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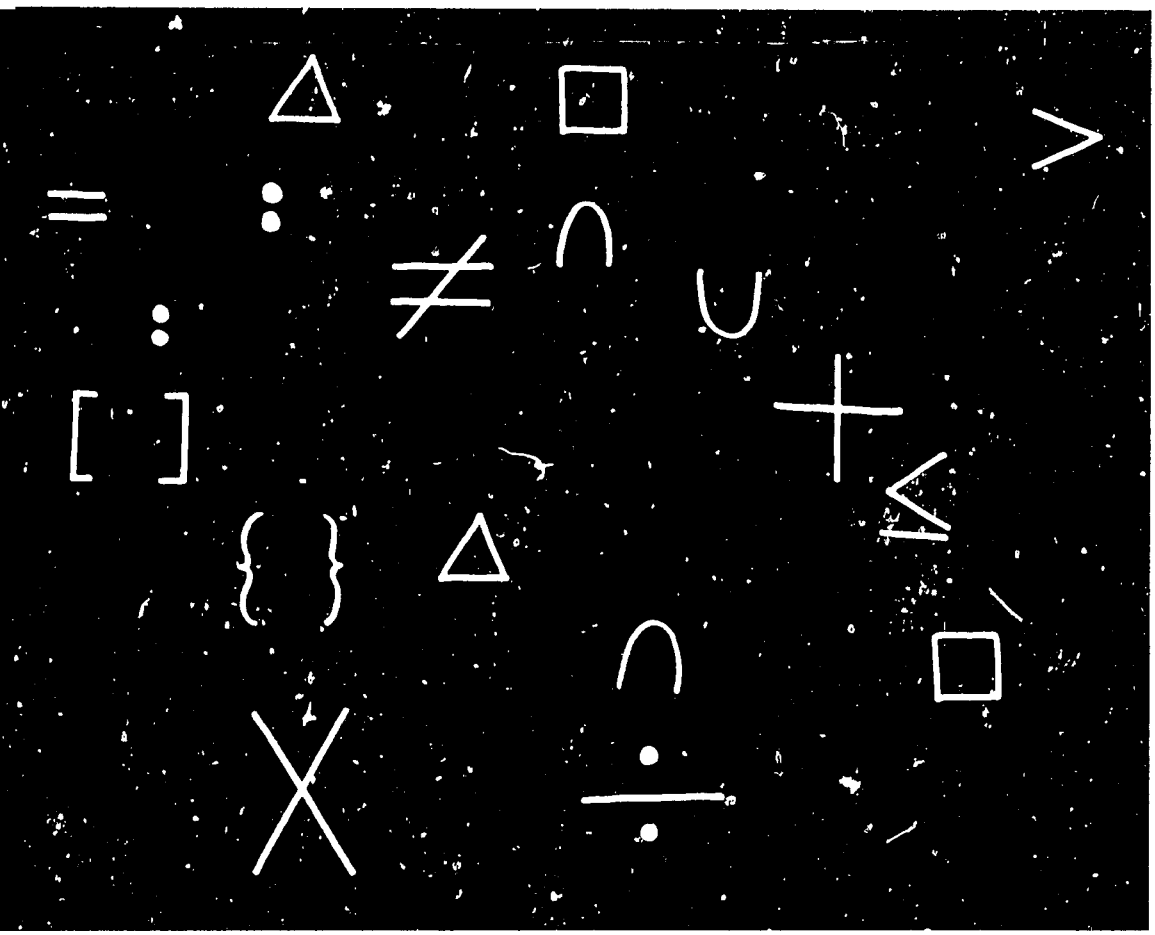

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Prepared at the 1965
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The Mombasa Mathematics Workshop was directed by Professor W. T. Martin of Massachusetts Institute of Technology, Cambridge, Massachusetts and Mr. John Oyelese of the University of Ibadan, Ibadan, Nigeria.

Secondary C One manuscripts were written at the 1965 Mombasa Workshop by the following:

Chairman: Dr. Ross L. Finney

Princeton University, Princeton, New Jersey

Professor Jean M. Calloway

Kalamazoo College, Kalamazoo, Michigan

Professor Roy Dubisch

University of Washington, Seattle, Washington

Mr. Isaac Hunja

Kagumo School, Kiganjo, Kenya

Mr. Edward Kizza

University College, Nairobi, Kenya

Mr. Robert L. N. Offurum

King's College, Lagos, Nigeria

Mr. Joseph P. Pavlovich

Comprehensive High School, Abeokuta, Nigeria

Mr. Richard S. Presser

Elston Senior High School, Michigan City, Indiana

Dean Jerome M. Sachs

Chicago Teachers College North, Chicago, Illinois

Mr. Henry Swain

New Trier Township High School, Winnetka, Illinois

Mr. Robert P. Taylor

Tororo Girls' School, Tororo, Uganda

Mr. J. E. Modupe Taylor-Pierce

Secondary Technical School, Freetown, Sierra Leone

The Student Text and Teachers' Guide were completed and edited in the United States under the direction of Professor Donald E. Richmond, Williams College, Williamstown, Massachusetts. He was assisted by Dr. Ross Finney, Princeton University, Mr. Richard Presser, Elston Senior High School, Michigan City, Indiana, and Dean Jerome M. Sachs, Chicago Teachers College North, Chicago, Illinois.

PREFACE

This Teachers' Guide accompanies the text which was prepared at a study and writing workshop held during the summer of 1965 at Mombasa, Kenya, in which mathematicians and teachers from most English-speaking countries of Tropical Africa, the United States, and the United Kingdom participated.

The material is intended to be used for the Geometry part of a first-year text for a four-year programme in Secondary Mathematics. Much of the material was adapted from texts prepared at Entebbe, Uganda, during the summers of 1962 and 1963. However, alterations have been made as a result of experience gained in the use of the earlier texts.

I. Preliminary Nature of This Material

The material must be considered to be preliminary until it has been tried out in classrooms and modified in accordance with the experience gained from this experimentation. The Student Text and the Teachers' Guide for MATHEMATICS SECONDARY C ONE will be subject to correction and improvement in the light of suggestions from the teachers who use them.

This means that the teacher who uses this material has the responsibility, in addition to helping interpret the new material to the students, of helping identify the areas where improvement is needed. To guide us in future revisions, we hope that teachers will fill out Secondary Chapter Reports. (See sample Report on pages ix, x, xi, xii.) In partial reward for this burden, the teacher will have the satisfaction of knowing that he or she is taking part in an experiment which is of great potential value.

II. Emphasis on Mathematical Ideas

In recent years there has been accumulated much evidence that young students are far more interested in <u>mathematical ideas</u> than they have usually been given credit for, and that they are far more competent to deal with such ideas than the current curricula would suggest. A presentation of mathematics which puts its

emphasis on concepts rather than the rules of manipulation is likely to lead to far greater satisfaction on the part of the student, and will also lead to greater mathematical competence.

III. Content of the Text

The subjects of algebra and geometry are treated in two volumes; one is devoted to algebra and the other to geometry.

Geometry is treated informally. We try to be mathematically sound but at the same time the development is gradual with an effort to involve the students actively. Instead of just telling the ideas to the students outright, the teacher should use problems and careful class discussion to help the students recognize the ideas for themselves.

IV. The General Approach

The best education is the education that the student creates from his own direct efforts. The teacher should resist the temptation to tell the class exactly what to do and how to do it. It is indeed a great temptation, for by such means a class will appear to be proceeding at a rapid pace. If the teacher takes the time and effort to lead the student to think through the idea himself without telling him outright, there is considerably more assurance that the idea will be mastered and retained by the student and will become truly his own.

Here again the demands on the teacher are greater. The teacher, like the student, must be thinking at every moment, for it takes far more insight to lead than to tell. This Guide is intended primarily to assist the teacher in the actual conduct of the class. Methods are suggested which will encourage student imagination and generate student interest. In addition the Guide contains mathematical background and explanatory material beyond that given in the Text. Answers to the problems are provided.

It should be emphasized that this modified curriculum represents a great opportunity for teacher and student alike. In making use of it, the school participates in a great experiment to help develop a strong African educational system of which we can be proud. It is a massive joint undertaking, in which students and teachers work side by side with mathematicians of international eminence, from their own country and from a dozen others, to create within Africa something that will be of major significance for Africa itself, and in a large measure for all the world.

ENTEBBE MATHEMATICS SERIES

SECONDARY _____ REPORT, CHAPTER _____

Would you please fill out one of these forms for each chapter of Secondary material as you complete the chapter. Send it by air mail to educational Services Incorporated, African Education Program, 55 Chapel Street, Newton, Massachusetts 02158, U. S. A. Your comments will help us to improve and to rewrite the material for further use.

Please respond to *all* parts of this report.

Name of Teacher _____

Name of School _____

School Address _____

Date _____

Chapter Number _____ in Secondary Book _____ Volume _____

Chapter Title _____

Number of classes in which you teach the Entebbe Mathematics Workshop Secondary Materials _____

_____

Number of pupils using material _____ . Number of periods each class meets per week _____ .

Length of class period _____ minutes. Class periods used teaching the chapter _____.

The pupils of my class are of the following ability level (tick one of the following):

fast _____ , slow _____ , medium _____ , mixed ability _____ .

My pupils have had, on the average, _____ years of schooling before this present year.

To the best of my judgment this chapter is (tick one of the following): excellent _____,

good _____ , fair _____, poor _____ , or extremely poor _____.

Why did you rate the chapter in this way? (If you need more space than is provided here, please use an additional sheet.) _____

_____

_____

If you used any supplementary materials in teaching this unit, please describe them. _____

_____

_____

Please indicate the specific sections which need improvement and how *you* would recommend improving them.

Questions	Comments
1. Should this whole chapter be omitted? Why?	
2. Name specifically any ideas which you think ought to be in this chapter but which are not there. Why should they be included?	
3. Name specifically any topics in this chapter which you think should be omitted. Why should they be omitted?	
4. Give the section and page number of any part of this chapter where you think the explanation is not clear enough. What suggestions do you have on how to make it more clear?	
5. Indicate which sets of problems need more problems to give adequate practice. What kind of problems are needed?	
6. Indicate by page and number any problems which you think are too difficult and problems which you think are too easy.	

STUDENT TEXT

(continued)

Questions	Comments
7. Indicate any problems or illustrations which are ineffective because they use words or ideas which are unfamiliar to your students. For instance, a child who had never been to sea in a schooner would not much appreciate a problem about the area of the jib and the foresail and the mainsail. For the examples which are inappropriate for your children, can you suggest local situations which would be more appropriate in the example.	

TEACHERS' GUIDE

Questions	Comments
1. In what parts of the chapter were the suggestions on how to teach the material not sufficient to help you teach it effectively? What suggestions have you for improving those parts?	
2. In what parts of the chapter were the suggestions on teaching the material trivial and unnecessary?	

Questions	Comments
3. In some places in the Teachers' Guide there are discussions for the teacher of the mathematical background, going more deeply into the ideas than one would with the students. Indicate by section and page number: (a) which of these discussions were not clear enough to be helpful, (b) which were too brief and need to be expanded, (c) which were unnecessary for you and might be omitted, (d) where no such discussion was given where it was needed.	
4. Indicate any answers to problems which were not sufficiently clear or complete.	

Would you please write any other comments which you believe might be helpful in rewriting this chapter.

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CHAPTER 1

SOME ELEMENTARY IDEAS AND FIGURES

1-1 Introduction

The analogy that we make here between numbers in arithmetic and points is intended to get the pupil to realise that numbers and points are not physical objects that we can see or feel.

Just as we invent numbers to help us to classify sets of objects as to how many objects the sets contain and perform operations on these numbers, we invent points to help us to classify certain geometric figures and to study some of the properties they possess and the relations these figures have to the real world.

However, you should not overly stress these concepts and do not be concerned if your pupils do not readily accept the idea that points exist only in our minds. As they mature mathematically, these ideas will become much easier for them to accept.

1-2 Points

The important thing here is for the pupil to recognize that a point has position but not size. You might draw a picture like this on the blackboard:

- and ask which is a better picture of what we imagine a point to be.

Answers to Problems 1-2 | Student Text Page 3

1. The corner of a book, a desk, etc.; a corner of the room where the ceiling, front, and side walls come together; the tip of a pencil; the sharp end of a pair of compasses; etc.

2. The pupil might think that there is a finite number of points between any two points. You might tell him that the pictures we draw of points soon begin to overlap because the pictures have size, but if we imagine a point to have no measurable size and if we imagine there is at least one point between any two points, then there is no end to the number of points between two given points.

3. One; eight. Here, again, we are hoping the pupil will realise that there are an infinite number of points on the top cover or the edge of a book.

4. (a)

(b)

(c)

(d)

1-3 Straightness

Your pupils should already have an idea of what straightness means. This is another concept in mathematics that is usually taken as undefined because most pupils have an intuitive notion of straightness and it is impossible to define every concept. Some terms must remain undefined, and in geometry the concepts of point, line, plane, straight, and between fall into this category. We try, by referring to physical objects and pictures, to give the pupil an intuitive feeling for these concepts, but make no attempt at formal definitions.

Your pupils should test their rulers for straightness by using a folded paper or a stretched string and perhaps test one or two other objects for straightness, but you should not dwell on this subject very long.

Answers to Problems 1-3

Student Text Page 6

1. The edge where two vertical walls come together in a room; the edge of a table; an edge of a box; the edge of a metre stick or a yard stick; the edge of a desk; etc.

