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NOTE

This book is the product of a study and writing workshop conducted during the summer of 1962 at Entebbe, Uganda, with more than fifty mathematicians and mathematics teachers from Africa, the United States, and the United Kingdom in attendance.

In order that it might be tested in African schools as quickly as possible, the book was edited and produced with the utmost speed. As a consequence, both editing and production suffer from defects. More time might have made it possible to eliminate most of these, but it would also have made it impossible to try out this book during the 1962-63 academic year, and it was held that this latter need took priority over the other. The African Education Program can only hope that those who use this book will appreciate the circumstances under which it has been made available, and will be tolerant of its imperfections.

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TEACHERS' GUIDE :

VOLUME ONE

MATHEMATICS SECONDARY ONE ;

Introduction to Algebra and to the Language of Geometry

Prepared at the
Entebbe Mathematics Workshop

African Education Study
Educational Services Incorporated

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PREFACE

This Teachers' Guide, like the text which it accompanies, is one of the products of a writing and study workshop held during the summer of 1962 at Entebbe, Uganda, in which mathematicians and mathematics teachers from all English-speaking Tropical Africa, the United States, and the United Kingdom participated.

The workshop was intended to take the first steps toward the preparation of curricular materials for use in the teaching of mathematics in African schools. Ultimately it is to be hoped that the completed materials will cover the entire range of public schooling, from the earliest years through the Sixth Form, and that they will include audio-visual materials and collateral reading as well as text and teachers' guide. To date only a modest start has been made, represented by half a text and half a teachers' guide for the first year of secondary school; the equivalent for the first year of primary school; a volume of general curriculum recommendations; and a volume of suggestions for the production of non-textual teaching aids.

In making use of the text and teachers' guide for Secondary I, both teacher and class will best be served if there is some general acquaintance with the nature of the materials so far produced, and with the underlying principles of the total program.

I. Existing Materials are Preliminary

Materials produced for use in the schools must be considered to be preliminary until they have been tested in the schools, and reviewed in accordance with the results of such tests. This teachers' guide, for example, must be extensively corrected in the light of the experience of teachers. Over the next few years the task of experimentation and correction will be at least as important to the program as any new writing that may be done.

This signifies that the teacher who uses this material will face a twofold task: that of working with materials that are known to be subject to improvement, and that of helping identify the areas where

improvement is required. Yet the teacher who shoulders this burden will have the satisfaction of knowing that he or she is taking part in an experiment which is of great potential value to all of Africa, to all other areas faced with rapidly expanding educational systems, and indeed even to those countries with well-established systems which have failed to keep pace with modern developments.

II. Emphasis Is On Mathematical Ideas

In recent years there has been accumulated much evidence that young students are far more interested in mathematical ideas than they have usually been given credit for, and that they are far more competent to deal with such ideas than current curricula would suggest. A presentation of mathematics that puts its emphasis on concept rather than on the rules of manipulation is likely to lead to far greater satisfaction on the part of the student, and will also lead to a far greater degree of mathematical competence.

The teacher may be surprised to discover that some mathematics in the Secondary I text introduces concepts which are commonly first met only at a far more advanced level. Here again, the experience of recent years has led to the conviction that young students are capable of far more than they have usually been given credit for. The teacher should not draw away from the presentation of advanced concepts, or attempt to dilute them, for when materials are well-prepared,a high degree of confidence in the student is rarely misplaced.

III. Content of the Text

The major portion of the text is devoted to an explanation of the structure of arithmetic in terms of algebra. The method used is that of building up, on the basis of the student's primary work, a small list of basic properties. In terms of these basic properties, all the usual computational rules can be deduced. We do not, at this grade level, give a complete treatment; the student must meet the same ideas again later. What is intended, at this stage of the student's progress, is a stimulation of insight.

There is also a considerable amount of geometry, much of which is not found in most texts. Finally, applications of arithmetic, algebra, and geometry occur throughout the book.

IV. The General Approach

The best education is the education that the student creates out of his own direct efforts. The teacher must resist the temptation to tell the class what to do and how to do it. It is indeed a great temptation, for by such means a class will appear to be proceeding at a far more rapid pace. But if the teacher takes the time, and exerts the effort, to get the student himself to generate the right idea, there is considerably more assurance that the idea will be mastered and retained by the student, and will become truly his own.

Here again, the demands upon the teacher are greater. The teacher, like the student, must be thinking at every moment, for it takes far more insight to lead than to direct. This guide is intended primarily to assist the teacher in the actual conduct of the class, by methods which are chosen to encourage student imagination and to generate student interest. In addition, each chapter contains mathematical background and explanatory material beyond that given in the text. Answers are also provided for most of the problems, often in some detail.

V. Relationship to Other Material

The text presumes that the student has studied conventional materials in earlier grades. It is designed to prepare the student for further mathematics in either the conventional curriculum or in curricula which will be devised in the next few years to follow upon this text.

The text is designed also to conform to the requirements of present examinations.

Finally, it should be repeated that this modified curriculum represents a great opportunity for teacher and student alike. In making use of it, the school participates in a great experiment, out of which there will come an African educational system that will be in

the forefront of educational systems throughout the world. It is a massive joint undertaking, in which the student and the teachers work side by side with mathematicians of international eminence, from their own country and from a dozen others, to create within Africa something that will be of major significance for Africa itself, and in a large measure for all the world.

FIRST YEAR SECONDARY TEACHERS' GUIDE

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FIRST YEAR SECONDARY TEACHERS GUIDE

CHAPTER 1

NUMBERS, NUMERALS and NUMERATION

Commentary

The basic concepts underlying this chapter are not new to the pupil. It is the environment in which they are placed that is new. For example, in their daily experience, children do differentiate between an object and the name of it in their own language. When they are made aware of this, the transition to the difference between a number and its name (numeral) is facilitated. The idea of place value was dealt with throughout the primary grades. In this chapter, place value is reviewed as a property of the Hindu-Arabic system of numeration regardless of the base in which the numerals are written. It is stressed again in the presentation of the concept of an expanded numeral. The process of regrouping, already familiar as "carrying" and "borrowing" is looked at anew in studying the algorithms for addition and subtraction of numbers written in any base. Probably the only new idea is that of an exponent, or index.

It is hoped that this exposure will clarify and vivify the ideas considered. A study of numeration in other bases provides a background for eventual discussion of the properties of numbers. For instance, if a student defines an even number as one that "ends in 0, 2, 4, 6, or 8" the study of this chapter in which the even number, eight, is written $(11)_{seven}$ or $(13)_{five}$ should give him pause and make him aware of the need to revise his definition of an even number. The student should see clearly that the <u>numeral</u> does not affect the properties of the <u>number</u>. Every rational number may be written as a terminating or repeating decimal numeral. This property of the <u>number</u> is not affected by writing the numerals in another base, except that we do not use "decimal numeral" unless we are working in base 10. In

base 5 we would use "quintal numeral" and in base 7 we would use "septal numeral."

Better appreciation of the idea that every place value is a power of the base is a goal of this study.

$(4213)_{five}$ tells us that we have 3 ones, 1 five (first power of five) 2 twenty-fives (second power of five), and 4 one hundred twenty-fives (third power of five).

$(4213)_{seven}$ tells us that we have 3 ones, 1 seven(first power of seven), 2 forty-nines (second power of seven), and 4 three hundred forty-threes (third power of seven).

$(4213)_{ten}$ tells us that we have 3 ones, 1 ten (first power of ten), 2 hundreds (second power of ten), and 4 thousands (third power of ten).

In this treatment of numeration systems, the decimal system is considered at the end of the chapter, after a study of systems in other bases. It was thought that this arrangement would give the student a better perspective of the decimal system as one of many possible systems. The properties of place value and zero belong to base 2 numeration, base 5 numeration, base 7 numeration,etc.,and also to base 10 numeration. Since one of the major goals of this chapter is to make clear the procedures followed in performing the operations in the decimal system, it was decided to analyse these procedures when performing the operations in other bases first. This detailed analysis is necessary because the number facts in bases other than ten are not familiar to the students. Then it was demonstrated that these procedures are precisely what we had been doing in calculating in the decimal system. The student must realize that the operations are defined for the numbers,and therefore procedures for performing these operations must be the same no matter what the base in which the numerals are written. Realization that we group by fives in adding in base 5, by sevens in adding in base 7, by twelves in adding in base 12, should clarify the grouping by tens in adding in base 10. It should be stressed that the decimal system is an efficient one for our daily calculations. We do not need to be able to calculate with speed in any other base. We simply wish to be very clear about what we are doing when we calculate in base 10. Sometimes the facts of addition, subtraction,etc.,are memorized without proper

understanding of the steps taken in obtaining these facts. The disadvantages of this situation are obvious. A study of this chapter should help to overcome this difficulty.

The games described at the end of the Teachers' Guide provide an opportunity for highly motivated drill, especially in mental arithmetic. Conversion of a numeral from any base to base 10 is another activity which involves opportunity for much drill in arithmetic. Drill exercises are frequently extremely tedious. This chapter affords a pleasanter environment in which to practise the important skill of rapid and accurate calculation.

The method of guided discovery suggested throughout the Teachers' Guide will provide an atmosphere in which imagination and creativity will flourish.

1-1. Numbers and Numerals

Skills and Understandings

1. Recognition that in our everyday speech we are constantly using names for objects with full appreciation that these names are not the objects themselves.
2. Awareness that there is a difference between a number and a numeral for that number.
3. Realization that there are many numerals for the same number.
4. Ability to name a number by a numeral which involves operational signs.
5. Appreciation of the fact that the number itself is an abstraction, an idea, and not a physical object.

Teaching Suggestions

1. This introductory lesson provides an excellent opportunity for student participation in classroom discussion. Use this opportunity to set the tone of the mathematics classroom as an arena for discussions and a laboratory for discovery rather than a room where dicta are laid down. This recognition given at this time to a student's contribution in a discussion about the idea behind the word "independence" will help lay the groundwork necessary to develop the self-confidence and poise to do some original

thinking in mathematics and to express his thoughts before the group.

2. The lesson might be opened by writing the word "cow", on the blackboard and asking: "What is on the blackboard?" If the children answer "cow" then ask: "What colour is the cow?" etc., until the children realize that there is no cow on the blackboard, but rather the word "cow". Then write "Ngozi" on the blackboard and ask: "What is on the blackboard?" Discussion should now more quickly reach the point where the students recognize that Ngozi herself is not attached to the blackboard, but that her name is written there.

3. Write the numeral "2" on the blackboard and ask someone to come to the board and write one half of what is on the blackboard. If the student writes "1", then you write $\mathcal{L}$, showing one-half of the numeral "2". Write the Roman numeral XII on the blackboard. Ask for someone to divide it in half. If the student aswers "six", then you erase the bottom half of the numeral and show VII or seven.

   Write on the blackboard 13 and 8 and ask: "Which is larger?" Write on the blackboard 3 and .9846 and ask: "Which is larger?" These examples stress the importance of differentiating clearly between the numberal and the number it names. Also stress that when we use +, -, ×, or ÷, we are operating with numbers. If we add the number three to the number two, we get the number five. If we add the numeral 3 to the numeral 2, we might get something like "2" 3.

4. Ask each child to write a name for the number 4 on his paper. Have several come to the blackboard and write their numerals. If all write "4", then ask what someone's grandmother says when she means "four". Write this on the blackboard. Suggest the possibility of 5 - 1 as a name for four. Lead the discussion so that the students will see that any of the four operations (+, -, ×, ÷) may be used in the naming of a number. Use this opportunity to review and drill in exercises involving +, -, ×, ÷ of whole numbers. Ask: "What number is named by 157 + 349?" "Give two other numerals for the number named by 17829 ÷ 9," etc.

Recognize that the numerals 1 + 3 and 3 + 1 are <u>different</u> names for the same number.

If we wish to show that 9 + 3 and 6 + 6 name the same number, we write, 9 + 3 = 6 + 6.

5. Lead a class discussion on abstract ideas such as "independence", "honesty", "leadership", etc. Stress that we can only think about these ideas, but that we cannot see them or handle them. Suggest to the students that numbers are just such ideas. We cannot see them or handle them either. But we <u>can</u> see or handle 4 stones, or 4 sticks, or 4 beans, etc. We can also see the <u>numerals</u> which represent the numbers because the numerals are marks on paper or on the blackboard. We <u>can</u> trace the <u>word</u> "four" on the blackboard.

Answers to Problems 1-1, page 1-1

Answers

1. For example : 7 + 1, 8, VIII, 6 + 2; 16, 10 + 6, 17 - 1; 35, 30 + 5, 36 - 1; 216, 200 + 16, 200 + 10 + 6.
2. Ten, fifteen, two hundred forty-six, one hundred nine, four, nine, one thousand three hundred forty-eight.

1-2. History of Numerals

Skills and Understandings

1. Awareness that the symbols we use in writing numerals are a matter of convenience and usage.
2. Appreciation of a study of number systems of other civilizations as enrichment of one's general cultural background.
3. Appreciation of selection of Egyptian, Babylonian, and Roman systems of numeration to provide an example of:
   a) a system with no zero symbol, and no place value property;
   b) an ancient system in which groupings by tens was used;
   c) an ancient system in which the grouping is done by some number other than ten;
   d) a system which has a "positional" property;
   e) a system which employs additive and subtractive techniques for writing numerals.

4. Realization that the Hindu-Arabic system of numeration used today with its zero symbol and place value property is the result of experience and development over centuries.
5. Use of the symbol for zero as a place holder.
6. Importance of place value property in writing numerals for large numbers.
7. Difficulty in performing some calculations in systems which do not have the place value property.

Teaching Suggestions.

1. It is not intended that the student should commit to memory the symbols used in the ancient systems of numeration mentioned in the text. This is not essential to the achievement of the skills and understandings which are the goals of this section.
2. Making posters showing numerals in some of the ancient systems of numeration might be a suitable extra-curricular activity for an interested student.
3. Since the ancient systems of numeration are a matter of history and not conjecture, a suggested procedure in this section would be for the teacher to tell the class about these systems, stopping to stress a point, make an observation, discuss a particular item to answer a question. In considering the Egyptian numerals, there should be class discussion as to the possible origin of grouping by tens which is a property of that system. In the discussion, bring out that this is due to the fact that we have ten fingers; if man had eight fingers, the Egyptians would probably have grouped by eights.
4. As each ancient system is considered, conduct the discussion to bring out the particular properties of that system.

System	Properties
Egyptian	No "positional" or place value No zero Grouping by tens.
Babylonian	No zero Symbol for a larger number is to the left of a symbol for a smaller number Grouping by sixties.

System	Properties
Roman	Additive and subtractive procedures used in writing numerals No zero Grouping in many ways: fives, tens, fifties, hundreds, etc.

After considering the text materials, students might be required to give examples of numerals illustrating the properties of each symbol.

5. Calculations in systems of numeration which have no place value and no zero is difficult. It might be worthwhile working through a multiplication example using Roman numerals with the class, pointing out the disadvantages of the absence of these two important properties. More than one example is time consuming and of dubious value.

Example: Multiply:

```
CLXVII     V times I = V
     V     V times I = V
--------
DCCCXXXV   V + V = X  (This gives us the first X to the
                       right)
           V times V = V + V + V + V + V = XXV
           If we add XXV to the X we got above, we have
           so far, XXXV
           V times X = X + X + X + X + X = L
           Now we have LXXXV.
           L times V = CCL, which when combined with
           LXXXV gives  CCCXXXV.
           C × V = D.
           Final answer:  DCCCXXXV
```

Because of the cumbersome nature of the calculations with these numerals, calculations were actually done on an abacus.

6. The following is an example of the clumsiness of writing numerals for large numbers in a system with no place value property:

MMMMDCCCLXXVIII 4,878

Answers to Problems 1-2(a), page 1-2

1. 1111111;
2. 3; 22; 402; 1113; 20211.

Answers to Problems 1-2(b), page 1-3

1. ▽▽▽▽▽▽; ◁▽▽▽; ◁◁▽▽▽▽▽▽▽▽; ◁◁◁◁▽▽▽▽▽; ◁◁◁◁◁▽▽▽▽▽▽
2. 21; 34; 5; 11.

Answers to Problems 1-2(c), page 1-4

1. IV; XXI; CDLXXVIII; MCMLXII; XXXIX; LII.
2. 19; 8; 64; 226; 2980; 1752.

Answers to Problems 1-2(d), page 1-5

1. .17; 425; 1006; 20,012.
2. Eighty-seven; three thousand five hundred; thirteen thousand two; six hundred forty-nine; forty-eight thousand ten; thirty-three.

1-3. The Abacus and Base 5 Numerals

Skills and Understandings

1. Realization that when we count in base 5 we are simply using different names for the natural numbers, and there is a symbol in base 5 for <u>every natural number</u>.
2. Recognition that if we group by fives when counting, we need only five digits· 0, 1, 2, 3, 2, and every natural number may be named by some combination of these digits.
3. The numeral "5" does <u>not</u> occur when grouping by fives. The number five does occur, and its numeral in base five is (10) five.
4. Every group of 5 ones makes 1 five.
   Every group of 5 fives makes 1 twenty-five.
5. Symbols with more than 1 digit are read as a sequence of individual digits: i.e. $(124)_{five}$ is read "one, two, four, base five."
6. The numeral $(124)_{five}$ may <u>not</u> be read one hundred twenty-four base five, since the numerals "one hundred" and "twenty" are characteristic of the decimal system of numeration.

Teaching Suggestions

1. To introduce the notion of grouping by some number other than ten,

the teacher might begin the lesson by holding up a handful of stones or beans and asking: "Will the number of stones change if I put this handful on the desk?" "Will the number of stones change if I separate them into two heaps?" "How will the total number of stones change if I regroup them into several heaps on the desk?" Bring out the fact that no matter how we group the stones, the number of stones remains unchanged.

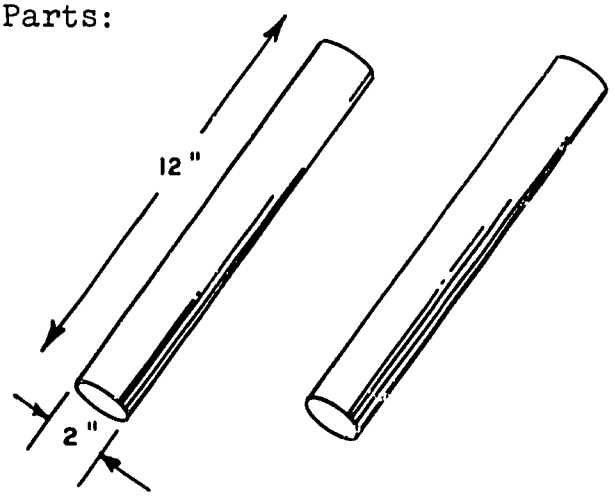
Give each child some stones (any number less than 25 to begin with) and ask: "How many stones do you have?" (Different children may have different numbers of stones.) "Separate the stones into groups of fives and see how many groups of fives you have and how many are left over." "Has the number of stones changed because of the regrouping?" Have children report their findings and write on the board:

Ngozi	17 stones	3 groups of five and 2 left over
Alimamy	9 stones	1 group of five and 4 left over,

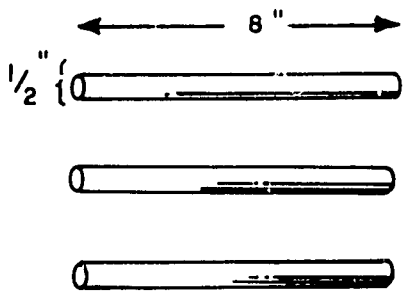
"Do you think we could group any number of stones by fives?" "We need more information before we can do this. Let us look at this counting frame or abacus which is specially made to simplify grouping by fives."

2. It would be desirable if every child could make an abacus of his own for use in counting in base 5. The following detailed description may assist the teacher in directing this class activity.

Parts:

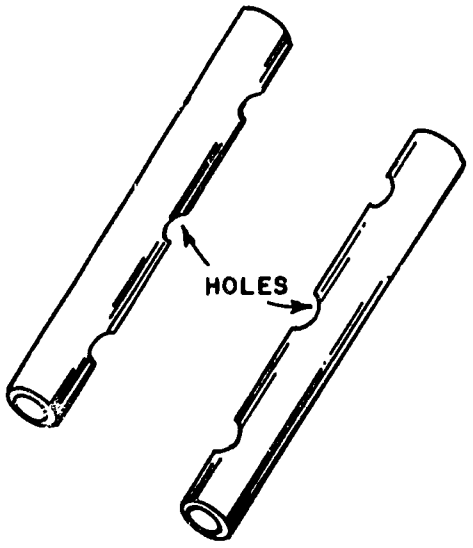


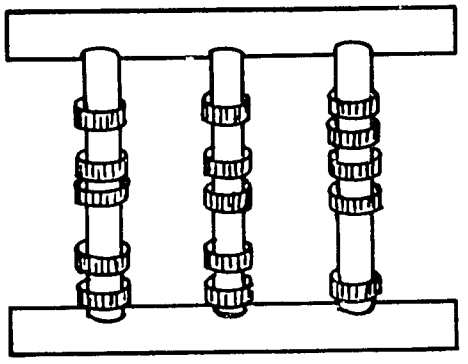
SIDE PIECES



CROSS PIECES

COUNTERS (ONLY 3 SHOWN)





ABACUS

(a) Side pieces: Cut off from a length of bamboo (about 2" outer diameter) two cylindrical pieces, about 12" long.

(b) Cross pieces: Cut off from another length of bamboo (about $\frac{1}{2}$" outer diameter) 3 pieces about 8" long.

(c) Counters: Cut off from another piece, 15 or 21 or 30 pieces, depending upon the number base you wish to use. The bamboo used must have an inner diameter just a little larger than the outer diameter of the lengths used for the cross pieces. This is necessary to prevent the counters from sliding too freely along the cross pieces. Each counter should be about $\frac{1}{2}$" long.

(d) Next cut three evenly spaced circular holes, about 3" apart, on one side of each of the side pieces. Each hole should be of a slightly smaller diameter than the outside diameters of the cross pieces.

<u>Directions for construction</u>:

a) Force the cross pieces into the holes in one side piece.

b) Slip 5 counters (or 7 or 10, etc.) over each cross piece.

c) Force the free ends of the cross piece into the holes in the other side piece.

Notes:

a) Do not use very young bamboo, or the whole thing will shrivel up in a few days. It might in fact be better after you have cut out the pieces listed above, to put all aside for several days. All necessary shrinkage should take place during this time.

b) All pieces should be taken from between joints of the bamboo used.

c) The number of cross pieces is not restricted to three; four cross pieces or five cross pieces, etc. may be used.

3. Provide opportunity for considerable practice in naming numbers by pushing up appropriate beads on the different rods and then writing the numerals in base 5.

4. When counting in base 5 is introduced, allow the students to recite aloud as they move the beads on the abacus: "one, two, three, four, 1 five, 1 five and 1 one, 1 five and 2 ones, 1 five and 3 ones, 1 five and 4 ones, 2 fives, ...4 fives and 4 ones, 1 twenty-five, etc.
Stress that each bead on the B rod is worth five times each bead on the A rod; each bead on the C rod is worth five times each bead on the B rod. Each bead on the C rod is worth_____ _____ times each bead on the A rod.

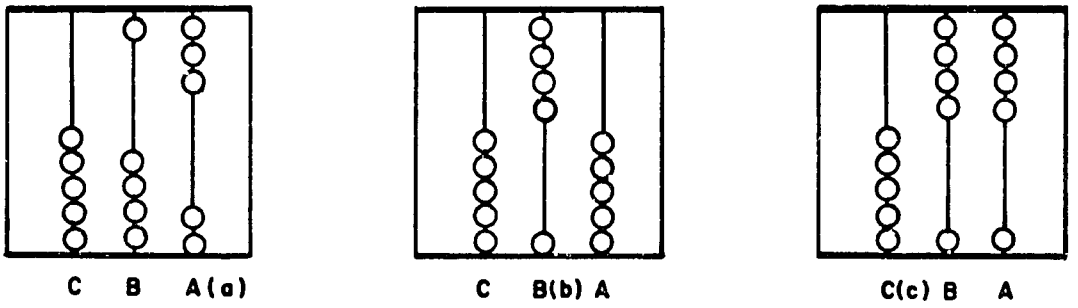
5. Proceed from counting aloud with the abacus to recording numerals for the numbers as indicated in Figure 1-3(e). Have the students read the numerals aloud to check that they are saying "one zero" for $(10)_{five}$ and not "ten"; "one one" for $(11)_{five}$ and not "eleven" and so forth. Then ask: "How many digits have we used to write numerals in base 5?" This technique of setting the stage for the students to make an observation by themselves sometimes takes more time than simply telling the class that there are 5 digits in base 5, and they are 0, 1, 2, 3, 4. But the general procedure of guided discovery produces in the classroom an atmosphere of inquiry which is stimulating, exciting, and conducive to fostering

creativity.

Answers to Problems 1-3(a), page 1-7

Answers

1. Seventeen
2. a) Push up one bead on the B rod and three beads on the A rod.
   b) Push up four beads on the B rod.
   c) Push up four beads on B rod and four beads on A rod.



Answers to Problems 1-3(b), page 1-9

1.

1,	2,	3,	4,	10,	11,	12,	13,	14,	20
21,	22,	23,	24,	30,	31,	32,	33,	34,	40
41,	42,	43,	44,	100,	101,	102,	103,	104,	110
111,	112,	113,	114,	120,	121,	122,	123,	124,	130
131,	132,	133,	134,	140,	141,	142,	143,	144,	200
201,	202,	203,	204,	210,	211,	212,	213,	214,	220
221,	222,	223,	224,	230,	231,	232,	233,	234,	240
241,	242,	243,	244,	300,	301,	302,	303,	304,	310
311,	312,	313,	314,	320,	321,	322,	323,	324,	330
331,	332,	333,	334,	340,	341,	342,	343,	344,	400

2. 4; 44; 400.

Answers to Problems 1-3(c), page 1-9

1. (a) 1 nickel, 1 cent, (b) 2 nickels, (c) 4 nickels, 3 cents, (d) 1 quarter, 1 nickel, 4 cents, (e) 1 quarter, 4 nickels, 3 cents.
2. (a) 57, (b) 98, (c) 24, (d) 62, (e) 230.

1-4. Counting in Other Bases

Skills and Understandings

1. Recognition of the fact that the numeral for the base in any system

of numeration is "10".

2. Realization that numeration in any base is essentially a way of grouping in that base.
3. Appreciation of the fact that though a number may be represented by different numerals in different bases, the number so represented remains the same.
4. To reinforce the understanding that a number is an idea and may have many numerals.
5. Recognition of the place of the student's own local system of numeration in any study of numeration systems.

Teaching Suggestions

1. It would be helpful to have a picture, or drawing of abaci for counting in various bases and if possible, an actual abacus in either base seven or base twelve.
2. Where there is an actual abacus, practice should be given to the pupils, in counting on it. If an actual abacus is not available, practice should be given on the blackboard with students picturing the numerals on an abacus designed for base seven counting.
3. Let each child provide himself with nineteen pebbles, beans, or beads. Have these grouped by fives. Ask: "How many groups of fives do you have, and how many are left over?" The correct answer will be 3 groups of fives and 4 ones left over. Say: "We write this as $(34)_{five}$ and read it "three four base five".
4. Now ask the pupils to put the pebbles together and then to regroup them by sevens. Ask the same question and proceed as in grouping by fives. The answer will be two groups of sevens and 5 ones left over, written as $(25)_{seven}$ and read "two five base seven".
5. Repeat the activity in paragraph 4 above, grouping by eights, tens, and twelves. The numerals will be $(23)_{eight}$ $(19)_{ten}$ and $(17)_{twelve}$. Many experiences such as those described above will help the pupil see and appreciate that the number "nineteen" can be represented by any of these symbols: $(34)_{five}$, $(25)_{seven}$, $(23)_{eight}$, $(19)_{ten}$, and $(17)_{twelve}$. This exercise will certainly strengthen the concept that a number is an idea that can be represented by many symbols.

6. An additional experience in regrouping on a slightly more abstract level is as follows:
Ask the students to make five sets of 19 X's in their exercise books. Ask them to group the first set by fives, the second set by sevens, the third set by eights, the fourth set by tens, and the fifth set by twelves.
For each set, ask: "How many groups of X's are there and how many are left over?" Ask the pupils to write their ansers in the various base numerals that are appropriate. The same answers as in paragraph 5 above should be obtained.
7. Elicit from the pupils the method of counting generally used in the traditional way in their own locality. This can be done, where possible, in their own vernacular. Then try to find out whether the base used is six, ten, twelve, or any other base.
In particular, be alert for cases where the system of numeration may be akin to the Roman system; that is, where both the notion of subtraction and/or addition is used. If such a method is in vogue, then again draw attention to the Roman system, and compare the numbers added or subtracted with what obtains in the Roman system.
8. The exercises are only a guide and a sample of what can be done. Construct several other examples using the local monetary system. the local units of measure, and the local system of numeration.
9. In the exercises at the end of the section, the pupils should be made to realize that groupings are done by sevens, twelves, or twenties, but the actual operations are done in the base 10 and <u>not</u> in bases 7, 12, or 20.

<u>Answers to Problems 1-4(a), page 1-11</u>

1.

1	2	3	4	5	6	10
11	12	13	14	15	16	20
21	22	23	24	25	26	30
31	32	33	34	35	36	40
41	42	43	44	45	46	50
51	52	53	54	55	56	60
61	62	63	64	65	66	100
101						

2. eight; twenty-two; forty; forty-eight.
3. 32; 62; 68.
4. (a) 6 weeks 1 day (b) 7 weeks 4 days
5. (a) 1 week 2 days (b) 1 week 2 days

Answers to Problems 1-4(b), page 1-12

1.

1	2	3	4	5	6	7	8	9	t	e	10
11	12	13	14	15	16	17	18	19	1t	1e	20
21	22	23	24	25	26	27	28	29	2t	2e	30
31	32	33	34	35	36	37	38	39	3t	3e	40
41	42										

2. Twelve; twenty-two; fifty.

Answers to Problems 1-4(c), page 1-12

1. (a) 12s. 6d. (b) 45s. 10d.
2. (a) 3s. 2d. (b) 5s. 5d.
3. (a) 18s. (b) 122s. 6d.
4. (a) 2s. 2d. (b) 3s. 7d.
5. 21s. 36s. 9d., 47s. 3d.
6. 24d., 33d., 42., 116d.
7. 2ft. 10in. 8. 2ft. 6in.
9. 11 doz. and 10. 10. 2 doz. 10 pencils.

Answers to Problems 1-4(d), page 1-14

1. £19 2s. 2. £47 9s. 3. £25 17s.
4. 7 tons 3 cwt. 5. 3 tons 14 cwt.

1-5. Simple Calculations in Bases 2, 5, and 12.

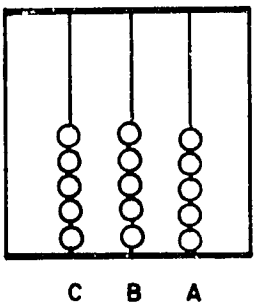
Skills and Understandings

1. Comprehension of the procedure used in adding and subtracting in any base in order to understand more fully what is done in addition and subtraction in base 10.
2. Awareness that calculations can be carried out when the numbers are named in bases other than 10.
3. Appreciation of the fact that computational skill in base 10 is all that is really necessary.
4. Ability to switch from calculations in one base to calculations in another, with understanding.

5. Recognition of the regrouping involved in "carrying" and "borrowing".

Teaching Suggestions

1. An abacus will be helpful in this section.
2. Remember that only simple calculations are to be done. Do not permit calculations to be done in base 10 with conversion of the answer to the base in which the problem is given.
3. Review the digits that are required for counting in base 5, and make sure that everybody remembers that there are only five, viz: 0, 1, 2, 3, 4.
4. To add 4 + 3 in base five, start with the abacus, preferably a large one which can be placed in front of the class. The figure below is a picture of the abacus.



Ask: "How shall we represent 4 on the abacus?" "How can we show 3 more?" Bring out that we start at rod A and move 4 beads up. To add 3, push up 3 more. But there is only one more bead on rod A. Push up the one bead on rod A. Since the five beads on A are equivalent to one bead on B, the five beads on A are pushed down and one bead on B is pushed up. Since two more must be added, push two beads up on A. The answer is 1 five and 2 ones, i.e. $(12)_{five}$. In mathematical symbols, we write what was done as:

$$4 + 3 = (4 + 1) + 2 = 1 \text{ five and } 2 \text{ ones} = (12)_{five}$$

5. Reverse the order of the addends, i.e. 3 + 4, and using the abacus again (if necessary) let the students demonstrate:

$$3 + 4 = (3 + 2) + 2 = 1 \text{ five and } 2 \text{ ones} = (12)_{five}$$

6. Allow the students to perform the following additions using their own abaci: 2 + 3, 4 + 2, 3 + 3, 4 + 4, 1 + 4 (all in base 5). Each example may then be done without the abacus, but using mathematical symbols:

$2 + 3 = 1$ five and 0 ones $= (10)_{five}$

$4 + 2 = (4 + 1) + 1 = (11)_{five}$

$3 + 3 = (3 + 2) + 1 = (11)_{five}$

$4 + 4 = (4 + 1) + 3 = (13)_{five}$

$1 + 4 = (10)_{five}$

7. Ask why the following procedure is not efficient: $4 + 4 = (4 + 2) + 2$. Bring out that we do not get a group of five by combining 4 and 2.

8. Do calculations of the type where no regrouping (carrying) is required:

Add:

$$\begin{array}{r} (13)_{five} \\ (21)_{five} \\ \hline (34)_{five} \end{array} \qquad \begin{array}{r} (22)_{five} \\ (11)_{five} \\ \hline (33)_{five} \end{array}$$

Use the abacus if the students find it helpful here.

9. In the following examples regrouping is done from the ones to the fives column only:

Add:

$$\begin{array}{r} (14)_{five} \\ (23)_{five} \\ \hline (42)_{five} \end{array} \qquad \begin{array}{r} (32)_{five} \\ (\ 4)_{five} \\ \hline (41)_{five} \end{array} \qquad \begin{array}{r} (24)_{five} \\ (14)_{five} \\ \hline (43)_{five} \end{array}$$

10. Have the students make an addition table in base five and use it to do further calculations.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

Addition Table in Base 5

11. If sufficient interest is aroused, try addition of 3 addends:

$3 + 4 + 3, \quad 1 + 2 + 4, \quad 4 + 4 + 4.$

Thus: $3 + 4 + 3 = 3 + (2 + 2) + 3 =$
$(3 + 2) + (2 + 3) = (20)_{five}$

$1 + 2 + 4 = 1 + 2 + (2 + 2) = (1 + 2 + 2) + 2$
$= (12)_{five}$

$4 + 4 + 4 = 4 + (1 + 3) + 4 =$
$(4 + 1) + (3 + 2) + 2 = (22)_{five}$

12. If 11. goes fairly well and if the brighter children are eager, try examples like

$$\begin{array}{r} (22)_{five} \\ (33)_{five} \\ (14)_{five} \\ \hline (124)_{five} \end{array}$$

<u>Addition in Base 12</u>

1. Carry out the same procedures as in base 5.
2. Limit examples to addition of two single digit numbers:

   8 + 7, 6 + 9, 5 + t, 3 + e, etc.

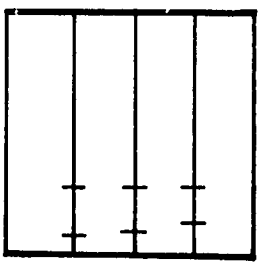
3. Let the students complete the addition table for base 12 numerals, and use the table in calculations.

+	0	1	2	3	4	5	6	7	8	9	t	e
0	0	1	2	3	4	5	6	7	8	9	t	e
1	1	2	3	4	5	6	7	8	9	t	e	10
2	2	3	4	5	6	7	8	9	t	e	10	11
3	3	4	5	6	7	8	9	t	e	10	11	12
4	4	5	6	7	8	9	t	e	10	11	12	13
5	5	6	7	8	9	t	e	10	11	12	13	14
6	6	7	8	9	t	e	10	11	12	13	14	15
7	7	8	9	t	e	10	11	12	13	14	15	16
8	8	9	t	e	10	11	12	13	14	15	16	17
9	9	t	e	10	11	12	13	14	15	16	17	18
t	t	e	10	11	12	13	14	15	16	17	18	19
e	e	10	11	12	13	14	15	16	17	18	19	1t

4. Stress that we will now group by twelves. Thus:
   $8 + 7 = 8 + (4 + 3) = (8 + 4) + 3 = (13)_{twelve}$.

<u>Addition in Base 2</u>

1. Carry out the same procedures as in base 5 and base 12.
2. Have the children draw a diagram of an abacus to calculate in base 2. If needed, construct a large abacus for classroom use.



3. Add in base 2:

   $1 + 1 + 1 + 1 + 1$, $\quad 1 + 1 + 1 + 1 + 1 + 1$,

$$\begin{array}{r} 1110 \\ \underline{11} \end{array} \qquad \begin{array}{r} 1011 \\ \underline{101} \end{array}$$

The first two examples can be done on the abacus. Ask the class why the third and fourth examples cannot be done on the abacus above. Bring out that the abacus is constructed to show only 3-place numerals while the third and fourth examples use 4-place numerals.

4. Bring out that there is frequent regrouping when calculating in base 2.

5. Stress that addition in base 2 is simple because there are only 4 addition facts: $0 + 0 = 0$, $0 + 1 = 1$, $1 + 0 = 1$, $1 + 1 = 10$. Numerals in base 2 are used in some electronic computers because the two digits, 0 and 1, can correspond to current off and current on.

Subtraction

1. Treat subtraction as finding the missing addend. So, if $(43)_{five} - (32)_{five} = x$, then $(43)_{five} = (32)_{five} + x$.

Example:

$$\begin{array}{r} (43)_{five} \\ -(32)_{five} \\ \hline (11)_{five} \\ \hline \end{array}$$

Think: Starting with the ones column, what must be added to 2 to give 3? The answer is 1. Then in the fives column, what must be added to 3 to give 4? The answer is 1.

2. Stress that the addition tables the students constructed may be used for subtraction.

Subtraction in Base 5

1. First do simple examples that do not involve regrouping:

$$\begin{array}{r} (34)_{five} \\ -(12)_{five} \\ \hline (22)_{five} \\ \hline \end{array} \qquad \begin{array}{r} (33)_{five} \\ -(11)_{five} \\ \hline (22)_{five} \\ \hline \end{array} \qquad \begin{array}{r} (44)_{five} \\ -(13)_{five} \\ \hline (31)_{five} \\ \hline \end{array}$$

Use the abacus here if needed.

2. Then do the following, using two digit numerals.

$$\begin{array}{r} (32)_{five} \\ +(14)_{five} \\ \hline \\ \hline \end{array}$$

Ask: "What must be added to 4 to give 2?" The students should realize that they do not know any such whole number. Some of them may tell you to borrow 1. Explain that the word borrow implies receiving an object which you plan to return. Since a number is an idea, it is not something that can be passed from hand to hand. What we really do, is to regroup the numeral. Thus we will regroup 3 fives and 2 ones to get 2 fives and 12 ones.

$$\begin{array}{r} (32)_{five} \\ -(14)_{five} \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} (2\ \ 12)_{five} \\ -(1\ \ \ 4)_{five} \\ \hline (1\ \ \ 3) \\ \hline \end{array}$$

Ask: "What do you add to 4 to give 12?" The answer is 3. "What do you add to 1 to give 2?" The answer is 1.

3. Encourage the students to do the regrouping mentally.
4. Discuss the example in the text:

$$\begin{array}{r} (211)_{five} \\ -43_{five} \\ \hline \\ \hline \end{array}$$

Subtraction in Base 12

1. Start with examples involving no regrouping and then go to those which require regrouping.
2. Confine examples to numbers of not more than two digits.
3. Be sure to give a few examples involving t and e.
4. Additional drill:

$$\begin{array}{r} (48)_{twelve} \\ -(17)_{twelve} \\ \hline (31)_{twelve} \\ \hline \end{array} \qquad \begin{array}{r} (te)_{twelve} \\ -(87)_{twelve} \\ \hline (24)_{twelve} \\ \hline \end{array} \qquad \begin{array}{r} (65)_{twelve} \\ -(38)_{twelve} \\ \hline (29)_{twelve} \\ \hline \end{array} \qquad \begin{array}{r} (e4)_{twelve} \\ -(5t)_{twelve} \\ \hline (56)_{twelve} \\ \hline \end{array}$$

$$\begin{array}{r} (82)_{twelve} \\ -(19)_{twelve} \\ \hline (65)_{twelve} \\ \hline \end{array}$$

Subtraction in Base 2

With all the foregoing, subtraction in base 2 should not present much difficulty. Carry out the procedures as given above. There are a sufficient number of examples in the text.

Multiplication and Division

These operations require considerable familiarity with the multiplication and division facts in each different base. Without this familiarity, the processes are difficult to perform and very time consuming. This is therefore left as an activity for only the interested and capable students.

Answers to Problems 1-5(a), page 1-15

1.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

2. 24; 3. 44; 4. 34; 5. 104; 6. 102;
7. 113; 8. 124; 9. 113; 10. 114; 11. 133.

Answers to Problems 1-5(b), page 1-16

1. 40; 2. 131; 3. 32; 4. 110; 5. 112.

Answers to Problems 1-5(c), page 1-17

1. 12; 2. 15; 3. 11; 4. 14; 5. 13;
7. 47; 8. 4e; 9. 11; 10. 82; 11. 125.

Answers to Problems 1-5(d), page 1-18

1. 11; 2. 101; 3. 1110; 4. 1000; 5. 1010010.

Answers to Problems 1-5(e), page 1-18

1. 1; 2. 4; 3. 4; 4. 11; 5. 21.

Answers to Problems 1-5(f), page 1-19

1.	114;	2.	203;	3.	10;	4.	323;	5.	334;
6.	11;	7.	e;	8.	8;	9.	t;	10.	1te;
11.	10;	12.	11;	13.	10100				

1-6. Decimal Numeration

Skills and Understandings

1. Awareness that the numeral 10 can be used to represent many numbers, depending on the base used.
2. Application of the earlier number study in order to re-examine the decimal notation.
3. Emphasis of the property of place value with expanded numerals.
4. Ability to expand numerals and to use exponents.

Teaching Suggestions

1. Ask the class to count the number of children present, the number of days before the end of term, and so on.

   Discuss the base used for this counting and the number of digits which were needed.

Explain that this system of working is already familiar to the class as it was used in the primary school. Emphasize that this system is the DECIMAL system because it uses ten digits and groups in tens. Revision work in other bases through a discussion of the numeral 10.

10 can mean two (in base two)
10 can mean five (in base five)
10 can mean seven (in base seven)
10 can mean twelve (in base twelve)
10 can mean ten (in base ten).

The numeral 10 can mean any number at all. Only in the last case can 10 be called "ten." In all the other cases we say "one-zero."

In each case, ask for the digits used.

"What does one-zero mean in the base seven?"

"What does one-one mean?"

"How many digits are used?"

"What is the largest digit?"

2. The abacus was used for grouping in the base five, and grouping in tens can be done in the same way.

In the base ten,

98 means 9 tens and 8 ones

99 means 9 tens and 9 ones

99 + 1 means 9 tens and 10 ones

But 10 ones make one group of tens, so

99 + 1 means 10 tens and 0 ones.

But 10 tens make one group of hundred, so

99 + 1 means 1 hundred and 0 tens and 0 ones, so

99 + 1 is 100.

After this build-up, take convenient and meaningful dates and distances, for example, and write the numerals on the blackboard and discuss the value of each digit.

Show that each numeral can be written in this way:

$$850 \text{ is } (8 \times 10 \times 10) + (5 \times 10) + (0 \times 1).$$

Explain that these are EXPANDED numerals.

3. Using the digits 1, 2, 3, 4, make four different numerals, e.g.

1	2	3	4
2	3	4	1
3	4	1	2
4	1	2	3

Discuss with the class what is represented by the "2" in each case.

Repeat this discussion for the other digits.

Now discuss the numeral 555 and emphasize the part of the numeral represented by each "5."

Explain the property of PLACE VALUE and go on to examine numerals for larger numbers, e.g., 1856, 1962, etc.

4. Demonstrate writing a numeral such as 1,234,567 on the blackboard and discuss its expanded form. Suggest that a shorter form for the expanded numeral would be very useful.

Introduce EXPONENT (INDEX) and practise the <u>use</u> of exponents in writing expanded numerals. Give oral practice in using numerals with exponents, for example,

$$1{,}234{,}567 \text{ is } 1 \times 10^6 + 2 \times 10^5 + 3 \times 10^4 + 4 \times 10^3 + 5 \times 10^2 + 6 \times 10^1 + 7 \times 1$$

5. Find out from the pupils the answer to the question asked in the last sentence of Section 1-6. The exponent of the 10 should be 7.

Answers to Problems 1-6(a), page 1-21

1. 8 hundreds
5 tens
6 ones

2. 9 tens
4 ones

3. 7 hundreds
2 tens
1 one

4. 3 hundreds
0 tens
3 ones

5. 9 hundreds
9 tens
9 ones

6. $(8 \times 10 \times 10) + (5 \times 10) + (6 \times 1)$
7. $(9 \times 10) + (4 \times 1)$
8. $(7 \times 10 \times 10) + (2 \times 10) + (1 \times 1)$
9. $(3 \times 10 \times 10) + (0 \times 10) + (3 \times 1)$
10. $(9 \times 10 \times 10) + (9 \times 10) + (9 \times 1)$.

Answers to Problems 1-6(b), page 1-21

1 (a). $(8 \times 10 \times 10 \times 10) + (2 \times 10 \times 10) + (4 \times 10) + (6 \times 1)$
(b). $(7 \times 10 \times 10 \times 10) + (0 \times 10 \times 10) + (1 \times 10) + (9 \times 1)$
(c). $(5 \times 10 \times 10 \times 10) + (0 \times 10 \times 10) + (0 \times 10) + (3 \times 1)$

2 (a). No.
(b). The "6" in (a) is 6 ones
The "6" in (b) is 6 thousands
The "6" in (c) is 6 tens
The "6" in (d) is 6 hundreds
(c). The "2" in (a) is 2 hundreds
The "2" in (b) is 2 ones
The "2" in (c) is 2 thousands
The "2" in (d) is 2 tens
(d). The "8" in (a) is 8 thousands
The "8" in (b) is 8 tens
The "8" in (c) is 8 hundreds
The "8" in (d) is 8 ones

(e). The "4" in (a) is 4 tens
The "4" in (b) is 4 hundreds
The "4" in (c) is 4 ones
The "4" in (d) is 4 thousands

Answers to Problems 1-6(c), page 1-23

1. $(5 \times 10^4) + (0 \times 10^3) + (1 \times 10^2) + (4 \times 10^1) + (9 \times 1)$
2. $(2 \times 10^2) + (2 \times 10^1) + (2 \times 1)$
3. $(3 \times 10^5) + (5 \times 10^4) + (7 \times 10^3) + (0 \times 10^2) + (0 \times 10^1) + (0 \times 1)$
4. $(8 \times 10^3) + (6 \times 10^2) + (6 \times 10^1) + (5 \times 1)$
5. $(9 \times 10^1) + (4 \times 1)$
6. 901 7. 55555 8. 680170

1-7. Conversions

Skills and Understandings

1. Appreciation that the property of place value applies to numerals in different bases.
2. Awareness that two numerals which look the same may represent different ideas (numbers), if different bases are used.
3. Ability to convert from any base to base ten and vice versa.

