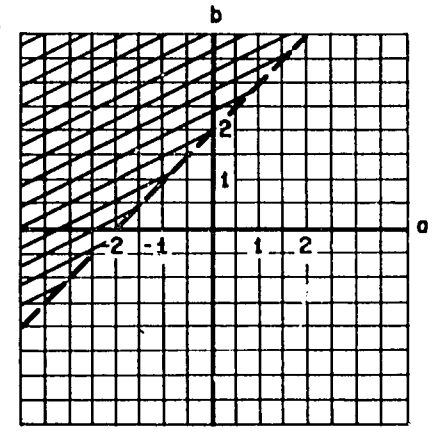
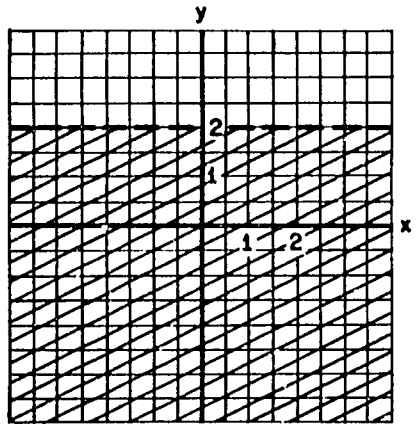
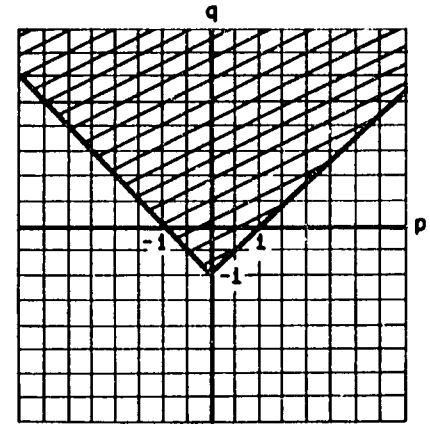


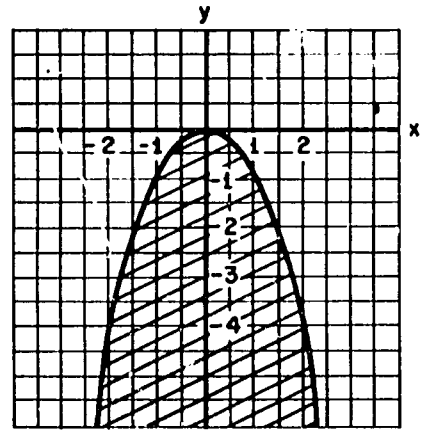
(c)



(e)



(f)



10. CHALLENGE PROBLEM

In a preceding chapter symmetry was discussed in relation to geometric figures. Look at your graphs for Problems 1-7 and see if you can answer the following questions:

(a) Which graphs have the x-axis as a line of symmetry?

(b) Which graphs have the y-axis as a line of symmetry?

(c) Which graphs have other lines of symmetry?

(d) Where you can, give the equation of each line of symmetry.

19-2 The Graph of $y = x^2$ and Tables of Squares and Square Roots

Section 19-1 dealt with the graphs of a variety of open sentences. This section deals in more detail with one of these, the graph of $y = x^2$, and its use in producing tables of squares and square roots.

You are already aware of the general shape of the graph of $y = x^2$ from working the exercise in section 19-1. You are aware also of its symmetry about the y-axis and the fact that it passes through the origin and exists only for non-negative values of y.

Class Activity

Let us draw the graph of $y = x^2$ for values of x from -5 to 5. When you draw your graph, take great care for you will need to take readings from the graph with considerable accuracy.

1. If $y = x^2$, find values of y for this table.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y											

What do you notice about the values of y for those values of x which are opposites of each other?

2. Decide on the position of each axis and the scales to be used so as to make the greatest use of your graph paper. Then draw the axes and plot the ordered pairs of numbers you have calculated.

3. Do the points you have plotted suggest a line or a curve? Would it be sensible to join the points by line segments? Additional points will be needed for a more accurate graph. Calculate the y-values when x is 0·5, 1·5, 2·5, 3·5, and 4·5, plot the points, and use symmetry to plot the points when x is -4·5, -3·5, ..., -0·5. Complete the graph by drawing a smooth curve through the points you have plotted.

A Table of Squares

One way of producing a table of squares is to use the multiplication process throughout. (You did this in obtaining the ordered pairs used for graphing $y = x^2$.) Another way of producing a table of squares is to use the graph of $y = x^2$ itself. Look at your graph of $y = x^2$ for values of x from -5 to 5 and see if you can find the value of $(3 \cdot 2)^2$. The exact square of 3·2 is 10·24. Were you able to find this answer from the graph? Again, see if you can find the value of $(1 \cdot 2)^2$ and $(3 \cdot 7)^2$ from the graph. Check by arithmetic to see how accurate your graph readings are. How could greater accuracy be obtained?

Class Activity

Draw a large graph of $y = x^2$ for values of x from 0 to 1. (Calculate at intervals of 0·1.) Use the graph to prepare a table of values of x^2 when x is 0·00, 0·05, 0·10, 0·15, 0·20, ..., 0·95, 1·00, estimating the values of x^2 to 3 decimal places.

A Printed Table of Squares

In practice we do not have to refer to a table of squares which we have made ourselves. Tables can be obtained giving an accuracy far greater than any we are likely to achieve by the graphical method. Compare the tables of squares you have prepared with the one in the back of this book.

At first glance it would appear that the table of squares in the book gives only the squares of numbers between 1 and 99. Decimal points are absent. The reason for this is simply explained.

When numbers such as 3·2, 32, 0·32, and 320 are squared, the numerals for the answers are all the same if one ignores the position of the decimal point.

$$\begin{aligned} 3{\cdot}2^2 &= 10{\cdot}24 \\ 32^2 &= 1024{\cdot} \\ 0{\cdot}32^2 &= 0{\cdot}1024 \\ 320^2 &= 102{,}400{\cdot} \end{aligned}$$

The printed table makes use of this property and leaves the placement of the decimal point to the reader.

Example 1. Find the square of 780.

We first note that

$$780 = 78 \times 10$$

Then

$$\begin{aligned} (780)^2 &= (78 \times 10)^2 \\ &= (78)^2 \times (10)^2 \end{aligned}$$

The table lists the square of 78 as 6084.

Thus

$$\begin{aligned} (780)^2 &= 6084 \times 10^2 \\ &= 608{,}400 \end{aligned}$$

Example 2. Find the square of 0·98.

We note that

$$\begin{aligned} 0\cdot98 &= 98 \times 0\cdot01 \\ (0\cdot98)^2 &= (98 \times 0\cdot01)^2 \\ &= (98)^2 \times (0\cdot01)^2 \end{aligned}$$

The table lists the square of 98 as 9604.

Thus

$$\begin{aligned} (0\cdot98)^2 &= 9604 \times 0\cdot0001 \\ &= 0\cdot9604 \end{aligned}$$

Problem 19-2a

Using the table of squares in the back of the book, find the squares of the following:

(a) 2·4 (d) 0·071

(b) 240 (e) 9·3

(c) 7100 (f) 0·00093

A Table of Square Roots

You will recall that if we have a number x such that $x^2 = y$, then x is said to be a square root of y. Since $7 \times 7 = 49$, 7 is a square root of 49. Since $(-7) \times (-7) = 49$, (-7) also is a square root of 49. We indicate the positive square root of y as $\sqrt{y}$ and the negative square root of y as $-\sqrt{y}$. Thus $7 = \sqrt{49}$ and $-7 = -\sqrt{49}$.

If $y = x^2$, is it true that $\sqrt{y} = x$? What if x is negative? In that case is $\sqrt{y} = -x$? Since $\sqrt{y} = x$ if $x \geq 0$ and $\sqrt{y} = -x$ if $x < 0$, is it correct to say that $y = x^2$ and $\sqrt{y} = |x|$ are equivalent equations?

Can we use the graph of $y = x^2$ to find square roots, that is, for every value of y on the graph, will the corresponding positive value of x be $\sqrt{y}$? What will the corresponding negative value of x be ?

Turn to the graph of $y = x^2$ that you used for finding the squares of numbers from -5 to 5. You know that a positive number has two square roots. Look at the graph and see how it shows that the square roots of 9 are 3 and -3, that is, when y is 9, x is 3 or -3. In the same way, find the square roots of 4, 16, and 2 from your graph.

In an earlier chapter we found the value of $\sqrt{2}$ to be approximately 1·414, expressed to 3 decimal places. How accurately were you able to estimate $\sqrt{2}$ from your graph? How could greater accuracy be obtained?

Problems 19-2b

1. Use your graph of $y = x^2$ from -5 to 5 to find values for the following table of the positive square roots (to 1 decimal place) of the integers from 1 to 25:

y	1	2	3	...	25
$\sqrt{y} = \lvert x \rvert$					

2. Use your graph of $y = x^2$ from 0 to 1 to find values for the following table of positive square roots to two decimal places:

y	0·1	0·2	0·3	...	0·9
$\sqrt{y} = \lvert x \rvert$					

A Printed Table of Square Roots

As with the squares of numbers, tables have been produced for square roots of numbers giving an accuracy far greater than that we can obtain by the graphical method. Look at the table of square roots given in the back of this book and compare it with the tables you made from your graphs.

You will notice that for each of the numbers from 1 to 99, there are two square roots listed; again, no decimal points are indicated. Let us see how to use the table.

We know that if we squared each of the following numbers, the decimal numerals for the answers would all be the same if we ignore the position of the decimal point:

4000, 400, 40, 4, 0·4, 0·04

Does the same thing occur when we take the <u>square root</u> of each? Try it and see.

You know that the square root of 36 is 6. How many of the following values of $\sqrt{n}$ can you find without doing additional arithmetic?

n	3600	360	36	3·6	0·36	0·036	0·0036
$\sqrt{n}$			6				

Since $\sqrt{3600} = \sqrt{36 \times 100} = \sqrt{36} \times \sqrt{100} = 6 \times 10$, you could see that $\sqrt{3600} = 60$.

To write $\sqrt{360}$ in terms of $\sqrt{36}$, we have to write $\sqrt{360} = \sqrt{36 \times 10} = \sqrt{36} \times \sqrt{10} = 6 \times \sqrt{10}$. This is not simple. Perhaps you noticed, however, that in the table there is a column marked $\sqrt{10n}$. When $n = 36$, $\sqrt{10n} = \sqrt{36 \times 10} = \sqrt{360}$. The table gives the digits 1897 for $\sqrt{10n}$ when n is 36. A decimal point is needed. Where should it be?

Since 360 is between 100 and 10,000, $\sqrt{360}$ must be between 10 and 100. This indicates that the approximation to $\sqrt{360}$ given by the table is 18·97.

A useful check is to observe from the table that $19^2 = 361$. This means that $\sqrt{361} = 19$, so $\sqrt{360}$ should be a bit less than 19, as we found.

Can you now find values for $\sqrt{n}$ in the rest of the table above without doing additional arithmetic? Use the facts that

$$3{\cdot}6 = 36 \times 10 \times \frac{1}{100}$$
$$0{\cdot}36 = 36 \times \frac{1}{100}$$
$$0{\cdot}036 = 36 \times 10 \times \frac{1}{10{,}000}$$
$$0{\cdot}0036 = 36 \times \frac{1}{10{,}000}$$

If you wanted to find $\sqrt{39}$ to four figures, would you look in the $\sqrt{n}$ column, or the $\sqrt{10n}$ column? In which column would you look to find $\sqrt{390}$? $\sqrt{3900}$? $\sqrt{3{\cdot}9}$?

Since 39 is listed in the n column, $\sqrt{39}$ is listed in the $\sqrt{n}$ column. We find $\sqrt{39} \approx 6{\cdot}245$. (The symbol $\approx$ means "is approximately".) Since $3900 = 39 \times 100$, we can write

$$\sqrt{3900} = \sqrt{39} \times \sqrt{100}$$
$$\sqrt{3900} \approx 6{\cdot}245 \times 10$$
$$\sqrt{3900} \approx 62{\cdot}45$$

The square root of 390 is listed in the $\sqrt{10n}$ column since $390 = 39 \times 10$. We find $\sqrt{390} \approx 19{\cdot}75$. Since $3{\cdot}9 = 39 \times 10 \times \frac{1}{100}$,

$$\sqrt{3{\cdot}9} = \sqrt{39 \times 10} \times \sqrt{\frac{1}{100}}$$
$$\sqrt{3{\cdot}9} \approx 19{\cdot}75 \times \frac{1}{10}$$
$$\sqrt{3{\cdot}9} \approx 1{\cdot}975$$

Problems 19-2c

1. Using the square root tables in the book, find the approximate square root to 4 figures of each of the following:

(a) 15	(d) 760
(b) 150	(e) 76000
(c) 1·5	(f) 0·00076

2. CHALLENGE PROBLEM

You have seen how to produce tables of squares and square roots either arithmetically or by using the graph of $y = x^2$. How could you produce a table of cubes and cube roots?

3. CHALLENGE PROBLEM

Produce tables of cubes and cube roots for numbers from -2 to 2 at intervals of 0·5. How many real cube roots has a number?

4. CHALLENGE PROBLEM

If you were to produce a printed table of cube roots for numbers from 10 to 99 which could also be used to find the cube roots of other numbers, how many columns would you need for each of the numbers from 10 to 99? You may wish to consider the following before you decide.

Since $10 \times 10 \times 10 = 1000$, the cube root of 1000 is 10. Which of the following cube roots would have the same digits with the decimal point differently placed?

$$\sqrt[3]{32},\ \sqrt[3]{320},\ \sqrt[3]{3200},\ \sqrt[3]{3\cdot 2},\ \sqrt[3]{0\cdot 32},\ \sqrt[3]{0\cdot 032},\ \sqrt[3]{320{,}000}$$

19-3 Other Graphs

Class Activity

Let us see what we can find out about the graph of $x^2 + y^2 = 25$ before we plot any points. Answer these questions about the coordinates of the ordered pairs in the truth set by examining the equation.

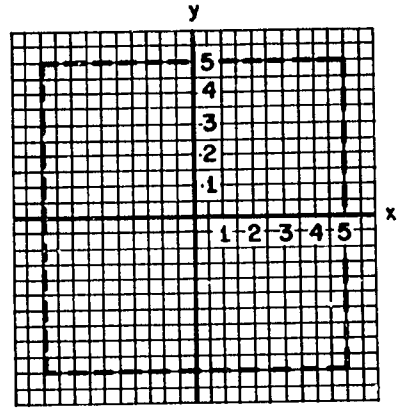
1. As x^2 becomes larger, does y^2 become larger or smaller?

2. How small can x^2 be? How small can y^2 be?

3. How large can x^2 be? Can it be 100? (If it were, then y^2 would be -75. Is this possible?)

4. If you answered the preceding questions correctly, you now know that for $x^2 + y^2 = 25$, $0 \leq x^2 \leq 25$. What, then, are the possible values of x in the truth set? Can x be negative? How small can x be? How large can x be? Can you answer the same questions about y?

5. Your answers to 4 should indicate to you that no points of $x^2 + y^2 = 25$ can be outside the 10 by 10 square with the origin as its center.



Draw axes on your graph paper with an appropriate scale. Use the same scale for both axes.

6. Find values for y and plot the points.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y											

You may wish to use your tables of square roots to estimate some of the y-values to one decimal place. For example, if x is -2, then:

$$(-2)^2 + y^2 = 25$$
$$4 + y^2 = 25$$
$$y^2 = 21$$
$$y = \sqrt{21} \text{ or } y = -\sqrt{21}$$

We can then put in the table 4·6 and -4·6 for values of y corresponding to -2 for x.

7. Can you find other ordered pairs for the graph of $x^2 + y^2 = 25$ without doing any more arithmetic? If x is 4·6 correct to one decimal place, do you know what values y can have? Mark any other points you can without doing any more arithmetic. What do you think the graph of $x^2 + y^2 = 25$ is?

8. $x^2 + y^2 = 25$ defines a locus algebraically. You have met this locus before in geometry. How was it defined there?

9. Describe the graphs of these inequalities in terms of the graph of $x^2 + y^2 = 25$.

(a) $x^2 + y^2 < 25$

(b) $x^2 + y^2 > 25$

(c) $x^2 + y^2 \leq 25$

(d) $x^2 + y^2 \geq 25$

Problems 19-3

For each of the following equations or inequalities sketch the graph. Try to find out as much as you can about the graph before you plot any points. You may wish to make use of your tables of squares and square roots.

1. (a) $x^2 + y^2 = 100$

(b) $x^2 + y^2 \geq 100$

(c) $x^2 + y^2 < 100$

2. $x^2 + 4y^2 = 100$

3. $4x^2 + y^2 = 100$

4. $x^2 - y^2 = 25$

5. $y^2 - x^2 = 25$

6. You met figures like the graphs in 1 (a), 2, 3, 4, and 5 in your work on loci. Do you remember the special name of each and how each was obtained?

7. Find lines of symmetry, if possible, for each of your graphs and write the equations of the lines of symmetry.

19-4 More Ways of Picturing Numerical Information

You have already met three ways of picturing numerical information. These are the pie chart, the bar chart and the histogram. Now we are going to examine two more ways.

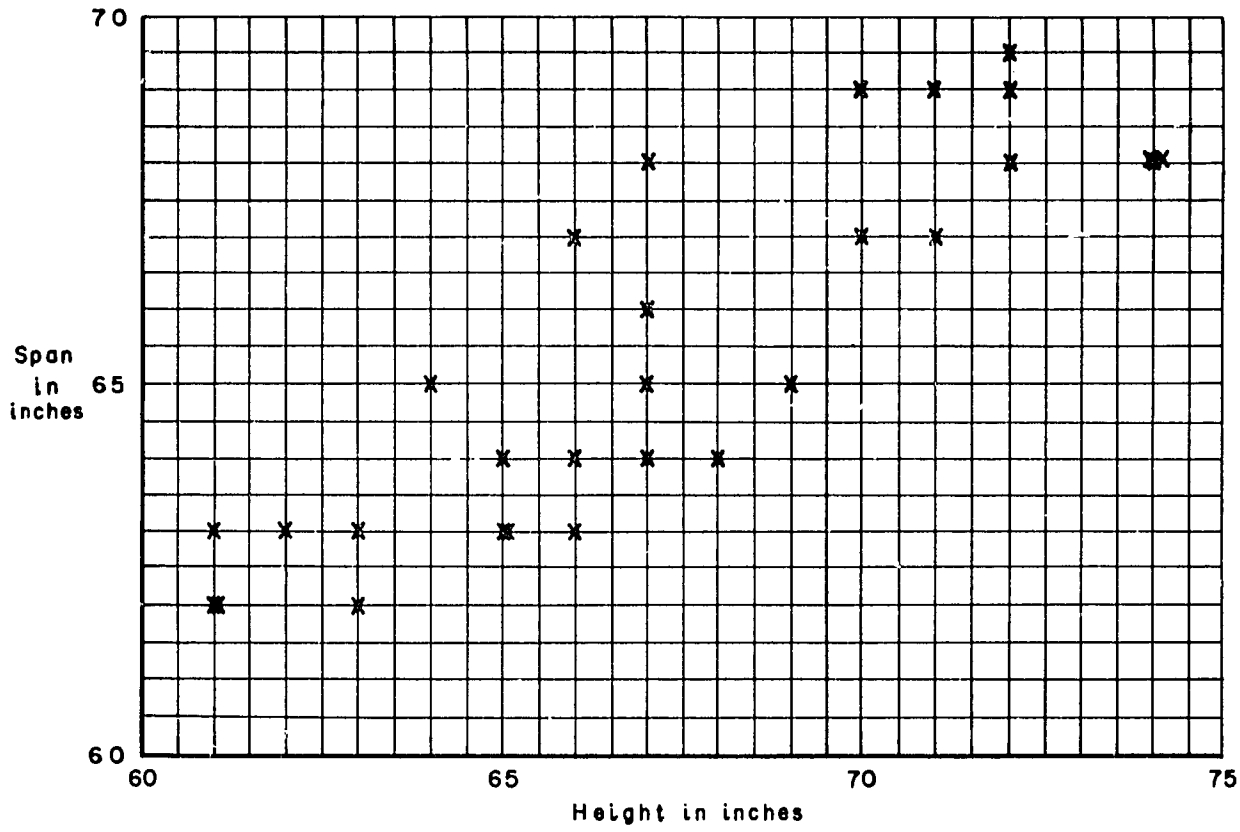
1. Scatter Diagram

In your work on the histogram, you drew the histograms of the heights and spans of 30 boys. These measures are given again below.

