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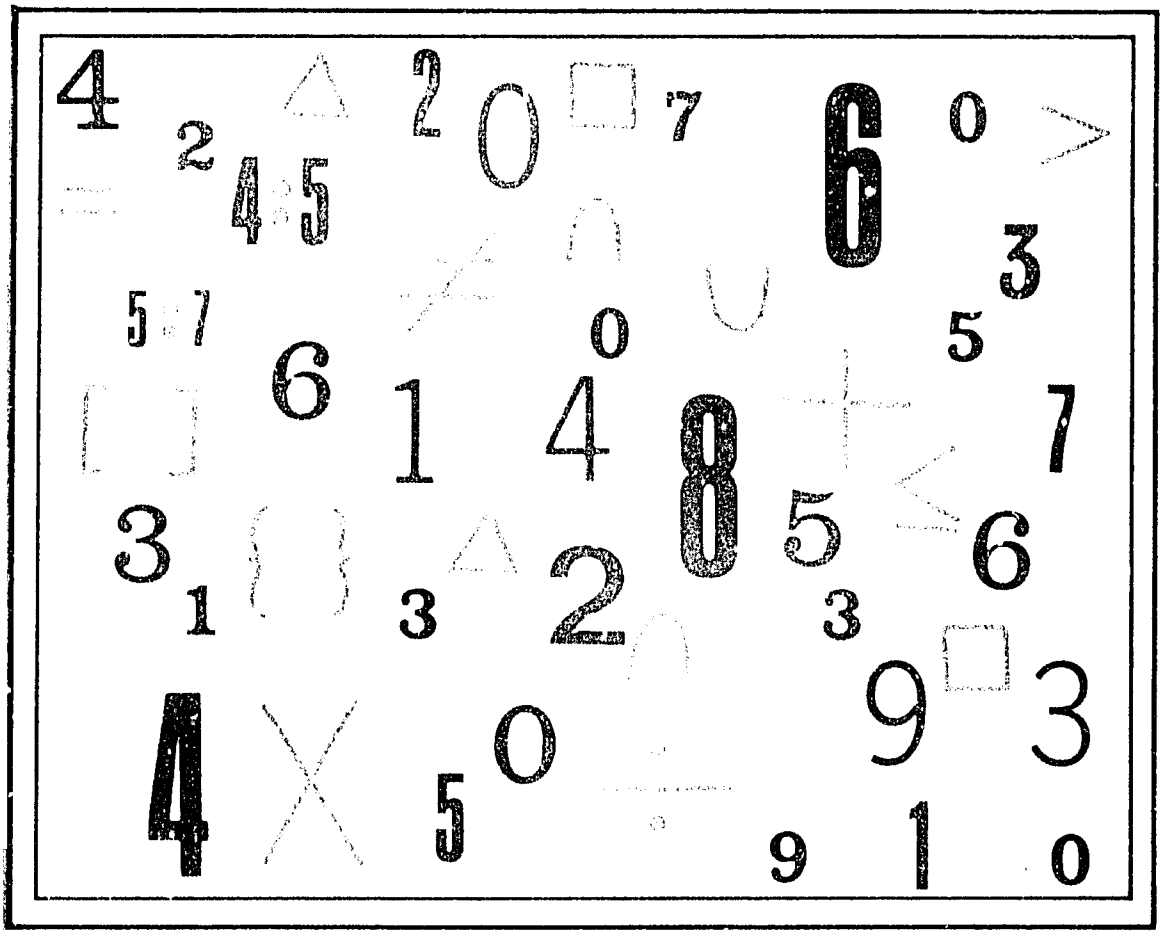

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SECONDARY C ONE

THE FIRST YEAR OF A FOUR YEAR SECONDARY COURSE

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Entebbe Mathematics Series

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TO THE STUDENT

This textbook has been specially prepared for students in the first year of a four year course leading to School Certificate. In using the textbook you have the opportunity to participate in the great experiment which was started in the summer of 1962 when more than fifty mathematicians and mathematics teachers gathered in Entebbe, Uganda, to begin the preparation of mathematics books as up-to-date as those in use anywhere else in the world. The writers came to Entebbe from all the countries in English-speaking Tropical Africa, from the United States, and from the United Kingdom to take part in this project. Now you and your teachers must do your share to find out whether this book is as good as it can be made. By using the textbook critically and making the most of the opportunities it offers to make the learning of mathematics an interesting and enriching experience, you will have played a part in the new growth of mathematics education in Africa.

You can expect to meet many familiar ideas, sometimes expressed in unfamiliar ways, and there will be some new ideas. Careful reading will be important because this book treats mathematics as a language designed to express certain kinds of ideas, and you must learn to read and speak this language.

You will learn why numbers behave as they do and find that algebra helps us more and more to understand arithmetic ideas. You will also explore some geometric ideas, observing properties of sets of points. You will even learn to make algebra and geometry work together, each making the other clearer.

Do not hesitate to ask questions. Talk about your work with your teacher and your fellow students. Above all, try to look upon mathematics as something that can be exciting and rewarding, for if you give it half a chance, that is exactly what it is.

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CHAPTER 1

THE LANGUAGE OF SETS

1-1 Sets

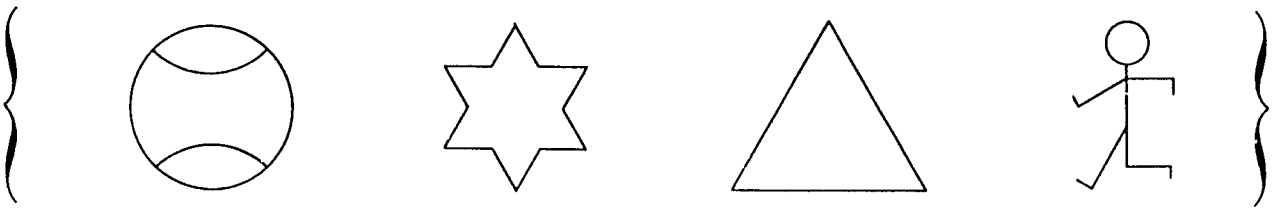
In our daily life if we wanted to describe a collection of things we might talk of a bundle of sticks, a flock of sheep, a herd of cattle, a class of boys or girls, and so on. Can you think of more names for collections of things? In mathematics, a collection of things is called a set.

We can talk of the set of books in the school library, the set of counting numbers less than ten, the set of boys in the school football team. Each of these sets is a set of similar things. A set can, however, contain any variety of objects. For example we may have, if we want, a set that is made up of the following things: a stick, a stone, a book, and a banana.

The things that make up the set are called members or elements of the set. For instance Uganda is a member of the set of independent African nations, a cat is a member of the set of all domestic animals. A dog is also a member of the latter set because a dog is a domestic animal. On the other hand a lion is not a member of the set of all domestic animals. Can you say why?

1-2 Ways of Describing Sets

When we are speaking of a specific set it is important that the set be described so clearly that no one can have any doubt about which objects are included in it. Let us discuss two ways of describing the set. One way is to list all the members of the set. For example if we wish to describe the set made up of the numbers 1, 3, 5, 7, 9, we can write this as $\{1, 3, 5, 7, 9\}$ which we read, "the set whose members are 1, 3, 5, 7, and 9." We use curly brackets to enclose the numerals representing the members of the set and to have the meaning, "the set whose members are . . . . "



(Figure 1)

Here is a picture of a set. The elements of the set are a football, a star, a triangle, and a man.

We may not always be able to list all the members in a set. There are several reasons for this. For example, consider the set of all odd numbers. No matter how many odd numbers we might list, there are always odd numbers not included in our list. We therefore list a few of the odd numbers and then put three dots after the last number listed to indicate that the set of all odd numbers goes on and on. Thus we write $\{1, 3, 5, 7, 9, \ldots\}$. Similarly the set of all counting numbers can be written $\{1, 2, 3, 4, 5, \ldots\}$.

If we wanted to name all the members of the set of whole numbers less than 10,000 which are exactly divisible by 3, it would be possible to list all the members. The first one is 3, the second one is 6, the third one is 9, and the last one is 9,999. It would take a long time, however, to list all the members of this set. Instead we shall indicate the list without writing all the members. Here again dots are used. Once the pattern is clear and the last member of the set is indicated, we use three dots to represent the missing numbers. We write this set as $\{3, 6, 9, 12, 15, \ldots, 9{,}999\}$.

In describing sets it is useful to use capital letters as names of the sets. For example,

$$A = \{1, 3, 5, 7, 9\}$$

$$B = \{3, 6, 9, 12, 15, \ldots, 9{,}999\}$$

$$C = \{1, 2, 3, 4, \ldots\}$$

If the number 5 is a member of the set A, we write "$5 \in A$", and read this "5 is a member of the set A" or "5 belongs to the set A". "$15 \in B$" means that 15 is an element of the set B.

Sometimes it is undesirable or impossible to indicate the members of a set by listing them. In this case we can give a word description of the set. For example,

A = the set of all countries in the world

B = the set of even counting numbers

C = the set of all palm trees in Africa

D = the set of numbers exactly divisible by three and less than 10,000.

The word description must be such that it is always possible to decide whether or not a given number or thing is a member of the set. Take the set of all even numbers. It is clear that 2, 4 and 6 are members of the set but 3, 5 and 7 are not members of the set. On the other hand if we speak of the set of all children in the village, it may not be at all clear who belongs to the set. Where do we draw the line between being a child and being an adult? What does "in the village" mean? Does a sixteen year old person who is visiting the village for a day belong to the set?

Give examples of sets in your class using word descriptions.

Problems 1-2

1. List the elements of the following sets.

   (a) The set of the continents of the world.

   (b) The set of History teachers in your school.

   (c) The set of months with names beginning with the letter J.

   (d) The set of days of the week.

   (e) The set of headmasters of your school.

   (f) The set of members of the staff of your school who are over 10 feet tall.

2. Describe the following sets. Which ones are difficult to describe?

   (a) {a, e, i, o, u}

   (b) {Mt. Kenya, Mt. Kilimanjaro, Mt. Ruenzori, Mt. Elgon}

   (c) {1, 2, 4, 6, 8, ...}

   (d) {lion, cat, tiger, cheetah}

   (e) {x, y, z}

3. In which of the following sets will it be difficult and in which will it be impossible to list all the members?

   (a) The set of all pupils in your school.

   (b) The set of all counting numbers exactly divisible by four.

   (c) The set of all cities in your country.

   (d) The set of all text-books in your class.

   (e) The set of all fish in Lake Victoria.

   (f) The set of all present mayors of Mombasa.

4. Here are some sets. For each part select the phrase which best describes the given set.

   (a) {2, 4, 6, 8, 10}

      i. The set of small even numbers.

      ii. The set of some even numbers.

      iii. The set of even counting numbers less than twelve.

   (b) {5, 10, 15, 20, 25, 30}

      i. The set of counting numbers less than 35.

      ii. The set of the first six counting numbers that are exactly divisible by five.

      iii. The set of all numbers obtained by counting by fives.

4. (c) {mango, pawpaw, orange, pineapple}

   i. The set of things we eat.

   ii. The set of fruits which are nice to eat.

   iii. The set of mango, pawpaw, orange and pineapple.

5. Answer these questions about the elements of a set.

   (a) Is a Nigerian a member of the set of all Africans?

   (b) Is an elephant a member of the set of all trees?

   (c) Is "t" a member of the set of all vowels of the English alphabet?

   (d) Are you a member of the set of all pupils in your school?

   (e) Is your headmaster a member of the set of all pupils in your school?

6. (a) List the members of the set of all oceans of the world.

   (b) List the members of the set of presidents of Ghana in 1965.

   (c) List the members of the set of pupils in your class who are members of Parliament (National Assembly) of your country.

   (d) List the set of all elephants in your classroom.

1-3 The Empty Set

Consider the answers which you gave to the last two parts of Problem 6 above. In both cases there are clearly no members in the set. Furthermore, consider the following sets.

A = the set of all squares with five sides.

B = the set of all baby boys each of whom weighed 1,000 pounds at birth.

C = the set of pupils in your class who are school teachers.

You will discover that set A has no members, set B has no members and set C has no members. The set which has no members is called the <u>empty set</u>. A way to represent the empty set is by { }. Therefore we write A = { }, B = { }, and C = { }. Another symbol which is sometimes used to represent the empty set is $\emptyset$.

Problems 1-3

1. Which of the following sets are the empty set?

   (a) The set of even numbers between 20 and 30.

   (b) The set of triangles with 4 sides.

   (c) The set of pupils in your class who are more than 40 years old.

   (d) The set of girls in your class.

   (e) The set of boys in your class.

   (f) The set of pupils in your class with green hair.

2. Give three more examples of sets which are empty.

1-4 Subsets

Let us look at the pairs of sets listed below and see if we can discover some new facts about them.

List 1	List 2
(a) The set of players in your school football team.	(a) The set of forwards in your school football team.
(b) $C = \{1, 2, 3, 4\}$	(b) $D = \{2, 3\}$
(c) The set of letters in the English alphabet.	(c) The set of vowels in the English alphabet.
(d) K = {pin, tree, book}	(d) N = {book}
(e) Y = {□ ▭ O △ ◇}	(e) Z = {O △ ◇ ▭}
(f) {5, 3, 7, 9}	(f) {9, 3, 7, 5}
(g) {7, 9}	(g) { }

In the first pair of sets, is each member of the set in List 2 also a member of the corresponding set in List 1? Answer this question for each of the pairs of sets. In each case list or describe the members of the set in List 1 which are not in the corresponding set in List 2. Do the same for the members of the set in List 2 which are not in the corresponding set in List 1.

You have observed that in each case every member of the set in List 2 is also a member of the corresponding set in List 1. Each set in the second list is called a <u>subset</u> of the corresponding set in the first list.

> If each member of set A is also a member of set B, then set A is called a <u>subset</u> of set B. We write $A \subseteq B$. The empty set is defined to be a subset of every set.

Notice the special case where $A = B$. It is true in that case that each member of A is also a member of B, so by the definition it is correct to say

that A is a subset of B and write $A \subseteq B$. In other words, every set is a subset of itself.

If $A \subseteq B$ and $A \neq B$ (that is, there are some members of B which are not in A), we call A a <u>proper subset</u> of B and sometimes write $A \subset B$ to emphasize this relation.

It seems a bit peculiar to say that { } is a subset of anything, but we find it necessary for later consistency to define { } as a subset of every set, as you saw above.

<u>Problems 1-4</u>

1. Copy the following statements and complete them.

   (a) {a, b} is a _____ of {a, b, c, d}.

   (b) If D = {2, 4, 6, 8, 10}, and E = {4, 6, 8}, then _____ is a subset of _____.

2. Y = {house, tree}
   V = {cat, house, tree}
   Which set is a subset of the other?

3. Is {a, b, c, d, e} a subset of {a, b, c}?
   Is {a, b, c, d, e} a subset of {a, b, c, d, e}?
   Explain your answers.

4. Consider the set {a, b, c}. Some examples of subsets of this set are {a, b}, {a, c}, {b}. Make a list of <u>all</u> the subsets of {a, b, c}. Did you find eight subsets? If not, go back and find the ones you missed.

5. List all the subsets of {a, b, c, d}. How many did you find?

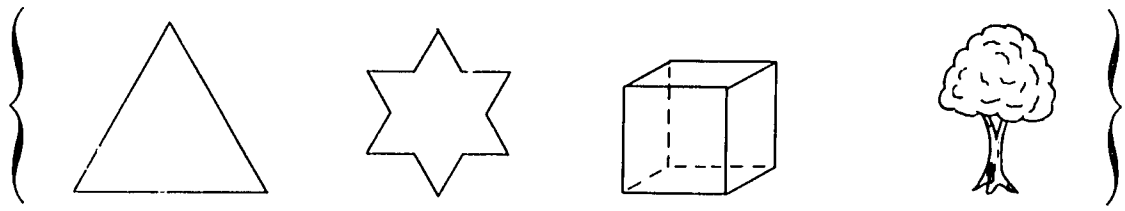
6. <u>Challenge Problem</u>

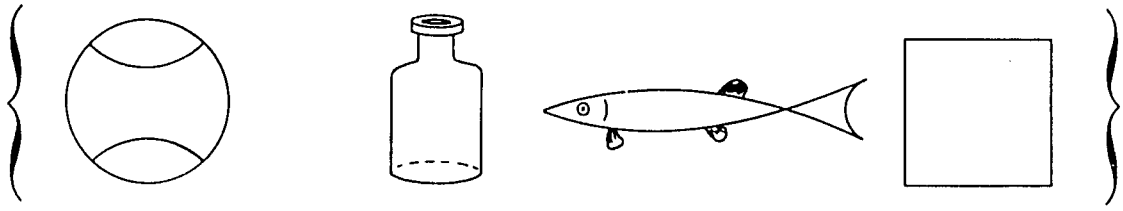
   (a) Determine how many subsets there are for a set of five members such as {a, b, c, d, e}.

   (b) A set of six members.

6. (c) Try to generalize from what you have done to give a rule for determining how many subsets there are in a set of n elements, where n is any positive integer. Test your rule for some particular cases.

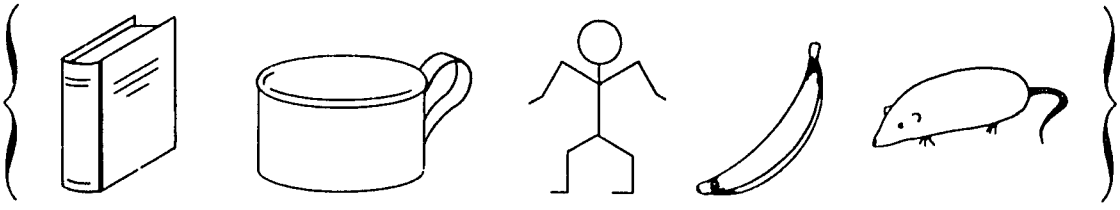
1-5 Matching of Sets

A = 

B = 

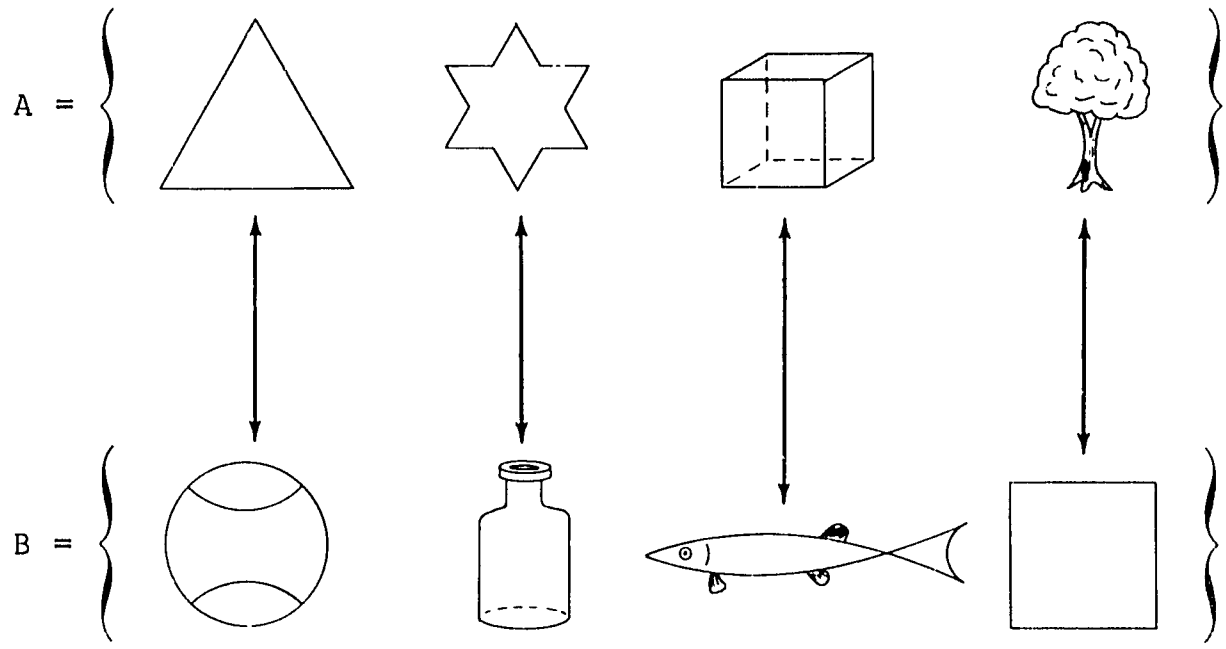
C = {1, 2, 3, 4}

D = {a, b, c}

E = 

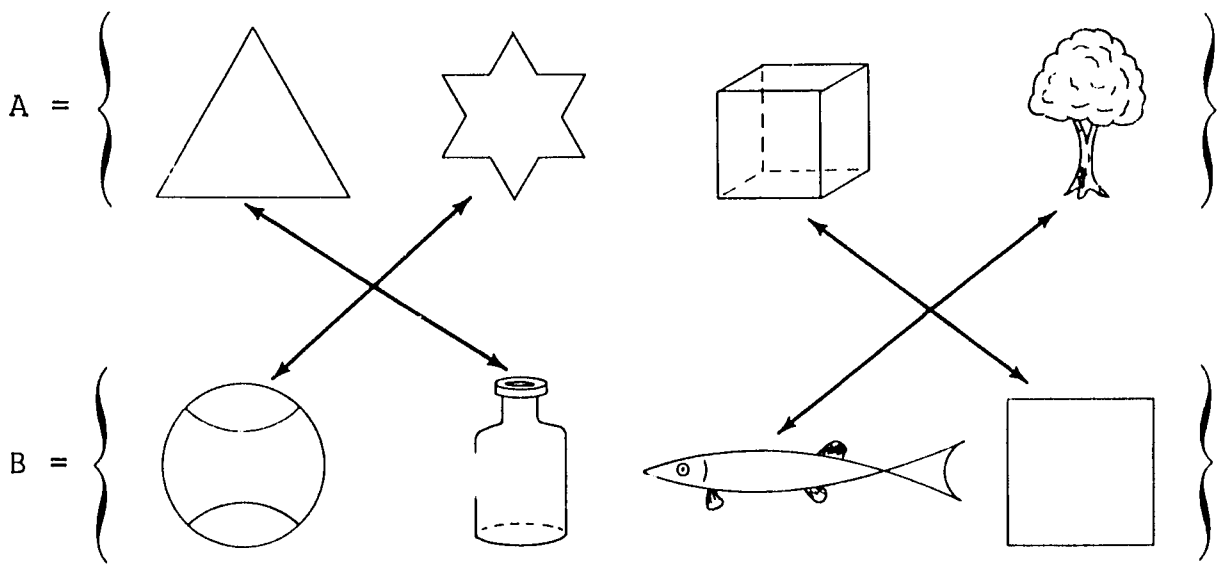
(Figure 2)

Consider the sets pictured above. They are all different because they have different members, and yet we see that A and B have something in common. We can find for each member of set B a partner in set A.



(Figure 3)

The double arrow shows how each member of A is matched with a member of B, and each member of B is matched with a member of A. This matching could be done in many different ways. Another way of doing it is shown below.



(Figure 4)

No matter which way we match the sets, each member of A is matched with exactly one member of B and each member of B is matched with exactly one member of A. We say that set A and set B <u>match exactly</u>. We have another name for an exact match between the members of two sets: a <u>one-to-one correspondence</u> between the sets. There is a one-to-one correspondence between the sets A and B.

Furthermore, A and C match exactly, and B and C match exactly. On the other hand, none of the sets A, B, and C matches exactly with either of the sets D and E. Why? We say that two sets are <u>equivalent</u>, if there is an exact match between them. Set A is equivalent to set B and we write $A \longleftrightarrow B$. Also set B is equivalent to set C and we write $B \longleftrightarrow C$. Set A is also equivalent to set C and we write $A \longleftrightarrow C$. You will notice that a set can be equivalent to more than one set. You will also notice that if we have any three sets P, Q, and R such that P is equivalent to Q and Q is equivalent to R, then P is equivalent to R.

<u>Problems 1-5A</u>

1. Which of the following pairs of sets can be matched exactly? Illustrate how this can be done in each case where exact matching is possible.

   (a) A = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

   B = {Ghana, Tanzania, Kenya, Liberia, Nigeria, Malawi, Zambia, Sierra Leone, Uganda, Ethiopia}

   (b) X = the set of all even counting numbers less than 20

   Y = the set of all even numbers which are greater than 40 and which are also less than 60

   (c) S = {1, 2, 3, 4, 5, ...}

   T = {5, 10, 15, 20, ..., 5000}

   (d) K = the set of all counting numbers

   L = the set of counting numbers whose members are divisible by 3

1. (e) P = the set of pupils in your class
   Q = the set of all feet in the same class

   (f) M = the set of pupils in your class
   N = the set of all noses in your class

2. (a) Give three examples of pairs of sets such that in each pair there is a one-to-one correspondence between the elements of the two sets.

   (b) Give three examples of pairs of sets such that in each pair the two sets do not match exactly.

3. Show by using arrows ($\longleftrightarrow$) that the set $\{a, b\}$ is equivalent to each of six subsets of the set $\{p, q, r, s\}$. To which subsets of $\{p, q, r, s\}$ is the set $\{a, b\}$ not equivalent?

<u>Equal Sets</u>

Let us look carefully at the following sets:

$A = \{1, 3, 4, 5, 2\}$

$B = \{3, 4, 1, 5, 2\}$

C = the set of the first five counting numbers.

Compare set A and set B. Do you notice that set A is equivalent to set B? Why? Did you also notice that the members of A are exactly the same as the members of B? The only difference between the set A and the set B is the order in which their members are listed. This difference is not important. We say that A and B are two names for the very same set. We write $A = B$, and we mean that A is <u>the same set</u> as B or set A is equal to set B.

Now look at set C. Is this set equal to set A and to set B? The members of C are the same as the members of A and are the same as the members of B. Therefore C is the same set as the sets A and B and we write $C = A$ and $C = B$.

Now look again at sets A, B, and C. Did you notice that A is a subset of B and B is a subset of A? It is, in fact, always true that if $A \subseteq B$ and $B \subseteq A$, then $A = B$. And, of course, if $A = B$ then $A \subseteq B$ and $B \subseteq A$. Do you see why these two statements are true?

Problems 1-5B

1. $A = \{2, 4, 6, 8\}$; B is the set of all even counting numbers less than 10. Is $A = B$?

2. In which of the following pairs of sets are the two sets equal?

   (a) $K = \{m, t, h, a, s\}$; $L = \{m, a, t, h, s\}$

   (b) The set of all toes on your left foot and the set $\{a, b, c, d, e\}$

   (c) The set of all eyes in the classroom and the set of all pupils in the classroom

   (d) The set of all odd counting numbers less than 10 and the set $\{1, 5, 7, 8\}$

   (e) The set of letters in the word "bundle" and the set $\{n, d, l, e, b\}$

   (f) The set of countries in East Africa and the set {Kenya, Nigeria, Tanzania}

3. Look again at the examples in Problem 2. In which pairs of sets are the two sets equivalent? Are all the pairs of equivalent sets also pairs of equal sets? Are all the pairs of equal sets also pairs of equivalent sets? Will equal sets always be equivalent sets?

4. Look at these sets carefully.

   $A = \{3, 5, 7, 9\}$ $B = \{5, 3, 9, 7\}$ $C = \{9, 7, 2, 5\}$
   $D = \{9, 7, 2, 3\}$ $E = \{5, 7, 9, 2\}$ $F = \{7, 3, 9, 5\}$

   They are all equivalent sets. Why?

   Which sets are not equal to set A? Write down three of them that are equal to set A.

1-6 Operations with Sets

We have learned some facts about sets. Now let us see how we can work with sets.

Study the following class list carefully.

Name	Age	Football team	School prefect	School choir
Agesa	13	✓		✓
Alli	14	✓	✓	
Juma	16	✓		
Mary	15		✓	✓
Kizza	16	✓		✓
Sarah	15			✓
Modu	13		✓	
Oyelese	15	✓		
Patel (Miss)	14			✓

Let us list the members of various sets of pupils in this class.

<u>Example 1</u>. List:

the set A of pupils who are 15 years of age and over,

the set B of pupils who are under 15 years of age,

the set C of pupils who are 14 years of age.

You will have written the following:

A = {Juma, Mary, Kizza, Sarah, Oyelese}

B = {Agesa, Alli, Modu, Patel}

C = {Alli, Patel}

What is the set whose members are in one or the other or both of the sets A and B? Let us call this set R. Then the set R looks like this.

R = {Agesa, Alli, Modu, Patel, Juma, Mary, Kizza, Sarah, Oyelese}

What is the set whose members are in one or the other or both of the sets A and C? Let us call this set S. The set S then contains the following members.

S = {Alli, Patel, Juma, Mary, Kizza, Sarah, Oyelese}

Let T be the set whose members are in one or the other or both of the sets B and C? Does your set T look like this?

T = {Agesa, Alli, Modu, Patel}

Is this set equal to set B? Why is this so?

Example 2. List:

the set F of pupils who play in the school football team,

the set P of pupils who are prefects of the school.

Have you written the following?

F = {Agesa, Alli, Juma, Kizza, Oyelese}

P = {Alli, Mary, Modu}

Let V be the set whose members are in one or the other or both of the sets F and P. Then the set V looks like this.

V = {Agesa, Alli, Juma, Kizza, Oyelese, Mary, Modu}.

We call the set R the union of set A and set B. You obtained the set R by writing the names of all the pupils who are members of A or of B or of both A and B. Similarly S is the union of A and C, T is the union of B and C, and V is the union of F and P.

For any two sets G and H, the set consisting of all those elements which are either members of G or members of H or members of both G and H is called the union of set G and set H. In symbols we write the union of G and H as "G $\cup$ H", which we read "G union H".

There is something interesting to be discovered in trying to answer the following questions about the union of two sets. Suppose you know how many members there are in each of the sets G and H. Can you tell how many members there will be in the union of G and H? Take a look at Example 1. How many members are there in set A? How many members are there in set B? How many members are there in set R, the union of A and B?

We see that these numbers have the familiar relation $9 = 5 + 4$. What happens when you ask the same questions in Example 2 about the sets F and P and their union V? This time the corresponding relation for the numbers of members of the sets would be $7 = 5 + 3$, which of course is false. How can you explain the difference between these two examples? Now ask yourself the same questions for sets A and C and their union S; sets B and C and their union T. It will be easier to explain these results if you use the following idea.

> Any two sets which have no members common to them are called <u>disjoint sets</u>.

Examples of disjoint sets are the sets A and B. A and C are also disjoint sets. However, the sets B and C are not disjoint and the sets F and P are not disjoint.

It is easy to see that if two sets G and H are disjoint, the number of elements in their union is equal to the sum of the numbers of elements in the two sets. But if the sets are not disjoint, the number of elements in their union is less than the sum of the numbers of elements in each set.

<u>Problems 1-6A</u>

1. Find the union for each of the following pairs of sets.

   <u>Example</u>: $X = \{1, 2, 3\}$; $Y = \{2, 3, 4\}$
   $X \cup Y = \{1, 2, 3, 4\}$

   (a) $R = \{5, 10, 15\}$; $T = \{15, 20\}$

   (b) $B = \{O\ \square\}$; $D = \{\Delta\}$

   (c) $A = \{\Delta\ \square\ O\}$; $B = \{\star\ O\ \Delta\ \square\}$

   (d) $E = \{a, b, c, d\}$; $F = \{c, d, e, f, g, h\}$

   (e) $J = \{a, b, c, d, e, f\}$; $K = \{\quad\}$

   (f) S is the set of pupils in your class who are girls; Y is the set of pupils in your class who are in the school choir.

(g) I is the set of pupils in your class who have two brothers; J is the set of pupils in your class who do not have two brothers.

(h) M is the set of the first five letters of the alphabet;
N = {a, b, c, d, e}

2. In the following statements some members of the sets are missing. Copy down the statements and complete them by supplying the missing members.

(a) {Δ □} ∪ {◇ } = { Δ O □ ◇ }

(b) {m, a} ∪ {d, , , } = {m, a, g, d, o, n}

(c) {n, } ∪ {a, , } = {n, y, a, l, i}
(Can this one be done in more than one way?)

(d) {1, 3, 5, 7, 9} ∪ {11, 13, , } = {1, 3, 5, 7, 9, 11, 13, 15}
(How many different answers are possible here?)

(e) {a, b, c, d} ∪ {a, , } = {a, b, c, d}
(How many different answers are possible?)

3. A = {2, 4, 8, 16}
B = {1, 3, 5}
C = {a, b, c, d, e}

(a) Find A ∪ B and B ∪ A. Is A ∪ B = B ∪ A?

(b) Find B ∪ C and C ∪ B. Is B ∪ C = C ∪ B?

(c) Find (A ∪ B) ∪ C and A ∪ (B ∪ C).
Is (A ∪ B) ∪ C = A ∪ (B ∪ C)?

We have seen how to form the union of two sets. Let us look again at Examples 1 and 2 on pages 14 and 15.

In <u>Example 1</u>,

A = {Juma, Mary, Kizza, Sarah, Oyelese}
B = {Agesa, Alli, Modu, Patel}
C = {Alli, Patel}

While you were forming T, the union of B and C, you found that Alli and Patel were members of B and also members of C. The set {Alli, Patel} is called the <u>intersection</u> of B and C.

> The <u>intersection</u> of set G and set H is the set of all those elements which are in both G and H. We write this as "$G \cap H$" which we read "G intersection H".

In <u>Example 2</u>,

F = {Agesa, Alli, Juma, Kizza, Oyelese}

P = {Alli, Mary, Modu}

Alli is a member of F and also of P. The set {Alli} is the intersection of F and P, that is, $F \cap P$ = {Alli}. What will be the intersection of A and B? Would you say that the intersection of A and C is the empty set? Why? What in general is the intersection of any two disjoint sets?

<u>Problems 1-6B</u>

1. Find the intersection for each of the pairs of sets in Problem 1 of 1-6A.

2. P = the set of all people in Africa.
   M = the set of all people in Malawi.
   Find the intersection of P and M.

3. A = {1, 3, 7, 4}; B = {2, 6, 8}; C = the set of all counting numbers; D = {4, 5, 6, 7, 8}; E = { }. Find $A \cap B$, $B \cap C$, $A \cap D$, $D \cap E$.

4. F = {1, 3, 5, 7}; G = {4, 14, 24}; H = {1, 2, 3, 4, 5, 6, 7, 8}
   Show that the following are true statements.

   (a) $F \cap G = G \cap F$, $G \cap H = H \cap G$, $F \cap H = H \cap F$.

   (b) $(F \cap G) \cap H = F \cap (G \cap H)$

   (c) $F \cap (G \cup H) = (F \cap G) \cup (F \cap H)$

(d) $F \cup (G \cap H) = (F \cup G) \cap (F \cup H)$

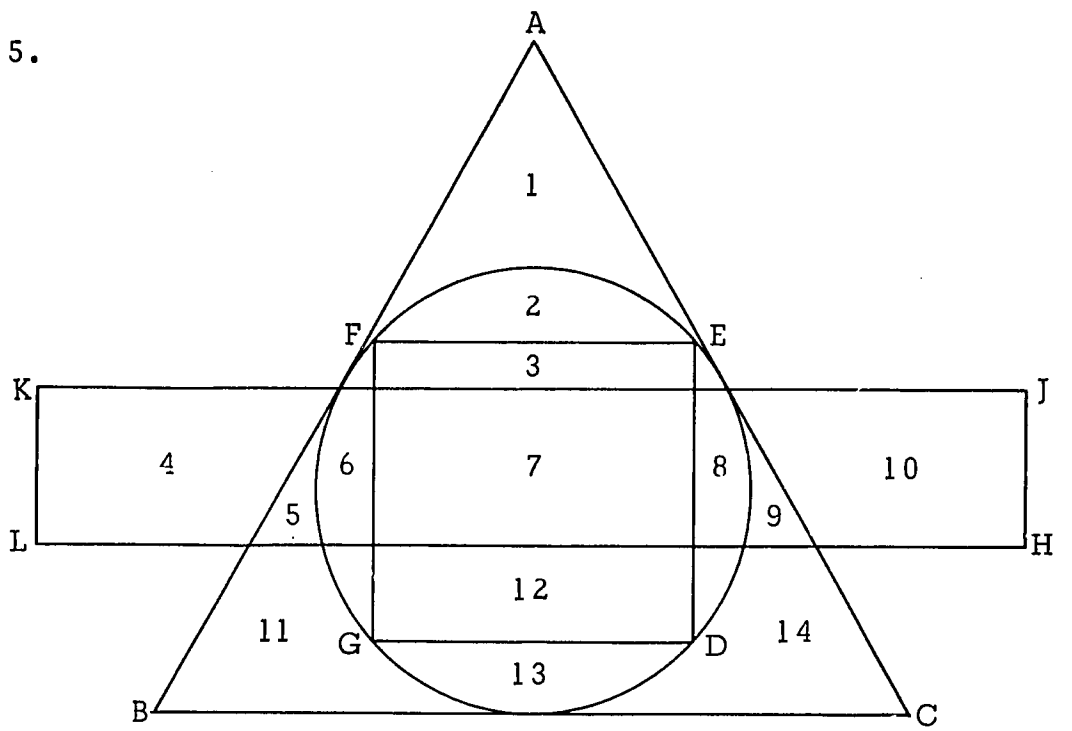
(e) $F \cap (F \cup G) = F \cup (F \cap G) = F$

(f) $F \cap F = F \cup F = F$

(g) $F \cap \{\quad\} = \{\quad\}$

(h) $F \cup \{\quad\} = F$

5.



Let O be the set of all numbers represented inside the circle DEFG. □ be the set of all numbers represented inside the square DEFG. ▭ be the set of all numbers represented inside the rectangle HJKL. Δ be the set of all numbers represented inside the triangle ABC.

From the figure answer the following questions.

<u>Example</u>. $O \cap ▭ = \{6, 7, 8\}$

(a) $O \cap □ =$?

(b) $O \cap \Delta =$?

(c) Is $O \subseteq \Delta$? Why?

(d) $▭ \cup □ =$?

5. (e) $\bigcirc \cup \square =$?

(f) Is $\square \subseteq \bigcirc$? Why?

(g) $\square \cap \triangle =$?

(h) $\bigcirc \cup \square =$?

(i) Is $\bigcirc \cup \square = \square \cup \bigcirc$?

(j) $\square \cap \square =$?

(k) $\square \cap \square =$?

(l) Is $\square \cap \square = \square \cap \square$?

(m) $\square \cup (\square \cap \bigcirc) =$?

(n) $(\square \cup \square) \cap (\square \cup \bigcirc) =$?

(o) Is $\square \cup (\square \cap \bigcirc) = (\square \cup \square) \cap (\square \cup \bigcirc)$?

(p) $\square \cap (\square \cup \bigcirc) =$?

(q) $(\square \cap \square) \cup (\square \cap \bigcirc) =$?

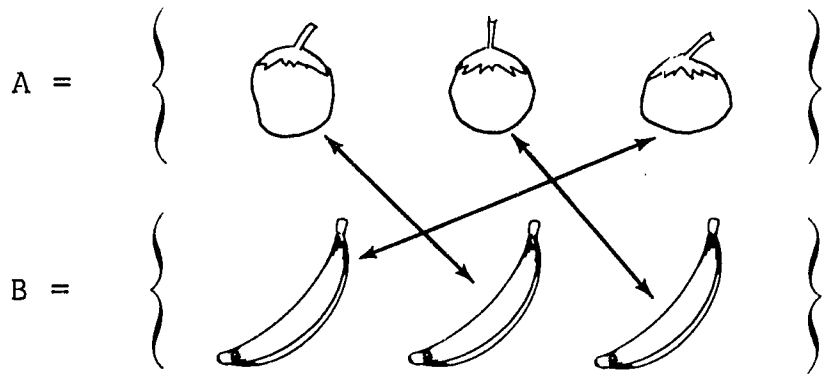
(r) Is $\square \cap (\square \cup \bigcirc) = (\square \cap \square) \cup (\square \cap \bigcirc)$?

CHAPTER 2

NUMBER

2-1 Comparison of Sets by Matching

You will remember that in Chapter 1, Section 1-5, we discussed at length facts about matching sets. Let us remind ourselves of these facts. Let us do this by looking at these sets of pawpaws and bananas.



We can match each banana with exactly one pawpaw and, in doing so, we also match each pawpaw with exactly one banana. You remember that these two sets are said to match exactly or to be equivalent sets. We notice that there are just as many bananas as there are pawpaws. Any other matching of each member of set A to one member of set B will be an exact match of set A and set B. How many more such matchings can you find? Draw pictures of some of them.

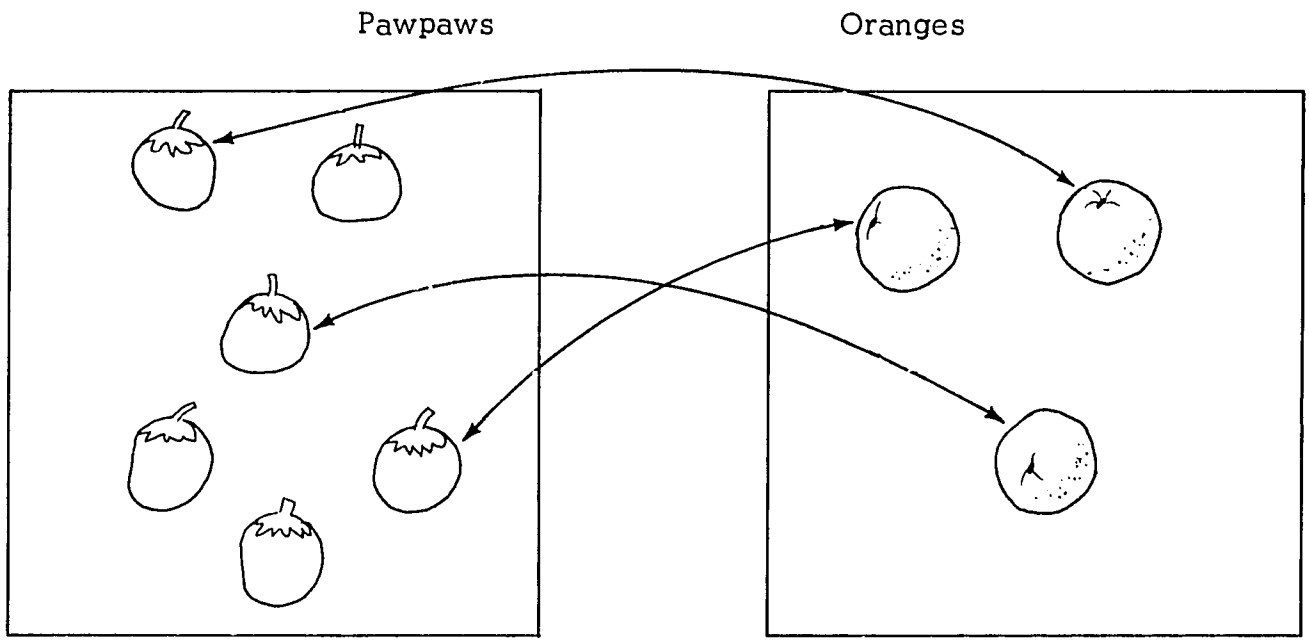
Again examine the following sets:

$$A = \{p, q, r, s, t\}$$

$$B = \{k, l, m, n, o\}$$

Do set A and set B match exactly? Give reasons for your answer. Try to match each member of set A with exactly one member of set B in as many different ways as you can.

Now look at these sets of oranges and pawpaws:



Match each orange with exactly one pawpaw. Some pawpaws are left over. Show two other ways in which you can match each orange with one pawpaw, and notice that in each case there were some pawpaws left over. These sets therefore do not match exactly. We say that the set of pawpaws has <u>more members</u> than the set of oranges. We may also say that the set of oranges has <u>fewer members</u> than the set of pawpaws.

