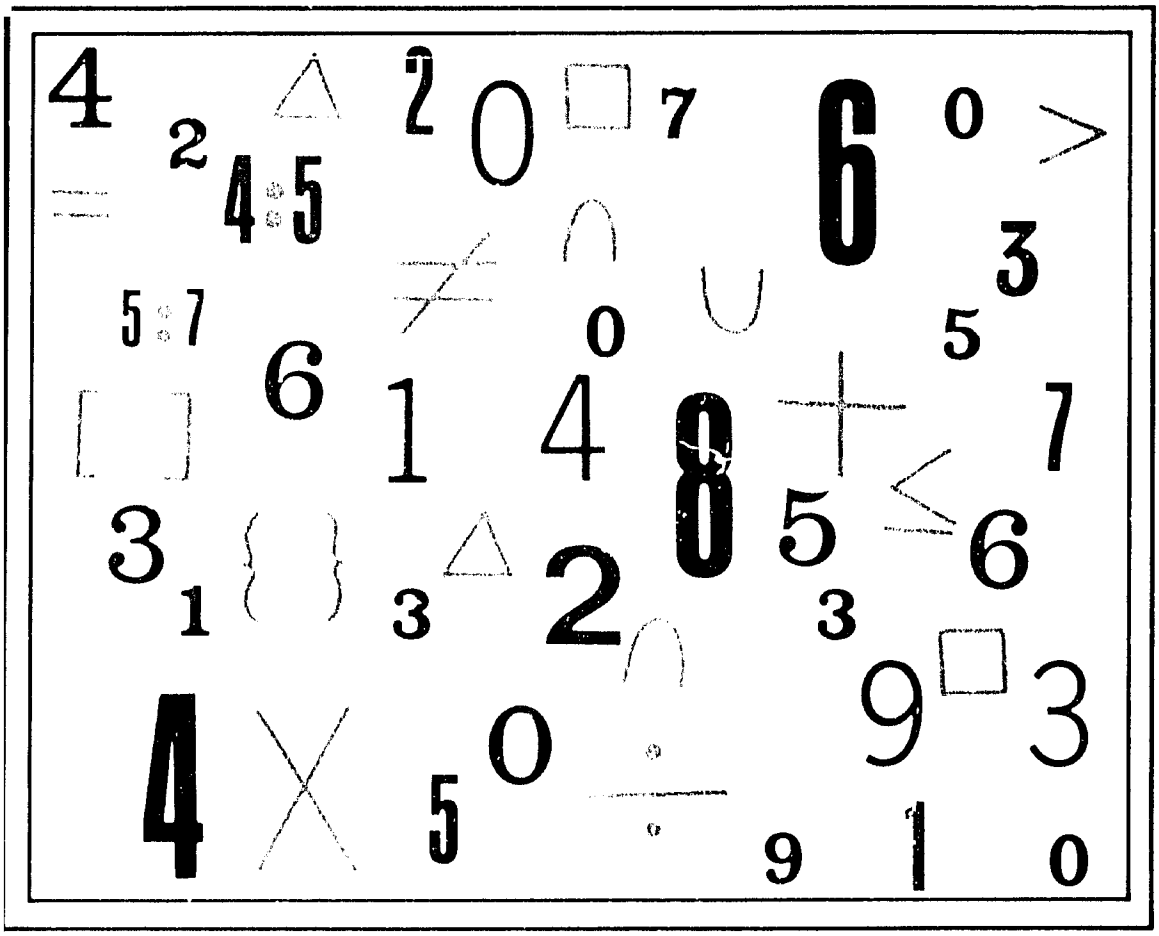
PN-AAX-991 4cbm
c.2

SECONDARY C ONE GEOMETRY

student



ntebbe Mathematics Series

Educational Services Incorporated
Newton, Massachusetts, U.S.A.

ENTEBBE MATHEMATICS SERIES

Primary One
Revised Preliminary Edition
Pupil Book: One volume
Teachers' Guide: Two volumes

Primary Two
Preliminary Edition
Pupil Book: Two volumes
Teachers' Guide: Two volumes

Primary Three
Preliminary Edition
Pupil Book: Two volumes
Teachers' Guide: Two volumes

Primary Four
Preliminary Edition
Pupil Book: One volume
Teachers' Guide: One volume

Primary Five
Preliminary Edition
To be written at 1966 Workshop

Entebbe Mathematics Teachers' Handbook, Primary I-III, Preliminary Edition

FIVE YEAR COURSE

Secondary One
Preliminary Edition
Student Text: One volume
Teachers' Guide: Three volumes

Secondary Two
Preliminary Edition
Student Text: Three volumes
Teachers' Guide: Three volumes

Secondary Three
Preliminary Edition
Student Text: Algebra—One volume
Geometry—One volume
Teachers' Guide: Algebra—One volume
Geometry—One volume

Secondary Four
Preliminary Edition
Student Text: Algebra—One volume
Geometry—One volum :
Teachers' Guide: Algebra—One volume
Geometry—One volume

Secondary Five
Preliminary Edition
To be written at 1966 Workshop

FOUR YEAR COURSE

Secondary C One
Preliminary Edition
Student Text: Algebra—One volume
Geometry—One volume
Teachers' Guide: Algebra—One volume
Geometry—One volume

Secondary C Two
Preliminary Edition
To be written at 1966 Workshop

Basic Concepts of Mathematics, An Introductory Text for Teachers, Preliminary Edition

Volume I —*Structure of Arithmetic*—Chapters 1-21
Volume II —*Structure of Arithmetic*—Chapters 22-46
Volume III—*Foundations of Geometry*—Chapters. 47-57
Volume IV—*Measurement, Functions and Probability*—Chapters 58-68

GEOMETRY
Student
Text

ENTEBBE MATHEMATICS SERIES

Secondary C One
The First Year of a Four Year Course
Preliminary Edition

Prepared at the 1965
Mombasa Mathematics Workshop

Published for the
AFRICAN EDUCATION PROGRAM
of
EDUCATIONAL SERVICES INCORPORATED
Newton, Massachusetts
by
SCIENCE RESEARCH ASSOCIATES, INC.
a subsidiary of International Business Machines Corporation
Chicago, Illinois

ACKNOWLEDGEMENTS

This portion of a Student Text for Secondary C One Mathematics and its associated Teachers' Guide was prepared during the summer of 1965 at Mombasa, Kenya, as part of a program of curriculum study being conducted by the African Education Program of Educational Services Incorporated.

Funds for the Program have been provided by the United States Government through the International Cooperation Administration, the Agency for International Development, and the Department of State; by the Ford Foundation; and by the Louis and Pauline Cowan Foundation, Inc. The Secondary C One books are being published pursuant to A.I.D. - financed contract No. AID/res-21 with Educational Services Incorporated.

The Mombasa Mathematics Workshop was directed by Professor W. T. Martin of Massachusetts Institute of Technology, Cambridge, Massachusetts and Mr. John Oyelese of the University of Ibadan, Ibadan, Nigeria.

Secondary C One manuscripts were written at the 1965 Mombasa Workshop by the following:

Chairman: Dr. Ross L. Finney

Princeton University, Princeton, New Jersey

Professor Jean M. Calloway

Kalamazoo College, Kalamazoo, Michigan

Professor Roy Dubisch

University of Washington, Seattle, Washington

Mr. Isaac Hunja

Kagumo School, Kiganjo, Kenya

Mr. Edward Kizza

University College, Nairobi, Kenya

Mr. Robert L. N. Offurum

King's College, Lagos, Nigeria

Mr. Joseph P. Pavlovich

Comprehensive High School, Abeokuta, Nigeria

Mr. Richard S. Presser

Elston Senior High School, Michigan City, Indiana

Dean Jerome M. Sachs

Chicago Teachers College North, Chicago, Illinois

Mr. Henry Swain

New Trier Township High School, Winnetka, Illinois

Mr. Robert P. Taylor

Tororo Girls' School, Tororo, Uganda

Mr. J. E. Modupe Taylor-Pierce

Secondary Technical School, Freetown, Sierra Leone

The Student Text and Teachers' Guide were completed and edited in the United States under the direction of Professor Donald E. Richmond, Williams College, Williamstown, Massachusetts. He was assisted by Dr. Ross Finney, Princeton University, Mr. Richard Presser, Elston Senior High School, Michigan City, Indiana, and Dean Jerome M. Sachs, Chicago Teachers College North, Chicago, Illinois.

TO THE STUDENT

This textbook has been specially prepared for students in the first year of a four year course leading to School Certificate. In using the textbook you have the opportunity to participate in the great experiment which was started in the summer of 1962 when more than fifty mathematicians and mathematics teachers gathered in Entebbe, Uganda, to begin the preparation of mathematics books as up-to-date as those in use anywhere else in the world. The writers came to Entebbe from all the countries in English-speaking Tropical Africa, from the United States, and from the United Kingdom to take part in this project. Now you and your teachers must do your share to find out whether this book is as good as it can be made. By using the textbook critically and making the most of the opportunities it offers to make the learning of mathematics an interesting and enriching experience, you will have played a part in the new growth of mathematics education in Africa.

You can expect to meet many familiar ideas, sometimes expressed in unfamiliar ways, and there will be some new ideas. Careful reading will be important because this book treats mathematics as a language designed to express certain kinds of ideas, and you must learn to read and speak this language.

You will learn why numbers behave as they do and find that algebra helps us more and more to understand arithmetic ideas. You will also explore some geometric ideas, observing properties of sets of points. You will even learn to make algebra and geometry work together, each making the other clearer.

Do not hesitate to ask questions. Talk about your work with your teacher and your fellow students. Above all, try to look upon mathematics as something that can be exciting and rewarding, for if you give it half a chance, that is exactly what it is.

CONTENTS

Chapter 1 Some Elementary Ideas and Figures

1-1 Introduction 1

1-2 Points 2

1-3 Straightness 3

1-4 Line Segments 7

1-5 Rays and Straight Lines 11

1-6 Flatness and Planes 15

1-7 Regions 18

1-8 Paths 20

1-9 Polygons and Polygonal Regions 22

1-10 Drawing a Circle 25

Chapter 2 Plane Figures

2-1 Introduction 31

2-2 Pairs of Line Segments 31

2-3 Perpendicular Bisector 36

2-4 Pairs of Straight Lines 44

2-5 Rays and Angles 56

2-6 Comparing Angles 60

2-7 Angles Made by Straight Lines 62

2-8 Alternate Interior Angles 66

2-9 Comparing Angles Without Folding 69

2-10 Bisecting an Angle 75

Chapter 3 Polygons and Other Simple Closed Paths

3-1 Introduction 79

3-2 Polygons 79

3-3 Triangles 83

3-4 Quadrilaterals 90

3-5 Plane Paths, Circles and Ellipses 98

3-6 Simple Closed Paths and Regions in a Plane 103

Chapter 4 Figures in Space

4-1 Introduction 111

4-2 Surfaces and Solids 111

4-3 Pyramids 118

4-4 Points, Lines and Planes in Space 121

4-5 Prisms and Cylinders 129

4-6 Closed Surfaces 136

4-7 Conclusion to the First Four Chapters 138

Chapter 5 Length

5-1 Counting and Measuring 141

5-2 Measurement of Length 143

5-3 Origins of Standard Units 147

5-4 Standardizing Units of Measure 149

5-5 Measuring Distance 151

5-6 Using a Ruler 157

5-7 A Property of Length 161

Chapter 6 Area

6-1 The Size of a Region 165

6-2 Measurement of Area 167

6-3 Measurement of the Areas of Right-Angled Triangles 171

6-4 Measurement of the Areas of Non-right-angled Triangles 173

6-5 Areas of Other Polygons 178

6-6 Areas of Other Regions 181

Chapter 7 Volume

7-1 Measurement of Volume 185

7-2 Volumes of Prisms 189

7-3 A Direct Measurement of Volume 194

Chapter 8 Angle Measure

8-1 Measurement of Angles 197

8-2 The Unit Angle 200

8-3 The Addition Property of Angles 205

8-4 Congruent Angles 208

8-5 Relations Between Angles 210

8-6 Angles of Polygons 216

Chapter 9 Related Measurements and a Backward Look

9-1 Relations Between Measurements 219

Conclusion 225

Chapter 10 Symmetry

10-1 Symmetric Geometric Figures 229

10-2 A Symmetry Test 231

10-3 Making Symmetrical Figures 234

10-4 Congruence 235

10-5 Symmetry in Space 240

CHAPTER 1

SOME ELEMENTARY IDEAS AND FIGURES

1-1 Introduction

The world is full of physical objects. Touching or handling such objects enables us to get an idea of their shapes and sizes. We are able to tell the differences between smooth and rough surfaces, small and large objects. We notice that some of these objects have ends, corners, edges, sides, etc. Some of their parts are straight, some are flat, some are round.

Touching and seeing enable us to tell the differences between the shapes and sizes and forms of the physical objects around us, and thus we are able to classify certain of these objects according to the characteristics they exhibit. In the same way we learn first to tell apart the objects in a set and then to count the objects. Thus both counting, the earliest part of arithmetic, and simple figures, the earliest part of geometry, arise from handling physical objects and from the impressions our minds get from these actions. In our study of geometry we shall try to develop geometric figures as ideas that come from the physical objects that we encounter in the world around us.

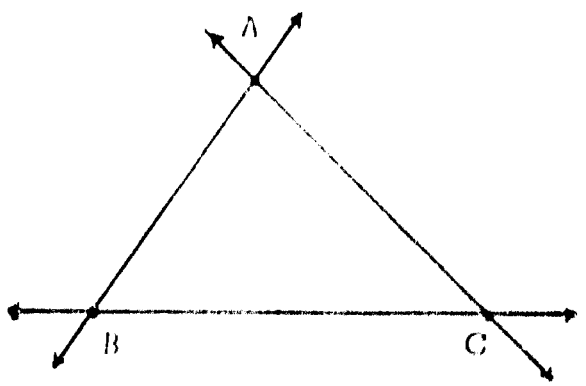
The idea of a number in arithmetic is a mathematical idea that grew out of a need to classify certain sets of objects as to how many objects they contained. But no one has ever seen, touched, felt, or smelled a number. In geometry, the ideas of point, line, plane, and space are mathematical ideas that grew out of a need to classify certain sets of figures and to measure their boundaries, or the regions bounded by the figures. But no one has ever seen or touched a point, line, or a plane. Just as in arithmetic you studied numbers and the operations on them, in geometry you will study points, lines, and planes and how they relate to each other.

1-2 Points

Let us begin with a simple figure in geometry, a point. Consider the following: the tip of a pin or a needle; the end of a sharpened stick or pencil; the corner of a box or a piece of paper; a grain of sand. All of these suggest what we mean by a point. Is the period you see at the end of the last sentence a point? Is the dot you make on a paper with a sharpened pencil a point? Is the hole you make in a paper with the sharp tip of a pin a point? If you answered yes to the last three questions, you have an idea of what we imagine a point to be, but it is incorrect to say that a period, a dot, or a pin hole <u>are</u> points. Points, like numbers in arithmetic, are a creation of the human mind. They do not exist in the same way that trees, houses, and animals exist. They are not physical objects that we can see or feel. The smallest dot you can make with your pencil is not a point. It is simply a picture of what we, in our minds, imagine a point to be.

This is the idea that we wish to convey about points. We will think of a point as having <u>position</u>, but <u>not size</u>. Thus the pictures of points that we draw on our papers, the blackboard, or the ground are simply an aid in helping us to see and remember the position of certain points as we try to develop new ideas from them, but we will not think of a point as having any measurable size. When we say "draw a point", "mark a point", "consider point P", etc., we will mean "draw a <u>picture</u> of a point", "mark a <u>picture</u> of a point", "consider the <u>picture</u> of point P", etc.

When we want to represent a point in a figure, we will make a small dot and label it with a capital letter. For example, in the figure below the letters A, B, and C are used to label certain points.



Problems 1-2

1. Find some physical objects in your room that resemble what we imagine a point to be.

2. In the figure above, how many points do you think are between points B and C? How many points do you think are "inside" the figure?

3. How many points are at the tip of one corner of this book? How many points are at the tips of all the corners of this book? How many points do you think are on the top cover of this book? How many points do you think are on an edge of the top cover?

4. Make a drawing that would represent:

   (a) A set of two points.

   (b) A set of five points.

   (c) A set of nine points.

   (d) A set of one point.

1-3 Straightness

Let us now move to another idea we get from physical objects — the idea of straightness. Which of the following objects would you call "straight"?

(a) The edge of this book.

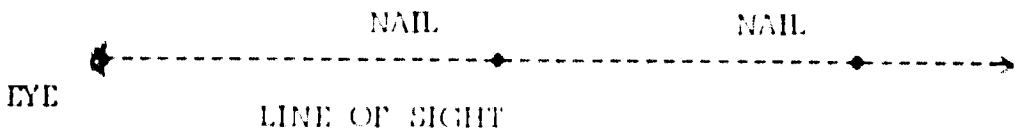
(b) The fold in a paper.

(c) The edge of a banana leaf.

(d) A stretched string.

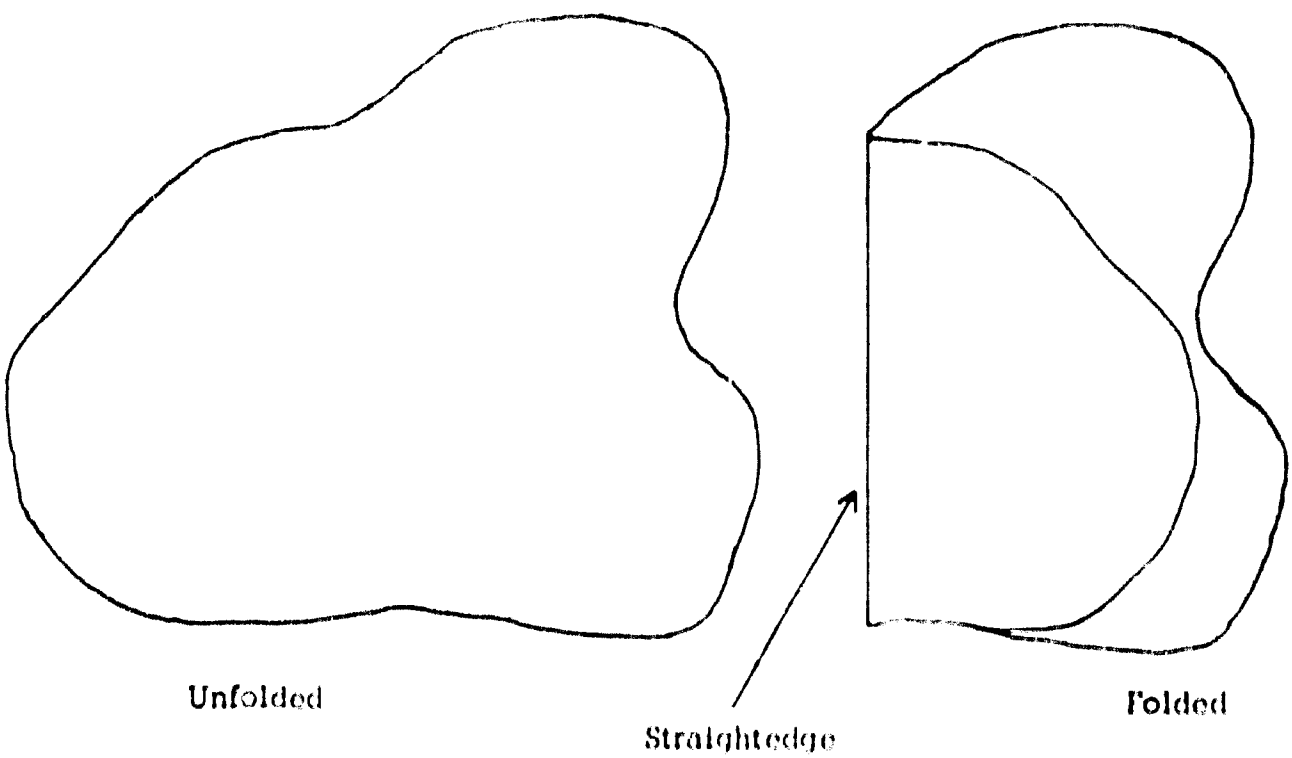
(e) The trail of a snail.

If your answers were (a), (b), and (d) you already have an idea of what is meant by "straight". The most natural way of deciding whether something "looks straight" is to sight along it. If you consider two fixed nails and arrange your "line of sight" so that the two nail points coincide in your vision; that is, one nail hides the other, we say that this "line of sight" is straight.

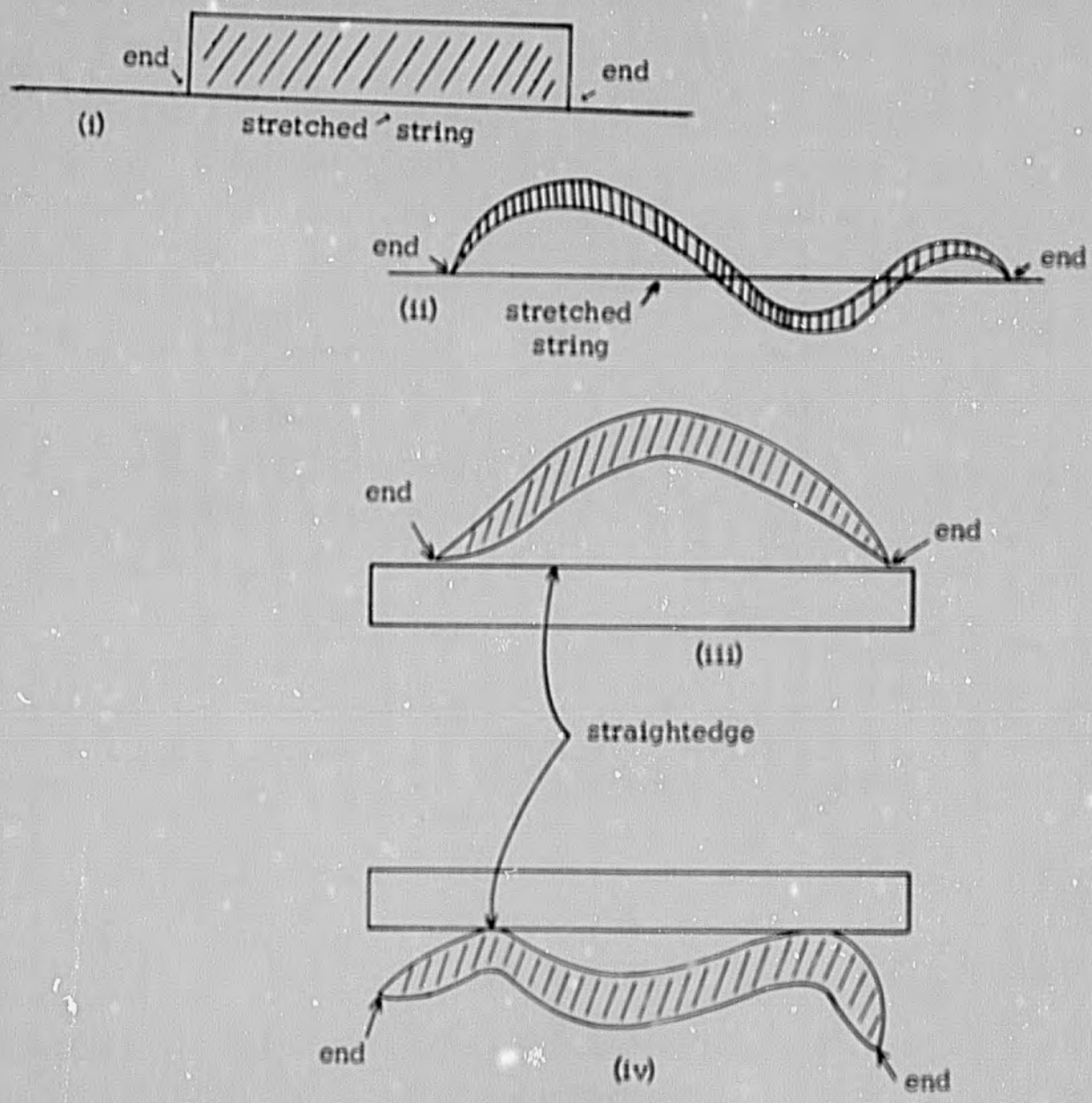


Thus we have one test for straightness — line of sight. But we can also test for straightness by fitting an object which we already know to be straight (for example, a stretched string or the edge of a ruler) along the ends of the object to be tested.

An easy and useful way to prepare a straightedge is to fold a piece of paper of any shape.



If the straightedge fits all along the object to be tested, the object is straight. Here are pictures of some tests.



In the fourth picture, the test is not being made correctly. The straightedge is not fitted at the ends of the drawing. In the other pictures, the test is made correctly and only the first object tested is straight.

To sum up: an object is straight if whenever we fit next to it something we know to be straight, the two fit exactly. Thus we can test straightness by a line of sight, a stretched string, or a straightedge such as a ruler or a folded paper.

Problems 1-3

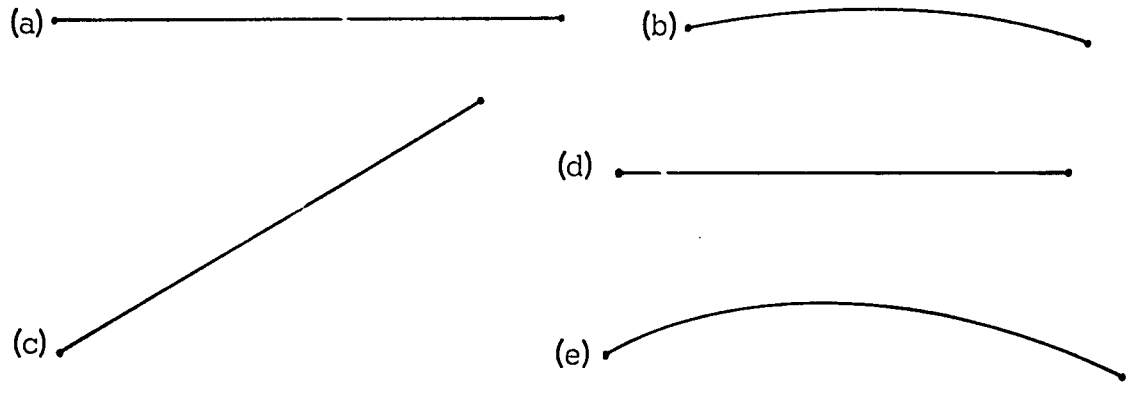
1. Name five objects, besides those mentioned in the text, that you would call straight.

2. Find three things near you that you believe to be straight. Test them first by sighting, and then by using a folded piece of paper for a straightedge.

3. Test two edges of this book for straightness.

4. Find or make something you think is not straight. Test it for straightness.

5. Here are some drawings. Test them for straightness, first by glancing at them, and second by using a folded paper for a straightedge.

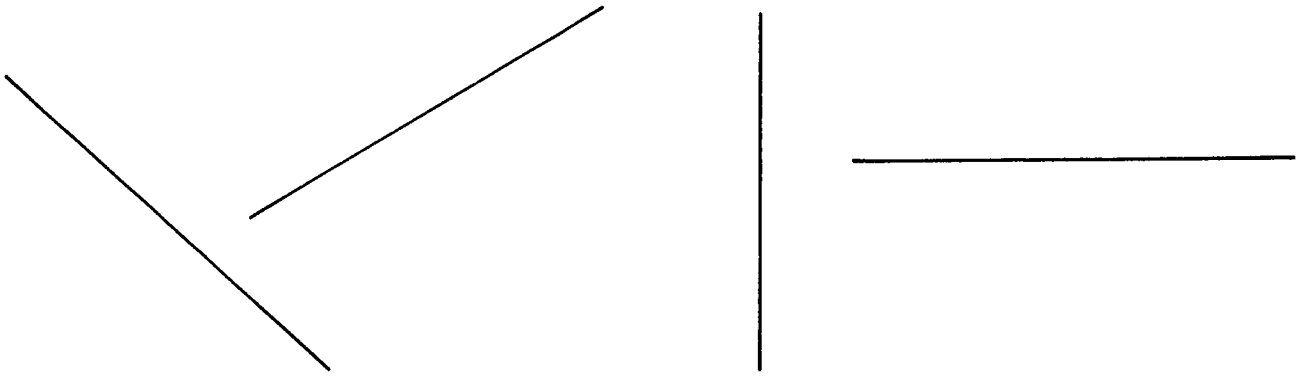


6. Test the edge of your ruler for straightness.

7. Mark two points on your paper and name them A and B. By sighting from A to B, mark two points on the line of sight from A to B and name them C and D. Cover A and B with small pieces of paper. On the line of sight from C to D mark another point and name it P. Do you think P is on the same line of sight from A to B? Uncover A and B and test your accuracy.

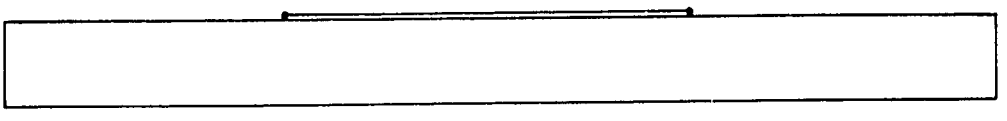
1-4 Line Segments

If you take straightedges and trace them you will make drawings that may look like these:



Straight drawings like these we will call pictures of line segments. We can be sure a drawing is straight if we make it by tracing a straightedge. A line segment, like a point or a line of sight, cannot be touched or seen. It, too, is an invention of the mind and does not exist in the way that a ruler or a folded paper exists. The pictures we drew above are pictures of what we imagine a line segment to be. Later, you will learn that a line segment has a measurable length, that is, a definite number representing its length, but it has no measurable width.

If we choose any two points, there is a line of sight from the first point to the second. If we put a straightedge (ruler) to these two points,

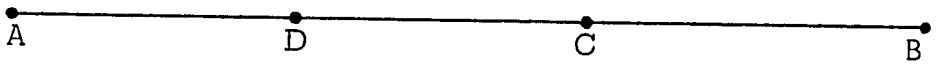


and trace along the edge from either point to the other, we have traced the line of sight from one point to the other. The resulting drawing is a picture of a line segment. The two given points are called its endpoints and we say that the line segment connects the two endpoints. If the two points are named, say P and R, we use the name $\overline{PR}$ or $\overline{RP}$ for the line segment having these endpoints.

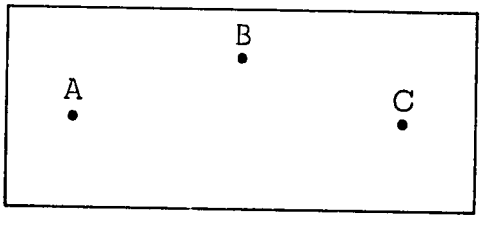
If A and B are any two points, how many line segments do you think can be drawn from A to B? (Remember, points have position, but not size.) How many "lines of sight" are there from A to B?

A line segment contains many other points besides its endpoints. Any dot you make on a line segment shows a picture of a point, and there are many dots that can be made. However, our pictures of points (dots) on the line segment soon begin to overlap because the pictures we draw of points have size. If we imagine points as having no size, do you think we would ever run out of points on a line segment, or be able to count all of them? It is a geometric fact that a line segment has an unlimited or infinite number of points on it.

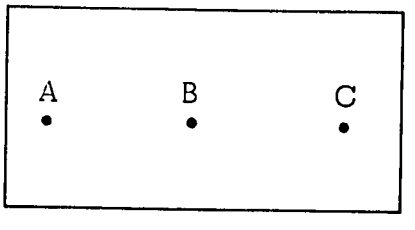
Suppose that we begin with a line segment $\overline{AB}$ and mark two more points, C and D, on $\overline{AB}$.



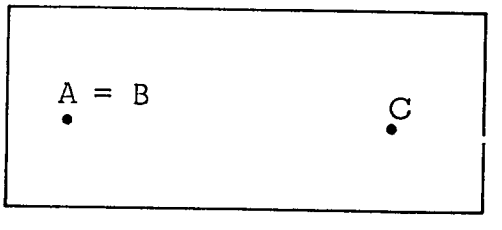
Both C and D are on $\overline{AB}$ and we say that each of them is <u>between</u> A and B. The point D is on the line segment $\overline{AC}$ and so it lies <u>between</u> A and C. The point C is on the line segment $\overline{DB}$ and so it lies <u>between</u> D and B. In general, we say that one point is <u>between</u> two others if no two of the points are the same and all three lie on the same line of sight.



B is <u>not</u> between A and C



B is between A and C



B is <u>not</u> between A and C

We can now define a <u>line segment</u> as the set of points consisting of <u>two points</u> and <u>all the points between them</u>.

Problems 1-4

1. Draw a line segment $\overline{AB}$ and mark points C, D, and E on it as shown.

Write down as many true statements as possible of the kind shown below:

(a) E is between A and C.

2. In the figure for Problem 1, which of the following statements are true and which ones are not true?

(a) A is between B and E.

(b) C is not between A and D.

(c) E is between A and D.

(d) E is between A and C.

(e) C is not between E and B.

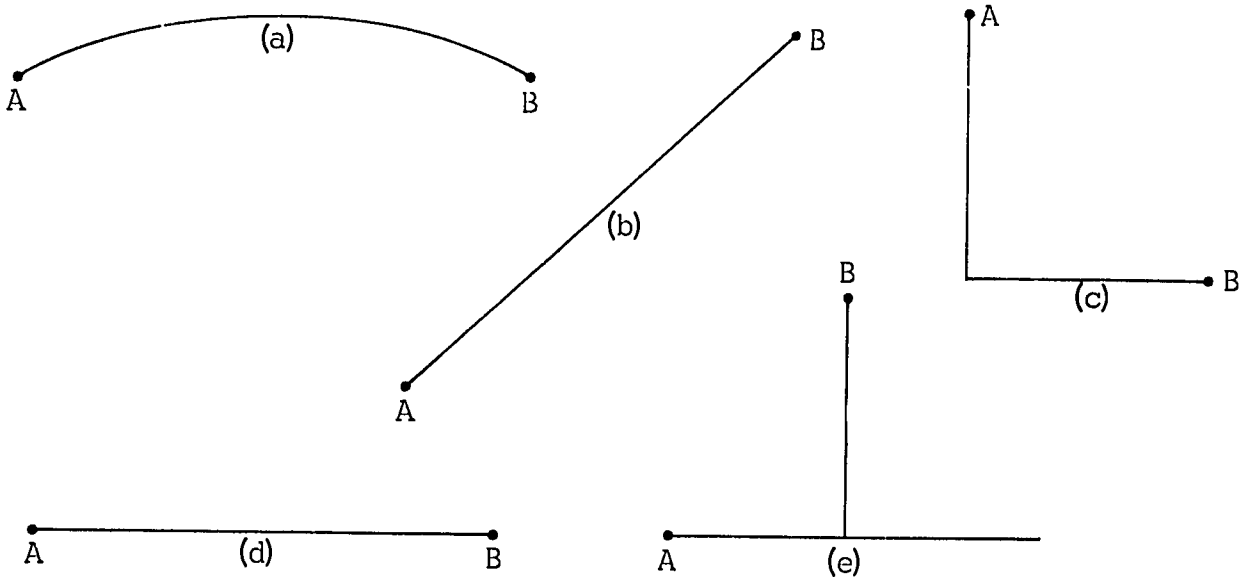
(f) B is between D and E.

(g) D is between E and B.

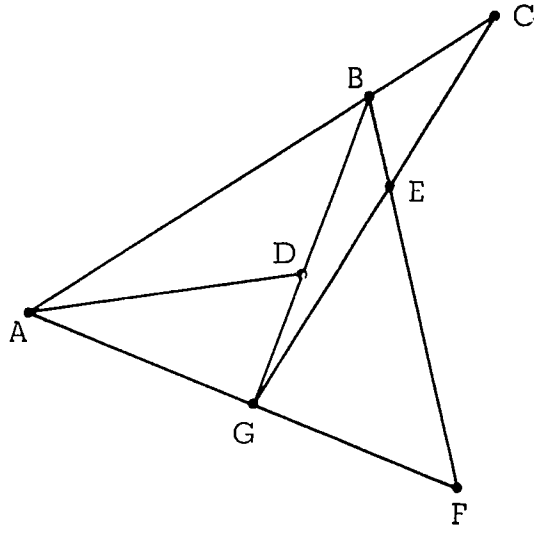
(h) E is between D and B.

(i) C is between E and D.

3. Which of the following drawings show a picture of line segment $\overline{AB}$?



4. In the figure on the right, B is on $\overline{AC}$, D is on $\overline{BG}$, E is on $\overline{BF}$ and on $\overline{CG}$, and G is on $\overline{AF}$.



(a) How many line segments have the endpoint A? D? E?

(b) Which point is an endpoint of the least number of segments?

(c) Which points are the endpoints of the greatest number of line segments?

(d) How many more line segments can be drawn using the given points?

5. How many line segments can be drawn in the following manner?

(a) When three points are used, if they are not in the same line segment.

(b) When four points are used, no three in the same line segment.

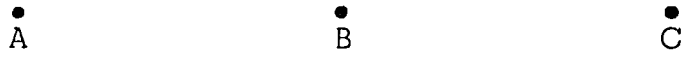
(c) When five points are used, no three in the same line segment.

(d) When six points are used, no three in the same line segment.

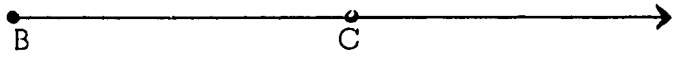
(e) Can you guess, from your answers above, how many line segments can be drawn when seven points are used, no three in the same line segment, without drawing a picture?

1-5 Rays and Straight Lines

Consider two points A and C with point B between them as in the figure below.

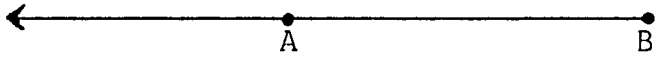


If we place our eye at point B and look toward point C, there is a line of sight from B <u>through</u> C which extends indefinitely on beyond C from B. The following drawing shows this idea.



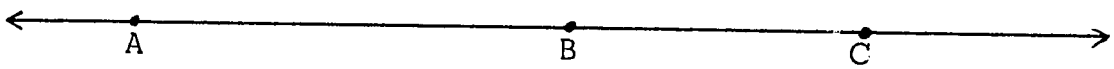
The arrow shows that the line of sight continues beyond point C <u>indefinitely</u>. Such a figure is called a <u>ray</u> and in this case it is the <u>ray from</u> B <u>through</u> C. We indicate this by writing $\overrightarrow{BC}$ (or $\overleftarrow{CB}$).

If we return to our first figure and imagine a line of sight from B through A, we would get a figure like this:



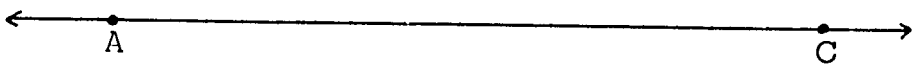
This is a picture of the ray $\overleftarrow{AB}$ or $\overrightarrow{BA}$. (Both of these symbols are read "ray BA".)

Let us now put these two rays (ray $\overrightarrow{BA}$ and ray $\overrightarrow{BC}$) opposite each other. We get a figure like this:



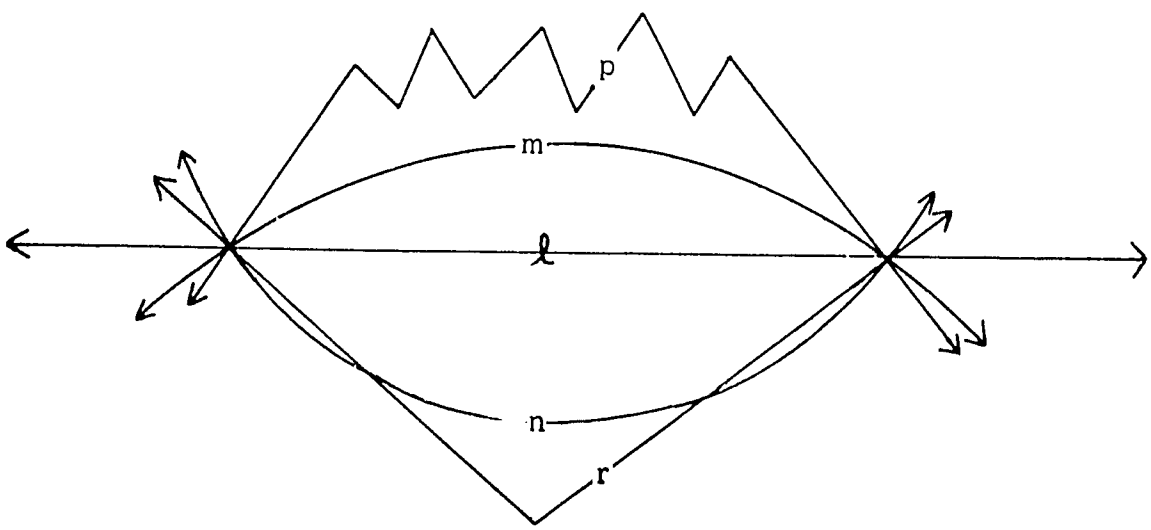
The ray from B through A goes on and on without end and the ray from B through C goes on and on without end; the two arrows remind us of this fact. We say that two rays are <u>opposite</u> each other if they have <u>only</u> their endpoints in common and lie on the <u>same</u> line of sight.

If we remove the common endpoint (B) from the two opposite rays pictured above, we get a picture like this:



We call such a figure as this a straight line. It is the straight line through points A and C, or more briefly $\overleftrightarrow{AC}$. It can be thought of as the union of the two rays $\overrightarrow{AC}$ and $\overrightarrow{CA}$.

Remembering that a point has no size, how many <u>straight lines</u> do you think two points determine? The picture below shows several "lines" through the points A and B. These lines have been named with lower case letters for reference.



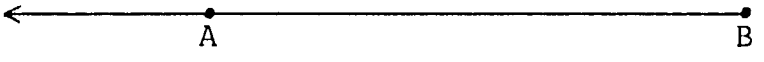
Which of the above lines would you call straight? If your answer was line ℓ then you already know what is meant by a straight line. We say that a line is straight if it lies entirely on the line of sight through two points. Since there is just the one line of sight through two points, we say that <u>two points determine exactly one straight line</u>. Hereafter, when we say "line $\overleftrightarrow{AB}$" we shall mean the one and only one <u>straight</u> line through points A and B.

To Summarize:

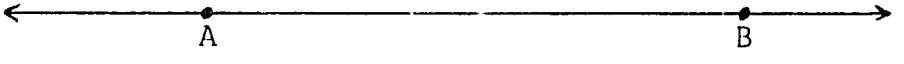
(1) This is the line segment $\overline{AB}$ or $\overline{BA}$, having the endpoints A and B.

(2) This is the ray from A through B, or $\overrightarrow{AB}$.

(3) This is the ray from B through A, or $\overrightarrow{BA}$.



(4) This is the line $\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$ containing the points A and B.



(5) Given any two points A and B,

(a) There is exactly one line segment from A to B or from B to A. (Line segment $\overline{AB}$ or $\overline{BA}$.)

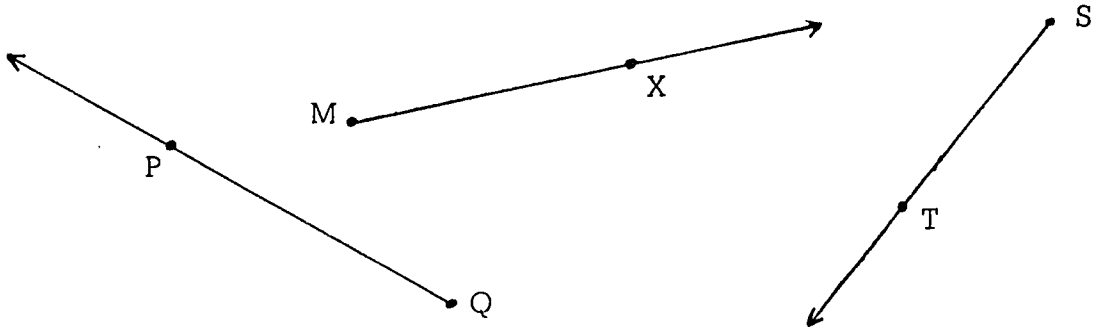
(b) There is exactly one ray through B having A as endpoint. (Ray $\overrightarrow{AB}$.)

(c) There is exactly one ray through A having B as endpoint. (Ray $\overrightarrow{BA}$.)

(d) There is exactly one straight line containing the points A and B. (Line $\overleftrightarrow{AB}$ or $\overleftrightarrow{BA}$.)

Problems 1-5

1. Here are some rays. Give the full name and short name for each.

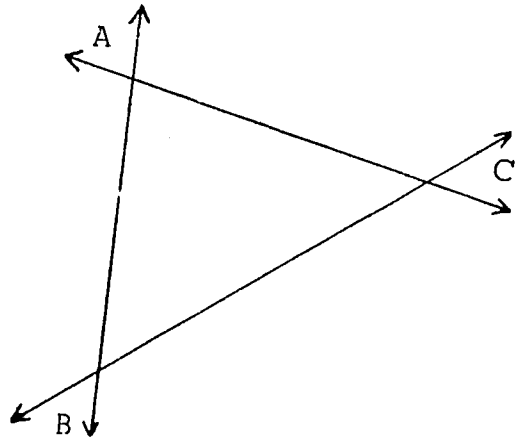


2. In the figure on the right,

   (a) Name three straight lines.

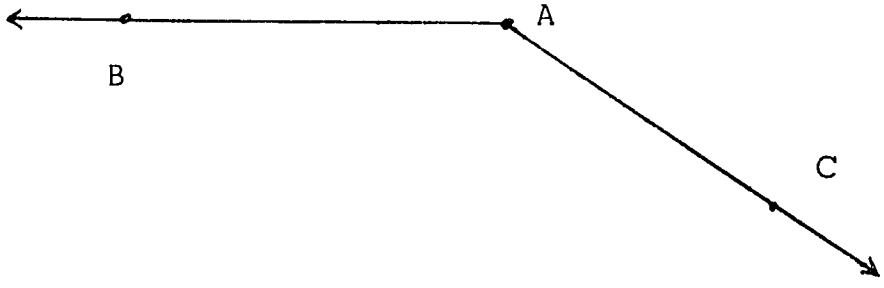
   (b) Name three line segments.

   (c) Name six rays.



3. We said that line $\overleftrightarrow{AB}$ could be thought of as the <u>union</u> of the two rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$. What could we call the <u>intersection</u> of these two rays?

4. The following picture shows two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ having the same endpoint. Do these rays form a straight line? Why?



5. Are the two rays pictured in Problem 4 <u>opposite</u> rays? Why?

6. A ray could be defined as the union of points A and B, all points between A and B, and all points beyond A from B on line $\overleftrightarrow{AB}$. Does this set define ray $\overrightarrow{AB}$ or does it define ray $\overrightarrow{BA}$?

7. Which of the following statements are true?

A ray has
(a) one endpoint.
(b) two endpoints.
(c) many endpoints.
(d) no endpoint.

A line has
(a) one endpoint.
(b) two endpoints.
(c) many endpoints.
(d) no endpoint.

A line segment has
(a) one endpoint.
(b) two endpoints.
(c) many endpoints.
(d) no endpoint.

1-6 Flatness and Planes

Which of the following surfaces do you think are flat?

(a) A table top.

(b) The surface of the earth.

(c) A pane of glass.

(d) The blackboard.

(e) The surface of a ball.

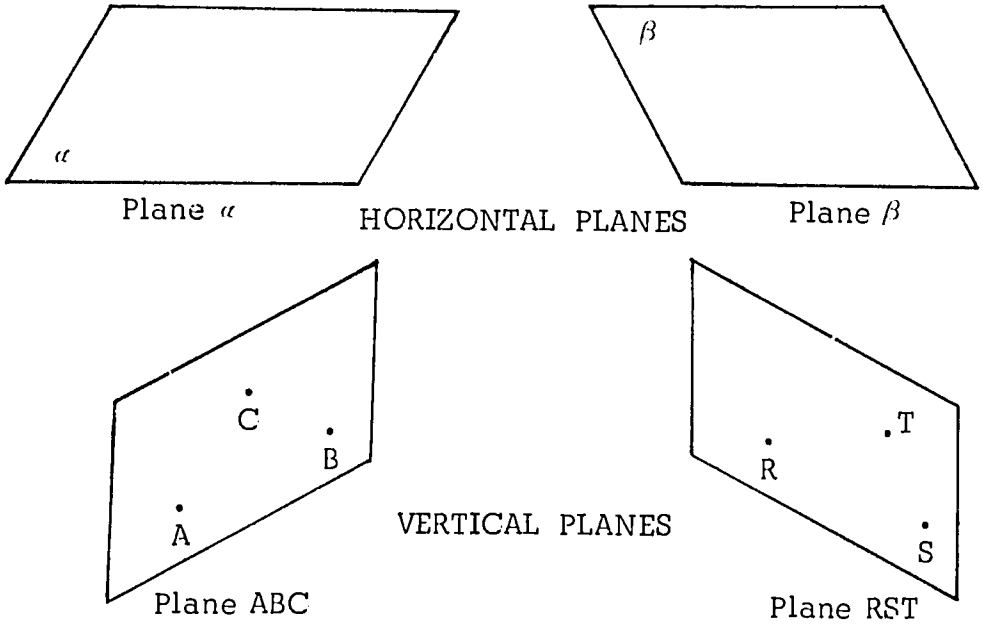
If your answers were (a), (c), and (d), then you already have some idea of what is meant by flatness. We say that a surface is flat if, when a straightedge is placed along the surface in any direction, every point of the straightedge

touches the surface. If you have a table in your room (if not, your desk top will do just as well), use a stretched string or a long straightedge and test it for flatness. You may find that it isn't quite flat in all directions, but is approximately so. If you have a pane of glass to test, you will probably find that it more nearly approximates a flat surface than the table or desk top.

Now imagine a flat surface such as a table top extending indefinitely far in every direction. This is the idea we wish to convey of a plane. We think of a plane as a flat surface which extends indefinitely in all directions. Just as a straight line is unbounded (extends indefinitely) in either direction, a plane is unbounded in all directions. We think of a plane as being a set of points just as we think of lines, rays, and line segments as being sets of points. Whereas a straight line has infinite length and no width, a plane has infinite length and width, but no depth or thickness.

When we say that a table top is a <u>horizontal</u> plane or that the blackboard is a <u>vertical</u> plane, we mean that the table top is <u>part</u> of a horizontal plane or that the blackboard is <u>part</u> of a vertical plane. Name some other surfaces in your classroom that are parts of vertical planes and name some that are parts of horizontal planes. If we were to raise a table slightly so that its top was no longer level or horizontal, we would have an example of an <u>inclined</u> plane — one that is neither horizontal nor vertical.

We sometimes draw pictures like the following to represent planes.



As the figures above suggest, we sometimes denote a plane with Greek letters (α, β, γ, etc.) or by naming three points (not all on the same straight lines) that are in the plane. We say that three points not all on the same straight line determine exactly one plane.

Problems 1-6

1. Mark two points A and B on the plane of your paper and draw the line $\overleftrightarrow{AB}$.

   (a) Do you think that every point of line segment $\overline{AB}$ also lies in the plane of the paper?

   (b) Do you think that every point of line $\overleftrightarrow{AB}$ lies in the plane of the paper?

   (c) Could there be another plane, different from the plane of your paper, that also contains every point of line $\overleftrightarrow{AB}$?

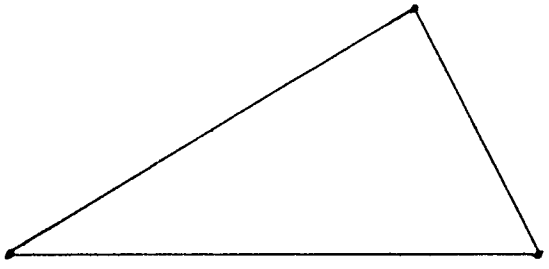
2. Look at the corner of your classroom where the plane of the side wall meets (or intersects) the plane of the front wall. What kind of geometric figure is formed by the intersection of these two walls? Do you see that the edge where the two walls (planes) meet is what we think of as a line segment? Do you think two planes will always determine a straight line when they intersect? Now look at the plane of the ceiling where it meets the planes of the front and side walls. How many points do all three of these planes have in common?

3. If two points of a line lie in a given plane, do you think all the points of the line lie in that plane?

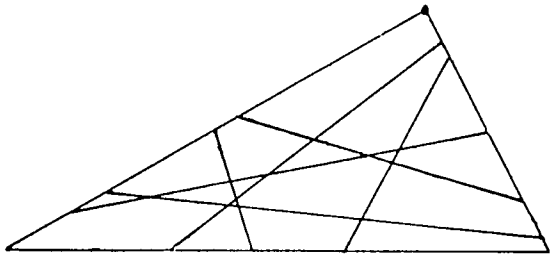
4. Could a line have only one point in common with a given plane?

1-7 Regions

Let us mark on a piece of paper three points which do not lie on the same straight line and draw the three line segments joining them. We get a figure like this. It is called a triangle. The triangle consists of the union of the three line segments, but not the region enclosed by them.



Because the paper or blackboard on which we draw it can be considered to be flat, the triangle is a part of a plane and we can draw line segments with endpoints on the triangle. Here are six examples of such line segments.



Suppose that we draw all such line segments. We will get a figure like this.

This is the flat surface that is bounded by the original three line segments which formed the triangle and, along with the sides of the triangle, is called a triangular region. If every line segment with endpoints on the triangle is extended to a straight line, the collection of the points on all the straight

lines will make a plane. Any three points anywhere that do not lie on any one straight line determine three things:

(a) a triangle

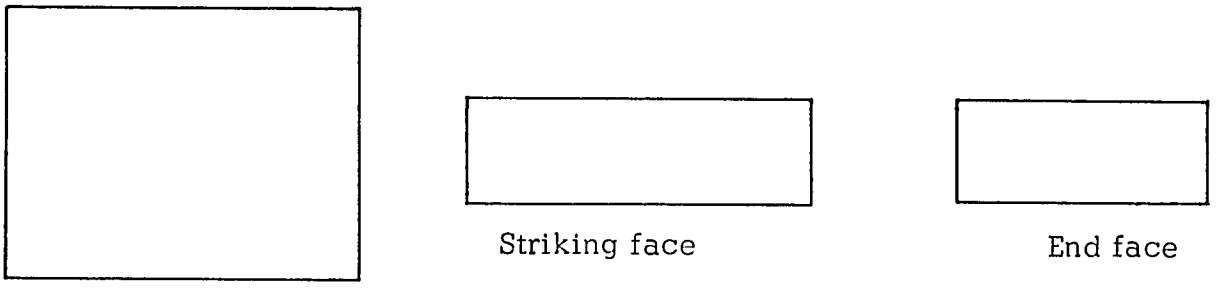
(b) a triangular region

(c) a plane.

The triangular region is full of straightness and the plane is full of straight lines. You may have a desk or a table which does not stay firm and steady when you lean on a part of its top and at times you have to place a wedge under one of its legs to keep it steady. Why is this? But have you noticed that a three-legged stool always sits firm and steady on any horizontal surface? Why is this?

Can you make a toy table from a small piece of cardboard and one pin? Would you succeed with two pins? Now try three pins and then four pins. What do you notice? Can you explain the conclusion you arrive at as a result of this exercise?

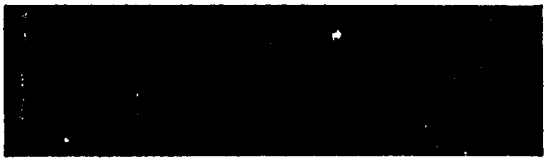
If you place a matchbox on a piece of paper, you can trace three different kinds of figures. They should look like this:



You could use a brick or a textbook to obtain the same kind of drawings on a large piece of paper. Each drawing will be composed of four line segments and if all the corners are square, it is called a rectangle.

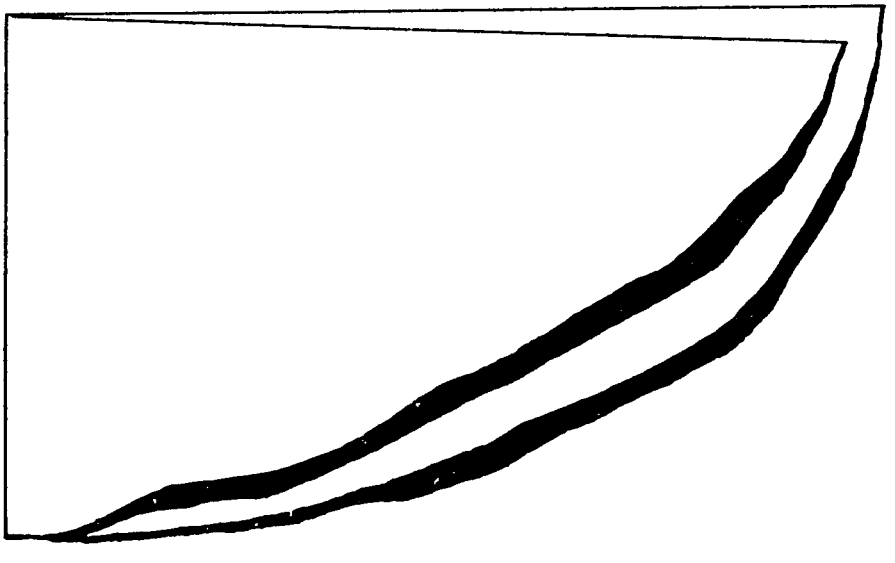
The four line segments of the rectangle are called the <u>sides</u> or <u>edges</u> and the four points which are the endpoints of the line segments are called the <u>vertices</u> of the rectangle.

If you fill the inside of the rectangle with line segments, the drawing will look like this:



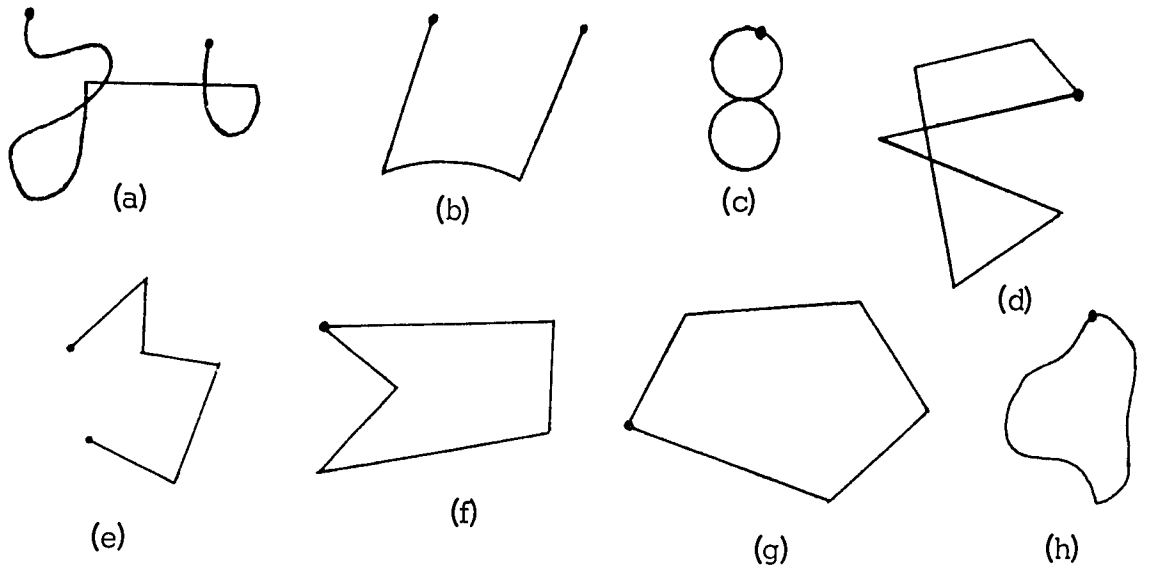
This is a rectangular region. The region includes the rectangle, and the sides and the vertices of the rectangle are also considered to be the sides and vertices of the region.

Note: A square corner can be obtained by folding a piece of paper to get a straightedge and then folding the paper again so that the two portions of the straightedge meet exactly.



1-8 Paths

Mark any point A anywhere on your paper and place the tip of your pencil at that point. Trace out any drawing you like without lifting your pencil from the paper. Drawings of this kind may be obtained.



All such drawings are called paths. Note that (a), (c), and (d) cross themselves at least once. Paths (a), (b), and (e) do not end where they started. Paths (d), (e), (f), and (g) consist entirely of line segments. Paths (f), (g), and (h) do not cross themselves and go back to the starting point.

A path that never crosses itself is a simple path. (b), (e), (f), (g), (h).

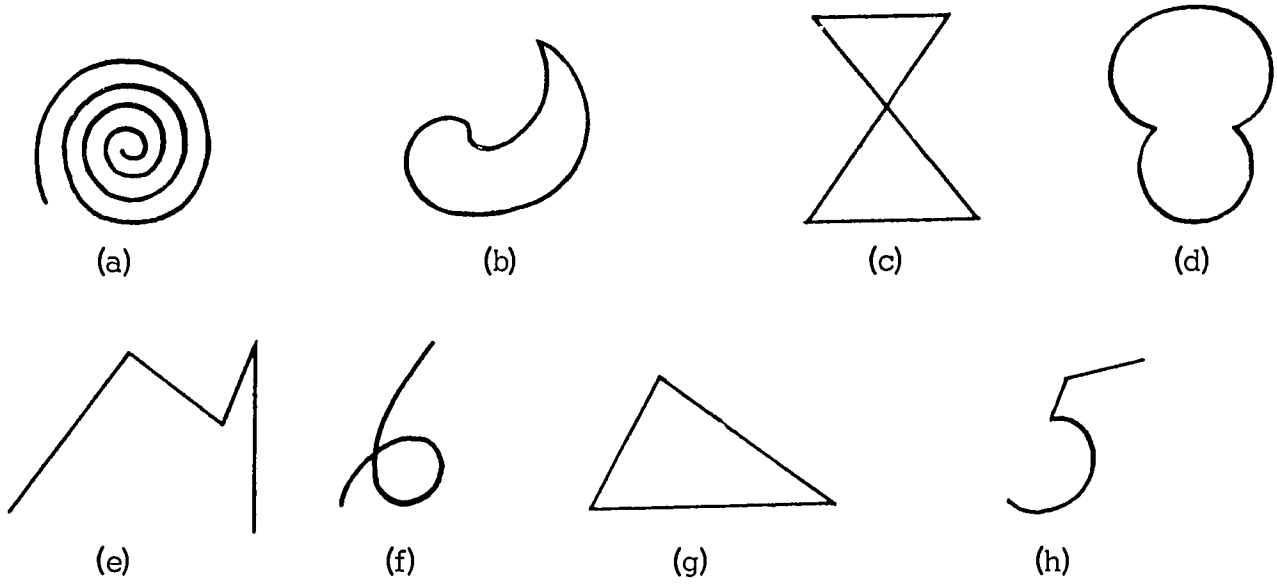
A path that does not cross itself and does not go back to the starting point is a simple open path. (b), (e).

A path that never crosses itself and goes back to the starting point is called a simple closed path. (f), (g), (h).

A simple closed path consisting entirely of line segments no two of which (with the same endpoint) lie on the same straight line is a polygon. (f), (g).

Problems 1-8

1. Which of the following are (i) simple paths, (ii) simple open paths, (iii) simple closed paths, (iv) polygons?



2. Draw three different simple open paths.

3. Draw three different simple closed paths so that two are polygons and one is not.

4. Draw two paths which are not simple paths.

1-9 Polygons and Polygonal Regions

Cut out a triangular region from cardboard. By tracing along the straight edges, you will get a triangle like Figure (i) below.

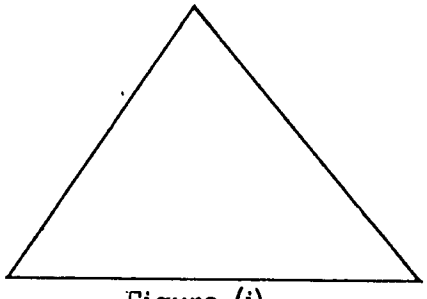


Figure (i)

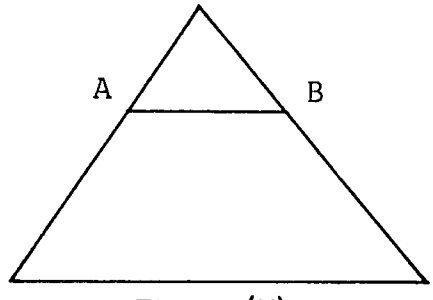


Figure (ii)

Now cut along a line segment $\overline{AB}$ as in Figure (ii) and if you trace out the edges of the remaining piece of cardboard you get a drawing like Figure (iii). Now cut along a line segment $\overline{CD}$ as in Figure (iv) and tracing along the edges of

the remaining figure you get a drawing like Figure (v). This process of cutting the cardboard along line segments could be repeated many times and the resulting plane figures would look like the following ones.

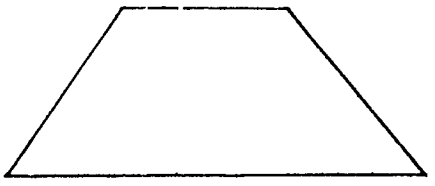


Figure (iii)

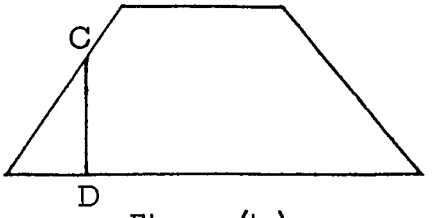


Figure (iv)

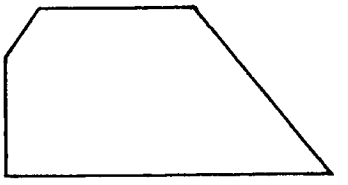


Figure (v)

You can see that what we get are plane figures with three, four, or five sides and if we go on cutting we can get a plane figure with 100 sides or more.

Such figures are called <u>polygons</u>. A polygon is a closed plane figure consisting of three or more line segments placed end to end successively such that:

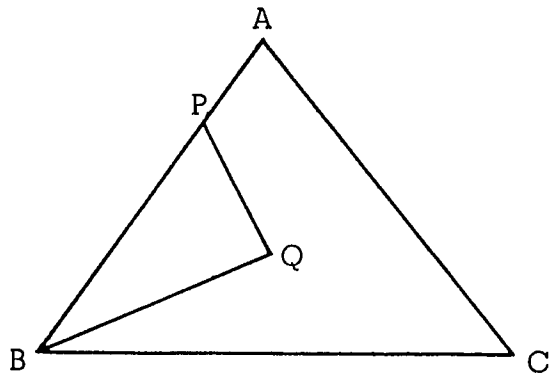
(a) No two segments cross each other except at an endpoint.

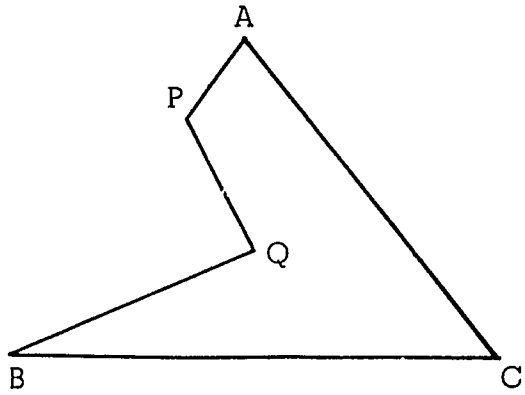
(b) No two segments with a common endpoint lie on the same straight line.

The common endpoints are called <u>vertices</u> (singular: vertex) and the line segments are called sides.

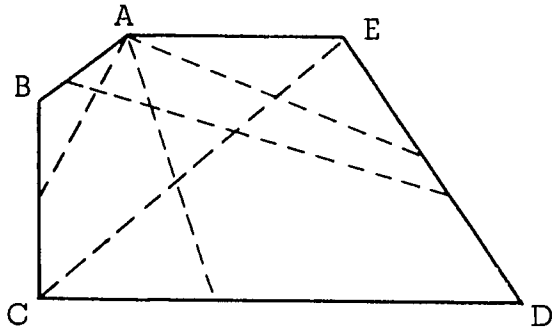
Some polygons have names according to the number of sides they have. We know already that a 3-sided polygon is called a triangle. A 4-sided polygon is called a <u>quadrilateral</u>. A 5-sided polygon is a <u>pentagon</u>. A 6-sided polygon is a <u>hexagon</u>. An 8-sided one is called an <u>octagon</u> and a 10-sided one is called a <u>decagon</u>.

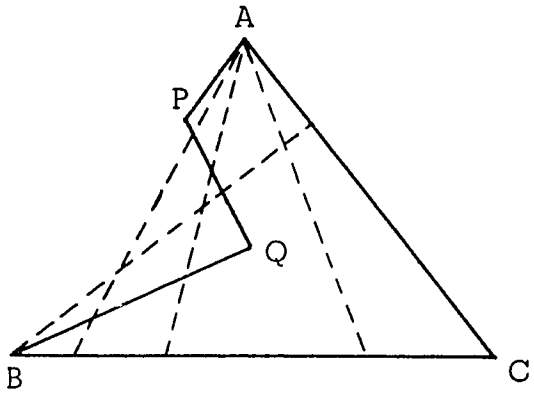
If we have a triangular piece of cardboard and we cut it along $\overline{PQ}$ and $\overline{BQ}$ as shown on the left, we would get a figure like the one shown on the right.





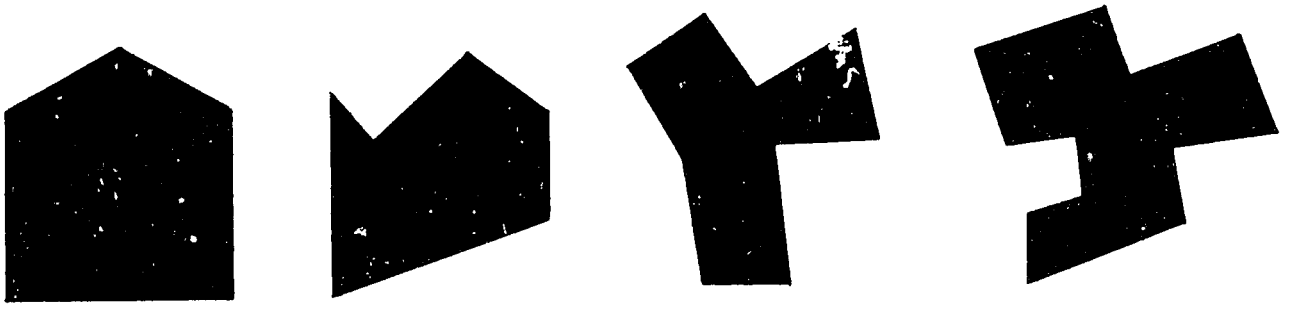
This is a five-sided polygon which is different from the polygon in Fig. (v). Let us find out the ways in which they are different.