5. (a), (c), and (d) are straight.

7. P is on the line of sight from A to B.

1-4 Line Segments

The important thing here is that your pupils realise that a line segment is a set of points, it is straight in the sense that all of its points lie on the same line of sight, and it has two endpoints.

The concept of betweenness is introduced here. We say that one point is between two other points if no two of the points are the same and all three points are on the same line of sight. You should draw some pictures of sets of three points on the blackboard, some of which are on the same line of sight, and some of which are not, and ask your pupils to tell which pictures show one point between the other two. The idea of betweenness is a simple one and the pupil can always rely on the pictures he sees to decide when one point is between two others.

Using the undefined terms point and between, we can now define a line segment $\overline{AB}$ as the set consisting of points A and B and all points between A and B.

Answers to Problems 1-4

Student Text Page 9

1. Five distinct points on a line segment determine ten betweenness relations. They are:

1. (a) E is between A and C.

   (b) E is between A and D.

   (c) E is between A and B.

   (d) C is between A and D.

   (e) C is between A and B.

   (f) C is between E and D.

   (g) C is between E and B.

   (h) D is between A and B.

   (i) D is between E and B.

   (j) D is between C and B.

2. (a) Not true.

   (b) Not true.

   (c) True.

   (d) True.

   (e) Not true.

   (f) Not true.

   (g) True.

   (h) Not true.

   (i) True.

3. (b) and (d).

4. (a) 5; 3; 4.

   (b) D.

   (c) B and G. (Each is the endpoint of 6 segments.)

   (d) 5. (They are line segments $\overline{AE}$, $\overline{CD}$, $\overline{CF}$, $\overline{DE}$, and $\overline{DF}$.)

5. (a) 3 points, not all on the same line segment, determine 3 line segments.

(b) 4 points, no three on the same line segment, determine 6 line segments.

(c) 5 points, no three on the same line segment, determine 10 line segments.

(d) 6 points, no three on the same line segment, determine 15 line segments.

(e) The pupil should see that for each additional point, he can draw as many new line segments as he had points before. Thus if he adds a seventh point, he can draw 6 new line segments, giving him a total of 21.

1-5 Rays and Straight Lines

In this section we again use the concept of a line of sight to develop the ideas of ray and line. The important ideas are that a line has no endpoints and extends indefinitely in both directions and that a ray has one endpoint and extends indefinitely in one direction.

Emphasise that rays and lines are sets of points. We could define ray $\overrightarrow{AB}$ as the set of points consisting of points A and B, all points P such that P is between A and B, and all points X such that B is between A and X. However, the phrase "ray $\overrightarrow{AB}$ is the ray from A through B" is easier for the pupil to remember and is suggested very nicely by the symbol for a ray, $\overrightarrow{AB}$.

Although there are many different "lines" through two given points, there is just the one "line of sight" through them, and hence just the one straight line. Since we are primarily interested in straight lines, emphasise to your pupils that when we use the word "line" in this text, we shall mean "straight line" unless we specify otherwise.

Be sure that the pupils understand what is meant by opposite rays. We shall use this concept again in a later chapter when we define a straight angle. We say that two rays are opposite rays if they have only their endpoints in common and lie on the same line of sight.

Answers to Problems 1-5 | Student Text Page 14

1. The ray from Q through P or ray $\overrightarrow{QP}$; the ray from M through X or the ray $\overrightarrow{MX}$; the ray from S through T or ray $\overrightarrow{ST}$.

2. (a) $\overleftrightarrow{AB}$; $\overleftrightarrow{BC}$; $\overleftrightarrow{CA}$.

   (b) $\overline{AB}$; $\overline{BC}$; $\overline{CA}$.

   (c) $\overrightarrow{AB}$; $\overrightarrow{BC}$; $\overrightarrow{CA}$; $\overrightarrow{BA}$; $\overrightarrow{CB}$; $\overrightarrow{AC}$.

3. The line segment $\overline{AB}$ or $\overline{BA}$.

4. No; they are not on the same line of sight.

5. No; they are not on the same line of sight.

6. This set defines ray $\overrightarrow{BA}$.

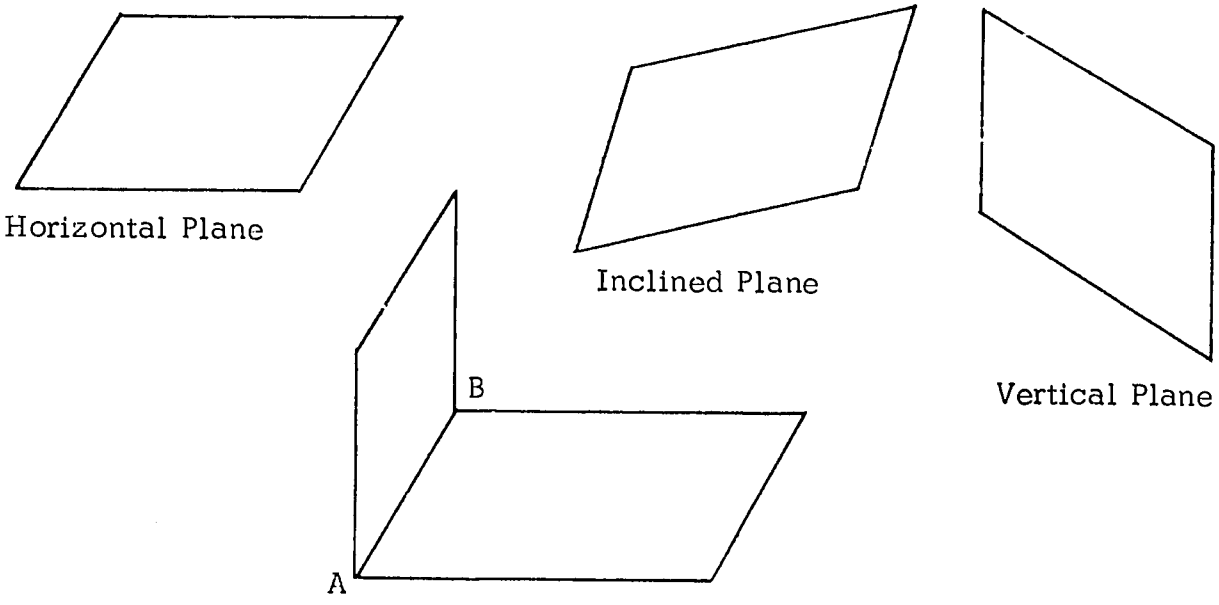
7. A ray has (a) one endpoint.
   A line has (d) no endpoint.
   A line segment has (b) two endpoints.

1-6 Flatness and Planes

The idea of flatness is important to the pupils' concept of a plane. Emphasise that a surface need not be level to be flat. Most of your pupils will already know what we mean by a flat surface, so that you needn't spend much time discussing this concept. They should use their rulers or a stretched string to test one or two surfaces for flatness.

What is important here is that they recognize that any flat surface can represent a plane and that a plane is a set of points.

Some of your pupils may not be able to visualize a plane as easily as they could visualize a ray or a line. You should have them draw pictures of vertical, horizontal, and inclined planes, and have them draw a picture of a vertical plane and a horizontal plane meeting or intersecting each other as in the figures below.



Vertical and Horizontal Planes
Intersecting in Line $\overleftrightarrow{AB}$

Emphasise that the pictures we draw of planes are necessarily bounded, and hence represent only parts of planes. A plane itself has infinite length and width, but no thickness.

Answers to Problems 1-6 | Student Text Page 17

1. (a) Yes.

   (b) Yes.

   (c) Yes.

2. A line segment; Yes; One (The three planes have just the corner point in common.)

3. Yes.

4. Yes. (If a line pierces a plane such as a pin pierces a paper, the line and the plane have only the one point in common.)

1-7 Regions

In this section we define a triangle, a triangular region, a rectangle, and a rectangular region. Be sure that your pupils understand that a triangle is a set of points and it is the set of points that lie on the three line segments that form the triangle. Too often, pupils think of the triangle as being the set of points that are in the interior of the triangle. Similarly for a rectangle or any polygon.

If A, B, and C are any three points not on the same straight line, we could define triangle ABC as the union of the three line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.

A triangular region is a set of points consisting of the points in the interior of a triangle and the points on the triangle. Similar definitions could be given for a rectangle and a rectangular region.

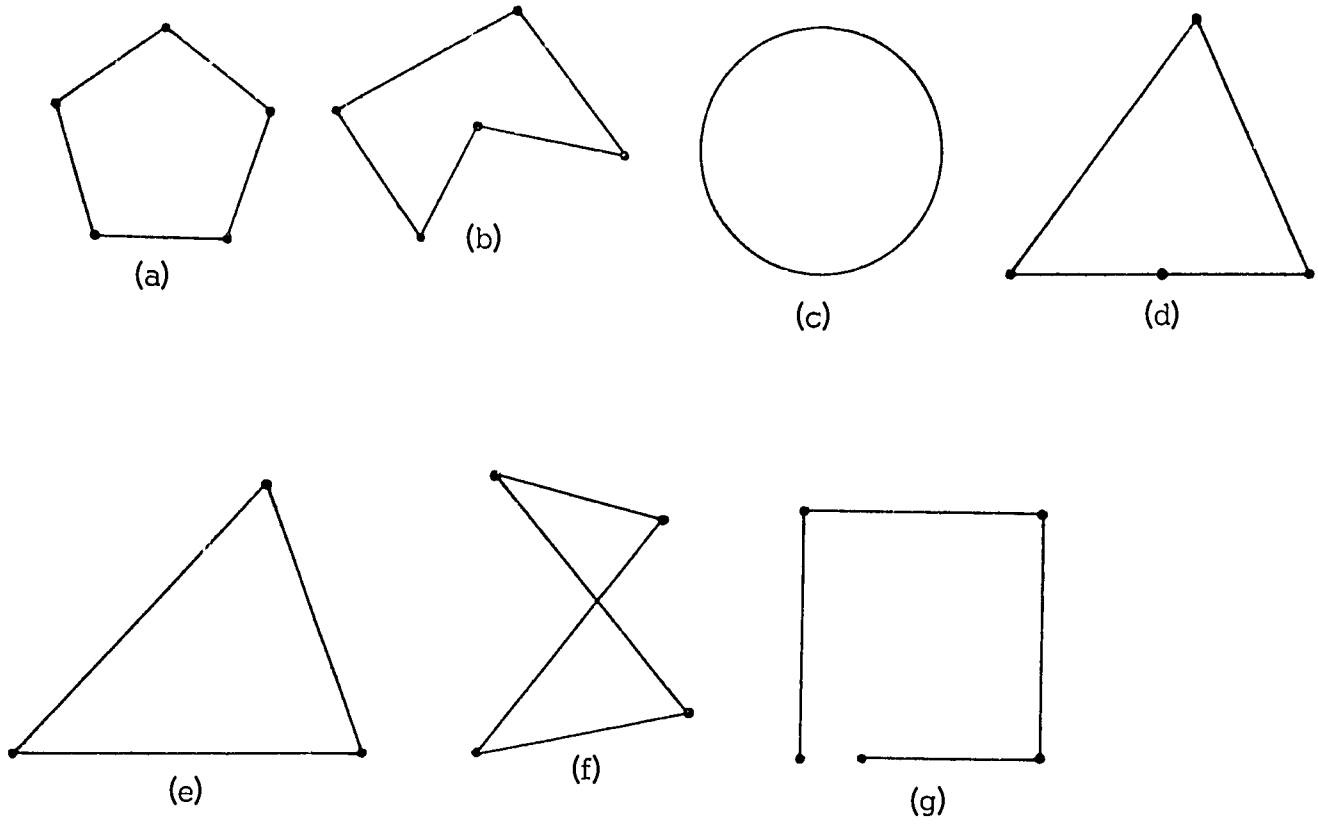
1-8 Paths

We include this section so that the pupil will know what is meant by a simple open path and a simple closed path. We are primarily interested in those simple closed paths that consist entirely of line segments placed end to end successively such that:

(a) No two segments cross each other except at an endpoint.

(b) No two segments with a common endpoint lie on the same straight line.

We call a path such as the one described above a simple polygon, or more briefly, a polygon. Your pupils should draw several simple closed paths — some of which are polygons and some that are not. You could draw some paths on the blackboard like those below, and ask your pupils which of these are pictures of polygons.



For those that are not polygons, you should have the pupil tell why they are not. For example, in the figures above, (c) is not a polygon because it does not consist of line segments; (d) is not a polygon because two of its line segments with the same endpoint lie on the same straight line; (f) is not a polygon because two of its line segments (sides) cross each other; and (g) is not a polygon because the path is not closed.

Answers to Problems 1-8

Student Text Page 21

1. (i) (a), (b), (d), (e), (g), (h)

   (ii) (a), (e), (h)

   (iii) (b), (d), (g)

   (iv) (g)

1-9 Polygons and Polygonal Regions

Your pupils should do the cutting and tracing suggested in this section. If each cut is made along one line segment whose endpoints are on adjacent sides of the polygon, each new polygon will be convex and will have one more side than the previous one.

It is convenient to name polygons of 3, 4, 5, 6, 8, and 10 sides, but your pupils may have trouble remembering these names. They should use the names triangle and quadrilateral for polygons of three and four sides respectively, but we often refer to a polygon with more than 4 sides as a "5-gon" or "5-sided polygon"; a "6-gon" or "6-sided polygon", etc.