Again consider the set $C = \{1, 3, 6, 8\}$ and the set $D = \{9, 11, 13, 16, 19, 20\}$. It is easy to discover that these sets do not match exactly. Why? Does set C have as many members as set D? Does set C have fewer members than set D?

It becomes clear then that when we match the members of a set P with the members of a set Q, we discover that one and only one of the following relations holds:

Set P has <u>exactly as many members</u> as set Q.

Set P has <u>fewer members</u> than set Q.

Set P has <u>more members</u> than set Q.

Problems 2-1

1. Take the following pairs of sets of objects and try to match their members in such a way that each member of one set is matched with exactly one member of the other. When you have compared two sets, make sentences about them using the phrases as many members, fewer members, and more members (the matching should be done without counting).

   (a) The set of coins in your pocket and the set of toes on your right foot.

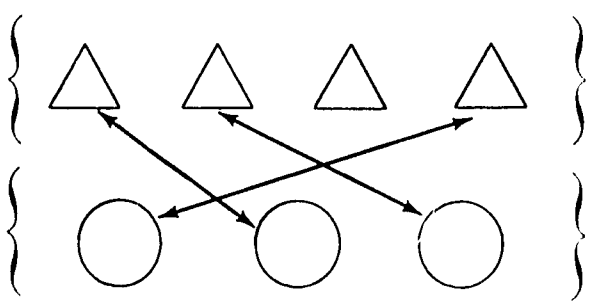
   (b) The set {O □ ← ⬭ ↓ ▽} and the set of teachers in your school.

   (c) The set of presidents in Africa and the set of independent African states.

   (d) The set of ministers in your government and the set of all universities in your country.

2-2 Sets in Their Natural Order

Compare the following sets:



The set of triangles has more members than the set of circles. In fact, since there is exactly one triangle that is not matched with a circle, we say that the set of triangles contains one more element than the set of circles. If we wish

to list the sets in order depending on which one contains more members, we would place the set of circles first, followed by the set of triangles. We then say that the sets are placed <u>in their natural order</u>.

Now consider the following sets of triangles.

{ Δ Δ Δ }

{ Δ Δ }

{ Δ Δ Δ Δ Δ }

{ Δ }

{ Δ Δ Δ Δ Δ Δ Δ Δ }

{ }

{ Δ Δ Δ Δ Δ Δ Δ }

{ Δ Δ Δ Δ Δ Δ Δ Δ Δ }

{ Δ Δ Δ Δ Δ Δ }

{ Δ Δ Δ Δ }

Can you see that no set of triangles matches any other set exactly? Let us rearrange the sets in their natural order, beginning with the set that has fewest members. We note that there is an empty set. Because the empty set has the fewest members of all, it comes at the beginning of the new arrangement. The set consisting of one triangle has one more member than the empty set, so it comes next. By continuing to choose the set which has one more member than the set before it, we place the sets in natural order as shown below.

{ }

{ Δ }

{ Δ Δ }

{ Δ Δ Δ }

{ Δ Δ Δ Δ }

{ Δ Δ Δ Δ Δ }

{ Δ Δ Δ Δ Δ Δ }

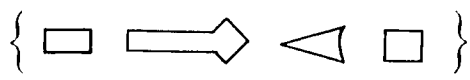
{ Δ Δ Δ Δ Δ Δ Δ }

{ Δ Δ Δ Δ Δ Δ Δ Δ }

{ Δ Δ Δ Δ Δ Δ Δ Δ Δ }

Problems 2-2

1. Place the following sets in natural order.

(a) 

(b)

(c)

(d)

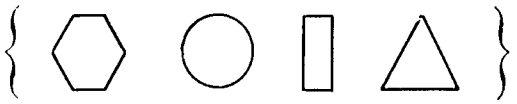
(e) {A, B, C, D, E, F, G, H}

(f) {p, q, r, s, t, u, v}

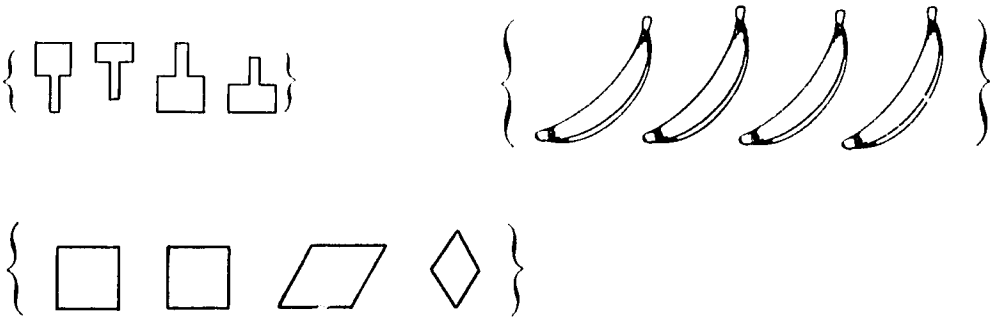
(g)

2-3 Number as a Property of Sets

Suppose that we start with some set, for example, the set pictured here:



Think of sets which are equivalent to this set; that is, sets which can be matched exactly with it. Here are pictures of some such sets.



Each of these sets has exactly as many members as the first set. In fact all sets that are equivalent to the first set will have exactly as many members as the first one.

Problems 2-3A

1. For each of the following sets, make up two sets which match these sets exactly.

   (a) {a, b, c, d, e}

   (b) { O △ □ }

   (c)

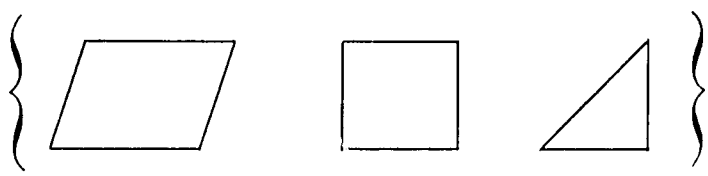
We have discovered that equivalent sets have something in common: each such set has exactly as many members as any other. We say that all of these sets have the same <u>number</u> of elements. The number we associate with the set

{a, b, c, d} is named <u>four</u>. This same number <u>four</u> is associated with all the other sets which are equivalent to this set. All these sets have one thing in common. It is the property of <u>four-ness</u>.

Let us see what we have discovered in this section. We have seen that with a <u>set</u> of objects, there is associated a <u>number</u> telling how many members the set contains. Each number has several names; for example the symbol "4" and the word "four" are both names for the same number. We call these names <u>numerals</u>.

<u>Problems 2-3B</u>

1. The number attached to the set



is called <u>three</u> and is represented by the numeral 3. Picture other sets containing 3 members.

2. What is a name for the number attached to the set which in the natural order comes just after the set in problem 1?

3. What is the number attached to any set which comes just before the set in problem 1 in the natural order? Picture a few such sets,

---

2-4 Identifying the Whole Numbers

You have seen how to place sets in natural order. We give a name to the number of members in each set, and we write the numeral which represents the number.

Set	English Name of Number	Numeral
{ }	Zero	0
{ □ }	One	1
{ □ □ }	Two	2
{ □ □ □ }	Three	3
{ □ □ □ □ }	Four	4
{ □ □ □ □ □ }	Five	5
{ □ □ □ □ □ □ }	Six	6
{ □ □ □ □ □ □ □ }	Seven	7
{ □ □ □ □ □ □ □ □ }	Eight	8
{ □ □ □ □ □ □ □ □ □ }	Nine	9

The numbers 0, 1, 2, 3, 4, 5, and so on are called <u>whole numbers</u>. The set {0, 1, 2, 3, 4, 5, ...} is the set of whole numbers. We give these numbers a special name because as you will see later there are other kinds of numbers.

2-5 Order in the Set of Whole Numbers

When <u>sets</u> are placed in natural order as above, we also say that the corresponding <u>whole numbers</u> are themselves in <u>natural order</u>. Thus 0, 1, 2, 3 are in natural order. On the other hand 3, 6, 5, 4 are not in natural order; the natural order is 3, 4, 5, 6.

Problems 2-5A

Here are some sets:

$\{a, b, c, e\}$

$\{b, c\}$

$\{a, b, d\}$

$\{e\}$

1. Find the whole numbers attached to each of these sets.

2. Place the sets in natural order. The corresponding numbers will also be in natural order.

We have seen how a set of objects has a whole number attached to it. Suppose the number of members in set A is m, and the number of members in set B is n.

(i) If A has exactly as many members as B, the number of members in A equals the number of members in B. We express this fact by writing $m = n$. In this case m and n are two numerals (names) for the same number.

(ii) If A has fewer members than B, we say that the number of members in A is less than the number of members in B. We express this fact by writing $m < n$.

(iii) If A has more members than B, we say that the number of members in A is greater than the number of members in B. We express this fact by writing $m > n$.

Of course if A has less members than B, then B has more members than A and accordingly if $m < n$, then $n > m$. For example if $2 < 5$, then $5 > 2$. Furthermore, to say that $m < n$ is the same as saying that the whole number n comes after the whole number m in the natural order. Since 3, 4, 5, 6 are in natural order, $6 > 5$, $6 > 4$, $6 > 3$, $5 > 4$, $5 > 3$, $4 > 3$.

Problems 2-5B

Here are four sets:

{a, b, c, d} {□ O Δ} { } {a, c, e, f}

1. Find the numbers of members in each set.

2. Write as many statements as you can, using these numbers and the symbols $<$, $=$, $>$.

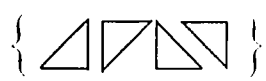
2-6 Counting and the Counting Numbers

We have found that when sets are in their natural order, each set has one more member than the set before it. Let us form sets of whole numbers and place them in their natural order. {1} is a set whose one member is the number 1; {1, 2} is a set consisting of two members, the numbers 1 and 2 in their natural order; {1, 2, 3} is a set containing three members, the numbers 1, 2, and 3 in their natural order; and so on. {1, 2, 3, ..., 27} is a set with twenty-seven members, the numbers 1, 2, 3, ..., 27 in their natural order. Notice that the last number listed in each of these sets is exactly the number of elements in the set. Similarly, the set {1, 2, 3, ..., n} is the set of n whole numbers 1, 2, 3, ..., n in natural order. The last number is n, and n is the number of members in the set. These sets of numbers are called counting sets. The numbers in each counting set are counting numbers in natural order, starting with 1. The complete set of counting numbers is the set {1, 2, 3, 4, 5, ...} and this set is also called the set of natural numbers. Counting sets are represented below.

Counting Sets	Number of members
{1}	1
{1, 2}	2
{1, 2, 3}	3
{1, 2, 3, 4}	4
{1, 2, 3, 4, 5}	5
{1, 2, 3, 4, 5, 6}	6
{1, 2, 3, 4, 5, 6, 7}	7
.	
.	
.	
{1, 2, 3, 4, 5, 6, 7, ..., n}	n

2-7 Finding the Number of Members in a Set

When we use a counting set, we use the natural order of the numbers in that set. That is we count "one, two, three, ...". Now you may ask, what do we actually do when finding the number of members in any given set? For example, take the set of triangles given below.



{1, 2, 3, 4}

Match each triangle with exactly one number in the counting set starting with 1, continuing in natural order. The set of triangles and the counting set match exactly when the last triangle is matched with the last number in the counting set. This number which is the largest number in the counting set, tells us the number of triangles in the set.

This is what you did when you first learned how to count. For example if you wanted to count the number of elements in a set of stones, or a set of sticks, or a set of bananas, you would say "one, two, three, four", and so on. When you said "one" you were matching the number 1 with one of the objects, as you said "two" you were matching the number 2 with another object, etc. The last number you named was matched with the last remaining object in the set. This last number told you the number of objects in the set.

2-8 Equivalence Using Counting Sets

You can use counting sets to tell whether two sets are equivalent or not. Remember we said that two sets A and B are equivalent when they match exactly. We can now state the idea of equivalence in another way, using counting sets.

Let us suppose we have a set A of triangles and a set B of squares. In order to tell whether the two sets are equivalent, find the counting set which matches exactly the set A. See whether the same counting set also matches exactly the set B. If it does, then we can conclude that because set A is equivalent to the counting set, and set B is equivalent to the counting set, the set A is equivalent to set B.

Consider the example below:

A = {△ △ △ △}

Counting Set {1, 2, 3, 4}

B = {□ □ □ □ }

We see that set A is equivalent to the counting set {1, 2, 3, 4} and set B is equivalent to the same counting set, so A and B are equivalent sets.

Problems 2-8

Find by counting, whether set A and set B are equivalent.

1. A: { □ ○ ⇦ ▽ ▭ ⬭ }

   B: The set of fingers on your left hand.

2. A: The set of consonants in the word "relation".

   B: The set of sides of a square.

2-9 Ways of Representing Numbers

The number four cannot be seen as we can see a stone or a banana. It is an idea. To make it seem real we represent it by a word or a symbol. Thus we can use the word four or the symbol 4, which we have called a numeral, to represent the idea of four-ness. Just as there are different words for the number four in different languages, there are different systems of numerals which have been used. The ancient Egyptians would have written |||| instead of 4. The Romans would have written IV.

Here are some of the numerals that were used by the early Egyptians. They built up their system in groups of ten as follows.

Number	Egyptian numeral	Object represented by numeral
One		Stroke or staff
Ten		Heel bone
One hundred		Scroll or coiled rope
One thousand		Lotus flower
Ten thousand		Pointing finger
Hundred thousand		Polliwog
One million		Astonished man

In the Egyptian system, the order in which the symbols are arranged does not matter. Thus, ∩∩ | , |∩∩ , ∩|∩ are all numerals for twenty-one. Other examples of Egyptian numerals are as follows:

	7	23	456	1821
Egyptian Numerals	|||||||	∩∩|||	𓍢𓍢𓍢𓍢 ||| ∩∩∩∩∩ |||	𓆼 𓍢𓍢𓍢𓍢 𓍢𓍢𓍢𓍢 ∩∩|

In the Babylonian system which was used about 4,000 years ago, there were only two symbols, 𒁹 and 𒌋 . 𒁹 was used to represent <u>one</u> and 𒌋 was used to stand for <u>ten</u>. These two symbols were used repeatedly to write numerals from 1 to 59. Examples of Babylonian numerals are as follows.

	6	23	39	45
Babylonian Numerals	𒁹𒁹𒁹 𒁹𒁹𒁹	𒌋𒌋𒁹𒁹𒁹	𒌋𒌋𒌋 𒁹𒁹𒁹 𒁹𒁹𒁹 𒁹𒁹𒁹	𒌋𒌋𒌋𒌋𒁹𒁹𒁹𒁹𒁹

In the Babylonian system unlike the Egyptian system a symbol for a larger number always came before a symbol for a smaller one. The same two symbols were used to represent numbers greater than 59 but the method was confusing. For example 𒁹 could represent either sixty or three thousand six hundred, as well as one.

Nowadays the system of numerals we use is called the Hindu-Arabic system. It has ten symbols 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, which we call digits. We are able to represent any number, however large, by using these symbols and an especially clever idea, that of <u>place value</u>. As you know, we write digits in the ones place, the tens place, the hundreds place, the thousands place, and so on. The value of any symbol depends both on the symbol and on the place it occupies in the row of digits. Thus in the numeral 3234, the first 3 represents three thousands and the second 3 represents three tens.

Hundred Thousands	Ten Thousands	Thousands	Hundreds	Tens	Ones
		3	2	3	4

The number 3234 would be written in the Egyptian system as

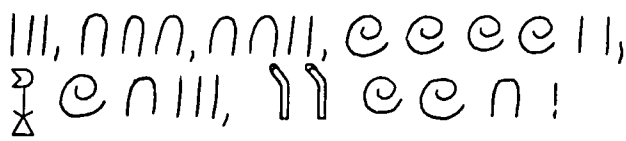
where the symbols could be arranged in any order.

Problems 2-9

1. Write Egyptian numerals for the following:

   six, fourteen, three hundred fifty-six, three thousand twenty.

2. What numbers are represented by the following Egyptian numerals?



3. Write Babylonian numerals for the following:

   ten, twenty-five, forty-nine, thirty-six.

4. (a) Write Roman numerals for the following:

   fifty, one hundred, five hundred, one thousand, three thousand five hundred fifty-six.

   (b) What numbers are represented by the following Roman numerals?

   LX, XXXIX, XLIII, CCLXIV, MCXCVIII, MDCL.

CHAPTER 3

NUMERATION SYSTEMS

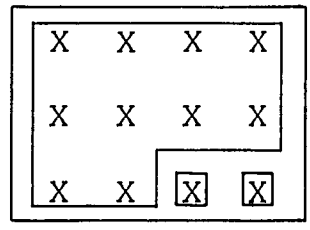
3-1 Base Ten Numeration

You have seen that our system of numeration is called the Hindu-Arabic system of numeration. This system was first used by the Hindus of India over a thousand years ago, and was carried to Europe by the Arabs. The symbols which we use today are not exactly like the ones first used by the Hindus. They have changed over the years. The final form of the symbols as we know them today is

0, 1, 2, 3, 4, 5, 6, 7, 8, 9

There are ten of these symbols. We call them digit symbols or simply digits.

We use these digit symbols to represent numbers. When we write a numeral for a number greater than nine, we may be thinking of a set with more than nine members. A numeral for the number of elements in such a set is suggested by grouping the members of the set in tens. For example, a set with twelve members can be thought of as containing 1 group of ten members, and 2 groups of one member,



and the number of elements in the set is expressed by the numeral 12. The place or position of the digit 1 in the numeral 12 tells us that there is one group of ten numbers. The place of the digit 2 tells us that there are two groups of one member. Thus 12 means 1 ten and 2 ones.

In the same way, we think of a set of ten members as containing 1 group of ten members and 0 groups of one member, and we write this as 10. We write

Previous Page Blank

numerals for numbers from ten to ninety-nine as two-digit numerals - the digit on the left telling us the number of groups of ten, and the digit on the right telling us the number of ones.

99 means 9 tens and 9 ones. The next counting number is one more than 99, and this consists of ten (10) groups of tens and 0 groups of ones. We therefore group in ten tens (10×10) or hundreds. We consider a hundred as 1 hundred and 0 tens and 0 ones. 444 means 4 hundreds and 4 tens and 4 ones.

Every digit of a numeral in the Hindu-Arabic system tells a number of ones, or a number of tens, or a number of ten tens or hundreds, or a number of ten hundreds or thousands, and so on. We say that our system of numeration is a base ten system and that each digit has a place value. The Latin word for ten is decem. From this we get the English adjective decimal. Our (Hindu-Arabic) system of numeration is called a decimal system because it is based on ten.

Problems 3-1

1. What is the name given to ten hundreds?

2. What does each of the digits in the following numerals mean?

(a) 35 (b) 649 (c) 438 (d) 1,325,678 (e) 507
(f) 3500 (g) 48,010 (h) 87 (i) 13,002 (j) 500

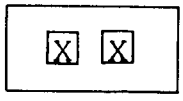
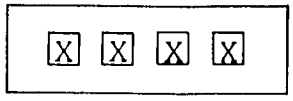
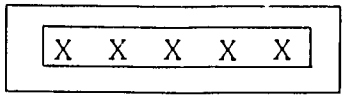
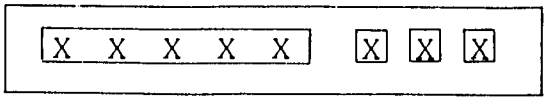
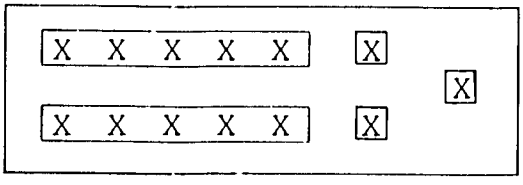
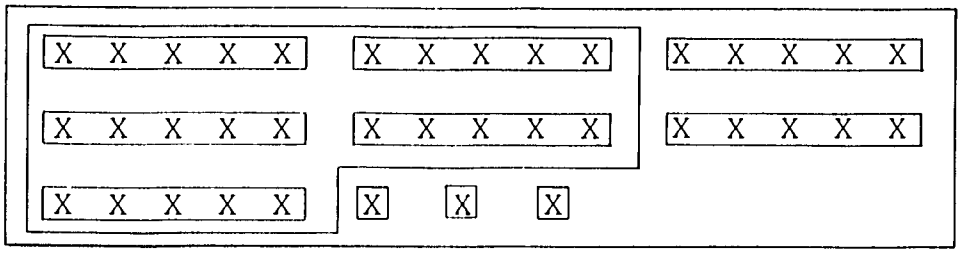
3-2 Base Five Numeration

When we group a set of elements in tens, we can represent the number of elements by a base ten numeral. It is not only in tens that we can group elements. We can arrange them in groups of other sizes also. As an example, let us study groupings in fives. We will therefore write numerals for our numbers in base five.

What numerals shall we use in our base five numeration? We may use the same symbols as for base ten for the number of members in sets containing zero, one, two, three, or four elements. Our first five numerals are 0, 1, 2, 3, 4. Now

however, we group by fives, and we think of a set containing five elements, as made up of 1 group of five members and 0 groups of 1 member. We write this as $(10)_{five}$ and read one-zero base five. We shall agree to use the numerals 0, 1, 2, 3, 4 for base five numerals.

We illustrate base five numerals in the following table. Study it carefully.

Sets	Base Five Numeration	Base Ten Numeration
 two ones	2	2
 four ones	4	4
 1 group of five and 0 groups of one	10	5
 1 group of five and 3 groups of one	13	8
 2 groups of five and 3 groups of one	23	13
 1 group of five fives, 2 groups of five, and 3 groups of one	123	38

We read $(23)_{five}$ as two-three base five. Also we note that $(123)_{five}$, read one-two-three base five, and $(38)_{ten}$, read three-eight base ten, are two names for the same number.

In $(123)_{five}$, the 3 on the extreme right indicates the number of ones, the 2 second from right indicates the number of fives and the 1 third from right indicates the number of five fives (twenty-fives).

When we count in base five, we say one, two, three, four, one-zero, one-one, one-two, one-three, and so on. We must be very careful not to read, for example, $(24)_{five}$ as twenty-four. Twenty-four means two tens and four ones, and not two fives and four ones.

Problems 3-2

1. Copy and complete the chart below for numerals in base five from 1 to $(100)_{ten}$.

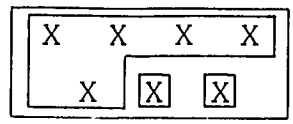
1	2	3	4	10	11	12	13	14	20
21									
41									

2. Use the chart in Problem 1 to answer the following question.

What are the largest one-digit, two-digit and three-digit numbers listed in your chart?

3-3 Addition in Base Five

Suppose we wish to add 4 and 3 in base five. If we group a set of (4 + 3) members in fives, we can represent this as



This shows that we have 1 group of five and 2 groups of one. We represent the sum as $(12)_{five}$. We could have considered the addition as follows: (4 + 3) = (4 + 1) + 2 = 1 five and 2 ones or $(12)_{five}$. Similarly (4 + 4) = 1 five and 3 ones = $(13)_{five}$.

Example Add 44 and 33 in base five.

$$\begin{array}{r} 44 \\ \underline{+\ 33} \end{array}$$

We may think out the addition as follows:

4 ones + 3 ones = 1 five and 2 ones; we write 2 in the ones place and add the 1 five to the 3 fives and the 4 fives.

$$\begin{array}{r} {}^{1} \\ 44 \\ \underline{+\ 33} \\ 2 \end{array}$$

The sum of 1 five, 3 fives, and 4 fives is 1 five five (twenty-five) and 3 fives; we write the 3 in the fives place and the 1 in the twenty-fives place.

$$\begin{array}{r} {}^{1} \\ 44 \\ \underline{+\ 33} \\ 132 \end{array}$$

Thus the sum of $(44)_{five}$ and $(33)_{five}$ is $(132)_{five}$.

We might also have arranged the work as follows:

$$\begin{array}{rcccc} 44 & = & (40)_{five} & + & 4 \\ +\ 33 & = & (30)_{five} & + & 3 \\ \hline & = & (120)_{five} & + & (12)_{five} \end{array} = (132)_{five}.$$

Problems 3-3

1. Copy and complete the addition table in base 5.

+	0	1	2	3	4
0					
1					
2					
3					
4					

2. Add in base 5

(a) $\begin{array}{r} 13 \\ +\ 11 \\ \hline \end{array}$ (b) $\begin{array}{r} 23 \\ +\ 21 \\ \hline \end{array}$ (c) $\begin{array}{r} 10 \\ +\ 24 \\ \hline \end{array}$ (d) $\begin{array}{r} 31 \\ +\ 21 \\ \hline \end{array}$ (e) $\begin{array}{r} 41 \\ +\ 21 \\ \hline \end{array}$

3. Add in base 5

(a) $\begin{array}{r} 24 \\ +\ 11 \\ \hline \end{array}$ (b) $\begin{array}{r} 23 \\ +\ 43 \\ \hline \end{array}$ (c) $\begin{array}{r} 14 \\ +\ 13 \\ \hline \end{array}$ (d) $\begin{array}{r} 22 \\ +\ 33 \\ \hline \end{array}$ (e) $\begin{array}{r} 23 \\ +\ 34 \\ \hline \end{array}$

4. Copy these sentences. Write symbols in the parentheses to make true sentences.

(a) $(\ 3)_{five} + (\ \ )_{five} = (10)_{five}$

(b) $(11)_{five} + (\ \ )_{five} = (13)_{five}$

(c) $(10)_{five} + (\ \ )_{five} = (44)_{five}$

(d) $(41)_{five} + (34)_{five} = (34)_{five} + (\ \ )_{five}$

(e) $(32)_{five} + (\ \ )_{five} = (40)_{five} + (32)_{five}$

3-4 Subtraction in Base Five

When we say 1 + 3 = 4, we mean that the sum of 1 and 3 is 4. However, we might ask what number must be added to 1 to get 4? From the equation above we see that the answer is 3. This is what we do when we subtract. To subtract 1 from 4 we ask what number must be added to 1 to get 4.

Problems 3-4A

Do the following subtractions in base 5.

1. $\begin{array}{r} 10 \\ -\ 4 \end{array}$	2. $\begin{array}{r} 12 \\ -\ 3 \end{array}$	3. $\begin{array}{r} 24 \\ -\ 13 \end{array}$	4. $\begin{array}{r} 42 \\ -\ 21 \end{array}$	5. $\begin{array}{r} 23 \\ -\ 11 \end{array}$

Study the following examples carefully. They show how we regroup in subtraction in base 5. Be sure you understand the steps in the calculation before you do the next set of exercises.

Example 1. Subtract in base 5.

$$\begin{array}{r} 31 \\ -\ 14 \end{array}$$

Write $(31)_{five} = (20)_{five} + (11)_{five}$ since we cannot subtract 4 from 1.

$$\begin{array}{rcl} (14)_{five} & = & (10)_{five} + 4 \\ \hline & & (10)_{five} + 2 = (12)_{five} \end{array}$$

Hence $\begin{array}{r} 31 \\ -\ 14 \\ \hline 12 \end{array}$ in base five

Check $\begin{array}{r} 12 \\ +\ 14 \\ \hline 31 \end{array}$

Example 2. Subtract in base 5.

$$\begin{array}{r} 211 \\ -\ 43 \end{array}$$

$$(211)_{five} = (100)_{five} + (100)_{five} + (11)_{five}$$

$$(\ 43)_{five} = \underline{\qquad\qquad (\ 40)_{five} + 3}$$

$$(100)_{five} + (\ 10)_{five} + 3 = (113)_{five}$$

Hence
$$\begin{array}{r} 211 \\ -\ 43 \\ \hline 113 \end{array}$$

Check
$$\begin{array}{r} 113 \\ +\ 43 \\ \hline 211 \end{array}$$

Problems 3-4B

Subtract in base 5. Check by addition.

1. $\begin{array}{r} 142 \\ -\ 23 \\ \hline \end{array}$ 2. $\begin{array}{r} 210 \\ -\ 2 \\ \hline \end{array}$ 3. $\begin{array}{r} 100 \\ -\ 40 \\ \hline \end{array}$ 4. $\begin{array}{r} 412 \\ -\ 34 \\ \hline \end{array}$ 5. $\begin{array}{r} 402 \\ -\ 13 \\ \hline \end{array}$

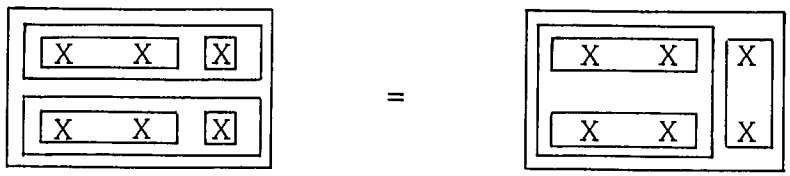
3-5 Base Two Numeration

Some of you may know what "computers" are. These are machines which are used to do calculations, many of which are long and complicated. A computer can record only two digits, 0 and 1; therefore the computer uses a base two system of numeration. Let us consider how numbers may be written in the base 2 system.

A set of two members is considered as 1 group of two members and 0 groups of one member, and is written 10. A set of three members is considered as 1 group of two members and 1 group of 1 member. This is written as 11. A set of four members is considered as 1 group of (2×2) members, 0 groups of two members, and 0 groups of one member. The number of elements in this set is written as 100. Thus we count in base two as follows: 1, 10, 11, 100, 101, 110, 111, 1000, 1001, etc.

Example Add 11 and 11 in base two.

We can represent this addition by the following pictures:



$(11)_{two}$ is represented by 1 group of two and 1 group of one. If we put two such pictures together and regroup we get 1 group of two two's (four), 1 group of two, and 0 groups of one.

We may also arrange the work as follows:

$$\begin{array}{rcrcr} 11 & = & 10 & + & 1 \\ \underline{11} & = & \underline{10} & + & \underline{1} \\ & & 100 & + & 10 \end{array} = 110;$$

that is, $(11)_{two} + (11)_{two} = (110)_{two}$.

Problems 3-5

Add in base 2

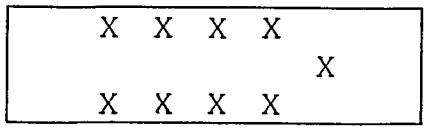
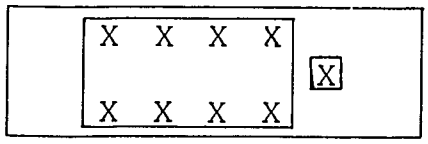
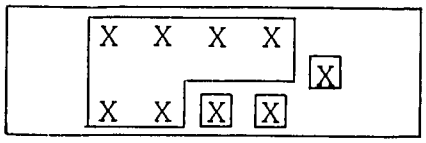
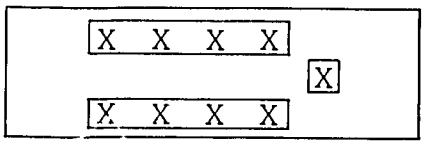
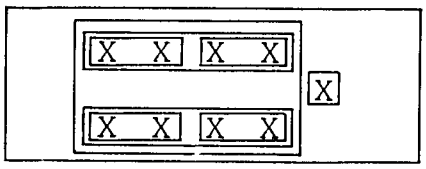
1.	10 + 1	2.	10 1 1 + 1	3.	1001 + 101	4.	101 + 11	5.	110111 + 11011

Subtract in base 2

6.	11 - 1	7.	111 - 11	8.	11011 - 1011	9.	1001 - 110	10.	1000 - 111

3-6 Numeration in Other Bases

From the above, you can see that numeration may be done in other bases by grouping in the number of the base being considered. Study the following table.

Set	Base of Numeration System	Numeral In That Base
	10	$(9)_{ten}$
	8	$(11)_{eight}$
	6	$(13)_{six}$
	4	$(21)_{four}$
	2	$(1001)_{two}$

A set of nine elements may be grouped as 1 group of eight members and 1 group of one member. The number of elements in this set may be written as $(11)_{eight}$. The set may also be grouped as 1 group of six members and 3 groups of one member. The number of elements in the set may be written as $(13)_{six}$. The set may be grouped as 2 groups of four members and 1 group of one member. The number of elements may be written as $(21)_{four}$. The set may be written as 1 group of $(2 \times 2 \times 2)$ members, 0 group of (2×2) members, 0 group of two members, and 1 group of one member. The number of elements in the set may be written as $(1001)_{two}$.

Problems 3-6

1. Express the number eight as a numeral in

   (a) base four (b) base three (c) base seven

2. Express the number $(19)_{ten}$ in

   (a) base eleven (b) base thirteen

3-7 The Decimal System of Numeration

You have seen that the position of each digit in the decimal system of numeration gives us special information. Consider the number 444.

The "4" on the right means 4 ones, and may be written (4×1). The "4" second from right means 4 tens and may be written (4×10), and the "4" third from the right means 4 hundreds and may be written $4 \times 10 \times 10$ (since $100 = 10 \times 10$). Thus $444 = (4 \times 10 \times 10) + (4 \times 10) + (4 \times 1)$.

Numerals like $(4 \times 10 \times 10) + (4 \times 10) + (4 \times 1)$ are called expanded numerals. The expanded numeral for 863 is $(8 \times 10 \times 10) + (6 \times 10) + (3 \times 1)$. The expanded numeral for 1005 is $(1 \times 10 \times 10 \times 10) + (0 \times 10 \times 10) + (0 \times 10) + (5 \times 1)$.

Problems 3-7A

Write expanded numerals for each of the following numbers.

1.	94	6.	5
2.	303	7.	50
3.	999	8.	500
4.	7019	9.	5000
5.	5003	10.	0

The position or place value of every digit in our system of numeration is important. The place value of every digit in the decimal system of numeration is 10 times the place value of the next digit on the right. Study the following chart showing place values.

PLACE VALUE CHART

hundred millions	ten millions	millions	hundred thousands	ten thousands	thousands	hundreds	tens	ones
9	8	2,	4	0	6,	2	1	5

When reading numerals for large numbers, we separate the digits in threes by commas. The numeral on the chart is read nine hundred eighty-two million, four hundred six thousand, two hundred fifteen. The expanded numeral for this number 982,406,215 may be written as follows:

$(9 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10) +$
$(8 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10) +$
$(2 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10) +$
$(4 \times 10 \times 10 \times 10 \times 10 \times 10) +$
$(0 \times 10 \times 10 \times 10 \times 10) +$
$(6 \times 10 \times 10 \times 10) +$
$(2 \times 10 \times 10) +$
$(1 \times 10) +$
(5×1)

Do you observe any pattern? Do you see how tiresome it is to keep writing 10's? Mathematicians have agreed to write 10^3, which is read ten cubed or ten to the third power, for $10 \times 10 \times 10$. In the same way 10^6, read ten to the sixth power or simply ten to the sixth, means $10 \times 10 \times 10 \times 10 \times 10 \times 10$. Also 5^4, read five to the fourth, stands for $5 \times 5 \times 5 \times 5$. The numbers 3, 6, and 4 above are called <u>exponents</u>. The expanded numeral for 982,406,215 may now

be written $(9 \times 10^8) + (8 \times 10^7) + (2 \times 10^6) + (4 \times 10^5) + (0 \times 10^4) + (6 \times 10^3) + (2 \times 10^2) + (1 \times 10) + (5 \times 1)$.

Do you see a pattern in which the exponents occur?

Problems 3-7B

Use exponents to write the following base 10 numerals in expanded form.

1.	513	6.	900,412
2.	50,149	7.	81,871,354
3.	222	8.	1,000,000
4.	357,000	9.	10,010,101
5.	8665	10.	10,000,000,001

Write simple numerals for each of the following numbers.

11. $(9 \times 10^2) + (0 \times 10^1) + (1 \times 1)$

12. $(5 \times 10^4) + (5 \times 10^3) + (5 \times 10^2) + (5 \times 10^1) + (5 \times 1)$

13. $(6 \times 10^5) + (8 \times 10^4) + (0 \times 10^3) + (1 \times 10^2) + (7 \times 10^1) + (0 \times 1)$

14. $(5 \times 10^2) + (1 \times 1)$

3-8 Expanded Numerals in Any Base

What is the expanded numeral for $(14)_{five}$? We know that $(14)_{five}$ means 1 five and 4 ones. Since the numeral for 1 five in base five is 10, we find that in base five, $14 = (1 \times 10) + (4 \times 1)$. This is the expanded numeral in base 5 for 14. In the same way, the expanded numeral in base 5 for $(34)_{five}$ is $(3 \times 10) + (4 \times 1)$. What is the expanded numeral in base five for $(234)_{five}$? We know that each digit has a place value; thus

2 3 4

4 ones

3 fives

2 five fives or 2 twenty-fives.

In base 5 we write a five as 10, a five times five as 100 or 10^2. Hence the expanded numeral for $(234)_{five}$ is $(2 \times 10^2) + (3 \times 10) + (4 \times 1)$. It is worth noting that although the two numerals stand for different numbers, the expanded numeral for $(234)_{ten}$ in base ten has exactly the same form as the expanded numeral for $(234)_{five}$ in base five. Is this accidental? It seems less surprising when we notice that the numeral 10 always represents the base number in any numeration system; similarly 100 always represents base number times base number or the base number squared, etc. The conclusion we draw is that expanded numerals always have the same form regardless of the number base.

Problems 3-8

1. Write the expanded numerals for the following numbers in base five.

   (a) $(12)_{five}$

   (b) $(412)_{five}$

   (c) $(13,421)_{five}$

2. Write the expanded numeral for $(16)_{seven}$ in base seven.

3. Write the expanded numeral for $(1101)_{two}$ in base two.

3-9 Changing Numerals Written in any Base to Base Ten

If we wish to change a numeral written in any base to base 10, we first write it as an expanded numeral in the original base, and then translate it into base 10.

Example 1. Write $(4324)_{five}$ in base 10.

$$
\begin{aligned}
(4324)_{five} &= (4 \times 10^3) + (3 \times 10^2) + (2 \times 10) + (4 \times 1) && \text{in base 5} \\
&= (4 \times 5^3) + (3 \times 5^2) + (2 \times 5) + (4 \times 1) && \text{in base 10} \\
&= 600 + 75 + 10 + 4 && \text{in base 10} \\
&= 689 \text{ in base 10.}
\end{aligned}
$$

Example 2. Write $(111111001)_{two}$ in base 10.

$(111111001)_2 = (1 \times 10^8) + (1 \times 10^7) + (1 \times 10^6) + (1 \times 10^5) +$

$(1 \times 10^4) + (1 \times 10^3) + (0 \times 10^2) + (0 \times 10) + (1 \times 1)$ in base 2

$= (1 \times 2^8) + (1 \times 2^7) + (1 \times 2^6) + (1 \times 2^5) +$

$(1 \times 2^4) + (1 \times 2^3) + (0 \times 2^2) + (0 \times 2) + (1 \times 1)$ in base 10

$= 256 + 128 + 64 + 32 + 16 + 8 + 1$ in base 10

$= 505$ in base ten.

Problems 3-9

Write as base 10 numerals.

1. $(321)_{five}$
2. $(13)_{five}$
3. $(103)_{five}$
4. $(57)_{eight}$
5. $(1011)_{two}$
6. $(321)_{nine}$
7. $(321)_{four}$
8. $(11022)_{three}$

3-10 Rewriting a Numeral From Base Ten to Another Base

Rewriting a base 10 numeral in another base may be done by regrouping. Thus 9 = 1 five and 4 ones, and is written 14 in base 5.

Larger numbers in base 10 may be written in other bases by repeated division.

Example 1.

Suppose we wished to write $(72)_{ten}$ in base 5. If we divide 72 by 5 we find that 72 = 14 fives and 2 ones left over. We know at once that the digit in the ones place is 2. To find the digit in the fives place we group the 14 fives

into groups of twenty-five (5×5) and groups of five. We find the number of such groups by dividing 14 by 5. Thus 14 fives = 2 twenty-fives and 4 fives. Now we know that the digit in the fives place is 4, and the digit in the (5×5)'s place is 2. Hence $(72)_{ten} = (242)_{five}$.

We found by dividing that $72 = (2 \times 5^2) + (4 \times 5) + (2 \times 1)$ in base 10

$= (2 \times 10^2) + (4 \times 10) + (2 \times 1)$ in base 5

$= 242$ in base 5.

The method of repeated division may be expressed as follows:

5 | 72

5 | 14 fives and 2 ones left over

5 | 2 twenty-fives and 4 fives left over

0 one-hundred twenty-fives and 2 twenty-fives left over

Hence $(72)_{ten} = (242)_{five}$.

<u>Example 2</u>.

Write $(1000)_{ten}$ in base 5.

We use the method of repeated division by five.

5 | 1000

5 | 200 fives and 0 ones left over

5 | 40 twenty-fives and 0 fives left over

5 | 8 one-hundred twenty-fives and 0 twenty-fives left over

5 | 1 six hundred twenty-fives and 3 one-hundred twenty-fives left over

0 three thousand twenty-fives and 1 six hundred twenty-fives left over

Hence $(1000)_{ten} = (13000)_{five}$.

Example 3.

Write $(19)_{ten}$ in base 2.

We use the method of repeated division by two.

2	19	
2	9	twos and 1 one left over
2	4	fours and 1 two left over
2	2	eights and 0 fours left over
2	1	sixteen and 0 eights left over
	0	thirty-twos and 1 sixteen left over

Hence $(19)_{ten} = (10011)_{two}$.

Problems 3-10

Rewrite in the bases shown.

1. $(15)_{ten} = (\quad)_{five}$
2. $(134)_{ten} = (\quad)_{five}$
3. $(746)_{ten} = (\quad)_{eight}$
4. $(26,495)_{ten} = (\quad)_{seven}$
5. $(2341)_{ten} = (\quad)_{five}$
6. $(645)_{ten} = (\quad)_{six}$
7. $(5555)_{ten} = (\quad)_{five}$
8. $(11)_{ten} = (\quad)_{two}$
9. $(123,456)_{ten} = (\quad)_{nine}$

3-11 Revision Problems

1. How many digits are used in
   (a) base ten numeration?

1. (b) base five numeration?

   (c) base six numeration?

   (d) base seven numeration?

   (e) base two numeration?

   (f) base b numeration (where b is a counting number greater than 1)?

2. In decimal numeration the place value of the digit 4 in 431 is 10^2. What is the place value of the digit 4 in

   (a) 1234?

   (b) 12413?

   (c) 40,000,001?

3. Write expanded numerals for each of the following:

   (a) 12,312

   (b) $(321)_{five}$

   (c) $(11010)_{two}$.

4. Work in base 5

   (a) Add 314
   22

   (b) Subtract 201
   13

5. Copy and fill in the following base five multiplication table.

×	0	1	2	3	4	10
0						
1						
2						
3						
4						
10						

6. Work in base 7:

Add	153	Subtract	624
	242		335
	605		

7. Find decimal numerals for these numbers:

(a) $(1321)_{five}$

(b) $(316)_{seven}$

(c) $(512)_{six}$

8. Fill in the blanks:

(a) $(123)_{ten}$ = ($)_{five}$

(b) $(32)_{five}$ = ($)_{ten}$ = ($)_{two}$

(c) $(40)_{five}$ = ($)_{two}$

(d) $(35)_{six}$ = ($)_{five}$

9. Write the following in words and then write down expanded numerals for each of them.

(a) 15,348,721 (b) 57,843,287 (c) 102,030,405

10. Does a counting number in base 10 remain a counting number when expressed in base 5?

CHAPTER 4

THE NATURAL NUMBERS

4-1 Natural Numbers and Whole Numbers

In Chapter 2 you learned that the set of counting numbers is $N = \{1, 2, 3, 4, \ldots\}$. This set of counting numbers is also called the set of natural numbers. If we add the number zero to this set, we have another set of numbers which we have called the set of whole numbers. You have learned that the set of whole numbers may be written as $W = \{0, 1, 2, 3, \ldots\}$.

In this chapter we shall be concerned only with the natural numbers. We shall now make the agreement that every time we use the word "number" in this chapter, we shall mean natural number.