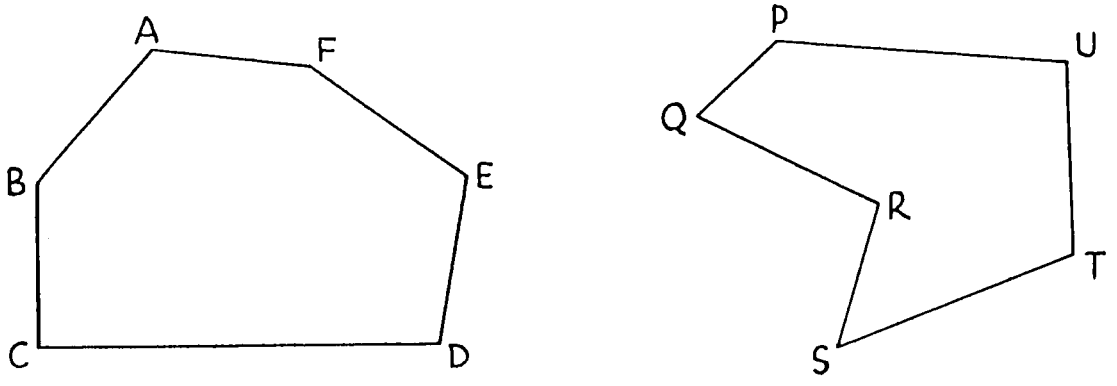
If we draw any line segments whose endpoints are on the sides of the two polygons, we notice that all the line segments of the first polygon lie inside the polygon, but in the second polygon some of the line segments may pass along the outside of the polygon. Polygons of the first kind, where all line segments whose endpoints are on two different sides of the polygon always lie inside the polygon, are called <u>convex polygons</u>. Any polygon which does not satisfy this property is not convex.

If the inside of any polygon, whether convex or not, is filled with straight line segments, the region so obtained is called a <u>polygonal region</u>. The following are polygonal regions:



Problem 1-9

Draw two hexagons ABCDEF and PQRSTU, the first one being convex and the second one not convex.



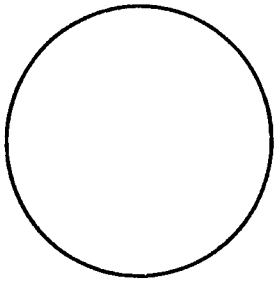
Draw the line segments $\overline{BF}$, $\overline{BE}$, $\overline{BD}$, $\overline{RU}$, $\overline{RP}$ and $\overline{RT}$ (these are called diagonals of the polygons). If none of these line segments intersect and all are in the interiors of the polygons, the polygons are subdivided into triangles.

Draw any polygon, whether convex or not, and show that it can be subdivided into triangles.

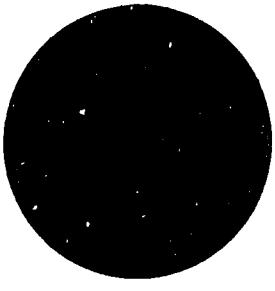
1-10 Drawing a Circle

Some objects are partly flat and partly round. Examples are bottles, some tins, sticks, poles and coins.

If we trace around the edge of a round coin we get a figure like this one:



This figure is called a circle and if the inside of the circle is filled with line segments, we get a circular region.

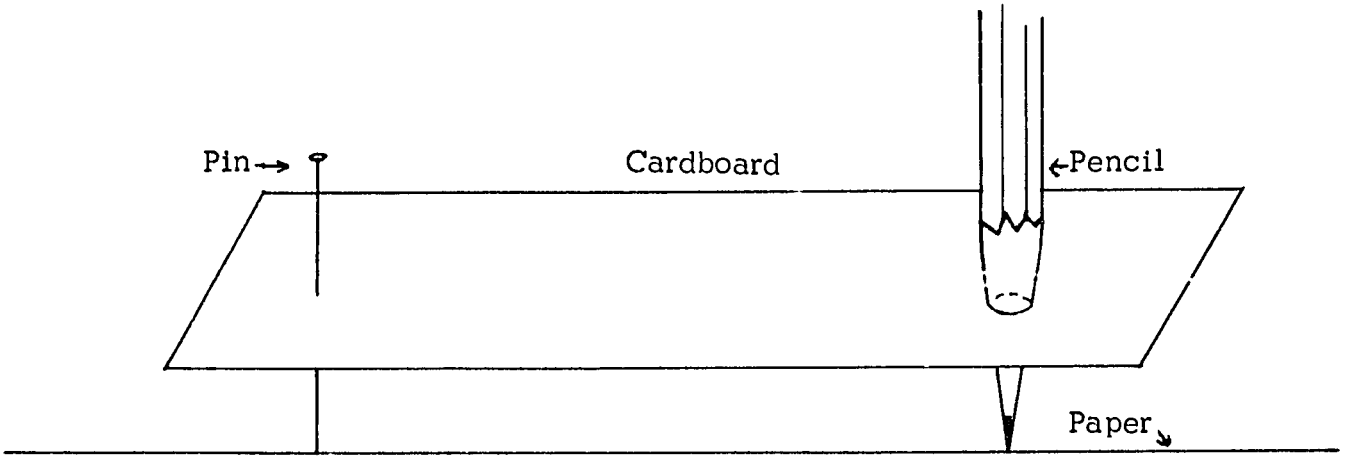


We could obtain the same kind of figures by tracing around the top or bottom of a round tin or the round bottom of a bottle. But these figures are usually only approximations to a circle. How can we obtain a figure that is a better approximation to a circle?

You may have fixed one of your heels on the ground, turned around on the heel and traced out a rough approximation to a circle in the dust with the end of your big toe. You may also have seen a tethered goat, sheep or donkey and seen how it goes around and around eating the grass in a certain circular region.

These two examples give us the idea of what is required to draw a circle. By tying a string to two sticks with pointed ends and pushing one into the ground, the other stick can be used to trace out a circle.

The same idea can be used to trace out a circle on paper by using a piece of cardboard in which two holes have been made.



A pin is pushed through the cardboard and into the paper on which the circle is to be drawn. A pencil is pushed through the other hole and by using the pencil to move the cardboard around, a circle should be obtained.

The two important principles in the drawing of a circle are:

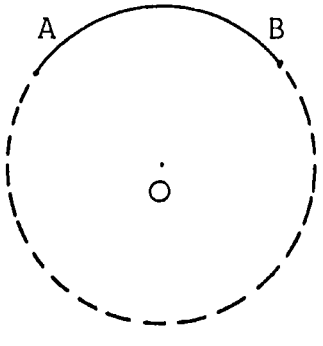
(a) to have a fixed point called the <u>centre</u> of the circle, and

(b) to have a fixed distance between the centre and the moving point. This distance is called the <u>radius</u> of the circle.

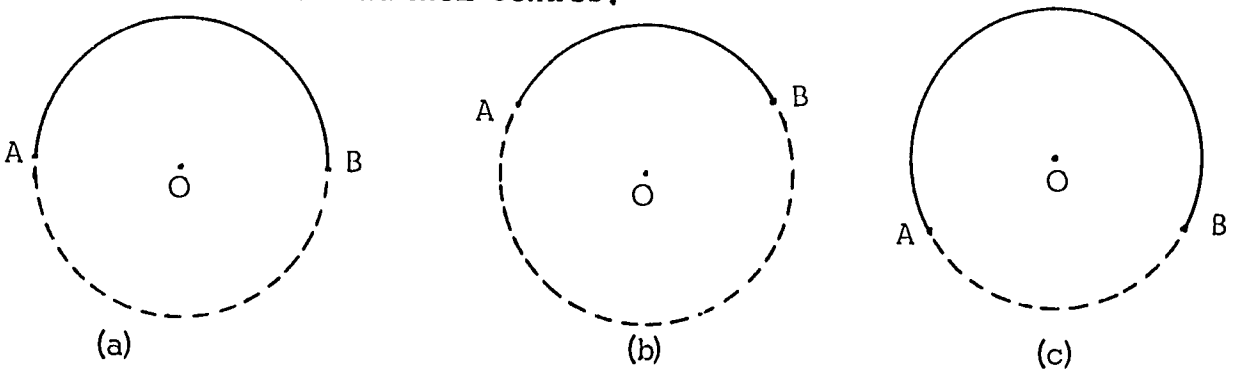
A circle then is the set of all points in a plane at a fixed distance from a given point in the same plane.

A pair of compasses is a convenient instrument for drawing a circle because the radius can be adjusted and it is possible to take any accurate radius from a ruler.

When you study constructions in Chapters 2 and 3 you will be asked to use a pair of compasses to draw only a part of a circle as in the figure below.

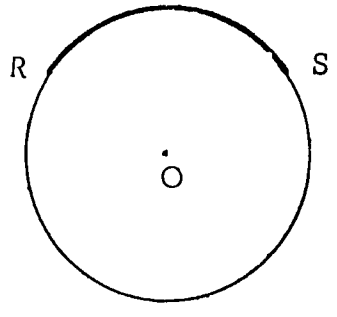


We call that part of the circle shown in solid line in the figure a <u>circular arc</u> or arc AB. We sometimes use the symbol $\widehat{AB}$ to denote a circular arc, and this is read "arc AB". The <u>centre of an arc</u> is the centre of the circle which contains the arc. The figure below shows three arcs of different sizes and their centres.



Semi-circular Arc Minor Arc Major Arc

In Figure (a) the arc $\widehat{AB}$ consists of half the circle, and is called a semi-circular arc or semi-circle. In Figure (b) the arc $\widehat{AB}$ is smaller than a semi-circle, and is called a <u>minor arc</u>. Figure (c) shows an arc $\widehat{AB}$ that is larger than a semi-circle, and is called a <u>major arc</u>. If we take any two points on a circle, such as R and S in the figure below, they determine two arcs. They are the minor arc $\widehat{RS}$ and the major arc $\widehat{RS}$.



In order to avoid confusion, when we say "draw arc $\widehat{RS}$", we shall mean the <u>minor</u> arc $\widehat{RS}$. If we intend you to draw the <u>major</u> arc $\widehat{RS}$, we shall say "draw major arc RS".

Problems 1-10

1. Use your compasses to draw three circles of different sizes. Are these circles paths? Are they simple paths? Are they closed paths or open paths?

2. Use your compasses to draw a semi-circular arc, a minor arc, and a major arc. Are these arcs simple paths? Are they closed paths or open paths?

3. With your ruler, mark two points A and B on your paper two inches apart. With a radius that is a little more than one inch, use your compasses to draw two circles, one with centre at A and the other with centre at B. Do these two circles intersect or cross each other? If so, how many times? Now take a radius that is a little less than one inch and draw two circles with centres at A and B as before. Do these circles cross each other?

4. With your ruler mark two points R and S that are three inches apart on your paper. Tell what radius you would need to set on your compasses so that circles with centres at R and S would:

   (a) Intersect in two points.

   (b) Touch each other in just one point.

   (c) Not intersect each other at all.

CHAPTER 2

PLANE FIGURES

2-1 Introduction

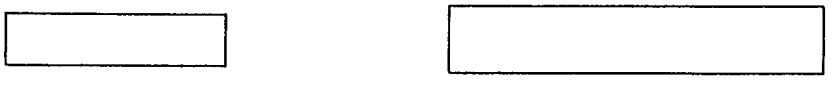
In Chapter 1 you saw how geometric ideas can come from considering objects in the world around us. You thought about straightness and flatness, and about such plane figures as line segments, rays, lines, triangles, rectangles, circles and arcs.

In this chapter, you will see what can be done with pairs of line segments, pairs of rays, and pairs of lines. We shall talk about congruence, and you will find some easy ways to copy line segments and angles.

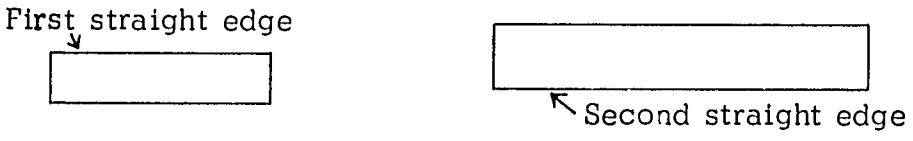
As you read this chapter, bear in mind that the figures we draw are plane figures. We may compare figures that lie in different planes, for example an angle drawn on the blackboard and an angle made by rays on a flat piece of paper. But we shall not consider space figures here. In particular, the figure made up of two lines that do not lie in a plane will not be considered until Chapter 4.

2-2 Pairs of Line Segments

To see what pairs of line segments are like, let us look at two straight edges. This is a good place to start because we draw line segments with straight edges. If we have two pieces of wood or cardboard that look like this,

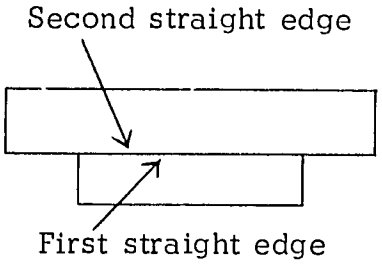


then we have many straight edges. Here are two:



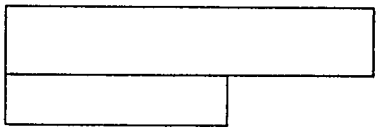
Previous Page Blank

If we fit the two edges together, we could get something like this:



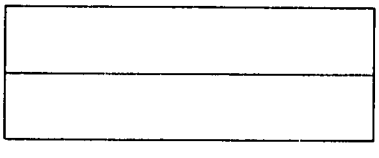
We notice that where the edges meet there are no holes or gaps. There is no difference in their straightness. Try it yourself with two straight edges.

If we slide one of these edges along the other until their ends just meet, they will look something like this:



We say that the edges do not fit exactly. The second straight edge is <u>longer</u> than the first, and the first is <u>shorter</u> than the second.

Another way straight edges can fit is like this:



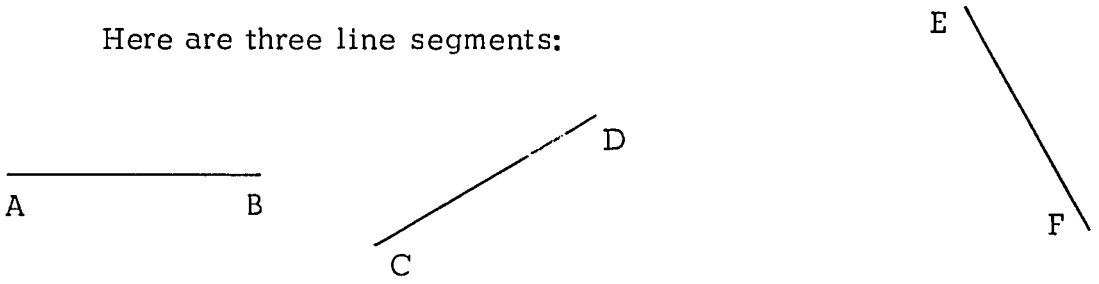
In your classroom can you find two straight edges that fit this way? We say that straight edges that fit this way <u>fit exactly</u>. Their endpoints fit together.

> Whenever two straight edges fit together so that endpoints fit endpoints, we say <u>they fit exactly</u>. We also say <u>they are congruent</u>. Congruent is another word for fit exactly.

Here is a pair of congruent line segments:

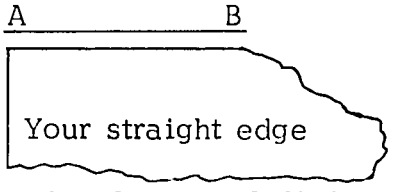
The page can be folded over so that the segments fit exactly. Try it after tracing the segments on a sheet of thin paper.

Here are three line segments:

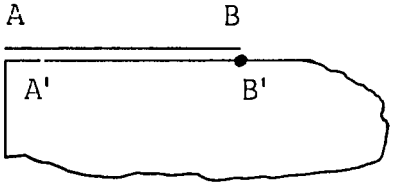


We cannot make any two of these fit together without damaging the page. How can we tell if any two of them are congruent? For example, how can we tell if either $\overline{CD}$ or $\overline{EF}$ is congruent to $\overline{AB}$? One way is to cut $\overline{AB}$ out of the page and to try to fit it to $\overline{CD}$ and to $\overline{EF}$. But isn't there some other way? Yes, there are several.

One other way, and it is a good way, is to make a straight edge that fits $\overline{AB}$, and then try to fit the edge to $\overline{CD}$ and $\overline{EF}$. For example, take the edge of a piece of paper (or some other straight edge with a sharp corner) and fit it to $\overline{AB}$. Fit one endpoint of the edge at A.



Then mark the point on the edge that fits B. Call the point you mark B', and call the endpoint that fits A, A', like this:



The part of the straight edge that starts at A' and ends at B' is a copy of $\overline{AB}$. If this copy fits $\overline{CD}$ then $\overline{AB}$ and $\overline{CD}$ are congruent. If it does not, then $\overline{AB}$ and $\overline{CD}$ are not congruent.

Try now to fit your edge to $\overline{CD}$. Can you fit it so that A' fits C at the same time that B' fits D? What do you conclude about $\overline{AB}$ and $\overline{CD}$?

Now try to fit your edge to $\overline{EF}$. Can you fit it so that A' fits E at the same time that B' fits F? What do you conclude?

Here is another line segment:

Is it congruent to $\overline{AB}$, longer than $\overline{AB}$, or shorter than $\overline{AB}$? Now you know how to test line segments for congruence.

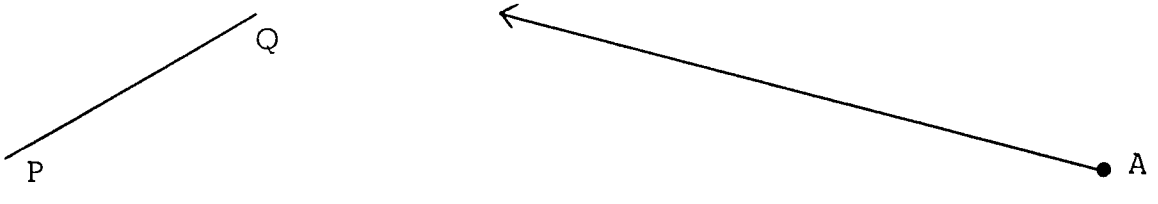
> To test two line segments for congruence, fit a copy of one to the other. If the copy fits the other exactly, the two line segments are congruent. If the copy does not fit the other exactly, the line segments are not congruent: one segment is shorter and the other is longer.

Suppose that a friend asked you to show him a pair of congruent line segments. Do you think you could make him a pair? How?

Now let's see if you can do this: Look at this line segment:

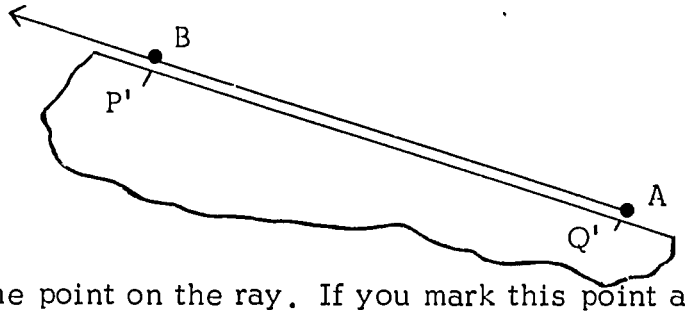
Can you make another line segment that is congruent to it? How?

Here is another question. On a separate piece of paper, draw a line segment and a ray like this:



Can you find a point B on the ray so that $\overline{AB}$ is congruent to $\overline{PQ}$? If you don't know how all ready, you can read the next paragraph to find out.

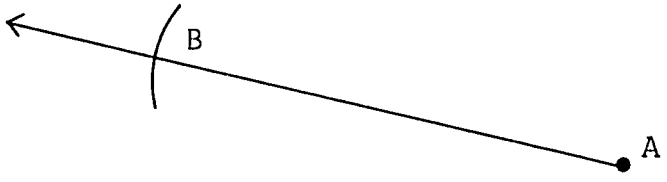
Make a copy of $\overline{PQ}$ on a straight edge, calling the endpoints of the copy P' and Q'. Then fit your edge to the ray, with Q' at A.



P' is then matched with some point on the ray. If you mark this point and call it B you will have made a line segment $\overline{AB}$ congruent to $\overline{PQ}$. Do you agree? Why?

Do you know how to find B with a pair of compasses? If you do, you may skip the next paragraph.

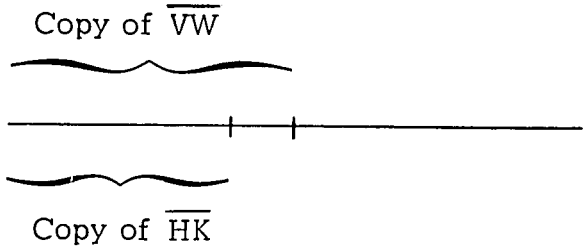
Fit the compass points to P and Q. This sets the compasses. Without changing this setting, put the pin point at A and draw an arc that cuts the ray. The arc cuts the ray at B.



If you don't believe it, carry out the construction with your compasses, and compare $\overline{P'Q'}$ with your new $\overline{AB}$.

Marking line segments on a ray leads us to a good way to compare two line segments: make a copy of each and compare the two copies. You could do this for the line segments here

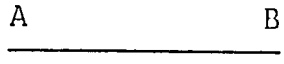
by copying them on a third segment like this:



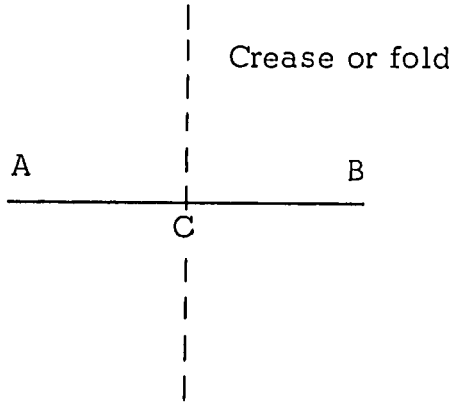
How is this like using a ruler to compare lengths?

2-3 Perpendicular Bisector

In this section we are going to show you with pictures and with words how to construct a very special kind of line segment. To begin with, draw a line segment $\overline{AB}$ in the middle of a clean piece of paper.



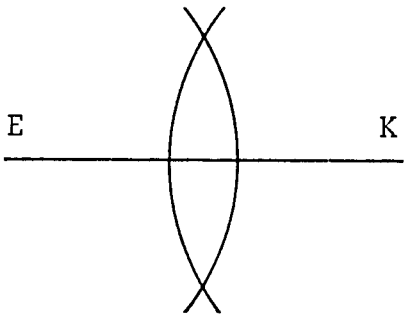
Then fold the paper so that the two endpoints are on top of each other. Press the paper flat to make a crease. Unfold the paper and mark the point where the crease cuts $\overline{AB}$. Call this point C. What can you say about the line segments $\overline{AC}$ and $\overline{CB}$? They are congruent because they match each other in folding. They fit exactly!



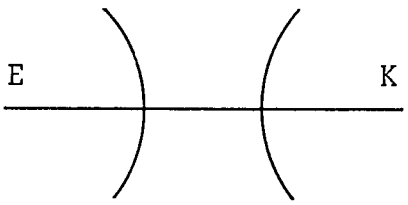
Since $\overline{AC}$ and $\overline{CB}$ are congruent, we can think of C as being half way between A and B. We call C the <u>midpoint</u> of $\overline{AB}$. Now put that piece of paper aside for a few minutes while we show you another way to find the midpoint of a line segment.

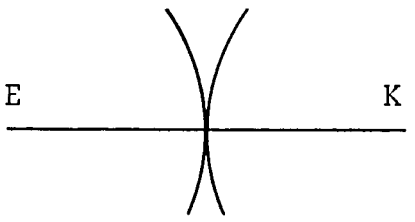
In the middle of another clean sheet of paper draw another line segment.

Set your compasses, and with E and K as centres, draw arcs that meet each other in two points, like this



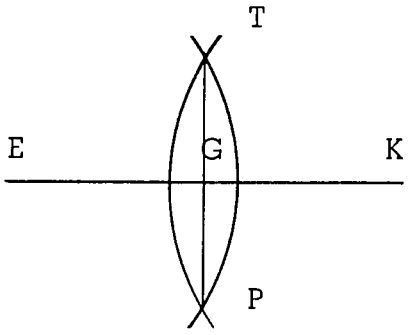
Be sure to use the same setting of your compasses for both arcs. If your arcs do not meet, or if they meet at exactly one point,





then change your compasses to a wider setting, and draw two arcs again.

Name the points where the arcs meet in your drawing T and P. Then draw $\overline{TP}$, and call the point where $\overline{TP}$ crosses $\overline{EK}$, G.



What can you say about G? Does it look as if $\overline{EG}$ and $\overline{GK}$ are congruent? Fold your paper along $\overline{TP}$. Hold your paper up to the light. Do you see that $\overline{EG}$ and $\overline{GK}$ fit exactly? They will indeed fit exactly if you have done your drawing and folding carefully. G is the midpoint of $\overline{EK}$ because $\overline{EG}$ and $\overline{GK}$ are congruent.

We say that $\overline{TP}$ is a <u>bisector</u> of $\overline{EK}$ because it cuts $\overline{EK}$ at its midpoint.

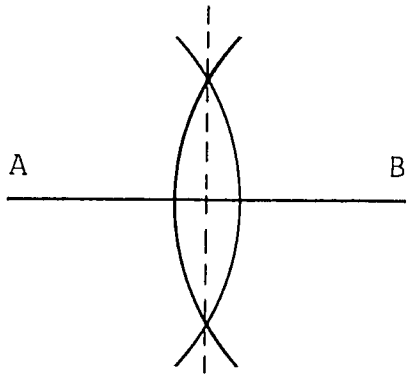
Keeping your paper folded along $\overline{TP}$, make another fold, this time along $\overline{EG}$. This fold makes a square corner at G. Do you see that? Unfold your paper and look at the creases. Do you see that $\overline{EG}$ and $\overline{TG}$ make a square corner? What other square corners do you see?

If you have any doubt about where the square corners are in your drawing, take a moment to check your observations with a square corner made from another piece of paper. But as soon as you are convinced that you know where the square corners are, go on with this lesson.

Whenever two line segments that cross make square corners, we say that the line segments are <u>perpendicular</u>.

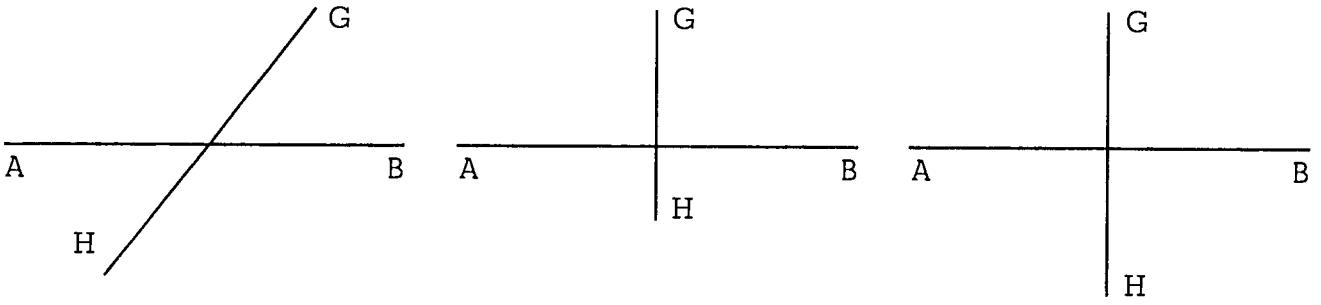
We see that the bisector $\overline{TP}$ is perpendicular to $\overline{EK}$. For this reason, we call $\overline{TP}$ a <u>perpendicular bisector</u> of $\overline{EK}$.

Now go back to the piece of paper on which you drew $\overline{AB}$. Remember how you folded this paper to find the midpoint C of AB. With A and B as centres, use your compasses once more to draw arcs that meet each other in two points. As before, make sure that you use the same setting for drawing both arcs.



Do the points where the arcs meet lie on your fold? If you have drawn carefully, they will. Draw the line segment that joins the two points. Do you see that it lies right in the fold? The fold you made is a perpendicular bisector of $\overline{AB}$.

To make your ideas of <u>bisector</u> and <u>perpendicular</u> clearer, think about these three drawings:



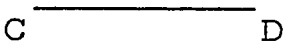
a. b. c.

In a., $\overline{GH}$ is a bisector of $\overline{AB}$ but is not a perpendicular bisector of $\overline{AB}$. Also, $\overline{AB}$ is a bisector of $\overline{GH}$ but is not a perpendicular bisector of $\overline{GH}$. In b., $\overline{GH}$ is a perpendicular bisector of $\overline{AB}$. $\overline{AB}$ does not bisect $\overline{GH}$, but does make square corners with $\overline{GH}$. In c., each of the two line segments is a perpendicular bisector of the other.

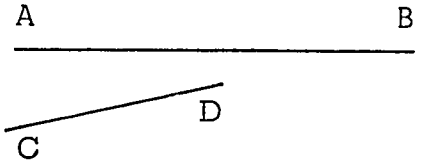
<u>Problems 2-3</u>

1. Suppose that we have traced a straightedge and made the following line segment.

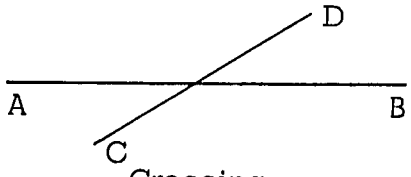
If we choose another straightedge and trace it on the same paper, we shall have a second line segment like this:



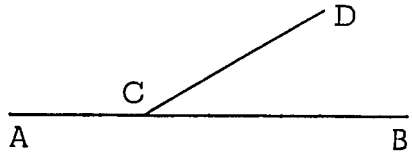
There are many kinds of positions in which $\overline{CD}$ might be drawn. One kind is that in which the two line segments have no point in common; they are non-intersecting.



If two line segments have one or more points in common they are said to <u>intersect</u>. There are positions in which the two line segments intersect in exactly one point. Here are some examples.

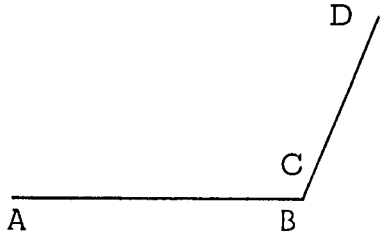


Crossing



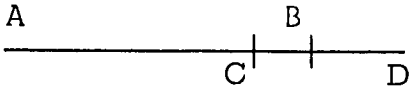
Touching

Aligned end to end



Connected end to end

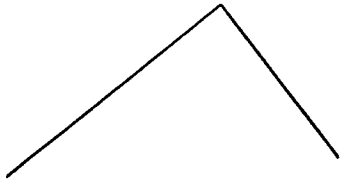
Finally there are positions in which they intersect in more than one point.



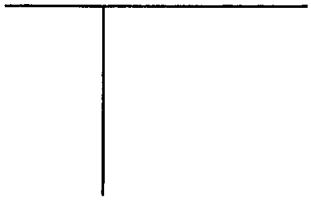
Use two paper straightedges to show each of these positions in turn.

2. In terms of where the points of intersection occur, try to describe accurately in words the differences between the three cases "crossing", "touching" and "connected end to end". Are "aligned end to end" and "making a square corner" special cases of "connected end to end"?

3. In the text we said what it means for line segments that cross to be perpendicular: they make square corners. Line segments can also make square corners like this:



Connected end to end



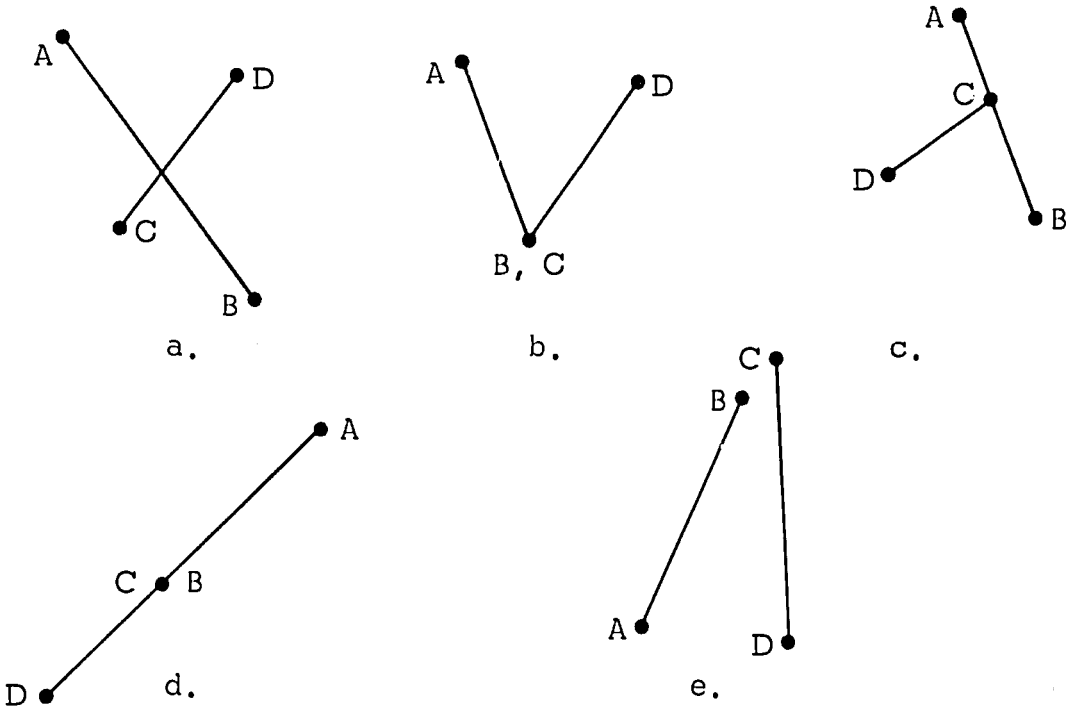
Touching

We now extend our definition of perpendicular to include any two line segments that make a square corner:

> Line segments are perpendicular if they make one or more square corners.

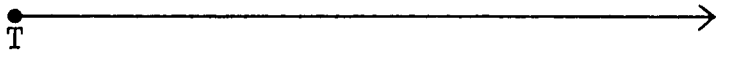
How many different numbers of square corners do you think two line segments can make?

4. Here are some pairs of line segments. In each case, use the phrase from Problem 1 that most accurately describes the relationship of the pair.

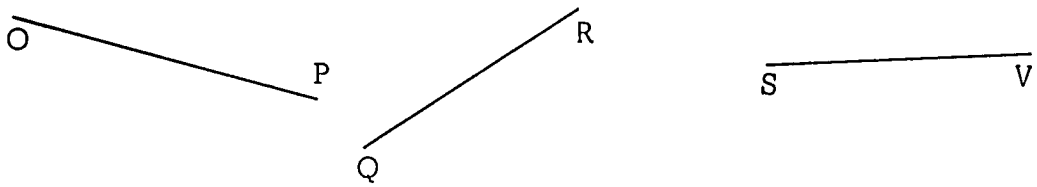


5. Test each pair in Problem 4 for congruence.

6. Draw a ray on a piece of paper, and call its endpoint T.

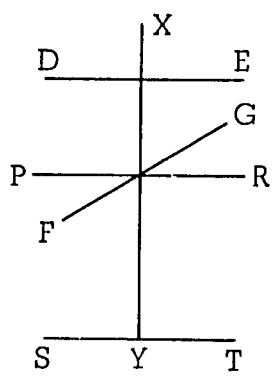


On your ray draw three line segments, with endpoints at T, that are congruent to these three line segments.



Which of these three given segments is the longest? Which is the shortest? (Do not use a ruler.)

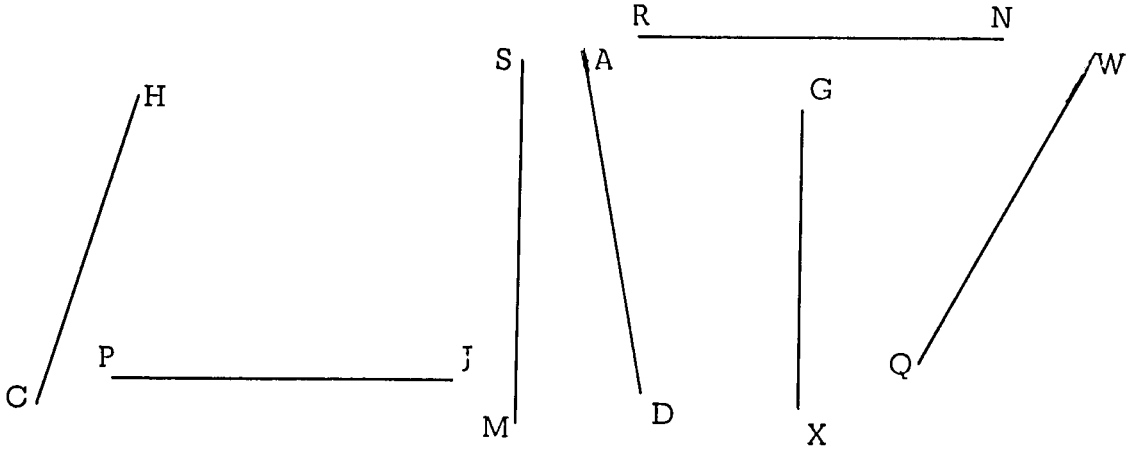
7. In the following picture, try to find a line segment that is a perpendicular bisector of $\overline{XY}$; of $\overline{DE}$; of $\overline{FG}$.



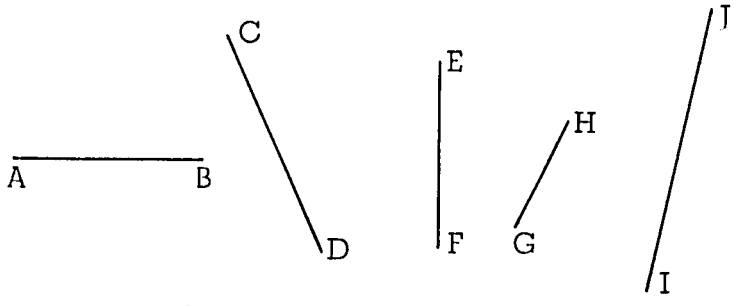
8. Make a paper straightedge and bisect it by folding. Bisect each resulting half of the straightedge in the same way.

9. Copy each of these line segments and use a ruler and compasses to construct a perpendicular bisector of each copy.

10. Among these line segments, which pairs are congruent pairs?

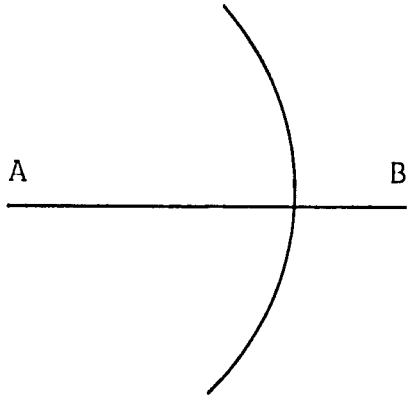


11. Without using a ruler, compare these line segments for length. Say which is the longest, the next longest and so forth.



12. When we constructed perpendicular bisectors with compasses, we made sure that we drew the two arcs with the same setting of the compasses. Something interesting will happen if you use different settings for the two arcs. See if you can discover what it is.

Start with a line segment $\overline{AB}$ drawn on a piece of paper. With A as centre, draw an arc so:



Then change your compasses to a smaller setting and draw an arc with centre B that intersects the first arc in two points. Connect these two points with a line segment. What do you see? If you don't believe it, test it with a square corner. Where is the midpoint of $\overline{AB}$?

13. Copy each of these line segments and find the midpoint of your copy:

(a) _____

(b) _____

(c)

_____

2-4 Pairs of Straight Lines

We saw in the preceding section that line segments can intersect in many different ways. In this section we shall see how lines intersect.

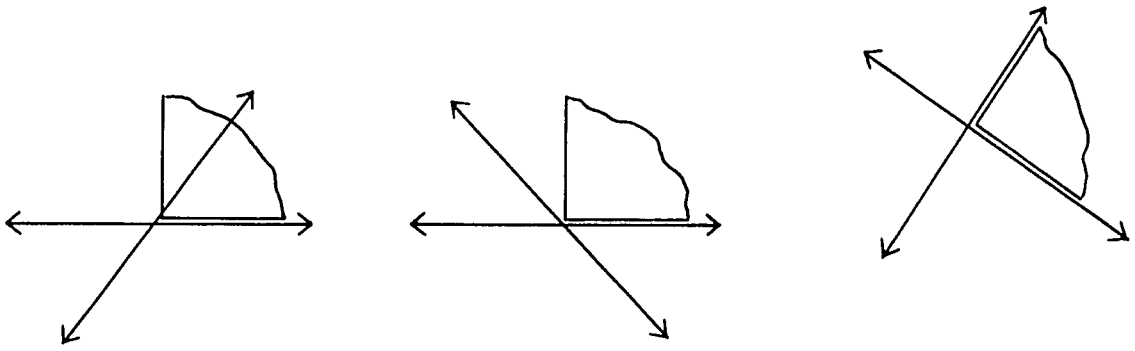
Two straight lines in a plane have no point in common, or have one point in common, or else are the same straight line. To see this, suppose that two straight lines have more than one point in common. Choose two of these common points. Then each of our lines goes through these two points. But through two points there is just one straight line. So the two straight lines are the same.

After this, when we say "two straight lines" we shall mean two that are not the same line. We can now say that

> Given any two lines in a plane, either they do not intersect or they intersect at just one point.

If two lines intersect, we call the point where they intersect the point of intersection.

If two straight lines intersect, we can try to fit a square corner at the intersection. Here are three things we might find:

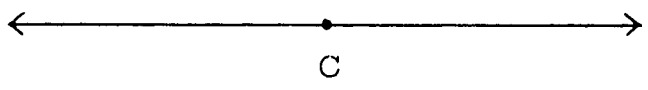


In each case, we put the tip of the square corner at the point of intersection, and fit one straight edge of the corner to one of the straight lines. We then see whether the other straight edge fits the other straight line. If it does, we call the two straight lines <u>perpendicular</u> (just as we did for line segments). In the picture above on the right the two lines are perpendicular. In the other two pictures the lines are not perpendicular.

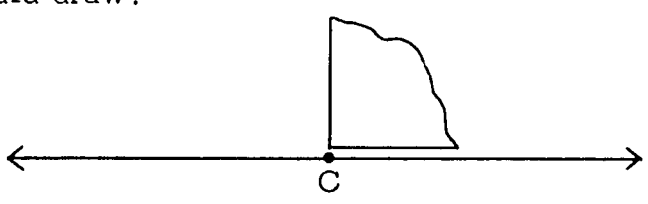
> Two straight lines are perpendicular if
> 1) they intersect, and
> 2) a square corner fits exactly at the intersection.

If two lines are perpendicular, and if their point of intersection has been named, say, "Q", we say that they are <u>perpendicular at Q</u>.

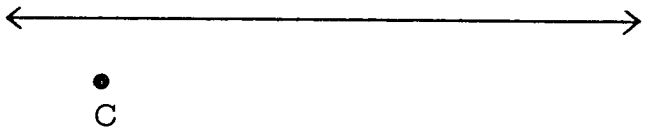
Now let us see how to make perpendicular lines. On a piece of paper, draw a line and a point chosen on the line:



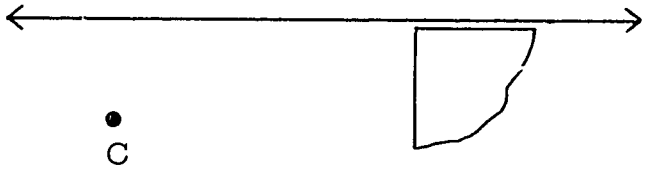
Do you think that you can draw a line that is perpendicular to your line at C? If you don't see how, would placing a square corner in this position help you to see what you could draw?



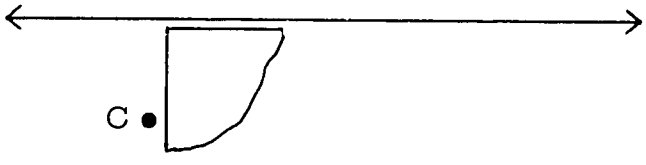
Now suppose that we are given a line, and a point C <u>not</u> on the line:



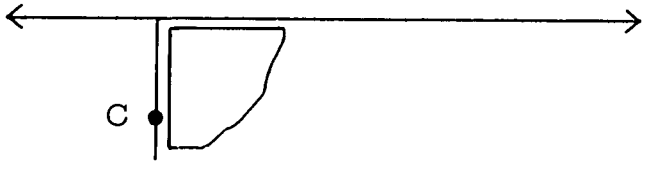
Ho ould we draw another line that is perpendicular to the given line and goes through C? Easy. We fit one edge of the square corner to the given line, like this,



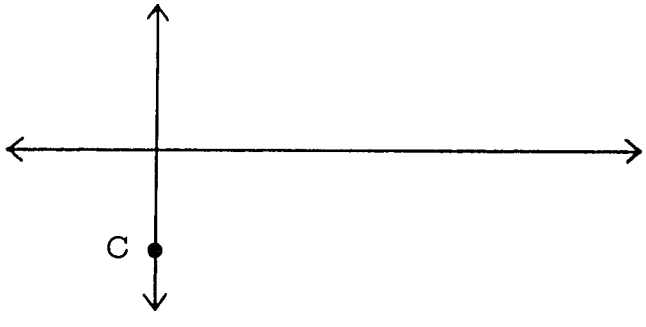
and then slide the square corner along the line, (keeping it fitted to the line) until the other straight edge of the square corner comes to the given point:



Here we stop, and trace the edge that is at the point. This makes a line segment



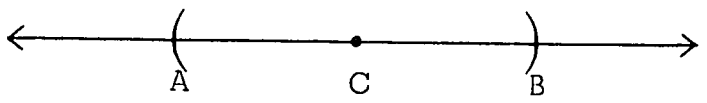
and the line segment determines a straight line



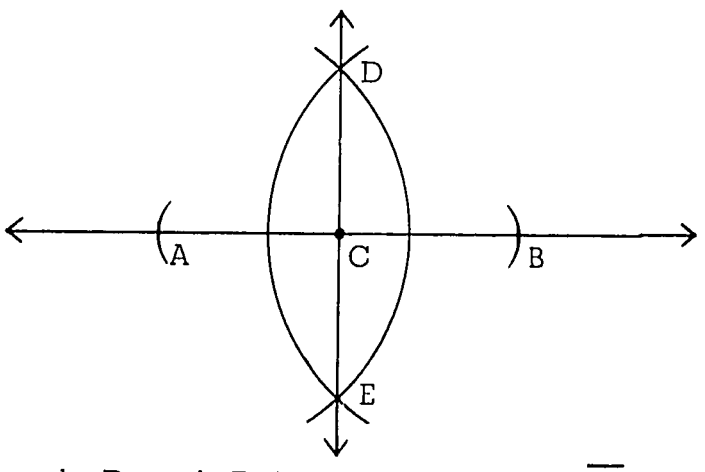
that passes through C and is perpendicular to the line we started with.

There is another way to draw perpendicular lines that is worth knowing about, in case you don't have a very good square corner handy, or in case the one you have is not a good size to work with. All you need for this new way is a straight edge and a pair of compasses.

Start once more with a line and a point C on it. Put the point of your compasses at C and cut the line with an arc on each side of C. Your picture looks like this:



(We have let A and B be the names of the points at which the arcs cut the line.) Now open your compasses some more, and with A and B as centres, draw two more arcs, the way we did when we bisected line segments. They intersect at two points, D and E:

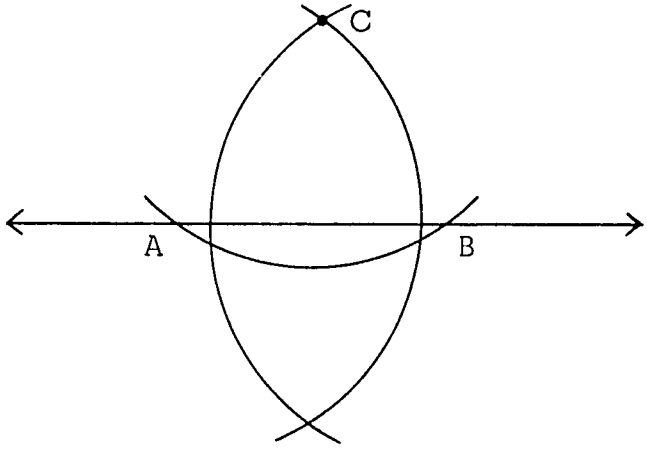


The straight line through D and E is perpendicular to $\overline{AB}$, the given line at C. Why?

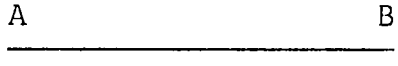
There are several things you can do here to convince yourself that $\overleftrightarrow{DE}$ is perpendicular to $\overleftrightarrow{AB}$. One is to remember the way you learned to construct perpendicular line segments. The line segment $\overline{DE}$ is perpendicular to $\overline{AB}$. Since $\overline{DE}$ and $\overline{AB}$ make square corners, so do $\overleftrightarrow{DE}$ and $\overleftrightarrow{AB}$. The lines, after all, are just extensions of these segments.

Another thing you can do to convince yourself that $\overleftrightarrow{DE}$ and $\overleftrightarrow{AB}$ are perpendicular is to fold your paper along $\overleftrightarrow{DE}$. If you then fold your paper along $\overleftrightarrow{CA}$ you will have made a paper square corner whose edges lie along $\overleftrightarrow{CA}$ and $\overleftrightarrow{CD}$.

We have used a straight edge and compasses to draw a line perpendicular to a given line through a specified point C on the given line. Now start with a line and a point C not on the line. Open your compasses enough so that when you put the point at C you can draw an arc that cuts the line in two points. Call these two points A and B. Then, without bothering to change the setting on your compasses, make big arcs with A and B as centres. These arcs intersect at two points, one of which is C itself. The line through C and the other point will be perpendicular to $\overline{AB}$. Why?

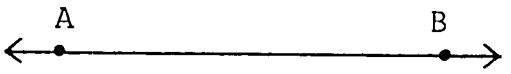


You have now seen enough examples to understand the difference between the line segments we draw and the pictures we draw when we think of lines. Line segments are drawings that we make with straight edges. Line segments have endpoints. If we draw a line segment $\overline{AB}$ joining two points A and B,



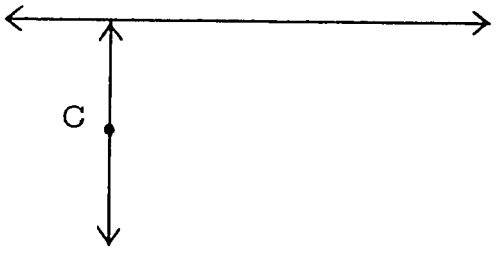
we can think of the line $\overleftrightarrow{AB}$ as an extension of the segment $\overline{AB}$. $\overline{AB}$ is just a segment of this line (that's where the words "line segment" come from).

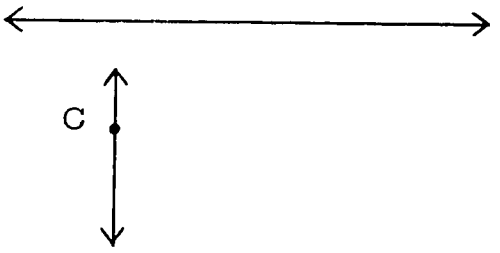
But we can't draw all of the line. We shall never be able to draw more than part of it. When we draw a picture for someone, then, how do we show him that we are thinking of the line $\overleftrightarrow{AB}$ and not just the segment $\overline{AB}$? You already know. You make the segment a little longer, so that it extends past A and B, and then you put arrow heads on it to show that what you have in mind is the full line.



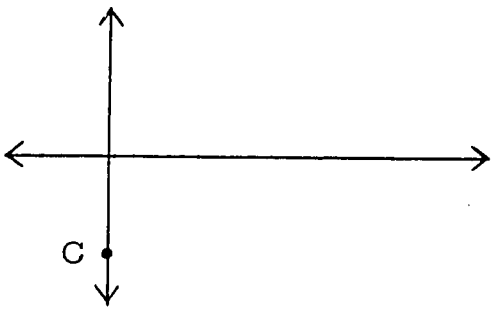
Thus, when we made a drawing for the line through D and E (page 47) we drew a line segment that extended past D and E, and then put arrow heads on it to show you that we were thinking of the line $\overleftrightarrow{DE}$.

There is another thing that we did that we want to make sure that you noticed. When we used our set square to make a drawing for the line through C perpendicular to the given line, we did not draw pictures like these:





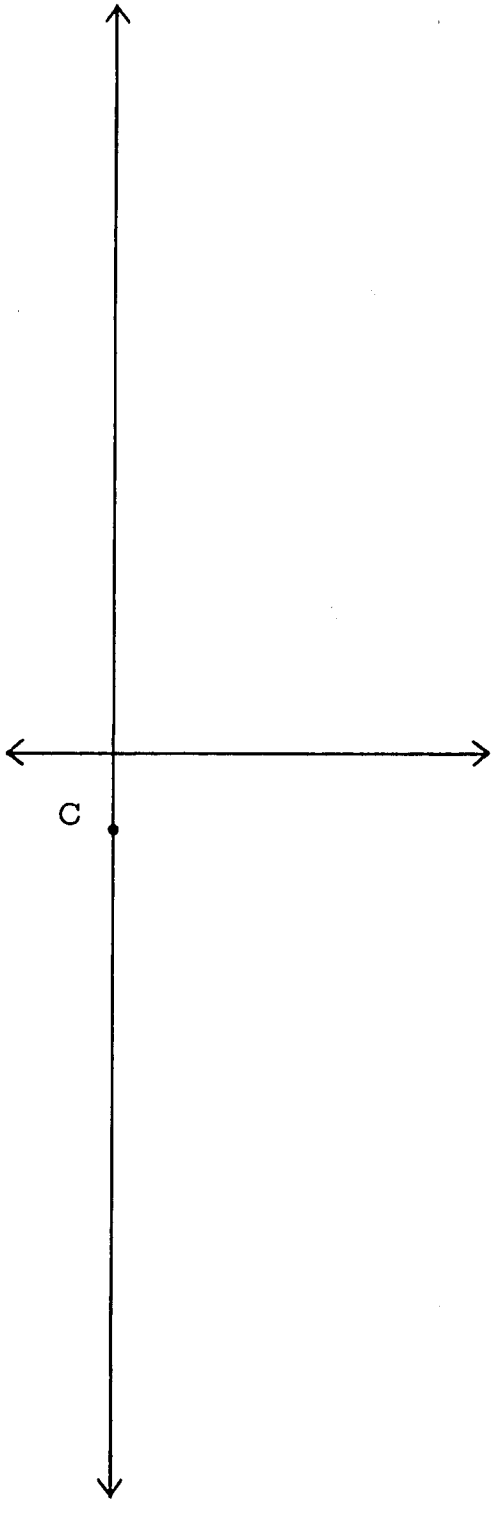
What we drew was like this:



This drawing shows more clearly the intersection of the two lines.

> When we draw part of a line in a geometric picture, we shall always draw enough of the line to show how the line intersects the other lines and arcs in the picture.

It is, of course, possible to draw too much as well as too little. We could have drawn this, for example:

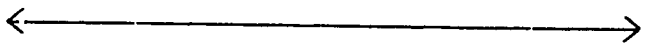


This picture, too, shows the intersection of the two lines. But it uses up a whole page, and takes more time to draw, and it doesn't show us any more about the lines than the previous figure does.

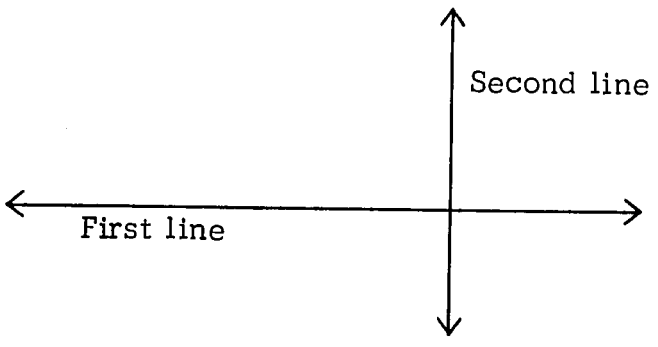
When we draw part of a line in a geometric picture, we draw only enough to show clearly the relationship of the line to the other parts of the picture. When we have drawn enough of the line, <u>we say that we have drawn the line</u>. We have, after all, drawn all of the line that we need to draw.

Now that we have said what we mean when we say "draw a line", we can explore another important relationship that lines can have.

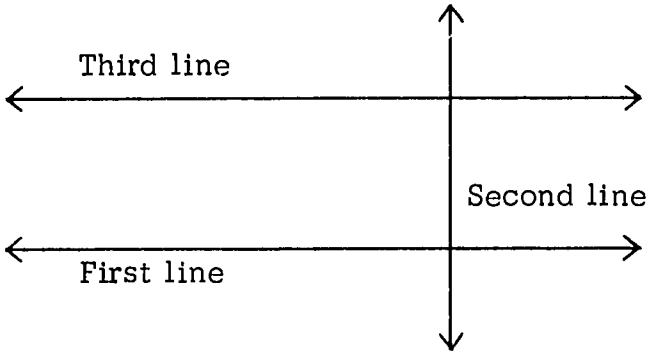
Two straight lines in a plane are parallel if there is another straight line that is perpendicular to each of them. Since you know how to draw perpendicular lines, it will be easy for you to draw parallel lines. For example, start with a line like this on a piece of paper:



Draw a second line that is perpendicular to it (this is easily done with your set square):

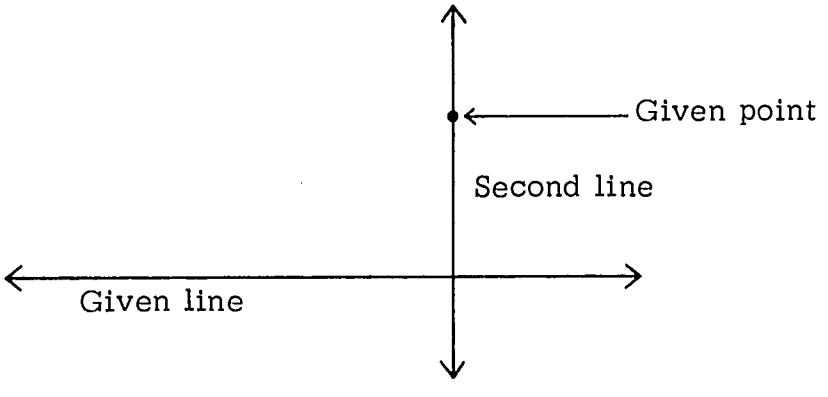


Then draw a third line that is perpendicular to the second line:

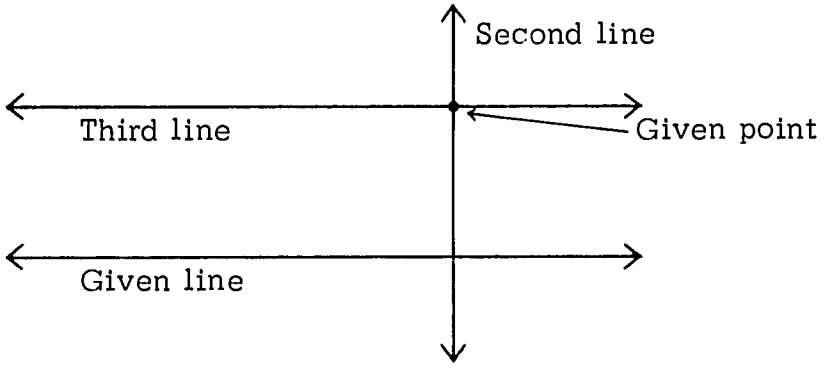


The first and third lines are parallel since the second is perpendicular to each of them.

You can do even more. You can be given any straight line and any point not on it and proceed to find another line that is parallel to the given line and goes through the given point. Here is how: Through the given point construct a line perpendicular to the given line:

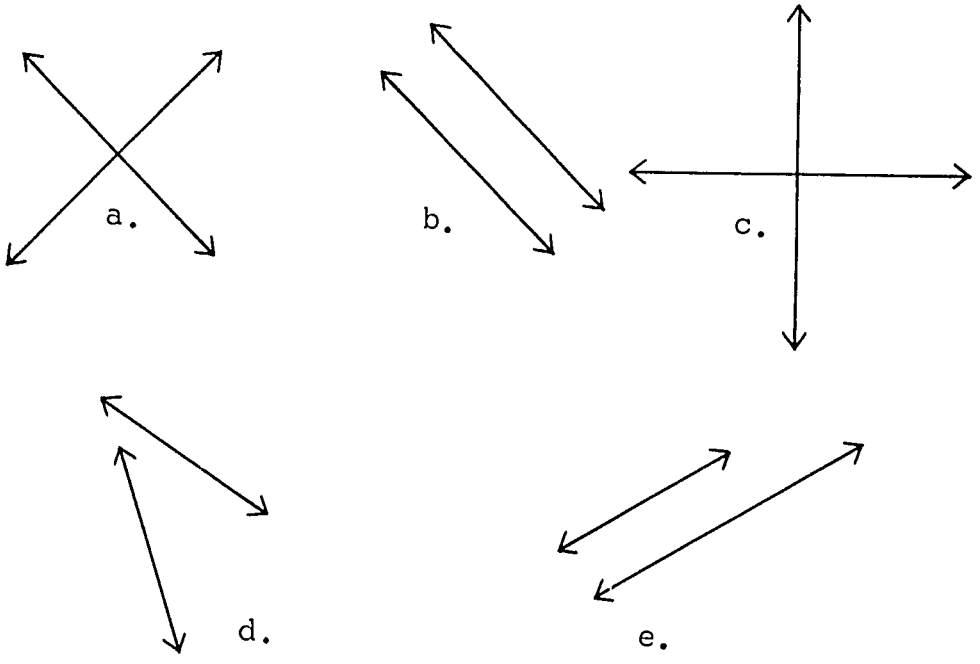


Then construct a third line that is perpendicular to the second line at the given point. Your result will look like this:



Problems 2-4

1. First by sight, then second using a square corner, test each of the following pairs of straight lines for being perpendicular and for being parallel.



2. In Problem 1, which pairs of lines do you think are intersecting? Remember that we can never draw all of any straight line; two straight lines may intersect although the parts of them shown in a drawing may not intersect.

3. It is a fact that two straight lines in a plane that are parallel cannot intersect. This being so, try to answer these two questions.

   (a) Can two parallel lines in a plane be perpendicular?

   (b) Can two perpendicular lines in a plane be parallel?

4. This exercise will lead to a discovery. In order to make the discovery you will have to draw a line that is perpendicular to one of two parallel lines. This will mean drawing perpendicular lines a number of times, and we should like to give you a word of advice about constructing perpendiculars. Whenever you can, use your set square instead of your compasses. You have a good set square, and with a good set

square you can draw perpendicular lines faster and with more accuracy than you can with compasses and a straight edge. Now for the discovery.

On a clean piece of paper, draw two parallel lines. Mark two points on each line, and give them names, so that one of the lines is $\overleftrightarrow{TP}$ and the other is $\overleftrightarrow{IH}$. Through T draw a line perpendicular to $\overleftrightarrow{TP}$. Be sure to draw enough of this line to show its intersection with $\overleftrightarrow{IH}$. Call the point of intersection B. What do you think is the relation between $\overleftrightarrow{BT}$ and $\overleftrightarrow{IH}$? What do the corners at B look like?

Now through P draw a line perpendicular to $\overleftrightarrow{TP}$. Let F be the name of the point where this line intersects $\overleftrightarrow{IH}$. What do the corners at F look like?

Now fold your paper so that $\overleftrightarrow{TP}$ and $\overleftrightarrow{IH}$ are right on top of each other. This puts B on top of T and F on top of P. Hold your paper up to the light. What do you see?

Do you see that the square corners that you constructed at T fit exactly the corners at B? Do you see that the corners at F fit exactly with the square corners that you made at P?

Now unfold your paper, and pick any point you like on $\overleftrightarrow{IH}$. Call it Y, for "your point". Through Y draw a line perpendicular to $\overleftrightarrow{IH}$. What can you say about the relationship between this new line and $\overleftrightarrow{TP}$? What is the relationship between this new line and $\overleftrightarrow{PF}$? $\overleftrightarrow{BT}$?

By now you should have some very definite thoughts about what happens when you draw a line perpendicular to one of two parallel lines. Here is what <u>we</u> think:

A line that is perpendicular to one of two parallel lines is also perpendicular to the other.

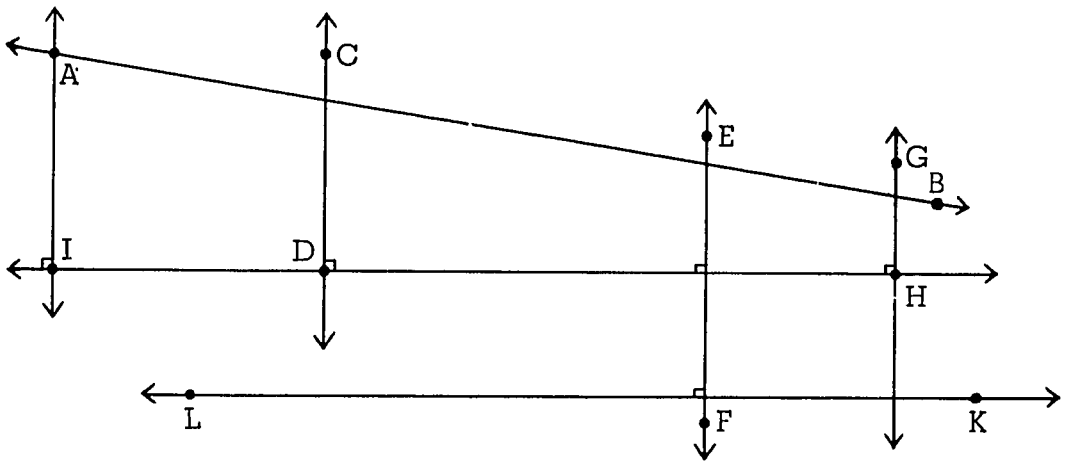
5. As you worked your way through Problem 4, you may have wished for a way to keep track of corners that you knew were square corners. In the picture below, you will see that we have drawn a little mark ⌐ in some of the corners. This little square corner is the mark that we use when we know that two intersecting lines are perpendicular. For example, we have put this mark at I, D, H, and at two other points to indicate that the lines crossing there are perpendicular (make square corners).

As you study the picture below, see if you can answer these two questions:

(a) Which pairs of straight lines in the following figure are pairs of perpendicular lines?

(b) Which are pairs of parallel lines?

Be careful here! We haven't marked <u>all</u> the square corners, and we haven't drawn all of the intersections.



6. Do you think that the following statement is a true statement?

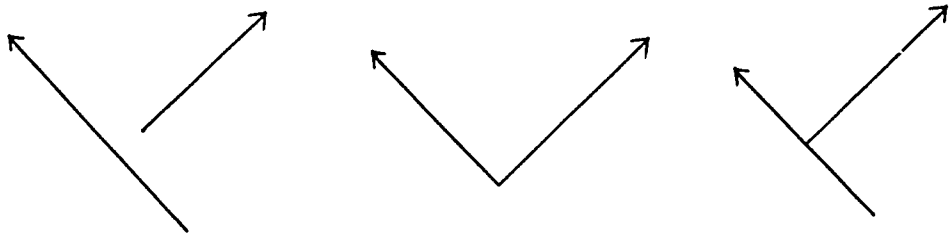
A line that is parallel to one of two parallel lines is parallel to the other one also.

2-5 Rays and Angles

Now let us investigate pairs of rays in a plane. There are several ways in which a pair of rays may intersect (we'll have a close look at these in an exercise below), but one is especially important. When two rays have the same endpoint, but no other point in common, we say that the two rays make an angle. That is:

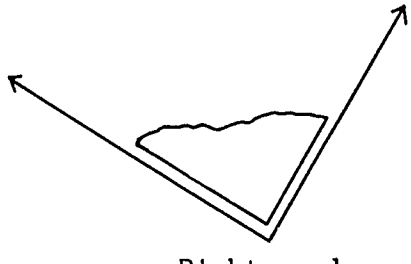
> An angle consists of, or is a combination of, two rays that have the same endpoint, but no other point in common.

Of the three pairs of rays pictured below, only the middle pair is an angle.



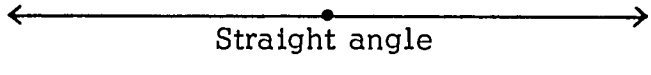
If two rays make an angle, we shall also say that they are connected end to end. The common endpoint is called the vertex of the angle, and the two rays are the sides or edges of the angle.

There are two special kinds of angles — the one in which a square corner fits the sides and the vertex,



Right angle

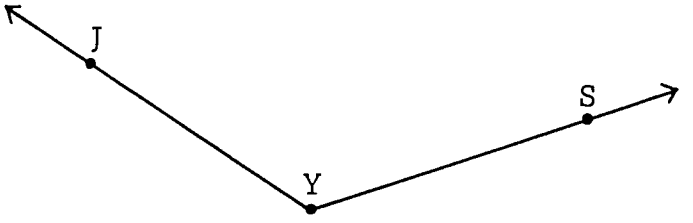
and the one in which the two rays make a straight line:



Straight angle

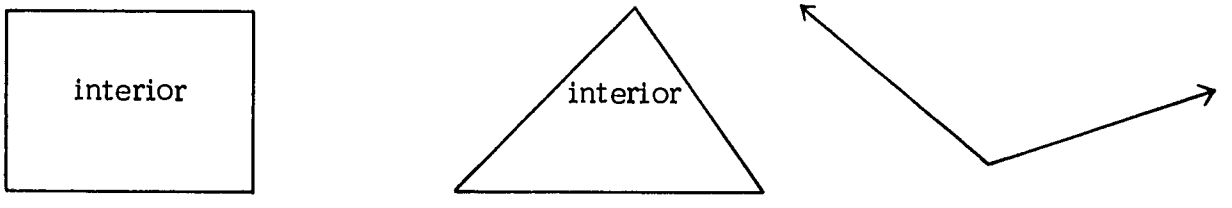
The first of these is called a <u>right angle</u>, and the second a <u>straight angle</u>. Rays that make a straight angle are <u>opposite rays</u>.

Here is a good way to name angles. Suppose that we have two rays that make an angle, and the rays are already named (if they are not, we can name them). The vertex has a letter name, perhaps Y, and the sides bear names like $\overrightarrow{YJ}$ and $\overrightarrow{YS}$. We could say "the angle $\overrightarrow{YJ}$ and $\overrightarrow{YS}$". This is quite all right, but can we be briefer? How about "$\overrightarrow{YJ}$ - $\overrightarrow{YS}$"?

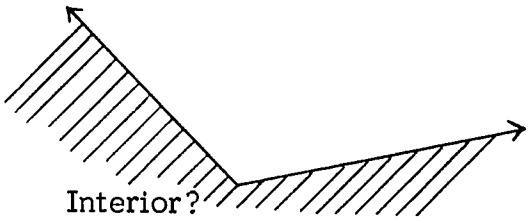


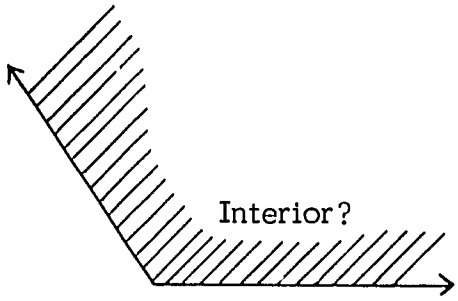
This would work, but it, too, can be shortened. What do you think about "$\overleftrightarrow{JYS}$"? It is short enough, but it suggests that J, Y, and S lie on a line, and they very well may not. How can we avoid this? Just bend the double-headed arrow. This gives "$\overleftarrow{\wedge}$JYS", the top of which looks like an angle. But we don't stop even here, for mathematicians cut it down even more: to $J\hat{Y}S$. Another name for $J\hat{Y}S$ is $S\hat{Y}J$. It doesn't matter in what order you write the letters as long as you make sure to put the vertex letter in the middle. So if you see the symbols "$Z\hat{X}Q$" or "$Q\hat{X}Z$", you will know that they stand for an angle that consists of the two rays $\overrightarrow{XZ}$ and $\overrightarrow{XQ}$.