Height	Span	Height	Span	Height	Span
66	63	67	66	66	64
65	63	69	65	65	63
62	63	71	69	61	62
63	63	70	69	61	63
72	68	72	69	61	60
66	67	61	62	65	64
71	67	61	62	74	68
70	67	63	62	74	68
67	64	72	69	67	65
67	68	68	64	64	65

All measurements are in inches.

One of the questions we may ask about this information is whether there is any connection between the heights and spans of the boys. The following picture of this information may help us to answer this.



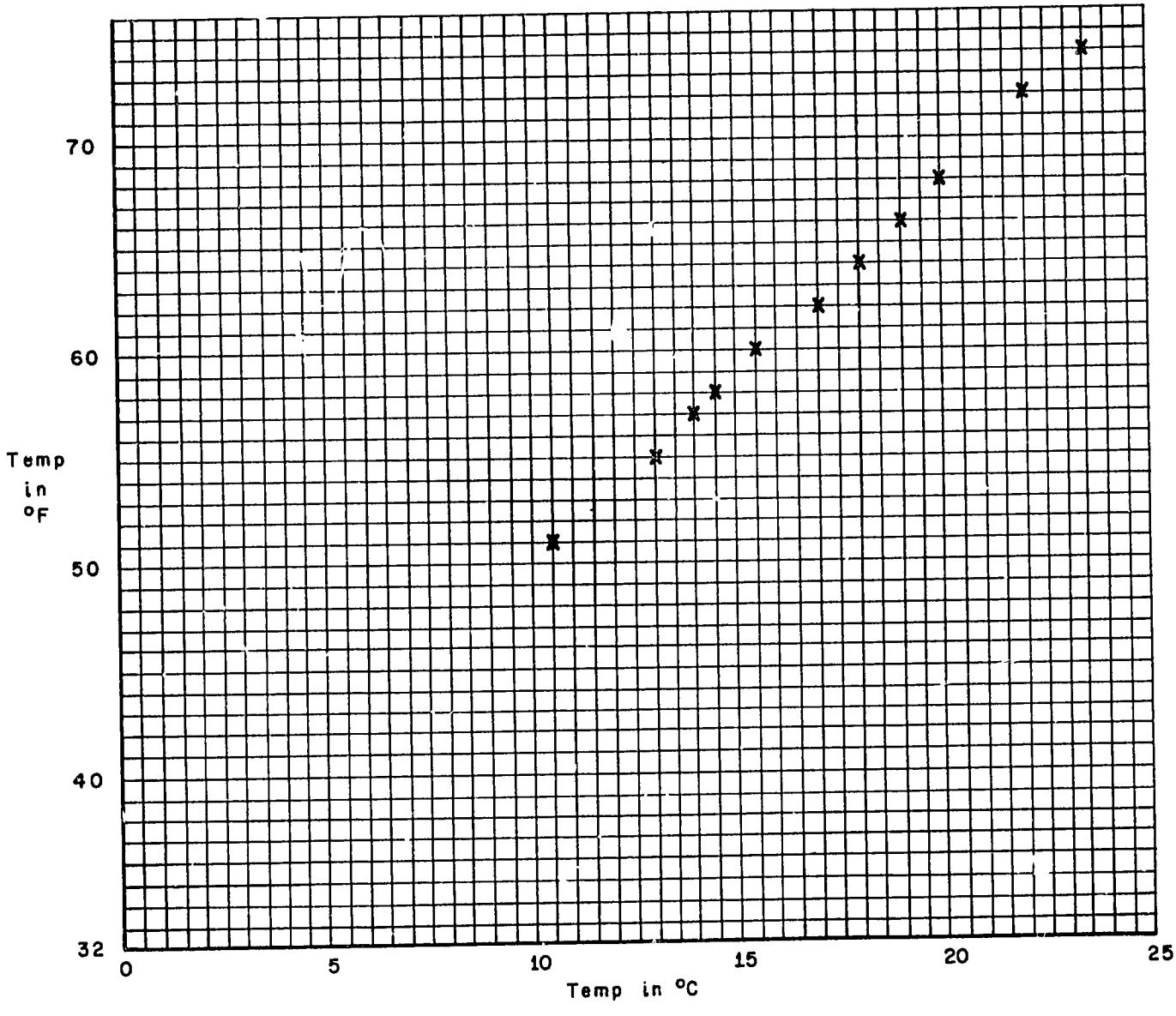
This is known as a <u>scatter diagram</u>. Why do you think it is called a scatter diagram?

How has it been constructed? What do you think the diagram tells us? What can you say about the span of a boy 66 in. tall?

Now look at the information shown below. These are temperatures taken during a day, by two boys. One used a centigrade thermometer, and the other a Fahrenheit thermometer. They obtained these temperatures.

y (temp. in ^{o}F.)	51	55	57	58	60	62	64	66	68	72	74
x (temp. in ^{o}C.)	10.5	13	14	14.5	15.5	17	18	19	20	22	23.5

You can see below the scatter diagram for these two sets of measurements.



How is this scatter diagram different from the previous one?

The points of this second diagram approximate a straight line so closely that we should expect to find the equation of a straight line relating x and y.

Do you know from your work in science what this equation should be? Draw this line.

Why do not all the points lie on the line?

Summary

When we have two sets S_1, S_2 of numbers which can be matched in ordered pairs, we can investigate the relation between S_1 and S_2 using a scatter diagram. In some cases, such as our second example, the scatter diagram is very nearly a set of points on a straight line. In these examples, we can usually find an equation connecting the members of the sets. This is the case in many science experiments. It is not often, however, that we get such a close approximation to a straight line as in our second example. Experimental error can produce much scattering.

Class Activity

1. Measure the diameters and circumferences of a number of circular objects. Plot their scatter diagram.

2. Measure the heights and spans of the pupils in your class, and plot their scatter diagram.

3. Plot the scatter diagram for the results of any of your science experiments which are suitable.

4. Plot a scatter diagram of the marks your class obtained in two subjects, e.g., mathematics and English.

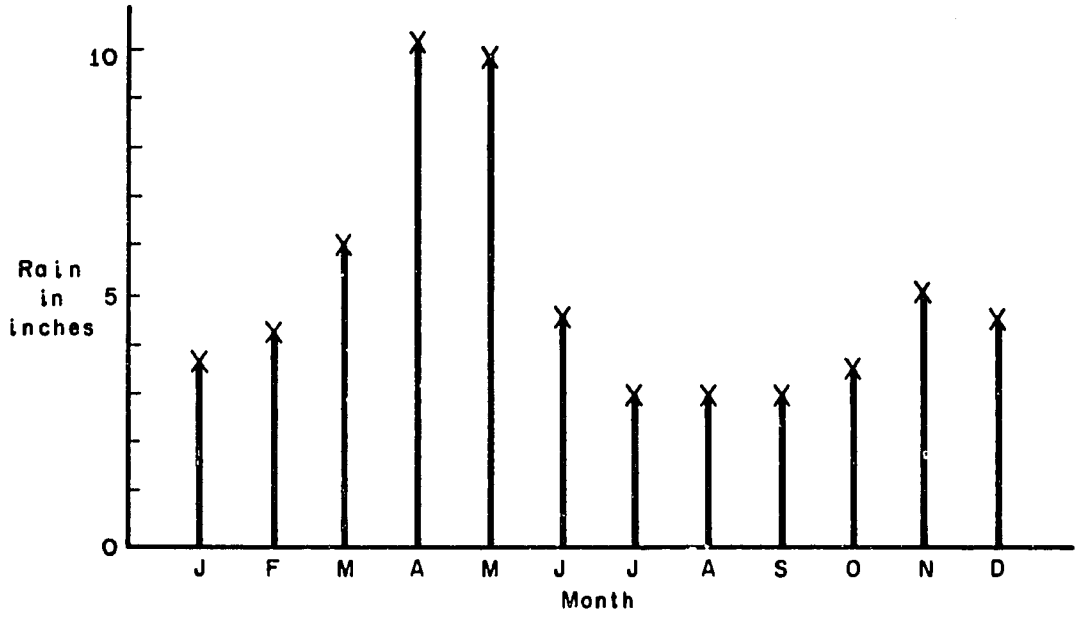
---

2. Continuous Line Chart

Now look at the average monthly rainfall figures for Entebbe which are given again below:

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Rain in In.	3·6	4·2	6·0	10·2	9·7	4·6	3·0	3·0	3·0	3·5	5·1	4·5

Here is a bar graph for the information:

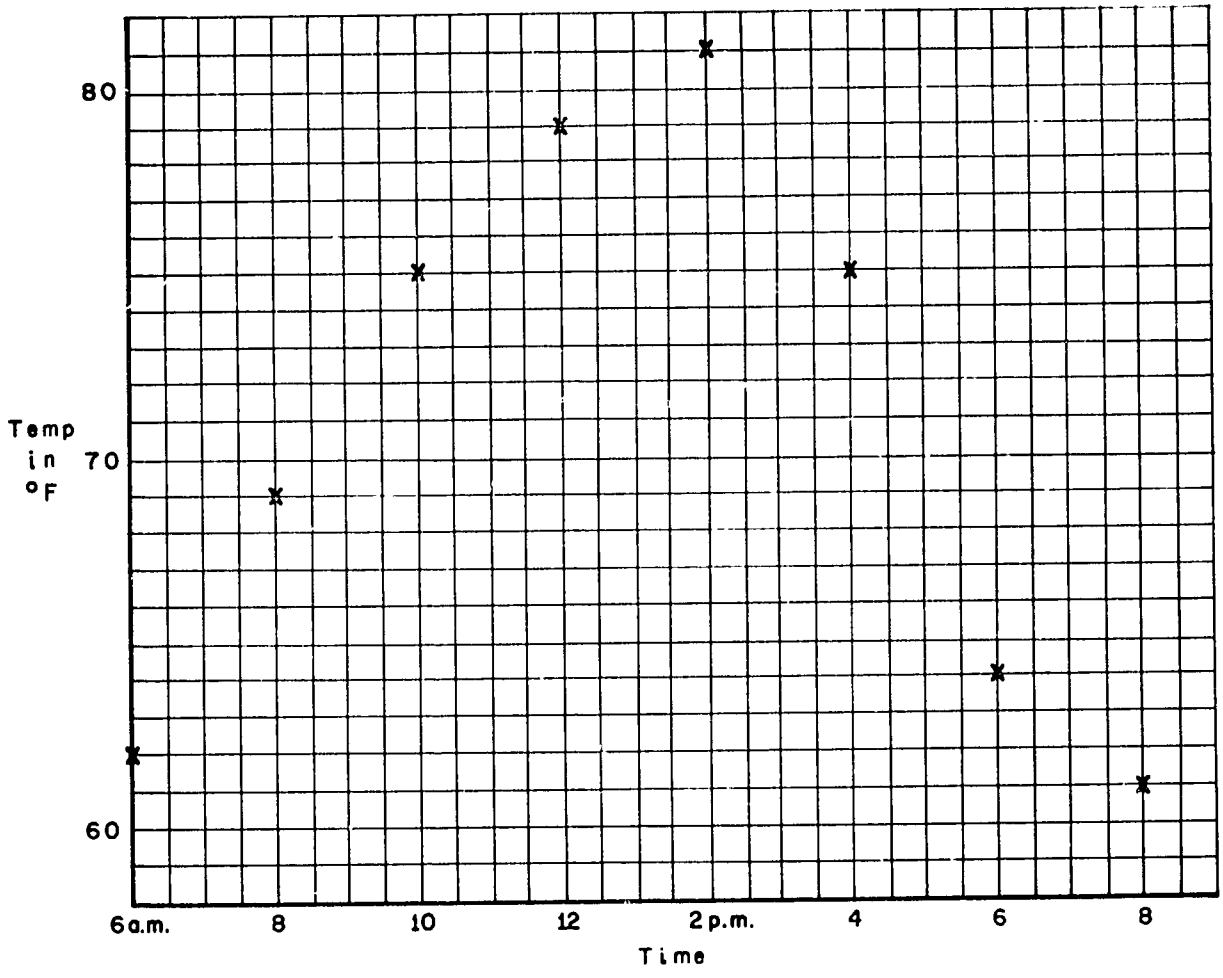


How many points are on this graph? Why can we plot no more points than this? With this sort of graph, there is no room to give additional information which might be available to us. Every drop of rain fell during one of the twelve months, and the monthly total only is recorded. We have no indication on the graph, for example, how much rain fell on 10 April, or even during the first 10 days of April.

In contrast, now look at the information below. These are temperatures as they were measured every 2 hours during a day.

Time	6 a.m.	8 a.m.	10 a.m.	12 noon	2 p.m.	4 p.m.	6 p.m.	8 p.m.
Temp. in $^{\circ}$F	62	69	75	79	81	75	64	61

We now plot these measurements:



In this example we could easily put more points on the graph by taking the temperature every hour, or even more frequently.

The main difference between this graph and the rainfall graph is that here we are able to plot the temperature at each moment of the day, provided this information is available to us. As our experience indicates that temperature does not change radically in a very short time, the graph allows us to make reasonable estimates of the temperature at different times. For example, what do you think the temperature was at 11 a.m.? The points we have plotted may be joined by a curve and, if we draw a reasonable curve, the curve should very nearly approximate the temperature at each moment throughout the day. In fact, if we could have recorded the temperature at each instant, we would have obtained a smooth curve. We call such a graph a <u>continuous curve graph</u>. Such a graph gives us a complete

picture of the behaviour of the temperature during the day. By contrast, the bar graph gives a rather incomplete picture of the rainfall during the year.

Give some other examples of continuous curve graphs.

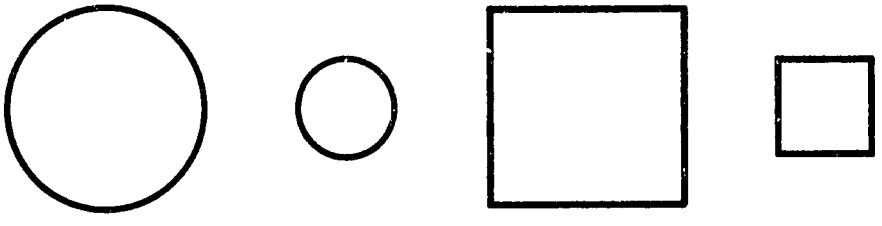
CHAPTER 20

SIMILAR FIGURES

20-1 Shapes of Plane Figures

In an earlier chapter you learnt that two plane figures are congruent if they have the same shape and size.

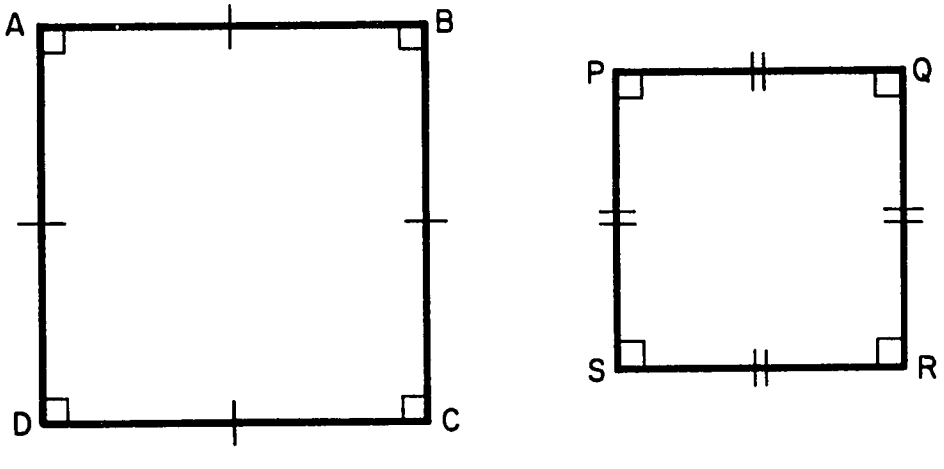
The two circles below, while they are not of the same size, appear to have the same shape. Similarly, the two squares shown below have the same shape.



If the smaller of each of these pairs of figures is stretched uniformly in all directions from its centre, it can be made congruent to the larger, or if the larger is compressed uniformly in all directions towards its centre, it can be made congruent to the smaller.

What properties can we discover about figures which have the same shape?

Let us consider a pair of squares of different sizes.



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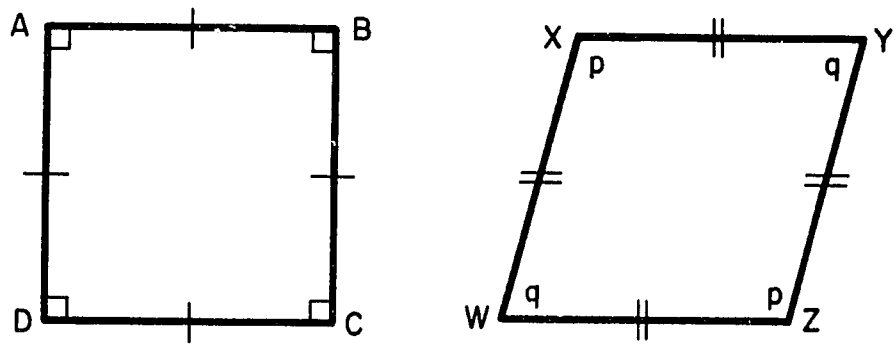
You will recall that a square has all its sides congruent and all its angles right angles. Just as with congruent figures, we can establish a <u>one-to-one correspondence</u> between the parts of the two figures. When we write ABCD $\longleftrightarrow$ PQRS, we have indicated the following correspondences between the parts of the squares:

$\hat{A}$	$\longleftrightarrow$	$\hat{P}$	$\overline{AB}$	$\longleftrightarrow$	$\overline{PQ}$
$\hat{B}$	$\longleftrightarrow$	$\hat{Q}$	$\overline{BC}$	$\longleftrightarrow$	$\overline{QR}$
$\hat{C}$	$\longleftrightarrow$	$\hat{R}$	$\overline{CD}$	$\longleftrightarrow$	$\overline{RS}$
$\hat{D}$	$\longleftrightarrow$	$\hat{S}$	$\overline{DA}$	$\longleftrightarrow$	$\overline{SP}$

What relationships do you observe between corresponding angles of the two squares? Can you see a relationship between their corresponding sides? Measure them and compare them.