Set of Natural Numbers	$N = \{1, 2, 3, \ldots\}$
Set of Whole Numbers	$W = \{0, 1, 2, 3, \ldots\}$

Problems 4-1

1. What is the intersection of N and W?
2. What is the union of N and W?
3. What is the smallest natural number? Is there a largest?
4. What is the smallest whole number? Is there a largest?
5. Is the set of natural numbers a subset of the set of whole numbers? Why?
6. Is the set of whole numbers a subset of the set of natural numbers? Why?

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4-2 Factors of Numbers

If we consider the product of two numbers, say 3 and 12, then $3 \times 12 = 36$. We say that 3 and 12 are both factors of the product 36. We also call 3 and 12 divisors of 36, since $36 \div 3 = 12$ and $36 \div 12 = 3$. The number 36 is said to be a multiple of 3 and 12.

More generally, if a, b, and n are natural numbers such that $a \times b = n$, a and b are called factors or divisors of n, while n is said to be a multiple of a and b.

What are other factors of 36? Do you agree that 1, 2, 4, 6, 9, 18 and 36 are the other factors of 36? Notice that 1 is a factor of 36. Is 1 a factor of every number? Why? Notice also that 36 is a factor of 36. Is every number a factor of itself? Why?

What is the set of all factors of 36? Do you agree that the set of all factors of 36 is $\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$? We say that the set of factors of any number is the set of its different factors.

Problems 4-2

1. List the set of factors of each of the following numbers:

   6; 12; 18; 24; 36; 48; 54; 64; 72; 81; 96; 100.

2. Is there a number which is a factor of every number? If so, what number?

3. In the following, each product represents a number. List the set of factors of that number.

   (a) $2 \times 3 \times 5$

   (b) $2 \times 3 \times 7$

   (c) $2 \times 2 \times 3$

   (d) $2 \times 2 \times 2$

   (e) $3 \times 3 \times 3$

   (f) $3 \times 5 \times 5$

   (g) $5 \times 5 \times 7$

   (h) $3 \times 3 \times 5 \times 7$

   (i) $3 \times 5 \times 5 \times 7$

   (j) $3 \times 5 \times 7 \times 7$

4-3 Prime Numbers

Consider the number 2. The set of factors of 2 is {1, 2}. Also, the set of factors of 3 is {1, 3}, and the set of factors of 5 is {1, 5}. The set of factors of 4 is {1, 2, 4}, and the set of factors of 6 is {1, 2, 3, 6}. Let us make a list of sets of factors of some more numbers:

Set of factors of 7 is {1, 7}
Set of factors of 8 is {1, 2, 4, 8}
Set of factors of 9 is {1, 3, 9}
Set of factors of 10 is {1, 2, 5, 10}
Set of factors of 11 is {1, 11}
Set of factors of 1 is {1}

We see that some numbers have only two factors, the number itself and 1, while other numbers have more than two factors. A number which has <u>exactly two</u> different factors, the number itself and one, is called a <u>prime number</u>. Check to see that the first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29.

Those numbers that have more than two different factors are called <u>composite numbers.</u> The set of composite numbers is {4, 6, 8, 9, 10, . . .}.

What can we say about the number 1? We notice that 1 is neither in the set of prime numbers, nor in the set of composite numbers; the set of factors of 1 is simply {1}. Hence the number 1 is <u>not</u> prime, and it is <u>not</u> composite.

In our discussion above, into how many different sets have we put the numbers? We see that there are three such sets: The set of prime numbers, the set of composite numbers, and the set consisting of the number 1.

1. A natural number which has exactly two different factors, itself and 1, is called a <u>prime number</u>.

2. A natural number which has more than two different factors is called a <u>composite number</u>.

3. The number 1 is neither composite nor prime.

A method for finding the prime numbers less than any given number was suggested by a Greek mathematician named Eratosthenes (276? - 195? B. C.). This method consists of sifting out those numbers which are not prime numbers and is called the sieve of Eratosthenes. One way of describing the sieve is as follows:

1. Write all the natural numbers from 2 to the given number.

2. Cross out all the numbers that have 2 as a factor, except 2 itself.

3. Cross out all the remaining numbers that have 3 as a factor, except 3.

4. The next number which is not crossed out is 5; now cross out all the remaining numbers that have 5 as a factor, except 5 itself, and so on. After this process is completed, the numbers remaining in the list are all prime. Why?

In finding the prime numbers less than 50, the work would look like this:

2, 3, ~~4~~, 5, ~~6~~, 7, ~~8~~, ~~9~~, ~~10~~, 11, ~~12~~, 13, ~~14~~, ~~15~~, ~~16~~,
17, ~~18~~, 19, ~~20~~, ~~21~~, ~~22~~, 23, ~~24~~, ~~25~~, ~~26~~, ~~27~~, ~~28~~, 29,
~~30~~, 31, ~~32~~, ~~33~~, ~~34~~, ~~35~~, ~~36~~, 37, ~~38~~, ~~39~~, ~~40~~, 41, ~~42~~,
43, ~~44~~, ~~45~~, ~~46~~, 47, ~~48~~, ~~49~~, ~~50~~.

One stroke (4) is used for numbers having 2 as a factor.
Two strokes (9) are used for other numbers having 3 as a factor.
Three strokes (25) are used for other numbers having 5 as a factor.
Four strokes (49) are used for other numbers having 7 as a factor.

Those numbers which are not crossed out are the prime numbers less than 50. This set is {2, 3, 5, 7, 11, 17, 19, 23, 29, 31, 37, 41, 43, 47}.

Problems 4-3

1. Use the sieve of Eratosthenes to find all the prime numbers less than 100.

2. Find the largest prime number less than 300; less than 500.

3. Find two prime numbers whose product is an even number. Must one of these numbers be 2? Why?

4. Find two prime numbers whose product is an odd number. Can one of the numbers be 2? Why? Can any two prime numbers other than 2 be used? Why?

5. We notice that

$4 = 2 + 2$	$10 = 3 + 7$
$6 = 3 + 3$	$12 = 5 + 7$
$8 = 3 + 5$	$14 = 3 + 11$

We see that all the even numbers greater than 2 and less than or equal to 14 can be written as the sum of <u>two</u> prime numbers. Extend this list to include all even numbers less than or equal to 50. Do you think that every even number greater than 2 can be expressed as the sum of two primes?

6. 3 and 5, 5 and 7, 11 and 13, 17 and 19, etc. are called <u>twin</u> primes because they are prime numbers which differ by 2. Find the next four sets of twin primes.

7. Could you use the sieve of Eratosthenes to find all the prime numbers less than 500? less than 1,000? less than 1,000,000? Do you think there are prime numbers greater than 1,000,000? Do you think there is a <u>last</u> prime number? How many prime numbers do you think there are?

4-4 Prime Factorization of Natural Numbers

In the previous section we grouped the set of natural numbers into three different subsets: the set of prime numbers, the set of composite numbers, and the set consisting of the number 1. Let us now consider the set of composite numbers and see if we can write each of these as a product of prime numbers. Consider the number 24. Is 24 in the set of composite numbers? Write 24 as the product of prime numbers. Have you written $24 = 2 \times 2 \times 2 \times 3$? You might have first said $24 = 4 \times 6$; then, since $4 = 2 \times 2$ and $6 = 2 \times 3$, you found that $24 = 2 \times 2 \times 2 \times 3$. Because all of these factors of 24 are prime numbers, we cannot obtain any factors smaller than these. Why? When we express a number as a product of prime factors, we then say that we have found the prime factorization of that number. Also, when a number is written as the product of prime numbers it is said to be factorized completely. Why don't we speak of the prime factorization of a prime number?

> When we express a number as a product of prime factors, then we say that we have found the prime factorization of that number.

Problems 4-4

1. Find the prime factorization of each of the following composite numbers: 10, 15, 16, 18, 20, 36, 48, 72, 144, 256.

2. Factorize completely each of the following:
   9, 12, 21, 24, 30, 42, 108, 125, 216, 648.

4-5 The Fundamental Theorem of Arithmetic

Consider the composite number 60. If we try to find the prime factorization of 60, we might first write $60 = 5 \times 12$. But $12 = 4 \times 3$, and $4 = 2 \times 2$. Therefore, $60 = 5 \times 2 \times 2 \times 3$ is a prime factorization of 60. But suppose we started with the factors 6 and 10 of 60 and first write $60 = 10 \times 6$. Will we

obtain the <u>same</u> prime factorization? $10 = 2 \times 5$, and $6 = 2 \times 3$. Hence $60 = 2 \times 5 \times 2 \times 3$, and we see that $2 \times 5 \times 2 \times 3 = 5 \times 2 \times 2 \times 3$; the prime factorizations are the same, but the factors are in a different order. Do you think that a similar result will be true for <u>any</u> composite number?

The fact that we obtain the same prime factors of 60, no matter how we carry out the factorization, is an example of what is called the <u>Fundamental Theorem of Arithmetic:</u> Every composite number can be factorized as a product of prime numbers in exactly one way, except for the order of the factors. Since the prime factorization can be done in only one way, we can say that the prime factorization is unique.

> <u>Fundamental Theorem of Arithmetic:</u>
>
> The prime factorization of any composite number is unique, except for the order of the factors.

<u>Problems 4-5</u>

Factorize completely each of the following numbers by two different methods, and show that the prime factors obtained in each are the same.

<u>Example</u> $30 = 6 \times 5 = 2 \times 3 \times 5$

$30 = 3 \times 10 = 3 \times 2 \times 5$

But $2 \times 3 \times 5 = 3 \times 2 \times 5$

30, 98, 81, 80, 45, 100, 330, 92, 126, 372, 390, 258, 231, 910.

4-6 Greatest Common Factor

Consider the numbers 36 and 90. The set of all factors of 36 is $A = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$, and the set of all factors of 90 is $B = \{1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90\}$. Do you agree? What is the set $A \cap B$, the set of all factors common to both A and B? Do you see that $A \cap B = \{1, 2, 3, 6, 9, 18\}$? What is the <u>largest</u> of these common factors?

We see that 18 is the largest number which belongs to $A \cap B$, and we call 18 the <u>greatest common factor</u> (GCF) of 36 and 90. The greatest common factor is also called the greatest common divisor (GCD).

We see then that the greatest common factor of two numbers is the largest number which is a factor of both the given numbers.

Suppose we were asked to find the greatest common factor of 120 and 192. We could proceed as in the previous paragraph, but the amount of work involved is quite long and rather tedious. Is there another method that we could use to find the greatest common factor of 120 and 192? In the previous section we noted that any composite number could be factorized uniquely as the product of prime factors except for the order of the factors. Let us use this fact to find the GCF of 120 and 192. $120 = 2^3 \times 3 \times 5$ and $192 = 2^6 \times 3$ (We recall from Section 3-7 that 2^3 means $2 \times 2 \times 2$, 2^6 means $2 \times 2 \times 2 \times 2 \times 2 \times 2$, etc.). What different prime factors are common to both of these factorizations? How many times is 2 a factor of 120? of 192? How many 2's are common factors of 120 and 192? Is 3 a factor of both 120 and 192? Is 5 a factor of both numbers?

We see then that the common factors of 120 and 192 are 2^3 and 3. Therefore the GCF of 120 and 192 is $2^3 \times 3 = 24$.

You will notice that what we actually did in the example above was to find the different prime factors which occurred in the prime factorization of <u>both</u> numbers. We then took each different prime factor as many times as it occurred as a common factor; that is, the <u>least</u> number of times it appears as a factor in either of the given numbers. The GCF was then the product of these common prime factors, each to the smallest power to which it occurs in either of the prime factorizations.

Let us look at another example. Suppose we wish to find the GCF of 56 and 84. $56 = 2^3 \times 7$. $84 = 2^2 \times 3 \times 7$. The common prime factors are 2 and 7. Since 2 occurs as a factor of 56 three times, but only twice as a factor of 84, 2^2 is the largest power of 2 which can enter into the GCF. 3 is not a common factor, but 7 is. Therefore $2^2 \times 7 = 28$ is the GCF.

If both numbers are prime, what will be their GCF? For example, find the GCF of 13 and 17. Clearly 17 is not a factor of 13, nor is 13 a factor of 17. The only factor which 13 and 17 have in common is 1. We can easily see that this is the case if we use our first method. The set of factors of 17 is $\{1, 17\}$; of 13 is $\{1, 13\}$. $\{1, 17\} \cap \{1, 13\} = \{1\}$, and the GCF of 13 and 17 is 1. A similar result would hold for any two distinct prime numbers. In general, if the greatest common factor of two numbers is 1, we say that the numbers are <u>relatively prime</u>.

Suppose we were asked to find the GCF of 9 and 16. Using the second method we first observe that the prime factorization of 9 is $9 = 3^2$ and that the prime factorization of 16 is $16 = 2^4$. Clearly there are no prime factors common to both 9 and 16. What then is the GCF? Since the number 1 is the only factor common to 9 and 16, the GCF of 9 and 16 is 1 and we say that 9 and 16 are relatively prime. Use the first method to check this result.

Now suppose we were asked to find the GCF of three or more numbers. Can we use the same methods? Let us find the GCF of 9, 12, 18, and 24. Using the second method, we have $9 = 3^2$, $12 = 2^2 \times 3$, $18 = 2 \times 3^2$, and $24 = 2^3 \times 3$. Since 2 is not a factor of 9, 3 is the only common prime factor. The least number of times 3 occurs in the prime factorizations of the four numbers is once, so we conclude that 3 is the GCF of 9, 12, 18, and 24. Use the first method to check that indeed 3 is the GCF of 9, 12, 18, and 24.

The <u>Greatest Common Factor</u> of two or more natural numbers is the largest number which is a factor of all the given numbers.

To find the Greatest Common Factor:

1. List all the factors of each number, and select the largest common factor.

- or -

2. Find the prime factorization of each number. The greatest common factor is obtained by taking each prime factor the least number of times it occurs in each of the given numbers. The greatest common factor is the product of these prime factors, if there are any; otherwise, the greatest common factor is 1.

Problems 4-6

1. Use the first method to find the GCF of each of the following pairs of numbers:

(a) 18, 45	(g) 96, 36	(m) 210, 56
(b) 15, 60	(h) 165, 99	(n) 11, 13
(c) 15, 27	(i) 8, 9	(o) 7, 19
(d) 12, 42	(j) 15, 16	(p) 546, 1521
(e) 3, 5	(k) 24, 108	(q) 36, 243
(f) 5, 7	(l) 425, 200	(r) 48, 256

2. Use the second method to find the GCF in each of the number pairs in Problem 1 above.

3. Use whatever method you think easier to find the GCF in each of the following:

(a) 28, 54	(f) 20, 21	(k) 35, 420, 245
(b) 16, 56	(g) 42, 273	(l) 24, 156, 180, 204
(c) 36, 162	(h) 12, 42, 72	(m) 4, 5, 7, 9
(d) 4, 7	(i) 16, 72, 104	(n) 30, 126, 105, 255
(e) 48, 108	(j) 4, 5, 9	(o) 4, 7, 9, 25

4-7 Least Common Multiple

You will recall that when we multiply a given number by any other number, the product is called a multiple of the given number. For example, the set of multiples of 3 may be written $\{3, 6, 9, 12, \ldots\}$. Let us consider the numbers 8 and 12. The set of all multiples of 8 is $A = \{8, 16, 24, 32, 40, 48, \ldots\}$, and the set of all multiples of 12 is $B = \{12, 24, 36, 48, \ldots\}$. We see that the set of all multiples that are in both A and in B is $A \cap B = \{24, 48, \ldots\}$. The smallest of these common multiples is 24. We shall call 24 the least common multiple (LCM) of 8 and 12.

We can generalize this idea to say that the least common multiple of two natural numbers is the smallest number that is a multiple of both the given numbers.

Suppose we were asked to find the LCM of 18 and 21. We could proceed as in the previous example, but the amount of work involved may be quite long and rather tedious. Is there another method which we could use to find the LCM of 18 and 21? Let us once again write the prime factorization of each of these numbers; the prime factorization of 18 is $18 = 2 \times 3^2$ and the prime factorization of 21 is $21 = 3 \times 7$. What prime factors from the prime factorizations of 18 and 21 are needed to make the resulting product of these factors a multiple of both 18

and 21? We shall need to use all of the different prime factors which occur in the factorizations of 18 and 21; that is, 2, 3, and 7. Moreover we shall need to use each prime factor to the highest power to which it occurs in either of the two factorizations. Hence $2 \times 3^2 \times 7 = 126$ is the LCM of 18 and 21. Use the first method to check this result.

Notice that what we have done in the above example was to take each prime factor the <u>greatest</u> number of times it occurs in any one of the numbers. The least common multiple is the product of these prime factors.

Will these same methods work for three or more numbers? Let us find the LCM of three numbers, say 9, 12, and 21, using the second method. The prime factorization of 9 is $9 = 3^2$; of 12 is $12 = 2^2 \times 3$; and of 21 is $21 = 3 \times 7$. The factor 3 occurs twice in the prime factorization of the number 9, the factor 2 occurs twice in the prime factorization of the number 12, and the factor 7 occurs once in the prime factorization of the number 21. Hence the product of these prime factors taken the proper number of times is $3^2 \times 2^2 \times 7 = 252$, and 252 is the LCM of 9, 12 and 21. Do you agree with this result? Use the first method to check that 252 is indeed the LCM of 9, 12 and 21.

But suppose the two given numbers were themselves prime? What would be the LCM then? Would our methods still work? For example, what is the LCM of 5 and 7? Using the first method, we have the set of all multiples of 5 as $A = \{5, 10, 15, 20, 25, 30, 35, 40, \ldots\}$, and the set of all multiples of 7 as $B = \{7, 14, 21, 28, 35, 42, \ldots\}$. The set of all multiples in $A \cap B$ is $\{35, 70, \ldots\}$, and the least common multiple is 35. But $5 \times 7 = 35$. Do you think that the LCM of two prime numbers will always be the product of those given prime numbers?

What about two numbers such as 7 and 9? What is the LCM of these numbers? Do you think the LCM of 7 and 9 is 63? Why? Check this result by using the first method.

The <u>Least Common Multiple</u> of two or more natural numbers is the <u>smallest</u> natural number that is a multiple of each of the given numbers.

To find the Least Common Multiple:

1. List the set of multiples of each of the given numbers, and select the smallest common multiple

- or -

2. Find the prime factorization of each of the numbers. The least common multiple is obtained by taking each prime factor the <u>greatest</u> number of times it occurs in any one of the given numbers. The least common multiple is the product of these prime numbers.

<u>Problems 4-7</u>

1. Use the first method to find the LCM of each of the following pairs of numbers:

(a) 4, 6	(g) 9, 15	(m) 15, 20
(b) 4, 9	(h) 8, 18	(n) 20, 25
(c) 6, 9	(i) 8, 9	(o) 10, 15
(d) 8, 15	(j) 15, 16	(p) 11, 13
(e) 3, 5	(k) 6, 15	(q) 16, 24
(f) 5, 7	(l) 9, 12	(r) 18, 24

2. Use the second method to find the LCM in each of the number pairs in Problem 1 above.

3. Use whatever method you think easier to find the LCM of each of the following:

(a) 10, 12	(g) 16, 36	(m) 2, 3, 10
(b) 14, 6	(h) 15, 25	(n) 6, 8, 12, 18
(c) 8, 12	(i) 7, 9	(o) 3, 5, 7, 11
(d) 12, 18	(j) 5, 12	(p) 4, 8, 16, 20
(e) 3, 5	(k) 8, 12, 18	(q) 2, 3, 4, 5
(f) 3, 8	(l) 4, 9, 15	(r) 5, 12, 18, 20

4. Find the GCF and the LCM and the product of the GCF and the LCM for each of the following pairs:

(a) 12, 18 (b) 7, 28 (c) 12, 35 (d) 96, 120

What do you notice about the product of the GCF and the LCM for each of these pairs of numbers? Do you think this is accidental?

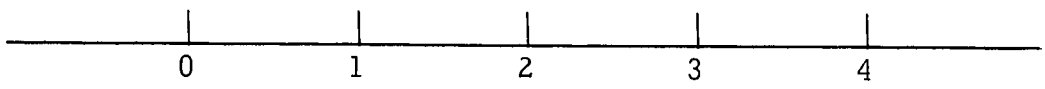
CHAPTER 5

RATIONAL NUMBERS AND THE NUMBER LINE

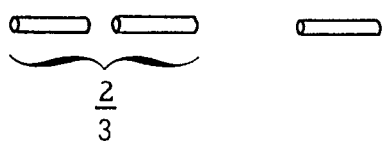
5-1 The Number Line

A very useful way of picturing numbers is to look at what is called the number line. In the picture below we have drawn a part of the number line.

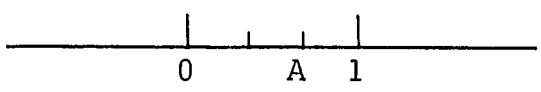
We have chosen a point to match or correspond to the number 0 and a point to correspond to the number 1. Any two points may be chosen to correspond to the numbers 0 and 1 except that it has been agreed to take the point corresponding to 1 to the right of the point corresponding to 0. Now points corresponding to 2, 3, 4, ... and so on are found by taking the distance between the points marked 1 and 2, 2 and 3, 3 and 4, and so on the same as the distance between our original points for 0 and 1. This is shown in the picture below.



What about the point mid-way between the points corresponding to 0 and 1? Would it not be sensible to match it with the number $\frac{1}{2}$? What about $\frac{2}{3}$? You know that we can think of this fraction as representing two of three equal parts of something. For example, $\frac{2}{3}$ might refer to two parts of a piece of sugar cane that has been divided into three equal parts.

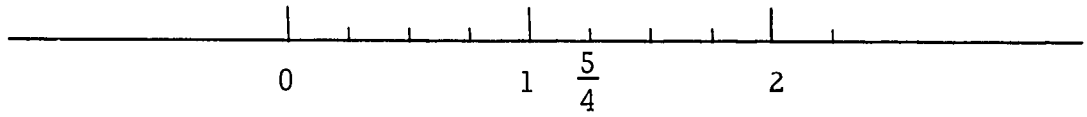


Suppose that we call the line segment from 0 to 1 the unit interval. Think of this unit interval as being divided into 3 equal parts and consider the point A at the end of the first two parts as in the picture below.



It would seem satisfactory to let the point A correspond to the number $\frac{2}{3}$.

What about the point corresponding to the fraction $\frac{5}{4}$? Could we not locate this point as shown in the picture below?



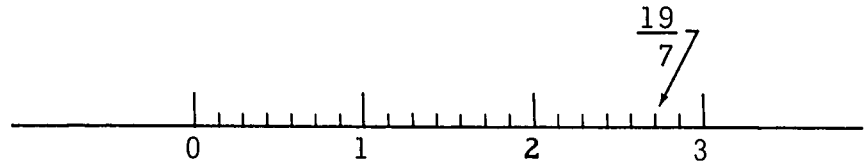
Suppose that we have such a fraction as $\frac{217}{729}$. We do not want to go to the trouble of dividing the unit interval into 729 equal parts and then counting off 217 of them to find the point corresponding to $\frac{217}{729}$, but do you not feel that this could be done and that there is a point on the number line corresponding to $\frac{217}{729}$?

Do you see how, given any fraction, you can picture it on the number line? We may think of this fraction as being written as $\frac{a}{b}$

where a is any whole number $(a \in \{0, 1, 2, 3, \ldots\})$

and b is any counting number $(b \in \{1, 2, 3, \ldots\})$.

Now we divide the unit interval into b equal parts and take a of these parts along the number line to reach the point corresponding to $\frac{a}{b}$. The picture below shows how this works for the fraction $\frac{a}{b}$ when $a = 19$ and $b = 7$



What happens if $b = 1$? Then we do not need to divide our unit interval at all and need only lay off a unit intervals. Do you see that $\frac{1}{1} = 1$, $\frac{2}{1} = 2$, $\frac{3}{1} = 3$ and, in general that $\frac{a}{1} = a$ if a is any counting number?

What happens if $a = b$? This means that we divide our unit interval into b equal parts and then take b of them. For example, if $a = b = 3$ we divide our unit interval into 3 equal parts and then take 3 of them. Clearly in such cases we reach the point corresponding to 1. We have $1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3}$ and so on.

What about fractions such as $\frac{0}{1}$, $\frac{0}{2}$ and so on? We remember that the numerator of a fraction tells us how many parts of the divided unit segment to take (or how many times to take the whole unit segment, if the denominator is 1). To take 0 of these parts means that we remain at the point corresponding to 0. We therefore have $0 = \frac{0}{1} = \frac{0}{2} = \frac{0}{3}$ and so on.

We have seen that it makes sense to talk about a fraction with a numerator of 0. What about 0 as a denominator? Remember that the denominator of a fraction tells us into how many equal parts we divide the unit interval. When we have a denominator of 2 we must have 2 equal line segments which together make up the unit interval. When we have a denominator of 3 we must have 3 equal line segments which together make up the unit interval. If we were to have a denominator of 0 we would have to have 0 segments which together make up the unit interval! This is certainly not possible and this is one reason why we do not allow the denominator of a fraction to be zero. You will see another reason in a later chapter when the relation of fractions to division is discussed. Remember that the <u>denominator of a fraction cannot be zero</u>.

> Any number which can be represented by a fraction in the form $\frac{a}{b}$ where $a \in \{0, 1, 2, 3, \ldots\}$ and $b \in \{1, 2, 3, \ldots\}$ is called a <u>rational number</u>.

$\frac{0}{1}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{217}{729}$, $\frac{19}{7}$, $\frac{3}{3}$ and so on are all rational numbers. Later we shall see that there are other kinds of rational numbers. From now on we shall almost always use the term rational number rather than the word fraction as you used in primary school. Indeed, later on we shall sometimes use fractions to represent numbers that are not rational numbers at all!

<u>Problems 5-1</u>

1. Draw a picture of the number line and locate and label on it the points corresponding to the following numbers.

(a) 5 (c) $\frac{3}{4}$ (e) $\frac{7}{2}$

(b) $\frac{1}{3}$ (d) 0 (f) $2 \cdot 4$

2. Do as in Problem 1 on a new picture of the number line.

(a) $2\frac{2}{3}$ (c) $\frac{25}{6}$ (e) $\frac{6}{1}$

(b) $\frac{16}{8}$ (d) $\frac{283}{283}$ (f) $\frac{0}{3}$

3. Why is there no point on the number line corresponding to the symbol $\frac{4}{0}$?

4. The fraction $\frac{7}{3}$ represents a rational number. Since the expression $2\frac{1}{3}$ represents the same number as $\frac{7}{3}$, we know that $2\frac{1}{3}$ also represents a rational number. All of the following expressions represent rational numbers. Rewrite each of them in the form $\frac{a}{b}$ where a is a whole number and b is a counting number.

(a) $6\frac{1}{2}$ (d) $4 \cdot 3$ (g) $0 \cdot 83$

(b) $1\frac{7}{8}$ (e) $9\frac{3}{11}$ (h) $8763 \cdot 1$

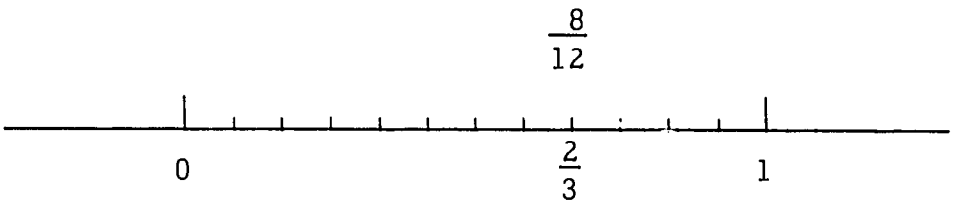
(c) 23 (f) 0 (i) $4\frac{6}{8}$

5-2 Equivalent Rational Numbers

Suppose that we compare the rational number $\frac{1}{2}$ with the rational number $\frac{2}{4}$. To locate the point corresponding to $\frac{1}{2}$ on the number line we divide the unit interval into 2 equal parts and then take 1 of them. To locate the point corresponding to $\frac{2}{4}$ we divide the unit interval into 4 equal parts and take 2 of them. Do you see that we reach the same point in both cases? You see that "$\frac{1}{2}$" and "$\frac{2}{4}$" are just different names for the same rational number. We have $\frac{1}{2} = \frac{2}{4}$.

What about $\frac{2}{3}$ and $\frac{8}{12}$? If we first divide the unit interval into 3 equal parts and take 2 of them, we locate the point corresponding to $\frac{2}{3}$.

Since $12 = 3 \times 4$ we shall have 12 equal parts if we take each of the 3 equal parts and divide it further into 4 equal parts.



Taking 8 of these 12 parts gives us the point corresponding to $\frac{8}{12}$. You see that it is the same point as the one corresponding to $\frac{2}{3}$, since $8 = 2 \times 4$.

Do you see that, in general, taking a of b equal parts of the unit interval takes us to the same point as taking $m \times a$ of $m \times b$ equal parts of the unit interval? We can put this result in symbols as follows:

> Basic principle for fractions:
>
> If a is a whole number and b and m are counting numbe
> then $\frac{a}{b} = \frac{m \times a}{m \times b}$.

We have for example,

$$\frac{2}{3} = \frac{1 \times 2}{1 \times 3} = \frac{2 \times 2}{2 \times 3} = \frac{3 \times 2}{3 \times 3} = \frac{4 \times 2}{4 \times 3} = \frac{5 \times 2}{5 \times 3}$$

and so on. In other words, $\frac{2}{3}$, $\frac{4}{6}$, $\frac{8}{12}$, $\frac{10}{15}$, and so on are all names for the same rational number.

This basic principle is very useful. It can be used for example, to replace $\frac{2}{3}$ by $\frac{8}{12}$ in a problem since the multiplier 4 gives $4 \times 2 = 8$ and $4 \times 3 = 12$.

$$\frac{2}{3} = \frac{4 \times 2}{4 \times 3} = \frac{8}{12}$$

It can also be used, of course, to replace $\frac{8}{12}$ by $\frac{2}{3}$ because we can factorize 8 and 12 and write

$$\frac{8}{12} = \frac{4 \times 2}{4 \times 3} = \frac{2}{3}$$

There is a standard method by which we can change fractions to lowest terms as we did when we wrote $\frac{8}{12} = \frac{2}{3}$. This method uses the idea of greatest common factor (GCF) that you studied in Chapter 4. For example, in the fraction $\frac{6}{21}$ the GCF of the numerator, 6, and the denominator, 21, is 3.

We can write 6 and 21 in terms of this factor 3 to give

$$\frac{6}{21} = \frac{3 \times 2}{3 \times 7} = \frac{2}{7}$$

by the use of the basic principle.

When a rational number is written in the form $\frac{a}{b}$ where the GCF of a and b is 1, we say that the rational number is written in <u>lowest terms</u>. $\frac{2}{3}$ is in lowest terms since the GCF of 2 and 3 is 1, but $\frac{8}{12}$ is not in lowest terms and $\frac{35}{85}$ is not in lowest terms. Why?

Do you see that it is always possible to write a given rational number in lowest terms? All we have to do is to find the GCF of the numerator and denominator of the given rational number and then use the basic principle. For example, to express $\frac{9}{24}$ in lowest terms we first note that the GCF of 9 and 24 is 3. Then we write

$$\frac{9}{24} = \frac{3 \times 3}{3 \times 8} = \frac{3}{8}$$

We have seen that it is possible to write a rational number in many different ways. Thus we have, for example,

$$\frac{3}{8} = \frac{9}{24} = \frac{6}{16} = \frac{300}{800} = \frac{21}{56}$$

and so on. Fractions which represent the same number, such as $\frac{3}{8}$ and $\frac{9}{24}$, are often called <u>equivalent</u> fractions and we talk about sets of equivalent fractions. For example,

$$\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \ldots\}$$

is a set of equivalent fractions and the set

$$\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}\}$$

is another set of equivalent fractions.

Problems 5-2

1. (a) Represent the number $\frac{10}{6}$ on the number line, showing the 6 equal parts of the unit interval.

   (b) On the same picture represent the number $\frac{5}{3}$, showing the 3 equal parts of the unit interval.

   (c) What does your picture demonstrate about $\frac{10}{6}$ and $\frac{5}{3}$?

2. Use the basic principle for fractions to show that the following are pairs of equivalent fractions.

   (a) $\frac{3}{2}, \frac{9}{6}$

   (b) $\frac{2}{5}, \frac{14}{35}$

   (c) $\frac{2}{8}, \frac{1}{4}$

   (d) $\frac{12}{10}, \frac{36}{30}$

   (e) $\frac{5}{4}, \frac{65}{52}$

   (f) $\frac{11}{9}, \frac{165}{135}$

3. Change the following fractions to lowest terms.

   (a) $\frac{12}{15}$

   (b) $\frac{5}{30}$

   (c) $\frac{27}{45}$

   (d) $\frac{48}{32}$

   (e) $\frac{39}{26}$

   (f) $\frac{20}{4}$

4. For each of the following fractions find an equivalent fraction which has the denominator indicated.

   Example: $\frac{2}{5}$, denominator 35.

$$\frac{2}{5} = \frac{2 \times 7}{5 \times 7} = \frac{14}{35}$$

(a) $\frac{3}{4}$; denominator 20

(b) $\frac{7}{2}$; denominator 8

(c) $\frac{5}{6}$; denominator 600

(d) $\frac{12}{42}$; denominator 7

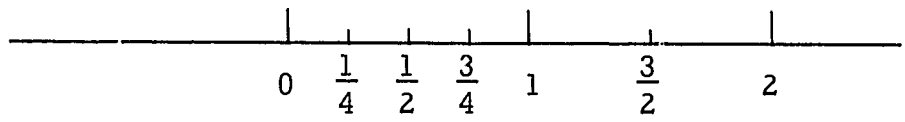
(e) 9; denominator 4

(f) $\frac{4}{13}$; denominator 169

5. Write a set of eight fractions all equivalent to the fraction $\frac{5}{2}$.

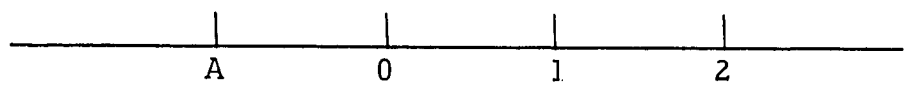
5-3 Negative Rational Numbers

Look again at our number line as shown below. Does it not look rather unbalanced?



All of the labels are attached to points to the <u>right</u> of the point corresponding to 0! What about points to the <u>left</u> of 0?

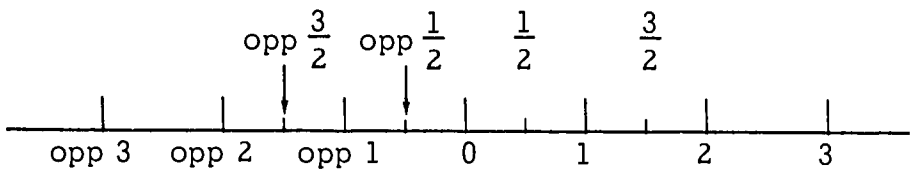
Consider, for example, the point labeled A in the picture below where A is as far to



the left of 0 as 1 is to the right of 0. What would be a suitable label for the point A? In ordinary language we might well say that A and 1 are on opposite sides of 0 and since A and 1 are the same distance from 0, we might agree to call A "the opposite of 1" or "opposite 1".

Using the same language we would then label the point two unit intervals to the left of 0 as "opposite 2". What would we then use for a label for a point 3 unit intervals to the left of 0? For a point $\frac{3}{2}$ of a unit interval to the left of 0?

Below is a picture of the number line labeled with the numerals for some numbers and their opposites. We have used the abbreviation "opp" for "opposite".



We shall think of the points to the left of 0 as corresponding to a new kind of number whose properties we shall consider in a later chapter. In writing numerals for these new numbers we often write opposite 2 as -2, opposite 3 as -3 and so on. There is a certain danger, however, in doing this. The symbol "-" is also used to indicate subtraction as, for example, in $5 - 3 = 2$. When we use the symbol "-" in connection with opposites, however, the situation is quite different. In subtraction we start with <u>two</u> numbers as, for example, 5 and 3 in $5 - 3$, and a third number is obtained. For opposites we start with <u>one</u> number and obtain another, its opposite.

Those rational numbers that correspond to points to the right of 0 are called <u>positive</u> rational numbers. Those rational numbers that correspond to points to the left of 0 are called <u>negative</u> rational numbers. Sometimes, in fact, we read -2 as "negative two", $-\frac{3}{2}$ as "negative three halves", and so on. The number 0 is neither a positive nor a negative number. We might say it is "neutral"!

In the language of sets we can write

$$\{\text{rational numbers}\}$$

$$= \{\text{positive rational numbers}\} \cup \{\text{negative rational numbers}\} \cup \{0\}$$

Among the rational numbers are some special ones such as

$$\frac{3}{3}, \frac{4}{2}, -\frac{3}{3}, -\frac{4}{2} \text{ and so on.}$$

We do know that $\frac{3}{3} = 1$, $\frac{4}{2} = 2$, $-\frac{3}{3} = -1$, $-\frac{4}{2} = -2$ and so on. The numbers 1, 2, 3, ... are what we have been calling the counting numbers. Another name

for a member of this set of counting numbers is <u>positive integer</u>. As you might guess, -1, -2, -3, ... are called <u>negative integers</u>. Any number which is either 0, a positive integer, or a negative integer is called an <u>integer</u>.

We can write

$$\{\text{integers}\} = \{\text{positive integers}\} \cup \{\text{negative integers}\} \cup \{0\}.$$

In your work in mathematics from now on you will find many uses for both positive and negative rational numbers. Here are just three examples of how negative numbers might be used.

<u>Example 1</u>: In a thermometer using the centigrade scale, 0 represents the freezing point of water. We can speak of temperatures above zero and temperatures below zero. Twenty degrees above zero is a comfortable temperature. (It is the same as 68 degrees Fahrenheit.) Twenty degrees below zero is not at all a comfortable temperature. (It is the same as 4 degrees below zero Fahrenheit.) We can write 20°C. to indicate the above zero temperature and $(-20)^{\circ}$C. to indicate the below zero temperature.

<u>Example 2</u>: We can use negative numbers to indicate debts. A balance of 100 shillings would mean that you had 100 shillings at hand, a balance of -100 shillings would mean that you were in debt 100 shillings.

<u>Example 3</u>: Heights of parts of the earth are usually measured as distances above sea level. What would a height of -100 feet represent? Would it not be natural to suppose that we are thinking of a depth of 100 feet below the sea level?

<u>Problems 5-3</u>

1. Show on a picture of the number line the points which correspond to the following numbers. Label each point.

   (a) 3

   (b) opposite 3

   (c) $\frac{2}{3}$

   (d) opposite $\frac{2}{3}$

   (e) opposite 5

   (f) 0

2. Do as in Problem 1 on a new picture of the number line.

(a) -4

(b) 4

(c) $-\frac{5}{3}$

(d) $-2\cdot5$

(e) $-2\cdot8$

(f) $-2\cdot1$

3. The symbol "-" is read "negative" or "minus" depending on its use. When it denotes subtraction we say "minus". "10 - 2" is read "ten minus two". When "-" denotes a number corresponding to a point to the left of 0, we say "negative". "-9" is read "negative nine". Supply the correct word, "minus" or "negative" in each of the following expressions.

(a) -23

(b) $\frac{3}{8} - \frac{1}{8}$

(c) $-\frac{49}{5}$

(d) -1

(e) $63 - 28$

(f) $2\cdot5 - 2\cdot4$

4. For each of the following numbers tell which of the following names can correctly be used. In some cases more than one of the names is correct.

positive integer
negative integer
zero
positive rational number
negative rational number
whole number
counting numbers

(a) $\frac{2}{5}$

(b) -7

(c) $-7\cdot5$

(d) 43

(e) 0

(f) -58

(g) $-\frac{5}{9}$

(h) $\frac{43}{5}$

(i) $2\frac{7}{8}$

(j) -700

(k) $48\cdot862$

(l) $-\frac{11}{2}$

5. Give some examples of how positive and negative rational numbers can be used in talking about

   (a) profit and loss in a business

   (b) positions north and south of the equator (latitude)

   (c) a game in which you may lose points as well as win points.

6. Challenge Problem

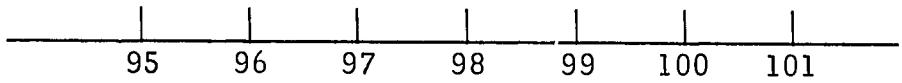
   Describe some applications of your own choosing and different from the ones above, in which positive and negative numbers are helpful.

5-4 Inequalities

Suppose that you are comparing your weight with that of a friend. You weigh yourself and find that you weigh 100 pounds. Your friend weighs himself and finds that he weighs 98 pounds. If someone asks you about the weights of you and your friend you surely would be likely to say more than just "My weight is not the same as that of my friend". Would you not be more likely to say "I weigh more than my friend" or "My friend weighs less than I do"?

The ideas of inequality and comparison are just as important to study as is the idea of equality. Mathematicians have, indeed, invented special symbols to write about such ideas. For example, $2 \neq 3$ is read "2 is not equal to 3"; $100 > 98$ is read "100 is greater than 98"; and $98 < 100$ is read "98 is less than 100". We can say that $\neq$ is the "not equal" symbol, $>$ is the "greater than" symbol, and $<$ is the "less than" symbol. Just as statements such as $3 + 2 = 5$ are called equalities, so statements such as $3 + 2 \neq 6$, $6 > 3 + 2$, and $3 + 2 < 6$ are called inequalities. Most of the time, however, when we use the word "inequality" we are thinking of inequalities involving $>$ or $<$ rather than those involving $\neq$.

How can we picture inequalities involving > or < on the number line? Think of the points corresponding to 100 and to 98 on the number line.



You can see that the point corresponding to 100 is to the right of the point corresponding to 98. Can you see that if a and b are positive numbers such that a > b, then the point corresponding to a is to the right of the point corresponding to b?

On the other hand, can you see that if a and b are positive numbers such that a < b, then the point corresponding to a is to the left of the point corresponding to b?

So far we have considered only positive numbers. What about inequalities involving 0 or negative numbers? We know that the point corresponding to 2 lies to the right of the point corresponding to 0. Does it not make sense to say that 2 > 0? If you have 2 shillings and your friend has 0 shillings, you certainly have more money than your friend does.

We also know that the point corresponding to 0 is to the right of the point corresponding to -1. Does it make sense to write 0 > -1? Remember how we thought of numbers in connection with temperatures. 3 > 2 can be thought of as saying that a temperature of 3°C. is warmer than a temperature of 2°C. Is a temperature of 0°C. warmer or colder than a temperature of $(-1)^{\circ}$C.? It certainly is warmer. (Although neither temperature would be very comfortable!) Think of 0 > -1 in terms of money. If a person has no money at all is he still not better off than a person who owes 1 shilling? Think of distances above and below sea level. Is not sea level (0) higher than a depth of 1 foot below sea level (-1)?

In the same way you can see that it makes sense to say that

$-1 > -2$	or	$-2 < -1$
$-5 > -8$	or	$-8 < -5$
$-\frac{15}{2} > -10$	or	$-10 < -\frac{15}{2}$
$2 > -5$	or	$-5 < 2$
$0 > -7$	or	$-7 < 0$

and so on. In general we can say

> If a and b are rational numbers, then $a > b$ means that the point corresponding to a is to the right of the point corresponding to b. If $a > b$ then $b < a$ and if $b < a$, then $a > b$.

Sometimes we want to consider the possibility that for two rational numbers a and b, <u>either</u> $a > b$ <u>or</u> $a = b$. We write

$$a \geqq b$$

to mean

$$a > b \text{ or } a = b.$$

Similarly we say that $a \leqq b$

means that $a < b$ or $a = b$.