Just as a triangle and a rectangle have interior regions, so does any angle that is not a straight angle. Look at these three figures:

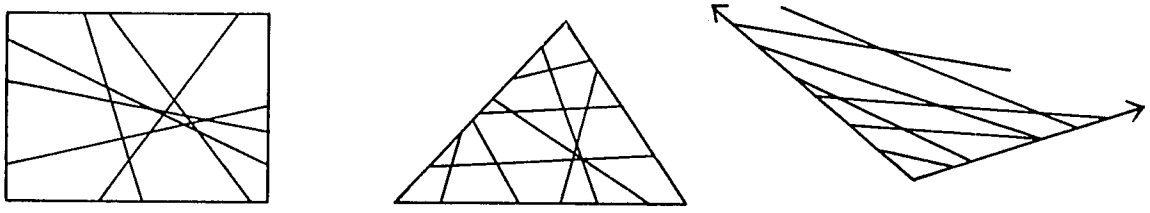


What should we regard as the interior of the one that is an angle? There are only two possibilities:





As you may have guessed, it has been agreed that the right-hand one should be the interior. Does this choice have anything in common with the interior of a triangle and the interior of a rectangle (even though the rectangle and the triangle are "closed" and the angle is "open")? All three have sides, and this gives us a common way of forming the interior: <u>draw all line segments that have their endpoints on the sides of the figure</u>.



<u>Problems 2-5</u>

1. Draw, if possible, a pair of rays that

   (a) do not intersect;

   (b) do not intersect and are not parallel (two rays are <u>parallel</u> if the lines of which they are part are parallel);

   (c) intersect at a single point;

   (d) intersect at a single point and are not an angle;

   (e) intersect at exactly two points;

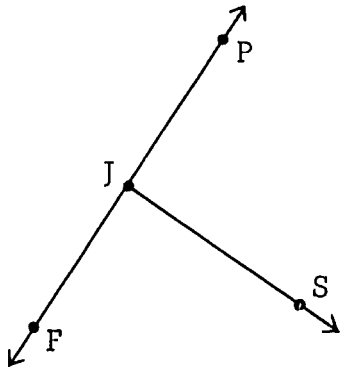
   (f) have in common a line segment;

   (g) make a right angle;

   (h) make an angle but do not intersect.

If, in any of these cases you think that there is no such pair, give your reasons for thinking so.

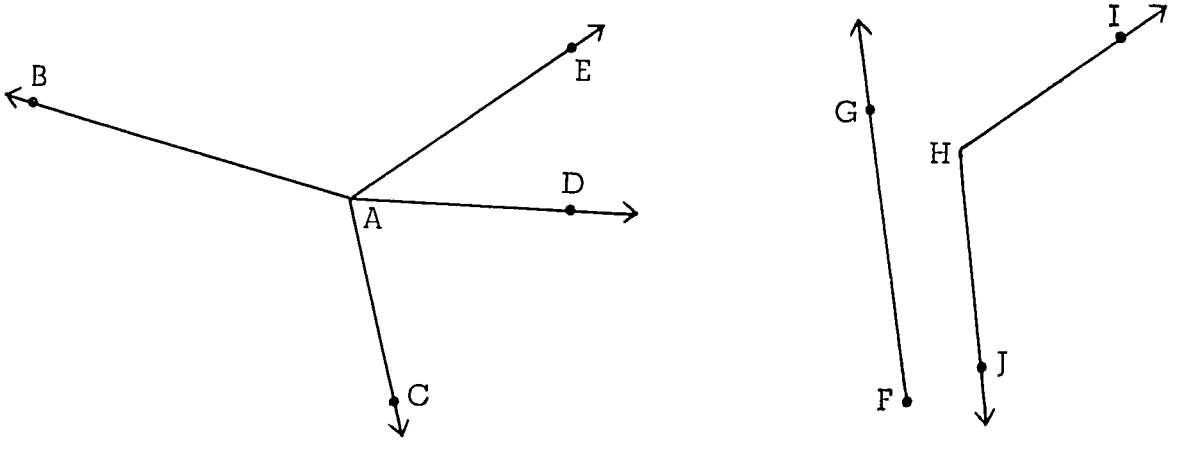
2. Here are three rays, $\overrightarrow{JP}$, $\overrightarrow{JS}$, and $\overrightarrow{JF}$. How many angles can you name from them? How many right angles do they make?



3. If, in Problem 2, you added a ray opposite $\overrightarrow{JS}$, how many right angles and how many straight angles could you make?

4. Here are seven rays.

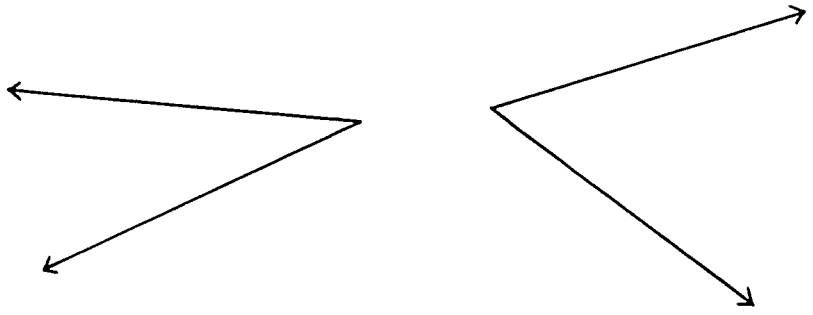
(a) How many angles do these rays make?



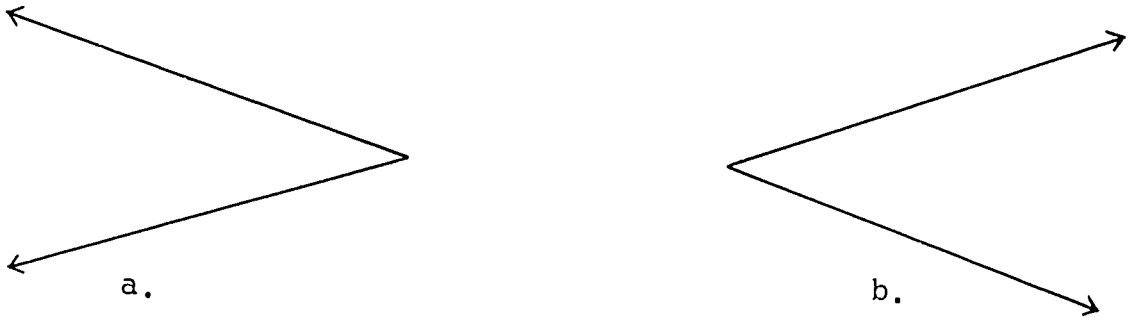
5. On a piece of tracing paper, copy the rays that start at A. Then shade the interiors of $B\hat{A}C$ and of $D\hat{A}E$.

2-6 Comparing Angles

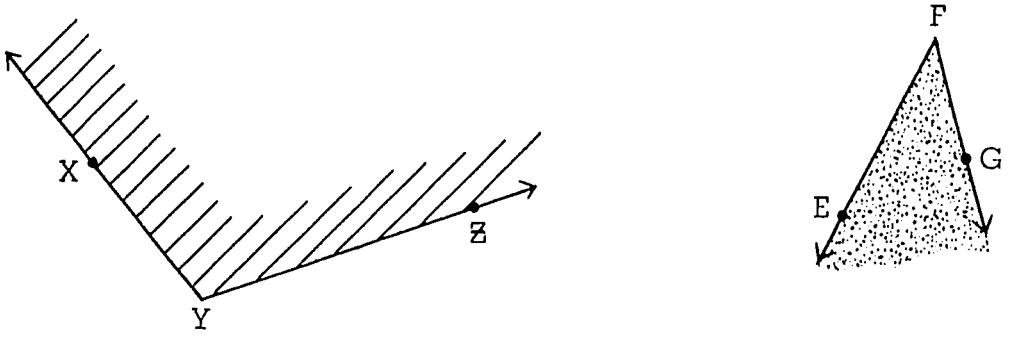
How can you tell when one angle opens more than another, or when two angles have the same size? Sometimes you can tell just by looking. Look at these angles:



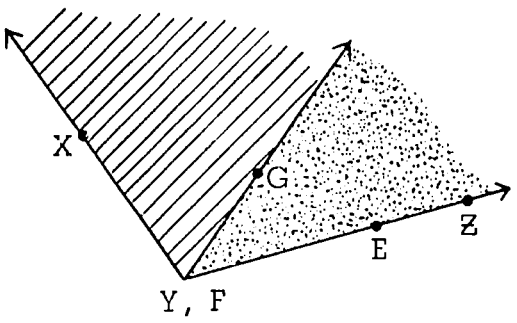
You wouldn't have any trouble saying which is larger, would you? But now look at these:

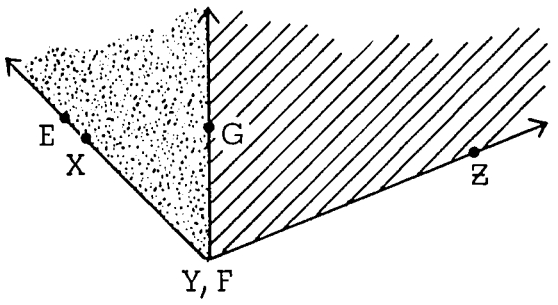


Do you think one of them is larger than the other, or would you say that they have the same size? If we could bring them closer together it would be easier to compare them. In fact, a good way to compare angles (when you can do it) is to try to put one right on top of the other. Move the angles together so that the interiors overlap and a side of one angle fits exactly on top of a side of the other angle. If we did this with the two angles $X\hat{Y}Z$ and $E\hat{F}G$ shown here



they would fit together in one of four ways. Here are two of the ways:



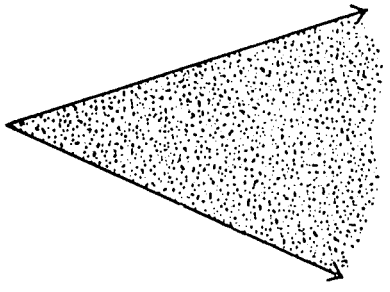


Can you think of the other two ways?

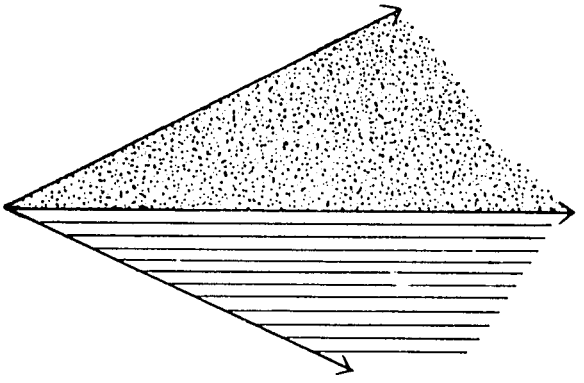
In each of the four it is clear that the interior of $\widehat{EFG}$ is contained in the interior of $\widehat{XYZ}$, and that the interior of $\widehat{XYZ}$ is not contained in the interior of $\widehat{EFG}$. It is clear also that we want to consider $\widehat{EFG}$ as being <u>smaller than</u> $\widehat{XYZ}$, and $\widehat{XYZ}$ as <u>larger</u> than $\widehat{EFG}$.

Whenever we put two angles together with their interiors overlapping, and with a ray of one angle fitted exactly to a ray of the other angle, one of two things happens:

(a) The other two rays fit exactly, so that the angles fit exactly. In this case the angles have the same size and we call them <u>congruent</u>.

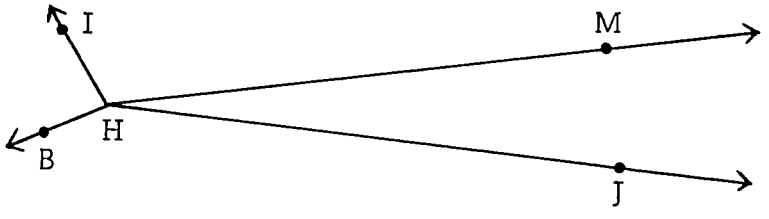


(b) A ray of one angle lies in the interior of the other angle. In this case the angles are not congruent. One angle is <u>smaller</u>, and the other is <u>larger</u>.



Problems 2-6

1. Copy the drawing below by tracing it on thin paper.

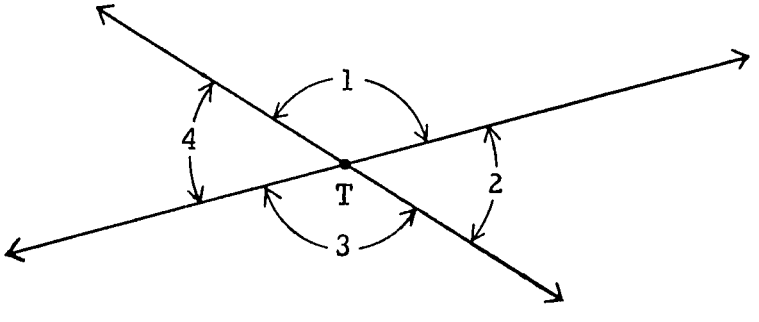


Before folding your paper, say which of your angles you think is larger, your copy of $I\hat{H}B$ or your copy of $M\hat{H}J$. Now check your statement by folding. Were you right?

2-7 Angles Made by Straight Lines

After you read this section you will see that the world around us is full of congruent angles.

On a clean piece of paper, draw two intersecting lines, and call their point of intersection T.



The lines make four rays, all starting at T. The rays, taken in pairs one from each line, make four different angles. Do you see them in your drawing? We've numbered ours 1, 2, 3, and 4. These <u>four</u> angles are called <u>the angles made by the two lines</u>. A side of each of these angles comes from each line. It is true that there are also two straight angles in the picture, but each comes from one line alone, and we do not consider them to be angles that are made by the <u>two</u> lines.

Angles 1 and 3 are called <u>vertically opposite angles</u> because they have a common vertex and because their rays are opposite pairs.

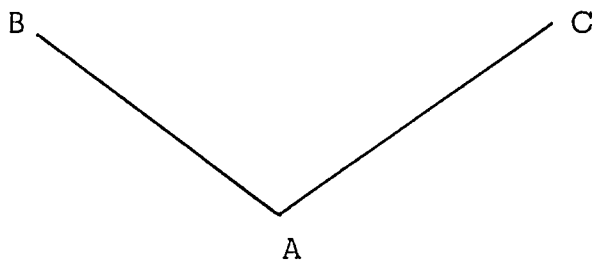
Do you think that 2 and 4 are <u>vertically opposite</u>? Why?

Do you think that 1 and 3 are congruent? Do they look congruent? Try to fit them together by folding your paper. What do you conclude?

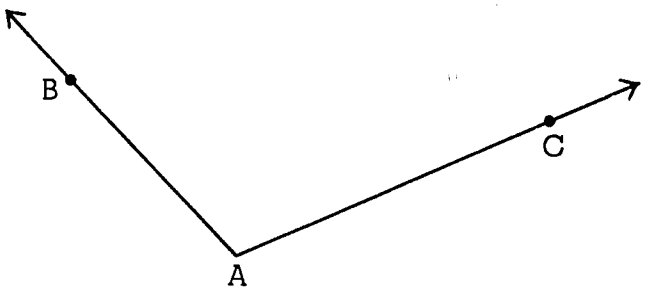
Is 2 congruent to 4? Test them by folding. What do you say?

Vertically opposite angles are congruent.

Before looking at the Problems below, let us briefly revise line segments. If two line segments have a common endpoint, the pair <u>is not an angle</u>, because an angle is a pair of <u>rays</u> with a common endpoint. But given two line segments $\overline{AB}$ and $\overline{AC}$ with a common endpoint,

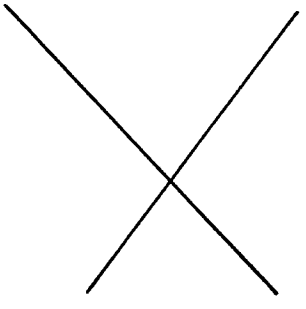


we can make an angle from them by drawing the two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

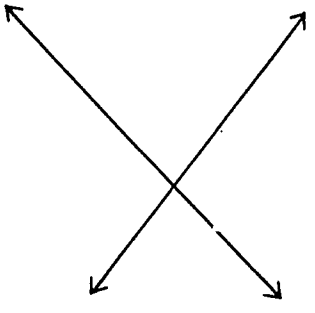


This angle, $B\hat{A}C$, is called <u>the angle made by</u> $\overline{AB}$ and $\overline{AC}$.

Likewise, when two line segments cross we say that they make four angles (the angles made by the two lines containing the segments):

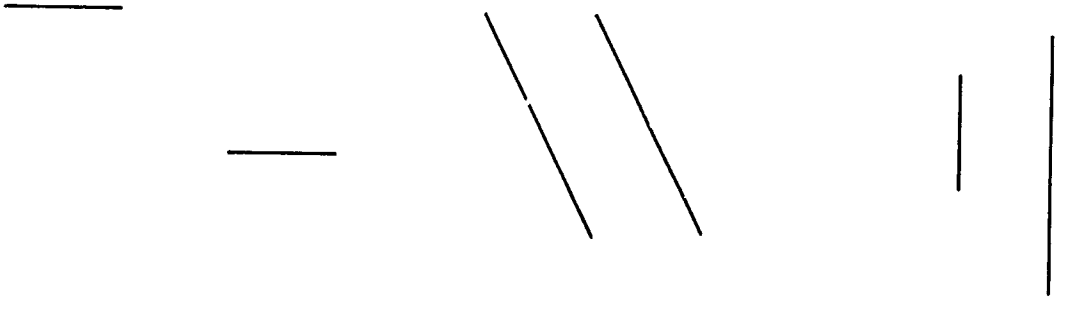


Crossing line segments



The four angles made by the crossing line segments

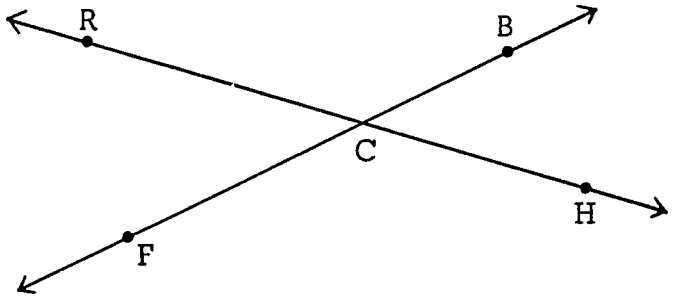
If two line segments lie on lines that are parallel, we call the line segments themselves parallel. Here are three pairs of <u>parallel line segments</u>:



<u>Problems 2-7</u>

1. What would you say it ought to mean for two rays to be parallel? For a ray and a line segment to be parallel? For a line and a line segment to be parallel?

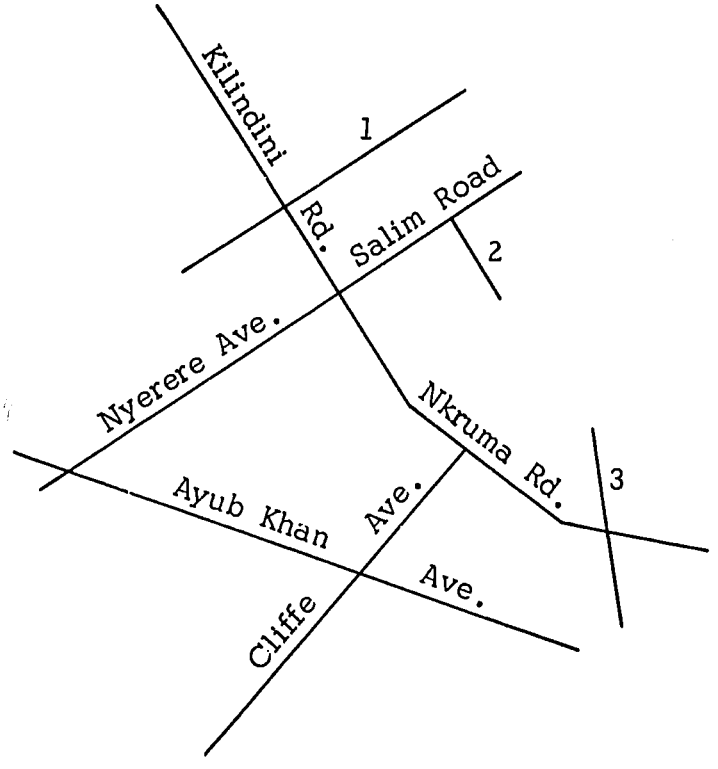
2. In the picture below, name the vertically opposite angles:



3. The line segments below are part of a map of Mombasa. Some of the streets are named, and others are numbered so that it will be easier for you to refer to them when you think about the questions.

(a) How many pairs of perpendicular line segments can you find? How many parallel ones?

(b) At each intersection, what angles are congruent?



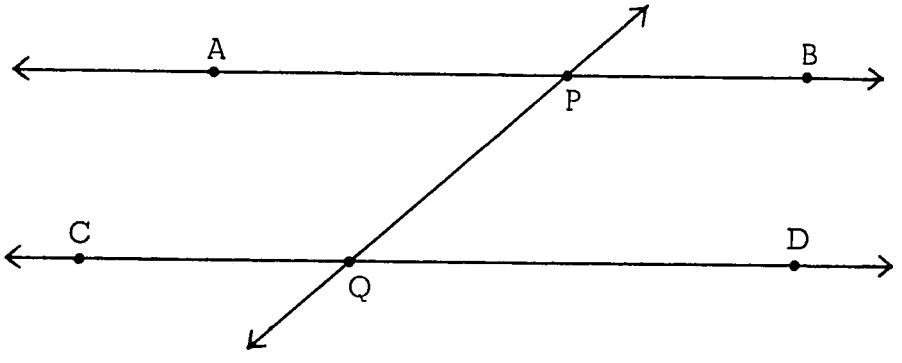
4. Think about where you have seen things that suggest vertically opposite angles. Do you know two paths that are straight where they cross? If you were to draw part of a fisherman's net would any of the line segments in your drawing make vertically opposite angles? If you were to draw a map of your school compound and grounds, would you have made any vertically opposite angles? Does your classroom building have roof supports that cross?

5. Look around your classroom for parallel and perpendicular line segments. For example, look at the windows. Are the sides parallel? (Look at one of the cross pieces — what kind of angle does it make with both sides?) Are the cross pieces parallel? How do you know?

2-8 Alternate Interior Angles

There is a famous geometric figure in which congruent angles always appear. Let's see if you can find them.

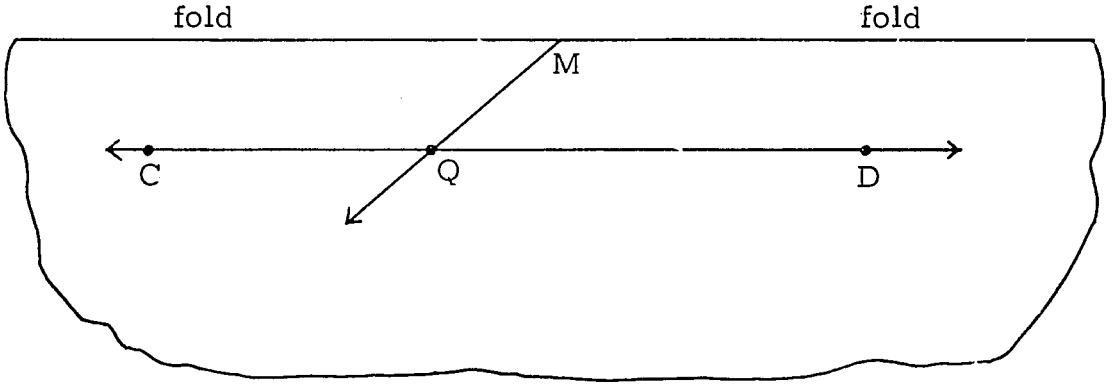
On a clean, thin sheet of paper draw two parallel lines (thin because you will want to look through four layers of it, and clean so that no other marks can get in the way of your picture). Then draw a line that cuts across them both:



A line like this is a transversal of the parallel lines. The transversal makes four angles with each of the parallel lines. Can you name them? Each angle at P is congruent to another angle at P, and each angle at Q is congruent to another

angle at Q. Now, do you see any angle at P that you think might be congruent to an angle at Q? Can you think of a way to fit them together by folding?

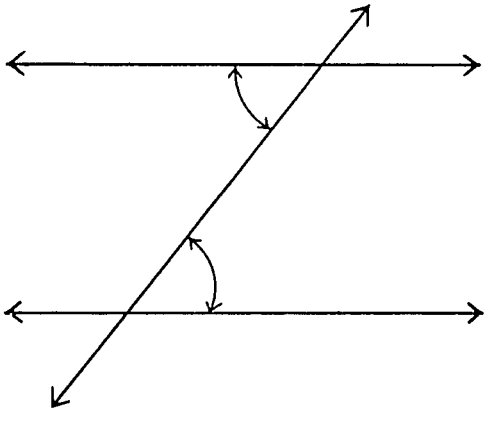
Here is one way. Fold your paper so that $\overleftrightarrow{CD}$ lies on top of $\overleftrightarrow{AB}$. Then your paper, on one side, looks like this:



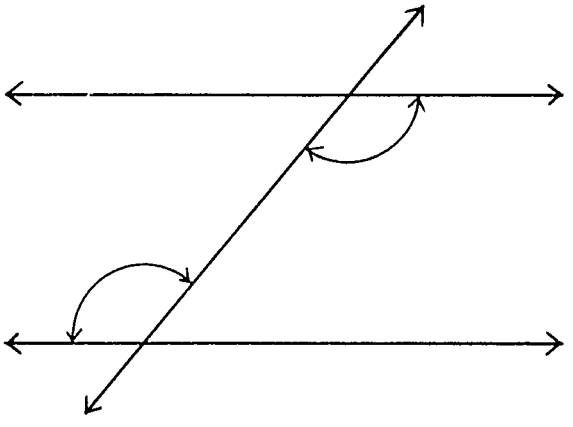
We have called the point where $\overleftrightarrow{PQ}$ crosses the fold M. Fold the paper again to make a square corner at M. If you hold your paper up to the light what do you see? If you cannot see clearly enough through all that paper, unfold your paper and draw the lines more heavily. Then fold and have another look. What do you conclude about $C\hat{Q}P$ and $Q\hat{P}B$?

What do you think about the angles $A\hat{P}Q$ and $P\hat{Q}D$? When you fold your paper, do they fit exactly? What do you conclude?

Do you see that the four angles we have been talking about, $P\hat{Q}D$, $A\hat{P}Q$, $Q\hat{P}B$, and $C\hat{Q}P$, are the only angles in the picture that have an edge containing $\overline{PQ}$? These four angles are called the <u>interior angles</u> formed by the transversal and the parallel lines. $A\hat{P}Q$ and $P\hat{Q}D$ are called <u>alternate interior angles</u>. $B\hat{P}Q$ and $P\hat{Q}C$ are also <u>alternate interior angles</u>.



Alternate interior angles



Alternate interior angles

Do you see that alternate interior angles have different vertices? Do you see that the interiors of alternate interior angles lie on opposite sides of the transversal?

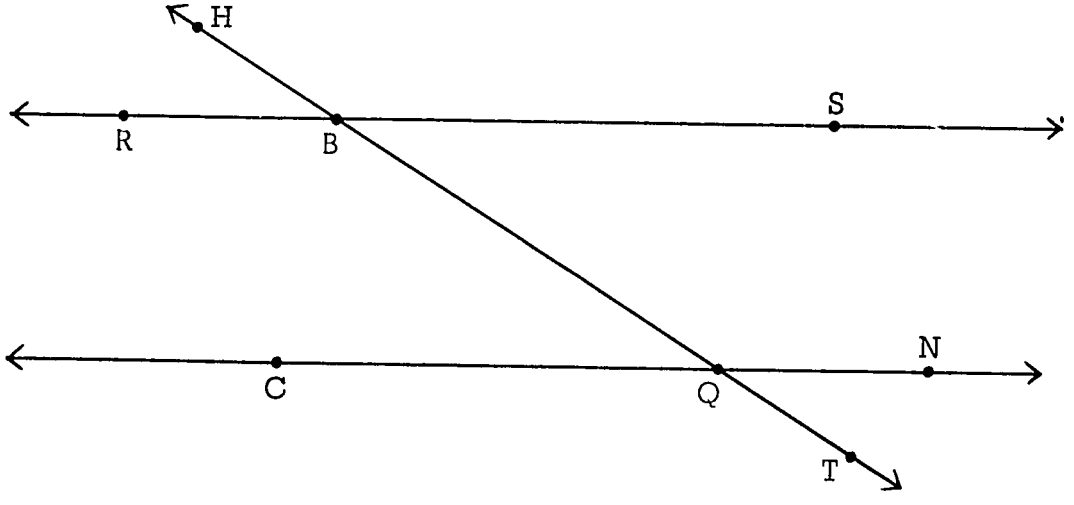
> A transversal to two parallel lines makes two pairs of alternate interior angles. The angles within each pair are congruent.
>
> In short:
>
> When a transversal crosses two parallel lines, the alternate interior angles are congruent.

Problems 2-8

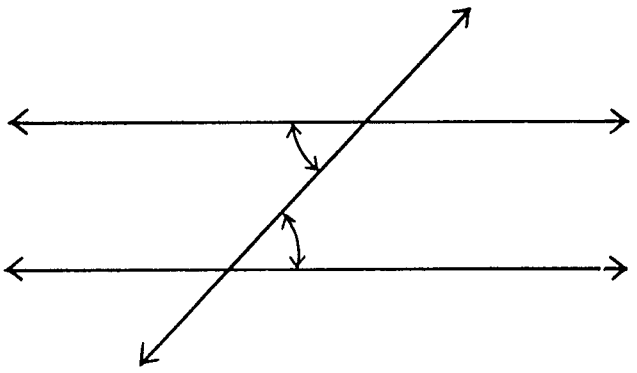
1. Here is a picture of a transversal crossing two parallel lines.

   (a) Use your knowledge of vertically opposite angles, and of alternate interior angles, to name all the angles congruent to $H\hat{B}R$.

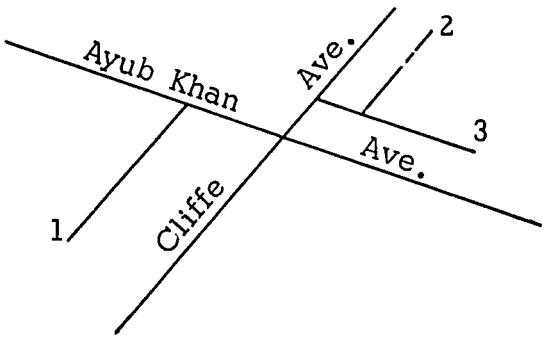
   (b) What angles are congruent to $B\hat{Q}N$?



2. On tracing paper make a copy of the figure below; and on your paper shade the interiors of the alternate interior angles marked with arcs.



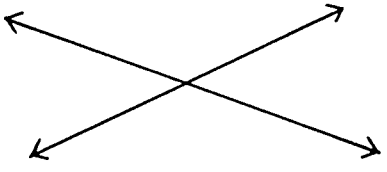
3. In Mombasa, let us look at the streets near the intersection of Cliffe Avenue and Ayub Khan Avenue:

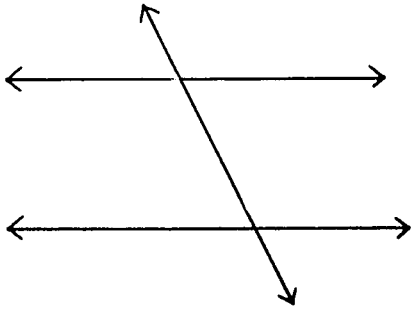


Remembering what we mean by angles made by line segments, what congruent angles can you find on the map?

2-9 Comparing Angles Without Folding

By now you know what angles are, and how lines and line segments make angles. You also know how to compare angles on paper whenever you can fold one on top of the other. You have used this method of comparison to investigate angles made by crossing lines in figures like these:





Your study of these figures led you to identify congruent angles made by objects in the world around you.

But what about angles that do not belong to one of these figures? What about angles that you cannot move about? How are you going to compare them?

Do you remember how we compared line segments? We made a copy of one and compared the copy with the other. If we could make copies of angles, then we could compare one angle with another by comparing a copy of it with the other. Then, instead of saying that two angles are congruent if they fit together exactly, we could say

> Two angles are congruent if a copy of either one fits the other exactly.

We could also say

> One angle is larger (smaller) than another if a copy of it is larger (smaller) than the other.

How, then, can we copy angles?

Some angles are very easy to copy. Right angles, for example. Your set square is a copy of every right angle. Given two right angles, your set square, which is a copy of one, always fits the other. <u>Any two right angles are congruent</u>. Do you agree?

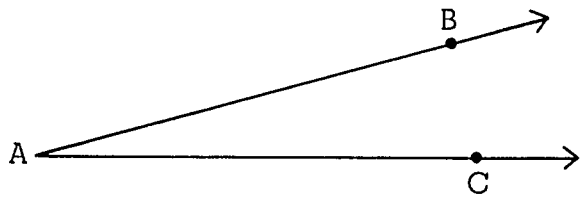
Would you say that any two straight angles are congruent? Why?

What about angles that aren't right angles or straight angles? How can we make copies of them?

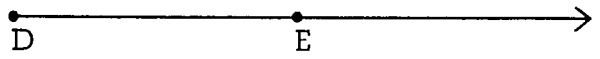
One way is to put thin paper over the angle and copy the angle onto the paper by tracing. Another way is to cut a model of the angle out of a sheet of paper. Another way, and it is a way that always works, is the way we shall

show you now. If you learn to copy angles this way, you will be able to copy <u>any</u> angle you wish.

Let us each start with an angle drawn on a piece of paper. Ours looks like this:

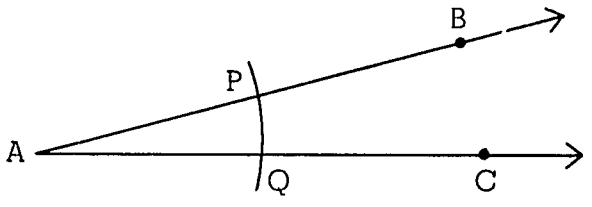


Yours may look different, but that doesn't matter, because we are going to show you with pictures and with words how to copy <u>your</u> angle. So, start with your angle $B\hat{A}C$ on a piece of paper, and on another piece of paper draw a ray:

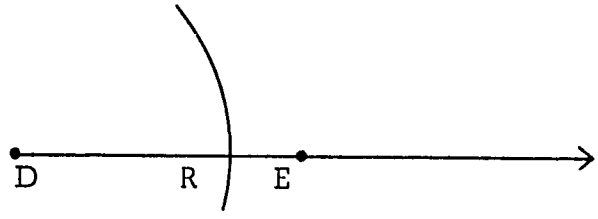


We will show you how to construct an angle which is congruent to $B\hat{A}C$ and has $\overrightarrow{DE}$ as one side.

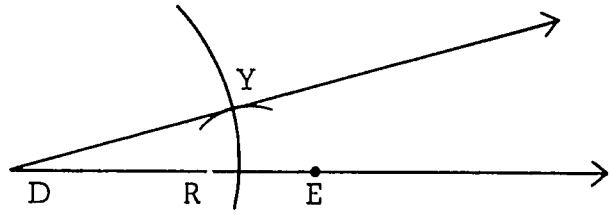
Take your compasses, and with vertex A as centre draw an arc that cuts both $\overrightarrow{AB}$ and $\overrightarrow{AC}$:



(We have called the points where the arc cuts the rays P and Q.) Leaving your compasses unchanged, put the point at D and draw an arc that cuts $\overrightarrow{DE}$. To be safe, draw this arc even longer than the one that cuts $B\hat{A}C$. (After you have had some practice, you will know in advance just how long to make this second arc.)

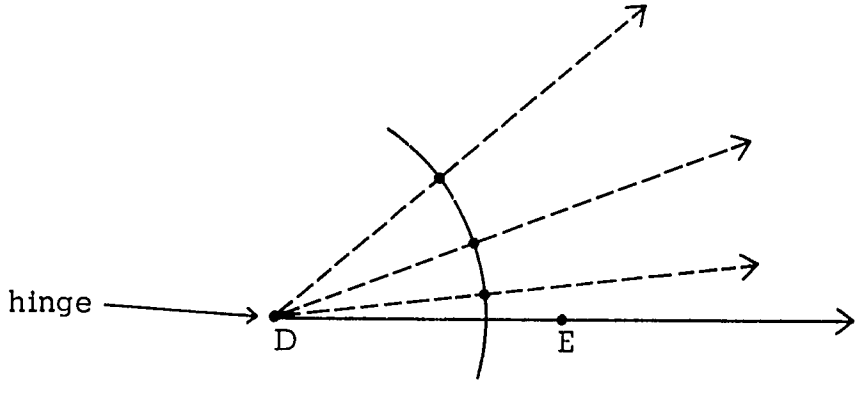


Returning to $B\hat{A}C$, set your compasses so that one point fits on Q at the same time that the other fits on P. Without changing this setting, put the pin point at R and draw an arc that cuts the first arc you drew there. Call the point of intersection Y, if you like, and draw the ray $\overrightarrow{DY}$.



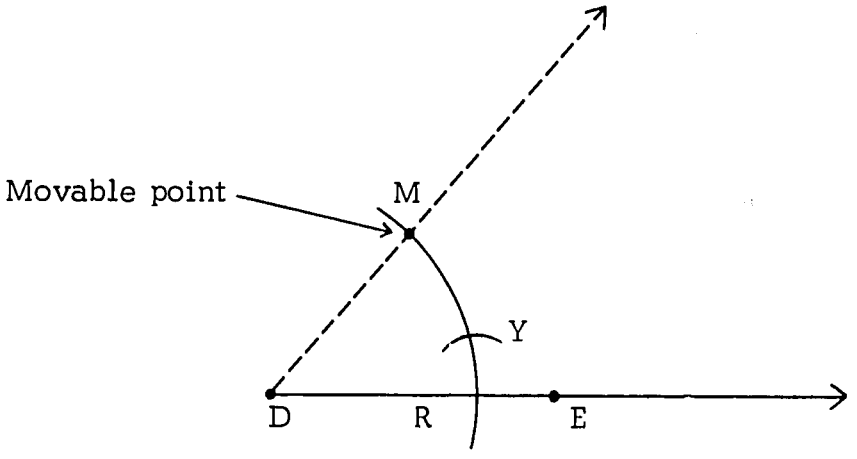
The angle $Y\hat{D}E$ is congruent to $B\hat{A}C$.

Here is a way to think about this construction that we think you will like. Imagine that while the ray $\overrightarrow{DE}$ is fixed, we have attached to it a movable ray that is hinged at D. This movable ray can take any position you like. You can open and close the angle the way you open and close a door.



Now draw the arc through R. Each time the movable ray takes on a new position it cuts the arc in a different point. What point does the ray pass when it makes an angle congruent to $A\hat{B}C$?

Now think of the picture in a slightly different way. Start with $\overrightarrow{DE}$ and with the arc through R. On the arc think of a point M that can move along the arc in either direction. Think of the ray $\overrightarrow{DM}$. As M moves, so does $\overrightarrow{DM}$. When M stops, $\overrightarrow{DM}$ stops. M holds $\overrightarrow{DM}$ just the way a door stopper holds a door.



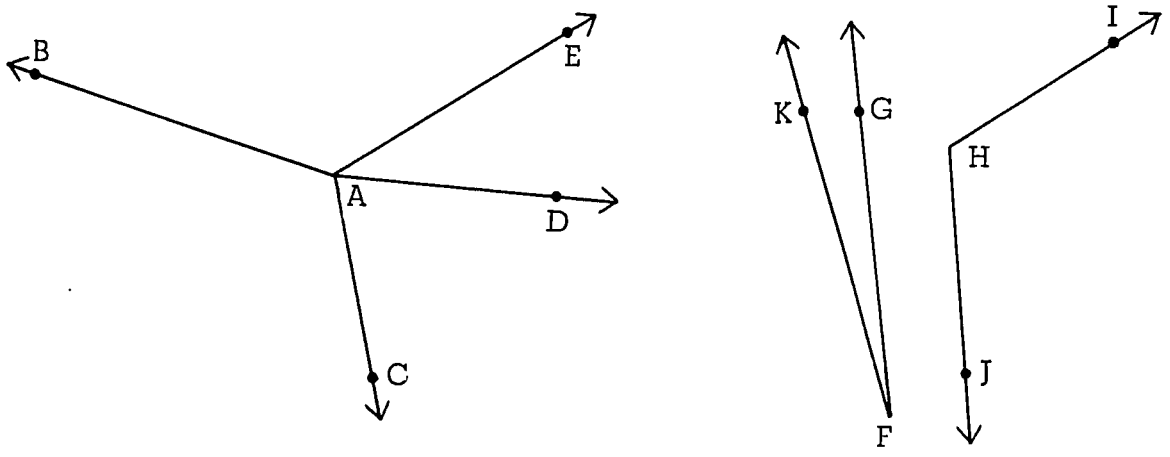
How do you know when to stop M to make $E\hat{D}M$ congruent to $A\hat{B}C$? You stop M when the arc MR is just the same size as the arc PQ. You stop M at Y.

> Now that we can copy angles, we can compare two angles by comparing one with a copy of the other. The copy is an angle that we can move about. To compare it with the other angle, all we have to do is to see what happens when we try to fit it to the other angle.

Problems 2-9

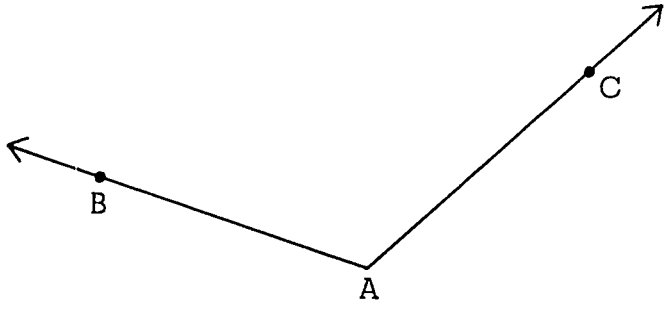
1. At the beginning of Section 2-6 we showed you two angles that are hard to compare by eye. Trace one of them now on thin paper and compare it with the other. What do you conclude?

2. Judging only by sight, list the angles below in order of decreasing size.

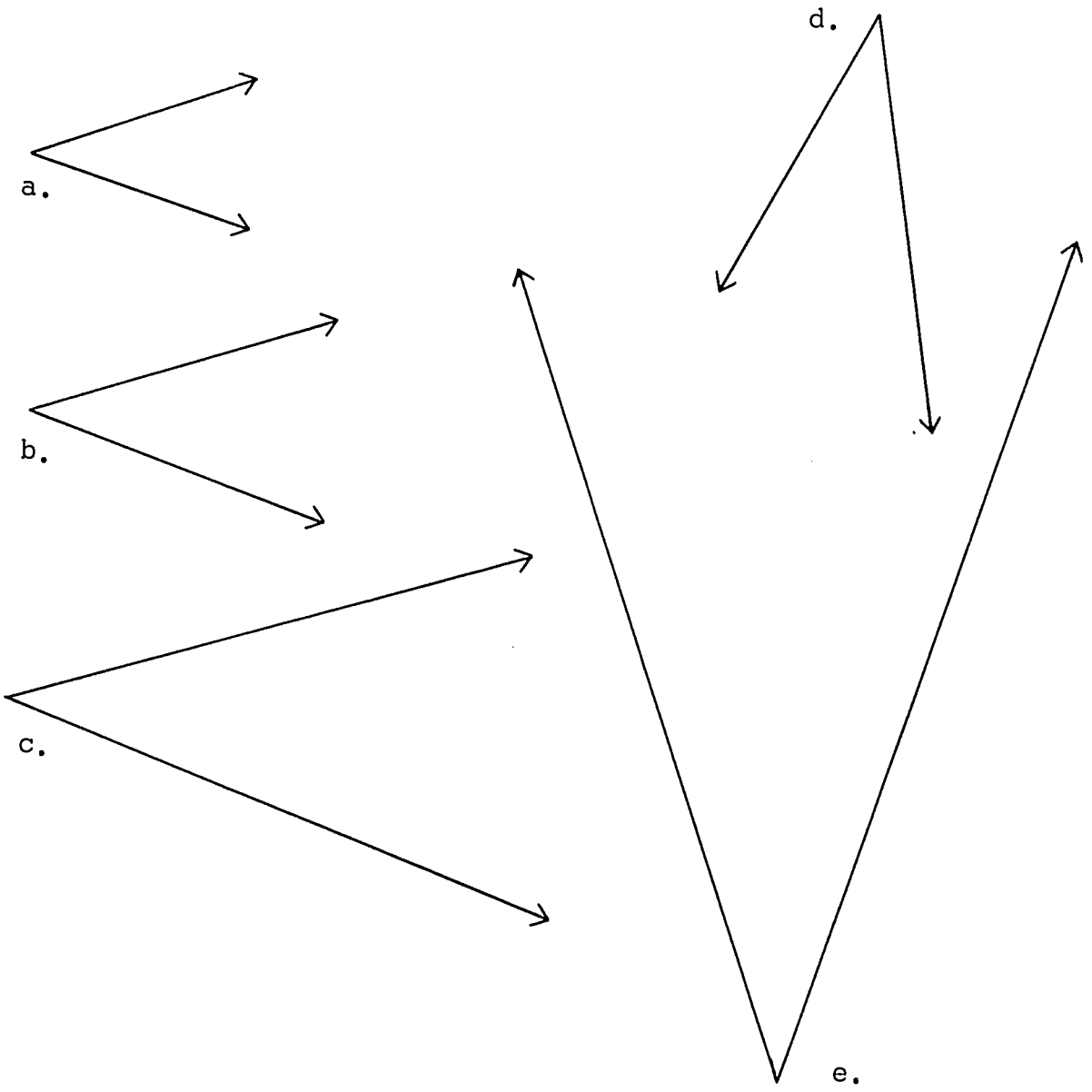


3. Do you remember that to compare line segments you could make a copy of each and then compare the copies? You can compare angles that way, too. For example, draw a ray, and on it construct an angle congruent to $B\hat{A}C$ of Problem 2. On the same ray construct also another angle, this one congruent to $I\hat{H}J$ of Problem 2. Which of these two angles in Problem 2 is larger?

4. Copy the following angle, $B\hat{A}C$, by tracing it on a piece of thin paper laid over it. On another piece of paper, draw a ray with endpoint D and use your compasses to construct an angle congruent to $B\hat{A}C$ and having your ray as one side. Put your thin paper copy over the compass copy, and compare them. Does the thin paper copy give you a way to show that the compass copy and $B\hat{A}C$ are congruent?



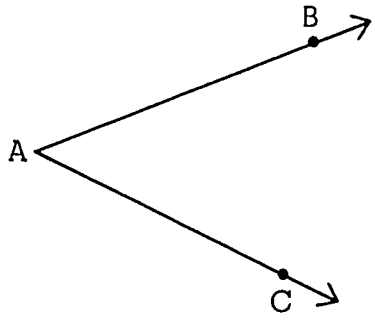
5. Compare the following angles. What do you conclude?



2-10 Bisecting an Angle

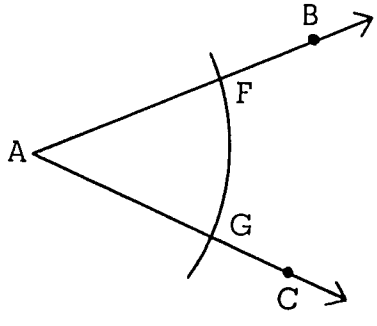
There is another important construction with angles, called bisection. You bisect an angle by dividing it into adjacent angles that are congruent, just as you bisect a line segment by dividing it into two line segments that are congruent.

To bisect the angle $B\hat{A}C$,

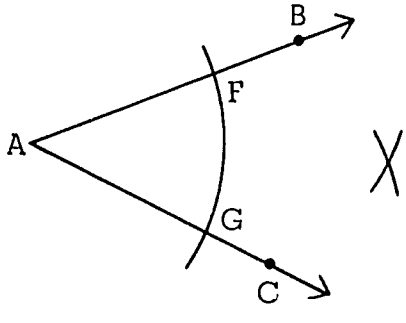


we find a third ray $\overrightarrow{AM}$ that lies in the interior of $B\hat{A}C$ in such a way that $B\hat{A}M$ and $C\hat{A}M$ are congruent. Do you know how to do this? Do you know how to do this with a straight edge and a pair of compasses? If you do, then just look at the next few pictures to make sure that we do what you think we will do. If the pictures tell you only what you know already, then go on to Problems 2-10. If you are in some doubt about what we are going to do, however, you should read the next few paragraphs.

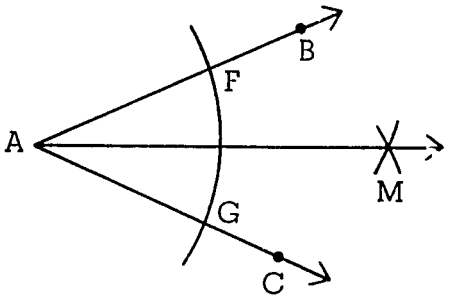
Here is how to bisect $B\hat{A}C$. With the point A as centre, draw an arc that cuts both $\overrightarrow{AB}$ and $\overrightarrow{AC}$. In the picture below, F and G are the names of the points at which the arc cuts the rays.



With the same setting, and with F and G as centres, make two more arcs that intersect away from A like these:



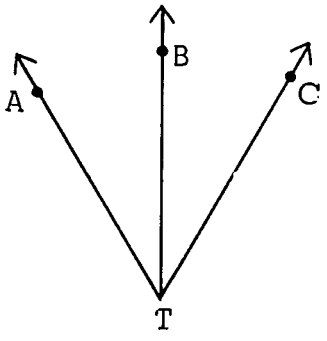
Let M be the point of intersection of these two arcs, and draw the ray $\overrightarrow{AM}$.



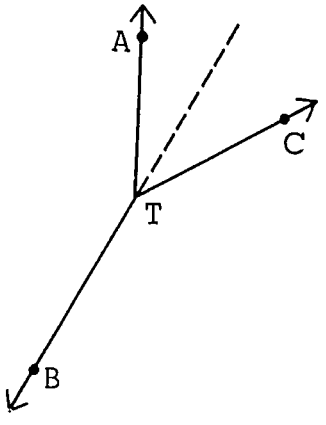
If you fold your paper along $\overrightarrow{AM}$, you will find that $\overrightarrow{AC}$ fits exactly on $\overrightarrow{AB}$. Try it. Do you see that $B\hat{A}M$ and $C\hat{A}M$ are congruent? $\overrightarrow{AM}$ <u>is the bisector of</u> $\widehat{BAC}$.

<u>Problems 2-10</u>

1. In each of the following, say how you could use paper folding to compare $A\hat{T}B$ and $C\hat{T}B$. If they were congruent, how would you know? If they were not congruent, how would you identify the larger one?



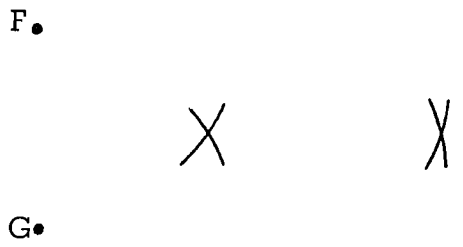
(a)



(b)

2. On the paper draw three angles and bisect each. Test each bisection by paper folding.

3. When we showed you how to bisect an angle, we used the same setting of our compasses for the two intersecting arcs (with centres F and G) as we did for the larger arc we drew first. What do you think would have happened if we had changed the setting before we drew the two intersecting arcs? On a piece of paper, draw two points like F and G. Set your compasses, and draw two intersecting arcs, one with centre F and one with centre G. Change the setting, and on the same side of F and G draw two more intersecting arcs. So far you have something like this:



Now, by changing the setting three or four times more, add some more pairs of arcs to your picture. What pattern do you see? Draw some pairs of arcs on the other side of F and G. What happens? What can you conclude about the way we made our angle bisector? Did we need the same setting for all three arcs?

CHAPTER 3

POLYGONS AND OTHER SIMPLE CLOSED PATHS

3-1 Introduction

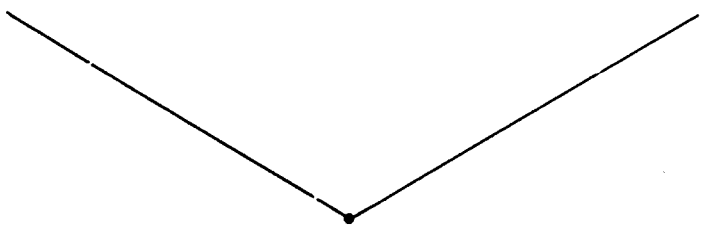
Now that you know how line segments can make angles, and what perpendicular and parallel mean, it is easy to describe polygons. The simplest polygons are triangles and quadrilaterals. We shall see that congruence of triangles and quadrilaterals is related in a very interesting way to congruence of angles and line segments.

Besides polygons, there are other simple closed paths that are easy to draw on a plane: circles and ellipses. Of course circles and ellipses are different from polygons. They are not made of line segments and they have no corners. They are rounded and curved. But polygons, circles and ellipses have a property in common with all simple closed paths on a plane: they separate the plane into two pieces of which they are the common boundary.

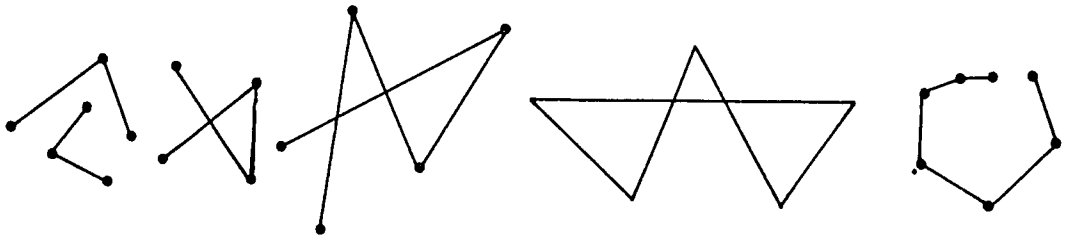
3-2 Polygons

In Chapter 1 we looked at polygons. Some of them, for example triangles and quadrilaterals, may have been familiar to you. We shall now look at polygons more closely.

Do you remember what it means for two line segments to be connected end-to-end? They have one endpoint in common, like this:

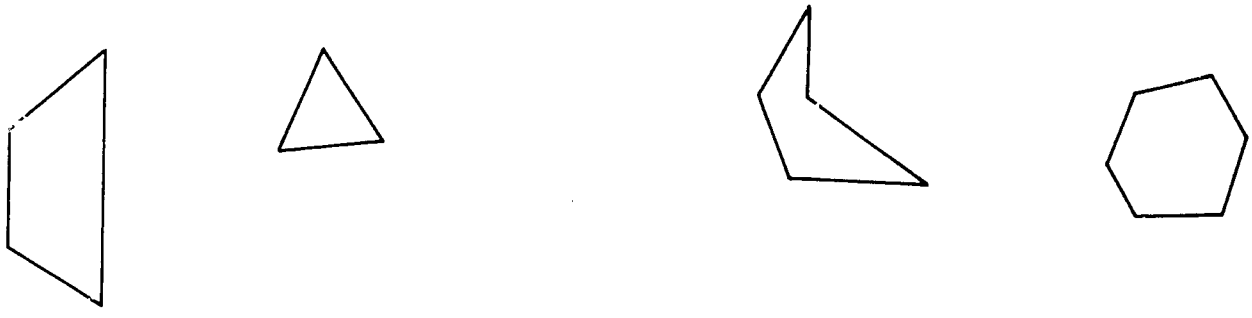


We can begin to describe polygons by saying that a polygon is a figure made of line segments connected end-to-end. You can see that this is only a beginning, because these figures



are <u>not</u> polygons even though they are made of line segments connected end-to-end.

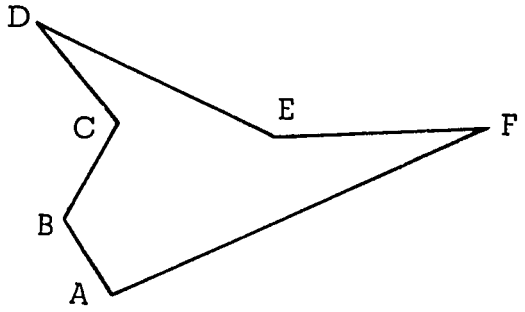
These figures <u>are</u> polygons:



Do you see that in each polygon each endpoint belongs to exactly two line segments? Do you see that no line segments cross? What do you see when you look at the figures that we said were not polygons?

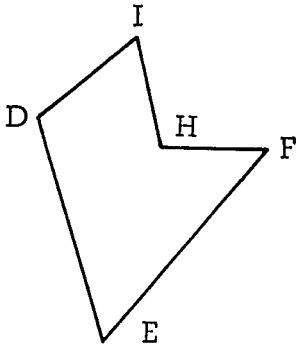
> A polygon is a figure made of line segments connected end-to-end. Each endpoint belongs to exactly two line segments, and no two segments cross each other.

The endpoints of the line segments of a polygon are called <u>vertices</u> of the polygon. The line segments are called <u>sides</u> of the polygon. Two sides of a polygon are <u>adjacent</u> if they have a vertex in common. Two sides of a polygon are called <u>opposite</u> if they have no vertex in common. In the six-sided polygon below $\overline{AB}$ and $\overline{BC}$ are adjacent.

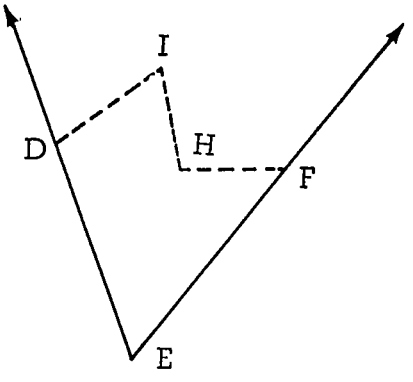


We also say that $\overline{AB}$ is adjacent to $\overline{BC}$ and that $\overline{BC}$ is adjacent to $\overline{AB}$. Is there another side of the polygon that is adjacent to $\overline{AB}$? The side $\overline{CD}$ is opposite $\overline{AF}$. Are any other sides of the polygon opposite $\overline{AF}$?

We have already seen how two line segments connected end-to-end make an angle. If two sides of a polygon are adjacent, the angle they make is called <u>an angle of the polygon</u>. The angle $D\hat{E}F$ is an angle of the polygon pictured below:



The polygon

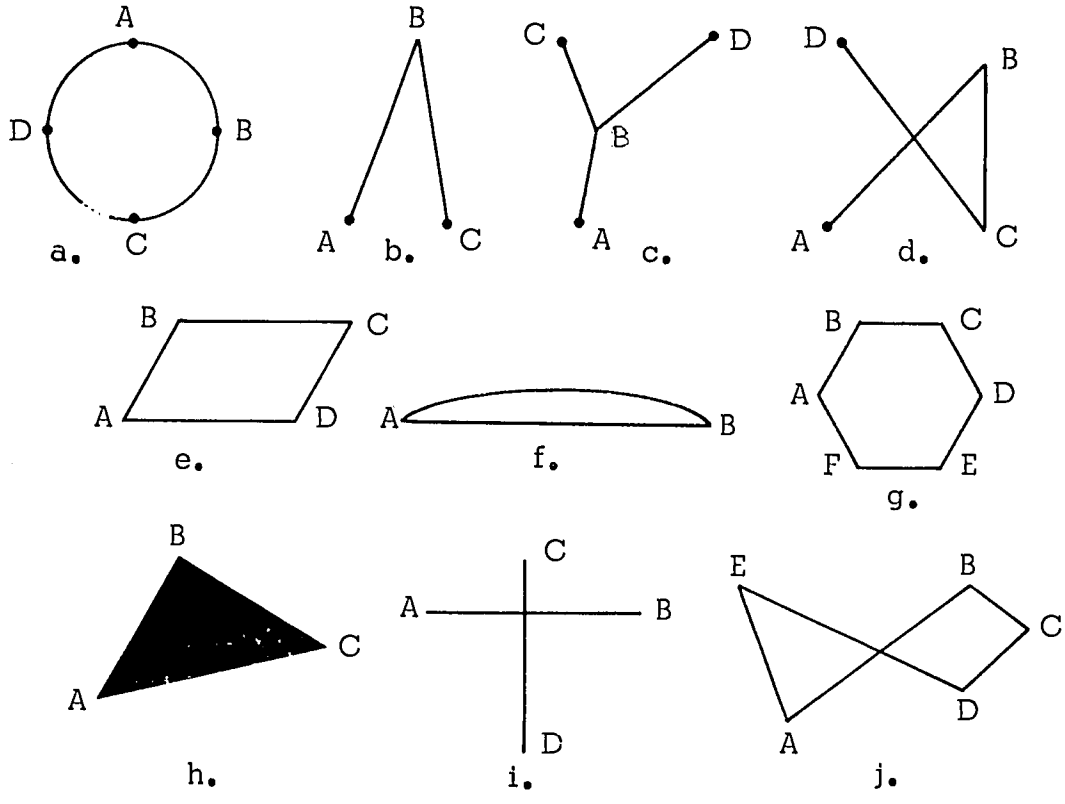


The angle $D\hat{E}F$

Can you name other angles of this polygon? How many angles does the polygon have? How many vertices does the polygon have? How many sides does the polygon have? What idea do you get? Draw some polygons and test your idea.

Problems 3-2

1. Which of the following figures are polygons?



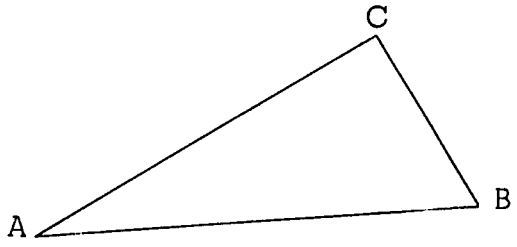
2. For each figure of Problem 1 that is a polygon, name the vertices, sides, and angles. Which pairs of sides are opposite and which pairs are adjacent?

3. In Problem 1 there are seven figures that are not polygons. Pick two of them, and say in each case what you could add or take away from the figure to make it a polygon.

4. Try to recall what it means for a figure to be a closed path. Do you think that polygons are closed paths? Why?

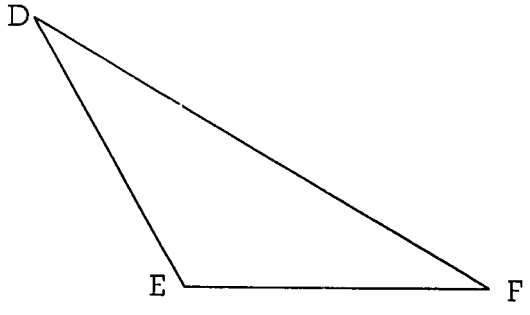
3-3 Triangles

A triangle is a three-sided polygon. A triangle is also a closed path. Triangles are usually named by their vertices. For example, if the vertices of a triangle have been named A, B and C



a good name for the triangle is "triangle ABC". Sometimes, instead of writing the name of a triangle in words, we write it in symbols: ΔABC. The symbols ΔABC are read "triangle ABC". We use the symbols because they can be written quickly and because they are easy to recognize.

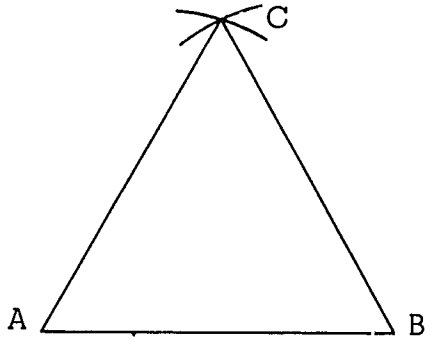
The idea of naming a triangle by its vertices leads to a way of naming the angles of a triangle that we think you will like. In ΔDEF below we can call $E\hat{F}D$ <u>the angle at</u> F or simply <u>the angle</u> F.



This suggests writing $\hat{F}$ instead of $E\hat{F}D$, and reading "angle F" or "the angle at F" whenever you see $\hat{F}$. This lovely short-hand lets us write $\hat{E}$ and $\hat{D}$ for the other two angles of the triangle.

Perhaps you know some of the interesting kinds of triangles already. One is the kind in which all three sides are congruent. We call such a triangle <u>equilateral</u>.

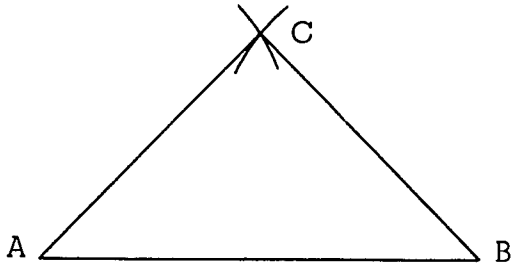
It is easy to make an equilateral triangle. You can start with a line segment $\overline{AB}$ drawn on a clean, thin sheet of paper (clean and thin so that you can test your triangle by folding). Set your compasses with one point at A and the other at B. Then, with A and B as centres, draw arcs like these:



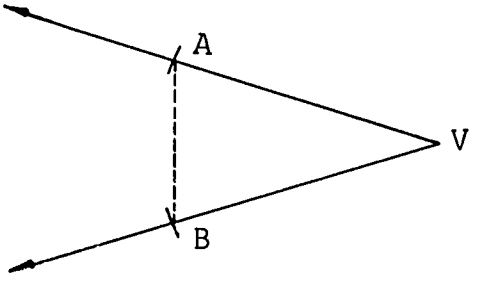
Call the point where the arcs meet C, and draw $\overline{AC}$ and $\overline{BC}$. Fold your paper to see whether $\overline{AC}$ and $\overline{BC}$ are both congruent to $\overline{AB}$. What do you conclude?

Another interesting kind of triangle is one in which two sides (but not necessarily all three sides) are congruent. If two sides of a triangle are congruent, the third side is called a <u>base</u> of the triangle, and the triangle is called <u>isosceles</u>. Every equilateral triangle is isosceles (do you see why?).

It is just as easy to make an isosceles triangle as it is to make an equilateral triangle. You can start, as before, with a line segment $\overline{AB}$ drawn on a sheet of paper. Then set your compasses so that they reach from A beyond the midpoint of $\overline{AB}$. Draw arcs with centres at A and B, and complete the triangle as in the picture below:

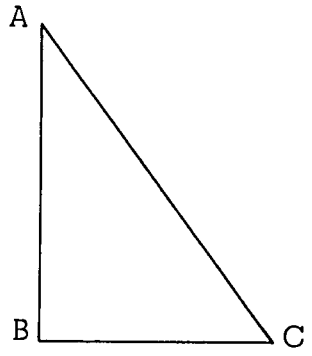


Here is a quicker way to make isosceles triangles. Start with an angle:



With one point of your compasses at the vertex V mark off congruent segments $\overline{VA}$ and $\overline{VB}$ on the two sides of the angle. If you then connect A and B with a line segment, you will find that ΔABC is isosceles.

A third important kind of triangle is one which has a right angle for one of its angles. It is easy to construct right-angled triangles. One way is to draw a line segment $\overline{AB}$ and to construct a line segment $\overline{BC}$ perpendicular to $\overline{AB}$ at B.



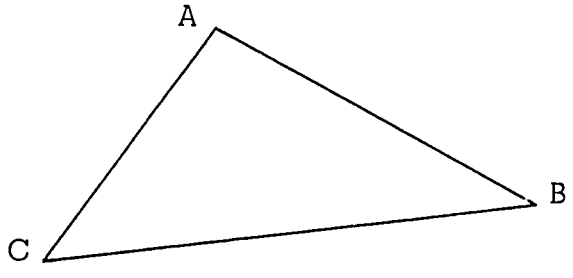
Triangle ABC is then a right triangle. Another way is to draw a right angle, and then draw a line segment from a point on one ray to a point on the other.

Problems 3-3A

1. As an isosceles triangle, how many bases does an equilateral triangle have?

2. Draw two line segments, $\overline{AB}$ and $\overline{XY}$, with $\overline{XY}$ longer than $\overline{AB}$. Then draw a triangle that has one side congruent to $\overline{AB}$ and each of the other two sides congruent to $\overline{XY}$. What kind of triangle have you drawn? What kind would it be if $\overline{AB}$ and $\overline{XY}$ were congruent?

3. In ΔABC below, $\overline{AB}$ is said to be <u>the side opposite</u> $\hat{C}$ and $\hat{C}$ is said to be <u>the angle opposite</u> $\overline{AB}$. What side do you think is opposite $\hat{B}$? What angle do you think is opposite $\overline{BC}$?

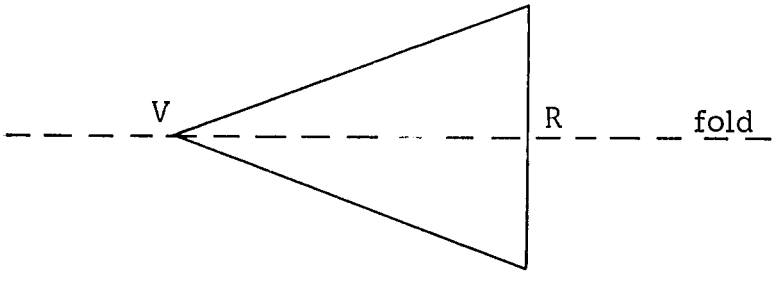


What does it mean for a side and an angle of a triangle to be opposite?

4. Draw an isosceles triangle on a sheet of thin paper. Fold the paper so that one of the congruent sides lies on top of the other. What can you say about the angles that are opposite these two congruent sides? Do you agree with the statement below?

If two sides of a triangle are congruent then the angles opposite them are congruent.

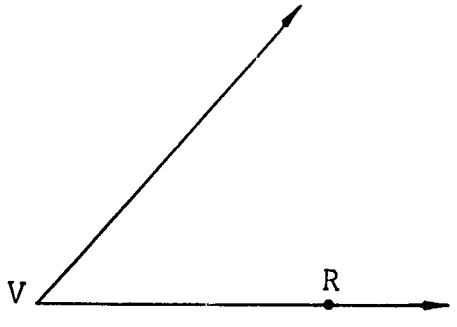
5. Now unfold the paper you used in Problem 4, and let V be the vertex that lies on the fold. Let R be the point where the line of the fold crosses the side opposite V.



How is $\overrightarrow{VR}$ related to $\hat{V}$? Do you see anything special about R? What do you conclude?

6. What can you say about the angles of an equilateral triangle?

7. This problem is an invitation to discover what a triangle with two congruent angles is like. On a thin piece of paper, draw an angle that is smaller than a right angle:



Call the vertex V, and on one of the rays pick a point R. Then draw a ray $\overrightarrow{RK}$ with $V\hat{R}K$ congruent to $\hat{V}$. Of course there are four possible rays you could draw for $\overrightarrow{RK}$ (do you see that?), and for our construction you should draw the one that intersects the other ray of $\hat{V}$. Call the point of intersection S. Then ΔSVR has two congruent angles. What do you think about the sides of ΔSVR that lie opposite $\hat{V}$ and $\hat{R}$? Can you tell by folding your paper? Do you agree with the statement below?