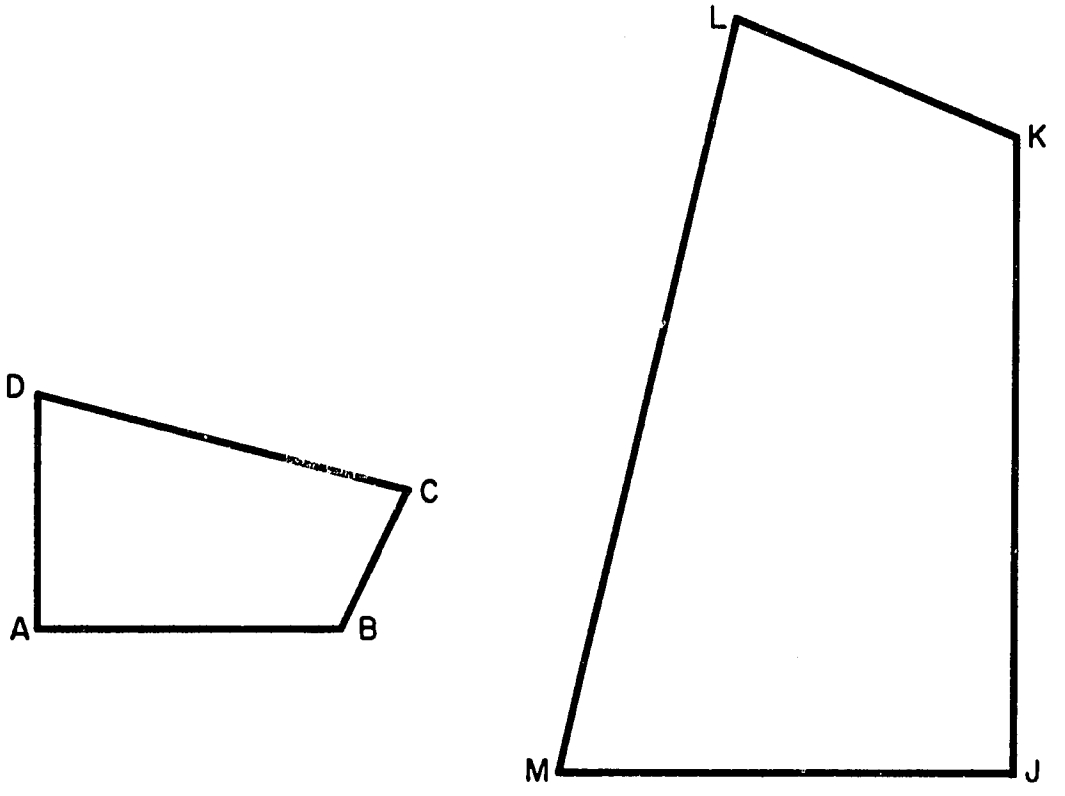
For the quadrilaterals below a correspondence ABCD $\longleftrightarrow$ XYZW can be established.



Square Rhombus

All the sides of each quadrilateral are congruent, and the angles are as shown. Have the quadrilaterals the same shape? Why?

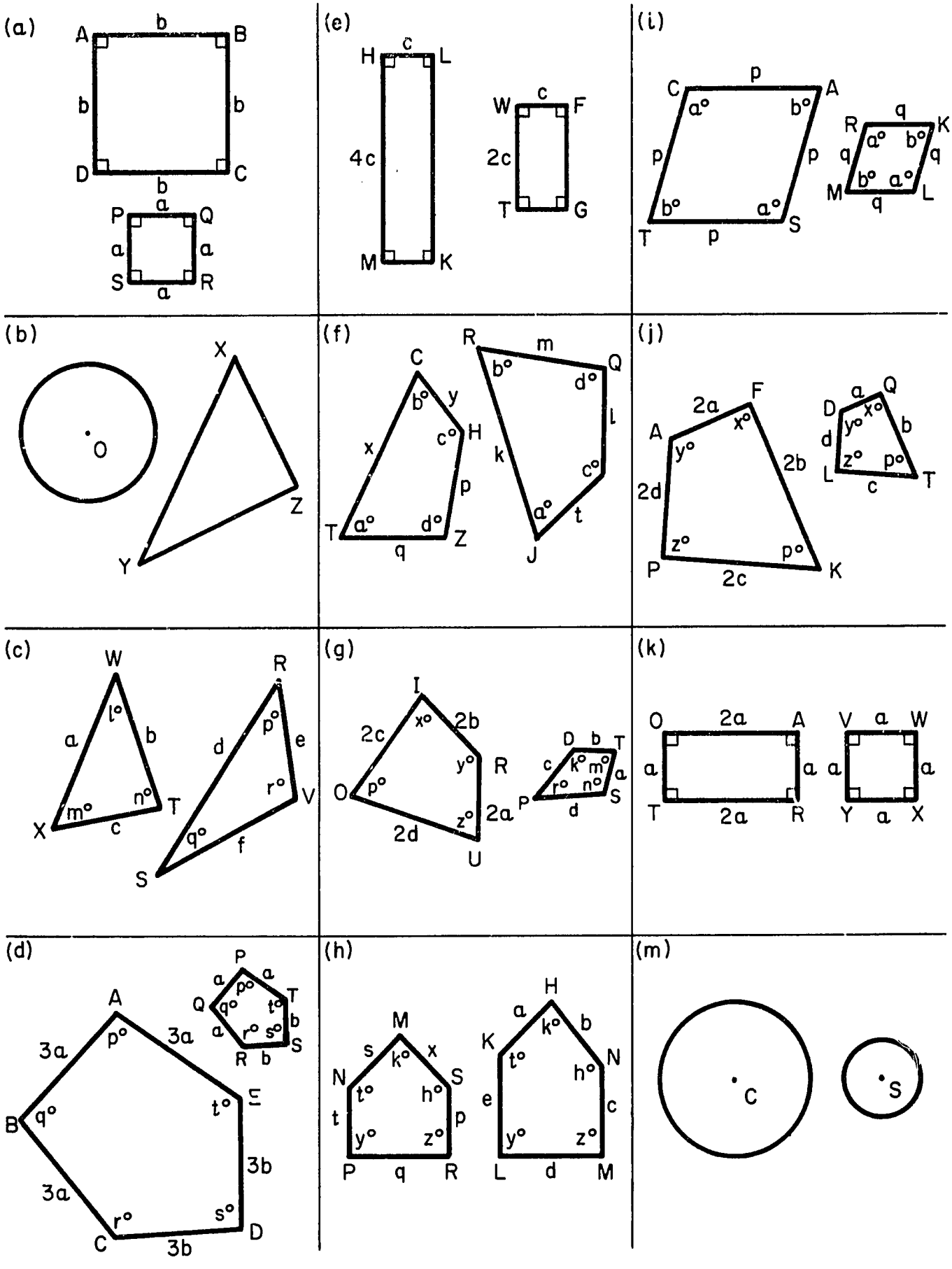
The following figures are of the same shape under the correspondence ABCD $\longleftrightarrow$ JKLM.



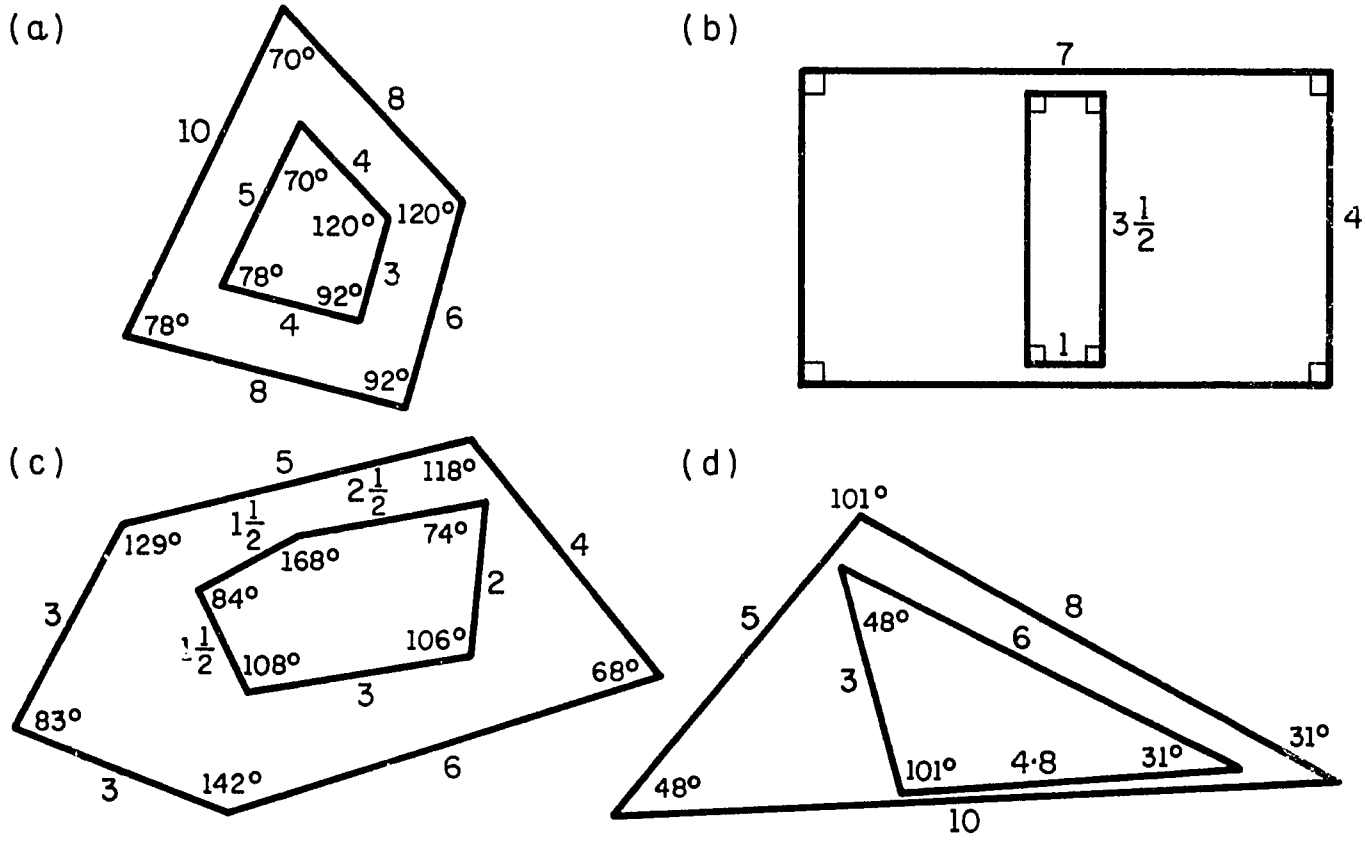
Measure the corresponding angles and the corresponding sides. What relations do you observe between measures of corresponding parts?

Class Activity

1. Examine carefully the pairs of figures below and say which pairs you think have the same shape under a suitable correspondence between the parts of the figures. The letters a, b, etc., on the figures represent measures of sides and angles. State the correspondence in the cases where the figures have the same shape.


(a)
A b B
b b
D b C
P a Q
a a
S a R
(b)
O
X
Y Z
(c)
W
a l° b
X m° n° T
c
R
d p° e
r° V
S q° f
(d)
A
3a p° 3a
B q° t° E
3a 3b
C r° s° D
3b
P
a p° a
Q q° t° T
a r° s° b
R b S
(e)
H c L
4c
M K
W c F
2c
T G
(f)
C
b° y
x c° H
p
T a° d° Z
q
R m Q
b° d°
l
k c°
a° t
J
(g)
I
2c x° 2b
y° R
O p°
z° 2a
2d
U
D b T
c k° m° a
r° n° S
P d
(h)
M
s k° x
N t° h° S
t p
y° z°
P q R
H
a k° b
K t° N
h°
e c
y° z°
L d M
(i)
C p A
a° b°
p p
b° a°
T p S
R q K
a° b°
q q
b° a°
M q L
(j)
2a F
A x° y°
2b
2d
P z° p° K
2c
D a Q
x°
y° b
d
z° p°
L c T
(k)
O 2a A
a a
T 2a R
V a W
a a
Y a X
(m)
C
S

2. Examine these figures carefully.



In each case do the figures have the same shape? Why?

You should now be able to tell whether two simple polygons have the same shape.

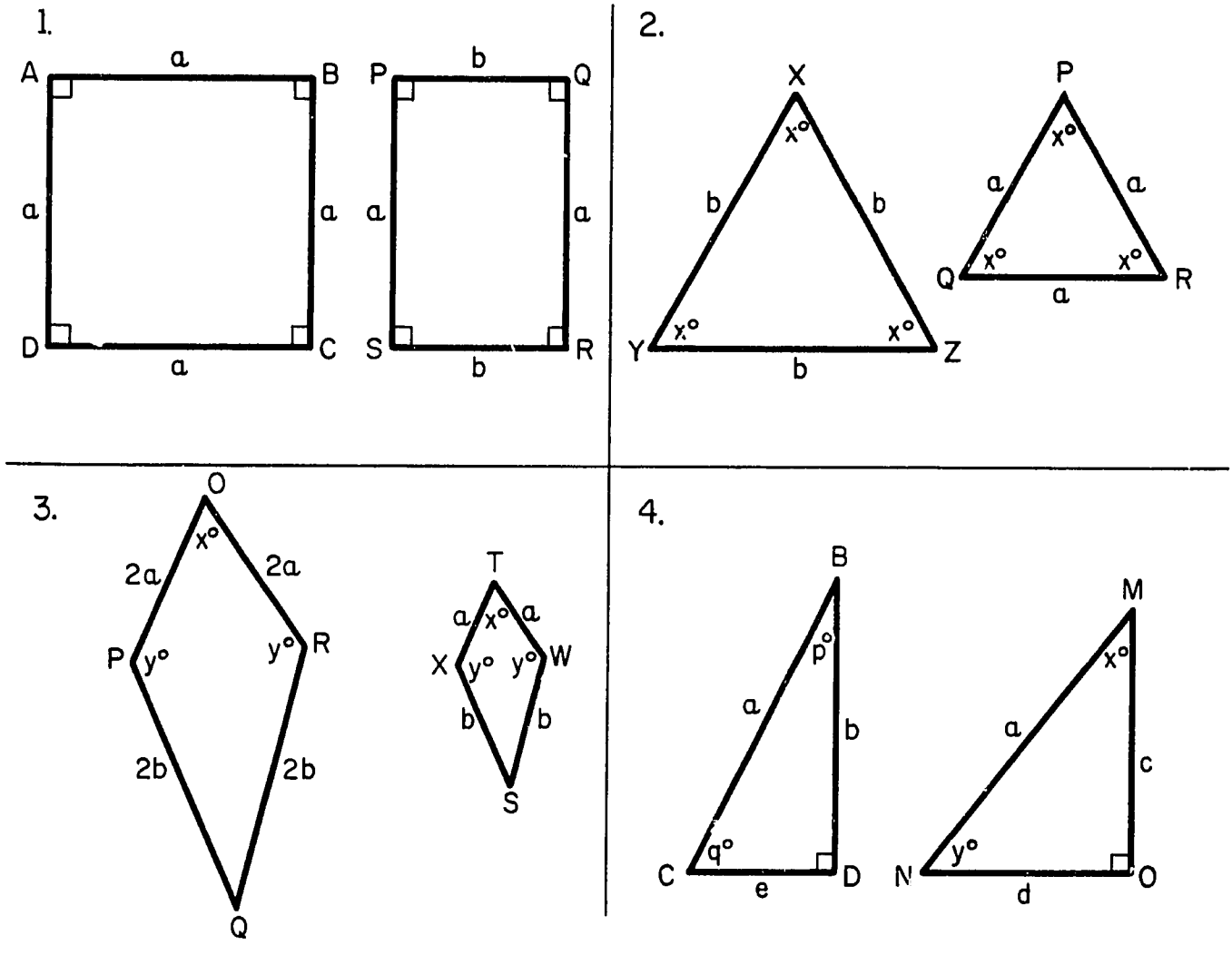
Two polygons have the same shape if the angles of one are congruent to the corresponding angles of the other and the ratios of the measures of corresponding sides are equal.

Figures with the same shape are called <u>similar</u> figures.

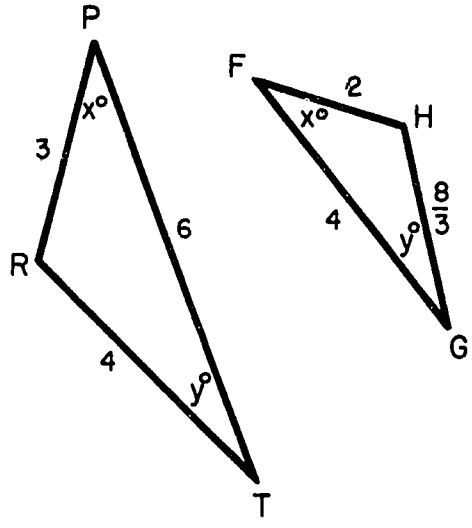
Are congruent polygons similar? What is the ratio of the measures of corresponding sides of two congruent polygons?

Problems 20-1

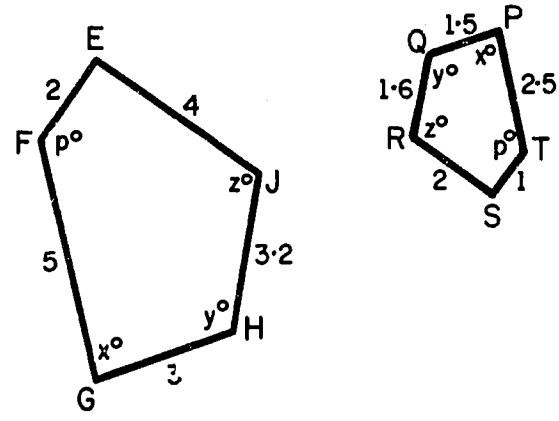
State which of the following pairs of figures are similar. In each case give a reason for your answer and indicate the correspondence in the cases of similarity.