Teaching Suggestions

1. At this stage, your class knows that the numeral $(123)_{five}$ tells us about groups of ones, fives, and twenty-fives. The numeral $(123)_{ten}$ tells us about groups of ones, tens, and hundreds, but in writing expanded numerals for these we have

$(123)_{five}$ is $(1 \times 10^2 + 2 \times 10^1 + 3 \times 1)_{five}$ (here 10 means five)

and $(123)_{ten}$ is $1 \times 10^2 + 2 \times 10^1 + 3 \times 1$ (here 10 means ten)

Similarly $(123)_{seven}$ is $(1 \times 10^2 + 2 \times 10^1 + 3 \times 1)_{seven}$ (here 10 means seven)

Discuss this build-up with the class carefully and establish the fact that the identical appearance of these numerals is brought about by the PLACE VALUE property.

2. Application of the property of place value can now be made by converting numerals in bases two, five, seven, and twelve to base ten. In using the property, remember that

$(234)_{five}$ means that we have 4 ones
and 3 fives
and 2 twenty-fives

In Base Ten
$4 \times 1 = 4$
$3 \times 5 = 15$
$2 \times 25 = 50$
69

So $(234)_{five} = (69)_{ten}$

When the children begin to make these conversions, do not insist on thorough memory work because speed and the ability to do these examples are less important than an appreciation of the underlying principles. If necessary, the tables made already can be used.

3. In converting a numeral from base ten to one in another base, the procedure used must be explained clearly.

Here is an example of one such conversion:

Convert $(37)_{ten}$ to a base three numeral.

(a) Represent $(37)_{ten}$ using strokes as follows:

///////////////////////////////////////

(b) Group them in threes as follows:

///|///|///|///|///|///|///|///|///|///|///|///|/

Notice that there are 12 groups of threes and 1 one left over.

(c) Group the threes in groups of three as follows:

///|///|///|///|///|///|///|///|///|///|///|///|/

Notice that there are 4 groups of nines and 0 threes left over after grouping the threes.

(d) Group the nines in groups of three as follows:

///|///|///|///|///|///|///|///|///|///|///|///|/

Notice that there is 1 group of twenty-seven and 1 nine left over after grouping the nines.

Having the property of place value in mind, observe that $(37)_{ten}$ = 1 group of twenty-seven + 1 group of nines + 0 group of three + 1 one. That is

$$(37)_{ten} = (1101)_{three}$$

4. A quick way or a short-cut method of doing the same conversion, especially with large numbers, is by repeated division by 3 as follows: [Note that three is the new base in which the numeral is to be expressed.]

```
3 | 37
  3 | 12 R1   i.e. 12 threes        and 1 one left over
    3 | 4 R0   i.e.  4 nines         and 0 threes left over
    3 | 1 R1   i.e.  1 twenty-seven and 1 nine left over
        0 R1   i.e.  0 eighty-ones and 1 twenty-seven left over
```

Thus the respective remainders in the repeated division give the numbers of ones, threes, nines, and twenty-sevens in the new numeral; therefore, using the property of place value the new numeral can be written.

Thus $(37)_{ten} = (1101)_{three}$

Be sure to have them check by computing in base ten. Thus $(1101)_{three} = 1 \times 27 + 1 \times 9 + 0 \times 3 + 1 = 37$

5. Notice that the division done in 4 is equivalent to the grouping done in 3. The regrouping is clearest in the check.

6. Observe, therefore, that in the conversion of a numeral from base ten to any other base, a short-cut method is to carry out a repeated division of the base ten number using the new base as divisor; then the respective remainders give you: the number of ones, sevens, forty-nines, etc. if you are converting to base seven, or the number of ones, twelves, one hundred forty-fours etc., if you are converting to base twelve, and so on.

7. Here are three more examples to illustrate this:

(i) Convert $(1619)_{ten}$ to a base seven numeral.

```
7 | 1619
  7 | 231 R2   i.e.  2 ones
    7 | 33 R0   i.e.  0 sevens
      7 | 4 R5   i.e.  5 forty-nines
          0 R4   i.e.  4 three hundred forty-threes
```

i.e. $(1619)_{ten} = (4502)_{seven}$

(ii) Convert $(16231)_{ten}$ to a base twelve numeral

```
12 |16231
 12 |1352   R7  i.e. 7 ones
  12 |112   R8  i.e. 8 twelves
   12 | 9   R4  i.e. 4 one hundred forty-fours
        0  R9  i.e. 9 one thousand seven hundred twenty-eights
```

i.e. $(16231)_{ten} = (9487)_{twelve}$

(iii) Convert $(19)_{ten}$ to a base two numeral

```
2 |19
 2 | 9   R1  i.e. 1 ones
  2 |4   R1  i.e. 1 twos
  2 |2   R0  i.e. 0 fours
  2 |1   R0  i.e. 0 eights
      0  R1  i.e. 1 sixteens
```

i.e. $(19)_{ten} = (10011)_{two}$

Answers to Problems 1-7(a), page 1-24

1. $(86)_{ten}$ 2. $(47)_{ten}$ 3. $(8)_{ten}$ 4. $(655)_{ten}$
5. $(11)_{ten}$ 6. $(1371)_{ten}$ 7. $(116)_{ten}$ 8. $(57)_{ten}$
9. $(121)_{ten}$ 10. $(262)_{ten}$

Answers to Problems 1-7(b), page 1-24

1. $(21)_{eight}$ 2. $(54)_{seven}$ 3. $(100)_{seven}$
4. $(50)_{twenty}$ 5. $(22)_{three}$ 6. $(110)_{two}$

Answers to Problems 1-7(c), page 1-26

1. $(1352)_{eight}$ 2. $(926)_{twelve}$ 3. $(140150)_{seven}$
4. $(33331)_{five}$ 5. $(1010110)_{ten}$ 6. $(14470)_{nine}$
7. $(2211)_{three}$ 8. $(2553)_{six}$ 9. $(41t0)_{eleven}$
10. $(402854)_{twelve}$

Answers to Problems 1-8, page 1-26

1. (a) 341 (b) 133
2. (a) 4t5 (b) 9t
3. (a) 1333 (b) 256
4. (a) 1692; 24820; 54874.
   (b) 93; 974; 89.
   (c) 11575; 280908; 7743600.
   (d) 451; 9004; $2041\frac{22}{39}$
5. (a) $(3746)_{ten} = (5122)_{nine}$
   (b) $(110110)_{two} = (44)_{ten}$
   (c) $(332)_{four} = (222)_{five}$
6. 1, 2, 3, 4, 5, 6, 7, 8, 9, t, e, -, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 1t, 1e, $\underline{1}$, 20, 21
7. $(134)_{six} = (58)_{ten}$
8. Two Answers:

Boy's Age	Father's Age	Birthday
12 yrs.	41 yrs.	30th (base5)
10 yrs.	29 yrs.	20th (base4)

9. 1' 5".
10. £ 2 12s. 9d.
11. 1s. 6d.
12. 6 dozen
13. £ 2 17s. 8d.
14. 4' 10"
15. £ 1 5s. 6d.
16. 24 packages
17. 20 doz.
18. 6d
19. 13 weeks and 1 day
20. 1 ton 6 cwt
21. 9 yr.

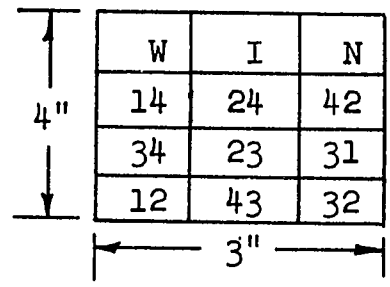
1-9 GAMES

1. WIN

This game is to be played in class during the work on conversion from numerals in base 10 to numerals in other bases. The game is best played with groups of six using conversions to base 5 - 9, the aim being to achieve rapid mental conversions to 2-digit numerals.

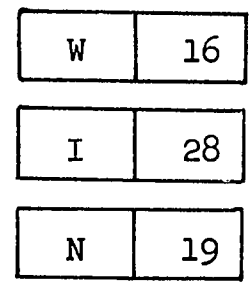
APPARATUS It is possible to use one set of apparatus for work with all bases. It would be more convenient, however, if a set of material were made for each base, in this way:

1. PLAYERS' CARDS



Five kinds of players' cards are needed, each kind being marked with two-digit numerals in the bases 5, 6, 7, 8, or 9. There must be six cards of each kind. All cards of a given kind must use different sets of numerals, although the same numeral can be used on two cards.

2. CALL CHITS



A set of 20 to 50 call chits is made with base 10 numerals marked on them. In addition, the chits are divided into 3 sets - one set is marked W, one set I, and one N.

Ensure that the numerals on the call chits correspond to the numerals on the players' cards and that no base 10 numeral is repeated.

3. <u>CHIT BOX OR BAG</u> is used to hold the call chits.
4. PEBBLES, BEANS or 1 INCH SQUARES are used by pupils. Each child needs 9.
5. If you have followed the directions above, you should have
   1 chit box or bag,
   1 set of 9 pebbles, beans or squares for each child,
   1 set of players' cards and call chits for base 5,
   1 set " " " " " " " " 6,
   1 set " " " " " " " " 7,
   1 set " " " " " " " " 8,
   1 set " " " " " " " " 9.
6. In preparing players' cards in other bases, do not use numerals with more than 2 digits. For bases 5, 6, and 7, some repetition of numerals is necessary, but not for bases 8 and 9.
   Remember that
   a) the numerals on the call chits are all in base 10.
   b) the numerals on the players' cards are all in bases 5 or 6 or 7 or 8 or 9.

<u>CLASS ORGANIZATION</u>

1. The class is divided into groups of six, consisting of five players and one caller.
   (if the children in the class do not make exact groups of 6, then make some groups of 7 and some of 6 children.)
2. Issue to each child in the class a set of nine pebbles, beans,or squares.
3. Select a caller for each group. Give each caller the call chits in the bag appropriate for the number base used for his group. Have the caller of each group distribute the cards to the five players. Make sure that each group understands the number base to be used.
4. Explain the way the game is played. Then have one of the groups begin to play in order that all the class can understand. Have the class play the game once to make sure that all know the rules. Then begin the contest.

PROCEDURE

1. The Caller begins the game by taking a chit from the bag and reading it aloud to the class: e.g.

W	9

"W nine"

The players then convert 9 to base five (assuming the conversion is from base ten to base five) to get 14. They then examine their "W" column to see if "14" is below "W".

If a player has 14 below W, he covers 14 with a pebble or bean.

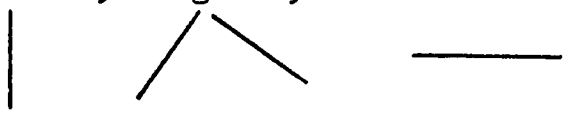
2. Without waiting too long, the caller then takes another chit from the bag and reads it aloud to the class.

N	23

"N twenty-three"

Once again the players convert this to a base five numeral and get 43. They examine their "N" columns to see if 43 is below N. If a player has 43 below N, he covers it, as before.

3. The game continues in this way until a player has three pebbles, beans, or squares covering numerals in straight lines in this way, i.e. vertical, diagonal, or horizontal.



The player then calls WIN and is the winner.

4. The teacher must look at the chits which have been drawn by the caller to check the winner's card.

5. At the end, all the winners are made into a group and another game is played to find the champion.

2. Game on Conversion of Numerals in Bases 3, 4, 5, 6, 7, 8, 9 to Numerals in Base 10 and vice versa.

1. The class is divided into groups of 10.

2. Each person in each group is given three cards having the same numeral written on it. The numerals in each set of three are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; i.e. 1st person has 0, 0, 0; 2nd person 1, 1, 1; 3rd person 2, 2, 2; and so on.

3. The teacher acts as caller and calls a numeral in any of the bases mentioned above or writes it on the blackboard and states to what base the numeral is to be converted:

   e.g. (i) $(24)_{seven} = (\quad)_{ten}$

   (ii) $(12)_{ten} = (\quad)_{five}$

4. Each person in each group works out the conversion individually and then the members in each group which have the digits of the new numeral must run up to the front of the class and arrange themselves in order showing their cards. The 1st group to do that correctly earns a point.

   Thus in example (i) above, the person having cards 1 and 8 in each group should try to run up quickly in front of the class and show 18.

   In (ii), the person having a set of twos should mave up and show two twos.

5. This is continued until any group earns 5 or 10 points, and the 1st group to do that wins the game.

6. Numerals having more than three digits should not be used.

FIRST YEAR SECONDARY TEACHERS' GUIDE

CHAPTER 2

OPEN SENTENCES

2-1. The Set of Whole Numbers

Skills and Understandings

1. In mathematics we refer to a collection of objects as a set of objects. Each object is a member of the set. It should be emphasized that the members of a set do not have to resemble each other in any way. We may, if we wish, consider the set consisting of the planet Mars, a lion, and the numeral 4. Each member of the set must be different and distinct from other members. We can consider the set consisting of Ngozi and a lion - this is a set with two members. But the set consisting of Ngozi and Ngozi is not a set with two members. It is a set consisting of one member. It might be worthwhile here to mention the empty set - for example, the set of lions in this classroom.
2. To show that a collection of objects is to be considered a set, we list the members and enclose the list in curly brackets.
3. When there are so many members in the set that it is impossible to list them all, we list a few and then write. Thus for the set of all even numbers, we write $\{2,4,6,8,\ldots\}$.
4. The set of whole numbers is closed under addition, i.e. the sum of two whole numbers is a whole number.
5. The set of whole numbers is closed under multiplication, i.e; the product of any two whole numbers is a whole number.

Teaching Suggestions

1. This is probably the student's first contact with algebraic concepts. Therefore the presentation might well be very informal. The students should nevertheless be made aware that the observations they make as a result of examining several examples are only observations. They should not believe they have proven anything. They should appreciate that they have made a good guess, and they

may have to abandon their guess when a single counterexample is produced.

Example of a counterexample: Suppose a child writes the equation $\triangle + \boxed{2} = 2 \times \triangle$ and say he thinks this is always true. He supports his observation as follows:

$$\triangle + \boxed{3} = 2 \times \triangle$$
$$6 = 6$$

A counterexample will be an example which shows that the student's observation that the equation is always true is incorrect.

Thus:
$$\triangle + \boxed{2} = 2 \times \triangle$$
$$5 = 6$$

2. Take every possible opportunity to encourage the children to take an active part in the class discussion. Try to create a classroom atmosphere in which the students feel free to ask questions, interact with each other or even with the teacher. Be patient if answers to questions are slow in coming. Try to refrain from explaining everything yourself.
3. A brief review of the set of whole numbers should bring out important characteristics of the elements of this set. Ask: "If I name a whole number, can you name the next larger whole number?" "Can you always do this?" "Can you name the next smaller whole number?" "Can you always do this?" "By how much do successive whole numbers differ?" "What is the next larger whole number to zero?" "What is the next smaller whole number to zero?" This discussion should result in the following conclusions:
   (a) The set of whole numbers has a smallest member - zero.
   (b) The set of whole numbers has no largest member.
   (c) The set of whole numbers is formed by starting with zero and adding 1 successively, thus: 0, 0 + 1, 0 + 1 + 1, 0 + 1 + 1 + 1.
   (d) In doing problems such as 2-1, 1 (b), it should be noted that the <u>order</u> of the members does not change the set. $\{11, 12, 13, 14\}$ is the same set as $\{14, 11, 13, 12\}$.
4. Give some practice in naming sets: i.e. a set of dishes, the set of students in this class, the set of books for a Junior Secondary I student, the set of three cans of paint: red, blue, and yellow, etc.
5. If, when we combine any two elements of a set, we <u>always</u> get some

element in the set, we say the set is closed with respect to the rule of combination. This property of the set is called closure. Give the class sufficient experience with addition and multiplication to enable them to make a good guess about closure for the set of whole numbers with respect to addition and multiplication. It should be noted that we can combine any element with itself.

6. Ask: "Can we always subtract any two whole numbers and get a difference which is a whole number?" "What is the answer when you subtract 4 from 10?" "What is the answer then you subtract 10 from 4?" Bring out that sometimes we may subtract in the set of whole numbers and remain in the set - but not always. We say the set of whole numbers is not closed with respect to subtraction.
7. Carry on a similar discussion for division. By questioning lead the children to the realization that sometimes we may divide one whole number by another and get a whole number, but not always.
   Example: $10 \div 5 = 2$ (a whole number)
   $5 \div 10$ does not name a whole number
   We say the set of whole numbers does not have the property of closure with respect to division.
8. Another illustration of a set not closed under an agreed rule of combination is as follows: Let the members of the set be 3 pots of paint (or 3 sticks of colored chalk), red, yellow, and blue. The rule for combining is simply to mix one color with another. If we mix any 2 colors, we get a color other than red, blue, or yellow,and hence we have a pot of paint that is not one of the 3 members of our original set.

Answers to Problems 2-1, Page

1. (a) $\{0, 1, 2, 3, 4, 5, 6\}$
   (b) $\{11, 12, 13, 14\}$
   (c) $\{12, 14\}$
   (d) $\{21, 22, 23, \ldots\}$
2. (a) $\{2, 4, 6, 8, \ldots\}$
   (b) Yes
   (c) Yes
3. (a) $\{1, 3, 5, 7, \ldots\}$
   (b) No
   (c) Yes

4. (a) Not closed (b) Closed (c) Closed (d) Not closed (+)
   (a) Not closed (b) Closed (c) Closed (d) Closed (×)
5. No
6. (a) $\{0\}$, set of multiples of 3, etc.
   (b) $\{1\}$, powers of 2, etc. (c) $\{1, 2, 3\}$ etc.

2-2. Sentences

Skills and Understandings

1. The = symbol tells us that we have two names for the same thing. In a number sentence, the = symbol tells us that we have two names for the same number.
2. A number sentence may be true or false.
3. The symbol > is read "is greater than."
4. The symbol < is read "is less than."

Teaching Suggestions

1. The children are familiar with the idea of a sentence in their language-study classes. They know that a sentence tells us something, expresses a whole idea "When the wind blew..." is not a sentence, because the idea is not completed. But: "When the wind blew, the leaves on the tree shook." is a sentence.
2. Point out that sentences may tell us something about numbers. If such sentences are written in mathematical language (using mathematical symbols) we say we are dealing with a <u>number</u> sentence.
   Example: English sentence: The sum of three and two is five.
   Number sentence: $3 + 2 = 5$
   Encourage the children to come to the blackboard and write number sentences. Ask: "Is $12 \div 4$ a number sentence?" "What is missing?"
3. Discuss with the class that just as we have English sentences, some of which are true and some of which are false, so we have number sentences, some of which are true and some of which are false. "Can you think of an English sentence which is false?" Write some on the blackboard. "Now let us look at an example of a number sentence which is false: $(34)_{five} = (24)_{seven}$
   Put on the board two columns of symbols:

A	B
$3 + 149$	$3 + 149$
$86 \div 2$	$86 \div 2$
17×3	17×3
etc.	etc.

A (cont.)	B (cont.)
"Complete the sentence to make it a true number sentence."	"Complete the sentence to make it a false number sentence."
Example: $3 + 149 = 152$	Example: $3 + 149 = 197$

4. Ask: "If $2 = 3$ is a false number sentence, how might we compare 2 and 3 to make a true sentence?" When some child says "2 is less than 3" or "3 is greater than 2" then introduce the symbols:

   $>$ is greater than

   $<$ is less than

   Call to the attention of the class that the pointed end of this symbol always faces the smaller number, and the open end faces the larger number, if the sentence is true.

   $2 < 3$ is a true sentence

   $2 > 3$ is a false sentence

Answers to Problem 2-2 (a), Page

1.	(a)	True	(e)	False
	(b)	False	(f)	True
	(c)	False	(g)	True
	(d)	False	(h)	True

Answers to Problems 2-2 (b), Page

1.	(a)	True	(h)	True
	(b)	True	(i)	False
	(c)	False	(j)	False
	(d)	False	(k)	True
	(e)	True	(l)	False
	(f)	False	(m)	False
	(g)	False	(n)	True

2-3. Open Sentences

Skills and Understandings

1. A sentence in which there is at least one symbol like $\square$ in which we may write any numeral is called an open sentence. An open sentence is neither true nor false. Naturally, we eventually will abandon boxes and use variables. At that time, for example, the open sentence $2 + \square = 7$ becomes $2 + x = 7$. But it appears that boxes are easier for a pupil to grasp initially. Unfortunately,

from a strictly logical viewpoint, it is not clear whether boxes are variables or physical objects into which one puts numerals. These subtleties will, however, not bother the students. Whenever a student sees a sentence containing a box, he is to ask himself the question: "What numerals inserted in the box make the sentence true?"

Teaching Suggestions

1. This topic might be introduced as follows: Write on the board a sentence like: "Today is his birthday." and ask "Is this true or false?" The class will say that they cannot tell because they do not know to whom the "his" refers. Rewrite the sentence putting in the names of different boys in the class. "Now you can decide if the sentence is true or false?" A sentence like "Today is his birthday." is called an open sentence. It can be made true or false by writing specific names in place of the word "his."
2. Number sentences may be open sentences too. Consider the sentence $2 \times \square = 14$. It says: "Two times some number whose name we will put in the box is fourteen." "Is it true or false?" "Does it tell you what to put in the box?" No! You may put in the numeral for any whole number you wish. Encourage the students to suggest numbers; have them come to the board and write:

$$2 \times \boxed{3} = 14$$
$$2 \times \boxed{9} = 14$$
$$2 \times \boxed{7} = 14$$

etc. In each case have the child decide if his choice made the sentence true or false. The sentence $2 \times \square = 14$ is called an open sentence. An open sentence is neither true or false.
3. Point out that whatever numbers make an open sentence true are members of the <u>truth set</u> of the open sentence. At this time the approach to finding the truth set should be most informal. The technique involves familiarity with the number facts and good, intelligent guessing.
4. In discussing the truth sets for inequalities, stress that we are considering only whole numbers as possible members of the truth set.

5. The starred exercise in Problems 2-3 is to be left for the best students.

Answers to Problem 2-3, Page

1.	(a)	True	(g)	True
	(b)	False	(h)	Open $\{5\}$
	(c)	Open $\{4\}$	(i)	Open $\{3\}$
	(d)	True	(j)	True
	(e)	Open $\{8\}$	(k)	Open $\{7\}$
	(f)	True	(l)	False
2.	(a)	$\{6\}$	(i)	$\{21\}$
	(b)	$\{9\}$	(j)	$\{20\}$
	(c)	$\{10\}$	(k)	$\{2\}$
	(d)	$\{2\}$	(l)	$\{5\}$
	(e)	$\{8\}$	(m)	$\{3\}$
	(f)	$\{1\}$	(n)	$\{2\}$
	(g)	$\{2\}$	(o)	$\{1\}$
	(h)	$\{5\}$	(p)	$\{3\}$
			(q)	$\{20\}$

3. (a) If the smaller number is $\square$,
then $\square + (\square + 5) = 59$.
(b) If the rectangle is $\square$ inches wide,
then $(\square + 3) + \square = 13$.
(c) If there are $\square$ pupils in the smaller school,
then $\square + (3 \times \square) = 216$.

2-4 Rule for Substitution

Skills and Understandings

1. Whatever number you put in one box in an open sentence must be put in every other box in that sentence.
2. Open sentences may have many symbols like boxes, triangles, diamonds, etc.
3. The numeral written in a box may be the same or different from the numeral written in a triangle or a diamond, etc.
4. All the boxes in an open sentence must have the same numeral written in them; all the triangles in an open sentence must have

the same numeral written in them; etc.

Teaching Suggestions

1. It is a natural step to go from $2 + \square = 6$ to $\triangle + \square = 6$ or to $\Diamond + \triangle + \square = 6$. Permit the children to compose their own open sentences using more than one shape. Have them write numerals in the interiors of the shapes to make the sentence true. They should realize that although there is only 1 number in the truth set of $2 + \square = 6$, there are many pairs of whole numbers in the truth set of $\triangle + \square = 6$. They are

$\triangle$	$\square$
0	6
1	5
2	4
3	3
4	2
5	1
6	0

2. It is important for the student to realize that the same numeral may be written inside different shapes: i.e.

$$\Diamond_{2} + \triangle_{2} + \boxed{2} = 6$$
$$\Diamond_{1} + \triangle_{1} + \boxed{4} = 6$$

or different numerals may be written inside the different shapes:

$$\Diamond_{1} + \triangle_{2} + \boxed{3} = 6$$

However, once the decision is made, the same numeral must be written in the same shape throughout thatopen sentence:

$$\triangle + \square + \square = 2 \times \triangle$$
$$\triangle_{2} + \boxed{1} + \boxed{1} = 2 \times \triangle_{2}$$

Is this sentence true or false?

Answers to Problems 2-4 (a), Page

1. $\{4\}$
2. $\{6\}$
3. $\{3\}$
4. $\{2\}$
5. $\{7\}$
6. $\{3\}$
7. $\{5\}$
8. $\{5\}$
9. $\{1\}$

Answers to Problems 2-4 (b), Page

1.	4	4.	4
2.	18	5.	1
3.	4	6.	8

2-5 Revision

The starred problems are to be left for the best students.

Answers to Problems 2-5, Page

2. and 3.	(a)	True	(e)	Open 4
	(b)	True	(f)	Open 5
	(c)	Open $\{8\}$	(g)	Open 3
	(d)	Open $\{12\}$	(h)	False
4.	(a)	$\{6\}$	(f)	$\{6\}$
	(b)	$\{14\}$	(g)	$\{3\}$
	(c)	$\{4\}$	(h)	$\{3\}$
	(d)	$\{6\}$	(i)	$\{4\}$
	(e)	$\{2\}$		
5.	(a)	5	(d)	10
	(b)	5	(e)	4
	(c)	8		

6.

	The set of whole numbers			The set of whole numbers	
(a)	greater than	9	(g)	less than	3
(b)	less than	8	(h)	greater than	3
(c)	less than	5	(i)	less than	3
(d)	greater than	5	(j)	less than	8
(e)	less than	4	(k)	greater than	2
(f)	greater than	3	(l)	greater than	2

7. (a) If there are $\square$ children in 1B, then $\square + (\square + 6) = 70$.
   (b) If Ojo is $\square$ years old, then $(\square \times 2) + \square = 30$.
   (c) If the winner received $\square$ votes, then $\square + (\square - 200) + (\square - 300) = 2500$.

2-6. Identities

Skills and Understandings

1. An identity is an equation which is true for every numeral we might write in the boxes, or triangles or diamonds, etc. as long as the Rule for Substitution is obeyed.

Teaching Suggestions

1. By now the children should have an idea of how to show that a given number makes an open sentence true or false. Choose a number, write its numeral in the box and then see if the equation is true or false. "Consider the equation: $(2 \times \square) + 4 = 22$. In seeking the truth set, let us write 3 in the box. Since we have 22 to the right of the = sign, what do you expect to get to the left of the = sign, if 3 is a member of the truth set?" "Suppose when you write 3 in the box you do not get a name for the number 22 to the left of the = sign. What will this tell you?" Write on the blackboard:

$$\begin{aligned} (2 \times \square) + 4 &= 22 \\ (2 \times \boxed{3}) + 4 &= 22 \\ 6 + 4 &= 22 \\ 10 &= 22 \end{aligned}$$

   Since 10 = 22 is a false equation, the number 3 does not belong to the truth set. Be sure that the students are convinced that this procedure does tell us whether a number is in the truth set or not. The idea may be very evident to us, but it is not always so clear to the children.

2. Ask: "Can anyone make up an open sentence that will be true whatever numbers we name in the boxes, etc., so long as the Rule for Substitution is obeyed?" If answers are too long in coming, put an open sentence on the board which is not an identity and ask if it is an identity, and how can they tell. If this still gets no response, then put a group of symbols like $\square + 3 =$ on the board and ask the students to complete the sentence so that it will be an identity. From here they should be able to go on and compose identities themselves.

3. If the students suggest an equation which is really <u>not</u> an identity, then provide a counterexample. Soon the children themselves will

be able to do this in evaluating each other's work.

4. If no child suggests an identity with more than one shape, then you give an example of such an identity to point this out to the class. Ask: "Is this equation an identity?" $\triangle + \square = \square + \triangle$

Answers to Problems 2-6, Page

1. (a), (c), (d), (e), (f), (g), (i), (j).
3. One

Answers to Problems 2-7, Page

1. (a) 9
   (b) 12
   (c) ($\square$ + 2)
2. (a) none
   (b) one
   (c) nine
3. Show these points on number line:

(a) 4	(i) 5,6,7...
(b) 2	(j) 1,2,3,4,5
(c) 0	(k) 0,1,2
(d) 3	(l) 0,1,2
(e) 3	(m) 0,1,2
(f) 4	(n) 0,1
(g) 6	(o) 0,1,2
(h) 0,1,2,3,4	(p) 0,1,2

4. $\square$ + 6, $\square$ + 9, $\square$ + 30.
   9,12, 33
5. 1, 5, 17. No. No. No.
6. No.
7. $\triangle = 0$

FIRST YEAR SECONDARY TEACHERS' GUIDE

CHAPTER 3

PROPERTIES OF NUMBER SYSTEMS

3-1. The Properties of Zero and One

Skills and Understandings

1. $\square + 0 = \square$ no matter what number is named in the box.
2. $\square \times 1 = \square$ no matter what number is named in the box.

Remark: Zero also has an important multiplicative property, namely, that $0 \times \square = 0$. This can be proved using the additive property of 0 and the distributive law as follows:

$$0 \times \square = (0 + 0) \times \square = (\times \square) + (0 \times \square)$$

and so $0 \times \square = 0$. But do not bring this up in class now. We remark on this here to make it clear that the property of zero listed is the only special one necessary.

Teaching Suggestions

1. The students will have met the identity $\square + 0 = \square$ in the exercises on identities in the last chapter. Here we wish to point up the significance of this particular identity. Ask: "What does this identity tell us?" If there is no answer, ask: "What does it tell us about the number zero?" Only when nothing is responded should you suggest the correct answer, and then let the students show how it works.
2. Develop the property of one similarly.

3. If the students have not already made up the identity $\square \times 0 = 0$, then it should be brought up here as an important property of zero.
4. Ask: "Is there any number other than zero in the triangle which makes the equation true: $\square + \triangle = \square$?" "Is there any number other than one in the triangle which makes the equation $\square \times \triangle = \square$ true?" Bring out the uniqueness of zero and one as the identity elements. Mathematicians call zero the identity for addition and one the identity for multiplication. This language need not be introduced unless the teacher considers the class is ready for it.

Answers to Problems 3-1, page 3-1

1. (e), (f), (h), (j)
2. (a) No
   (b) No
3. (a), (d), (f), (g), are identities
   (b) $\boxed{7} = 3 \times \boxed{7}$ (false)
   (c) $(3 \times \boxed{5}) + 2 = (4 \times \boxed{5}) + 1$ (false)
   (e) $\boxed{5} + \triangle 2 = \boxed{5} + \boxed{5}$ (false)

3-2. The Commutative Property

Skills and Understandings

1. Pairs of numbers added in either order give the same sum:

$$\triangle + \square = \square + \triangle$$

2. Pairs of numbers multiplied in either order give the same product:

$$\square \times \triangle = \triangle \times \square$$

Teaching Suggestions

1. Introducing the commutative properties through a discussion of identities provides a new and refreshing approach. The children have been using these properties in their calculations in the primary grades. Now they should be made to

realize that these properties may be regarded as special members of the set of identities rather than as isolated generalities.

2. The aim of the first set of examples in 3-2 is to help the pupil to recognize that what we are doing is adding pairs of numbers in two ways and that either way gives the same sum. Do not tell them at this point that pairs of numbers may be added in either order. Let this come from them when they are ready.

3. In seeking the truth sets for equations like: $\square + 5 = 5 + 7$, the students will probably naturally use commutativity and conclude that $\square = 7$. If so, then all is well. But a student might add 5 and 7, getting 12, and then ask himself what must he add 5 to in order to get 12. If this happens, give him examples involving larger numbers so the arithmetic is harder, so as to push him to find the short-cut; for example, $479 + 983 = \square + 479$.

4. After the students have worked the second set of examples in 3-2 and observed that $\square + 3 = 3 + \square$ is an identity, write on the board: $\square + 4 =$ and ask the class to complete the equation so that it will be an identity. Then call for other examples of similar identities. List these on the board:

$$\square + 4 = 4 + \square$$
$$\square + 3 = 3 + \square$$
$$\square + 7 = 7 + \square$$
$$\square + 10 = 10 + \square$$

and so forth, on and on.

Ask: "How do these identities differ from each other?" Lead the class to see that the only real difference in all of these identities are the numbers, 3, 4, 7, 10, etc., so that we could say

$$\square + \text{any number} = \text{that number} + \square$$

Ask: "Do we have some mathematical symbol which we might write instead of the words 'any number'?" If the children do not suggest using some shape like the box, then you do,

and write

$$\square + \square = \square + \square$$

Discuss with the group that if we obey the Rule for Substitution, this identity will not really say what we mean, because all four numerals we write will have to be the same. This is not so in the list of identities for which we are seeking some pattern. The use of some shape other than the box should now be forthcoming and the final form of the identity may be: $\square + \triangle = \triangle + \square$.

5. Provide opportunity for practice in writing numerals for numbers in the identity just written.
6. Tell the class that this is such an important identity, we have a special name for it: The commutative property of addition.
7. Use similar procedures to develop the commutative property of multiplication. The conclusions from the children will probably come much more readily now.

Class Activity 3-2, Page 3-2

1. (a), (c), (d), (f) are true
2. (a) {2}
   (b) {7}
   (c) {738}
   (d) {67}
   (e) {all whole numbers} an identity

Class Activity 3-2, Page 3-3

1. {57}
2. {14}
3. {71}
4. {7}
5. {615}

3-3. The Associative Properties

Skills and Understandings

1. The sum of three addends is the same whether we combine

the first two and add the third to the sum, or combine the second and third and add the sum to the first.

$$(\square + \triangle) + \diamond = \square + (\triangle + \diamond)$$

2. The product of three numbers is the same whether we multiply the first two and then multiply the product by the third or multiply the second and the third and then multiply the first by the product.

$$(\triangle \times \square) \times \diamond = \triangle \times (\square \times \diamond)$$

Teaching Suggestions

1. Clearly the verbalization of this property is both complicated and clumsy. The advantage of writing it in mathematical symbols is obvious. If the children can write it in symbols and use it correctly, then it is not important for them to be able to say it in English.
2. Introduce the lesson by writing the sum of three addends on the board: i.e. 6 + 4 + 123, 99 + 1 + 72, 374 + 75 + 25, 69 + 8 + 2, etc. In some of these it is a short cut to add the first two numbers, and then combine with the third; and in others, it is a short-cut to add the second two numbers and then combine with the first. Do enough of these exercises so that the class appreciates this.
3. Introduce the use of brackets to show which two addends you wish to add first. Put a list of the sums of three addends on the board and ask the children to include those they have decided to add first in brackets. For the purposes of this lesson do not give examples like 13 + 96 + 7 in which it is a simplification to add the 13 and 7 first. We are concerned only with grouping the addends and do not wish at this time to include rearranging them, using the commutative property. This should come at the end of the lesson.
4. Now proceed to develop the awareness of the associative property of addition by following much the same procedure used in recognizing the commutative property. First might come the decision as to whether the two sums (5 + 7) + 39 and 5 + (7 + 39) are the same. Then do exercises in finding

the truth sets of equations like:
$(4 + 6) + \diamond = 4 + (6 + 8)$, $(\square + 189) + 365 = 72 + (189 + 365)$. When the students can give the truth set on sight, without actually performing any arithmetic processes, then they are using the associative property.

5. If the class has responded well to the lesson, you could ask at once if anyone can suggest an identity which will be true for any three numbers we might add. If there is no response, then backtrack a little and ask the class to write identities like:

$$(\triangle + 3) + 15 = \triangle + (3 + 15)$$
$$(16 + 121) + \diamond = 16 + (121 + \diamond)$$
$$(7 + \square) + \triangle = 7 + (\square + \triangle)$$
$$(\square + 5) + \triangle = \square + (5 + \triangle)$$

6. Tell the class that the special name for this property is the associative property of addition and we write it in symbols as:

$$(\square + \triangle) + \diamond = \square + (\triangle + \diamond)$$

7. By similar procedures make the students aware of the associative property of multiplication.
8. Provide drill in writing numerals for numbers in place of the boxes, triangles, diamonds, etc. in the identities which describe the associative property of addition and of multiplication.
9. Introduce problems in mental calculation which can be done very rapidly by making use of the commutative and associative properties:

   i.e. $35 + 72 + 65$
   $17 \times 25 \times 4$
   $30 \times 169 \times 2$
   $347 + 583 + 417$
   etc.

10. Point out to the class that they use these properties every time they check an addition problem by adding first in one direction and then in the other. They also use the properties when they regroup a set of addends to simplify the calculation.

Add and Check

$$\begin{array}{r} 6 \\ 9 \\ 10 \\ \underline{7} \end{array}$$

$[(6 + 9) + 10] + 7$
$[(7 + 10) + 9] + 6$

Add:

$$\begin{array}{r} 9 \\ 2 \\ 3 \\ 8 \\ 7 \\ \underline{1} \end{array}$$

Answers to Problems 3-3(a) Page 3-5

1. (a) $(257 + 86) + 843 = 343 + 843 = 1186$
   $257 + (86 + 843) = 257 + 929 = 1186$
   (b) $(1277 + 964) + 59 = 2241 + 59 = 2300$
   $1277 + (964 + 59) = 1277 + 1023 = 2300$
   (c) $(182 + 63) + 94 = 245 + 94 = 339$
   $182 + (63 + 94) = 182 + 157 = 339$
2. $(\square + \triangle) + \diamond = \square + (\triangle + \diamond)$
   (a), (b), and (d) are identities.

Answers to Problems 3-3(b), Page 3-6

1. (a) $3 + (59 + 7) = 3 + (7 + 59) = (3 + 7) + 59 = 10 + 59 = 69$
   (b) $14 + 70 + 6 = 14 + (70 + 6) = 14 + (6 + 70) = (14 + 6) + 70 = 20 + 70 = 90.$
   (c) $26 + (187 + 74) = 26 + (74 + 187) = (26 + 74) + 187 = 100 + 187 = 287.$
   (d) $45 + (239 + 55) = 45 + (55 + 239) = (45 + 55) + 239 = 100 + 239 = 339.$
   (e) $24 + (157 + 26) = 24 + (26 + 157) = (24 + 26) + 157 = 50 + 157 = 207.$
   (f) $17 + 243 + 83 = 17 + (243 + 83) = 17 + (83 + 243) = (17 + 83) + 243 = 100 + 243 = 343.$

☆2. $\square + \triangle = \triangle + \square$
$(\square + \triangle) + \diamond = \square + (\triangle + \diamond)$

Answers to Problems 3-3(c), Page 3-7

1. $(23 \times 4) \times 6 = 92 \times 6 = 552$
   $23 \times (4 \times 6) = 23 \times 24 = 552$
2. $(45 \times 27) \times 12 = 1215 \times 12 = 14580$
   $45 \times (27 \times 12) = 45 \times 324 = 14580$
3. $(563 \times 47) \times 256 = 26461 \times 256 = 6,774,016$
   $563 \times (47 \times 256) = 563 \times 12032 = 6,774,016$

Answers to Problems 3-3(d) Page 3-7

1. (a) $2 \times (57 \times 5) = 2 \times (5 \times 57) = (2 \times 5) \times 57 = 10 \times 57 = 570$
   (b) $5 \times (371 \times 20) = 5 \times (20 \times 371) = (5 \times 20) \times 371 =$
   $100 \times 371 = 37100$
   (c) $12 \times (15 \times 4) = 12 \times 60 = 720$
   (d) $4 \times (893 \times 25) = 4 \times (25 \times 893) = (4 \times 25) \times 893 =$
   $100 \times 893 = 89300$
   (e) $(6 \times 41) \times 5 = (41 \times 6) \times 5 = 41 \times (6 \times 5) = 41 \times 30 = 1230$
   (f) $5 \times (72 \times 10) = 5 \times (10 \times 72) = (5 \times 10) \times 72 = 50 \times 72 = 3600$
   (g) $40 \times (13 \times 25) = 40 \times (25 \times 13) = (40 \times 25) \times 13 =$
   $1000 \times 13 = 13,000$
   (h) $15 \times (6 \times 77) = (15 \times 6) \times 77 = 90 \times 77 = 6930$

☆2. $\square \times \triangle = \triangle \times \square$
   $(\square \times \triangle) \times \diamond = \square \times (\triangle \times \diamond)$

3-4. Equivalent Equations

Skills and Understandings

1. Equivalent equations are equations with the same truth set.
2. If the same number is added to the number represented on each side of the = symbol in an equation, the truth set of the new equation is the same as the truth set of the original equation. Thus:
   $5 + 2 + \triangle = 16 \div 2$ has the same truth set as $(5 + 2 + \triangle) + 11 = (16 \div 2) + 11$.
3. If the same number is subtracted from the number represented in each side of the = symbol in an equation, the truth set of the new equation is the same as the truth set of the

original equation: Thus
$5 + 2 + \triangle = 16 \div 2$ has the same truth set as $(5 + 2 + \triangle) - 6 = (16 \div 2) - 6$.

4. If the number represented on each side of the = symbol in an equation is multiplied by the same non-zero number, then the new equation has the same truth set as the original equation. Thus
$5 + 2 + \triangle = 16 \div 2$ has the same truth set as $(5 + 2 + \triangle) \times 10 = (16 \div 2) \times 10$.
5. If the number named on each side of the = symbol in an equation is divided by the same non-zero number, then the new equation has the same truth set as the original equation. Thus
$5 + 2 + \triangle = 16 \div 2$ has the same truth set as $\frac{5 + 2 + \triangle}{4} = \frac{16 \div 2}{4}$.
6. We may use the operations of addition, subtraction, multiplication, and division to obtain an equivalent equation.
7. A very simple form of an equation is $\square$ = the numeral for some number or $\square = 5$ or $\square = 17$, etc.
8. The student should be able to tell which procedure he is using to get a simpler equivalent equation.
9. In seeking the truth set of an equation, the student should continue to find simpler equivalent equations until he finds an equivalent equation whose truth set he can tell on sight.

Teaching Suggestions

1. This topic is of importance for two major reasons. One is that the technique of writing equivalent equations is essential in proving theorems in algebra, and the other is that this technique provides a procedure for finding the truth set of an equation.
2. Once again the approach to a new topic is intuitive and inductive (making an observation as a result of recognizing a pattern from several examples). Throughout this section, whenever suitable, remind the students that their observations hold only until a counterexample can be produced.

3. Put an open sentence like: $\square + 4 = 10$ on the board and ask the students to write another equation with the same truth set. If no one is able to do this, then ask: "If $\square + 4 = 10$, then how much is $\square + 5$?" When they say 11, write on the board $\square + 5 = 11$. "Do these two equations have the same truth set?" "Now give another equation with the same truth set." Record these on the board. If the children do not suggest by themselves equivalent equations which are derived by use of the properties already studied, then you suggest one to get them started. There should be some like the following listed:

$$\square + 0 = 18 \quad \text{and} \quad \square = 18$$
$$\triangle \times 1 = 57 \quad \text{and} \quad \triangle = 57$$
$$\triangle \times 6 = 42 \quad \text{and} \quad 6 \times \triangle = 42$$
$$35 + (6 + \square) = 196 \quad \text{and} \quad (35 + 6) + \square = 196, \quad \text{etc.}$$

4. To bring out that we can get an equivalent equation not only by using the properties but by other methods, write on the board:

$$\square + 4 = 10$$
$$\square + 5 = 11$$

Ask: "Are these equivalent equations; do they have the same truth set?" "Is there any objection to using the numeral $4 + 1$ for 5?" or "$10 + 1$ for 11?" "Then I may write:

$$\square + (4 + 1) = 10 + 1$$

or $(\square + 4) + 1 = 10 + 1$?" "Is this equation equivalent to $\square + 4 = 10$?" "Can you see how I might have started with $\square + 4 = 10$ and found the equivalent equation $\square + 5 = 11$?" "Compose other equations equivalent to $\square + 14 = 183$, using the method we just recognized."

5. If the students really understand what has been happening here, they will themselves suggest the possibility of composing equivalent equations using subtraction, multiplication and division. However, if the students do not suggest such procedures, then go through the same type of development as suggested for addition. Put on the board:

$$\square + 3 = 12$$
$$\square + 2 = 11.$$

Write 2 as 3 - 1, and 11 as 12 - 1. Ask: "Do the following equations have the same truth set:

$$\square + 3 = 12 \text{ and}$$
$$(\square + 3) - 1 = 12 - 1?"$$

Compose many sets of equivalent equations using subtraction. Multiplication and division should go rather rapidly now.

6. The problem of multiplication and division by non-zero numbers may need to be discussed at this point. "Let us use multiplication to compose an equation equivalent to $\square = 6$." List on the board whatever examples the children suggest, i.e.

$$\square \times 7 = 6 \times 7$$
$$\square \times 4 = 6 \times 4$$
$$\square \times 1{,}572 = 6 \times 1{,}572 \text{ etc.}$$

"What is the truth set of each of these equivalent equations?" The answer will be given quickly $\{6\}$. "Suppose we use multiplication by zero. Let us see what we would get. $\square \times 0 = 6 \times 0$." "Is 6 in the truth set of this equation?" "Is there any other number in the truth set of this equation?" Bring out that the truth set of this particular equation is the set of all numbers that the children know. Write on the board

$$\square = 6 \quad \{6\}$$
$$\square \times 0 = 6 \times 0 \quad \{\text{all the numbers}\}$$

Clearly the truth sets are not the same. Furthermore every equation that looks like $\square \times 0 = 6 \times 0$, (list a few on the board

$$\square \times 0 = 7 \times 0$$
$$\square \times 0 = 8 \times 0$$
$$\square \times 0 = 9 \times 0, \text{ etc})$$

has the same truth set, $\{\text{all the whole numbers}\}$. Since multiplication by zero does not lead to equivalent equations, we must not allow multiplication by zero when trying to find equivalent equations.

7. If the question arises, point out that division by zero, if permitted, would lead us to contradictions of previously made agreements. Thus: we have said that if $\frac{18}{2} = 9$, then

it must also be true that $18 = 9 \times 2$. Now consider $\frac{18}{0} = \square$. It must follow that $18 = 0 \times \square$. But we already have an identity which tells us that $0 = 0 \times \square$. We cannot have both $0 = 0 \times \square$ and also $18 = 0 \times \square$. Therefore, we exclude division by zero.

8. The four techniques for finding equivalent equations may be used successfully to write simpler equations with the same truth set. The simpler the equation, the easier it is to identify the truth set. Therefore, we will practice writing simpler equivalent equations. There are no fixed criteria for determining when an equation is simpler than another. Let the children decide this for themselves. You may suggest two equivalent equations and ask which seems simpler to the class:

$$\square + 1{,}489 = 2{,}589$$
$$\square = 1{,}100$$

Ask: "Which method was used to go from the first to the second equation?" "For which equation can you find the truth set more readily?"