If two angles of a triangle are congruent, then the sides opposite these angles are congruent.

8. If Modupe said, "Every triangle with two congruent angles is isosceles", would you believe him?

9. If Abbiw said, "I have an isosceles triangle with angles of three different sizes", would you believe him?

10. Draw a right-angled triangle that has one side congruent to the following line segment:

_____

11. Draw an isosceles right-angled triangle.

12. In Chapter 2 you found out that lines do not intersect if they are parallel (see 2-4). Does this information help you to answer the question: Is it possible for a triangle to have two right angles?

13. Can an equilateral triangle have a right angle? Explain.

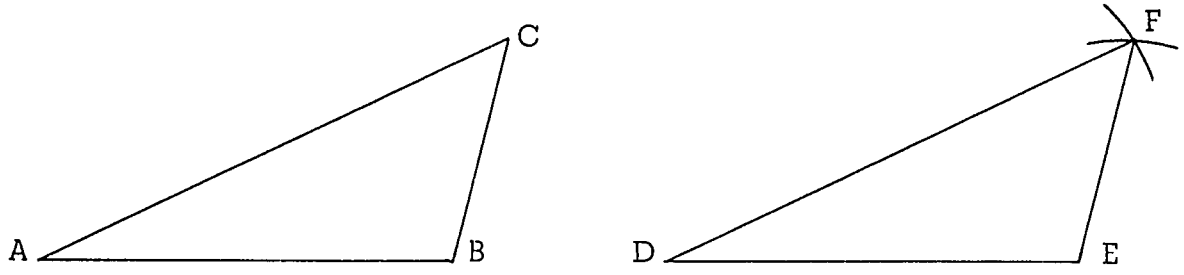
There remains for us to study an important relationship that might exist between two triangles: it is possible for two triangles to fit together perfectly. We shall say, just as we did about line segments and about angles, that

Two triangles are congruent if one fits the other exactly.

Of course this isn't a very good definition for triangles that you cannot move about, and we shall agree also upon a definition that is a good test for congruence:

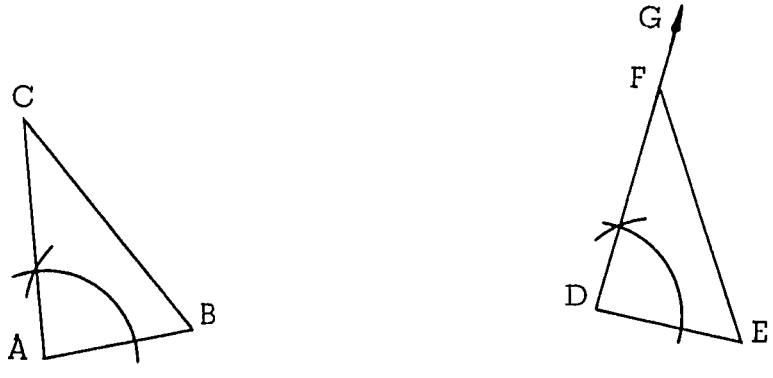
> Two triangles are congruent if <u>a copy of one</u> fits the other exactly.

Do you know how to make a triangle that is congruent to a given triangle? One way is to cut out the region bounded by the first triangle and trace around its sides to form a second triangle. Another is to copy the given triangle with tracing paper. Here are three other ways. You will see that they use drawing tools and methods with which you are already familiar. The first is probably the easiest way to copy a triangle with straightedge and compasses. As we describe it, keep an eye on the following picture.



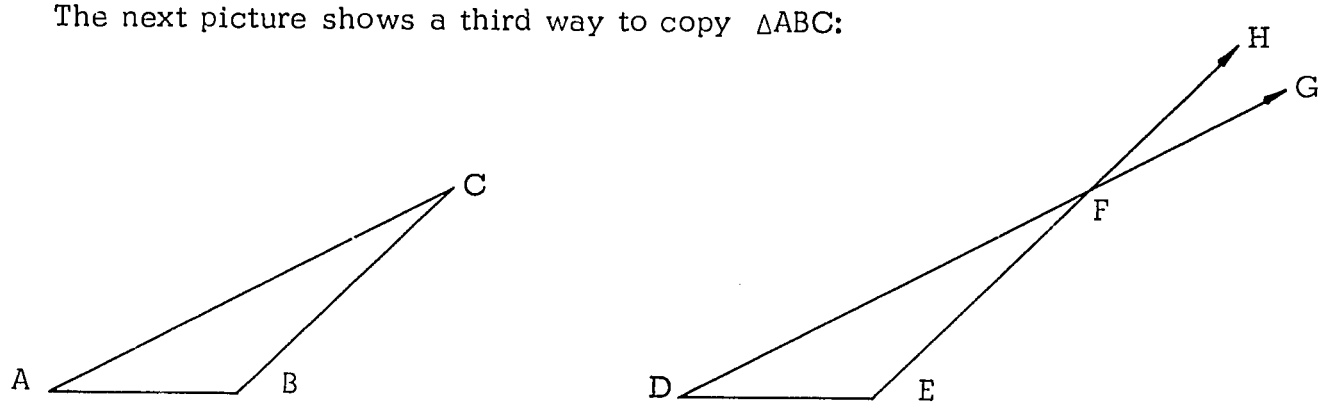
If ΔABC is the triangle to be copied, you can begin by making a line segment $\overline{DE}$ congruent to $\overline{AB}$. With D as centre, construct an arc with radius the length of $\overline{AC}$. Then with E as centre, construct an arc with radius the length of $\overline{BC}$. Draw enough of each arc so that they intersect. If the point of intersection is called F, then ΔDEF is congruent to ΔABC.

Another way to copy ΔABC is suggested by the following picture:



Here is how we drew ΔDEF. We first drew $\overline{DE}$ congruent to $\overline{AB}$. Then we copied $B\hat{A}C$ by drawing a ray $\overrightarrow{DG}$ so that $E\hat{D}G$ is congruent to $B\hat{A}C$. We then marked off a segment $\overline{DF}$ on $\overline{DG}$ congruent to $\overline{AC}$. Finally we drew $\overline{EF}$.

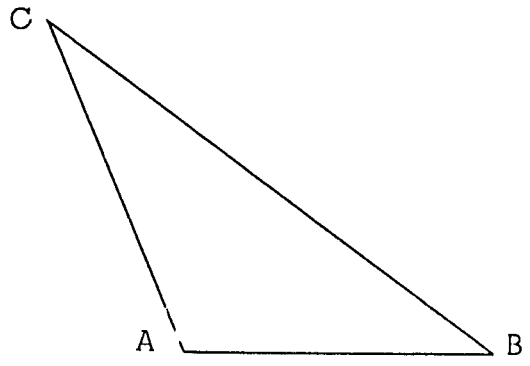
The next picture shows a third way to copy ΔABC:



Here is how we drew ΔDEF. First, we drew $\overline{DE}$ congruent to $\overline{AB}$. Then we drew $G\hat{D}E$ at D congruent to $C\hat{A}B$, and drew $D\hat{E}H$ at E congruent to $A\hat{B}C$. The rays $\overrightarrow{DG}$ and $\overrightarrow{EH}$ met at the point we called F.

Problem 3-3B

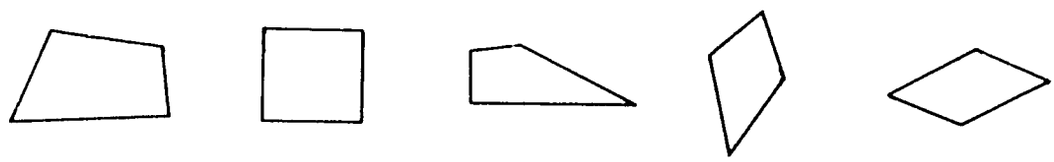
1. Here is a triangle:



Make a copy of it by each of the three drawing methods we just described. Then make a tracing-paper copy of one of your drawings and compare it with ΔABC and the other two copies.

3-4 Quadrilaterals

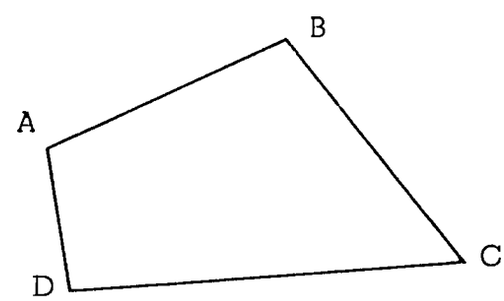
A quadrilateral is a four-sided polygon. These are quadrilaterals,



and these are not:

Among the figures above, can you find a four-sided closed path that is not a quadrilateral?

Do you remember what it means for two sides of a polygon to be adjacent? In the quadrilateral below, which pairs of sides are adjacent, and which are opposite?

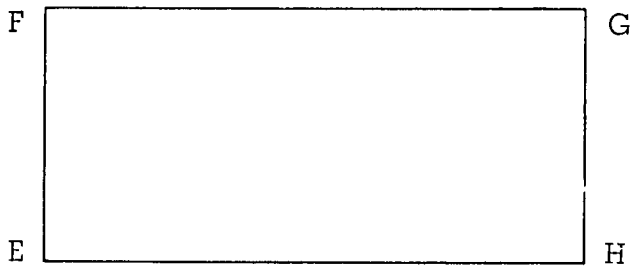


In a polygon, two angles that have a common side are called adjacent. Two angles that are not adjacent are called opposite. In the quadrilateral ABCD above, which angles are adjacent and which are opposite? How many pairs of adjacent angles are there? How many pairs of opposite angles did you find?

Quadrilaterals can be named by their vertices. The only thing that you have to be careful to do when you name a quadrilateral is to make sure that you write down the names of the vertices in an order in which they are connected.

For example, CDAB and CBAD are names for the quadrilateral above, but ACBD and ABDC are not. In how many different ways can you order the letters A, B, C and D? How many of these orderings are names for the quadrilateral above? How many are not?

In Chapter 1 you saw that you could make a figure similar to this



by tracing a book on a flat piece of paper, and you learned that this figure is called a <u>rectangle</u>. Now that you know more geometry, we can be clearer about what a rectangle is:

A <u>rectangle</u> is a quadrilateral with four right angles.

In rectangle EFGH, opposite sides are parallel. Have you any reason to believe that this is true of all rectangles? (Think of what it means for two line segments to be parallel, and then you'll see.)

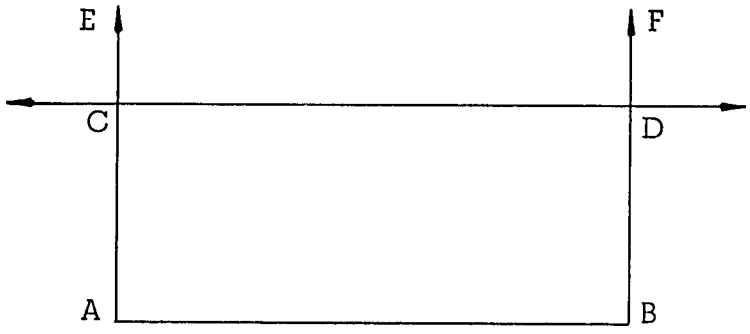
In a rectangle, opposite sides are parallel.

In rectangle EFGH, it looks as if opposite sides are congruent. Can you think of how you might fold a sheet of paper to learn whether this is true of other rectangles?

In a rectangle, opposite sides are congruent.

Here is a quick way to draw a rectangle. Draw a line segment on a piece of paper (in the picture below we call ours $\overline{AB}$). Then use your set square to draw rays $\overrightarrow{AE}$ and $\overrightarrow{BF}$ perpendicular to $\overline{AB}$, so that they lie on the same side of

$\overline{AB}$. Pick a point C, different from A, on $\overrightarrow{AE}$. Through C draw a line perpendicular

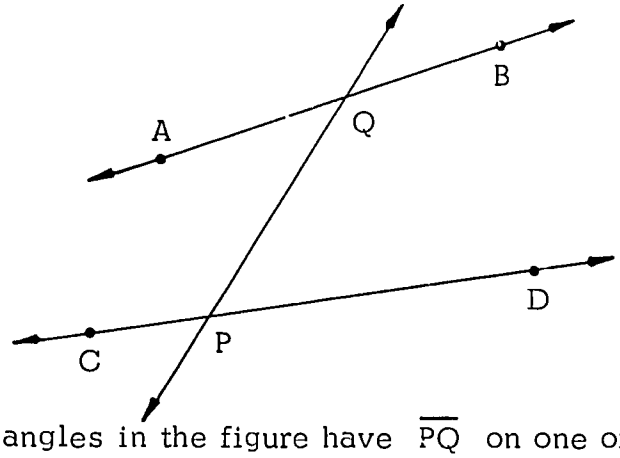


to $\overrightarrow{AE}$. If you give the name D to the point where this line crosses $\overrightarrow{BF}$, you will find that $\overleftrightarrow{CD}$ and $\overrightarrow{BF}$ are perpendicular at D. Why? Can you tell without folding your paper? Since all four angles of ABDC are right angles, ABDC is a rectangle.

Before going on, we need to think once more about transversals. Do you remember what a transversal of two parallel lines is? What do you know about such transversals? To extend our notion of transversal (so that we can talk about the transversal to any two lines whether they are parallel or not) we say

A <u>transversal</u> of two lines is a third line that intersects them both.

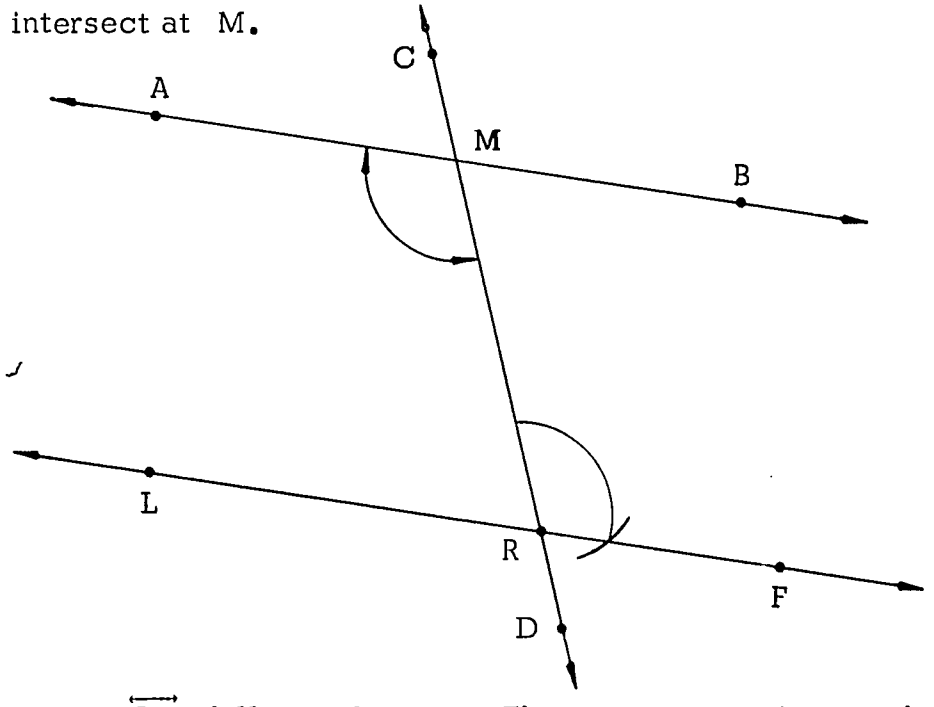
A transversal makes four angles with each of the lines it intersects. In the picture below, $\overleftrightarrow{PQ}$ is a transversal of $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.



Four of the eight angles in the figure have $\overline{PQ}$ on one of their sides. Can you name the four? These four are called <u>the interior angles</u> formed by the transversal and the two lines. Two interior angles are called <u>alternate interior angles</u> if they

have different vertices and no common interior points. In the figure above, which pairs of angles are alternate interior angles?

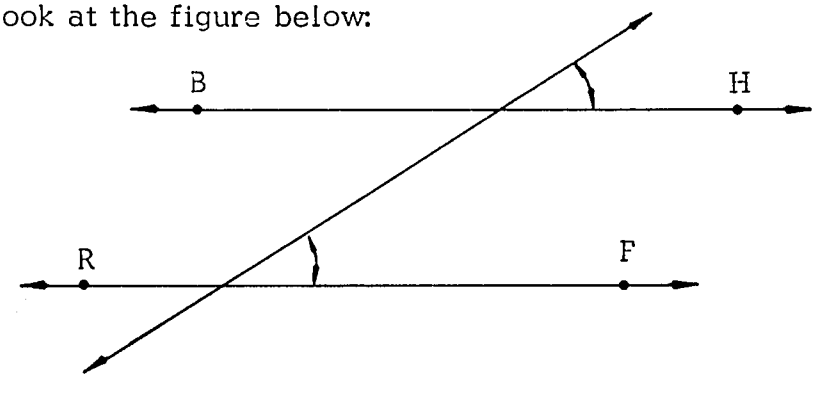
Now let's see what we can discover about lines and their transversals. On a clean, thin sheet of paper, draw two lines that intersect. We call ours $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, and they intersect at M.



Let R be a point on $\overleftrightarrow{CD}$ different from M. Then, as in our picture, draw a line $\overleftrightarrow{LF}$ through R in such a way that $A\hat{M}R$ and $F\hat{R}M$ are congruent. Now look at your picture. What idea do you get about $\overleftrightarrow{AB}$ and $\overleftrightarrow{LF}$? How can you test your idea? Would it help to draw a line perpendicular to $\overleftrightarrow{LF}$? Did you discover that

> Two lines in a plane are parallel if they are cut by a transversal in such a way that two alternate interior angles are congruent.

Now look at the figure below:

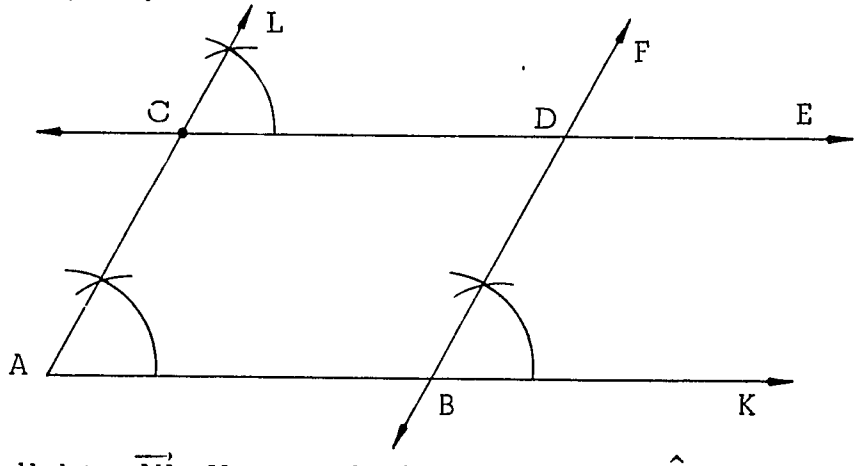


Suppose that the two marked angles are congruent. Do you then have any reason to believe that $\overleftrightarrow{BH}$ and $\overleftrightarrow{RF}$ are parallel?

You now know enough about parallel lines to learn how to make a parallelogram.

> A <u>parallelogram</u> is a quadrilateral in which opposite sides are parallel.

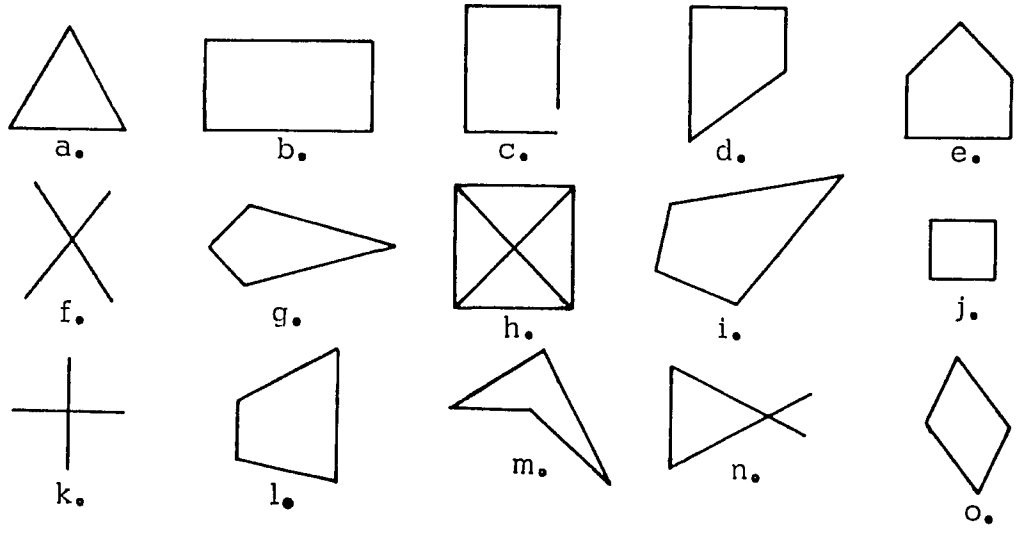
Every rectangle is a parallelogram, but, as you will see, it is easy to make a parallelogram that is not a rectangle. Start with an angle $L\hat{A}K$ that is not a right angle. Through a point C on $\overrightarrow{AL}$ draw



a line $\overleftrightarrow{CE}$ parallel to $\overrightarrow{AK}$. You can do this by making $L\hat{C}E$ congruent to $L\hat{A}K$. Through a point B on $\overrightarrow{AK}$, draw a line $\overleftrightarrow{BF}$ parallel to $\overrightarrow{AL}$. Let D be the point where $\overleftrightarrow{CE}$ and $\overleftrightarrow{BF}$ intersect. Then ABDC is a parallelogram.

<u>Problems 3-4A</u>

1. Which of these figures are quadrilaterals?



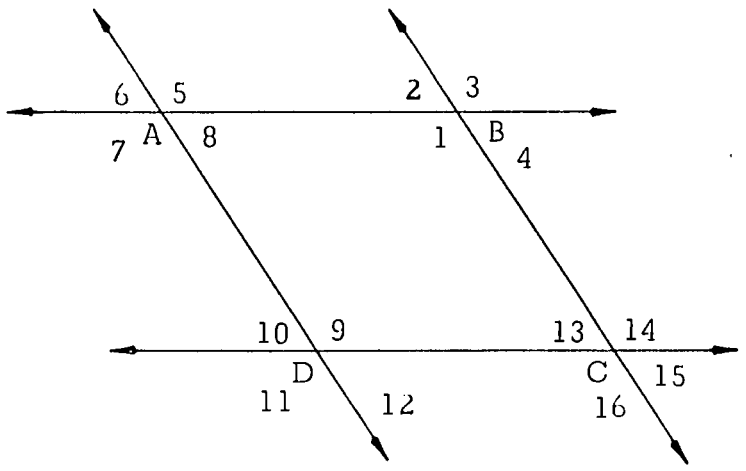
2. A <u>square</u> is a rectangle with four congruent sides. In Problem 1, which figures are squares? Which are parallelograms? Which are rectangles?

3. Isaac says he has a quadrilateral with only three right angles in it. Do you believe him?

4. Is every square a parallelogram? How do you know?

5. Abu has a secret rectangle he will not show to his friend Ben. Ben thinks it is a square, but he is not sure, and Abu will not tell him. But Ben learns from Abu that two adjacent sides are congruent. Is this enough for Ben to be sure that Abu has a square?

6. In the figure below we have drawn a parallelogram ABCD. We have also drawn the lines that contain the sides of ABCD and we have numbered the sixteen angles these lines make.



Which of these angles are congruent to angle 1? Which are congruent to angle 8? Can you say anything interesting about opposite angles in ABCD? Do you think you have enough evidence to decide whether the following statement is true?

In a parallelogram, opposite angles are congruent.

7. Can you find a quadrilateral with all four sides congruent that is not a square? How would you draw one?

8. Draw a rectangle on paper by the method of the text. Cut out the piece of paper bounded by the rectangle, and fold it to show that its opposite sides are congruent.

9. Can a quadrilateral have exactly one right angle? exactly two?

10. Draw a parallelogram on paper, and cut out the region bounded by the parallelogram. Try to decide by folding it whether opposite sides of your parallelogram are congruent. What do you conclude? What statement would you put in the box below?

11. In a rectangle, opposite sides are congruent. Let us turn this around, and say, "If opposite sides of a quadrilateral are congruent, the quadrilateral is a rectangle." Is this statement true? Why?

12. If all four sides of a quadrilateral are congruent, must the angles all be congruent?

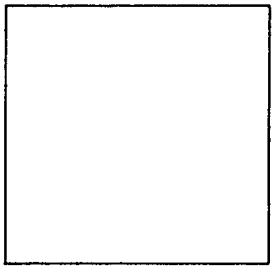
Quadrilaterals can be congruent, just as line segments, angles and triangles can. If you were writing this text, what definition of congruence would you write in the box below?

Two quadrilaterals are congruent if . . .

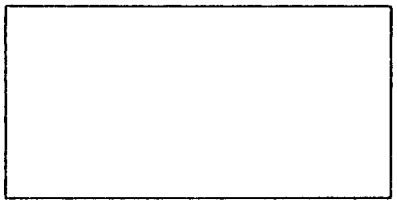
Just as with triangles, whenever we have a quadrilateral that fits exactly on top of another, the fitting makes a pairing between sides of one and sides of the other. Each side of the lower quadrilateral is paired with the side of the upper quadrilateral that is fitted on it. These two sides are congruent line segments. Likewise, each angle of the bottom quadrilateral is paired with the angle of the top one that fits on it. The angles paired by the fitting are congruent. The study of congruence of quadrilaterals is more complicated than that of triangles and we shall omit it here, except for a few problems below.

Problems 3-4B

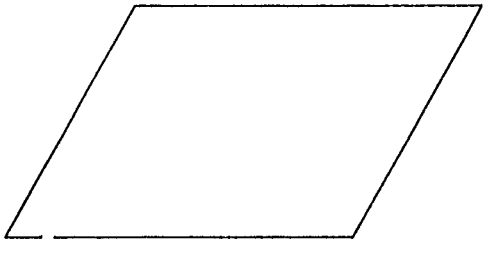
1. Using compasses and straightedge, draw a square congruent to this one:



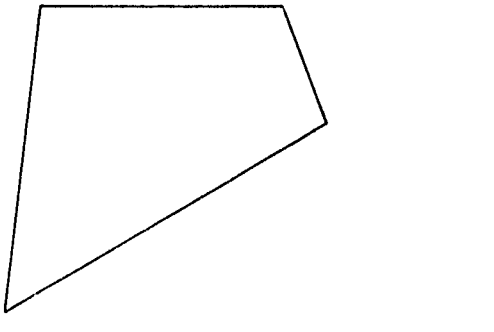
2. Using compasses and straightedge, draw a quadrilateral congruent to this one:



3. Using a straightedge and compasses, draw a quadrilateral congruent to the following one:



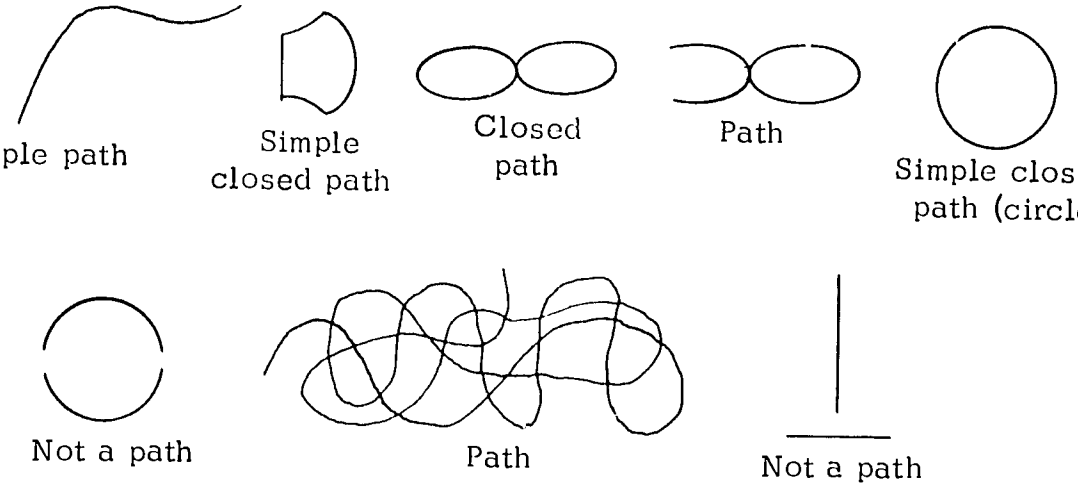
4. Tell in your own words how you would construct a quadrilateral congruent to this one:



_____

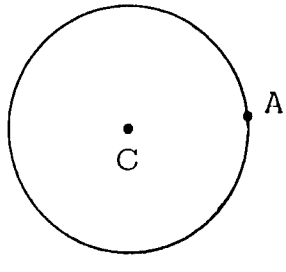
3-5 Plane Paths, Circles and Ellipses

Polygons are special cases of what, in Chapter 1, we called paths. You can make a path by running a pencil over a piece of paper without letting the pencil jump or skip. You can draw a path on a blackboard by running a piece of chalk over the blackboard without letting it skip or jump. If the drawing ends at its starting point, the path is closed. If the path never crosses itself, it is a simple path. A quadrilateral, for example, is both simple and closed. Some of the figures below are paths, and some are not:



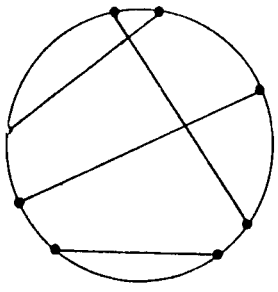
You already know that a circle is a simple closed path, and that you can draw one with a pair of compasses. Now that you know about congruence also, we can be clearer still about what circles are.

When we draw a circle, one point of the compasses stays fixed at a point we call the centre of the circle. The other point of the compasses moves around, marking the points that we call points of the circle. Let us call the centre C. Then if A is a point on the circle, $\overline{CA}$ is congruent to the line segment whose endpoints are the points of the compasses.

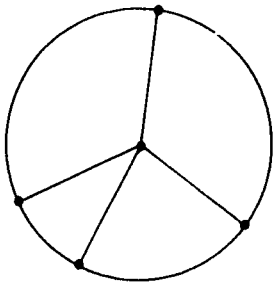


Also, if A is a point on the paper with $\overline{CA}$ congruent to the line segment joining the compasses' points, then the moving point will pass through A when we draw the circle.

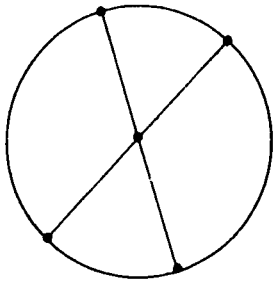
This leads us to a general definition of a circle. Let C be a point in a plane, and let $\overline{XY}$ be a line segment. Think of all line segments in the plane of the circle that are congruent to $\overline{XY}$ and have one endpoint at C. If you draw the endpoints of all these line segments, you will have drawn a <u>circle with centre</u> C. Any line segment that has one endpoint at C and one endpoint on the circle is called a <u>radial segment</u> or a <u>radius</u> of the circle. The word <u>radius</u> is also used for the length common to all the radial segments. A line segment that has both endpoints on the circle is a <u>chord</u> of the circle. A chord through the centre of the circle is called a <u>central chord</u>. Central chords are all congruent. Their common length, which is twice the radius of the circle, is called the <u>diameter</u> of the circle.



chords

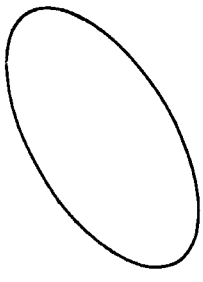


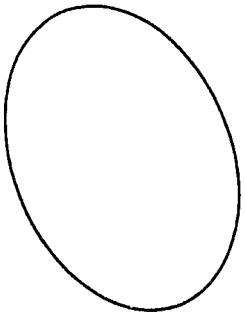
radial segments

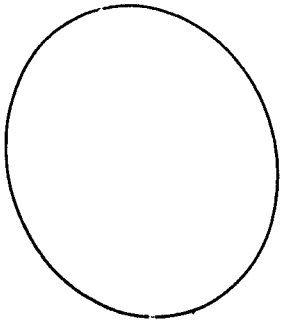


central chords

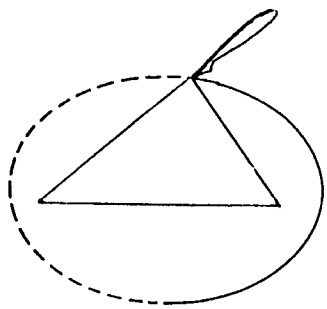
Another simple closed path worth mentioning here is the <u>ellipse</u>. Here are some ellipses:



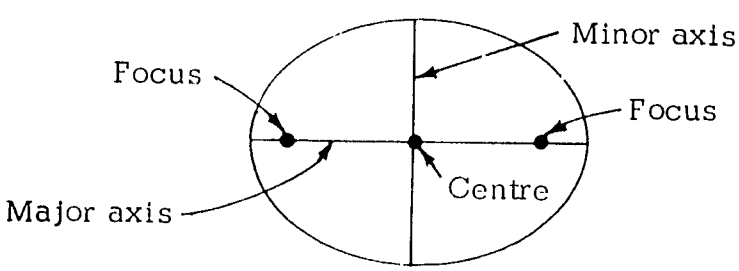




It is easy to draw an ellipse if you have a piece of string and two pins. Mark two points on a piece of paper, and stick a pin into the paper at each point. Tie together the ends of the piece of string to form a loop. Place the loop over the pins and draw it tight with a pencil. Keeping the string tight, trace a closed curve. This will be an ellipse.



The points where you put the pins are not points on the ellipse, but each is called a focus (plural foci) of the ellipse. The midpoint of the line segment joining the foci is the centre of the ellipse. A segment that joins two points of the ellipse is a chord of the ellipse, and a chord is a central chord if it passes through the centre of the ellipse. The centre is the midpoint of each central chord, but (unlike a circle) an ellipse has central chords of many different lengths. The longest central chord is the one that passes through both foci. It is called the major axis of the ellipse. The shortest central chord is the one perpendicular to the major axis. It is called the minor axis of the ellipse.



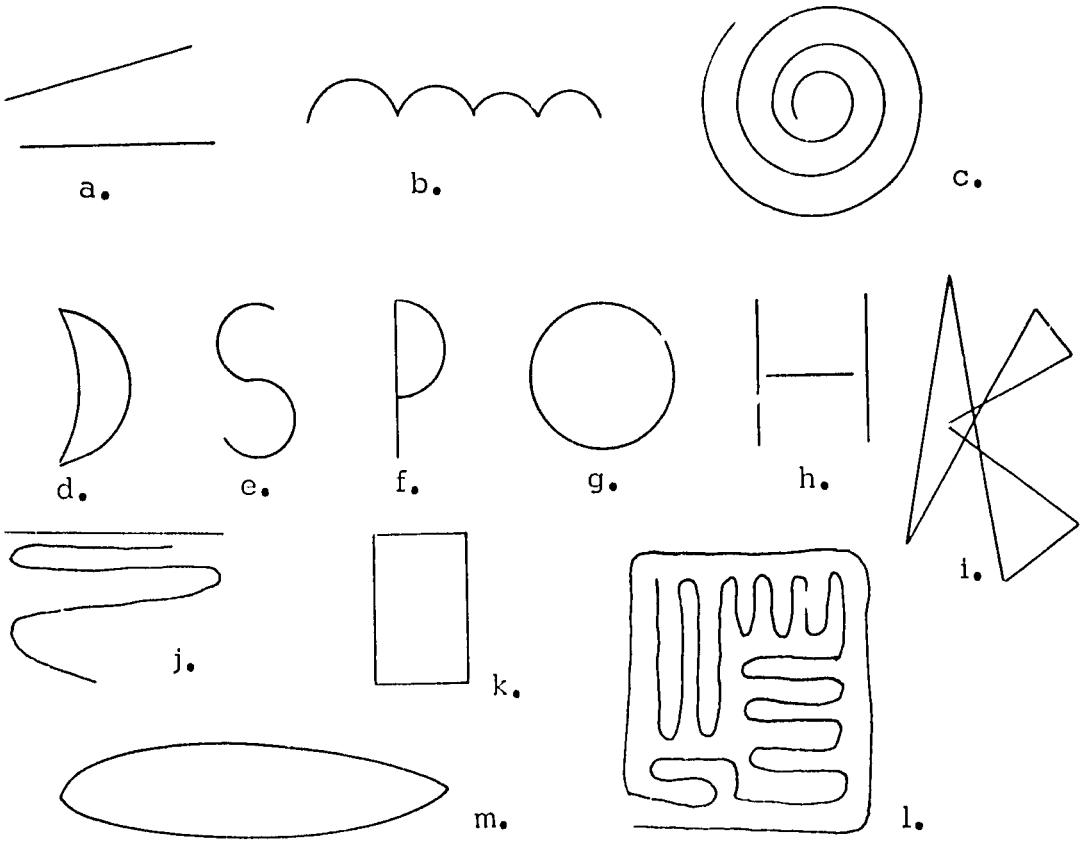
For many years it was believed that the sun moved around the earth, tracing out a circle with the earth as centre. Then people learned that the earth moves around the sun, but they still thought that the path was a circle (this time with centre at the sun). In the early part of the 17th century, a German scientist,

Johannes Kepler, discovered that the path of the earth is not a circle, but an ellipse that has one focus at the sun.

The orbit of our moon is an ellipse with one focus at the centre of the earth. Halley's comet, last seen in 1910, moves about the sun on an ellipse so large that the comet takes 76 years to trace it out. Satellites that are put into orbit around the earth by rocket travel on paths that are ellipses with one focus at the centre of the earth.

Problems 3-5

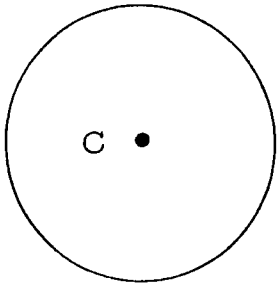
1. Which of the drawings below are paths? For those that are paths, say whether they are closed, and whether they are simple.



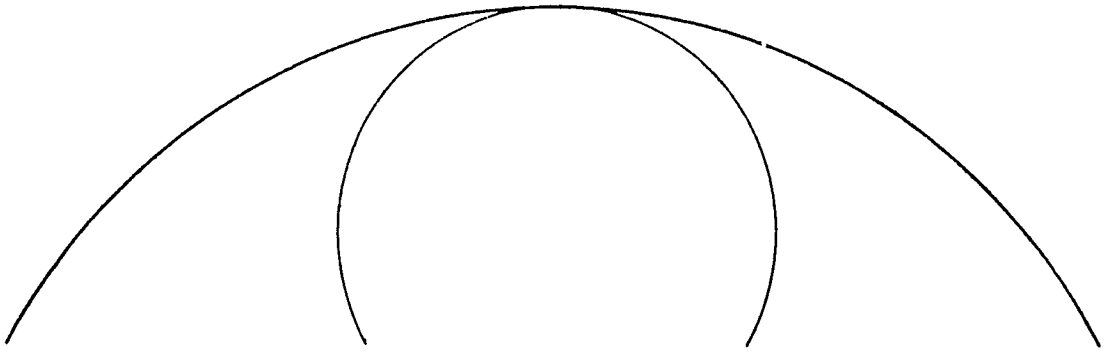
2. Draw two points on a piece of paper. Using a straightedge and compasses, try to draw a circle through these two points.

3. If you can do Problem 2, try this one. If three points in a plane do not lie together on a line, there is exactly one circle that passes through the three points. Draw three such points and see if you can also draw the circle.

4. Below we have drawn a circle, and its centre, C. How would you draw a circle congruent to ours?



5. Open your compasses wide, and draw a large circle on a piece of paper. Cut out the disc (circular region) bounded by the circle. Cut carefully, to make the edge smooth. Then close your compasses and draw a small circle, and cut out the region bounded by it. If you put one disc on top of the other so that their edges (the circles) just touch, you will get something like this:



The small circle curves more than the large one. If you trace the edges of the discs with your fingers, you can feel a difference in the two discs. You can describe this difference by saying that <u>the smaller circle has greater curvature</u>.

6. Draw an ellipse with a long major axis and a short minor axis.

7. Draw an ellipse with major and minor axes that are nearly of the same length.

8. Now put the pins you have been using right next to each other, and try to draw an ellipse in the usual way. What does your ellipse look like?

9. Draw a large ellipse on a piece of paper. Mark its centre. Then cut out the region bounded by the ellipse. Now draw some of the central chords, and fold your paper to satisfy yourself that the centre of the ellipse is also the midpoint of each of these chords. If you haven't done so yet, draw the major axis of the ellipse. Then use your compasses and straight edge to draw the minor axis of your ellipse.

10. If Edward said, "The major axis of an ellipse is a perpendicular bisector of the minor axis", would you believe him?

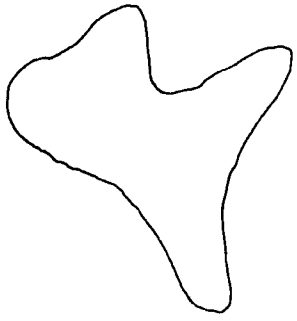
11. If Rebecca said, "Two chords of an ellipse always bisect each other", would you believe her?

12. If Rebecca changed her mind and said, "Two central chords of an ellipse always bisect each other", would you believe her?

3-6 Simple Closed Paths, and Regions in a Plane

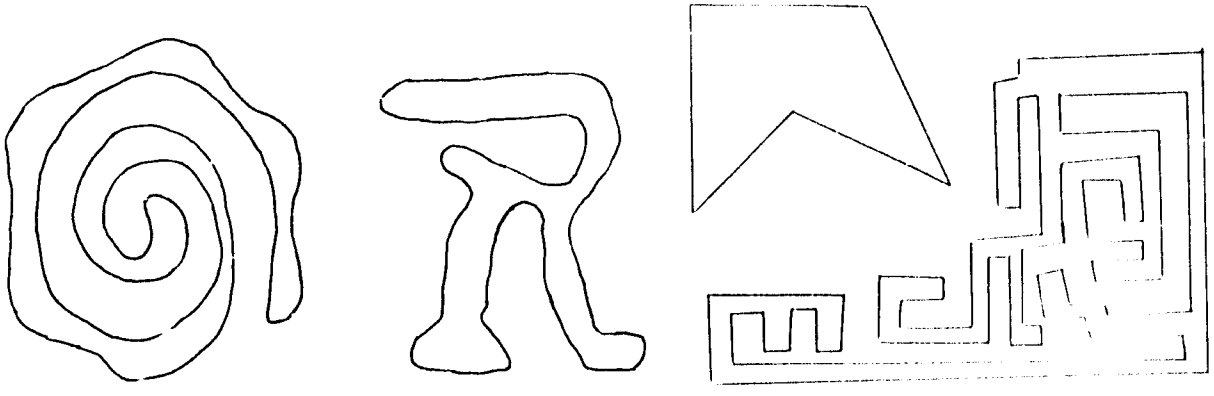
In some of the questions above, we asked you to cut out regions. We asked you to cut out the region bounded by an ellipse, and, earlier, we asked you to cut out two circular regions (discs). We didn't tell you what a region was, we just expected you to know. After all, you had no trouble cutting out the piece of paper bounded by one of your rectangles, or the piece bounded by one of your parallelograms.

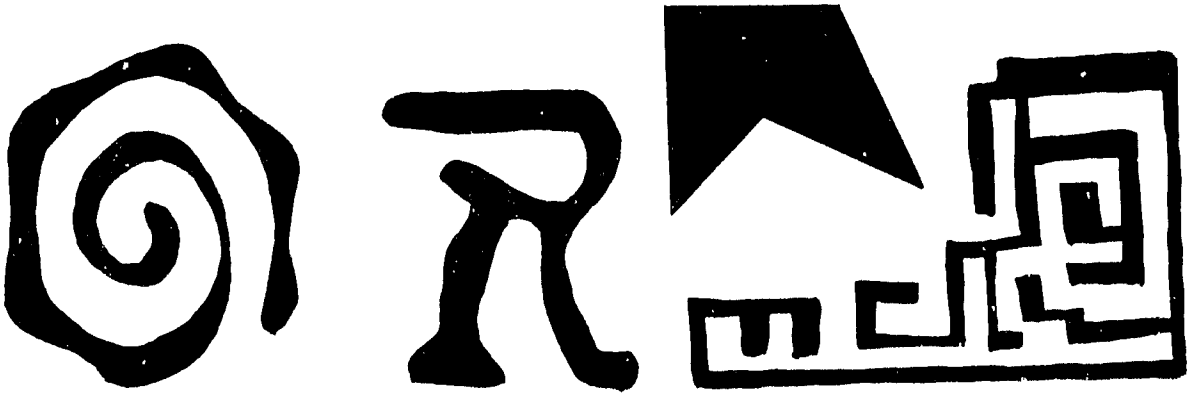
The idea of <u>region</u> is easy to grasp. Every time you draw a simple closed path in a plane, the path you draw separates the plane into two pieces: the piece inside the path, and the piece outside the path.



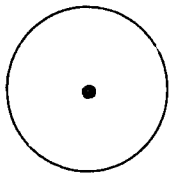
It is clear from the drawing that you can consider the path to be the boundary of each piece. But the pieces themselves are quite different. One piece is small, and the other is very large indeed. One piece contains no lines at all, and the other contains lines. The small piece and the path together make up what we call the <u>region bounded by the path</u>. The small piece itself, the piece inside the path, is called the <u>interior</u> of the region.

Here are some simple closed paths and the regions they bound:



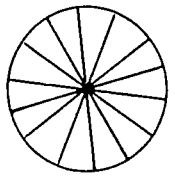


You can draw the region bounded by a circle in a plane by drawing all chords of the circle. You can also draw the region by drawing all radial segments of the circle:



Circle

Circle with some radial



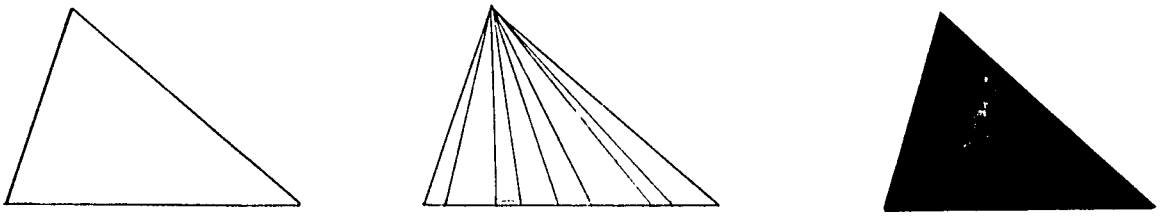
segments filled in

Circle with all radial segments

filled in; result is a circular region.

If C is the centre of the circle, then the points you draw in the interior of the region are precisely those that are inside the circle: the points A whose segment $\overline{AC}$ is shorter than a radial segment.

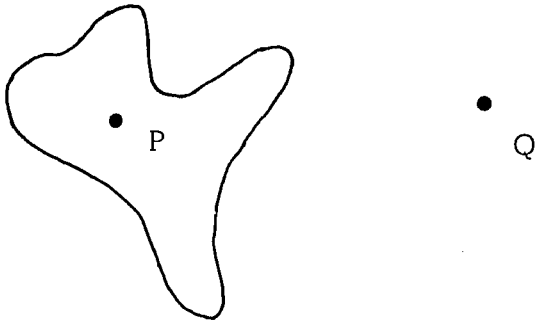
You can draw the region bounded by a triangle in a plane by drawing all chords of the triangle (line segments joining two points of the triangle). You can also draw the region by choosing a vertex of the triangle and then drawing all line segments that join that vertex to the side opposite it:



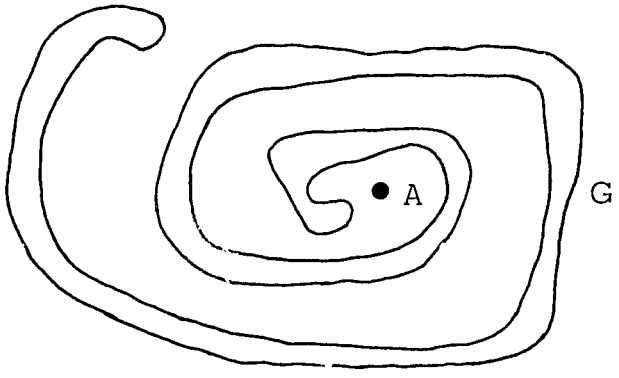
A point lies in the interior of the triangular region if it lies in the region but not on the triangle itself.

When we say "a simple, closed path in the plane separates the plane into two pieces", we use the word <u>separate</u> because it is more suggestive than "cut" or "divide". The path separates a point inside the path from a point outside the path. The boundary of a playing field separates a player inside the field from a spectator outside the field. Neither can walk to the other without crossing the boundary of the field. Likewise, you cannot make a path from a point inside a simple closed path to a point outside the simple closed path without crossing the

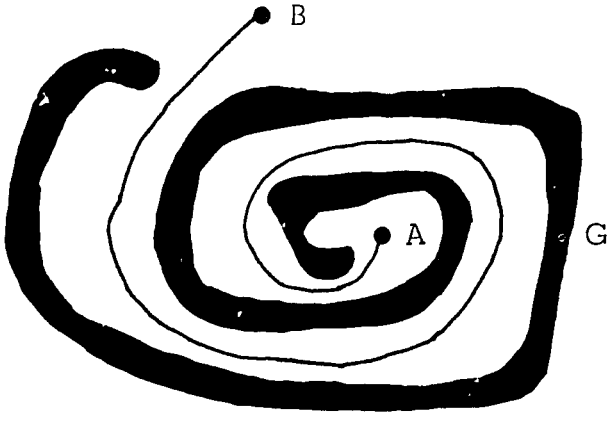
simple closed path. In the picture below, any path you draw that starts at P and ends at Q must cross the simple closed path.



Look at the simple closed path below. The point A is not on G itself, so it must be inside G or outside G. Can you tell which?

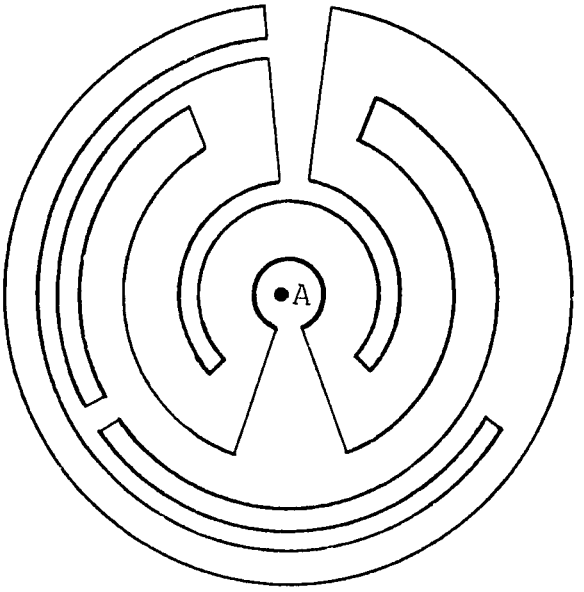


The point A is outside! Here is how we know. We can pick a point B that we know is outside G, and we can connect A to B with a path that does not cross G.



In the picture we have shaded the region bounded by G to make it clearer, still, that A lies outside G. If A were inside G there would be no such path from A to B.

Have you ever seen a puzzle like this?



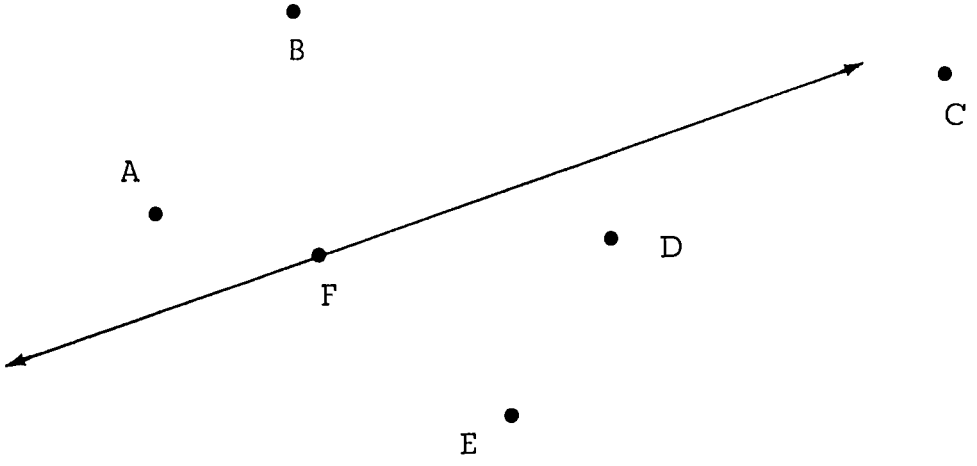
The picture represents a garden. The simple closed path is supposed to be a hedge that you cannot cross. Starting from A, can you find your way out of the garden? That is, is A outside the path formed by the hedge?

There are two last comments we would like to make. The first is that a region is often named by the kind of path that bounds it. For example, a region bounded by a circle is a <u>circular region</u>, a region bounded by a triangle is called a <u>triangular region</u>, and a region bounded by a square is a <u>square region</u>.

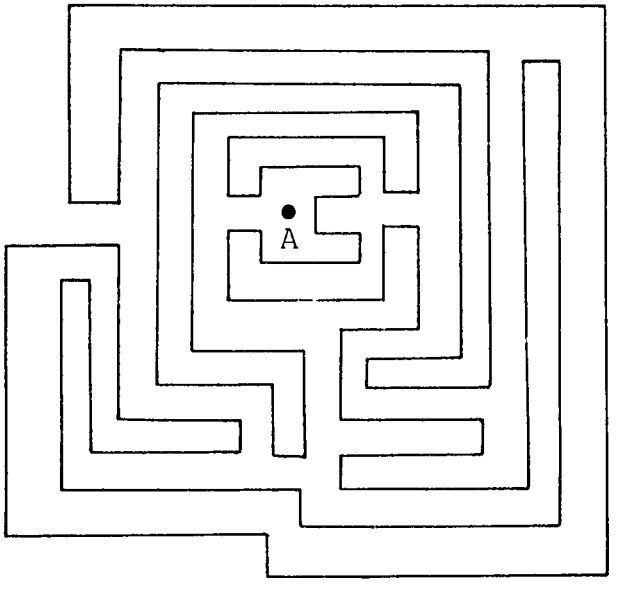
The second is about lines. A line in a plane separates the plane into two pieces, each of which is called a <u>side</u> of the line. If a point A on the plane is not on the line, then it is on one of the two sides. There is no particular reason to call one side the "inside" and the other side the "outside", but you can tell whether two points are on the same side of the line: the line segment joining them does not cross the line.

Problems 3-6

1. When two points lie on opposite sides of a line in a plane, we say they are separated by the line. We say that they are separated because you cannot make a path from one to the other without crossing the line. In the figure below, which pairs of points are separated by the line? Which pairs of points are on the same side of the line?

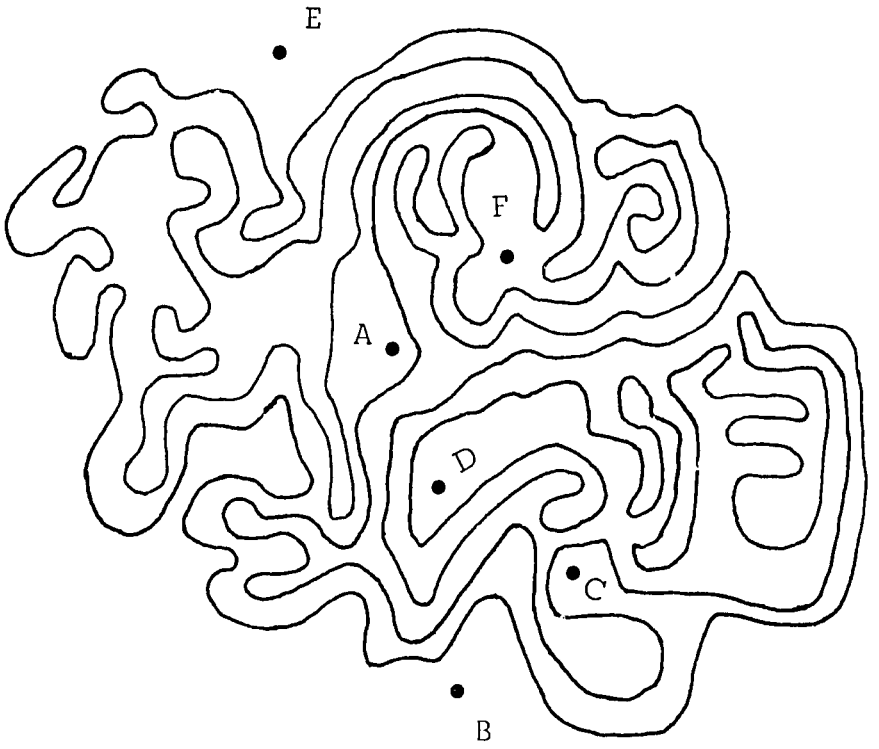


2. Below are a simple (!) closed path and a point A not on the path. Copy them with tracing paper, and use your copy to show that A is outside the path.



3. To do Problem 2 you copied a simple closed path. Shade in the region bounded by the path in your copy.

4. In the figure below, B lies outside the simple closed path, and C lies inside the path. On which side is A? On which side is D?



5. In the figure for Problem 4, how many times would you cross the path if you went along a straight line from B to A? from B to D? from B to C? from E to A? from E to D?

What pattern do you see? Can you use your discovery to decide whether F lies in the region bounded by the path?

6. Suppose that C is the centre of a circle in a plane, and that A is a point on the plane. What do you have to know about $\overline{AC}$ in order to say that A is

(a) in the region bounded by the circle?

(b) in the interior of the region?

(c) on the circle?

(d) outside the circle?

CHAPTER 4

FIGURES IN SPACE

4-1 Introduction

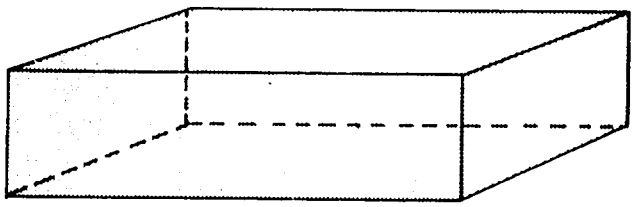
Many objects in the world around us are solid, and some have surfaces that are made of plane regions fitted together in interesting ways. Now that you know about regions, we can tell you about such surfaces and solids.

Some solids, like balls, have surfaces that are entirely rounded. Others, like circular cylinders and circular cones, have surfaces that are partly flat and partly rounded.

The surface of a solid object in space is somewhat like a simple closed path in a plane: it separates space into two pieces of which it is the common boundary.

4-2 Surfaces and Solids

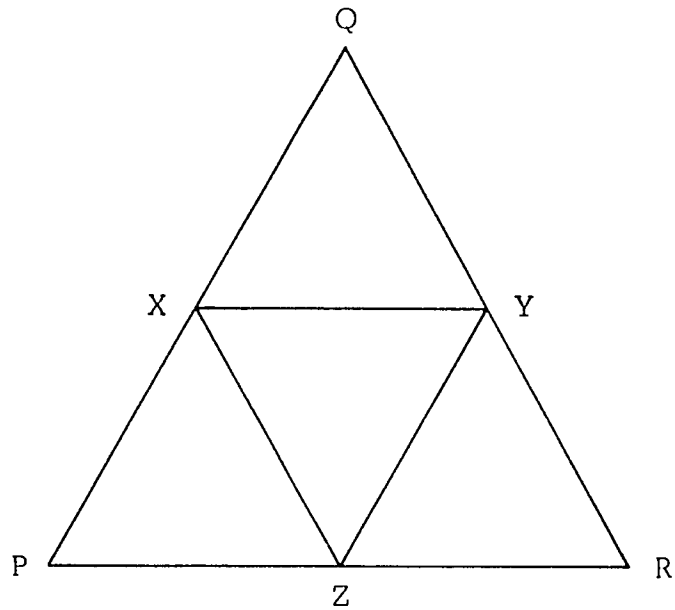
Imagine that you have a small, closed cardboard box filled with sand. You can think of this as a brick. If you pour the sand out through a hole and then cover up the hole, you are left with the cardboard. This is the surface, or "skin", of the brick. It looks something like this:



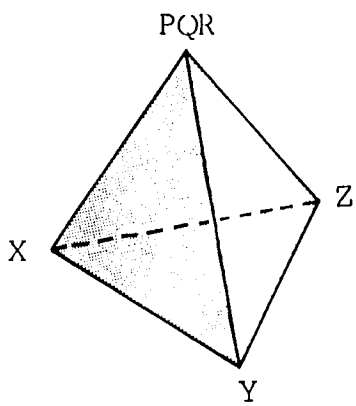
The surface of the brick has six faces, each a rectangular region. Each face meets four other faces, one at each of its edges. Two faces of a brick that do not meet are called opposite. A brick also has twelve straight edges. Can you count them in the picture? Each edge is shared by two faces. The eight corner points are called vertices of the brick. Each vertex belongs to three edges and to three faces. A brick is an example of a solid figure.

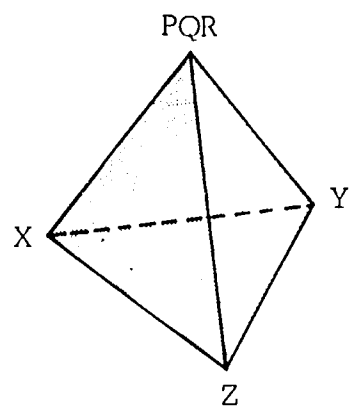
Previous Page Blank

Here is another example of a solid figure. Draw a large equilateral triangle PQR on a sheet of thin cardboard. Mark the midpoints X, Y and Z of the edges $\overline{PQ}$, $\overline{QR}$ and $\overline{RP}$. Draw $\overline{XY}$, $\overline{YZ}$ and $\overline{ZX}$.



The triangular region PQR is now divided into four smaller triangular regions. Do you see anything special about them? Cut out the triangular region PQR. Fold it along $\overline{XZ}$, $\overline{YZ}$ and $\overline{XY}$ to bring P, Q and R together. The result looks like one of these (depending on which way you fold):

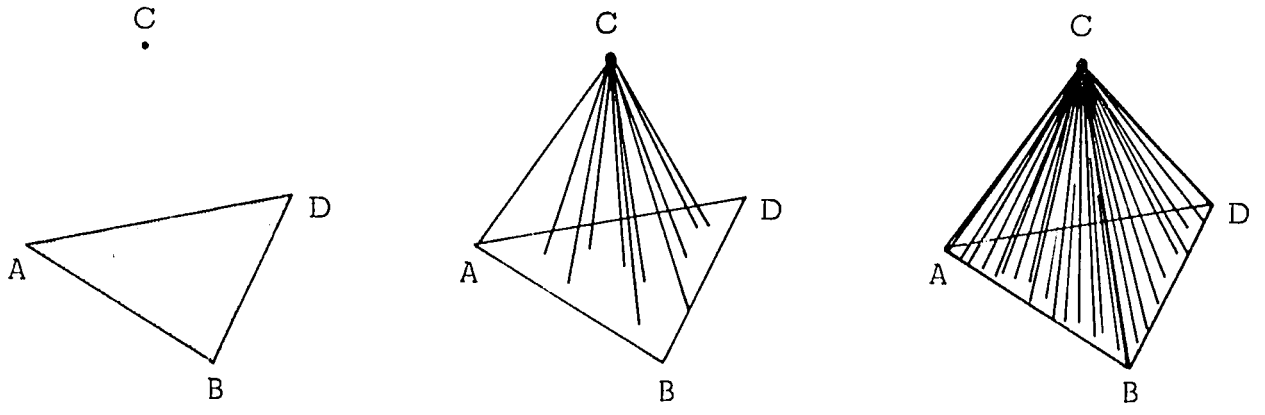




If you fill the cardboard with sand to make the figure solid, you have a <u>tetrahedron</u>. The cardboard itself is the surface of the tetrahedron. The tetrahedron has four faces, each a triangular region. It also has six edges and four vertices. Can you name them?

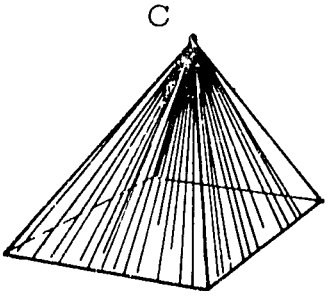
The four faces of the tetrahedron we have just made are congruent, but it is possible to have a tetrahedron whose faces are not all congruent (as you will see in Section 4-3).

Here is a way to think about tetrahedrons that we think you will like. Think of a triangular region ABD on a plane, and a point C not on the plane. Then think of the line segments that have one end at C and the other end in the triangular region.

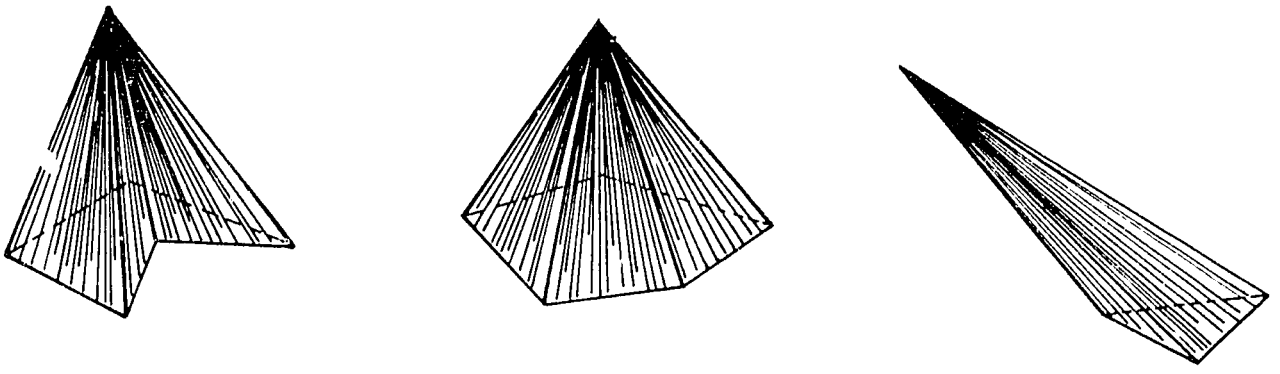


All the segments together make up a tetrahedron whose vertices are A, B, C and D. Can you name the faces of the tetrahedron?

If we had started with a square region, and joined all of its points with line segments to a point C not in the plane of the region, we would get a solid figure like this:



With other polygonal regions we would get figures like these:

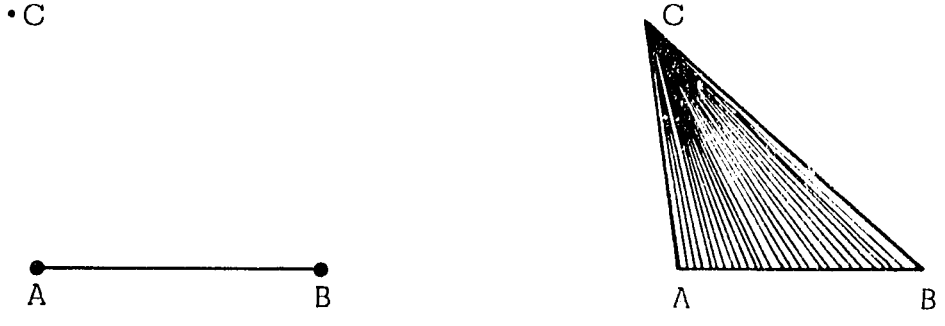


Such figures are called <u>pyramids</u> (see below).

Whenever you start with a figure B and join all possible points on B with line segments to a point C, the new figure is called <u>the join of</u> B <u>with</u> C. The figure B itself is called the <u>base</u> of the join.

A line segment is the join of two points:

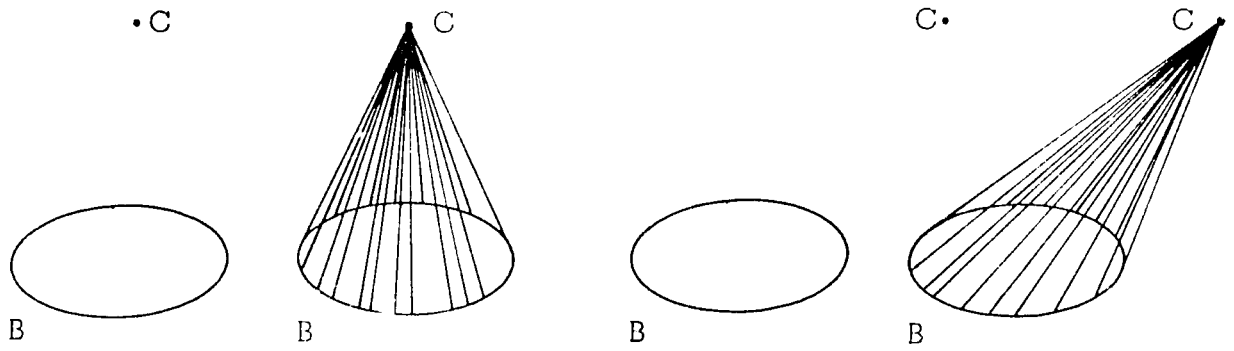
A triangular region is the join of a line segment with a point that is not on the line of the segment.



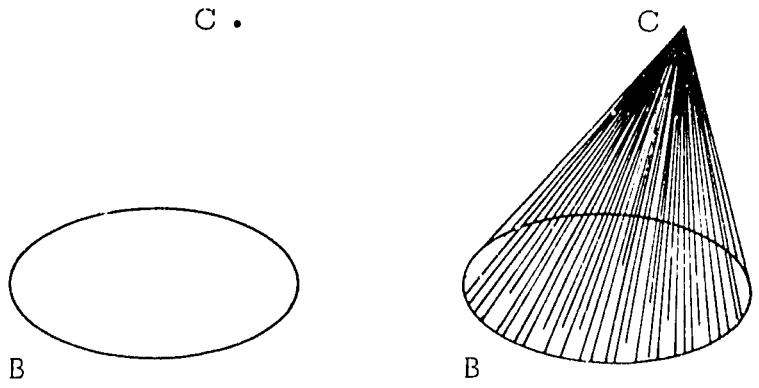
A tetrahedron is the join of a triangular region and a point that is not on the plane of the region.

A pyramid is the join of a polygonal region and a point not on the plane of the region.

A circular region is the join of a circle with its centre. The join of a circle with a point not on the plane of the circle is called a <u>circular cone</u>.