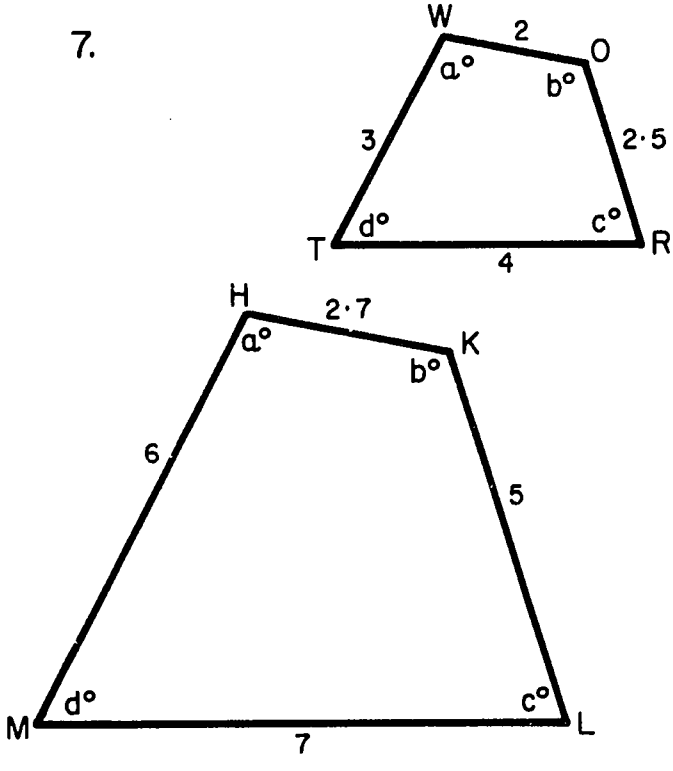
5.


P
x°
3
6
R
4
y°
T
F
2
x°
H
4
y°
8/3
G

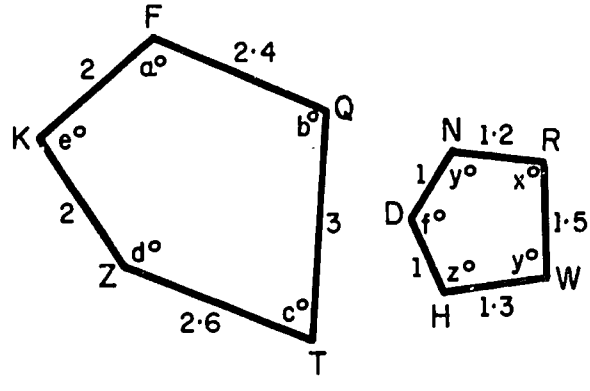
6.


E
2
4
F
p°
z°
J
5
3·2
y°
H
x°
3
G
Q
1·5
P
x°
y°
1·6
2·5
R
z°
p°
T
2
1
S

7.


W
2
O
a°
b°
3
2·5
T
d°
c°
R
4
H
2·7
K
a°
b°
6
5
M
d°
c°
L
7

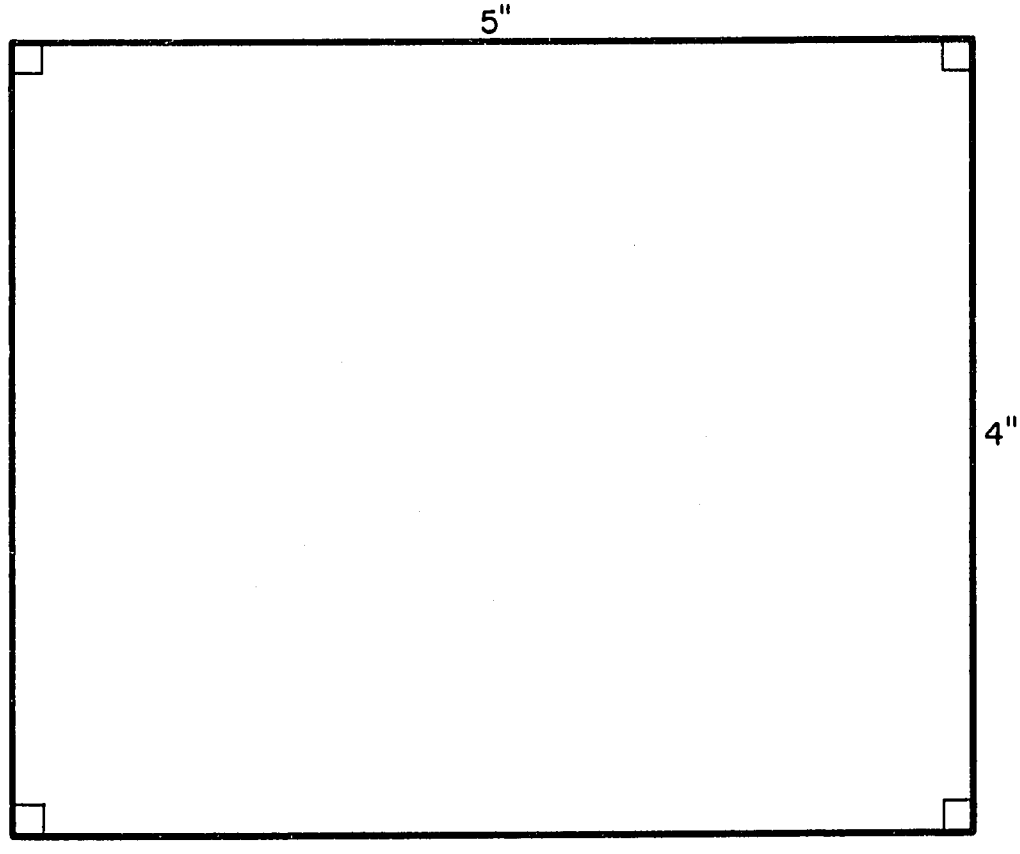
8.


F
2
2·4
a°
Q
K
e°
b°
2
3
Z
d°
2·6
c°
T
N
1·2
R
1
y°
x°
D
f°
1·5
1
z°
y°
W
H
1·3

20-2 Scale Drawing -- Plans and Maps

Similar figures are used when a surveyor or architect draws a plan of a piece of land. The plan must have the same shape as the piece of land it represents but has to be smaller in size because the paper on which he draws the plan is much smaller than the land itself.

Let us consider the plan of a rectangular piece of land which is 30 feet long and 24 feet wide. One possible plan will be that shown below; why do we say one possible plan?



Observe that the plan is rectangular. What condition for similarity between the plan and the piece of land is therefore satisfied?

Is the other condition for similarity satisfied? To compute the ratios of the measures of corresponding sides, everything must be expressed in terms of the same unit of measure. Convert the dimensions of the piece of land to inches and

then find the ratio of the length of the plan to the length of the piece of land. Do the same for the widths. Are the ratios equal? Are the polygons similar?

In this example you have found that the ratio of the measures of the corresponding sides of the plan and land is $\frac{5}{30 \times 12} = \frac{1}{72}$. If the plan is drawn 10 inches long and 8 inches wide, what is the ratio of corresponding sides? You can see that the ratios of corresponding sides of the plan and land will be different for different sizes of plans. These ratios are what are known as the <u>scales</u> of the plans. The architect usually chooses a scale which will give as large a plan as possible for a given piece of paper.

Suppose that the paper on which the architect makes the plan is 32 inches long and 27 inches wide. He usually leaves a margin around the edge of the paper. If the width of the margin is at least 1 inch, what will be the length and width of the largest plan of the land he can draw? What scale will he use?

The making of maps also is based on the principle of similar figures. Thus, there is a one-to-one correspondence between the points on the map and the actual points in the town which the map represents. The angles between roads on the map are congruent to the corresponding angles between the roads in the town itself, and the ratios of corresponding lengths are the same.

Again, this ratio between corresponding distances on the map and in the town which the map represents is called the scale of the map.

Scales are usually given in the form 1:b where b is a positive integer. For example, 1:200 would mean that a line segment of length 1 inch on the plan or map represents a line segment of length 200 inches on the land. In that case, 1 inch on the plan represents about how many feet on the load?

The corresponding lengths could be 1 foot and 250 feet, 1 yard and 200 yards, and so on. Sometimes the scale is given in the form, "1 inch represents 1 mile." In this case what will be the ratio of the measures of corresponding line segments?

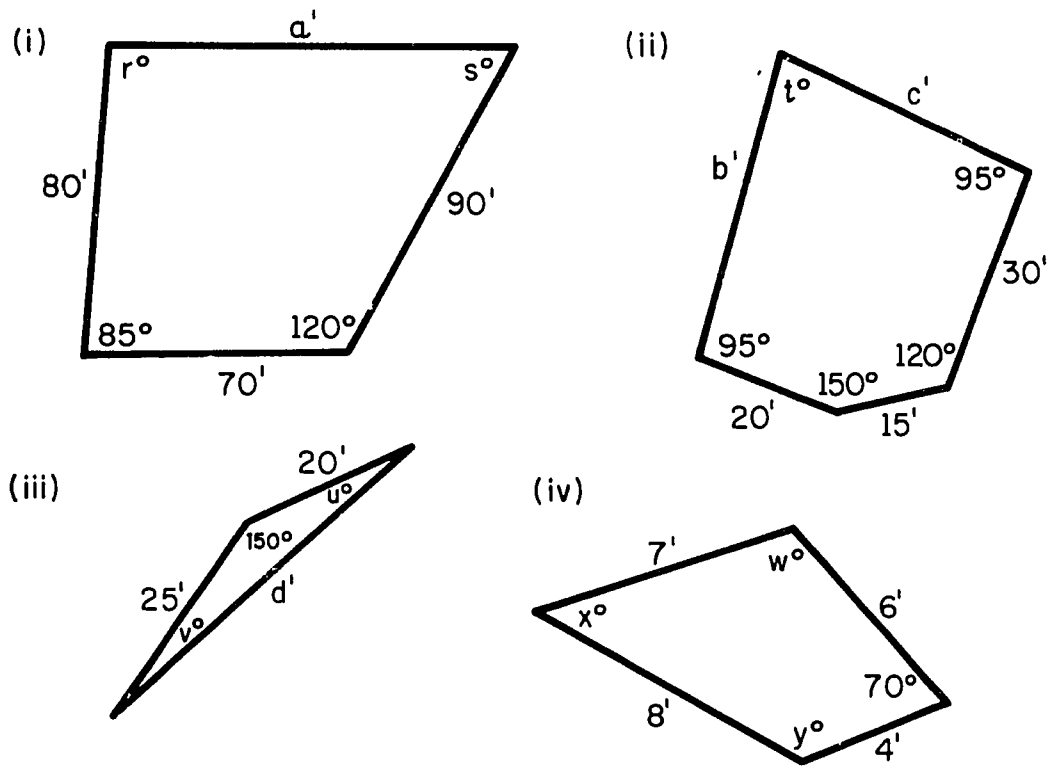
Drawings of plans and maps may also be referred to as scale drawing.

Problems 20-2

1. Using a margin of 1 inch on the left, for each of the following, draw the largest plan you can on your paper. State your scale in each case.

(a) A cricket pitch, rectangular in shape, whose length is 66 feet and width 6 ft.

(b) Plots of flat land whose boundaries have shapes and dimensions as given below.



Find the size of the unspecified sides and angles from your plans.

2. We hope that you noticed from Problem 1 that you do not need to know the size of all the angles and sides of a figure before your plan can be drawn. Can you, in the case of quadrilaterals, list the minimum information required for drawing a plan?

3. The height of a tower is 84 feet and a point on the level ground is 106 feet from the top of the tower. Draw a scale diagram showing the point on

the ground, the foot of the tower and the top of the tower. Determine from your diagram the distance of the point on the ground from the foot of the tower.

4. A ladder 10 feet long leans against a vertical wall. The point at which the ladder meets the ground is 6 feet from the wall. Use a scale drawing to find how high up the wall the ladder reaches.

5. The scale of a map of a town is 1:100. A road on the map has a length of $6\frac{3}{4}$ inches. What will be the length of the corresponding road in the town? Give your answer in feet.

6. On the map of a certain place a line segment joining two points is 5 inches long. The distance between the actual points represented is 600 feet. What is the scale of the map?

7. The length and width of a rectangular field on the map of a certain town are 5 inches and 4 inches respectively. If the scale of the map is 1:400, (a), what is the actual length of the field? (b) What is the actual width of the field? (c) Find the area of the field in square feet.

20-3 Similar Triangles

If there is a correspondence between two triangles such that the angles of one are congruent to the corresponding angles of the others, do you suspect some further interesting property of these triangles?

Class Activity

1. Draw a triangle two of whose angles are of 80°, 60°. What will be the size of the third angle?

2. How many such triangles can you draw?

3. Draw any two such triangles which are not of the same size.

4. Measure as accurately as you can the lengths of the sides of each triangle and compute the ratios of the measures of corresponding sides. What do you observe about these ratios?

5. Do you think you will have the same results with any two triangles with two angles of 80° and 60°?

6. Will such triangles be similar?

If your constructions and measurements were carefully done you have discovered that for any two triangles if the angles of one are congruent to the corresponding angles of the other, then the ratios of the measures of corresponding sides are equal and the triangles are therefore similar.

Can we similarly draw some conclusion about two triangles in which we know only that the ratios of the measures of the corresponding sides are equal? Let us experiment.

Class Activity

1. Draw any triangle ABC to occupy about half of your paper.

2. Draw another triangle PQR on another paper such that, $PQ = 2AB$, $QR = 2BC$ and $RP = 2CA$.

3. Measure as accurately as you can the size of each angle of both triangles. What do you notice about corresponding angles?

4. Repeat Question 2 but this time make $PQ = \frac{1}{3}AB$, $QR = \frac{1}{3}BC$ and $RP = \frac{1}{3}CA$.

5. Measure again the angles of the two triangles. What do you notice again about corresponding angles?

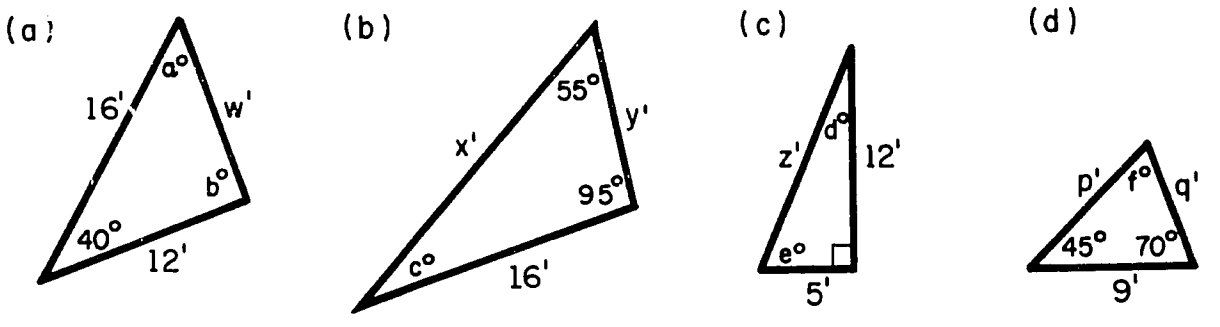
6. Repeat Questions 1 and 2 but use a different ratio between corresponding sides. Do you always have the corresponding angles congruent?

You probably have discovered that if the ratios of corresponding sides of two triangles are equal then the corresponding angles of the triangles are congruent, and the triangles are therefore similar.

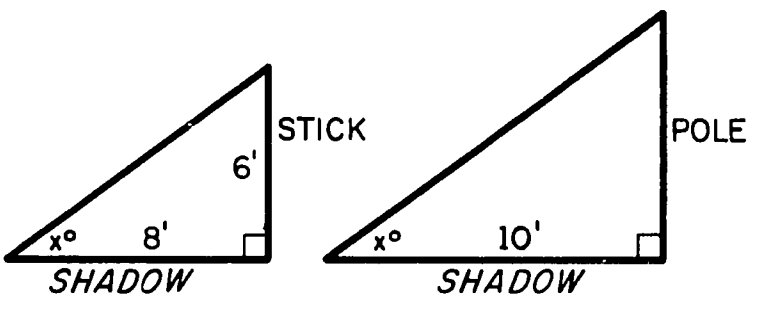
> 1. Two triangles are similar if the angles of one are congruent to the corresponding angles of the other. (The ratios of the measures of corresponding sides will be equal as a result of the congruent angles.)
>
> 2. Two triangles are similar if the ratios of the measures of corresponding sides are equal. (The corresponding angles will be congruent as a result of the equal ratios.)

Problems 20-3

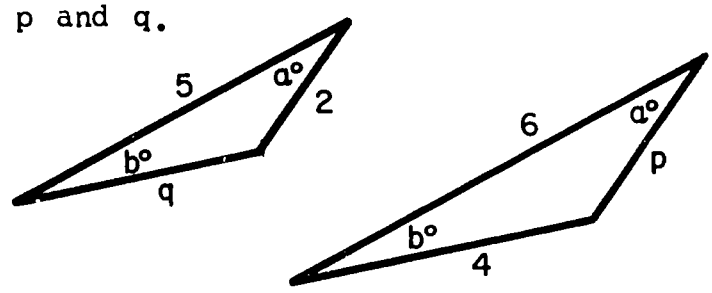
1. In each of the figures below, find the sizes of the unknown sides and angles by scale drawing and measurement. The letters a, b, etc., represent measures of sides and angles.



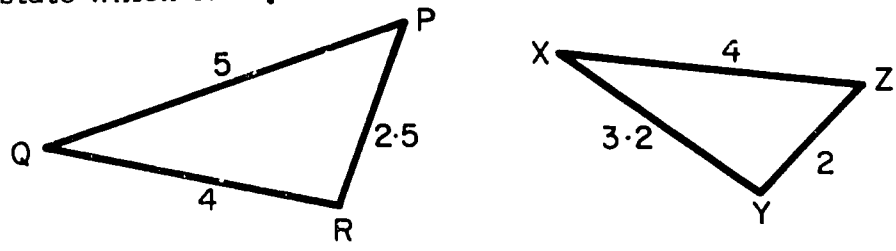
2. A straight stick is 6 ft. long and is erected vertically in the ground. Its shadow is 8 ft. long at a particular time of the day. Find the height of a pole erected in the same way and whose shadow at that same time is 10 ft. long.



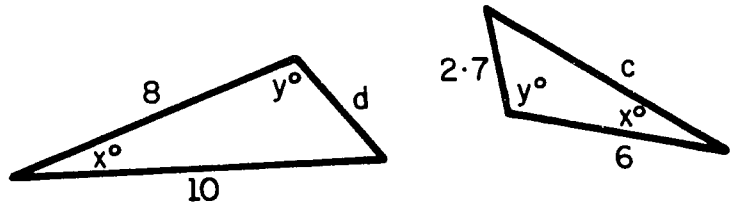
3. (a) Find p and q.



(b) Determine without measuring whether any angles are congruent and, if so, state which ones.



(c) Find c and d.



4. In the figure below, $a_1 = a_2$; $b_1 = b_2$

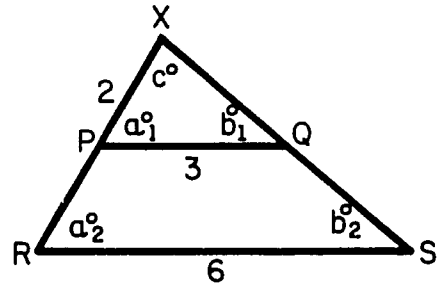
(a) Write the names of two similar triangles.

(b) Write the ratios of corresponding line segments in the two triangles. Are these ratios equal?