The use of the word "or" in mathematics sometimes causes trouble. Suppose that you know that Modupe is a Sierra Leonan. You also know, then, that he is not a Nigerian. Look at the following sentences:

1. Modupe is a Sierra Leonan.

2. Modupe is a Nigerian.

3. Modupe is a Sierra Leonan or Modupe is a Nigerian.

There is no question but that sentence 1. is a true sentence and that sentence 2. is a false sentence. What about sentence 3.? You might think that it is false since you know that Modupe is a Sierra Leonan and not a Nigerian. But you are certainly not telling a lie if you say to a friend of yours "Modupe is a Sierra Leonan or Modupe is a Nigerian". You are simply not telling your friend all that you know about Modupe's nationality! When the clauses of your sentence are joined by "or", the sentence is true whenever at least one of the clauses is true. Sentence 3. is indeed a true sentence.

Now consider the three sentences

1. $3 = 3$

2. $3 > 3$

3. $3 > 3$ or $3 = 3$.

Again you know that sentence 1. is certainly true and that sentence 2. is certainly false. What about sentence 3.? Again you might think it is false because you know that 3 is equal to 3 and that 3 is not greater than 3, but again you are certainly not telling a lie if you say "$3 > 3$ or $3 = 3$". You are just not telling all you know. You are saying that at least one of the two statements "$3 > 3$" and "$3 = 3$" is true.

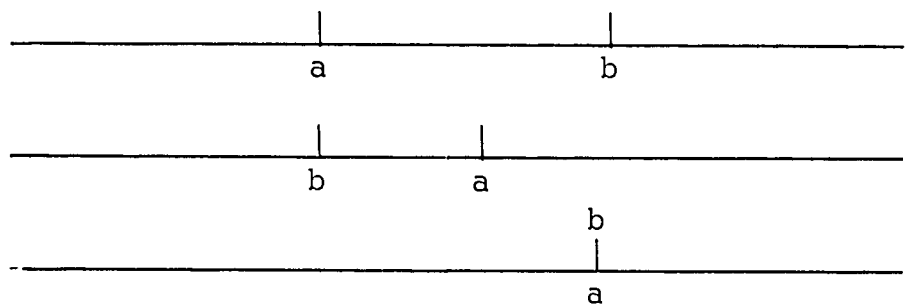
In other words, we shall agree to regard $3 \geqq 3$ as a true statement. In the same way we regard $3 \geqq 2$ as a true statement. Since $3 > 2$ is true, the statement "$3 > 2$ or $3 = 2$" is true even though we know that $3 = 2$ is false.

What about a statement such as $2 \geqq 3$? This means that either $2 > 3$ or $2 = 3$ and this will be true only if at least one of the two clauses is true. Since $2 > 3$ and $2 = 3$ are both false sentences, we conclude that $2 \geqq 3$ is a false sentence. It is not true that $2 \geqq 3$.

It is quite common to abbreviate "$\geqq$" to "$\geq$" and "$\leqq$" to "$\leq$". We shall do so here.

> If a and b are rational numbers, then $a \geq b$ means that $a > b$ or $a = b$, and $a \leq b$ means that $a < b$ or $a = b$.

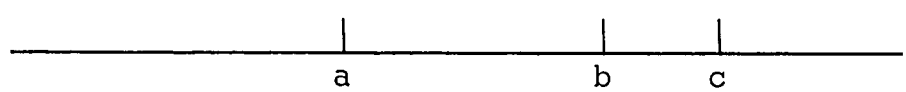
A look at the number line will make it easy for us to accept two basic assumptions about the relations "greater than" and "less than".



If a and b are any two rational numbers, it seems geometrically necessary that the point corresponding to a must be either to the left of or to the right of or the same as the point corresponding to b. There is no other possibility and exactly one of these must be true.

> For any two rational numbers a and b exactly one of the following relations is true:
>
> $$a < b, \quad a > b, \quad a = b.$$

Look again at the number line and answer the following question. If a, b, and c are rational numbers such that on the number line a is to the left of b and b is to the left of c, how is a related to c? Does it seem necessarily true that a is to the left of c?



> For any three rational numbers a, b, and c,
>
> if $a < b$ and $b < c$, then $a < c$.

These two properties will be accepted as assumptions about inequalities. We shall have use for them later. Since they have to do with the order in which the rational numbers come, they are usually referred to as <u>properties of order</u>.

<u>Problems 5-4</u>

1. Which of the following inequalities are true?

(a) $3 < 30$	(g) $-\frac{1}{3} > -\frac{2}{3}$	(m) $-\frac{5}{3} \leq -\frac{7}{3}$
(b) $-3 < -30$	(h) $-3\frac{1}{2} < -3$	(n) $-\frac{8}{18} \geq -\frac{4}{9}$
(c) $5 > -2$	(i) $-4 \cdot 2 < -4 \cdot 3$	(o) $\frac{4}{5} > \frac{4}{6}$
(d) $2 > -2$	(j) $\frac{5}{12} < \frac{10}{24}$	(p) $-\frac{1}{8} < -\frac{1}{9}$
(e) $-3 < -2$	(k) $6 \geq 2$	(q) $\frac{8}{3} \neq -\frac{8}{3}$
(f) $\frac{2}{3} > -\frac{4}{3}$	(l) $25 \leq 24$	(r) $\frac{8}{3} \neq \frac{88}{33}$

2. In each of the following tell which of the symbols "<", ">", or "=" placed between the given numerals will make a true sentence.

(a)	$\frac{25}{2}$	25	(f) $\frac{14}{8}$	$\frac{7}{4}$
(b)	-7	6	(g) $-\frac{14}{8}$	$\frac{7}{4}$
(c)	7	-6	(h) $-\frac{14}{8}$	-2
(d)	-7	-6	(i) $-\frac{5}{9}$	$-\frac{100}{180}$
(e)	$-3 \cdot 8$	$-4 \cdot 0$	(j) $-\frac{1}{95}$	$-\frac{1}{96}$

3. If John is taller than Joseph and Joseph is taller than James, what conclusion do you draw about John and James? What property of order did you use in reaching your conclusion?

4. If Mary insists that she is not older than Margaret and that also she is not younger than Margaret, what conclusion must we reach about Mary's age and Margaret's age? What property of order did you use?

5. Make up more examples such as Problems 3 and 4 illustrating the two properties of order which we saw above.

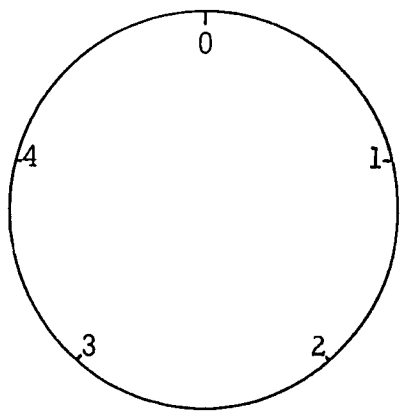
CHAPTER 6

CLOCK ARITHMETIC

6-1 Introduction

We are going to play a kind of game that will turn out to be much like arithmetic. This game will help you to understand arithmetic and algebra.

You will remember how we constructed the number line in Chapter 5 to represent numbers visually. Now let us see what happens if we arrange numerals on a circle like a clock. To keep it simple suppose that we decide to use only the five numbers 0, 1, 2, 3, 4 with their numerals arranged as in the picture below.



We are going to explore a strange kind of arithmetic using only these numbers.

6-2 Addition

Let us think of the circle as a kind of clock with five hours. We shall add the numbers just as we add hours on a clock. When we pass 0 we begin counting all over again. Using this idea we can invent a strange type of addition. We shall use the symbol $\oplus$ for this addition to show that it comes from the circle, and also to show that it is not ordinary addition. We shall call it <u>clock addition</u>. Examples will show how the game is played.

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$1 \oplus 2 = 3$	Because 1 o'clock plus two hours is 3 o'clock.
$2 \oplus 3 = 0$	Because 2 o'clock plus 3 hours is 0 o'clock. (Look at our circle.)
$3 \oplus 3 = 1$	Because 3 o'clock plus 3 hours is 1 o'clock.
$3 \oplus 4 = 2$	Why?
$4 \oplus 4 = 3$	Why?

Find all possible sums of any two of the numbers in this strange addition. Make an addition table. Part of such a table is shown below. Copy it and fill in the empty squares. We read the first number of the sum from the column at the left and the second number from the top row.

$\oplus$	0	1	2	3	4
0					
1			3		
2				0	
3				1	2
4					3

Examine your addition table. Did you find in the table a number which does not belong to the set $\{0, 1, 2, 3, 4\}$? Of course not.

> If we take any two numbers, a and b, such that
> $a \in \{0, 1, 2, 3, 4\}$ and $b \in \{0, 1, 2, 3, 4\}$,
> then $(a \oplus b) \in \{0, 1, 2, 3, 4\}$.

We say that the set of numbers $\{0, 1, 2, 3, 4\}$ is <u>closed</u> under clock addition. This property is known as the <u>closure property of addition</u>. It tells us that we stay within the set $\{0, 1, 2, 3, 4\}$ when we do clock addition.

Look at your table again. Do you find that the following statements are true?

$$0 \oplus 1 = 1 \oplus 0$$

$$0 \oplus 2 = 2 \oplus 0$$

$$1 \oplus 2 = 2 \oplus 1$$

$$2 \oplus 3 = 3 \oplus 2$$

$$3 \oplus 4 = 4 \oplus 3$$

Yes, they are. Write from your addition table all the possible statements similar to the ones above. Do you conclude the following?

> If we take any two numbers, c and d, such that $c \in \{0, 1, 2, 3, 4\}$ and $d \in \{0, 1, 2, 3, 4\}$, then $c \oplus d = d \oplus c$.

Why? Have you tried every possible case? We say that clock addition is <u>commutative</u>. This property is called the <u>commutative property of addition</u>. It tells us that in clock addition we can take the sum in either order and obtain the same result.

If you were asked to express $1 \oplus 2 \oplus 3$ as one of the numbers 0, 1, 2, 3, 4 how would you do it? Is the answer 1? Which did you add first, the 1 and the 2 or the 2 and the 3? Does it matter which you do first? If we use brackets to show us which two numbers we shall add first, would you agree that

$$(1 \oplus 2) \oplus 3 = 1 \oplus (2 \oplus 3)?$$

$$2 \oplus (4 \oplus 3) = (2 \oplus 4) \oplus 3?$$

Why? It would take some time to test it for all possible choices of the three numbers, but if you did try them all, you would find that this property is true in every case.

> If we take any three numbers, a, b, and c, that are all members of {0, 1, 2, 3, 4}, then
> $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

We say that clock addition is <u>associative</u>. This property is called the <u>associative property of addition</u>. It tells us that in clock addition we can group the terms as we like without changing the sum.

<u>Problems 6-2A</u>

1. Express each of the following sums as one of the numbers 0, 1, 2, 3, 4.

   (a) $3 \oplus (4 \oplus 4)$; $(3 \oplus 4) \oplus 4$

   (b) $(2 \oplus 3) \oplus 2$; $2 \oplus (2 \oplus 3)$; $(2 \oplus 2) \oplus 3$

   (c) $(4 \oplus 2) \oplus (3 \oplus 1)$; $(3 \oplus 2) \oplus (1 \oplus 4)$; $1 \oplus (3 \oplus 4) \oplus 2$

   (d) $1 \oplus 1 \oplus 1 \oplus 1$; $4 \oplus 4 \oplus 4$; $2 \oplus 2 \oplus 2$; $3 \oplus 3 \oplus 3 \oplus 3$

   (e) $0 \oplus 1 \oplus 2 \oplus 3 \oplus 4$

   (f) In part e what rearrangement of the terms makes it easiest to find the result?

2. Find the numbers which make each of the following equations a true statement by testing all the numbers 0, 1, 2, 3, 4. Is there only one number which makes each equation a true statement?

   (a) $x \oplus 1 = 1$

   (b) $x \oplus 1 = 3$

   (c) $x \oplus 1 = 0$

   (d) $x \oplus 2 = 4$

   (e) $x \oplus 2 = 0$

   (f) $x \oplus 2 = 2$

   (g) $x \oplus 3 = 0$

   (h) $x \oplus 3 = 1$

   (i) $x \oplus 3 = 3$

   (j) $x \oplus 4 = 4$

   (k) $x \oplus 4 = 2$

   (l) $x \oplus 4 = 0$

   (m) $x \oplus 0 = 2$

   (n) $x \oplus 0 = 4$

   (o) $x \oplus 0 = 1$

   (p) $x \oplus 0 = 0$

3. Perform the following subtractions. Remember that for ordinary subtraction $a - b = c$ means that $a = b + c$. Therefore if $\ominus$ denotes clock subtraction, it seems desirable to agree that $a \ominus b = c$ means $a = b \oplus c$.

<u>Example</u>. To find $4 \ominus 3$ we can write $4 \ominus 3 = x$, which means $4 = 3 \oplus x$. The solution of this equation is 1; therefore $4 \ominus 3 = 1$.

(a) $3 \ominus 0$

(b) $3 \ominus 1$

(c) $3 \ominus 2$

(d) $3 \ominus 3$

(e) $3 \ominus 4$

(f) $1 \ominus 0$

(g) $1 \ominus 1$

(h) $1 \ominus 2$

(i) $1 \ominus 3$

(j) $1 \ominus 4$

Do you think that the set of numbers $\{0, 1, 2, 3, 4\}$ is closed under clock subtraction? Is it commutative? Give reasons in each case.

In Problem 2 you found that the equations

$$x \oplus 0 = 0$$

$$x \oplus 1 = 1$$

$$x \oplus 2 = 2$$

$$x \oplus 3 = 3$$

$$x \oplus 4 = 4$$

all have the same solution, 0. Have we enough evidence to support the following statement? Why?

> There is a unique number 0 such that for any number a which is a member of $\{0, 1, 2, 3, 4\}$, $0 \oplus a = a \oplus 0 = a$.

The number 0 is called the <u>identity element</u> for clock <u>addition</u>. This property is called the <u>identity property of addition</u> or the <u>property of zero</u>. It tells us that in clock addition, a number is not changed by adding zero.

Also in Problem 2 you found that

$$x \oplus 1 = 0 \text{ has the solution } 4,$$

$$x \oplus 3 = 0 \text{ has the solution } 2.$$

Why is the following statement true?

> For every number a such that $a \in \{0, 1, 2, 3, 4\}$ there is a unique number b in $\{0, 1, 2, 3, 4\}$ such that $a \oplus b = b \oplus a = 0$.

We call the number b the <u>opposite</u> of a or the <u>additive inverse</u> of a, and the property stated above is known as the <u>inverse property of addition</u> or simply the <u>property of opposites</u>. It tells about pairs of numbers whose sum is zero in clock arithmetic. For example, we can say that 4 and 1 are opposites of each other, since $4 \oplus 1 = 0$.

<u>Problems 6-2B</u>

1. Clock arithmetic is used in telling time. There are twelve numerals on the clock. Find the following sums on this 12 hour clock.

1. (a) $2 \oplus 5$

   (b) $5 \oplus 7$

   (c) $5 \oplus 8$

   (d) $7 \oplus 8$

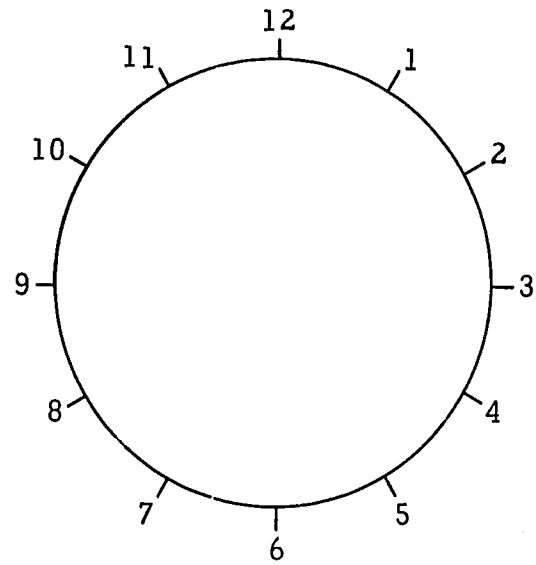
   (e) $10 \oplus 4$

   (f) $11 \oplus 11$

   (g) $1 \oplus 12$

   (h) $12 \oplus 11$

   (i) $10 \oplus 2$



Which is the identity element in this clock addition? Why? Is there an inverse for each clock number? Verify your answer.

2. Make addition tables for clock arithmetic using the set of

   (a) four numbers: $\{0, 1, 2, 3\}$

   (b) three numbers: $\{0, 1, 2\}$

   (c) two numbers: $\{0, 1\}$

   For each part find which of the properties of clock addition we have been discussing are satisfied, and which ones are not satisfied.

6-3 Multiplication

We shall now consider in our set $\{0, 1, 2, 3, 4,\}$ an operation which we call clock multiplication, denoted by $\otimes$.

In ordinary arithmetic 4×3 can be represented as the sum $3 + 3 + 3 + 3$. We shall define $4 \otimes 3$ in clock arithmetic in a similar way.

$$4 \otimes 3 = 3 \oplus 3 \oplus 3 \oplus 3 = 2$$

Have we done the clock addition correctly? If so, then we know that $4 \otimes 3 = 2$.

<u>Examples</u>.

Since $2 \otimes 3$ means $3 \oplus 3$, $\quad 2 \otimes 3 = 1$.

Since $1 \otimes 2$ means 2, $\quad 1 \otimes 2 = ?$.

Since $3 \otimes 4$ means $4 \oplus 4 \oplus 4$, $\quad 3 \otimes 4 = ?$.

Since $4 \otimes 0$ means $0 \oplus 0 \oplus 0 \oplus 0$, $\quad 4 \otimes 0 = ?$.

As in ordinary arithmetic, the special case $0 \otimes 4$ cannot be thought of as a sum, but we shall define it, as you might expect, to have the value 0; that is, $0 \otimes 4 = 0$.

You can now make a multiplication table for clock multiplication. Part of such a table is shown below where the products already obtained have been filled in.

$\otimes$	0	1	2	3	4
0					0
1			2		
2				1	
3					2
4	0				

Make a copy of this table and fill in the missing numbers.

Explain why can you say immediately that the following property is true.

> If a and b are both elements of $\{0, 1, 2, 3, 4\}$, then $(a \otimes b) \in \{0, 1, 2, 3, 4\}$.

We say that the set of numbers $\{0, 1, 2, 3, 4\}$ is <u>closed</u> under clock multiplication, and this property is called the <u>closure property of multiplication</u>. It tells us that we stay within the set $\{0, 1, 2, 3, 4\}$ when we do clock multiplication.

From your table find the following:

(a) $2 \otimes 3$

(b) $3 \otimes 2$

(c) $2 \otimes 4$

(d) $4 \otimes 2$

(e) $1 \otimes 3$

(f) $3 \otimes 1$

(g) $0 \otimes 3$

(h) $3 \otimes 0$

You found that

$$2 \otimes 3 = 3 \otimes 2$$

$$2 \otimes 4 = 4 \otimes 2$$

$$1 \otimes 3 = 3 \otimes 1$$

$$0 \otimes 3 = 3 \otimes 0$$

Does this pattern hold for any two numbers you might choose from the numbers 0, 1, 2, 3 and 4? Give a few examples other than the ones above to show this.

For any two numbers a and b which are elements of the set $\{0, 1, 2, 3, 4\}$, $a \otimes b = b \otimes a$.

We say that clock multiplication is <u>commutative</u>. This is known as the <u>commutative property of multiplication</u>. It tells us that in clock multiplication we can take the product in either order and obtain the same result.

Do you think that there will be an <u>associative property of multiplication</u>? Let us find out.

Do the following clock multiplications.

(a) $4 \otimes (3 \otimes 2)$

(b) $(4 \otimes 3) \otimes 2$

(c) $(2 \otimes 3) \otimes 3$

(d) $2 \otimes (3 \otimes 3)$

(e) $(1 \otimes 1) \otimes 2$

(f) $1 \otimes (1 \otimes 2)$

(g) $1 \otimes (3 \otimes 2)$

(h) $(1 \otimes 3) \otimes 2$

What did you discover? You must have found that

$$4 \otimes (3 \otimes 2) = (4 \otimes 3) \otimes 2,$$

$$(2 \otimes 3) \otimes 3 = 2 \otimes (3 \otimes 3),$$

$$(1 \otimes 1) \otimes 2 = 1 \otimes (1 \otimes 2),$$

$$1 \otimes (3 \otimes 2) = (1 \otimes 3) \otimes 2.$$

It seems likely that this pattern will continue for any three numbers you pick from the numbers 0, 1, 2, 3 and 4. As a matter of fact, it does.

If a, b, and c are members of $\{0, 1, 2, 3, 4\}$, then $(a \otimes b) \otimes c = a \otimes (b \otimes c)$.

We say that clock multiplication is <u>associative</u>. We call this the <u>associative property of multiplication</u>. It tells us that in clock multiplication we can group the factors as we like without changing the product.

<u>Problems 6-3A</u>

Perform the following clock operations.

1. (a) $2 \otimes 1$; $1 \otimes 2$

(b) $3 \otimes 0$; $0 \otimes 3$

(c) $1 \otimes 0$; $0 \otimes 1$

(d) $4 \otimes 3$; $3 \otimes 4$

(e) $0 \otimes 2$; $2 \otimes 0$

(f) $0 \otimes 0$

2. $2 \otimes [2 \otimes (2 \otimes 2)]$

3. $(4 \otimes 2) \otimes 3$; $4 \otimes (3 \otimes 2)$; $(3 \otimes 4) \otimes 2$

4. $3 \otimes 1 \otimes 3$; $4 \otimes 0 \otimes 2$; $4 \otimes 4 \otimes 4$

5. (a) $3 \otimes (4 \oplus 1)$
   (b) $3 \otimes (2 \oplus 3)$
   (c) $4 \otimes (3 \oplus 0)$
   (d) $1 \otimes (1 \oplus 1)$
   (e) $(3 \otimes 4) \oplus (3 \otimes 1)$
   (f) $(3 \otimes 2) \oplus (3 \otimes 3)$
   (g) $(4 \otimes 3) \oplus (4 \otimes 0)$
   (h) $(1 \otimes 1) \oplus (1 \otimes 1)$

Did you notice when you did Problem 5 that each result on the left hand side was equal to the corresponding result on the right hand side? For example was $3 \otimes (4 \oplus 1) = (3 \otimes 4) \oplus (3 \otimes 1)$? The answer is yes. In fact the property shown by Problem 5 is true for any three numbers chosen from the numbers 0, 1, 2, 3, and 4. We state this property as follows:

> For any three numbers a, b, and c
> which are elements of $\{0, 1, 2, 3, 4\}$,
> $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$.

We say that clock multiplication is <u>distributive</u> with respect to clock addition. This is known as the <u>distributive property of multiplication with respect to addition</u>. Notice that this property involves <u>both</u> operations, addition and multiplication. It tells us how to change certain products to sums and sums to products.

<u>Example 1</u>. The product $2 \otimes (a \oplus 3)$

can be written as the sum $(2 \otimes a) \oplus (2 \otimes 3)$

or $(2 \otimes a) \oplus 1$

<u>Example 2</u>. The sum $(2 \otimes 4) \oplus (2 \otimes 1)$

can be written as the product $2 \otimes (4 \oplus 1)$

or $2 \otimes 0$

or 0

<u>Problems 6-3B</u>

1. Find the numbers which make each of the following equations a true statement by testing the numbers 0, 1, 2, 3, 4. You may use the multiplication table if you choose.

(a) $1 \otimes x = 3$ (d) $4 \otimes x = 3$ (g) $1 \otimes x = 1$

(b) $2 \otimes x = 3$ (e) $4 \otimes x = 4$ (h) $0 \otimes x = 0$

(c) $3 \otimes x = 3$ (f) $2 \otimes x = 2$ (i) $3 \otimes x = 0$

2. Find the numbers which make each of the following equations a true statement.

(a) $0 \otimes x = 1$

(b) $1 \otimes x = 1$

(c) $2 \otimes x = 1$

(d) $3 \otimes x = 1$

(e) $4 \otimes x = 1$

3. Use the distributive property to change the following products to sums. Check your results by finding the values of both forms.

(a) $4 \otimes (1 \oplus 3)$ (c) $1 \otimes (3 \oplus 2)$

(b) $2 \otimes (4 \oplus 4)$ (d) $3 \otimes (1 \oplus 1)$

4. Use the distributive property to change the following sums to products. Check by finding the values of both forms.

(a) $(4 \otimes 3) \oplus (4 \otimes 4)$

(b) $(3 \otimes 2) \oplus (3 \otimes 0)$

(c) $(3 \otimes 3) \oplus (3 \otimes 3)$

(d) $(2 \otimes 3) \oplus (2 \otimes 4)$

5. Use the distributive property to change the following sums to products and the products to sums.

(a) $(3 \otimes 2) \oplus (3 \otimes a)$

(b) $(y \otimes 4) \oplus (y \otimes 2)$

(c) $2 \otimes (1 \oplus a)$

(d) $4 \otimes (a \oplus b)$

(e) $(3 \otimes x) \oplus (3 \otimes x)$

(f) $(a \otimes 3) \oplus (a \otimes 2)$

6. Solve the following equations by first applying the distributive property and then using your multiplication table to find the number x which makes the equation a true statement.

<u>Example</u>

$$(x \otimes 3) \oplus (x \otimes 4) = 1$$

$$x \otimes (3 \oplus 4) = 1$$

$$x \otimes 2 = 1$$

Since $3 \otimes 2 = 1$, $x = 3$.

(a) $(x \otimes 3) \oplus (x \otimes 3) = 2$

(b) $(x \otimes 2) \oplus (x \otimes 1) = 4$

(c) $(x \otimes 3) \oplus (x \otimes 2) = 2$

(d) $(x \otimes 4) \oplus (x \otimes 4) = 0$

(e) $(x \otimes 4) \oplus (x \otimes 1) = 0$

7. Define clock division by saying that $a \div b = c$ means that $a = b \otimes c$. Explain why the following are true.

<u>Example</u>. $2 \div 4 = 3$ means that $2 = 4 \otimes 3$, and the latter is true. Therefore we conclude that the former is true.

(a) $1 \div 2 = 3$	(d) $3 \div 4 = 2$	(g) $4 \div 3 = 3$
(b) $2 \div 3 = 4$	(e) $4 \div 4 = 1$	(h) $3 \div 2 = 4$
(c) $1 \div 3 = 2$	(f) $1 \div 4 = 4$	(i) $1 \div 1 = 1$

Would you say that clock division is commutative? Why?

In Problem 1 above you found that the following equations all have the same solution, 1.

$$1 \otimes x = 1$$
$$2 \otimes x = 2$$
$$3 \otimes x = 3$$
$$4 \otimes x = 4$$

Is it true also that $0 \otimes 1 = 0$?

> There is a unique number 1 such that for any number a which is an element of $\{0, 1, 2, 3, 4\}$
> $$a \otimes 1 = 1 \otimes a = a.$$

The number 1 is called the <u>identity element</u> or the <u>unit</u> for clock <u>multiplication</u>. This property is called the <u>identity property of multiplication</u> or the <u>property of one</u>. It tells us that in clock arithmetic multiplying by 1 leaves the number unchanged.

In Problem 2 above you saw the following:

$1 \otimes x = 1$ has the solution 1

$2 \otimes x = 1$ has the solution 3

$3 \otimes x = 1$ has the solution 2

$4 \otimes x = 1$ has the solution 4

This shows that in clock arithmetic if $a \neq 0$, there is always a unique number b for which $a \otimes b = 1$. Why did we say $a \neq 0$? Are there any numbers such that $0 \otimes x = 1$? Will b ever turn out to be 0? Why?

If $a \in \{1, 2, 3, 4\}$, then there is a unique number b in the set $\{1, 2, 3, 4\}$ such that $a \otimes b = b \otimes a = 1$.

We call the number b the <u>multiplicative inverse</u> of a or the <u>reciprocal</u> of a. This is called the <u>inverse property of multiplication</u> or the <u>reciprocal property</u>. It tells us about pairs of numbers whose product is 1 in clock arithmetic.

It is not difficult to see from the definition of clock division that if a and b are reciprocals, then $b = 1 \div a$, since $a \otimes b = 1$ and $b = 1 \div a$ mean the same thing.

<u>Problems 6-3C</u>

1. (a) Does clock multiplication in the set $\{0, 1, 2\}$ have the commutative property?

   (b) Is it commutative in the set $\{0, 1, 2, 3\}$?

   (c) Can you solve the equation $2 \otimes x = 1$ in these arithmetics?

   (d) Does every non-zero number have a reciprocal in these arithmetics?

   (e) Give reasons for all your answers.

Challenge Problem

2. Discuss the following operation:

$$a \circ b = (2 \times a) + b$$

where "×" and "+" indicate ordinary multiplication and addition respectively.

For example, $3 \circ 4 = (2 \times 3) + 4 = 10$.

Make a table for this operation for the numbers 0, 1, 2, 3, 4 as suggested below, and answer the following questions.

o	0	1	2	3	4
0				3	
1	2				
2		5			
3					10
4					

(a) Is the operation commutative? That is, is it true that $a \circ b = b \circ a$ for all a and b in {0, 1, 2, 3, 4}? Why?

(b) Is the set {0, 1, 2, 3, 4} closed under this operation? Why?

(c) If we used this operation in the set {0, 1, 2, 3, ...}, that is, the set of all whole numbers, would the closure property be true? Why?

(d) Is this operation associative in {0, 1, 2, 3, 4}? Why?

(e) Is there a unique identity element in the set {0, 1, 2, 3, 4} for this operation? If so, what is it?

(f) Is there a unique inverse element for each element of {1, 2, 3, 4}? Explain.

(g) In what ways is this operation different from clock addition in the same set?

Challenge Problem

3. Let $a * b = (3 \otimes a) \oplus b$ where a and b are in $\{0, 1, 2, 3, 4\}$ and $\otimes$ and $\oplus$ denote clock multiplication and addition respectively. Answer parts a - g of Problem 2.

6-4 Summary of Basic Properties

Let us use C to represent the set $\{0, 1, 2, 3, 4\}$. We have discovered the following properties about the numbers 0, 1, 2, 3, and 4 under clock arithmetic.

1. Closure property of addition.

   If $a \in C$ and $b \in C$, then $(a \oplus b) \in C$.

2. Closure property of multiplication.

   If $a \in C$ and $b \in C$, then $(a \otimes b) \in C$.

3. Commutative property of addition.

   If $a \in C$ and $b \in C$, then $a \oplus b = b \oplus a$.

4. Commutative property of multiplication.

   If $a \in C$ and $b \in C$, then $a \otimes b = b \otimes a$.

5. Associative property of addition.

   If $a \in C$, $b \in C$, and $c \in C$, then

   $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

6. Associative property of multiplication.

   If $a \in C$, $b \in C$, and $c \in C$, then

   $(a \otimes b) \otimes c = a \otimes (b \otimes c)$.

7. Distributive property of multiplication with respect to addition.

   If $a \in C$, $b \in C$, and $c \in C$, then

   $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$.

8. <u>Identity property of addition</u> or <u>property of zero</u>.

   There is a unique number 0 such that $0 \in C$ and, if $a \in C$, then $a \oplus 0 = 0 \oplus a = a$.

9. <u>Identity property of multiplication</u> or <u>property of one</u>.

   There is a unique number 1 such that $1 \in C$ and, if $a \in C$, then $a \otimes 1 = 1 \otimes a = a$.

10. <u>Inverse property of addition</u> or <u>property of opposites</u>.

    If $a \in C$, then there is a unique number b such that $b \in C$ and $a \oplus b = b \oplus a = 0$.

    The number b is called the additive inverse of a or the <u>opposite</u> of a.

11. <u>Inverse property of multiplication</u> or <u>property of reciprocals.</u>

    If $a \in C$ and $a \neq 0$, then there is a unique number b such that $b \in C$ and $a \otimes b = b \otimes a = 1$. The number b is called the <u>multiplicative inverse</u> of a or the <u>reciprocal</u> of a.

You will meet these properties again later. Watch for them.

CHAPTER 7

THE ARITHMETIC OF RATIONAL NUMBERS

7-1 Addition of Rational Numbers

This is not an entirely new topic. You have already studied addition of fractions in primary school. There you first learnt how to add fractions with the same denominator. It is easy to see that

$$\frac{3}{7} + \frac{2}{7} = \frac{3 + 2}{7} = \frac{5}{7}$$

$$\frac{14}{25} + \frac{12}{25} = \frac{14 + 12}{25} = \frac{26}{25}$$

We saw on the number line that one-seventh is one of the seven equal parts of the unit interval. $\frac{3}{7} + \frac{2}{7}$ would be 3 + 2, or 5, of these "one-sevenths," so $\frac{3}{7} + \frac{2}{7} = \frac{5}{7}$.

> If $\frac{a}{c}$ and $\frac{b}{c}$ are positive rational numbers, then
>
> $$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$$

Suppose now that we want to add two positive fractions, such as $\frac{2}{3}$ and $\frac{4}{5}$, that have different denominators. We can do this problem easily if we make use of equivalent fractions. You remember that you learnt in Chapter 5 that the fraction $\frac{a}{b}$ is equivalent to the fraction $\frac{a \times c}{b \times c}$ where c is any counting number. For example,

$$\frac{2}{3} = \frac{2 \times 2}{3 \times 2} = \frac{4}{6} \qquad \frac{3}{5} = \frac{3 \times 2}{5 \times 2} = \frac{6}{10}$$

$$\frac{2}{3} = \frac{2 \times 3}{3 \times 3} = \frac{6}{9} \qquad \frac{3}{5} = \frac{3 \times 3}{5 \times 3} = \frac{9}{15}$$

$$\frac{2}{3} = \frac{2 \times 4}{3 \times 4} = \frac{8}{12} \qquad \frac{3}{5} = \frac{3 \times 4}{5 \times 4} = \frac{12}{20}$$

$$\frac{2}{3} = \frac{2 \times 5}{3 \times 5} = \frac{10}{15}$$

and so on.

and so on.

Let us form the set

$$A = \left\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \frac{12}{18}, \ldots\right\}$$

of fractions equivalent to $\frac{2}{3}$ and the set

$$B = \left\{\frac{3}{5}, \frac{6}{10}, \frac{9}{15}, \frac{12}{20}, \ldots\right\}$$

of fractions equivalent to $\frac{3}{5}$. We see that $\frac{10}{15}$ is in A and $\frac{9}{15}$ is in B and we can write

$$\frac{2}{3} + \frac{3}{5} = \frac{10}{15} + \frac{9}{15} = \frac{10 + 9}{15} = \frac{19}{15}.$$

Can we always be sure of finding equivalent fractions with the same denominator? Suppose that we have two fractions $\frac{5}{6}$ and $\frac{7}{4}$ and wish to add them. We shall try to find two fractions which have the same denominator and are equivalent to $\frac{5}{6}$ and $\frac{7}{4}$, respectively.

Look at the set of multiples of 6

$$C = \{6, 12, 18, \ldots\}$$

and compare it with the set of multiples of 4

$$D = \{4, 8, 12, 16, \ldots\}.$$

Do you see that 12 is the smallest number that is in both C and D? In Chapter 4 we called 12 the least common multiple or LCM of 4 and 6. We can use it to write

$$\frac{5}{6} + \frac{7}{4} = \frac{5 \times 2}{6 \times 2} + \frac{7 \times 3}{4 \times 3} = \frac{10}{12} + \frac{21}{12} = \frac{10 + 21}{12} = \frac{31}{12}.$$

You should know from Chapter 4 how to find the LCM of any set of counting numbers. You can use this LCM to help you add positive rational numbers.

For example, to find the sum

$$\frac{7}{6} + \frac{3}{10} + \frac{5}{12}$$

we first find the LCM of 6, 10, and 12. Since

$$6 = 2 \times 3$$
$$10 = 2 \times 5$$
$$12 = 2 \times 2 \times 3$$

we see that the LCM of 6, 10, and 12 is $2 \times 2 \times 3 \times 5$, or 60. In each fraction we multiply the numerator and the denominator by whatever number is necessary to make the denominator 60.

$$\frac{7}{6} + \frac{3}{10} + \frac{5}{12} = \frac{7 \times 10}{6 \times 10} + \frac{3 \times 6}{10 \times 6} + \frac{5 \times 5}{12 \times 5}$$

$$= \frac{70}{60} + \frac{18}{60} + \frac{25}{60}$$

$$= \frac{70 + 18 + 25}{60}$$

$$= \frac{113}{60}$$

The LCM of the denominators of a set of fractions is called the <u>least common denominator</u> (LCD) of the set of fractions. Thus the LCD of the fractions $\frac{7}{6}$, $\frac{3}{10}$, and $\frac{5}{12}$ is 60.

In adding fractions it is customary to give the answer in simplest form. You remember from Chapter 5 that a fraction is said to be in simplest form, or lowest terms, if the numerator and denominator of the fraction have 1 as their greatest common factor. Do you see that $\frac{5}{7}$, $\frac{26}{25}$, $\frac{22}{15}$, $\frac{19}{12}$, and $\frac{113}{60}$ are all in simplest form? If we add $\frac{15}{26}$ and $\frac{13}{26}$ we have

$$\frac{15}{26} + \frac{13}{26} = \frac{28}{26}$$

and our answer is not in simplest form. 28 and 26 have greatest common factor 2, so we can simplify as follows:

$$\frac{28}{26} = \frac{14 \times 2}{13 \times 2} = \frac{14}{13}$$

Given any two positive rational numbers, $\frac{a}{b}$ and $\frac{c}{d}$, we can write

$$\frac{a}{b} + \frac{c}{d} = \frac{a \times d}{b \times d} + \frac{c \times b}{d \times b} = \frac{(a \times d) + (c \times b)}{b \times d}$$

which gives

$$\boxed{\frac{a}{b} + \frac{c}{d} = \frac{(a \times d) + (c \times b)}{b \times d}}$$

For example,

$$\frac{5}{6} + \frac{3}{4} = \frac{(5 \times 4) + (6 \times 3)}{6 \times 4} = \frac{20 + 18}{24} = \frac{38}{24}$$

$$= \frac{19 \times 2}{12 \times 2} = \frac{19}{12}.$$

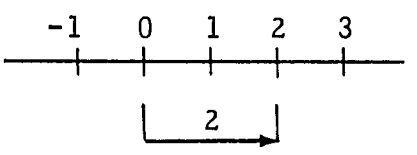
It is much better, however, to add positive rational numbers without using the formula just given. The way of adding by using the LCD and thinking about equivalent fractions is much better. When you do a problem in addition of fractions by using equivalent fractions, you know at each step <u>why</u> you are doing what you do. You do not have to memorize a formula that you can easily forget later.

<u>Problems 7-1A</u>

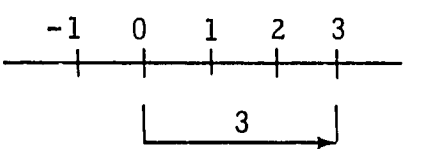
Find the following sums. Give your answer in simplest form.

1. $\frac{2}{3} + \frac{1}{5}$	5. $\frac{1}{6} + \frac{1}{4}$	9. $\frac{2}{3} + \frac{3}{2}$
2. $\frac{1}{6} + \frac{7}{9}$	6. $\frac{3}{10} + \frac{3}{4}$	10. $\frac{1}{4} + \frac{5}{4} + \frac{3}{8}$
3. $\frac{3}{7} + \frac{5}{2}$	7. $\frac{5}{6} + \frac{1}{3}$	11. $\frac{1}{7} + \frac{2}{3} + \frac{5}{2}$
4. $\frac{4}{1} + \frac{2}{9}$	8. $\frac{5}{1} + \frac{1}{6}$	12. $\frac{0}{3} + \frac{7}{8} + \frac{1}{4}$

To help discover how we shall add rational numbers when they are not necessarily positive, let us consider addition of rational numbers on the number line. Here are pictures of 2 and 3 on the number line using arrows.

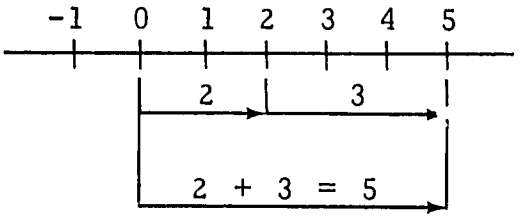


The number line arrow for 2



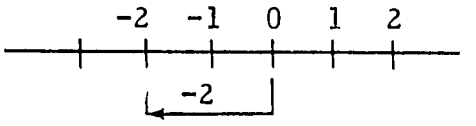
The number line arrow for 3

The addition of 2 and 3 can then be pictured as follows:



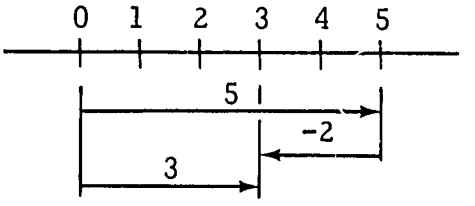
We show the addition of one positive number to another positive number by moving to the right along the number line.

Think now of how we may picture negative numbers on the number line. Using an arrow we can picture -2 as follows:



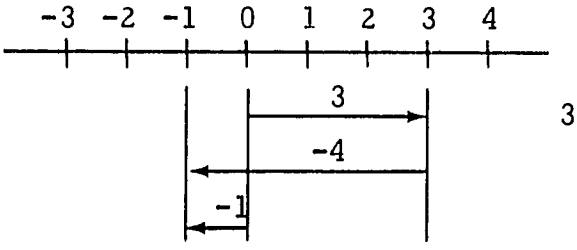
The arrow that corresponds to -2 is the same length as that for 2 but goes in the opposite direction, from 0 to -2 instead of 0 to 2. Arrows for positive numbers go to the right; arrows for negative numbers go to the left.

What would be a reasonable meaning to give to the sum 5 + (-2)? Would it not be reasonable to say that the point corresponding to 5 + (-2) is found by starting at 0, moving 5 units to the right and then 2 units to the left? If we do this we find that 5 + (-2) = 3 as shown in the picture below.



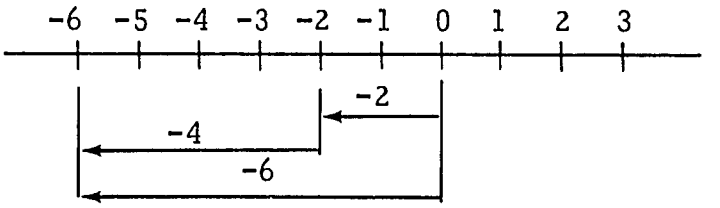
5 + (-2) = 3

To add 3 and -4 in the same way we can draw a picture like the following:

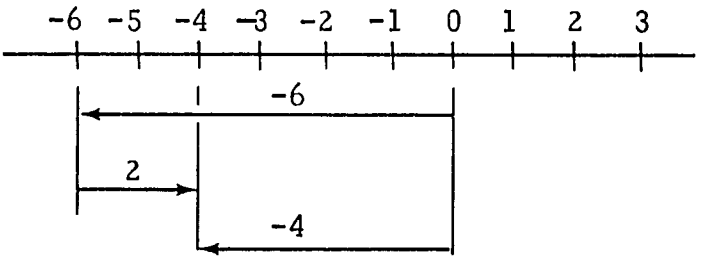


3 + (-4) = -1

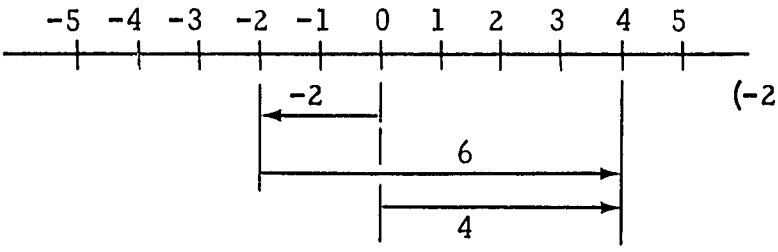
Here are some more examples of addition of positive and negative rational numbers.