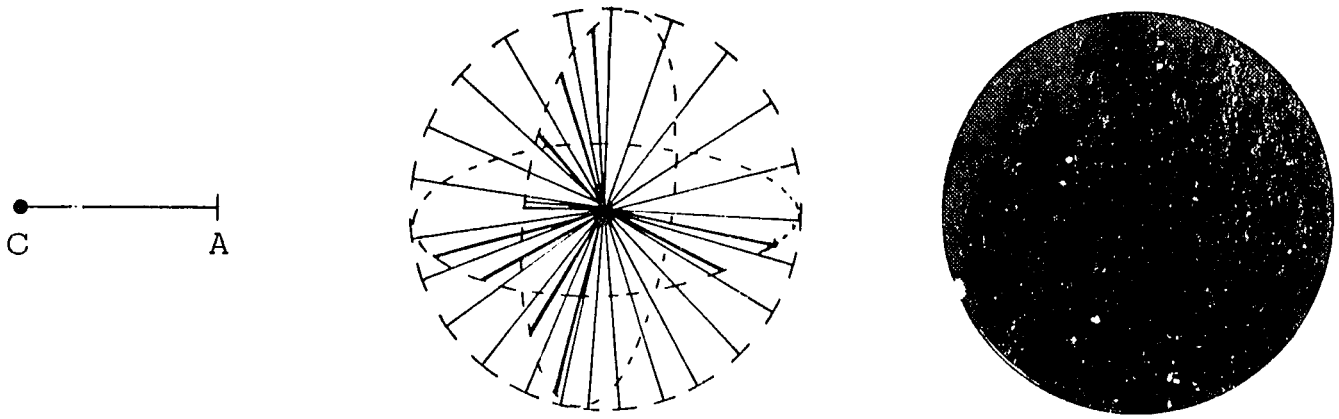


The join of a circular region with a point not on the plane of the region is called a <u>solid circular cone</u>.



Thus, the surface of the solid circular cone is made of a circular cone and a circular region that have the same bounding circle.

Do you know that balloons and hollow balls are <u>spheres</u>? We can give a good description of spheres with geometry. Start with a point C and a line segment $\overline{CA}$. Imagine <u>all</u> line segments in space that have C as an endpoint and that are congruent to $\overline{CA}$.

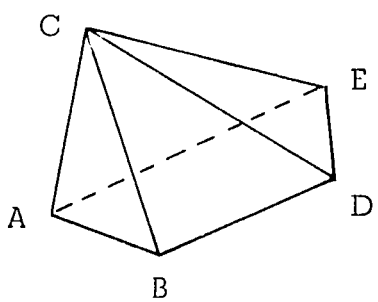


The result is a figure called a solid ball. The surface of a solid ball is made of the endpoints (other than C) of the line segments. The surface of a solid ball is called a sphere.

The point C is called the centre of the sphere, even though it is not a point on the sphere. Each line segment joining C to a point on the sphere is a radius of the sphere. The radii are all congruent, of course, and their common length is called the radius of the sphere. A line segment that joins two points of the sphere is called a chord of the sphere. A chord that passes through the centre of the sphere is called a diameter of the sphere. The diameters of a sphere are all congruent, and their common length, which is twice the radius of the sphere, is called the diameter of the sphere. The solid ball that is bounded by the sphere is the join of the sphere with the sphere's centre.

Problems 4-2

1. Here is the outline of a pyramid:



Its base is the four-sided region ABDE. The face CBD is the join of C with $\overline{BD}$. Which other faces are joins with C?

2. An orange is an example of a solid ball. Its skin is a sphere. What other examples of spheres and solid balls can you think of?

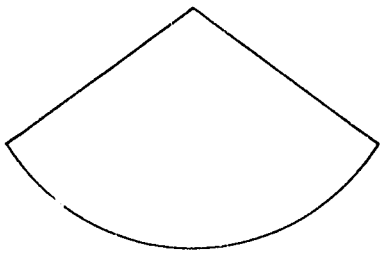
3. Can you find any examples of pyramids? Can you find any roof-tops that look like pyramids or cones?

4. Draw a line segment and a point not on the line of the segment. Then draw the join of the segment with the point.

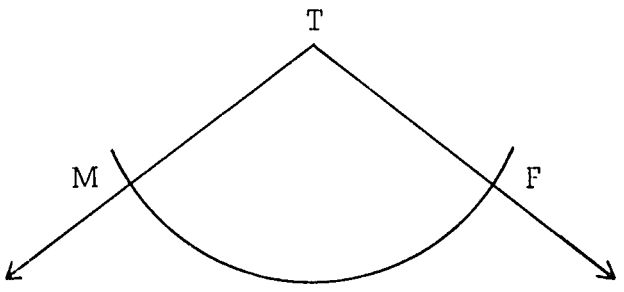
5. Draw the join of a circle with its centre.

6. Patricia says that you can think of a tetrahedron as a join in four different ways. Do you believe her?

7. Make a very large region like this out of paper:



Draw the boundary of the region by drawing an angle and an arc whose centre is at the vertex of the angle:



Then all you have to do is to cut around the closed path TMF.

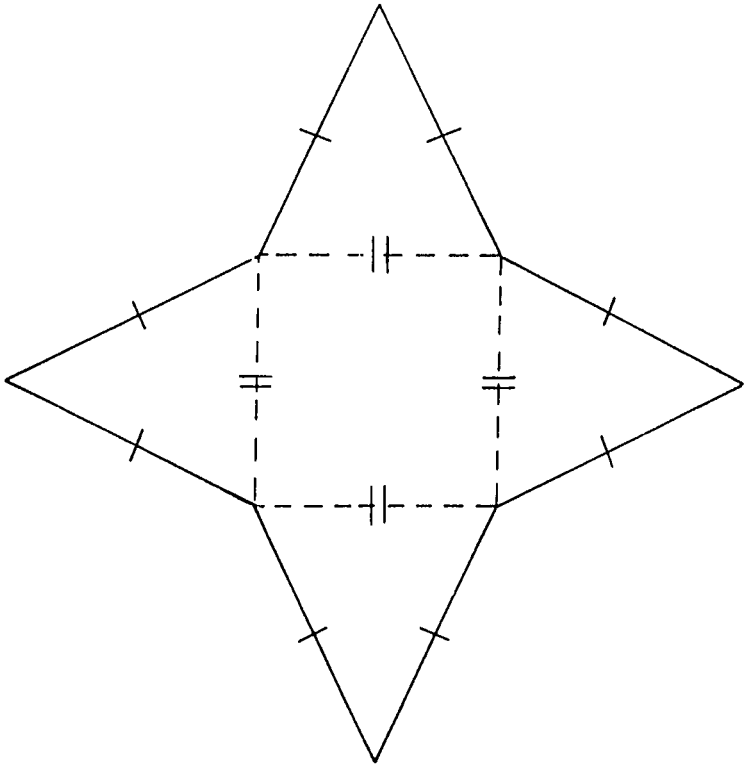
Once you have cut out your region, do not fold it, but bend it around so that you can tape $\overline{TM}$ to $\overline{TF}$. (Use cellotape, or gummed paper.) What kind of figure have you made?

Set the cone upright on a piece of cardboard, and trace the base circle of the cone. Cut out the cardboard disc, and tape its boundary to the base of the cone. What kind of solid is bounded by the surface you have just made?

4-3 Pyramids

To get a better idea of what pyramids are like, let us make models of some. All you will need are thin cardboard sheets, a straight-edge, a razor blade or a pair of scissors, and cellotape or gummed paper.

First, draw on one of your pieces of cardboard a figure like this (but make your drawing bigger than ours — it will be easier to handle if you do):



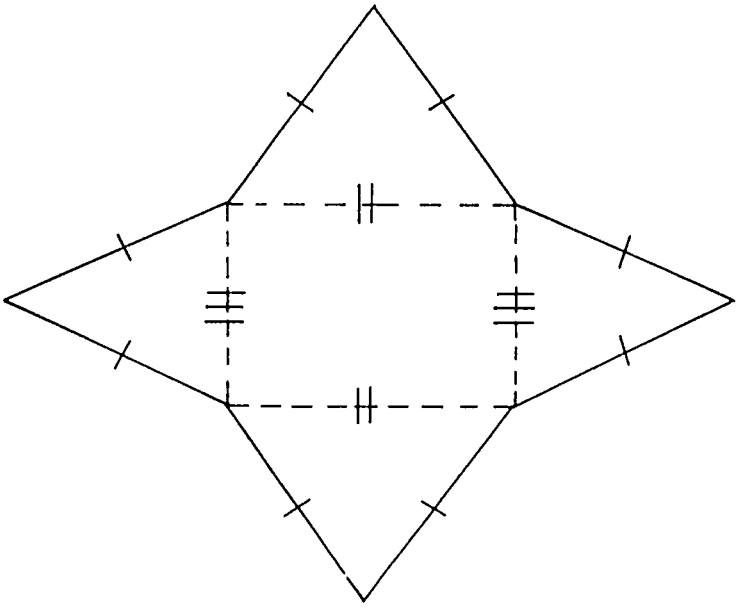
The figure is composed of a square and four congruent isosceles triangles. Line segments that are supposed to be congruent are marked alike. Each of the four sides of the square has two marks: the sides of the isosceles triangles all have one mark.

To draw your figure, draw the square first. Then draw the isosceles triangles with straight-edge and compasses. Use the same setting of your compasses for all four triangles. Why?

Now cut your figure (all in one piece!) from the cardboard. Make folds along the dotted lines to bring the vertices of the triangles together. Seal the loose edges of the figure with cellotape.

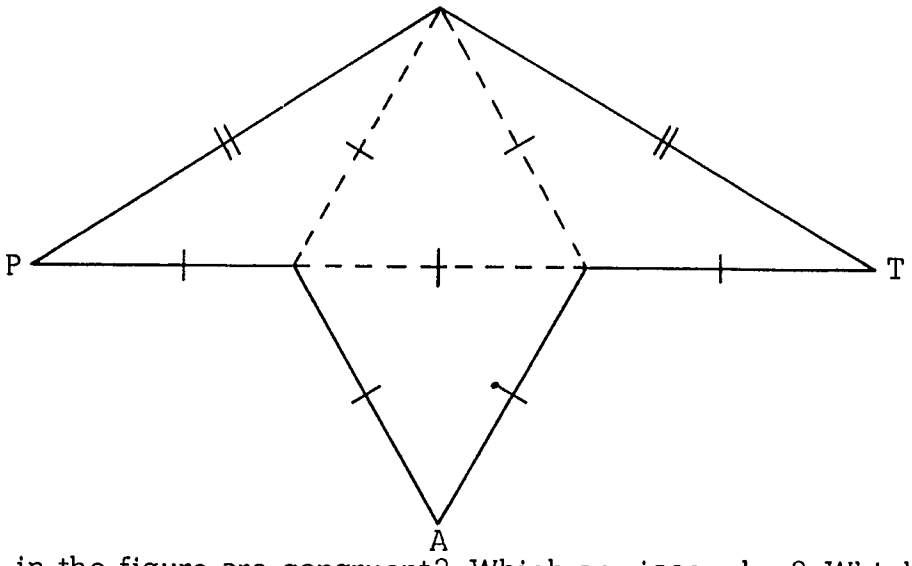
Your figure is now the surface of a pyramid. How would you describe the pyramid as a join? What is its base? How many faces does it have? Don't forget that the base is a face. Count the edges and vertices.

If you make the next figure, you will know an example of a pyramid that has a rectangular base that is not a square base. The figure to draw on your cardboard is one like this (only make yours bigger):



The rectangle has an isosceles triangle on each of its four sides. Opposite isosceles triangles are congruent. The eight sides of the triangles are all congruent. The markings tell you all you need to know about the congruences. Cut out your figure, fold it along the dotted lines, bring the triangle vertices together, and tape down the loose edges. You now have the surface of a rectangular pyramid. Can you describe the pyramid as a join? Count the faces, edges, and vertices of the pyramid.

Now draw on a piece of cardboard a large figure like this one:



Which triangles in the figure are congruent? Which are isosceles? Which are equilateral? Cut your figure out, fold it along the dots to bring P, A and T together, and tape the loose edges. What have you made?

Problems 4-3

1. Copy and complete the following table.

No. of sides of base of pyramid	3	4	5		18	N			
F, number of faces of pyramid	3 + 1	5		9					15
E, number of edges of pyramid	3 × 2	8	10		36			18	
V, number of vertices of pyramid	3 + 1	5	6	9			7		
F - E + V	2								

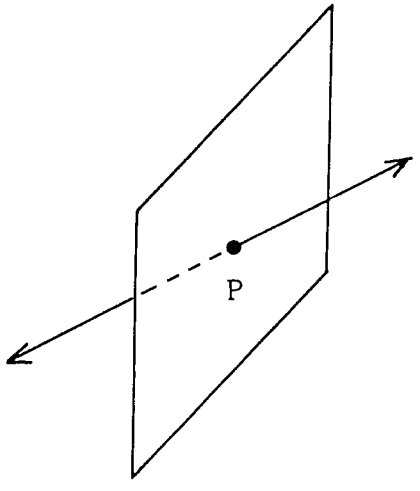
The number F - E + V is called the Euler number of the pyramid. Leonhard Euler, a Swiss mathematician, was the first person to record the amazing fact that you have just discovered about the

number $F - E + V$, although it had previously been noticed by the French mathematician, René Descartes.

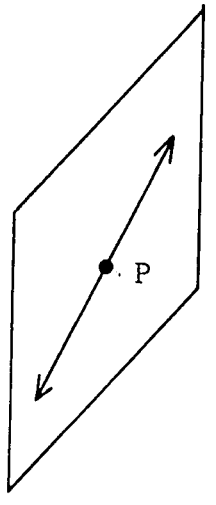
4-4 Points, lines and planes in space

Let us return briefly to points, lines and planes. We want to see how lines and planes may intersect in space, and how they may be perpendicular or parallel.

If a plane contains two points of a straight line, then the plane contains all of that straight line. Thus, if a line and a plane are known to have a point P in common, there are two possibilities:



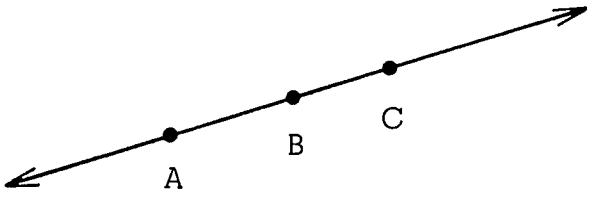
P is their only common point



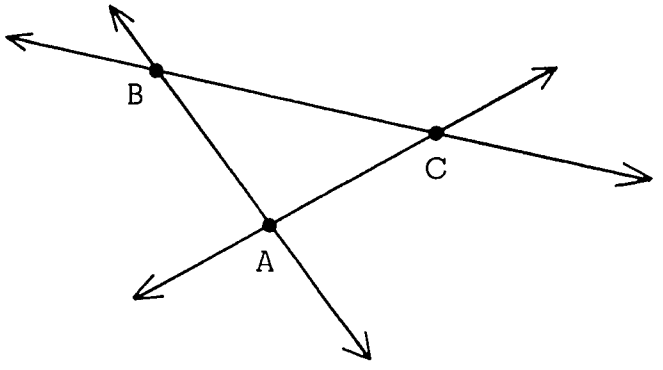
The plane contains every point of the line

Given any two points A and B in space, there is exactly one line segment whose endpoints are A and B. That is why we can call it <u>the</u> line segment $\overline{AB}$. Also, there is exactly one straight line through A and B, which we call $\overleftrightarrow{AB}$. The line $\overleftrightarrow{AB}$ is composed of the two rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$.

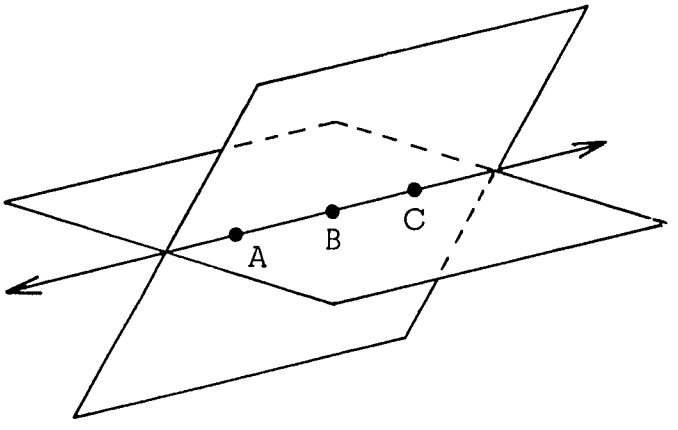
Given any three points A, B and C in space, either all three lie on some line, like this,



or there is no line that contains more than two of the points, and they look like this:

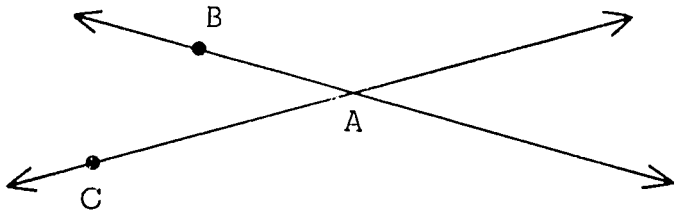


In the first case there will be many planes that contain the line on which A, B and C lie.



Can you think of examples to show this? In the second case, as we saw in Chapter 1, there is just one plane through the three points. You can see this by balancing a book on the tips of three fingers. The plane of the book-cover is the plane through the points of your finger tips.

Suppose that two lines in space intersect at a point A.



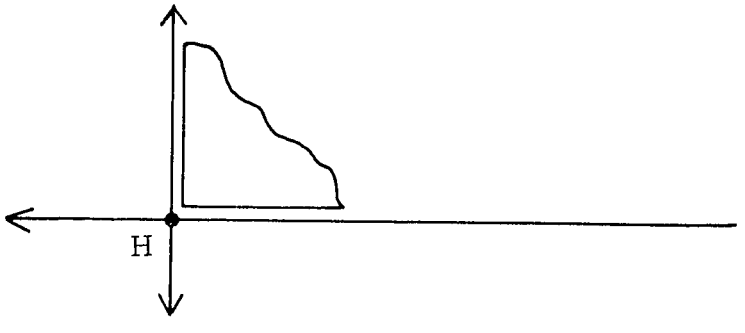
If a point different from A is chosen from each line, B and C in our picture, then the plane of A, B and C contains both lines. Why? Also, the plane of A, B and C is the only plane that contains the two lines. Why?

If two lines intersect, there is a plane that contains them both, and this is the only plane that contains them.

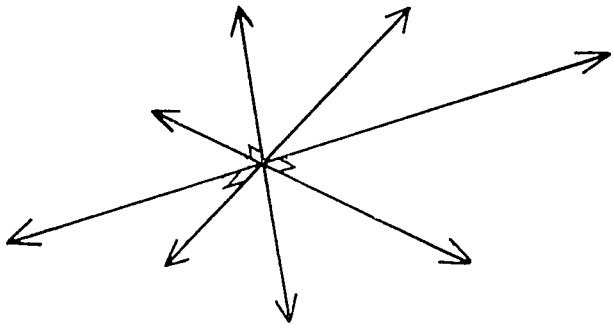
This plane is called <u>the plane of the two lines</u>, or <u>the plane determined by the two lines</u>.

Look at the floor of your room, and choose two straight edges of the floor that meet at some corner. How is the plane of the floor related to the two edges?

We say that two lines in space are perpendicular if they intersect in a right angle. You know, of course, that given a point H on a line in a plane there is only one line in that plane perpendicular to the given line at H:

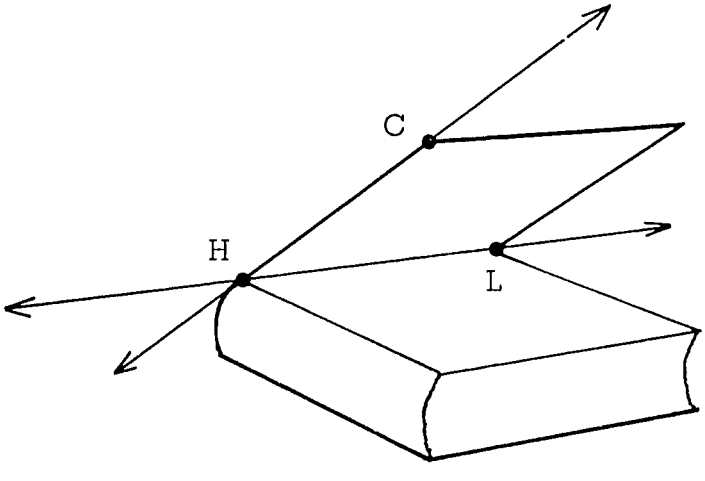


You know how to draw that line with a square corner. But in space there are <u>many</u> lines perpendicular to the given line at H. There is one in each plane that contains the given line. Perhaps the picture below will help you to imagine them.



Set on your desk a book with a stiff, flat cover. Open and close the cover a few times. Do you see that the cover is hinged at the binding of the book? The plane of the cover passes through the line of the hinge. As you open and close the cover, the hinge line doesn't move, but the plane swings on the line just as a door swings on the line of its hinges. The bottom edge of the cover determines a line that is perpendicular to the hinge line, for each position of the cover.

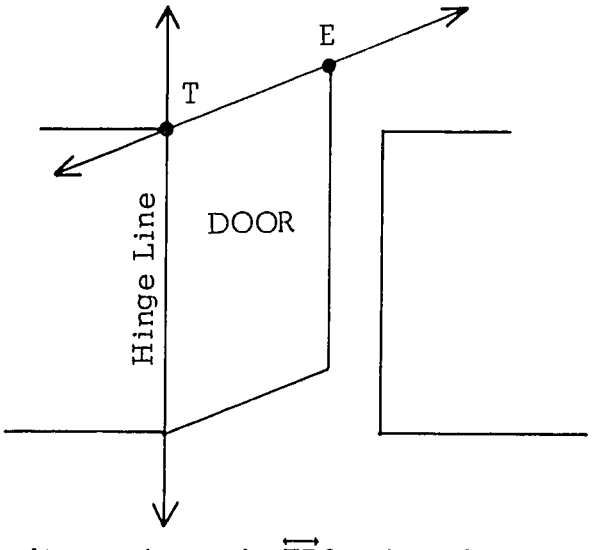
In the picture below, H is the point where the hinge line $\overleftrightarrow{HL}$ and this perpendicular line $\overleftrightarrow{HC}$ meet.



No matter how you swing the cover, $\overleftrightarrow{HC}$ is perpendicular to $\overleftrightarrow{HL}$. For each position of the plane of the cover, $\overleftrightarrow{HC}$ is the only line in that plane perpendicular to $\overleftrightarrow{HL}$ at H.

Have you noticed that, as you swing the cover of the book, $\overleftrightarrow{HC}$ sweeps out another plane? How would you say that this plane is related to $\overleftrightarrow{HL}$?

If you weren't sure about the relation between $\overleftrightarrow{HL}$ and the plane swept out by $\overleftrightarrow{HC}$, think for a minute about doors and their hinge lines. Let $\overleftrightarrow{TE}$ be the line of the top edge of your classroom door. Suppose that we have named the line so that T is also the point where the hinge line meets $\overleftrightarrow{TE}$:



What angle does the hinge line make with $\overleftrightarrow{TE}$? What if you move the door? Are the two lines always perpendicular? As you swing the door, $\overleftrightarrow{TE}$ sweeps out a plane. The hinge line meets this plane at T. Would you say that the hinge line is perpendicular to every line in that plane that passes through T? What word would you put in the box below:

The hinge line is $\boxed{}$ to the plane at T.

Here is another way to get an idea of what it means for a line to be perpendicular to a plane. On a flat piece of paper, mark a point P. Put the tip of your pencil at P. The line of your pencil then passes through the plane of your

paper at P. If you wiggle the pencil about while holding the tip at P, the line moves with it. Can you find a position in which the pencil stands up straightest from the paper? When the pencil is in this position, we say that the line is perpendicular to the plane at P.

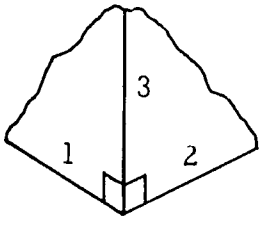
Here is another way to think about perpendicularity. Tie a weight to a string, and hold the string so that the weight hangs just above the floor of your room. What idea do you get about the line of the string and the plane of the floor? The word "perpendicular" comes, in fact, from old words for "hang straight".

These examples all give good ideas about what it means for a line to be perpendicular to a plane. They also lead to a definition:

> A line is perpendicular to a plane at a point P on the plane, if it is perpendicular to every line in the plane that passes through P.

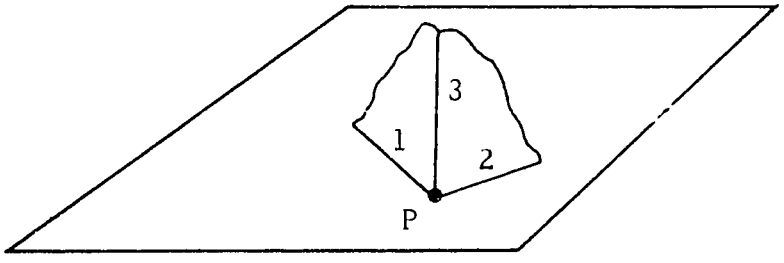
But how are we going to <u>test</u> for perpendicularity? If you think about one more example, you will see that there is a simple test that uses only a square corner! It is a test that always works.

Make a square corner out of paper, but open it up half way, like this:



In our picture we have numbered three edges. Edges 1 and 2 are part of the original straight edge of the paper. Edge 3 makes a square corner with both 1 and 2.

Mark a point P on a flat sheet of paper. Stand your half-opened square corner on the paper with the corner at P.



Make sure to stand it, as we have, with edges 1 and 2 in the plane of the paper. Do you see that the line of edge 3 is perpendicular to (at least) two lines in the plane that pass through P? Which two?

Now turn the square corner around on the paper, keeping edges 1 and 2 on the plane and the corner at P. What happens to the line of edge 3 as you do this? Does it move? If your paper is flat, and if you really do keep the corner at P, then the line doesn't move. Do you see that?

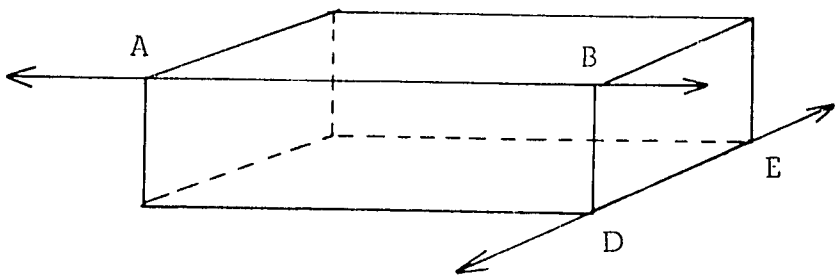
Now suppose that you have a line on your paper through P. Without moving line 3 you can shift the square corner around so that edge 1 lies along the line. That means that the line of edge 3 is perpendicular to the line in the plane. Is the line of edge 3 perpendicular to every line in the plane through P? What can you conclude from that? Would you agree that

> A line is perpendicular to a plane if it is perpendicular to two lines in the plane.

Look at the line segment in which two walls of your room meet. Test it to see if its line is perpendicular to the plane of the floor. If a line segment and a plane region meet at a point, we say the line segment is perpendicular to the region if its line is perpendicular to the plane of the region.

Two <u>planes</u> are said to be <u>parallel</u> if there is a line that is perpendicular to them both. Two plane regions are called parallel if they lie in parallel planes. Can you find parallel plane regions in your classroom? How can you test opposite walls of your classroom to see if they are parallel?

We do not call two <u>lines</u> in space parallel just because they lie in parallel planes. For example, look at a brick whose faces are all rectangular regions.



Then opposite faces of the brick are parallel. Why? Thus $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ lie in parallel planes, but we do not call the lines themselves parallel. The line $\overleftrightarrow{BD}$ is perpendicular to both $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ but we still do not call $\overleftrightarrow{AB}$ and $\overleftrightarrow{DE}$ parallel. In order to be parallel, lines must lie in the same plane.

> Two lines in space are <u>parallel</u> if there is a plane that contains them both, and if they are parallel in this plane.

<u>Problems 4-4</u>

1. Find a pair of straight edges whose lines

   (a) are parallel.

   (b) are perpendicular.

   (c) intersect but are not perpendicular.

   (d) do not intersect but are not parallel.

2. Find a pair of straight edges whose lines intersect. Describe the plane determined by these two lines.

3. The plane of a book cover is a "swinging plane". Find another swinging plane and point out the line it swings around. Put the plane in several positions to show different planes through that line.

4. What examples of parallel planes can you think of?

5. Find two plane regions that look as if they might be parallel, and test them to see whether they are.

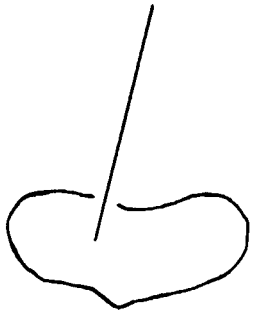
6. Do you know any trees that look as if they are perpendicular to the ground? Find other things in the world around you that suggest lines perpendicular to planes.

7. Ask a friend to hold a pencil in a way that he or she thinks is perpendicular to a flat piece of paper. Test the pencil with your set square. Is it perpendicular to the plane of the paper?

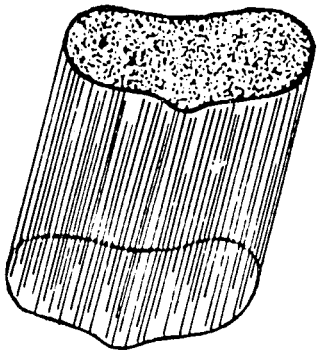
8. When a mango falls out of a tree, along what kind of path does it travel?

4-5 Prisms and cylinders

There is another kind of solid figure that we think you will like. Start with a plane region, and imagine a line segment with one endpoint in the region, like this:

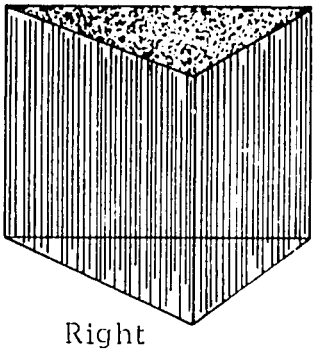


Then imagine <u>all</u> line segments congruent to and parallel to the given line segment. Imagine also that they have one endpoint in the region and that they lie on the same side of the region as the given line segment, like this:

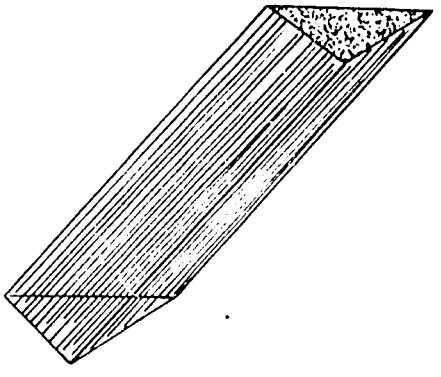


These line segments make a solid figure called a <u>cylinder</u>. The region is the <u>base</u> of the cylinder. Each of the line segments is a <u>directrix</u> of the cylinder.

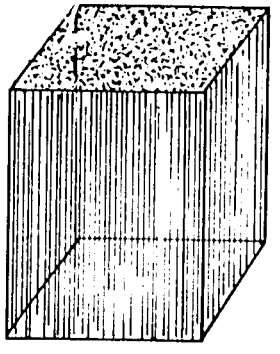
You can make cylinders with many different kinds of bases:



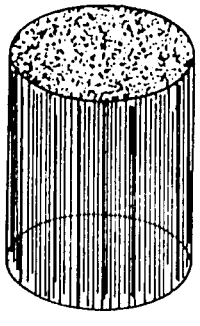
Right triangular prism



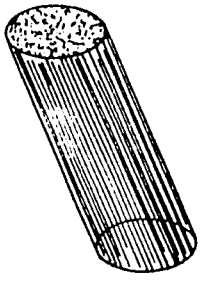
Triangular prism



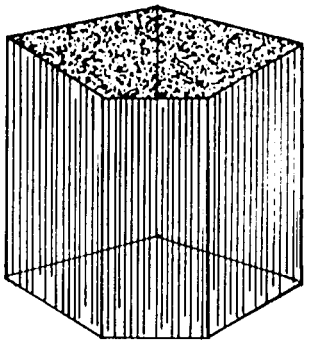
Rectangular prism



Right circular cylinder



Circular cylinder



Right pentagonal prism

It is common to name cylinders after their bases. A <u>circular cylinder</u> is one with a circular base.

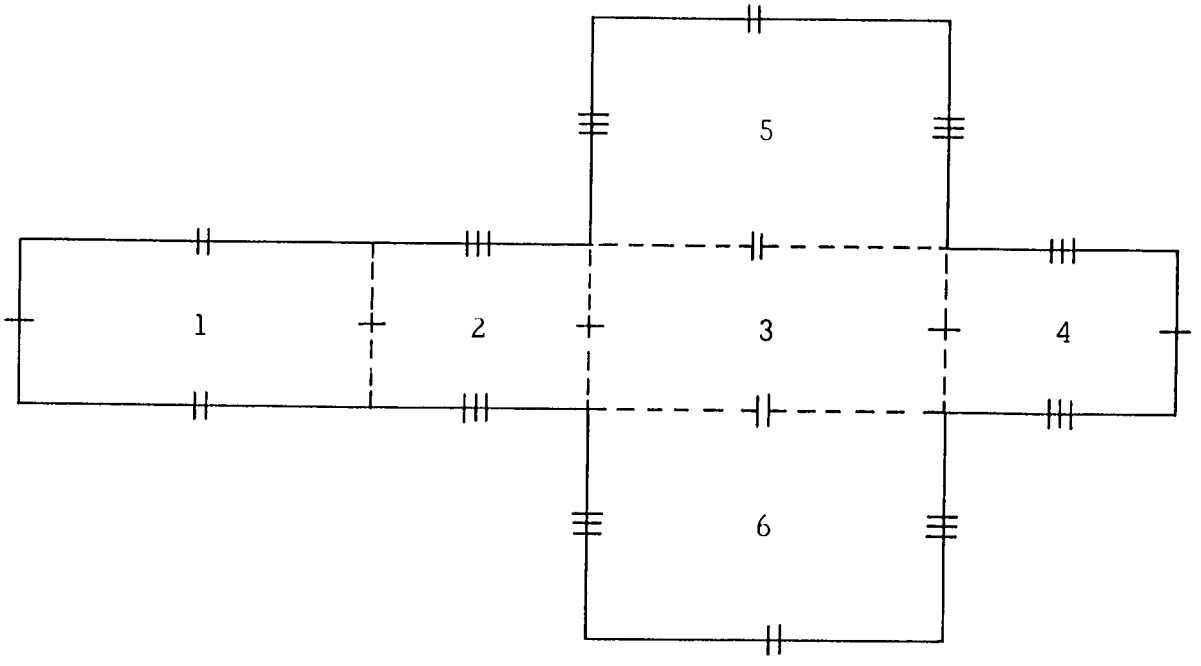
A cylinder whose base is a polygonal region is called a prism. Thus, a triangular prism is a cylinder whose base is a triangular region. A cylinder whose directrices are all perpendicular to the base is called a right cylinder. Can you guess where the name comes from? One of the triangular prisms in the picture above is a right triangular prism. The other is not. Can you see the difference between the two circular cylinders in the picture?

We can now give a clear description of what we mean by a brick: a brick is a right rectangular prism.

Let us now build some models. As before, you will need thin cardboard, a razor blade or a pair of scissors, and cellotape or gummed paper.

Problems 4-5

1. On a piece of cardboard draw a large figure like this:

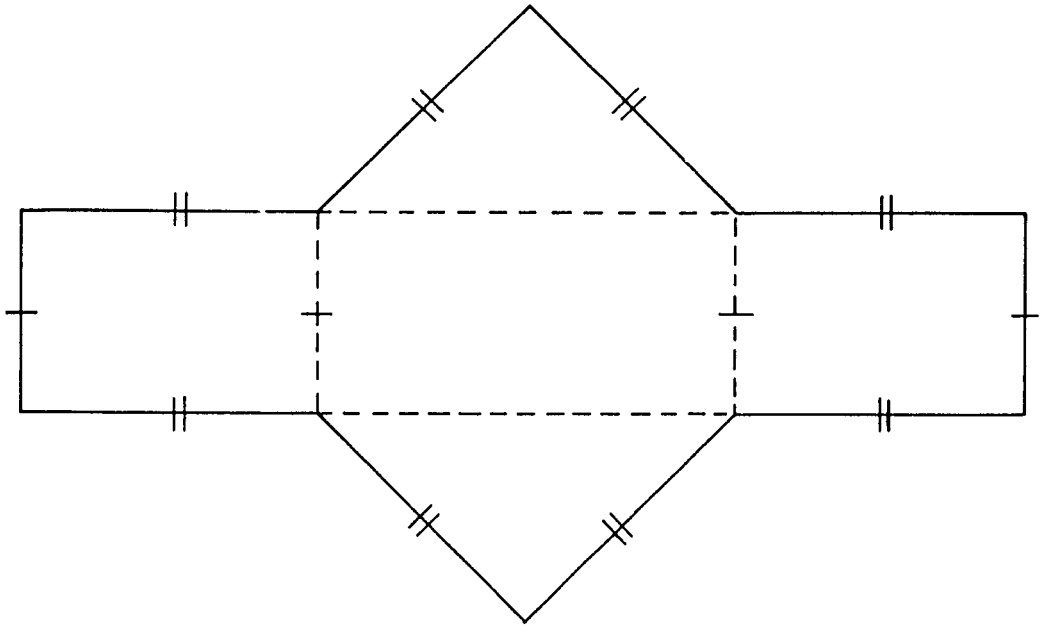


The rectangular regions 1 and 3 are to be congruent; so are 2 and 4; and 5 and 6. Edges that are supposed to be congruent are marked in the usual way.

Cut out the figure. Fold the cardboard along the dotted lines, and tape the edges together to make the surface of a right rectangular prism.

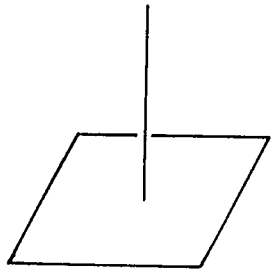
How many faces does the prism have? How many edges and vertices are there? Which faces of the prism could you consider to be a base of the prism? Which pairs of faces would you call adjacent? Which pairs would you call opposite? Are pairs of opposite faces related in an interesting way? How do you know?

2. Next, draw a large figure on cardboard that looks like this:

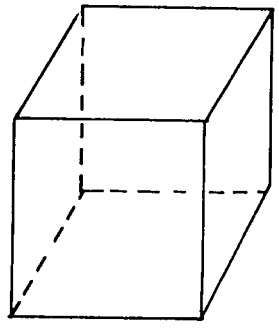


As you can see, the figure is made of three rectangles and two triangles. What can you say about the triangles? Which of the rectangles are congruent? Cut out your figure. Fold it along the dotted lines and tape it together to make the surface of a triangular prism. Is the prism a right prism? Which of the faces could you call a base of the prism? What can you say about these faces? How do you know? Count the faces, edges and vertices of the prism.

3. A right, rectangular prism that has all of its edges congruent is a <u>cube.</u> You can make a cube by starting with a square region and making all the directrices congruent to a side of the square (and perpendicular to the square).

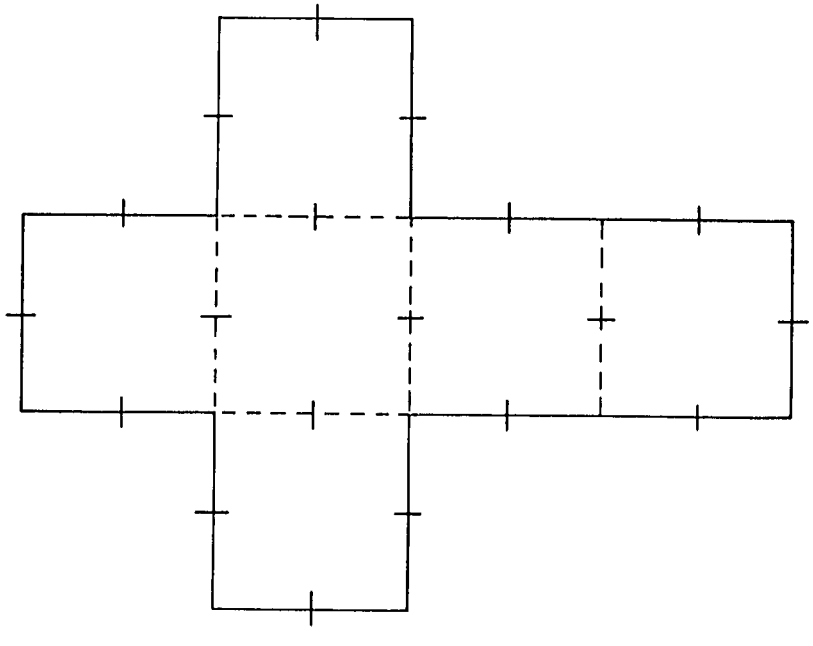


Square region and directrix



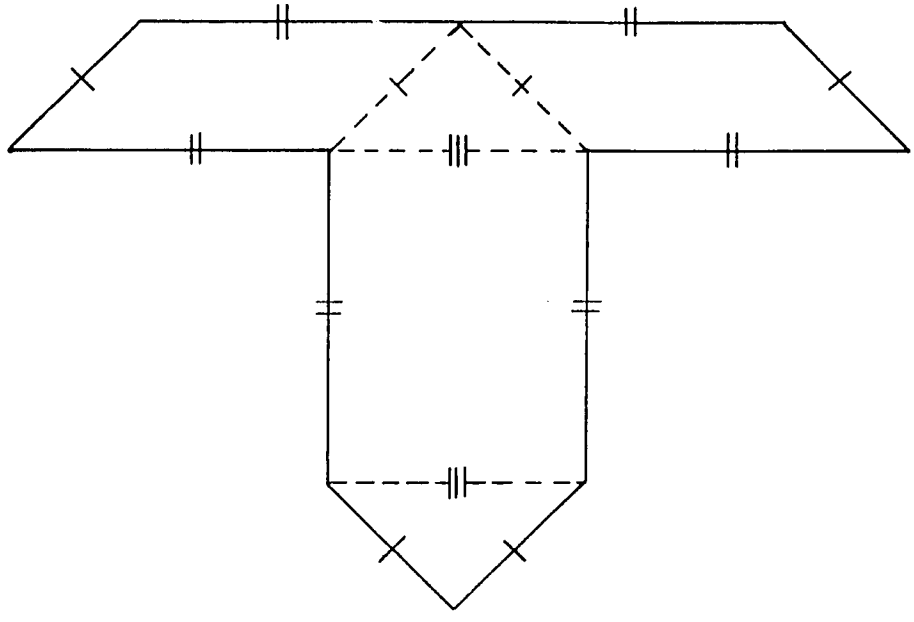
Outline of cube

To make the surface of a cube, start with a large figure like this, drawn on cardboard.



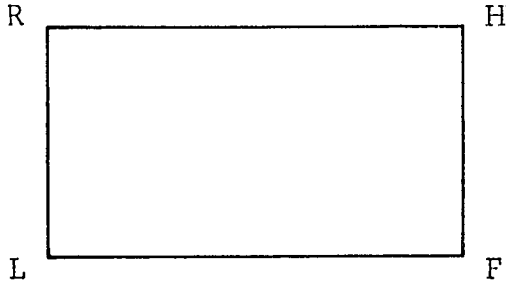
As usual, cut out the figure, fold it along the dotted lines, and tape the free edges together. Count the faces, edges and vertices. How many of these faces could be a base of the cube? Do you notice anything special about opposite faces of the cube?

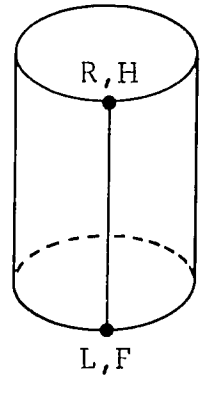
4. The next figure may look peculiar, but draw it anyhow to see what happens:



You should, of course, draw it on cardboard, and make your figure larger than ours. The figure consists of two parallelograms, two triangles and a rectangle. Which pairs of regions are congruent? Cut out your figure, fold it along the dotted lines, and tape the free edges to make the surface of a cylinder. What kind of cylinder is it? Which faces could be a base?Call one of them a base; what can you say about the face opposite it?

5. Take a rectangular sheet of paper (do you know what we mean by that?). Bend your paper (do not fold it) to bring a pair of opposite edges together. If you tape them so that they stay together, you will get an open tube like this:





There are many objects in the world whose surfaces or walls are, in part, like this open tube. What ones can you think of?

6. Sometimes an open tube like the one you made in Problem 5 is called an open cylinder, "cylinder" because it is very much like a cylinder (do you see how?) and "open" because it is open at the ends and is not solid. What do you need to add to the tube to make it into the surface of a right cylinder? Do that now.

7. What examples of circular cylinders can you think of?

8. Copy and complete the following table.

No. of sides of base of prism	3	4		8	10	N			
F, number of faces of prism	3 + 2	6		10			7		
E, number of edges of prism	3 × 3	12	18						21
V, number of vertices of prism	3 × 2	8	12					22	
F - E + V									

9. What can you say about the Euler number of pyramids and prisms?

10. Which of the following statements would you say are true?

(a) Each face of a prism is a base of the prism.

(b) The base of a cylinder and the face opposite it are parallel.

(c) In a cube, each edge is perpendicular to two faces.

(d) In a right rectangular prism, opposite faces are parallel.

(e) A prism always has an even number of faces.

(f) A pyramid always has an even number of faces.

11. Henry says that all prisms have an even number of vertices. Do you believe him?

12. Modupe says that whenever two edges of a right rectangular prism meet, they are perpendicular. Do you agree?

13. Abbiw says that when two edges of a right prism meet they are perpendicular. Do you believe him?

14. Rebecca says that the face opposite the base of a prism is congruent to the base. What do you think?

4-6 Closed Surfaces

A closed surface is the surface of a solid figure. The surface of a cylinder is a closed surface. The surface of a pyramid is a closed surface. So is the surface of a cone.

A single face of a prism is not a closed surface. A tube like the one you made in Problem 5 above is not a closed surface. A closed surface has no holes. A box is closed when its top is on, and is not closed when the top is off or partly opened. Water won't leak from a closed bottle. Sand won't pour from a closed box. Look around your room and name some surfaces that are closed.

A closed surface separates space into two parts: the part inside the surface, and the part outside the surface. (Where have you heard something like this before?) The outside is the large piece: it contains lines, and even planes, that do not meet the surface. The inside contains no lines or planes. It does, of course, contain plane regions and segments of lines. A point in space lies either inside the surface, outside the surface, or on the surface. The surface is the boundary of both the inside and the outside.

A water tank separates water inside from air outside. A closed bottle separates its contents from the outside: if it didn't, you could shake the contents out. What containers can you think of that are like closed surfaces? What do they keep in? What do they keep out? What about a bicycle tyre? How about a safe for keeping food? What is that supposed to keep out?

The way to describe "keep in" and "keep out" in geometry is to say that a closed surface separates points inside the surface from points outside the surface. No path can start at a point inside the surface and end at a point outside the surface without intersecting the surface at some point. A fly outside the food safe cannot get into the safe. Any path from where the fly is to a point inside the safe is intercepted by the walls of the safe.

Surfaces that are not closed do not separate space. You can get from one point not on the surface to any other point not on the surface without touching the surface. If you leave the safe open the fly can get in!

Closed surfaces aren't the only objects that separate space. Can you guess what else does? Planes, of course. Each plane separates space into two parts. (What does this remind you of?) When we described cylinders, we talked about line segments that lay on the same side of a plane region. What we meant was on the same side of the plane of the region. Any point that is not on the plane lies on one side or the other. It is easy to tell when two points lie on the same side of the plane. Do you know how?

Exercise 4-6

1. Would you say that a circle separates space? That is, given a circle in space, and two points not on the circle, do you think there is a path from one point to the other that does not meet the circle?

2. Would you say that a line separates space?

3. Do you think that a sphere separates space?

4. Do you think that a solid ball separates space? Given two points that are not points of the ball, is there a path from one to the other that does not meet the ball?

5. Do you think that a tyre tube separates space? How about a light bulb? What does it keep out? Do you know why light bulbs are made that way?

6. Does your classroom separate space?

7. Does a football separate space? What does it keep in?

4-7 Conclusion for the First Four Chapters

Geometry begins as a description of the world around us. We see shapes and patterns. We find objects that are alike. Some have straightness in common, others flatness; some are curved, others rounded. Still others have edges and corners. We give names to classes of objects that look or feel alike, for example, points and straight edges. From light, shadows, and lines of sight we get our first ideas of lines and rays. From flat objects, we get our first ideas of planes. From looking around, and from moving, we get ideas about something we call space. The earth turns. What in? Space. The sun rises and sets. Where is it? Way off in space. The moon revolves around the earth. The stars shine. Where are they all? In space. Your room is part of space. You can move around in it. You can draw points in it, make line segments, and imagine planes that contain the walls. Rays are interesting things to think about. They make angles, and there are examples of angles everywhere. The idea angle is a useful one because it helps you to describe things you see in the world. There are regions everywhere. Maps are full of regions (countries) bounded by closed paths (borders). Nigeria is divided into regions. Can you name them? Perhaps your country is divided into provinces or regions, too. The idea of a rectangle, is a useful one. So is the idea of a solid and of a surface. Think of how much better a description you can give of your room, and of the objects you use and handle every day, now that you have ideas like these at your command. Of course it wont't help you to use words like angle, prism, ray, and parallel when you talk to a person who doesn't know them. But you will find that most educated people do know them, because they find them useful.

One of the interesting problems that arise in our world is that of comparing the sizes of things. Do these two bottles hold the same amount? Has

this piece of land the same size as that one? How big do I need to make this door? How far is it to the market? These are questions we all hear, and they are questions that we can understand and answer clearly merely by learning a little more geometry! If you want to see how, go on to the next chapter.

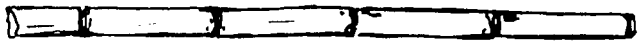
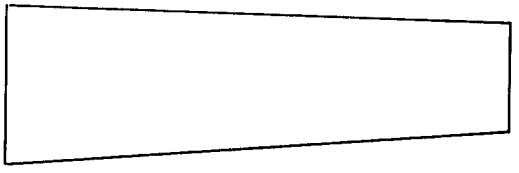
CHAPTER 5

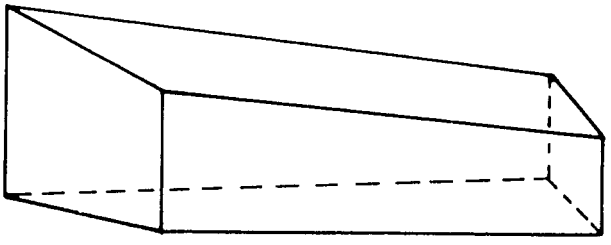
LENGTH

5-1 Counting and Measuring

You and another boy represented your class in a singing competition. Your class was first so your teacher decided to give each of you a prize. He put his hand in his pocket and drew out some shilling pieces, 4 in all. This was the prize. How do you share it?

Suppose that he gave you a piece of cloth instead. How do you share it? The first sharing involves counting. You ask, "How many has my partner? How many have I?" The second involves measuring. You ask, "How much cloth has he? How much have I?" The answer to a question which involves counting is always a natural number such as 1, 2, 3, 4, ... What kind of numbers answer questions involving measuring? Before we can answer this question we must discuss what we do when we make a measurement.

We measure many different things. How we measure depends upon the type of thing that we are measuring. If you are sharing a piece of sugar cane like this  you ask how long it is. If your piece is just as long as your partner's piece then you would have equal shares. But if you are sharing a piece of cloth like this  the question, "how long", is not enough. Dividing the cloth so that your piece is just as long as your partner's piece would not mean equal shares. To get equal shares you would have to consider width as well. If we are sharing a piece of cake like this



we would have to ask, "how long, how wide, how thick?"

Previous Page Blank

The first question, "how long", applies to measurement on a line. The question, "how long, how wide", applies to measurement in a plane. The third question, "how long, how wide, how thick", applies to measurement in space. Let us begin with measurement on a line.

Suppose that your teacher asked each member of the class in turn to try to find the distance between a pair of opposite walls in your classroom. Kwese walks from one wall to the other counting his strides. He announces, "The distance is 8 strides".

Kofi steps heel-to-toe from one wall to the other counting his steps. He says, "The distance is 17 steps".

Kwame counts how many times he must use the span of his hand from one wall to the other. He tells the class, "The distance is 24 spans". Notice the different numbers (8, 17, 24), even though all three measured the same distance. Which one of them was right? Were they all right? How can 8 strides, 17 steps, 24 spans all be measurements of the same distance?

In each measurement there are two parts, the number and what is used for measurement:

Number	What is used for measurement
8	Strides
17	Steps, heel-to-toe
24	Spans

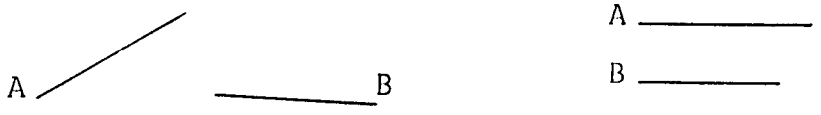
The number is called the <u>measure</u>; what is used for measurement is called the <u>unit</u>. Suppose that someone is asked to find the distance between a pair of opposite walls of a hall. He comes back and says, "The distance is 14." Has he given us all the information that we need? Is the distance 14 strides? or 14 steps? or 14 spans? It might be 14 of any unit. So it is necessary when one gives a measurement to state the <u>measure</u> and the <u>unit</u>.

Problems 5-1

Choose a convenient distance in your classroom and measure it in strides, steps and spans. Suggest other possible units that you could choose, using your fingers, arms, or legs. Compare your measurements with other measurements of the same distance.

5-2 Measurement of Length

We can sometimes compare lengths by moving objects about so that they lie side by side. Suppose that you have two sticks like those shown in the picture at the left and you wish to know which is longer or if they have the same length. If you put them side by side as in the picture on the right with one end of stick B directly below an end



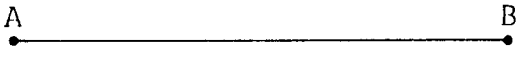
of stick A, you can easily decide whether one is longer or whether they are equally long.

Sometimes you cannot move the objects to make this comparison. Is one room in your school longer than another room? Is one edge of a table longer than the opposite edge? Is one field longer than another? Whenever we wish to compare lengths which cannot be compared directly, we choose some kind of a measuring stick which we can easily apply to both. You recall that one boy found that the distance between a pair of opposite walls in his classroom was 17 heel-to-toe steps. Suppose that he wished to compare this distance with the distance between the other pair of opposite walls in the same room. He could not easily make this comparison if he used steps for one distance and strides for the other. Suppose that he decided to use steps for both. If the second distance was 19 steps he could tell which was the larger distance and how much larger. We would say that we were using a step as a unit of length and length was measured by the number of copies of this unit that we needed.

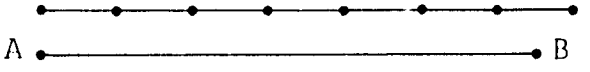
Suppose that each of the pupils in your class measured the distance between the same pair of opposite walls in your classroom with heel-to-toe steps. Would the results be the same? Suppose one boy made this measurement today and the same measurement a year from today; would the results be the same? If you agree that it does not seem likely that the room changes size, you can see that a step is not the best measuring unit. Perhaps we could get a good stout stick and use it. We could mark the end of one stick length on the floor in chalk and then use that mark as starting point for another stick length. This would allow us to make many copies of our unit. The stick would be a better unit since we would not have different numbers of copies when different pupils did the measuring nor would we get different numbers now and a year from now.

The stick could still cause trouble, however. One class might use one stick in its classroom and another class might use a different stick in its classroom. If the sticks were not equal in length we could not compare the results. If a stick got lost then all measurements would have to be done again with a new stick. It would be much easier if many people used the same stick and if there were many exact copies of this stick which could be used wherever they were needed.

Suppose that we choose a convenient unit for measuring along a line on this page. We can take this line segment $\overset{u}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet}$, which is called a centimetre, as our unit. We wish to measure the distance from A to B using this unit.



We make many copies of u. (Are all of the copies equally long?) Then we line up copies of u, end to end, along $\overline{AB}$.



We see that $\overline{AB}$ is longer than 6 copies of our unit centimetre and not as long as 7 copies of the same unit.

We can use different units. Suppose that we agree to use the line segment $\overset{v}{\bullet\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\bullet}$, called an inch, as our unit. To measure $\overline{CD}$, we make many copies of v and line them up, end to end, along $\overline{CD}$.

We can see that $\overline{CD}$ is longer than 5 copies of our unit inch and not as long as 6 copies of that same unit.

Problems 5-2

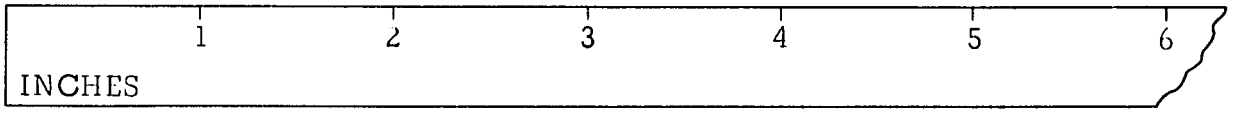
1. Use copies of the inch unit along $\overline{AB}$ and fill in the following blanks: $\overline{AB}$ is longer than_____copies of the inch unit and not as long as _____copies of the same unit.

2. Use copies of the centimetre unit along $\overline{CD}$ and fill in the following blanks: $\overline{CD}$ is longer than_____copies of the centimetre unit and is not as long as_____copies of that same unit.

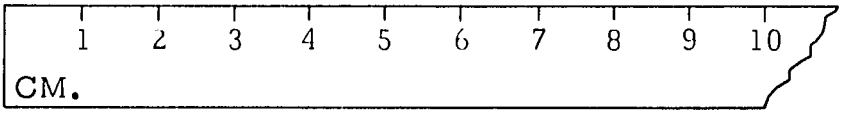
3. Make copies of a foot unit and use them along the edge of a table or blackboard. Write a statement about the measure of this edge like the statements in Problems 1 and 2.

4. Use your copies of the centimetre unit along the length and width of your desk. Fill in the following blanks: My desk is longer than____ copies of the centimetre unit and not as long as_____copies. It is wider than_____copies of the centimetre unit and not as wide as _____copies.

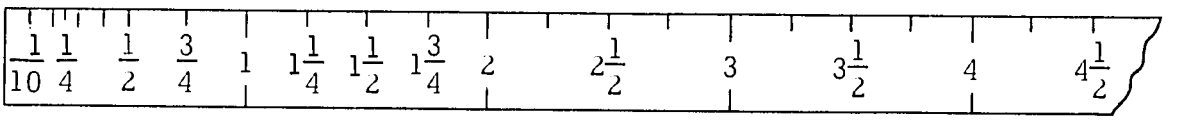
A ruler is a straight edge marked with many copies of some unit. A ruler marked in inches would look like this:



The number for any mark is the number of copies of the unit from the end of the ruler to that mark. A ruler marked in centimetres would look like this:

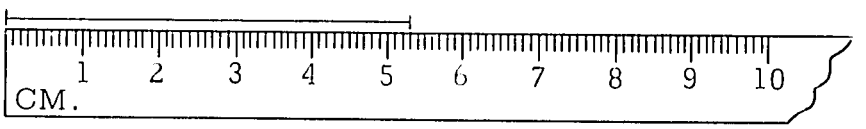


You are already familiar with the number line which looks very much like the edge of a ruler. As with the number line we can subdivide our units into equal parts, using as fine a division as we may need, but keeping in mind that practically we cannot make the marks too close together. If we try to subdivide again and again we may find that the pencil or ink marks begin to run into one another.



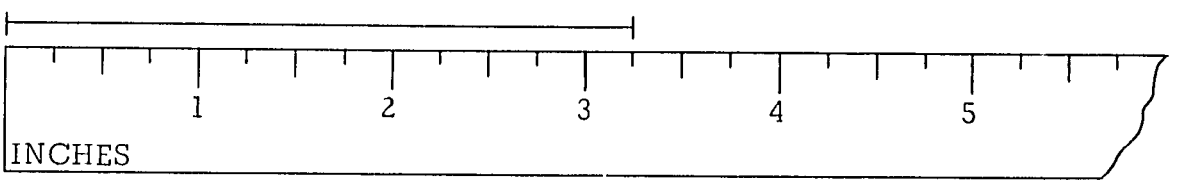
When we subdivide our unit on the ruler, we are really just choosing a smaller unit for our measurement. If we use the half marks on the ruler we are talking about a smaller unit so that two copies of this one make the old unit. If we use the tenth marks we are using a smaller unit so that ten copies of this one make up the old unit.

To measure a line segment we put the end of the ruler at one end of the line segment and mark the point on the ruler opposite the other end of the line segment. Then using the markings on the ruler we can read off the number of copies of the unit from one end of the line segment to the other.

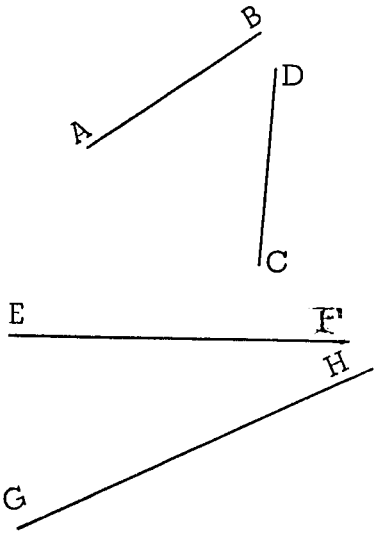


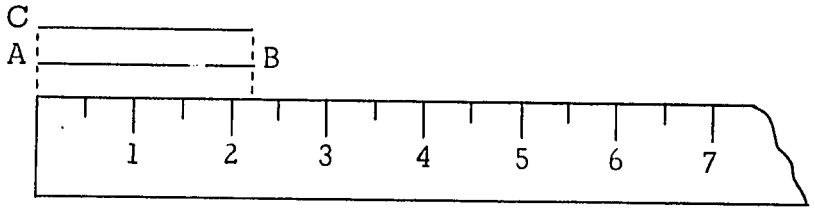
We can say that this line segment has length 5·3 cm. We are using 0·1 cm as our unit and noting that we have 53 copies of this unit, which we write as 5·3 cm.

We can say that the following line segment is longer than 3 copies of our inch unit and not as long as 4 copies or, using the $\frac{1}{4}$ inch division as a unit we can say that it is $3\frac{1}{4}$ inches long.

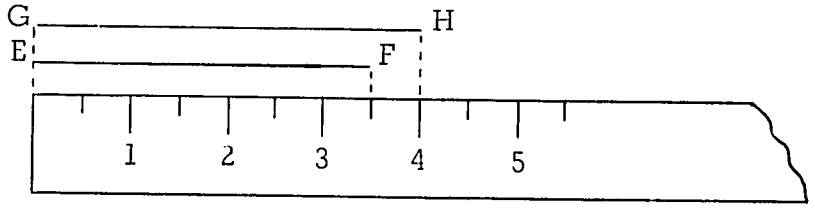


If we wish to compare two lengths now we can mark them off on a ruler to find out if they are equal in length and if not, which one is longer and how much longer. Are the lengths of the sticks AB and CD equal? Are the lengths of the sticks EF and GH equal?





These two sticks are of equal length.



These two sticks are not of equal length.

The longer one is GH. It is about $\frac{1}{2}$ unit longer.

We use rulers like the ones in the pictures to measure length. Instead of choosing our own unit each time we generally use rulers already marked in units which many other people use and understand. This is like getting many people to agree on a certain stick and having copies of it available wherever it is needed. These standard units of length are all units you have seen before, such as the inch, the foot, the yard, the mile, the centimetre, the metre, etc. You can always mark the straight edge of a stiff piece of paper or cardboard or wood with many copies of a unit you choose for yourself, but if you expect to compare your measurements with those made by others using other rulers, then a <u>standard unit</u> seems desirable.

5-3 Origins of Standard Units

Long ago people had no need for great accuracy when they measured distances or heights. They did not have the types of instruments that we use for measurement today. Their measures were based on body units, and so were very crude. For example, the distance from a man's elbow to the tip of his fingers was called a <u>cubit</u>. (Look up the measurements of Noah's Ark in a Bible.) The distance from a man's nose to the tip of his finger when his hand was outstretched was called a "yard". In fact, one early king of England

declared that the distance from his nose to the tip of his finger when his hand was outstretched was the official "yard". A cubit is about half a yard. When a carpenter wanted a piece of wood one yard long, he measured with a rope the distance from his nose to his finger tips, with his hand outstretched. He then laid this length of rope on the wood and cut the desired piece.

You can see that a "yard" for one person differs from a "yard" for another person, unless, of course, the two persons are of the same size. Do you think that a pygmy's "yard" measures the same as a "yard" for an average-sized man, say in East Africa or in West Africa? The distance between a man's fingertips when his arms are outstretched is about two yards. This distance was called a "fathom". Today this distance is officially 6 feet, or 2 yards. In many parts of Africa, even today, some people take the distance from midchest to the fingertips as one yard. Do you think that this is a convenient unit of measure in country areas where there are no rulers or tape measures, or where the people know nothing of such tools?

An "inch", long ago, was the width of a man's thumb. A "foot" was the length of a man's foot. Can you see again that this distance was not always the same for different persons?

Even by these crude units of measures, people soon found that a foot contained about 12 inches, and that about 3 feet made one yard. Of course, it is not very convenient to express very long distances in inches, or sometimes even in feet or yards. It is more convenient to use much larger units. The Romans, for example, used 1000 "paces", or double steps, as a unit when they measured long distances. The pace was about 5 feet. Our word "mile" comes from "mille" in the Latin expression "mille passum", which means "a thousand paces". ("Mille" is the Latin word for 1,000.)

Another unit which we use today is the "rod". The rod was about 16 feet long, and was the unit used to lay out farm lands. Sometimes when farm lands lay side-by-side, arguments arose as to who owned what land. Can you think of possible reasons for such arguments? At this time one English king declared

that the official rod was the length equal to 16 men's feet. To get this rod he ordered his officers to go to a certain church on a particular Sunday, take the first 16 men who came out of the church, and measure a rod by having them stand so that the toe of one man just touched the heel of the man in front of him. This length, which was measured by a rope, was divided into 16 equal parts. One part became the official "foot" of England at that time.

As people travelled more they soon found that their measures were different from those of people with whom they came into contact. As arguments arose over measures, people became more and more convinced that some standard measures were necessary, even in the same country. A king of England, for example, had an iron bar made which was one yard long. A third of this bar was called a "foot", and one thirty-sixth of it was called an "inch". This was a very good idea, but since only the king and his friends could use the iron bar, arguments over measures continued in the country. There are many stories about the origin of some other measures that we use today. Read about them in an encyclopedia.

5-4 Standardizing Units of Measure

Disagreements over measures continued, and they were so frequent that finally a group of French scientists called a conference of representatives from many countries to establish international units of measure. The conference decided to abandon the old body units and to base their units on the distance from the North Pole to the Equator. This was how the metric system of measures was established. The "metre" is the basic unit of measure in this system, and is defined as one ten-millionth of the distance from the North Pole to the Equator. The centimetre we have been using is one one-hundredth of a metre. More recently an international conference of scientists defined the metre in relation to light waves.

The metric system is the system used by scientists all over the world. This system is also in general use in most countries except the English-speaking countries, which use the British system of measures. In this system the "foot" is the basic unit of measure. The inch that we have been using is, of course, one-twelfth of a foot.

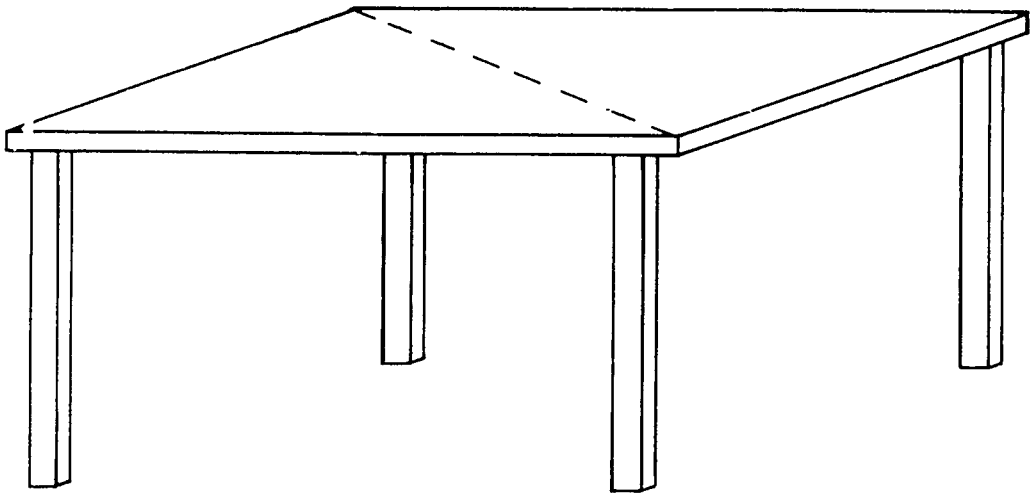
Problems 5-4

What unit would you choose to measure each of the following lengths?

1. The length of the shortest road from Mombasa to Nairobi in Kenya.
2. The length of your index finger.
3. The height of your house.
4. The distance from the earth to the sun.
5. The width of your desk.
6. Would an inch be a reasonable unit for 1? for 2? for 3? for 4? for 5?
7. Would a metre be a reasonable unit for 1? for 2? for 3? for 4? for 5?
8. Would a mile be a reasonable unit for 1? for 2? for 3? for 4? for 5?
9. Would a centimetre be a reasonable unit for 1? for 2? for 3? for 4? for 5?
10. Would one-tenth inch be a reasonable unit for 1? for 2? for 3? for 4? for 5?

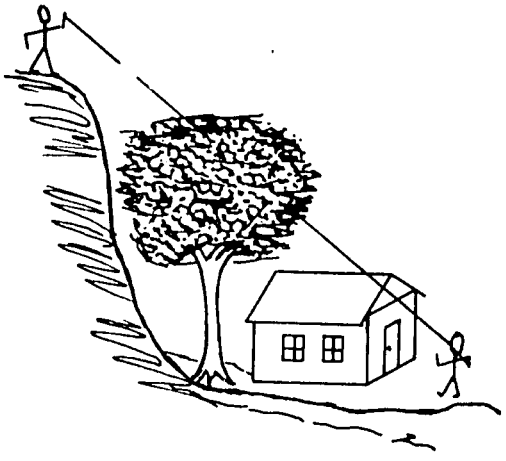
5-5 Measuring Distance

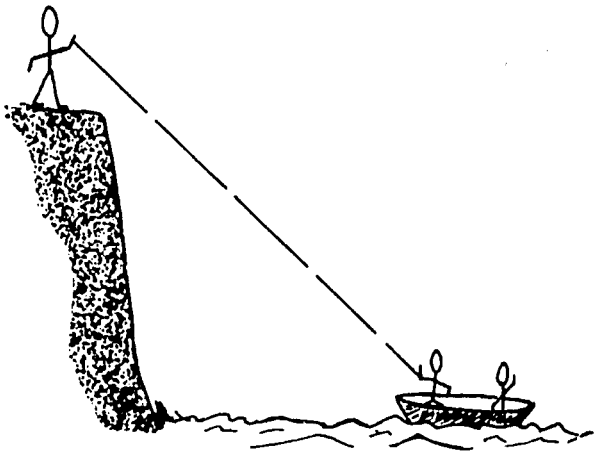
Suppose that you wish to measure the distance from one corner of a table top to the opposite corner as shown in the drawing. If you had a long enough ruler you could lay it on the table top from corner to corner and read the measurement on the ruler.



If your ruler was not long enough you could find a straight stick and mark the length you wish to measure on the stick and apply copies of your unit to the stick. You could also stretch a string from one corner to the other with the help of one of your classmates and ask another classmate to trace the path of the string on the table top with chalk. Then you could apply copies of your unit to the chalk line segment.