Be sure your pupils understand the difference between convex polygons and those that are not convex. We sometimes say a convex polygon is one in which all of the vertices point outward or away from the interior of the polygon. A polygon that is not convex is one in which at least one vertex points inward or toward the interior of the polygon.

A polygonal region is a set of points consisting of the points that are in the interior of the polygon along with the points that are on the sides of the polygon.

Answers to Problem 1-9 | Student Text Page 25

One easy way to subdivide a convex polygon into non-overlapping triangles is to begin at any vertex and draw all the diagonals from that vertex. This will not always work with a polygon that is not convex. However, any non-convex polygon can be subdivided into non-overlapping triangles by drawing certain of its diagonals.

1-10 Drawing a Circle

Most of your pupils may already be familiar with the methods for drawing circles and may have already used a pair of compasses to do so. If you have no compasses available, a "homemade" compass similar to the one described in the student text can be made with a pin, a piece of cardboard, and a pencil.

Emphasise that a circle is a set of points but that the centre of the circle is not a point of the circle. The centre is in the interior of the circle and is necessary in defining the circle, but does not belong to the set we call the circle.

A circular region is the set of points in the interior of the circle along with the set of points on the circle.

A circular arc or simply an arc is a set of points and is part of a circle. Have your pupils draw several arcs of different sizes. They should then name the arc as being semi-circular, minor, or major. Have them draw two minor arcs that intersect in only one point, and then two minor arcs that intersect in two points.

Answers to Problems 1-10

Student Text Page 29

1. A circle is a simple closed path.

2. A circular arc is a simple open path.

3. If the radius is taken to be greater than one inch (greater than half the distance between the two points), the two semi-circular arcs with centres at A and B will intersect in two distinct points. If the radius is taken to be less than one inch (less than half the distance between the two points), the two semi-circular arcs will not intersect.

4. (a) The radius must be greater than $1\frac{1}{2}$ inches (greater than half the distance between the two points).

   (b) The radius must be equal to $1\frac{1}{2}$ inches.

   (c) The radius must be less than $1\frac{1}{2}$ inches.

CHAPTER 2

PLANE FIGURES

2-1 Introduction

This is the first of three chapters on informal geometry. No attempt is made to build up geometry as a system of statements that can be proved from a few assumptions. Rather the student is encouraged to use paper folding, copying of line segments and similar methods to discover relations which exist between different line segments, angles and so on. He is also taught to use a pair of compasses to construct line segments perpendicular to a given line segment, to construct parallels, and to bisect an angle or a line segment.

Our view is that geometry is rooted in experience and that at this stage the student should be taught to organize this experience for himself and not be presented with a ready-made system. We often ask the question "Why?" but we do not expect a long argument for an answer. We are satisfied when the student says something which shows understanding. An answer should be convincing to him and to the other students. It is important that the student gain confidence in the use of his own mind. If we ask for proofs of the obvious or require him to memorize proofs constructed by others which he cannot imagine himself thinking of, we destroy his confidence in his own mental powers.

As we have said, the relations discovered in these chapters are not deduced logically from a basic set of assumptions. However, the interested reader will find in Basic Concepts, Volume III, an introduction to deductive thinking and a systematic ordering of this material which may be useful as a background for teaching.

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2-2 Pairs of Line Segments

If we have two line segments we can ask whether they can be made to fit exactly or whether one of them is larger than the other. This question can be answered by laying one segment along the other or by copying one segment and moving the copy so that a comparison can be made.

We can ask for a line segment which is congruent to a given one but has a given endpoint. We may also require the new line segment to have a definite direction, that is, lie along a given ray drawn from this endpoint. These problems can be solved by using a copy of the given line segment. They can also be solved by using a pair of compasses.

It should be noted that nowhere in this chapter do we compare line segments by measuring them with a ruler. The idea of measurement is postponed until Chapter 5. In terms of the ideas of the present chapter we can say that a certain line segment is longer than another but we cannot say how many times longer it is. We can arrange segments in order of length but we cannot as yet give a number to the length of a segment by choosing a unit of measure. Comparison comes before measurement, both historically and logically.

2-3 Perpendicular Bisector

We learn how to divide a line segment into two congruent pieces. By paper folding and also by using compasses we discover how to find the midpoint of the segment. These constructions give a line segment that not only bisects the given one but makes square corners with it. That is, we get not only a bisector but a perpendicular bisector. Of course, we do not allow ourselves to use a ruler. However, it is worth mentioning that we can bisect a line segment more accurately by paper folding than by the use of a ruler.

Answers to Problems 2-3 | Student Text Page 39

2. If the line segments are "crossing", the point of intersection is not an endpoint of either segment. If they are "touching", the point of intersection must be an endpoint of at least one of them. If they are "connected end to end", their point of intersection must be an endpoint of each of them.

   Yes.

3. 1, 2, or 4

4. (a) crossing
   (b) connected end to end
   (c) touching
   (d) aligned end to end
   (e) non-intersecting

5. The line segments in (b) are congruent. None of the other pairs are congruent.

6. $\overline{OP}$ is the longest segment, $\overline{QR}$ the shortest.

7. $\overline{PR}$ is perpendicular bisector of $\overline{XY}$;
   $\overline{XY}$ is perpendicular bisector of $\overline{DE}$, $\overline{PR}$ and $\overline{ST}$;
   and $\overline{FG}$ has no perpendicular bisector in the figure.

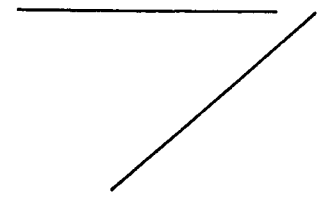
10. $\overline{HC}$, $\overline{SM}$, $\overline{GX}$ and $\overline{RN}$ are not congruent to any of the others. $\overline{PJ}$, $\overline{AD}$, and $\overline{WQ}$ are congruent to each other.

11. Listed from shortest to longest: $\overline{HG}$, $\overline{EF}$, $\overline{AB}$, $\overline{CD}$, $\overline{IJ}$.

12. The two line segments are perpendicular. The midpoint of $\overline{AB}$ is to the left of the point of intersection.

2-4 Pairs of Straight Lines

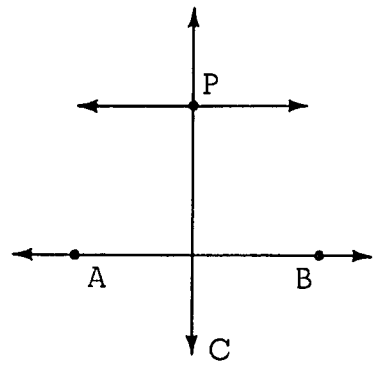
Two line segments in a plane may or may not intersect. But if they do not intersect, this may be because they are too short. If they are extended the



resulting segments may very well intersect. For example, this is true in the figure shown. We are naturally led to consider the result of extending a line segment both ways without end. In this way we come to talk about lines.

Of course, we cannot draw lines. We can draw segments of lines. In practice we draw sufficiently long segments to show the intersections in which we are interested. It is worth noticing that even though we cannot <u>draw</u> a whole line, the <u>idea</u> of a line is useful because it makes things simpler.

It may happen that two line segments in a plane are so placed that no matter how far we extend them there will be no intersection. We then say that the <u>lines</u> to which they belong are parallel. This is fine, but we need a way to decide whether two given lines are parallel or not. For example, if we have a line $\overleftrightarrow{AB}$ and a point P not on it, how can we draw a line through P so that we are sure that it will not intersect $\overleftrightarrow{AB}$?

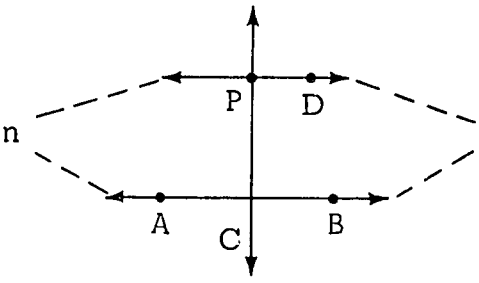


The way this problem is met is to define parallel in a different way. Two lines in a plane are said to be parallel if there is a line to which each of them is perpendicular. This gives us a practical way of solving the problem because there is a simple way of drawing a line $\overleftrightarrow{PC}$ through P perpendicular to $\overleftrightarrow{AB}$ and there is a simple way of drawing a line through P perpendicular to $\overleftrightarrow{PC}$. These constructions are described in this section.

Now if two lines are parallel in the sense that they lie in the same plane and are each perpendicular to a certain line, it is a fact that the lines are parallel in the sense that they do not intersect. This is stated in Problem 3 but not proved.

It will be of interest to explain how this statement might be justified. Let $\overleftrightarrow{PD}$ and $\overleftrightarrow{AB}$ each be perpendicular to $\overleftrightarrow{PC}$. If $\overleftrightarrow{PD}$ and $\overleftrightarrow{AB}$ did meet

off the paper on one side of $\overleftrightarrow{PC}$, then by folding a larger piece of paper over $\overleftrightarrow{PC}$ there should be a corresponding intersection on the other side because the figure is completely symmetrical about $\overleftrightarrow{PC}$. But then we would have two lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{PD}$ intersecting in two different points. This we regard as impossible.



We have not included this argument in the text but it could be given if an alert student raises a question.

Answers to Problems 2-4

Student Text Page 53

1. (a) perpendicular
   (b) parallel
   (c) perpendicular
   (d) neither
   (e) parallel

2. (a), (c) and (d)

3. (a) No, for perpendicular lines form a square angle, and therefore must intersect.
   (b) No, for parallel lines may never intersect.

4. $\overleftrightarrow{ET}$ and $\overleftrightarrow{IH}$ are perpendicular.
   The corners look like square corners.
   Square corners.
   $\overline{BF}$ is congruent to $\overline{TP}$, the corner at B fits the corner at T, and the corner at F fits the corner at P.
   The new line and $\overleftrightarrow{TP}$ are perpendicular,
   the new line and $\overleftrightarrow{PF}$ are parallel,
   and the new line and $\overleftrightarrow{BT}$ are parallel.

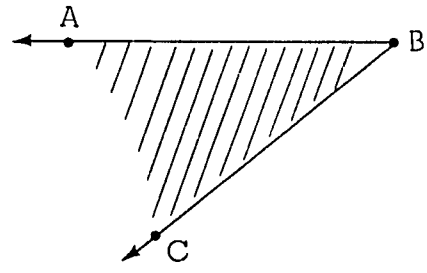
5. (a) $\overleftrightarrow{AI}$, $\overleftrightarrow{CD}$, $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ are perpendicular to $\overleftrightarrow{IH}$, and are also perpendicular to $\overleftrightarrow{LK}$.
   (b) $\overleftrightarrow{AI}$, $\overleftrightarrow{CD}$, $\overleftrightarrow{EF}$ and $\overleftrightarrow{GH}$ are all parallel to each other, and $\overleftrightarrow{IH}$ and $\overleftrightarrow{LK}$ are also parallel.

6. Yes, it is true, because then all three of the lines can be shown to be perpendicular to the same line, and therefore are all parallel to each other.

2-5 Rays and Angles

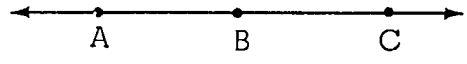
An angle is a figure which consists of two rays with a common endpoint. A right angle and a straight angle are important special cases.

The student is introduced to the notation which will be used for an angle and to the idea of the interior of an angle.



The definition of angle has two consequences of which the teacher should be aware. One is that angles greater than a straight angle are excluded. Thus $\hat{ABC}$ is the angle whose interior is shaded in the figure. Some definitions of angle permit us to speak of the angle outside the shaded region. We shall not do this.

The second point is that, if we apply the directions for finding the interior of an angle to the special case of a straight angle, we obtain a curious result. If we draw all segments that have their endpoints on the sides as described in the text, we obtain



simply the line consisting of the two sides (or rays). If the student should raise this point the simplest way out would be to say that in this special case we could use a different definition of interior and agree that by the interior of $\hat{ABC}$ we shall mean the region on one side of the line $\overleftrightarrow{AC}$, choosing the side to suit the purpose we have in mind.

It should be noted that measurement of angles in terms of a unit, for example the degree, is not discussed in this chapter but is postponed until Chapter 8.

Answers to Problems 2-5 | Student Text Page 58

1. (e) is impossible, because, if the rays intersect at two points, they must lie along one line of sight and therefore must have many points in common.

   (h) is impossible, because, to make an angle, they must have their endpoints in common.

2. Three angles can be named from the three rays: $P\hat{J}S$, $F\hat{J}S$ and $F\hat{J}P$. There are two right angles: $P\hat{J}S$ and $F\hat{J}S$.

3. Four right angles are made by the rays and two straight angles.

4. (a) eight

2-6 Comparing Angles

Earlier we considered comparing line segments so that two segments can be said to be congruent or, alternatively, one segment can be said to be larger than another. Similarly, we can compare angles. Either two angles are congruent to each other or one angle is larger than another. We decide which is true by fitting one ray of one angle exactly to a ray of the other so that the angle interiors overlap. If the other two rays fit exactly, the angles are said to be congruent. If the remaining ray of one angle lies in the interior of the other angle, the first angle is smaller than the second.

See also Section 2-9 where we use a copy of one angle to make the comparison.

Answers to Problems 2-6 | Student Text Page 62

1. $I\hat{H}B$ is larger.

2-7 Angles Made by Straight Lines

When two lines intersect they make four angles that are congruent in pairs. Vertically opposite angles are congruent. The student is expected to see this from the picture. No proof is required.