(c) Calculate XR and PR.

(d) Calculate $\frac{XQ}{XS}$ and $\frac{XQ}{QS}$.

(e) Compare $\overline{XQ}$ and $\overline{QS}$.



5. In the figure below, $\overline{BC}$ is parallel to $\overline{DE}$.

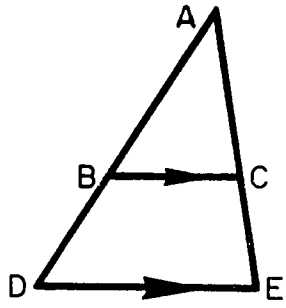
(a) Name the congruent angles.

(b) Name the similar triangles.

(c) Write the ratios of corresponding sides. Are these ratios equal?

(d) If AB = 3, BD = 2 and BC = 2, find DE.

(e) If, in addition, AE = 4·5, find AC.



6. In the figure below, $\overline{QT}$, $\overline{RU}$ and $\overline{SV}$ are parallel.

(a) Name the angles that are congruent.

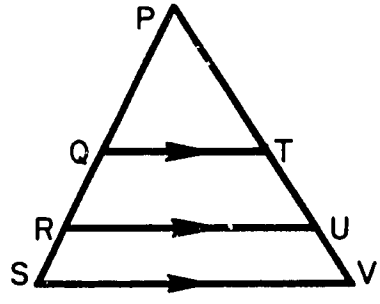
(b) Name the triangles that are similar.

(c) Use the properties of these similar triangles to fill in the blanks in the following:

(i) $\frac{PQ}{PR} = \frac{?}{PU} = \frac{QT}{?}$

(ii) $\frac{PR}{PS} = \frac{RU}{?} = \frac{?}{PV}$

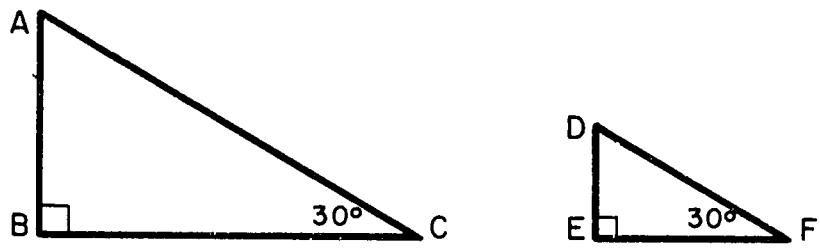
(iii) $\frac{PQ}{?} = \frac{PT}{PV} = \frac{?}{SV}$



20-4 Similar Right-Angled Triangles

Class Activity

1.



(a) Calculate the size of $\hat{A}$ and the size of $\hat{D}$.

(b) How is $\triangle ABC$ related to $\triangle DEF$?

(c) Is $\frac{AB}{BC} = \frac{DE}{EF}$? Why?

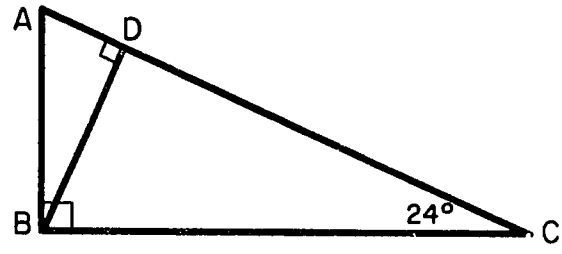
2. Draw two right-angled triangles, PQR and XYZ, where $\hat{Q}$ and $\hat{Y}$ are right angles and $\hat{R} = \hat{Z}$.

(a) Are the triangles that you have drawn the only ones you can draw satisfying the given conditions?

(b) How are $\triangle PQR$ and $\triangle XYZ$ related? You may now be able to conclude that:

If an acute angle of a right-angled triangle is congruent to an acute angle of another right-angled triangle, then the triangles are similar.

3.



(a) In the figure above, calculate the sizes of the following angles:

(i) $B\hat{D}C$ (ii) $D\hat{B}C$ (iii) $D\hat{B}A$ (iv) $B\hat{A}D$

(b) How many right-angled triangles are there in the figure?

(c) Name all the triangles in the figure which are similar? Why are they similar?

4. Draw triangle XYZ with a right angle at Y. From Y, draw the altitude $\overline{YH}$ of the triangle, perpendicular to $\overline{XZ}$.

Let $x = YZ$, $y = XZ$, $z = XY$, $p = XH$, $q = HZ$

(a) Name some similar triangles in the figure. Explain why they are similar.

(b) Since $\triangle XYH$ is similar to $\triangle XZY$, $\frac{p}{z} = \frac{z}{?}$

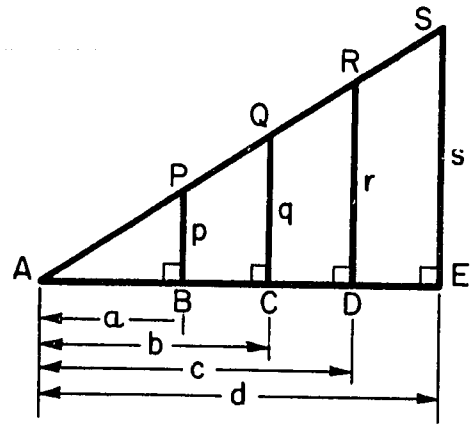
Hence $z^2 = \underline{?} \times \underline{?}$

Since $\triangle YZH$ is similar to $\triangle XZY$, $\frac{q}{x} = \frac{x}{?}$

Hence $x^2 = ? \times ?$

By adding the expressions for z^2 and x^2 which you now have, you can show, with a little simplification, that $z^2 + x^2 = y^2$. Write the steps of the simplification and explain each step. What name do you give to this property of right-angled triangles?

Problems 20-4



Using the above figure:

1. Name all the right-angled triangles.

2. Are these right-angled triangles similar? Why?

3. Show that $\frac{a}{b} = \frac{p}{q}$, $\frac{a}{c} = \frac{p}{r}$, $\frac{a}{d} = \frac{p}{s}$.

4. Show that

   (a) $\frac{a}{b} = \frac{p}{q}$ and $\frac{q}{b} = \frac{p}{a}$ are equivalent equations.

   (b) $\frac{a}{c} = \frac{p}{r}$ and $\frac{r}{c} = \frac{p}{a}$ are equivalent equations.

   (c) $\frac{a}{d} = \frac{p}{s}$ and $\frac{s}{d} = \frac{p}{a}$ are equivalent equations.

5. Can you see from (4) that

$$\frac{p}{a} = \frac{q}{b} = \frac{r}{c} = \frac{s}{d} \ ?$$

CHAPTER 21

TRIGONOMETRY

21-1 Direct and Indirect Measurement

You may have seen some tall building or some tall trees, or some street electric poles, and wondered how high they were. If you need to find the height of an electric pole, you may, at some risk, use a ladder to get to the top of the pole, and with the help of a rope loaded at the end, measure the height of the pole. You would thus have made a direct measurement.

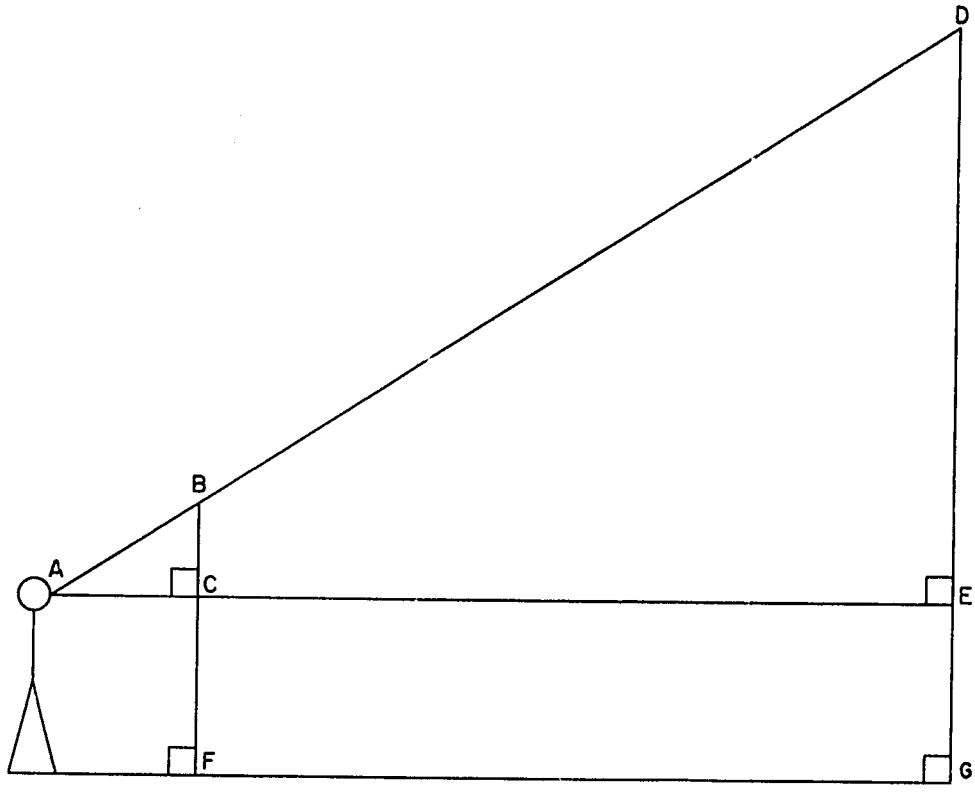
It may not always be practicable, or possible, for you to obtain heights by direct measurements. Can you see what difficulties will arise if you wish to determine the height of a hill or the height to which a kite is flown, by direct measurement? Fortunately, it is possible to determine heights by indirect measurement. We shall examine one method of doing this.

Class Activity

1. Select a tree, or post, or building on level ground, whose height you wish to measure.

Obtain a long staff (a Scout's staff will do).

Hold the staff vertically at arms length, with its end on the ground, and retreat until you can just see the top of the object you are measuring in line with the top of the staff. You would then have a situation which may be represented as below.



$\overline{DG}$ represents the tree, $\overline{BF}$ the staff, $\overline{BC}$, the portion of the staff above your eye level.

2. (a) Obtain the length of $\overline{AE}$, $\overline{AC}$, $\overline{BC}$, and by means of a scale drawing, find the height of the tree.

(b) Explain why $\triangle ABC$ and $\triangle ADE$ are similar.

(c) Fill in the blanks.

(i) $\frac{DE}{BC} = \frac{AE}{\quad}$

(ii) $\frac{BC}{AC} = \frac{\quad}{AE}$

(d) Find DE. What is the height of the tree?

3. (a) The angle $E\hat{A}D$ is called the <u>angle of elevation</u> of the top of the tree. Can you give a reason why the word "elevation" is used here? Notice that one side of the angle is <u>horizontal</u>.

(b) From your scale drawings measure the angle of elevation of the top of the tree from the point A.

4. (a) Select a building and find its height (i) by scale drawing, (ii) by the use of ratios of the sides of similar triangles.

(b) From your scale drawing, find the angle of elevation of the top of the building.

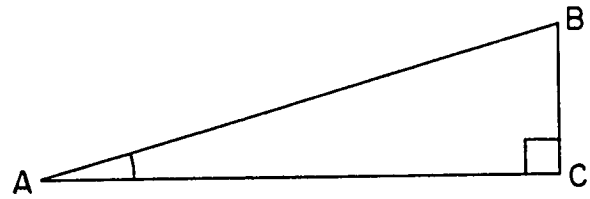
5. The angle of elevation of the top of a tower from a point on the ground 200 feet from the foot of the tower is 60°. By means of a scale drawing, find the height of the tower.

21-2 Tangents

You have seen that problems on heights of buildings may be solved by use of the properties of similar triangles. We will now see how some of these properties when generalized are of great use in solving these and many other problems.

It will be convenient first to give names to the sides of a right-angled triangle since we shall need to refer to them often.

We already have a name for the longest side, the hypotenuse. In triangle ABC, if $\hat{C}$ is the right angle, then $\overline{AB}$ is the hypotenuse.



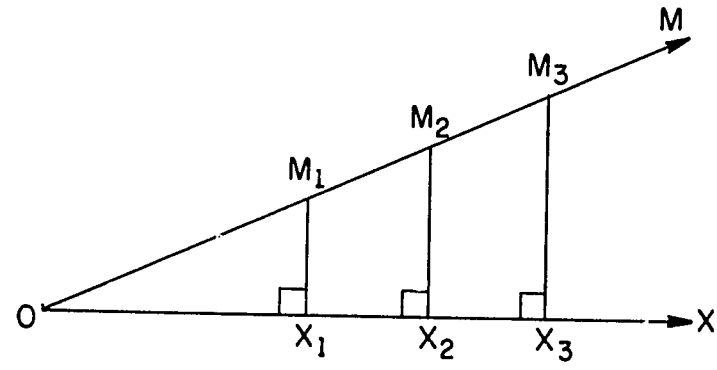
If we are interested in $\hat{A}$, it is natural to think of $\overline{CB}$ as being on the opposite side of the triangle and to think of $\overline{AC}$ as being next to or adjacent to $\hat{A}$. Hence we say that $\overline{CB}$ is the <u>side opposite $\hat{A}$</u> and $\overline{AC}$ is the <u>side adjacent to $\hat{A}$</u>.

<u>Problems 21-2a</u>

1. $\triangle XYZ$ is a right-angled triangle where $\hat{Y}$ is the right angle.
   (a) What is the side opposite $X\hat{Z}Y$?
   (b) What is the side adjacent to $X\hat{Z}Y$?
   (c) What is the side opposite $Y\hat{X}Z$?
   (d) What is the side adjacent to $Y\hat{X}Z$?

Class Activity

Draw the ray $\overrightarrow{OX}$ as shown. Let $\overrightarrow{OM}$ be a ray such that $X\hat{O}M$ is acute. Mark off points X_1, X_2, X_3, ... along $\overrightarrow{OX}$, and draw segments from these points perpendicular to $\overrightarrow{OX}$ to meet $\overrightarrow{OM}$ at M_1, M_2, M_3, ...



Show that $\triangle OX_1M_1$, $\triangle OX_2M_2$, $\triangle OX_3M_3$, ... are similar triangles.

Explain why $\dfrac{X_1M_1}{OX_1} = \dfrac{X_2M_2}{OX_2} = \dfrac{X_3M_3}{OX_3} = \ldots$

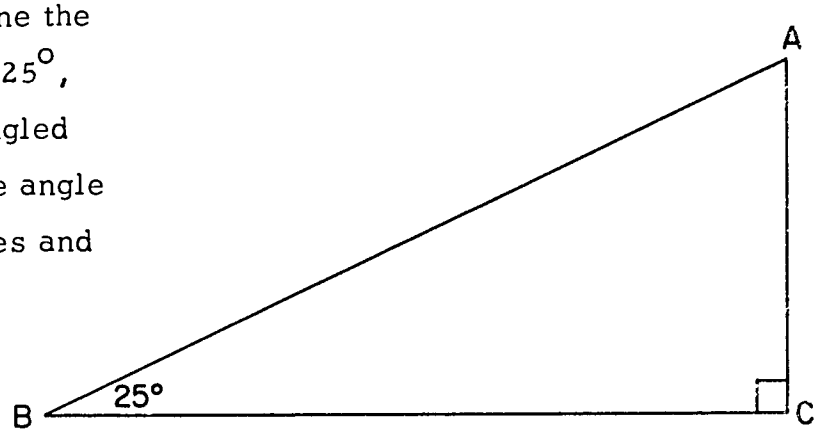
We have seen above that for a given angle $X\hat{O}M$, $\dfrac{X_1M_1}{OX_1} = \dfrac{X_2M_2}{OX_2} = \dfrac{X_3M_3}{OX_3} = \ldots$, that is, these are all names for the same ratio whenever the points X_1, X_2, X_3, ... are chosen on $\overrightarrow{OX}$. It is clear that the value of this ratio depends on $X\hat{O}M$. We call this ratio the <u>tangent of $X\hat{O}M$</u> and abbreviate it $\tan X\hat{O}M$.

$$\text{Hence } \tan X\hat{O}M = \frac{X_1M_1}{OX_1} = \frac{X_2M_2}{OX_2} = \frac{X_3M_3}{OX_3}$$

$$= \frac{\text{measure of side opposite } X\hat{O}M}{\text{measure of side adjacent to } X\hat{O}M}$$

The tangent is one of several ratios in triangles which are called <u>trigonometric ratios</u>. Trigonometry is the study of such ratios and has much to do with measurement in triangles.

Suppose we wish to determine the tangent of any angle, an angle of 25°, say. We can construct a right-angled triangle, such that the size of one angle is 25°. We then measure the sides and find the ratio.



In the figure above, the lengths of $\overline{CA}$ and $\overline{BC}$ are approximately 4·7 cm. and 10 cm., respectively.

$$\tan A\hat{B}C = \frac{\text{measure of side opposite } A\hat{B}C}{\text{measure of side adjacent to } A\hat{B}C} = \frac{CA}{BC} \approx \frac{4.7}{10} = 0.47$$

We write $\tan 25^\circ \approx 0.47$, where $\tan 25^\circ$ means the tangent of any angle of size 25°. The number 0·47 is an approximate value, subject to the errors of measurement.

Notice that we found it convenient to make $\overline{BC}$ 10 cm. long.

Problems 21-2b

1. By drawing and measurement, find approximate values for the tangents of angles with size 10°, 20°, 30°, 40°, 50°. Use a protractor to draw the angles.

2. Draw a segment $\overline{OX}$ 10 cm. long. Construct a perpendicular to $\overline{OX}$ at X. At O, with ruler and compasses only, construct angles of 15°, 30°, 45°, 60°, 75°, to meet the perpendicular at X at the points A, B, C, D, E. Find, correct to two decimal places,

(a) $\tan 15^\circ$
(b) $\tan 30^\circ$
(c) $\tan 45^\circ$
(d) $\tan 60^\circ$
(e) $\tan 75^\circ$

3. Using ruler and compasses, construct an equilateral triangle ABC of side 2 inches. Construct the altitude $\overline{AD}$ perpendicular to $\overline{BC}$.

(a) What is the size of $A\hat{B}C$?

(b) What is the length of $\overline{BD}$?

(c) Calculate the length of $\overline{AD}$.

(d) Calculate tan 60° and tan 30°.

4. Using ruler and compasses, construct a right-angled isosceles triangle ABC, where $\hat{B}$ is the right angle, such that the length of $\overline{AB}$ is 2 inches. Calculate tan 45°. Keep your drawings for Problems 3 and 4 for later use.