Answers to Problems 3-4(a), Page 3-8

1. (a) $\square + 5 = 13$
   (b) $\square + 1 = 9$
   (c) $\square + 10 = 18$
   (d) $\square + 1 = 9$
   (e) $\square + 4 = 12$
   (f) $7 + \square = 15$
   (g) $7 + \square = 15$
   (h) $\square + 5 = 13$
2. (a) $9 \times \square = 45$
   (b) $\square \times 1 = 5$
   (c) $6 \times \square = 30$
   (d) $12 \times \square = 60$
   (e) $100 \times \square = 500$

Answers to Problems 3-4(b), Page 3-10

1. (a), (c), (e) are sets of equivalent equations.
2. These are some samples of possible answers. There are, of course, many answers.
   (a) $7 + \square = 14$
   $10 + \square = 17$
   $506 + \square = 513$

(b) $\square + 6 = 8$

$\square + 27 = 29$

$\square + 1{,}462 = 1{,}464$

(c) $269 + \square = 565$

$300 + \square = 596$

$1{,}267 + \square = 1{,}563$

(d) $\square + 5 = 31$

$\square + 10 = 36$

$\square + 50 = 76$

Answers to Problems 3-4(c) Page 3-11

1. $9 \times \square = 63$ is obtained from $3 \times \square = 21$ using the method: "If the same number is <u>multiplied</u>, etc."
   $3 \times \square = 21$ is obtained from $9 \times \square = 63$ using the method: "If the same number is <u>divided</u>, etc."
2. $1 + \square = 2$ is obtained from $64 + \square = 65$ using the method: "If the same number is <u>subtracted</u>, etc."
   $64 + \square = 65$ is obtained from $1 + \square = 2$ using the method: "If the same number is <u>added</u>, etc."
3. $5 \times \square = 20$ is obtained from $15 \times \square = 60$ using the method: "If the same number is <u>divided</u>, etc."
   $15 \times \square = 60$ is obtained from $5 \times \square = 20$ using the method: "If the same number is <u>multiplied</u>, etc."
4. $18 + \square = 32$ is obtained from $6 + \square = 20$ using the method: "If the same number is <u>added</u>, etc."
   $6 + \square = 20$ is obtained from $18 + \square = 32$ using the method: "If the same number is <u>subtracted</u>, etc."
5. $\square + 2 = 11$ is obtained from $\square + 8 = 17$ using the method: "If the same number is <u>subtracted</u>, etc."
   $\square + 8 = 17$ is obtained from $\square + 2 = 11$ using the method: "If the same number is <u>added</u>, etc."

Answers to Problems 3-4(d), Page 3-12

1. Divide (a) by 3 to get (b)
   Multiply (b) by 3 to get (a)
3. Subtract 6 from (a) to get (b)
   Add 6 to (b) to get (a)

4. Add 100 to (a) to get (b)
   Subtract 100 from (b) to get (a)
6. Subtract 183 from (a) to get (b)
   Add 183 to (b) to get (a)
7. Add 6 to (a) to get (b)
   Subtract 6 from (b) to get (a)

Answers to Problems 3-4(e), Page 3-13

1. (a) Subtract 16 from the number named on each side of the equation.
   (b) Divide the number named on each side of the equation by 7.
   (c) Subtract 17 from the number named....
   (d) Add 10 to the number named on each side,...
   (e) Add 35 to the number named on each side....
   (f) Divide the number named on each side of the equation by 83.
   (g) Subtract 124 from the number named....

2. (a)	$\square = 51$	(a)	$\{51\}$
(b)	$\square = 9$	(b)	$\{9\}$
(c)	$36 = \square$	(c)	$\{36\}$
(d)	$\square = 30$	(d)	$\{30\}$
(e)	$202 = \square$	(e)	$\{202\}$
(f)	$\triangle = 3$	(f)	$\{3\}$
(g)	$59 = \diamond$	(g)	$\{59\}$

3-5. The Use of Letters in Writing Open Sentences

Skills and Understandings

1. Mathematicians use letters of the alphabet instead of boxes, triangles, diamonds etc. in writing open sentences.
2. Each letter must represent the same number wherever that letter appears in the open sentence.
3. All the procedures which we used for writing simpler equivalent equations involving boxes, etc., may be used for open sentences written with letters.
4. The multiplication symbol between two letters or between a letter and a numeral may be omitted.

Teaching Suggestions

1. Having the experience of writing open sentences using $\square$, etc., the students will make the transition to the use of letters very readily.
2. In view of the fact that there has been no teaching or practice given to the problem of translating an example from English into mathematical symbols, it might be advisable to leave the starred problems for the best students.

Answers to Problems 3-5(a), Page 3-14

1. There are many possible answers. These are samples.

$$a \times 7 = 21$$

2. $2 \times b = 54$
3. $19 = x + 11$

Answers to Problems 3-5(b), Page 3-15

1. There are many possible answers. These are samples.

$$a + b = 9 + c$$

2. $54 + m = n + p$
3. $q \times r = 7 \times t$

Answers to Problems 3-5(c), Page 3-15

1. (a) $(2 \times a) + 7 = 21$
   (b) $3 \times a = 19 + (2 \times a)$
   (c) $(a \times 5) + b = (b \times 5) + a$
   (d) $71 \times a = 39 + c + b$
   (e) $a \times b = (c \times b) + 55$
2. (a) $a + 0 = a$
   (b) $a \times 1 = a$
   (c) $a + b = b + a$
   (d) $a \times b = b \times a$
   (e) $(a + b) + c = a + (b + c)$
   (f) $(a \times b) \times c = a \times (b \times c)$
3. (a) $\square + (2 \times \diamond) + \triangle = 59$
   (b) $\square \times \triangle = 21$

(c) $\diamond + 39 = \square + 5$

(d) $\diamond + \triangle = 37 + \square$

4. (a) {26}

(b) {10}

(c) {8}

(d) {12}

(e) {3}

5. (a) 12

(b) 12

(c) 5

(d) 1

6. (a) $a = 48$

(b) $y = 7$

(c) $x = 3$

(d) $z = 240$

(e) $b = 13$

7. (a) $(x + 5) + x = 15$

(b) $2x + x = 39$

Answers to Problems 3-5(d), Page 3-17

1. (a) $3 \times y = 6$

(b) $(4 \times a) + (4 \times 6) = 4 \times (a + 6)$

(c) $(x \times y) + (a \times b)$

(d) $m \times (3 + 7) = (3 \times m) + (7 \times m)$

(e) $(4) \times (5) - (8 \times y)$

2. (a) $10t + u$

(b) $10\ (t + u)$

(c) $7 \times 9 = (7 \times 6) + (7 \times 3)$

(d) $yw - yz$

(e) $(a + 5)\ 6 = a6 + (5 \times 6)$

(f) $25 \times 4 = 100$

3-6. Domain of a Variable

Skills and Understandings

1. The letters used to represent numbers in open sentences are called variables.

2. Open sentences are made true or false when definite values are given to the variables.
3. We may restrict to a particular set the set of numbers from which we may choose a value of the variable. This particular set is called the domain of the variable.
4. When no member of the domain of the variable is also in the truth set of an open sentence, then the truth set for the given domain is the empty set. We show the empty set by $\{\ \}$.

Teaching Suggestions

1. Up to this point the domain of the variable for any open sentence was understood to be the set of whole numbers. In discussing the truth set for the open sentence: $3(x + 5) = 21$, any whole number was represented in place of x. Ask:"Would the truth set have changed if we had chosen the set of all even numbers as the domain of the variable?" "Would the truth set have changed if we had chosen the set of whole numbers below 100 as the domain of the variable?" "Would the truth set have changed if we had chosen the set of odd numbers as the domain of the variable?"
2. The last question might serve as an introduction to the notion of the empty set. The answer to that question is "Yes," the truth set would change. As a matter of fact there is no number in the domain which is also in the truth set. Then the truth set is the empty set. To help understand the notion of the empty set ask: "What is the set of all students in this class who are over eight feet tall?" "What is the set of all students in this class who can run a mile in 45 seconds?" Then let the students make up examples of the set with no members - the empty set.
3. Considerable practice should be provided in finding the truth set of an equation when the domain has a finite number of members. It should be stressed that only the members of the domain may be used in seeking the truth set. It should be brought out at every opportunity that in one domain an

equation may have a truth set which is the empty set, but that if we change the domain of the variable there may be members in the truth set. Example: If the domain of the variable is the set of whole numbers, then the truth set for the equation $2a = 5$ is the empty set. If the domain of the variable is made the set of all rational numbers, then the truth set is $\frac{5}{2}$.

4. When an open sentence has more than one variable, the domain of each variable must be clearly indicated. The domains, in this case, need not be the same.

5. In recording the members of the truth set for an equation with two variables, we may list the numerals in the table as we did in the previous chapter, or we may write number pairs: Example: $x = y + 4$. Find the truth set if the domain of x is the set of whole numbers, and the domain of y is the set $\{0, 2, 3, 5, 8, 9\}$. Ask: "What numerals may be written in place of x?" "What numerals may be written in place of y?"

x	y
4	0
6	2
7	3
9	5
12	8
13	9

$(4, 0)$
$(6, 2)$
$(7, 3)$
$(9, 5)$
$(12, 8)$
$(13, 9)$

The first number in the number pair is a value of x, while the second number in the number pair is a value of y.

The truth set may then be written: $\{(4,0), (6,2), (7,3), (9,5), (12,8), (13,9)\}$

Answers to Problems 3-6(a), Page 3-18

1. (a) $\{5\}$
   (c) $\{4\}$
   (f) $\{4\}$
   (g) $\{3, 4\}$
   (i) $\{3, 4, 5, 6\}$
2. (d) $\{10, 11, 12\}$
   (e) $\{12\}$
   (i) $\{9, 10, 11, 12\}$

Answers to Problems 3-6(b), Page 3-19

1. (a) $\{11\}$
   (b) $\{1\}$
   (d) $\{0, 1, 2, 3, 4\}$
   (f) $\{3\}$
   (g) $\{10, 11, 12, \ldots\}$
   (h) $\{0, 1, 2, 3, \ldots\}$
   (i) $\{2, 3, 4\}$
   (j) $\{0, 1, 2, 3, \ldots\}$
   (k) $\{\ \}$

Answers to Problems 3-6(c), Page 3-19

1. (a) $\{24\}$
   (b) $\{3\}$
   (c) $\{\ \}$
   (d) $\{0, 1, 2, 3, \ldots\}$
   (e) $\{0, 1, 2, 3, \ldots\}$
   (f) $\{\ \}$
   (g) $\{0, 1, 2, 3, \ldots\}$
   (h) $\{\ \}$
   (i) $\{0, 1, 2, 3, \ldots\}$
2. (a) $\{2, 4, 6, 8, 10\}$
   (b) $\{2, 4, 6, 8, 10\}$
   (c) $\{2, 4, 6, 8, 10\}$
   (d) $\{\ \}$
   (e) $\{\ \}$
   (f) $\{4, 6, 8, 10\}$

Answers to Problems 3-6(d), Page 3-20

1. (a) $\{(0, 3), (1, 2)\}$
   (b) $\{(1, 3)\}$
   (c) $\{(1, 3)\}$
   (d) $\{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3)\}$
   (e) $\{(1, 2)\}$
   (f) $\{\ \}$
   (g) $\{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3)\}$

(h) $\{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3)\}$
(i) $\{\ \}$
(j) $\{(0, 2), (0, 3), (1, 1), (1, 2), (1, 3)\}$
(k) $\{(1, 1)\}$

3-7. The Distributive Property

Skills and Understandings

Another important identity is called the distributive property. It tells us that $a(b + c) = ab + ac$. It involves two operations, multiplication and addition. This particular identity says that multiplication is distributive over addition.

Teaching Suggestions

1. The exercises in the text are carefully graded to enable the children to state the identity which is the distributive property. The steps taken parallel those followed in the development of the identities for the commutative and associative properties.
2. In motivating this lesson, you might give the children mental calculations of the kind: 14×12, 57×106, 63×41, etc., which are much simplified if the child knows how to apply the distributive property:

$$14 \times 12 = 14\ (10 + 2) = 140 + 28 = 168$$
$$57 \times 106 = 57\ (100 + 6) = 5700 + 342 = 6042$$
$$63 \times 41 = 63\ (40 + 1) = 2520 + 63 = 2583$$

If any of the children give the answers rapidly, fine. If not, then do not tell how to apply the property as though it were just another rule to follow. Tell the class that the property they are going to study will help them to do these problems with ease. Bring the problems up again at the end of the lesson.

3. After the distributive property is developed and practice in its use is given, then it might be of some value to discuss with the class whether addition is distributive over multiplication. Let the class decide whether the following

is an identity:

$$a + (bc) = (a + b)\ (a + c)$$

4. Show the class how we use the distributive property in multiplying 43 by 25. We will use the numeral 40 + 3 for 43, and 20 + 5 for 25. (20 + 5) (40 + 3) = 20 (40 + 3) + 5 (40 + 3) = 800 + 60 + 200 + 15 = 1075. Note that the distributive property was used several times, and the commutative and associative properties were used in adding in the last step. Show the class this same example done in the usual way:

$$\begin{array}{r} 43 \\ \underline{\times 25} \\ 215 \\ \underline{86} \\ 1075 \end{array}$$

and ask, "Why do I put the 6 under the 1 rather than under the 5?" "Am I multiplying by 2 or 20?" etc. Have the students point out the corresponding numbers when the example is done using the traditional method and the method which shows the use of the distributive property.

5. There are any number of applied problems which illustrate the distributive property. Example: Every child in a class brought 10 cents (or pence) to contribute to a local charity. There are 24 boys and 16 girls in the class. How much was contributed? This might be done by adding 24 + 16 and multiplying the sum by 10. Or it might be done by multiplying 10 × 24 and 10 × 16 and adding the products. Both procedures give the same answer because the distributive property holds:

$$10\ (24 + 16) = (10 \times 24) + (10 \times 16)$$

Let the children compose problems which illustrate the distributive property.

<u>Answers to Problems 3-7(a), Page 0-00</u>

1. (a) True
   (b) False
   (c) True
   (d) True
   (e) False
   (f) True

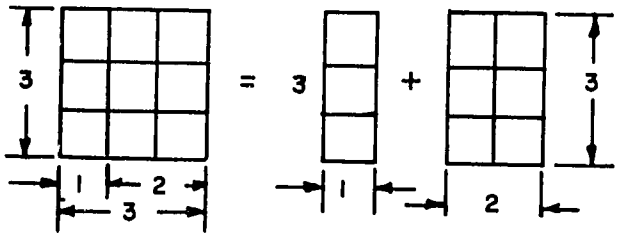
2. (a) $\{8\}$ (c) $\{16\}$
   (b) $\{6\}$ (d) $\{4\}$
3. The following open sentences are identities:
   (b) (d) (e)

Answers to Problems 3-7(b), Page 3-22

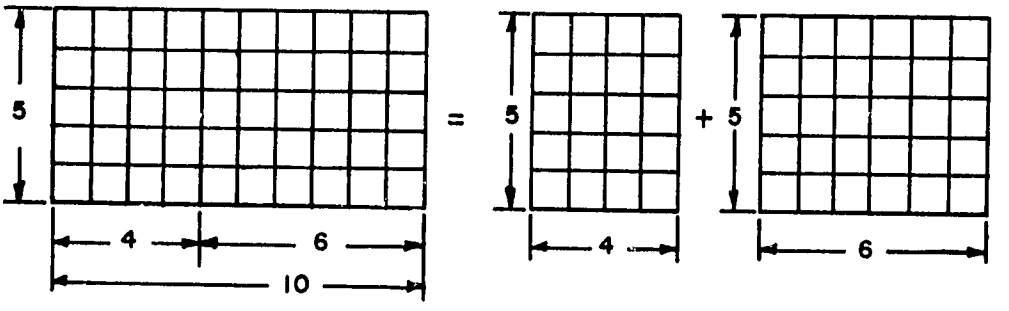
1. $\square \times (\triangle + \diamond) = (\square \times \triangle) + (\square \times \diamond)$
2. (a) 90 (d) 1620
   (b) 7100 (e) 340
   (c) 400

Answers to Problems 3-7(c), Page 3-24

1. (a)



(b)



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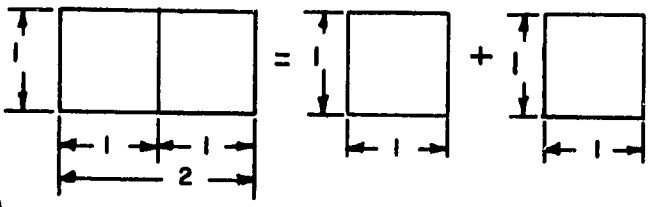
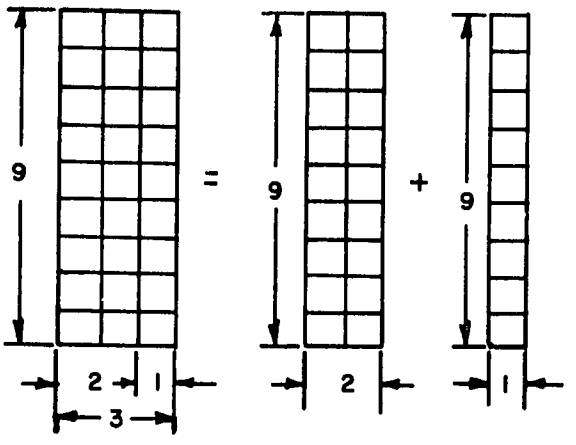


(d)



2. Yes. It is true for all values of a, b, and c.
3. Altogether Isa has 80 balls. He has 50 white balls and 30 red balls. This is an example of the distributive property, i.e.,

$$10\ (5 + 3) = 50 + 30$$

4. There are 10 balls in each box; 70 balls altogether; 21 red balls and 49 white.
5. Tunde has (br) red balls and (bw) white ones. There are (r + w) balls in each box. Altogether Tunde has b (r + w) balls, i.e.,

$$b\ (r + w) = br + bw.$$

6. There are xz girls in the whole school and xy boys. There are x (z + y) or(xz) + (xy) children in the school.

3-8. <u>Revision</u>

<u>Answers to Problems 3-8, Page 3-25</u>

1. $\square + 0 = \square$

   $\square \times 1 = \square$

   $\square + \triangle = \triangle + \square$

   $\square \times \triangle = \triangle \times \square$

   $(\square + \triangle) + \diamond = \square + (\triangle + \diamond)$

   $(\square \times \triangle) \times \diamond = \square \times (\triangle \times \diamond)$

   $\square \times (\triangle + \diamond) = (\square \times \triangle) + (\square \times \diamond)$

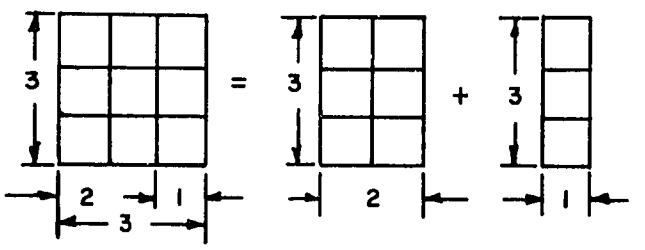
2. An identity is an open sentence true for all values of the variables, if the Rule for Substituting is obeyed.
3. (a) Identity. The third property above.
   (b) Identity. The sixth.
   (c) Identity. The second and the seventh.
   (e) Identity. The sixth.
   (i) True. The second.
   (j) True. The first.
4. (a) Yes.
   (b) Yes.
   (c) Because they are true for all values of a, b, and c.
5. (a) Two equations which have the same truth set.
   (b) Adding, subtracting, multiplying, and dividing the number named on both sides of the equal symbols by the

same number. Multiplication and division by non-zero numbers only.

6. (a) They both have same truth set $\{16\}$
   (b) They both have same truth set $\{9\}$
   (d) They both have same truth set $\{3\}$
   (e) They both have same truth set $\{9\}$
   (f) They both have same truth set $\{4\}$
   (h) They both have same truth set $\{9\}$
   (i) They both have same truth set $\{9\}$
   (j) They both have same truth set $\{3\}$

7. (a) $\square = 13$
   (b) $\triangle = 4$
   (c) $35 = a$
   (d) $b = 15$
   (e) $5 = x$
   (f) $a = 4$
   (g) $2 = b$
   (h) $a = 5$

8. (a)



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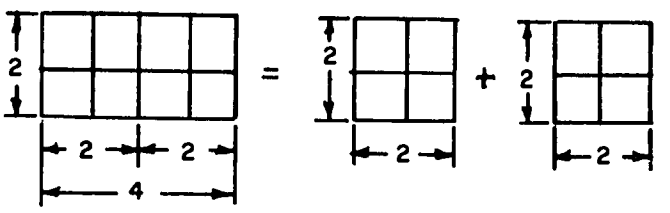

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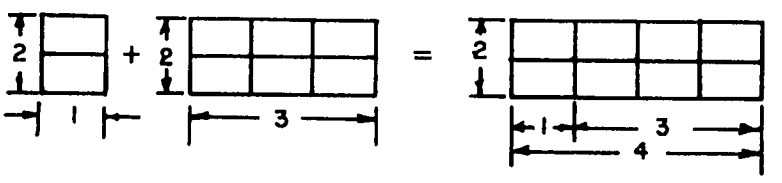
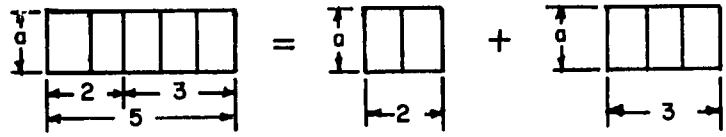
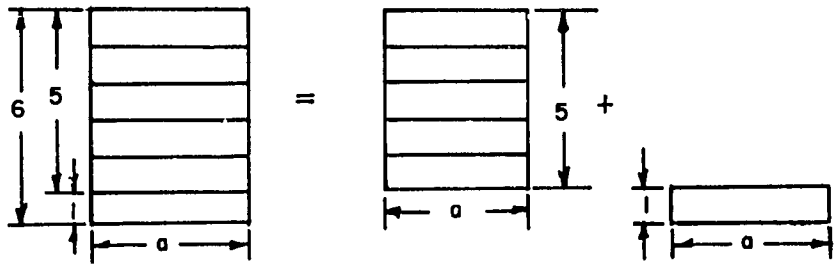


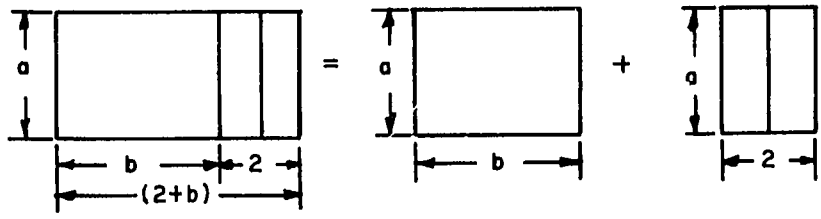
9. (a)


a
2
3
5
=
a
2
+
a
3

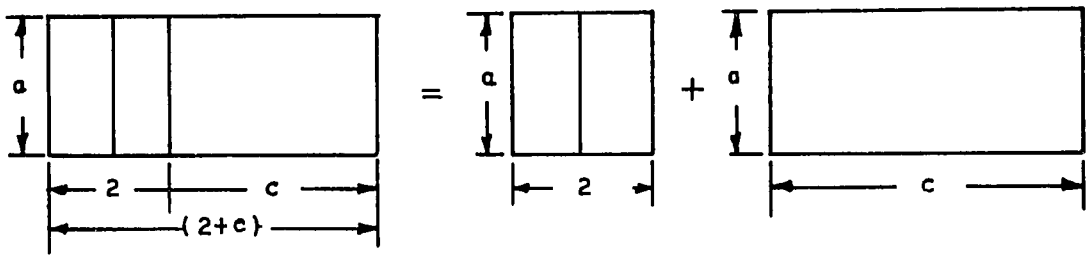
(b)


6
5
a
=
5
a
+
a

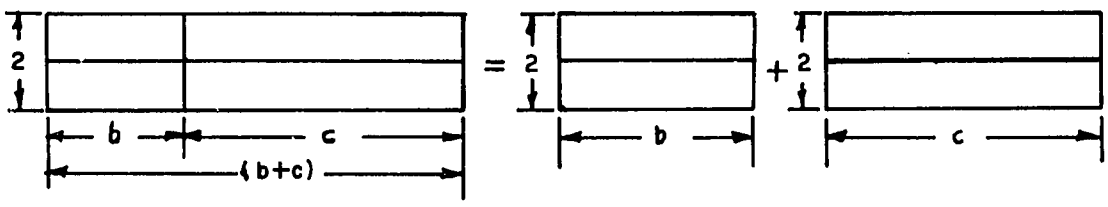
(c)


a
b
2
(2+b)
=
a
b
+
a
2

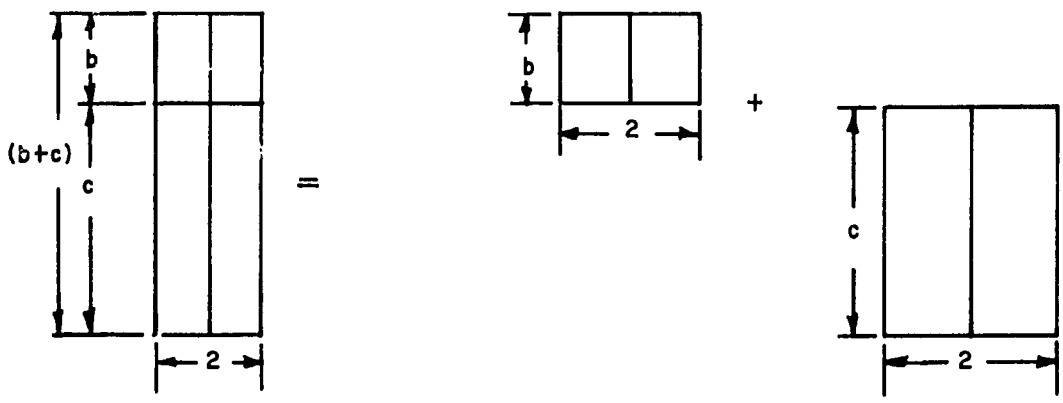
(d)


a
2
c
(2+c)
=
a
2
+
a
c

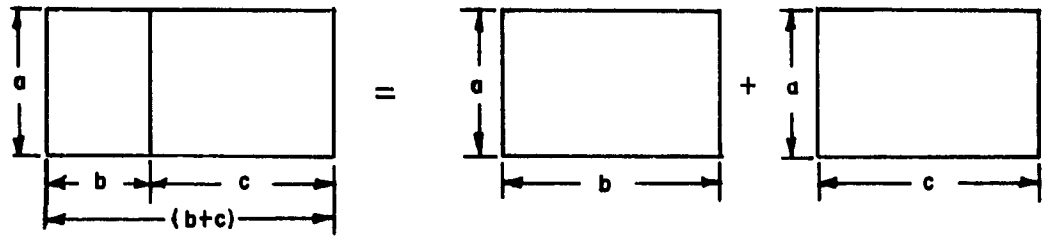
(e)


2
b
c
(b+c)
=
2
b
+
2
c

(f)


b
(b+c)
c
2
=
b
2
+
c
2

(g)

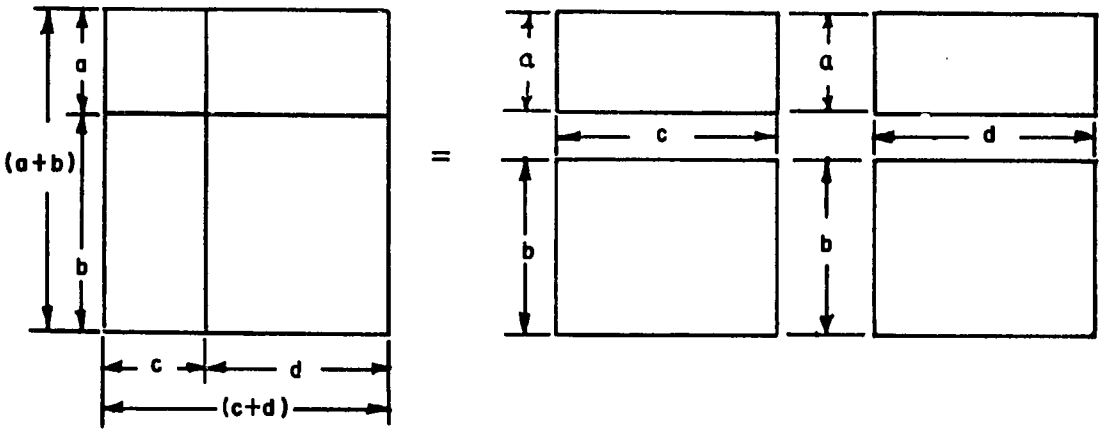


10. (a) Any whole number multiplied by 2 is even.
    (b) Yes.
    (c) Yes: $(2a + 1)$.

☆11. $2a + 2b = 2(a + b)$, using the last identity, (distributive property).

☆12. $(2a + 1) + (2b + 1) = 2a + 2b + 2$
$= 2(a + b + 1)$
using associative, distributive, and commutative properties.

☆13. Yes.



FIRST YEAR SECONDARY TEACHERS' GUIDE

CHAPTER 4
WHOLE NUMBERS

Commentary.

In this chapter we are concerned with factorization and divisibility properties of whole numbers, topics which are a part of what is called number theory. It is important to keep always in mind that only whole numbers are being considered.

To see why we want to consider only whole numbers, suppose, for the moment, that we can use fractions (or, more precisely, numbers represented by fractions.) Now let us ask what factors 12 might have. We find that

$$12 = 1 \times 12 \text{ and } 12 = 2 \times 6, \text{ etc.}$$

where we have whole numbers only. But if we allow all possible numbers, then we have

$$\begin{aligned} 12 &= \frac{1}{2} \times 24 \\ &= \frac{1}{3} \times 36 \\ &= \frac{1}{8} \times 96 \\ &= \frac{1}{7} \times 84 \\ &= \frac{1}{a} \times 12a \\ &= \frac{2}{a} \times 6a \qquad \text{etc.} \end{aligned}$$

Every rational number would then be a factor of 12. Since this would be uninteresting, we exclude it as a possibility and allow only whole numbers as factors. It is of the first importance that each student appreciate this restriction.

Skills and Understandings

1. In order to compute using fractions, it is necessary that the student be able to find the LCM or the GCF of two whole numbers. These skills are essential for reduction of fractions to lowest terms and for easy addition of numbers represented by fractions. This is traditional material and must be taught.

2. The student should really grasp that the names LCM and GCF are natural descriptions of rather simple ideas. In the long run, understanding of the ideas is of far greater importance than facility in computing LCM or GCF. The teacher should continuously stress ideas by answering questions in terms of the original concept rather than explaining a computational method.

3. In items 1 and 2, above, we have commented on LCM and GCF. But these ideas are dependent on the simpler concepts of factor and prime number. These concepts are developed in Sections 4-1, 4-2, 4-3 and deserve the strongest emphasis.

4-1. Factorization.

Here there is one central idea: among all the whole numbers, to find those which are factors of a given number. Here there is no special technique to be taught.

Methodology.

To factorize a whole number, n, one has only to decide which of the whole numbers x, such that $1 \leqq x \leqq n$, divide exactly into n. Thus there is a finite problem to be solved which can be solved given enough time and patience.

You might ask the class, "How many numbers must you try to find all the factors of 70?" If a student says, for example, that 2 is a factor, ask him if this gives him another factor.

There are interesting things that can arise here. Suppose the student has observed that $70 = 35 \times 2$. You should ask him if he thinks it is necessary to return to 70 to find other factors. He should respond with some remark to the effect that he could just look at 35. Perhaps he would then say that 5 is also a factor of 70; i.e. $70 = 2 \times (5 \times 7)$. Then you could ask him how he would find the factor 10.

This will give him a chance to show the use of the associative property: $2 \times (5 \times 7) = (2 \times 5) \times 7$ so that we get factors of 10 and 7.

At this point you should ask (if some student has not already done so) how long this successive factorization can go on. Try to bring out that one has a finite number of possibilities. This leads up naturally to Section 4-2 and the notion of prime numbers.

Answers to Problems 4-1, page 4-2

1.

(a)

a	b
1	30
2	15
3	10
5	6
6	5
10	3
15	2
30	1

(b)

a	b
1	14
2	7
7	2
14	1

(c)

a	b
1	13
13	1

(d)

a	b
1	24
2	12
3	8
4	6
6	4
8	3
12	2
24	1

(e)

a	b
1	3
3	1

(f)

a	b
1	1

(g)

a	b
1	96
2	48
3	32
4	24
6	16
8	12
12	8
16	6
24	4
32	3
48	2
96	1

(h)

a	b
1	2
2	1

(a) $\{1, 2, 3, 5, 6, 10, 15, 30\}$ (b) $\{1, 2, 7, 14\}$ (c) $\{1, 13\}$

(d) $\{1, 2, 3, 4, 6, 8, 12, 24\}$ (e) $\{1, 3\}$ (f) $\{1\}$
(g) $\{1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96\}$ (h) $\{1, 2\}$

2. $a = 0$ and b arbitrary, or a arbitrary and $b = 0$. All numbers are factors of zero.
3. Because $1 \times a = a$ for all a. No.
4. Yes. Because even numbers, by definition, all have 2 as a factor.

Note: The teacher probably should give more problems than those in the text.

4-2. Prime Numbers.

By far the most important idea here is the definition of prime number. Do not attempt to stress any particular method for finding prime numbers. Begin by calling the attention of the class to the answers to Problem 4-1, and get them to notice that some whole numbers have but two numbers in their set of factors - themselves and 1.

If the teacher has had the students look at more examples like those in 4-1, then the student will already know several prime numbers. Be sure that all understand that because of convenience we do not make 1 a prime number. (We wish to be able to state the Fundamental Theorem of Arithmetic in as neat a form as possible. This leads us to exclude 1 from the set of prime numbers.)

Once the idea of prime number is clear, and students can find prime numbers by direct computation, then the teacher can discuss the sieve of Eratosthenes. Do not drill on this, however. Let the students use it a few times. That should suffice. In connection with the sieve, you should try to get some of the students to see that one does not have to keep crossing out as long as might be expected. Thus, to find the prime numbers less than 50, one has only to consider whole numbers less than or equal to 50. In this case, therefore, we need cross out only the multiples of 2, 3, 5, 7. What is left are the prime numbers less than 50. In order to bring this out for the student, you can point out that if one factor of a large number is rather small, then the other will be on the large side. Another question could be, "When we have crossed out all multiples of 2, 3, and 5, how far do we know that we have left only prime numbers?" The answer is that we do know up to 23, and indeed up to 47.

Answers to Problems 4-2, page 4-4

1. The set of primes less than 100 is $\{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97\}$
2. Any even number different from 2 would be of the form $2 \times \square$, where $\square$ represents a number bigger than 1. But then $2 \times \square$ is a number which can be factorized.
3. No.
4. No, because the sum would be even and greater than 2. According to Problem 2, the sum would not be prime.
5. Yes. There are many examples. Thus $7 = 5 + 2$, $13 = 11 + 2$, $19 = 17 + 2$, etc., as one can see from Problem 1. There is a famous unsolved problem related to this question: "Are there infinitely many prime numbers which differ by 2?" Primes which differ by 2 are called <u>twin primes</u>. So the question can be phrased, "Are there infinitely many twin primes?" This is an astonishingly difficult question and if given to the class should be intended only to stimulate interest and not in order to have the students try to solve it. The question is also of interest because one finds, even at this elementary level, hard questions which have resisted all efforts to solve them.

4-3. Prime Factorization.

Here the teacher should strive for the student to <u>understand</u> the Fundamental Theorem of Arithmetic and not necessarily have him be able to <u>recite</u> it. Numerous examples are the key to success here.

Begin by factorizing small whole numbers and arranging the prime factors in order of magnitude; e.g. $24 = 2 \times 2 \times 2 \times 3$. Then you might examine larger numbers and systematically go through the list of prime numbers until the end is reached. Thus, for example, to factorize 360 completely we first observe that 360 is even and so has a factor, 2.

$$360 = 2 \times 180$$

Now we need only worry about the second factor. It too is even, so

$$360 = 2 \times (2 \times 90) = 2 \times 2 \times 90$$

Now we look at 90, which is also even, so

$$360 = 2 \times 2 \times 2 \times 45$$

At last we have an odd factor, 45. Now we pick the next prime number, 3, and see if it is a factor of 45. As it happens to be, we get

$$360 = 2 \times 2 \times 2 \times 3 \times 15$$

Finally turning to 15 we have $15 = 3 \times 5$, so the complete factorization of 360 is

$$360 = 2 \times 2 \times 2 \times 3 \times 3 \times 5.$$

No other factorization into primes is possible. Of course, rearrangement of the factors gives the same factorization.

Answers to Problems 4-3, page 4-5

1. $30 = 2 \times 15 = 2 \times 3 \times 5$. One can obtain this factorization in several ways; e.g., $30 = 6 \times 5 = (2 \times 3) \times 5$.
2. (a) $2 \times 5 \times 13$ (b) $2 \times 2 \times 2 \times 2 \times 2 = 2^5$
   (c) $3 \times 3 \times 11 = 3^2 \times 11$ (d) $43 = 43$ (e) $49 = 7 \times 7 = 7^2$
   (f) $72 = 2 \times 2 \times 2 \times 3 \times 3 = 2^3 \times 3^2$
3. No. One is not a prime number.
4. No, because no product of prime numbers is zero.

4-4. Greatest Common Factor.

This is quite straight-forward. However, it pays to make the following obvious remark: there are but finitely many factors of any whole number(except 0,) and ,therefore,for any two numbers there is some definite number of factors of each. A few of these numbers may be factors of both (maybe only the number 1.) These are the common factors. The largest of these is the GCF.

Answers to Problems 4-4(a), page 4-6

1. 1, 3 2. 1, 2 3. 1 4. 1 5. 1 6. 1, 3
7. Factors of 18 are: 1, 2, 3, 6, 9, 18. Factors of 48 are: 1, 2, 3, 4, 6, 8, 12, 16, 24, 48. The common factors are: 1, 2, 3, 6.
8. The common factors are 1, 2, 4.

Answers to Problems 4-4(b), page 4-7

1. (a) $\{1, 5\}$ (b) $\{1, 2\}$ (c) $\{1\}$ (d) $\{1\}$
   (e) $\{1, 2, 3, 6\}$ (f) $\{1, 2, 3, 4, 6, 12\}$ (g) $\{1\}$

Answers to Problems 4-4(c), page 4-7

1. (a) $\{1, 2, 3, 6\}$ GCF = 6 (b) $\{1, 2, 3, 6\}$ GCF = 6
   (c) $\{1, 5\}$ GCF = 5 (d) $\{1, 2, 4\}$ GCF = 4
   (e) $\{1, 2\}$ GCF = 2 (f) $\{1\}$ GCF = 1
   (g) $\{1, 2, 3, 6\}$ GCF = 6

The teacher may want to supply more problems of this type.

2. 8 and 40, or 24 and 40, etc.
3. 9, 27, 36 for example.
4. 1
5. 1
6. That other number.
7. (a) GCF = 2, $\frac{1}{2}$ (b) GCF = 6, $\frac{1}{3}$
   (c) GCF = 3, $\frac{2}{5}$ (d) GCF = 7, $\frac{2}{5}$
   (e) GCF = 6, $\frac{4}{5}$ (f) GCF = 1, $\frac{7}{9}$
   (g) GCF = 3, $\frac{12}{25}$ (h) GCF = 8, $\frac{7}{8}$

4-5. Least Common Multiple.

This is standard material presented in terms of set intersection. In this case the set of all multiples of a given number is an infinite set. The set of all common multiples is also an infinite set. It is a fundamental property of whole numbers that in any set of them there is a least one.

Answers to Problems 4-5(a) page 4-9

1. 3, 6, 9, 12, 15, ...
   5, 10, 15, 20, 25, ...
   11, 22, 33, 44, 55, ...
   20, 40, 60, 80, 100, ...
2. $\{1, 2, 3, ..\}$, that is all whole numbers $\neq 0$.

3. $\{0\}$
4. (a) $\{12, 24, 36, 48, \ldots\}$ (b) $\{15, 30, 45, 60, \ldots\}$
   (c) $\{9, 18, 27, 36, \ldots\}$ (d) $\{6, 12, 18, 24, \ldots\}$
5. Two answers. 1, 21 and 3, 7.
6. Yes. Their product is a multiple of both. Yes. Infinitely many.

Answers to Problems 4-5(b), page 4-10

1. $\{24, 48, 72, \ldots\}$, LCM = 24
2. $\{18, 36, 54, \ldots\}$, LCM = 18
3. $\{30, 60, \ldots\}$, LCM = 30
4. $\{18, 36, \ldots\}$, LCM = 18
5. $\{6, 12, 18, \ldots\}$, LCM = 6

4-6. Alternative Methods for GCP and LCM.

The methods here are those commonly used for finding GCF and LCM. While the definitions of GCF and LCM serve to make perfectly clear the ideas involved, the definitions do not supply the most convenient computational method. The methods here depend on being able to factorize completely into prime factors, a skill which for relatively small numbers is not hard to attain.

Answers to Problems 4-6(a), page 4-11

1. (a) $36 = 2 \times 2 \times 3 \times 3$, $64 = 2 \times 2 \times 2 \times 2 \times 2 \times 2$, $72 = 2 \times 2 \times 2 \times 3 \times 3$. Therefore, the GCF $= 2 \times 2 = 4$.
   (b) $18 = 2 \times 3 \times 3$, $28 = 2 \times 2 \times 7$, $42 = 2 \times 3 \times 7$. Therefore GCF = 2.
   (c) GCF = 9
   (d) GCF = 7
   (e) GCF = 24
2. A prime factor of the GCF must be a prime factor of each of the given numbers, and a prime factor of each of the given numbers must be a prime factor of the GCF.

   The LCM is also easy to find as soon as we have each of the given numbers factorized into prime factors. Another example which shows that the LCM may get large enough to make the use of the definition lengthy, is the following.

Find the LCM of 36 and 35. We have

$$36 = 2 \times 2 \times 3 \times 3$$
$$35 = 5 \times 7$$

Therefore,

$$\text{LCM of 35 and 36} = 2 \times 2 \times 3 \times 3 \times 5 \times 7 = 1260$$

<u>Answers to Problems 4-6(b), page 4-12</u>

1. (a) 36 (b) 180 (c) 252 (d) 1008 (e) 48
2. Each prime factor of each of the given numbers also occurs in the LCM at least as many times as it occurs in any one of the given numbers.
3. If any one of the prime factors were to be left out, then the number obtained could not be a multiple of one (at least) of the given numbers. Therefore, the number obtained by the method in the text is as small as possible, and so is the Least Common Multiple.

4-10

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TEACHERS' GUIDE VOLUME TWO

MATHEMATICS SECONDARY ONE

Introduction to Algebra and to the Language of Geometry

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Entebbe Mathematics Workshop

African Education Study
Educational Services Incorporated

FIRST YEAR SECONDARY TEACHERS' GUIDE

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CHAPTER 5

SETS OF POINTS

Mathematical Background.

This chapter is an introduction to the concept of a point and sets of points. Sets of other things have already been studied, such as the set of whole numbers. Many interesting sets can be constructed from the set of whole numbers, as the set of even numbers, the set of odd numbers, the set of numbers greater than 15, and the set of numbers greater than 2 and less than 12. Later, other sets of numbers will be studied, such as the set of integers and the set of rational numbers.

Interesting sets which will be studied in this chapter are lines, angles, polygons, circles, spheres, and many more. These are all sets of points. The idea that a line, an angle, a triangle, or any other geometric object is a set of points is fundamental in the approach of geometry given here. A geometric object is specified by describing the set of points which forms it.

Thus a line segment with end points A and B is defined as the set of points consisting of the points A and B and all the points between them. A circle is defined as the set of points in a plane at a given distance from a given point. A virtue of this approach is that it distinguishes clearly between the objects themselves and measurements associated with them. That is, one must keep in mind that the line segment AB is to be regarded as a particular set of points, while the measure of line segment AB is a number. The statement "line segment AB equals line segment CD" says something about two sets of points; that is, it means that every point in line segment AB is in line segment

CD and also that every point in line segment CD is in line segment AB. This necessarily means that the set of points forming line segment AB and the set of points forming the line segment CD are not different sets, but the same.

If line segment AB and line segment CD are not the same set of points but two distinct sets of points, we could say, "the length of line segment AB is equal to the length of line segment CD;" or the "length of line segment AB is not equal to the length of line segment CD" as the case may be. Here we are associating a number with length, and we can say whether numbers are equal or not equal. Since the word equal cannot be used when speaking of two distinct line segments (as they are not the same set of points), we use the word congruent instead, in case their measures are the same; or, we could say, "the measure of line segment AB is equal to the measure of line segment CD" and write it $m\,\overline{AB} = m\,\overline{CD}$.

Obviously a pupil can never confuse the weight or the age or the height of a boy with the boy himself. Similarly, a line is different from its measure. We distinguish between an angle and its measure, which is a number associated with the angle. Yet unless the distinction is emphasized, the student is likely to concentrate on the measure of an angle, forgetting what the angle actually is. Consequently, in this chapter we begin with the study of what an angle is, leaving the measure of an angle to a later chapter.

Notice that associated with a given circle are several sets of points: (1) the circle itself, i.e., the set of points traced by a compass, (2) the set of points inside the circle, called the interior of the circle, (3) the set of points outside the circle, called the exterior of the circle. The set of points consisting of all points which are either on the circle or in the interior of the circle is called the circular region. That is, we say that the union of these two sets is a region.

A knowledge of the distinction between a circle and a circular region is important when area is discussed. That is, the area of a circular region is πr^2 [r being the radius of the circle;] whereas, the area of the circle itself is zero, since a circle is the set of points which can be traced by the compass. If we care to measure the circle, we shall associate a number with it which is called the length of the circle, or its circumference.

The student will also meet in this chapter many words used in mathematics with a meaning slightly different from that used in ordinary language. It may seem strange to them to regard line segments and triangles as special kinds of curves, or to think of parallel lines as two lines in a plane whose intersection is empty. The reason for such seeming distortions of language is that it actually simplifies our mathematical language. For example, suppose we wish to say that a triangle, a quadrilateral, a hexagon, a circle, etc., separate the plane into two parts, the interior and the exterior; it would be simpler language to say a simple closed curve does this. The latter expression can be used provided we define all such figures as simple closed curves, which we do.

Introduction.

The purpose of this chapter on "Sets of Points" is to help the children get a proper concept of points, lines, planes, and space. A major objective is achieved when the children have come to realize that "space is the set of all points everywhere."

It is advisable to use as many concrete illustrations as possible, but care must be taken to prevent the children from getting too tied up with the concrete materials. Abstractions should be made as quickly as possible, though one might return to the concrete things when necessary.

Our aim is to make an informal approach to geometry. There is no measurement, and our treatment is qualitative rather than quantitative. An appropriate approach is to use the discovery method wherever possible. Usually the children will give the expected result.

It is intended that the sections headed Problems in this chapter be used for pupil assignments, whereas those headed Class Activity are for group work during the class period.

Intersection and Union of Sets.

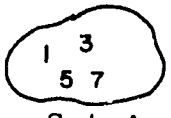
Since many of the geometric figures are described in this chapter in terms of union and intersection of sets of points, a review of these two operations seems important.

Intersection of Sets.