Strictly speaking, intersecting line segments do not make angles with each other. However, since they determine pairs of rays which are angles, we agree to call these pairs of rays the angles which the line segments make with each other. Similarly, by extending line segments into lines we can talk about parallel line segments when we mean that they are parts of parallel lines.

The method of naming angles used in the first figure in the section is a convenient one, but it should not be misunderstood. The numbers do not label the interiors of the angles. The interior of an angle is not part of the angle. The angle is the figure consisting of the rays. The use of the arrows pointing to the rays should help to keep this clear. Later these arrows may be omitted if there is no danger of misunderstanding. (See, for example, Problem 6 in Chapter 3, Section 3-4.)

Answers to Problems 2-7 | Student Text Page 64

1. Two rays should be parallel if each of them is a part of one of two distinct parallel lines.
   They are parts of parallel lines.
   They are parts of parallel lines.

2. $R\hat{C}B$ and $F\hat{C}H$ are vertically opposite.
   $R\hat{C}F$ and $B\hat{C}H$ are vertically opposite.

3. (a) 1, Nyerere and Salim are perpendicular to Kilindini.
   2 is also perpendicular to Salim.
   2 is parallel to Kilindini.
   1 is parallel to Salim and Nyerere.

(b) At each intersection, the vertically opposite angles are congruent. That is, vertically opposite corners of the street match each other.

5. Other examples are the sides of the door, parallel to each other and perpendicular to the top and bottom edges; the edges of a wall, parallel to each other and perpendicular to the base and top edge; and the side edges of a page in a book, parallel to each other and perpendicular to the top and bottom edges.

2-8 Alternate Interior Angles

When two parallel lines are met by a transversal, the pairs of alternate interior angles are congruent. This important fact is made plausible by paper folding.

In the problems, the student combines this result with his knowledge that vertically opposite angles are congruent to find other pairs of congruent angles in the figure made by two parallel lines and a transversal.

It is a fact not mentioned here that if two lines are met by a transversal so that alternate interior angles are equal, then the two lines are parallel. This will be discovered in Section 3-4.

Answers to Problems 2-8

Student Text Page 68

1. (a) $S\hat{B}Q$, $B\hat{Q}C$ and $N\hat{Q}T$

   (b) $H\hat{B}S$, $R\hat{B}Q$ and $C\hat{Q}T$

3. The smaller angle formed by 2 and 3, the smaller angles formed by Cliffe and Khan, by Cliffe and 3, and the smaller angle formed by 1 and Khan are all congruent.

The larger angles formed by 2 and 3, by Cliffe and 3, by Khan and Cliffe, and by 1 and Khan are all congruent.

2-9 Comparing Angles without Folding

In Section 2-6 we compared two angles directly by trying to fit one to the other. In Section 2-8 we showed by paper folding that certain angles are congruent. In this section we can compare two angles by making a copy of one of them and using the copy to see if it fits the second angle or not. A similar method was used to compare two line segments. To carry it out with angles we must know how to copy an angle. A way to do this with compasses is described.

Answers to Problems 2-9

Student Text Page 73

1. b. is larger.
2. Probably in this order: $B\hat{A}D$, $B\hat{A}E$, $B\hat{A}C$, $I\hat{H}J$, $C\hat{A}E$, $D\hat{A}C$, $E\hat{A}D$, $K\hat{F}G$.
3. $B\hat{A}C$ is very slightly larger.
4. Yes, it shows that they match exactly.
5. They are all the same size.

2-10 Bisecting an Angle

We learned how to divide a line segment into two adjacent congruent segments. We now learn to bisect a given angle, that is, to divide the angle into two adjacent congruent angles. The construction with compasses is simple but important.

Answers to Problems 2-10

Student Text Page 77

1. Fold the paper along $\overrightarrow{TB}$. If $\overrightarrow{TA}$ and $\overrightarrow{TC}$ coincide, then the angles are congruent. If they are not congruent, then the smaller angle will be seen to fall entirely inside the larger. That is, the second side of the smaller angle will be contained in the interior of the larger.

3. Changing the setting will not change the fact that the intersection of the arcs lies on the bisector of the angle. The pattern revealed by the intersections of the arcs is that they all lie on one line, as long as the same setting is used for the two arcs for one intersection. It does not matter whether the same setting is used for the first large arc.

CHAPTER 3

POLYGONS AND OTHER SIMPLE CLOSED PATHS

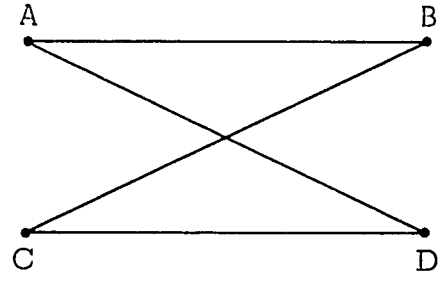
3-1 Introduction

We have met polygons in Chapter 1 and classified them by the number of sides. Now we take a closer look and discuss their angles and interiors. We devote particular attention to triangles and quadrilaterals and notice the consequences of assuming that certain sides are congruent, perpendicular or parallel.

We also study figures called simple closed paths, which share with polygons the property of separating the plane into interior and exterior regions. An important class of simple closed paths consists of circles and ellipses.

3-2 Polygons

The idea of a polygon will be familiar to the student. (It might be well to revise Sections 1-8 and 1-9 in Chapter 1.) We now introduce some of the words that are used to describe polygons, like vertices, sides, angles, adjacent and opposite sides and angles.



The problems should make clear how line segments may be connected to form a polygon. Thus ABCD is not a polygon. It is true that line segments are connected end to end and that each endpoint belongs to exactly two line segments. But $\overline{AD}$ intersects $\overline{BC}$, so that the path is closed, but not simple.

It may be pointed out that our definition of angle and interior has an odd consequence. In the figure at the bottom of page 81 (Student Text) the polygon is not convex, and the interior of $\hat{IHF}$ is outside of the polygon. Our definition does not allow us to talk about angles larger than a straight angle. It seems

Previous Page Blank

strange to call our $\widehat{IHF}$ an angle of the polygon. The cure is to use a more complicated definition of angle. However, this does not seem worthwhile at the present time, since almost all polygons which will be encountered are convex (see Chapter 1, Section 1-9).

Answers to Problems 3-2

Student Text Page 82

1. e. and g. are polygons.

2. The vertices of e. are A, B, C and D; the angles are $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$; the sides are $\overline{AB}$, $\overline{BC}$, $\overline{CD}$ and $\overline{DA}$.

   The vertices of g. are F, E, D, C, B, and A; the angles are $\hat{F}$, $\hat{E}$, $\hat{D}$, $\hat{C}$, $\hat{B}$, and $\hat{A}$; the sides are $\overline{BC}$, $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FA}$, and $\overline{AB}$.

   In e. $\overline{CD}$ and $\overline{BA}$, and $\overline{BC}$ and $\overline{AD}$ are pairs of opposite sides. $\overline{AB}$ is adjacent to $\overline{BC}$ and $\overline{AD}$; $\overline{BC}$ is adjacent to $\overline{CD}$ and $\overline{AB}$; $\overline{CD}$ is adjacent to $\overline{DA}$ and $\overline{BC}$; and $\overline{AD}$ is adjacent to $\overline{AB}$ and $\overline{CD}$.

   In g. $\overline{AB}$ is opposite to $\overline{CD}$, $\overline{DE}$ and $\overline{EF}$;
   $\overline{BC}$ is opposite to $\overline{AF}$, $\overline{EF}$ and $\overline{DE}$;
   $\overline{CD}$ is opposite to $\overline{AB}$, $\overline{AF}$ and $\overline{EF}$;
   $\overline{DE}$ is opposite to $\overline{AB}$, $\overline{BC}$ and $\overline{AF}$;
   $\overline{EF}$ is opposite to $\overline{AB}$, $\overline{BC}$ and $\overline{CD}$;
   and $\overline{AF}$ is opposite to $\overline{BC}$, $\overline{CD}$ and $\overline{DE}$.
   $\overline{AB}$ is adjacent to $\overline{BC}$ and $\overline{AF}$,
   $\overline{BC}$ is adjacent to $\overline{AB}$ and $\overline{CD}$,
   $\overline{CD}$ is adjacent to $\overline{BC}$ and $\overline{DE}$,
   $\overline{DE}$ is adjacent to $\overline{CD}$ and $\overline{EF}$,
   $\overline{EF}$ is adjacent to $\overline{DE}$ and $\overline{AF}$,
   and $\overline{AF}$ is adjacent to $\overline{EF}$ and $\overline{AB}$.

4. b. would be a polygon if line segment $\overline{AC}$ were added. c. would be a polygon if $\overline{AB}$ were removed and $\overline{CD}$ were added.
   h. would be a polygon if all interior points were removed.

5. Polygons are paths because their line segments are connected end-to-end. One can draw a polygon without lifting the pencil. Polygons are closed paths because the "starting point" of the figure is the endpoint of two line segments. Thus there is a line segment that "comes back" to the starting point and makes the path a closed one.

3-3 Triangles

Triangles are the simplest and most important kinds of polygons, and this section should be studied thoroughly.

Triangles may be classified first of all by the number of congruent sides. If two sides are congruent, the triangle is called isosceles; if all three sides are congruent, it is called equilateral. If no two sides are congruent, the triangle is called scalene, although the word is not used in the text at this point.

An important fact brought out in Problem 4 is that if two sides of a triangle are congruent, the angles opposite them are congruent. In Problem 7 we learn that conversely if two angles of a triangle are congruent, the sides opposite to them are congruent. In particular, an equilateral triangle is equiangular and an equiangular triangle is equilateral.

It is possible to have one right angle in a triangle, and triangles of this sort are important enough to have a name. Further names, like obtuse-angled triangle and acute-angled triangle, will be introduced in Chapter 8.

Two triangles are congruent if they fit exactly, or if a copy of one fits the other exactly. The text shows how to construct a copy of a given triangle in three different ways.

Answers to Problems 3-3A | Student Text Page 85

1. An equilateral triangle has three bases as an isosceles triangle, because each side is adjacent to two equal sides.

2. An isoceles triangle, an equilateral triangle

3. $\overline{AC}$, $\hat{A}$.

4. Yes.

5. $\overrightarrow{VP}$ divides $\hat{V}$ into two equal angles. R is the midpoint of the base. The conclusion is that the bisector of the vertex angle of an isosceles triangle bisects the base.

6. The angles must be equal to each other, because the base angles of an isosceles triangle are equal, and an equilateral triangle has three bases as an isosceles triangle.

7. Yes.

8. Yes, because of the answer to Problem 7.

9. No, because the base angles must be equal.

12. It is impossible for a triangle to have two right angles, because the two parallel sides would not meet. If two sides did not meet, then there would be an endpoint belonging to only one line segment, and the figure would not be a polygon at all.

13. No, because then there would be <u>three</u> right angles, and a triangle cannot have even <u>two</u> right angles.

3-4 Quadrilaterals

Quadrilaterals are four-sided polygons. Rectangles are an important kind of quadrilateral in which all the angles are right angles. We discover that opposite sides are parallel and also congruent. But this turns out to be true for any parallelogram, rectangle or not.

The student is taught how to construct rectangles. In order to construct parallelograms that are not rectangles, we need the fact that, if two lines are cut by a transversal so that a pair of alternate interior angles are congruent,

then the two lines are parallel. Compare Chapter 2, Section 2-8.

The problems are very important. Some new results are discovered; for example, the fact that opposite angles of a parallelogram are congruent. Enough knowledge has now been built up so that the student should begin to feel that geometry has some system and order to it. It does not consist of a list of unrelated statements.

The section concludes with a brief treatment of the congruence of quadrilaterals.

Answers to Problems 3-4A

Student Text Page 94

1. b., d., g., i., j., l., m. and o.

2. j. is a square.
   b., j. and o. are parallelograms.
   b. and j. are rectangles.

3. No, because if there are three right angles, then there are two parallel line segments perpendicular to a third, with a fourth line segment also perpendicular to one of the first two. Then this fourth line segment must be perpendicular to the other one of the first two line segments, also, and there are now four right angles in the quadrilateral.

4. Yes, because it is a rectangle, which is a parallelogram.

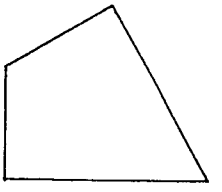
5. Yes, because opposite sides of a rectangle are congruent and therefore all sides must be congruent to the two congruent adjacent sides.

6. 1 is congruent to 3, 5, 7, 9, 11, 14 and 16.
   8 is congruent to 2, 4, 6, 10, 12, 13 and 15.

   Opposite angles are congruent, because, for instance, 9 is congruent to 14 since they are corresponding angles, 14 is congruent to 3 since they are corresponding angles, and 3 is congruent to 1 since they are vertical angles. Therefore 9 is congruent to 1.

7. Yes, there are parallelograms that have four congruent sides and that are not squares. To construct such a parallelogram, draw a ray $\overrightarrow{AQ}$ and mark a line segment $\overline{AB}$ on $\overrightarrow{AQ}$. Draw a ray at B that is not perpendicular to $\overline{AB}$. Mark a point C on the ray so that $BC = AB$. Then at A draw ray $\overrightarrow{AR}$ so that $R\hat{A}B$ is congruent to $C\hat{B}Q$. Then mark a point D on $\overrightarrow{AR}$ so that $AD = AB$. Then draw $\overline{DC}$. ABCD is a parallelogram with congruent sides.