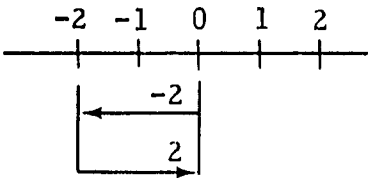
$(-2) + (-4) = -6$



$(-6) + 2 = -4$



$(-2) + 6 = 4$



$(-2) + 2 = 0$

We see by the use of the number line that addition of rational numbers is rather simple. We do not really need to use the number line in doing such additions. It is enough to keep in mind the positions of the numbers on the number line in order to carry out additions. We see that:

When both numbers are positive, the sum is positive as in

$$5 + 3 = 8$$

When both numbers are negative, the sum is negative as in

$$(-5) + (-3) = -8$$

When one number is positive and one number is negative, it is the number farther from 0 that determines whether the sum is positive or negative. If the numbers are the same distance from 0, the sum is 0.

For example:

In $(-5) + 3 = -2$, the sum is negative because -5 is farther from 0 than 3 is, and -5 is negative.

In $5 + (-3) = 2$, the sum is positive because 5 is farther from 0 than -3 is, and 5 is positive.

In $2 + (-2) = 0$, the sum is 0 because 2 and -2 are the same distance from 0.

Our examples have been with integers because it is easy to picture them on the number line. Since we do not really need the pictures, we can also easily add rational numbers that are not integers by thinking out our problems as we did for integers. For example,

$$\frac{4}{7} + \left(-\frac{3}{7}\right) = \frac{1}{7}$$

because we can think of moving to the right from 0, a distance of $\frac{4}{7}$ and then back to the left, a distance of just $\frac{3}{7}$. Since $\frac{4}{7} > \frac{3}{7}$ this leaves us $\frac{1}{7}$ to the right of 0. In the same way we see that

$$-\frac{3}{5} + \frac{1}{4} = -\frac{3 \times 4}{5 \times 4} + \frac{1 \times 5}{4 \times 5} = -\frac{12}{20} + \frac{5}{20} = -\frac{7}{20}$$

because we can think of moving to the left from 0, a distance of $\frac{12}{20}$ and then back to the right, a distance of $\frac{5}{20}$. Since $\frac{5}{20} < \frac{12}{20}$ this leaves us $\frac{7}{20}$ to the left of 0.

Problems 7-1B

1. Find the following sums and draw a picture on the number line.

(a) $9 + (-5)$

(b) $10 + (-7)$

(c) $(-5) + (-2)$

(d) $5 + (-10)$

(e) $(-7) + 5$

(f) $3 + (-7)$

2. Find the following sums:

(a) $(-7) + 16$

(b) $(-3) + (-9)$

(c) $15 + (-19)$

(d) $(-21) + 16$

(e) $(-5) + (-20)$

(f) $(-7) + 7$

3. Find the following sums:

(a) $(-\frac{1}{2}) + \frac{1}{3}$

(b) $(-\frac{3}{4}) + (-\frac{1}{2})$

(c) $(-\frac{1}{5}) + \frac{2}{3}$

(d) $\frac{2}{3} + (-\frac{1}{6})$

(e) $(-\frac{9}{16}) + \frac{9}{16}$

(f) $\frac{1}{4} + (-\frac{2}{3})$

4. A company reports income for the first six months of a year as follows:

January	shs.	5000	profit	April	shs.	1000	profit
February	shs.	2000	profit	May	shs.	4000	loss
March	shs.	6000	loss	June	shs.	3000	loss

(a) How could you use positive and negative numbers to represent these income figures?

(b) What is the total income for the six-month period?

(c) What is the total income for the first three months of the year?

(d) What is the total income for the four-month period: March, April, May, and June?

5. A boy rows upstream at a speed of 4 miles per hour against a current of 2 miles per hour.
   (a) How could you use positive and negative numbers to represent these speeds?
   (b) What represents his actual speed upstream?

6. An airplane traveling at an altitude of 13,000 feet makes a climb of 5000 feet, followed by a descent of 3000 feet.
   (a) Represent the plane's final altitude as a sum of positive and negative numbers.
   (b) What is the plane's altitude after the descent?

7-2 Properties of Addition of Rational Numbers

You have long been familiar with certain basic properties of addition for positive rational numbers. For example, as a very young child in primary school you soon saw that you needed to learn only half of the "addition facts". Once you learnt the sums in the first column below, you could use those results to know the sums in the second column.

$7 + 1 = 8$	$1 + 7 = 8$
$7 + 2 = 9$	$2 + 7 = 9$
$7 + 3 = 10$	$3 + 7 = 10$
$7 + 4 = 11$	$4 + 7 = 11$
and so on.	and so on.

Perhaps at first you were not quite sure, for example, that $7 + 2 = 2 + 7$, and counted out on your fingers to make sure. (Very young children really do this!) As your experience with numbers grew you began to feel more and more confident that, for <u>any two whole numbers</u> a and b, it is true that $a + b = b + a$.

Will this property still be true for numbers other than whole numbers? Verify, by actually computing them, that it is true for the positive rational numbers in the following examples.

$\frac{3}{8} + \frac{5}{8}$	$\frac{5}{8} + \frac{3}{8}$
$\frac{5}{4} + \frac{7}{2}$	$\frac{7}{2} + \frac{5}{4}$
$2\frac{1}{4} + 9\frac{3}{4}$	$9\frac{3}{4} + 2\frac{1}{4}$
$5.6 + 1.3$	$1.3 + 5.6$
$\frac{5}{6} + \frac{4}{9}$	$\frac{4}{9} + \frac{5}{6}$

It would take a great deal of work to check, for example, that

$$\frac{1,932,495}{2,735,964} + \frac{700,291}{31,592,314} = \frac{700,291}{31,592,314} + \frac{1,932,495}{2,735,964}$$

by actually performing these additions. Do you not feel that these two sums are equal without actually performing the additions? Do you agree that $a + b = b + a$ is true for <u>any two positive rational numbers</u> a and b?

The property that addition of two numbers may be performed in either order is so important in mathematics that a special name has been given to it. You met this name, the <u>commutative property of addition</u>, in Chapter 6 when you learnt about a new kind of arithmetic.

Do you feel that the commutative property still holds if we include negative rational numbers? Try some. Compute the following pairs of sums and see whether they are equal.

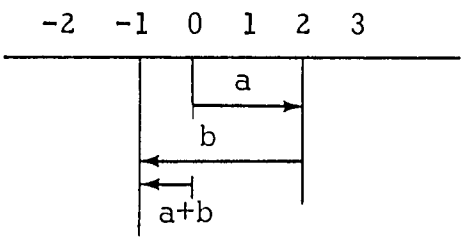
$(-3) + 2$	$2 + (-3)$
$\frac{7}{8} + (-\frac{5}{8})$	$(-\frac{5}{8}) + \frac{7}{8}$
$(-\frac{3}{10}) + (-\frac{7}{20})$	$(-\frac{7}{20}) + (-\frac{3}{10})$

We shall not ask you to work out the sums

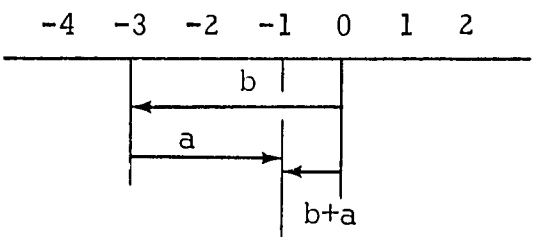
$$(-\frac{1,932,495}{2,735,964}) + \frac{700,291}{31,592,314} \quad \text{and} \quad \frac{700,291}{31,592,314} + (-\frac{1,932,495}{2,735,964}),$$

but by now you probably believe that they are equal.

Thinking of the number line will further convince you that addition of rational numbers is indeed commutative. For example, suppose a is a positive number such as 2 and b is a negative number such as -3. Then a + b can be thought of as being obtained by moving a units to the right from 0 and then back b units to the left. Would it make any difference if we first moved b units to the left from 0 and then moved a units to the right to get b + a? See the picture below.



a units to the right and then b units to the left



b units to the left and then a units to the right

If you think through addition of rational numbers in terms of movement to left and right on the number line, you should see that addition of rational numbers is indeed commutative. For <u>any two rational numbers</u>, a and b,

$$a + b = b + a.$$

Another basic property of addition of positive rational numbers is illustrated by the facts that

$$(2 + 3) + 7 = 5 + 7 = 12$$

and also

$$2 + (3 + 7) = 2 + 10 = 12.$$

In general, for <u>any three positive rational numbers</u> a, b, and c, it seems to be true that $(a + b) + c = a + (b + c)$. This property also was discussed in connection with the clock arithmetic of Chapter 6 and was called the <u>associative property of addition</u>. It is another basic property that continues to hold for addition of any rational numbers. Thus, for example,

$$(-2) + \left[4 + (-8)\right] = (-2) + (-4) = -6$$

and also

$$\left[(-2) + 4\right] + (-8) = 2 + (-8) = -6.$$

Verify by computation that the following equations are true.

$$\frac{3}{5} + \left[(-\frac{7}{5}) + \frac{19}{5}\right] = \left[\frac{3}{5} + (-\frac{7}{5})\right] + \frac{19}{5}$$

$$\left[(-\frac{1}{2}) + (-\frac{1}{3})\right] + \frac{1}{4} = (-\frac{1}{2}) + \left[(-\frac{1}{3}) + \frac{1}{4}\right]$$

Can you see, by thinking of addition of rational numbers on the number line, that the associative property of addition holds for rational numbers? For any three rational numbers a, b, and c,

$$(a + b) + c = a + (b + c).$$

Another basic property of addition of positive rational numbers is illustrated by the fact that $7 + 0 = 0 + 7 = 7$. Once again, think of addition of rational numbers in terms of movement on the number line with 0 representing no movement at all. You can see that this property also carries over to the set of all rational numbers. That is,

$a + 0 = 0 + a = a$ for all rational numbers a.

We call 0 the <u>identity element</u> for addition and this is the <u>identity property of addition</u>.

The last basic property of addition that we shall find useful and necessary is closely related to the nature of negative numbers. In which of the following equations can you find a <u>positive</u> rational number which will make the statement true?

(a) $x + 3 = 5$

(b) $x + 5 = 3$

(c) $\frac{3}{4} + x = \frac{1}{4}$

(d) $x + \frac{15}{8} = 1$

You see that there are equations which cannot be solved with positive numbers only. The last three equations are true only when x has the values (-2), $(-\frac{1}{2})$, and $(-\frac{7}{8})$, respectively. Before we had negative numbers we could not have solved these equations. Now that we have negative rational numbers available, we know that every equation of the form $x + p = q$ has a solution for any rational numbers p and q.

In particular we are interested in equations of the form $x + p = 0$ where the sum of the two numbers is 0, such as:

$$x + 3 = 0 \qquad x + \frac{2}{3} = 0$$

$$x + (-6) = 0 \qquad x + (-\frac{7}{8}) = 0$$

You see that in our set of rational numbers there always is a number which makes such an equation true.

$x + 3 = 0$	is true for $x = -3$
$x + (-6) = 0$	is true for $x = 6$
$x + \frac{2}{3} = 0$	is true for $x = -\frac{2}{3}$
$x + (-\frac{7}{8}) = 0$	is true for $x = \frac{7}{8}$

In general we observe that, for every rational number a there is a rational number b such that

$$a + b = 0$$

We say that b is the <u>opposite</u> or <u>additive inverse</u> of a. Do you see that 2 is the opposite of -2, $-\frac{15}{16}$ is the opposite of $\frac{15}{16}$, $\frac{15}{16}$ is the additive inverse of $-\frac{15}{16}$, and so on? What is the opposite of 0? Do you see that every rational number has an opposite? Can you see by picturing opposites on the number line why the word "opposite" is used?

Recall that we used this word "opposite" when we first introduced the negative numbers. There we spoke of negative numbers being the opposites of positive numbers. We can now say that

if a number is positive, its opposite is negative;

if a number is negative, its opposite is positive;

if a number is zero, its opposite is zero;

the sum of a number and its opposite is zero.

We shall now have still another use for the symbol "-". If a is any rational number, we shall represent its opposite by "$-a$" which we shall read "the opposite of a." This means that

if a is positive, then $-a$ is negative, and

if a is negative, then $-a$ is positive.

Notice that $-a$ is not necessarily a negative number.

Let us summarize these basic properties relating to the addition of rational numbers.

1. The <u>commutative property of addition</u>.

   For all rational numbers a and b,

   $$a + b = b + a.$$

2. The <u>associative property of addition</u>.

   For all rational numbers a, b, and c,

   $$(a + b) + c = a + (b + c).$$

3. The <u>identity property of addition</u> or the <u>property of zero</u>.

   For all rational numbers a,

   $$a + 0 = 0 + a = a.$$

4. The <u>inverse property of addition</u> or the <u>property of opposites</u>.

   For any rational number a, there is a unique rational number b such that

   $$a + b = b + a = 0.$$

   We call b the <u>opposite</u> of a and write $b = -a$.

<u>Problems 7-2</u>

1. Without doing any arithmetic, tell which of the sums in the left-hand column below is equal to which of the sums in the right-hand column. Tell why you know this. (For example, $\left[(-7) + 6\right] + (-2) = \left[6 + (-7)\right] + (-2)$ by the commutative property of addition.)

1. (a) $(-2) + (3 + 5)$
   (b) $1 + \left[(-2) + (-3)\right]$
   (c) $-2 + \left[2 + (-1)\right]$
   (d) $-4 + (3 + 5)$
   (e) $-1 + \left[(-1) + 3\right]$
   (f) $8 + \left[4 + (-4)\right]$
   (g) $\left[(-1) + 3\right] + 1$
   (h) $\left[(-2) + 3\right] + 5$
   (i) $\left[(-2) + 2\right] + (-1)$
   (j) $(8 + 4) + (-4)$
   (k) $1 + \left[(-3) + (-2)\right]$
   (l) $(3 + 5) + (-4)$
   (m) $\left[(-1) + (-1)\right] + 3$
   (n) $1 + \left[(-1) + 3\right]$

2. Compute each of the sums in Problem 1.

3. Sometimes we can recognize that two expressions are equal by more than one application of the basic properties. For example:

$$(86 + 35) + 14 = 86 + (35 + 14) =$$

$$86 + (14 + 35) = (86 + 14) + 35$$

by the associative property of addition and then the commutative property of addition, and then the associative property again.

Show in this manner why in each of the following the two expressions are equal.

(a) $5 + (6 + 7) = (7 + 6) + 5$

(b) $\left[(-8) + 3\right] + (-4) = (-8) + \left[(-4) + 3\right]$

(c) $\frac{3}{5} + \left[(-\frac{2}{5}) + \frac{1}{5}\right] = \frac{1}{5} + \left[\frac{3}{5} + (-\frac{2}{5})\right]$

(d) $\left[(-\frac{5}{8}) + (-\frac{3}{5})\right] + (-2) = (-\frac{3}{5}) + \left[(-2) + (-\frac{5}{8})\right]$

4. Draw a figure with arrows, as in the text, to picture

why $(-2) + 3 = 3 + (-2)$;

why $(-4) + 2 = 2 + (-4)$;

why $(-3) + (-1) = (-1) + (-3)$.

5. Find the opposite of each of the following rational numbers and picture each number and its opposite on the number line.

(a) 8

(b) $-\frac{3}{4}$

(c) $\frac{9}{23}$

(d) $-\frac{14}{5}$

(e) 0

(f) $\frac{2}{3}$

6. For each of the following incomplete equations find the number which will make the equation true.

(a) $(-3) + _____ = (-3)$

(b) $\frac{5}{9} + _____ = 0$

(c) $(-\frac{5}{11}) + 0 = _____$

(d) $_____ + (-94) = 0$

(e) $239 + (-239) = _____$

(f) $_____ + \frac{45}{46} = \frac{45}{46}$

(g) $(-16 \cdot 8) + _____ = 0$

7. In each part of Problem 6 tell what basic property is illustrated.

7-3 Applications of the Properties of Addition

You may be wondering why we bother to list the basic properties.

The importance of the properties lies in the fact that as we go on in algebra, we shall find that most of the changes we make in algebraic expressions as we solve problems can be explained and made meaningful with the help of these few properties and a few more for multiplication. By being thoroughly familiar with these properties you will have a kit of tools which will open the door to understanding in a great deal of algebra.

Here are some examples of such applications of the properties of addition.

Application 1. The usual method for adding whole numbers depends heavily on the commutative and associative properties of addition. For example, consider the problem of adding 24 and 37. Recall that $24 = 20 + 4$ and $37 = 30 + 7$. We can write

$$\begin{aligned} 24 + 37 &= (20 + 4) + (30 + 7) \\ &= (20 + 30) + (4 + 7) && \text{(Why?)} \\ &= 50 + 11 \\ &= 50 + (10 + 1) \\ &= (50 + 10) + 1 && \text{(Why?)} \\ &= 60 + 1 \\ &= 61 \end{aligned}$$

What basic properties are used at each point where it says "Why?"?

We now can see the reason behind the "rule" which is sometimes given for adding 24 and 37: "4 + 7 is 11; put down 1 and carry 1" and so on.

Application 2. Suppose that you are asked to find the sum $45 + 97 + 55$. Does this mean $(45 + 97) + 55$ or $45 + (97 + 55)$? The associative property of addition tells us that it does not matter which of these interpretations we give. The two ways of writing the sum give the same result. Now we can rewrite $45 + (97 + 55)$ as $45 + (55 + 97)$, and finally as $(45 + 55) + 97$. Why? In this form we can see the answer very quickly. Because $45 + 55 = 100$, the result is 197.

Application 3. Add $2\frac{3}{4}$ and $7\frac{5}{8}$. If you recall that $2\frac{3}{4} = 2 + \frac{3}{4}$ and $7\frac{5}{8} = 7 + \frac{5}{8}$, you can write

$$
\begin{aligned}
2\tfrac{3}{4} + 7\tfrac{5}{8} &= (2 + \tfrac{3}{4}) + (7 + \tfrac{5}{8}) \\
&= (2 + 7) + (\tfrac{3}{4} + \tfrac{5}{8}) && \text{(Why?)} \\
&= 9 + (\tfrac{6}{8} + \tfrac{5}{8}) && \text{(Why?)} \\
&= 9 + \tfrac{11}{8} \\
&= 9 + (\tfrac{8}{8} + \tfrac{3}{8}) \\
&= 9 + (1 + \tfrac{3}{8}) \\
&= (9 + 1) + \tfrac{3}{8} && \text{(Why?)} \\
&= 10 + \tfrac{3}{8} \\
&= 10\tfrac{3}{8}
\end{aligned}
$$

(Of course, you do not need to write down all of these steps in doing such problems.)

Application 4. We can use these properties to take another look at addition of positive and negative rational numbers. Let us assume, as before, that we know how to add positive rational numbers. We shall pretend that we do not know how to add two negative rational numbers or a positive rational number and a negative rational number. Then if we assume that the four basic properties of addition hold for the set of all rational numbers, we can find out how to add negative rational numbers, and our results will be the same as those we already know.

How can we use the basic properties to find the sum $(-4) + (-3)$? It is certainly reasonable to guess that $(-4) + (-3) = -7$. What do we know about -7? We know that it is that unique number which when added to 7 gives 0:

$$7 + (-7) = 0.$$

Hence if it turns out that

$$7 + \left[(-4) + (-3)\right] = 0$$

we will know that $(-4) + (-3)$ <u>must</u> be -7. Let us see.

$$\begin{aligned} 7 + \left[(-4) + (-3)\right] &= (4 + 3) + \left[(-4) + (-3)\right] && \text{(Why?)} \\ &= \left[4 + (-4)\right] + \left[3 + (-3)\right] && \text{(Why?)} \\ &= 0 + 0 && \text{(Why?)} \\ &= 0 && \text{(Why?)} \end{aligned}$$

Hence $7 + \left[(-4) + (-3)\right]$ <u>is</u> 0 and so $(-4) + (-3)$ <u>is</u> -7. Our guess was correct!

Consider now the sum $4 + (-3)$.

$$\begin{aligned} 4 + (-3) &= (1 + 3) + (-3) \\ &= 1 + \left[3 + (-3)\right] && \text{(Why?)} \\ &= 1 + 0 && \text{(Why?)} \\ &= 1 && \text{(Why?)} \end{aligned}$$

Hence $4 + (-3) = 1$.

What about the sum $4 + (-7)$?

$$\begin{aligned} 4 + (-7) &= 4 + \left[(-4) + (-3)\right] && \text{(Why?)} \\ &= \left[4 + (-4)\right] + (-3) && \text{(Why?)} \\ &= 0 + (-3) && \text{(Why?)} \\ &= -3 \end{aligned}$$

Hence $4 + (-7) = -3$.

<u>Application 5</u>. The expression $-(-2)$ would be read "the opposite of the opposite of 2" or "the opposite of negative 2". What does it mean?

By the inverse property of addition the sum of 2 and its opposite is zero.

$$2 + (-2) = 0$$

or by the commutative property,

$$(-2) + 2 = 0$$

Also by the inverse property the sum of (-2) and <u>its</u> opposite is zero.

$$(-2) + \left[-(-2)\right] = 0$$

If we compare the last two equations we see that 2 and -(-2) must both be the additive inverse of (-2). Since (-2) has only one additive inverse, the numbers 2 and -(-2) must be the same number, that is, $-(-2) = 2$.

In words, we can say that the opposite of the opposite of 2 is 2. Does this not look reasonable when you look at the number line picture of 2, -2, and -(-2)?

If a is any rational number, what do you think we can say about -(-a)?

<u>Problems 7-3</u>

1. As in Application 1, show the reasoning behind the additions $29 + 34 = 63$ and $125 + 397 = 522$.

2. As in Application 2, find an easy way of doing the following additions:

   (a) $35 + 164 + 65$

   (b) $7 + 9 + 3 + 1 + 6 + 18 + 4$

   (c) $9{,}998 + 2{,}344 + 1{,}002$

   (d) $\frac{5}{9} + \frac{11}{2} + \frac{4}{9} + \frac{1}{2}$

   (e) $(-\frac{3}{4}) + \frac{7}{8} + \frac{73}{8} + \frac{11}{4}$

   (f) $(-3 \cdot 7) + (-12 \cdot 9) + 8 \cdot 0 + (-9 \cdot 3) + 3 \cdot 9$

3. Perform the following additions:

(a) $3\frac{5}{8} + 5\frac{7}{12}$

(b) $7\frac{2}{3} + 6\frac{3}{5}$

(c) $8\frac{2}{3} + 9\frac{3}{4} + 2\frac{1}{3}$

(d) $5\frac{1}{4} + 8\frac{2}{3} + 3\frac{3}{4} + 7\frac{2}{5}$

4. Use a procedure as in Application 4 to show that

(a) $(-3) + (-2) = -5$

(b) $(-4) + (-5) = -9$

(c) $6 + (-3) = 3$

(d) $5 + (-7) = -2$

(e) $(-5) + 3 = -2$

(f) $(-9) + 14 = 5$

5. Find a positive rational number equal to $-(-9)$; to $-(-\frac{15}{16})$.

6. What can you say about $-\left[-(-9)\right]$?

About $-\left[-(-a)\right]$ where a is any rational number?

7-4 Multiplication of Rational Numbers

You have studied multiplication of fractions in primary school. Perhaps you examined a sequence of products in which some products which we accept as correct suggest that certain other products ought to be correct. For example,

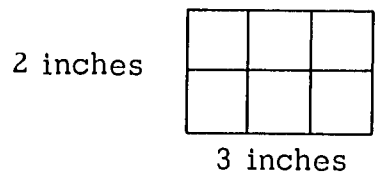
$$2 \times \frac{1}{5} = \frac{2}{5}$$
$$2 \times \frac{3}{5} = \frac{6}{5}$$
$$\frac{1}{3} \times \frac{12}{5} = \frac{4}{5}$$
$$\frac{2}{3} \times \frac{12}{5} = \frac{24}{15}$$

These examples suggest the following general principle for multiplying positive rational numbers.

> If $\frac{a}{b}$ and $\frac{c}{d}$ are two positive rational numbers, then
>
> $$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$$

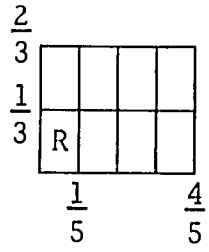
One way to justify this principle is to consider the area of a rectangle.

A rectangle with base 3 inches and height 2 inches



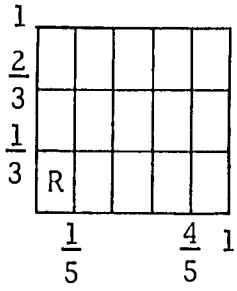
has area 6 square inches. This is immediately clear by counting squares. More generally, if the base is a inches and the height b inches the area is $a \times b$ square inches, where a and b are any whole numbers.

Suppose that the base a and the height b are fractions (positive rational numbers). It would be very pleasant if the area should still be $a \times b$ square inches. This will be true if we multiply fractions by the principle which we have stated.



Consider, for example, a rectangle with base $\frac{4}{5}$ inches and height $\frac{2}{3}$ inches.

There are 8 rectangles like R in the original rectangle. Then the area of the original rectangle must be 8 times the area of R. But a 1-inch square contains $5 \times 3 = 15$ rectangles like R so that the area of R must be $\frac{1}{15}$ square inches. Consequently the original rectangle is $\frac{8}{15}$ square inches. But if we follow our method of multiplying fractions,



$$\frac{2}{3} \times \frac{4}{5} = \frac{2 \times 4}{3 \times 5} = \frac{8}{15}.$$

We have seen that $\frac{8}{15}$ square inches is just the area that we should get.

We have chosen particular fractions for a and b, but it is easy to see that the result is a general one.

How can we multiply rational numbers which are not positive? For example, what will be the value of $3 \times (-2)$? Remember that when you first studied multiplication of whole numbers you found $3 \times 2 = 6$ by thinking of 3×2 as $2 + 2 + 2$, the sum of three 2's. Suppose that we wish to continue this pattern. We could say that $3 \times (-2)$ is $(-2) + (-2) + (-2)$, the sum of three (-2)'s. That would give us

$$3 \times (-2) = (-2) + (-2) + (-2) = -6.$$

What, then, would be the value of $5 \times (-3)$? $2 \times (-29)$?

What about $(-2) \times 3$? It does not make sense to talk about the sum of negative two 3's! Let us instead think for a moment of $(-2) \times 3$ as compared with $3 \times (-2)$. Just as we find it useful in ordinary arithmetic to know that $2 \times 3 = 3 \times 2$, so it would seem desirable here to have $3 \times (-2) = (-2) \times 3$. Let us agree, then, that

$$(-2) \times 3 = 3 \times (-2) = -6$$

What is the value of $(-5) \times 4$? $(-3) \times 25$?

Let us next ask about the product $(-2) \times 0$. Since we know that $a \times 0 = 0$ for any positive rational number a, it seems desirable to agree that $(-2) \times 0 = 0$ and that the product of zero and any rational number is zero. What is the value of $(-5) \times 0$? $0 \times (-18)$?

We are left with the problem of deciding about the product $(-2) \times (-3)$. Let us look for a pattern into which this case might fit. Consider the chain of products

$$(-2) \times 3 = -6$$

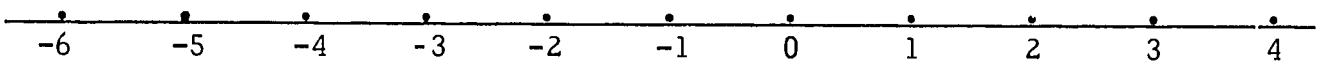
$$(-2) \times 2 = -4$$

$$(-2) \times 1 = -2$$

$$(-2) \times 0 = 0$$

We see that, as we <u>decrease</u> the multiplier by 1 (from 3 to 2 to 1 to 0) we <u>increase</u> the product by 2 (from -6 to -4 to -2 to 0). (Remember that $-6 < -4$, $-4 < -2$, and $-2 < 0$.) On the number line our chain looks like this:

$(-2)\times 3$	$(-2)\times 2$	$(-2)\times 1$	$(-2)\times 0$
$= -6$	$= -4$	$= -2$	$= 0$



Suppose that we wish to continue the pattern which we see here. In the sequence 3, 2, 1, 0 what would be the next number? (Remember that they are decreasing by 1.) In the sequence -6, -4, -2, 0 what would be the next number? (Remember that they are increasing by 2.)

$$(-2) \times 3 = -6$$

$$(-2) \times 2 = -4$$

$$(-2) \times 1 = -2$$

$$(-2) \times 0 = 0$$

$$(-2) \times (-1) = 2$$

If we continue the sequence another step, we would have

$$(-2) \times (-2) = 4.$$

On the basis of this exploration and the desire to retain patterns already existent, let us agree that

$$(-2) \times (-1) = 2$$

$$(-2) \times (-2) = 4$$

$$(-2) \times (-3) = 6$$

and so on.

What, then, is $(-3) \times (-5)$? $(-10) \times (-4)$?

Our discussion above leads us to the following agreement for multiplication of rational numbers. We have used integers in the examples for simplicity, but we shall wish to extend the agreement to all rational numbers.

If a is any rational number, then $a \times 0 = 0$.

For example, $2 \times 0 = 0$; $(-\frac{15}{16}) \times 0 = 0$; $0 \times 0 = 0$.

If a and b are any positive rational numbers, then $(-a) \times b = a \times (-b) = -(a \times b)$.

For example, $(-\frac{5}{3}) \times \frac{4}{7} = \frac{4}{7} \times (-\frac{5}{3}) = -(\frac{5}{3} \times \frac{4}{7}) = -\frac{5 \times 4}{3 \times 7} = -\frac{20}{21}$.

If a and b are any positive rational numbers, then $(-a) \times (-b) = a \times b$.

For example, $(-\frac{5}{3}) \times (-\frac{4}{7}) = \frac{5}{3} \times \frac{4}{7} = \frac{20}{21}$,

$$(-2\tfrac{1}{2}) \times (-4\tfrac{3}{5}) = (-\tfrac{5}{2}) \times (-\tfrac{23}{5}) = \tfrac{5}{2} \times \tfrac{23}{5} = \tfrac{23}{2}.$$

<u>Problems 7-4</u>

1. Which of the following sentences are true?

   (a) The product of two negative rational numbers is positive.

   (b) The product of a negative rational number by zero is zero.

   (c) The product of zero by a negative rational number is negative.

   (d) The product of a negative rational number by a positive rational number is always negative.

   (e) If the product of two rational numbers is positive, then one of them is positive and the other is negative.

   (f) If the product of two rational numbers is positive, then both of the numbers are positive.

2. Find the following products:

(a) $\frac{2}{3} \times \frac{3}{4}$

(b) $\frac{14}{15} \times \frac{6}{7}$

(c) $(2\frac{1}{12}) \times (1\frac{3}{5})$

(d) $(-\frac{14}{15}) \times \frac{6}{7}$

(e) $(-\frac{14}{15}) \times (-\frac{6}{7})$

(f) $3 \cdot 8 \times (-0 \cdot 5)$

(g) $\frac{14}{15} \times (-\frac{6}{7})$

(h) $(-\frac{6}{7}) \times (-\frac{14}{15})$

(i) $(-\frac{17}{3}) \times (\frac{11}{3})$

(j) $(-\frac{35}{6}) \times (-\frac{10}{21})$

(k) $(-\frac{2,148,214}{5,732,195}) \times 0$

(l) $(-2 \cdot 5) \times (-50)$

7-5 Other Fractions for Rational Numbers

So far we have thought of rational numbers as being written in the form $\frac{a}{b}$ or $-\frac{a}{b}$, where a and b are whole numbers and $b \neq 0$. (Do you remember why we must have $b \neq 0$?) Thus we have been writing numerals such as $\frac{2}{3}$, $-\frac{2}{3}$, $\frac{6}{5}$, and $-\frac{6}{5}$ for rational numbers. We have not yet given meaning to fractions such as $\frac{-1}{2}$. It turns out that we shall want to consider $\frac{-1}{2}$, $\frac{1}{-2}$, $\frac{-1}{-2}$, $\frac{-2}{3}$, $\frac{2}{-3}$, $\frac{-2}{-3}$, and so on, as representing rational numbers also.

Consider $\frac{-5}{2}$. If we agree that the same properties of addition should apply to the number which this represents, we have

$$\frac{5}{2} + \frac{-5}{2} = \frac{5 + (-5)}{2} = \frac{0}{2} = 0$$

We already know, however, that if a number is added to $\frac{5}{2}$ and the result is 0, then that number is the opposite of $\frac{5}{2}$, which is $-\frac{5}{2}$. We know this by the inverse property of addition:

$$\frac{5}{2} + (-\frac{5}{2}) = 0,$$

and since the additive inverse is unique, we see that $\frac{-5}{2}$ is just another way to write $-\frac{5}{2}$.

$$\frac{-5}{2} = -\frac{5}{2}$$

Consider now $\frac{5}{-2}$. Remember that in Chapter 5 you learnt that

$$\frac{5}{2} = \frac{5 \times 2}{2 \times 2} = \frac{5 \times 3}{2 \times 3} = \frac{5 \times 4}{2 \times 4} \text{ and so on.}$$

That is, you learnt that you could obtain a set of fractions equivalent to a given fraction by multiplying the numerator and denominator of the given fraction by the same positive integer. It would be natural to wish that this same principle would apply when the multiplier is a negative integer. We would then have

$$\frac{5}{-2} = \frac{5 \times (-1)}{(-2) \times (-1)} = \frac{-5}{2}.$$

Since we have already seen that $\frac{-5}{2} = -\frac{5}{2}$, we now have

$$\frac{-5}{2} = \frac{5}{-2} = -\frac{5}{2}.$$

Finally, if we apply this procedure to $\frac{-5}{-2}$ we have

$$\frac{-5}{-2} = \frac{(-5) \times (-1)}{(-2) \times (-1)} = \frac{5}{2}.$$

In every case if the numerator or the denominator of a fraction is a negative integer, the fraction still represents a rational number: $\frac{-2}{3} = -\frac{2}{3}$, $\frac{2}{-3} = -\frac{2}{3}$, $\frac{-2}{-3} = \frac{2}{3}$, and so on. Similarly, how can we rewrite $\frac{7}{-8}$? $\frac{-7}{8}$? $\frac{-7}{-8}$? In general, we make the following agreement.

If a and b are positive integers, then
$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b}$ and $\frac{-a}{-b} = \frac{a}{b}$.

We see from this that any rational number can now be written in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$. For example, $-\frac{5}{9}$ can be written $\frac{-5}{9}$ or $\frac{5}{-9}$. In final answers, however, the form $-\frac{5}{9}$ is preferred to $\frac{-5}{9}$ or $\frac{5}{-9}$.

Problems 7-5

1. Write each of the following rational numbers in the form $\frac{-a}{b}$ where a and b are positive integers:

(a) $-\frac{2}{3}$ (b) $\frac{4}{-5}$ (c) $-\frac{3}{7}$ (d) $\frac{5}{-16}$

2. Write each of the following rational numbers in the form $\frac{a}{-b}$ where a and b are positive integers:

(a) $-\frac{3}{7}$ (b) $\frac{-4}{9}$ (c) $-\frac{5}{24}$ (d) $\frac{-7}{20}$

3. Write each of the following rational numbers in the form $\frac{a}{b}$ where a and b are positive integers:

(a) $\frac{-3}{-4}$ (b) $-\frac{-5}{7}$ (c) $\frac{-9}{-10}$ (d) $-\frac{8}{-15}$

4. Write each of the following rational numbers in the form $\frac{a}{b}$ or $-\frac{a}{b}$ where a and b are positive integers.

(a) $\frac{-5}{19}$ (b) $\frac{3}{-25}$ (c) $\frac{-84}{-11}$ (d) $\frac{-678}{5}$

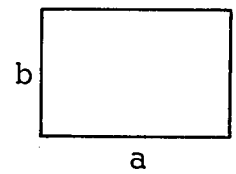
7-6 Properties of Multiplication of Rational Numbers

Do you remember the basic properties of addition of rational numbers? If not, you should revise them here. Do you think that multiplication of rational numbers has similar properties?

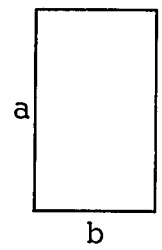
We have seen that if a rectangle has a base a inches and height b inches where a and b are any two positive rational numbers, then the area is $a \times b$ square inches, if we multiply by the method that we have agreed upon.

If a and b are whole numbers

$$a \times b = b \times a.$$



Is this true if a and b are any two positive rational numbers? Of course, because the area of the rectangle is not altered if we turn the rectangle so that the base and height are interchanged. The statement $a \times b = b \times a$ is true for <u>all positive rational numbers</u> a and b.



Why should we stop there? Suppose that we have <u>any</u> two rational numbers. For example, we know that

$$(-\tfrac{4}{5}) \times \tfrac{2}{3} = -(\tfrac{4}{5} \times \tfrac{2}{3}) \qquad \text{(Why?)}$$

and

$$\tfrac{2}{3} \times (-\tfrac{4}{5}) = -(\tfrac{2}{3} \times \tfrac{4}{5}) \qquad \text{(Why?)}$$

Since $\frac{4}{5} \times \frac{2}{3} = \frac{2}{3} \times \frac{4}{5}$ we have $-(\frac{4}{5} \times \frac{2}{3}) = -(\frac{2}{3} \times \frac{4}{5})$ (Why?)

and hence

$$(-\tfrac{4}{5}) \times \tfrac{2}{3} = \tfrac{2}{3} \times (-\tfrac{4}{5}).$$

Show in a similar manner that

$$(-\tfrac{4}{5}) \times (-\tfrac{2}{3}) = (-\tfrac{2}{3}) \times (-\tfrac{4}{5})$$

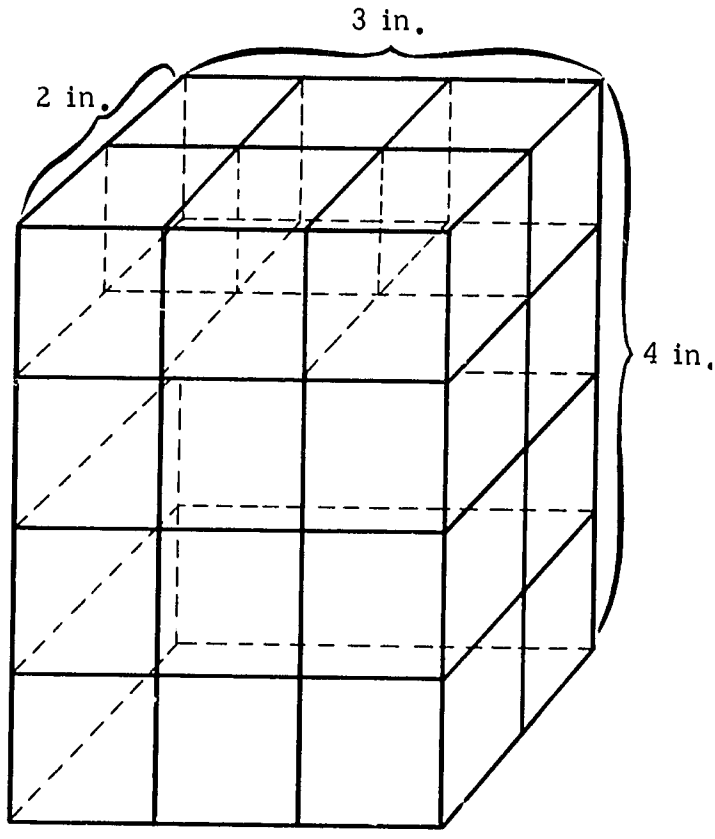
Are you ready to agree now that multiplication of rational numbers is <u>commutative</u>? We can state this property as follows:

For any two rational numbers a and b,
$a \times b = b \times a.$

Shall we have the associative property for the multiplication of rational numbers? Is it true that

$$(a \times b) \times c = a \times (b \times c)?$$

This is surely true if a, b and c are whole numbers. Is it true if they are any positive rational numbers?



An easy way to see that this is true is to consider the volume of a rectangular box. The volume in cubic inches is the area of the base in square inches times the height in inches. In the figure this is $(2 \times 3) \times 4 = 24$ cubic inches. This is the same as $2 \times (3 \times 4)$ cubic inches, because we can use the 3×4 face as the base and 2 inches as the height and obtain the same volume. The same sort of thing will be true no matter what the dimensions of the box are. Therefore,

$(a \times b) \times c = a \times (b \times c)$

for any positive rational numbers a, b and c.

The same conclusion can be drawn if some or all of the numbers a, b and c are negative. Consider the following products

$$\left[(-2) \times 5\right] \times (-4) \text{ and } (-2) \times \left[5 \times (-4)\right]$$

$$(-\tfrac{2}{3}) \times \left[(-\tfrac{8}{5}) \times (-\tfrac{1}{7})\right] \text{ and } \left[(-\tfrac{2}{3}) \times (-\tfrac{8}{5}) \times (-\tfrac{1}{7})\right]$$

Are the products on the left the same as those on the right? The only new feature is the possibility that one product might be negative while the other product is positive. Is this possible? No. We conclude that the <u>associative</u> property holds for all rational numbers.

For any rational numbers a, b, and c,
$(a \times b) \times c = a \times (b \times c)$.

You remember that 0 has the property that $a + 0 = a$ for any rational number a. That is, adding 0 to any rational number gives us that same number again. Is there some number that acts in a similar way for multiplication? That is, is there a rational number b such that, for any rational number a, it is true that

$$a \times b = a?$$

What is suggested by the following equations?

$$2 \times x = 2$$

$$\frac{7}{4} \times x = \frac{7}{4}$$

$$(-5) \times x = -5$$

$$(-\frac{3}{8}) \times x = -\frac{3}{8}$$

Of course, 1 is the value of x which makes each of them true.

For all rational numbers a, $a \times 1 = a$.

We call 1 the identity element for multiplication.

In our study of addition of rational numbers you saw that, for any rational number a, there is a unique rational number b such that $a + b = 0$. What would be the corresponding property for multiplication? We have seen that 1 acts for multiplication much as 0 acts for addition and so we would want, for any rational number a, to be able to find a rational number b such that

$$a \times b = 1.$$

Can we do this? By what rational number can we multiply $\frac{2}{3}$ to obtain the number 1? Suppose that it is $\frac{c}{d}$.

Then we would want

$$\frac{2}{3} \times \frac{c}{d} = 1.$$

Perhaps you see that $c = 3$ and $d = 2$ will work, since

$$\frac{2}{3} \times \frac{3}{2} = \frac{2 \times 3}{3 \times 2} = 1.$$

The number by which we multiply $\frac{2}{3}$ to obtain 1 is $\frac{3}{2}$.

What number b would make $a \times b = 1$ if a is $\frac{2}{9}$? $\frac{-3}{4}$? $\frac{12}{-5}$? $-\frac{6}{7}$? Do you see a pattern? Explain why each of the following is true.

$$\frac{2}{9} \times \frac{9}{2} = 1 \qquad \frac{12}{-5} \times \frac{-5}{12} = 1 \qquad 13 \times \frac{1}{13} = 1$$

$$\frac{-3}{4} \times \frac{4}{-3} = 1 \qquad (-\frac{6}{7}) \times (-\frac{7}{6}) = 1 \qquad (-\frac{1}{5}) \times (-5) = 1$$

The pattern is that if our rational number a is in the form $\frac{c}{d}$, then b can be written as $\frac{d}{c}$ because $\frac{c}{d} \times \frac{d}{c} = \frac{c \times d}{d \times c} = 1$.

There is a restriction which we must not overlook. What about the rational number 0? We know that

$$0 \times b = 0$$

is true for any rational number b, so it can never be true that

$$0 \times b = 1.$$

Thus we cannot say that for <u>any</u> rational number a there is a rational number b such that $a \times b = 1$. We <u>can</u> say the following:

For any rational number a ($a \neq 0$), there is a unique rational number b such that $a \times b = 1$.

We call b the <u>reciprocal</u> of a or the <u>multiplicative inverse</u> of a. Thus

$\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$.

$\frac{-5}{8}$ is the reciprocal of $\frac{8}{-5}$.