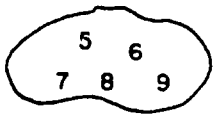
The intersection of two sets is a third set made up of those

members common to the two sets.

Consider these two sets of whole numbers,

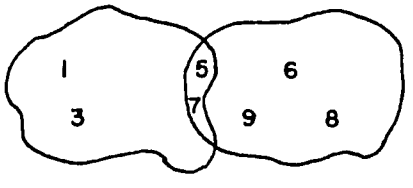


Set A



Set B

In Set A, the whole numbers 1, 5, 3, 7 are the <u>members</u> or the <u>elements</u> of this set; the whole numbers 5, 7, 6, 8, 9 are members of Set B. In Set A there are four members, while in Set B there are five members. The intersection of Set A and Set B is another set which includes those numbers found in Set A and also in Set B. Hence, the intersection includes 5 and 7. This can be interpreted diagrammatically in the following manner:



Set A Set B

The formal notation for sets given in Chapter 2 should be recalled:

$$\text{Set A} = \{1,\ 3,\ 5,\ 7\},\ \text{Set B} = \{5,\ 8,\ 6,\ 7,\ 9\}$$

This is read, "the Set A is equal to the set of whole numbers 1, 3, 5, 7," or simply "the set of numbers 1, 3, 5, 7."

The <u>symbol</u> for intersection is $\cap$, and will be new to the student. We use this symbol to write the intersection of Set A and Set B:

$$\text{Set A} \cap \text{Set B or simply A} \cap \text{B},$$
$$\text{or } \{1,\ 5,\ 3,\ 7\} \cap \{5,\ 7,\ 8,\ 9,\ 6\} = \{5,\ 7\}$$

The <u>intersection of two sets of points</u> may be illustrated in the following manner:

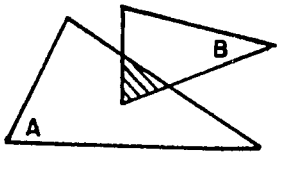


Figure 1

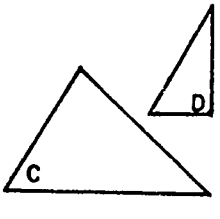


Figure 2

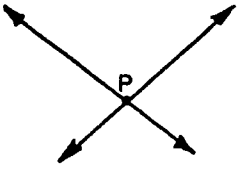


Figure 3

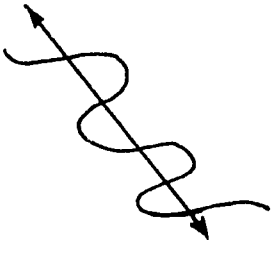
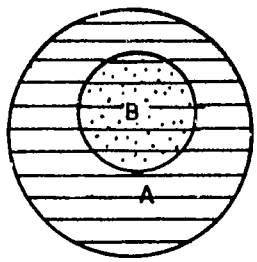


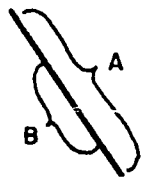
Figure 4

In Figure 1, Set A and Set B are the points enclosed by the two triangles. The shaded portion shows the intersection of Set A and Set B.

In Figure 2, there are no points in Set C that are also in Set D. The intersection of Set C and Set D is a set with no members. Recall that this set is <u>the empty set</u> or <u>the null set</u>, $\{\ \}$.

The intersection illustrated in Figure 3 contains a single point (point P), while the intersection in Figure 4 contains five distinct points.





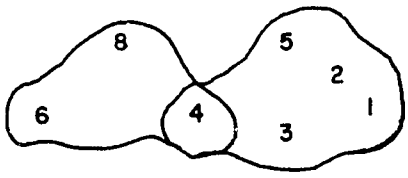
These figures illustrate a set of points A and a set of points B. Observe that every member of Set B is also a member of Set A. Hence, $A \cap B = B$. In such a case as this, Set B is said to be a <u>subset</u> of Set A.

<u>Union of sets</u>

Consider the following sets.

$$\text{Set E} = \{4, 6, 8\}, \quad \text{Set N} = \{1, 2, 3, 4, 5\}$$

These may be illustrated in this manner,



Set E Set N

The <u>union</u> of Set E and Set N is the set which contains every number that is either in Set E or in Set N or in both. The union of these two sets is $\{6, 8, 4, 5, 2, 1, 3\}$. Notice how this differs from the intersection of the two sets.

The symbol for union is $\cup$.

We use this symbol to write the union of Set E and Set N.

$$\text{Set E} \cup \text{Set N or simply E} \cup \text{N}$$

or $\{6, 4, 8\} \cup \{1, 3, 2, 4, 5\} = \{6, 8, 4, 5, 2, 1, 3\}$.

Next consider the union of two sets which do not have a common member as was the case with Set E and Set N.

Such sets may be illustrated in this manner,

Set X Set Y

Set X and Set Y are called <u>disjoint</u> sets since they have no member in common. Their intersection is the empty set.

The union of Set X and Set Y is written,

Set X $\cup$ Set Y or X $\cup$ Y

or $\{0, 2, 4, 6, 8\} \cup \{1, 3, 5, 7\} = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$

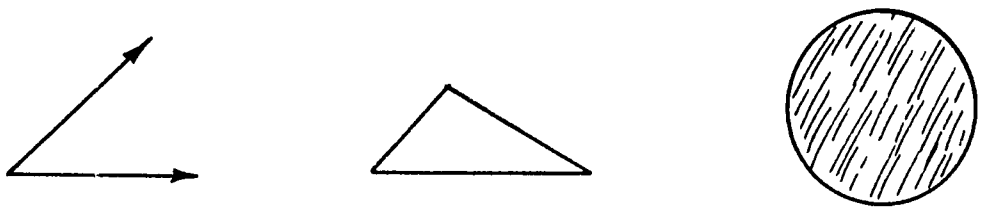


Figure 1 Figure 2 Figure 3

The <u>union of sets of points</u> is used in defining many geometric figures.

In Figure 1 the drawing represents an angle which is defined as the union of two distinct rays having the same end point.

Figure 2 is a triangle defined as a simple closed curve formed by the union of three line segments.

In Figure 3 we have a circular region which is defined as the union of the circle (the boundary) and the interior of the circle. The circle and its interior are described as two different sets of points.

5-1. <u>A Point and a Set of Points</u>.

It is important that children get a notion of what is meant by a point in the study of mathematics. This is a concept that we do not define. It is an idea invented by mathematicians used in describing a line, a plane, an angle, and other geometric figures.

The concept of a point is an abstract idea often found difficult for students to fully appreciate. In developing an understanding of

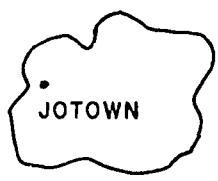
this concept, the teacher should use many illustrations selected from the experiences of the student. Suggestions are a star in the sky, a speck on the wall, a dot on the blackboard, and the corner of the window. The students should be encouraged to give examples other than those given in the textbook and those supplied by the teacher. He must be made to realize that these are not mathematical points, but are illustrations taken from the physical world.

A point may be likened to a number in that both are abstract concepts. Both may be represented by marks; a number by a numeral and a point by a dot. A numeral may be erased, but the number idea remains; a dot may be erased but the point or idea remains. Actually, each mark is nothing more than a "glob" or chunk of chalk when made on the blackboard as [3 •]. We would certainly never consider that these marks are a number and a point, any more than one would say that the name "John" written on the blackboard is the boy himself.

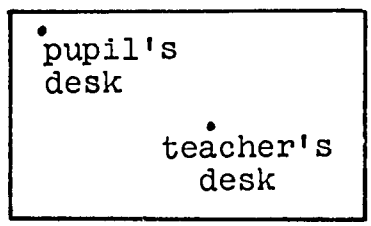
The student should be encouraged to make a very small dot when representing a point. Possibly this will discourage him from associating size with a point, which he may have developed through the use of physical illustrations. There should be considerable discussion of this topic before the students are asked to do Problems 5-1.

Answers to Problems 5-1(a) page 5-1

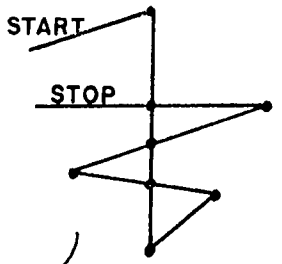
1. In this problem the student is given an opportunity to use knowledge he learned about his country in geography and place it in a mathematical setting. Obviously, the drawing of the map and the location of the capital city will depend on the country in which the student lives.
   Here a hypothetical country and its capital have been used to illustrate what is expected.



2. The drawing to represent the floor of the room and the position of the teacher's desk and that of the pupil's will depend on the particular classroom of each student.

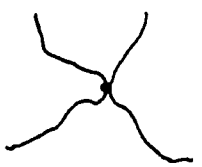


3. In case the student has crayons available, he could use one colour for the dots marking the places where the ant changes



its direction; and another colour could be used to mark the place where it passed a second time.

4. The dot shown is where the two boys could meet.



Problems 3 and 4 have been included here to give the students an idea of what is meant by "a set of points." In previous work the students have learned about sets of things and sets of numbers. Even though the student has had some experience with sets, it may be wise to give some time for a short revision. The following discussion may help you in this.

Point out that a group or collection of things is called a set of those things. For example, a tennis racket, a tennis ball, a tennis net, and a football are all used on the playground; so this can be called a set of playground equipment. Other examples of sets are: all the dishes on the table; all the counting numbers including zero. Point out also that it is possible to consider a set of elements which have nothing in common with each other.

Each dish in the set of dishes is called a member or an element of the set; each whole number is called a member or element of the set of whole numbers. The number of members in a set is determined by counting the members. As you see, there may be any number of members in a set or infinitely many members. In fact, a set may have five members, three members, one member, or no members.

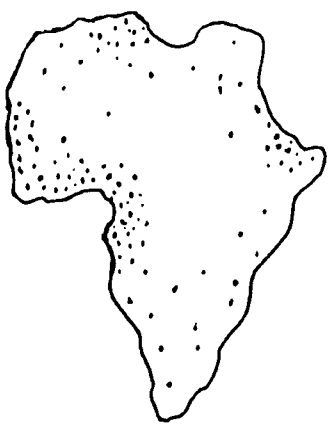
To explain to the students that a set may have any number of members or no members, use the following example. Have a bowl containing five bananas before the class. Point out that there are five members in this set of bananas. Suggest that two of the bananas be eaten, ask how many members there are in the set now. Repeat this until one banana is left. Ask how many elements there are in the set. This emphasizes that it is possible to have a set with only one member. After this banana is eaten, ask the students how many members there are now. Explain that the set (you might say "bowl" at first, then later say "set") is empty; so we call a set with no members, the <u>empty</u> set or the

null set.

The set of whole numbers is a good example to give for a set whose membership is infinite, since there is no largest whole number. This set prepares the student for the idea that a plane is a particular infinite set of points in space.

Answers to Problems 5-1(b), page 5-2

1 and 2. A fairly accurate map should be drawn by each student. It is certainly not expected that the student refer to a map of Africa and locate all the towns on that map. Rather, it is hoped that the student use his imagination and place many dots over his map, as he should realize that there are many towns in Africa not shown on a map.



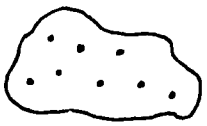
3. Yes, there are a great many.

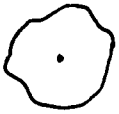
Answers to Problems 5-1(c), page 5-2

1.

(Note: a line may be drawn around each set as shown. This makes one think of each as a set.)

2.

3. 

4. 

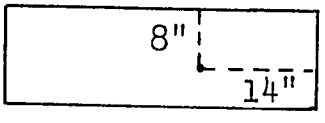
5.

6.

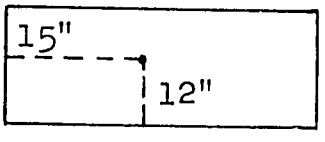
5-2. Plane.

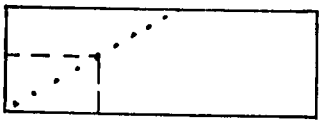
Class Activity, page 5-3

The teacher may find it convenient to use other measurements different from the given ones in this activity.

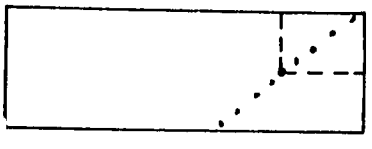
1. 

Only one point

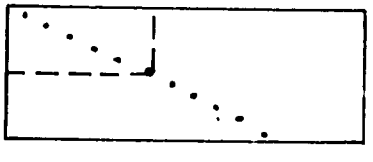
2. 

3. 

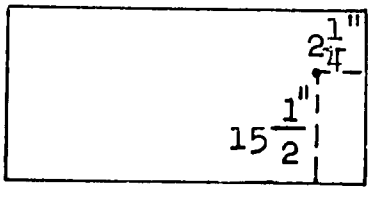
Endlessly many points

4. 

Endlessly many points

5. 

Endlessly many points

6. 

7. (a) A pupil might give:

3" from the bottom, 5" from the right

or 4" from the bottom, 2" from the left

or ... from the bottom, ... from the top (This may give no point)

... from the top ... from the left

etc.

(b) Yes

(c) An unlimited number.

What the exercises will bring out is that a plane consists of endlessly many points. Help the pupils to realize that a plane has no boundary and a plane does not bend.

Class Activity, page 5-4

Ceiling, floor, top of a table, the surface of the blackboard, the sheet of paper, etc.

We should constantly impress on the students that these are not planes but representations of planes.

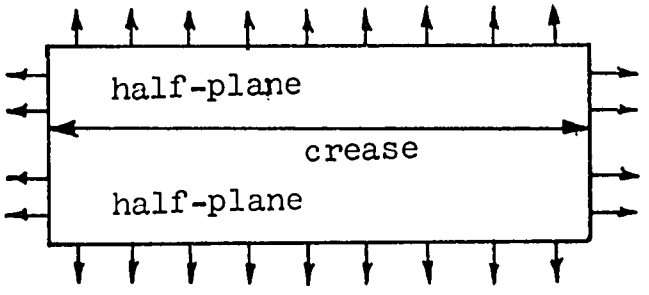
5-3. Line and Half-Plane.

Class Activity, page 5-4

The aim of this activity is to explain clearly the concept of lines and half-planes. The line is always straight and extends indefinitely. It has no end points. For this reason the arrows are attached at both ends to show that it continues both ways. A plane can be separated into two half-planes by any line in the plane. The line is not included in either half-plane.

1. -3. Yes, the points are members of the set of points of the plane.

4. The object of the question is to show that a line has no end points but extends indefinitely. Later the pupils will find that the full significance of the arrows at the ends of the representation of a line is merely to indicate the endless nature of the line.

5. 

The line separates the remaining points of the plane into two different sets of points, namely half-planes.

6. (a) No

(b) Yes

7. (a) On the opposite side of the line.

(b) On the same side of the line.

The aim in questions 6 and 7 is to show that the line is a set of points between the two half-planes. Consequently, the plane consists of three sets of points, viz., one half-plane, the line, and the other half-plane.

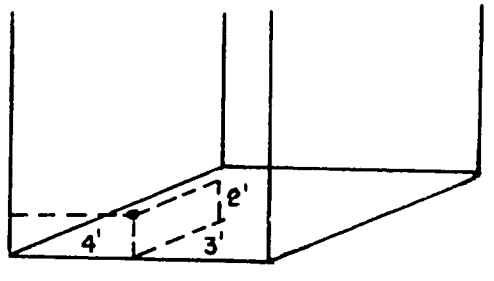
In the paragraph that follows it is the term "union" of sets of points that must be understood (see introduction).

5-4. Space.

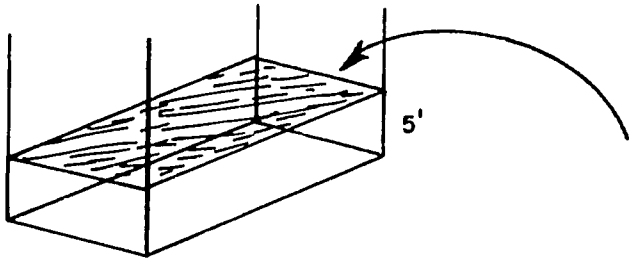
The teacher should explain the telescope and its uses if necessary. When we view with a telescope, we can see a bigger depth of stars. The feeling of depth is important to give a better concept of space. The objective is achieved if by the end of this section the students have come to realize that "space is the set of all points everywhere."

Class Activity, page 5-6

1. Examples are: the top of the teacher's table, the blackboard, the floor of the room.
2. The edges of the blackboard. The intersection of two adjacent walls. The intersection of two sheets of paper in a book.
3. The point is located in the room. (It is preferable that a student comes out and locates the point in the room. Note that there are several possibilities, depending on the pair of adjoining walls chosen.)



4. This has a solution similar to 5-2 Problem 7, page 5-3. Here, however, we are considering points in space not restricted to a plane.
5. Space
6.



A horizontal plane.
An endless number of points.

7. A straight line which is the intersection of the plane 48" from the ceiling and the plane 24" from one wall.

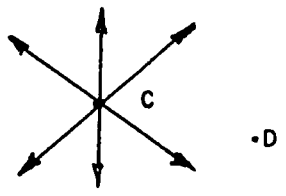
5-5. Line and Line Segment.

It must be emphasised that there is a difference between line and line segment. A line is always straight and extends indefinitely both ways. A line segment is always straight but has two end points. Consequently, their pictorial representations are ←——→ and ———, and their symbols are $\overleftrightarrow{AB}$ and $\overline{AB}$ respectively.

Some students may object when we tell them to "draw a line" and say that all we can draw is a representation of a line. They are quite right. We should tell them so and explain that when we say "draw a line," we mean "draw a representation of a line."

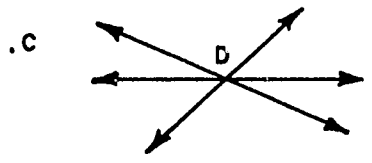
Answers to Problems 5-5(a), page 5-7

1. (a)



Endlessly many

(b)

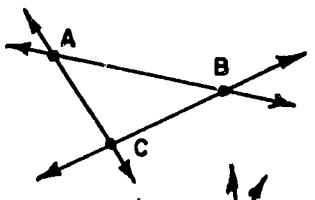


Endlessly many

(c)

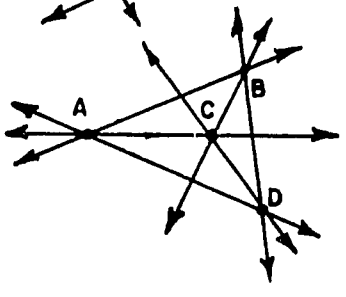
Only one straight line can be drawn through any two points.

2.



Only three

3.



There are six possible lines.

4. (a) and (b). Similarly there are 10 possible lines.
   (c) There are 8 lines.

☆5. (a) The pattern for determining the number of lines can be seen by thinking of drawing lines through one of the points to all the other points. If there are 6 points, this can be done in 5 ways. If we do this through each of the 6 given points, we get a count of 6×5. In doing this, however, we have counted each line twice, once from each of the two points on it. We therefore divide by 2 to obtain the correct count.

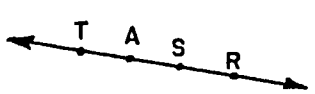
$$\frac{6 \times 5}{2} = 15$$

(b) $\frac{8 \times 7}{2} = 28$ (c) $\frac{11 \times 10}{2} = 55$

(d) $\frac{n(n-1)}{2}$

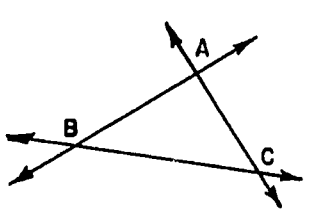
6. Both are correct because they contain the same set of points.

7. (a) $\overleftrightarrow{RT}$, $\overleftrightarrow{TR}$, $\overleftrightarrow{RA}$, $\overleftrightarrow{AR}$, $\overleftrightarrow{AT}$, $\overleftrightarrow{TA}$
   (b) 6 more



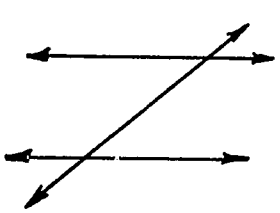
Answers to Problems 5-5(b), page 5-9

1.



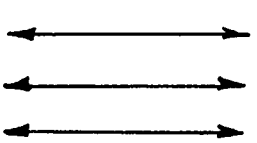
Point A, Point B, Point C

2.



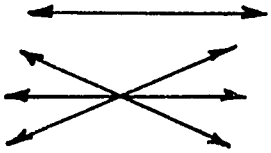
Two of the lines do not intersect, i.e., they are parallel.

3.

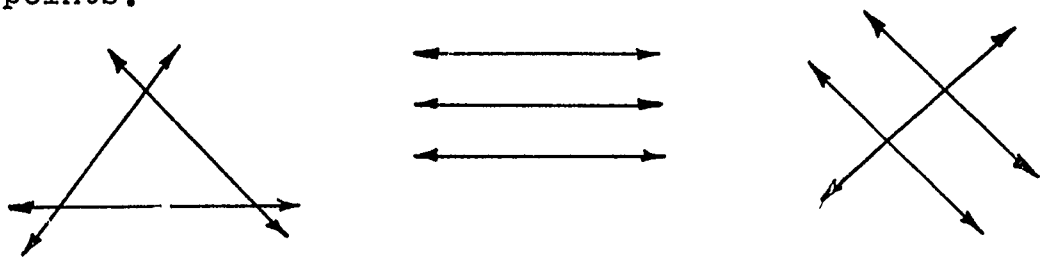


The three lines are parallel

4.



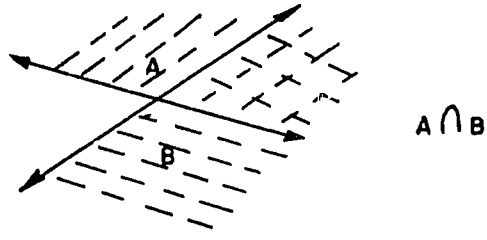
5. It is not possible to draw more than one line through two points.

6. 

Since the intersection of three lines is the set of points on all three lines, any one of these three examples would have the empty set for the intersection. Compare this with Problem 3 where we are interested in their intersections two at a time.

7. (a) Two parallel half-planes; or two half-planes which are separated by a line. (The separating line is not part of either half-plane and therefore is not the intersection of the half-planes.)

(b) Two crossing half-planes



8. Only the line parallel to the separation line.

Class Activity, page 5-10

1. N.B. The hands are to be moved about in space, the length of the stretched string between the fingers should not change, i.e., the hands or fingers should not move along the string.

2. (a) O A P

(b) Yes, many.

(c) Yes, many.

(d) Many.

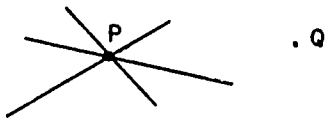
3. O P

Class Activity, page 5-10

1. AD, DC, BC, CG, FG, EH, DH, AE, HG, EF.
2. Intersection of walls, floors and wall, ceiling and walls.
3.

   A line extends indefinitely and has no end points; a line segment has two end points.

Answers to Problems 5-5(c), page 5-11

1. (a) 

   Endlessly many

   (b)

   Endlessly many

   (c) P ._____.Q One line segment.

2. A._____X_____.B

   (a) Yes.
   (b) $\overline{AX}$.
   (c) Point X.
   (d) $\overline{AB}$.
   (e) $\overline{AB}$.
   (f) Yes.
   (g) No.

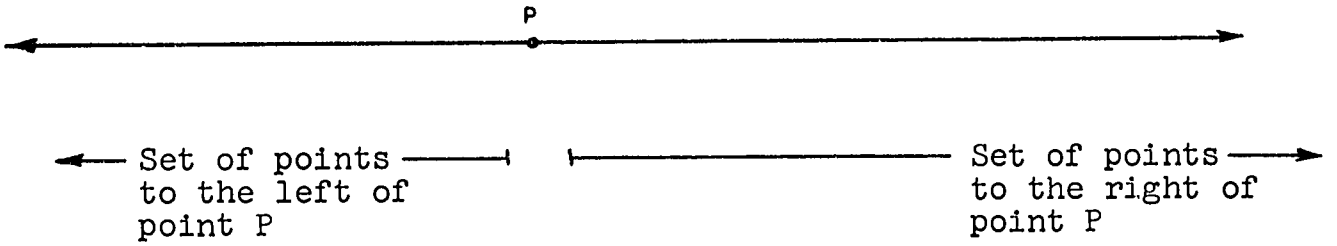
3. (a) $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{AD}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{BD}$, $\overleftrightarrow{CD}$.

   (b) $\overline{AB}$, $\overline{AC}$, $\overline{AD}$, $\overline{BC}$, $\overline{BD}$, $\overline{CD}$.

5-6. Ray and Half-Line.

A ray and a half-line are described as sets of points on a line. Since a line is a particular set of points in a plane (or in space),

it is possible to choose a point on the line and specify it as point P.



The point P is said to separate the other points of the line into a set of points to the right of point P and a set of points to the left of point P as shown in the diagram. The set to the right of point P is called a <u>half-line</u>; likewise, the set to the left of point P is called a <u>half-line</u>. This may seem strange to call each of these sets <u>half-lines</u> when it appears in the drawing that the set to the right is longer. One must keep in mind, however, that a line extends indefinitely; hence if length is considered, one "side" of point P is just as "long" as the other "side."
In this respect, the name half-line should be acceptable.

Some students may object to the fact that the sum of two half-lines is not the whole line because the point of division is not in either half-line. This is contrary to the usual meaning of "half" where the sum of two halves is the whole. Explain that the word "half-line" is just a name for the geometric concept we have described.

It is important that students have an understanding of separation and betweenness. The Problems 5-6(a) are designed to develop this understanding.

Point P <u>separates</u> points X and Y if a line segment drawn from X to Y passes through point P. Point P then is <u>between</u> X and Y.

<u>Answers to Problems 5-6(a), page 5-12</u>

1. Yes, the ant must go through point P when travelling along the path from point B to point A.
2. No.
3. (a) Point P is between points A and B.
   (b) No.
4. Explain to the students that "the A side of the stick" means that part of the line in which is the point A.

(a) The A side of the stick.

This problem may arouse much interest on the part of the student. Suggest that the ant starts at B and goes across the stick once. Now ask, "On which side of the stick will it be?" Next it goes across twice, then three times, etc. The student should soon generalize that when the ant goes across the stick an odd number of times, it is always on the side opposite from where it started; an even number of crossings brings the ant back to his starting position. Do not point this out to the student. Let him <u>discover</u> this.

(b) Twice.

A <u>ray</u> is described as the set of points of a half-line together with the separation point. In other words, a ray is the union of these two sets. In this figure, the ray includes

point P and all the points to the right of point P. Another ray in this figure is the union of point P and all the points to the left of point P.

Note the difference between a half-line and a ray. A half-line does not have an end point, whereas the ray has an end point.

When we specify or name a ray, the end point is always given first followed by the name of a point in the half-line.

For example: The ray in figure (a) <u>is</u> ray AX. The ray symbol is, $\longrightarrow$, and is placed over these <u>letters</u> as $\overrightarrow{AX}$. This is read, ray AX. The ray in figure (b) is written $\overrightarrow{YB}$ and read ray YB. Notice that the ray symbol always points to the right no matter in what direction the ray is pointing.

<u>Answers to Problems 5-6(b), page 5-13</u>

1. (a) $\overrightarrow{OD}$

$\overrightarrow{OY}$

$\overrightarrow{OB}$ or $\overrightarrow{OC}$ (Either name given is correct, since O is the end point and both B and C are points in the same half-line which is a part of the ray.)

$\overrightarrow{OE}$

$\overrightarrow{OX}$

$\overrightarrow{OA}$ or $\overrightarrow{OF}$ (The same can be said of these names as has been said above concerning $\overrightarrow{OB}$ and $\overrightarrow{OC}$.)

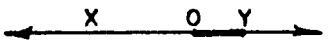
$\overrightarrow{AF}$, $\overrightarrow{CB}$, $\overrightarrow{DE}$ or $\overrightarrow{DO}$, $\overrightarrow{ED}$ or $\overrightarrow{EO}$, $\overrightarrow{XY}$ or $\overrightarrow{XO}$, and $\overrightarrow{YX}$ or $\overrightarrow{YO}$

(b) $\overleftrightarrow{DE}$ or $\overleftrightarrow{ED}$ (These name the same set of points.)

$\overleftrightarrow{XO}$, $\overleftrightarrow{OX}$, $\overleftrightarrow{XY}$, $\overleftrightarrow{YX}$, $\overleftrightarrow{OY}$, $\overleftrightarrow{YO}$ (These name the same set of points.)

(c) Point O is the intersection of $\overrightarrow{OB}$ and $\overrightarrow{OF}$, since it is the point common to both sets.

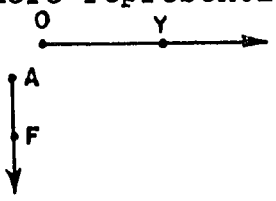
(d) The intersection is the line segment OY ($\overline{OY}$), since this set of points is common to $\overrightarrow{OY}$ and to $\overrightarrow{YX}$. (It may help students to see this by using coloured chalk. Use one colour to draw $\overrightarrow{OY}$ and another colour to draw $\overrightarrow{YX}$. Ask them to observe the portion which has both colours, as shown by the heavy portion in this figure.)



(e) $\overrightarrow{OF}$

(f) $\overrightarrow{OX}$ is not the same as $\overrightarrow{XO}$. (The end point of $\overrightarrow{OX}$ is O, whereas the end point of $\overrightarrow{XO}$ is X. Hence, the two are different rays)

(g) The empty set. (There are no points common to both sets. Observe the figure here representing $\overrightarrow{AF}$ and $\overrightarrow{OY}$.)



(h) $\overline{AO}$. (Emphasize this by drawing a figure on the blackboard.)

Note: Other problems for discussion may be created by the teacher and students, based on Figure 5-6(b), page 5-13

2. A <u>ray</u> has (a) one end point.

A <u>line</u> segment has (b) two end points.

A <u>line</u> has (d) no end points.

3. (a) $\overrightarrow{BA}$, $\overrightarrow{BX}$, $\overrightarrow{BC}$, $\overrightarrow{BE}$ (or $\overrightarrow{BY}$), $\overrightarrow{ED}$, $\overrightarrow{EY}$, $\overrightarrow{EF}$, $\overrightarrow{EB}$ (or $\overrightarrow{EX}$). (Such questions might be asked at this time: Is $\overrightarrow{BE}$ the same as $\overrightarrow{EB}$? What is another name for $\overrightarrow{AB}$? What is the intersection of $\overrightarrow{AB}$ and $\overrightarrow{BC}$? What is the union of $\overrightarrow{BA}$ and $\overrightarrow{BC}$?

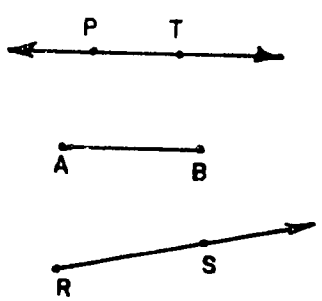
(The answers to these questions in the order asked are: no; $\overrightarrow{AC}$; $\overrightarrow{BC}$; $\overleftrightarrow{AC}$.)

(b) $\overleftrightarrow{AB}$ ($\overleftrightarrow{BA}$, $\overleftrightarrow{AC}$, $\overleftrightarrow{CA}$, $\overleftrightarrow{BC}$, $\overleftrightarrow{CB}$ are other names which may be given.) $\overleftrightarrow{DE}$ (other names may be given using any two of the letters on the line as was done above.) $\overleftrightarrow{XY}$ (other names may be given also.)

(c) $\overline{AB}$ (or $\overline{BA}$), $\overline{AC}$, $\overline{BC}$, $\overline{BX}$, $\overline{XE}$, $\overline{XY}$, $\overline{EY}$, $\overline{BY}$, $\overline{BE}$, $\overline{DE}$, $\overline{DF}$, $\overline{EF}$. (Other names may be given each of these line segments by reversing the order of the letters, as $\overline{BC}$ is another name for $\overline{CB}$.)

Note: At this point, the teacher might ask: "In which of these, a ray, a line, and a line segment, may one reverse the letters used and have the new name denote the same geometric figure?" That is, does $\overrightarrow{AB}$ name the same ray as $\overrightarrow{BA}$? No. Does $\overleftrightarrow{XY}$ name the same line as $\overleftrightarrow{YX}$? Yes. Does $\overline{RS}$ name the same line segment as $\overline{SR}$?. Yes. The student should realize that the letters used in naming a ray cannot be reversed and still name the same ray; whereas, those letters naming either a line or a line segment may appear in any order.

4.



$\overleftrightarrow{PT}$ or $\overleftrightarrow{TP}$

$\overline{AB}$ or $\overline{BA}$

$\overrightarrow{RS}$

5-7. Angles.

We have defined an angle as "the union of two distinct rays having the same endpoints." By this definition we are excluding the zero angle. This should not worry us now since we do not need this zero angle until much later. There is, of course, an alternative approach. We can define an angle as the "union of two distinct rays not in the same line, having the same endpoint." The straight angle (180°)is also excluded by this method. Either definition has difficulties which can be overcome later.

The merit of treating angles in either of these ways is to ensure that angles are not confused with their measure.

Class Activity, page 5-14

1. $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are rays in the given picture. It is clear that other rays can be named with any other point on the given rays as end point. However, here the intent is to identify those rays whose union forms the angle represented.
2. B is the endpoint of each ray.
   For purposes of simplicity, we will not consider straight angles in the following problems.

Answers to Problems 5-7(a), page 5-14

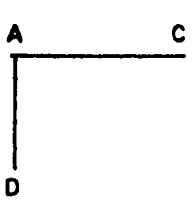
1. The vertices are R, O, A, C.
2. The arms are: $\overrightarrow{RT}$, $\overrightarrow{RW}$; $\overrightarrow{OZ}$, $\overrightarrow{OY}$; $\overrightarrow{OX}$; $\overrightarrow{AE}$, $\overrightarrow{AC}$; $\overrightarrow{CD}$, $\overrightarrow{CB}$.

Answers to Problems 5-7(b), page 5-15

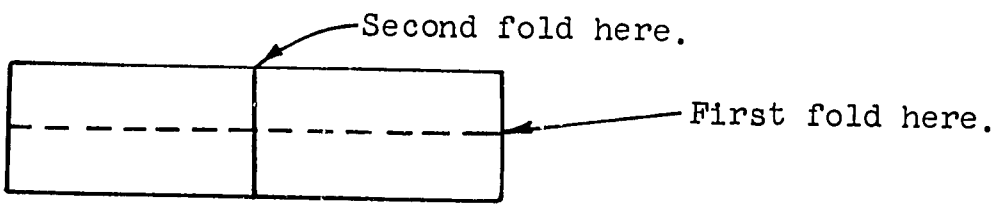
1. (a) $\angle$BAC or $\angle$CAB.
   (b) $\angle$AOB, $\angle$BOC, $\angle$AOD, $\angle$DOC;
   (c) $\angle$GWH, $\angle$GWZ, $\angle$FZE, $\angle$WXA, $\angle$ZWX, $\angle$HWX, $\angle$FZW, $\angle$WZY, $\angle$EZC, $\angle$WXY, $\angle$AXB, $\angle$YXB, $\angle$ZYX, $\angle$ZYD, $\angle$DYC, $\angle$XYC;
   (d) $\angle$WPX, $\angle$WPY, $\angle$YPZ, $\angle$XPZ.
   Other names for these angles may be given by the students; however, the middle letter must be the one which names the vertex.
2. Neither $\overline{PW}$ nor $\overline{PX}$. A ray is a set of points having only one end point, hence it would be impossible to measure a ray. Consequently, comparing lengths of rays has no meaning.

Class Activity, page 5-15

It is most important in this section that the students understand well that line segments are often thought of as rays in naming angles. It is a practical custom.

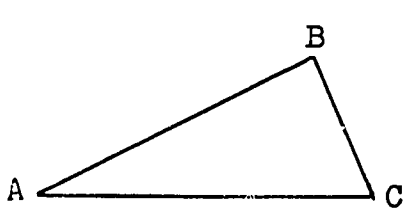
1. The angles at A are: $\angle$BAC, $\angle$BAD, $\angle$CAD. Another possible order for naming the angles can be used, e.g., $\angle$CAB instead of $\angle$BAC. Help children understand that the vertex letter must be the middle letter.
2. There are 24 angles altogether.
3. 

4. We have to appeal to the matching idea because we have not introduced measure of angles. (a) and (c) are representations of a right angle.
   Fold a rectangular sheet of paper so that opposite edges coincide.
5. Repeat the process using the other pair of opposite edges of the folded paper.



   Other ways may be presented by the students which are equally acceptable.
6. The angles with the corners of the room as vertices and the intersection of walls as arms, and many more. We may repeat that the teacher must make certain that there is no confusion in thinking of line segments as rays. It will suffice if it is pointed out that for want of space what is usually done is to represent rays by line segments.

5-8. Triangles.



It is important that the student think of the union of $\overline{AB}$, $\overline{BC}$, $\overline{CA}$ as the $\triangle$ ABC. That is, the triangle is the set of points which can be traced by marking with a pencil along the line segments. It would be well to point out that a triangle is actually a kind of line; that is, the kind just traced. Do not make the mistake of cutting out such a figure and saying, "This is a triangle." Actually, you have a triangular region which will be discussed in section 5-9. The triangle is only the boundary of this shape which you have cut out.

Answers to Problems 5-8, page 5-17

1. (a) $\triangle$ RVT. These letters may be given in any order and the same triangle will be named.

(b) $\triangle$ABC.
(c) $\triangle$ XCA.
(d) $\triangle$ MOP.
(e) $\triangle$ DZG and $\triangle$ AZB.
(f) $\triangle$ RPQ and $\triangle$ PSQ.
(g) $\triangle$ MNO and $\triangle$ MNP.
(h) $\triangle$ PZA and $\triangle$ ZAX.
(i) $\triangle$ DAC and $\triangle$ ABC.
(j) $\triangle$ XZW, $\triangle$ ZWY, $\triangle$ XOZ, $\triangle$ ZOW, $\triangle$ YOW.
(k) $\triangle$ AMB, $\triangle$ ABN, $\triangle$ AMN, $\triangle$ BMN, $\triangle$ MPN, $\triangle$ APM, $\triangle$ BPN, $\triangle$ APB.

2. (a) M, N, P.
(b) $\overline{MN}$, $\overline{NP}$, $\overline{MP}$. The student should recognize that the sides of the triangle are line segments, and should refer to them as such.

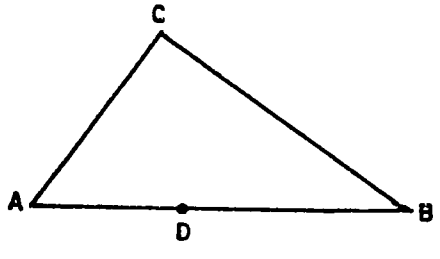
3. $\triangle$MNP, $\triangle$MPN, $\triangle$NMP, $\triangle$NPM, $\triangle$PMN, $\triangle$PNM.

5-9. Interior and Exterior of a Triangle.

At this time it is sufficient for students to have an intuitive idea about inside and outside (interior and exterior) of a triangle by observing a drawing of a triangle. The Class Activity exercises are designed to fortify this intuitive notion which students have and lay the foundation for a more formal development.

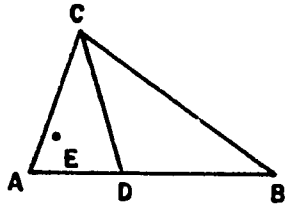
Class Activity, page 5-18

1.



A point D of the triangle means a member of the set of points forming the triangle, hence must be located on one of the line segments. In this case, between points A and B.

2.

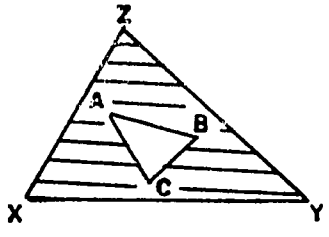


3. No, point E is not in the interior of $\triangle$ BCD.
4. Yes, point E is in the interior of $\triangle$ ABC.
5. (a) Only the points D and I are in the interior of $\triangle$ ABC. Points A, B, G, H, C are not in the interior of $\triangle$ ABC. In fact, they are in the set of points which form the triangle.
   (b) (1) There is no point specified with a letter in the drawing that satisfies both of these conditions.
   (2) Point D.
   (3) Point I.
   (4) Points G and H.
   (5) Points A, B, C, G, H.
   (6) Points D, E, F, G, H.
   (7) Points A, B, G, H, C, D, E, F.
   (8) Point K.
   (9) Points F, E.
   (10) Point L.
   (11) Points F, K, E, L, A, B, C.

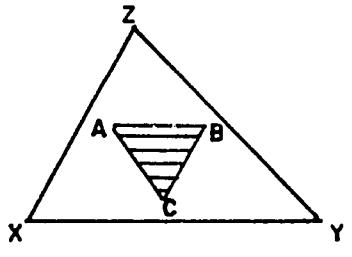
At this time, a review of intersection and union might be helpful. Refer to the discussion of this in the Introduction at the beginning of this chapter, page 5-3.

Class Activity, page 5-20

1. (a) $\overline{DG}$.
   (b) $\overline{DH}$.
   (c) It is the empty set, since there are no points in this intersection.
   (d) $\angle$ DGH. Do not confuse this with the interior of $\angle$ DGH.
   (e) Point D.
   (f) Point H.
2. (a)

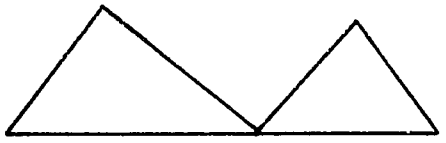


(b)

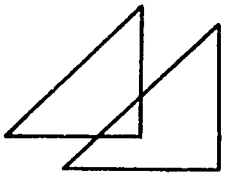


3. The intersection is the empty set.

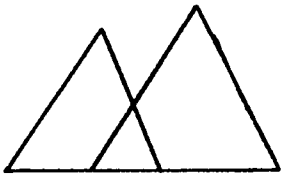
4. (a)



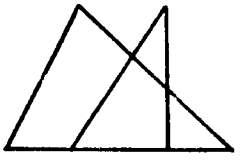
(b)



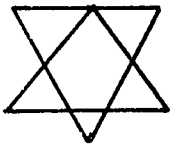
(c)



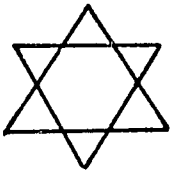
(d)



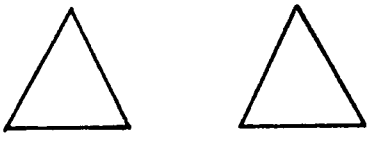
(e)



(f)



(g)

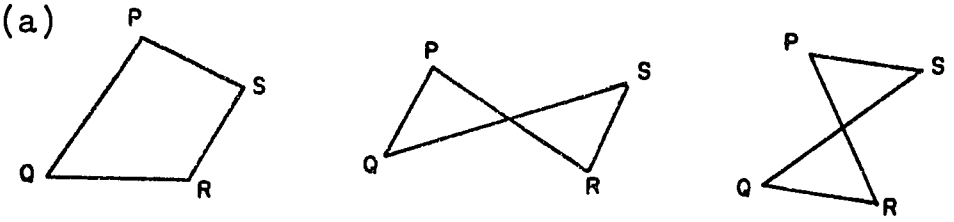


5. Six points

Answers to Problems 5-10(a), page 5-21

1. (a) DGFE names the same quadrilateral as DEFG.
   (b) DFGE names the same quadrilateral as DEGF.

2. (a)

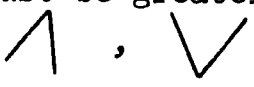


There are only three possibilities.

(b) PQRS, PQSR, PRQS.

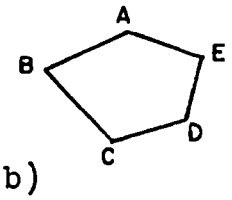
Answers to Problems 5-10(b), page 5-22

1. (a) True.
   (b) False.
   (c) False.
   (d) True.
   (e) False.
   (f) True.
2. (a) There are 16 rectangles shown.
   (b) ADLI, ABFE, BFGC, CGHD, HGKL, GFJK, FJIE, AEGC, EIKG, AEHD, EILH, CDLK, BCKJ, ABJI, BDHF, HLJF.
   (c) The following rectangles are squares: ABFE, BCGF, CDHG, GHLK, FGKJ, EFJI.
   (d) They are squares because the sides are all equal.

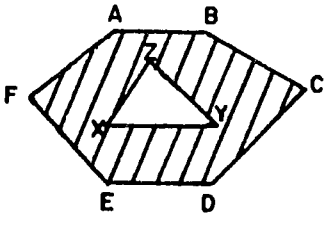
Plenty of practice in drawing polygons is required here. The number of sides must be greater than two. This condition excludes figures such as  . The sides must be line-segments. This excludes such figures as .

Answers to Problems 5-10(c), page 5-23

1. (a), (c), (d), (f), (g), and (h).
2. A triangle, a square, a rectangle, are always simple polygons.
3.



4. (a) (b)

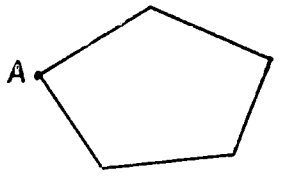


Notice, no point of the triangle can also be a point of the hexagon if the triangle is to be inside of the hexagon. Notice that $\triangle$XYZ is included in the shaded portion.

(c) The set of points indicated by the shading is the intersection of the exterior of the triangle and the interior of the hexagon.

Class Activity, page 5-24

1. The aim here is to continue some practice in polygon drawing and to emphasize the idea that a polygon is a simple closed curve.

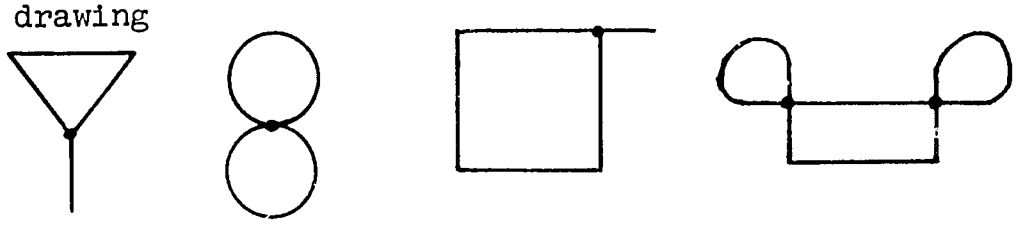


(a) A can be reached again without lifting the pencil.

(b) A is the only point reached twice. Attention is drawn to various shapes of simple closed curves shown. There is always room for more. The students may be invited to draw their own pictures different from these.

Class Activity, page 5-25

1. The tracing can be easily done.
2. All the curves can be traced without lifting the pencil.
3. In drawing

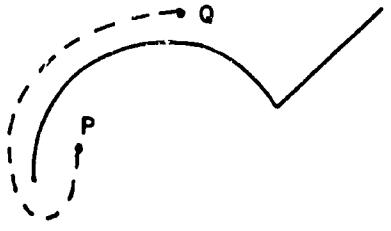


The points clearly marked are passed through more than once.

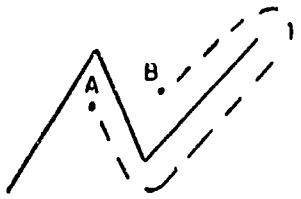
4. (a)

(b) The line segment | and arcs S are arcs. With several examples the idea of arcs can be firmly fixed.