9. Yes, yes.



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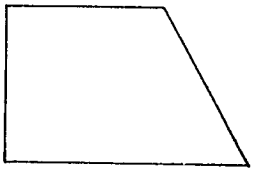


are examples.

10. Opposite sides of a parallelogram are congruent.

11. No, because there are many parallelograms that are not rectangles, and these parallelograms have congruent opposite sides.

12. No, because the quadrilateral you constructed in Problem 7 has congruent sides but not congruent angles.

3-5 Plane Paths, Circles and Ellipses

A path that does not cross itself is called simple even if, as in Problem 18, it has many twists and turns. We are concerned in this section and the next with simple closed paths. These figures, like polygons, separate the plane into two regions, interior and exterior. The most important simple closed paths are circles and ellipses.

It might be interesting to add the following remark at the end of the section. When a stone is thrown into the air, if it were not for the effect of the friction of the air, the stone would follow a path which is an arc of an ellipse with focus at the earth's centre.

Answers to Problems 3-5 | Student Text Page 101

1. b., c., d., e., f., g., i., k., l. and m. are paths.
   d., g., i. and m. are closed.
   b., c., d., e., f., g., k., l. and m. are simple.

2. This is possible by naming the points A and B, taking any point C on the perpendicular bisector of $\overline{AB}$, setting the compasses so that they reach from C to A, and drawing the circle with radius $\overline{CA}$, with C as centre.

3. Name the three points A, B, and C. Draw the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$. Name the point of intersection of the bisectors D. Then draw the circle with radii $\overline{DC}$ and $\overline{DB}$, and centre D.

4. Choose a point A on the circle. Set the compasses so that they reach from C to A. Then draw a circle with the compasses at this setting.

8. It looks like a circle.

10. Yes, it is a perpendicular bisector.

11. No, they do not always bisect each other.

12. Yes.

3-6 Simple closed paths and regions in a plane

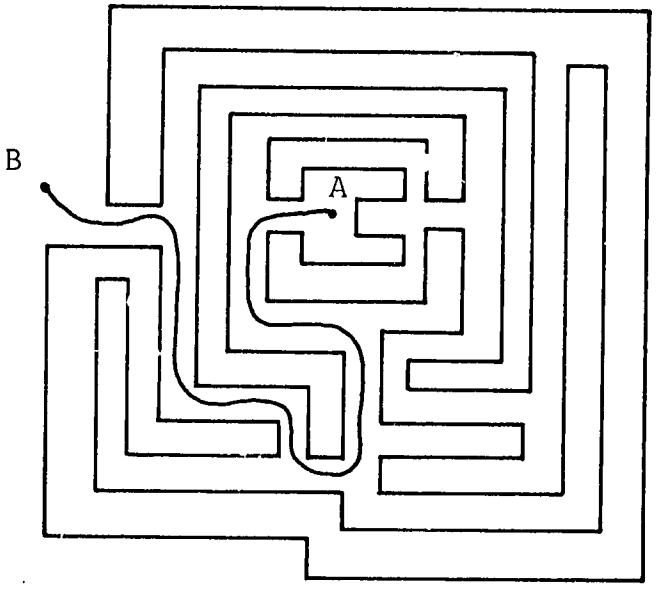
This section requires little comment. The test whether two points are both outside or one inside and one outside (Problem 5) should prove interesting. You might illustrate the even-odd rule with some other simple cases.

Answers to Problems 3-6

Student Text Page 108

1. A and E, A and D, A and C, B and E, B and D, and B and C are pairs that are separated by the line. A and B, E and D, E and C, D and C are pairs that are on the same side of the line.

2.



B is obviously outside the given path and A is connected to it by a path that does not cross the given path.

4. A is inside, D is outside.

5. B to A: 7 times
B to D: 4 times
B to C: 3 times
E to A: 3 times
E to D: 6 times

If the points lie on the same side of the path, the line segment connecting them crosses the path an even number of times. Otherwise, it crosses the path an odd number of times. The line segment connecting B and F crosses the path 8 times. Therefore F is outside the path.

6. AC is less than or equal to the radius
AC is less than the radius
AC is equal to the radius
AC is greater than the radius

CHAPTER 4

FIGURES IN SPACE

4-1 Introduction

This chapter is concerned with important figures in space — "bricks", tetrahedrons, pyramids, cones, prisms, cylinders and spheres. It also considers relations between points, lines and planes in space. As in previous chapters, the treatment is informal and is tied to physical experience. Since space figures are often hard to visualize and also hard to draw, it is important to construct models to assist the students' imagination. It may be desirable to help the students in constructing these models by making it a class activity.

4-2 Surfaces and Solids

This material is mainly descriptive and needs little comment. The join of a figure with a point is easy to understand and should be a useful unifying idea.

The students may need help with the constructions which are described.

Answers to Problems 4-2 | Student Text page 116

1. CDE is the join of $\overline{DE}$ with C,
   CEA is the join of $\overline{AE}$ with C,
   and CAB is the join of $\overline{AB}$ with C.

2. A cricket ball is a solid ball. Its surface is a sphere. A round balloon is an example of a sphere.

6. Yes, for if its vertices are A, B, C and D, it may be thought of as the join of A with ΔBCD, the join of B with ΔACD, the join of C with ΔABD, or the join of D with ΔABC.

7. The surface of a solid circular cone

4-3 Pyramids

The students should enjoy constructing these models and filling out the table in Problem 1. The Euler number will be met again in connection with prisms (Section 4-5, Problem 8).

Answers to Problems 4-3

Student Text page 120

1.

No. of sides of base	3	4	5	8	18	N	6	9	14
F, no. of faces	3 + 1	5	6	9	19	N + 1	7	10	15
E, no. of edges	3 × 2	8	10	16	36	2N	12	18	28
V, no. of vertices	3 + 1	5	6	9	19	N + 1	7	10	15
F - E + V	2	2	2	2	2	2	2	2	2

4-4 Points, Lines and Planes in Space

This is a very important section. The suggested experiments with a book and the classroom door should develop the students' ability to picture the set of planes through a given line.

The definition in the box on page 126 is clear but impractical, since it requires testing the perpendicularity of a line L with <u>every</u> line through its intersection P with the plane. There are of course an infinite number of lines through P in the plane. Fortunately, it is necessary to test the

perpendicularity of L with only <u>two</u> lines through P. It is possible to find the perpendicular to a plane at a point P by using a square corner that has been partly opened. This is a very simple trick.

We next take up parallel planes. You will remember that we said that two lines in a plane are parallel if they are each perpendicular to a given line. In the same way, we now say that two planes in space are parallel if they are each perpendicular to a given line.

In Chapter 2, Section 2-4, Problem 4, we showed that if a line is perpendicular to one of two parallel <u>lines</u>, it is also perpendicular to the other. It is similarly true that if a line is perpendicular to one of two parallel <u>planes</u>, it is perpendicular to the other. However, we do not show this in the text. You may have occasion to mention it, but we have no direct use for it.

In the final two paragraphs we talk about a brick. This figure has not been defined. The definition comes in Section 4-5 where a brick is defined as a right rectangular prism. It does not follow without some argument that <u>all</u> the faces of a brick are rectangles, but we assume that this is the case. With this assumption we see that the base and top lie in parallel planes. Since $\overleftrightarrow{BD}$ is perpendicular to two lines in each plane, $\overleftrightarrow{BD}$ is perpendicular to each plane and therefore the planes are parallel by definition. Note that we are here using our simple test for perpendicularity of a line to a plane.

<u>Answers to Problems 4-4</u> | Student Text page 128 |

1. Possible answers are:

   (a) the side edges of a door

   (b) the edge of a table and a leg that adjoins it

   (c) the two sides of the letter A

   (d) a side edge of a window and the bottom edge of a window on another wall

2. One example might be two pencils lying on a desk. They determine the plane that contains the surface of the desk top.

3. A notebook cover can be swung about its hinge.

4. The ceiling and the floor of a room are often parallel.

6. Poles for carrying telephone lines or electric wires.
   The legs of many tables.

8. The path is perpendicular to the plane of the ground underneath the tree.

4-5 Prisms and Cylinders

The general concept of cylinder is introduced. Some authors restrict the term cylinder to the case where the base has a curved boundary. However, we take a prism to be a kind of cylinder also — a cylinder whose base is a polygonal region. This seems simpler, since the underlying idea of a cylinder is the same whether the base has a curved boundary or not.

This is an important set of problems and they should be answered with care. Note that in Problem 4 we cannot prove that the face opposite the base is parallel to it, but the model makes it plausible. Perhaps it is well for the student to realize that there is more to geometry than he has yet learned. The evidence of the model could be supplemented by the following remarks. A vertical edge (directrix) of a triangular prism is perpendicular to the base. Is it also perpendicular to the top? Would the final figure be any different if the parallelograms were attached to the bottom triangle?

Answers to Problems 4-5 | Student Text page 131

1. 6 faces, 12 edges and 8 vertices. Any face might be a base.
   1 is adjacent to 2, 4, 5 and 6
   2 is adjacent to 1, 3, 5 and 6
   3 is adjacent to 2, 4, 5 and 6
   4 is adjacent to 1, 3, 5 and 6
   5 is adjacent to 1, 2, 3 and 4

6 is adjacent to 1, 2, 3 and 4

1 is opposite to 3

2 is opposite to 4

5 is opposite to 6

Pairs of opposite faces are parallel and congruent. The reason for the fact that they are parallel is actually rather complex to state, but the student understands well enough if he notices that there is a line that is in a face adjacent to both, and that is perpendicular to both, or if he notices that opposite faces are in some sense perpendicular to a single face that joins them. It is also indicative of understanding if the student notices that there are four parallel segments of equal length joining the vertices of a pair of opposite faces.

2. The prism is a right prism.

   Either of the triangles is a base.

   The faces are parallel. The reason given for this could be similar to that in Problem 1.

   There are 5 faces, 9 edges, and 6 vertices.

3. 6 faces, 12 edges and 8 vertices. Any face could be a base.

   Opposite faces are congruent and parallel.

4. It is a triangular prism. Either triangular face can be considered a base. The face opposite the base is congruent and parallel to it.

6. The circular bases must be added in order to make it the surface of a cylinder.

7. Unsharpened pencils, wood poles, many table legs.

8.

No. of sides of base	3	4	6	8	10	N	5	11	7
F, no. of faces	3 + 2	6	8	10	12	N + 2	7	13	9
E, no. of edges	3 × 3	12	18	24	30	3N	15	33	21
V, no. of vertices	3 × 2	8	12	16	20	2N	10	22	14
F - E + V	2	2	2	2	2	2	2	2	2

9. The Euler number of a pyramid or prism is always 2.

10. (a) false (d) true

(b) true (e) false

(c) true (f) false

11. Yes, for each vertex of the base has another vertex of the prism attached to it by one of the line segments parallel to the directrix. Thus, if the base has N vertices, the prism has 2N vertices.

12. Yes

13. No, because a triangular prism may be a right prism, and may have two edges of the base that meet in an acute angle.

14. Yes, because the face opposite is matched exactly with the base, by means of the line segments that are parallel to the directrix.

4-6 Closed Surfaces

Closed surfaces in space are like closed paths in the plane. A closed surface separates space into an inside part and an outside part. There are surfaces that are not closed. The most important are planes. Although a plane

does not separate space into an inside and an outside, it does of course separate space into two parts just as a line separates a plane into two parts. There are two sides of a plane. Two points lie on the same side of a plane when the line segment which connects them does not pierce the plane.

A sphere with a hole cut in it is not closed and it does not separate space into two parts. This is analogous to a circle with an arc cut out.

Answers to Problems 4-6 | Student Text page 137

1. No, because in space a point not on the circle may always be connected with another point not on the circle by going over, under, or around the circle.

2. No, because in space one can always go over or under a line, just as one jumps over or crawls under a horizontally stretched rope.

3. Yes, for if one were inside a sphere, for instance, one could not get outside without breaking the sphere.

4. No, for if the ball is solid and one is not at a point of the ball, one can get to any other point not in or on the ball just by going over, under, or around the ball.

5. Yes. Yes, it keeps out air so that the filament will not burn.

6. Yes, if the doors and windows are closed.

7. Yes, it keeps in air.

4-7 Conclusion

This requires no comment.

CHAPTER 5

LENGTH

This is the first of five chapters devoted to measurement. It is concerned with the measurement of length. Later chapters will consider the measurement of area, volume and angle. In each case the first step is the choice of a unit. To measure length we choose a unit of length. To measure area we choose a unit of area. To measure volume we choose a unit of volume.

5-1 Counting and Measuring

To measure the length of a line segment we choose a particular line segment and assign to it the number 1. We call this particular line segment a unit of length. It may be a stride, a step or a span as in this section. Later we introduce the more customary units — inch, foot, yard, centimetre and metre — and explain the reasons for doing so.

The report of a measurement includes two items, a number and a unit. The number is called the measure. It tells the number of units that must be combined to match the quantity to be measured. How many strides or paces combine to give the length of the classroom? When we have answered this question we have measured its length (or the distance from one side to the other).

In doing Problem 5-1, different students will probably get somewhat different results because one student's stride is not the same as another's. This points up the need for a standard unit. It is also true that the distance may not come out to be a whole number of units. This leads to the idea of subunits and approximations to be discussed later.

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5-2 Measurement of Length

Here we describe the process of measurement in some detail. There must be a way of putting units together to match the thing to be measured. In the case of length we can put copies of the unit end to end. With a student's paces there is the difficulty that one of his paces may not be an exact copy of another. By using a stick length as a unit, we overcome this difficulty. We could overcome the same difficulty by choosing as a unit the length of the teacher's forearm. But obviously it would be inconvenient to apply the arm to the floor again and again.