$\frac{9}{-4}$ is the reciprocal of $\frac{-4}{9}$.

$\frac{1}{2}$ is the reciprocal of 2.

6 is the reciprocal of $\frac{1}{6}$.

$-\frac{7}{8}$ is the reciprocal of $-\frac{8}{7}$.

4 is the reciprocal of 0•25.

Zero has no reciprocal.

Can you explain all of these reciprocals?

Let us summarize these basic properties relating to multiplication of rational numbers.

1. The <u>commutative property of multiplication</u>.

   For all rational numbers a and b,

   $a \times b = b \times a$.

2. The <u>associative property of multiplication</u>.

   For all rational numbers a, b, and c,

   $(a \times b) \times c = a \times (b \times c)$.

3. The <u>identity property of multiplication</u> or the <u>property of one</u>.

   For all rational numbers a, $a \times 1 = 1 \times a = a$.

4. The <u>inverse property of multiplication</u> or the <u>property of reciprocals</u>.

   For any non-zero rational number a, there is a rational number b such that $a \times b = b \times a = 1$.

   If $a = \frac{c}{d}$, then $b = \frac{d}{c}$.

<u>Problems 7-6</u>

1. Without doing any arithmetic, tell which of the products in the left-hand column below is equal to which of the products in the right-hand column. Tell why you know this. For example, $\left[(-\frac{3}{4}) \times 2\right] \times (-\frac{1}{2}) = (-\frac{3}{4}) \times \left[2 \times (-\frac{1}{2})\right]$ by the associative property of multiplication.

1. (a) $(-2) \times (5 \times 3)$

(b) $4 \times \left[(-3) \times (-5)\right]$

(c) $\left[\frac{2}{3} \times (-\frac{4}{5})\right] \times \frac{7}{10}$

(d) $\frac{3}{5} \times \left[(-2) \times \frac{2}{7}\right]$

(e) $\left[(-\frac{3}{8}) \times (-\frac{2}{9})\right] \times (-\frac{16}{15})$

(f) $\left[(-\frac{3}{10}) \times \frac{2}{9}\right] \times (-\frac{3}{4})$

(g) $\left[(-2) \times 5\right] \times 3$

(h) $\frac{7}{10} \times \left[\frac{2}{3} \times (-\frac{4}{5})\right]$

(i) $(-\frac{16}{15}) \times \left[(-\frac{3}{8}) \times (-\frac{2}{9})\right]$

(j) $\left[\frac{2}{9} \times (-\frac{3}{10})\right] \times (-\frac{3}{4})$

(k) $\left[4 \times (-3)\right] \times (-5)$

(l) $\frac{3}{5} \times \left[\frac{2}{7} \times (-2)\right]$

2. Compute each of the products in Problem 1 and express your answers in lowest terms.

3. $25 \times (73 \times 4) = (25 \times 4) \times 73$ because
$25 \times (73 \times 4) = 25 \times (4 \times 73)$
by the commutative property of multiplication
and $25 \times (4 \times 73) = (25 \times 4) \times 73$
by the associative property of multiplication.

Show in this manner why the following equations are true.

(a) $(5 \times 7) \times 2 = 7 \times (5 \times 2)$

(b) $\left[(-3) \times 9\right] \times (-4) = (-4) \times \left[9 \times (-3)\right]$

(c) $\frac{3}{5} \times (\frac{2}{3} \times \frac{5}{3}) = (\frac{3}{5} \times \frac{5}{3}) \times \frac{2}{3}$

(d) $\left[(-\frac{1}{2}) \times 20\right] \times (-35) = 20 \times \left[(-\frac{1}{2}) \times (-35)\right]$

(e) $(-\frac{3}{4}) \times \left[(-\frac{1}{5}) \times (-\frac{7}{8})\right] = (-\frac{7}{8}) \times \left[(-\frac{1}{5}) \times (-\frac{3}{4})\right]$

4. Find the reciprocal of each of the following rational numbers.

(a) $\frac{4}{9}$	(d) 3	(g) $\frac{1}{2}$	(j) -1
(b) $\frac{-13}{7}$	(e) $-\frac{17}{8}$	(h) $-\frac{1}{3}$	(k) $2 \cdot 5$
(c) $-\frac{5}{-12}$	(f) -2	(i) 1	(l) 5000

5. Why is it permitted to write

$$3 \times \frac{1}{5} \times \frac{1}{3} \times 5 \times 9$$

without any brackets and still have it perfectly clear what number is intended?

6. In Problem 5 we could use the commutative and associative properties repeatedly to change

$$3 \times \frac{1}{5} \times \frac{1}{3} \times 5 \times 9$$

to

$$(3 \times \frac{1}{3}) \times (5 \times \frac{1}{5}) \times 9.$$

Why is this product equal to the number 9?

7. Do as is suggested by Problem 6 to find the following products simply.

(a) $\frac{3}{4} \times 13 \times 2 \times \frac{4}{3} \times \frac{1}{2}$

(b) $(-2) \times \frac{7}{8} \times 8 \times (-5)$

(c) $\frac{9}{7} \times (-\frac{5}{4}) \times (-\frac{7}{9}) \times 2 \times 4$

(d) $\frac{15}{4} \times (-3) \times (-4) \times \frac{7}{15} \times \frac{1}{3}$

(e) $0 \cdot 125 \times 17 \times 8$

8. For each of the following incomplete equations find the number which will make the equation true.

8. (a) $(-3) \times$ _____ $= (-3)$

(b) $\frac{5}{17} \times$ _____ $= 1$

(c) $1 \times (-\frac{2}{5}) =$ _____

(d) $(-\frac{1}{13}) \times$ _____ $= 1$

(e) _____ $\times (-\frac{17}{3}) = -\frac{17}{3}$

(f) $(-\frac{27}{5}) \times (-\frac{5}{27}) =$ _____

(g) $23 \times$ _____ $= 1$

(h) $0{\cdot}58 \times$ _____ $= 0{\cdot}58$

9. In each part of Problem 8 tell what basic property is illustrated.

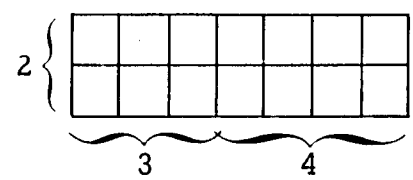
7-7 The Distributive Property

There is one more basic property of the arithmetic of rational numbers. This one concerns both addition and multiplication.

Consider the two expressions

$$2 \times (3 + 4) \text{ and } (2 \times 3) + (2 \times 4)$$

The first expression gives the number of square inches in a rectangle of base $(3 + 4)$ inches and height 2 inches. The second expression gives the sum of the units of area for the rectangle with base 3 inches and height 2 inches and the units of area for the rectangle with base 4 inches and height 2 inches. The results are the same.



Of course we can verify by arithmetic that $2 \times (3 + 4) = (2 \times 3) + (2 \times 4)$ is a true statement:

$$2 \times (3 + 4) = 2 \times 7 = 14$$

and

$$(2 \times 3) + (2 \times 4) = 6 + 8 = 14.$$

The pattern of the two cases is that in one case we added 3 and 4 first and then multiplied the result by 2; in the other case we first multiplied 3 and 4 each by 2 and then added the results. The result was 14 in each case. Does this pattern continue to give equal numbers?

We can generalize the geometrical argument to see that

$$a \times (b + c) = (a \times b) + (a \times c)$$

for any whole numbers a, b and c. Indeed, this is also true for any positive rational numbers a, b, and c, because, as we have seen, when we multiply fractions as we do, the area of a rectangle is still the length of the base times the height.

We shall assume that this continues to be true if a, b, and c are any three rational numbers. We can test the conclusion with examples.

$$\text{Is } (-\tfrac{3}{4}) \times \left[\tfrac{5}{8} + (-\tfrac{3}{2})\right] = \left[(-\tfrac{3}{4}) \times \tfrac{5}{8}\right] + \left[(-\tfrac{3}{4}) \times (-\tfrac{3}{2})\right]?$$

You should know that the number represented on each side of the equals sign is $\frac{21}{32}$. In the problems you will be asked to test out a few other cases.

We shall agree to accept the following basic property:

If a, b, and c are any three rational numbers, then $a \times (b + c) = (a \times b) + (a \times c)$.

This property is called the <u>distributive</u> property for multiplication with respect to addition. We shall usually refer to it simply as the distributive property. Observe carefully that the expression on the left indicates a <u>product</u>. The expression on the right indicates a <u>sum</u>. The expression on the left says to add b and c before doing the product. The expression on the right says to multiply b and c by a before doing the sum. The symbol "=" says that you obtain the same number whichever way you do it.

Problems 7-7

1. Express each of the following in the form $(a \times b) + (a \times c)$. That is, change the products to sums by the distributive property. For example, $\frac{1}{2} \times \left[(-\frac{1}{4}) + \frac{3}{2}\right] = \left[\frac{1}{2} \times (-\frac{1}{4})\right] + (\frac{1}{2} \times \frac{3}{2})$.

   (a) $2 \times \left[(-3) + 5\right]$

   (b) $\frac{4}{9} \times (\frac{3}{5} + \frac{10}{7})$

   (c) $\frac{2}{3} \times \left[(-\frac{3}{4}) + \frac{7}{8}\right]$

   (d) $(-\frac{4}{9}) \times \left[\frac{3}{5} + (-\frac{1}{3})\right]$

2. Express each of the following in the form $a \times (b + c)$. That is, change the sums to products by the distributive property.

   (a) $\left[2 \times (-5)\right] + \left[2 \times (-3)\right]$

   (b) $(\frac{1}{2} \times \frac{3}{5}) + (\frac{1}{2} \times \frac{7}{10})$

   (c) $\left[(-\frac{2}{3}) \times \frac{1}{5}\right] + \left[(-\frac{2}{3}) \times \frac{3}{7}\right]$

   (d) $\left[\frac{7}{5} \times (-\frac{3}{2})\right] + \left[\frac{7}{5} \times (-\frac{1}{2})\right]$

3. In Problems 1. (c) and 2. (c) verify by actual computation that the expression you obtained and the given expression represent the same number.

7-8 Applications of the Basic Properties

We saw a variety of uses of the basic properties of addition. The basic properties of multiplication of rational numbers and the distributive property are also useful in many different ways. Here are some examples of such applications.

Application 1. The usual method for multiplying whole numbers depends heavily on these properties. For example, consider the problem of multiplying 24 by 7. Recall that $24 = 20 + 4$. Then

$$\begin{aligned} 7 \times 24 &= 7 \times (20 + 4) \\ &= (7 \times 20) + (7 \times 4) \qquad \text{(Why?)} \\ &= 140 + 28 = 168 \end{aligned}$$

We can now see the reason behind the mechanical "rule" which is sometimes given for multiplying 24 by 7: "7×4 is 28; put down 8 and carry 2", and so on. We have really just used the distributive property. Perhaps you recognize the use of some other properties of multiplication and addition hidden in the last steps.

Application 2. To calculate $(52 \times 125) + (52 \times 75)$ by first multiplying 52 by 125, then multiplying 52 by 75, and then adding is far more work than using the distributive property first.

$$(52 \times 125) + (52 \times 75) = 52 \times (125 + 75) = 52 \times 200 = 10{,}400.$$

Application 3. We can use these properties to take another look at the multiplication of positive and negative rational numbers. Instead of defining multiplication of rational numbers and observing that the properties seem to be true, it is logically possible to assume that the properties are true and prove that multiplication involving negative rational numbers is forced to be the way it is.

Suppose that we already know how to multiply positive rational numbers and that $a \times 0 = 0$ for all rational numbers. Watch how the properties support our reasoning step by step, first to prove that $3 \times (-2) = -6$.

We can start by making use of the meaning of -2 in the inverse property of addition.

$$2 + (-2) = 0$$

Since this gives

$$3 \times \left[2 + (-2)\right] = 3 \times 0$$

and since by the <u>distributive property</u>

$$3 \times \left[2 + (-2)\right] = (3 \times 2) + \left[3 \times (-2)\right],$$

we have

$$(3 \times 2) + \left[3 \times (-2)\right] = 3 \times 0$$

Since $3 \times 2 = 6$ and $3 \times 0 = 0$, we now have

$$6 + \left[3 \times (-2)\right] = 0$$

Since by the <u>inverse property of addition</u> -6 is the <u>only</u> number which added to 6 will give 0, it must be true that

$$3 \times (-2) = -6.$$

Now to prove that $(-2) \times 3 = -6$, all we need is the <u>commutative property of multiplication</u>.

$$(-2) \times 3 = 3 \times (-2) = -6.$$

Finally we want to prove that $(-2) \times (-3) = 6$. Supply the reasons for the following steps.

$$3 + (-3) = 0$$

$$(-2) \times \left[3 + (-3)\right] = (-2) \times 0$$

$$(-2) \times \left[3 + (-3)\right] = \left[(-2) \times 3\right] + \left[(-2) \times (-3)\right]$$

$$\left[(-2) \times 3\right] + \left[(-2) \times (-3)\right] = (-2) \times 0$$

$$(-6) + \left[(-2) \times (-3)\right] = 0$$

Hence $\left[(-2) \times (-3)\right]$ must be the opposite of -6, that is,

$$(-2) \times (-3) = 6.$$

<u>Problems 7-8</u>

1. As in Application 1, show the reasoning behind the multiplications 6×32 and 9×54.

2. As in Application 2, find an easy way of doing the following calculations.

(a) $(124 \times 12) + (124 \times 88)$

(b) $(\frac{5}{7} \times \frac{3}{4}) + (\frac{5}{7} \times \frac{1}{4})$

(c) $\left[(-25) \times (298)\right] + \left[(-25) \times (102)\right]$

(d) $(27 \times 13) + \left[27 \times (-3)\right]$

(e) $12 \times (\frac{5}{6} + \frac{7}{4})$

(f) $\left[8 \times (-\frac{4}{5})\right] + \left[8 \times (-\frac{6}{5})\right]$

(g) $(-\frac{14}{3}) \times (\frac{9}{7} + \frac{3}{2})$

(h) $\left[(-39) \times (-\frac{17}{15})\right] + \left[(-39) \times (-\frac{8}{15})\right]$

3. Use the reasoning shown in Application 3 to prove that

(a) $(-4) \times 5 = -20$

(b) $6 \times (-7) = -42$

(c) $(-4) \times (-5) = 20$

(d) $(-6) \times (-7) = 42$

7-9 Revision of Properties of the Rational Number System

Let us collect the basic properties of the arithmetic of rational numbers. These are the properties which we shall use to explain a large part of what we do in algebra.

We begin by mentioning two basic properties that we have used without actually stating them. The first of these is the property that when we add two rational numbers the sum is always a rational number. The second is that when we multiply two rational numbers the product is always a rational number. We say that the set of rational numbers is <u>closed</u> under both addition and multiplication.

Let us use set language to write these properties as well as the other basic properties. Recall that, if A is a set, then $x \in A$ means that x is a member of the set A. We let R be the set of rational numbers.

1. <u>Closure property of addition</u>

   If $a \in R$ and $b \in R$, then $(a + b) \in R$.

2. <u>Closure property of multiplication</u>

   If $a \in R$ and $b \in R$, then $(a \times b) \in R$.

3. <u>Commutative property of addition</u>

   If $a \in R$ and $b \in R$, then $a + b = b + a$.

4. <u>Commutative property of multiplication</u>

   If $a \in R$ and $b \in R$, then $a \times b = b \times a$.

5. <u>Associative property of addition</u>

   If $a \in R$, $b \in R$, and $c \in R$, then

   $$(a + b) + c = a + (b + c).$$

6. <u>Associative property of multiplication</u>

   If $a \in R$, $b \in R$, and $c \in R$, then

   $$(a \times b) \times c = a \times (b \times c).$$

7. <u>Distributive property of multiplication with respect to addition</u>

   If $a \in R$, $b \in R$, and $c \in R$, then

   $$a \times (b + c) = (a \times b) + (a \times c).$$

8. <u>Identity property of addition</u> or <u>property of zero</u>

   There is a unique number 0 such that $0 \in R$ and, if $a \in R$, then $a + 0 = 0 + a = a$.

9. <u>Identity property of multiplication</u> or <u>property of one</u>

   There is a unique number 1 such that $1 \in R$ and, if $a \in R$, then $a \times 1 = 1 \times a = a$.

10. <u>Inverse property of addition</u> or <u>property of opposites</u>

If $a \in R$, then there is a unique number b such that $b \in R$ and $a + b = b + a = 0$.

11. <u>Inverse property of multiplication</u> or <u>property of reciprocals</u>

If $a \in R$ and $a \neq 0$, then there is a unique number b such that $b \in R$ and $a \times b = b \times a = 1$.

Recall now the clock arithmetic of Chapter 6. Suppose that we let $S = \{0, 1, 2, 3, 4\}$ and that we consider the arithmetic of a clock with five "hours". Do you see that the statements 1-11 just given are almost the same as the statements 1-11 given in Chapter 6? All you have to do to make the statements above the same as those in Chapter 6 is to replace "R" by "S", and "+" by "$\oplus$" and "×" by "$\otimes$".

The sets R and S are certainly very different and the arithmetic of the two sets is different, but the two arithmetics have much in common, namely properties 1-11. Systems in which properties 1-11 hold are so important that mathematicians have given them a special name. They are called <u>fields</u>. You have seen that the rational numbers form a field under addition and multiplication and also that the "clock" numbers 0, 1, 2, 3, 4 form a field under "clock" addition and multiplication.

If these are fields, are there systems of numbers which are <u>not</u> fields? Do you remember exploring the clock arithmetic which uses the set of numbers $\{0, 1, 2, 3\}$? You found the following multiplication table.

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

What is the identity element for multiplication in this system? If the identity element is 1, then if the system is a field there should be a number b in the system for which $2 \times b = 1$. Is there such a number? Look at your multiplication table. Since there is not, the system is not a field. The inverse property of multiplication fails.

You saw also that some peculiar operations such as $a \circ b = (2 \times a) + b$ and $a * b = (3 \otimes a) \oplus b$ do not have the property of commutativity. Such operations could not be used in a field.

We do not need to have peculiar operations, though, to fail to have a field. Consider the set of all the whole numbers with the operations of ordinary addition and multiplication.

1. Does this system have closure for both addition and multiplication? Yes. (Why?)
2. Does it have commutativity? Yes. (Why?)
3. Does it have associativity? Yes. (Why?)
4. Does the distributive property hold? Yes. (Why?)
5. Is there an identity element for both addition and multiplication? Yes. (What are they?)
6. Is there an additive inverse (opposite) for every element? No! There is no whole number b, for example, for which $8 + b = 0$.

The system therefore is not a field. The inverse property of multiplication also fails to hold. Why?

<u>Problems 7-9</u>

Examine the following systems and decide for each one whether or not it is a field. For each one which is not a field, indicate which properties are not true.

1. The set of positive integers with the operations of ordinary addition and multiplication.

2. The set of all integers with the operations of ordinary addition and multiplication.

3. The set of all negative integers with the operations of ordinary addition and multiplication.

4. The set of positive rational numbers and zero with the operations of ordinary addition and multiplication.

5. The set $\{0, 1\}$ with the operations of clock addition and clock multiplication.

6. The set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ with the operations of clock addition and clock multiplication.

7. The set $\{a, b, c\}$ with the operations defined by these tables:

+	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

×	a	b	c
a	a	a	a
b	a	c	b
c	a	b	c

7-10 Subtraction of Rational Numbers

You already know how to do some subtraction problems with rational numbers. For example, you know that $5 - 2 = 3$, $5 - 5 = 0$, $\frac{9}{5} - \frac{3}{5} = \frac{9 - 3}{5} = \frac{6}{5}$, $\frac{7}{12} - \frac{2}{9} = \frac{7 \times 3}{12 \times 3} - \frac{2 \times 4}{9 \times 4} = \frac{21}{36} - \frac{8}{36} = \frac{21 - 8}{36} = \frac{13}{36}$.

Subtraction is related to addition. The subtractions that you have just seen can be checked by doing additions as shown below:

Subtraction	Check by addition
$5 - 2 = 3$	$3 + 2 = 5$
$5 - 5 = 0$	$0 + 5 = 5$
$\frac{9}{5} - \frac{3}{5} = \frac{6}{5}$	$\frac{6}{5} + \frac{3}{5} = \frac{9}{5}$
$\frac{7}{12} - \frac{2}{9} = \frac{13}{36}$	$\frac{13}{36} + \frac{2}{9} = \frac{7}{12}$

In fact, the check is merely applying the definition of subtraction which you had in arithmetic.

$5 - 2 = a$ means that $a + 2 = 5$

$\frac{9}{5} - \frac{3}{5} = a$ means that $a + \frac{3}{5} = \frac{9}{5}$

and so on.

In general, $x - y = a$ means that $a + y = x$. The difference a is that number which must be added to y to give x.

Consider the following <u>sums</u>.

$$5 + (-2) = 3$$
$$5 + (-5) = 0$$
$$\frac{9}{5} + (-\frac{3}{5}) = \frac{6}{5}$$
$$\frac{7}{12} + (-\frac{2}{9}) = \frac{13}{36}$$

We observe that $5 + (-2)$ gives the same result as $5 - 2$ and we can make a similar observation in each of the other examples. In fact, we can show that

$$5 - 2 = 5 + (-2)$$

without doing any computation at all, by using the definition of subtraction and some of our basic properties.

$5 - 2 = \left[5 + (-2)\right]$ will be true <u>if</u> $\left[5 + (-2)\right] + 2 = 5$ (Why?)

We know, however, that

$$\left[5 + (-2)\right] + 2 = 5 + \left[(-2) + 2\right] \quad \text{(By what property?)}$$

$$= 5 + 0 \quad \text{(By what property?)}$$

$$= 5 \quad \text{(By what property?)}$$

Since $\left[5 + (-2)\right] + 2 = 5$ <u>is</u> true, we have proved that 5 <u>minus</u> 2 is the same number as 5 plus the opposite of 2.

Similarly $\frac{9}{5} - \frac{3}{5} = \frac{9}{5} + (-\frac{3}{5})$

because $\left[\frac{9}{5} + (-\frac{3}{5})\right] + \frac{3}{5} = \frac{9}{5}$. (Explain.)

In the set of positive rational numbers the subtraction $3 - 5$ could not be done. Will the idea we are exploring give us a way to do this subtraction in the set of all rational numbers? Could we say that

$$3 - 5 = 3 + (-5)?$$

Let us see. Is it true that $\left[3 + (-5)\right] + 5 = 3$? Since this <u>is</u> true we can say that $3 - 5 = 3 + (-5)$ is true and we have found that $3 - 5 = -2$. (Where did the -2 come from?)

What about $2 - (-3)$? To do as above we would write "2 <u>minus</u> (-3)" as "2 <u>plus the opposite</u> of (-3)". What is the opposite of (-3)? Can we write $-(-3)$ as 3?

$$2 - (-3) = 2 + 3 = 5$$

because $(2 + 3) + (-3) = 2$.

Can we perhaps generalize this procedure for subtraction to apply to <u>all</u> rational numbers? This would be a great convenience because it would mean that any subtraction can be rewritten in terms of addition and we do not have to learn any new rules for the process of subtraction.

We <u>can</u> generalize the idea because the reasoning we did in the examples above did not at any point depend on what particular numbers we were subtracting. Suppose a and b are <u>any</u> two rational numbers. We wish to prove that

$$a - b = a + (-b).$$

This will be true by the definition of subtraction if we can show that

$$\left[a + (-b)\right] + b = a.$$

We know, however, that

$$\begin{aligned} \left[a + (-b)\right] + b &= a + \left[(-b) + b\right] && \text{(Why?)} \\ &= a + 0 && \text{(Why?)} \\ &= a && \text{(Why?)} \end{aligned}$$

Hence $a - b = a + (-b)$ is true.

Since we put no restriction on a and b except that they are rational numbers, we see that it is always permitted to rewrite "<u>a minus b</u>" as "<u>a plus the opposite of b</u>".

> Subtracting the rational number b from the rational number a is the same as <u>adding</u> the opposite of b to a. In symbols: $a - b = a + (-b)$.

<u>Problems 7-10</u>

1. Perform the following subtractions and check your answer by addition. For example, $2 - (-3) = 2 + 3 = 5$; check: $5 + (-3) = 2$.

1. (a) $7 - 3$
   (b) $5 - 8$
   (c) $(-5) - 8$
   (d) $(-5) - 3$
   (e) $8 - (-3)$
   (f) $5 - (-7)$
   (g) $(-6) - (-8)$
   (h) $(-8) - (-4)$
   (i) $(-\frac{3}{4}) - \frac{7}{4}$
   (j) $(-\frac{4}{5}) - (-\frac{4}{5})$
   (k) $\frac{3}{2} - (-\frac{2}{3})$
   (l) $\frac{5}{9} - \frac{4}{3}$
   (m) $(-\frac{7}{8}) - \frac{5}{6}$
   (n) $3\frac{2}{3} - 1\frac{7}{8}$
   (o) $4\frac{2}{7} - (-\frac{5}{14})$
   (p) $(-0.84) - (-0.46)$
   (q) $864 - (-136)$

2. Is the set of rational numbers closed under the operation of subtraction?

3. Is subtraction of rational numbers a commutative operation? Is it an associative operation?

4. Is multiplication of rational numbers distributive over subtraction? That is, is it true that for all rational numbers a, b, and c, $a \times (b - c) = (a \times b) - (a \times c)$?

7-11 Division of Rational Numbers

We looked at the meaning of subtraction in terms of addition to help us find out how to subtract rational numbers. In the same way we can look at the meaning of division in terms of multiplication to help us to find out how to divide one rational number by another.

What do we mean when we say $6 \div 2 = 3$? Do we not mean that $3 \times 2 = 6$? In general, if a, b, and c are numbers, and $b \neq 0$, then

$$c \div b = a$$

means the same as

$$a \times b = c.$$

Thus

$$6 \div 2 = 3 \text{ because } 3 \times 2 = 6$$

$$\frac{6}{5} \div 3 = \frac{2}{5} \text{ because } \frac{2}{5} \times 3 = \frac{6}{5}$$

$$\frac{8}{3} \div \frac{2}{3} = 4 \text{ because } 4 \times \frac{2}{3} = \frac{8}{3}$$

Consider the following <u>products</u>.

$$6 \times \frac{1}{2} = 3$$

$$\frac{6}{5} \times \frac{1}{3} = \frac{6 \times 1}{5 \times 3} = \frac{2 \times 3}{5 \times 3} = \frac{2}{5}$$

$$\frac{8}{3} \times \frac{3}{2} = \frac{8 \times 3}{3 \times 2} = \frac{4 \times 2 \times 3}{2 \times 3} = 4$$

Notice that $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{2}$ are the reciprocals of the numbers 2, 3, $\frac{2}{3}$ in the divisions above. Notice also that the answers are the same as the corresponding answers above.

Let us show that $\frac{6}{5} \div 3 = \frac{6}{5} \times \frac{1}{3}$ without any computation by using the definition of division.

$\frac{6}{5} \div 3 = (\frac{6}{5} \times \frac{1}{3})$ will be true <u>if</u> $(\frac{6}{5} \times \frac{1}{3}) \times 3 = \frac{6}{5}$ (Why?)

We know, however, that

$$(\frac{6}{5} \times \frac{1}{3}) \times 3 = \frac{6}{5} \times (\frac{1}{3} \times 3) \qquad \text{(By what property?)}$$

$$= \frac{6}{5} \times 1 \qquad \text{(By what property?)}$$

$$= \frac{6}{5} \qquad \text{(By what property?)}$$

Since $(\frac{6}{5} \times \frac{1}{3}) \times 3 = \frac{6}{5}$ <u>is</u> true, we know that

$$\frac{6}{5} \div 3 = \frac{6}{5} \times \frac{1}{3} \text{ is true.}$$

We have proved that $\frac{6}{5}$ <u>divided</u> by 3 is the same number as $\frac{6}{5}$ <u>multiplied by the reciprocal</u> of 3.

You probably already see the pattern

$$\frac{8}{3} \div \frac{2}{3} = \frac{8}{3} \times \frac{3}{2}$$

because $$(\frac{8}{3} \times \frac{3}{2}) \times \frac{2}{3} = \frac{8}{3}.$$ (Explain.)

Naturally we shall want to extend this pattern to all rational numbers.

$$(-\frac{3}{7}) \div (\frac{4}{5}) = (-\frac{3}{7}) \times \frac{5}{4} = -\frac{15}{28}$$

because $$\left[(-\frac{3}{7}) \times \frac{5}{4}\right] \times \frac{4}{5} = -\frac{3}{7}$$

or $$(-\frac{15}{28}) \times \frac{4}{5} = -\frac{3}{7}.$$ (Verify this one by computation.)

What about $(-\frac{1}{5}) \div (-\frac{3}{20})$? What is the reciprocal of $(-\frac{3}{20})$? Since $(-\frac{3}{20}) \times (-\frac{20}{3}) = 1$ (Why?), we can use $(-\frac{20}{3})$ as the reciprocal of $(-\frac{3}{20})$. (Why?)

$$(-\frac{1}{5}) \div (-\frac{3}{20}) = (-\frac{1}{5}) \times (-\frac{20}{3}) = \frac{4}{3}$$

Is it true that $\frac{4}{3} \times (-\frac{3}{20}) = -\frac{1}{5}$?

Let us generalize this viewpoint of division to all rational numbers (except zero).

Suppose a and b are any two rational numbers ($b \neq 0$). We wish to prove that

$$a \div b = a \times \frac{1}{b}.$$

This will be true by the definition of division if we can show that

$$(a \times \frac{1}{b}) \times b = a.$$

It is true, though, that

$$\begin{aligned} (a \times \frac{1}{b}) \times b &= a \times (\frac{1}{b} \times b) && \text{(Why?)} \\ &= a \times 1 && \text{(Why?)} \\ &= a && \text{(Why?)} \end{aligned}$$

Hence $$a \div b = a \times \frac{1}{b} \text{ is true.}$$

In words, a <u>divided</u> by b is the same number as a <u>multiplied by the reciprocal</u> of b.

> <u>Dividing</u> the rational number a by the rational number b ($b \neq 0$) is the same as <u>multiplying</u> a by the reciprocal of b. In symbols: $a \div b = a \times \frac{1}{b}$.

We can now express any division of rational numbers in terms of multiplication and we shall not have to learn a new process to do division.

Since $a \div b = a \times \frac{1}{b}$

and since $a \times \frac{1}{b} = \frac{a \times 1}{b} = \frac{a}{b}$,

we see that $a \div b$ and $\frac{a}{b}$ represent the same number.

We can use fractions to represent division. Thus $\frac{3}{8}$ means $3 \div 8$ and $\frac{\frac{3}{5}}{\frac{2}{3}}$ means $\frac{3}{5} \div \frac{2}{3}$. In the future we shall use the fraction form more than we shall use the symbol "÷".

<u>Problems 7-11</u>

1. Perform the following divisions.

Examples: $$\frac{4}{9} \div \left(-\frac{7}{6}\right) = \frac{4}{9} \times \left(-\frac{6}{7}\right) = -\frac{4 \times 6}{9 \times 7}$$

$$= -\frac{4 \times 3 \times 2}{3 \times 3 \times 7} = -\frac{8}{21}$$

$$\left(-\frac{5}{21}\right) \div (-50) = \left(-\frac{5}{21}\right) \times \left(-\frac{1}{50}\right) = \frac{5 \times 1}{21 \times 50}$$

$$= \frac{5 \times 1}{21 \times 5 \times 10} = \frac{1}{210}$$

1. (a) $\frac{2}{3} \div \frac{4}{7}$

(b) $(-\frac{3}{5}) \div \frac{9}{5}$

(c) $(-\frac{8}{7}) \div (-\frac{4}{3})$

(d) $1 \div \frac{7}{12}$

(e) $(-\frac{18}{5}) \div \frac{3}{10}$

(f) $(-5) \div (-\frac{3}{4})$

(g) $\frac{9}{2} \div (-2)$

(h) $(-\frac{3}{4}) \div 5$

(i) $(-\frac{3}{7}) \div (-3)$

(j) $27 \div (-\frac{15}{7})$

2. Is division of rational numbers a commutative operation? Is it an associative operation?

3. Is division of rational numbers distributive over addition? That is, is it true that for all rational numbers a, b, and c we have $a \div (b + c) = (a \div b) + (a \div c)$? (Assume that $b \neq 0$, $c \neq 0$, and $b + c \neq 0$.)

7-12 Revision Problems

1. Perform the following computations. Look for ways to shorten your work by use of the commutative, associative, and distributive properties. Express your answers in lowest terms.

(a) $\frac{3}{-7} + (-\frac{5}{3})$

(b) $\frac{3}{-7} - (-\frac{5}{3})$

(c) $\frac{0}{9} \cdot \frac{-4}{-6}$

(d) $\frac{-6}{-3} - \frac{2}{(-3)}$

(e) $\left[\frac{2}{3} + \frac{-3}{10}\right]$

(f) $\frac{-3}{4} \div \left[\frac{2}{3} + \frac{-3}{10}\right]$

(g) $\left[\frac{1}{2} - \frac{2}{3}\right] \div \left[\frac{3}{4} + \frac{-1}{2}\right]$

(h) $\left[\frac{-5}{3} \times \frac{6}{-10}\right] \div \left[\frac{5}{4} - \frac{2}{8}\right]$

(i) $\left[\frac{2}{3} \div \frac{-4}{5}\right] \times \frac{-4}{5}$

(j) $\left[(-\frac{1}{5}) \div \frac{2 \times 4}{1 - 6}\right] \times \frac{8}{-5}$

1. (k) $\left[(-\frac{2}{3}) \div \frac{5}{7}\right] \times \frac{-1}{2}$

(l) $(-\frac{2}{3}) \div \left[\frac{5}{7} \times \frac{-1}{2}\right]$

(m) $\left[(-\frac{2}{3}) \times (\frac{-4}{5})\right] + \left[(-\frac{2}{3}) \times \frac{4}{5}\right]$

(n) $\left[\frac{-2}{3} + \frac{9}{16}\right] + \frac{-2}{-3}$

(o) $\left[\frac{-5}{-3} \times \frac{-3}{17}\right] \div \frac{5}{-9}$

(p) $\frac{-5}{-3} \times \left[\frac{-3}{17} \div \frac{5}{-9}\right]$

(q) $(\frac{-3}{5} + \frac{-4}{15}) - (\frac{4}{-15} + \frac{3}{-5})$

(r) $(\frac{-9}{5} \times \frac{27}{35}) \times (\frac{-5}{-9} \times \frac{-35}{27})$

2. Do you think that $(a \times b) \div c = a \times (b \div c)$ for all rational numbers a, b, and c with $c \neq 0$? Why? Try writing the divisions in fraction form.

3. Do you think that $(a \div b) \times c = a \div (b \times c)$ for all rational numbers a, b, and c with $b \neq 0$ and $c \neq 0$? Why?

4. It is true that $(a + b) \div c = (a \div c) + (b \div c)$ for all rational numbers a, b, and c with $c \neq 0$? If you rewrite this in fraction form, you have

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}.$$

Do you remember seeing this before? Let us see whether we can now prove it. Supply the reasons for the following steps.

$$\begin{aligned} \frac{a + b}{c} &= (a + b) \times \frac{1}{c} && \text{(Why?)} \\ &= (a \times \frac{1}{c}) + (b \times \frac{1}{c}) && \text{(Why?)} \\ &= \frac{a}{c} + \frac{b}{c} && \text{(Why?)} \end{aligned}$$

CHAPTER 8

OPEN SENTENCES

8-1 Statements

Consider the sentence $8 + 5 = 13$. This is a sentence which relates numbers. It tells us that $(8 + 5)$ and 13 are numerals for the same number. $8 + 5 = 13$ is a true sentence, whereas the sentence $8 + 6 = 13$ is not true. We call the latter sentence a false sentence. We see that each of these two sentences is either true or false; such sentences are called statements. A statement is a sentence which is either true or false, but not both. Most of the time we shall be interested in statements that are true.

Many mathematical statements, such as $8 + 5 = 13$, involve the symbol "=", meaning "is" or "is equal to". There are other symbols which we can use to relate numbers. For example, the symbol "$\neq$" means "is not" or "is not equal to". Thus $8 + 6 \neq 13$ is a true statement, whereas $8 + 5 \neq 13$ is a false statement. Do you agree?

We have already learnt the meaning of the symbols "<" (is less than); ">" (is greater than); "$\leq$" (is less than or equal to); and "$\geq$" (is greater than or equal to). We can also use these symbols to make statements. For example, $5 \geq 3 + 1$, $\frac{-2}{3} < \frac{1}{3}$, $0 \geq \frac{-4}{5}$, and $-7 > -9\frac{1}{5}$ are all true statements, whereas $5 < 3 + 1$, $\frac{-2}{3} \geq \frac{1}{3}$, $0 \leq -\frac{2}{3}$, and $-7 > -5$ are all false statements. Do you agree?

Problems 8-1

1. Which of these statements are true? Which are not true?

(a) $4 + 8 = 10 + 5$

(b) $-\frac{2}{3} + \frac{2}{3} \neq -5 + 5$

(c) $\frac{12}{13} + \frac{14}{13} \geq 2$

(f) $\frac{3}{4} - \frac{1}{4} < -\frac{1}{2}$

(g) $\frac{-2}{3} - \left(-\frac{1}{3}\right) > -\frac{1}{3}$

(h) $\left(\frac{4}{7}\right)\left(\frac{7}{8}\right) \neq \frac{1}{2}$

1. (d) $\frac{-2}{3} - \frac{4}{3} \leq -\frac{2}{3}$ (i) $\left(-\frac{3}{4}\right)\left(\frac{1}{2}\right) \leq -\frac{1}{2}$

(e) $\left(-\frac{4}{5}\right)\left(-\frac{5}{6}\right) \geq \frac{4}{5}$ (j) $\left(-\frac{1}{2}\right)\left(\frac{2}{3}\right) \geq \frac{1}{2}$

2. Which of these statements are true? Which are false?

(a) $48 \cdot 7 \times 93 \cdot 8 = 93 \cdot 8 \times 48 \cdot 7$

(b) $48 \cdot 7 + 0 > 48 \cdot 7$

(c) $(48 \cdot 7 + 26 \cdot 9) + 93 \cdot 8 \neq 48 \cdot 7 + (26 \cdot 9 + 93 \cdot 8)$

(d) $48 \cdot 7 + 93 \cdot 8 \geq 93 \cdot 8 + 48 \cdot 7$

(e) $48 \cdot 7\,(26 \cdot 9 + 93 \cdot 8) < \left[48 \cdot 7 \times 26 \cdot 9\right] + \left[48 \cdot 7 \times 93 \cdot 8\right]$

(f) $48 \cdot 7 \times 1 \neq 48 \cdot 7$

(g) $(48 \cdot 7 \times 26 \cdot 9) \times 93 \cdot 8 \leq 48 \cdot 7 \times (26 \cdot 9 \times 93 \cdot 8)$

(h) $48 \cdot 7 + (-48 \cdot 7) = 0$

(i) $48 \cdot 7 \times \frac{1}{48 \cdot 7} > 1$

8-2 Variable

In the above paragraph we looked at sentences which related numbers, and since in each case the sentence was either true or false, we called the sentences statements. Now consider the sentence $\square + 2 = 5$. Is this sentence true? Is it false? Of course you cannot say until you know what the symbol $\square$ means. If we agree that the box represents the number $-\frac{3}{4}$, then the sentence is false. If $\square$ represents 3, the sentence is true. Is the sentence true if $\square$ represents 10? -3? How many possible numbers are there which $\square$ might represent? What if $\square$ represents "rectangle"? Is the sentence true or false? We see that the sentence $\square + 2 = 5$ becomes a statement if a numeral is substituted for $\square$, but is meaningless if something else is substituted for $\square$.

A symbol such as $\square$ which is used to represent any member of a given set is called a <u>variable</u>. In most of our work in algebra the set whose members the variable represents is usually a set of numbers; however in other connections we may want to speak of variables which represent members of a set of points, a set of lines, a set of geometric figures, etc. In this chapter it is understood that the sets whose members the variables represent are sets of numbers. Various symbols such as Δ, $\Diamond$, $\square$, or more often letters of the alphabet such as x, y, and z, are used as variables.

<u>Problems 8-2</u>

In each of the following sentences, decide whether the sentence is true if the variable has the suggested value.

1. $7 + \square = 12$; let $\square$ be 5.
2. $-\frac{2}{3} + x \neq -1$; let x be $-\frac{1}{3}$.
3. $y + (-3) = 2$; let y be -1.
4. $t + 9 \neq 11$; let t be $-\frac{2}{5}$.
5. $5 + \Delta < -\frac{3}{4}$; let Δ be $-5\frac{3}{4}$.
6. $\Diamond - \frac{2}{3} \geq 0$; let $\Diamond$ be $\frac{2}{3}$; let $\Diamond$ be $\frac{1}{3}$.
7. $\left[2 \times m\right] - 3 = 5$; let m be 4; let m be 5.
8. $\left[\frac{2}{3} \times n\right] + \frac{3}{4} = -\frac{1}{4}$; let n be $-\frac{3}{4}$; let n be $-\frac{3}{2}$.
9. $\frac{4}{5} - x = -\frac{1}{5}$; let $x = -1$; let $x = 1$.
10. $\left[3 \times y\right] - \left(-\frac{3}{4}\right) = \frac{9}{4}$; let $y = \frac{1}{2}$; let $y = \frac{3}{8}$.

8-3 Domain of the Variable

In the problems above, we considered sentences of the form $7 + \square = 12$, and were asked if the sentences were true for specified values of $\square$. If $\square$

represents 5, we agree that the sentence is true. If $\square$ represents 10, the sentence is false. How do we know what objects the variable may represent? In this chapter we have agreed that the variables represent members of a set of numbers; however the particular set of numbers has not been specified for the symbol $\square$. $\square$ might represent any rational number. Of those, the number 5 makes the sentence true and all the others make the sentence false.

Suppose that we agree that $\square$ may represent only a number from the set $D = \{1, 2, 3, 4, 5, 6\}$. Then again the sentence $7 + \square = 12$ is true when $\square$ is 5 and false when $\square$ is 1, 2, 3, 4, or 6. In this case $\square$ may not represent any other number. This set of numbers, any one of which the variable may represent, is called the <u>domain</u> of the variable.

Let us consider the sentence $7 + \square < 12$ and let the domain of the variable be $\{1, 2, 3, 4, 5, 6, 7\}$. What set of numbers from this domain will make the sentence $7 + \square < 12$ true? Will the number 1 make the sentence true? Yes, since $7 + 1 < 12$. Will 2 make the sentence true? Yes. Will 3, will 4? Yes. Will 5 make the sentence true? $7 + 5 < 12$ is clearly a false statement. The numbers 6 and 7 will also give false statements. Hence the set of numbers <u>from the given domain</u> which will make the sentence $7 + \square < 12$ true is $\{1, 2, 3, 4\}$.

Suppose that we change the domain to $\{-3, -2, -1, 0, 1, 2, 3\}$. From this domain, what set of numbers will make $7 + \square < 12$ true? Do you agree that <u>every</u> number in this domain will give us a true statement? Try each number in the domain and check that indeed every number <u>from this domain</u> will make $7 + \square < 12$ true.

Now suppose that the domain D is the set of all rational numbers. From this set D, what numbers will make $7 + \square < 12$ true? Will $4\frac{15}{16}$ make the sentence true? Will $\frac{99}{100}$ make the sentence true? Will $\frac{-21}{23}$ make the sentence true? Do you agree that every rational number less than 5 will make this sentence, $7 + \square < 12$, true? All other rational numbers will make the sentence false.

We see then that the set of numbers which will make any sentence like this into a true statement will depend not only on the sentence itself, but also on the domain D of numbers which we can use as values of the variable. We can now define variable more carefully.

> A variable is a symbol which represents any member of a specified set called the <u>domain</u> of the variable. The elements of the domain are called <u>values</u> of the variable.