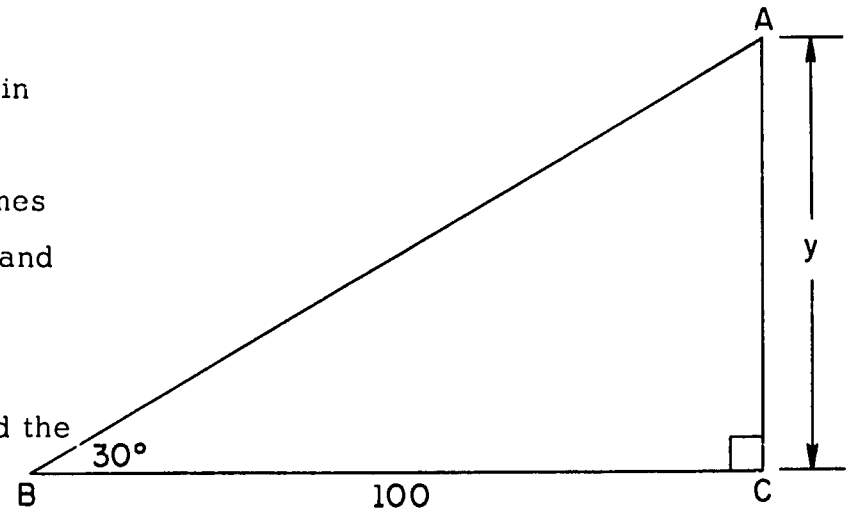
5. From your results in the previous questions, complete the following table, and save it for use in the problems to follow.

size of the angle	tangent of the angle
10°	
15°	
20°	
25°	
30°	
40°	
45°	
50°	
60°	
75°	

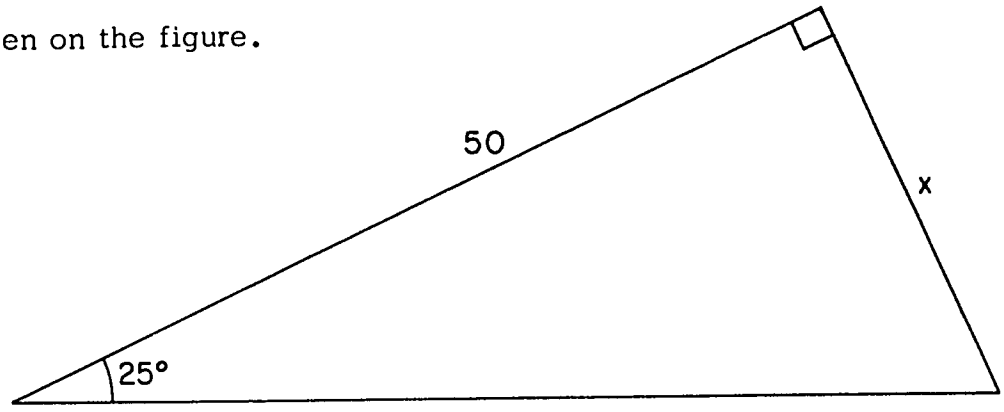
6. Use steps (a) and (b) to calculate the measure of $\overline{AC}$ in the figure.

(a) Write two other names for tan $A\hat{B}C$, using the figure and your table.

(b) Solve the equation $\frac{y}{100} = \tan 30^\circ$ and hence find the measure of $\overline{AC}$.



7. Find x from the information given on the figure.

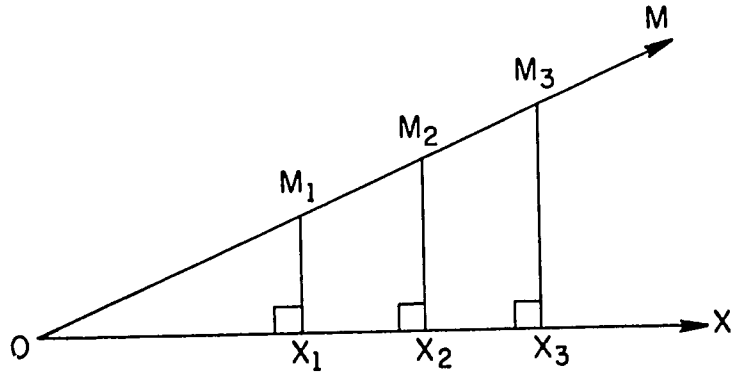


8. Find the height of a tower if the angle of elevation of the top of the tower from a point on the level ground 60 ft. from the foot of the tower is 75°.

9. A section of a tent is an isosceles triangle with sides making an angle of 40° with the ground. Its greatest width is 20 ft. Find its height.

21-3 Sines and Cosines

Let us consider again the figure we used to introduce the idea of the tangent of an angle. $\triangle OX_1M_1$, $\triangle OX_2M_2$, $\triangle OX_3M_3$, ... are similar triangles. Can you tell why? Hence $\dfrac{X_1M_1}{OM_1} = \dfrac{X_2M_2}{OM_2} = \dfrac{X_3M_3}{OM_3} = \ldots$



That is, for a given angle $X\hat{O}M$, no matter where the points X_1, X_2, X_3 are chosen on $\overrightarrow{OX}$, this ratio is the same number. We call this ratio the <u>sine of $X\hat{O}M$</u> and we abbreviate it sin $X\hat{O}M$.

$$\text{Hence } \sin X\hat{O}M = \frac{X_1M_1}{OM_1} = \frac{X_2M_2}{OM_2} = \frac{X_3M_3}{OM_3} = \ldots$$

$$= \frac{\text{measure of side opposite } X\hat{O}M}{\text{measure of hypotenuse}}$$

From this figure, we see also that $\frac{OX_1}{OM_1} = \frac{OX_2}{OM_2} = \frac{OX_3}{OM_3} = \ldots$ Can you say why?

We call this ratio the <u>cosine of $X\hat{O}M$</u>, and we abbreviate it cos $X\hat{O}M$.

$$\text{Hence } \cos X\hat{O}M = \frac{OX_1}{OM_1} = \frac{OX_2}{OM_2} = \frac{OX_3}{OM_3} = \ldots$$

$$= \frac{\text{measure of side adjacent to } X\hat{O}M}{\text{measure of hypotenuse}}$$

The ratios tangent, sine, and cosine are three of the trigonometric ratios.

<u>Problems 21-3a</u>

1. From your drawings of Problem 21-2b, Problems 3 and 4, calculate
   - (a) sine of 30°
   - (b) cosine of 30°
   - (c) sine of 60°
   - (d) cosine of 60°
   - (e) sine of 45°
   - (f) cosine of 45°

2. From a drawing find approximate values for
   - (a) sin 20°
   - (b) cos 20°
   - (c) sin 40°
   - (d) cos 40°

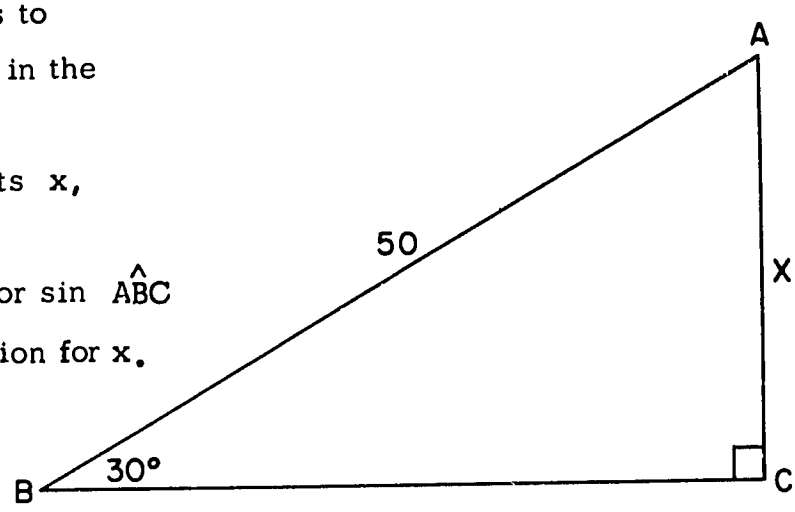
3. Using your answers to Problems 1 and 2 complete the following table and keep it for further use.

size of the angle	sine of the angle	cosine of the angle
20°		
30°		
40°		
45°		
60°		

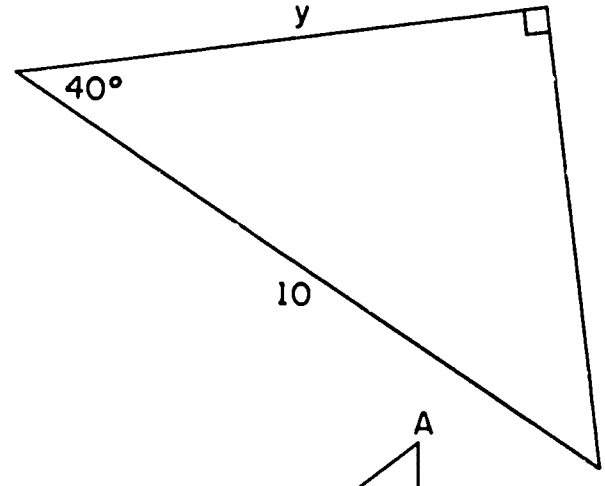
4. Use the following steps to help you find the value of x in the figure.

(a) What ratio connects x, 50, and $A\hat{B}C$?

(b) Write two names for sin $A\hat{B}C$ and solve the resulting equation for x.



5. Using the information given in the figure, find the value of y.



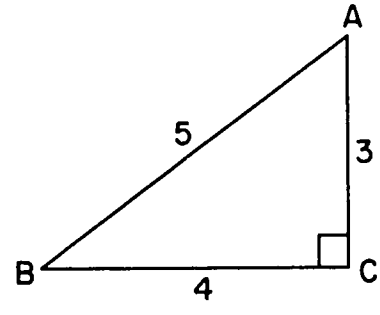
6. Using the figure, write ratios for

(a) sin $\hat{B}$ (d) sin $\hat{A}$

(b) cos $\hat{B}$ (e) cos $\hat{A}$

(c) tan $\hat{B}$ (f) tan $\hat{A}$

Do you notice any interesting results? State them.



7. Write in terms of lengths of sides (a, b, c) the following ratios:

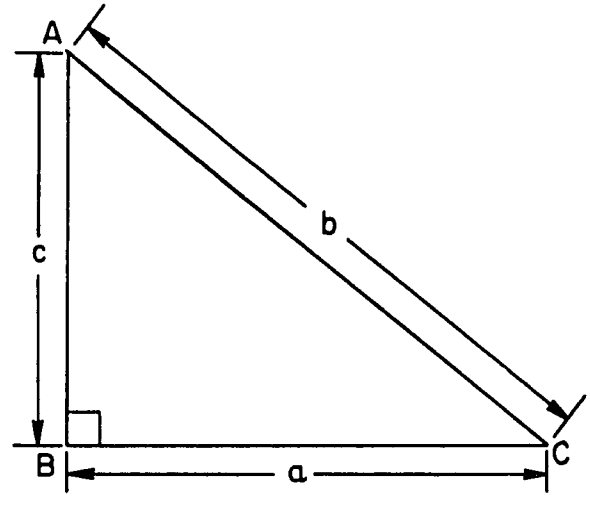
(a) sin $\hat{C}$ (e) $\frac{\sin \hat{A}}{\cos \hat{A}}$

(b) cos $\hat{A}$ (f) tan $\hat{A}$

(c) sin $\hat{A}$ (g) $\frac{\sin \hat{C}}{\cos \hat{C}}$

(d) cos $\hat{C}$ (h) tan $\hat{C}$

Do you notice any interesting results? State them.



8. Draw any right-angled triangle ABC where $\hat{B}$ is a right angle. Let b be the measure of $\overline{AC}$, let a be the measure of $\overline{BC}$, and let c be the measure of $\overline{AB}$. In terms of a, b, and c, write:

(a) $\sin \hat{A}$ (c) $\cos \hat{A}$

(b) $(\sin \hat{A})^2$ (d) $(\cos \hat{A})^2$

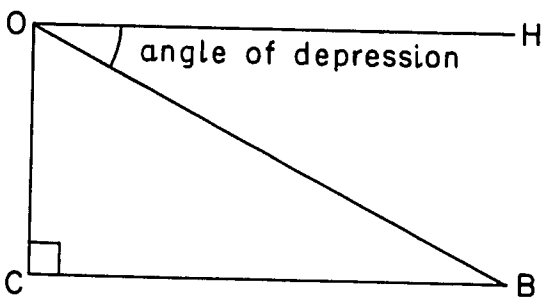
We abbreviate $(\sin \hat{A})^2$ as $\sin^2 \hat{A}$. Similarly we write $\cos^2 \hat{A}$ and $\tan^2 \hat{A}$ instead of $(\cos \hat{A})^2$ and $(\tan \hat{A})^2$. Write the value of $\sin^2 \hat{A} + \cos^2 \hat{A}$.

You now probably have discovered that

1. The sine of an angle is the same number as the cosine of the complementary angle.
2. The cosine of an angle is the same number as the sine of the complementary angle.
3. $\tan \hat{A} = \dfrac{\sin \hat{A}}{\cos \hat{A}}$
4. $\sin^2 \hat{A} + \cos^2 \hat{A} = 1$

Angle of Depression

When an observer at the top of a cliff observes a boat at sea, the angle formed by a horizontal ray and the ray connecting his eye to the boat is called the angle of depression. Angle $H\hat{O}B$ is an angle of depression.



Problems 21-3b

1. A ladder, 18 feet long, leans against a wall and makes an angle of 60° with the ground which is horizontal. How high up the wall does the ladder reach? How far from the wall is the foot of the ladder?

2. A man starts at O, and walks to a point P, 300 yd. north-east of O. If he reached P by first walking due north and then due east, how far would he have to walk in each direction?

3. The diagonal of a rectangle is 10 cm. long, and makes an angle of 60° with one of the sides. Find the lengths of the sides of the rectangle.

4. From the top of a building 60 feet high a man watches people walking along the street. If the angle of depression of the foot of a passer-by is 60° how far is the passer-by from the foot of the building?

5. Let P be the point (10, 10) on the number plane. What is the size of the angle $\overline{OP}$ makes with the x-axis, where O is the origin?

6. If Q is the point $(\sqrt{3}, 1)$, and O is the origin, what is the size of the angle $\overline{OQ}$ makes with the y-axis?

7. The shadow of an electric pole is 12 feet long when the angle of elevation of the sun is 60°. Find the length of the shadow when the angle of elevation of the sun is 45°.

8. A pendulum 12" long swings to and fro through an angle of 20° to each side. How high does the lower end of the pendulum rise?

21-4 Graphs of the Trigonometric Ratios

Let us study the behavior of the trigonometric ratios, $\sin x^\circ$, $\cos x^\circ$ and $\tan x^\circ$. We shall plot the graphs of these ratios so that we may observe their behavior more clearly.

Class Activity

1. (a) On squared paper draw a circle with centre O and radius 10 units.

(b) Draw the horizontal and vertical axes $\overrightarrow{OX}$, $\overrightarrow{OY}$. Let the circle intersect $\overrightarrow{OX}$ at A.

(c) Draw $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$, ..., $\overrightarrow{OR}$ to cut the circle at B, C, D, ... R so that the size of $X\hat{O}B$, $X\hat{O}C$, $X\hat{O}D$ ..., $X\hat{O}R$ is 5°, 10°, 15°, ..., 85° respectively.

(d) Write the coordinates of the points B, C, D, ..., R.

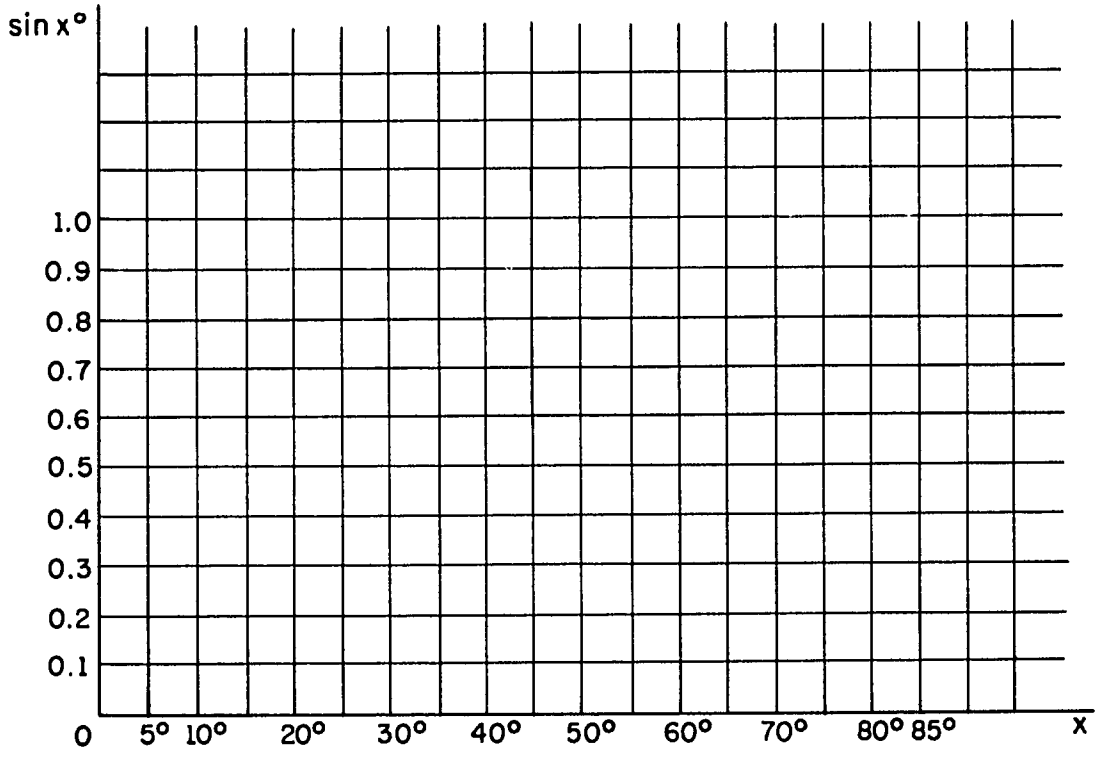
(e) We can now make a table of sines and cosines, since we note that $\sin X\hat{O}C = \dfrac{\text{y-coordinate of C}}{10}$. (Why?)

$\cos X\hat{O}C = \dfrac{\text{x-coordinate of C}}{10}$. (Why?)

Make a table like this and fill in the values of the sines and cosines.

size of the angle	sine of the angle	cosine of the angle
5°		
10°		
15°		
20°		
25°		
30°		
35°		
40°		
45°		
50°		
55°		
60°		
65°		
70°		
75°		
80°		
85°		

2. (a) Plot the values you have obtained for $\sin x^{\circ}$ $(0 < x < 90)$ on squared paper. Set up the coordinate axes like this:



(b) Do you observe any pattern?

(c) Join your points to obtain a smooth curve.

3. From your graph find approximate values of the following:

(a) $\sin 23^{\circ}$

(b) $\sin 41^{\circ}$

(c) $\sin 9^{\circ}$

(d) x if $\sin x^{\circ} = 0 \cdot 5$

(e) y if $\sin y^{\circ} = 0 \cdot 4$

(f) z if $\sin z^{\circ} = 0 \cdot 7$

21-5 Tables

Your graph has given you a means of finding approximately the sine of an acute angle, but the graph cannot give very accurate values. For situations where more accurate values are necessary, printed tables, called standard tables are available. Obtain one of these tables from your teacher or from the library and compare the numbers obtained from your graph with the numbers in the table.