The standard inch and centimetre units are introduced. At the end of the Section the importance of using standard units is discussed, but it might be helpful to lead up to this topic by asking at the beginning why it is useful to agree on the same unit for everyone.

Answers to Problems 5-2 Student Text page 145

1. 2; 3

2. 13; 14

It should be clear from these problems that measurements do not usually "come out even". It is therefore useful to subdivide units. This is usually done on a ruler. A ruler is simply an instrument in which units of length, say inches, are placed end to end in a permanent way. It is a piece of a number line that one can carry about. Just as fractions (or rational numbers) are shown on the number line, they are indicated on the ruler.

5-3 Origins of Standard Units

5-4 Standardizing Units of Measure

These sections give a brief history of the units commonly used today and should interest the student.

Answers to Problems 5-4 | Student Text page 150

1. mile
2. centimetre or inch
3. foot or metre
4. mile
5. foot

5-10. The students should suggest appropriate answers to these problems.

5-5 Measuring Distance

To measure the distance between two points we must measure the length of the line segment that connects them. Sometimes this line segment exists at first only in our imagination, as in the case of the diagonal of a table top. By using a stretched string or a sufficiently long ruler, we can obtain a line segment to measure. But in the case of the distance of a boat from the top of a cliff, stretching a string might be difficult. In fact, many important measurements are indirect; they require the use of instruments and calculations. This subject cannot be pursued very far at this stage.

In measuring the distance from a point to a plane or a point to a line or the distance between parallel lines or planes, there must be an agreement about what to measure. We know how to find the length of a line segment, but what line segment should we measure? It is desirable that everyone should get the same answer. This happy result will come about if we agree that we want the length of the shortest line segment from point to plane, point to line, line to line and plane to plane.

The student should be brought to agree to the summary in the box. If possible, he should discover the results for himself. However, the case of

the skew lines may be difficult for him to see. This is a very useful exercise in developing his ability to visualize relations in space.

5-6 Using a Ruler

This section discusses the idea that measurements are approximate and that in a report of a measurement we include one and only one "guess" digit. Problems 5-6 should illustrate the agreement about writing the results of a measurement.

Answers to Problems 5-6

Student Text page 160

		doubtful
1.	$6{\cdot}5 < 6{\cdot}57 < \underline{6{\cdot}6}$	7
2.	$\underline{3{\cdot}1} < 3{\cdot}12 < 3{\cdot}2$	2
3.	$27 < 27{\cdot}5 < 28$	5
4.	$8 < 8{\cdot}5 < 9$	5
5.	$8{\cdot}0 < 8{\cdot}05 < 8{\cdot}1$	5
6.	$\underline{21{\cdot}0} < 21{\cdot}01 < 21{\cdot}1$	1
7.	$7{\cdot}5 < 7{\cdot}58 < \underline{7{\cdot}6}$	8
8.	$\underline{9{\cdot}2} < 9{\cdot}20 < 9{\cdot}3$	0

In 1, 2, 6, 7 and 8 we have underlined one of the extremes to show that the reported measurement favors this result.

9.	$1\frac{1}{4}$ in.	3·1 cm.
10.	$4\frac{3}{16}$ in.	10·7 cm.
11.	$\ell > \frac{5}{16}$ in., $\ell < \frac{3}{8}$ in.	·9 cm.
12.	$1\frac{5}{8}$ in.	4·1 cm.

5-7 A Property of Length

The main point to be brought out is that length is additive. That is, if a given line segment is subdivided by marking points between the endpoints, the sum of the lengths of the new segments is equal to the length of the original line segment.

Answers to Problems 5-7A

Student Text page 162

5. 1, 2, 3, 4. The number of points needed is one less than the number of segments desired.

6. n + 1 segments

Answers to Problems 5-7B

Student Text page 163

$\overline{AB}$ and $\overline{CD}$ are congruent.

The following pairs of segments are not congruent:
$\overline{AB}$ and $\overline{EF}$, $\overline{AB}$ and $\overline{GH}$, $\overline{EF}$ and $\overline{GH}$, $\overline{CD}$ and $\overline{EF}$, $\overline{CD}$ and $\overline{GH}$.

CHAPTER 6

AREA

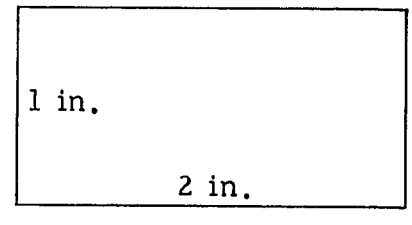
6-1 The Size of a Region

The student has an intuitive idea of area. He can compare the size of a football field with that of a tennis court merely by looking. He also realizes that if two regions are congruent (fit exactly), the areas are the same. However, he should realize that visual comparisons are not always sufficient. Regions may have different shape but the same area. Sometimes appearances are untrustworthy, as in the case of the tomato garden. It is interesting to point out that the 14×16 garden can be changed to the 10×20 garden by increasing one side by 4 ft. and decreasing the other side by 4 ft. Increasing and decreasing the sides by the same amount does <u>not</u> give a figure with the same area as before.

6-2 Measurement of Area

To measure the length of a line segment we must choose a line segment as a unit of length. To measure the area of a region we must choose a region as a unit of area, for example, a square region one inch on a side. We call this unit a square inch.

Instead of using a square of a certain size as a unit of area, we could, if we like, choose a region of a different shape as a unit. For example, we could choose a rectangular region as shown in the figure. But a square region is more convenient, even for measuring the areas of rectangles. Suppose, for example, that we try to use this rectangular region as our unit of area and that we wish to find the area of another rectangle. (We commonly say "area of a rectangle" instead of the longer "area of a rectangular region".) How many of our rectangular units would fit



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into the rectangle 5 inches by 3 inches of Problem 3 of 6-2? Have the students try to answer this question, and lead them to see how much more convenient the square unit is.

When we use the square unit there is a natural and simple connection between the lengths of the sides of the rectangle and the area. The area is connected with the number of squares that we can line up along each of the sides, as in Problem 4. With our rectangular unit there would not be this simple connection. Problems 6-2 should lead the student to the rule: Area of a rectangle = length of base × height.

It is a fact that this rule works even when the sides of the rectangle are not measured in whole numbers of units. The discussion in the text tries to bring this out.

It is useful for the teacher to know that 1 inch = 2·54 centimetres and that 1 square inch is therefore 2·54 × 2·54 square centimetres, that is, approximately 6·45 square centimetres.

Answers to Problems 6-2 | Student Text page 169

3. (a) 15. 104.

   (b) More than 84 sq. cm. and less than 104 sq. cm. The answer is not exact.

4. 8 sq. in.

5. Multiply the length and the width to find the area.

6. 35. No. 40. >, <.

6-3 Measurement of the Areas of Right-angled Triangles

The principal idea here is that a right-angled triangle has $\frac{1}{2}$ the area of a rectangle whose sides are the segments which meet at the vertex of the right angle. The student should make a rectangular region out of paper as suggested and cut along a diagonal.

Answers to Problems 6-3 | Student Text page 171

1. 24 sq. cm. The two triangles are congruent and therefore have the same area, 12 sq. cm. In each triangle, one angle is an angle of the rectangle and therefore a right angle.

2. AC = $1\frac{1}{4}$ in. BC = $1\frac{1}{4}$ in. Area = $\frac{25}{32}$ sq. in.

3. 13·6 sq. cm.

4. 23·14 sq. in.

5. The measures of the bases and the heights are the same.

6. Yes. The measures of the bases are the same and the measures of the heights are the same.

7. 18 sq. in.

(a) Yes, 9, (b) 12, Yes, 6, (c) 6, Yes, 3.

9. (a) 18 sq. cm., (b) 25 sq. cm.

10. 6 sq. in.

6-4 Measurement of the Area of a Non-right-angled Triangle

If the student answers the questions in the text he will easily convince himself that the area of <u>any</u> triangle is $\frac{1}{2}$ (length of base × height) or, as we usually say, $\frac{1}{2}$ (base × altitude).

There are, of course, three bases and three altitudes in every triangle. Therefore, there are 3 different ways of calculating the same area. This fact makes it easy to find one altitude from another altitude and the lengths of the sides. Problems 6-4B illustrate how this can be done.

Answers to Problems 6-4A

Student Text page 175

1. Each of these triangles has the area 25 sq. cm.

2. The results should agree approximately.

   (a) 3 sq. in., (b) $17\frac{1}{2}$ sq. cm.

3. 12 sq. cm.

4. $(2 \times 3) + \frac{1}{2})(3 \times 2) + \frac{1}{2}(3 \times 2) = 6 + 3 + 3 = 12$ sq. cm.
   The answer is the same.

Answers to Problems 6-4B

Student Text page 177

1. Base BC and altitude AD
   Base AB and altitude CS
   Base BC and altitude AT
   Base AC and altitude BQ
   Base AC and altitude BH; base BC and altitude BA
   Base BC and altitude AN; base AB and altitude CD
   Base BC and altitude AP; base AC and altitude BX;
   base AB and altitude CY

2. $\frac{1}{2}(BE \times 6) = \frac{1}{2}(8 \times 5)$, $BE = 6\frac{2}{3}$ in.

3. $\frac{1}{2}(QB \times 6) = \frac{1}{2}(4 \times 12)$, $QB = 8$ cm.

4. $\frac{1}{2}(BE \times 6) = \frac{1}{2}(4 \times 9)$, $BE = 6$ cm.

6-5 Areas of Other Polygons

By drawing diagonals it is shown that the area of any polygon is the sum of the areas of certain triangles. Sometimes it is more convenient to express the area as the sum of the areas of rectangles and triangles. Problems 6-5 fully illustrate these facts.

Answers to Problems 6-5

Student Text page 179

1. Area of rectangle ACDF = 20 sq. in.
   Area of ΔABC = 4 sq. in.
   Area of ΔFED = 2 sq. in.
   Total area = 26 sq. in.

2. Area of rectangle ACDF = area of ΔACD + area of ΔCDF = 10 sq. in. + 10 sq. in. Total area is 26 sq. in.

3. The plots have the same area. If w is the width or altitude, this area is 8w sq. yds.

4. The area is unchanged in each case.

5. Area of the goal = 16 × 8 = 128 sq. ft.
   Area of the field = 70 × 120 = 8400 sq. yds.

6. (a) Area of roof = 400 sq. ft.

   (b) Area of door and windows = 48 sq. ft.

   (c) Area of 2 sides = 432 sq. ft.
   Area of 2 ends = 392 sq. ft.
   824 sq. ft.

   - Area of door and windows - 48 sq. ft.
   776 sq. ft.

7. 96 sq. cm.

8. 112 sq. in.

6-6 Areas of Other Regions

It is impossible to fit together triangles and rectangles to obtain a region with curved boundaries. Consequently, we must be satisfied with an approxi-

mation. We can count the number of squares that lie entirely inside the region. We can also count the number of squares that will completely cover the region. This will include some squares that are partly inside and partly outside. Of course, the approximations to the area found in this way will be better if we use smaller squares.

Circular regions are the most important regions with curved boundaries, and most of the problems are concerned with their areas.

Answers to Problems 6-6

Student Text page 183

1. (a) 9

   (b) 33

   (c) 33, 9

   (d) 21 sq. units, approximately

   (e) $20\frac{1}{4}$ sq. units, approximately

   The answers to (d) and (e) may vary greatly, depending on the way the student counts the half squares and quarter squares when he makes his estimate.

2. (a) 36 is a likely answer.

   (b) 60 is a likely answer.

   (c) The student may get as close to the actual area measurement as $48\frac{1}{2}$ sq. in., if he uses quarter squares to estimate. The correct area is 16π, or about 50.3 sq. in.

3. (a) 284 sq. cm., (b) 328 sq. cm.

   (c) The radius is $10 \cdot 16$ cm.
   The correct area $= \pi(10 \cdot 16)^2$ which is approximately $103 \cdot 2\pi \approx 324$ sq. cm.

4. (a) The large squares are 1 inch on a side. There are 4 squares entirely inside, and 16 squares cover the circular region completely. The area appears to be about 12 sq. inches.

4. (b) The small squares are one tenth of an inch on a side. If you add the total number of square units in the picture, not counting some of the edge pieces because they are too small and counting others as full pieces to make up, you find a total of 1244 unit squares. Obviously 100 such square units make an inch-square unit, and so this gives the area of the circle as 12·44 sq. in. (The correct value $\approx$ 12·57.)

CHAPTER 7

VOLUME

7-1 Measurement of Volume

To measure the length of a segment, we use a unit which is a segment. To measure the area of a region, we use a unit which is a region. To measure the volume of space, we use a unit which is a space, the space enclosed by a cube 1 inch on a side, for example.

To develop a feeling for volume, it might be useful to take a box like that in which chalk comes and pack into it cubes 1 inch on a side. The students might make a number of such cubical blocks. Or even without the box, the blocks could be put together to form rectangular prisms whose volume can be found by counting the number of blocks. See if the students can arrive at the rule that the volume of a rectangular box or prism is length × width × height, or the area of the base × the height.

Answers to Problems 7-1 | Student Text page 189

1. (a) 30 cu. cm., (b) 66 cu. ft.

2. $30 = \ell \times 5 \times 2$. Therefore, $\ell = 3$ ft.