<u>Problems 8-3</u>

In each of the following sentences, what numbers from the domain D make the sentence a true statement? What numbers from D will make the sentence a false statement?

1. $\square + \frac{2}{3} = \frac{5}{3}$ $\quad D = \{0, 1, 2, 3\}$
2. $\square + \frac{2}{3} < \frac{5}{3}$ $\quad D = \{0, 1, 2, 3, 4\}$
3. $\square + \frac{2}{3} > \frac{5}{3}$ $\quad D = \{0, 1, 2, 3, 4, 5\}$
4. $\square + \frac{2}{3} \geq \frac{5}{3}$ $\quad D = \{\text{natural numbers}\}$
5. $\square + \frac{2}{3} \leq \frac{5}{3}$ $\quad D = \{\text{integers}\}$
6. $\square + \frac{2}{3} \geq \frac{5}{3}$ $\quad D = \{\text{positive rational numbers}\}$
7. $\square + \frac{2}{3} \leq \frac{5}{3}$ $\quad D = \{\text{positive rational numbers}\}$
8. $x - 5 = \frac{1}{2}$ $\quad D = \left\{\frac{11}{2}, 6, \frac{13}{2}, 7, \frac{15}{2}, 8\right\}$
9. $x - 5 < \frac{1}{2}$ $\quad D = \left\{4, \frac{9}{2}, 5, \frac{11}{2}, 6, \frac{13}{2}\right\}$
10. $x - 5 > \frac{1}{2}$ $\quad D = \{4, \frac{9}{2}, 5, \frac{11}{2}, 6, \frac{13}{2}, 7, \frac{15}{2}, 8\}$
11. $x - 5 \geq \frac{1}{2}$ $\quad D = \{\text{positive rational numbers}\}$
12. $x - 5 \leq \frac{1}{2}$ $\quad D = \{\text{positive rational numbers}\}$
13. $x - 5 \leq \frac{1}{2}$ $\quad D = \{\text{rational numbers}\}$

14. $x - 5 \leq \frac{1}{2}$ D = {negative rational numbers}

15. $x - 5 \leq \frac{1}{2}$ D = {natural numbers}

8-4 Open Sentences and Truth Sets

In our problems in 8-3 above, we considered sentences of the form $x - 5 \leq \frac{1}{2}$ in which we were given a specified domain D, and were asked to find the numbers from the set D which will make the given sentence into a true statement. Sentences such as $x - 5 \leq \frac{1}{2}$ which contain a variable such as x, y, □, or Δ are called open sentences. An open sentence is neither true nor false but becomes a statement which is either true or false when we give a value to the variable. Remember that we agreed to take values of the variable only from a specified set which we called the domain, and we choose the domain so that for any value of the variable the open sentence will become a statement. The set of numbers from the domain which will make a given open sentence a true statement is called the truth set, or solution set, of that open sentence. For example what is the truth set of the open sentence $x - 5 < \frac{1}{2}$ if the domain is {0, 1, 2, 3, 4, 5, 6, 7, 8}? Do you agree that the truth set is {0, 1, 2, 3, 4, 5}? Every element of this solution set, if it is used as the value of the variable, changes the open sentence into a true statement.

Sometimes there is no number from the domain which will make the sentence true. For example, suppose that we want to find the solution set of $x + 2 < 0$ if the domain of the variable x is the set of natural numbers. Clearly any natural number added to 2 will give a number greater than zero, not less than zero. In this case, then, the solution set is the empty set, which we write as { }. Another sentence for which the truth set is the empty set is the equation $x^2 + 2 = 1$ where the domain of x is the set of rational numbers. Can you explain why? (x^2 means the product of x by x.)

Open sentences which use the symbol "=" are called equations, whereas open sentences which use the symbols $\neq$, $>$, $<$, $\geq$, or $\leq$ are called inequalities.

Let us summarize all of these ideas:

1. Sentences in which variables appear are called <u>open sentences</u>.

2. Open sentences in which the symbol "=" is used are called <u>equations</u>.

3. Open sentences in which the symbols $\neq$, $>$, $<$, $\geq$, or $\leq$ are used are called <u>inequalities</u>.

4. The set of numbers which make an open sentence into a true statement is called the <u>truth set</u>, or <u>solution set</u>, of the open sentence.

5. Each number in the truth set is called an <u>element</u> or <u>member</u> of that truth set or a <u>solution</u> of the open sentence.

6. If the solution set of a sentence has no members, then the solution set is said to be <u>empty</u>.

<u>Problems 8-4</u>

1. Determine the truth set of each of the following equations and inequalities for the domain specified:

(a) $\square - \frac{4}{5} = \frac{1}{5}$ $\quad D = \{0, 1, 2, 3, 4\}$

(b) $\square - \frac{4}{5} > \frac{1}{5}$ $\quad D = \{0, 1, 2, 3, 4\}$

(c) $\square - \frac{4}{5} \geq \frac{1}{5}$ $\quad D = \{0, 1, 2, 3, 4, 5\}$

(d) $\square - \frac{4}{5} < \frac{1}{5}$ $\quad D = \{0, 1, 2, 3\}$

(e) $\square - \frac{4}{5} \leq \frac{1}{5}$ $\quad D = \{0, 1, 2, 3, 4, 5\}$

(f) $x - \frac{4}{5} = \frac{1}{5}$ $\quad D = \{\text{integers}\}$

(g) $x - \frac{4}{5} \geq \frac{1}{5}$ $\quad D = \{\text{integers}\}$

(h) $x - \frac{4}{5} > \frac{1}{5}$ $\quad D = \{\text{integers}\}$

(i) $x - \frac{4}{5} < \frac{1}{5}$ $\quad D = \{\text{integers}\}$

(j) $x - \frac{4}{5} \leq \frac{1}{5}$ $\quad D = \{\text{integers}\}$

1. (k) $y - \frac{4}{5} = \frac{1}{5}$ D = {rational numbers}

   (l) $y - \frac{4}{5} > \frac{1}{5}$ D = {rational numbers}

   (m) $y - \frac{4}{5} \geq \frac{1}{5}$ D = {rational numbers}

   (n) $y - \frac{4}{5} < \frac{1}{5}$ D = {rational numbers}

   (o) $y - \frac{4}{5} \leq \frac{1}{5}$ D = {rational numbers}

2. Find the solution set for each of the following open sentences for the domain specified.

   (a) $3x > 10$ D = {2, 4, 6, 8}

   (b) $\frac{x}{3} = 10$ D = {rational numbers}

   (c) $3 + x \leq 10$ D = {positive integers}

   (d) $3 - x > 10$ D = {positive integers}

   (e) $3 - x > 0$ D = {negative integers}

   (f) $\frac{3}{x} < 0$ D = {rational numbers}

   (g) $\frac{3}{x} = 0$ D = {rational numbers}

   (h) $x + 3 \neq 10$ D = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

   (i) $3x = 10$ D = {rational numbers}

   (j) $x - 3 \geq 10$ D = {1, 3, 5, 7, 9, 11, 13, 15}

3. Find the truth set for each of the following open sentences for the domain specified.

   (a) $x = 9$ or $x = -9$ D = {rational numbers}

   (b) $x = 9$ and $x = -9$ D = {rational numbers}

   (c) $x + 1 = 9$ or $x + 1 = -9$ D = {integers}

   (d) $x^2 + 2 > 0$ D = {rational numbers}

   (e) $x^2 \leq 0$ D = {rational numbers}

3. (f) $x^2 < 0$ D = {rational numbers}

(g) $x \geq 0$ or $x \leq 0$ D = {rational numbers}

(h) $x \geq 0$ and $x \leq 0$ D = {rational numbers}

(i) $x + 3 < 2$ D = {natural numbers}

(j) $6x + 1 > 0$ D = {whole numbers}

(k) $x^2 = 100$ D = {rational numbers}

(l) $(x - 2)(x - 3) = 0$ D = {integers}

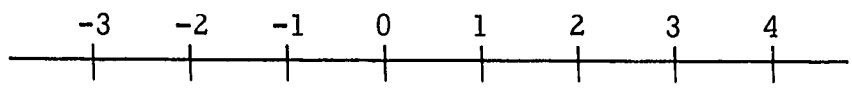
(m) $x > 3$ and $x < 6$ D = {integers}

(n) $x > 3$ and $x < 2$ D = {integers}

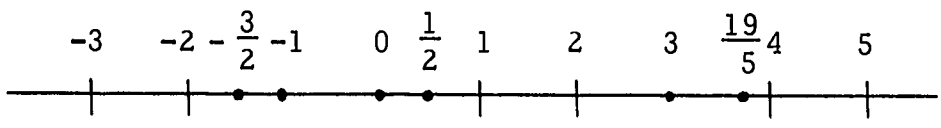
(o) $x > 3$ or $x < 2$ D = {integers}

8-5 Graphs of Solution Sets

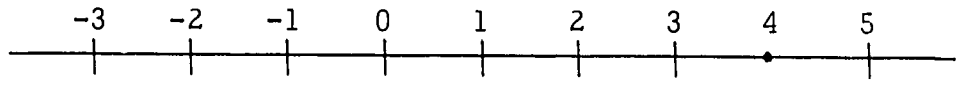
You learnt in Chapter 5 that corresponding to every rational number we can locate a point on the number line. The number is called the coordinate of the point and the point is called the graph of the number. A picture of part of the number line looks like this.



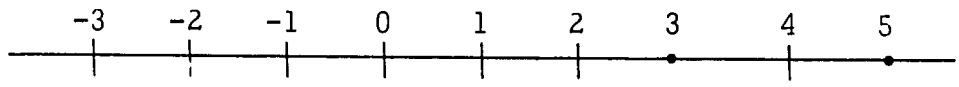
We can represent a set of numbers graphically by showing heavy dots at the points with coordinates which are numbers in the set. For example, the graph of the set $\left\{-\frac{3}{2}, -1, 0, \frac{1}{2}, 3, \frac{19}{5}\right\}$ will look like this:



It is sometimes helpful to visualize the truth set of an open sentence by representing it on the number line. For example, if the domain of y is the set of integers, the truth set of $y + 11 = 15$ is $\{4\}$. Why? The graph of $\{4\}$ is

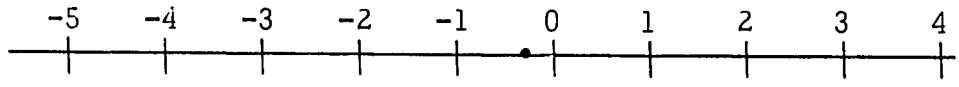


Verify that the following graph represents the truth set of the sentence $(x - 3)(x - 5) = 0$ in the domain {integers}.



If the domain of y is the set of rational numbers, what is the solution set of $y + \frac{9}{4} = 2$?

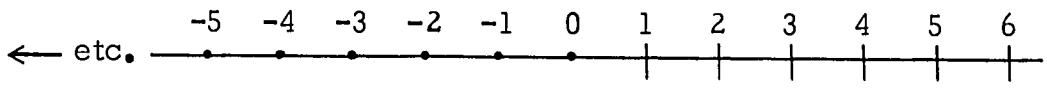
Does this graph show it reasonably well?



You learnt that the number line shows at a glance which of two numbers is the larger or the smaller. For example, $2 < 5$, and the point with coordinate 2 lies to the <u>left</u> of the point with coordinate 5. Again $-2 > -5$, and the point with coordinate -2 lies to the <u>right</u> of the point with coordinate -5.

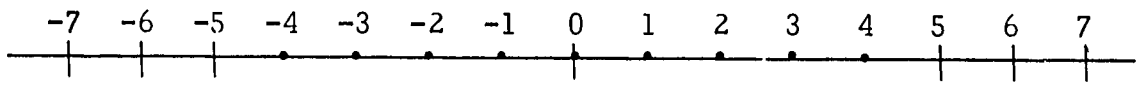
If a and b are rational numbers and $a > b$, then the graph of a lies to the right of the graph of b on the number line. If $a < b$, then the graph of a lies to the left of the graph of b. This characteristic of the number line will be helpful in connection with representing truth sets of inequalities.

If the domain of x is the set of integers, what is the solution set of $x - \frac{4}{5} < \frac{1}{5}$? The solution set is $\{0, -1, -2, -3, \ldots\}$. Notice the three dots; they indicate that the solution set contains <u>all</u> integers smaller than 0. We cannot draw the complete graph of this solution set, but we make a graph which looks like this:

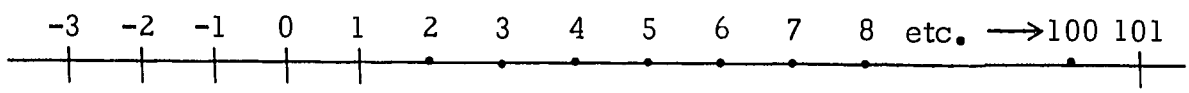


Notice that we have solid dots as far to the left as our drawing continues and then show "$\longleftarrow$ etc." to indicate that the graph continues in this manner without end to the left.

If the domain of x is the set of integers, what is the solution set of the open sentence, $x > -5$ and $x < 5$? Why does its graph look like this?



If the domain of x is $\{0, 1, 2, 3, \ldots, 100\}$, what is the solution set of $2x > 3$? We represent it this way on the number line.



The "etc. $\longrightarrow$" means to continue in the same manner to the right and the final dot at 100 means to stop there.

<u>Problems 8-5</u>

1. Graph each of the following sets on the number line.

   (a) $\{1, 2, 3, 4\}$

   (b) {natural numbers greater than 6}

   (c) {integers x such that $x \leq -2$}

   (d) {integers between, but not including, -4 and 3}

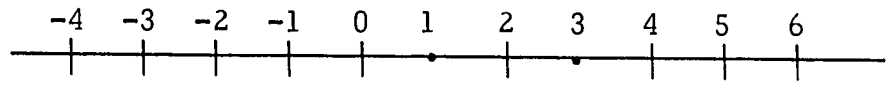
   (e) {integers x such that $x \neq -1$, $x \neq 0$, $x \neq 1$}

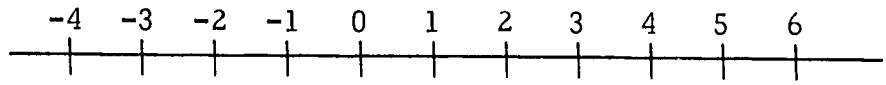
   (f) $\{-5, -\frac{7}{2}, \frac{11}{4}\}$

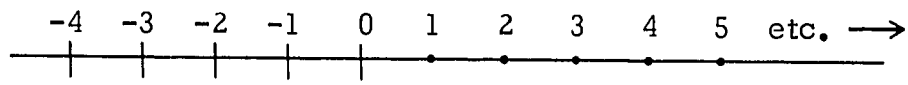
   (g) $\{\quad\}$

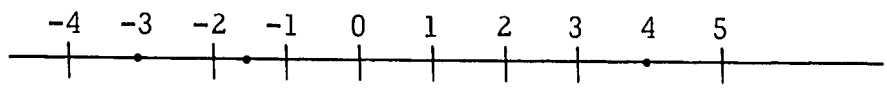
   (h) {whole numbers less than 10}

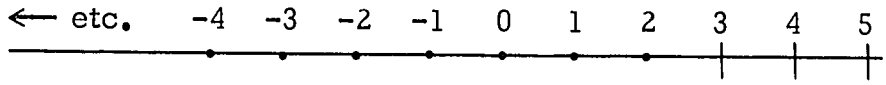
2. For each of the following graphs write the corresponding set.

(a) 

(b) 

(c) 

(d) 

(e) 

3. Determine the solution set of each of the following open sentences for the domain specified, and graph your solution set on the number line:

(a) $x + 5 = 3$ D = {whole numbers}

(b) $x + 5 > 3$ D = {whole numbers}

(c) $x + 5 \geq 3$ D = {whole numbers}

(d) $x + 5 \leq 3$ D = {whole numbers}

(e) $x + 5 < 3$ D = {whole numbers}

(f) $x + 5 \neq 3$ D = {whole numbers}

(g) $x + 5 = 3$ D = {integers}

(h) $x + 5 > 3$ D = {integers}

(i) $x + 5 \geq 3$ D = {integers}

(j) $x + 5 \leq 3$ D = {integers}

(k) $x + 5 < 3$ D = {integers}

(l) $x + 5 \neq 3$ D = {integers}

8-6 Compound Sentences

Which of the following statements do you think are true?

1. $(3 + 2 = 5)$ and $(6 > 4)$.

2. (Red is a colour) and (Wednesday is a day of the week).

3. $(3 \times 2 = 5)$ and $(4 > 6)$.

4. $(2 + 2 = 4)$ and $(5 < 0)$.

5. (All men have white hair) and (Tuesday comes after Monday).

6. (Nigeria is in South America) and (there are 15 months in the year).

Each of these sentences is made up of two statements joined by the connecting word "and". We call these sentences compound sentences. When "and" is the word connecting the two statements we mean that for the sentence to be true both statements must be true. Is it true that $3 + 2 = 5$? That $6 > 4$? Since both of these statements are true, Sentence 1 above is true. Show why Sentence 2 also is true. What about Sentence 3? Is $3 \times 2 = 5$ a true statement? Is $4 > 6$ true? Certainly since both of these are false, we would agree that Sentence 3 is false. In Sentence 4, although $2 + 2 = 4$ is true, $5 < 0$ is not true. Since "and" means that both parts must be true, we say that Sentence 4 is a false sentence. Explain why Sentence 5 and Sentence 6 are also false.

It is frequently helpful to use open sentences which are compound sentences using the word "and". For example if we want to describe the set of integers between 5 and 11, we can write the open sentence $x > 5$ and $x < 11$, in which the domain of x is the set of integers. Is the number 6 greater than 5 and at the same time less than 11? What about 13? It is greater than 5. Why is it not in the truth set of this compound sentence? List all the numbers which are in the truth set. Graph this truth set on the number line. Do you see how the points of the graph are between 5 and 11? The compound sentence "$x > 5$ and $x < 11$" is often abbreviated "$5 < x < 11$". When this form is used it is understood to mean the same thing as the compound sentence "$5 < x$ and $x < 11$".

Problems 8-6A

Find the truth set of each of the following sentences if the domain of the variable is the set of integers. Graph each truth set on the number line.

1. $y < 8$ and $y > 0$
2. $50 < x < 60$
3. $-4 < a < 2$
4. $x < 8$ and $x > 6$
5. $-100 < y < 100$
6. $y > 20$ and $y < 5$
7. $a < 3$ and $a > 3$
8. $-2 < x < -1$
9. $y < 12$ and $y < 6$
10. $x > -3$ and $x = 6$
11. $y < 5$ and $y = 6$
12. $a > -8$ and $a > -2$

Which of the following sentences do you think are true?

1. $(3 + 2 = 5)$ or $(6 > 4)$
2. (Red is a colour) or (Wednesday is a day of the week).
3. $(3 \times 2 = 5)$ or $(4 > 6)$
4. $(2 + 2 = 4)$ or $(5 < 0)$
5. (All men have white hair) or (Tuesday comes after Monday).
6. (Nigeria is in South America) or (There are 15 months in the year).

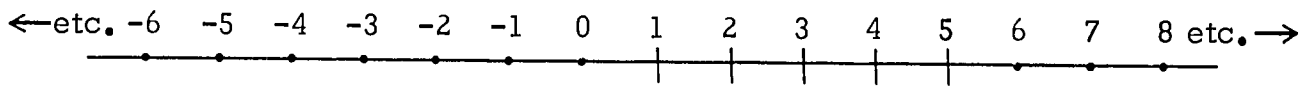
Again we have compound sentences, but this time the connecting word is "or". In mathematics "or" between two statements means that either the first statement is true, or the second statement is true, or both statements are true. Sentence 6 above is clearly false because neither one of the statements is true. Are there any more false compound sentences in the list? Why is Sentence 4 true?

You have already seen compound sentences such as "$x \geq 3$", which we agreed would mean "$x > 3$ or $x = 3$". If the domain of the variable is the set of integers, is 5 in the truth set? 19? 3? 1? Draw the graph of this truth set.

In the sentence

$$y > 5 \text{ or } y < 1 \qquad D = \{\text{integers}\}$$

the number 6 makes the first clause true but the second one false. Is 6 in the truth set of the compound sentence? The graph of the sentence would look like this.



Notice that "$y > 5$ or $y < 1$" cannot be abbreviated to "$5 < y < 1$". If we wrote it the latter way, "and" would be understood between the clauses, which is not what we mean this time.

Problems 8-6B

Find the truth set of each of the following sentences if the domain of the variable is the set of integers. Graph each truth set on the number line.

1. $x < 10$ or $x > 12$
2. $y \leq -4$
3. $a > 2$ or $a = 0$
4. $a > 2$ or $a = 5$
5. $x > -1$ or $x < -3$
6. $x > -1$ or $x < -1$
7. $y < 8$ or $y > 2$
8. $y < -3$ or $y < 3$
9. $x > -4$ or $x > -2$
10. $a = 6$ or $a = 1$

8-7 Equivalent Equations

Thus far in this chapter we have been dealing with open sentences whose truth sets were self-evident. You have been able to guess at numbers in the truth set and test whether or not your guess is correct. Suppose, however, you were given an equation such as:

$$\frac{2}{3}x - \frac{3}{4} = -\frac{3}{2}x + \frac{4}{5}$$

This equation is much more complicated than any which we have had thus far, and it would probably take you a long time to guess its solution set!

What we need now are some methods by which we can find the solution set for this and for other more complicated equations. In the next section we shall find methods to obtain the solution sets of inequalities.

Let us consider a few simple examples first. What is the truth set of each of the following equations?

$$x + 4 = 10$$

$$3 + y = 9$$

$$z + 1 = 7$$

$$13 + \square = 19$$

Although these equations are all different, there is something that is the same for all of them. What is it? They have the same truth set, namely $\{6\}$. Consider the following equations:

$$\frac{2}{3}x + 7 = 5$$

$$2x + 21 = 15$$

$$2x = -6$$

$$x = -3$$

These equations are all different, but there is something which they too have in common. Do you see that $\{-3\}$ is the truth set of each of these equations? Since they all have the same truth set, we shall call them <u>equivalent</u> equations.

Equations with the same truth set are called <u>equivalent equations</u>.

Let us see what changes can be made in an equation which will give a new equation equivalent to the old one. If by making such changes we can obtain equivalent equations which are simpler, there is more hope of being able to find the truth set. We need only find the truth set of the simpler equation.

Consider the equation $x + 5 = 13$. This says that when x has a value in the truth set, $(x + 5)$ and 13 are two names for the same number. If we add 3 to that number, then $(x + 5) + 3$ and $13 + 3$ are two names for the new number. We can therefore write

$$(x + 5) + 3 = 13 + 3$$

or

$$x + 8 = 16$$

and this new equation must be true for the same value of x which makes the first equation true. We can equally well start with the second equation, $x + 8 = 16$ and by adding -3 to the number represented by $x + 8$ and 16 obtain the equation

$$(x + 8) + (-3) = 16 + (-3)$$

or

$$x + 5 = 13$$

which is our first equation. This shows that any value of x which makes the second equation true will also make the first equation true. The two equations must therefore have the same truth set because we have shown that

1. every member of the first truth set is a member of the second truth set, and

2. every member of the second truth set is a member of the first truth set.

The two equations are therefore equivalent.

Suppose instead of adding 3 to the number represented by the two names $(x + 5)$ and 13, we add 12. What new equation results? If you add -12 to both sides of the new equation do you get the old one back again? When $x = 8$, $(x + 5)$ and 13 are names for the same number. Is it true also that when $x = 8$, $(x + 5) + 12$ and $13 + 12$ are names for the same number? Are the equations $x + 5 = 13$ and $x + 17 = 25$ equivalent equations? Why?

We might equally well have added 2 or 10 or -5 or -1 and by the same reasoning we still would have obtained a second equation which is equivalent to the first one. What would be the new equation in each of these cases?

We could describe the change we made each time by saying we added the same number to the number represented on both sides of the equation. This is usually abbreviated to "added the same number to both sides of the equation". We could also say "subtracted" because subtracting would be adding the opposite.

Let us summarize what we have discovered.

> If the same number is added to both sides of an equation, the new equation is equivalent to the old one.

As you might expect, there is a similar property for multiplication. Consider the equation $3x = 12$. For what value of x are $3x$ and 12 names for the same number? Let us multiply that number by 6. The new number can be represented by $6(3x)$ or by $6(12)$. Since these represent one number, we can write $6(3x) = 6(12)$ or $18x = 72$. Does the value of x which makes $3x = 12$ true also make $18x = 72$ true? If we start with $18x = 72$, what can we multiply by to obtain the original equation $3x = 12$? Since this multiplication procedure can be done both ways, we know that any solution of the first equation is a solution of the second equation <u>and</u> any solution of the second equation is a solution of the first equation, so the equations are equivalent. What equivalent equation would you obtain if you multiply both sides of $3x = 12$ by 2? by -8? by $\frac{1}{4}$? by $\frac{1}{3}$?

If you multiply both sides of $3x = 12$ by the number zero, you obtain the equation $0 = 0$. Can you find a number by which you can multiply both sides of $0 = 0$ to take you back to $3x = 12$? Is $0 = 0$ an open sentence? What is the truth set of $3x = 12$? Is the sentence $0 = 0$ equivalent to the open sentence $3x = 12$? As you see, zero is a very exceptional number. If we multiply both sides of an equation by zero, we do <u>not</u> expect to obtain an equivalent equation.

> If both sides of an equation are multiplied by the same non-zero number, the new equation is equivalent to the old one.

We could also have said "divide" because that would be the same as multiplying by the reciprocal of the number, which exists because the number is non-zero.

Our examples have been very simple. We could see the truth sets almost at a glance. In less obvious equations these changes to other equivalent equations will still be easy, even though we don't know the truth sets. For example, $\frac{5}{8}x = 7$ is equivalent to

$$(\tfrac{5}{8}x)8 = (7)8$$

or

$$5x = 56.$$

$$17y - 30 = 72 \text{ is equivalent to}$$

$$(17y - 30) + 30 = 72 + 30$$

or

$$17y = 102$$

These changes will lead to easy ways to find the truth sets, as we shall see in the next section.

Problems 8-7

1. Which of the following pairs are pairs of equivalent equations? For each case in which the equations are equivalent tell whether you saw the equivalence by

   (1) adding the same number to both sides. For example,

$$17a + 39 = -19 \text{ is equivalent to}$$

$$(17a + 39) + 19 = (-19) + 19$$

or

$$17a + 58 = 0$$

(2) multiplying both sides by the same non-zero number. For example,

$$37x = 23 \text{ is equivalent to}$$

$$(37x)\frac{1}{37} = (23)\frac{1}{37}$$

or

$$x = \frac{23}{27}$$

(3) demonstrating that the truth sets of the two equations are the same. For example,

$$y + 23 = 30 \text{ is equivalent to}$$

$$y + 14 = 21$$

because the truth set of $y + 23 = 30$ is $\{7\}$ and the truth set of $y + 14 = 21$ is also $\{7\}$.

(a) $x + 4 = 9;\ x = 9 - 4$

(b) $2x - \frac{3}{8} = \frac{3}{4};\ 2x = \frac{9}{8}$

(c) $7y = 19;\ y = 12$

(d) $\frac{3}{5}x = 12;\ 3x = 60$

(e) $\frac{3}{4}a + \frac{5}{8} = \frac{35}{8};\ a = 5$

(f) $x + (-3) = 7;\ x = 7 + 3$

(g) $x + \left(-\frac{3}{4}\right) = -\frac{5}{8};\ x = \frac{3}{4} + \left(-\frac{5}{8}\right)$

(h) $\frac{1}{2}a = 20;\ a = 10$

(i) $y - 8 = 0;\ 2y + 17 = 33$

(j) $7x + 12 = 2;\ 7x = 14$

(k) $3x = 182;\ x = \frac{182}{3}$

2. In each of the following parts there are several equivalent equations. In each one tell which would be the easiest of them to use in finding the truth set, and tell how that equation was obtained from the first one listed.

2. (a) $x - 9 = 73$

$x - 16 = 66$

$x = 82$

$x + 9 = 91$

(b) $y + \frac{2}{3} = \frac{5}{6}$

$y = \frac{5}{6} - \frac{2}{3}$

$y + 1 = \frac{7}{6}$

$y + \frac{2}{3} - \frac{5}{6} = 0$

(c) $6a = 13$

$3a = \frac{13}{2}$

$6a - 13 = 0$

$a = \frac{13}{6}$

(d) $\frac{1}{4}x = 4$

$x = 16$

$\frac{1}{4}x - 4 = 0$

$\frac{1}{16}x = 1$

(e) $\frac{3}{5}y = \frac{2}{7}$

$3y = \frac{10}{7}$

$\frac{7}{2} \times \frac{3}{5}y = 1$

$y = \frac{2}{7} \times \frac{5}{3}$

(f) $-8x = 19$

$-64x = 19 \times 8$

$x = 19\left(-\frac{1}{8}\right)$

$-x = 19 \times \frac{1}{8}$

8-8 Truth Sets Found by Using Equivalent Equations

In the previous section you learnt that two equations are equivalent if they have the same truth set. You also learnt that if we add the same number to both sides of an equation or subtract the same number from both sides of an equation, the resulting equation will be equivalent to the original equation. We can also multiply or divide both sides of an equation by the same non-zero number, and the resulting equation will be equivalent to the original equation.

If you were given a rather complicated equation, do you think you could now write a succession of simpler, equivalent equations until you get an equation whose truth set is easily seen?

Consider the following example: Find the truth set of $\frac{1}{2}x - 3 = 1$. If we can find an equivalent equation in the form

$$x = \text{a number},$$

the truth set will be seen immediately. Eliminating the -3 in $\left(\frac{1}{2}x - 3\right)$ would be one step toward obtaining x alone. What could we do to $\left(\frac{1}{2}x - 3\right)$ to eliminate the -3? The inverse property of addition suggests adding 3 because $(-3) + 3 = 0$. We do so and obtain the equivalent equation

$$\left(\frac{1}{2}x - 3\right) + 3 = 1 + 3$$

or

$$\frac{1}{2}x = 4.$$

Now what can we do to $\frac{1}{2}x$ to obtain x? Can we use an inverse property again? Can we multiply $\frac{1}{2}$ by a number which will give 1 as the product? What is that number? Why is it helpful to multiply both sides of the equation by 2? We obtain the equivalent equation

$$2\left(\frac{1}{2}x\right) = 2(4)$$

or

$$x = 8.$$

What is the truth set of this last equation? Therefore what is the truth set of the original equation?

We could show what we have done by writing the following sequence of equivalent equations. Tell why each one is equivalent to the one before it.

$$\frac{1}{2}x - 3 = 1$$

is equivalent to $\left(\frac{1}{2}x - 3\right) + 3 = 1 + 3$ (Why?)

which is equivalent to $\frac{1}{2}x + (-3 + 3) = 4$ (Why?)

which is equivalent to $\frac{1}{2}x = 4$ (Why?)

which is equivalent to $2\left(\frac{1}{2}x\right) = 2 \times 4$ (Why?)

which is equivalent to $\left(2 \times \frac{1}{2}\right)x = 8$ (Why?)

which is equivalent to $x = 8$ (Why?)

Here is another example.

$$5x + 8 = 3 - 2x$$

is equivalent to $(5x + 8) + 2x = (3 - 2x) + 2x$ (Why?)

which is equivalent to $7x + 8 = 3$ (Why?)

which is equivalent to $(7x + 8) - 8 = 3 - 8$ (Why?)

which is equivalent to $7x = -5$ (Why?)

which is equivalent to $\frac{1}{7}(7x) = \frac{1}{7}(-5)$ (Why?)

which is equivalent to $x = -\frac{5}{7}$ (Why?)

The truth set is $\left\{-\frac{5}{7}\right\}$.

You may enjoy looking at a more complicated equation such as some we shall be solving later. As we become more familiar with the method of equivalent equations we shall do some of the steps mentally and not write them, and we shall understand, without saying so every time, that all the equations in the sequence are equivalent. Notice in this example how each equation is a little simpler than the one before. Can you explain in each step why the equations are equivalent?

$$3x - \frac{5}{3} = 8 + \frac{5}{3}x + \frac{10}{3}$$

$$3\left(3x - \frac{5}{3}\right) = 3\left(8 + \frac{5}{3}x + \frac{10}{3}\right)$$

$$9x - 5 = 24 + 5x + 10$$

$$9x = 24 + 5x + 10 + 5$$

$$9x = 39 + 5x$$

$$4x = 39$$

$$x = \frac{39}{4}$$

The solution set is $\left\{\frac{39}{4}\right\}$.

Problems 8-8

Find the truth set of each of the following equations if the domain of the variable is the set of rational numbers.

1. $3x - 13 = 26$
2. $2x - 7 = 32 - x$
3. $4y - 9 = -5$
4. $8 - 3a = 11$
5. $14 = 8 - 2x$
6. $\frac{1}{2}y + 3 = 9$
7. $\frac{1}{4}(8x + 4) = 3$
8. $\frac{1}{3}(3x + 9) = x + 3$
9. $2(2x + 3) = x + 1$
10. $\frac{1}{3}(x + 4) = 20$
11. $\frac{2}{5}a + 9 = 0$
12. $y - 18 = 7y + 3$
13. $18 - 3x + x + 3 = 0$
14. $\frac{3}{5}y - \frac{2}{5} = \frac{4}{5}$
15. $\frac{1}{2}(5x - 4) = x + \frac{11}{2}$
16. $\frac{4}{3}x = \frac{7}{3} + x$

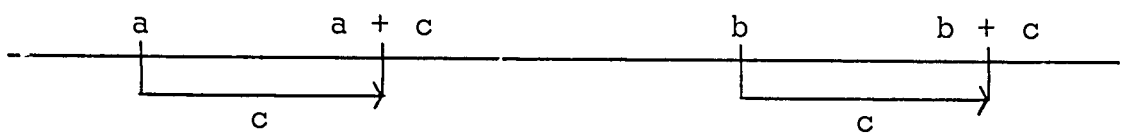
8-9 Equivalent Inequalities

Just as any two equations with the same truth set are called equivalent equations, so any two inequalities with the same truth set are called <u>equivalent inequalities</u>. Again, it will be a great help in finding truth sets of inequalities if we know what changes can be made to produce simpler, equivalent inequalities.

For this purpose we need first to observe some properties of inequalities which we shall use. For each of the following pairs of inequalities is the one on the left true? Is the one on the right true? What was done to the one on the left to obtain the one on the right?

$3 + 2 < 6$	$(3 + 2) + 7 < 6 + 7$
$10 \times 6 > 20$	$(10 \times 6) + 5 > 20 + 5$
$\frac{25}{3} < 10$	$\frac{25}{3} + (-2) < 10 + (-2)$
$8(3 + 7) \geq 80$	$8(3 + 7) + 1 \geq 80 + 1$
$14 \leq 15$	$21 \leq 22$
$\frac{9}{8} \geq \frac{3}{4}$	$\frac{1}{8} \geq -\frac{1}{4}$
$3 \cdot 29 < 4$	$0 \cdot 29 < 1$

If you observed that all the sentences on the left are true, that each sentence on the right was obtained by adding the same number to both sides of the inequality on the left, and that each of the resulting sentences is true, you observed one of the basic properties of inequalities. We can visualize this property on the number line by seeing that if $a < b$, the point a is to the left of the point b. If now we move a certain distance from a and the same distance from b in the same direction the new points will have the same order as the old ones did; that is, the first will still be to the left of the second. We shall accept this property as a fundamental assumption about inequalities.



> If a, b, and c are numbers and $a < b$, then $a + c < b + c$.

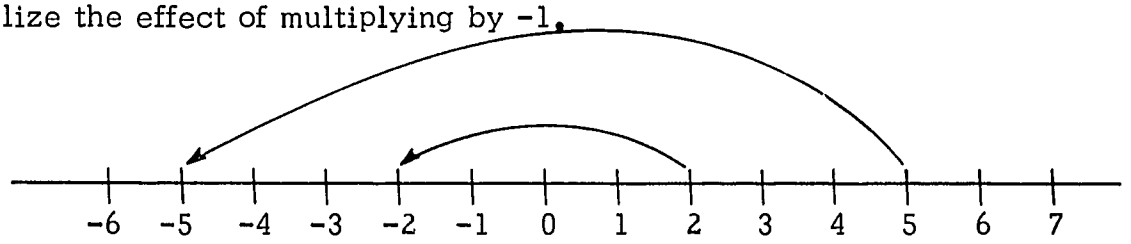
Properties of inequalities are often referred to as properties of <u>order</u> because the relations $<$ and $>$ show the order of the numbers. We shall call this property the <u>addition property of order</u>.

Do you suppose that there is a multiplication property of order? For each of the following pairs of sentences is the one on the left true? Is the one on the right true? What was done to the one on the left to obtain the one on the right?

$6 + 2 < 12$	$(6 + 2)5 < (12)5$
$\frac{2}{3} \leq \frac{5}{3}$	$\frac{2}{3} \times 3 \leq \frac{5}{3} \times 3$
$-3 > -4$	$(-3) \times 2 > (-4) \times 2$
$6 > 4$	$6 \times (-2) > 4 \times (-2)$
$-5 < 5$	$(-5) \times (-1) > (5) \times (-1)$
$-7 \leq -2$	$14 \geq 4$
$-\frac{3}{5} > -\frac{1}{5}$	$3 > 1$

Were you alert enough to notice that some of the statements are not true? Which ones are true? Which ones are false? Look at the true ones. In some of the statements on the right the direction of the inequality sign has been reversed from its direction in the statement on the left. What number was used as the multiplier in those cases?

If you have observed carefully, you see that when we multiplied both sides of a true inequality by a <u>positive</u> number, a true inequality was obtained without any further change. When we multiplied by a <u>negative</u> number, however, it was necessary to change > to < or to change < to > in order to obtain a true inequality. This reversal of direction seems less surprising when we use the number line to visualize the effect of multiplying by -1.



The statement $2 < 5$ is true. Multiplying 2 and 5 by -1 gives their opposites, -2 and -5. The arrows on the figure help us see that the new numbers are in reverse order. Draw a similar figure for the statement $-4 < -1$; for $5 > 1$; for $-2 < 3$.

We state as another basic assumption about inequalities the <u>multiplication property of order</u>.

If a, b and c are numbers and $a < b$, then $ac < bc$ if $c > 0$, and $ac > bc$ if $c < 0$.

From here on, we can reason about equivalent inequalities much as we did about equivalent equations.

If $x - 9 < 13$ is true for some value of x, then $(x - 9) + 3 < 13 + 3$,

$$\text{or} \qquad x - 6 < 16,$$

is true for the same value of x, because of the addition property of order. Could we have added to both sides of the inequality some other number such as 12 or -72 or any other number we wanted and still know that the new inequality is true for the same value of x? What number would you add to obtain a very simple inequality?

To know that the new inequality is equivalent to the old one we must be able to reverse the process. Is there a number which we can add to both sides of $x - 6 < 16$ to obtain the original inequality, $x - 9 < 13$? What is that number? Why is it true then that any number x which makes the second inequality true will also make the first one true? Since we now know that any number that makes either inequality true also makes the other one true, the two inequalities have exactly the same truth set and they are equivalent. Of course it would have been more helpful to add 9 to both sides and thus to find out that

$$x - 9 < 13$$

$$\text{and} \qquad x < 22$$

are equivalent equations.

In general, by the reasoning we have used we can conclude that if the same number is added to both sides of an inequality, the resulting inequality is equivalent to the original inequality.

Problems 8-9A

Which of the following pairs are pairs of equivalent inequalities?

1. $x + \frac{1}{2} < 2\frac{1}{2}$; $x + 1 < 3$

2. $x - 3 > 2$; $x - 4 > 1$

3. $x + 2 \leq 1$; $x - 4 \leq -3$

4. $x - 3 \geq 5$; $x - 3 \geq 2$

5. $x + 7 < \frac{11}{5}$; $x < \frac{11}{5} - 7$

6. $x - 4 > 2$; $4 - x < -2$

7. $x - 5 > 6$; $10 - x < 12$

8. $x + 2 \geq 4$; $2 \leq x$

9. $x - 4 < -\frac{3}{2}$; $x < \frac{5}{2}$

10. $x + \frac{2}{3} \geq \frac{5}{3}$; $1 \leq x$

When it comes to multiplying both sides of an inequality by the same number to obtain an equivalent inequality the reasoning is very much the same, but there are two precautions we must bear in mind.

1. We cannot multiply by zero, since if we multiply by zero we no longer have an open sentence and therefore cannot have an equivalent open sentence. (If $x > 3$ and we multiply by zero, $0 > 0$ is no longer an open sentence but simply a false statement.)

2. If we choose to multiply both sides by a negative number, we must remember to reverse the direction of the inequality to obtain an equivalent open sentence.

Thus $-8y > 24$ and $8y < -24$ are equivalent open sentences, but $-2y < -30$ and $y < 15$ are not.

Since subtracting is adding the opposite and dividing is multiplying by the reciprocal, we can include subtraction and division in our statement. Let us now summarize our conclusions.

The following operations on an inequality give an equivalent inequality:

1. Adding the same number to both sides or subtracting the same number from both sides.
2. Multiplying or dividing both sides by the same <u>positive</u> number.
3. Multiplying or dividing both sides by the same negative number and changing > to < or < to >.

Can you write similar statements about equivalent inequalities that use $\leq$? that use $\geq$?

<u>Problems 8-9B</u>

1. Which of the following inequalities are true?

(a) $3 > 2$

(b) $3(-1) > 2(-1)$

(c) $3(-1) < 2(-1)$

(d) $-3 < -2$

(e) $(-3)(-1) < (-2)(-1)$

(f) $-2 < 3$

(g) $(-2)(-1) > 3(-1)$

(h) $(-2)(-1) < 3(-1)$

(i) $2 < -3$

(j) $4 < 5$

(k) $4(2) < 5(2)$

(l) $4(-2) < 5(-2)$

(m) $4\left(-\frac{1}{2}\right) < 5\left(-\frac{1}{2}\right)$

(n) $4\left(-\frac{1}{2}\right) > 5\left(-\frac{1}{2}\right)$

2. Which of the following are pairs of equivalent inequalities?

(a) $3x > 5;\ x > \frac{5}{3}$

(b) $3x > 5;\ (-3)x < -5$

(c) $\frac{1}{2}x < 6;\ -2x > -24$

(d) $3x < 2;\ x < \frac{2}{3}$

(e) $2 - x > 4;\ x < -2$

(f) $2x + 4 < 6;\ 1 > x$

2. (g) $\frac{2}{3}x > \frac{1}{2}$; $x > \frac{3}{4}$ (i) $\frac{2}{3} + \frac{x}{4} < -\frac{1}{3}$; $x < -4$

(h) $\frac{1}{2} - 2x > \frac{3}{2}$; $x < -\frac{1}{2}$ (j) $\frac{2}{3} - \frac{x}{4} < -\frac{1}{3}$; $x > 4$

8-10 Solution Sets of Inequalities

If you were given a rather complicated inequality, do you think you could write a succession of simpler, equivalent inequalities until you got an inequality whose truth set was easily seen? Is this simple inequality equivalent to the original inequality? Will the truth set of the simple inequality be the same as the truth set of the more complicated one?

Consider the inequality

$$\frac{7}{2}x - \frac{4}{3} > \frac{3}{4}x + \frac{2}{3}$$

where the domain is the set of rational numbers. This inequality is equivalent to

$$\frac{7}{2}x > \frac{3}{4}x + \frac{2}{3} + \frac{4}{3} \qquad \text{Why?}$$

which is equivalent to $\frac{11}{4}x > 2$ Why?

which is equivalent to $x > 2 \times \frac{4}{11}$ Why?

which is equivalent to $x > \frac{8}{11}$.

Hence the truth set of the original inequality is the set of all rational numbers greater than $\frac{8}{11}$.

As another example, consider the inequality

$$-\frac{2}{3}x + 3 \geq 0$$

where the domain of x is the set of rational numbers.