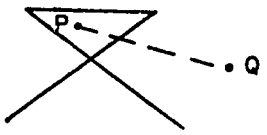
5. (a)



(b)



6. (a)



(b)

7. (a) AB, BD, BC, AD, DC, BDC, BDA, BCA, AC, BACB.
   (b) BACB is a simple closed curve. Also BADB and BDCB.
   (c) AB, BD, BC, AD, DC, BDC, BDA, BCA, AC, are arcs. There are of course other simple closed curves not named, e.g., BADB.

5-11. Interior, Exterior, and Regions.

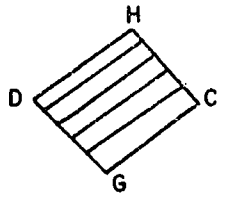
In this section we associate with each simple closed curve a corresponding region, consisting of the set of points which are either on the curve or in the interior of the curve. That is, a region of a simple closed curve is the union of the set of points forming the curve and the set of points in the interior of the curve. The curve is called the boundary of the region. The boundary is between the interior and the exterior of the simple closed curve, that is, it separates these two sets of points.

Answers to Problems 5-11(a), page 5-26

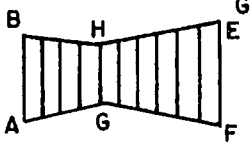
1. The boundary of the country drawn will depend on where the student lives.
   (a) Yes
   (b) Yes
   (c) Yes
   (d) No
2. (a) No
   (b) Yes
   (c) Yes
   (d) No

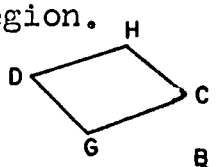
Answers to Problems 5-11(b), page 5-28

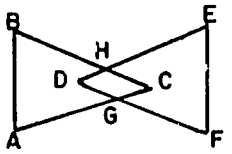
1. (a) Yes
   (b) Yes. Quadrilateral region, polygonal region.

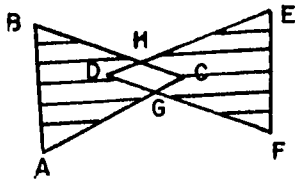


2. (a) Quadrilateral region DGCH.
   (b) Region whose boundary is AGFEHB. This is called a polygon region.
   (c) Points H and G.
   (d) The two triangles as shown.
   (e) $\overline{DH}$.
   (f) Point H.
   (g) Points G and H.
   (h) The shaded region is the union.



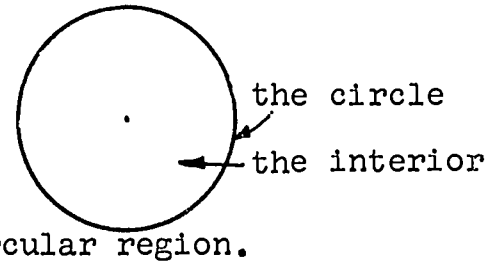






5-12. Circles.

This section is to introduce the student to the set of points which is called a circle. A circle is a simple closed curve. In order to describe this set of points, one must use the notion of measure. Here radius is thought of as a number which is associated with some unit of measure.

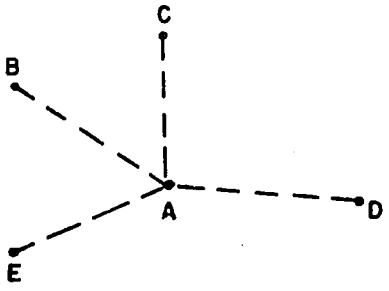


It is important for the student to know that the simple closed curve is the circle; that it separates the interior from the exterior and that the union of the circle and its interior is the circular region.

The student will need to be taught how to use a compass on paper and on the blackboard. A string tied to a small piece of chalk makes a very adequate blackboard compass.

Class Activity, page 5-28

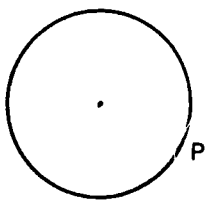
1, 2, 3.



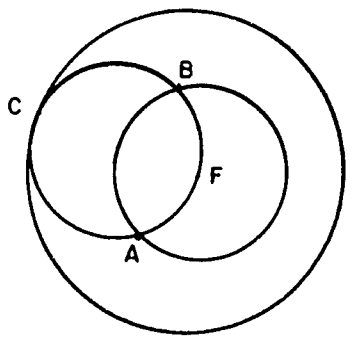
4. The points one foot from A are unlimited in number. Yes, more than 10. Yes, more than 1000.
5. This is a simple construction with the compass. The center is A and the radius is one foot. Yes, a circle is a simple closed curve.

Answers to Problems 5-12, page 5-29

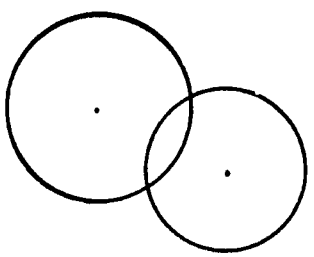
1. (a), (b)



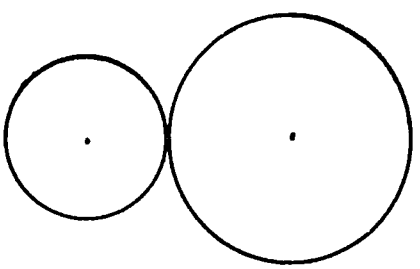
(c), (d)



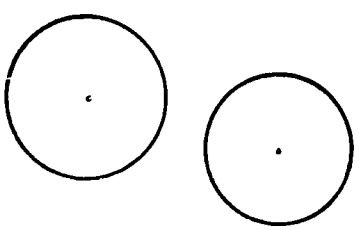
2. (a)



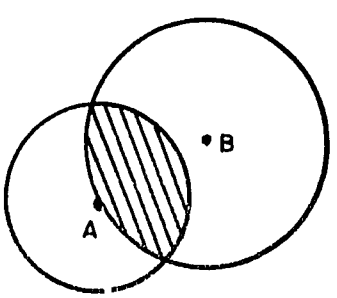
(b)



(c)

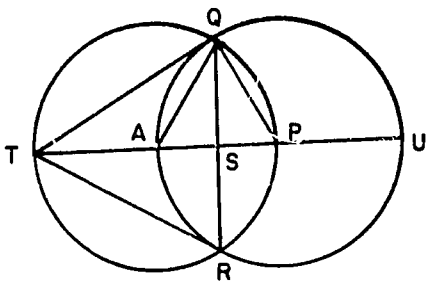


3. No.
4. (a), (b), (c)



(d) Point B must be located $1\frac{1}{2}$ inches or more from point A. $1\frac{1}{2}''$ is the least distance. There is no greatest distance.

5. (a), (b), (c), (d), (e), (g), (i), (k)



(f) The measure of a line segment is indicated as shown, $m\overline{QA} = 1$ inch, and read "the measure of $\overline{QA}$ is 1 inch."
$m\overline{QP} = 1$ inch.
$m\overline{AP} = 1$ inch.
These measures are all the same.

(h) $m\overline{AS} = \frac{1}{2}''$
$m\overline{SP} = \frac{1}{2}''$

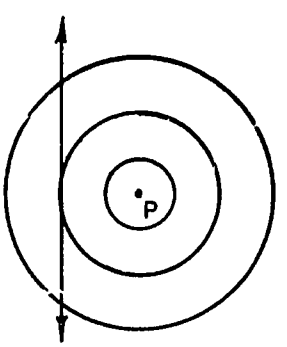
(j) $m\overline{TS} = 1\frac{1}{2}''$
$m\overline{TU} = 3''$

(l) Yes
$m\overline{TQ} = m\overline{TR}$. Each is approximately $1\frac{11}{16}''$ in length.

(m) Yes. There are five $\angle TQP$, $\angle QSP$, $\angle QST$, $\angle RSP$, $\angle RSA$.

(n) They measure the same, approximately $1\frac{11}{16}''$.

6. (a), (b)



(c) No.

(d) The null set or the empty set.

(e) Only one circle can be drawn. Many circles can be drawn. Many circles. No circle will intersect the line in three or more distinct points.

7. Figure (a) is constructed by drawing a circle with any length for the radius. Select a point on the circle to be used as a centre and with the same radius draw an arc that intersects the original circle. Using each of these points of intersection as centres and the same radius, continue as before until the picture is completed.
   Figure (b) is constructed by drawing a circle with any radius. Take any point on the circle as a centre and the radius as before, draw another circle. With one of the points of intersection as a centre and the same radius draw the third circle. Draw the line segments as shown in the diagram.
8. This can prove to be a very fascinating exercise for the students. Encourage them to use shading and crayons. This is also a beneficial exercise in that it gives needed practice in the use of ruler and compass and often supplies information used later in the study of geometry.

5-13. Lines and Planes.

The aim of this section is to introduce the student to interesting properties of lines and planes through the use of physical objects. Here we discuss the intersection of lines, of lines and planes, and of planes. Other properties that the student discovers is that a plane is determined by a line and a point not in the line, by two intersecting lines, and by two parallel lines. These properties are not formally stated, and we see no need to expect the student to do so at this time.

Frequently one encounters difficulty when introducing concepts through the use of physical objects. For example, Exercise 6(c) and (d) may cause some difficulty. In this exercise one expects the student to conclude that only one plane can pass through a line and a point not in the line, that is, a plane is determined by a line and a point not in the line. Yet, in using the pages of his book, he may insist that another page can be held by his thumb and forefinger. Here, you must divert his attention from the physical objects to the plane being a set of points, and a set of points occupies a unique position in space.

In this section it may be necessary to emphasize again the difference between the use of the verb intersect and the noun intersection. For

example, two lines may not intersect, in which case their intersection is the empty set. That is, lines (or planes) may or may not intersect, but they always have an intersection. The use of these words is often confusing to students. When difficulty is met over the use of intersect and intersection, one must be prepared to discuss this frequently whenever these two words are used.

Class Activity, page 5-31

Exercises 1-3 need no comment.

4. By tilting the needle many different positions can be observed, the conclusion being that there are infinitely many lines that can pass through point O.
5. Here the students should discover by manipulating the two needles that skew lines are neither parallel nor do they intersect. Another illustration could be the line where two walls meet and the line where a third wall meets the floor.
6. (a) The intersection of the planes represented by the pages is a line represented by the spine of the book.
   (b) Yes. An unlimited number.
   (c) and (d) No. Only one plane can pass through a given line and a given point not in the line. Refer to the discussion on this given at the beginning of this section.
7. This is to be demonstrated by the students with class discussion of each position. Remind the students that the cardboard is merely a representation of a portion of a plane. Hence, in any position in which the cardboards represent planes, one must always be aware that these sets of points extend on indefinitely.
8. (a) Many
   (b) To support the cardboard, the needles must be either parallel or intersecting. Remember that the needles represent lines so we cannot support the cardboard on the point of the needle.

Answers to Problems 5-13, page 5-33

1. The pairs of parallel planes are HABGF and IJCDE, ABCJ and HIDG, GBCD and HIJA.

2. HABGF, JABC, HAJI } intersect in point A.

HABGF, GBCD, ABCJ } intersect in point B.

BCJA, JCDEI, BCDG } intersect in point C.

JCDEI, BCDG, FGDE } intersect in point D.

FGDE, IJCDE, IHFE } intersect in point E.

FHABG, FGDE, FHIE } intersect in point F.

ABGFH, CDGB, EDGF } intersect in point G.

FHABG, FHIE, IHAJ } intersect in point H.

IJCDE, IHAJ, IHFE } intersect in point I.

IEDCJ, ABCJ, IHAJ } intersect in point J.

3. HABGF, HAJI } intersect in line HA.

HABGF, JABC } intersect in line AB.

HABGF, DCBG } intersect in line BG.

HFEI, HFGBA } intersect in line HF.

EFGD, HFGBA } intersect in line FG.

GBCD, ABCJ } intersect in line BC.

HAJI BAJC	intersect in line AJ.	FGDE BGDC	intersect in line GD.
FHIE AHIJ	intersect in line HI.	HAJI DCJI	intersect in line JI.
GBCD EIJCD	intersect in line CD.	ABCJ IEDCJ	intersect in line CJ.
EIJCD EIHF	intersect in line EI.	EDGF EDCJI	intersect in line ED.
EFGD EFHI	intersect in line FE.		

4. In the figure we have represented actually line segments. We have, however, been referring to them as lines formed by the intersection of planes.

AH AB AJ	intersect in point A.	BA BG BC	intersect in B.
CB CJ CD	intersect in C.	DC DI DE DG	intersect in D.
GF GH GB GD	intersect in G.	HG HF HA HI	intersect in H.

IH, IE, ID, IJ	intersect in I.	EI, ED, EF	intersect in E.
FE, FG, FH	intersect in F.	JC, JI, JA	intersect in J.

5. AJ and DC do not intersect and they are not parallel. Others may be selected. Lines in space which are not parallel and do not intersect are called <u>skew lines.</u>

5-14. Spheres.

The main idea in this section is that a sphere, as we think of it in mathematics, is not solid as it is so frequently thought of by the students. To emphasize this concept of a sphere such representations as a globe and a tennis ball are used. When an orange is used, point out that the whole orange, the part one eats and the rind, does not represent what is meant by a sphere; however, the rind around the orange may serve as a good representation.

Class Activity, page 5-34

1. (b) The surface formed by the rotating circle is a sphere. The student will be better able to think of a surface if he twirls the straight wire between the palms of his hands very fast.

   (c) Besides the objects mentioned in the textbook, the student should mention other things. A soap bubble is an excellent representation of a sphere.

2. The intent of this exercise is for the students to get the idea that a sphere is the union of all the great circles represented by the rubber bands.

3. From the description that a circle is a set of points in a plane, each of which is given distance from a given point, we then evolve a description of a sphere as being a particular

set of points in space. See the description given following exercise 3(b).

4. A circle is described as a set of points in a plane, whereas a sphere is described in the same way except that the set of points is in space.
5. The intersection is a circle, a point, or the empty set. Be sure that the students do not think of the intersection as a circular region. This would be the case if we describe the intersection of a plane with the sphere and its interior.

5-15. Pyramids and Cones.

This section is an extension of the study of figures in space. A cone can have any simple closed curve and its interior as the base of the cone. If the base is a circular region, the cone is a circular cone. We frequently use the word "cone" to mean circular cone. If the base is a polygonal region, the cone is a pyramid. Pyramids are named triangular, rectangular, square, hexagonal, etc., according to the polygonal regions of their bases.

You may not want to mention the general viewpoint of cone to the class, but just point out that the circular cone differs from the pyramids in having a circular region for its base.

Class Activity, page 5-36

1. By bringing the same points (E) together a rectangular pyramid is obtained.
2. (a) Line segments CE, DE, and BE respectively.
   (b) They are called the edges.
   (c) The apex E is the intersection of the triangular regions AEC, CED, DEB, BEA; segments EA, EC, ED, EB.
3. For a circular cone the region is circular; for pyramids the regions are polygonal. In each case, the region observed will be the same shape as the base.

Answers to Problems 5-15, page 5-37

1.

	Number of		
	Faces	Edges	Vertices
Triangular Pyramid	3 + 1 = 4	3 × 2 = 6	3 + 1 = 4
Rectangular Pyramid	4 + 1 = 5	4 × 2 = 8	4 + 1 = 5
Pentagonal Pyramid	5 + 1 = 6	5 × 2 = 10	5 + 1 = 6
Hexagonal Pyramid	6 + 1 = 7	6 × 2 = 12	6 + 1 = 7

2. The number of vertices and the number of faces are the same for each pyramid. The number of each can be found by adding 1 to the number of sides of the polygonal region which forms the base. To find the number of edges, multiply the number of sides of the base by 2.
As the number of sides of the base in increased by 1, the number of edges increases in the series (beginning with a triangular base) 6, 8, 10, 12 for pyramids with faces whose numbers increase in the series 4, 5, 6, 7. Hence, the numbers in these two series might be matched as (4,6), (5,8), (6,10), (7,12).

3.

Number of Faces	Number of Edges	Number of Vertices
7	12	7

The pattern is the same as for pyramids. Observe the matching of the numbers (7, 12) in the above discussion. This figure is actually two pyramids placed together, base to base, then the top of one "cut" off.

5-16. Prisms and Cylinders .

The base of a cylinder can be any plane region bounded by a simple closed curve. If the region is a circular region, the cylinder is a circular cylinder. We frequently mean "circular cylinder" when we say just "cylinder." If the base is a polygonal region, the cylinder is

a prism. The student should learn to recognize cylinders and the various kinds of prisms--triangular, rectangular, square, hexagonal, etc. He should bear in mind, however, that no physical object accurately represents mathematical ideas.

Class Activity, page 5-38

1. The region DEF is the other triangular face.
2. Other rectangular faces are the regions FCBE, FDAC.
3. These faces are called regions by definition - the union of the sets of points forming the boundary and the sets of points forming the interior.
4. A rectangular prism has six faces, two of which are often referred to as the bases.
5. A circular cylinder differs from a prism because it has two circular bases.
6. (a) In the folded picture the three F's are the same point and the three C's are the same point.
   (b) This will be easy to do if the measurements are correctly done. Notice in the figure which line segments must have the same length.
   (c) The line segments which must be the same length are: $\overline{EB}$, $\overline{FC}$, and $\overline{DA}$; $\overline{FD}$ and $\overline{CA}$; $\overline{BC}$ and $\overline{EF}$.
7. (a) The vertices of the rectangular prism are: A, B, C, D, E, F, G, H.
   (b) The edges are: $\overline{AB}$, $\overline{BG}$, $\overline{GF}$, $\overline{FE}$, $\overline{ED}$, $\overline{DA}$, $\overline{AH}$, $\overline{HG}$, $\overline{HE}$, $\overline{CD}$, $\overline{CF}$, $\overline{CB}$.
   (c) The faces are: EDAH, EHGF, DCBA, CBGF, ABGH, DCFE.
8. The figure for this exercise is self-explanatory. First, fold along AD, BC; then along DC, AB; and finally along FG.
9. Other models can be made if we change the bases into other polygonal regions, e.g., triangular region, hexagonal region, pentagonal region, etc. Note that the faces other than the bases will still be rectangular.
10. It is instructive to try constructing cylinders with the height of the rectangle less than six times the radii of the circles.

11. (a) The intersection is empty when the prism and the plane have no point in common.

(b) The intersection is a single point when the plane intersects the prism at a vertex and nowhere else.

(c) The intersection is a line segment if the plane intersects the prism along one edge and nowhere else.

(d) In a triangular prism the intersection by a plane through the lateral faces will be a triangle. In any prism if the plane intersects a base and two adjacent lateral faces and nowhere else, a triangle can be obtained.

(e) If the base of the prism is a quadrilateral, the intersection by a plane through the lateral faces will be a quadrilateral.

(f) In a triangular prism if the base is in the intersecting plane, the intersection is a triangular region.

(g) In a rectangular prism if the base is in the intersecting plane, the intersection is a rectangular region. In any right prism, if one of the lateral faces is in the intersecting plane, the intersection is a rectangular region.

(h) If a circular cylinder is intersected by a plane parallel to a base, the intersection is a circle.

(i) If the intersecting plane contains the base of a circular cylinder, the intersection is a circular region.

It is hoped that in these problems, the students will demonstrate these intersections by using models. They may find other peculiar intersections by taking the intersecting plane in various positions. The surface of the water in the tilted cylinder is an ellipse. It becomes a circle when the can is not tilted.

FIRST YEAR SECONDARY TEACHER'S GUIDE

CHAPTER 6

THE NUMBERS OF ARITHMETIC

Commentary

This long chapter is an extremely important one for it has three rather different purposes -- no one of which can be neglected at the expense of the others. These objectives are:

I. To revise the arithmetic skills and appliations with which the student is already familiar. In other words, to make sure that he can compute rapidly and accurately, and that he can apply this skill.

II. To deepen the student's understanding of the reasons underlying the computational rules with which he is familiar.

III. To prepare the student for further work in algebra by beginning to use algebra in a discussion of arithmetic.

Let us consider these three aspects in somewhat greater detail. With respect to Objective I. there can be no disagreement. Certainly all should calculate quickly and accurately. What we wish to point out here is that the best computers are invariably persons who have a fine grasp of the meaning of the different computational processes. Therefore Objectives II and III will serve to strengthen a student's computational skills as well as his ability to apply them to concrete problems. It will be important for the teacher to be alert to individual differences among the students and to assign additional computations to some -- but only after the reasons underlying the process in question are clear to all.

It is in Objective II that this text differs most from traditional ones -- and it is this objective which forms the central topic of the chapter. The deeper understanding of arithmetic computation is brought about principally by emphasis on the algebraic structure of arithmetic -- where by algebraic structure is meant the statement of a few basic properties of numbers and their systematic

use to discover new (to the student) methods of computation or to justify old ones. Already, in Chapter 3, the student has met with a small list of Basic Properties. Now, at this stage, these same Basic Properties are directly used to show why the algorithms he has learned are correct. (An algorithm is just a rule of procedure, or a process. For example, to divide fractions the student has learned to "invert the denominator and multiply" -- but he may not know why.) We shall be much concerned with establishing, by algebraic means, the methods by which we compute.

Besides this emphasis on structure there is much geometric, or physical, interpretation of arithmetic operations. Because the ideas of number were evolved from our experience with the real world it is wise continually to refer to this physical origin. The most important of these interpretations is the number line, an interpretation which the student will meet again and again in the future, and one which can be of direct use to him when he leaves school. The teacher should insist that every student grasp at least one of these ways of looking at number processes, that is the structural or the geometric. Only when arithmetic processes are understood, and felt to be reasonable, can the student apply them in new and different situations.

Objective III is achieved almost automatically during the course of Objective II. Once computing processes are thoroughly understood there is almost no difficulty in solving simple equations in one unknown. The structure that has been introduced in arithmetic contains the beginnings of algebra. Furthermore, this simple algebra is a great help to both student and teacher in solving the usual word problems.

A word must be said here about language. There is in this chapter a change from what the student is used to. The numbers we deal with are called rational numbers instead of fractions. Actually, we discuss in this chapter only the non-negative rational numbers, but we shall not use the word"non-negative" at this time in the text because we have not yet discussed negative numbers. They occur in Chapter 7. So we call the numbers of the chapter just rational numbers, and later, when we learn about negative numbers, we shall find that there are some more rational numbers. We keep the word fraction for the symbol, or numeral. (Thus $\frac{3}{5}$ and $\frac{6}{10}$ are different numerals, or names, for the same number.) This precision in language will be convenient

in later work where the word fraction is used for the indicated quotient of two numbers.

Finally, a word about methodology. Every effort should be made to get the student to participate in class discussion and in discovery. Ways of achieving this are suggested in this manual for each section -- along with the language the student must learn in order to communicate his ideas. Students should be encouraged to make conjectures and look ahead. Often the teacher can promote this attitude by well selected questions. The student should not be passive in class. He should be thinking. Encouragement of mental arithmetic and use of oral problems in class can help to keep the students alert.

In outline, this chapter is arranged as follows:

1. The (non-negative) rational numbers are defined as numbers which can be written as quotients of whole numbers. It is then stated that these numbers have the Basic Properties.

   The numbers and some of their properties are illustrated geometrically, in particular, on the number line.
2. The computation rules for multiplication and addition are now developed.
3. Subtraction and division are treated as operations inverse to addition and subtraction. Their computational rules are then easily obtained from the rules for addition and multiplication.
4. Geometric interpretation in the number line is emphasized throughout. Inequalities are treated by use of the number line.
5. Applications occur in many sections and are treated from an algebraic point of view. There is a long revision section.

6-1. Introduction.

The main purpose of this section is to remind the students that whole numbers are not enough to solve all problems. The numbers which the student calls fractions arise naturally in real problems. For example, if three students are to be divided equally between two teams then we are faced with an impossible situation. But if three cakes are to be divided equally between two people then each can

receive $1\frac{1}{2}$, or $\frac{3}{2}$, cakes. Furthermore, the fact that the two portions give the whole 3 cakes is expressed by the equation

$$(\star) \qquad 2 \times \frac{3}{2} = 3.$$

In ways such as this, the student has learned about fractions and has learned to accept them as a natural extension of the system of whole numbers.

The point of view we wish to emphasize here is that the basic property is described by equations like ($\star$). Thus, for example, the numbers $\frac{5}{3}$, $\frac{7}{8}$, $\frac{16}{7}$ are such that

$$3 \times \frac{5}{3} = 5, \qquad 8 \times \frac{7}{8} = 7, \qquad 7 \times \frac{16}{7} = 16.$$

In other words, fractions (or rather the numbers represented by them) satisfy equations of the form

$$bx = a$$

where a and b are whole numbers, and $b \neq o$. The solution of this equation is $\frac{a}{b}$, because $b \times \frac{a}{b} = a$.

The teacher might begin the discussion by first considering a problem or two like the people and cakes problems described above. Equations like ($\star$) should come out of this discussion. Then you might put several equations on the board, some with solutions which are whole numbers and some which are not, for example, $3x = 11$. In this example he should see that $x = \frac{11}{3}$ because $3 \times \frac{11}{3} = 11$, and <u>there is no other number with this property</u>.

At this point try to help the pupils see that no other number will work. This is important for the logical development of number properties even though the treatment in this text is a mixture of logic and intuition. It may be of interest to the teacher to see a proof that there is only one number in the truth set of the equation $3x = 11$. Suppose that x and y are numbers such that

$$3x = 11 \quad \text{and} \quad 3y = 11.$$

Then, by subtracting, we have

$$3x - 3y = 0$$
$$3(x - y) = 0$$

Therefore, $x - y = 0$ and so $x = y$. Do <u>not</u> present this to the class. They do not have the necessary algebra. To convince the student that $3x = 11$ has only one solution just make a remark such as, "If I multiply 3 by different numbers, then I get different answers."

Now have the student read up to Problem 6-1(b). Perhaps have the class do Problems 6-1(a) and (b) orally in class. Call on individual students and require them to tell not only the solution but why it is so.

Answers to Problems 6-1(a), page 6-2

1. $x = 21 \div 3 = 7$. Check: $3 \times 7 = 21$.
2. $a = 30 \div 6 = 5$.
3. $y = 50 \div 5 = 10$.
4. $a = 72 \div 9 = 8$.
5. $x = 13 \div 13 = 1$.
6. $y = 51 \div 17 = 3$.

Answers to Problems 6-1(b), page 6-2

1. (a) 6 because $7 \times \frac{6}{7} = 6$.
   (b) 72 because $2 \times \frac{72}{2} = 72$.
   (c) 2 because $5 \times \frac{2}{5} = 2$.
   (d) 23 because $25 \times \frac{23}{25} = 23$.
   (e) 14 because $7 \times \frac{14}{7} = 14$.
2. (a) $\frac{5}{2}$
   (b) $\frac{12}{7}$
   (c) $\frac{4}{3}$
   (d) $\frac{64}{132} = \frac{16}{33}$
   (e) $\frac{51}{147} = \frac{17}{49}$
   (f) $\frac{39}{13} = 3$
   (g) $\frac{11}{4}$
   (h) $\frac{15}{15} = 1$
   (i) $\frac{b}{a}$ where a must be $\neq 0$.

3. (a) $13x = 7$.

   (b) $4x = 10$ or $2x = 5$.

   (c) $11x = 1$.

   (d) $8x = 5$.

4. Yes, because for example $1 = \frac{2}{2} = \frac{13}{13}$, or $2 = \frac{2}{1} = \frac{4}{2}$, etc.

5. Many are possible, for example, $2x = 8$, $3y = 51$.

6. Many are possible, for example, $2x = 7$, $3y = 50$.

7. (a) $\frac{3}{10}$, (b) $\frac{1}{17}$, (c) 1.

6-2. Rational Numbers

Begin by explaining the words rational number. You might tell the class that we are just using different words for what they are familiar with. You will have to be careful now, and for some time to come, to always say rational number when you are referring to the number. When you refer to the symbol, the word fraction is the proper one.

Now be sure that the class knows the statement in the box. It is vital that everyone knows that

$$b \times \frac{a}{b} = a$$

and that we shall use this relation to prove the computational rules.

Be sure that the class sees that among the rational numbers are all the whole numbers. In other words, every whole number is a rational number. For example,

$$3 = \frac{3}{1} = \frac{6}{2} = \frac{51}{17}, \text{ etc.}$$

$$15 = \frac{15}{1} = \frac{45}{3} = \frac{225}{15}, \text{ etc.}$$

$$0 = \frac{0}{1} = \frac{0}{2} = \frac{0}{7}, \text{ etc.}$$

But there are also rational numbers which are not whole numbers. For example,

$$\frac{1}{3}, \frac{7}{2}, \frac{13}{18}, \frac{42}{13}, \text{ etc.}$$

The student knows, of course, that different fractions may represent the same number, but at this stage it is not necessary to dwell on this. The subject of equivalent fractions (that is, representing the same number) is treated in Section 6-6.

At this point (and this point may not occur on the first day) go on to the geometric interpretation of fractions. Tell the students that we are <u>not</u> proving anything here. We are just reminding the students of facts familiar from primary school, and that the pictures are only supposed to make clear the physical origin of mathematical ideas.

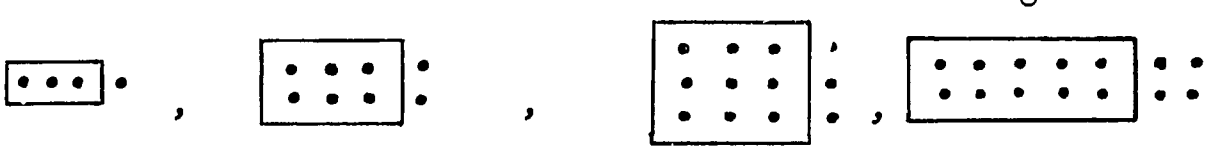
<u>Answers to Problems 6-2, page 6-4</u>

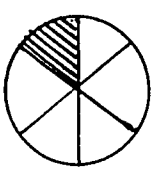
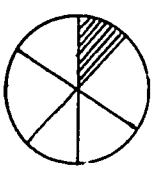
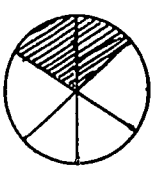
1. $1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{11}{11}$, etc.; $3 = \frac{3}{1} = \frac{9}{3} = \frac{6}{2}$, etc.; $7 = \frac{7}{1} = \frac{14}{2}$, etc.

2. $\frac{1}{3} = \frac{2}{6} = \frac{3}{9} = \frac{17}{51}$, etc.; $\frac{1}{2} = \frac{2}{4} = \frac{5}{10}$, etc.; $\frac{5}{4} = \frac{10}{8} = \frac{15}{12}$, etc.

3.

$\frac{1}{6}$ $\frac{2}{12}$ $\frac{3}{4}$ $\frac{7}{8}$

4. 

5.  and  "make" 

6-3. <u>The Number Line</u>.

This section is meant only to illustrate geometrically the rational numbers and the order relations between them. There is no question here of proof. The student is to use his knowledge of arithmetic as well as his feeling for geometry to answer the questions.

In the first set of problems we recall for the students the one-to-one correspondence between whole numbers and certain points of the line. The main understandings here and in the rest of the section are:

(a) The one-to-one correspondence between numbers and points.

(b) The use of the number line to learn, or to visualize, properties of numbers.

The word "coordinate" should be learned (or re-learned) here. The coordinate of a point is the number associated with that point.

Answers to Problems 6-3(a), page 6-5

1. One step to the right of the point corresponding to 7.
2. Yes. It is one step to the right of the point corresponding to 6,999,999.
3. The point corresponding to 5.
4. No. For example, the points corresponding to 5 and 6.
5. Yes, the point corresponding to the smaller number.
6. Yes, the point corresponding to 0.
7. No.
8. Yes.
9. Yes.

You might ask the class the additional question:

10. What number properties are illustrated on the line?

This question can lead to some discussion. There are many things which can arise. For example:

(a) The line shows which numbers are larger or smaller than others. This anticipates the work of Section 6-13.

(b) The line shows which fractions correspond to the same point. This anticipates the work of Section 6-6 on equivalent fractions.

(c) The line can be used to show addition of fractions as in Section 6-7.

Now begins the process of locating a point on the number line corresponding to each rational number. Do not get involved with any methods of constructing such points. Thus, for example, in locating a point for $\frac{2}{3}$ on the number line we talk about dividing a line segment into three equal parts. It should be understood that we are not concerned with the mechanical process of constructing the point, but rather with accepting the existence of such a point. When the student

is locating a point on the line, a careful estimate of the position is sufficient. If anyone seems worried about how to find the point, say, "There certainly must be a point $\frac{1}{7}$ of the way from 2 to 3. We'll label that point $\frac{15}{7}$, and this approximate position is sufficient to remind us that the point is there."

A few properties of rational numbers can be observed on the number line now without using any operations or order relations. After we know more about addition of rational numbers and about inequalities, the number line will help illustrate more properties.

In Problem 6-3(b), page 6-6, the point corresponding to $\frac{7}{2}$ is obtained in two ways. Do not ask for any <u>proof</u> that one gets the same point both ways. The student should just observe that he does.

Answers to Problems 6-3 (c), page 6-7

3. Three-tenths of the way between 968 and 969.
4. 1, $\frac{2}{2}$, $\frac{3}{3}$, etc.; $\frac{5}{2}$ $\frac{15}{6}$, etc. $\frac{24}{14}$, $\frac{36}{21}$, etc.; 2, $\frac{4}{2}$, $\frac{8}{4}$, etc.
5. A, B, C, D have coordinates $\frac{4}{5}$, 3, $\frac{11}{5}$, $\frac{7}{5}$, respectively.
6. $\frac{8}{10}$, $\frac{12}{15}$, $\frac{16}{20}$, $\frac{20}{25}$, $\frac{24}{30}$, $\frac{28}{35}$, etc.
7. There are infinitely many <u>names</u>, but only <u>one number</u> associated with <u>each point</u>.

The last problem leads up to the next discussion in the text. The student must see that there are (infinitely) many fractions representing each point. There is nothing to be proved here. The student's knowledge of arithmetic and feeling for geometry should permit him to answer all questions.

Answers to Problems 6-3 (d), page 6-8

1. There are many answers. For example, in part (a) $5\frac{1}{2}$, $5\frac{1}{3}$, $5\frac{2}{3}$, 5.1, etc.; (b) $\frac{7}{8}$; (c) 19.85; (d) $\frac{3}{4}$; (e) 850.001; (f) $\frac{53}{16 \times 17}$, for example; (g) $\frac{80}{8 \times 6}$, for example; (h) $\frac{1}{2{,}000{,}000}$, for example.
2. Yes. For example, the arithmetic mean, or average, of the two. The student might answer: Take the smaller and add to it one-half of the difference between the numbers.
3. Yes. Essentially the same answer as for Problem 2.

4. Infinitely many. You might expect the student to answer, "as many as you want." You could argue that

$$\frac{25}{1,000,000} = \frac{25,000,000}{1,000,000,000,000} \quad \text{and}$$

$$\frac{26}{1,000,000} = \frac{26,000,000}{1,000,000,000,000},$$ so that there are at least 999,999 rational numbers between these numbers. Between any two rational numbers there is always another, and in fact infinitely many rational numbers.

5. There is none. This question can promote considerable discussion. An excellent answer would be, "There couldn't be a smallest number greater than 0 because $\frac{1}{2}$ of that number would be smaller."
6. There is none.
7. There is none. This is quite similar to Problem 5.
8. Yes.

6-4. Basic Properties

In Chapter 3 the student studied the commutative, associative, and distributive principles for the whole numbers. It is essential that the student understand that these general principles, or rules, must still be valid for rational numbers, for in a given computation problem one cannot in advance be sure that only whole numbers are involved so that it would be impossible to proceed unless the same principles applied to all numbers.

Ask a student to write on the blackboard a particular case of the commutative law for addition. He might write

$$\frac{1}{2} + \frac{1}{3} = \frac{1}{3} + \frac{1}{2}$$

Similarly, ask students to either write on the board, or tell you how to write on the board, illustrations of the other basiic laws. In order to drill them on the terms, you can ask questions such as, "What principle states that

$$\frac{1}{2} \times \left(\frac{2}{3} \times \frac{5}{6}\right) = \left(\frac{1}{2} \times \frac{2}{3}\right) \times \frac{5}{6}?"$$

Be sure before proceeding that the class has mastered the words commutative, associative, and distributive.

"Commutative" is derived from the verb "commute," which means to interchange.

"Associative" is clear, as you just assoicate the factors differently.

"Distributive" is used because multiplication distributes over addition.

It should be observed that both addition and multiplication are binary operations. In other words, for both addition and multiplication, from two numbers another number is "produced," called the sum, or product, as the case may be. The principles apply to these binary operations. Later, when these principles are thoroughly in hand we shall omit parentheses when no confusion can arise. For example, later on, we shall write simply $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ in place of either $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$ or $\frac{1}{2} + \frac{1}{3} + \frac{1}{4}$. Because they are equal there is no need to use the brackets unless one wishes to pay attention to the order of carrying out the addition.

In many of the problems in the next set, the student is asked to compute a sum or a product. Strictly speaking, such problems are out of place because we have not yet established in the text the rules for adding or for multiplying. The point of view which we take here is that we wish the student to recall and review his methods -- and that by computing a few simple cases he will be convinced of the commutative, associative, and distributive properties.

Answers to Problems 6-4(a), and page 6-9

1. (a) Sum = 4. True.
   (b) $\frac{25}{4}$ True.
   (c) False.
   (d) $\frac{8}{9}$ True.
   (e) $\frac{99}{6}$ True.
   (f) $\frac{13}{8}$ True.
2. (a) $\{17\}$
   (b) $\{\frac{5}{17}\}$

(c) $\left\{\frac{7}{18}\right\}$

(d) $\left\{\frac{1}{4}\right\}$

(e) {all rational numbers}

(f) {12}

3. That addition is commutative.

4. (a) $\frac{28}{3}$ True

(b) $\frac{15}{8}$ and $\frac{5}{8}$ False

(c) $\frac{1}{77}$ True

(d) 1 True

(e) 77 and 7 False

(f) 5 True

5. (a) $\left\{\frac{2}{3}\right\}$

(b) $\left\{\frac{4}{93}\right\}$

(c) $\left\{\frac{9}{5}\right\}$

(d) $\left\{\frac{3}{7}\right\}$

(e) $\left\{\frac{1}{3}\right\}$

(f) {all rational numbers}

6. That multiplication is commutative.

7. (a) 2 True

(b) $\frac{1}{30}$ True

(c) 1 True

(d) $\frac{5}{3}$ True

(e) 10 and 73 False

8. (a) $\left\{\frac{1}{2}\right\}$

(b) $\left\{\frac{1}{3}\right\}$

9. That addition and multiplication are associative.
10. (a) 5. The right side is simpler.
    (b) 1. Both sides about the same.
    (c) $\frac{43}{38}$. Left side perhaps simpler.
11. The distributive property.
12. (a) {0}
    (b) {0}
    (c) {0}
    (d) {1}
    (e) {1}
    (f) {1}
13. The identity properties of zero and one.
14. (a) {0}
    (b) {0}
    (c) {0}
    (d) {0}
    (e) {all rational numbers}.
15. That $0 \times a = 0$ for all rational numbers a.
16. For parts (a) through (g) the truth set = {all rational numbers}.
    for (h) the truth set is empty = { }.
17. (a) Commutativity of addition.
    (b) Commutativity of multiplication.
    (c) Associativity of addition.
    (d) Associativity of multiplication.
    (e) Distributivity of multiplication over addition.
    (f) Identity property of 0.
    (g) Identity property of 1.
18. In each case the truth set is {1}.
19. All are 1.
20. {All rational numbers different from 0}.

The table at the bottom of page 12 summarizes the basic properties. At this stage all students should be convinced that these properties are valid. A student may not think they are very important. That appreciation should come later. He must know them, however.

In the next sections we shall use these properties along with the relation

$$b \times \frac{a}{b} = a,$$

to establish all the familiar rules.

The next set of problems should present no difficuties. They are preparatory for the sections which follow.

Answers to Problems 6-4 (b), page 6-13

1. (a) $\frac{2}{3} + \frac{1}{2}$

   (b) $\frac{2}{3} \times \frac{1}{2}$

   (c) $\left(\frac{1}{2} \times 2\right) \times \frac{2}{3}$

   (d) $\left(\frac{1}{3} + \frac{1}{5}\right) + \frac{1}{7}$

   (e) $\frac{1}{8} + \left(\frac{2}{3} + \frac{1}{3}\right)$

   (f) $\frac{2}{3} \times \left(\frac{4}{5} \times \frac{1}{2}\right)$

   (g) $\frac{1}{3} \times \left(\frac{1}{2} + \frac{1}{2}\right)$

   (h) $\left(\frac{2}{3} + \frac{3}{4}\right) \times \frac{3}{5}$

   (i) $\frac{13}{12} + \frac{7}{3}$

   (j) $\left(\frac{1}{2} \times \frac{1}{4}\right) + \left(\frac{1}{3} \times \frac{1}{4}\right)$

2. (a) $\frac{3}{7}$

   (b) $1\frac{17}{19}$

   (c) $\frac{17}{13}$

   (d) $1\frac{5}{12}$

   (e) $\frac{5}{12}$

(f) $\frac{2}{13}$

(g) u

(h) 0

(i) 17

(j) $1\frac{2}{15}$

(k) $\frac{c}{d}$

3. (a) 1

(b) $\frac{3}{4}$

(c) 0

(d) all rational numbers

6-5. Multiplication.

In this section we establish the rule for multiplying two rational numbers by first considering easy special cases.

First we prove that $a \times \frac{1}{b} = \frac{a}{b}$. Note that $\frac{a}{b}$ is the number

such that $$b \times \frac{a}{b} = a,$$

and that $\frac{1}{b}$ is the number such that $b \times \frac{1}{b} = 1$.

It is necessary to prove that the product $a \times \frac{1}{b}$ is the same number as $\frac{a}{b}$. Our point of view here is that both

$$a \times \frac{1}{b} \text{ and } \frac{a}{b}$$

satisfy the same equation, namely

$$bx = a.$$

You might begin the discussion by asking the class what equation has the solution $\frac{5}{9}$. (You would want the answer $9x = 5$). Then you might ask what equation has the solution $\frac{1}{9}$. ($9x = 1$). Now ask what equation has the solution $5 \times \frac{1}{9}$. If a student answers "$9x = 5$", ask him how he knows this. He might give the following answer:

$$9 \times \left(5 \times \frac{1}{9}\right) = 5 \times \left(9 \times \frac{1}{9}\right)$$ after use of commutativity and associativity

$$= 5 \times 1 = 5$$

Should this happen, you should praise him, and ask how this proves that $5 \times \frac{1}{9} = \frac{5}{9}$. The answer, of course, is that both satisfy the same equation, $9x = 5$, and that this equation has only one solution.

Answers to Problems 6-5 (a) page 6-15

1. (a) Because $\frac{8}{7}$ is the number which satisfies $7x = 8$.
(b) $7 \times \frac{8}{7}$ and 8 name the same number. We have multiplied that number by $\frac{1}{7}$. (c) By commutativity of multiplication. (d) By associativity. (e) Because $7 \times \frac{1}{7} = 1$ and $\frac{8}{7} \times 1 = \frac{8}{7}$.

2. The steps are given only for part (a). The other parts are similar.

$9 = 5 \times \frac{9}{5}$ because $\frac{9}{5}$ satisfies $9 = 5x$.

$9 \times \frac{1}{5} = (5 \times \frac{9}{5}) \times \frac{1}{5}$ multiplying by $\frac{1}{5}$.

$= (\frac{9}{5} \times 5) \times \frac{1}{5}$ commutativity.

$= \frac{9}{5} \times (5 \times \frac{1}{5})$ associativity.

$= \frac{9}{5} \times 1 = \frac{9}{5}$ because $5 \times \frac{1}{5} = 1$.

The teacher should be aware that other steps could be taken.

For example: $9 \times \frac{1}{5} = \frac{1}{5} \times 9 = \frac{1}{5} \times \left(5 \times \frac{9}{5}\right)$

$= \left(\frac{1}{5} \times 5\right) \times \frac{9}{5} = 1 \times \frac{9}{5} = \frac{9}{5}$

3. (a) $\left\{\frac{9}{5}\right\}$

(b) $\left\{\frac{3}{17}\right\}$

(c) $\{5\}$

(d) $\left\{\frac{2}{5}\right\}$

(e) $\left\{\frac{1}{17}\right\}$

(f) $\{7\}$

(g) $\left\{\frac{1}{3}\right\}$

(h) $\{24\}$

4. (a) $5 \times \frac{1}{9}$

(b) $7 \times \frac{1}{43}$

(c) $50 \times \frac{1}{3}$

(d) $23 \times \frac{1}{23}$ (or $1 \times \frac{1}{1}$ also.)

(e) $4 \times \frac{1}{4}$, etc.

(f) $1 \times \frac{1}{1}$, etc.

(g) $14 \times \frac{1}{16}$ (or $7 \times \frac{1}{8}$)

(h) $2 \times \frac{1}{1}$

(i) $a \times \frac{1}{b}$

We now turn to products of the form $\frac{1}{a} \times \frac{1}{b}$, where $\underline{a}$ and $\underline{b}$ are whole numbers. The argument is the same as before. We prove that $\frac{1}{a} \times \frac{1}{b}$ and $\frac{1}{a \times b}$ satisfy the same equation.

You might ask the class, "What equations are satisfied by the numbers $\frac{1}{3}$, $\frac{1}{5}$, and $\frac{1}{15}$?" The equations are

$$3x = 1, \ 5x = 1, \text{ and } 15x = 1.$$

Then ask "What equation is satisfied by $\frac{1}{3} \times \frac{1}{5}$?" If someone says $15x = 1$, you can ask him why. A possible answer would be

$$15 \times \left(\frac{1}{3} \times \frac{1}{5}\right) = (5 \times 3) \times \left(\frac{1}{3} \times \frac{1}{5}\right)$$

$$= \left(3 \times \frac{1}{3}\right) \times \left(5 \times \frac{1}{5}\right) \quad \text{after several steps}$$

$$= 1 \times 1 = 1$$

This would prove that $\frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$ because both satisfy an equation with

only one solution.

This discussion in the text is similar to the above.

Answers to Problems 6-5 (b), page 6-16

1. Because $6 \times \frac{1}{6} = 1$.

The next sentence is $1 \times 1 = 1$.

The missing steps are:

$$1 = \left(7 \times \frac{1}{7}\right) \times \left(9 \times \frac{1}{9}\right) = 7 \times \left(\frac{1}{7} \times \left(9 \times \frac{1}{9}\right)\right) \text{ associativity}$$

$$= 7 \times \left(\left(\frac{1}{7} \times 9\right) \times \frac{1}{9}\right) \text{ associativity}$$

$$= 7 \times \left(\left(9 \times \frac{1}{7}\right) \times \frac{1}{9}\right) \text{ commutativity}$$

$$= 7 \times \left(9 \times \left(\frac{1}{7} \times \frac{1}{9}\right)\right) \text{ associativity}$$

$$= (7 \times 9) \times \left(\frac{1}{7} \times \frac{1}{9}\right) \text{ associativity}$$

Therefore $\frac{1}{7} \times \frac{1}{9} = \frac{1}{7 \times 9}$ because

$$(7 \times 9) \times \frac{1}{7} \times \frac{1}{9} = 1$$

$$(7 \times 9) \times \frac{1}{7 \times 9} = 1$$

So both $\frac{1}{7} \times \frac{1}{9}$ and $\frac{1}{7 \times 9}$ satisfy the same equation and must be the same number.