Class Activity

1. (a) On the same axes on which you plotted the graph of $\sin x^\circ$ $(0 < x < 90)$ plot the graph of $\cos x^\circ$ $(0 < x < 90)$.

   (b) Obtain values of $\cos x^\circ$ from your graph and compare them with values in a printed table of cosines.

2. (a) Using the circle drawn in the previous Class Activity, draw a line at A parallel to the y-axis. Let $\overrightarrow{OB}$, $\overrightarrow{OC}$, ..., $\overrightarrow{OR}$ cut this line at B', C', ... R'.

   (b) $\tan X\hat{O}C' = \frac{AC'}{10}$ (Why?) Make a table of values of $\tan x^\circ$ as you did for $\sin x^\circ$ and $\cos x^\circ$.

   (c) Plot the graph of $\tan x^\circ$ $(0 < x < 90)$, using values from your table.

   (d) Obtain values of $\tan x^\circ$ from your graph, and compare these with values in a printed table of tangents.

Problems 21-5

1. Find the height of a tower if the angle of elevation of the top is 34° at a point 50 ft. from its foot.

2. The diagonal of a rectangle makes an angle of 39° with its largest side. Find its width if its length is 50 cm.

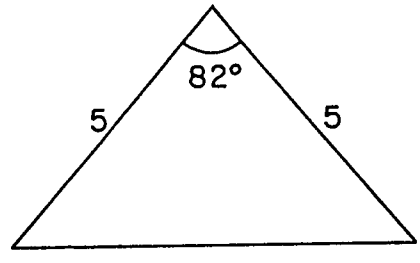
3. The length of a shadow of a 40 foot tree is 20 feet. What is the approximate size of the angle of elevation of the sun?

4. A 30-foot ladder leans against a vertical wall with its top at a height 25 feet from the ground. What is the approximate size of the angle that the ladder makes with the horizontal?

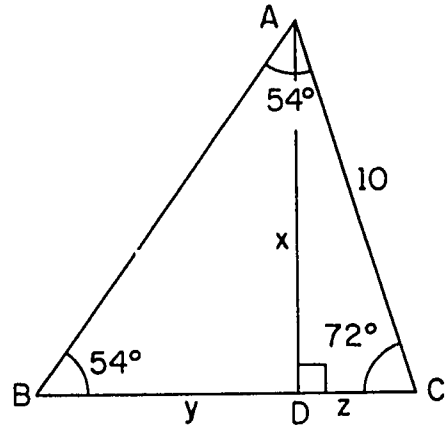
5. A hill slopes upwards at an angle of 14° with the horizontal. What height does a man rise when he has travelled 100 yards up the hill?

6. An airplane flies from Entebbe Airport on a course 52° east of due north for half an hour at an average speed of 300 miles an hour. How far is it (a) north , (b) east of the airport?

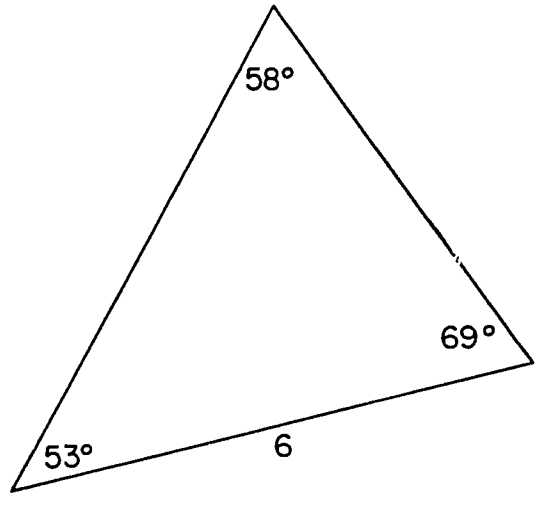
7. Calculate the area of the triangle in the figure on the right.



8. Calculate x, y, z, and hence find the area of the triangle shown.



9. Calculate the area of the triangle at the right.



CHAPTER 22

AREAS OF SIMILAR FIGURES

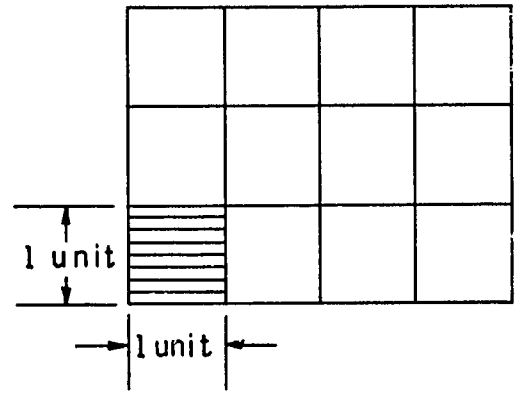
22-1 Areas of Basic Figures

We have seen that in two similar figures the ratios of the measures of corresponding sides are equal. Recall that if two figures are similar the larger one is the smaller magnified, that is, looked at through a magnifying glass. Similar figures have the same shape but may be different in size.

In this section we are going to consider the ratio of areas of two similar figures.

1. You may have met figures like this.

What does this tell you about the area of a rectangular region?



If the sides of a rectangle are of length 4 in. and 3 in. what is the area of the rectangular region?

Find the area of the rectangular region whose sides are of length:

(a) 7 in., 5 in.

(b) 2 cm., 3 cm.

(c) 4 miles, 2 miles

(d) 6 ft., 2 ft.

(e) a cm., b cm.

In general, if the sides of a rectangle are of length b units and h units and the number of square units in the area is A, can you express A in terms of b and h? From the examples above you may have seen that the <u>area of a rectangular region</u> is given by

$$\boxed{A = bh}$$

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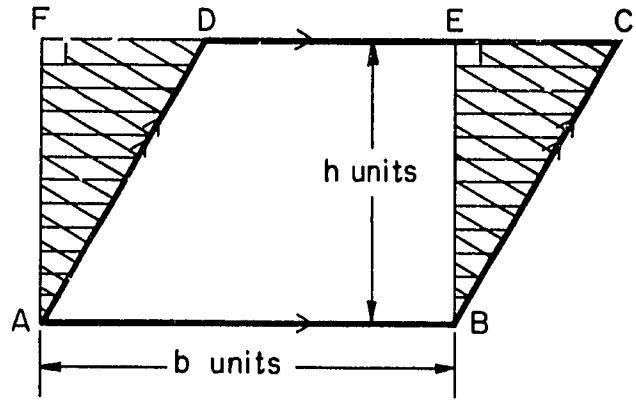
2. You may also have met this figure.



(a) Name the parallelograms in the figure.

(b) Which of the parallelograms is a rectangle?

(c) What can you say about the triangles AFD and BEC?

(d) Are they similar?

(e) Are they congruent?

(f) Are they equal in area?

(g) What is the union of the quadrilateral region ABED and the triangular region AFD ?

(h) What is the union of the quadrilateral region ABED and the triangular region BEC ?

(i) What can you say about the rectangular region ABEF and the parallelogram region ABCD?

(j) What is the area of the rectangular region ABEF?

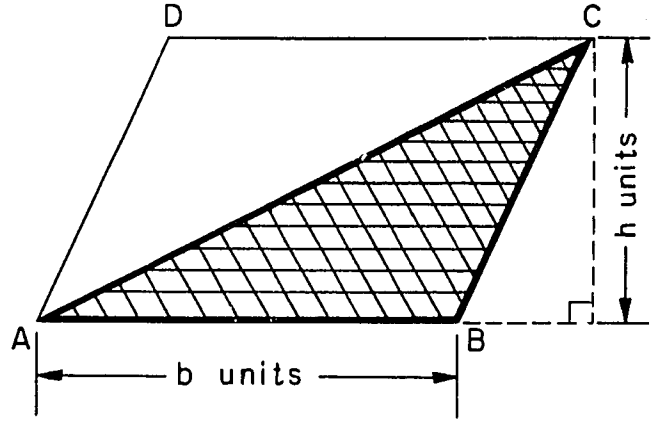
(k) What is the area of the parallelogram region ABCD?

(l) What does this tell you about the area of a parallelogram region?

Note that the rectangle and the parallelogram have a common side $\overline{AB}$, and that their heights are equal. We can say that the <u>area of a parallelogram region</u> is given by

$$\boxed{A = bh}$$

3. Do you remember this figure?



What is the area of the region enclosed by

(a) the parallelogram ABCD?

(b) the triangle ABC?

If the number of square units in the area of a triangular region is A, and if the base is b units and the height is h units, can you express A in terms of b and h ?

Our observations indicate that for a triangular region

$$\boxed{A = \frac{1}{2}bh}$$

We have so far tried to be careful about distinguishing between a triangle (which is a union of segments) and a triangular region (which includes also the points in the interior of the triangle). The word "area" applies to regions. In common practice the word "triangle" is used to mean either the triangle or the triangular region and we understand which meaning is intended from the way it is used. Thus when we say "area of a triangle", we really mean "area of a triangular region". The same understanding applies to the words "rectangle" and "parallelogram". We shall hereafter use the shorter form, omitting the word "region", but we shall still want to remember clearly the difference between a triangle and a triangular region when we say, "the area of a triangle is $\frac{1}{2}bh$".

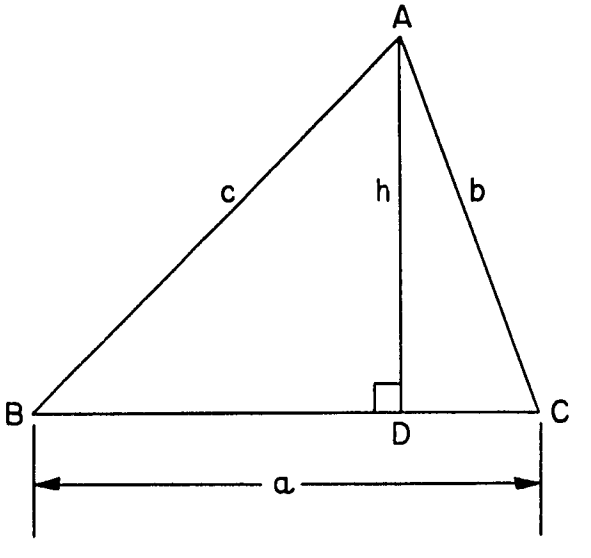
4. We can express the area of a triangle in another way, using trigonometry.

Using the figure at the right, find

(a) h in terms of b and $\hat{C}$.

(b) The area A of the triangle ABC in terms of a and h.

(c) An expression for A when the value for h obtained in (a) is substituted in (b).



You have probably obtained

$$\boxed{A = \frac{1}{2}ab \sin \hat{C}}$$

Problems 22-1

1. Draw triangles with sides having the following lengths. State what you notice about them and calculate their areas.

(a) 3 in., 4 in., 5 in.

(b) 5 in., 12 in., 13 in.

(c) 4·5 in., 6 in., 7·5 in.

(d) 7 in., 24 in., 25 in.

2. A triangle ABC is such that the length of $\overline{BC}$ is 6 cm., the length of $\overline{CA}$ is 8 cm., and the length of the altitude $\overline{AD}$ is 4 cm. How many triangles with these measurements but different shapes can you draw? What is the length of the altitude from B in each of them?

3. The area of a triangle is 24 square inches. If the altitudes are 4 in., 5 in., 6 in. in length, find the perimeter.

4. Find the area of a triangle ABC if the unit of length is an inch and
   (a) AC = 5, AB = 6, size of $\hat{A}$ is 30°.
   (b) BC = 6, AC = 5, AB = 5.
   (c) AB = 4, BC = 7, size of $\hat{B}$ is 60°.

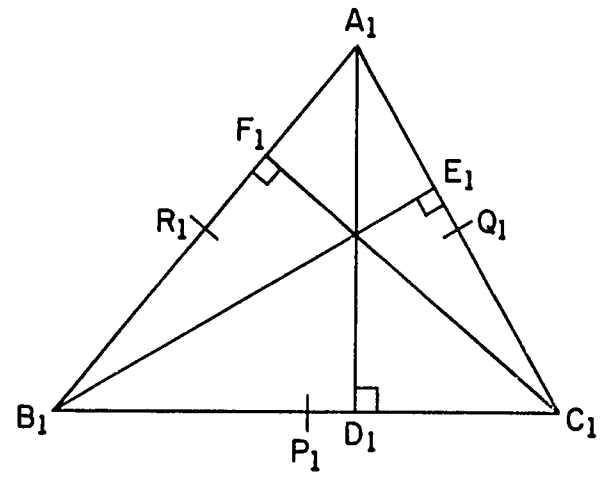
5. Use the sine graph to find the area of the following triangles:
   (a) $\triangle ABC$, given BC = 8, CA = 7, size of $\hat{C}$ is 40°.
   (b) $\triangle ABC$, given AB = 4, AC = 5, size of $\hat{A}$ is 62°.

22-2 Ratios in Similar Triangles

Class Activity

Draw a line segment, $\overline{B_1C_1}$.

With $\overline{B_1C_1}$ as common side, draw on the same side of $\overline{B_1C_1}$ angles of 50° and 60° at B_1 and C_1 respectively, and let the other sides meet at A_1.



Let the midpoint of the sides $\overline{B_1C_1}$, $\overline{C_1A_1}$ and $\overline{A_1B_1}$ be P_1,

Q_1, and R_1 respectively, and let the feet of the altitudes, $\overline{A_1D_1}$, $\overline{B_1E_1}$, and $\overline{C_1F_1}$ be D_1, E_1, and F_1.

Draw another triangle $A_2B_2C_2$ with $\hat{B}_2$ and $\hat{C}_2$ of size 50° and 60° respectively but with $\overline{B_2C_2}$ twice as long as $\overline{B_1C_1}$. As before, let $\overline{A_2D_2}$, $\overline{B_2E_2}$, and $\overline{C_2F_2}$ be the altitudes, and P_2, Q_2, and R_2 the midpoints of $\overline{B_2C_2}$, $\overline{C_2A_2}$, and $\overline{A_2B_2}$ respectively.

Make yourself a table like the one at the right. Measure $\overline{A_1B_1}$ and $\overline{A_2B_2}$ in your triangles, both in inches and in centimetres. Enter the measures in the table as well as their ratio $\frac{A_1B_1}{A_2B_2}$. Do the same for tables (b) and (c) below.

(a)	A_1B_1	A_2B_2	$\frac{A_1B_1}{A_2B_2}$
Inches			
Cm.			

(b)	A_1D_1	A_2D_2	$\frac{A_1D_1}{A_2D_2}$
Inches			
Cm.			

(c)	A_1P_1	A_2P_2	$\frac{A_1P_1}{A_2P_2}$
Inches			
Cm.			

Construct similar tables for the measures of the following pairs of line segments:

(d) $\overline{B_1C_1}$, $\overline{B_2C_2}$

(e) $\overline{B_1E_1}$, $\overline{B_2E_2}$

(f) $\overline{B_1Q_1}$, $\overline{B_2Q_2}$

(g) $\overline{C_1A_1}$, $\overline{C_2A_2}$

(h) $\overline{C_1F_1}$, $\overline{C_2F_2}$

(i) $\overline{C_1R_1}$, $\overline{C_2R_2}$

What can you say about the last columns of the tables? You observe that they all show approximately the same number, $\frac{1}{2}$.

Repeat the above activity, but this time use the triangles $A_1B_1C_1$ and $A_3B_3C_3$ where the length of $\overline{B_3C_3}$ is three times that of $\overline{B_1C_1}$, the angles remaining the same as before. The last columns also show approximately the same number, $\frac{1}{3}$.

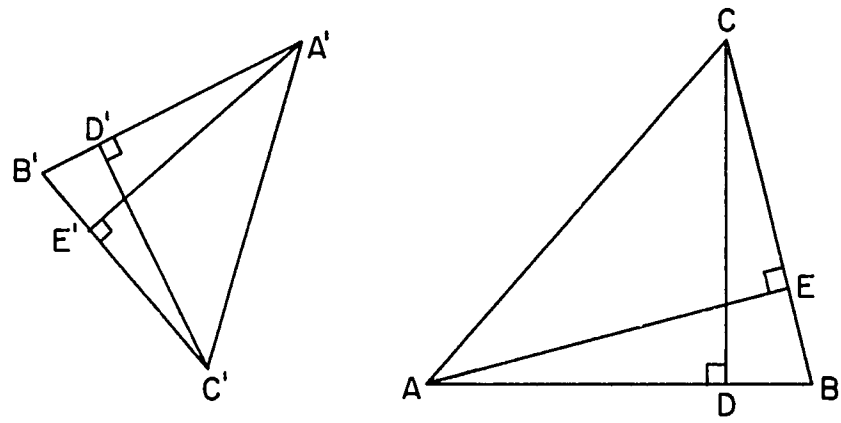
Notice that when we used the subscript 2, we used it to indicate that the side $\overline{B_2C_2}$ was two times as long as $\overline{B_1C_1}$, and that when the subscript was 3 we made $\overline{B_3C_3}$ three times as long as $\overline{B_1C_1}$.

The results illustrate the idea that:

If two triangles are similar, corresponding line segments are proportional.

Problems 22-2

1. $\triangle ABC$ is similar to $\triangle A'B'C'$ and altitudes are drawn as shown.



(a) If $AB = 12$ and $A'B' = 3$, find $\dfrac{AE}{A'E'}$.

(b) If $BC = 7$ and $B'C' = 1$, find $\dfrac{BA}{B'A'}$.

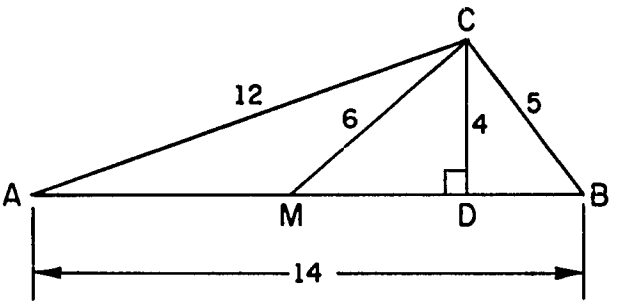
(c) If $C'D' = 3$ and $CD = 4$, find $\dfrac{C'A'}{CA}$

(d) If $A'E' = 6$ and $AE = 18$, find $\dfrac{AC}{A'C'}$.

(e) If $AC = 2$ and $A'C' = \frac{3}{2}$, find $\dfrac{C'D'}{CD}$.

(f) If $A'B' = 350$ and $AB = 750$, find $\dfrac{C'D'}{CD}$.

2. $\triangle PQR$ is similar to $\triangle ABC$, $\overline{CD}$ and $\overline{RT}$ are altitudes, and M and N are the mid-points of $\overline{AB}$ and $\overline{PQ}$ respectively.



AB = 14 CD = 4

BC = 5 CM = 6

CA = 12

(a) If PQ = 7, find PR.

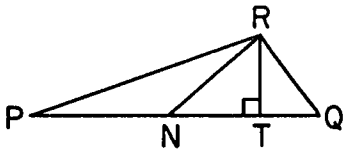
(b) If PR = 3, find RN.

(c) If PQ = 2, find RT.

(d) If RQ = 10, find RT.

(e) If RN = 1, find RQ.

(f) If PQ = 10, find TR.



22-3 Ratios of Areas of Similar Triangles

We now go on to find the ratio of the areas of two similar triangles.