3. (a) 1728 cu. in.

   (b) 54 cu. ft.

4. 24 cu. ft.

7-2 Volumes of Prisms

We show that the volume of a prism with a right triangle as base is one half the volume of a box of the same height. This is done in a manner similar to the way it was found that the area of a right triangle is one half that of a rectangle. We see, therefore, that $V = \frac{1}{2}(\ell \times w \times h)$ which is the area of the triangular base multiplied by the height.

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The idea may be made clearer to the student as follows. Let him take a cardboard box and draw parallel diagonals along the top and bottom faces. If he cuts the box along these diagonals and the vertical edges that join their ends, he will separate the box into two right-triangular prisms (of course with one side face of each prism missing). The student can see that the two prisms are congruent.

The case where the base of the prism is a triangle but not a right triangle is next discussed. Again the volume is the area of the base $\times$ the height. The same rule gives the volume of a prism with a base bounded by a polygon with any number of sides. The argument which shows this should be studied carefully.

<u>Answers to Problems 7-2</u>

Student Text page 193

1. $\frac{1}{2}(3 \times 4) \times 6 = 36$ cu. in.

   The base is a right triangle whose area is $\frac{1}{2}(3 \times 4)$ sq. in.

2. $V = \frac{1}{2}(a \times b) \times h$ cu. cm.

3. $V = \frac{1}{2}(BG \times AC) \times CD = 165$ cu. cm.

4. The area of the base is

   $\frac{1}{2}(AP \times BF) + (BC \times BF) + \frac{1}{2}(CD \times CE) =$

   $\frac{1}{2}(3 \times 3) + (9 \times 3) + \frac{1}{2}(4 \times 3) = 37\frac{1}{2}$ sq. in.

   Volume = Area of base $\times$ height $= 37\frac{1}{2} \times 8 = 300$ cu. in.

5. $V = (5 + 4 + 3 \cdot 5) \times 7 = 87 \cdot 5$ cu. cm.

6. Use the foot as a unit of length.

   Area of base $= \frac{1}{2}(1 \times \frac{2}{3}) = \frac{1}{3}$ sq. ft.

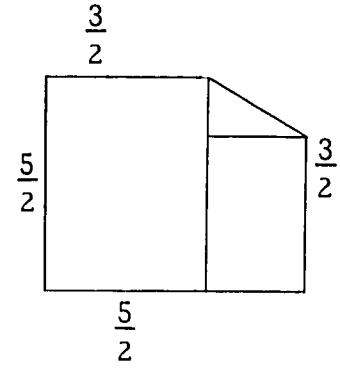
   $V = \frac{1}{3} \times 20 = 6\frac{2}{3}$ cu. ft.

7. Use the foot as a unit of length.

Area of end $= (\frac{4}{3} \times \frac{1}{6}) + \frac{1}{2}(\frac{2}{3} \times \frac{4}{3}) = \frac{2}{9} + \frac{4}{9} = \frac{2}{3}$ sq. ft.

$V = \frac{2}{3} \times 30 = 20$ cu. ft.

8. The end can be subdivided into two rectangles and a triangle with dimensions in feet as shown. The area is



$(\frac{5}{2} \times \frac{3}{2}) + (\frac{3}{2} \times 1) + \frac{1}{2}(1 \times 1) = 5\frac{3}{4}$ sq. ft.

$V = (5\frac{3}{4}) \times 4 = 23$ cu. ft. for one bin.

115 cu. ft. for five bins.

CHAPTER 8

ANGLE MEASURE

8-1 Measurement of Angles

The same principles apply to the measurement of angles as to the measurement of length, area and volume. To measure an angle we must choose a unit of the same sort, that is, an angle. The unit commonly used is the degree. This will be introduced in Section 8-2. But we could choose as a unit any convenient pair of rays through a point. In the text u is such a unit.

Answer to Problem 8-1 — Student Text Page 199

$m(A\hat{B}C) = 2$ $m(D\hat{E}F) = 3$

$m(X\hat{O}Y) = 5$ $m(M\hat{N}P) = 6$

8-2 The Unit Angle — Student Text Page 203

The usual degree unit is introduced as $\frac{1}{180}$ part of a straight angle and the use of a protractor marked in degrees is explained. The student should acquire some skill in using this convenient instrument.

Answers to Problems 8-2A

1.	42	6.	27
2.	40	7.	108
3.	117	8.	24
4.	20	9.	131
5.	145	10.	17

Previous Page Blank

8-3 The Addition Properties of Angles

The addition property in question is stated in the box at the end of the section.

Answer to Problem 8-3 | Student Text Page 207

Make a square corner as described in Chapter 1. The right angle measures 90 degrees. Fold over so that the edges of the square corner coincide. This will give an angle with degree measure 45. Now fold again for a measure of $22\frac{1}{2}$.

8-4 Congruent Angles

Congruent angles have the same degree measure, and angles with the same degree measure are congruent.

Answers to Problems 8-4 | Student Text Page 209

1. Yes, because the angles are congruent.
2. Yes, because the angles are congruent.
3. $A\hat{O}C$ is congruent to $B\hat{O}D$ and $A\hat{O}D$ to $B\hat{O}C$. Vertically opposite angles are congruent.
4. Yes; when the intersecting lines are perpendicular.
5. A rectangle, a parallelogram, an isosceles triangle, a regular polygon (see Section 8-6).

8-5 Relations between Angles

The most important result is that the sum of the degree measures of the angles of a triangle is 180. It is also important that the student learn the meanings of the supplement of an angle, of corresponding angles, and of alternate interior and exterior angles.

Answers to Problems 8-5A

Student Text Page 211

1. (a) 170 (b) 63 (c) 98 (d) 90

   (e) 15 (f) 158 (g) $111\frac{1}{2}$ (h) $48\frac{3}{4}$

   These are the degree measures of the supplements of the given angles.

2. (a) $m(A\hat{O}B) = m(C\hat{O}D)$; $m(A\hat{O}D) = m(B\hat{O}C)$

   $m(B\hat{Q}P) = m(A\hat{Q}R) = m(Q\hat{R}D) = m(C\hat{R}S)$

   $m(A\hat{Q}P) = m(B\hat{Q}R) = m(Q\hat{R}C) = m(D\hat{R}S)$

   (b) The following angles are supplementary.

$A\hat{O}B$ and $A\hat{O}D$	$A\hat{Q}P$ and $P\hat{Q}B$	$Q\hat{R}C$ and $C\hat{R}S$
$A\hat{O}B$ and $B\hat{O}C$	$A\hat{Q}P$ and $A\hat{Q}R$	$Q\hat{R}C$ and $Q\hat{R}D$
$D\hat{O}C$ and $A\hat{O}D$	$B\hat{Q}R$ and $P\hat{Q}B$	$S\hat{R}D$ and $S\hat{R}C$
$D\hat{O}C$ and $C\hat{O}B$	$B\hat{Q}R$ and $A\hat{Q}R$	$S\hat{R}D$ and $D\hat{R}Q$

3. $m(P\hat{C}A) + m(A\hat{C}B) + m(B\hat{C}Q) = 180$

   $m(P\hat{C}A) = m(C\hat{A}B)$

   $m(B\hat{C}Q) = m(A\hat{B}C)$

   $m(A\hat{C}B) + m(C\hat{A}B) + m(A\hat{B}C) = 180$

4. An angle in an equilateral triangle has a degree measure of 60. In a triangle the sides opposite congruent angles are congruent.

Answers to Problems 8-5B

Student Text Page 214

1. (a)

	AB	BC	CA	$m(\hat{A})$	$m(\hat{B})$	$m(\hat{C})$	$m(\hat{A})+m(\hat{B})+m(\hat{C})$
1	4·76	5	2·54	80	30	70	180
2	3	1·8	2·8	36	65	79	180
3	2	3	4	47	104	29	180
4	3	4	5	53	90	37	180
5	5	6	7	57	$78\frac{1}{2}$	$44\frac{1}{2}$	180
6	6	6	6	60	60	60	180
7	6	8	10	53	90	37	180
8	5·65	8	5·65	90	45	45	180

(b) The angle sum has degree measure 180.

(c) 1, 2, 5 and 6

(d) 4, 7 and 8

(e) 3

(f) 6 and 8

(g) 6

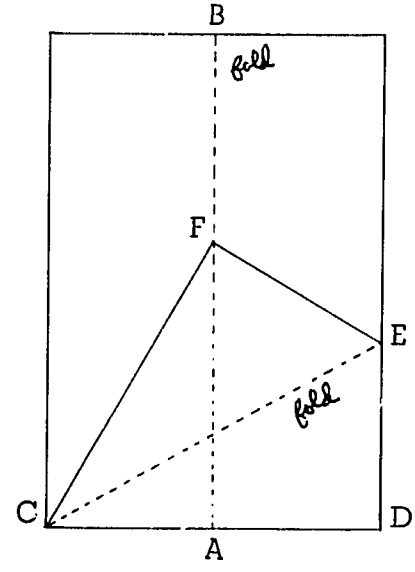
(h) 1, 2, 3, 4, 5 and 7

2.

	AB	BC	CA	$m(\hat{A})$	$m(\hat{B})$	$m(\hat{C})$	Type of $\triangle$
1	3	5	3	113	$33\frac{1}{2}$	$33\frac{1}{2}$	isosceles — obtuse-angled
2	4	2·83	2·83	45	45	90	isosceles — right-angled
3	2	3	4	47	104	29	scalene — obtuse-angled
4	4	4	4	60	60	60	equilateral — equiangular
5	5	3·7	5·7	40	80	60	scalene — acute-angled
6	3	4	5	53	90	37	scalene — right-angled
7	3	5·2	3	120	30	30	isosceles — obtuse-angled

3. Fold a rectangular piece of paper in half along the segment $\overline{AB}$. Now fold along a segment $\overline{CE}$ so that D falls on $\overline{AB}$ at a point which we call F. Fold along $\overline{CF}$.

Then ΔCDF is equilateral and $m(F\hat{C}D) = 60$, $m(C\hat{F}A) = 30$ and $m(B\hat{F}E) = 120$. By folding we can make $\overline{FA}$ fall on $\overline{FC}$ and obtain an angle with degree measure 15.



8-6 Angles of Polygons

We introduce the idea of an exterior angle of a triangle and obtain the result that the measure of an exterior angle is equal to the sum of the measures of the opposite interior angles.

The problems bring out some interesting results about the sums of the angle measures of exterior and interior angles of polygons. Regular polygons are introduced.

Answers to Problems 8-6 Student Text Page 216

2.

	AB	BC	CA	$m(\hat{A})$	$m(\hat{B})$	$m(\hat{C})$	$m(A\hat{C}X)$	$m(\hat{A}) + m(\hat{B})$
1	4	1·46	3·25	20	50	110	70	70
2	1·55	3	3	75	75	30	150	150
3	2	5	4·7	87	70	23	157	157
4	4	2·3	4·6	30	90	60	120	120

3. (a) 1 (b) 2

The last column in the table should read 1, 2, 3, 4, 5, 6, 7 and 8.

(c)

Number of sides	Sum
3	180
4	360
5	540
6	720
7	900

4. The sum of the degree measures of $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ and $\hat{e}$ is 360. For a hexagon the sum of the measures of the exterior angles is 360; for a heptagon, 360. The same result applies for a polygon of n sides.

5. (a) and (d) are regular.

6. 135

7. 10; 8

CHAPTER 9

RELATED MEASURES AND A BACKWARD LOOK

9-1 Relations Between Measurements

The reader is reminded that the areas and volumes of simple figures depend upon one or more lengths. However, the main purpose of this short chapter is to introduce the student to some simple inequalities among the lengths of the sides and the measures of the angles of a triangle. He should learn that in any triangle the angle opposite a longer side is larger than the angle opposite a shorter side. He should also discover that the sum of the lengths of any two sides of a triangle is greater than the length of the remaining side. Therefore, it is not always possible to draw a triangle whose sides have three given lengths. For example, there is no triangle whose sides have the lengths 1 in., 2 in., and 5 in. because 1 + 2 is not greater than 5.

Conclusion

This section should bring out the general principles which apply to the measurement of lengths, areas, volumes and angles. It summarizes many of the points made in the previous four chapters.

Answers to Problems 9-1A

Student Text Page 219

1. (a) $2\frac{3}{16}$ in.
   (b) 5·5 cm.

2. BC = 3·1 cm., AB = 2·5 cm., PQ = 2·5 cm., QR = 2·9 cm.
   (a) 7·75 sq. cm.
   (b) 7·25 sq. cm.

Answers to Problems 9-1B

Student Text Page 221

1. (a) $m(\hat{A}) + m(\hat{B}) + m(A\hat{C}B) = 180$
   (b) $m(A\hat{C}D) = m(\hat{A}) + m(\hat{B})$

2. b is larger because b = a + c.

3. The larger angle is opposite the larger side.

4. The right angle is the greatest angle. The line segment marked x is the shortest. $\overline{PS}$ is the shortest line segment that can be drawn from P to $\overleftrightarrow{QR}$.

Answers to Problems 9-1C

Student Text Page 224

5. Impossible. No triangle.

9. Impossible. No triangle.

10. Impossible. No triangle.

CHAPTER 10

SYMMETRY

10-1 Symmetric Geometric Figures

Your pupils should test many geometric figures for plane symmetry with respect to a line by paper folding in this section. In addition to tracing those figures in the text, they should draw some figures of their own and test them for symmetry. With a little experience, they will soon be able to recognize symmetry by inspection. Symmetry should stand out as a rare, beautiful, and intriguing phenomenon — and a useful one.