2. The answer is given only for part (d). The others are similar. The reasons are as in Problem 1.

$$1 = \left(a \times \frac{1}{a}\right) \times \left(b \times \frac{1}{b}\right)$$

$$= a \times \left(\frac{1}{a} \times \left(b \times \frac{1}{b}\right)\right)$$

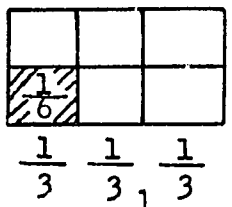
$$= a \times \left(\left(b \times \frac{1}{a}\right) \times \frac{1}{b}\right)$$

$$= a \times \left(b \times \left(\frac{1}{a} \times \frac{1}{b}\right)\right)$$

$$= (a \times b) \times \left(\frac{1}{a} \times \frac{1}{b}\right)$$

Therefore

$$\frac{1}{a} \times \frac{1}{b} = \frac{1}{a \times b}$$

3. 

4.

$$\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$$

5. (a) $\left\{\frac{1}{60}\right\}$

(b) $\left\{\frac{1}{10}\right\}$

(c) $\left\{\frac{1}{80000}\right\}$

(d) $\left\{\frac{1}{25}\right\}$

(e) $\left\{\frac{1}{7}\right\}$ (f) $\left\{\frac{1}{20}\right\}$

(g) $\{1\}$ (h) $\left\{\frac{1}{ab}\right\}$

Finally, with the two special cases $a \times \frac{1}{b}$ and $\frac{1}{a} \times \frac{1}{b}$ disposed of, it is an easy matter to prove that

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

by use of the basic properties. The discussion in the text omits several steps, and you may want to omit these in class. A full treatment would be as follows:

We have already seen that $\frac{2}{3} = 2 \times \frac{1}{3}$ and $\frac{7}{5} = 7 \times \frac{1}{5}$ so

$$\frac{2}{3} \times \frac{7}{5} = \left(2 \times \frac{1}{3}\right) \times \left(7 \times \frac{1}{5}\right)$$

$$= 2 \times \left[\frac{1}{3} \times \left(7 \times \frac{1}{5}\right)\right] \qquad \text{associative property}$$

$$= 2 \times \left[\left(7 \times \frac{1}{5}\right) \times \frac{1}{3}\right] \qquad \text{commutative property}$$

$$= 2 \times \left[7 \times \left(\frac{1}{5} \times \frac{1}{3}\right)\right] \qquad \text{associative property}$$

$$= (2 \times 7) \times \left(\frac{1}{5} \times \frac{1}{3}\right) \qquad \text{associative property}$$

$$= 14 \times \frac{1}{15} \qquad \text{We proved } \frac{1}{a} \times \frac{1}{b} = \frac{1}{ab}.$$

$$= \frac{14}{15} \qquad \text{We proved } a \times \frac{1}{b} = \frac{a}{b}.$$

Answers to Problems 6-5 (c), page 6-17

1. The solution is given for part (a). The others are similar=

$$\frac{5}{9} \times \frac{4}{11} = \left(5 \times \frac{1}{9}\right) \times \left(4 \times \frac{1}{11}\right) = (5 \times 4) \times \left(\frac{1}{9} \times \frac{1}{11}\right) \qquad \text{after several steps.}$$

$$= 20 \times \frac{1}{99}$$

$$= \frac{20}{99}$$

(b) $\frac{3}{32}$

(c) $\frac{18}{4} = \frac{9}{2}$. At this stage the answer $\frac{18}{4}$ is sufficient and is all that can be stated on the basis of the text development. Only after Section 6-6 will we have discussed how to simplify fractions. Of course if some pupils answer $\frac{9}{2}$, on the basis of earlier experience, accept the answer without comment

(d) $\frac{440}{483}$

(e) $\frac{14}{24} = \frac{7}{12}$

(f) $\frac{ac}{bd}$

2. (a) $\frac{14}{27}$

(b) $\frac{14}{27}$

(c) $\frac{13 \times 28}{53 \times 75}$

(d) $\frac{17 \times 18}{17 \times 19}$ or $\frac{18}{19}$

(e) $\frac{12}{12}$

(f) $\frac{75}{12}$

(g) $\frac{75}{12}$

(h) $\frac{24}{36}$

3. Many answers are possible for each part.

(a) $\frac{6}{7} \times \frac{1}{5}$ or $6 \times \frac{1}{35}$, etc.

(b) $\frac{3}{2} \times \frac{3}{2}$

(c) $\frac{5}{9} \times 3$

(d) $\frac{50}{20} \times \frac{10}{10}$

(e) $\frac{7}{11} \times \frac{7}{11}$

(f) $\frac{3}{7} \times \frac{1}{2}$

(g) $\frac{11}{17} \times 2$

(h) $\frac{106}{53} \times \frac{1}{3}$

(i) $14 \times \frac{1}{18}$

(j) $\frac{3}{2} \times \frac{37}{23}$

(k) $\frac{a}{c} \times \frac{b}{d}$

4. $\frac{3}{5} \times \frac{3}{4} = (3 \times 3) \times \left(\frac{1}{5} \times \frac{1}{4}\right)$. Then $\frac{1}{5} \times \frac{1}{4}$ gives $\frac{1}{20}$ on the line and there are 9 of these steps.

6-6. Equivalent Fractions.

In this section we show, using what we have proved about multiplication, that different fractions can represent the same number. We simply use the fact that $\frac{b}{b} = 1$ and use the identity property of 1: $\frac{a}{b} \times 1 = \frac{a}{b}$.

It is important that the students know the meaning of equivalent fractions and see the convenience of having a word to describe fractions which represent the same number. Thus, for example $\frac{7}{9} = \frac{14}{18}$ because the numbers are the same, the fractions $\frac{7}{9}$ and $\frac{14}{18}$, although different, are equivalent.

Answers to Problems 6-6 (a), page 6-19

In these problems it is the proof that is important. That the fractions represent the same numbers (are equivalent) will be familiar to all from primary school arithmetic. Now they can see why. The work can be arranged in different ways. The way shown in Problem 1 is just a suggestion.

1. $\frac{4}{6} = \frac{2 \times 2}{3 \times 2} = \frac{2}{3} \times \frac{2}{2}$ because of the rule for multiplication

   $= \frac{2}{3} \times 1$ because $\frac{2}{2} = 1$

   $= \frac{2}{3}$ property of 1

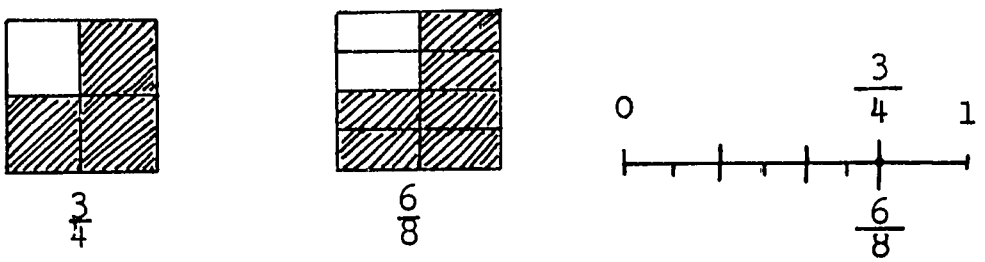
   Therefore the fraction $\frac{4}{6}$ is equivalent to the fraction $\frac{2}{3}$

2. $\frac{21}{14} = \frac{3 \times 7}{2 \times 7} = \frac{3}{2} \times \frac{7}{7} = \frac{3}{2} \times 1 = \frac{3}{2}$. Etc.

With respect to the geometric illustrations of equivalent fractions there is nothing to prove. The student is just supposed to draw the appropriate figures and observe that both fractions are pictured as the same part of a whole, or as tags for the same point on the number line.

Answers to Problems 6-6(b), page 6-20

1.



and similarly for other problems.

— — —

The next problems are concerned with the notion of changing fraction to lowest terms. Observe that we are concerned here with the <u>form of the fraction</u>. In other words, we are concerned with the <u>numeral</u>, or <u>name</u>, of the number. Be sure that each student can decide whether or not a given fraction is in lowest terms and whether he can change the fraction to lowest terms.

Answers to Problems 6-6(c), page 6-21

1. $\frac{3}{5}$

2. $\frac{3}{2}$

3. $\frac{1}{11}$

4. $\frac{23}{41}$

5. $\frac{3}{1}$ In this problem, since we want a fraction for the answer, we leave it as $\frac{3}{1}$ rather than 3. However, if a student gives the answer 3, do no more than remark, "Yes, and a fraction for this is $\frac{3}{1}$."

6. $\frac{28}{21}$

7. $\frac{3}{5}$

8. $\frac{xy}{c}$ Here we cannot go farther because we do not know what x, y, c are. Sometimes, of course, $\frac{xy}{c}$ would not be in the lowest terms, for example, when $x = 4$, $y = 3$,

$c = 10$. Then $\frac{xy}{c} = \frac{4 \times 3}{10} = \frac{6}{5}$. But in other cases $\frac{xy}{c}$ would be in the lowest terms. For example, when $x = 2$, $y = 3$, $c = 7$.

The next problems are preparatory to the next section on addition of rational numbers.

Answers to Problems 6-6(d), page 6-21

1. (a) $\frac{1}{2} = \frac{12}{24}$

   (b) $\frac{7}{12} = \frac{21}{36}$

   (c) $\frac{6}{27} = \frac{18}{81}$

   (d) $\frac{3}{5} = \frac{3x}{5x}$

2. This problem is not actually necessary for the discussion of addition to follow, and so could be omitted. However, it is useful to suggest to the student that the numerator of a fraction can also be changed.

   (a) $\frac{35}{63}$, (b) $\frac{2}{12}$, (c) $\frac{44}{76}$, (d) $\frac{ay}{cy}$

6-7. Addition.

In this section the main idea is that the rule for adding rational numbers, represented by fractions, is a simple consequence of the distributive property.

Once the rule for adding has been established, the number line gives an excellent illustration. In the example in the text we simply take steps of length $\frac{1}{8}$.

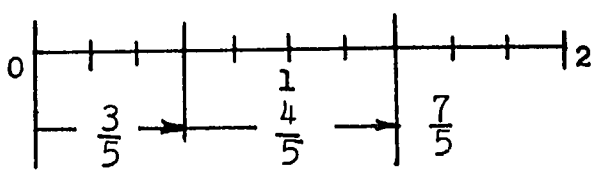
Answers to Problems 6-7(a), page 6-23

1. (a) We have shown that $\frac{a}{b} = a \times \frac{1}{b}$.
   (b) The distributive property.
   (c) Because $3 + 2 = 5$.
   (d) Because $a \times \frac{1}{b} = \frac{a}{b}$.

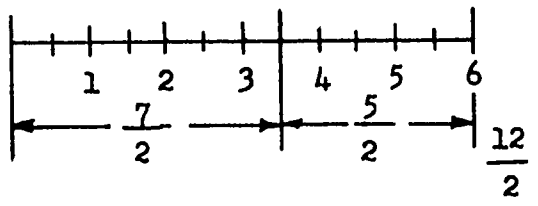
2. (a) Because $\frac{a}{b} = a \times \frac{1}{b}$.
   (b) The distributive property.
   (c) Because $5 + 7 = 12$.
   (d) Because $a \times \frac{1}{b} = \frac{a}{b}$.
   (e) Factorization of 12 and 9.
   (f) Because of the rule for multiplication.
   (g) Because $\frac{3}{3} = 1$.
   (h) Property of 1.

3. (a) $\frac{11}{3}$
   (b) $\frac{4}{5}$
   (c) $\frac{3}{2}$
   (d) $\frac{9}{4}$
   (e) $\frac{7}{3}$
   (f) 1
   (g) 4
   (h) 1
   (i) $\frac{5}{6}$
   (j) $\frac{7}{4}$

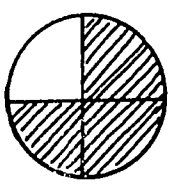
4. (a)



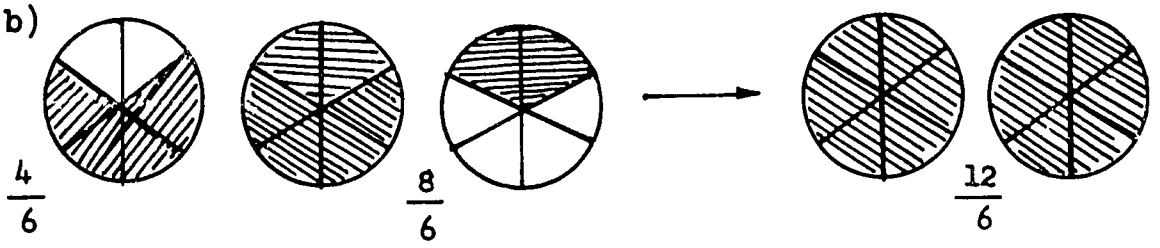
(b)



5. (a)



(b)



6. $\frac{a}{b} + \frac{b}{c} = a \times \frac{1}{c} + b \times \frac{1}{c} = (a + b) \times \frac{1}{c} = \frac{a + b}{c}$

Answers to Problems 6-7(b), page 6-25

1. $\frac{7}{6} + \frac{3}{10} = \frac{70}{60} + \frac{18}{60} = \frac{88}{60} = \frac{22 \times 4}{15 \times 4} = \frac{22}{15}$. This is less convenient because numerator and denominator in $\frac{88}{60}$ are larger than necessary.

2. (a) 7
   (b) 2
   (c) $\frac{7}{7}$
   (d) $\frac{2}{2}$

3. Because the LCM of the denominators is 45 we use 1 in the forms $\frac{3}{3}$ and $\frac{5}{5}$.

4. (a) $\frac{13}{15}$
   (b) $\frac{17}{18}$
   (c) $\frac{41}{14}$
   (d) $\frac{38}{9}$
   (e) $\frac{5}{12}$
   (f) $\frac{21}{20}$
   (g) $\frac{7}{6}$
   (h) $\frac{31}{6}$
   (i) $\frac{13}{6}$

5. (a) $\frac{15}{8}$
   (b) $\frac{53}{10}$
   (c) $\frac{139}{42}$
   (d) 0
   (e) $\frac{9}{8}$

(f) $\frac{3}{8}$

(g) $\frac{4}{3}$

(h) $\frac{241}{168}$

(i) $\frac{113}{168}$

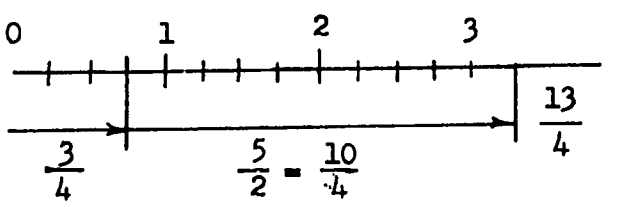
(j) $\frac{29}{21}$

6. (a) Because $33 = 3 \times 11$ and $6 = 3 \times 2$ The LCM of 33 and 6 is $3 \times 11 \times 2$.
   (b) Property of 1.
   (c) Equivalent forms for 1.
   (d) Rule for multiplication.
   (e) Number facts.
   (f) Because $a \times \frac{1}{b} = \frac{a}{b}$.
   (g) Distributive law.
   (h) Number facts.
   (i) Because $a \times \frac{1}{b} = \frac{a}{b}$.
   (j) Number facts.
   (k) Rule for multiplication.
   (l) Because $\frac{3}{3} = 1$.
   (m) Property of 1.

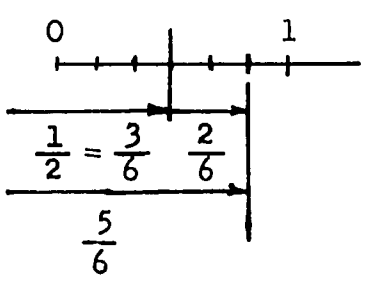
7. $$\frac{a}{b} + \frac{c}{d} = \frac{a}{b} \times \frac{d}{d} + \frac{c}{d} \times \frac{b}{b}$$
$$= \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad + bc}{bd}$$

Answers to Problems 6-7(c), page 6-27

1.



2.



At this stage of our study of addition we have dealt always with fractions and not with numerals called "mixed numbers." Observe that this is not a well-selected name because what we are talking about is not the number but the symbol for the number.

The first thing to recall is the meaning of these symbols. For example, $2\frac{3}{4} = 2 + \frac{3}{4}$. The student will easily change this to the fraction $\frac{11}{4}$. Now you can show him how this follows from the rule for adding numbers.

$2\frac{3}{4} = 2 + \frac{3}{4}$	by the meaning of $2\frac{3}{4}$
$= \frac{2}{1} + \frac{3}{4}$	meaning of $\frac{2}{1}$
$= \frac{8}{4} + \frac{3}{4}$	equivalent form of $\frac{2}{1}$
$= \frac{11}{4}$	the rule for addition.

Answers to Problems 6-7(d), page 6-28

1. (a) $\frac{101}{21} = 4\frac{17}{21}$

   (b) $\frac{349}{35} = 9\frac{34}{35}$

   (c) 54

   (d) $\frac{57}{10} = 5\frac{7}{10}$

   (e) $\frac{325}{24} = 13\frac{13}{24}$

   (f) $\frac{129}{16} = 8\frac{1}{16}$

2. $1\frac{3}{4} = 1 + \frac{3}{4}$ because $1\frac{3}{4}$ is just an abbreviation of $1 + \frac{3}{4}$. That $1 + \frac{3}{4} = \frac{7}{4}$ follows from the rule for addition.

3. $58\frac{1}{6}$ feet = 58' 2".

4. (a) $\frac{161}{8} = 20\frac{1}{8}$

   (b) $7\frac{13}{20}$

   (c) 6

   (d) $15\frac{1}{5}$

   (e) $21\frac{1}{4}$

   (f) $29\frac{9}{16}$

5. (a) $x = 0$, $y = 10$; $x = 1$, $y = 9$; $x = 2$, $y = 8$; etc.
   (b) $x = 0$, $y = 10$; $x = 1$, $y = 7$; $x = 2$, $y = 4$; $x = 3$, $y = 1$.
   (c) $x = 1$, $y = 2$.

6-8. Subtraction.

Subtraction is treated, in the same way as addition, by a distributive principle. Although the appropriate principle,

(☆) $$ax - bx = (a - b)x$$

has not been established in the text, it should be acceptable to the student. If not, demonstrate with particular numbers.

It might be desirable to do the example in the text another way, as follows:

Let

$$\frac{7}{3} - \frac{2}{3} = x$$

Then $\frac{7}{3} = x + \frac{2}{3}$ and by multiplying by 3,

$$7 = 3\left(x + \frac{2}{3}\right) = 3x + 2 \quad \text{by the distributive principle}$$

Then $5 = 3x$ and

$$x = \frac{5}{3}.$$

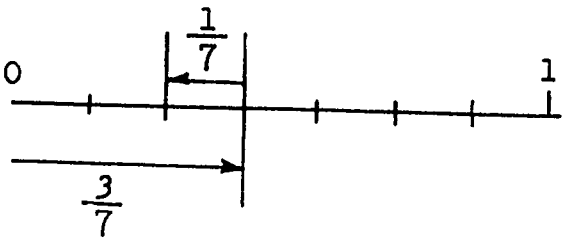
So we conclude that $\frac{7}{3} - \frac{2}{3} = \frac{5}{3}$.

Other problems could be done the same way, but this may be enough to convince the student of the distributive principle (☆).

Answers to Problems 6-8(a), page 6-29

1. (a) $14 \times \frac{1}{20} - 7 \times \frac{1}{20} = (14 - 7) \times \frac{1}{20} = 7 \times \frac{1}{20} = \frac{7}{20}$
   (b) $5 \times \frac{1}{6} - 2 \times \frac{1}{6} = (5 - 2) \times \frac{1}{6} = 3 \times \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$
   (c) $\frac{1}{4}$
   (d) 4
   (e) 0
   (f) 1

2. (a)



Answers to Problems 6-8(b), page 6-30

1. (a) $\frac{1}{8}$

   (b) $\frac{7}{15}$

   (c) $\frac{1}{6}$

   (d) $\frac{23}{24}$

   (e) $\frac{53}{10}$

   (f) $\frac{149}{72}$

   (g) 0

2. $\frac{11}{12}$ hours or 55 min.

3. $2\frac{1}{4}$ pounds or 36 ounces.

4. $15\frac{1}{15}$ inches

5. 9 inches.

6. (a) Property of 1.
   (b) Different names for 1.
   (c) Rule for multiplication.
   (d) Multiplication facts.
   (e) Because $a \times \frac{1}{b} = \frac{a}{b}$.
   (f) Distributive property for subtraction.
   (g) Subtraction fact.
   (h) Because $a \times \frac{1}{b} = \frac{a}{b}$.
   (i) Factorization of 35 and 42.
   (j) Rule for multiplication.
   (k) Different name for 1.
   (l) Property of 1.

6-9. Reciprocals.

The purpose of this brief section is to present one idea -- that of reciprocals. Once this has been established, the next section, on division, can proceed smoothly.

It is important to show the class that zero has no reciprocal. Suppose that R is the reciprocal of 0, then according to the definition of a reciprocal

$$0 \times R = 1.$$

But this is impossible because $0 \times R = 0$ for all numbers R.

Answers to Problems 6-9(a), page 6-32

1. (a) $\frac{1}{17}$
   (b) 12
   (c) 1
   (d) $\frac{1}{25}$
   (e) $\frac{4}{3}$
   (f) 276
   (g) $\frac{1}{1000}$
   (h) $\frac{3}{7}$

2. Parts (a), (c), (d), (e), (g), (h), are reciprocals because the products are 1 in each case.

Answers to Problems 6-9(b), page 6-33

1. (a) $\frac{9}{5}$
   (b) $\frac{2}{5}$
   (c) $\frac{1}{100}$
   (d) $\frac{5}{17}$
   (e) 72
   (f) $\frac{4}{19}$
   (g) 500
   (h) $\frac{921}{7}$
   (i) $\frac{10}{171}$
   (j) 4
   (k) $\frac{20}{7}$
   (l) $\frac{1}{100}$

6-10. Division.

The first set of problems just reminds the student how division is defined in terms of multiplication. That is

$$a \div b = c \text{ means that } a = b \times c.$$

Answers to Problems 6-10(a), page 6-33

1. (a) 4
   (b) 11
   (c) 63
   (d) 17
2. (a) $4 = 17y$
   (b) $35 = 244x$
   (c) $37 = 37t$
   (d) $0 = 28u$
   (e) $19 = 20r$
   (f) $46 = 5x$
   (g) $\frac{1}{2} = \frac{1}{4}x$
   (h) $\frac{3}{5} = \frac{4}{7}y$
   (i) $\frac{7}{15} = \frac{7}{15}a$
   (j) $\frac{2}{3} = \frac{4}{15}b$

Now that any division problem has been changed into a multiplication problem, it is easy, using reciprocals, to find the desired quotient. Observe that in the text the commutative and associative principles are used. Full details of the text example are as follows:

If

$$x = \frac{3}{4} \div \frac{1}{2}$$

then

$$\frac{1}{2}x = \frac{3}{4} \qquad \text{(definition of division)}$$

$$\left(\frac{1}{2}x\right) \times 2 = \frac{3}{4} \times 2$$

$$\left(x \times \frac{1}{2}\right) \times 2 = \frac{3}{4} \times 2 \qquad \text{(commutativity)}$$

$$x \times \left(\frac{1}{2} \times 2\right) = \frac{3}{4} \times 2 \qquad \text{(associativity)}$$

$x \times 1 = \frac{3}{4} \times 2$ (2 and $\frac{1}{2}$ are reciprocals)

$x = \frac{3}{2}$ (property of 1 and multiplication of $\frac{3}{4}$ by 2)

Answers to Problems 6-10(b), page 6-34

The solution to Problem 1 is given. The others are similar. If $x = \frac{2}{3} \div \frac{7}{11}$, then $\frac{7}{11}x = \frac{2}{3}$. So $\left(\frac{7}{11}x\right) \times \frac{11}{7} = \frac{2}{3} \times \frac{7}{11}$, or $x\left(\frac{7}{11} \times \frac{11}{7}\right) = \frac{2 \times 11}{3 \times 7}$, or $x = \frac{22}{21}$.

The equation in the box on page 6-35 has been justified by the preceding problems. It states the familiar rule, invert the divisor and multiply.

Should not everyone fully understand the rather algebraic method given above for establishing the rule for division, then the next method may seem easier.

Answers to Problems 6-10(c), page 6-36

1. (a) $\frac{3}{2}$
   (b) 5
   (c) $\frac{325}{243}$
   (d) $\frac{22}{41}$
   (e) $\frac{48}{25}$
   (f) $\frac{63}{26}$
   (g) 207
   (h) $\frac{3}{8}$
   (i) ay
   (j) $\frac{a}{d}$
2. (a) $\frac{35}{12}$
   (b) $\frac{105}{4}$

3. (a) 6

(b) $\frac{83}{40}$

(c) 6

(d) $\frac{3}{8}$

(e) $\frac{17}{10}$

(f) $\frac{5}{4}$

(g) $\frac{36}{5}$

(h) $\frac{23}{30}$

4. $\frac{22}{7}$

5. $\frac{1}{6}$

6. $\frac{15}{32}$

7. $\frac{2}{9}$

8. $\frac{27}{2}$

9. $\frac{1}{3}$

6-11. Revision.

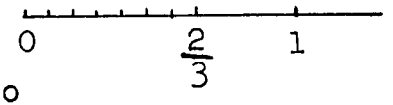
In addition to providing necessary drill, this section should give the teacher an opportunity to ask students reasons for various small steps they take in solving problems. Only by inquiring, and listening to students' explanations, can the teacher determine what steps are confusing. It should be noted that your questions may have many answers. Almost any explanation which shows insight can be satisfactory.

Urge the students to do as many of the arithmetic steps mentally as they can, even when doing written work. Naturally, neat, accurate written work is important, yet accuracy is to a large extent dependent on mental computation in order to avoid excessive writing, with its attendant clerical errors. For this

reason do not insist that a student write <u>all</u> the intervening steps. He should write only what is necessary for him. On the other hand, there should be no skimping of adequate explanation of the symbols and formulas necessary for the problem.

Begin the discussion with a brief look at the examples of computational techniques in the box in the text. Ask students to supply reasons for the methods. In addition and subtraction ask for interpretations on the number line. The reasons given can be of several kinds. What is important is that each student has some way of looking at the computation which is meaningful to him. Examples of explanations of division might be as follows:

(a) $\frac{2}{3} \div 4 = \frac{2}{12}$ because



if I divide each third into fourths I get 12ths. So I get $\frac{2}{12} = \frac{1}{6}$.

(b) To divide $\frac{2}{3}$ by 4 means I want a number x such that $\frac{2}{3} = 4x$. If I make $x = \frac{1}{4} \times \frac{2}{3}$, it works because $\frac{2}{3} = 4 \times \left(\frac{1}{4} \times \frac{2}{3}\right) = \left(4 \times \frac{1}{4}\right) \times \frac{2}{3}$.

So I get $\frac{2}{3} \div 4 = \frac{1}{4} \times \frac{2}{3} = \frac{2}{12} = \frac{1}{6}$.

(c) $\frac{2}{3} + \frac{3}{4} = \frac{17}{12}$ because $\frac{2}{3} = \frac{8}{12}$ and $\frac{3}{4} = \frac{9}{12}$ and by counting the twelfths I get $\frac{17}{12}$. (The student might do this on the number line.)

(d) $\frac{6}{5} = \frac{42}{35}$ because $\frac{6}{5} = \frac{6}{5} \times 1 = \frac{6}{5} \times \frac{7}{7} = \frac{42}{35}$. (Here also the student may do this on the number line.)

(e) $\frac{2}{3} \times \frac{4}{7} = \frac{8}{21}$ because $\frac{1}{7}$ of $\frac{1}{3} = \frac{1}{21}$ so $\frac{1}{7}$ of $\frac{2}{3} = \frac{2}{21}$ and 4 times that is $\frac{8}{21}$. (The student may do this on the number line too.)

Having reviewed the computational rules, you should ask individual students to compute mentally the exercises from Problems 6-11(a). Expect quick responses. If an answer is given correctly and there are no puzzled looks, proceed. Try to make sure that

each student sees any short cut or short cuts that permit rapid calculation. Reserve the harder problems for the better students. Encourage short cuts. With some luck it may be possible to cover most of the mental arithmetic problems before the end of the hour.

Answers to Problems 6-11(a), page 6-38

1. $\frac{25}{33}$
2. $\frac{22}{15}$
3. $\frac{5}{6}$
4. $\frac{8}{9}$
5. $\frac{2}{9}$
6. $\frac{5}{18}$
7. $\frac{3}{7}$
8. $\frac{2}{3}$
9. 0
10. 0
11. $\frac{3}{7}$
12. 12
13. $\frac{80}{3}$
14. 15
15. 15
16. $\frac{3}{80}$
17. $\frac{3}{4}$
18. $\frac{1}{2}$
19. $\frac{xs}{yt}$
20. $\frac{2ac}{br}$
21. $\frac{19}{18}$
22. $\frac{11}{18}$
23. $7\frac{1}{2}$
24. $\frac{2}{3}$
25. $\frac{1}{6}$
26. 0
27. $\frac{12}{35}$
28. $\frac{y + x}{xy}$
29. $\frac{1}{24}$
30. $\frac{3}{7}$
31. $\frac{1}{b}$
32. $1\frac{1}{8}$
33. $\frac{5}{6}$
34. $\frac{5a}{6}$
35. $2\frac{1}{12}$
36. $1\frac{11}{12}$
37. $\frac{1}{30}$
38. $1\frac{1}{6}$
39. $7\frac{1}{6}$
40. $6\frac{5}{6}$
41. $\frac{1}{2}$
42. $1\frac{3}{4}$
43. 4
44. $\frac{6}{7}$
45. $\frac{9}{17}$
46. $\frac{3}{2}$
47. $\frac{2}{3}$
48. 1
49. 12
50. $\frac{xb}{ya}$
51. 0
52. $\frac{1}{7}$
53. $\frac{1}{7}$
54. $\frac{4}{7}$
55. $\frac{4}{3}$
56. 1
57. $\frac{1}{2}$

58. $2\frac{3}{8}$

59. $\frac{3}{4}$

60. $\frac{3}{4}$

61. 1

62. 0

63. 0

In Problems 6-11(b) simple equations are to be solved by using the definitions of addition and subtraction. The teacher should review briefly the definitions and illustrate them with an example or two. After the students have worked some of the problems, have one or two put on the blackboard by students, and explained. Be sure that each student can explain how he uses the definition. Thus, to solve the equation

$$\frac{7}{2}x = \frac{14}{13}$$

a student might say, "From the definition of division I know that

$$x = \frac{14}{13} \div \frac{7}{2} = \frac{14}{13} \times \frac{2}{7} = \frac{4}{13}\text{."}$$

Some students may want to use algebraic techniques. For example, to solve

$$\frac{7}{2}x = \frac{14}{13}$$

a student might say, "I multiply the number on both sides of the equation by $\frac{2}{7}$ and get

$$\frac{2}{7} \times (\frac{7}{2}x) = \frac{2}{7} \times \frac{14}{13}$$
$$(\frac{2}{7} \times \frac{7}{2})\, x = \frac{14}{13},$$

so $x = \frac{4}{13}$." If he does this, compliment him, and ask if he sees how the definition of division applies too.

Answers to Problems 6-11(b), page 6-39

1. (a) $\frac{1}{2}$ (b) $\frac{1}{8}$ (c) $\frac{20}{17}$ (d) $\frac{803}{243}$ (e) $\frac{37}{35}$ (f) $\frac{23}{4}$ (g) $\frac{103}{60}$ (h) 1 (i) $\frac{13}{30}$ (j) $\frac{319}{45}$ (k) $\frac{1}{60}$ (l) $\frac{c - a}{b}$

(m) $\frac{3b}{a}$

(n) $\frac{89}{48}$

(o) $\frac{1}{10}$

(p) $\frac{a}{5}$

(q) $\frac{5}{12}$

(r) $\frac{25}{12}$

(s) $\frac{35}{6}$

(t) $\frac{112}{9}$

(u) $\frac{16}{27}$

(v) $\frac{11}{12}$

(w) $\frac{1}{6}$

(x) $\frac{1}{4}$

2. (a) $\{1\}$.
   (b) {all rational numbers}.
   (c) {all rational numbers}.
   (d) The empty set, $\{\ \}$.
   (e) The empty set, $\{\ \}$.
   (f) {all rational numbers different from 0}.

Problems 6-11(c) are written drill problems. The teacher should circulate in the class and keep asking why the computing methods work. Some students can carry out the entire calculation mentally. Encourage this. Others may leave out one or more steps. Tell the student that he is working for speed and accuracy and need not have written work showing every step. For example, a student might write

$$17\frac{1}{4} \div \left(\frac{13}{6} + \frac{5}{3}\right) = \frac{69}{4} \times \frac{6}{23} = \frac{9}{2},$$

which is satisfactory.

Another equally satisfactory solution to the same problem is

$$17\frac{1}{4} \div \left(\frac{13}{6} + \frac{5}{3}\right) = \frac{\frac{69}{4}}{\frac{13}{6} + \frac{5}{3}} \times \frac{12}{12}$$

Where 12 is the LCM of all the denominators.

$$= \frac{69 \times 3}{26 + 20} = \frac{23 \times 3 \times 3}{23 \times 2}$$

$$= \frac{9}{2}$$

For future work it is desirable to encourage students to use the fractional symbol for division; that is, $\frac{a}{b}$ in place of $a \div b$.

Answers to Problems 6-11(c), page 6-40

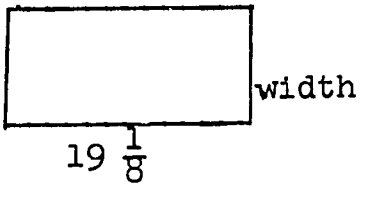
1. $\frac{19}{4}$	10. $\frac{5}{9}$	19. $\frac{1}{12}$
2. $\frac{19}{4}$	11. $\frac{55}{8}$	20. $\frac{7}{60}$
3. $\frac{22}{15}$	12. $\frac{3}{4}$	21. $\frac{113}{30}$
4. $\frac{22}{15}$	13. 0	22. $\frac{26}{15}$
5. 0	14. $\frac{83}{24}$	23. $\frac{1}{4}$
6. $\frac{32}{15}$	15. $\frac{1}{7}$	24. $\frac{2}{3}$
7. 5	16. 0	25. $\frac{1}{2}$
8. $\frac{1}{12}$	17. $\frac{11}{15}$	26. $\frac{2}{3}$
9. $\frac{8}{9}$	18. $\frac{145}{24}$	

Finally, Problems 6-11(d) give applications of the four fundamental operations to many of the standard problems. Be sure that the student, in writing, explains any symbols he may introduce. The work of each student should be clear to the reader when the problem is at hand. Not all students need arrange their work the same way. Consider the following problems and solutions.

Problem: A rectangle has a perimeter of $48\frac{1}{2}$ inches. Its length is $19\frac{1}{8}$ inches. Find the width.

Student 1. the width in inches

$$= \frac{1}{2} \times (48\frac{1}{2} - 2 \times 19\frac{1}{8})$$
$$= \frac{1}{2} \times (\frac{97}{2} - \frac{153}{4})$$
$$= \frac{1}{2} \times \frac{41}{4} = \frac{41}{8} = 5\frac{1}{8}$$

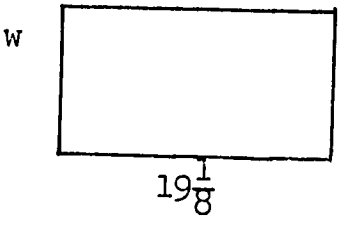


Student 2. If w is the width in inches then

$$2w \div 2 \times 19\tfrac{1}{8} = 48\tfrac{1}{2}$$

$$2w = \frac{97}{2} - \frac{153}{4} = \frac{41}{4}$$

$$w = \frac{41}{8} = 5\tfrac{1}{8}$$



Hence, the rectangle is $5\frac{1}{8}$ inches wide.

The student should always draw a figure when appropriate.

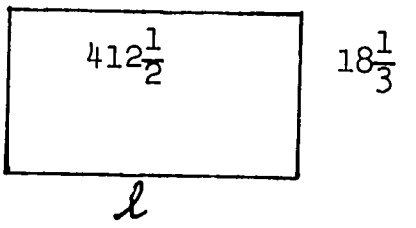
Problem: The area of a rectangle is $412\frac{1}{2}$ square feet and its width is 18'4". How long is it?

Student 3. If the length is ℓ feet, then

$$\ell \times (18\tfrac{1}{3}) = 412\tfrac{1}{2}$$

$$\ell \times \frac{55}{3} = \frac{825}{2}$$

$$\ell = \frac{825}{2} \div \frac{55}{3} = \frac{\overset{15}{\cancel{825}}}{2} \times \frac{3}{\underset{1}{\cancel{55}}} = \frac{45}{2}$$



The length is 22'6".

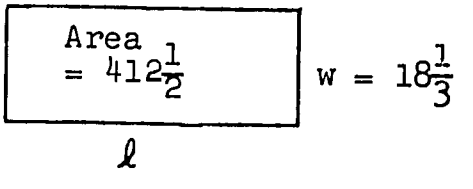
Student 4. If the rectangle is ℓ feet long, then

$$\ell = (412\tfrac{1}{2}) \div (18\tfrac{1}{3}) = \frac{825}{2} \div \frac{55}{3} = \frac{825}{2} \times \frac{3}{55} = \frac{45}{2}$$

Observe that in both solutions there is an "if - then" statement. Another way of wording would be to say, "Let ℓ be the length of the rectangle in feet. Then etc." It is highly desirable for the student to become accustomed to this kind of language which is necessary for clear communication. As soon as one uses the letters to represent numbers, then one is bound, in all honesty, to say explicitly what each letter represents. Later the teacher may wish to allow this understanding to be described by a figure. In the above problem the arithmetic would be clear enough from the picture at the right.

Then

$$\ell = (412\tfrac{1}{2}) \div (18\tfrac{1}{3}) = \frac{45}{2}$$



But now one needs to interpret this number as the measure of the longer side <u>in feet</u> by saying, "The rectangle is $22\frac{1}{2}$ feet long."

Finally the teacher should impress upon the students that in many of the problems some quantity is involved which is given, in a certain way, by a <u>formula</u>. For example: Area of a rectangle = length × width = $\ell \times w$, and perimeter of a rectangle = $2\ell + 2w$. Once one knows these formulas, the problem is solvable by perhaps several methods. For example, if the perimeter p is given, along with the width w, of a rectangle, then one might write either

$p = 2\ell + 2w$ so $2\ell = p - 2w$, and $\ell = (p - 2w) \div 2$ or
$p = 2\ell + 2w$ so $\frac{p}{2} = \ell + w$, and $\ell = \frac{p}{2} - w$.

Both are satisfactory and no set method should be prescribed.

<u>Answers to Problems 6-11(d), page</u>

1. If p is the perimeter in inches, then $p = 2 \times (12\frac{1}{3} + 10\frac{1}{2}) = 45\frac{2}{3}$.
2. If w is the width in inches then $38\frac{1}{2} = 2 \times (13\frac{1}{2} + w)$. Therefore, $38\frac{1}{2} = 27 + 2w$, so $11\frac{1}{2} = 2w$, and $w = 5\frac{3}{4}$.
3. $2 \times (14\frac{1}{12} + w) = 52\frac{2}{3}$, so $14\frac{1}{12} + w = 26\frac{1}{3}$, and $w = 26\frac{1}{3} - 14\frac{1}{12} = 12\frac{1}{4}$. One cannot say anything about the area because lengths of other sides are not known.
4. If d is the distance travelled, in miles, then $d = \frac{3}{2} \times 38\frac{1}{2} = \frac{231}{4} = 57\frac{3}{4}$.
5. If r is the speed, in miles per hour, then $128\frac{1}{2} = (2\frac{2}{3}) \times r$. Therefore, $r = \frac{257}{2} \div \frac{8}{3} = 48\frac{3}{16}$.
6. If r is the speed, in miles per hour, then $9.6 = r \times \frac{1}{5}$, so $r = 48$.
7. If the time of travel is t hours, then $30\frac{1}{3} = (59\frac{1}{2}) \times t$. Therefore $t = 30\frac{1}{3} \div 59\frac{1}{2} = \frac{26}{51}$. The time of travel, in minutes, is $\frac{26}{51} \times 60 = 30\frac{10}{17}$.

8. Cost per item, in shillings $= 22\frac{2}{3} \div 17 = \frac{4}{3}$ s = 1s. 4d. Twenty items cost £1.6s.8d.

9. If ℓ is the length of the third side, in inches, then $\ell + 13\frac{1}{2} + 17\frac{1}{3} = 52\frac{1}{6}$ and $\ell = 21\frac{1}{3}$.

10. If x is the other number then $(7\frac{3}{4})\,x = 27\frac{1}{8}$, so $x = \frac{217}{8} \div \frac{31}{4} = \frac{7}{2}$

11. If x is the number to be added, then $\frac{3}{4} + x = \frac{8}{9}$, $x = \frac{5}{36}$.

12. $b + g$, $\frac{b}{b+g}$, $\frac{27}{38}$.

13. $\frac{1}{8}$

14. 312. It is also interesting to use a little algebra on this problem. Let x be the number of pages in the book. Then $\frac{1}{4}x = 78$ and $x = 312$. Here is another equivalent equation: $\frac{3}{4}x + 78 = x$.

15. Length in feet, $= 168\frac{3}{4} \div 12\frac{1}{2} = 13\frac{1}{2}$.

16. $36 \div \frac{4}{5} = 45$.

17. Cost per yard, in shillings, $= 46\frac{2}{3} \div 2\frac{1}{2} = 18\frac{2}{3}$, so cost per yard = 18s. 8d.

18. Width, in yards, $= 6 \times 4840 \div 120 = 242$. The perimeter in yards, $= 724$.

19. Area of the path in square feet, $= 39\frac{1}{2} \times 29\frac{1}{2} - 30\frac{1}{2} \times 20\frac{1}{2} = 540$. Or, an easier way: area $= 9 \times 30\frac{1}{2} + 9 \times 20\frac{1}{2} + 9 \times 9 = 540$.

20. (a) 100 metres; (b) 4500 metres or $4\frac{1}{2}$ kilometres, (c) 3200 seconds or $53\frac{1}{3}$ minutes.

21. Difference of the perimeters, in inches $= (6\frac{1}{4} + 7\frac{1}{8} + 9\frac{1}{16}) - (8\frac{1}{8} + 5\frac{1}{4} + 6\frac{1}{2}) = 1 - 1 + 3\frac{1}{16} - \frac{1}{2} = 2\frac{9}{16}$.

22. $\frac{1}{4}$.

23. $\frac{2}{3} \times 34\frac{1}{2} = 23$ s. = £1.3 s.

24. If he had x shillings at first, then at the end of the second week he had spent $\frac{1}{10}x + \frac{1}{6}\left(\frac{9}{10}x\right) = \frac{1}{4}x$. Therefore $\frac{3}{4}x = 42$, in shillings. Hence $x = 56$ s. = £2.16s.

25. $6\frac{1}{3} + 5\frac{1}{4} + 10\frac{1}{6} = 21\frac{3}{4}$ yds.

26. £1512

27. $63 \div 1\frac{3}{4} = 36$.

28. $(522 \div 9) \times 6\frac{1}{4} = 362\frac{1}{2}$ s. = £18. 2s. 6d.

29. If x is the other number, then $\frac{9}{4}x = 7\frac{1}{2}$, so $x = \frac{10}{3}$.

6-12. Ratio and Proportion.

The objectives of this section are two in number. One is to introduce the student to the language. The second is to justify our use of the term "rational number."

It is vitally necessary that all students know the definitions of ratio and "x is proportional to y." Once the definitions are clear the solution of the problems becomes an easy exercise in arithmetic. Nothing beyond the definitions needs to be memorized.

The first set of problems should be done orally in class although you may allow students to use paper and pencil as aids to computation.

Answers to Problems 6-12(a), page

1. $\frac{1}{2}$ 2. 0 3. 0 4. $\frac{5}{7}$ 5. $\frac{11}{15}$

6. $\frac{34}{13}$ 7. $\frac{17}{29}$ 8. $\frac{1}{6}$ 9. $\frac{x+1}{y}$ 10. $\frac{x+y}{a+b}$. Note that because $a \neq 0$ then $a + b \neq 0$ no matter what b is. Therefore we can divide by $a + b$. 11. $\frac{3}{2}$ 12. $\frac{y}{x}$

Now that the meaning of ratio is clear to the class, ask the students what the following ratios are: (Write on the board)

The ratio of 3 to 2 = ?
The ratio of 5 to 7 = ?
The ratio of 26 to 52 = ?
The ratio of 26 to 73 = ?

Then ask the class what kind of number we get if we take the ratio of two whole numbers (the second not zero). If someone answers "rational numbers", you have gotten the desired response. If this does not occur, you can point out that these are "ratio-numbers" and so we call them rational numbers. Mention, in so many words, that rational numbers can be expressed as ratios of whole numbers.

Should anyone in the class remark that the ratio of any two numbers (that the class knows about) is a rational number, then you should mention that later on in the term (and again next year) they will meet numbers which cannot be expressed as ratios of whole numbers. If no one brings up this question, you should not mention it. Irrational (that is non-rational) numbers are mentioned briefly in Chapter 8, on decimals. At that time the question of the existence of numbers other than rational numbers will be more natural.

Now begin with the discussion of the standard problems on proportion. In the text we have defined the meanings of is proportional to and constant of proportionality by giving examples. Observe that we have, in this introduction, not given a formal definition of proportion as it is not necessary for the applications here. Later on, of course, the student must know that:

A proportion is a statement that two ratios are equal.
Thus,

$$\frac{a}{b} = \frac{c}{d}, \quad \frac{2}{3} = \frac{4}{6}, \quad \frac{7}{19} = \frac{21}{57}, \text{ and } \quad \frac{a}{b} = \frac{ax}{bx}$$

are all proportions.

All the student needs is to understand the phrase is proportional to. A sound, formal definition is difficult to give at this stage because the students have not yet met the concept of function. If y is proportional to x then y is a function of x of a very special kind. In other words y depends on x in the very special way: $y = kx$, where k is some fixed number. Therefore, at this time, do not push these problems too hard. However, all students should be able to do

the first 5 or 6 problems.

Answers to Problems 6-12(b), page 6-46

1. $2\frac{1}{2}$ miles. 10 miles. $d = \frac{5}{2}t$. 2. Rate per year $= 7\frac{1}{5} \div 120 = \frac{6}{100}$. Interest $= \frac{6Pt}{100}$. 3. If t is the time, in years, then $243 = 1200 \times (4\frac{1}{2} \div 100)\ t = 54t$. So $t = 4\frac{1}{2}$. $I = \frac{4\frac{1}{2}}{100} Pt = \frac{9}{200} Pt$.
4. Cost, in shillings $= \frac{24}{5}x$. Cost, in pounds, $= \frac{6}{25}x$.
5. $\frac{x}{y} = \frac{12}{3} = 4$. So $x = 4y$ and $y = \frac{1}{4}x$. 6. $a = 2$. 7. 168 lbs.
8. 40,000 seconds $= \frac{4000}{6}$ minutes $= 11\frac{1}{9}$ hours. 9. $x = \frac{6}{y}$.
10. $x = \frac{5}{2}y$ and $y = \frac{2}{3}z$. Therefore $x = \frac{5}{3}z$. So x is proportional to z. 11. 30 miles. 12. 2,000,000 inches $= \frac{3125}{99}$ miles or between 31 and 32 miles.