Suppose that you wish to know the distance from a mark on one wall of your classroom, say near the ceiling, to a mark on the opposite wall, near the floor. You could still stretch the cord, if it were long enough, and if there were two people to hold it tight between the marks on the walls. You could then stretch the cord on the floor or outside on the ground and apply your ruler to it. If you wish to know the distance from a place on the top of a hill to a place in the valley below or the distance from a place at the top of a cliff to a boat in the water you may find that you cannot get a cord long enough or that you cannot stretch it tight between the two places.

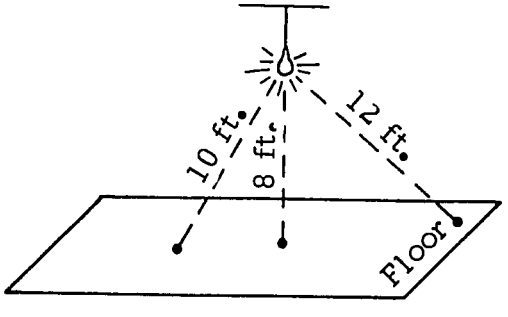




You can still imagine a cord stretched tight between the places and you can still imagine applying a ruler to this cord. Later on you will learn more about measurements such as these which cannot be made directly.

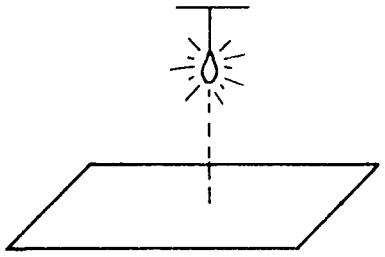
From these examples you can see what it means to measure the distance between points in space. You think of a straight line segment from one point to the other and copies of a standard measuring unit applied to this line segment to find its length. In our real examples the points were places and the line segment was along a straight edge or a stretched cord.

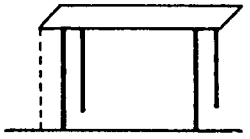
Now suppose that you ask how high a light is above the floor. Is this question different from those we asked about distances from place to place? The light can be thought of as a point in space. The floor can be thought of as a plane. If distance is measured from one point to another, which point on the floor shall we use? If we use different points we may get line segments of different lengths as the picture shows. Unless we can agree on a meaning which leads to a single result, this measurement will not be very useful. Is there a point on the floor such that the segment between the light and this point is shorter than the segment between the light and any other point on the floor?

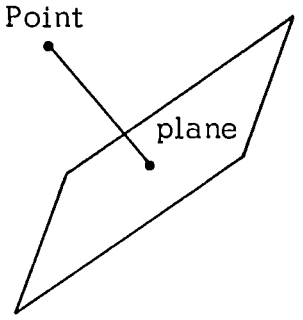


Would you agree that the shortest segment is between the light and a point on the floor directly below the light? Can you apply this idea to get the shortest distance from any point in space to any plane in space?

Measure the distance from the corner of a table top to several different points on the floor. Can you find the point on the floor so that this distance is shortest? Suppose that two of your classmates stretch a string from the corner to the point on the floor for which the distance is shortest. Does this string make a square corner with the floor in all directions?



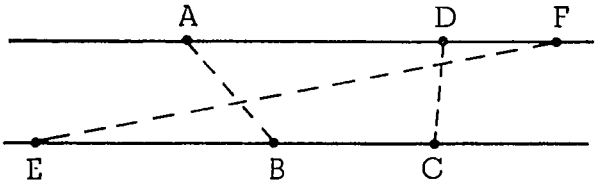




Can you fill in the blank in the following statement? The shortest line segment from a point to a plane is the one which is _____ to that plane.

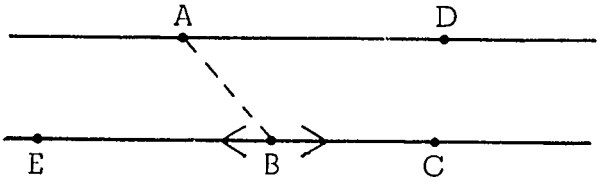
How would you find the shortest line segment from a point to a line? For example you can make a dot on the blackbcard with chalk and ask how far it is from the left edge. Can you see that the shortest line segment from a point to a line is the one which is perpendicular to the given line?

Now suppose that you ask for the distance between the inner edges of the two rails of a railroad track. Each edge can be thought of as a straight line so you are really trying to find the length of a line segment which has its endpoints on two parallel lines. If we pick a point on one straight line and a point on the other we can talk about the length of the line segment between these two points. But there are many possible pairs of points; a few of them are shown in the picture.



How can we choose a pair so that the length of the line segment will mean the same thing to different people? We can again agree that we want the shortest

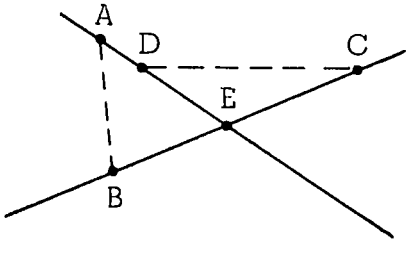
line segment. Does this settle the matter? Look at the next picture. Measure



the distance from A to any point B on the other line. For what position of B will this distance be least? You can find this position by trial and error. Measure the length of the line segment from A to any point B on the other line. Move B a little to the right. Does the length of the line segment from A get larger or smaller? Continue to move B by small amounts in the direction which makes the length of the line segment from A get smaller. Does the length continue to get smaller? Do you find that after several moves the length seems to get larger again? When that happens move back in the opposite direction but not as far as the previous position. Find the shortest length. Try this process again starting at point C on the other line and using any point on the first line. Do you end up with the same shortest length? Start at D. Start at E. Is the shortest length the same no matter where you start?

You will remember that any line perpendicular to one of two parallel lines is perpendicular to the other. By using a square corner, draw a perpendicular to one of the two parallel lines representing the inner edges of the rails. Measure the distance between the parallel lines on this perpendicular. Can you state a relation between a line segment of shortest length between two parallel lines and one that is a perpendicular to the two parallel lines?

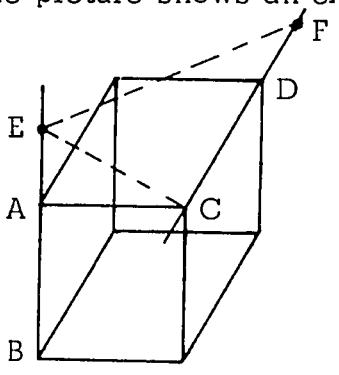
Suppose that we have two lines which are not parallel, what do we mean by the distance between them? We have seen that to give "distance" a single meaning we often have to act as though we had said "shortest distance". Let



us agree that we always mean shortest distance. If two lines are not parallel, they may intersect. What is the shortest distance between a pair of intersecting lines? We can pick any point on one line, such as A, and any point on the other line, such as B, and measure the length of the line segment between them. What points shall we pick, one on each line,

so that the line segment is shortest? Does the segment from A to B look shorter than the segment from C to D? Is the segment from A to B the shortest or can you find a shorter one? Can you fill in the following blank with a number? The shortest distance between two intersecting lines is _____ units. (Does it matter which units you specify in this case?)

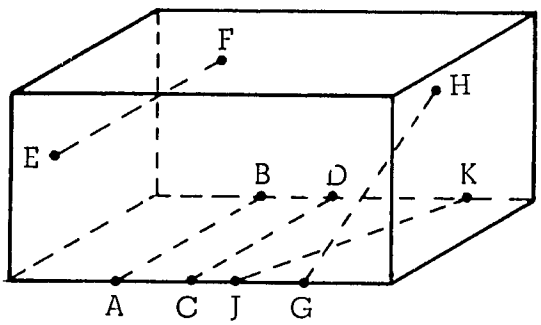
If two lines are not parallel and do not intersect, we call them <u>skew</u> lines. The picture shows an example. The edges $\overline{AB}$ and $\overline{CD}$ of the box extended are not parallel and do not intersect. What about the shortest distance between them? If you pick any point on $\overleftrightarrow{AB}$, say E, and any point on $\overleftrightarrow{CD}$, say F, you can measure the distance between these points.



Suppose that we keep E fixed at first and ask where we should move F so that we have the shortest distance from E to $\overleftrightarrow{CD}$. We know that the shortest distance from a point to a line is measured along a perpendicular to the given line. Of all the line segments between E and a point of $\overleftrightarrow{CD}$, which is perpendicular to $\overleftrightarrow{CD}$? Find a box or make one of cardboard. Fasten straight sticks, rulers, or stiff wire to show $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ as extensions of the segments $\overline{AB}$ and $\overline{CD}$. Pick a point like E and tie a cord to the stick at E. Stretch it to meet $\overleftrightarrow{CD}$ at various points and test perpendicularity with a square corner. Of all the line segments between E and a point of $\overleftrightarrow{CD}$ which is perpendicular to $\overleftrightarrow{CD}$?

Now keep C fixed and find the position of E so that the distance from C to $\overleftrightarrow{AB}$ shall be shortest. This line segment will be perpendicular to $\overleftrightarrow{AB}$. What line segment gives the shortest distance between C and a point of $\overleftrightarrow{AB}$? Is this line segment also perpendicular to $\overleftrightarrow{CD}$? Fill in the following blank. The shortest distance from one of two skew lines to the other is measured along a line which is _____ to both.

Now let us return to the problem of measuring the distance from one wall to the opposite wall in your classroom. Does it matter where the measurement begins? Will you get the same result if you measure from A to B and then from C to D? Will this be the same result you will get if you measure from E to F? From G to H? From J to K? If we again agree that we want the shortest distance, which of these measurements would we use? Can you convince yourself that the shortest distance between two parallel planes is measured along any line perpendicular to both planes? Try to find the shortest distance between the opposite walls of a room or the opposite faces of a box and use a square corner to test whether the line of this shortest distance is perpendicular to both planes.



Problems 5-5

1. Measure the distance from one corner of the door of your classroom to an opposite corner. Measure the height of the door. Measure the width of the door.

2. Measure the distance from a corner of a desk to the nearest wall. Measure the distance from the left front corner of your desk to the front wall. How high is the top of your head above the floor when you are sitting at your desk?

3. What is the distance between the left and right inside edges of a window in your classroom? What is the distance between opposite edges of a table in your classroom?

We have discussed the measurement of many distances and agreed that we shall always mean "shortest distance" even if we do not use the word "shortest". A summary of what we have discovered follows.

1. The distance between two points is the length of the line segment which has these two points as endpoints.

2. The distance from a point to a line is the length of the line segment between the point and the line on the perpendicular to the line.

3. The distance from a point to a plane is the length of the segment between the point and the plane on a line perpendicular to the plane.

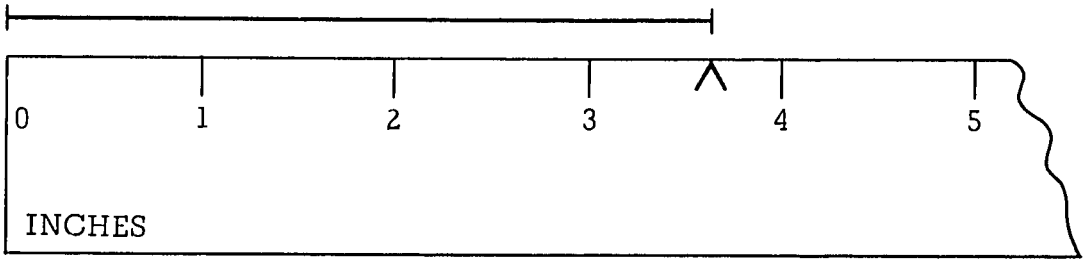
4. The distance between two parallel lines is the length of the segment between the parallel lines on any perpendicular.

5. The distance between intersecting lines is zero.

6. The distance between skew lines is the length of the segment between the lines on a line perpendicular to both.

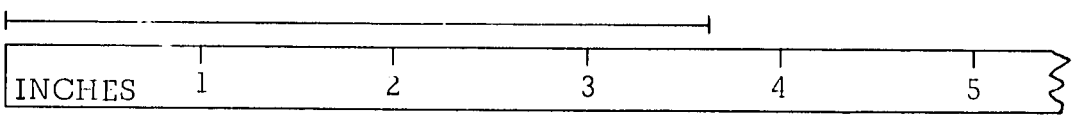
5-6 Using a Ruler

Suppose that we have a straight stick and we want to measure its length with a ruler. We line up one end of the stick with the end of the ruler which corresponds to the 0 on the number line. We find a mark or make one on the ruler at the point opposite the other end of the stick as in the drawing. Now we need only read the number of units and subdivisions of units for that point to get the length of the stick.

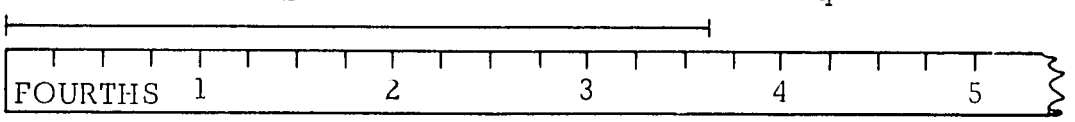


When measuring the length of a segment, we try first to read the length to the nearest mark on the ruler. This involves making a decision since it is not likely that the end of the stick we are measuring will fall exactly at one of the marks. Even if it appears to do so, we know that this appearance depends upon the eyesight and the judgment of the measurer. We also know that the measuring instrument is man-made and subject to human error. Thus the measurement of length or any other measurement may not be exact. When we know that our measurement is not exact we call the number we get an approximation. We say that the number is an approximation to the length of the object that we are measuring.

Suppose that we are measuring a stick as in the drawing. If we have a ruler marked only in inches we can see that the stick is more than 3 inches in length and less than 4 inches. We can decide that it is closer to 4 inches than



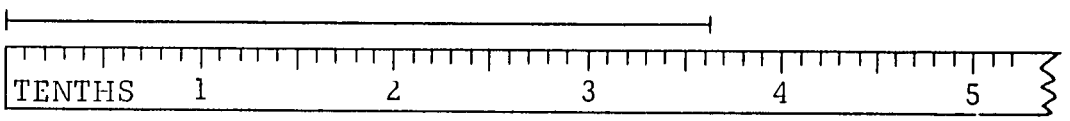
to 3 inches and even estimate its length as a number between 3 and 4. Most rulers are subdivided into units smaller than inches. Suppose that we have the same stick and a ruler marked in fourths of inches. The next picture shows that the stick is more than $3\frac{1}{2}$ inches in length and less than $3\frac{3}{4}$ inches. We can now



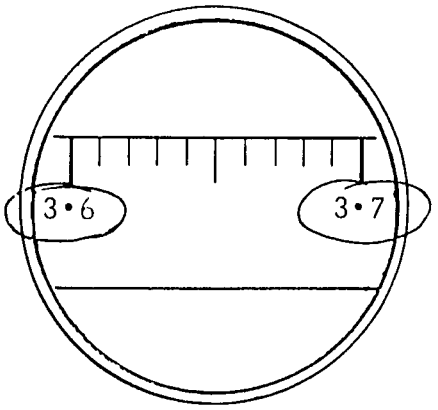
write that the length L measured in inches is between $3\frac{1}{2}$ and $3\frac{3}{4}$ as follows:

$$L > 3\frac{1}{2}, \quad L < 3\frac{3}{4}.$$

Could we be even more accurate than that? We often use a ruler which is divided into tenths of inches. Suppose that we apply such a ruler to our stick as shown in the next picture.



The picture shows that the length of the stick in inches is between 3•6 and 3•7 and it appears to be closer to 3•6. Most of the time when we use a ruler like this we do not stop with the closest mark on the ruler but we try to locate a point on the ruler that is opposite the end of our stick. Suppose that we guess that this measurement is about 3•62 inches. We know that the measurement L in inches is between 3•6 and 3•7, that is, $L > 3{\cdot}6$ and $L < 3{\cdot}7$. We think of dividing this tenth of an inch into ten equal parts, each a hundredth of an inch in length, as in the drawing. This drawing is made as though we were looking through a magnifying glass at the end of the stick and the part of the ruler between 3•6 and 3•7. The numbers and the ten divisions are not on the real ruler but are only in our imaginations. The marks above the 3•6 and the 3•7 are <u>real marks</u> on the ruler. The other marks shown in this drawing are <u>guess marks</u> and are not on the real ruler.



We do this in our imaginations only and we guess that the stick will end at a point about the second hundredth beyond the 3•6. When we do this we are sure of all the digits except the last one to the right. The others should be correct if we have been careful but the last one is the result of a guess or estimate.

When we write the result of a measurement, digits, except the last, ought to be reliable. Only the last digit is questionable.

If we write the length of the stick as 3•62 inches we are sure that the measurement is between 3•6 and 3•7 inches so we have confidence in the first two digits (3 and 6). The final digit (2) is, as you have seen, the result of a guess and subject to all the errors possible in the measurer and the measuring instrument. We have no great confidence that this final digit is a 2 instead of a 1 or a 3.

To draw a line or to make a measurement you sometimes ignore the finer subdivisions of your measuring instrument depending upon the accuracy required. To draw a line 2•3 inches long you would probably use a ruler marked in tenths of inches. The width of a goal in football is supposed to be 24 feet. Measure it yourself to the nearest inch. Does it make sense to measure the width of the goal to the nearest $\frac{1}{2}$ inch? To the nearest $\frac{1}{10}$ inch? To the nearest $\frac{1}{64}$ inch?

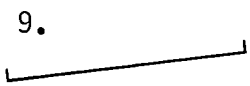
Problems 5-6

For each of the following recorded measurements write a statement using > and a statement using < to show the digits in which you have confidence. Then, in each case, list any digits which are doubtful.

1.	6•57 inches	2.	3•12 feet	3.	27•5 cm.
4.	8•5 inches	5.	8•05 inches	6.	21•01 cm.
7.	7•58 inches	8.	9•20 inches		

In problem 8 why did we write 9•20 instead of just 9•2? If we wrote 9•2 we would be sure of the 9 but the digit 2 would be questionable. In writing 9•20 we claim to be sure of the 9•2 but the digit 0 is questionable. You can see that 9•20 inches is a more accurate measurement than 9•2 inches. Measure each of the following lengths in inches, guessing the second decimal place. Make measurements in centimetres also. Is it harder to guess the second decimal place for inches or for centimetres?

9.



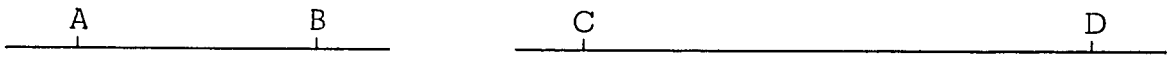
10.

11.

12.

5-7 A Property of Length

You will remember that a line segment is the part of a line between two points called the endpoints of the segment. Since we know how to measure the distance between two points, we can measure the length of a line segment. For example look at the following picture.



Suppose that the line segment with endpoints A and B has the length 1.23 inches. When we write the length of a line segment we shall use the letters for the end points with no bar above, for example AB = 1.23 inches. Measure its length in centimetres. Suppose that the line segment with endpoints C and D has length 6.7 cm., that is, CD = 6.7 cm. Measure its length in inches.

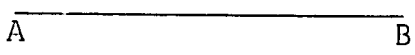
Now look at the picture below showing three points marked on a straight line. There are three possible line segments whose length we could measure. Suppose AB = 2.15 inches, and BC = 3.37 inches. Would you have to measure with a ruler to find the length of the segment $\overline{AC}$? Use the ruler to verify that AC = 5.52 inches.

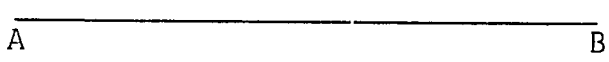
Next consider the line segment shown below. Its length is 4.75 inches. Mark 3 points anywhere on the segment between the endpoints A and B. Label the one nearest A with the letter C, the next to the right with the letter D, and the one nearest B with the letter E. Find the lengths of $\overline{AC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EB}$. Add these lengths. Compare the sum with AB = 4.75 inches. Does it matter if the points C, D, and E divide the segment $\overline{AB}$ into 4 separate segments of equal length? In general do you expect that when a segment

is subdivided into segments that the length of the entire segment is the sum of the lengths of the separate segments?

Problems 5-7A

1. Divide each of the following segments into four equal parts. Measure each part and measure the whole segment. Show that the length of the segment is the sum of the lengths of the parts.

   (a) 

   (b) 

2. Divide each of the segments in problem 1. into four segments, no two of which are equal. Measure each segment. Show that the sum of the lengths of the segments is the length of the original segment.

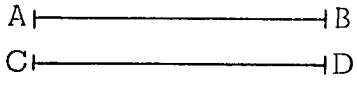
3. Repeat the process suggested in problem 2 using five segments for each given segment.

4. Draw a line segment of any length that you choose. Divide it into as many segments as you wish, equal or otherwise. Is the length of the original segment the sum of the lengths of its parts?

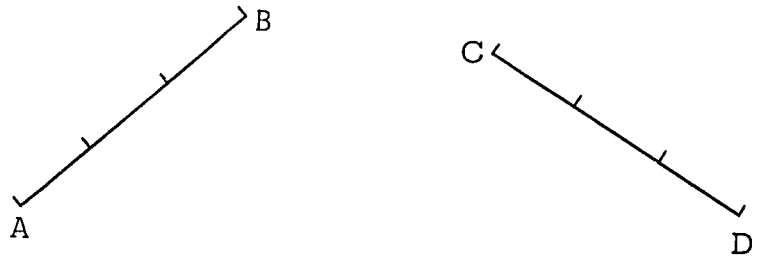
5. Given a line segment, how many points must you mark to divide it into 2 segments? How many to divide it into three segments? Four segments? Five segments?

6. Given $\overline{AB}$ and n points marked on it between the endpoints. Into how many segments is the segment $\overline{AB}$ divided?

If two line segments have the same length then we can put them side by side and the ends will line up as in the following picture.

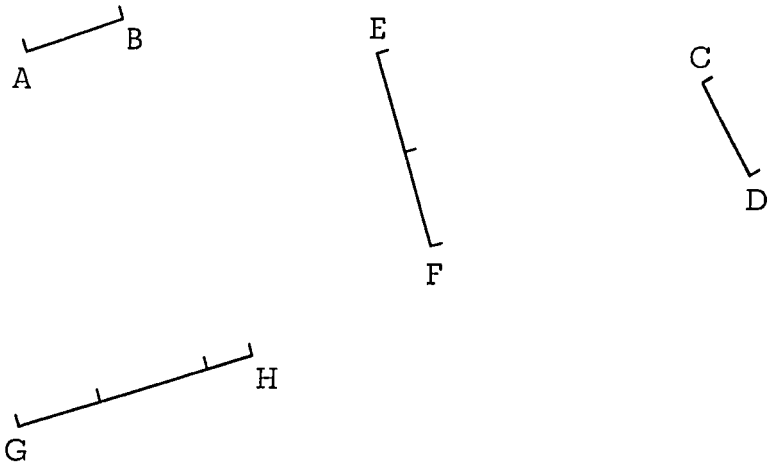


If line segments have the same length and so can be fitted to one another exactly, we call them <u>congruent</u> segments. Example: Segments $\overline{AB}$ and $\overline{CD}$ are congruent segments.



<u>Problems 5-7B</u>

Which of the following segments are congruent segments? Which are not congruent?

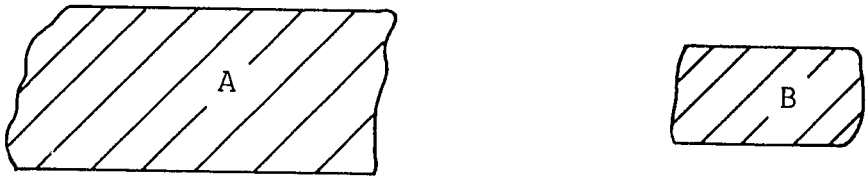


CHAPTER 6

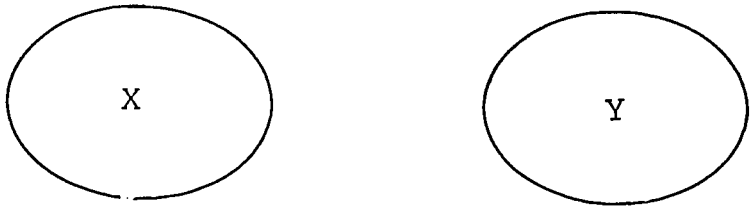
AREA

6-1 The Size of a Region

If we enter a school compound and look at the playing fields, our eyes will tell us that the football field is larger than the tennis court; or again if we enter a building we may judge that the floor of a bedroom is smaller that that of the sitting room. When we look at the region A and at the region B in the picture below, our eyes tell us that the size of A is greater than that of B. If we lift B and place it on A, we can do so with B lying entirely inside A and not covering all of A.



Suppose that we could lift a region X and place it over another region Y. If X and Y fit exactly, that is, are congruent, we say that they have the same size and therefore the same measurement. In this case the measurement is called the area. Hence if the measure of X is n, then the measure of Y, using the same unit of measure, is also n.

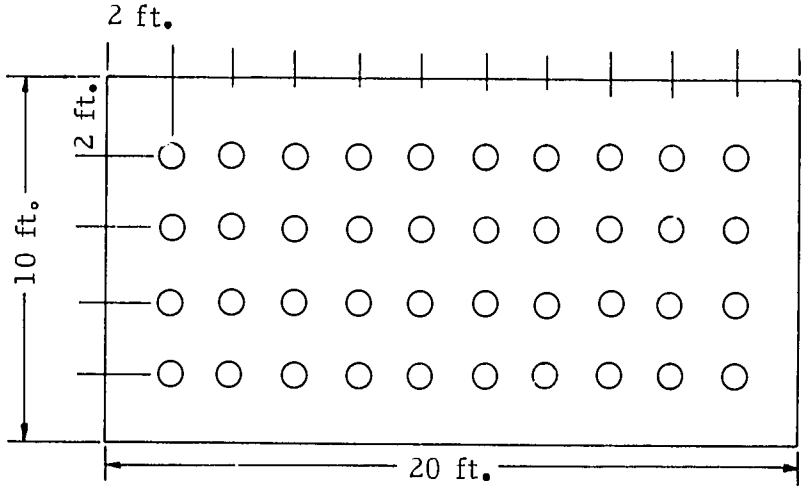


Now if the areas of two disjoint regions are known, the area of the union of the regions is the sum of their areas.

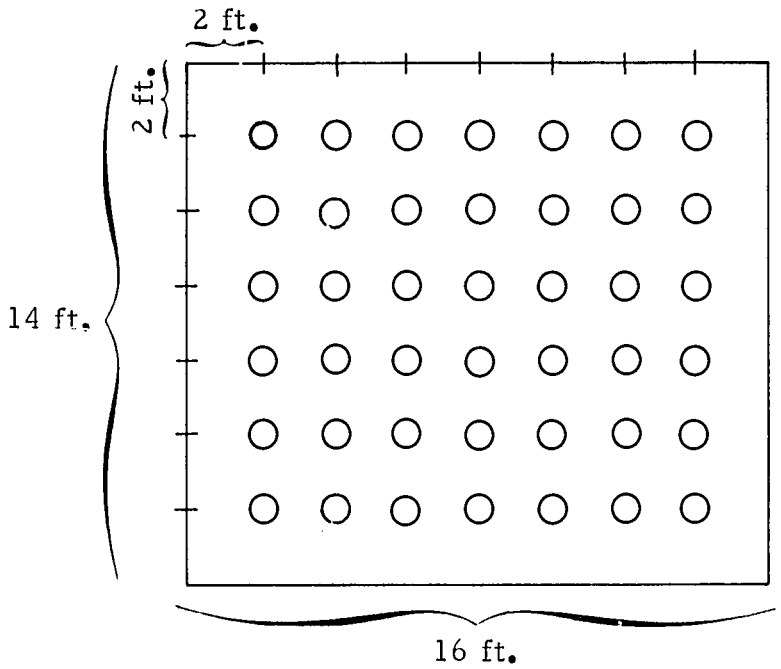
Previous Page Blank

If we compare C and D in the following picture, our eyes may tell us that they seem to have the same size. But this is not a sure way to tell. Sometimes we cannot see easily that one region

is larger than another or that two regions have the same size. Let us look at an example. A man is planning to plant tomato plants in a small garden. The rows are to be 2 feet apart and the plants in each row are to be 2 feet apart. The seeds are to be planted no nearer than 2 feet to the edges of the garden. There are two possible places for the garden. One is 20 feet long and 10 feet wide and the other is 16 feet long and 14 feet wide. Can he plant the same number of tomato plants in each garden? The "distance around" is 60 feet in each case. Does this mean that the two gardens have the same size? The following pictures will help you to decide.



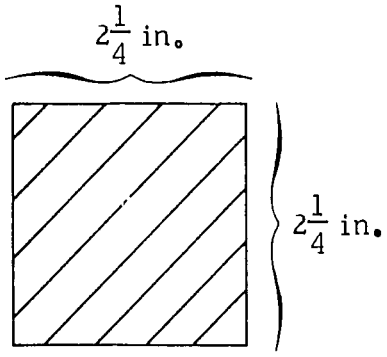
In this garden there will be 4 rows with nine plants in each row or 36 plants.



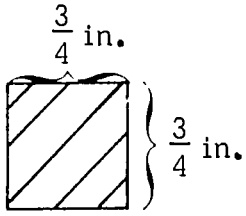
In this garden there will be 6 rows with 7 plants in each row, or 42 plants. You can see that the sum of the length and width does not tell you which is the larger garden.

6-2 Measurement of Area

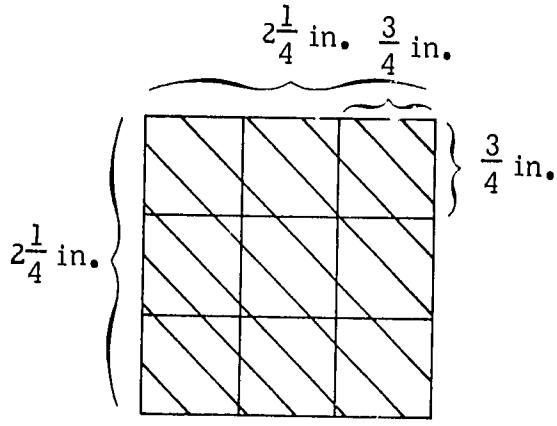
In order to measure the length of a line segment we use a unit which is a line segment. Suppose that we wish to measure the region inside the rectangle in the following picture.



Can we use a unit which is a line segment? Of course we cannot. We need a unit which is of the same nature as what we are measuring; that is, a unit which is itself a region, say something like this:

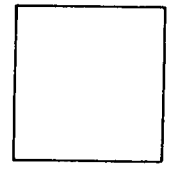


Then we can make many copies of this unit region and use them to cover the rectangular region.

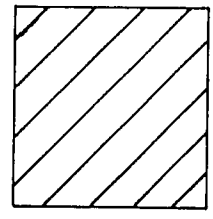


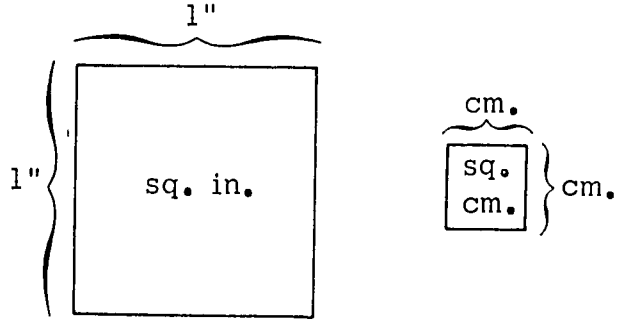
The result of our measurement in this case will be 9 units.

Before we can discuss further what units are used for measuring area, let us talk about the <u>square</u>. A square is a rectangle whose sides are all congruent. Since it is a rectangle the corners are all square corners, but a square is a special type of rectangle — the four sides are congruent.



The square together with its interior is called a square region. We use square regions as units for measuring area. If each side of the square is 1 inch in length, the unit is a "square inch". If each side of the square is 1 centimetre in length the unit is a "square centimetre". We also have the square foot, the square yard, the square metre, the square mile, etc.





Problems 6-2

1. Construct a unit square inch on a paper.

2. Construct a unit square centimetre on a paper.

3. Construct two rectangles, each 5 inches by 3 inches.

   (a) How many copies of the unit square inch can cover the rectangular region?

   (b) How many copies of the unit square centimetre can cover the rectangular region? What is the area of the region? Do you obtain an exact answer?

4. Construct a rectangle 4 inches by 2 inches. Mark square inch units along one side. How many are there? Mark square inch units along an adjacent side. How many are there? Without subdividing the whole rectangle, say what the area is in square inches.

5. What rule would you use to find the area of a rectangle?

6. Draw a rectangle 3 inches by 2 inches. Fit as many square centimetre units as possible in its region. How many can you fit in? Is 35 square centimetres the area of the rectangle? Use more square centimetres until you have covered the whole region. How many were necessary? Is 40 square centimetres the area of the rectangle? Make this sentence true with $<$ or $>$: The area of the rectangle is _____ 35 square centimetres and _____ 40 square centimetres.

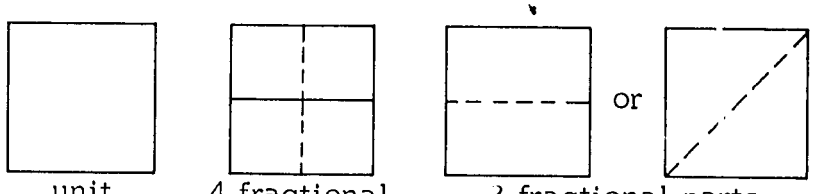
Fold each of the units which fell outside the rectangle into two congruent parts. Fit their halves and estimate the area of the rectangle. Measure the side of the rectangle in centimetres and apply the rule you stated in problem 5 to estimate the area.

The idea of the rule of problem 5 can be stated as follows: The area of a rectangle is the product of the length and the width, both measured in the same units. This area is measured in square units. It is easy to see that this rule works when the

length and width are whole numbers of units. The folding of the square centimetres in problem 6 should make it seem reasonable that we should use the same rule even when the sides of the rectangle are not measured in whole numbers of units.

From the above problems we can see two things.

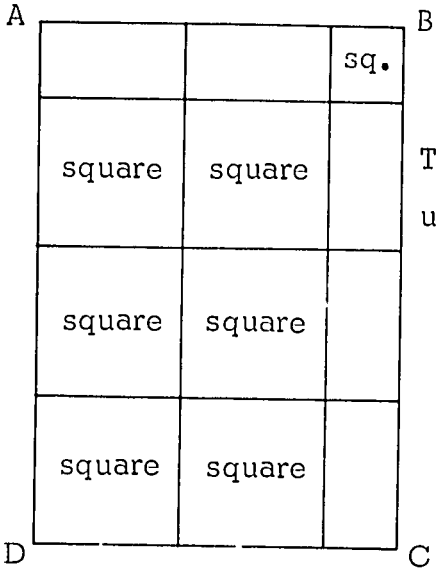
First: To help us to get the area of a region where the unit copies do not fit exactly, we can subdivide the unit and use fractional parts. See the following figures.



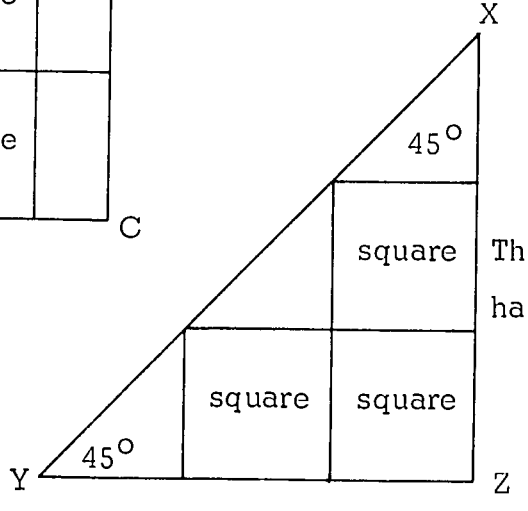
unit

4 fractional parts, each $\frac{1}{4}$ unit

2 fractional parts each $\frac{1}{2}$ unit



The area of ABCD is 6 units and 5 half units and 1 quarter unit = $8\frac{3}{4}$ units.

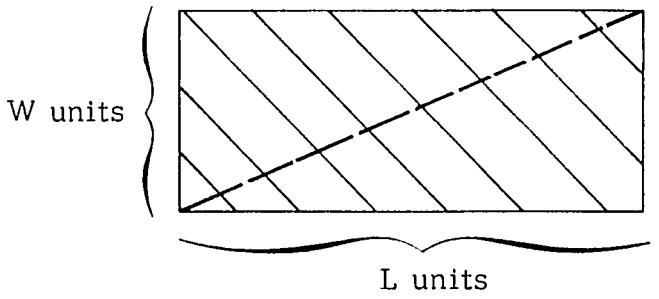


The area of ΔXYZ is 3 units and 3 half units = $4\frac{1}{2}$ units.

Second: We have a rule for finding the area of a rectangle. We measure the lengths of the adjacent sides. If the measures are L and W, then the measure of the area of the rectangle is $L \times W$.

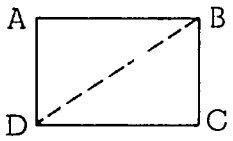
6-3 Measurement of the Areas of Right-angled Triangles

Now that we know how to find the areas of rectangles, we can use this knowledge to enable us to find the areas of right-angled triangles. Suppose that we have a rectangle of length L units and width W units as in the picture. We know that the area of this rectangle is $L \times W$ square units. If we draw a diagonal line from one corner to an opposite corner then we have subdivided the rectangle into two right-angled triangles. Can we find the area of each of these triangles without putting in as many unit squares as we can and then parts of unit squares as we did in problem 6 of the last exercise? We can do so if we notice that the rectangle has been cut in half by the diagonal line that we have drawn. If you were to make this rectangle out of paper and cut it with scissors, you would have the same total area as before. We see that the area of either of the two congruent triangles is half of the area of the rectangle.

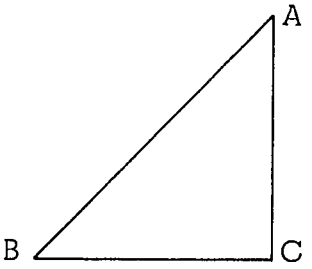


Problems 6-3

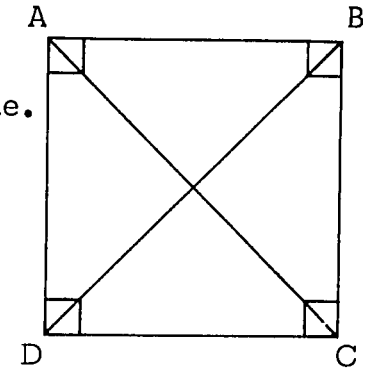
1. Draw a rectangle like ABCD which is 6 cm. by 4 cm. What is the area of the rectangle? Draw a diagonal and cut the rectangle along the diagonal into two parts. Compare the areas of the triangular regions ABD and CDB by fitting one on the other. Which is the larger area? Are they of the same size? What is the area of the triangle ABD? What is the area of the triangle BCD? How do you know that these triangles are right-angled triangles?



2. In the triangle, the angle of C is a right angle. What is the length of $\overline{AC}$? What is the length of $\overline{BC}$? What is the area of the triangle?



3. In a triangle PQR, the angle at Q is a right angle.
The length of $\overline{PQ}$ = 4 cm.
The length of $\overline{QR}$ = 6•8 cm.
What is the area of the triangle?

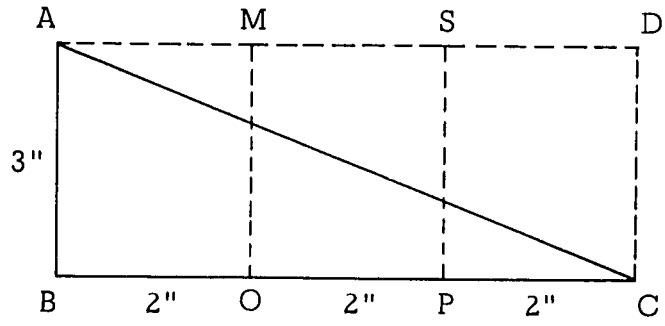


4. In a triangle XYZ, the angle at Y is a right angle.
The length of $\overline{XY}$ = 5•2 in.
The length of $\overline{YZ}$ = 8•9 in.
What is the area of the triangle?

5. Show that the area of ΔADC = area of ΔBDC.

6. Is area of ΔABC equal to area of ΔADC in problem 5? Why?

7.



Find the area of the rectangle ABCD.

(a) Is area of ΔABC equal to area of ΔADC? Find the area of ΔABC.

(b) Find the area of rectangle ABPS. Is area of ΔABP equal to area of ΔAPS? Find the area of ΔABP.

(c) Find the area of the rectangle ABOM. Is area of ΔABO equal to area of ΔAOM? Find the area of ΔABO.

8. Consider a given ΔABC, BC = 4 cm., AD = height = 2 cm. Draw a rectangle on the base $\overline{BC}$, whose area is 4 × 2 sq. cm. We notice that the area of ΔABC is equal to one half the area of this rectangle, with the same base and height.

Area of a triangle $= \frac{1}{2}h \times b$, where h = height and b = base.

9. Find the areas of the following triangles.

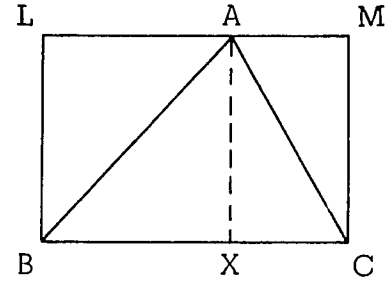
   (a) base = 9 cm., height = 4 cm.

   (b) base = 10 cm., height = 5 cm.

10. What is the area of ΔAOC in problem 7?

6-4 Measurement of the Areas of Non-right-angled Triangles

Consider the triangle ABC in which none of the angles is a right angle. We can think of it as standing on $\overline{BC}$ so we call $\overline{BC}$ the base of the triangle. Through A, draw $\overline{AX}$ perpendicular to $\overline{BC}$, and call AX the height or the altitude of the triangle. Now construct a rectangle with $\overline{BC}$ as one side and the point A on the opposite side. Is there a triangle in the picture which has the same area as triangle ABX? If so, name this triangle: _____. Is there a triangle in the picture which has the same area as triangle AXC? If so, name this triangle: _____.



Complete the following statements:

The area of triangle ABX is _____ of the area of rectangle _____.

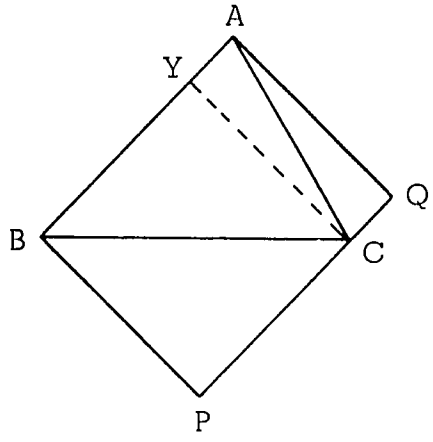
The area of triangle AXC is _____ of the area of rectangle _____.

The area of triangle ABC is _____ of the area of rectangle _____.

The area of rectangle LMCB = AX × B__.

Therefore, the area of triangle ABC = ___ × AX × B__.

What would happen if we considered another side of this same triangle, say $\overline{AB}$, as the base? If we do this, then we must draw a line segment $\overline{CY}$ through C which is perpendicular to $\overline{AB}$. Then CY is the height for this base. We can again construct a rectangle, but this time we use $\overline{AB}$ as one side and have the opposite side through C. We can ask the same kind of questions that we asked before. Is there a triangle which has the same area as triangle CBY? Is there a triangle which has the same area as triangle CAY? Complete the following statements:



The area of triangle CBY is _____ of the area of rectangle _____.

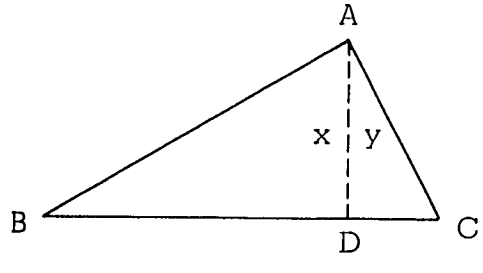
The area of triangle CAY is _____ of the area of rectangle _____.

The area of triangle ABC is _____ of the area of rectangle _____.

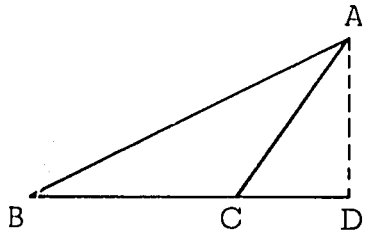
The area of rectangle PQAB = CY × A__.

Therefore, the area of triangle ABC = ___ × CY × A__. Can you see from the two cases that the areas of the rectangles LMCB and PQAB must be equal? Can you make the drawing and the argument, using the third side of the triangle as the base? Clearly the area of the triangle does not change when we use one side or another to call the base. So there are three ways of getting the area measure of a single triangle.

Look at the following drawings. Suppose that you know that the area of ΔABD is x square units and the area of ΔACD is y square units. What is the area of ΔABC?

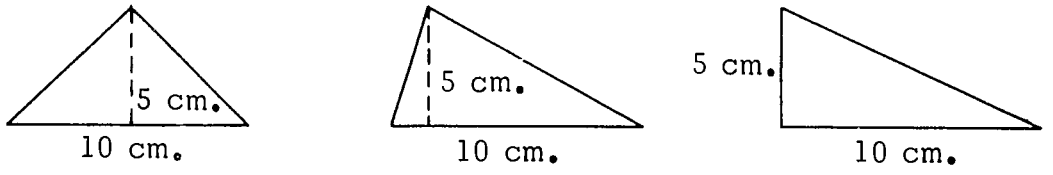


Look at the next drawing. Suppose that you know that the area of ΔABD is x square units and the area of ΔACD is y square units. What is the area of ΔABC?



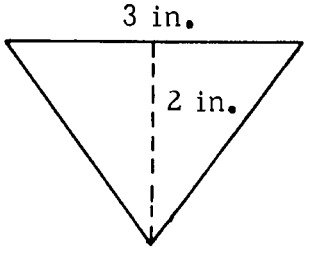
Problems 6-4A

1. Draw a triangle with base 10 cm. and height 5 cm. Find the area of this triangle. Can you draw another triangle of a different shape with this same base and same height? Will this triangle have the same area? Can you draw a right triangle with the sides of the right angle equal to 10 cm. and 5 cm.? What will the area of this triangle be?

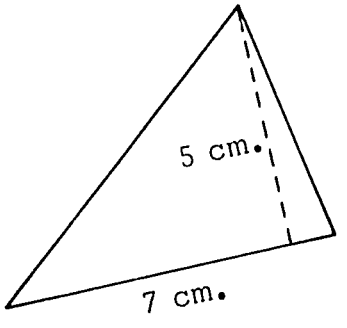


2. Draw triangles like those shown. Measure any lengths which are not given but which you need to find the area. Find the areas. Then for each triangle, use a different side as the base, make the necessary measurements and find the areas again. Do the results agree fairly well?

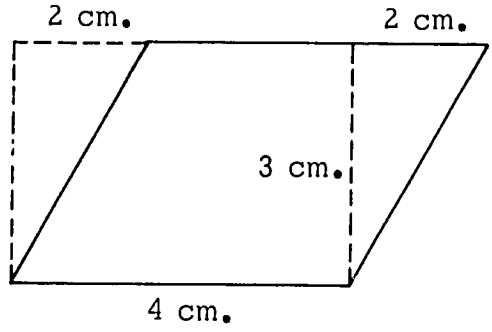
(a)



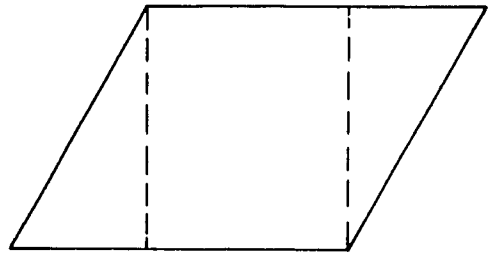
(b)



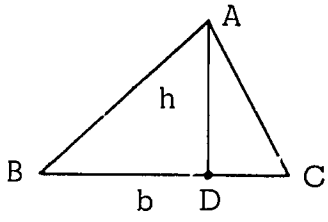
3. The picture shows a parallelogram. The dotted lines show a way of cutting it into two pieces and then putting the two pieces together in a way which will make a rectangle. What is the area of the parallelogram?



4. The picture shows the same parallelogram as in problem 3. Instead of cutting it and making a rectangle to find the area, find the area by finding the area of the rectangle in the centre and the areas of the triangles at the ends and adding all three areas. How does the answer compare with the answer to problem 3?



Consider the triangle ABC. Suppose that $\overline{AD}$ is perpendicular to $\overline{BC}$, BC = b units and AD = h units.



The area of ΔABC is the sum of the areas of right triangles ABD and ACD.

Area of triangle ABD $= \frac{1}{2}BD \times h$

Area of triangle ACD $= \frac{1}{2}DC \times h$

Area of triangle ABC $= \frac{1}{2}BD \times h + \frac{1}{2}DC \times h.$

Using the distributive property we have

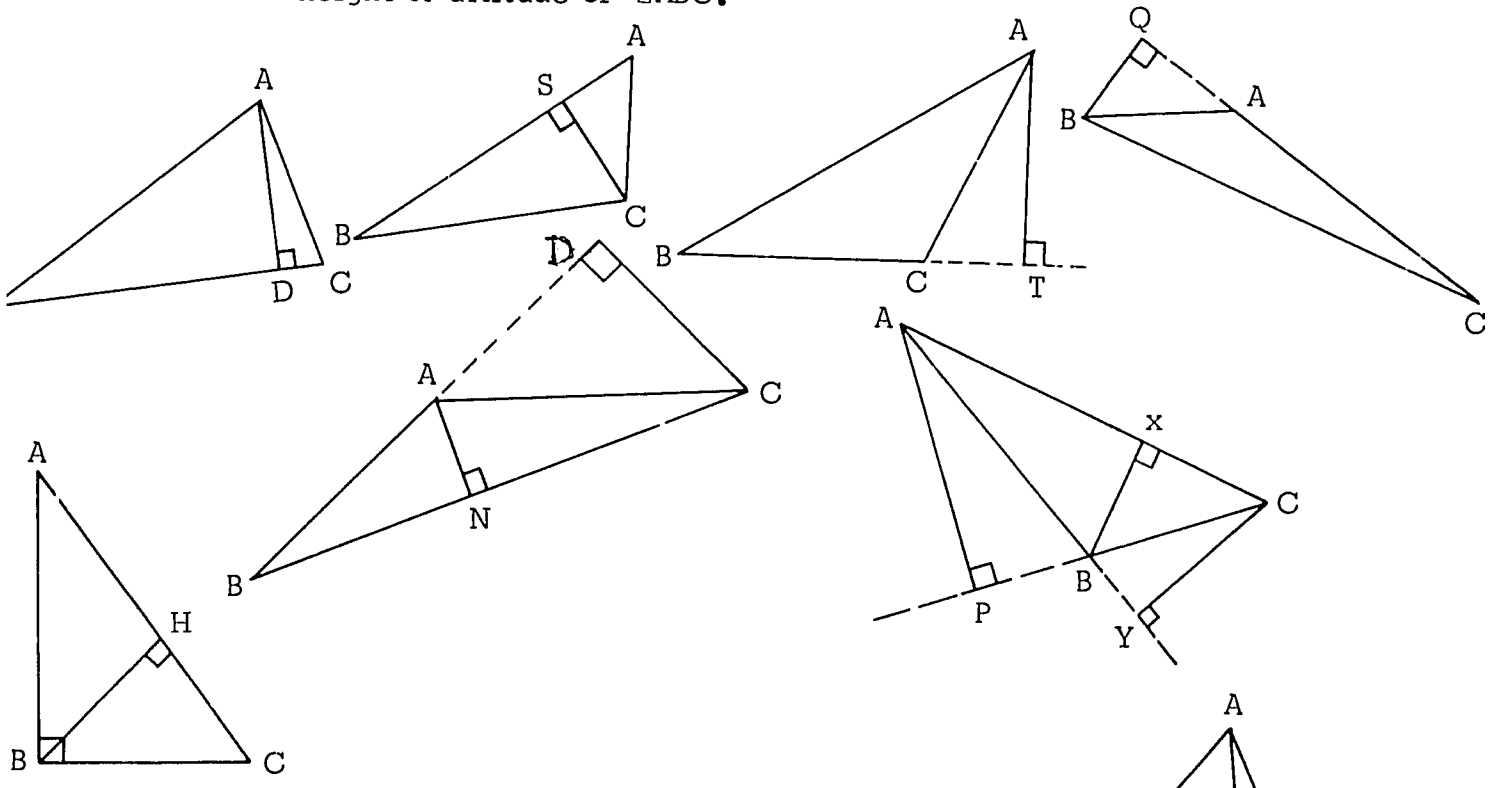
Area of triangle ABC $= \frac{1}{2}h\ (BD + DC)$.

But $BD + DC = b$ so we can write: Area of triangle ABC $= \frac{1}{2}h \times b$ or $\frac{1}{2}b \times h$. We can state in general

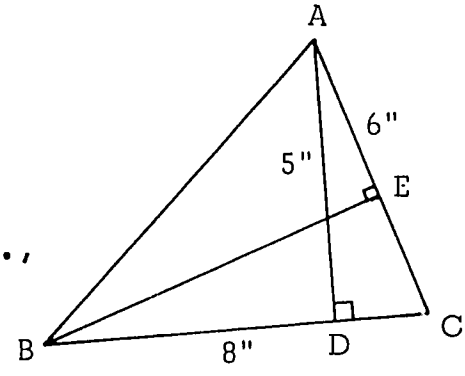
The area of a triangle is $\frac{1}{2} \times$ base $\times$ height.

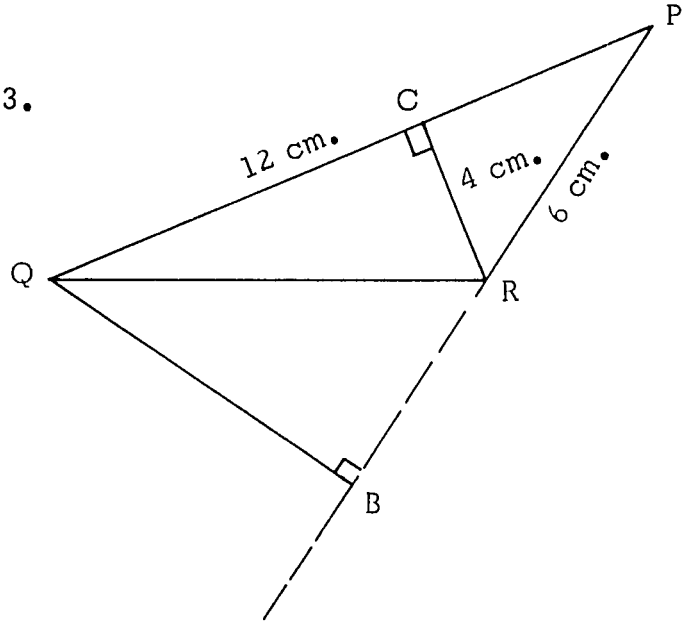
Problems 6-4B

1. In the following figures write down the base and the corresponding height or altitude of ΔABC.



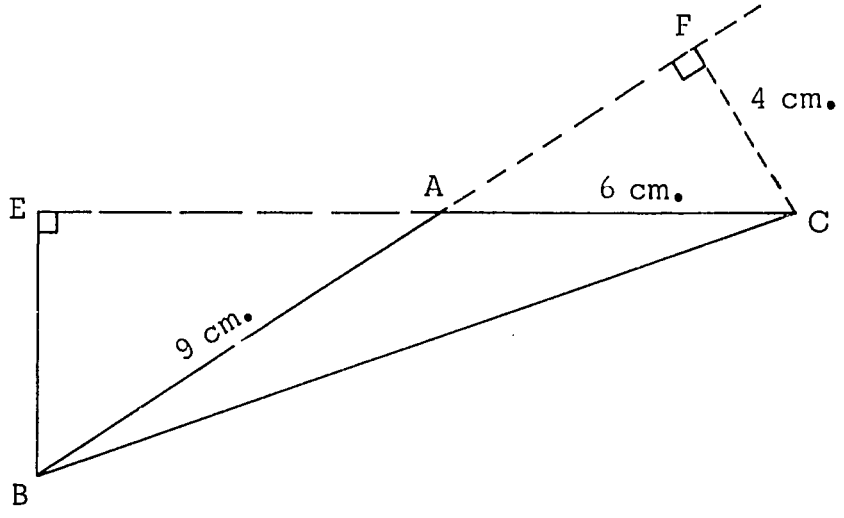
2. In the given figure, $\overline{AD}$ and $\overline{BE}$ are altitudes of triangle ABC. If BC = 8 in., AC = 6 in., AD = 5 in., find BE.



3. 

In the given figure, $\overline{CR}$ and $\overline{QB}$ are altitudes of ΔPQR. If PQ = 12 cm., PR = 6 cm., CR = 4 cm., find QB.

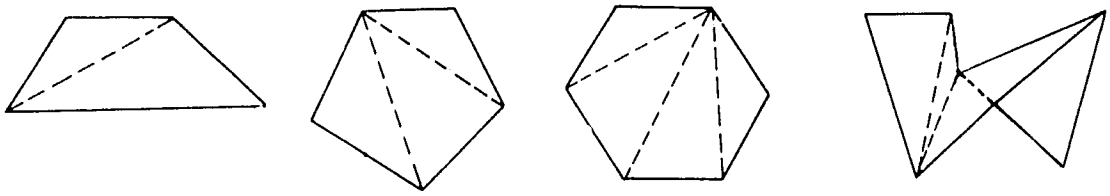
4. In the given figure, $\overline{BE}$ and $\overline{CF}$ are altitudes of ΔABC. If AB = 9 cm., AC = 6 cm., CF = 4 cm., find BE.



6-5 Areas of Other Polygons

Do we have enough information about area to be able to find the area inside closed polygons like those shown? What we do know about area? We know how to find the area of a rectangle and we know how to find the area of any triangle if we can make the necessary measurements of length. How will this help us? As the dotted lines show, you can cut all of the regions inside

these polygons into triangular pieces. Then, making the needed measurements you can find the area for each piece. Adding all

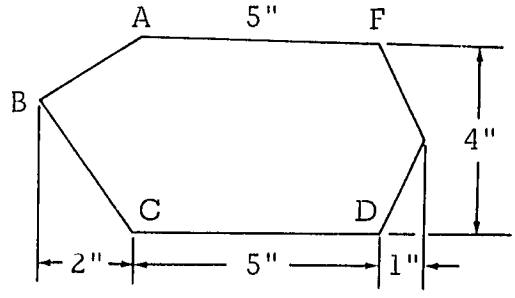


these areas together will give the area inside each polygon. We see, in general, that the area of any polygonal region can be obtained in the following way:

> Divide the region into triangular regions and find the area of each region. The area of the polygonal region is the sum of the areas of the triangular regions.

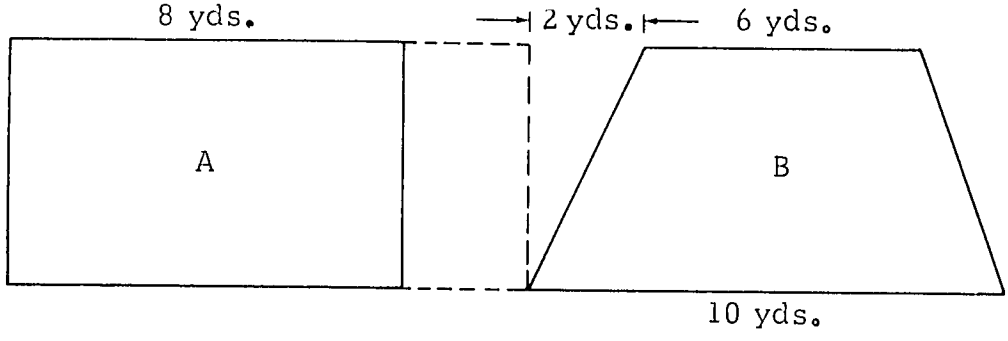
Problems 6-5

1. Divide the inside of this polygon into three regions, one rectangular and two triangular, and find the area of the region enclosed by the polygon. (It is given that $\overline{AF}$ is parallel to $\overline{CD}$ and $\widehat{ACD}$ is a right angle.)



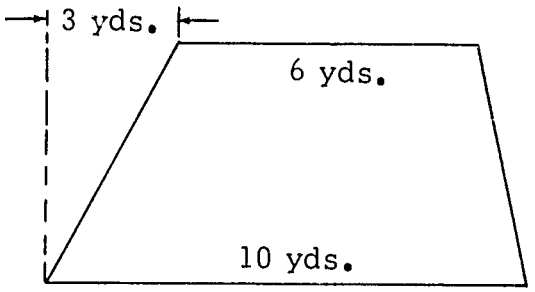
2. Find the area of the region in the picture of Problem 1, by dividing it into four triangular regions. How do the answers compare?

3. The picture shows two flower plots in a school. The teacher asked Okello to look after plot A and Bassey to look after plot B. Which pupil has the larger plot to look after?

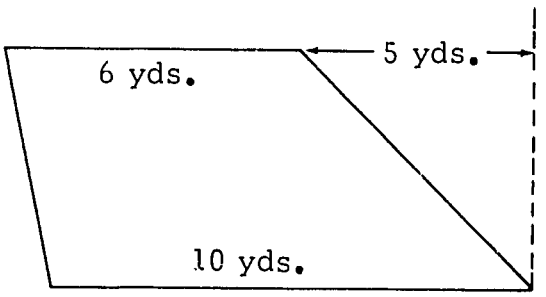


4. Would it change the area of plot B if the width were unchanged but the plot was shaped like (a) below? Like (b) below?

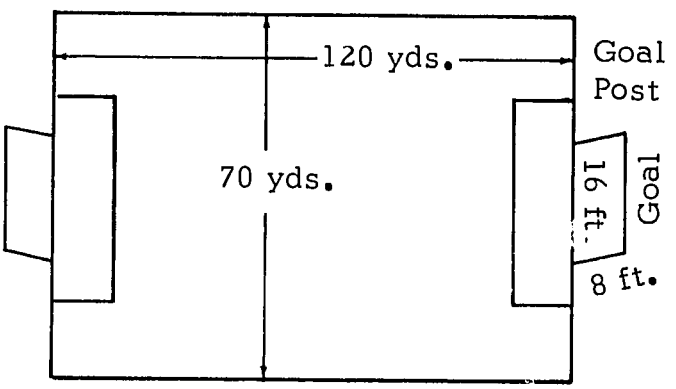
(a)



(b)



5. Here is a football field. What is the area of the goal? What is the area of the field?

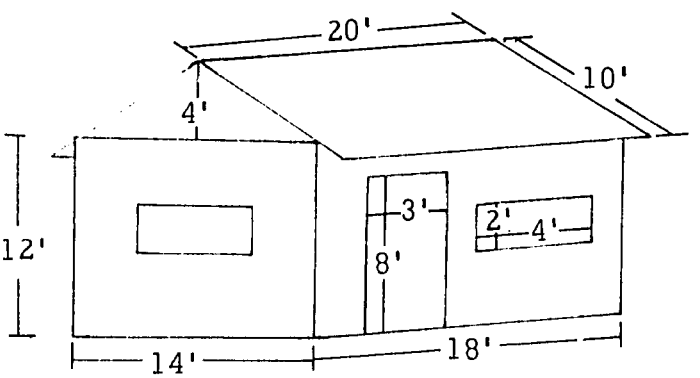


6. This house has three congruent windows and one door. What is

(a) the total area of the roof?

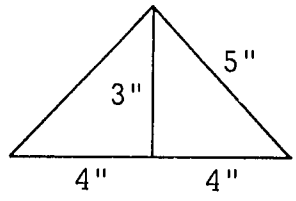
(b) the total area of the door and windows?

(c) the total area of the walls, excluding door and windows?



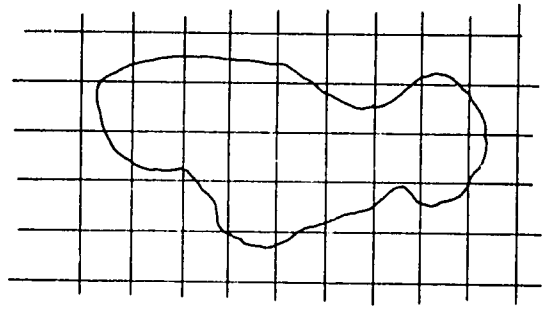
7. Find the surface area of a prism whose base is a triangle 3 by 4 by 5 cm. and height 6 cm.

8. Find the surface area of a pyramid on a square base of side 8 inches if each of the other edges is 5 inches. The perpendicular distance from the vertex to each edge of the base is 3 inches.



6-6 Areas of Other Regions

Suppose that we have a region which is not enclosed by a polygon, say an irregular region like the one shown in the next picture.



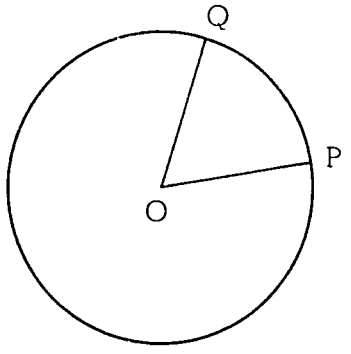
Is there any way to find the area enclosed? Since our ability to find area by using rules extends only to rectangles and triangles and to regions which can be divided into rectangles and triangles, we really cannot hope to get a simple rule to find such an area as this one. We can, of course, see how many unit squares we can fit entirely inside and then how many unit squares we need to cover the region completely. This will tell us that the area is larger than one whole number of units (the number of unit squares which will fit entirely inside) and less than another whole number of units (the number of unit squares you need to cover the region completely). We can also subdivide unit squares and

try to see how many half units we can put inside and how many we need to cover. If we used quarter units we might come closer still to the area.

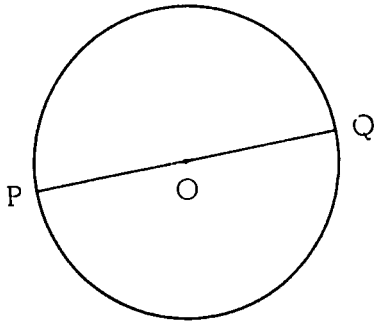
The Circle

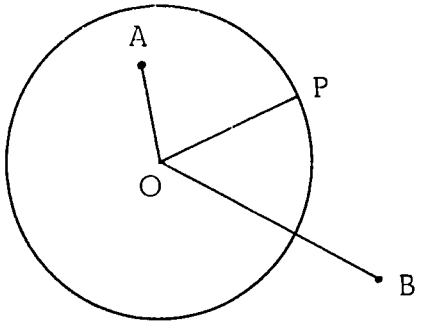
The circle is an interesting geometric figure. You can draw one as follows. Tie the ends of a piece of string together to form a loop. Then put a tack into a piece of paper resting on your desk. With one end of the loop around the tack, use a sharp pencil to pull the loop tight. Keeping the string tight, the point of the pencil will trace a circle with centre O at the tack.

As the pencil moves from a point P to a point Q, its distance from O remains unchanged. For example, in the figure, OQ = OP. Any position of the string, $\overline{OP}$, is called a radius of the circle.



In the circle on the right, the radius $\overline{OP}$ and the radius $\overline{OQ}$ lie along the same line segment $\overline{POQ}$. Such a line segment is called a diameter of the circle. Its length is twice the length of a radius. So if d is the length of a diameter and r the length of a radius, then d = 2r.





Consider the circle above. The length OA is clearly less than r. We say that the point A is in the interior of the circle. Also, it is clear that the length OB is greater than r. Such a point as B is said to be exterior to the circle. Of course, for a point P on the circle, the length OP is equal to r.

As you know, we call a segment like $\overline{OP}$ a radius of the circle. But sometimes we also call r (the length of $\overline{OP}$) the radius of the circle. For example, we can speak of a circle of radius 3 inches.

Problems 6-6

1. Use the drawing above of the irregular-shaped region for the following questions:

   (a) How many square units are entirely in the region?

   (b) How many square units completely cover the region?

   (c) The area enclosed is less than _____ square units and greater than _____ square units.

   (d) Using half squares, estimate the area to the nearest half square.

   (e) Using quarter squares, estimate the area to the nearest quarter square.

2. Draw a circle whose radius has length 4 inches.

   (a) How many square-inch units will fit completely inside?

   (b) How many square-inch units will you need to cover it completely?

   (c) Estimate the area of this circle as closely as you can.

3. Repeat Problem 2, using square centimetre units. Estimate the area of the circle in square centimetres.

4. Construct a circle, whose radius has length 2 inches, on graph paper with thick and thin line rulings.

   (a) Use the square regions formed by the thick lines as units to estimate the area of the circle.

4. (b) Use the square regions formed by the thin lines as units to estimate the area of the circle.

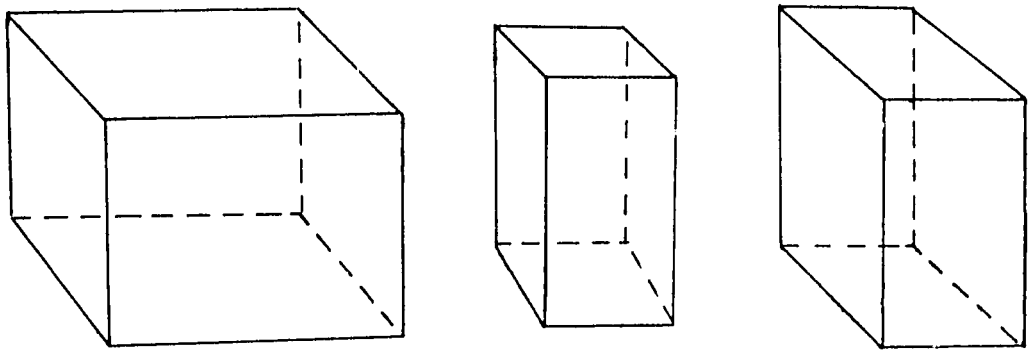
Which of the two estimates do you think is a closer approximation to the area of the circle?

CHAPTER 7

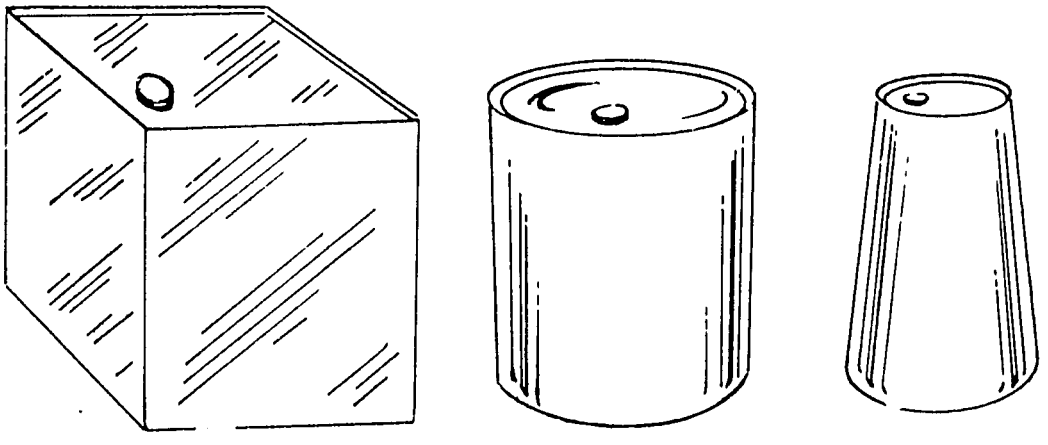
VOLUME

7-1 Measurement of Volume

The pictures show some empty boxes. Suppose that you were going to the market to buy rice. Which box would you take to carry as much rice as possible?