Class Activity

As before, let $A_1B_1C_1$ and $A_2B_2C_2$ be two similar triangles, with $\overline{B_2C_2}$ twice as long as $\overline{B_1C_1}$. Let us abbreviate "area of triangle $A_1B_1C_1$" by the symbol "area $\triangle_1$", which we shall read "area triangle one". Express area $\triangle_1$ in terms of B_1C_1 and A_1D_1; express area $\triangle_2$ in terms of B_2C_2 and A_2D_2; and write the ratio of the areas.

$$\text{area } \triangle_1 = \frac{1}{2} \times B_1C_1 \times A_1D_1$$

$$\text{area } \triangle_2 = \frac{1}{2} \times B_2C_2 \times A_2D_2$$

$$\frac{\text{area } \triangle_1}{\text{area } \triangle_2} = \frac{\frac{1}{2} \times B_1C_1 \times A_1D_1}{\frac{1}{2} \times B_2C_2 \times A_2D_2} = \frac{B_1C_1}{B_2C_2} \times \frac{A_1D_1}{A_2D_2}$$

What do you know about the ratios $\frac{B_1C_1}{B_2C_2}$ and $\frac{A_1D_1}{A_2D_2}$?

Since these are ratios of the measures of corresponding line segments in similar triangles, they are equal and in this case their value is $\frac{1}{2}$.

What is this going to give us for the ratio of the areas, $\frac{\text{area } \Delta_1}{\text{area } \Delta_2}$?

$$\frac{\text{area } \Delta_1}{\text{area } \Delta_2} = \frac{B_1C_1}{B_2C_2} \times \frac{A_1D_1}{A_2D_2} = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

Thus when the ratio of corresponding sides is $\frac{1}{2}$, the ratio of the areas is $\frac{1}{4}$. What will be the ratio of the areas of similar triangles when the ratio of corresponding sides is $\frac{1}{3}$?

$$\frac{\text{area } \Delta_1}{\text{area } \Delta_3} = \frac{\frac{1}{2} \times B_1C_1 \times A_1D_1}{\frac{1}{2} \times B_3C_3 \times A_3D_3} = \frac{B_1C_1}{B_3C_3} \times \frac{A_1D_1}{A_3D_3}$$

$$= \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$$

Using the same kind of reasoning, find each of the following ratios:

$$\frac{\text{area } \Delta_2}{\text{area } \Delta_1}, \quad \frac{\text{area } \Delta_3}{\text{area } \Delta_1}, \quad \frac{\text{area } \Delta_2}{\text{area } \Delta_3}, \quad \frac{\text{area } \Delta_1}{\text{area } \Delta_4}, \quad \frac{\text{area } \Delta_k}{\text{area } \Delta_1}$$

This discussion illustrates the fact that:

If two triangles are similar, the ratio of their areas is equal to the ratio of the squares of the measures of corresponding segments.

Problems 22-3

1. In $\triangle ABC$, D and E are the mid-points of $\overline{AB}$ and $\overline{AC}$ respectively and $\overline{DE}$ is parallel to $\overline{BC}$. What can you say about triangles ADE and ABC ? Find the ratio of the area of triangle ADE to that of triangle ABC.

2. X and Y are points on $\overline{AB}$ and $\overline{AC}$, respectively, of $\triangle ABC$ such that $\overline{XY}$ is parallel to $\overline{BC}$. Find the ratio of the area of $\triangle AXY$ to that of $\triangle ABC$ if:

(a) $\frac{AX}{AB} = \frac{1}{4}$

(b) $\frac{AY}{AC} = \frac{2}{3}$

(c) $\frac{AX}{XB} = \frac{1}{2}$

(d) $\frac{XY}{BC} = \frac{3}{5}$

(e) $\frac{AY}{AC} = \frac{h}{k}$

(f) $\frac{AX}{XB} = \frac{k}{2}$

(g) $\frac{AY}{YC} = \frac{h}{k}$

3. Triangles ABC and XYZ are similar.

(a) If $\frac{XY}{AB} = \frac{3}{2}$ and the area of $\triangle ABC$ is 8 sq. in., find the area of $\triangle XYZ$.

(b) If the area of $\triangle XYZ$ is 4 times the area of $\triangle ABC$ and $\overline{YZ}$ is 6 ins. long, find the length of $\overline{BC}$.

4. CHALLENGE PROBLEM.

$\triangle ABC$ is a triangle right-angled at A. Points X, Y, Z are taken outside the triangle such that triangles ABX, CAY, BCZ, are similar. Prove that the sum of the areas of triangles ABX and CAY is equal to the area of triangle BCZ.

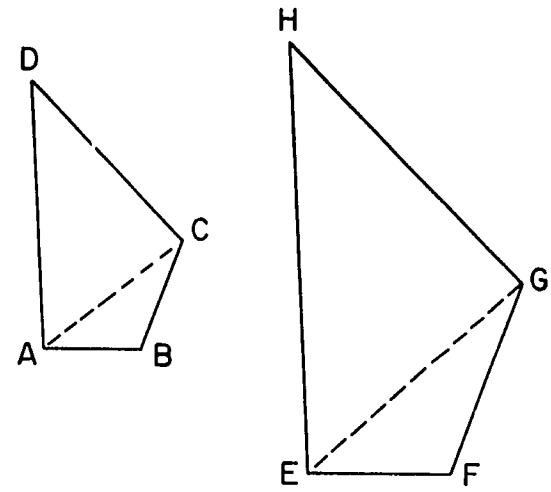
22-4 Ratios of Areas of Other Similar Polygons

Class Activity

Here are two similar quadrilaterals ABCD, EFGH. For these two figures write all you know about

(a) corresponding angles;

(b) corresponding sides.



If you draw $\overline{AC}$ and $\overline{EG}$, what can you say about

(a) triangles ABC and EFG ?

(b) the ratio $\frac{AC}{EG}$?

(c) the ratio of the area of $\triangle ABC$ to the area of $\triangle EFG$?

(d) the ratio of the area of $\triangle ADC$ to the area of $\triangle EHG$?

What do you think will be true about the ratio of the areas of the two quadrilaterals?

Suppose $\frac{AB}{EF} = k$. Write all other ratios of lengths in the two figures which are equal to k.

You have learnt in the preceding section that the ratio of the area of $\triangle ABC$ to the area of $\triangle EFG$ is k^2.

$$\text{Since} \quad \frac{\text{area } \triangle ABC}{\text{area } \triangle EFG} = k^2,$$

$$\text{area } \triangle ABC = k^2 (\text{area } \triangle EFG).$$

Because of the common sides $\overline{AC}$ and $\overline{EG}$, the ratio of corresponding sides of $\triangle ADC$ and $\triangle EHG$ is the same number k as the ratio of corresponding sides of the first two triangles. Thus

$$\frac{\text{area } \triangle ADC}{\text{area } \triangle EHG} = k^2, \text{ and}$$

$$\text{area } \triangle ADC = k^2 (\text{area } \triangle EHG).$$

It follows that

$$\text{area } \triangle ABC + \text{area } \triangle ADC = k^2 (\text{area } \triangle EFG) + k^2 (\text{area } \triangle EHG)$$
$$= k^2 (\text{area } \triangle EFG + \text{area } \triangle EHG)$$

$$\frac{\text{area } \triangle ABC + \text{area } \triangle ADC}{\text{area } \triangle EFG + \text{area } \triangle EHG} = k^2$$

$$\frac{\text{area } ABCD}{\text{area } EFGH} = k^2$$

Thus the ratio of the areas of the similar quadrilaterals is the square of the ratio of measures of any pair of corresponding sides.

Compare this result with what we learnt about ratios of areas of similar triangles. Would it make any difference if we took <u>any</u> pair of similar polygons? From the above discussion we see that

The ratio of the areas of two similar polygons is equal to the ratio of the squares of the measures of corresponding sides.

Table of Squares and Square Roots

n	n^2	$\sqrt{n}$	$\sqrt{10n}$
1	1	1000	3162
2	4	1414	4472
3	9	1732	5477
4	16	2000	6325
5	25	2236	7071
6	36	2449	7746
7	49	2646	8367
8	64	2828	8944
9	81	3000	9487
10	100	3162	1000
11	121	3317	1049
12	144	3464	1095
13	169	3606	1140
14	196	3742	1183
15	225	3873	1225
16	256	4000	1265
17	289	4123	1304
18	324	4243	1342
19	361	4359	1378
20	400	4472	1414
21	441	4583	1449
22	484	4690	1483
23	529	4796	1517
24	576	4899	1549
25	625	5000	1581
26	676	5099	1612
27	729	5196	1643
28	784	5292	1673
29	841	5385	1703
30	900	5477	1732

n	n^2	$\sqrt{n}$	$\sqrt{10n}$
31	961	5568	1761
32	1024	5657	1789
33	1089	5745	1817
34	1156	5831	1844
35	1225	5916	1871
36	1296	6000	1897
37	1369	6083	1924
38	1444	6164	1949
39	1521	6245	1975
40	1600	6325	2000
41	1681	6403	2025
42	1764	6481	2049
43	1849	6557	2074
44	1936	6633	2098
45	2025	6708	2121
46	2116	6782	2145
47	2209	6856	2168
48	2304	6928	2191
49	2401	7000	2114
50	2500	7071	2236
51	2601	7141	2258
52	2704	7211	2280
53	2809	7280	2302
54	2916	7348	2324
55	3025	7416	2345
56	3136	7483	2366
57	3249	7550	2388
58	3364	7616	2408
59	3481	7681	2429
60	3600	7746	2449

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n	n^2	$\sqrt{n}$	$\sqrt{10n}$
61	3721	7810	2470
62	3844	7874	2490
63	3969	7937	2510
64	4096	8000	2530
65	4225	8062	2550
66	4356	8124	2569
67	4489	8185	2588
68	4624	8246	2608
69	4761	8307	2627
70	4900	8367	2646
71	5041	8426	2665
72	5184	8485	2683
73	5329	8544	2702
74	5476	8602	2720
75	5625	8660	2739
76	5776	8718	2757
77	5929	8775	2775
78	6084	8832	2793
79	6241	8888	2811
80	6400	8944	2828
81	6561	9000	2846
82	6724	9055	2864
83	6889	9110	2881
84	7056	9165	2898
85	7225	9220	2915
86	7396	9274	2933
87	7569	9327	2950
88	7744	9381	2966
89	7921	9434	2983
90	8100	9487	3000

n	n^2	$\sqrt{n}$	$\sqrt{10n}$
91	8281	9539	3017
92	8464	9592	3033
93	8649	9644	3050
94	8836	9695	3066
95	9025	9747	3082
96	9216	9798	3098
97	9409	9849	3114
98	9604	9899	3130
99	9801	9950	3146

GLOSSARY

VOLUME 3*

Absolute value (page 235): $|a| = a$ if $a \geq 0$, $|a| = -a$ if $a < 0$. The absolute value of a is the distance on the number line from the point representing a to the point 0.

Addition, basic properties of (page 14): Fundamental properties of addition from which other properties may be derived.

Additive inverse (page 12): For any number x, the additive inverse is that number a such that $x + a = 0$. See opposite.

Angle of depression (page 332): The angle formed by a horizontal ray and the line of sight of an observer looking down.

Angle of elevation (page 324): The angle formed by a horizontal ray and the line of sight of an observer looking upward.

Arcs, major and minor (page 139): The larger and smaller parts of a circle determined by two points on the circle.

Area (page 339): A number assigned to a plane region as a measure of its size. The number is determined by the standard region which is chosen as the unit of measure.

Associative property of addition (page 14): $a + (b + c) = (a + b) + c$

Associative property of multiplication (page 22): $a(bc) = (ab)c$

Central angle (page 142): An angle with its vertex at the centre of a circle.

Chord (page 140): A segment with its end points on a circle.

Commutative property of addition (page 14): $a + b = b + a$.

Commutative property of multiplication (page 22): $ab = ba$.

Congruent angles (page 97): Angles which have the same size.

* The glossary for Volume 3 includes the words from the glossary for Volume 1 and Volume 2.

Congruent circles (page 97): Circles which have radii of the same length.

Congruent figures (page 104): Figures which have the same size and shape; figures which can fit exactly according to at least one correspondence.

Congruent line segments (page 97): Line segments which have the same length.

Constant of proportionality (page 270): The constant k in the relation $y = kx$ which indicates that y is proportional to x.

Continuous line chart (page 300): A kind of chart on which information is represented by means of a curve.

Converse (page 179): A statement in which the hypothesis and conclusion of a previous statement have been interchanged.

Coordinate axes (page 243): The two number lines in the number plane by which coordinates of a point in the plane are determined.

Coordinates (page 243): The ordered pair of numbers which is associated with a point of the number plane.

Correspondence, one-to-one (page 98): A matching of members of one set with members of another set such that each member of either set matches exactly one member of the other set.

Cosine of an angle (page 332): The ratio $\frac{\text{side adjacent to the angle}}{\text{hypotenuse}}$ in a right-angled triangle.

Decimal notation (page 181): Numeration for real numbers using the base ten.

Decimal places (page 188): The positions of the digits after the decimal point in a decimal numeral.

Difference (page 16): The number resulting from the operation of subtraction.

Direct measurement (page 323): Measurement by applying the measuring device directly to the object to be measured.

Distributive property (page 22): $a(b + c) = ab + ac$.

Division (page 27): The operation in $a \div b$ or $\frac{a}{b}$. $a \div b = x$ $(b \neq 0)$ means that $bx = a$.

Domain of a variable (page 31): The set of numbers from which the value of the variable may be chosen.

Envelope (page 163): The curve determined by a set of lines or curves and touching each line or curve in the set.

Equivalent equations (page 32): Equations which have the same truth set.

Equivalent inequalities (page 38): Inequalities which have the same truth set.

Expanded form of a decimal numeral (page 181): Decimal numeral written out showing the powers of ten.

Field (page 80): A number system having the properties of (1) closure for addition and multiplication, (2) commutativity for addition and multiplication, (3) associativity for addition and multiplication, (4) distributivity, (5) additive and multiplicative identities, (6) additive and multiplicative inverses.

Formula (page 47): A representation of a number in terms of one or more variables.

Graph of an open sentence in two variables (page 254): The set of points on the number plane which represents the truth set of the open sentence.

Hexagon (page 122): A six-sided polygon.

Images (page 125): Two sets of points which are symmetrical to each other about a line, point or plane.

Indirect measurement (page 323): Calculation of the measure of an object by means of other, related measurements, without actually applying a measuring device to the original object.

Inscribed angle (page 144): An angle with its vertex on a circle and subtended by an arc of the circle.

Inscribed polygon (page 123): A polygon each of whose vertices lies on a circle.

Intercept on the y-axis (page 263): The second coordinate of the point where the line intersects the y-axis.

Irrational number (page 216): A number whose decimal numeral is neither terminating nor repeating. The number is not rational.

Linear equation in two variables (page 254): An equation, the graph of which is a straight line. Its equation can be written in the form $ax + by + c = 0$.

Locus (page 160): The set of all points which have a given property.

Minus (page 17): Indicates subtraction of the two numbers mentioned, the second from the first.

Multiplication, basic properties of (page 22): Fundamental properties of multiplication from which other properties may be derived.

Multiplicative inverse (page 28): Two numbers a and b are multiplicative inverses of one another if $ab = 1$. Same as reciprocal.

Negative numbers (page 3): The numbers which correspond to points to the left of zero on the number line.

Non-repeating decimal numeral (page 216): A decimal numeral that neither terminates nor repeats.

Octagon (page 122): An eight-sided polygon.

One (page 22): That number which, multiplied by any number x, gives the same number x as the result.

Open sentence (page 6): Sentence containing a variable.

Opposites (page 6): Two numbers are opposites if they correspond, on the number line, to points on opposite sides of 0 and equally far from 0. Also, the opposite of 0 is 0. The opposite of x is denoted by $-x$. Opposites have the basic property: $x + (-x) = 0$.

Ordered pair of numbers (page 56): Two numbers of which one is designated as first and the other second, written (x, y).

Origin (page 243): The point at the intersection of the two axes in the number plane.

Percentage (page 196): A fraction with 100 as its denominator.

Positive numbers (page 2): The numbers which correspond to points to the right of zero on the number line.

Proportional (page 270): y is proportional to x if $y = kx$ and k is a non-zero real number.

Rate (page 196): The numerator of a fraction written as a percentage (with denominator 100).

Rational number (page 3): A number which can be expressed as the quotient of two integers.

Real field (page 234): The real numbers are a field under the operations of addition and multiplication.

Real numbers (page 222): The set of real numbers is the union of the set of rational numbers and the set of irrational numbers.

Reciprocal (page 28): Two numbers a and b are reciprocals of one another if $ab = 1$. Same as multiplicative inverse.

Regular polygon (page 123): A polygon in which all sides are congruent and all angles are congruent.

Repeating decimal numeral (page 210): A decimal numeral in which a block of digits repeats itself endlessly, each block occuring immediately after the preceding block.

Scatter diagram (page 298): A kind of chart on which information, usually numerical, is represented by means of disconnected points.

Side adjacent to an angle (page 325): The leg of a right-angled triangle which has the vertex of the angle as an end-point.

Side opposite an angle (page 325): The side of a triangle, usually a right-angled triangle, which does not have the vertex of the angle as an end point.

Similar polygons (page 311): Polygons with the same shape. The corresponding angles are congruent and the ratios of corresponding sides are equal.

Sine of an angle (page 330): The ratio $\frac{\text{side opposite the angle}}{\text{hypotenuse}}$ in a right-angled triangle.

Slope of a line (page 257): The ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$, for any two points on the line.

Smaller of two numbers (page 37): The one that must have a positive number added to it to give the other.

Solution of an equation in two variables (page 55): An ordered pair of numbers for which the equation is true.

Solution set (pages 31 and 58): See truth set.

Solve an equation or an inequality (page 31): Find the truth set.

Subtended by (page 144): An angle is subtended by a chord (or arc) if the ends of the chord (or arc) are on the sides of the angle, but not at the vertex.

Subtraction (page 16): The operation in $a - b$. $a - b = x$ means that $b + x = a$.

Symmetry about an axis (page 129): Two points are symmetrical about an axis if the axis is the perpendicular bisector of the segment joining the points.

Symmetry about a plane (page 135): Two points are symmetrical about a plane if the plane is perpendicular to and bisects the segment joining the points.

Symmetry about a point (page 132): Two points are symmetrical about a point if the point is the mid-point of the segment joining the points.

Tangent of an angle (page 326): The ratio $\frac{\text{side opposite the angle}}{\text{side adjacent to the angle}}$ in a right-angled triangle.

Trigonometric ratios (page 326): The ratios relating sides of a right-angled triangle. The common ones are called sine, cosine, tangent.

Truth set of an equation in one variable (page 31): The set of all numbers which make the equation true.

Truth set of a system of two equations in two variables (page 58): The set of all ordered pairs of numbers which make both equations true.

Variable (page 47): A symbol, usually a letter, used to represent any one of a given set of numbers.

Zero (page 13): That number which, added to any number x, gives the same number x as the result.