Answers to Problems 10-1A | Student Text Page 230

1. There are many geometric figures whose left halves are symmetric to their right halves. Examples are a square, rectangle, circle, equilateral triangle, isosceles triangle, regular pentagon, regular hexagon, etc. The pupil can fold paper, cut his own geometric design and trace it on his paper.

2. Square, rectangle, circle, regular hexagon, regular octagon, etc.

3. Triangle (scalene), parallelogram, quadrilateral, trapezium, etc.

Answers to Problems 10-1B | Student Text Page 231

1. A square has four lines of symmetry: the two diagonals and the two lines that bisect opposite sides.

2. A rectangle has two lines of symmetry (the two lines that bisect the opposite sides). The diagonals of a rectangle are not lines of symmetry. The pupil should fold his paper along the diagonal of a rectangle and see this for himself.

3. An equilateral triangle has three lines of symmetry, a regular pentagon has five, and a regular hexagon has six. (Your pupils should see a pattern here.)

10-2 A Symmetry Test

The test for symmetry with respect to a line is a simple one. We say that a figure is symmetric with respect to a line if, for every point A on the figure, there is a point B on the figure such that the line is the perpendicular bisector of $\overline{AB}$. Of course, not every point can be tested, but enough points can be tested to give us a good idea as to whether or not a figure is symmetric to a line. We need find only one pair of points failing the test to prove that the figure is not symmetric to the line being tested.

We have not discussed symmetry with respect to a point in the text. If your pupils should inquire about this, we say a figure is symmetric to a point P if, for every point A on the figure, there exists a point B on the figure such that P is the midpoint of $\overline{AB}$. Obviously, a figure could have only one point of symmetry, for if it had two such points, then the segment passing through these two points, with endpoints on the edge of the figure, would have two midpoints, and this is impossible. It is interesting to note that although some figures (such as a parallelogram) have no line symmetry, they do have point symmetry.

10-3 Making Symmetrical Figures

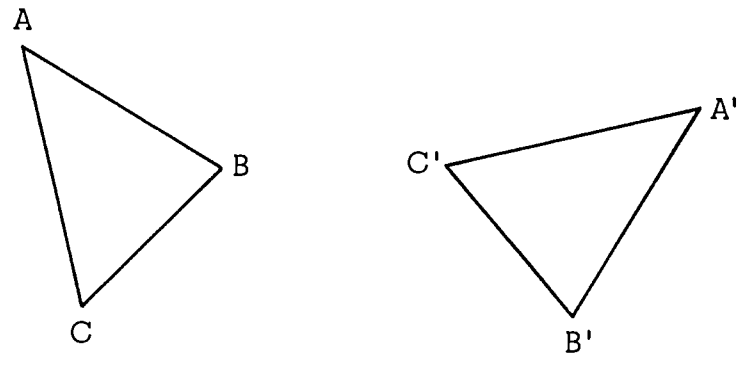
Your pupils will enjoy doing this. You might hold a contest to see which pupil could produce the most interesting or unusual symmetrical figure by cutting folded paper. You might also let them use a pair of compasses and see if they can produce a unique or unusual symmetrical figure by drawing circles and arcs.

10-4 Congruence

Your pupils should recall the paper folding that they did in Chapter 2 to test for congruence and compare this to the symmetry tests they have done in this chapter. The two ideas are closely related. If a figure is symmetric to a line, then the line divides the figure into two congruent parts.

For the equilateral triangle on Page 236 of the student text, if $\overleftrightarrow{AD}$ is a line of symmetry, then ΔABD is congruent to ΔACD and all the parts of the one triangle are congruent to all the corresponding parts of the other. A similar statement could be made for each of the other two lines of symmetry through B perpendicular to $\overline{AC}$ and through C perpendicular to $\overline{AB}$.

If a figure is symmetric to a line, there is at least one congruence that results from this symmetry, but if two figures are congruent, they may not have a line of symmetry as in the figure below, where ΔABC is congruent to $\Delta A'B'C'$ and there is no line of symmetry between them.



Answers to Problems 10-4

Student Text Page 236

1. (a) is the picture of the construction for a perpendicular bisector of a segment. The perpendicular bisector is the line of symmetry for the segment and also for the figure formed by the two intersecting arcs.

   (b) is a picture of two pairs of vertical angles. The figure has two lines of symmetry. They are the lines that bisect each pair of vertical angles. The vertical angles are congruent and the half-angles obtained from the angle bisectors are also congruent.

   (c) is a picture of an isosceles triangle. It has one line of symmetry from its vertex perpendicular to its base. This line of symmetry divides the isosceles triangle into a pair of congruent triangles whose corresponding parts are congruent.

   (d) is a picture of an equilateral triangle. It has three lines of symmetry and each one divides the triangle into a pair of congruent triangles whose corresponding parts are congruent.

   (e) is a picture of the construction of a perpendicular to a line from a point not on the line. The segment determined by the arc that intersects the line in two places has the perpendicular as a line of symmetry, as do the other figures formed by the intersecting arcs and line.

   (f) is a picture of the construction for the bisector of an angle. The angle bisector is the line of symmetry for the angle and the two angles formed by the bisector are congruent.

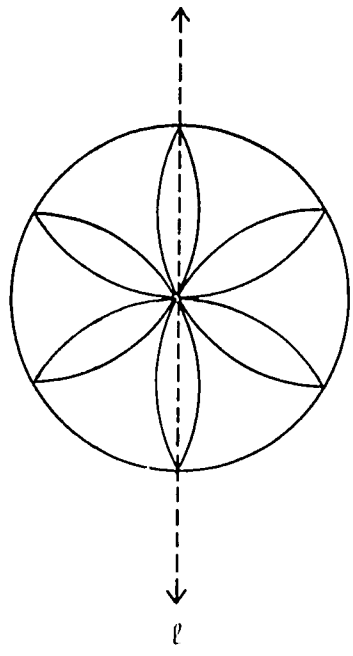
2. After the pupil has folded the paper along the dotted line and then along the line of symmetry, $A\hat{B}C$ will fall on top of $C\hat{D}E$ and the two angles are congruent.

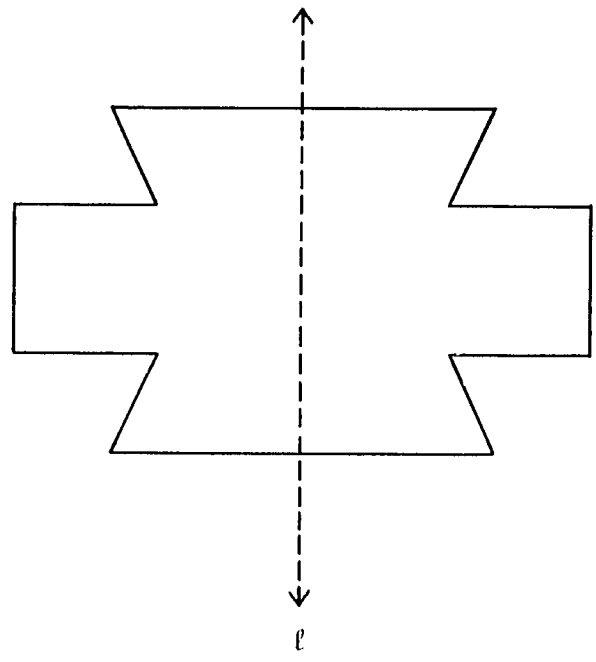
3. (a) $\overline{EC} \cong \overline{AC}$; $\overline{ED} \cong \overline{AB}$; $\overline{DC} \cong \overline{BC}$; $A\hat{C}B \cong E\hat{C}D$; $C\hat{A}B \cong C\hat{E}D$; $C\hat{B}A \cong C\hat{D}E$; $\triangle CAB \cong \triangle CED$. ("$\cong$" means "is congruent to")

   (b) No.

4. Figures (a), (c), (f), (g), (o) and (p) are not symmetric to any line.

5.





10-5 Symmetry in Space

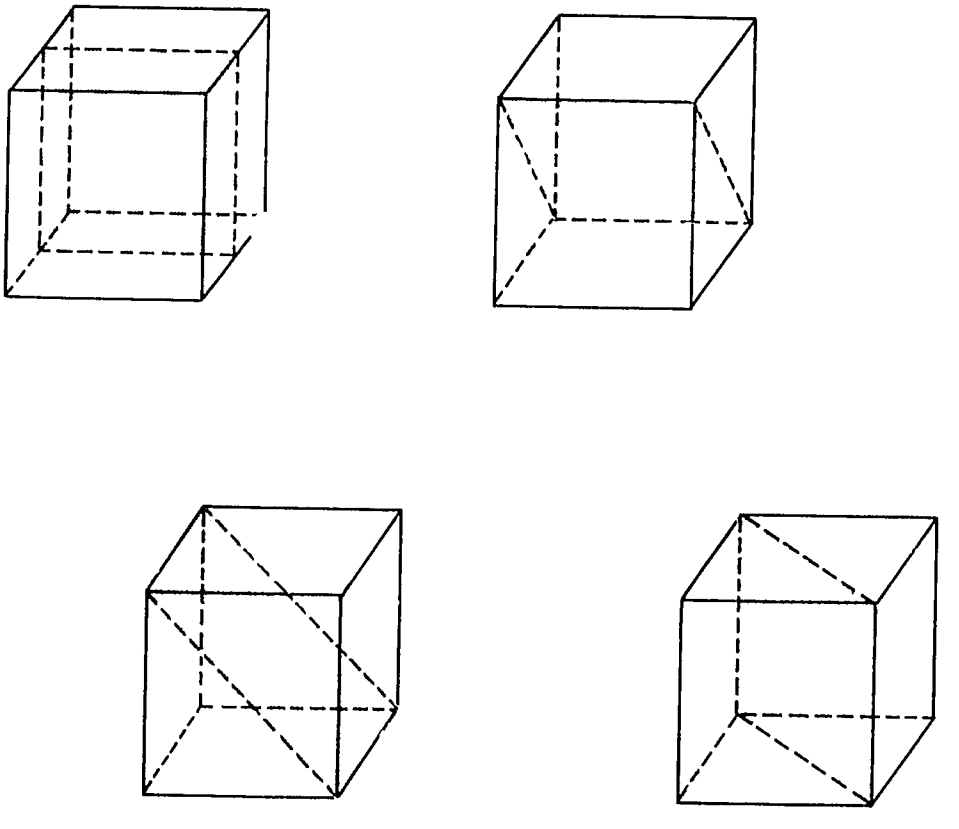
Symmetry in solid figures is not as easily determined or tested as is symmetry for plane figures. Since, when drawing three dimensional figures on a plane, some of the segments must be drawn shorter (or longer) than their actual lengths in order to obtain the right perspective, it may appear to the eye that certain figures do not possess symmetry when in fact they do.

The pupil must rely heavily on his knowledge of the properties possessed by some of the more familiar solid figures and the models he constructs of them in order to determine if there is symmetry to one or more lines and planes.

As your pupils work in this section they should use the models of solid

figures they constructed in Chapter 4 to help them to see what symmetry, if any, these solids possess. They may need to construct other models that actually show planes cutting through the models in order to see if a particular solid is symmetric to a particular plane.

The cubes pictured in the text show two planes of symmetry parallel to the faces and three planes of symmetry through the diagonals of opposite faces. The cube has one more plane of symmetry parallel to its faces and three more planes of symmetry through the diagonals of opposite faces. These are pictured below. The cube also has six more lines of symmetry than the three shown in the text. There is one more line of symmetry through the centres of opposite faces and five more lines passing through the midpoints of diagonally opposite edges. Thus a cube has a total of nine <u>planes</u> of symmetry and nine <u>lines</u> of symmetry.



The tests for symmetry with respect to a line and to a plane for a solid figure are easily described, but not easily made. A solid figure is symmetric to a line if for every point A on the figure there exists a point B on the figure such that the line is the perpendicular bisector of segment $\overline{AB}$.

Similarly, a solid figure is symmetric to a plane if for every point A on the figure there exists a point B on the figure such that the plane bisects segment $\overline{AB}$ and segment $\overline{AB}$ is perpendicular to the plane.

Although it is not mentioned in the text, solid figures may also have point symmetry. The sphere is a good example of a figure symmetric with respect to a point (the centre). We say that a solid figure is symmetric to a point P if for every point A on the figure there is a point B on the figure such that P is the midpoint of $\overline{AB}$.

Answers to Problems 10-5 | Student Text Page 243

1. (a) A rectangular solid has three planes of symmetry in general.

   (b) A parallelepiped (all of whose faces are parallelograms) has no plane of symmetry in general.

   (c) A sphere has infinitely many planes of symmetry.

2. (a) Three.

   (b) None.

   (c) Infinitely many.

3. (a) Yes; the plane midway between and parallel to the circular ends.

   (b) There is one line of symmetry through the centres of the two circular ends. In addition to this, there are infinitely many lines containing the diameters of the circle determined by the cutting plane described in (a) above, and each of these lines is a line of symmetry.

3. (c) Yes; it has infinitely many planes of symmetry through the diameters of its circular ends. It has one line of symmetry through the centres of its circular ends.

   (d) Yes; it has one plane of symmetry parallel to its bases and midway between them. It has no lines of symmetry.

   (e) No; No.

   (f) Six; none.

4. There are four planes of symmetry for the cube other than those shown on page 241 of the student text. They are shown on page 74 in the Teachers' Guide.

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