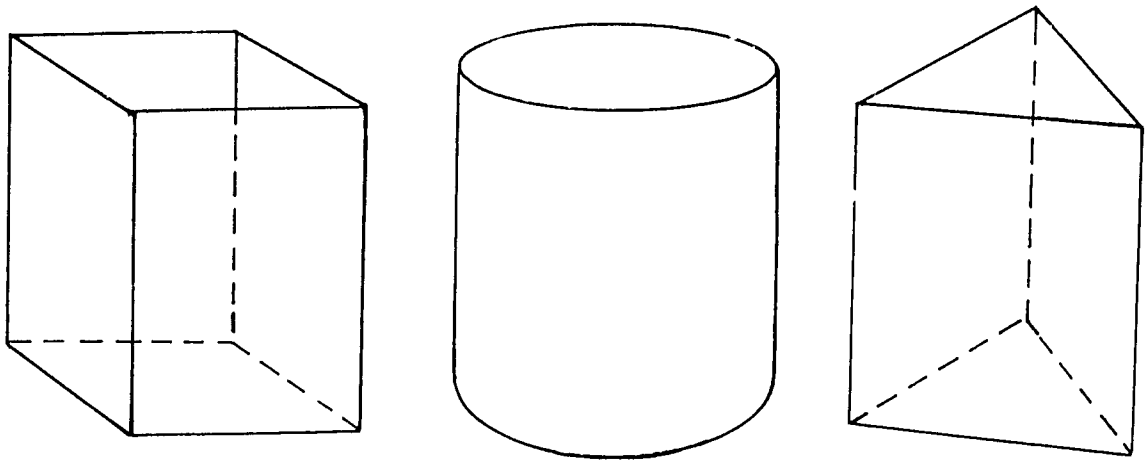


Opara, Mary and Anita are going to draw water. Opara is to carry the greatest quantity of water, Anita the least. Which of the vessels below should each use?



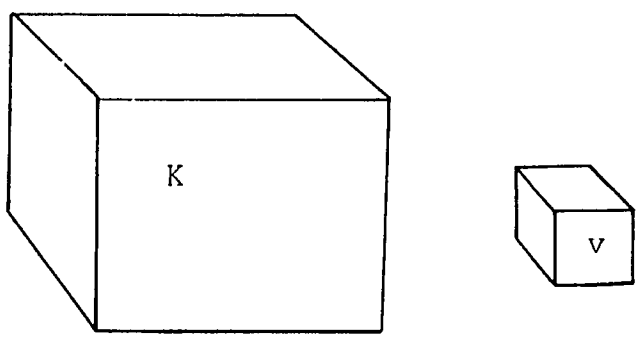
Have you thought why you made these choices? Can you see that the spaces enclosed by these vessels have different sizes?

Here are pictures of three containers. Can you say which of them encloses the greatest amount of space?

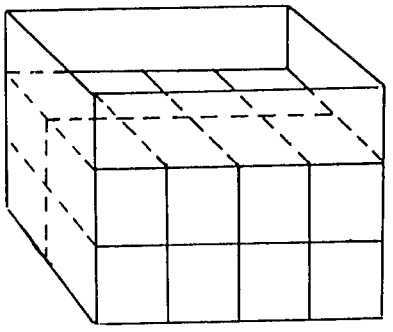


In order to compare the sizes of spaces enclosed in solid figures we need a unit of measurement. Using that unit we shall say that if two figures have the same measure of space enclosed then these figures have equal <u>volumes</u>.

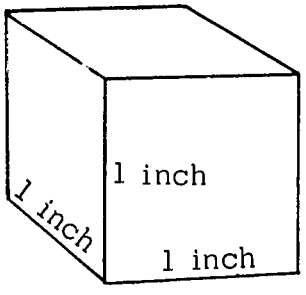
We have already seen that the unit for measuring something must be of the same nature as what is being measured. So to measure a figure in space we must also have a space unit. Suppose that we want to measure the volume of the box K.

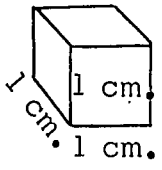


We may choose a unit v which is a cube. Then we find how many of these units must be used to fill the space in K.

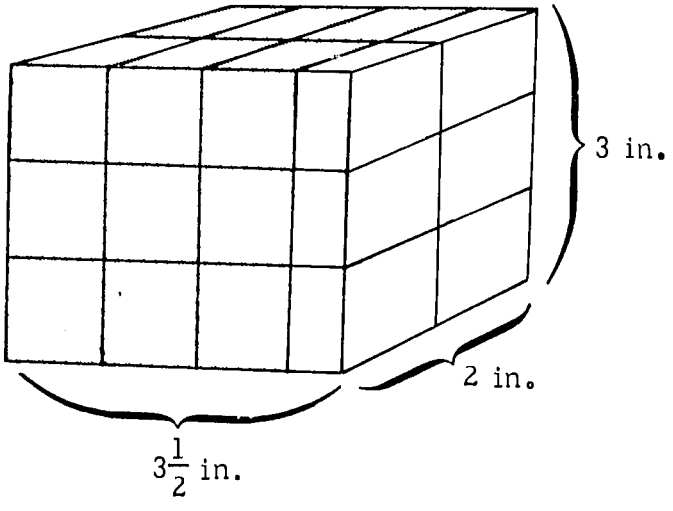


This number gives the measure of K with v as a unit. In practice we often use a cube whose edges are each one inch as a unit. We may also use a cube whose edges are 1 centimetre as a unit. We call the first unit a <u>cubic inch</u> and the second unit a <u>cubic centimetre</u>.

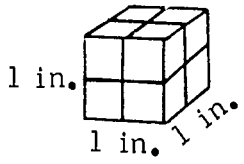




Suppose that we have a box which is $3\frac{1}{2}$ inches long, 2 inches wide, and 3 inches high. We cannot fill this box with unit cubes. If we try, we find that we can get 18 cubes in but they do not fill the box.



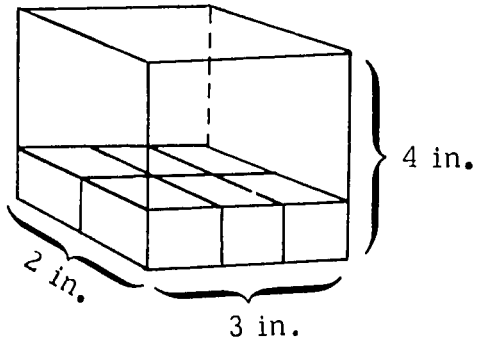
If we try to put six more blocks on the right they will not fit within the box. We can solve this problem by using a smaller unit cube. Begin with our inch cube and divide each edge in half. As you can see in the drawing, this divides the cube into 8 equal smaller cubes, each $\frac{1}{8}$ of the original, that is, each is $\frac{1}{8}$ cubic inch in volume. If we try to fill the box with these smaller cubes we find that we can do so with 18 unit cubes (which will be 18×8 or 144 small cubes) plus 24 more small cubes or a total of 168 small cubes. This means that the volume of the box is 168 ($\frac{1}{8}$ cubic inches), or $\frac{168}{8}$ or 21 cubic inches. You can see that the measure of this volume is "length times width times height" ($3\frac{1}{2} \times 2 \times 3$). (The 168 is the result of using a $\frac{1}{2}$ inch unit for length so that $3\frac{1}{2} \times 2 \times 3$ cubic inches is expressed as $7 \times 4 \times 6$ one-eighth cubic inches or 168 one-eighth cubic inches.)



If we have a box which we cannot fill with these smaller cubes, we can subdivide these again. If we cut each edge in half again we shall get a cube which has volume $\frac{1}{64}$ cubic inches. We can agree that it seems reasonable to expect the volume of a box to be (length $\times$ width $\times$ height) no matter what numbers we get from the measurement of length, width and height.

Suppose that we have a box with dimensions 2 by 3 by 4 inches. Then to find the volume, we find how many of the unit cubes can be used to cover the set of points of the interior.

The volume of this figure is 24 cubic inches. You need 24 of the unit cubes to fill the box.



In the same way, for any rectangular box the volume is given by the formula "$V = l \times w \times h$" where l, w, and h are length, width, and height.

Problems 7-1

1. Find the volume of a rectangular box when

(a) $\ell = 2$ cm.
$w = 3$ cm.
$h = 5$ cm.

(b) $\ell = 3$ ft.
$w = 4$ ft.
$h = 5\frac{1}{2}$ ft.

2. Find the length of a rectangular box when the volume = 30 cubic ft., $w = 5$ ft., $h = 2$ ft.

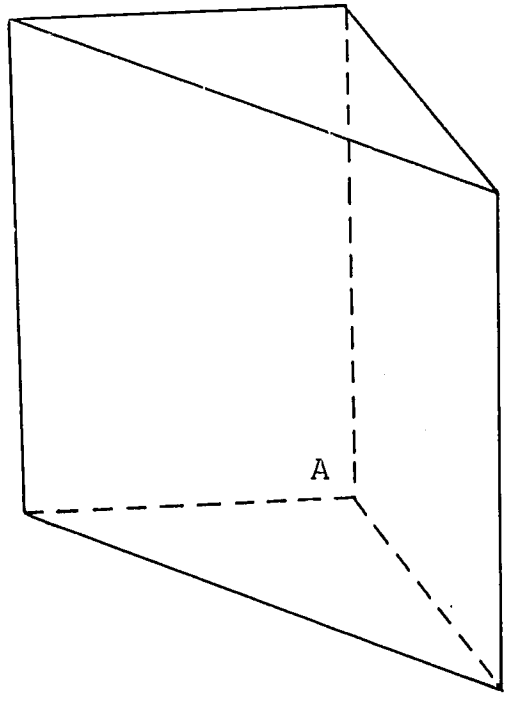
Of course other units can be used. A unit cube with each edge equal to 1 cm. will have a volume equal to 1 cubic cm. A cubic yard and other units can be similarly defined.

3. (a) How many cubic inches are there in 1 cubic foot?

(b) How many cubic feet are there in 2 cubic yards?

4. A rectangular prism has length equal to 4 feet, width equal to 2 feet, height equal to 3 feet. Find the volume.

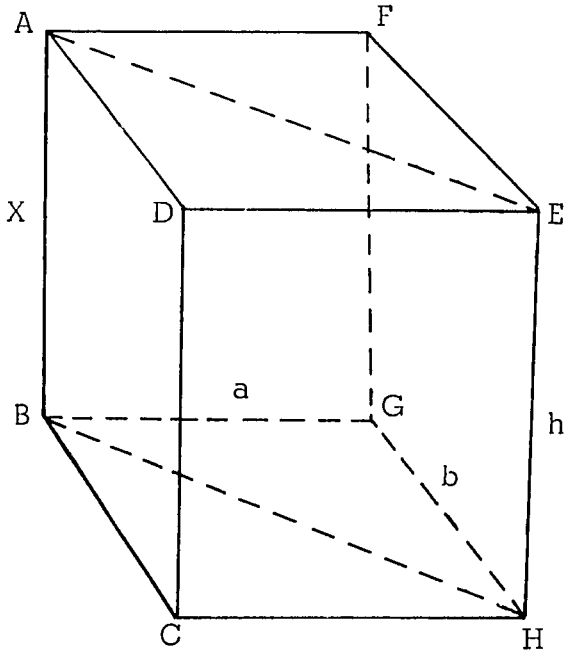
7-2 Volumes of Prisms

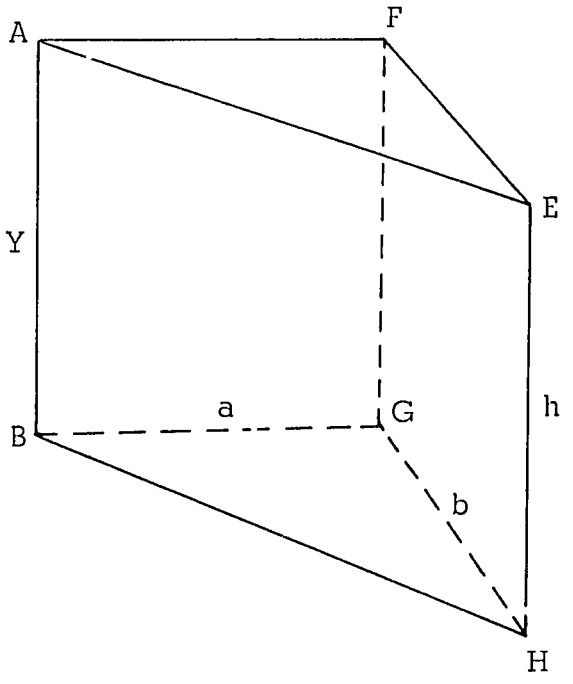


Here is a triangular prism. The base is a right-angled triangle with the right angle at A. Suppose that we want to find its volume. Can we fill the enclosed space with unit cubes? (A triangular prism has congruent triangles for top and bottom and all the side faces are rectangles.)

How accurate would the measurement be if we tried to fill the enclosed space with unit cubes?

Now consider the pictures below. X is a rectangular prism. (A rectangular prism has a rectangle for top and bottom as well as for all the side faces.) The prism X is split in two along the plane ABHE; the result is two triangular prisms, one of which is shown as Y. Compare the volume of X and that of Y.



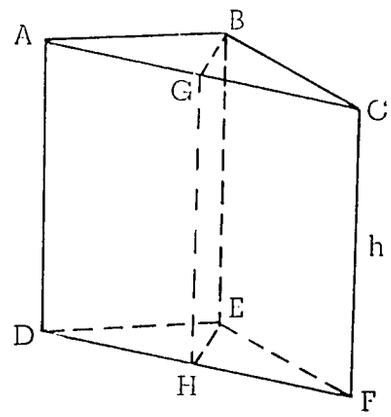


The volume of Y is one half the volume of X. The volume of X is $a \times b \times h$, so the volume of Y is $\frac{1}{2} \times a \times b \times h$ which can be written as $(\frac{1}{2}a \times b) \times h$.

The area of the right-angled triangle which is the base of this prism is $\frac{1}{2}a \times b$, so we know that for a triangular prism with a right-angled triangle as base,

$$\text{volume} = (\text{area of base}) \times \text{height}.$$

Suppose now that we have a prism with a triangular base which is <u>not</u> a right-angled triangle. How shall we find its volume? If we draw $\overline{EH}$ perpendicular to $\overline{DF}$ and $\overline{BG}$ perpendicular to $\overline{AC}$ and join G to H we divide our prism into two prisms. Suppose V is the volume of the prism with triangle DEF as base, Y is the volume of the prism with triangle DEH as base and Z is the volume of the prism with triangle EHF as base. Then we know that



$$V = Y + Z.$$

But since the triangle DEH is a right-angled triangle (why?) and the triangle EHF is a right-angled triangle (why?) we know that

$$Y = (\text{area of triangle DEH}) \times h$$

$$Z = (\text{area of triangle EHF}) \times h$$

and we can write

$$V = \left[(\text{area of triangle DEH}) \times h\right] + \left[(\text{area of triangle EHF}) \times h\right].$$

Using the distributive property we can write

$$V = h \times \left[(\text{area of triangle DEH}) + (\text{area of triangle EHF})\right].$$

We know that

$$(\text{area of triangle DEH}) + (\text{area of triangle EHF}) = \text{area of triangle DEF}.$$

Thus we have

$$V = h \times (\text{area of triangle DEF})$$

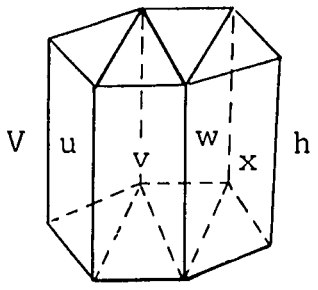
or

$$V = (\text{area of triangle DEF}) \times h.$$

In words this says that

The volume of any triangular prism is the product of the area of the base and the height.

What about a prism with some other polygon as its base? In the picture below you can see how such a prism can be divided into triangular prisms.



Since the volume of the entire prism is the sum of the volumes of the triangular prisms we can write

$$V = u + v + w + x.$$

For each of the triangular prisms the volume is the product of the area of the base and the height. Suppose the prisms of volume u, v, w, and x have bases of area a, b, c, and d. Then $u = a \times h$, $v = b \times h$, $w = c \times h$ and $x = d \times h$. Thus we can write

$$V = (a \times h) + (b \times h) + (c \times h) + (d \times h)$$

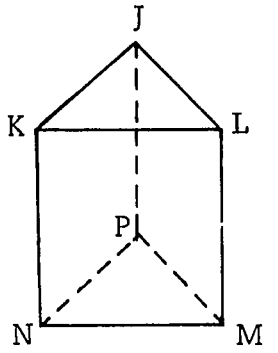
or

$$V = h(a + b + c + d).$$

But $(a + b + c + d)$ is the area of the base of the entire prism. Thus we find for prisms which have triangles as bases and for those which have other polygons as bases we get the same rule.

volume = (area of base) × height

Problems 7-2



1. Find the volume of the triangular prism if JP = KN = LM = 6 inches, PN = 4 inches, PM = 3 inches. $\widehat{NPM}$, $\widehat{KNP}$, $\widehat{KNM}$ are right angles.

2. Find the volume of the triangular prism shown above if

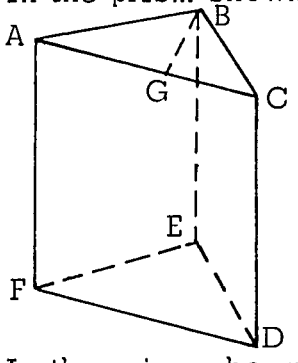
   JP = h centimetres = KN = LM

   PN = a centimetres

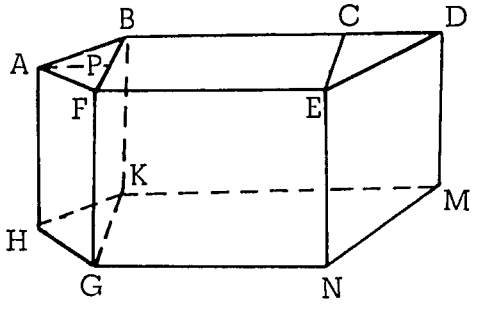
   PM = b centimetres

   $\widehat{NPM}$, $\widehat{KNP}$, $\widehat{KNM}$ are right angles.

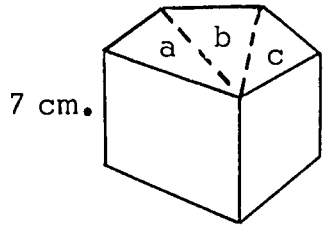
3. In the prism shown $\overline{BG}$ is perpendicular to $\overline{AC}$. If BG = 3 cm., AC = 10 cm., CD = 11 cm., what is the volume of the prism?



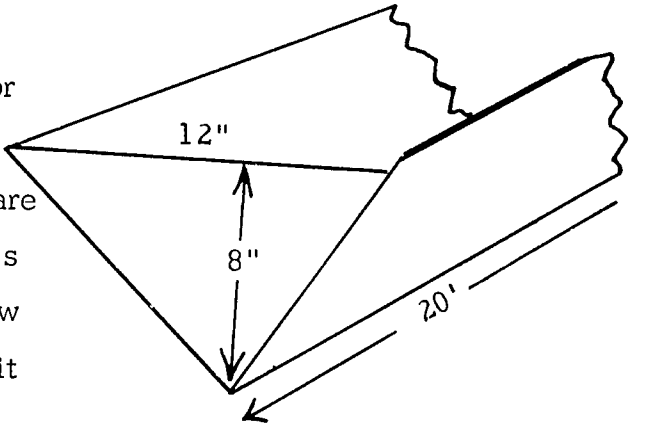
4. In the prism shown BCEF is a rectangle with BC = 9 inches and BF = 3 inches, CD = 4 inches, $\overline{AP}$ is perpendicular to $\overline{BF}$, AP = 3 inches, and DM = 8 inches. Find the volume of the prism.



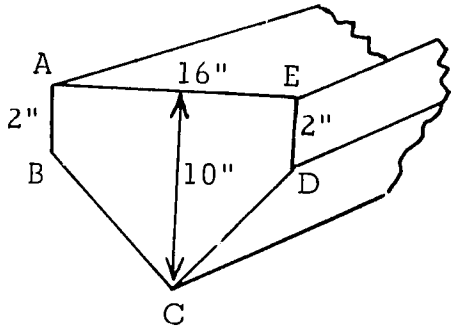
5. In the prism shown, the area a is 5 square cm., b is 4 square cm., and c is 3·5 square cm. What is the volume?



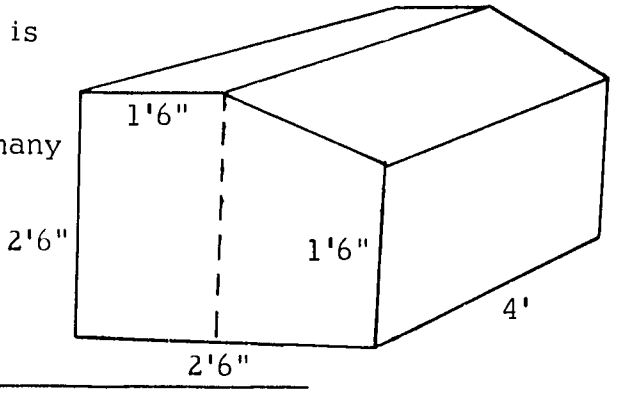
6. A horizontal drinking trough for cattle has a triangular cross-section whose measurements are as shown in the figure (' means feet and " means inches). How many cubic feet of water will it hold when full?



7. Repeat Problem 6 for a trough whose cross-section is as shown if $\overline{AB}$ and $\overline{ED}$ are vertical and $\overline{AE}$ is horizontal and the trough is 30 feet long.

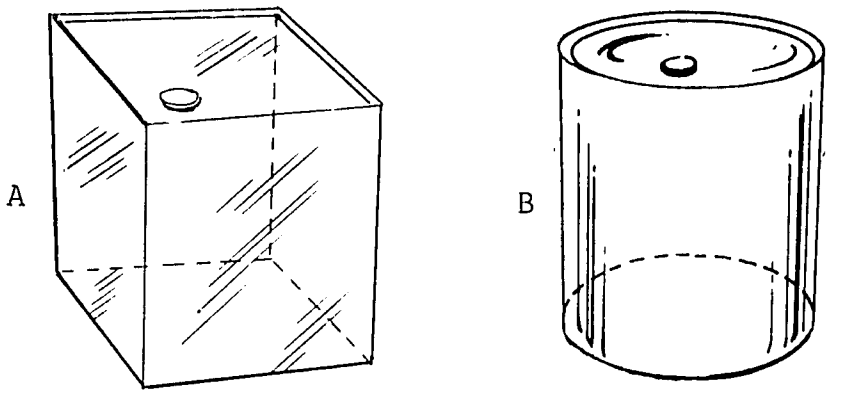


8. In a certain boarding school, maize meal or kitchen flour is stored in bins like the one shown in the figure. How many cubic ft. can be stored in 5 such bins?



7-3 A Direct Measurement of Volume

One of the ways of comparing volumes of empty vessels is by filling one with water and then transferring the contents to the other, if possible. For example if you fill A with water and then transfer the water to empty B and B is filled without any water remaining in A then you would say that A and B have the same volume. If B is full before A is empty then the volume



of A is greater than the volume of B; if A is empty before B is full then the volume of B is greater than the volume of A.

Now if the can X is the unit for measurement of volume then we can find how many times we would fill X with water and transfer to B, beginning with B empty, in order to fill B. This will give us the measure of the volume of B with X as unit.

For example, when you go to the market to buy rice, beans, peas, etc., how many cigarette tins would fill your bucket?

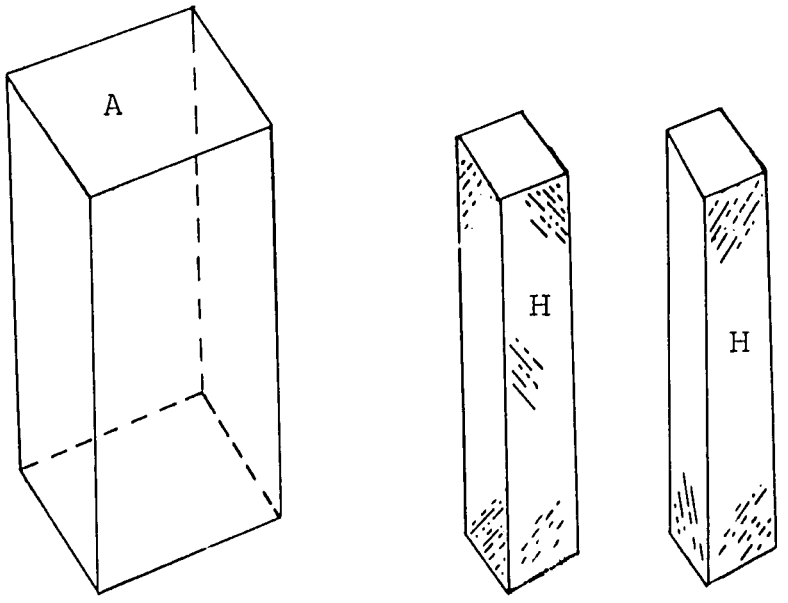
Your teacher can show you other unit vessels such as pints, gallons, etc.

CHAPTER 8

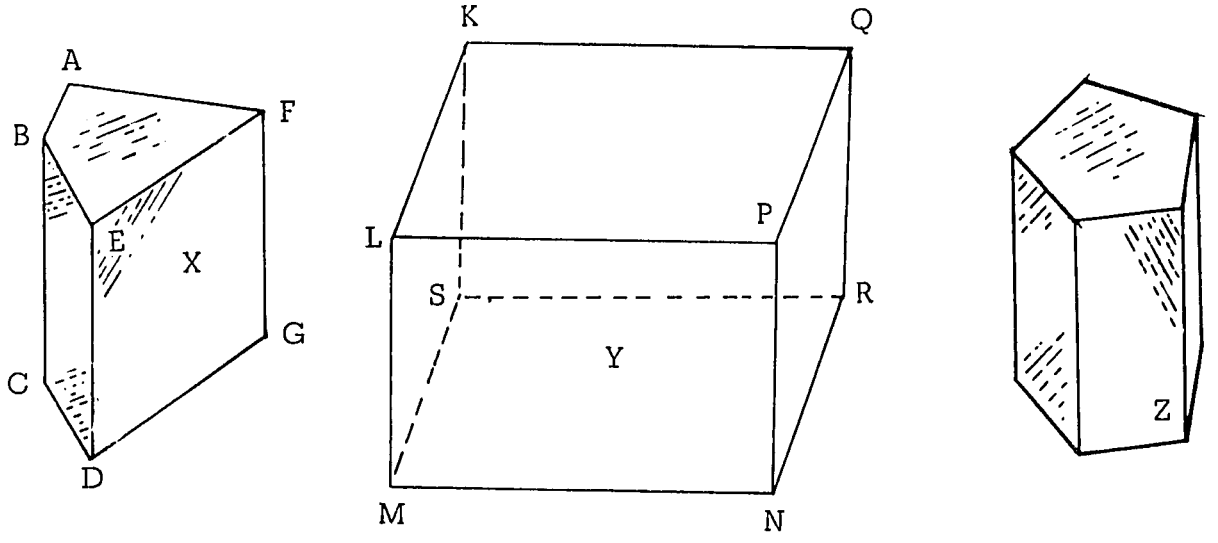
ANGLE MEASURE

8-1 Measurement of Angles

Here is a box A; we wish to pack blocks like those labeled H in it. In order to pack as many blocks as possible we would like to get the edge of a block along the edge of the box so that the faces will be against one another without leaving gaps. Can you do this, packing blocks H into box A?



Will this kind of packing be possible for every shape of block? Look at the block X and the empty container Y.

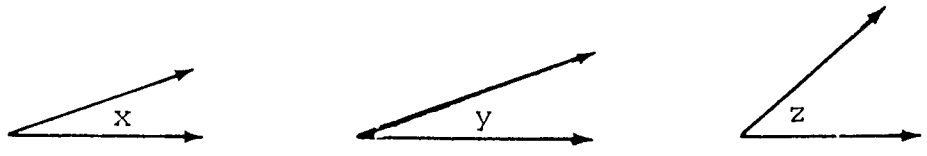


Previous Page Blank

Can block X be fitted into container Y so that edge $\overline{BC}$ lies along $\overline{LM}$? Give reasons for your answer. Can block X be fitted into Y so that the edge $\overline{FG}$ lies along $\overline{QR}$? Can block Z be fitted in so that any of its edges lie along any edge of the box Y?

The above exercise makes us realize that at times we need to compare the sizes of angles, or even to measure them.

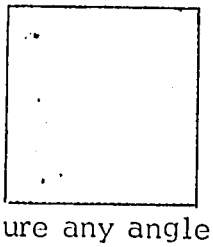
Here are three angles x, y and z.



Copy x by tracing it on semi-transparent paper and lay the copy on y. Are they congruent? If we find a way of measuring angles, will they have the same measure? Now lay the copy on z. Are x and z congruent? If not, which is the greater angle? If we found a way to measure angles, would these two have the same measure? If not, which would have the greater measure?

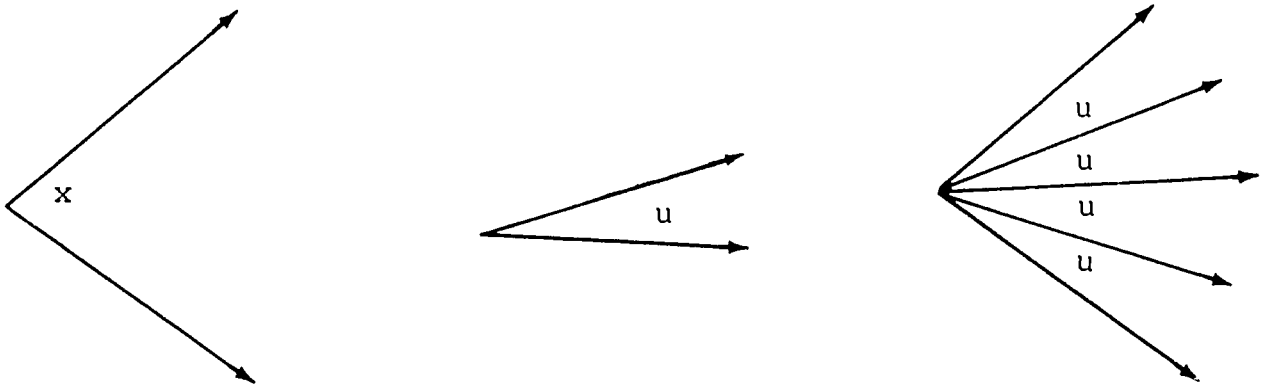
In order to find a unit for the measurement of angle, let us look again at the types of units we have used for other measurements. To measure a line segment we used a unit which is a line segment, for example, the inch •————• , or the centimetre •——•.

An angle is a pair of rays with a common endpoint. Can we use a line segment to measure it?

We also found that we could not use a line segment to measure a region. To measure a region we had to use a unit which is a region, for example, the square inch  or the square centimetre .

To measure any angle, what type of unit do you think we should have? To measure anything we must use a unit which is of the same kind as the thing to be measured. So to measure an angle we must use a unit which is an angle.

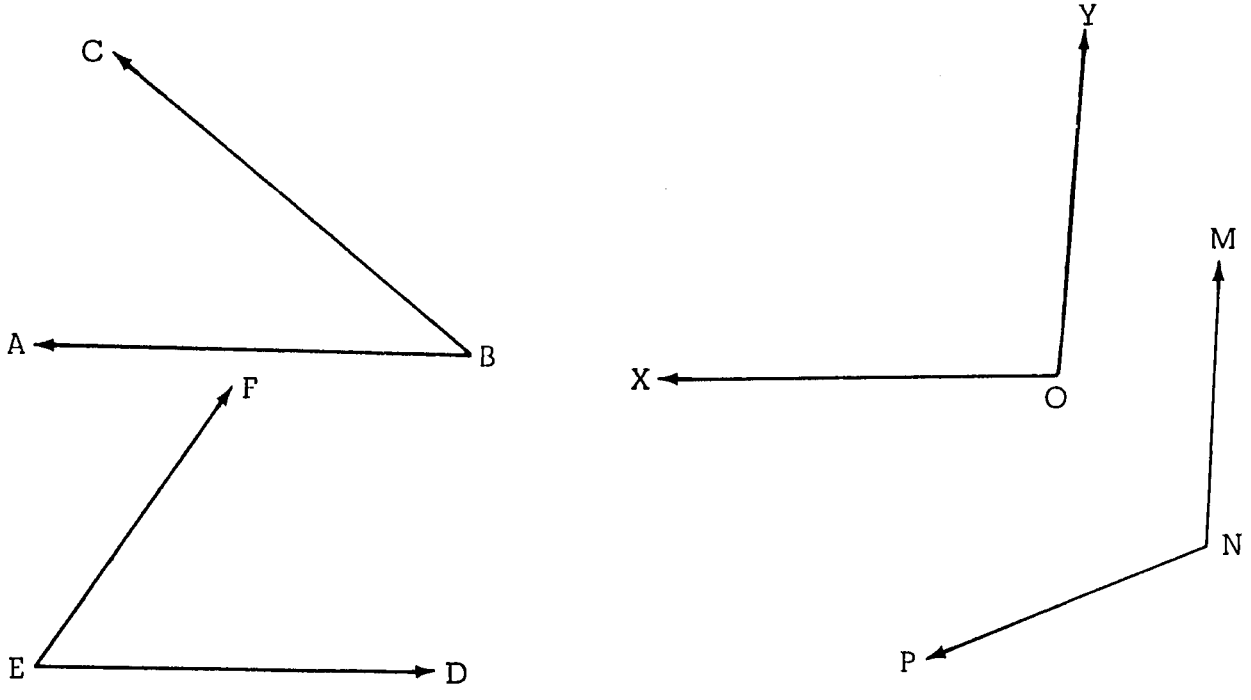
If we want to measure the angle x, we may choose the angle u as the unit angle for measurement. Then we find out how many copies of the angle u we need to make up the angle x.



If we need four copies, as shown in the picture, then the measure of angle x is 4, with u as the unit.

Problem 8-1

Cut out a unit of measure like the one above and measure each of these angles.

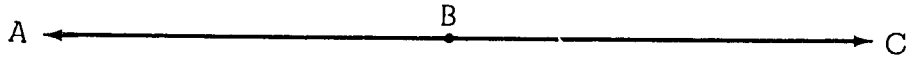


	Size in units
$A\hat{B}C$	
$X\hat{O}Y$	
$D\hat{E}F$	
$M\hat{N}P$	

8-2 The Unit Angle

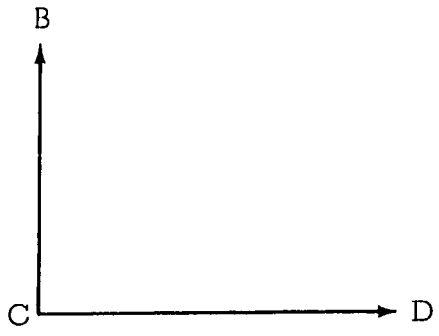
When discussing the measurement of length, area and volume, we learnt that the units chosen were arbitrary. For example, we could choose any line segment and call its length an inch. We could choose any line segment and call its length a foot. These units, as well as the metre or the mile, could have been chosen longer or shorter than those we now use. These choices would have altered the units of area and volume also. Units of different sizes could be equally useful if everyone agreed to use one or another of them. Of course the relationships between units, for example the relationship between inches and feet, might be easier to express for one choice than for another.

In a way, the selection of the unit for angle measurement is not arbitrary. The most natural angle unit is the one made by two rays that lie on the same straight line, as in the drawing, and this is called a straight angle.

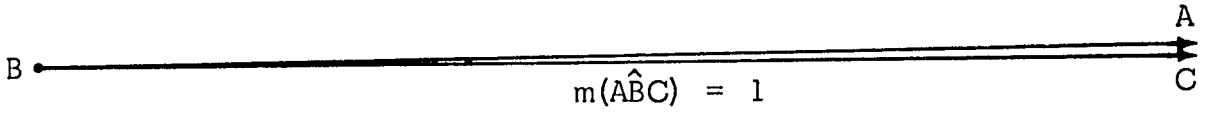


This would be a good unit for measuring angles but it is too big. We need therefore to choose a smaller unit so that many of these smaller units fit into the straight angle. We could choose a unit so that two of these make the straight angle or another unit so that ten of them make the straight angle. The unit we choose is still arbitrary in the sense that we can select one which fits

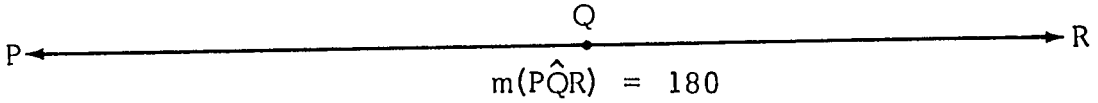
into the straight angle as many times as we please. The number we select is 180. That is, we choose a unit so that 180 of these units fit into the straight angle. We call this unit <u>one degree</u>, so that the degree measure of the straight angle is 180. We know that two right angles fit into a straight angle so that we can say that the degree measure of a right angle is 90. If the angle $B\hat{C}D$ is a right angle, we write this fact as follows:



$m(B\hat{C}D) = 90$, meaning that the degree measure of $B\hat{C}D$ is 90.

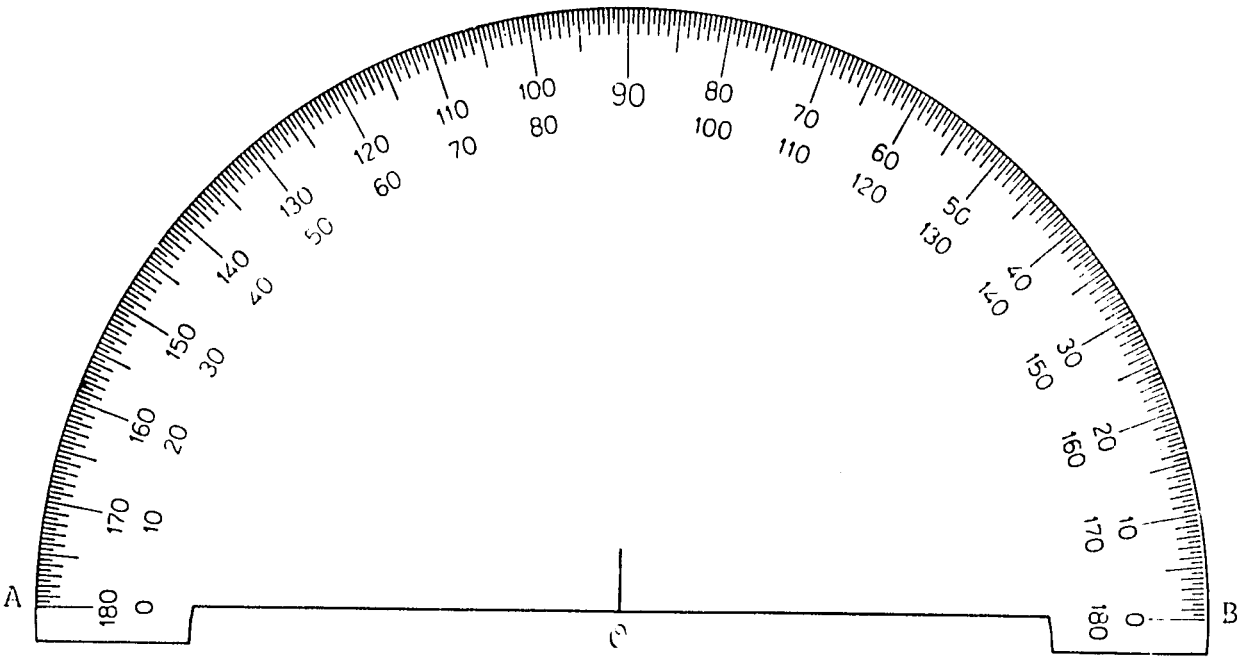


$m(A\hat{B}C) = 1$



$m(P\hat{Q}R) = 180$

We use an instrument called a <u>protractor</u> to measure angles. You have one in your instrument box. It looks like this:

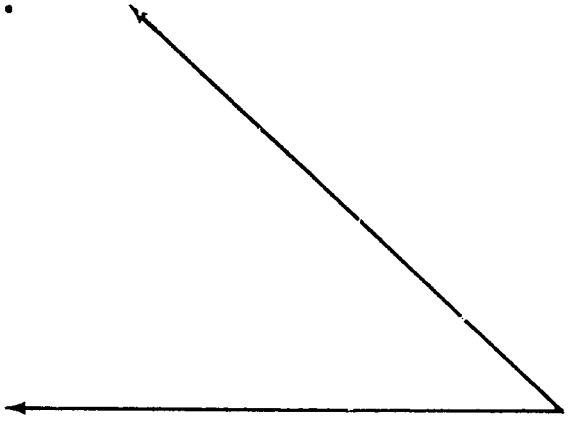


To use a protractor for measuring an angle, either the line segment $\overline{AC}$ or the line segment $\overline{CB}$ must be placed on top of one of the rays of the angle to be measured, and C must be placed over the vertex of the angle. The other ray, which can always be drawn so that it is longer than the radius of the protractor, tells us the degree measure of the angle. To find the degree measure we use one of the two scales along the protractor. If one ray of the angle is under $\overline{AC}$ we use the scale with the zero at A. If that ray is under $\overline{CB}$ we use the scale with the zero at B. Where the other ray crosses the protractor we read the appropriate scale as accurately as we can. We read it just as we read the number line on a ruler for the length of a line segment. This number line is not straight, however.

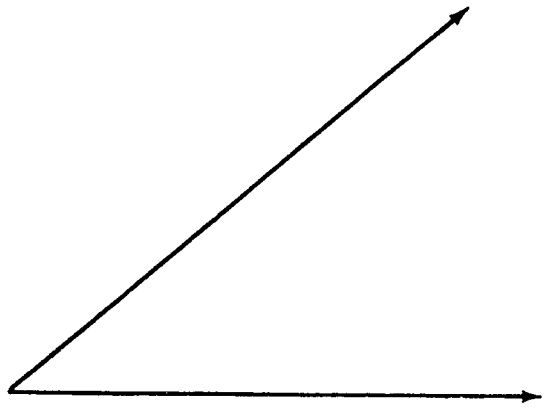
Problems 8-2A

Use your protractor to measure each of the following angles.

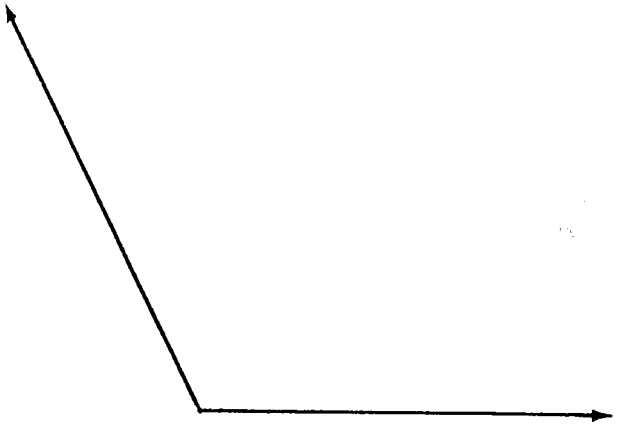
1.



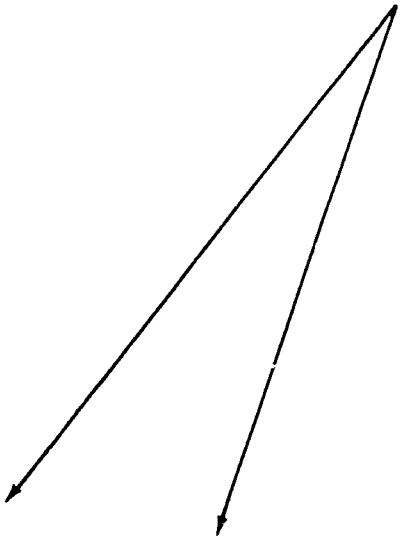
2.



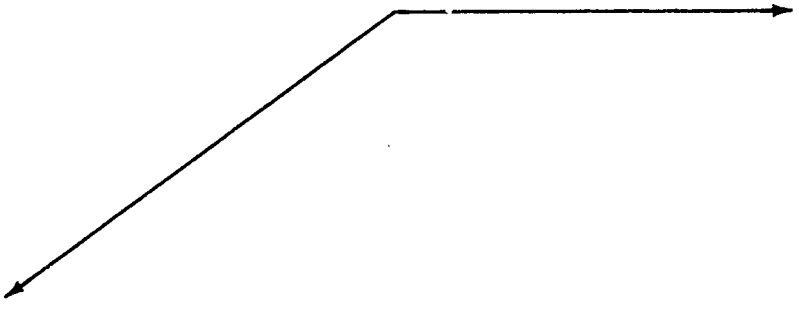
3.



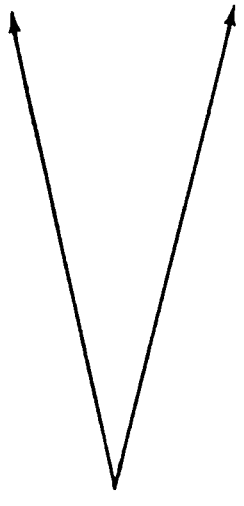
4.



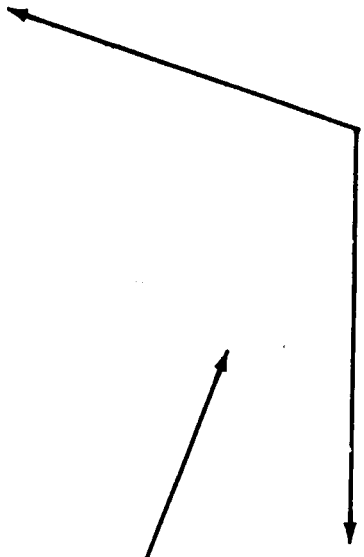
5.



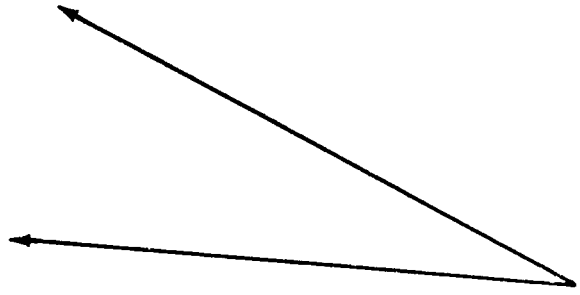
6.



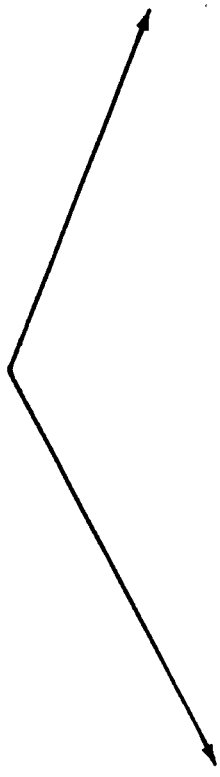
7.



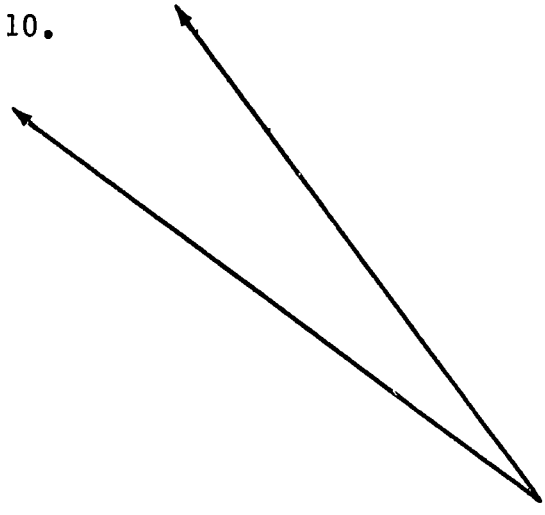
8.



9.



10.



When you use your protractor to measure an angle you are, in a sense, making a copy of the angle on the protractor. Thus your protractor has on it a copy of every possible angle. The copy on the protractor has a positive number assigned to it. (Do you see that every angle on the protractor has a positive number assigned to it?) This positive number is the degree measure of the copy. It is also the degree measure of the angle since we want all copies of an angle to have the same measure.

We have seen how we can find the degree measure of a given angle by using a protractor. Can we use the protractor to draw an angle if we know the degree measure? Of course we can. We draw any ray and put the protractor on our paper with either the segment $\overline{AO}$ or the segment $\overline{OB}$ on the ray, with O over the endpoint of the ray. Then using the appropriate scale we locate the number given as the degree measure of our angle. The second ray of our angle has the same endpoint as the first ray and goes through the point on our scale determined by the positive number which was the given degree measure of the angle.

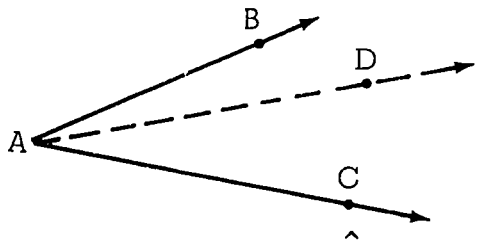
Problems 8-2B

Use your protractor to draw angles having the following degree measures.

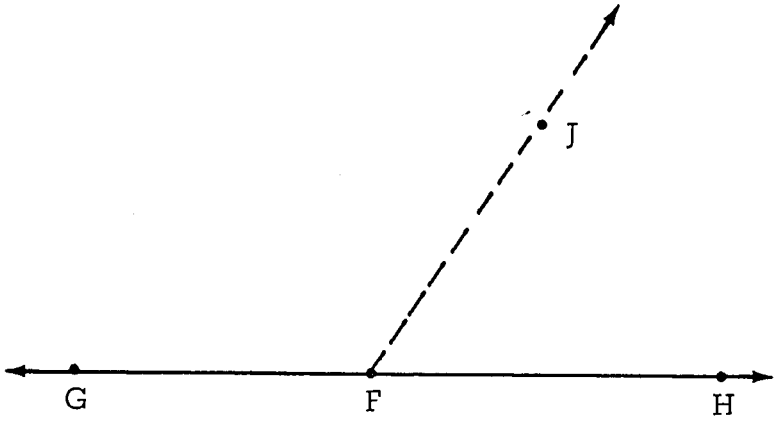
1.	60	2.	115
3.	21	4.	150
5.	95	6.	5
7.	56	8.	160
9.	45	10.	120

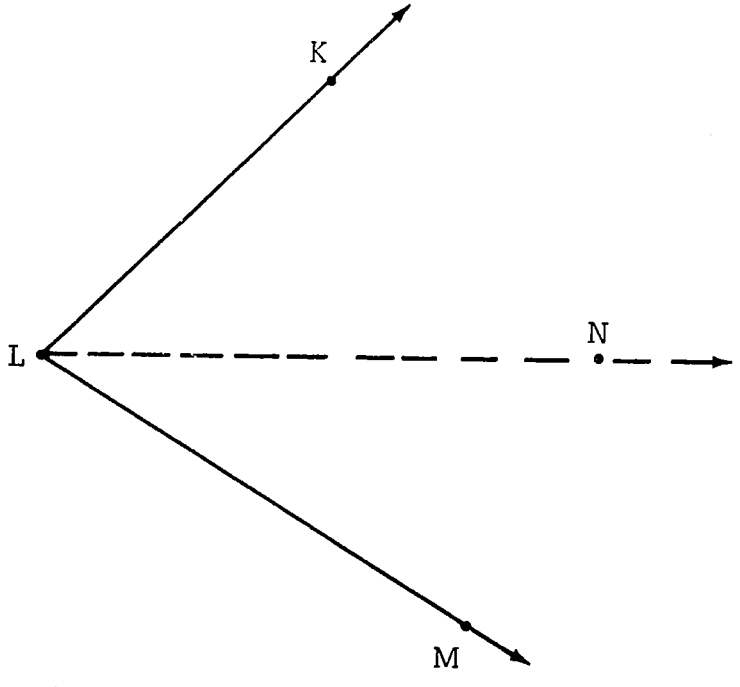
8-3 The Addition Property of Angles

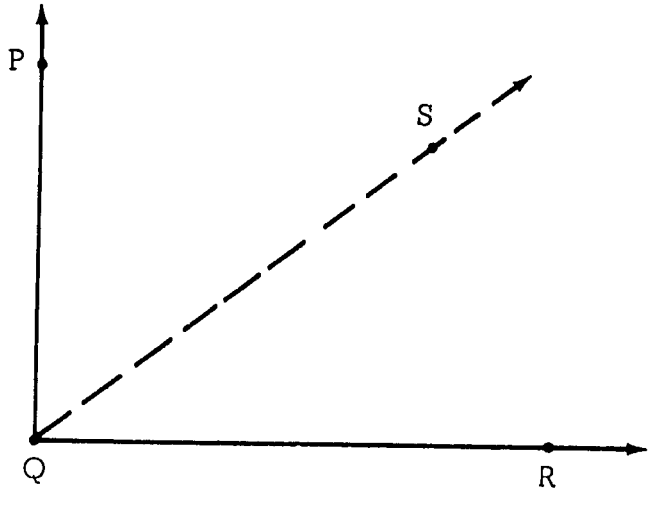
Suppose that we have an angle like $C\hat{A}B$, shown in the drawing. If we mark a point D in the interior of this angle, we can draw another ray, $\overrightarrow{AD}$. The ray $\overrightarrow{AB}$ and the ray $\overrightarrow{AD}$ make an angle. The ray $\overrightarrow{AC}$ and the ray $\overrightarrow{AD}$ make an angle. How is $m(B\hat{A}C)$ related to $m(B\hat{A}D)$ and $m(C\hat{A}D)$?



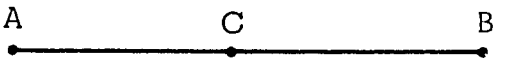
Test the relationship that you have discovered by measuring the angles in the drawings below. (The rays of the given angles are solid lines; the ray through the interior point of each given angle is a broken line.)



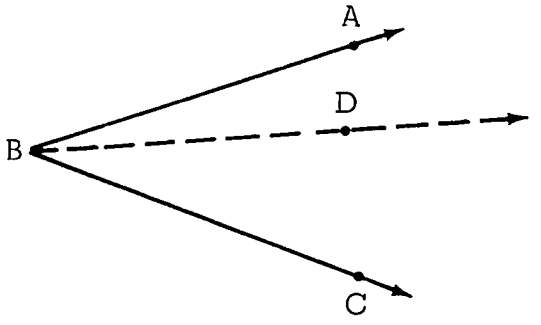




You recall that we showed that if a point C is marked anywhere between the points A and B of $\overline{AB}$, the length of $\overline{AB}$ equals the sum of the lengths of $\overline{AC}$ and $\overline{CB}$.



In the same way, you have just seen that if a point D is marked anywhere in the interior of $A\hat{B}C$, then $m(A\hat{B}C) = m(A\hat{B}D) + m(D\hat{B}C)$.



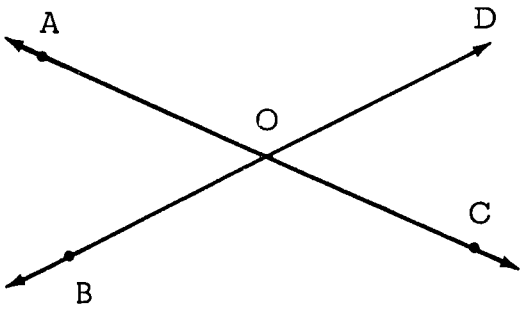
> If a point is chosen in the interior of an angle and the ray from the vertex to that point is drawn, the measure of the angle is equal to the sum of the measures of the angles formed by the interior ray and each of the original rays.

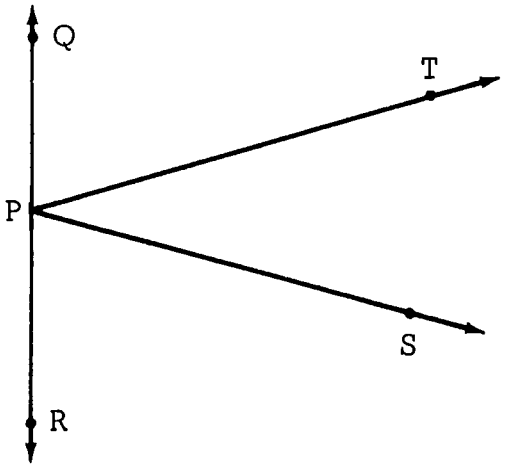
Problems 8-3

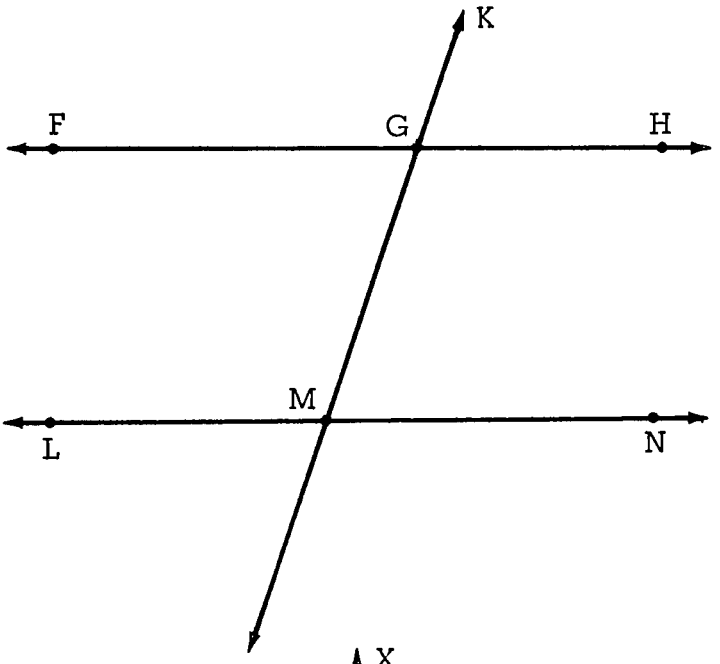
Fold paper to get angles of degree measure 90, 45, $22\frac{1}{2}$.

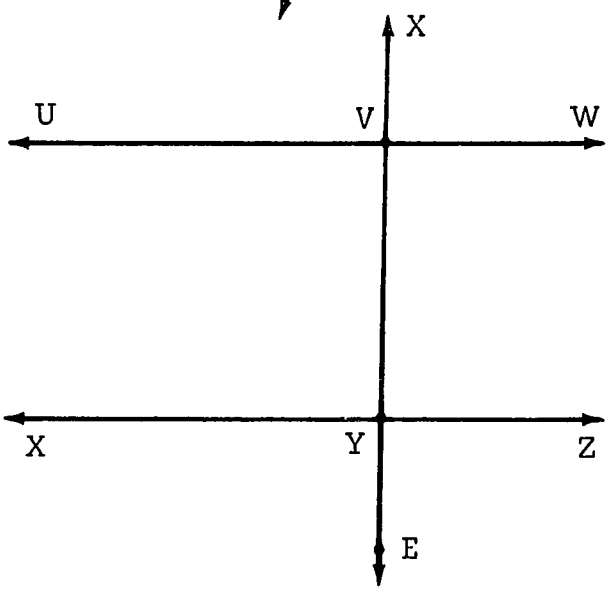
8-4 Congruent Angles

Measure all of the angles in the following figures.









Which angles in these figures have equal measures?

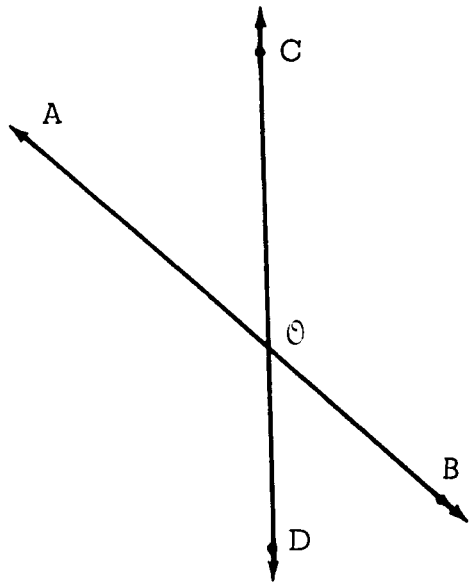
You recall that we measure length and area so that congruent line segments have the same length and congruent regions have the same area. Do you agree that the way we have chosen to measure angles will give us a similar property?

Congruent angles have the same degree measure.

Problems 8-4

1. Do the angles of an equilateral triangle have the same degree measure? Why?

2. Do the base angles of an isosceles triangle have the same degree measure? Why?



3. When any two lines intersect, you can be sure of having two pairs of congruent angles of degree measure less than 180. Which are they, and why are they congruent?

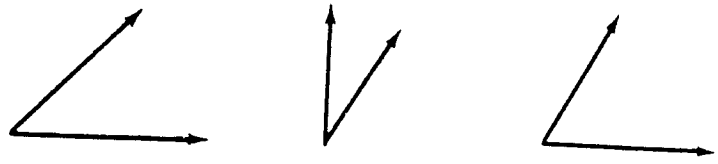
4. Is there any special case of two intersecting lines for which you get more congruencies than the two in Problem 3?

5. Describe some other figures in which there are pairs of angles having the same degree measure.

6. Choose a number between 0 and 180 and use your protractor to make two angles which have this number as degree measure. Are the angles congruent? Do you think that this will always be the case? Make a third angle with this same number as degree measure. Make two angles for another number. What do you think of the following statement?

Angles with the same degree measure are congruent.

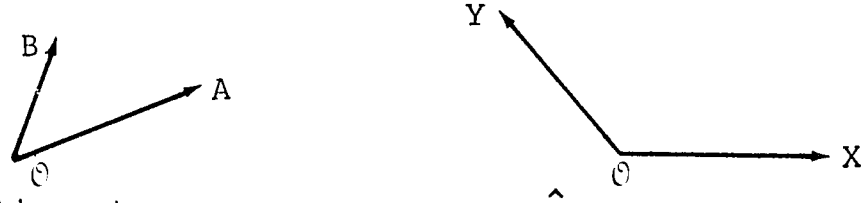
7. Now that you are able to use your protractor to make copies of angles, copy the following with your protractor.



If you don't believe that the angles you have just constructed are congruent to the above, check by superimposing a tracing paper copy or a ruler and compasses-copy.

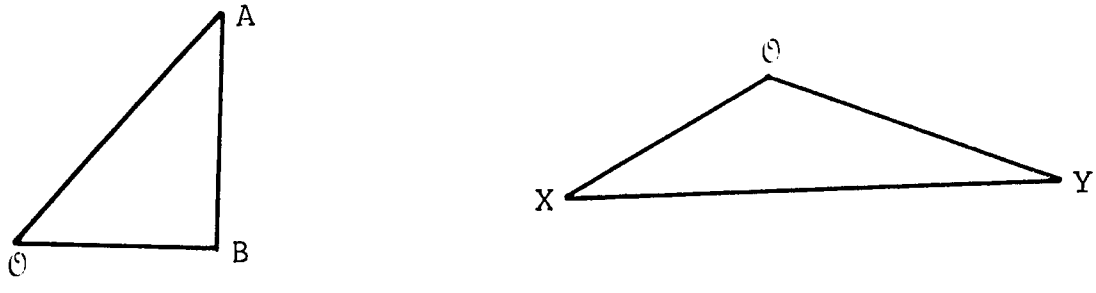
8. An angle is called <u>acute</u> if its degree measure is less than 90 and <u>obtuse</u> if its degree measure lies between 90 and 180.

Examples:



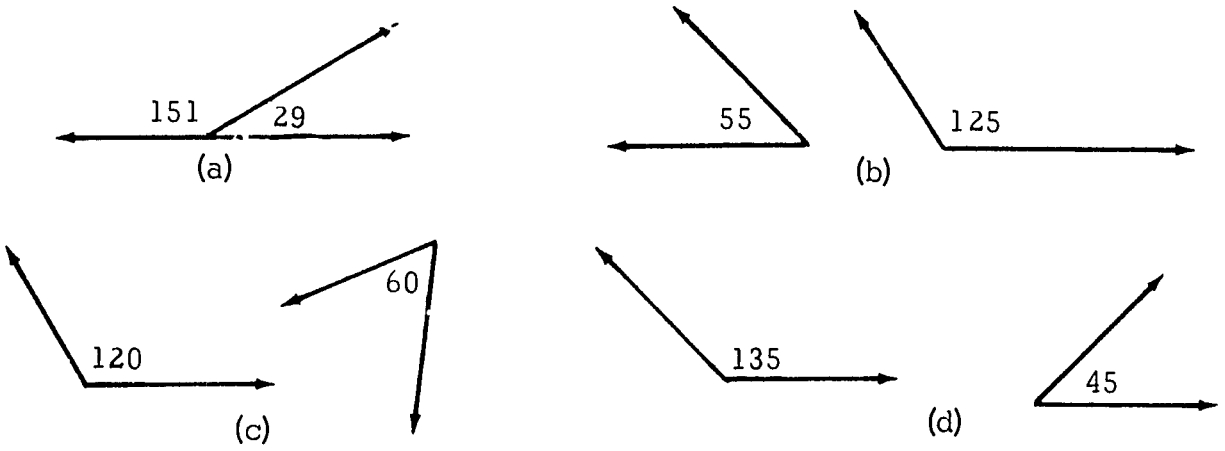
$A\hat{O}B$ is acute $X\hat{O}Y$ is obtuse .

Measure the angles in the following triangles and tell which are acute and which are obtuse.



8·5 Relations Between Angles

Two angles are called <u>supplementary</u> if the sum of their degree measures is 180. Angles need not have a ray or even a vertex in common in order to be supplementary. The following are pairs of supplementary angles. We say that the members of each pair are supplements of each other.



Problems 8-5A

1. What are the supplements of the angles whose degree measures are given below?

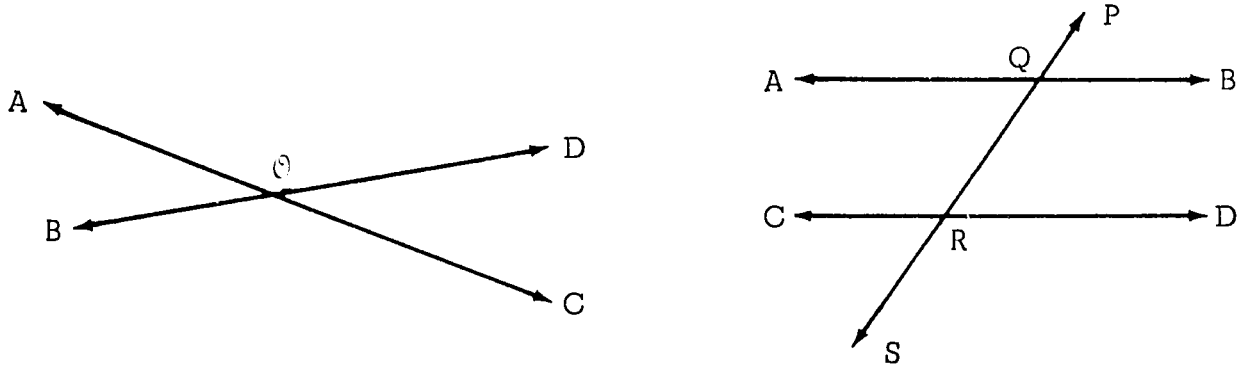
(a) 10 (b) 117

(c) 82 (d) 90

(e) 165 (f) 22

(g) $68\frac{1}{2}$ (h) $131\frac{1}{4}$

2. Measure all the angles in each of the following figures. Two of the lines of the second figure are parallel.



Which angles (a) have equal measure?

(b) are supplementary?

In the second figure above, there are angles which have been given special names.

The following are pairs of <u>corresponding angles</u>:

$P\hat{Q}A$ and $Q\hat{R}C$, $P\hat{Q}B$ and $Q\hat{R}D$

$A\hat{Q}R$ and $C\hat{R}S$, $B\hat{Q}R$ and $D\hat{R}S$

The following are pairs of <u>alternate interior angles</u>:

$A\hat{Q}R$ and $Q\hat{R}D$

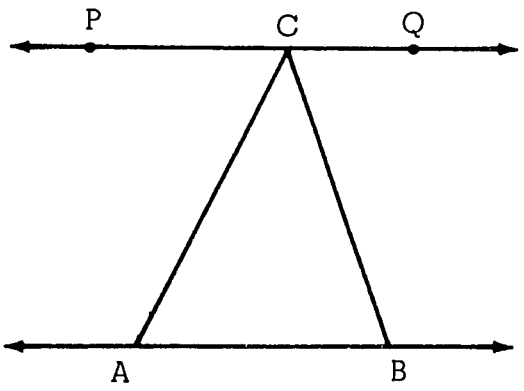
$B\hat{Q}R$ and $Q\hat{R}C$

The following are pairs of <u>alternate exterior angles</u>:

$P\hat{Q}A$ and $D\hat{R}S$

$P\hat{Q}B$ and $C\hat{R}S$

3. Consider triangle ABC. Draw a line through C which is parallel to $\overleftrightarrow{AB}$.



Can you determine the following sum?

$m(P\hat{C}A) + m(A\hat{C}B) + m(B\hat{C}Q) =$ _____?_____

What angle can you find whose measure is equal to the measure of $P\hat{C}A$? _____

What angle can you find whose measure is equal to the measure of $B\hat{C}Q$? _____

Using these two angles you have found and the above sum, can you find the value of this sum?

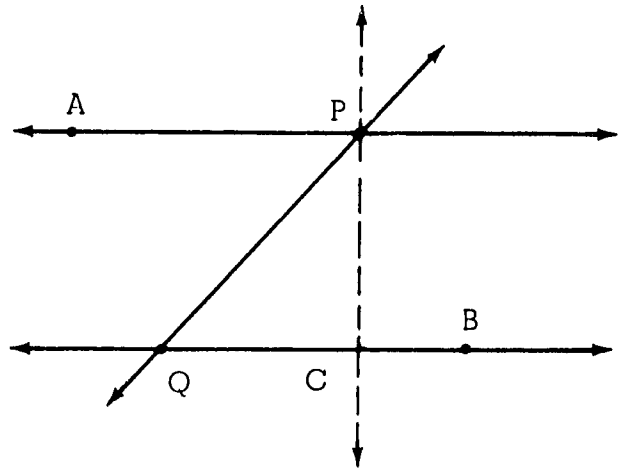
$$m(A\hat{C}B) + m(C\hat{A}B) + m(A\hat{B}C) = \underline{\qquad ? \qquad}$$

Can you fill in the following blank?

The sum of the degree measures of the three angles in a triangle is _____.

4. An equiangular triangle is one in which all three angles are congruent. What is the measure of each of these angles? Construct an equiangular triangle. Compare its sides by paper folding. Measure them with a ruler. What do you find? If we said: "In a triangle, congruent angles always have congruent sides opposite them," would you disagree or agree?