6-13. Inequalities.

In this section we examine the order relations "is less than" and "is greater than." The approach is entirely intuitive. The student already has a good idea as to which of two numbers is the smaller. For example, he will recognize that $\frac{5}{13}$ is less than $\frac{7}{13}$.

The section begins by giving a method for deciding which of two rational numbers is the smaller. The idea is simple. When the two numbers are represented as fractions with a common denominator, then the fraction with the smaller numerator represents the smaller number. You might begin the discussion by asking the class which one of each of the following pairs of numbers is the smaller.

$\frac{7}{8}$ and $\frac{6}{8}$

$\frac{9}{16}$ and $\frac{12}{16}$

$\frac{3}{4}$ and $\frac{4}{3}$ (A natural answer is that $\frac{3}{4}$ is less than 1 while $\frac{4}{3}$ is greater than 1)

$\frac{4}{5}$ and $\frac{5}{6}$

The last one may cause some discussion. Some students may suggest using decimals. Then $\frac{4}{5} = 0.8$ and $\frac{5}{6} = 0.8333\cdots$ and one can see that $\frac{5}{6}$ is the larger. If this happens, compliment the student, and ask if he sees another way. Try to get the class to discover that

$$\frac{4}{5} = \frac{24}{30} \text{ and } \frac{5}{6} = \frac{25}{30}$$

and so

$$\frac{4}{5} < \frac{5}{6} . \quad (\text{Or, } \frac{5}{6} > \frac{4}{5}).$$

In the course of this discussion someone may discover that we could also get fractions with equal numerators. In this example we would have

$$\frac{4}{5} = \frac{20}{25} \text{ and } \frac{5}{6} = \frac{20}{24}$$

Therefore

$$\frac{4}{5} < \frac{5}{6} .$$

When all see how to decide which number is the smaller, take up the illustration of this idea on the number line. Note that when the two fractions have a common denominator, the fraction with the smaller numerator must correspond to a point on the line to the left of the point corresponding to the other number. Thus, to plot the point corresponding to $\frac{39}{15}$ we take 39 "steps" of length $\frac{1}{15}$, while for $\frac{40}{15}$ we must take 40 steps.

Answers to Problems 6-13(a), page 6-49

1. (a) $\frac{7}{8} > \frac{3}{4}$ (b) $\frac{39}{23} > \frac{38}{23}$ (c) $\frac{25}{8} < \frac{25}{7}$ (d) $\frac{5}{4} > \frac{1}{2}$

(e) $\frac{2}{3} > \frac{7}{12}$ (f) $\frac{15}{50} = \frac{9}{30}$ (g) $\frac{15}{8} > \frac{37}{20}$ (h) $8 > \frac{500}{63}$

(i) $\frac{12}{17} > \frac{24}{35}$ (j) $\frac{4}{77} < \frac{6}{113}$ (k) $\frac{5}{3} > \frac{7}{5}$ (l) $\frac{2000}{7} < \frac{573}{2}$

(m) $\frac{23}{9} > 0$ (n) $\frac{200}{13} < \frac{500}{31}$ (o) $1 > \frac{876}{943}$.

2. Yes.

3.(a) $\frac{5}{4} < \frac{4}{3} < \frac{3}{2}$ (b) $\frac{7}{4} < \frac{11}{6} < \frac{17}{9}$ (c) $\frac{5}{11} < \frac{5}{9} < \frac{10}{17}$

(d) $4 < \frac{33}{8} < \frac{29}{7}$

4.(a) a. (b) d because if a = 0 the numbers are equal. Otherwise a would apply.

(c) d because x = 0 gives equality. (d) d.

(e) B. Note that $a \neq 0$. (f) b (g) d (h) d (i) c.

The next idea to consider is the so-called <u>Property of Trichotomy</u>, which says that given any two rational numbers, either they are equal or one is bigger than the other. That is, there are just three possibilities. The array of sentences in the text should suggest this property to the students. Try to get them to find this idea before you call their attention to it in the box on page 6-50

The next important property of inequalities is that of <u>transitivity</u>, -- which is given in the box on page 6-51. The students are, of course, familiar with the idea. You might try to get them to formulate it by means of some examples. If you do, you might begin with some obvious ones, such as

$$1\tfrac{2}{3} < 3\tfrac{1}{2} \quad \text{and} \quad 3\tfrac{1}{2} < 1762\tfrac{1}{2}$$

<u>Answers to Problems 6-13(b), page 6-51</u>

1. Mary is shorter than Ngozi. 2. Nothing on the basis of what is given. 3.(a) $\frac{37}{8} < \frac{27}{5}$ (b) $\frac{12}{17} < \frac{14}{13}$ (c) $\frac{125}{42} < \frac{130}{43}$

The last property of the relation "<" is that given in the box on page 6-52. There are other important facts about inequalities. You may wish to get the students to discover the following:

(☆) If $a < b$ then $a - c < b - c$

whenever the subtractions are possible. The teacher should know that (☆) remains true when we have negative numbers too.

Another important inequality which you might choose to discuss with the students is:

(☆☆) If $a < b$ and $c \neq 0$, then $ac < bc$. This has a peculiarity which should be noted, namely that c must be a positive number. Of course, we have only positive numbers and zero now, but <u>later on</u> the student must recognize that he must require that $c > 0$ rather than $c \neq 0$.

Of course, both (☆) and (☆☆) should be approached by way of numerical examples.

<u>Answers to Problems 6-13(c), page 6-52</u>

1. (a), (e), (h)

<u>Answers to Problems 6-13(d), page 6-53</u>

1. (a) >, (b) < Compare with 5, (c) >, (d) <, (e) =, (f) = .

2. Yes, it is true. 3. Yes. 4. $x + z > y + z$.

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TEACHERS' GUIDE VOLUME THREE

MATHEMATICS SECONDARY ONE

Introduction to Algebra and to the Language of Geometry

African Education Study
Educational Services Incorporated
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FIRST YEAR SECONDARY TEACHERS' GUIDE

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FIRST YEAR SECONDARY TEACHERS' GUIDE

CHAPTER 7

NEGATIVE NUMBERS

Commentary.

In this chapter our number system is extended to include the negative numbers. The first eight sections are devoted to the integers only and the last section to the negative rational numbers. A more detailed study of the complete set of rational numbers will be made in the second year.

The negative numbers are introduced as numbers needed to satisfy certain open sentences which have no solution in the numbers of arithmetic. The name "opposites" is given to any two numbers whose sum is zero.

We have assumed, without any attempt to prove them at this level, that the commutative properties of addition and multiplication, the associative properties of addition and multiplication, the property of one, properties of zero, and the distributive property hold for the negative numbers as well as for the numbers of arithmetic.

Addition and multiplication of integers are established using the basic properties. Subtraction and division are described in terms of addition and multiplication, respectively. We used the number line as a physical device for interpreting the results obtained, wherever possible. Another physical means of interpretation used here is that of gains and losses.

A more precise statement of the meaning of a is less than b is given in this chapter.

7-1. The Negative Integers.

At the end of this section the students should be able to

(a) Find the truth sets of open sentences such as $x + 7 = 1$, $x + 10 = 0$, $x + 25 = 2$, etc.

(b) Associate negative integers with points on the number line.

(c) Demonstrate such sentences as $-4 + 1 = -3$ on the number line and also interpret these sentences in terms of gains and losses.

(d) Name the opposite of any integer.

You may begin by asking students to find a number which will make the open sentence $x + 6 = 5$ true.

This need for new numbers should be presented to the students from three different viewpoints, as discussed in the text. We suggest you consider the following open sentences in the order given here:

$$x + 6 = 8$$
$$x + 6 = 7$$
$$x + 6 = 6$$
$$\text{and finally, } x + 6 = 5$$

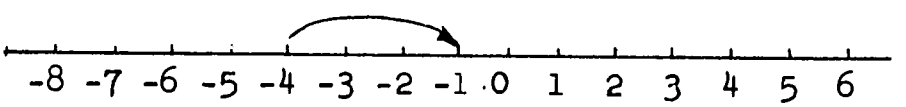
Consider other open sentences such as $x + 9 = 0$, $x + 4 = 2$, etc., and ask students to invent numbers which make them true and associate these new numbers with points on the number line.

Answers to Problems 7-1(a), page 7-3

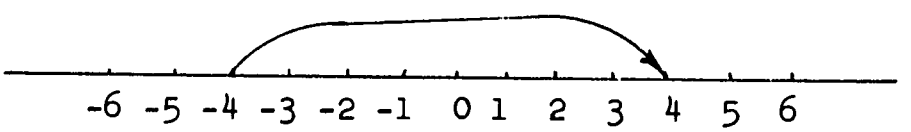
1. (a) -7
   (b) -1
   (c) -5
   (d) -4
   (e) -98
   (f) -60
   (g) -3
   (h) -5

2.

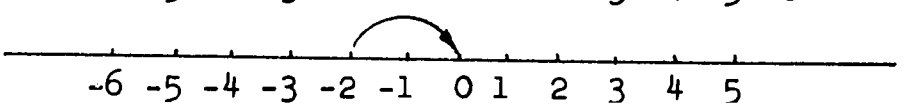
(a)



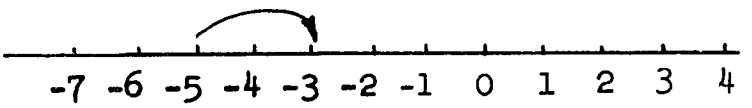
(b)



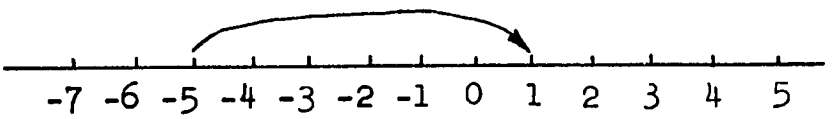
(c)



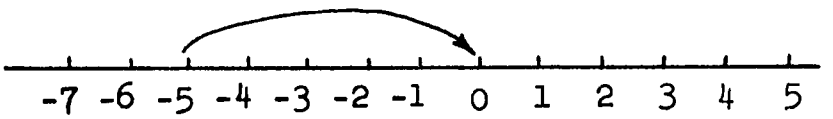
(d)



(e)



(f)



3. (a) You must have been 4 shillings in debt if after earning 3 shillings you are still 1 shilling in debt.
   (b) We must have been 4 shillings in debt if after earning 8 shillings we have only 4 shillings.
   (c) We must have been 2 shillings in debt if after earning 2 shillings we have no money.
   (d), (e), and (f) can be done in the same way.
4. (a) False (b) True
   (c) True (d) False
5. (a) -2 (b) 1 (c) -6 (d) -2

The idea of opposites is introduced in this section. The students must learn to name the <u>opposite</u> of any integer by means of the definition. The notation "-x" is used to denote "opposite of x." "-x" should <u>not</u> be read as "negative x," because students may erroneously think that -x is always a negative number.

<u>Answers to Problems 7-1(b), page 7-4</u>

1. (a) The opposite of 12 is -12.
   (b) 8
   (c) 93
   (d) -47
   (e) -1
   (f) 1
   (g) -7000
   (h) 0
   (i) 2

2. The integers between -5 and 3 are -4, -3, -2, -1, 0, 1, and 2.
3. Two numbers whose sum is zero are called opposites.
4. Yes.
5. Yes.
6. Yes. The opposite of zero is itself.

Answers to Problems 7-1(c), page 7-5

1. (a) $-a = 5$
   (b) $-y = 4$
   (c) $-x = 9$
   (d) $-b = -496$
   (e) $-m = 0$
2. (a) $-(-7) = 7$
   "-(-7)" means the opposite of negative 7.
   (b) $-(-93) = 93$
   (c) $-(-1) = 1$
   (d) $-(-x) = x$
   -(-x) should be read as "the opposite of the opposite of x." If x is a positive integer then -(-x) is a positive integer because the opposite of a positive integer is a negative integer and the opposite of this negative integer would be a positive integer. If x is a negative integer then -(-x) is a negative integer.

7-2. Basic Properties.

The purpose of this section is to have the students recognize the basic properties and distinguish one property from another.

Tell the students that all the basic properties of addition and multiplication with whole numbers still hold true for the negative numbers but draw as much as possible from the students what these properties are.

Stress the property of opposites, namely, $a + (-a) = 0$, as this property is very important later on.

At the end of the discussion of the problems in the text, students should be able to generalize the basic properties for any integers a, b, and c.

Answers to Problems 7-2(a) page 7-5

1.	(a) -3	Commutative property of addition
	(b) -7	Commutative property of addition
	(c) -19	Addition property of zero
	(d) -5	Associative property of addition
2.	(a) 5	Commutative property of multiplication
	(b) -6	Associative property of multiplication
	(c) -10	Multiplication property of 1
	(d) 0	Multiplication property of 0
	(e) -4	Distributive property
	(f) -8	Distributive property
	(g) -6	Distributive property

7-3. Addition.

Understandings :

1. Adding two integers using the basic properties.
2. Interpreting sums on the number line.
3. Making generalizations about conditions under which the sum of two integers is a positive number or a negative number.

The desire to retain all the basic properties dictates to us how to add integers.

Begin with the sum of a positive integer and a negative integer, as this type is easier for the students.

It might be desirable (depending on the class) as a preliminary exercise to have students write positive integers as the sum of two positive integers, one of them being given. For example: ask them to write 6 as the sum of two positive integers

(a) one of which is 1
(b) one of which is 3
(c) one of which is 4

The answers are: (a) 1 + 5; (b) 3 + 3; (c) 4 + 2. The point of this is that in an expression such as 6 +(-4), it would be helpful to have a 4 to combine with the (-4). Hence, writing 6 as 2 + 4 is helpful.

You may ask the students if there are any positive and negative integers whose sums they already know. Their response should be that,

from the property of opposites, sums such as 6 + (-6), 2 + (-2), 30 + (-30), etc., are all zero.

Ask students: What is the sum 6 + (-4)? Let us use the basic properties to find the sum 6 + (-4).

We know from the property of opposites that 4 + (-4) = 0. Looking ahead, we see that if the number 6 is written as 2 + 4, we can regroup the three numbers, using the associative property of addition, so that the 4 and -4 are added first and then the 2.

6 + (-4)	= (2 + 4) + (-4)	2 + 4 is another name for 6
	= 2 + [4 + (-4)]	Associative Property of Addition
	= 2 + 0	Property of Opposites
	= 2	Addition Property of Zero

Interpret the sum 6 + (-4) on the number line and in terms of gains and losses, as discussed in the students' text.

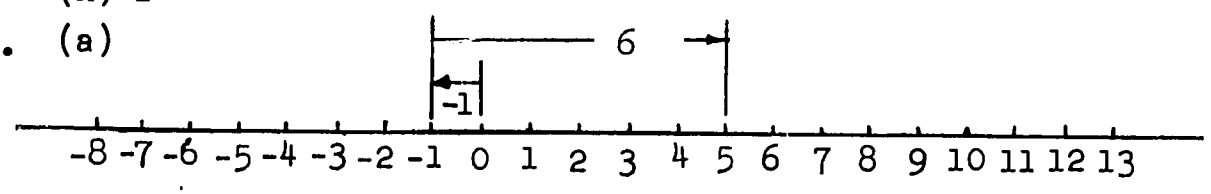
Sums such as -5 + 2 will be considered after we can add negative integers. Notice the positive integer 2 cannot be written as the sum of two positive integers, one of which is the opposite of -5.

Answers to Problems 7-3(a), page 7-7

1. (a) -1 + 6 = -1 + (1 + 5) 1 + 5 is another name for 6
   = (-1 + 1) + 5 Associative Property of Addition
   = 0 + 5 The Property of Opposites
   = 5 Addition Property of zero

   (b) 2 (we write 4 as 2 + 2)
   (c) 2
   (d) 3
   (e) 3
   (f) 7
   (g) 9
   (h) 1

2. (a)



Start at 0 and move 1 unit to the left and then 6 units to the right. The coordinate 5 of our final position is the sum -1 + 6.

(b), (c), (d), (e), (f), (g), and (h) should be done in a similar manner.

3. (a) A loss of 1 shilling followed by a gain of 6 shillings leaves a gain of 5 shillings.
   (b) A gain of 4 shillings followed by a loss of 2 shillings leaves a gain of 2 shillings.
   (c) A gain of 3 shillings followed by a loss of 1 shilling leaves a gain of 2 shillings.
   (d) A loss of 5 shillings followed by a gain of 8 shillings leaves a gain of 3 shillings.

   (e), (f), (g), and (h) can be interpreted in a similar way.

4. The students should observe that the number 2 cannot be written as the sum of two positive integers, one of which is the opposite of -5. However, a better student might write the number 2 as [5 + (-3)] and do this:

$$\begin{aligned} -5 + 2 &= -5 + [5 + (-3)] \\ &= (-5 + 5) + (-3) \\ &= 0 + (-3) \\ &= -3 \end{aligned}$$

   If this occurs, congratulate the student on his good work. Then explain that there will be another way to do this type of problem after we know how to add two negative integers.

Consider the sum of two negative integers. According to the basic properties, what must the sum (-4) + (-3) be?

You may ask the students the following:

(1) Is there anything interesting we know about the number -3?

(2) Is there anything interesting we know about the number -4?

The students should say: We know that -4 + 4 = 0 and -3 + 3 = 0 from the property of opposites.

Write the following sentence on the board:

$$[(-4) + (-3)] + (\quad\quad) = 0$$

Ask the students the following question:

If we could complete this sentence, what would we know about the sum $(-4) + (-3)$?

The reply (perhaps with some guidance if necessary) would be:

Then $(-4) + (-3)$ must be the opposite of the number in the bracket.

The next question is, "What number can we put in the bracket to make this sentence true?" If this is difficult for the students to see, suggest to them that we try $4 + 3$ since $-4 + 4 = 0$ and $-3 + 3 = 0$. Is $[(-4) + (-3)] + (4 + 3) = 0$ a true statement?

We can use the basic properties to show it is true. Students should give the reasons for each step.

$$
\begin{aligned}
[(-4) + (-3)] + (4 + 3) &= [(-4) + (-3)] + (3 + 4) && \text{Commutative Property of Addition} \\
&= \{(-4) + [(-3) + 3]\} + 4 && \text{Associative Property of Addition} \\
&= ((-4) + 0) + 4 && \text{Property of Opposites} \\
&= -4 + 4 && \text{Addition Property of Zero} \\
&= 0 && \text{Property of Opposites}
\end{aligned}
$$

Since it is true that $[(-4) + (-3)] + (4 + 3) = 0$, we know that $(-4) + (-3)$ must be the opposite of $(4 + 3)$, or

$$(-4) + (-3) = -(4 + 3) = -7$$

Interpret the sum on the number line and in terms of gains and losses.

During class discussions, students should observe that the sum of two negative integers is a negative integer.

Answers to Problems 7-3(b), page 7-9

1. (a) Is $[(-6) + (-2)] + (6 + 2) = 0$ a true statement?

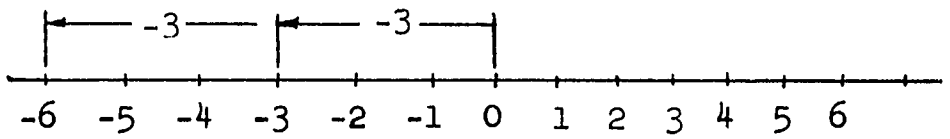
$$
\begin{aligned}
[(-6) + (-2)] + (6 + 2) &= [(-6) + (-2)] + (2 + 6) \\
&\quad \text{Commutative Property of Addition} \\
&= \{(-6) + [(-2) + 2]\} + 6 \\
&\quad \text{Associative Property of Addition} \\
&= (-6) + 0 + 6 \\
&\quad \text{Property of Opposites} \\
&= (-6) + 6 \quad \text{Addition Property of Zero} \\
&= 0 \quad \text{Property of Opposites}
\end{aligned}
$$

Since it is true $[(-6) + (-2)] + (6 + 2) = 0$, then $(-6) + (-2)$ must be the opposite of $(6 + 2)$ or

$$(-6) + (-2) = -(6 + 2) = -8$$

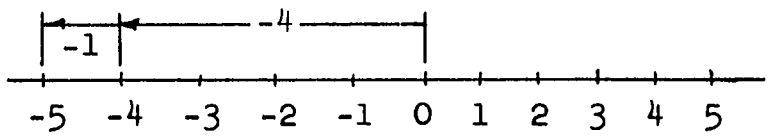
(b) -7
(c) -13
(d) -17
(e) -31
(f) -7
(g) -12

2. (a)

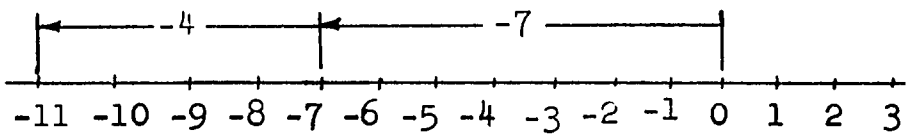


(to add -3, step 3 units to the left)

(b)



(c)



3. $-6 + (-9)$
4. $-10 + (-5)$
5. (a) $-2 + (-5)$
   (b) $-3 + (-4)$
   (c) $-6 + (-1)$
6. $-997 + (-3)$

We return now to the sum of a positive integer and a negative integer. Sums such as $-5 + 2$ are easily done now that we can write a negative integer (except -1) as the sum of two integers.

In this example we would write the number -5 as $(-3) + (-2)$ and follow the same procedure as when adding a negative and positive integer in an earlier part of this section.

Students should observe that the sum of a positive integer and negative integer is sometimes a positive number and sometimes a negative number. Have them think in terms of gains and losses and in terms of the number line to decide when the sum is positive and when it is negative.

Answers to Problems 7-3(c), page 7-10

1. (a)	$-7 + 3 = [(-4) + (-3)] + 3$	$(-4) + (-3)$ is another name for -7.
	$= (-4) + [(-3) + 3]$	Associative Property of Addition.
	$= (-4) + 0$	The Property of Opposites.
	$= -4$	Addition Property of Zero.

(b) -7 (c) -10 (d) -9 (e) -2 (f) -11 (g) -3 (h) -99

Answers to Problems 7-3(d), page 7-10

1.	(a)	Positive	(d)	Negative	(g) Negative
	(b)	Negative	(e)	0 (neither positive or negative)	(h) Negative
	(c)	Positive	(f)	Positive	(i) Negative

7-4. Subtraction.

Understandings:

1. $a - b = c$ means $b + c = a$.
2. To subtract the integer y from the integer x is the same as adding the opposite of y to x.
3. Subtraction can be interpreted on the number line.

Once the idea of "adding the opposite" is developed, it is desirable that students use this idea to perform subtraction. Every student must be able to interpret any subtraction problem as an addition problem.

You will find a detailed discussion of both approaches in the students' text.

Point out that it is always possible to subtract in the set of integers since every integer has an opposite.

Answers to Problems 7-4(a), page 7-11

1. (a) $9 - 5 = x$ means $5 + x = 9$. "What number added to 5 gives 9? The number is 4. Therefore, $9 - 5 = 4$.
   (b) $9 - (-5) = x$ means $-5 + x = 9$. We ask, "What number added to -5 gives 9?" The number is 14. Therefore, $9 -(-5) = 14$.
   (c) $-9 -5 = x$ means $5 + x = -9$. "What number added to 5 gives -9?" The number is -14. Therefore, $-9 -5 = -14$.
   (d) $-9 - (-5) = x$ means $-5 + x = -9$. "What number added to -5 gives -9?" The number is -4. Therefore, $-9 - (-5) = -4$.
   (e) $5 - 9 = x$ means $9 + x = 5$. "What number added to 9 gives 5?" The number is -4. Therefore, $5 - 9 = -4$.
   (f) $5 - (-9) = x$ means $-9 + x = 5$. "What number added to -9 gives 5?" The number is 14. Therefore, $5 - (-9) = 14$.
   (g) $(-5) - 9 = x$ means $9 + x = -5$. "What number added to 9 gives -5?" The number is -14. Therefore, $(-5) -9 = -14$.
   (h) $(-5) - (-9) = x$ means $-9 + x = -5$. "What number added to -9 gives -5?" The number is 4. Therefore, $(-5) - (-9) = 4$.

2. (a) 7 (d) 2
   (b) -16 (e) -9
   (c) 10 (f) -1

   In each case, the same number makes all three sentences true.

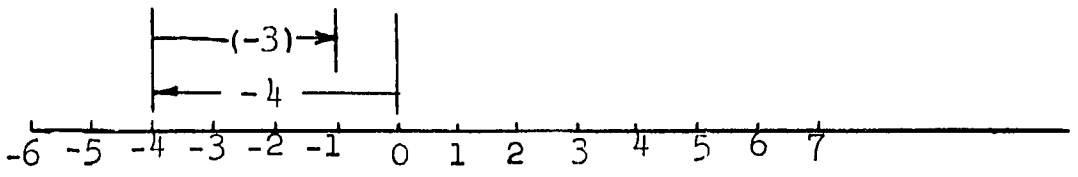
Answers to Problems 7-4(b), page 7-12

1. (a) $9 - 27 = 9 + (-27) = -18$
   (b) $(-9) - 27 = (-9) + (-27) = -36$
   (c) $9 - (-27) = 9 + 27 = 36$
   (d) $(-9) - (-27) = -9 + 27 = 18$
   (e) $(-27) - 9 = (-27) + (-9) = -36$
   (f) $(-27) - (-9) = (-27) + 9 = -18$
   (g) $27 -9 = 27 + (-9) = 18$
   (h) $27 - (-9) = 27 + 9 = 36$

The method used to show subtraction on the number line is essentially the same as that of picturing addition, since in each case to subtract a number we add its opposite.

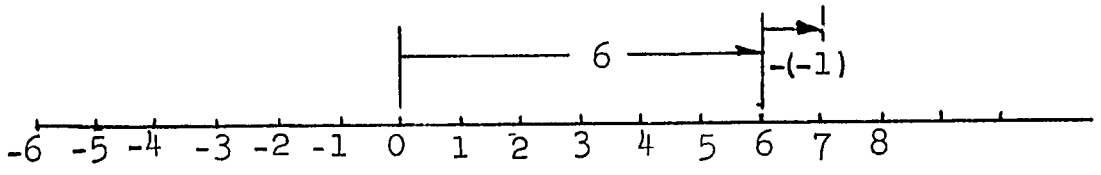
Answers to Problems 7-4(c) page 7-13

1.



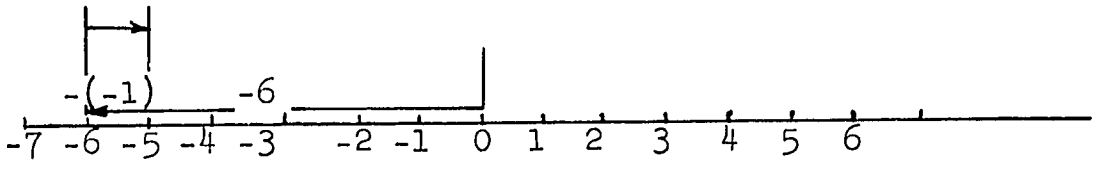
$\therefore -4 - (-3) = -1$

2.



$\therefore 6 - (-1) = 7$

3.

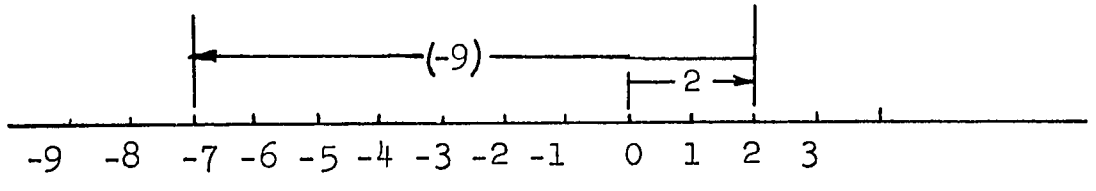


$\therefore -6 - (-1) = -5$

4.

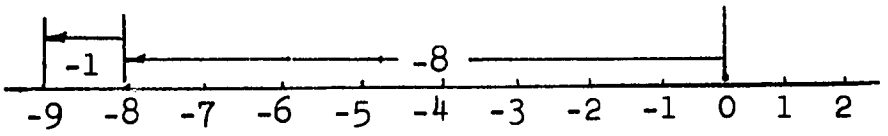
$\therefore -4 - (-4) = 0$

5.



$\therefore 2 - 9 = -7$

6.



$\therefore -8 - 1 = -9$

7-5. Practice with Addition and Subtraction.

Answers to Problems 7-5(a), page 7-14

1. (a) $-4 + (-2) = -(4 + 2) = -6$

(b) $-6 + (-11) = -(6 + 11) = -17$

(c) $8 + (-3) = (5 + 3) + (-3)$ — 5 + 3 is another name for 8

$= 5 + [3 + (-3)]$ — Associative Property of Addition

$= 5 + 0$ — The Property of Opposites

$= 5$ — Addition Property of Zero

(d) $-11 + (-13) = -(11 + 13) = -24$

(e) $11 + (-13) = 11 + [(-11) + (-2)]$

$= [11 + (-11)] + (-2)$ — Associative Property of Addition

$= 0 + (-2)$ — The Property of Opposites

$= -2$ — Addition Property of Zero

(f) 1

(g) -13

(h) $[(-2) + (-3)] + (-2) = (-5) + (-2) = -7$

(i) $-5 + 5 = 0$ — The Property of Opposites

(j) -2

2. (a) 6

(b) 20

(c) -6

(d) 11

(e) -4

3. (a) $8 - (-6) = 8 + 6 = 14$
   (b) $-9 - (-6) = -9 + 6 = -3$
   (c) $13 - (-12) = 13 + 12 = 25$
   (d) $-2 - (-2) = -2 + 2 = 0$
   (e) $5 - 5 = 5 + (-5) = 0$
   (f) $-16 - (-7) = -16 + 7 = -9$

4. (a) True
   (b) True
   (c) False
   (d) True
   (e) True

7-6. Multiplication.

Understandings:

1. Multiply integers using the basic properties.
2. Make generalizations about conditions under which the product of two integers is positive or negative.

Our rules for multiplication of integers come out of the desire to preserve the distributive property.

As soon as the students understand how to multiply integers using the basic properties, they should be permitted to find products without having to do this each time. The teacher should use his or her judgment in deciding when this should happen.

Begin by considering products such as $3 \times (-5)$.

Ask students: What is an important thing we know about the number -5?

$5 + (-5) = 0$ Property of Opposites

Suggest multiplying by 3, as this would introduce the product we want

$3 \times [5 + (-5)] = 3 \times 0$	$5 + (-5)$ and 0 are names for the same number.
$3 \times [5 + (-5)] = 0$	Multiplication Property of Zero
$3 \times 5 + 3 \times (-5) = 0$	Distributive Property

Ask students: What can you say about the product $3 \times (-5)$?

Since the sum of 3×5 and $3 \times (-5)$ is zero, we know $3 \times (-5)$ must be the opposite of 3×5, or

$$3 \times (-5) = -(3 \times 5) = -15$$

Students should observe that the product $3 \times (-5)$ is a negative number.

Answers to Problems 7-6(a), page 7-15

1. (a) $6 + (-6) = 0$ — Property of Opposites

$7 \times [6 + (-6)] = 7 \times 0$

$7 \times [6 + (-6)] = 0$ — Multiplication Property of Zero

$7 \times 6 + 7 \times (-6) = 0$ — Distributive Property

Since $7 \times 6 + 7 \times (-6) = 0$, then $7 \times (-6)$ must be the opposite of 7×6, or $7 \times (-6) = -(7 \times 6) = -42$.

(b) 256

(c) -198

(d) -6762

(e) 0

(f) -483

(g) -54

(h) -90

(i) -32

Now we consider the product of two negative integers. For example; $(-3) \times (-5)$:

$5 + (-5) = 0$	Property of Opposites
$(-3) \times [5 + (-5)] = (-3) \times 0$	$5 + (-5)$ and 0 are names for the same number
$(-3) \times [5 + (-5)] = 0$	Multiplication Property of Zero
$(-3) \times (5) + (-3) \times (-5) = 0$	Distributive Property
$-15 + (-3) \times (-5) = 0$	-15 is another name for $(-3) \times (5)$.

Since $-15 + (-3) \times (-5) = 0$, then $(-3) \times (-5)$ must be the opposite of -15, or $(-3) \times (-15) = -(-15) = 15$.

Observe that the product of two negative integers is a positive number.

Answers to Problems 7-6(b), page 7-16

1. (a) $5 + (-5) = 0$ — Property of Opposites

$-7 \times [5 + (-5) = -7 \times 0$ — $5 + (-5)$ and 0 are names for the same number

$-7 \times [5 + (-5)] = 0$ — Multiplication Property of Zero

$-7 \times 5 + (-7) \times (-5) = 0$ — Distributive Property

Since $-7 \times 5 + (-7) \times (-5) = 0$, then $(-7) \times (-5)$ must be the opposite of $(-7) \times 5$, or $(-7) \times (-5) = -(-35) = 35$.

(b) 35 (c) -35

(d) -35 (e) 35

(f) $(-3) \times (-5) \times (-1) = [(-3) \times (-5)] \times (-1) = 15 \times (-1) = -15$

(g) -72

(h) $(-4) \times (-2) \times (-1) \times (2) = [(-4) \times (-2)] \times [(-1) \times (2)]$
$= 8 \times (-2) = -16$

(i) -60 (j) 12

(k) 0

2. (a) -13 (d) 0

(b) -50 (e) 15

(c) -58

3. (a) -6 (g) -2

(b) -1 (h) 9

(c) -3 (i) -9

(d) -4 (j) -9

(e) -1 (k) 1

(f) 2 (l) 0

(m) 0

4. (a) True (f) True

(b) True (g) True

(c) False (h) True

(d) True (i) True

(e) False (j) True

7-7. Division.

Understandings:

(1) Students must interpret questions such as, "What is the quotient $(-10) \div (-5)$?" as another way of asking, "By what number should -5 be multiplied to get -10?"

(2) Students should know under what conditions the quotient of two integers is positive or negative.

Begin by recalling the following relation using specific examples:

For any whole numbers x and y($x \neq 0$) if there is a whole number z for which $x \times z = y$ then $\frac{y}{x} = z$.

We do not include any discussion of this development here as you can find it in the students' text.

Next, suggest to them that we can use the same reasoning to find the quotient of two integers.

For example, find the quotient $\frac{-15}{3}$.

Ask students: What is another way of asking, "What is $(-15) \div 3$?"

From this discussion we can conclude $\frac{-15}{3} = -5$ since $-5 \times 3 = -15$.

After many of these examples, students should realize that the above relation for whole numbers is also true for the integers, and can be stated as follows:

For any integers x and y ($x \neq 0$) if there is an integer z for which $x \times z = y$ then $\frac{y}{x} = z$.

Answers to Problems 7-7(a), page 7-18

1. (a) $\frac{35}{7} = 5$ since $7 \times 5 = 35$

   (b) $\frac{-21}{3} = -7$ since $3 \times (-7) = -21$

   (c) $\frac{-21}{-3} = 7$ since $-3 \times 7 = -21$

   (d) $\frac{21}{-3} = -7$ since $-3 \times (-7) = 21$

(e) $\frac{-8}{8} = -1$ since $8(-1) = -8$

(f) $\frac{-36}{-6} = 6$ since $-6 \times 6 = -36$

(g) $\frac{64}{-8} = -8$ since $-8 \times (-8) = 64$

(h) $\frac{-3}{-3} = 1$ since $-3 \times 1 = -3$

(i) $\frac{-1}{-1} = 1$ since $-1 \times 1 = -1$

(j) $\frac{1}{-1} = -1$ since $-1 \times (-1) = 1$

(k) $\frac{-1}{1} = -1$ since $1 \times (-1) = -1$

(l) $\frac{0}{-23} = 0$ since $-23 \times 0 = 0$

2. (a) 12
   (b) -1
   (c) -9
   (d) 9
   (e) -9
   (f) 9
3. (a) -63 since $9 \times (-7) = -63$
   (b) 12
   (c) 2 since $2 \times (-4) = -8$
4. Yes
5. Yes
6. No The quotient is always negative.
7. Yes
8. Yes
9. No

7-8. <u>Practice with the Four Operations</u>.

<u>Answers to Problems 7-8, page 7-19</u>

1. (a) $-8 \times 0 = 0$
   (b) $\frac{-72}{8} + 2 = -9 + 2 = -7$
   (c) $\frac{-72 + 2}{7} = \frac{-70}{7} = -10$
   (d) $2 - (-6) = 2 + 6 = 8$
   (e) 19
   (f) -50
   (g) -7
   (h) 0

2. (a) $x = 6$
   (b) $x = 789$
   (c) $x = -3$
   (d) $x = -4$
   (e) $x = -18$
   (f) $x = -27$
   (g) $x = 4$
   (h) $x = -1$

3. (a) -32
   (b) 30
   (c) 36
   (d) -7
   (e) 0
   (f) 40

4. (a) -14
   (b) 4
   (c) 6
   (d) 2
   (e) 0
   (f) -8

5. (a) -3
   (b) 3

6. $17 - 2 = 15$

7. $(-50) + (-25) = (-75)$ or
   $(-6) + (-3) = (-9)$

8. (a) $-10 = -1 + (-9)$
   (b) $-10 = 6 + (-16)$

9. (a) -4
   (b) 9

10. Give reasons for each step of the following simplification.
    (a) $[(-2) \times 6 - 7] + (-2) \times (-5) = [(-2) \times 6 + (-7)] + (-2) \times (-5)$

    Subtraction is same as adding the opposite

    (b) $= [(-2) \times 6 + (-2 \times -5)] + (-7)$

    Associative property and commutative property of addition

    (c) $= (-2) \times [6 + (-5)] + (-7)$

    Distributive property

    (d) $= (-2) \times (1) + (-7)$ 1 is another name for $6 + (-5)$

    (e) $= (-2) + (-7)$ Multiplication property of 1

    $= -9$

7-9. Inequalities.

Our aim is to give a more precise statement of the meaning of "a is less than b," using only the properties of the numbers, and to observe some of the properties of order of inequalities.

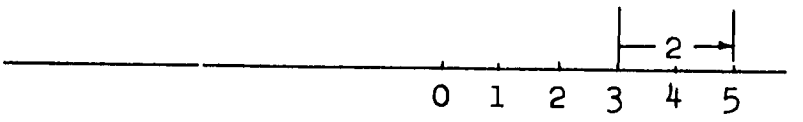
Begin by using the intuitive approach "is less than" means

"to the left of, on the number line" to determine which number is less than the other or which is greater. Give many examples and show the numbers on the line. Do not tell the class. Ask what they think about it. For example, ask about -2 and 3, -3 and -2, -8 and -1.

As soon as the students are familiar with the concept of order in the set of integers, proceed to develop with them a more precise statement of the meaning of "a is less than b" by observing several inequalities and their associated equations.

You may use this approach to get students to give associated equations for the inequalities which you wish to consider in the above discussion.

Let us look at $3<5$ on the number line



Ask students: How can we get from 3 to 5?

Answer: By stepping 2 units to the right, or $3 + 2 = 5$.

Likewise, we can get from -5 to -1 by stepping 4 units to the right, or $-5 + 4 = 1$.

It is recommended that the problems in this section be used as material for class discussion. The purpose of problems 7-9(b) is to lead students to observe some of the properties of inequalities.

Answers to Problems 7-9(a), page 7-21

1. If neither of these symbols is correct, the symbol "=" must be used.
   (a) $2 > (-1)$ because 2 is to the right of (-1) on the number line
   (b) $27 > -27$
   (c) $-5 > -6$
   (d) $1 > -300$
   (e) $-2 < -1$ because -2 is to the left of -1
   (f) $-18 < 0$

(g) $-1 < 1$
(h) $-19 > -20$
(i) $-30 < 30$
(j) $-30 > -50$

Answers to Problems 7-9(b), page 7-22

1. (a) $-9 < 30$ because $-9 + 39 = 30$
   (b) $-30 < -9$ " $-30 + 21 = -9$
   (c) $-28 < -19$ " $-28 + 9 = -19$
   (d) $-8 < -6$ " $-8 + 2 = -6$
   (e) $-1 < 1$ " $-1 + 2 = 1$
   (f) $-999 < 1$ " $-999 + 1000 = 1$
   (g) $-7 < 0$ " $-7 + 7 = 0$
2. Yes. Exactly one of these is true.
3. Yes. If a and b are integers, only one of the three relations is true. Either $a < b$ or $b < a$ or $b = a$.
4. If $a > b$ and $a + x = b$, x must be a negative number.
   Examples: We know $6 > 5$ and $6 + (-1) = 5$
   $100 > 25$ and $100 + (-75) = 25$
5. (a) $8 < 10$. Is $8 + 5 < 10 + 5$? We know 13 is another name for $8 + 5$ and 15 is another name for $10 + 5$.
   Since $13 + 2 = 15$, $13 < 15$ $\therefore$ $8 + 5 < 10 + 5$.
   Here the students should observe that the same number has been added to each number of the inequality.
   (b) Yes. $(-2) + 6 < 1 + 6$
   (c) Yes. $(-5) + (-3) < (-4) + (-3)$
   (d) Yes. $(-71) + (-2) < (-70) + (-2)$
   (e) No. $-2 + 16 > 1 + 9$. Note the numbers added are not the same.
6. (a) Yes. $(-1) + 3 > (-2) + 3$
   (b) Yes.
   (c) Yes.
7. (a) False
   (b) True
   (c) False
   (d) True

8. (a) We know that $3 \times 5 = 15$ and $4 \times 5 = 20$
   Since $15 + 5 = 20$, $15 < 20$; therefore, $3 \times 5 < 4 \times 5$
   (b) Yes.
   (c) Yes.
   (d) No. $3 \times (-8) = -24$ and $4 \times (-8) = -32$
   Which is less than the other -24 or -32?
   Since $-32 + 8 = -24$, $-32 < -24$ or $-24 > -32$.
   Therefore, $3 \times (-8) > 4 \times (-8)$.
   Note the order of the inequality is reversed.
   (e) No.
   (f) No.
   (g) No.
   (h) No. Note: observe the two numbers of the inequality are not multiplied by the same positive number.

9. (a) Yes.
   (b) No. $6 \times (-1) = -6$ and $2 \times (-1) = -2$
   Since $-6 + 4 = -2$, $-6 < -2$.
   Therefore, $6 \times (-1) < 2 \times (-1)$. Note that each number of the inequality is multiplied by the same negative number and the order is reversed.
   (c) Yes.
   (d) No. $24 < 56$. Same reason as (b).
   (e) No. Note: observe that the two numbers of the inequality are multiplied by different numbers.

10. (a) True (c) True
    (b) True (d) True

7-10. The Set of all Rational Numbers.

The students are now familiar with both the set of positive rational numbers and the set of negative integers. The main purpose of this section is to lead them to combine their knowledge of the two sets to compute with the rational numbers. They have, for the first time, a set which is an example of a field; that is, a set in which they can always add, subtract, multiply, and divide (except for division by zero).

Problems 7-10(a), page 7-25

1. (a) $\frac{7}{3}$ (e) $\frac{-1}{20}$

   (b) $\frac{6}{5}$ (f) $2\frac{3}{4}$

   (c) $\frac{-45}{2}$ (g) $4\frac{1}{2}$

   (d) 9 (h) 0

2. $\frac{-3}{5}$

3. $\frac{3}{11}$

4. Yes

5. Yes. (When the rational number is 0.)

Problems 7-10(b), page 7-26

1. (a) Addition property of zero.
   (b) Commutative property of addition.
   (c) Multiplication property of zero.
   (d) Distributive property.
   (e) Associative property of multiplication.
   (f) Property of opposites.
   (g) Multiplication property of one.
   (h) $\begin{cases}\text{Multiplication property of zero.}\\ \text{Multiplication property of one.}\end{cases}$

2. x is an integer.

   $\therefore$ Its opposite -x exists and is also an integer.

   $\therefore$ $(-x) + [-(-x)] = 0$.

   We infer that $-(-x) = x$; both are opposites of -x.

Answers to Problems 7-10(c), page 7-27

As stated in the text, students are expected to use the result: $a + (-b) = a - b$.

It is clear from the exercises whether this result is to be applied forwards or backwards.

1. (a) $\frac{1}{4}$ (b) $\frac{-11}{24}$

(c) $\left(-\frac{2}{3}\right) + \frac{3}{4} = \frac{3}{4} + \left(-\frac{2}{3}\right)$

$= \frac{3}{4} - \frac{2}{3}$

$= \frac{1}{12}$

(d) $-\frac{3}{14}$ (e) $\frac{3}{2} = 1\frac{1}{2}$

(f) $-1\frac{11}{15}$ (g) $-\frac{2}{3}$

(h) 0 (i) $\frac{3}{4}$

(j) $-\frac{35}{12}$

3. (a) Associative property of addition.
(b) Associative property of addition.
(c) Commutative property of addition.
(d) Associative property of addition.
(e) Associative property of addition.
(f) Property of opposites.
(g) Addition property of zero.

Problems 7-10(d), page 7-28

(a) $-\frac{17}{12} = -1\frac{5}{12}$ (d) $-4\frac{2}{15}$

(b) $-\frac{23}{24}$ (e) $-6\frac{17}{40}$

(c) $-\frac{37}{45}$ (f) $-2\frac{11}{24}$

Problems 7-10(e), page 7-29

1. (a) $\frac{1}{2}$
   (b) $\frac{4}{5}$
   (c) $5\frac{1}{5}$
   (d) $\frac{10}{3} = 3\frac{1}{3}$
   (e) $\frac{x}{y}$

Problems 7-10(f), page 7-30

1. (a) $-\frac{4}{5}$
   (b) $-\frac{4}{5}$
   (c) $\frac{4}{5}$
   (d) $\frac{4}{5}$
   (e) $-\frac{35}{2} = -17\frac{1}{2}$
   (f) $-\frac{110}{21} = 5\frac{5}{21}$
   (g) $14\frac{1}{6}$
   (h) $\frac{12}{7}$ ax
2. (a) True
   (b) True
   (c) False
   (d) True
   (e) False
   (f) False

Problems 7-10(g), page 7-31

(a) $\frac{1}{2}$
(b) $-\frac{1}{2}$
(c) $-\frac{1}{2}$
(d) $\frac{1}{2}$
(e) -8
(f) -1
(g) $\frac{2}{81}$
(h) $\frac{2}{3}$
(i) -1
(j) 0
(k) $-\frac{19}{13}$
(l) $-\frac{13}{19}$
(m) not defined
(n) $-\frac{3}{8}$
(o) $\frac{ad}{bc}$

Problems 7-10(h), page 7-31

(a) $\frac{4}{35}$

(b) (-1)

(c) $-\frac{51}{8}$

(d) $\frac{11}{2}$

(e) All non-zero rational x

(f) 1

(g) $\frac{25}{16}$

(h) All rational x.