5. Suppose that we have two lines met by a transversal so that alternate interior angles are congruent. What can we say about the two lines? Look at the following drawings. The lines $\overleftrightarrow{AP}$ and $\overleftrightarrow{QB}$ are met by the transversal $\overleftrightarrow{PQ}$ so that $m(A\hat{P}Q) = m(B\hat{Q}P)$. Draw a line through P which is perpendicular to $\overleftrightarrow{QB}$ and call the point where this line meets $\overleftrightarrow{QB}$,



the point C. Since triangle PCQ has a right angle at C, we know that $m(C\hat{Q}P) + m(C\hat{P}Q) = 90$. But since $m(C\hat{Q}P) = m(A\hat{P}Q)$, we know that $m(C\hat{P}Q) + m(A\hat{P}Q) = 90$. Thus $C\hat{P}A$ is a right angle and $\overleftrightarrow{PC}$ is perpendicular to $\overleftrightarrow{AP}$. Since $\overleftrightarrow{PC}$ is perpendicular to both $\overleftrightarrow{AP}$ and $\overleftrightarrow{QB}$, it follows that $\overleftrightarrow{AP}$ and $\overleftrightarrow{QB}$ must be parallel.

If two lines are met by a transversal so that alternate interior angles are congruent, the lines are parallel.

To revise the concepts that we have covered, let us do the following problems.

Problems 8-5B

1. Draw triangles ABC with the following data in inches and degrees:

	AB	BC	CA	$m(\hat{A})$	$m(\hat{B})$	$m(\hat{C})$	$m(\hat{A}) + m(\hat{B}) + m(\hat{C})$
1		5			30	70	
2	3			36	65		
3	2	3	4				
4	3	4	5				
5	5	6	7				
6		6			60	60	
7	6	8	10				
8		8			45	45	

(a) Measure the sides and angles of the triangles and fill in the table.

(b) What do you notice about the sum of the measures of the angles of any triangle?

(c) Which triangles have all their angles acute?

(d) Which triangles have a right angle?

(e) Which triangles have an obtuse angle?

(f) Which triangles have the measures of two sides the same?

1. (g) Which triangles have the measures of all three sides the same?

   (h) Which triangles have the measures of none of the sides the same?

   A triangle whose three angles are acute angles is an <u>acute-angled triangle</u>.

   A triangle having one obtuse angle is called an <u>obtuse-angled triangle</u>.

   A triangle having one right angle is called a <u>right-angled triangle</u>.

   A triangle having none of its sides congruent is called a <u>scalene triangle</u>.

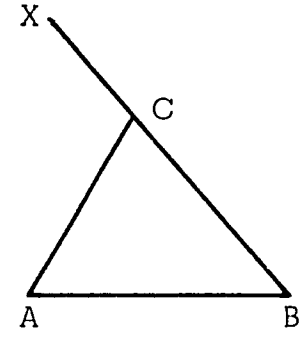
2. Draw triangles ABC with the measurements in inches and degrees given in the table below. Measure the sides and angles of the triangles and fill in the table.

	AB	BC	CA	$m(\hat{A})$	$m(\hat{B})$	$m(\hat{C})$	Type of triangle
1	3	5	3				
2	4			45	45		
3	2	3	4				
4	4	4	4				
5	5			40	80		
6	3	4	5				
7	3		3	120			

3. Fold paper to get angles of degree measure 120, 60, 30, 15.

8-6 Angles of Polygons

When a line segment is made longer (continued, extended), it is said to be <u>produced</u>.



In $\triangle ABC$, the side $\overline{BC}$ is produced to a point X. Consequently, $A\hat{C}X$ is formed whose interior is "outside" the triangle. $A\hat{C}X$ is called an <u>exterior</u> angle of the triangle.

$A\hat{C}B$ is adjacent to $A\hat{C}X$.

$A\hat{B}C$ and $B\hat{A}C$ are each <u>opposite interior</u> angles to $A\hat{C}X$. Do you agree or disagree with the following statement? The measure of the exterior angle is equal to the sum of the measures of the opposite interior angles.

<u>Problems 8-6</u>

1. (a) Draw a triangle.

   (b) Produce a side to form an exterior angle.

   (c) Give a name to the exterior angle.

   (d) Name the angle adjacent to this exterior angle.

   (e) Name the two opposite interior angles.

2. Draw $\triangle ABC$ with each of the following sets of data in inches and degrees. Produce $\overline{BC}$ to a point X. Complete the table.

	AB	BC	CA	$m(\hat{A})$	$m(\hat{B})$	$m(\hat{C})$	$m(A\hat{C}X)$	$m(\hat{A}) + m(\hat{B})$
1	4			20	50			
2		3	3			30		
3	2	5			70			
4	4			30	90			

What conclusion can you draw?

3. (a) Draw a polygon of 3 sides. Choose one vertex and join it to the other vertices. How many triangles are thus obtained?

(b) Draw a polygon of 4 sides. Choose one vertex and join it to the other vertices. How many triangles are thus obtained?

Repeat for polygons of more sides and fill in the following table.

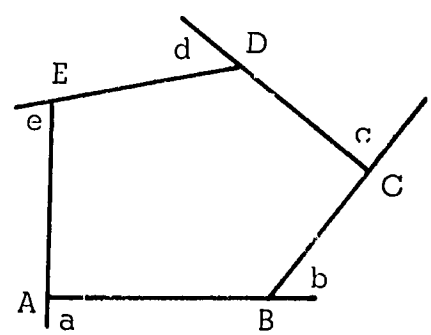
Name of Polygon	No. of Sides	No. of Triangles
triangle	3	
quadrilateral	4	
pentagon	5	
hexagon	6	
heptagon	7	
octagon	8	
nonagon	9	
decagon	10	

You will notice that the number of triangles is 2 less than the number of sides.

(c) We have already seen that the sum of the measures of the three angles of a triangle is 180.

What is the sum of the measures of the angles of a polygon of 3 sides, 4 sides, 5 sides, 6 sides, 7 sides?

4. Here is a pentagon whose sides are produced in order. The exterior angles are $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{e}$. Find by calculation the sum of the measures of these external angles. Verify your answers by measurement.

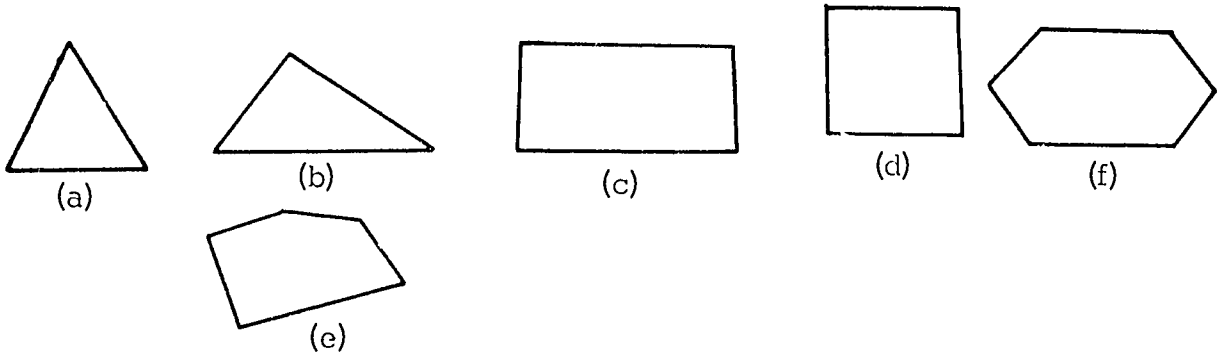


Repeat for a hexagon and a heptagon.

Because we speak of exterior angles of a polygon we therefore, for emphasis, speak of the interior angles of a polygon.

If the measures of all the sides and of all the angles of a polygon are equal, the polygon is said to be regular.

5. Which of these polygons are regular?



A regular triangle is said to be equilateral, or equiangular. A regular quadrilateral is called a square.

6. Find by calculation the measure of an interior angle of a regular octagon. Verify by measurement.

7. How many sides has a regular polygon whose interior angles each are angles of degree measure 144? Angles of degree measure 135?

CHAPTER 9

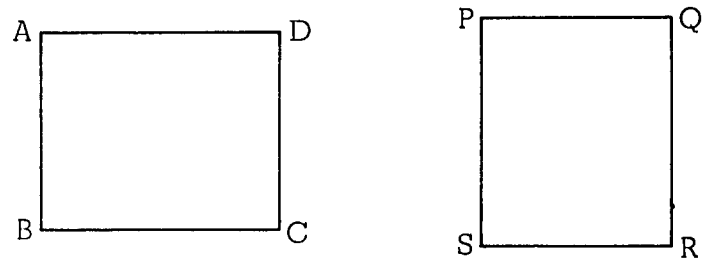
RELATED MEASUREMENTS AND A BACKWARD LOOK

9-1 Relations Between Measurements

Problems 9-1A

1. Measure the length of this line segment:
   State its length in (a) inches; (b) centimetres.

2. Measure $\overline{BC}$, $\overline{AB}$, $\overline{PQ}$, and $\overline{QR}$ in centimetres.
   (a) What is the area of the rectangle ABCD?
   (b) What is the area of the rectangle PQRS?



3. Get a box and measure the lengths of its edges. What is the volume of the box?

Notice that in Problems 2 and 3 you are able to find the area and volume of figures from the measurement of certain lengths. This shows that area and volume depend upon length. In many cases we need only a ruler to measure geometric figures, whether we want to measure length, area or volume. In Problem 2, measuring with a ruler was what we needed to compare the areas of the rectangles ABCD and PQRS.

The formula for the area of a rectangle is <u>length × width</u>. Both length and width are measurements of length. We get the measure of area by multiplying the numbers which are the measures of two lengths.

The formula for the area of a triangle is $\frac{1}{2}$ <u>base × height.</u> Both <u>base</u> and <u>height</u> are measurements of length. (The $\frac{1}{2}$ is not a measurement.) So again we get the measure of area by multiplying the measures of two lengths.

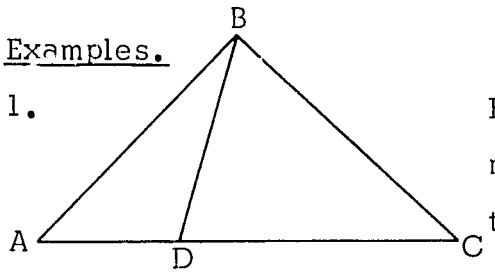
A measure of area involves the product of two measures of length, no more, no less.

The formula for the volume of a rectangular prism is <u>length × width × thickness</u>. To get a measure of volume we take the product of three measures of length. Notice that in order to measure many areas and volumes, the only tool we need is a ruler. Sometimes it is also helpful to know how to subdivide polygonal regions into triangle and rectangles.

In these chapters on measurement, we have found many equalities. For example, the statement that in a triangle sides of equal length are opposite angles of equal measure is a statement of equality about angle measures and lengths. There are also equalities relating angular measures. Here are two examples: The sum of the measures of the angles of any triangle is equal to 180. The measure of an exterior angle is equal to the sum of the measures of the opposite interior angles.

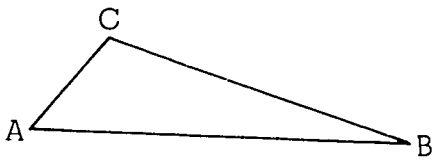
However, we are often interested in inequalities.

<u>Examples.</u>

1. 

How does $m(B\hat{A}D)$ compare with $m(B\hat{D}C)$? By measuring, we see that $m(B\hat{D}C) > m(B\hat{A}D)$.

2. In $\triangle ABC$, find AB, BC and AC. By measuring show that $AB > BC > AC$. By measuring show that $m(A\hat{C}B) > m(C\hat{A}B) > m(A\hat{B}C)$.

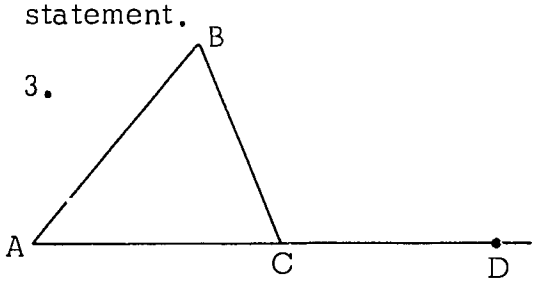


By noting that $A\hat{C}B$ is opposite $\overline{AB}$
and $C\hat{A}B$ is opposite $\overline{BC}$
and $A\hat{B}C$ is opposite $\overline{AC}$,

can you think of a rule for finding the largest angle in a triangle by measuring the sides of the triangle? What happens if two sides of a triangle are equal? What happens if three sides of a triangle are equal? Can you think of a rule to

find the largest side of a triangle by measuring the angles? Draw a triangle and see if your rules work on your triangle. In a right triangle, we call the side opposite the 90 degree angle, the hypotenuse of the right triangle. Draw a right triangle, and see what you can say about length of the hypotenuse as compared to the length of the other two sides of the triangle. Is your statement in Problem 2 above, true for all right triangles? If not, try to think of another statement.

3.



By measuring, see if you can answer the following:

$m(B\hat{C}D)$ ____?____ $m(B\hat{A}C) + m(A\hat{C}B)$

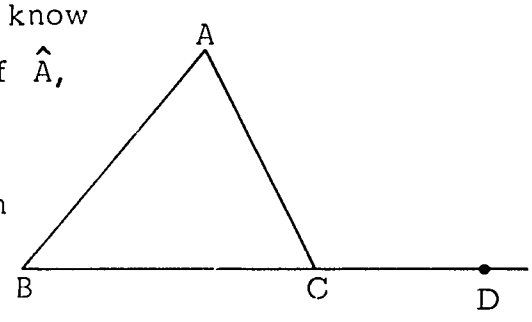
$m(B\hat{C}D)$ ____?____ $m(A\hat{B}C) + m(A\hat{C}B)$

In any triangle, how does the measure of an exterior angle compare with the sum of the measures of the supplement of the exterior angle and one of the opposite interior angle?

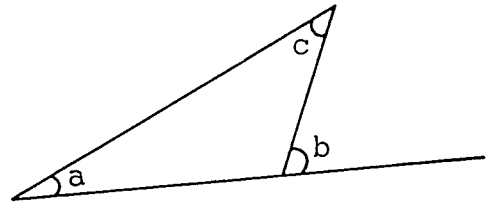
Problems 9 - 1B

1. (a) In the given figure, what do you know about the sum of the measures of $\hat{A}$, $\hat{B}$, and $A\hat{C}B$?

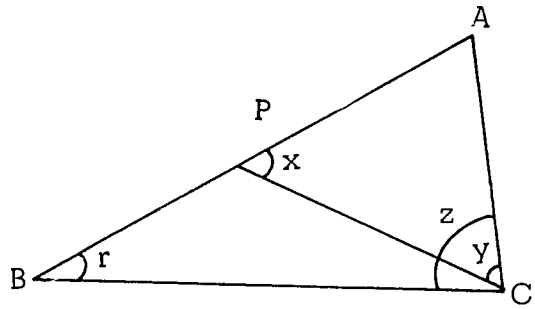
   (b) What is the relationship between the measure of $A\hat{C}D$ and the measures of $\hat{A}$ and $\hat{B}$?



2. In the following figure a, b, and c are the measures of the marked angles. Which is bigger, a or b? Give reasons for your answer.

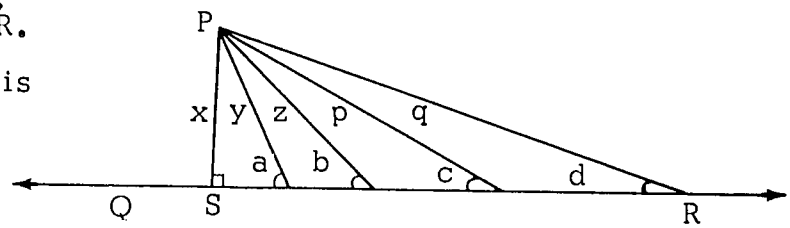


3. In ΔABC, $AB > AC$. P is taken on $\overline{AB}$ so that $AP = AC$. The triangle APC is therefore isosceles and $x = y$. But question 2 showed us that $x > r$. Therefore, $y > r$. But $z > y$. (Why?) It now follows that $z > r$.



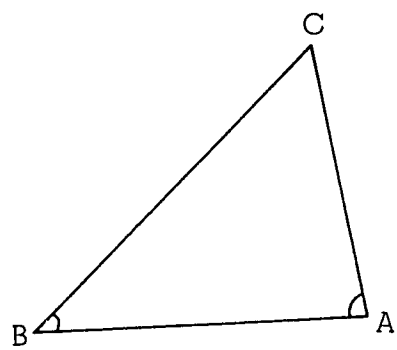
Can you state in your own words the relation between the angles and sides of a triangle if the sides are unequal?

4. In the given figure, $\overline{PS} \perp \overleftrightarrow{QR}$. Which of the angles marked is the greatest? Which of the marked line segments is the shortest? Can you draw another line segment from P to $\overleftrightarrow{QR}$ so that it is shorter than $\overline{PS}$?



Question 3 has shown that if two sides of a triangle are unequal, the greater angle is opposite the greater side. Is the converse true? Let us find out.

Suppose that we are given that in ΔABC, $\hat{A} > \hat{B}$. What can we say about the sides $\overline{BC}$ and $\overline{AC}$ which are opposite these two angles? Either $BC > AC$, or $BC = AC$, or $BC < AC$.

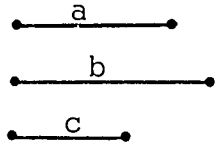


If $BC = AC$, the triangle is isosceles and then $\hat{A} = \hat{B}$. But we are given that $\hat{A} > \hat{B}$. Therefore, $BC \neq AC$.

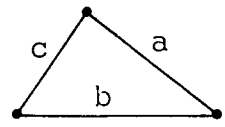
If $BC < AC$, $AC > BC$ and, therefore, $\hat{B} > \hat{A}$ as the argument of the previous question showed. But, remember, we were given that $\hat{A} > \hat{B}$ and so it is not true that $BC < AC$.

We have now shown that two of the possible relations between BC and AC are untrue. Therefore, the remaining one <u>must</u> be true, that is, BC > AC.

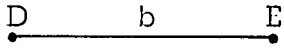
This now proves that if two angles of a triangle are unequal, the greater side is opposite the greater angle.

Here are three line segments, 

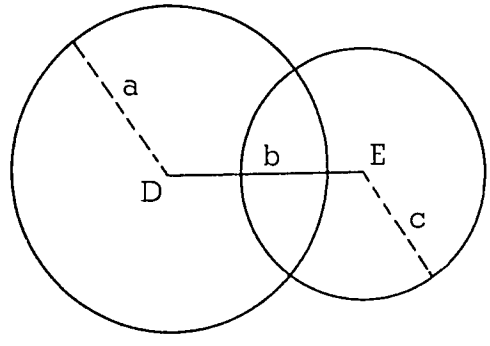
Can you make a triangle out of these line segments? Try to get some small straight sticks like match sticks and break them carefully to approximately these three lengths. Then see if you can put them together to form a triangle. Of course the position of the triangle might differ for different people. Here is one way to put the sticks together



You can draw a picture of this triangle by using paper and compasses instead of sticks. Begin by making a copy of any one of the given segments, say the longest one, b. Mark the ends of this segment with the letters D and E as shown.



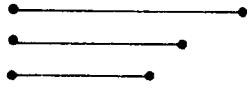
Now set the compasses to the length of another of the given segments, say a. With the point of the compasses at D draw a circle with radius equal to the length of a. Now set the compasses to the length of the third segment, c. With the point E as centre draw a circle with radius equal to the length of c.

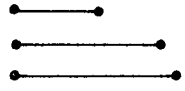


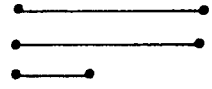
We want a point so that the segment joining that point to D has the same length as a and the segment joining that point to E has the same length as c. Can you find such a point? Can you find more than one? Can you draw two triangles whose sides are congruent to the given segments?

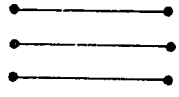
Problems 9-1C

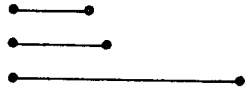
For each problem below draw a triangle whose sides are congruent to the given line segments.

1. 

2. 

3. 

4. 

5. 

For each problem below draw a triangle whose sides are congruent to the given line segments.

6. a = 2 inches
   b = 1.5 inches
   c = 1.75 inches

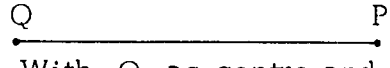
7. a = 3 cm.
   b = 4 cm.
   c = 5 cm.

8. a = 2 cm.
   b = 3 cm.
   c = 4 cm.

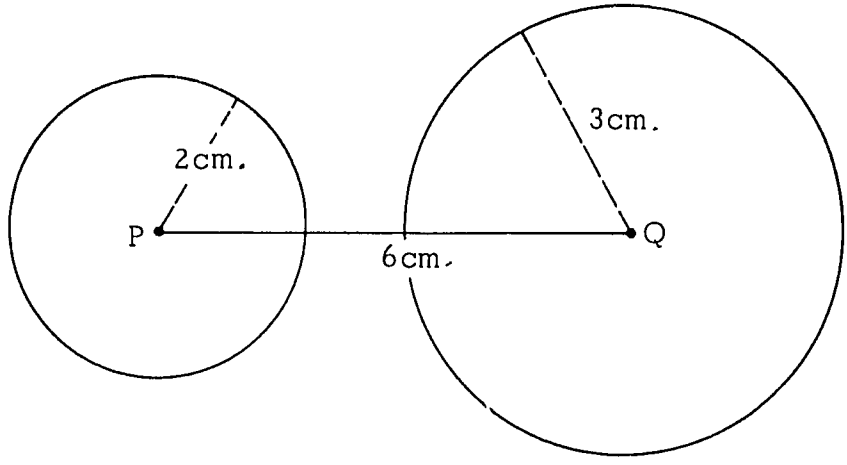
9. a = 2 cm.
   b = 4 cm.
   c = 6 cm.

10. a = 3 cm.
    b = 2 cm.
    c = 6 cm.

Did you have difficulty with problems 5, 9 and 10? What happens when you try to draw a triangle with sides 2cm., 3cm. and 6cm.? Draw the longest side first. Label its end points P and Q.



With P as centre and radius 2 cm. draw a circle. With Q as centre and radius 3 cm. draw a circle. Can you find a point which is 2 cm. from P? You can find many of them. Can you find a point which is 3 cm. from Q? Again there are many of these. Can you find a point which is both 2 cm. from P <u>and</u> 3 cm. from Q?



We do not get a triangle because $2 + 3 < 6$. In Problem 9 we do not get a triangle because $2 + 4 = 6$. Can you make a drawing to show why we do not get a triangle for the sides given in Problem 9?

In general we cannot have a triangle whose sides have lengths a, b and c units if the sum of any two lengths is less than the third or if the sum of any two is equal to the third. Thus if we <u>have</u> a triangle, we know that the sum of the lengths of any two sides must be greater than the length of the third side.

This useful relation is know as the <u>triangle inequality</u>. For example, $a + b > c$, $a + c > b$, $b + c > a$.

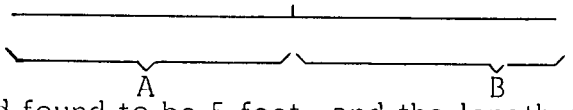
<u>Conclusion</u>

What we have been saying shows how we make a measurement. For length, for example, we choose a particular straight edge, call it "unit", give a special name to it (some names used are <u>inch, foot, metre, yard, mile, furlong</u>), make many copies of the unit, and measure any length or distance by putting these copies end to end until they stretch from one end of what we are measuring to the other end. We may connect the copies end to end in the form of a ruler, knotted string or tape measure, in order to measure more easily and accurately.

We follow the same idea in every measurement, whether we are measuring length, area or the size of angles. We choose for a unit a particular thing of the kind we are measuring (a certain straight edge, a certain square region, a certain angle) and make many copies of this unit. Then, to measure

something of the same kind (the edge of a board, the area of a floor, the size of an angle in a triangle) we put together copies of the unit end to end or edge to edge, until they cover as closely as possible the thing we are measuring. We then count the number of copies we have used and call this number the measure, in our unit, of what we are measuring. Sometimes the copies are already put together, as in a ruler, knotted string, or protractor.

This way of making a system of measuring is the first principle of measuring. The second principle is that measuring is additive. This really comes from the first principle, and means, for example, that if we tie together two strings, A and B, and the length of A from one end to the



knot is measured and found to be 5 feet, and the length of B is found to be 8 feet, then the length of the whole string when we measure it will be 13 feet. If five foot rulers end to end stretch exactly along A and eight foot rulers do the same along B, then these foot rulers together are all fitted end to end and stretch exactly along the whole length of the combined string. Since there are thirteen of them, 13 is the length of the string. We can state the second principle thus: if we have a system of measuring a certain kind of thing and two of these things are measured separately and then put together end to end or edge to edge to make a third thing of the same kind, then the measure of the third thing will turn out to be the sum of the measures of the first two things.

The third principle is that we make our units from material that won't change with time or movement; a wooden ruler changes very little in length from day to day and from place to place. As a result, our measurements of the length of an object will give the same answers (or almost the same) tomorrow that they give today, and the same answers when we measure the object in Lagos that they would give if we carried the object to Nairobi and measured it there, provided the object being measured doesn't change with movement or time.

The fourth principle is that any two objects that are congruent will have the same measure when measured with the same unit.

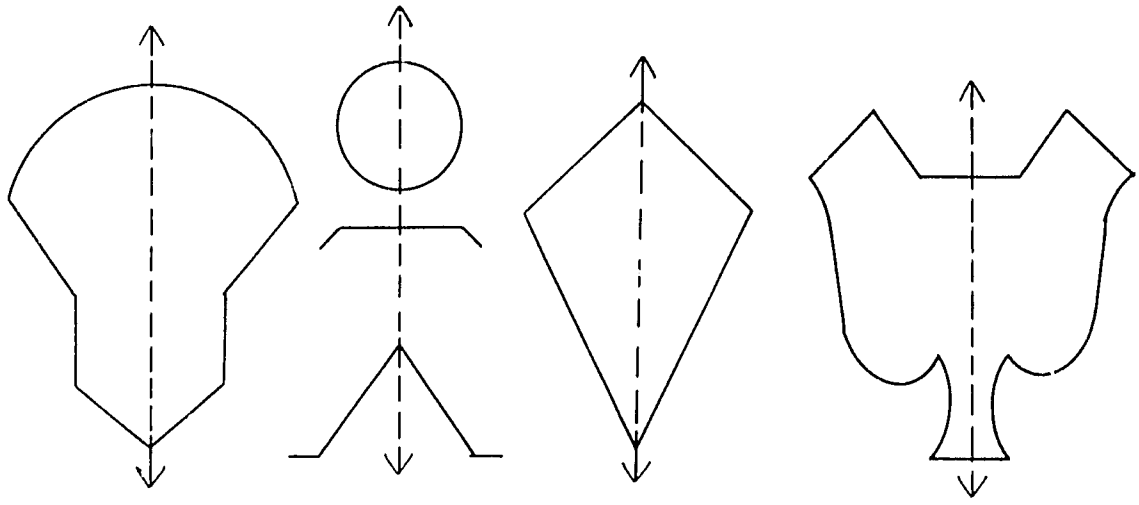
In this chapter we have developed informally, and just as described above, some standard ways of measuring line segments, angles, certain plane regions (rectangular, triangular, and so on) and certain solids.

CHAPTER 10

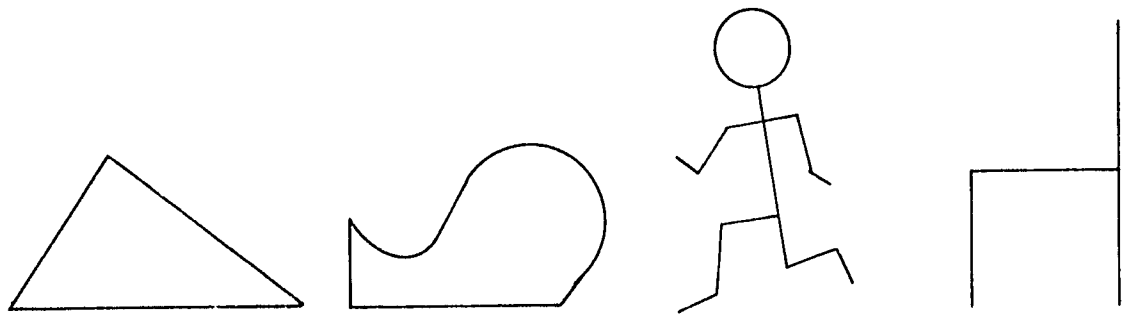
SYMMETRY

10-1 Symmetric Geometric Figures

Look at the four geometric figures pictured here. What do you notice about them?



Do you see that the left half of each figure is an exact copy of the right half? Here are some drawings of figures whose left halves are not exact copies of their right halves.



Previous Page Blank

In the case of the first four figures, we call each half a mirror image of the other half.

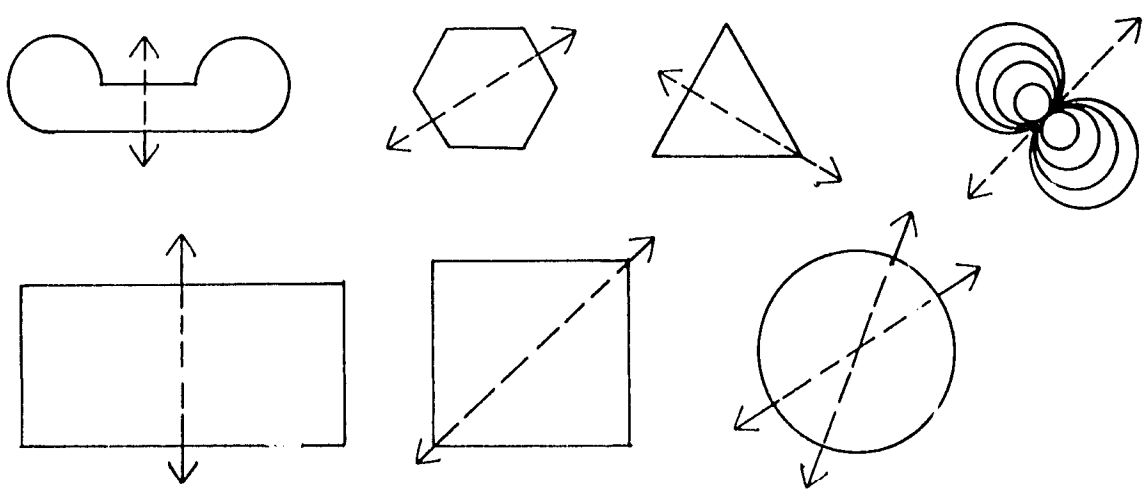
Problems 10-1A

1. Draw six geometric figures whose left halves are mirror images of their right halves.

2. Draw five geometric figures whose bottom halves are mirror images of their top halves.

3. Draw four geometric figures whose left halves are not mirror images of their right halves.

If you have a figure drawn on a piece of paper, there is a way that you can test it to see if its left half is the mirror image of its right half. Try to fold the figure in half along a straight line. This will fold the left half over on the right half, and if the two halves fit exactly over each other, then they are mirror images of each other. Trace the first four figures in this section on pieces of paper and try the folding experiment. If you have carefully traced the figures, the halves will fit on each other when you fold the paper. Here are some figures with folds shown as dotted lines. If you trace them and fold each figure along the line, the halves will fit over each other.



If you fold a figure into two halves that will fit over each other, the straight line along which you make the fold is called a line of symmetry for the figure. We say that the figure is symmetrical with respect to the line of symmetry. Notice that the circle shown above has more than one line of symmetry. Can you find other lines of symmetry for the circle? Where do all of its lines of symmetry intersect?

Problems 10-1B

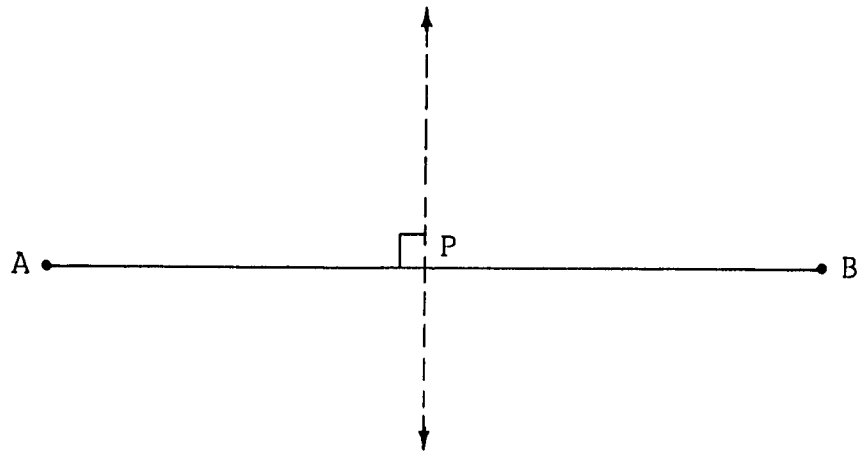
1. Find all the lines of symmetry for a square. How many lines of symmetry are there?

2. Find all of the lines of symmetry for a rectangle that is not a square. How many lines of symmetry are there?

3. How many lines of symmetry are there for an equilateral triangle? a regular pentagon? a regular hexagon?

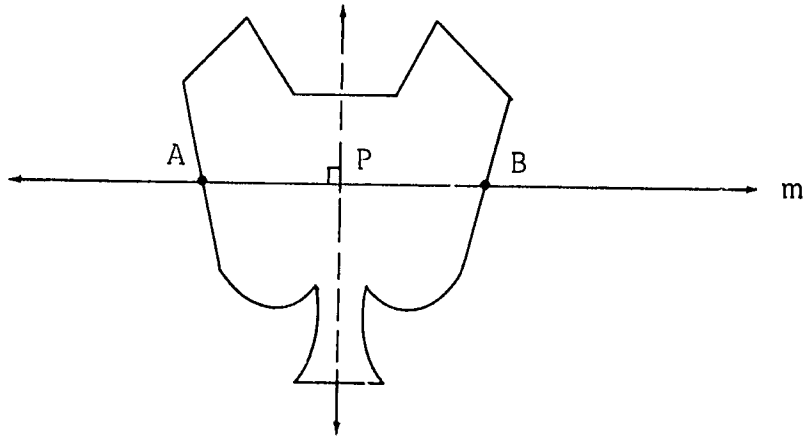
10-2 A Symmetry Test

One of the simplest of geometric figures is a line segment. You can see that the perpendicular bisector of a line segment is a line of symmetry for it.



Since the distances AP and BP are equal, the points A and B coincide when the fold is made.

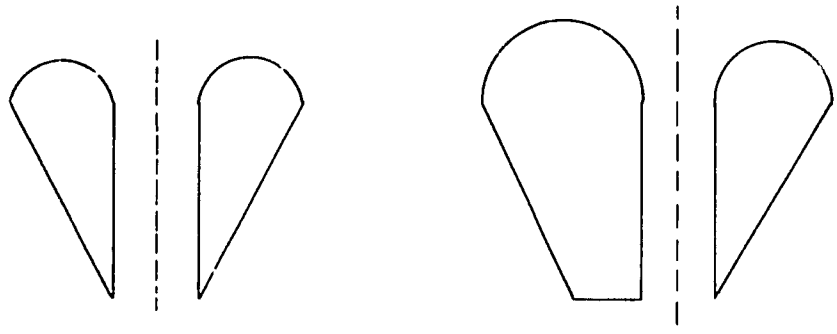
Let us look at one of our earlier figures together with a line of symmetry for it.



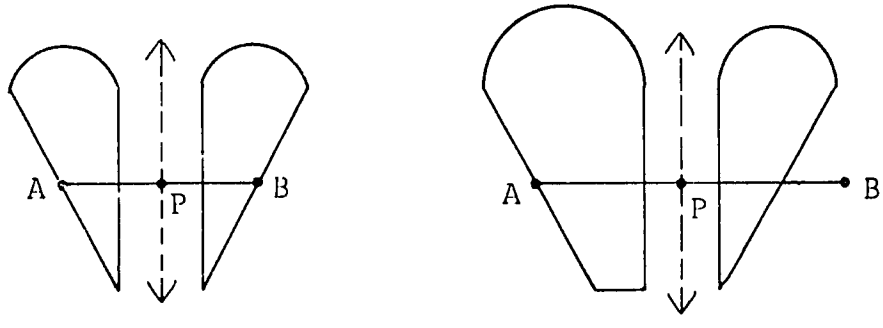
Suppose a point A is on the figure. If we construct the line m through the point A and perpendicular to the line of symmetry, this line will intersect the line of symmetry in a point P. Now let us fold the paper again. The two halves of the picture fit exactly on top each other, so that some point B of the figure must be on top of the point A. This will be a point in which the line $\overleftrightarrow{AP}$ intersects the other half of the figure. Trace the figure and try folding it. The points B, P and A are all on the straight line m, and since B and A coincide when the paper is folded, the distance BP must be equal to the distance AP. The line of symmetry for the entire figure is also a line of symmetry for the line segment $\overline{AB}$.

If you are given a figure and a straight line, you can test to see if the line is a line of symmetry by folding the paper that the figure is drawn upon.

But perhaps the figure is not drawn on a piece of paper. Perhaps it is scratched on a flat piece of wood. How would you test it? Imagine that these figures are scratched on a flat board.



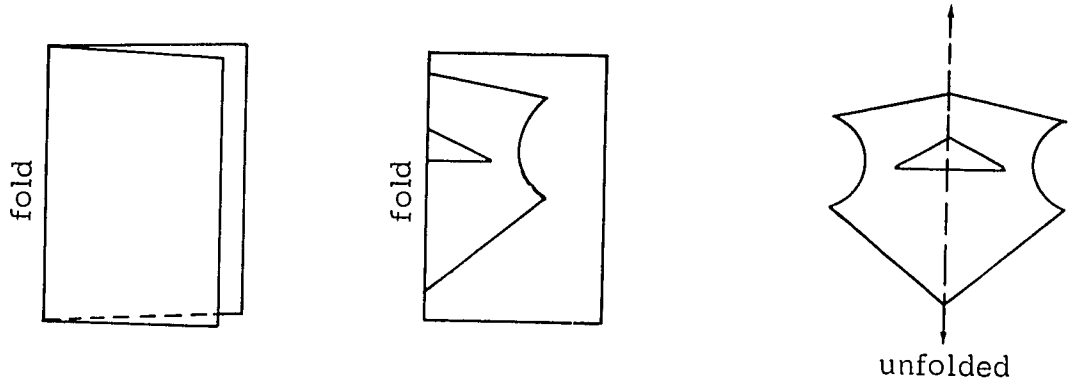
This is what you can do. Choose a point A on the figure, as shown below. Then construct the straight line segment $\overline{AB}$, which is perpendicular to the dotted line, intersects the dotted line in a point P and is such that the distances AP and BP are equal. If the dotted line is a line of symmetry, then B must be a point of the figure.



If B is always on the figure, no matter which point of the figure we choose for A, then the line is a line of symmetry. You could not test every point A, because the set of points on the figure is endless. But you can get a good idea whether or not the line is a line of symmetry for the figure. Of course, if a single pair of points A and B fails the test, the line is not a line of symmetry. The figure shown on the left has the dotted line shown as a line of symmetry. The figure shown on the right does not.

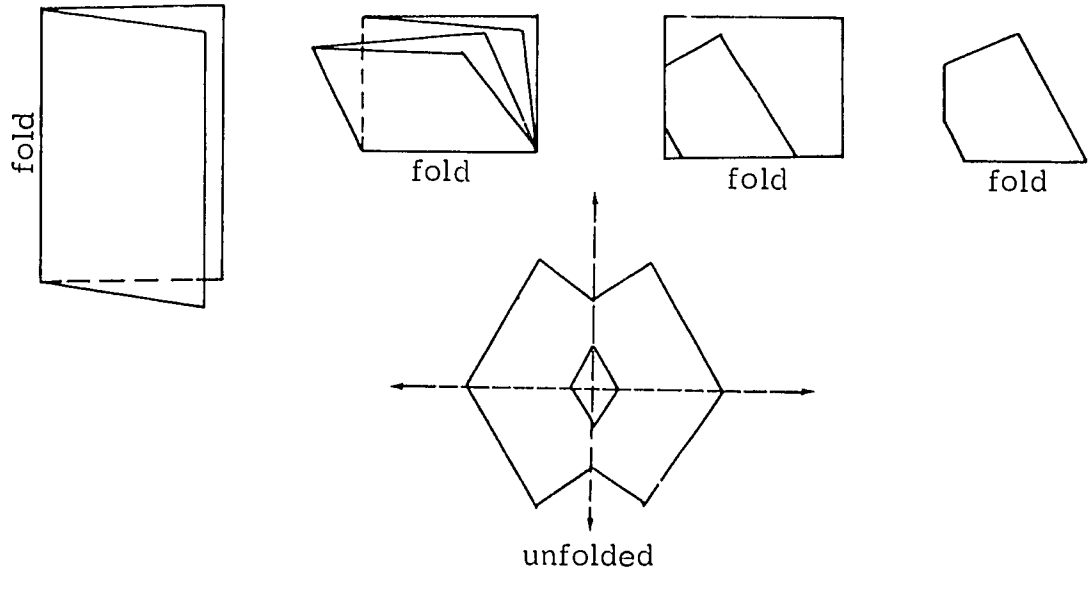
10-3 Making Symmetrical Figures

There are some interesting ways of making symmetrical geometric figures. Take a piece of paper and fold it in half.

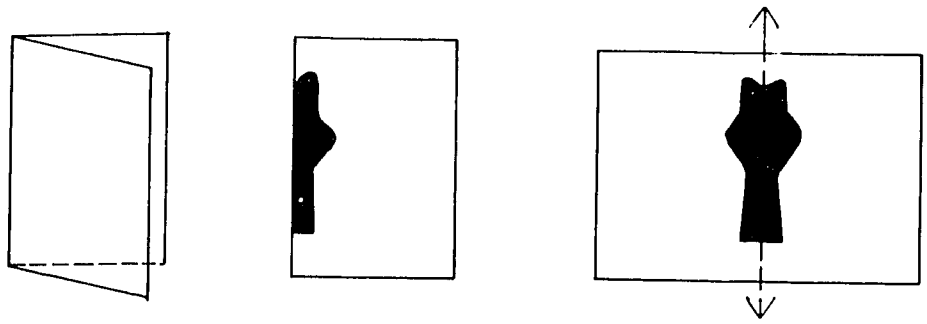


Then cut a design in the paper, starting at some point along the fold and returning to the fold with the cut. You can even make several cuts. When you open up the paper, the piece will have the fold as a line of symmetry. (Can you see why this is so by folding the paper together again?)

More interesting figures can be made by folding a paper twice or more before cutting. Each fold will then be a line of symmetry.



Another easy way of making complicated geometrical figures is this. Fold a piece of paper in half. Then put a drop of ink between the halves or on one half of the paper only and press them together. The ink will spread out on the paper and make the same blot on one half as on the other. When you open the paper, the fold will be a line of symmetry for the ink blots.



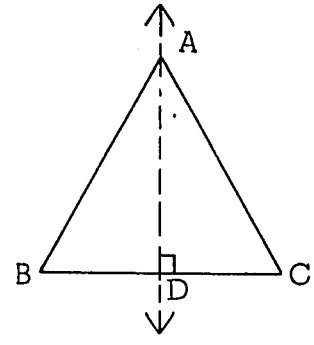
Problems 10-3

1. Make as interesting symmetrical figures as you can by folding and cutting paper.

2. By folding pieces of paper once in half and using drops of ink, produce some symmetrical ink blots. Can you see animals or plants in any of your figures?

10-4 Congruence

We have said that two figures are congruent if one can be made to fit exactly on the other. Recall that in Chapter 2 we folded paper to show that certain angles in equilateral and isosceles triangles were congruent. For example, in the equilateral triangle ABC below, if we draw $\overleftrightarrow{AD}$ perpendicular to $\overline{BC}$ and fold the paper along the line $\overleftrightarrow{AD}$, we find that every point of $A\hat{B}C$



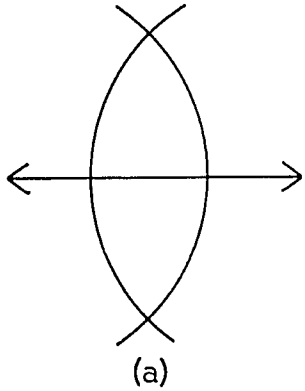
fits every point of $A\hat{C}B$, and so the two angles are congruent.

What relation does this have to symmetry? Is line $\overleftrightarrow{AD}$ a line of symmetry with respect to the equilateral triangle ABC? What other congruences between sides, or angles, or triangles can you find in the figure? Does the equilateral triangle have any other lines of symmetry besides line $\overleftrightarrow{AD}$? What congruences do these other lines of symmetry determine?

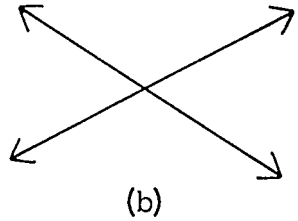
We have said that if a figure is symmetric to a line, then, when your paper is folded along the line of symmetry, every point of the figure that is on either side of the line will fit exactly on a point of the figure that is on the other side of the line. This idea of one figure fitting another exactly is what we mean by congruence. Thus we see that whenever a figure has one or more lines of symmetry, there will be one or more congruences resulting from this symmetry. Can you draw a picture of two congruent figures that do not have a line of symmetry?

Problems 10-4

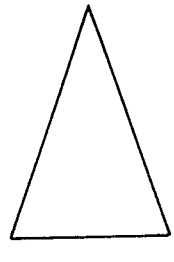
1. Consider the following figures:



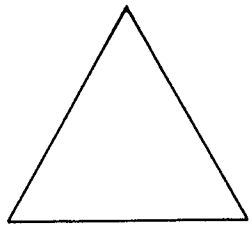
(a)



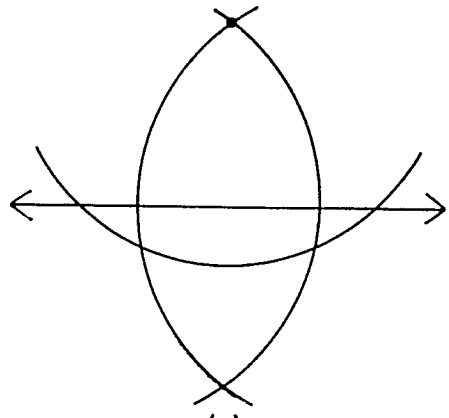
(b)



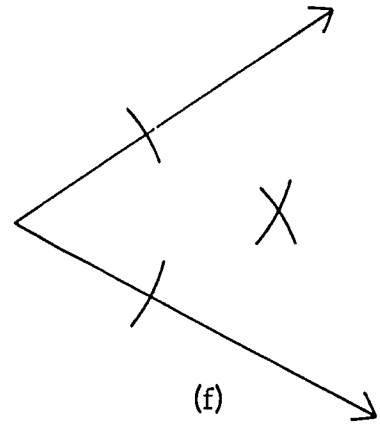
(c)



(d)



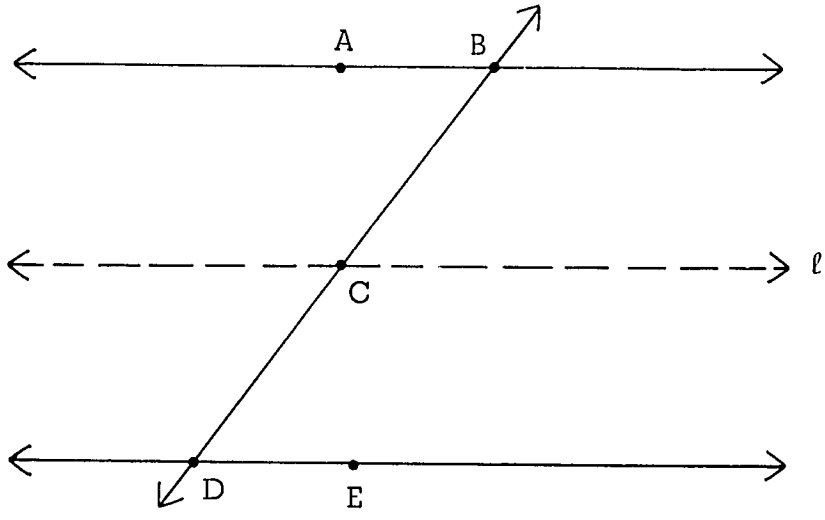
(e)



(f)

Where have you seen these figures before? What are they? What properties do they have? Draw each of the above figures on your own paper and then draw in the line or lines of symmetry for each of them. Name the resulting congruences for each figure.

2. We folded paper in Chapter 2 to show that a transversal to parallel lines makes certain pairs of angles congruent. In the picture itself

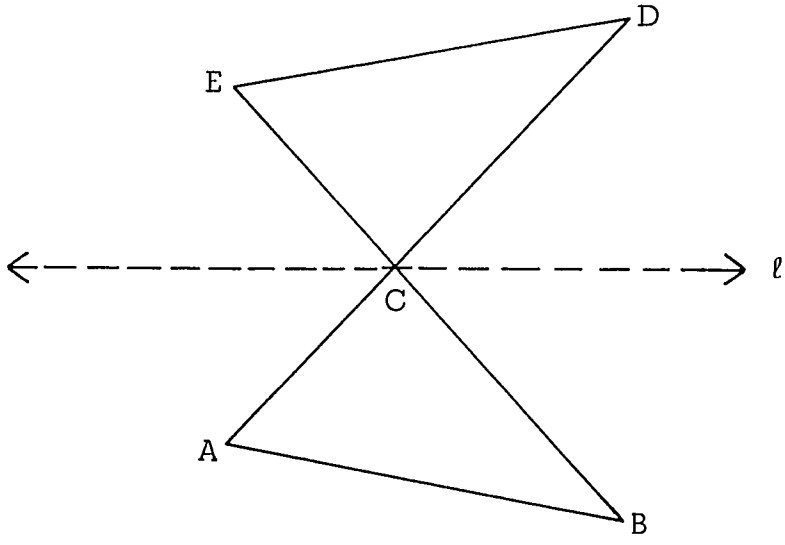


there is no symmetry, but after you have folded it along the dotted line symmetry appears. Trace the figure on thin paper, fold along the dotted line ℓ and then fold again along the line of symmetry for the resulting figure. Tell what happens to $A\hat{B}C$ and $C\hat{D}E$ after you have folded the paper the second time (along the line of symmetry).

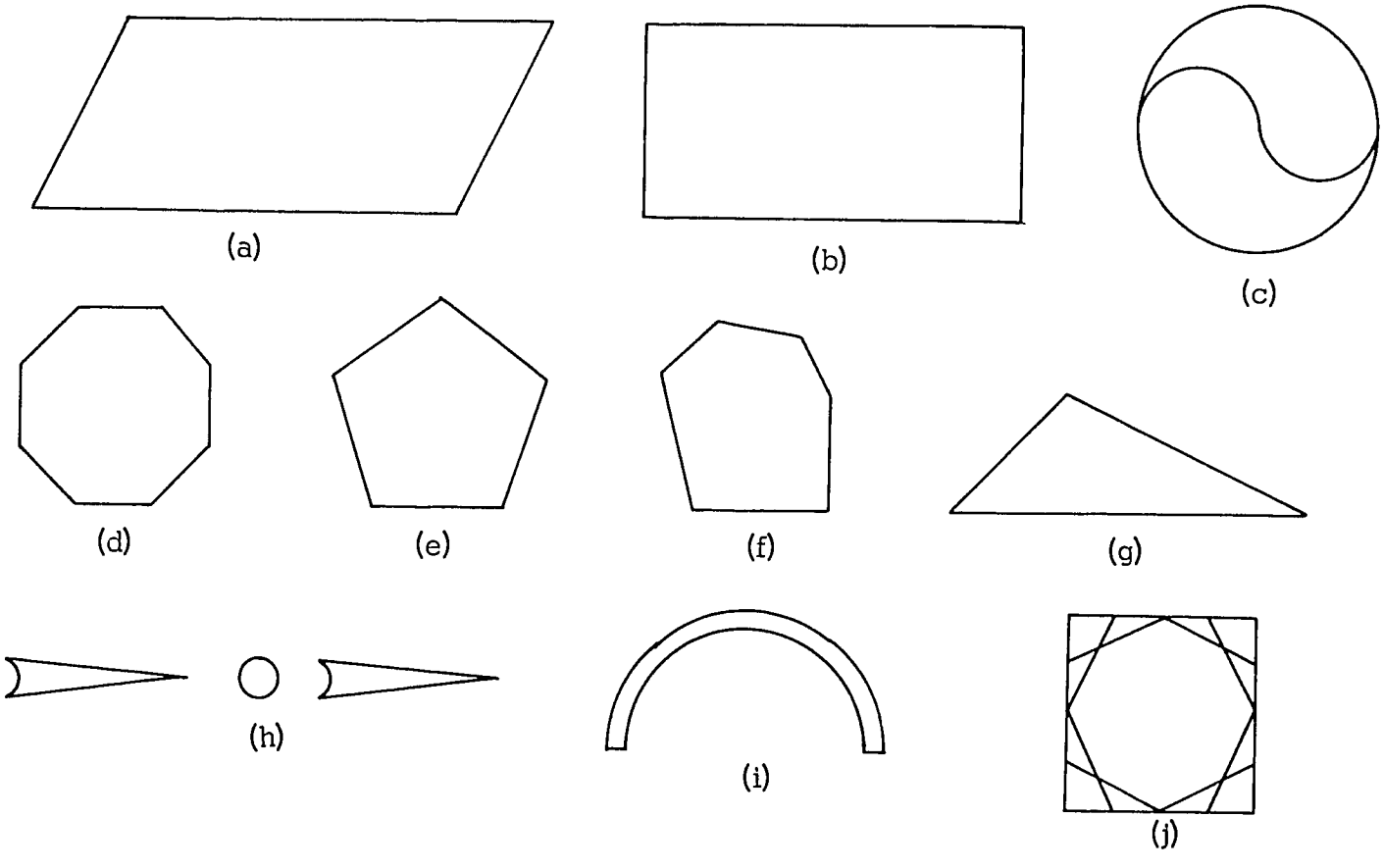
3. The figure below is symmetric to line ℓ.

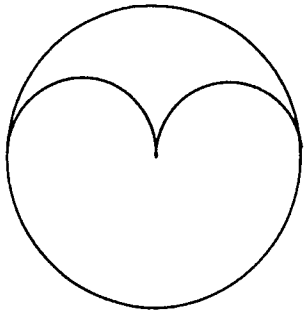
(a) Name three pairs of congruent segments, three pairs of congruent angles, and one pair of congruent triangles.

(b) Does the figure have a second line of symmetry?

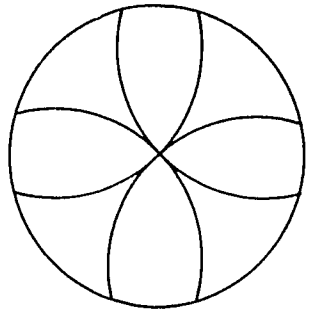


4. Tell which of the following figures do <u>not</u> show symmetry with respect to at least one line.



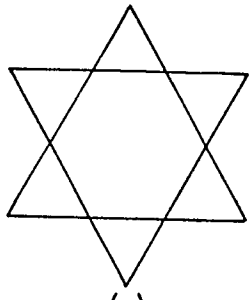


(k)

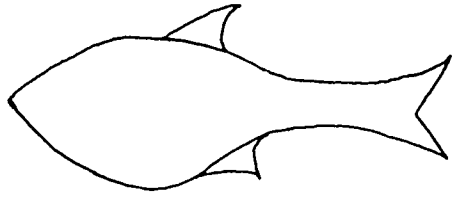


(ℓ)

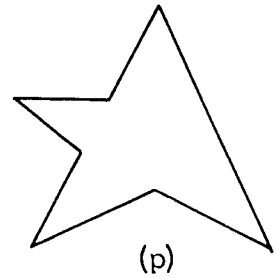
(m)



(n)

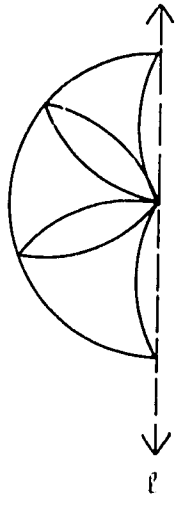


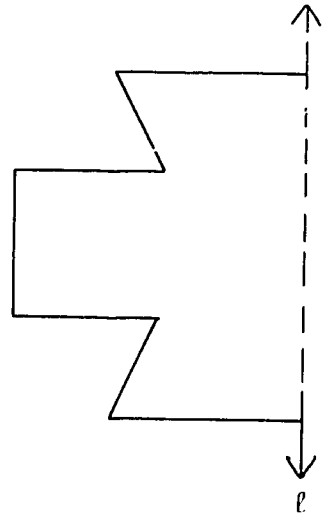
(o)



(p)

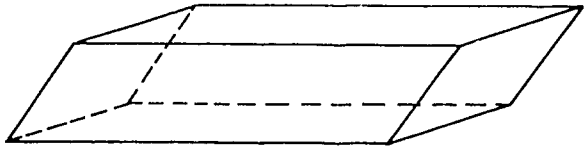
5. If we know that a figure is symmetric to a certain line, we can use this information as an aid in sketching the figure. Each of the pictures below shows the left-hand side of a figure that is symmetric to a line ℓ. Draw these pictures on your paper and then draw the right-hand side of each figure as accurately as you can.



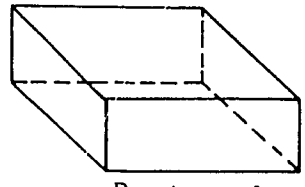


10-5 Symmetry In Space

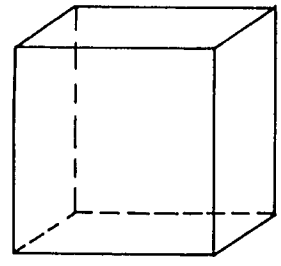
Thus far we have been examining plane figures for symmetry with respect to one or more lines. The pictures below show several solid figures or figures in three dimensions. Do you recognize any symmetry in them with respect to one or more lines? With respect to one or more planes?



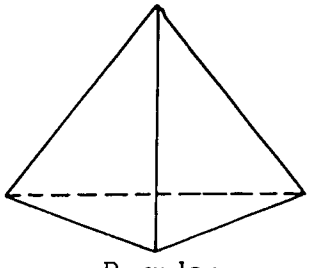
Parallelepiped



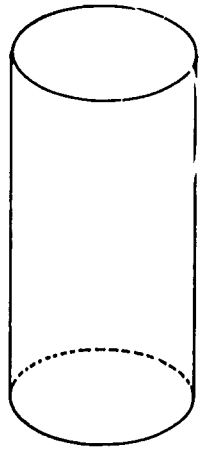
Rectangular Solid



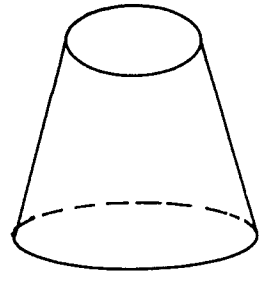
Cube



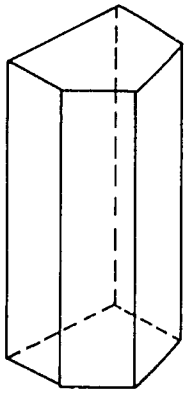
Regular Tetrahedron



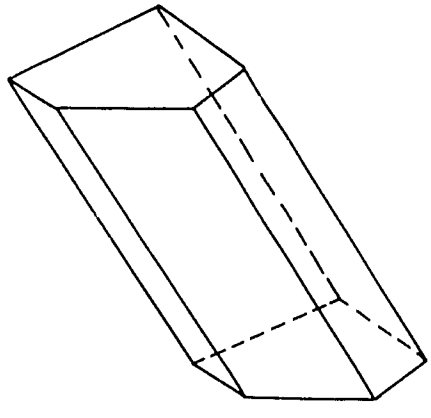
Right Cylinder



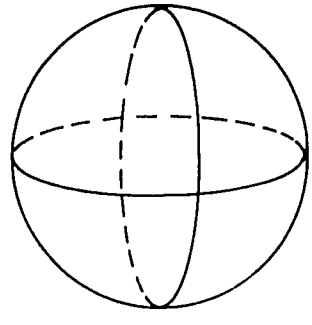
Frustum of Cone



Right Prism



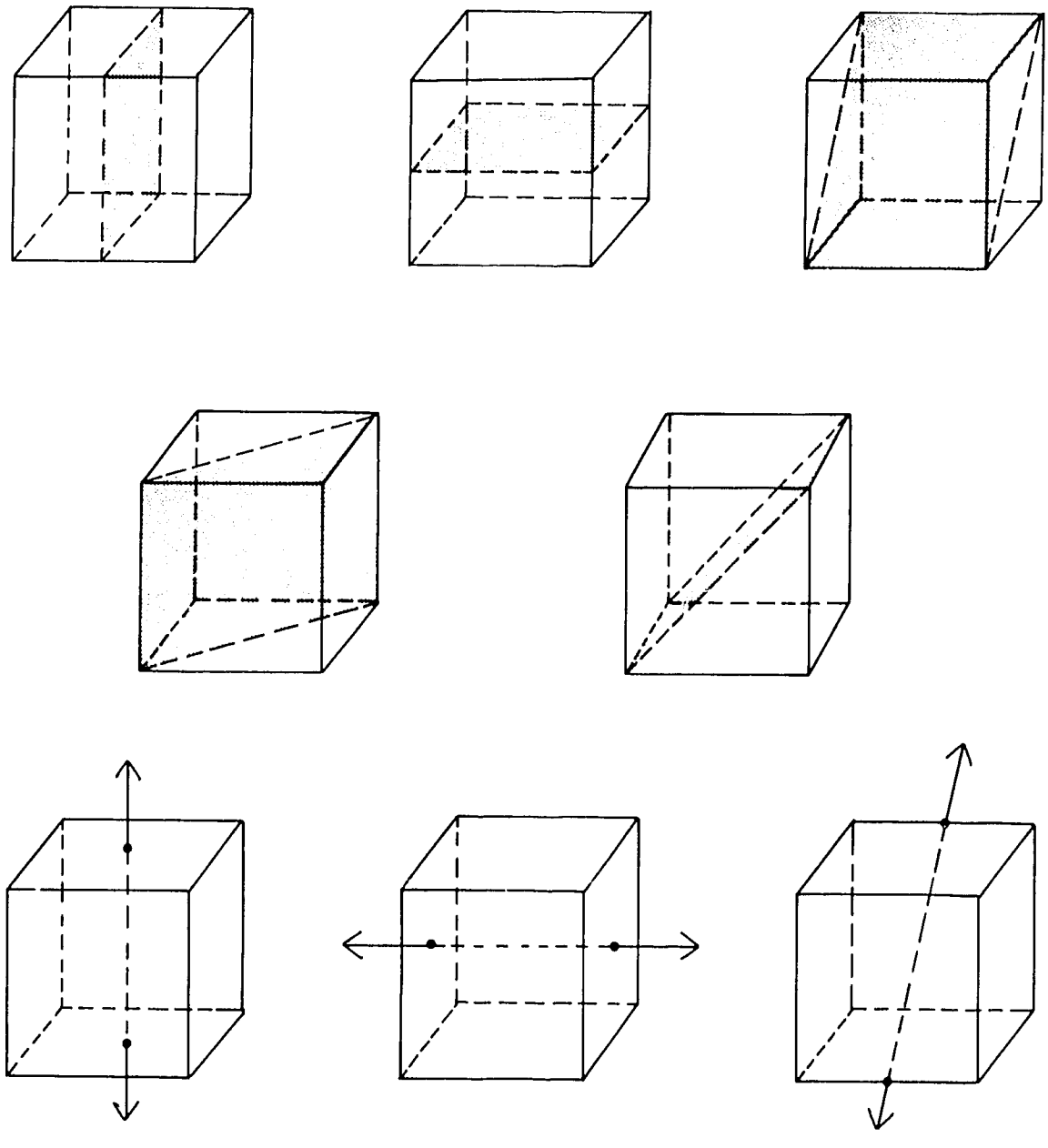
Oblique Prism



Sphere

If a figure is symmetric to a plane, what would have to be true about the parts of the figure on either side of the plane?

The figure below shows a cube being cut by a plane in several different ways.

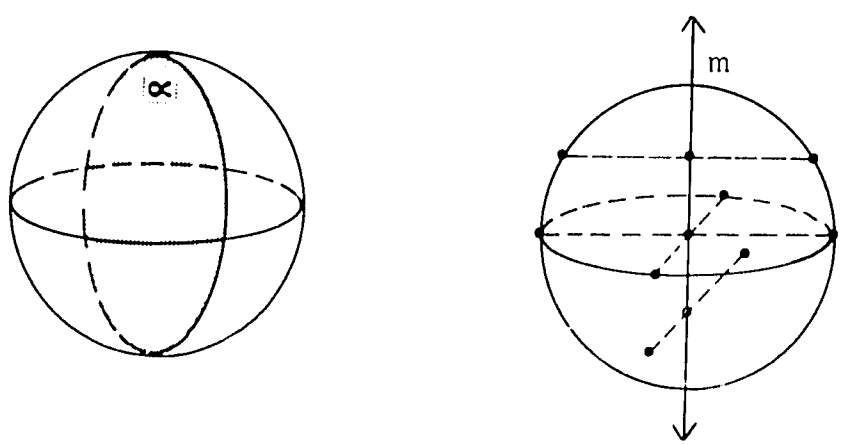


Would you say that the cube is symmetric to the plane in each case? The figure also shows a cube being pierced by a line in three different ways. Would you say that the cube is symmetric to the line in each case?

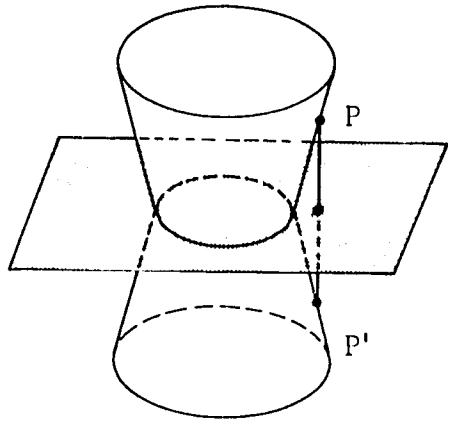
We say that a cube is <u>symmetric to a plane</u> if that part of the cube on one side of the plane is a mirror image of that part on the other side of the plane. We call the plane a <u>plane of symmetry</u>. Are there any other planes of symmetry with respect to the cube different from those pictured above? How many planes of symmetry does a cube have altogether?

In Section 10-2 we showed how to test a plane figure for symmetry with respect to a line. This same test for symmetry with respect to a line holds for a solid figure. A figure (solid or plane) is said to be symmetric with respect to a line if for every point A on the figure there exists a point B on the figure such that the line is the perpendicular bisector of segment $\overline{AB}$. Does a cube have any lines of symmetry other than those shown above? How many?

The figure below shows a sphere symmetric to a plane α and also a sphere symmetric to a line m.



What kind of test could be made for symmetry with respect to a plane? The figure below suggest a test for plane symmetry that is similar to the test for line symmetry. Try to describe this test before reading on.



A figure is said to be symmetric with respect to a plane if the plane bisects all line segments perpendicular to the plane and terminated by the figure. In the figure above, if P is any point of the figure, P' is said to be its symmetric point with respect to the plane if P' is on the figure, $\overline{PP'}$ is perpendicular to the plane, and $\overline{PP'}$ is bisected by the plane. Does the above figure have a line of symmetry? If you think it does, describe the location of this line. Does it have any other planes of symmetry? Describe them.

Problems 10-5

1. Use the models you constructed in Chapter 4 and determine how many different planes of symmetry the following solid figures have. (a) A rectangular solid. (b) A parallelepiped. (c) A sphere.

2. How many lines of symmetry does each of the figures in Problem 1 have?

3. Answer the following questions about the figures pictured on page 240.

   (a) Does the cylinder have any planes of symmetry other than those planes that contain diameters of the circular ends?

   (b) How many lines of symmetry can you find for the cylinder?

   (c) Does a frustum of a cone have any planes of symmetry? How many? Does it have any lines of symmetry? How many?

3. (d) Does the right prism have any planes of symmetry? How many? Does it have any lines of symmetry? How many?

   (e) Does the oblique prism have any planes of symmetry? Any lines of symmetry?

   (f) A regular tetrahedron has equilateral triangles for its four faces. How many planes of symmetry does it have? How many lines of symmetry?

4. Draw some pictures of cubes and shade in the planes of symmetry that were <u>not</u> shown in the figures on page 241.

AFRICAN EDUCATION PROGRAM
MATHEMATICS STEERING COMMITTEE

Professor W. T. Martin, Chairman
Massachusetts Institute of Technology
Cambridge, Massachusetts

Dr. Iya Abubakar
Ahmadu Bello University
Zaria, Nigeria

Mr. James L. Aldrich
Educational Services Incorporated
Newton, Massachusetts

Professor Edward G. Begle
Stanford University
Stanford, California

Professor David H. Blackwell
University of California
Berkeley, California

Mr. Hugh P. Bradley
Educational Services Incorporated
Newton, Massachusetts

Professor Jerome S. Bruner
Harvard University
Cambridge, Massachusetts

Professor Robert B. Davis
Webster College
Webster Groves, Missouri

Professor Robert P. Dilworth
California Institute of Technology
Pasadena, California

Professor Roy Dubisch
University of Washington
Seattle, Washington

Professor Vincent Haag
Franklin and Marshall College
Lancaster, Pennsylvania

Professor Clarence Hardgrove
Northern Illinois University
DeKalb, Illinois

Mr. D. K. Abbiw-Jackson
Kwame Nkrumah University of Science and Technology
Kumasi, Ghana

Mr. Christopher Modu
West African Examinations Council
Yaba, Lagos, Nigeria

Mr. Paul E. Mugambi
Makerere University College
Kampala, Uganda

Mr John O. Oyelese
University of Ibadan
Ibadan, Nigeria

Professor David A. Page
Educational Services Incorporated
Newton, Massachusetts

Professor B. J. Pettis
University of North Carolina
Chapel Hill, North Carolina

Mr. J. E. Phythian
University College
Dar es Salaam, Tanzania

Professor Walter Prenowitz
Brooklyn College
Brooklyn, New York

Professor Alfred L. Putnam
University of Chicago
Chicago, Illinois

Professor Donald E. Richmond
Williams College
Williamstown, Massachusetts

Professor Patrick Suppes
Stanford University
Stanford, California

Dr. Onyerisara Ukeje
University of Nigeria
Nsukka, Nigeria

Mr. Stephen White
Columbia Broadcasting System, Inc.
New York, New York

Dr. Awadagin Williams
University of Sierra Leone
Freetown, Sierra Leone

Dr. Grace Alele Williams
University of Lagos
Lagos, Nigeria

Professor Jerrold R. Zacharias
Massachusetts Institute of Technology
Cambridge, Massachusetts

Ex Officio:
Dr. James R. Killian, Jr.
Chairman of the Board
Educational Services Incorporated
Newton, Massachusetts

Printed in the United States of America