FIRST YEAR SECONDARY TEACHERS' GUIDE

CHAPTER 8

DECIMAL NUMERALS

Commentary.

This chapter is concerned with a particular way of representing numbers. Much of the work in earlier chapters has been concentrated on properties of the numbers, properties which are not dependent on the way the numbers are written. For example the commutative property remains valid however we represent numbers, so that 7 + 9 = 9 + 7 is true as in VII + IX = IX + VII.

Here we study one important representation of numbers. We study decimal numerals. We have regarded the word "decimal" as an adjective so that the numerals become decimal numerals, but do not worry too much if you or your students occasionally refer to the numerals simply as "decimals."

The main feature of the chapter is the clarification, by means of expanded notation, of the four fundamental operations of addition, subtraction, multiplication, and division. With this clarification, the work on percentage follows readily.

Finally, Sections 8-10 and 8-11 deserve special mention, for they are new to the student and require that he conceive of the whole infinite sequence of digits as representing exactly one number. These sections are important for they prepare the student for the work next year on the real number system.

8-1. Revision of Base 10 Numeration.

This section is a review of what was already done in Chapter 2. Hence, do this section in class as rapidly as possible. Begin by asking the class what several decimal numerals represent. Be sure that index notation is clear to all. Problems 1, 2, 3, 4 can probably be done orally, in class, or with students putting the work on the board. You may want to assign a few parts of problems 5

and 6 to the better students.

Answers to Problems 8-1, page 8-1

1. (a) p^2 (b) p^3 (c) p^5
2. (a) 375
   (b) 413
   (c) 403
   (d) 52569
   (e) 50569
   (f) 80007
3. (a) $2\ (10^2) + 9\ (10) + 6$
   (b) $9\ (10) + 8$
   (c) $9\ (10) + 9$
   (d) $1\ (10)^2$
   (e) $4\ (10^2) + 5\ (10) + 6$
   (f) $5\ (10^3) + 5\ (10^2) + 5\ (10) + 9$
   (g) $5\ (10^3) + 0\ (10^2) + 5\ (10) + 9$
   (h) $5\ (10^3) + 5\ (10^2) + 0\ (10) + 9$
   (i) $5\ (10^3) + 0\ (10^2) + 0\ (10) + 9$
   (j) $1\ (10^6)$
   (k) $2\ (10^4) + 3\ (10^3) + 4\ (10^2) + 5\ (10) + 6$
   (l) $2\ (10^4) + 4\ (10^2) + 5\ (10) + 6$
   (m) $2\ (10^4) + 4\ (10^2) + 5\ (10)$
   (n) $3\ (10^6) + 4\ (10^5) + 5\ (10^4)$
   (o) $1\ (10) + 4$
4. (a) $36_{seven} = 3\ (10)_{seven} + 6$ in expanded form
   (b) $= 27$ in base 10
5. (a) 39 (d) 38
   (b) 22 (e) 314
   (c) 5 (f) 1415
6. The pupils may choose any base for their solutions. The following are furnished merely as examples.
   (a) $50 = (71)_{seven} = (200)_{five} = (110010)_{two}$
   (b) $86 = (152)_{seven} = (321)_{five} = (1010110)_{two}$
   (c) $125 = (236)_{seven} = (1000)_{five} = (1111101)_{two}$
   (d) $(100)_{seven} = (144)_{five} = (110001)_{two}$

(e) $(10)_{two} = 2_{seven} = 2_{five}$
(f) $15_{twelve} = 32_{five} = (10001)_{two}$

8-2. Addition and Subtraction with Decimal Numerals.

This material too has been discussed in Chapter 2. Begin the discussion without reference to the text, by asking students to explain simple sums.
For example:

$$\begin{array}{r} 53 \\ 42 \\ \hline 95 \end{array} \qquad \begin{array}{r} 57 \\ 64 \\ \hline 121 \end{array} \qquad \begin{array}{r} 637 \\ 795 \\ \hline 1432 \end{array}$$

Keep asking questions until students, on their own, make use of expanded notation to explain their work. For example, in 57 + 64 = 121, you might ask a student how he got the "2." If he talks about "carrying one" get him to explain why the "one" is really "ten." Perhaps he may argue that 57 plus 60 is 117 and then 4 more gives 121.

$$\begin{array}{r} 57 \\ 60 \\ 4 \\ \hline 11 \\ 110 \\ \hline 121 \end{array}$$

While you would like everyone to discover for himself that

$$\begin{aligned} 57 + 64 &= [5(10) + 7] + [6(10) + 4] \\ &= 11(10) + 11 \\ &= 1\ (10^2) + 1\ (10) + 1\ (10) + 1 \\ &= 121, \end{aligned}$$

not everyone will think of doing it this way. However, when all are aware of how the notation might be used, then you can go to the text. Have the students explain the steps in the text and how the basic principles are used. Also show how these steps occur in the usual shorthand computational algorithm.

Answers to Problems 8-2, page 8 - 3

1. (a) $3\ (10) + 5 = 35$
   (b) $2\ (10) + 6 = 26$
   (c) $5\ (10) + 2 = 52$
   (d) $4\ (10) + 8 = 48$

(e) $5\ (10) + 6 = 56$
(f) $3\ (10^2) + 5\ (10) + 4 = 354$
(g) $3\ (10^2) + 8\ (10) + 4 = 384$
(h) $1\ (10^4) + 1\ (10^3) + 7\ (10^2) + 9\ (10) + 1 = 11791$
(i) $4\ (10^3) + 3\ (10^2) + 7\ (10) + 7 = 4377$
(j) $8\ (10) + 8 = 88$

2. (a) 25442 (b) 22220

3. (a) 34_{five} (b) 40_{five} (c) 100_{five}
(d) 225_{seven} (e) 43_{seven} (f) 1010_{two}
(g) 11010_{two} (h) 1000_{nine} (i) 8_{nine}
(j) 102_{three}

8-3. Multiplication with Decimal Numerals.

This material has hardly been mentioned in Chapter 2. Yet by this time a student should by himself be able to use expanded notation to explain the usual computational algorithm. It is of great importance that the student see how the distributive property is applied.

A good way to begin the discussion is to write a problem on the board such as is shown here. Then ask students to explain where each digit comes from in the computational algorithm. As soon as the class sees what the difficulties are, you can proceed to a study of the text - but do not stop class discussion just to get to the text. It is quite possible that some student may suggest something like this:

$$\begin{array}{r} 27 \\ 42 \\ \hline 54 \\ 108 \\ \hline 1134 \end{array} \quad \text{algorithm for } 42 \times 7$$

$$(\star) \quad \left\{ \begin{aligned} 42 \times 27 &= (40 + 2)\ (20 + 7) \\ &= 40 \times (20 + 7) + 2 \times (20 + 7) \\ &= [(40 \times 20) + (40 \times 7] \\ &\quad + [(2 \times 20) + (2 \times 7)] \\ &\quad [800 + 280] + [40 + 14] \end{aligned} \right. \quad \left\{ \begin{array}{l} \text{by the} \\ \text{distributive} \\ \text{property} \end{array} \right.$$

Now the problem is reduced to a problem in addition - a problem which has already been discussed in Section 8-2. At this point you can once more ask the class where the different digits in the computational algorithm come from.

Observe that the explanation in (☆) does not fully use the expanded form. This is quite all right. The purpose of this sec-

tion is to make perfectly clear to each student the reasons underlying the algorithm which he may have learned by rote. Once this has been achieved, there is no necessity for insisting that the student write out everything in expanded form.

For example, the problem shown in (☆) may also be written as follows:

$$\begin{aligned} 42 \times 27 &= [4(10) + 2] \times [2(10) + 7] \\ &= [8(10^2) + 4(10)] + [28(10) + 14] \\ &= [8(10^2) + 4(10)] + [2(10^2) + 8(10) + 1(10) + 4] \\ &= 10(10^2) + 13(10) + 4 \\ &= 1(10^3) + 1(10^2) + 3(10) + 4 \\ &= 1134. \end{aligned}$$

Remember: <u>Use of expanded notation is not a skill to be learned</u>. Its purpose is rather that of clarification.

<u>Answers to Problems 8-3, page 8-4</u>

1. (a) $1\ (10^2) + 4\ (10) + 8 = 148$
   (b) $2\ (10^2) + 0\ (10) + 7 = 207$
   (c) $4\ (10^2) + 3\ (10) + 7 = 437$
   (d) $1\ (10^2) + 2\ (10) + 1 = 121$
   (e) $2\ (10^3) + 1\ (10^2) + 9\ (10) + 6 = 2196$
   (f) $2\ (10^3) + 1\ (10^2) + 9\ (10) + 6 = 2196$
   (g) $2\ (10^3) + 2\ (10^2) + 3\ (10) + 2 = 2232$
   (h) $1\ (10^3) + 1\ (10^2) + 0\ (10) + 7 = 1107$
   (i) $6\ (10^3) + 0\ (10^2) + 2\ (10) + 7 = 6027$

8-4. <u>Division with Decimal Numerals</u>.

If pressed for time the teacher can omit this section, as a good discussion cannot be done quickly. Nevertheless, the teacher is urged to find the time because of all the methods learned in primary school, long division is by far the least well understood.

Begin the discussion by reminding the class of the definition of division as in the text. Then start doing a long division problem such as shown at the right. Ask the class what the 2 means, and what the 94 means.

```
     2
47/1081
   94
```

Ask why one selects 2 rather than some other digit. These questions should promote a fair amount of discussion and argument. It is likely that students will not keep in mind, or tell you, the aim of the whole process. That is, they will lose sight of the fact that they are trying to find a decimal numeral for the number x such that

$$1081 = 47x$$

Keep asking questions until it happens (it may not happen so you will have to tell the class, but try to be patient) that some student remarks that x must be less than 30 because

$$47 \times 30 = 1410$$

and $1410 > 1081$. Also x must be more than 20 because

$$47 \times 20 = 940$$

and $940 < 1081$. Now the class should be able to answer two of your earlier questions: the 2 in the methods stands for 20, and the 94 in the method stands for 940.

At this point you can either abandon your example of $1081 \div 47$ and go to the text, or you can continue on with the example. What you should do will depend on how the class feels. Hopefully, the class will want to continue. If so, ask a student to explain where the 141 comes from in the long division algorithm shown here. He should be able to tell you that 141 is the difference between 1081 and 940. In other words, he should know that he is looking for a number $\square$ such that

$$\begin{array}{r} 23 \\ 47\overline{)1081} \\ \underline{94} \\ 141 \\ \underline{141} \end{array}$$

$$1081 = 47 \times 20 + 47 \times \square, \text{ or}$$

$$141 = 47 \times \square$$

The problem now is a simpler one. Obviously, $\square = 3$ so

$$1081 = 47 \times 20 + 47 \times 3 = 47 \times 23, \text{ and}$$

$$1081 \div 47 = 23.$$

Now turn to the text and ask students to follow the two methods in the text and answer the questionsin the text.

Do not overteach the trial and error method. As soon as a student understands all the steps in the left columns and can do problems this way, then he should simply practise a bit of long division.

Answers to Problems 8-4, page 8-8

1. (a) between 10 and 20 (b) between 30 and 40
   (c) between 40 and 50 (d) between 300 and 400
   (e) between 30 and 40 (f) between 300 and 400
   (g) between 3000 and 4000

2. (a) If $98 \div 7 = x$, then $98 = 7x$, and $10 < x < 20$. Put $x = 10 + y$. Then $98 = 7(10 + y) = 70 + 7y$ so $28 = 7y$ and it is clear where the 28 in the long division comes from. Then y is clearly 4, so $x = 10 + y = 10 + 4$ and $98 \div 7 = 14$.

```
     1
7/98
 7
```

The 1 represents 10.

```
     1
7/98
 7
 28
```

```
     14
7/98
 7
 28
 28
```

(c) If $612 \div 17 = x$, then $612 = 17x$ and $30 < x < 40$. Put $x = 30 + y$. Then $612 = 17\ (30 + y)$
$= 510 + 17y$
so
$102 = 17y$
and it is clear where the 102 in the long division comes from. Then y is clearly 6, so $x = 30 + y = 30 + 6$ and $612 \div 17 = 36$.

```
     3
17/612
   51
```

The 3 represents 30.

```
     3
17/612
   51
   102
```

```
     36
17/612
   51
   102
   102
```

(g) If $2565 \div 35 = x$ then $2565 = 35x$ and $70 < x < 80$. Put $x = 70 + y$. Then $2565 = 35\ (70 + y)$
$= 2450 + 35y$
so
$115 = 35y$
and it is clear where the 115 in the long division comes from.

```
     7
35/2565
   245
```

The 7 represents 70.

```
     7
35/2565
   245
    115
```

```
     73
35/2565
   245
    115
    105
     10
```

Then $y = 3 + \frac{10}{35}$ because

$35\ (3 + \frac{10}{35}) = 105 + 10 = 115.$

Therefore, $x = 70 + 3 + \frac{10}{35} = 73\ \frac{2}{7}$

$2565 \div 35 = 73\ \frac{2}{7}$.

3. (a) 12 (b) 13
(c) $237\ \frac{31}{35}$ (d) $37\ \frac{75}{223}$
(e) 437 (f) 24
(g) $23\ \frac{46}{55}$ (h) 283

4. (a)
```
   12
3/41
  3
  11
  11
```

(b)
```
      43
23/2144
   202
    124
    124
```

(c)
```
      11
101/1111
    101
     101
     101
```

(d)
```
      11
11/1001
   11
    11
    11
```

(e)
```
      21
12/1022
   101
     12
     12
```

(f)
```
      63
31/2613
   246
    123
    123
```

8-5. <u>Rational Numbers in Decimal Notation.</u>

The students already know how to write whole numbers in decimal notation. The teacher may take them quickly through a few exercises of writing whole numbers in the expanded form. About four examples will do.

The teacher could then start a discussion on the place values of 3 in the following whole numbers: (a) 4300, (b) 430, (c) 43, (d) 43000, (e) 430000. Because of the work they have done already on decimal notation they should have no difficulty in explaining, for example, that the 3 in (a) is 10 times the 3 in (b) and

also that the 3 in (b) is 10 times the 3 in (c).

Ask the students to write the $\frac{43}{10}$ in the expanded form. If they are able to do so after some effort, the exercise may be extended to $\frac{43}{100}$.

They should then be asked to write the number $23\frac{43}{100}$ in the expanded form. They are able to state that it is

$$2\ (10) + 3\ (1) + 4\ (\tfrac{1}{10}) + 3\ (\tfrac{1}{10}^2)$$

Ask them whether they can say something about the place values of the digits 2, 3, 4, and 3 in order in the above expanded number. They should notice that the place value of each digit is 10 times the place value of the digit immediately on the right. Now tell them that we write the number $2\ (10) + 3\ (1) + 4\ (\frac{1}{10}) + 3\ (\frac{1}{10}^2)$ in decimal form as 23.43 and use the decimal point to separate the whole numbers from the number less than 1. It is read as "twenty-three point four three."

Answers to Problems 8-5, page 8-9

1. (a) 432.56
   (b) 0.456
   (c) 0.05630
   (d) 7000.005
   (e) 80000.0506
2. (a) $3\ (10^2) + 9\ (10) + 1 + 3\ (\frac{1}{10}) + 4\ (\frac{1}{10}^2) + 5\ (\frac{1}{10}^3)$

   (b) $2\ (10) + 7\ (1) + 0\ (\frac{1}{10}) + 1\ (\frac{1}{10}^2) + 2\ (\frac{1}{10}^3)$

   (c) $0\ (1) + 5\ (\frac{1}{10}) + 1\ (\frac{1}{10}^2) + 3\ (\frac{1}{10}^3)$

   (d) $5\ (10) + 0\ (1) + 0\ (\frac{1}{10}) + 0\ (\frac{1}{10}^2) + 7\ (\frac{1}{10}^3)$

   (e) $6\ (10^2) + 0\ (10) + 0\ (1) + 0\ (\frac{1}{10}) + 1\ (\frac{1}{10}^2) + 0\ (\frac{1}{10}^3) + 2\ (\frac{1}{10}^4)$

8-6. Addition and Subtraction.

The work which the students have already done in addition and

subtraction with whole numbers should simplify the work of this section.

Begin by setting a problem like the following

$$3.64 + 4.12.$$

Ask the students to suggest how this should be done. They may be led to recall what was done with whole numbers and this will suggest the expanded form.

Thus

$$\begin{aligned} 3.64 &= 3\ (1) + 6\ (\tfrac{1}{10}) + 4\ (\tfrac{1}{10}2) \\ 4.12 &= 4\ (1) + 1\ (\tfrac{1}{10}) + 2\ (\tfrac{1}{10}2) \\ \hline &= 7\ (1) + 7\ (\tfrac{1}{10}) + 6\ (\tfrac{1}{10}2) \end{aligned}$$

Hence the sum is 7.76. The use of the distributive law should be noted here as the teacher leads the students to see how each digit is obtained.

The students should soon get to the stage where the problem may be understood without using the expanded form.

The teacher should stress the importance of keeping the decimal points directly under one another when the addition is set out, and the problem may be set out again as follows:

$$\begin{array}{r} 3.64 \\ 4.12 \\ \hline 7.76 \end{array}$$

The teacher should also give examples of subtraction, e.g., 8.73 - 3.45

$$\begin{array}{r} 8.73 \\ -3.45 \\ \hline 5.28 \end{array}$$

Students should be encouraged to check the subtraction by addition.

The teacher may also observe that since $.20 = \frac{2}{10} + 0\ (\frac{1}{10}2)$, .2 and .20 are names for the same number.

<u>Answers to Problems 8-6, page 8-12</u>

1.	(a)	9.43	(b)	9.441	(c)	13.92	(d)	16.482
2.	(a)	1.11	(b)	6.09	(c)	1.981	(d)	0.1096
3.	(a)	11.31	(b)	11.31	(c)	7.77	(d)	7.77
4.	(a)	12.42 inches			(b)	3.2 inches.		

5. Yes; by reducing the time by 0.65 seconds.
6. (a) 981.7 (b) 9997.4.

8-7. Multiplication and Division by 10.

The students have seen the distributive property applied in Multiplication with Decimal Numerals (8-3). A good way to start will therefore be to set a problem like 3.2×10 for the children to do on the board in the expanded form.

The problem is done as follows:

$$\begin{aligned} 3.2 \times 10 &= [3\ (1) + 2\ (\tfrac{1}{10})] \times 10 \\ &= 3\ (10) + 2\ (1) \\ &= 32 \end{aligned}$$

At this stage the teacher could start a discussion with the students on place values of the digits in 3.2 and 32. In particular the teacher could ask them whether they notice what has happened to the place values of the digits in 3.2 after it is multiplied by 10.

They may say that the place values of the digits have changed. Ask them to tell how. Some may be able to say that each digit is now ten times what it was at first. That is a correct answer, but in order to draw attention to the decimal point, the teacher should lead them on to what they may have observed, namely that each digit has moved one place to the left.

Let them see what happens in this example which may be worked on the board: 72.65×10

Result $= 726.5$ There is, as it were, promotion for each digit.

Now ask them to tell what happens when a number is multiplied by 100 or 10^2. Lead them to make a general statement about multiplying a number by 10^n where n is a positive integer.

Division by 10.

This should, of course, follow from multiplication by 10. One example may be put on the board for the students:

$$24.38 \div 10 \text{ or } 24.38 \times \frac{1}{10}$$

They proceed as follows:

$$[2\ (10) + 4\ (1) + 3\ (\tfrac{1}{10}) + 8\ (\tfrac{1}{10}2)] \times \tfrac{1}{10}$$

$= 2\ (1) + 4\ (\frac{1}{10}) + 3\ (\frac{1}{10}2 + 8\ (\frac{1}{10}3)$, using the distributive property

$= 2.438$

Now ask them to tell what has happened to the digits. It is easy for them to see that the digits have moved one place to the right.

We may then find out from them whether there has been another "promotion" in this case. This, of course, will lead to their stating that there has been a demotion, i.e., the reverse process.

Then follow up by asking for a statement for dividing by 10^2, 10^3, and 10^n where n is a positive integer.

Answers to Problems 8-7, page 8-15

1.	110; 1100; 11000	11.	7.1; 0.71; 0.071
2.	11; 110; 1100	12.	0.73; 0.073; 0.0073
3.	0.1; 1; 10	13.	1.001; 0.1001; 0.01001
4.	0.12; 1.2; 12	14.	0.0102; 0.00102; 0.000102
5.	510.01; 5100.1; 51001	15.	10.051; 1.0051; 0.10051
6.	0.101; 1.01; 10.1	16.	0.1001; 0.01001; 0.001001
7.	9615.32; 96153.2; 961532	17.	61.9235; 6.19235; 0.619235
8.	80.01; 800.1; 8001	18.	0.8001; 0.08001; 0.008001
9.	0.77; 7.7; 77	19.	0.0017; 0.00017; 0.000017
10.	0.07; 0.7; 7	20.	0.0003; 0.00003; 0.000003

8-8. Decimal Places, Multiplication, and Division.

The purpose of this section is to show that multiplication and division involving rational numbers in decimal notation are the same as multiplication and division in decimal notation, except that "decimal places" come into the picture.

Start off by telling the students what is meant by decimal places and then get them to give the number of decimal places in the following, orally:

37.5, 0.24, 40.105, 0.0056, 149.35043.

The next step would be to pose the following problem: - What is

the truth set for the following equation?

$$2.54 = 254 \times \square$$

They should be able to give the answer $\left\{\frac{1}{100}\right\}$. From the above, they are able to see that we can write 2.54 as $254 \times \frac{1}{100}$. Proceed to work 4×2.54 on the board as follows:

$$\begin{aligned} 4 \times 2.54 &= 4 \times (254 \times \tfrac{1}{100}) \\ &= 1016 \times \tfrac{1}{100} \\ &= 10.16. \end{aligned}$$

The students may now do 1.24×3.2 in a similar way. The working will take this form:

$$\begin{aligned} 1.24 \times 3.2 &= (124 \times \tfrac{1}{100}) \times (32 \times \tfrac{1}{10}) \\ &= (124 \times 32) \times (\tfrac{1}{100} \times \tfrac{1}{10}) \quad \text{Why?} \\ &= 3968 \times \tfrac{1}{1000} \\ &= 3.968. \end{aligned}$$

Ask the students what they notice about the decimal places in the answers of both examples and lead them on through trial and error to the general statement: "When we multiply numbers in decimal notation, we find the product as with whole numbers and then fix the decimal point according to the sum of the decimal places in the factors."

Division.

We go about division first by doing two division examples on the chalk board with the students. The following will do:

```
1.      2.36
      4/9.44
       8
       1.4
       1.2
         24
         24

2.      0.0062
      6/0.0372
          36
           12
           12
```

They see that division is easy with whole number divisors.

The teacher may now ask them: "What should be done in a division like $9.44 \div 0.4$?" After some effort they might come out with the suggestion that if 0.4 were 4, the problem could be done easily. They may then be asked to put the problem in a form such that the divisor is a whole number.

If $94.4 \div 4$ is suggested, ask for the explanation. This may lead to

$$\frac{9.44}{0.4} = \frac{9.44}{0.4} \times \frac{10}{10}$$

We could now generalise by telling them that since it is easy to divide by whole numbers, we try to obtain whole numbers for division as was done above.

Answers to Problems 8-8, page 8-17

1. (a) 4; (b) 4; (c) 7.
2. (a) 0.026248
   (b) 220.81968
   (c) 0.0290184
   (d) 0.0813
3. (a) 65.62
   (b) 1122.5
   (c) 0.007
   (d) 1.3
4. 0.19525
5. 4.7 inches
6. Average for 100 yds. is 10.8 sec.
   Average for 200 yds. is 26.8 sec.
   Average for 440 yds. is 58.8 sec.
7. 342.72 miles
8. 394
9. 98.07 square metres
10. 3.05_{seven}

8-9. Rounding off.

Students may be asked to say whether the population of a country

at 8 a.m. on a certain day will be the same as the population of the country at 8 p.m. on the same day. This should lead to the fact that the population may change at every instant of the day, and that it is practically impossible to obtain the exact number of people in a country at any one time, since difficulties may arise by some being in the process of being born and others in the process of dying. Discussion on the practical means of estimating the population of a country should reveal that an approximation to the population is all that is really necessary or practicable.

Records of weights and measures are liable to error because of the nature of the instruments. The measurement that we obtain for the length of a line segment depends on the accuracy of the instrument used, which in turn depend on physical conditions. Thus the measurement we obtain for the length of a line segment can only be an approximation.

Suppose we wish to find the area of a rectangle whose dimensions are measured to be 12.5 inches, carried to 1 decimal place, and 9.8 inches, carried to 1 decimal place, respectively. The statement that the length is 12.5 inches means that it is anything between 12.45 inches and 12.55 inches. In the same way, the breadth is between 9.75 inches and 9.85 inches which makes the area between (12.45×9.75) and (12.55×9.85) inches, or 121.5 inches and 123.5 inches correct to 1 decimal place.

<u>Answers to Problems 8-9, page 8-20</u>

1. (a) 6,123,000 (b) 5,432,000 (c) 1,236,000 (d) 2,000,000
2. (a) 5 (b) 7 (c) 8
3. (a) 5.2 (b) 6.5 (c) 10.0 (d) 8.5
4. Correct to 2 decimal places
   (i) 1.23 (ii) 9652.85 (iii) 2615.21 (iv) 8.65
   Correct to 3 decimal places
   (i) 1.235 (ii) 9652.855 (iii) 2615.211 (iv) 8.651

5. Correct to 2 decimal places

(i) 0.33	(ii) 0.17	(iii) 0.14
(iv) 0.27	(v) 0.56	(vi) 0.20
(vii) 1.89	(vii) 11.57	(ix) 7.54
	(x) 5.32	

Correct to 3 decimal places

(i) 0.333	(ii) 0.167	(iii) 0.143
(iv) 0.273	(v) 0.556	(vi) 0.200
(vii) 1.889	(viii) 11.571	(ix) 7.538
	(x) 5.316	

8-10. Decimal Expression of Rational Numbers

The teacher should remind the students that the numberal 1 may be written as 1.000 etc., and in general a fraction can be represented as a decimal numeral by dividing the numerator by the denominator.

Answers to Problems 8-10(a) page 8-21

1. (a) 0.5
   (b) 0.25
   (c) 0.75
   (d) 0.375
   (e) 0.075
   (f) 0.1625

2. (a) $\frac{1}{4}$
   (b) $\frac{7}{4}$
   (c) $\frac{14}{5}$
   (d) $\frac{707}{500}$

Repeating Decimal Numerals

The expression $\frac{1}{3}$ or $\frac{1}{9}$ as a decimal numeral should reveal the essential value of the repeating decimal numeral. Ask the children to state the digit in the 10th or 20th position. When they

have discussed the repeating value of the numeral, they may wish to invent a way of denoting this. The teacher may use the notation of the class for some time, and gradually lead them to the standard notation.

Answers to Problems 8-10(b), page -

1.	$.\dot{1}$	2.	$.\dot{0}\dot{1}$
3.	$.\dot{0}0\dot{1}$	4.	$.\dot{6}$
5.	$.\dot{1}4285\dot{7}$	6.	$0.\dot{7}1428\dot{5}$
7.	$0.\dot{0}\dot{9}$	8.	$0.\dot{2}\dot{7}$
9.	$0.\dot{0}2\dot{7}$	10.	$0.\dot{0}7317\dot{}$

Terminating Decimals and Repeating Decimals.

The teacher may lead the children to see, by means of some examples, that terminating decimal numerals are formed only from rational numbers whose denominators are divisible only by 2 and/or 5. Begin by giving them examples of terminating decimal numerals like .58, .96, and .43217. If these decimal numerals are expressed as fractions, their denominators will be powers of 10 and would thus be divisible by 2 and/or 5. If a fraction has for a denominator a power of two or five, the denominator can easily be converted to a power of 10.

e.g. $\frac{1}{2^p5^q} = \frac{1}{2^p5^q} \times \frac{5^{p-q}}{5^{p-q}} = \frac{5^{p-q}}{2^p5^p} = \frac{5^{p-q}}{10^p}$ (p and q are integers and $p > q$)

Hence $\frac{1}{2^p5^q}$ can be expressed as a terminating decimal.

The fact that all rational numbers which cannot be expressed as terminating decimals can be expressed as repeating decimal numerals may be seen from examining the process of division carefully. Consider the expansion of $\frac{1}{37}$:

```
      0.027
37/1.000
     74
     260      1st remainder is 26
     259
       1      2nd remainder is 1.
```

The 1, however, corresponds to the original dividend, and the same

pattern will be followed if the division is continued. We know that the process will be repeated if we have a remainder which corresponds to the original dividend or a remainder repeats itself. Each remainder must be less than the divisor, and since the divisor is finite, a remainder must be repeated. If the divisor is n, there can be at most (n-1) different remainders.

The teacher should illustrate the discussion by expressing $\frac{1}{7}$ as a decimal numeral. The remainders in order are 3, 2, 6, 4, 5, 1, and the next remainder must be one of these numbers; it turns out to be 3.

The teacher should stress the nature of the repeating decimal numeral, emphasizing that $.\dot{3}\dot{7}$ means .37373737...and explaining carefully the use of the dots. The equation $10 \times .\dot{7}\dot{6} = 76.\dot{7}\dot{6}$ should then be discussed and this should lead on to the problems in the text.

Answers to Problems 8-10(c), page 8-23

1. (i) 100 (ii) 10 (iii) 1000 (iv) 100000 (v) 10000
2. (a) $34.5\dot{3}\dot{4}\dot{5}...$ (b) $3.45\dot{3}\dot{4}\dot{5}...$ (c) $345.\dot{3}\dot{4}\dot{5}$
3. (a) 99x (d) 8
   (b) 9x (e) 93476
   (c) 9999x (f) 7245

Answers to Problems 8-10(d), page 8-24

1. (a) $\frac{2}{3}$ (d) $\frac{1}{7}$
   (b) $\frac{7}{8}$ (e) $\frac{3}{11}$
   (c) $\frac{5}{33}$ (f) $\frac{16}{99}$
2. (a) $\frac{43}{7}$ (c) $\frac{617}{500}$
   (b) $\frac{74}{33}$ (d) $\frac{883}{55}$

8-11. Numbers which are not Rational.

The purpose of this section is to indicate the existence of numbers which cannot be expressed as terminating or repeating decimal numerals. The children should be led gently into this idea and the following example may be given them:

0.12345678910111213141 5... where the dots indicate that the next digits are 161718, etc., in the form of the counting numbers. This numeral does not terminate and does not repeat. The students should be able to make simple examples like 0.24681012... or 2.304005000600007... .

8-12. Percentages.

The aim of this section is to introduce that aspect of rational numbers which we call percentages. The definition of the term percentage can be easily understood by the students if they recall what they had earlier learned about fractions. They must have learned, in particular, that $\frac{7}{10}$ means that a whole thing is divided into ten equal parts and 7 of those parts are represented by the fraction $\frac{7}{10}$. Every part is a tenth part of the whole.

The teacher should then explain that a "per cent" means a hundreth part. He might at this stage ask one student to express "a per cent," or 1 per cent, as a rational number on the chalk board.

The teacher should at this stage comment on the word " per cent" and its derivation from the Latin phrase "per centum." It may be in order, after this, to get the students to write a few examples on the chalk board. The following called out by the teacher will do:

8 per cent, 10 per cent, 30 per cent,
34 per cent, 55 per cent, 89 per cent,
72 per cent, 92 per cent.

The following equivalent fractions are now written on the chalk board, $\frac{7}{10}$, $\frac{14}{20}$, $\frac{21}{30}$, $\frac{35}{50}$, $\frac{70}{100}$, to find out whether the students recognize them as equivalent fractions. If they do (and there is no reason why they should not) the first fraction can be discussed in terms of percentages.

A boy got 7 sums out of 10 correct. What percentage of the sums did he get correct? They know that the score of the boy is

8. $\frac{67}{100} = 67 \times \frac{1}{100} = 67\%$

9. $\frac{15}{100} = 15 \times \frac{1}{100} = 15\%$

10. $37 \times \frac{1}{100} = \frac{37}{100} = 37\%$

Answers to Problems 8-12(b), page 8-27

1. (a) $66\frac{2}{3}\%$
   (b) 50%
   (c) 75%
   (d) $133\frac{1}{3}\%$
   (e) 140%
   (f) 400%
   (g) $33\frac{1}{3}\%$
   (h) $166\frac{2}{3}\%$
   (i) 20%
   (j) 40%
   (k) 60%
   (l) 80%
   (m) 100%
   (n) $53\frac{1}{3}\%$
   (o) 120%

2. (a) $\frac{7}{200}$
   (b) $\frac{1}{3}$
   (c) $\frac{13}{20}$
   (d) $\frac{11}{200}$

3. (a) £ 4.2s.6d.
   (b) £ 4.4s.0d.
   (c) £ 50.8s.0d.

represented by the fraction $\frac{7}{10}$. To show that $\frac{7}{10}$ is the same as $\frac{70}{100}$ should not be difficult. They must then be told that if a fraction is transformed into one with 100 as its denominator, it is said to be expressed as a <u>percentage</u>; the numerator of the fraction is called the <u>rate</u>. Example:

> In the percentage $\frac{75}{100}$, 75 denotes the rate. The symbol [$\% = \frac{1}{100}$] should now be introduced. They should then be given a few examples, preferably those dealt with earlier, and asked to state the rate in each case, and how each should be written with the symbol $\%$.

<u>Answers to Problems 8-12(a), page 8-26</u>

The method of expressing a fraction as a percentage is already suggested in the discussion that went before this paragraph. The time is now opportune for the teacher to state the method in the general form as it appears in the text:

$$\frac{a}{b} = \frac{x}{100} = x \times \frac{1}{100} = x\%$$

1. $90\% = 90 \times \frac{1}{100} = \frac{90}{100}$

2. 30 per cent $= \frac{30}{100} = 30 \times \frac{1}{100}$

3. $\frac{5}{100} = 5 \times \frac{1}{100} = 5\%$

4. $7 \times \frac{1}{100} = \frac{7}{100} = 7\%$

5. $55\% = 55 \times \frac{1}{100} = \frac{55}{100}$

6. $100\% = \frac{100}{100} = 1$

7. $45\% = 45 \times \frac{1}{100} = \frac{45}{100}$

(d) £23.1s.0d.
(e) $187.57
(f) $12.48

4. 11 %
5. 16 %
6. $67\frac{23}{31}$ %

7. $78\frac{4}{7}$ %; 28.
8. 5,000
9. $18\frac{1}{3}$ %
10. 522
11. $31\frac{13}{17}$ %
12. 50 %

8-13. Percentages and Decimals.

The students already know how to convert fractions into percentages and vice versa. The aim of this section is to show how we can convert percentages into decimals and vice versa.

What the teacher may do here is as follows: The students are asked to express 45 % as a fraction in hundredths. The answer, of course, is $45 \times \frac{1}{100}$.

They may then be asked to express $\frac{45}{100}$ in decimal notation. They all know this from what they have done earlier, so the answer 0.45 is given. They then deduce that

$$45\% = 0.45.$$

It is likely that if they are asked to write 40 % as a decimal, some may put down 0.40. They should be reminded that in decimal notation 0.40 can also be written as 0.4.

The students have seen from the above that we adopted a method of conversion as follows:

Percentage ------> Fraction with denominator ----> Decimal.

Answers to Problems 8-13, page 8-29

1. (a) 32.5 %
   (b) 5 %
   (c) 230 %
   (d) 0.8 %
   (e) $85\frac{5}{7}$ %

2. (a) $0.\dot{3}$ (d) 0.125
(b) 0.005 (e) 2.5
(c) 0.015 (f) 0.75

8-14. Applications of Percentages.

The teacher may begin by discussing why anyone goes into business. Many reasons may be given for this, but it is generally true that a businessman wishes to make profit. He buys and sells, and hopes to sell for more than he buys. If he sells for more than he bought, then he makes a profit. If he sells for less than he bought, he takes a loss. It is possible that he may sell for exactly as he bought.

The teacher may here wish to give some oral exercises in profit and loss: (a) If I buy a dozen eggs for 1s.9d. and sell them at 2d. each, how much do I gain? I gain 3d. Since I sold the eggs for 12 × 2d. = 2s, then I made 2s. - 1s.9d. = 3d. profit.

(b) If I buy six exercise books for 3s.6d., how much do I gain or lose if I sell them at 5d. each? I lose 1s. since I sell them for 5 × 6d. = 30d. = 2s.6d. and 3s.6d. - 2s.6d. = 1s.

The teacher may then lead the class to the next topic by asking the class which is the better transaction, buying a book for 10s. and selling it for 10s.6d., or buying a book for 5s. and selling it for 5s.6d.?

The person who invested in the first book could have used the money to buy two of the second book and in that way gain 1s. Thus the second transaction is better than the first.

But not every set of transactions may be so clear. Which is the best transaction, buying a book for 9s. and selling it for 12s., or buying a book for 8s. and selling it for 10s., or buying a book for 10s. and selling it for 13s.6d.? We will be able to find the best transaction if we know the relationship between the ratios $\frac{3}{9}$, $\frac{2}{8}$, or $\frac{3\frac{1}{2}}{10}$.

Hence the ratio of the profit to the cost price (c.p.) gives us an adequate basis of comparison. We express this ratio as a percentage, so that comparison is easy.

Answers to Problems 8-14(a), page 8-31

1. $14\frac{2}{7}\%$
2. $62\frac{1}{2}$
3. $22\frac{8}{11}\%$ profit
4. £ 780
5. 8s.$7\frac{1}{2}$d.
6. £ 150
7. £ 366. 133s.4d.
8. $18\frac{3}{4}\%$
9. 4s.6d.
10. 2d.

Commissions and Discounts.

The teacher may prepare for this lesson by obtaining local material on commissions and discounts. Shops or newspaper offices may supply this. It will be of value if the students could have a practical notion at first hand of commissions and discounts.

Answers to Problems 8-14(b), page 8-33

1. £ 540
2. £ 6.17s.6d.
3. £ 600
4. £ 900
5. £ 36.8s.4d.
6. $14\frac{1}{16}\%$

Simple Interest.

Many students use the Post Office banking system. The Post Office uses compound interest to compute amounts. This interest system should be explained to the students, and also the reason that simple interest is studied as a prelude to compound interest.

Answers to Problems 8-14(c), page 8-34

1. (a) £ 45
   (b) £ 73.11s.$4\frac{4}{5}$d.
   (c) £ 62.10s.
   (d) 56 dollars 25 cents = $56.25
   (e) 100 francs.

2. 1 year 9 months
3. $3\frac{1}{2}$ %

4. $14\frac{2}{7}$ yrs.

5. £ 150
6. £ 675
7. £ 5.5s.
8. $12\frac{1}{2}$ %

9. £ 183.6s.8d.

Answers to Revision Problems, page 8-35.

1. (a) 35.63
   (b) 403.107
   (c) 1.0103
   (d) 0.01705
   (e) 9090.909
   (f) 58.087
   (g) 105.0077
   (h) 3.001003

2. (a) $1 + 2\left(\frac{1}{10}2\right) + 4\left(\frac{1}{10}3\right) + 9\left(\frac{1}{10}4\right)$

   (b) $5 + 3\left(\frac{1}{10}2\right)$

   (c) $3(10^4) + 7(10^3) + 3(10) + 5(1) + 4\left(\frac{1}{10}\right) + 4\left(\frac{1}{10^2}\right) + 9\left(\frac{1}{10^3}\right)$

   (d) $7\left(\frac{1}{10}3\right) + 5\left(\frac{1}{10}4\right)$

   (e) $8\left(\frac{1}{10}3\right) + 2\left(\frac{1}{10}4\right) + 1\left(\frac{1}{10}5\right)$

   (f) $1(10^3) + 1\left(\frac{1}{10}4\right)$

3. (a) $x = 1$
   (b) $x = 10^2 = 100$
   (c) $x = \frac{1}{10^3} = \frac{1}{1000}$
   (d) $x = \frac{1}{10}$

(e) $x = \frac{1}{10^7} = \frac{1}{10000000}$ (g) $x = 10^4 = 10000$

(f) $x = \frac{1}{10^4} = \frac{1}{10000}$ (h) $x = 10^2 = 100$

4. 4. (a) 2.2 inches (b) 2.5 inches
   (c) 1.0 inch
   5. £ 25.12s.6d.
   6. 220 yds.
   7. 437 yards 16 inches
   8. 547.5 miles
   9. (a) 7
      (b) 11
   10. (a) 8808.6 miles (b) 2602.9 miles
      (c) 3714.1 miles
   11. (a) 0.45 (b) 0.125
      (c) .075
   12. (a) .25 (b) .5
      (c) .85 (d) .025
      (e) 3.05 (f) 1.125

   13. 50.4 cwt.
   14. £ 13.15s.
   15. 3,279,900

FIRST YEAR SECONDARY TEACHERS' GUIDE

CHAPTER 9

MEASUREMENTS

Commentary.

This chapter is a very rough first draft, so that it is hoped that the teacher using this chapter will rely heavily on his own experience. Fortunately, geometry has traditionally been the strongest part of the curriculum and this fact makes it possible to supplement the material here from readily available sources.

The four main objectives of this chapter are as follows:

1. Recognition by the student of the basic differences in measurement of sets of distinct objects and sets which are continuous.
2. Recognition by the student that the geometric object (a set of points) is quite different from its measure (a number).
3. Student familiarity with formulas for the measures of some of the simpler geometric objects.
4. Recognition that measurement of real objects always results in an approximation. In fact, it does not make sense to speak of the exact measure of any real, continuous object.

Class Activity, page 9-4

1. The students may suggest such examples as the bananas in a basket, the flowers in a vase, the eggs in a basket, etc.

2. The teacher should have a basket of bananas on the desk.
   (a) Students should tell the number of bananas.
   (b) By counting.
   (c) The set of whole numbers.

Class Activity, page 9-4

2. (h) with the smallest unit
   (i) the smallest unit

3. (e) $\frac{1}{4}$"

Answers to Problems 9-5, page 9-7

1. Students should be able to deduce from the tabulated results that the sum of the measure of two sides of this triangle is greater than the measure of the third side. Students should be able to compare the results cm/in. with their neighbors. The expected result is 1 inch = 2.54 cm.

2. (c) Yes.
   (d) They meet two-thirds of the way from each vertex.

3. (c) $m\overline{YZ} = \frac{1}{2}m\overline{AB}$; $m\overline{XY} = \frac{1}{2}m\overline{BC}$; $m\overline{XZ} = \frac{1}{2}m\overline{AC}$

4. (a) $m\overline{AO} = m\overline{OB} = m\overline{OC}$
   (c) Yes.

5. $\frac{2}{1}$

6. $\frac{2}{1}$ and $\frac{2}{1}$

Answers to Problems 9-6, page 9-11

1. Yes.

2. It is the vertex.

3. Yes.

4. Cannot point out "angle B." B is the vertex of $\angle EBD$, $\angle DBA$, $\angle ABC$, $\angle EBA$, etc.

Answers to Problems 9-7, page 9-13

1. Yes.
2. Y and P.
3. Exterior of $\angle AXB$.
4. $\angle AXB$
5. No. Yes.
6. In the exterior or on the angle.

Answers to Problems 9-9, page 9-15

2. (a) $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$, $\overrightarrow{OD}$
   (b) 4
   (d) Same measure.
3. (b) $\angle ACD$ and $\angle DCB$
4. (b) 90°
   (c) Yes.
5. (c) Yes. $m\angle APD = m\angle BPC$
   $m\angle APB = m\angle DPC$
6. (b) 90°

Answers to Problems 9-10, page 9-17

1. (b) 180°
   (c) (i), (ii), (v), (vi)
   (d) (iv), (vii), (viii)
   (e) (iii)
   (f) (vi), (viii)
   (g) (vi)
   (h) (i), (ii), (iii), (iv), (v), (vii)
2. (a) isosceles, obtuse
   (b) right, isosceles
   (c) scalene, obtuse
   (d) equilateral, acute, equiangular
   (e) acute, scalene

2. (f) right, scalene
   (g) obtuse, isosceles

   Note: Teacher can fill in more data to make triangles of different types.

Answers to Problems 9-11, page 9-19

2. $m\angle ACX = m\angle A + m\angle B$
3. (a) 1
   (b) 2
   (c) 180°, 360°, 540°, 720°, 900°
4. 360°
5. (a), (d)
6. $\frac{6 \times 180}{8} = 135^\circ$
7. 10, 8

Answers to Problems 9-12, page 9-21

1. (a) e,g; f,h; a,c; b,d; yes
   (b) b,g; c,f; yes
   (c) e,d; a,h; yes
   (d) a,e; b,f; e,h; etc.

Answers to Problems 9-13, page 9-23

2. 6"
3. 5", 12"
4. 10", 10", 5"
5. 24"
6. 15"

Answers to Problems 9-14, page 9-25

1. 3.14
2. Yes. $3\frac{1}{7} = 3.14$ to nearest hundredth.

3. (a) $2 \times \frac{22}{7} \times 5 = \frac{220}{7}$ cm.

   (b) $2 \times (3.14) \times (3.2) = 20.1$ cm.

   (c) $2 \times (3.14) \times (14) = 87.92$ cm.

   (d) $14\frac{2}{3}$ cm.

   (e) 44.59 miles

   (f) $37\frac{5}{7}$ in.

   (g) $28\frac{2}{7}$ yards

4. 18, 14, 16

Answers to Problems 9-15, page 9-26

1. $\frac{7}{2} \times \frac{9}{2} = \frac{63}{4}$ square inches

2. 16

3. 12 sq. cm.

4. 16 sq. in.

5. Measures of bases and heights are the same.

6. Yes. Measures of bases same, measures of heights same.

7. (a) Yes, 9
   (b) 12, yes, 6
   (c) 6, yes, 3

9. (a) 18 sq. cm.

   (b) 25 sq. cm.

10. 6 sq. in.

Answers to Problems 9-16, page 9-28

1. (a) 16π sq. cm.

   (b) 36π sq. cm.

   (c) 144π sq. ft.

   (d) 36π sq. ft.

   (e) $\frac{25}{4}\pi$ sq. ft.

2. (a) $36\pi - 4\pi = 32\pi$ sq. ft.

   (b) $16 - 4\pi$ sq. in.

3. $\frac{3600}{30\pi} = \frac{120}{\pi} \approx 38$

Answers to Problems 9-17(a), page 9-29

1. $40 + 60 + 48 = 148$ sq. cm.

2. $2ab + 2bc + 2ac = 2\,(ab + bc + ac)$ sq. ft.

Answers to Problems 9-17(b), page 9-31

1. 84 sq. cm.

2. 112 sq. in.

3. $2 \times \pi\,(30)^2 + 2\pi \times 30 \times 6 = 2160\pi$ sq. cm.

4. (a) 4π sq. cm.

   (b) 17π sq. cm.

   (c) $\frac{876}{16}\pi$ sq. ft.

   (d) 4π sq. yd.

Answers to Problems 9-18(a), page 9-32

1. (a) 30 cubic cm.

   (b) 66 cubic ft.

2. 3 ft.

Answers to Problems 9-18(b), page 9-32

1. (a) $36 \times 36 \times 36 = 46{,}656$ cubic in.

   (b) 54

2. 24 cu. ft.

3. 36 cu. in.

2 3 4 5 6 7 8 9 10 11 12 13 14 15 RS 70 69 68 67 66 65 64