This is equivalent to $-\frac{2}{3}x \geq -3$ Why?

which is equivalent to $\frac{2}{3}x \leq 3$ Why?

which is equivalent to $x \leq \frac{3}{2} \times 3$ Why?

which is equivalent to $x \leq \frac{9}{2}$. Why?

Hence the solution set of $-\frac{2}{3}x + 3 \geq 0$ is

$$\left\{\text{rational numbers } x \text{ such that } x \leq \frac{9}{2}\right\}.$$

Problems 8-10

1. Determine the solution set of each of the following inequalities if the domain of the variable is the set of rational numbers.

(a) $x - 4 \leq 1\frac{1}{2}$

(b) $x - 3 \leq 2$

(c) $2x - 7 > 3$

(d) $2x + 1 < 1$

(e) $x + \frac{1}{2} \geq -\frac{3}{2}$

(f) $\frac{1}{3} > x - \frac{4}{5}$

(g) $(-2) + 5 + (-3x) \leq 4x + 7 + (-2x)$

(h) $-(2 + x) \leq 3 + (-7)$

(i) $\frac{2}{3} + y < 15$

(j) $3x \geq (8 + x) - 2$

2. Determine the solution set of each of the following inequalities if the domain of the variable is the set of rational numbers.

(a) $2x + 7 > \frac{5}{2}$

(b) $3x + 5 \leq \frac{1}{17}$

(c) $7 - 2x < 11$

(d) $\frac{4}{5} - \frac{1}{5}x \geq \frac{2}{7}$

2. (e) $\frac{2}{3} - \frac{1}{3}x > \frac{4}{3}x + \frac{2}{5}$

(f) $-2 + 3 \leq -5x$

(g) $-2 + (-4x) \geq -6$

(h) $(-4) + 7 < (-2x) + (-5)$

(i) $5 + (-2x) \leq 4x + (-3)$

(j) $\frac{2}{3} + \left(-\frac{5}{6}\right) < \left(-\frac{1}{6}\right) + (-3x)$

3. Determine the solution set of each of the following if the domain of the variable is the set of rational numbers.

(a) $3x - 13 > 26$

(b) $2x - 7 < 32 - x$

(c) $4x - 9 \geq -5$

(d) $14 \geq 8 - 2x$

(e) $\frac{1}{2}x + \frac{3}{4} < 2$

(f) $\frac{3}{4}x + \frac{1}{2} > \frac{17}{16}$

(g) $\frac{3}{2}y - \frac{2}{5} < \frac{4}{5}$

(h) $8 - 3x \geq 11$

(i) $\frac{1}{2}(5x - 4) \leq x + \frac{11}{2}$

(j) $18 > \frac{7}{4}x + \frac{21}{4}$

(k) $18 - 3x < x + 3$

(l) $\frac{4}{5}x - \frac{3}{4}x \leq \frac{7}{15}$

(m) $\frac{1}{3}(3x + 9) \geq x + 3$

(n) $\frac{1}{2}(2x + 3) > x + 1$

CHAPTER 9

WORD PROBLEMS

9-1 Problems and Open Sentences

I am thinking of a number. I add six to it. The result is ten. What number was I thinking of?

You will not find it difficult to discover the number that I had in mind. Suppose, however, that you wished to state the problem in another way, using fewer words but expressing the same idea. You might say:

A certain number plus six is ten. Find the number. You are probably saying that you could write an even shorter statement of the problem. Why not use a variable? If x is a rational number, then we could write

$$x + 6 = 10. \text{ Find } x.$$

We have now expressed the ideas of our problem in the form of an open sentence,

$$x + 6 = 10. \tag{1}$$

(We shall refer to this equation as Equation (1).)

What do we gain by expressing a problem in the form of an open sentence? Since we know how to solve some equations and inequalities, when we translate a problem into an open sentence, we are expressing it in a form from which we can sometimes find an answer to the problem.

We have also seen that an open sentence expresses a problem in a clear and concise way. As you gain more experience in meeting and solving problems, you will observe that many problems which look different at first, may look very much alike when they are expressed in terms of open sentences. Let us compare some problems.

A man travelled a certain distance from his home town and rested. He started again and travelled six miles more. He then found that he had travelled ten miles all together. What distance had he travelled before he rested?

Suppose the man had traveled y miles before he rested. Then the numbers of the problem are related by the equation

$$y + 6 = 10. \tag{2}$$

You will notice that equation (2) is equivalent to equation (1). Can you think of other problems which produce equations equivalent to equations (1) and (2)?

Consider the following problem:
A boy wanted to find how much water was inside a partly full ten-gallon drum. He took a full four-gallon can of water and emptied it into the drum. It was not full. He then emptied a further full two-gallon can of water into the drum, and it became quite full. How many gallons of water were in the drum before the boy started pouring water into it?

In this problem we know that:

the number of gallons of water in the drum at first
+ the number of gallons of water poured into the drum
= the number of gallons of water that the drum could hold.

Suppose there were z gallons of water in the drum at first.
Then

$$z + 4 + 2 = 10$$

$$z + 6 = 10 \tag{3}$$

You will notice that equation (3) is equivalent to equations (1) and (2). By solving one equation we could find the answers to all three problems.

There are many problems which produce equations which we can solve. Do not be put off because a problem contains many words. You will have to study each problem carefully, select variables to represent unknown numbers, and translate your problems into open sentences.

You will find that you can invent problems too. In the open sentences given below try to think of problem situations, such as the ones we just discussed, which will lead to those equations.

Problems 9-1

Invent word problems which could be translated into the following open sentences:

1. $x + 2 = 5$

2. $y + 1 = 9$

3. $3 \times z = 12$

4. $5 \times q = 43$

5. $p - 6 = 2$

6. $x + y = 10$

9-2 Solving Word Problems

We have said that it is often very useful to be able to translate a problem into an open sentence in order to be able to solve it. Let us look closely at an example of such a translation to see how it goes. Then we shall see how you can develop the skills by which you can solve such problems yourself.

A man paid 29 shillings for 11 books. Some of the books were geography books, and the rest were history books. If each of the geography books cost 3 shillings, and each of the history books cost 2 shillings, how many geography books did he buy?

You should make sure that you understand what the problem is about before you write anything. Can you visualize the two piles of books, 11 of them all together? Can you see the price tag "3 shillings each" on the geography books and "2 shillings each" on the history books? Can you see the man paying 29 shillings?

Have you understood the problem? Let us now try to solve it. We might try to solve it by guessing the answer! This can be quite a useful way of starting,

and we might learn something from guessing. Suppose our guess is that the man bought 5 geography books. How do we know whether our guess satisfies the problems? We must test our guess! We test it as follows:

If the man bought 5 geography books, then he must have bought 6 history books. (Why?)

Each of the geography books costs 3 shillings, so

5 geography books will cost 3×5 shillings.

Each of the history books costs 2 shillings, so

6 history books will cost 2×6 shillings.

We know that he paid 29 shillings for all the books. Hence if our guess is right, then

$$(3 \times 5) + (2 \times 6) = 29$$

or

$$15 + 12 = 29.$$

We know that this is a false statement, but we may learn something from our guess.

You might try a few more guesses which may lead you to a solution of the problem. Try to guess the answers to your problems, but remember, you must always test your guess.

Consider the problem again. Let us try to solve it by translating it into an open sentence. In order to do this, we must observe quite clearly the numbers that we do not yet know, and then select variables for them. We sometimes refer to these numbers that we do not yet know as unknowns.

The unknowns in this problem are:

(1) The number of geography books the man bought.

(2) The number of history books the man bought.

Are there any more unknowns? Let us represent these unknowns by variables as follows:

Suppose the man bought p geography books and q history books. Then we can write $(p + q)$ for the total number of books he bought. Since we know that he bought 11 books, we can write the open sentence

$$p + q = 11.$$

We call this an equation in two variables. The variables are p and q. So far you have learned to solve only equations in one variable. You will learn some methods of solving systems of equations in two variables in the next chapter.

This equation deals with only a part of the problem. We can write another equation which uses the fact that we know the total cost of the books. We know that

the cost in shillings of the geography books
+ the cost in shillings of the history books
= the total cost in shillings.

Use the pattern we saw when we guessed that there were 5 geography books.

p geography books at 3 shillings each will cost $(3 \times p)$ shillings.

q history books at 2 shillings each will cost $(2 \times q)$ shillings.

Do you see another equation coming?

$$(3 \times p) + (2 \times q) = 29$$

This also has two variables.

Since we <u>have</u> learned to solve some open sentences in <u>one</u> variable, it would be more productive for us at present if we could translate our problem into an open sentence in one variable. It is quite possible to do so in this problem. The unknowns, as was stated before, are

(1) the number of geography books bought,

(2) the number of history books bought.

If we know (1), can we find (2)? Remember what we did when we guessed that there were 5 geography books. We found the number of history books by taking $(11 - 5)$. If we know the number of books in one pile, we can find the number of books in the other pile by subtracting from the total number of books. In terms of a variable we can say:

If there are x geography books, then there are $(11 - x)$ history books.

The expression $(11 - x)$ is a symbol for a number in the same way that x is a symbol for a number. We say that $(11 - x)$ is a <u>formula</u> or a <u>mathematical phrase</u> or an <u>expression</u> for the number of history books in terms of the number of geography books.

We can also obtain formulas for the cost of the books.

One geography book costs 3 shillings, so
x geography books cost $(3 \times x)$ shillings.
One history book costs 2 shillings, so
$(11 - x)$ history books cost $\left[2 \times (11 - x)\right]$
shillings.

We can write "$3 \times x$" as "$3x$" and "$2 \times (11 - x)$" as "$2(11 - x)$". Since we know that the total cost of these books is 29 shillings, we are now ready to write

$$3x + 2(11 - 2) = 29.$$

This is an equation in one variable which you are able to solve and which will lead you to an answer to our problem.

The following equations are equivalent.

$$\begin{aligned} 3x + 2(11 - x) &= 29 \\ 3x + 22 - 2x &= 29 \\ x + 22 &= 29 \\ x &= 7 \end{aligned}$$

The man bought 7 geography books and 4 history books.

Let us check our answers as we checked our guesses.

He bought 11 books: $7 + 4 = 11$.
They cost 29 shillings: $(3 \times 7) + (2 \times 4) = 21 + 8 = 29$.

You will notice that we have been able to solve the problem using only one variable, and that it is not essential in this case to use more than one variable.

9-3 Formulas

You noticed in the previous section that we needed to write expressions or formulas for certain numbers in terms of other numbers. These formulas are very useful in building open sentences.

Example. The sum of two numbers is 20. If one of the numbers is x, write a formula for the other number. The other number is $(20 - x)$.

While our feeling for the two parts of a whole (that if you take away one part of the whole, the other part is left) makes us believe that this is a correct formula, we may want to test it for some specific values of the variable. We want the formula to be correct for any number x that we may choose. We cannot expect to check all possible values of x, but by trying some values we may have some further indications that the formula will work. For example, to test the formula $(20 - x)$ we see that:

If $x = 3$, then the other number is $(20 - 3)$ or 17.
The formula works because $3 + 17 = 20$.
If $x = 7$, then the other number is $(20 - 7)$ or 13.
The formula works because $7 + 13 = 20$.

There are many varieties of situations for which we need to write expressions to use in solving problems. We develop skill in writing these expressions or formulas largely by experience. You will need to read carefully and use good common sense.

Here are some samples.

1. John is 3 inches taller than Matthew. If Matthew is x inches tall, then John is $(x + 3)$ inches tall.

2. John is 3 times as tall as Matthew. If Matthew is x inches tall, then John is $(3x)$ inches tall.

3. John and Matthew together have 42 shillings. If John has y shillings, then Matthew has $(42 - y)$ shillings.

4. John's mother is 3 years younger than Matthew's mother. If Matthew's mother is a years old, then John's mother is $(a - 3)$ years old.

5. John's village has only half as many people as Matthew's village. If Matthew's village has p people, then John's village has $\left(\frac{1}{2}p\right)$ people.

6. The number of people in John's family is greater by 3 than the number of people in Matthew's family. If there are m people in Matthew's family, then there are $(m + 3)$ people in John's family.

7. John's age is less by 2 years than Matthew's age. If Matthew is x years old, then John is $(x - 2)$ years old.

8. John and Matthew have some long poles. If John's pole were decreased in length by 4 feet, it would be the same length as Matthew's pole. If John's pole is f feet long, then Matthew's pole is $(f - 4)$ feet long.

9. Matthew's house is x yards long. It is therefore $(3x)$ feet long.

10. John is a years old now. Four years ago his age was $(a - 4)$ years.

Now you do some.

Problems 9-3

1. The sum of two numbers is 100. If one of the numbers is x, write a formula for the other.

2. The product of two numbers is 20. If one of the numbers is n, write a formula for the other.

3. The difference of two numbers is 7. If one of the numbers is y, write a formula for the other.

4. Kizza and Abbiw have 30 bananas between them. If Kizza has p bananas, how many bananas has Abbiw?

5. If Awadagin has 12 more mangoes than Funlays, and Funlays has x mangoes, how many has Awadagin?

6. Give a formula for each of the following:

   (a) The number of inches in s feet.

   (b) The number of inches in m yards.

   (c) The number of cents in q shillings.

   (d) The number of feet in d miles.

   (e) The number of ounces in p pounds.

   (f) The number of inches in s feet and t inches.

   (g) The number of ounces in p lbs. and q oz.

7. Choose a number. Multiply it by 2 and subtract 5 from the product. Write a formula for the number obtained in terms of the number chosen. Since the variable has not already been indicated this time, you must be sure to show it clearly yourself. For example, "If n is the number chosen, then the number obtained is _____".

8. Olayunka and Olu have 20 pineapples between them. Write a formula for the number of pineapples that Olu has, in terms of the number that Olayunka has.

9. A positive number is half another number. Write a formula for the larger number in terms of the smaller number.

10. A positive number is one third as big as another number. Write a formula for the smaller number in terms of the larger number.

11. Given a number, add 10 to it, multiply the sum by 3, and subtract the given number from the resulting number. Write a formula for the number obtained in terms of the given number.

12. (a) A number is 5 less than x. Write a formula for the number.

    (b) Use the formula to find the number which is 5 less than 3.

13. (a) A number is 2 greater than the number k. Write a formula for the number.

    (b) What number is 2 greater than (-14)?

14. (a) A number is less than another number by $\frac{3}{4}$. Write an expression for the first of the numbers in terms of the other.

    (b) What number is less than $\left(-7\frac{1}{2}\right)$ by $\frac{3}{4}$?

15. (a) A number is greater than twice another number by 11. Write an expression for the larger number in terms of the smaller number.

    (b) What number is greater than twice the number $\left(-\frac{7}{16}\right)$ by 11?

16. (a) One number is greater than half another number by 15. Write a formula for the first number in terms of the other.

    (b) What number is greater than half of 7 by 15?

    (c) What number is greater than half of (-7) by 15?

17. Alfred is 5 years older than Kasule. Write a formula for Alfred's age in terms of Kasule's age.

18. A boy is 8 years younger than his sister. Write an expression for the boy's age in terms of his sister's age.

19. A father is four times as old as his son. Write a formula for the father's age in terms of his son's age.

20. A boy's age is one seventh the age of his father. Write a formula for the father's age in terms of the age of his son.

21. In a certain community there were two thirds as many girls as boys. Write a formula for the number of girls in terms of the number of boys.

22. The price of admission to a concert is 4 shillings per person. Write a formula for the total number of shillings in terms of the total number of tickets bought.

23. A man can paint a house in x days. Write a formula for the part of the house that he can paint in one day.

24. If a pipe fills $\frac{1}{6}$ of a swimming pool in one hour, write a formula for how much of the pool is filled in y hours.

25. Write a formula for the length of a second side of a triangle, if it is 2 inches less than half the length of the first side.

26. A plant grows a certain number of inches per week. It is now 10 inches tall. Write a formula giving the number of inches in its height five weeks from now.

27. Manuel rides his bicycle at the uniform speed of 8 miles per hour. Write an expression for the distance he travels in a given time.

28. An airplane flies 2000 miles at a uniform speed in h hours. Write an expression for the speed of the plane.

29. A boy walks 23 miles at a uniform speed. Write a formula for the time he walks in terms of the speed.

30. A rectangle is 10 feet longer than it is wide. Find a formula for the length of the rectangle in terms of its width.

31. A rectangular field is 25 yards long and w yards wide. Write an expression for the perimeter of the field.

32. The area of a rectangular garden plot is 600 sq. ft. and it is w ft. wide. Write an expression for the length of the plot.

33. Write a formula for the length of a rectangle if the length is five feet less than twice the width.

34. Write a formula for the perimeter of the rectangle described in Problem 33.

35. Write a formula for the area of the rectangle described in Problem 33.

36. If n is an even number, write a formula for the next consecutive even number.

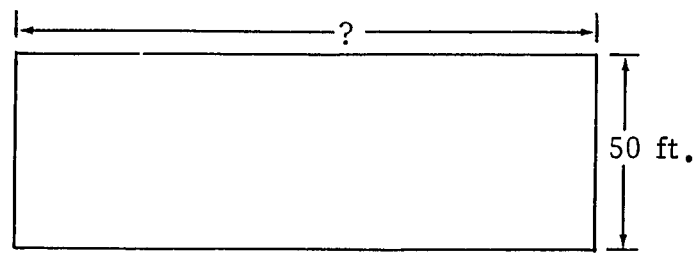
37. If m is an odd number, write a formula for the next consecutive odd number.

9-4 Writing Open Sentences

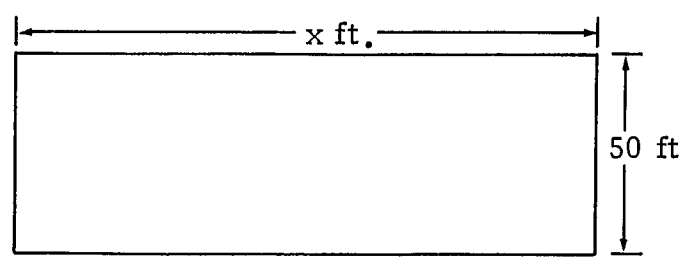
You have had some experience now in writing expressions for quantities which are involved in word problems. The next thing is to put together such expressions to make open sentences. Let us consider some problems and see how they can be translated into open sentences.

Example 1. A man has 260 ft. of fencing which he is going to put around a rectangular field which is 50 ft. wide. How long is the field?

As was suggested earlier, you need to read carefully and try to see clearly in your mind exactly what the problem is saying. In this case the first thing to do would be to draw a picture of the field and mark on it all the information you can.



Next you want to recognize what quantity or quantities are unknown. Since we want algebra to help us, we shall want to use a variable as a symbol for the unknown quantity in which we are most interested. In this case, clearly the length of the field is what we care about. Let us say, then, that the field is x feet long and show this on the picture.



Now we ask what else we know about the situation. Here is where you have to read most carefully and use plenty of common sense. The man is going to use 260 feet of fencing to put around the field. Picture the fence going around all four sides, and ask how you could express that distance in terms of your variable. Here is where your skill in writing formulas comes in.

What we really know here is that the perimeter of the field is 260 feet. What is the perimeter in terms of x? Since the perimeter includes the measure of all four sides, we would need

$$x + 50 + x + 50$$

which we might write

$$2(x + 50).$$

With this formula for the perimeter, can you now see how to write an open sentence for this problem? It will be just a statement of fact from the conditions of the problem.

In words: The perimeter is 260 feet.

Open sentence: $2(x + 50) = 260.$

Before we go on and finish the problem by solving the equation, let us do more practice just on setting up open sentences for problems.

<u>Example 2</u>. If 77 is added to a number, the sum is 8 times the number. Find the number.

Some number problems can be translated almost word for word into open sentences. Here, if n is the number, we might visualize the translation this way.

77	added to	the number	the sum is	8	times	the number.
77	+	n	=	8	×	n

<u>Example 3</u>. The sum of two numbers is 24. Twice the first plus half the second is 10. Find the numbers.

Some number problems are not so obvious. If you do not see the open sentence quickly, it may help to pretend that you know the answer and see what you would do to check it.

In this problem let us guess that the answer is 6 for the first number. What, then, is the second number? It is $(24 - 6)$, and to see the pattern we

should keep it as $(24 - 6)$ rather than change it to 18. Let us try to check these two numbers, 6 and $(24 - 6)$.

What is twice the first number? 2×6

What is half the second number? $\frac{1}{2}(24 - 6)$

The problem says "plus" between these, so we write

$$(2 \times 6) + \frac{1}{2}(24 - 6).$$

What does the problem say about this sum? It is equal to 10.

$$(2 \times 6) + \frac{1}{2}(24 - 6) = 10$$

$$12 + \frac{1}{2}(18) = 10$$

We do not need to go farther to see that this is not true, but if we go back and use a variable where we tried 6, we shall obtain our open sentence.

If x is the first number, then $(24 - x)$ is the second number.

$$2x + \frac{1}{2}(24 - x) = 10$$

is the open sentence.

<u>Example 4.</u> Akanji ran at a certain speed for one third of an hour and then walked for half an hour at the rate of 3 miles an hour. At the end of that time he had gone 4 miles. How fast did he run?

If you cannot write down the solution easily, try guessing. Let us guess that Akanji ran at 6 m.p.h. If he ran at 6 m.p.h., then he ran a distance of $(6 \times \frac{1}{3})$ miles. (He ran for one third of an hour.) When he walked for half an hour at 3 m.p.h., he traveled $(3 \times \frac{1}{2})$ miles. If our guess is true, then

$$(6 \times \frac{1}{3}) + (3 \times \frac{1}{2}) = 4.$$

We can verify that this is not true, but if Akanji ran at x miles per hour, then he ran a distance of $(x \times \frac{1}{3})$ miles. It is still true that he walked $(3 \times \frac{1}{2})$ miles. Since he traveled 4 miles in all, the open sentence is

$$(x \times \frac{1}{3}) + (3 \times \frac{1}{2}) = 4$$

or

$$\frac{1}{3}x + \frac{3}{2} = 4$$

Example 5. A boy is 20 years younger than his father and the sum of their ages is 36 years. How old is the father and how old is the son?

We write this one in the form you should use in making a clear presentation of solutions of word problems. You will soon be completing solutions in the same way this solution is completed. Notice how the variable is carefully described at the start.

If the father is y years old, then the son is $(y - 20)$ years old.

$$\begin{aligned} y + (y - 20) &= 36 \\ 2y - 20 &= 36 \\ 2y &= 56 \\ y &= 28 \end{aligned}$$

The father is 28 years old and the son is 8 years old.

Check: Boy is 20 years younger: $8 = 28 - 20$.

Sum of ages is 36: $28 + 8 = 36$.

Problems 9-4

Write open sentences which would help solve the following problems, being careful to give the meaning of the variable for each. Do not find the truth sets of the open sentences. We shall do this later.

1. A rectangular field is 25 feet wide and has an area of 6125 sq. feet. How long is the field?

2. A rectangle is 6 times as long as it is wide. Its perimeter is 144 inches. How wide is the rectangle?

3. The length of a rectangle is 4 feet greater than its width. The perimeter is 100 inches. Find the width.

4. The width of a rectangle is 10 inches less than the length. If the perimeter is 140 inches, find the width.

5. The length of a rectangle is 3 feet greater than the width. If the area is 70 sq. feet, find the width.

6. The sum of a number, $\frac{3}{8}$ of the number, and $\frac{5}{16}$ of the number is $7\frac{1}{2}$. Find the number.

7. I am thinking of a certain number. If I multiply it by 6, add (-18) to the product, and take one third of the sum, I get (-2). Of what number am I thinking?

8. The sum of two consecutive even numbers is 54. Find the numbers.

9. The sum of two consecutive odd numbers is 208. Find the numbers.

10. The sum of two numbers is ten, and their difference is two. Find the numbers.

11. A man bought 20 books. Some cost 8 shillings each and the others cost 3 shillings each. If he spent 110 shillings in all, how many of the 3 shilling books did he buy?

12. There are twice as many shilling pieces as there are pound notes in a box. If the total value of the money in the box is 154 pounds, how many shilling pieces are there in the box?

13. Divide 102 shillings among three boys A, B, C, so that A gets twice as much as B, and C gets $1\frac{1}{2}$ times as much as A.

14. Musoka bought three plates and three mugs. If each plate cost him three shillings more than each mug and if he paid twenty-four shillings all together, find the cost of each mug.

15. A father is now four times as old as his son. If the sum of their ages ten years ago was sixty, find their present ages.

16. A boy is three years younger than his sister. If his age three years ago was two thirds of her age at that time, what are their present ages?

17. A table is three times as long as it is wide. If it were 3 feet shorter and 3 feet wider, it would be a square. How long and how wide is it? (Draw two pictures of the table top.)

18. One number is greater than another by 15. If five times the larger number minus twice the smaller number is 3, what are the numbers?

19. A boy was asked to subtract 7 from a certain number and divide the result by 5. Instead, he subtracted 5 and divided the result by 7. His answer was 4 less than it should have been. What was the given number?

20. A man had four sons, each of whom (except for the youngest) is 2 years older than his next younger brother. If the sum of the boys' ages is 52 years, how old is the oldest son?

21. Invent problems of your own which would lead to the following open sentences:

(a) $p + 1 = 18$

(b) $q - 5 = 20$

(c) $3x + 8 = 35$

(d) $x(x - 3) = 70$

22. Translate the following problem into an open sentence and then translate back into a new problem of your own invention:
I weigh 3 pounds more than my brother does, and the total weight of the two of us is 185 pounds. How much does my brother weigh?

9-5 Problems Involving Inequalities

Problems which contain phrases such as "is more than", is shorter than", "is older than", may lead to open sentences which are inequalities.

Example 1. A man who is 40 years old is more than twice as old as his son. How old is the son?

If the son is x years old, then the open sentence is $40 > 2x$.

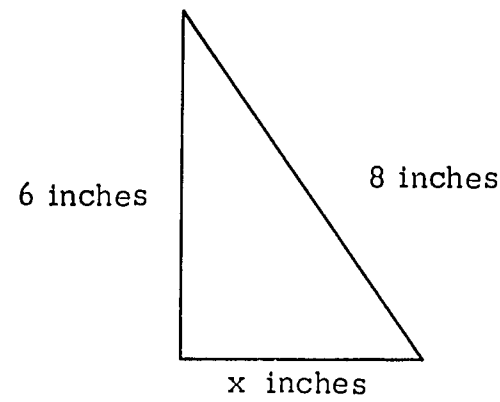
Example 2. Every number of a set of integers is greater than 40. What are the members of the set that are less than 43?

If y is a member of the set, y is an integer and $y > 40$.

If it is also true that y is less than 43, then $y < 43$. The numbers that are both members of the set and are less than 43 are given by the compound sentence: $40 < y < 43$, where y is an integer.

Example 3. Two sides of a triangle have lengths 6 inches and 8 inches. What is the length of the third side?

You may know that the sum of the lengths of two sides of a triangle is greater than the length of the third side. If the third side is x inches long, then $8 + 6 > x$. Also, $6 + x > 8$ so that $x > 2$. So $2 < x < 14$.



Problems 9-5

Write open sentences for each of the following problems, being careful to give the meaning of the variable for each.

1. Two sides of a triangle have lengths 10 inches and 14 inches. What is the length of the third side?

2. The smallest number in a set of integers is 512 and the largest is 999. What are the possible other members of the set?

3. The smallest number in a set of even numbers is 512 and the largest is 800. What are the possible members of the set?

4. The smallest number in a set of rational numbers is $-\frac{5}{16}$ and the largest is $\frac{5}{16}$. What are the possible members of the set?

5. A boy had only 100 cents in his pocket as he went to a concert where the admission ticket was 90 cents. If some of the money dropped on the way, and he was still able to pay the admission fee from the rest of the money, what amount could the boy have lost?

6. The local record for running a mile in a certain town is 4·1 minutes. If an athlete broke the record, in what time could he have run the mile?

7. In Problem 6, if the athlete did not break the record, in what time could he have run the mile?

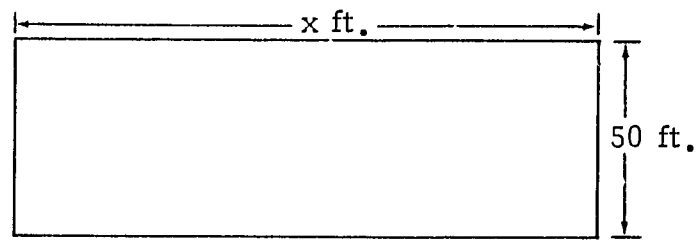
8. A student scores 70 and 76 marks in two tests. How many marks must he score in the third test to be put in Grade A if all pupils scoring an average of 80 or higher in three tests are put in Grade A?

9-6 Solving Word Problems

Let us go back to our first problem about the fence around the field and finish it in good form. We state the problem again.

A man has 260 ft. of fencing which he is going to put around a rectangular field which is 50 ft. wide. How long is the field?

If the field is x ft. long, then the perimeter is $(x + 50 + x + 50)$ ft.



$$x + 50 + x + 50 = 260$$
$$2x + 100 = 260$$
$$2x = 160$$
$$x = 80$$

The field is 80 ft. long.

Check: Total fencing is 260 ft.: $80 + 50 + 80 + 50 = 260$

Here are some of the points to remember about doing solutions to word problems efficiently and clearly.

1. Try to see clearly at the start what the problem is all about.

2. Look for the unknown quantities.

3. Show clearly what your variable represents, making sure that units, such as feet or pounds, are clearly indicated.

4. If necessary, write formulas for other unknown quantities in terms of the variable.

5. Write an open sentence which states a fact of the problem. Sometimes you can use direct translation word by word. Sometimes guessing an answer and checking it suggests the open sentence.

6. Use a sequence of equivalent equations to solve the open sentence.

7. Use the truth set of the open sentence to help answer clearly and fully the question or questions of the problem.

8. Check the answers by testing them in the conditions stated in the problem.

Problems 9-6

Complete the solution of some of the problems in Problems 9-4.

CHAPTER 10

EQUATIONS IN TWO VARIABLES

10-1 An Equation in Two Variables

Suppose that Mr. X and Mr. Y together have eight children. How could we express this fact in the language of algebra?

Since there are two numbers that we do not know, it seems natural to use two variables.

If Mr. X has x children and Mr. Y has y children, then

$$x + y = 8.$$

What are some numbers which will make this equation true? If Mr. X has 2 children, Mr. Y must have 6 children. We show this by saying:

If $x = 2$ and $y = 6$, the equation becomes
$2 + 6 = 8$, which is true.

What are some other pairs of numbers which make the equation true? List them in a table like this.

x	y
2	6
7	1
0	?
6	?
?	?

It takes two numbers, one for x and one for y, to satisfy (or make true) the equation $x + y = 8$. This pair of numbers is written in brackets thus:

$$(2, 6)$$

and we say that

(2, 6) <u>satisfies</u> the equation

or (2, 6) <u>is a solution</u> of the equation

or (2, 6) <u>is a member of the truth set of</u> the equation.

Since it is not the same thing for Mr. X to have 2 children and Mr. Y to have 6 children as it is for Mr. X to have 6 children and Mr. Y to have 2 children, we need a way to show that these are different situations. Your table suggests that (2, 6) and (6, 2) are two different symbols describing these two different situations. Since the order of the two numbers matters, we call (2, 6) <u>an ordered pair of numbers</u> and (6, 2) is a different ordered pair of numbers. How many different ordered pairs of numbers are there which express the number of children of Mr. X and Mr. Y?

In the symbol (6, 2), the first number written is a value of the first variable and the second number written is a value of the second variable. If x and y are the variables, it is customary to consider x the first variable and y the second variable regardless of the order in which they are written in the equation. Thus if we had $3y + x = 18$, the "first variable" would still be x.

Let us look at another example. Find some ordered pairs which satisfy the equation

$$2x + y = 10.$$

For instance (0, 10), (1, 8), (2, 6) are members of the truth set of the equation. Find five more members of the truth set.

Probably you had to suggest an ordered pair such as $(\frac{3}{2}, 7)$ or (8, -6). Is this all right? Since the numbers this time need not represent numbers of children, they need not be whole numbers and we may use any pair of rational numbers which makes the sentence true. Find five more solutions of the equation. How many ordered pairs do you think there are which will satisfy this equation? The pair (2, 6) does satisfy the equation. Does the pair (6, 2) satisfy it?

<u>Problems 10-1</u>

1. Show that the ordered pairs (1, 10), (5, 2), (0, 12), (8, -4) are members of the truth set of the equation $2x + y = 12$.

2. Show that the ordered pairs (6, 0), (2, 8), $(\frac{1}{2}, 11)$ satisfy the equation $2x + y = 12$.

3. Show that the ordered pairs (4, 6), (2, 5), (6, 1) are <u>not</u> members of the truth set of the equation $2x + y = 12$.

4. Find five additional ordered pairs which satisfy the equation $2x + y = 12$.

5. Which of the following ordered pairs satisfy the equation $3x + 4y = 7$?

$$(1, 0), (1, 1), (5, -2), (2, 2), (7, 3), (9, 5)$$

6. Complete the following ordered pairs to give solutions of the equation $x + 3y = 17$.

$$(\quad, 2), (\quad, 5), (\quad, 7), (\quad, 100), (\quad, -2),$$
$$(\quad, \tfrac{4}{3}), (\quad, -\tfrac{7}{3}), (\quad, \tfrac{1}{2}), (\quad, 3\cdot 5)$$

7. If you choose any number whatever for y in the equation $x + 3y = 17$, can you always find the corresponding value of x so that the ordered pair (x, y) is a member of the truth set? Explain how.

8. Complete the following ordered pairs to give solutions of the equation $2x - 3y = 8$.

$$(1, \quad), (7, \quad), (-2, \quad), (5, \quad), (\tfrac{1}{2}, \quad)$$

9. Choose any other value of x and show how the corresponding value of y can always be found to satisfy the equation $2x - 3y = 8$.

10. If x and y are restricted to whole numbers, that is, if the domain of x and of y is the set of whole numbers, verify that the truth set of $3x + y = 15$ can be written

$$\{(0, 15), (1, 12), (2, 9), (3, 6), (4, 3), (5, 0)\}$$

11. If the domain of x and of y is the set of whole numbers, how many ordered pairs will there be in the truth set of the equation

$$x - y = 2?$$

Could you write them all as we did in Problem 10?

12. CHALLENGE PROBLEM. Two whole numbers have the property that twice the larger plus three times the smaller is 24. Find the numbers.

10-2 A Problem in Two Variables

I am thinking of two numbers. The first number plus three times the second number is 18. Five times the first number minus three times the second number is also 18. Can you find the numbers?

Here there are two numbers which we do not know and two facts about these numbers are given. This suggests the possibility of using equations in two variables such as those in the preceding section.

If the first number is x and the second number is y, then

$$x + 3y = 18 \text{ and } 5x - 3y = 18$$

If we can find an ordered pair of numbers which satisfies both of these equations, we call this ordered pair a member of the truth set or a solution of the system of simultaneous equations. The word "simultaneous", which means "at the same time", tells us that both equations are to be satisfied by the same pair of numbers. The system is usually written with a bracket to indicate the word "and" between the two equations.

Thus to indicate

Find x and y if

$$x + 3y = 18$$
$$\text{and } 5x - 3y = 18$$

we usually write

$$\begin{cases} x + 3y = 18 \\ 5x - 3y = 18 \end{cases}$$

If $x = 12$ and $y = 2$, the first equation is true: $12 + (3 \times 2) = 18$ is true. The second equation is false: $(5 \times 12) - (3 \times 2) = 18$ is false. Hence $(12, 2)$ satisfies the first equation, but does <u>not</u> satisfy the system of equations.

If $x = 3$ and $y = -1$, the first equation is false: $3 + 3 \times (-1) = 18$ is false. The second equation is true: $(5 \times 3) - 3 \times (-1) = 18$ is true. Hence, $(3, -1)$ satisfies the second equation, but is <u>not</u> a solution of the system of equations.

One way you might find an ordered pair of numbers which satisfies <u>both</u> equations is to make two lists, one of ordered pairs which satisfy the first equation and one of ordered pairs which satisfy the second equation, and see whether the same ordered pair occurs in both lists.

Verify that the following ordered pairs of numbers are members of the truth set of $x + 3y = 18$.		Verify that the following ordered pairs of numbers are members of the truth set of $5x - 3y = 18$.	
$(0, 6)$	$(-3, 7)$	$(0, -6)$	$(-3, -11)$
$(3, 5)$	$(-6, 8)$	$(3, -1)$	$(-6, -16)$
$(6, 4)$	$(17, \frac{1}{3})$	$(6, 4)$	$(\frac{2}{5}, -\frac{16}{3})$
$(9, 3)$	$(-2, \frac{20}{3})$	$(9, 9)$	$(\frac{18}{5}, 0)$
$(12, 2)$		$(12, 14)$	
$(15, 1)$			
$(18, 0)$			
$(21, -1)$			

The ordered pair $(6, 4)$ is in both lists. That is, $6 + (3 \times 4) = 18$ and $(5 \times 6) - (3 \times 4) = 18$ are both true. Hence an answer to our original problem is that the first number is 6 and the second number is 4.

The ordered pair $(6, 4)$ is in the truth set of the <u>system of simultaneous equations</u>

$$\begin{cases} x + 3y = 18 \\ 5x - 3y = 18 \end{cases}$$

Problems 10-2

1. By making two lists as was done above, try to find an ordered pair of numbers in the truth set of the following system:

$$\begin{cases} 4x + 3y = 6 \\ 2x + 5y = -4 \end{cases}$$

2. (a) If the domain of x and of y is restricted to whole numbers, the truth set of the equation $3x + 2y = 18$ is finite. Find it.

   (b) Do the same for the equation $5x + y = 30$.

   (c) Name any ordered pairs which are in both of the truth sets in (a) and (b).

3. Follow the instructions of Problem 2 for the system of equations

$$\begin{cases} 2x + 5y = 35 \\ 6x + y = 23 \end{cases}$$

10-3 Systems of Simultaneous Equations

The method described in the preceding section is not very efficient. It depends on being able to make a good guess. How can we use familiar properties of numbers to find the truth set of a system of equations more easily?

Consider the system

$$\begin{cases} 5x + 3y = 9 \\ -5x + 2y = 1 \end{cases}$$

Our aim will be to find a simpler system which has the same truth set but which has an equation with only one variable. From there on we can use familiar methods.

We shall use the idea that the sum of two numbers is the same no matter what names you use for the numbers. If x and y represent a pair of numbers in the truth set of the system, then $\underline{5x + 3y}$ and $\underline{9}$ are two names for a number and $\underline{-5x + 2y}$ and $\underline{1}$ are names for another number. Since we can use either of the names, the sum of these numbers can be written either as $(5x + 3y) + (-5x + 2y)$ or as $9 + 1$, which means that

$$(5x + 3y) + (-5x + 2y) = 9 + 1$$

is true for the same pair of numbers x and y that we started with.

The advantage of this new equation is that when it is simplified, it has only one variable. We planned it that way.

$$\begin{aligned} (5x + 3y) + (-5x + 2y) &= 9 + 1 \\ 5y &= 10 \\ y &= 2 \end{aligned}$$

Now in the original system of equations we can replace one of the equations (either one) by the simpler equation $y = 2$.

$$\begin{cases} 5x + 3y = 9 \\ y = 2 \end{cases}$$

To make the second equation true y must be 2. Hence y must be 2 in the first equation also. Thus the first equation can be replaced by

$$\begin{aligned} & 5x + (3 \times 2) = 9 \\ \text{or} \quad & 5x = 3 \\ \text{or} \quad & x = \frac{3}{5} \end{aligned}$$

Our system has become

$$\begin{cases} x = \frac{3}{5} \\ y = 2 \end{cases}$$

The pair of numbers $(\frac{3}{5}, 2)$ satisfies this last system of equations. Does it satisfy the original system? We must check.

$$(5 \times \tfrac{3}{5}) + (3 \times 2) = 9 \text{ is true}$$

$$(-5) \times \tfrac{3}{5} + (2 \times 2) = 1 \text{ is true.}$$

Hence $\left\{ (\frac{3}{5}, 2) \right\}$ is the truth set of the given system.

Suppose we had used the equation $y = 2$ in place of the <u>first</u> equation of the original system instead of the second one, giving the system

$$\begin{cases} y = 2 \\ -5x + 2y = 1 \end{cases}$$

Solve this system and show that you still get the truth set $\left\{ (\frac{3}{5}, 2) \right\}$.

Let us consider another example. This time we shall obtain an equation in one variable by using subtraction instead of addition. Do you see why subtraction removes the variable y ? Give a reason for each of the following steps.

$$\begin{cases} 2x + 4y = -12 \\ 5x + 4y = -33 \end{cases}$$

$$(2x + 4y) - (5x + 4y) = (-12) - (-33) \qquad \text{(Why?)}$$

$$-3x = 21 \qquad \text{(Why?)}$$

$$x = -7 \qquad \text{(Why?)}$$

$$\begin{cases} 2x + 4y = -12 \\ x = -7 \end{cases} \qquad \text{(Why?)}$$

$$2(-7) + 4y = -12 \qquad \text{(Why?)}$$

$$4y = 2 \qquad \text{(Why?)}$$

$$y = \tfrac{1}{2} \qquad \text{(Why?)}$$

$$\begin{cases} x = -7 \\ y = \frac{1}{2} \end{cases} \qquad \text{(Why?)}$$

Check: $2 \times (-7) + (4 \times \frac{1}{2}) = -12$ is true

$5 \times (-7) + (4 \times \frac{1}{2}) = -33$ is true.

The truth set is $\left\{ (-7, \frac{1}{2}) \right\}$.

<u>Problems 10-3</u>

Find the truth set of each of the following systems of simultaneous equations. Decide carefully when to add and when to subtract.

1. $\begin{cases} 7x - 2y = 29 \\ 7x + y = 38 \end{cases}$

2. $\begin{cases} 10x - 3y = 36 \\ 8x + 3y = 18 \end{cases}$

3. $\begin{cases} 5x - 6y = 16 \\ 7x + 6y = 44 \end{cases}$

4. $\begin{cases} x + 6y = -5 \\ x - 9y = 0 \end{cases}$

10-4 More Systems of Simultaneous Equations

How can we solve the following system?

$$\begin{cases} 3x + 2y = 12 \\ 7x - 4y = 2 \end{cases}$$

Previously we could remove a variable by subtraction or addition because the coefficients of one of the variables in both equations were either the same or opposites. In this example we can bring about a similar situation by replacing the first equation by an equation in which the coefficient of y is 4. Do you see how to do this?

If the equation $3x + 2y = 12$ is true for some ordered pair of numbers, the equation $6x + 4y = 24$ will be true for the same pair of numbers. Why?

If we replace $3x + 2y = 12$ by the equivalent equation $6x + 4y = 24$, we obtain the system

$$\begin{cases} 6x + 4y = 24 \\ 7x - 4y = 2 \end{cases}$$

Can you complete the job of finding a solution? What is the truth set?

Sometimes, to avoid fractions, it is convenient to multiply each of the equations by a suitably chosen number in order to make the coefficients of one of the variables the same or opposites. For example:

$$\begin{cases} 7x + 4y = 43 \\ 5x + 6y = 26 \end{cases}$$

If we multiply in the first equation by 3 and in the second equation by 2, we obtain an equivalent system in which we can remove y by subtraction.

$$\begin{cases} 21x + 12y = 129 \\ 10x + 12y = 52 \end{cases}$$

Can you now complete the job of finding a solution? What is the truth set?

For the same problem, if you wished to make the coefficients of the x terms the same or opposites, what number would you multiply by in the first equation? What number would you multiply by in the second equation? Show that you obtain the same truth set no matter which variable you choose to eliminate.

Problems 10-4

For problems 1 through 7, find the truth set of each of the systems of simultaneous equations. In problem 1, solve first by eliminating the x variable, and then by eliminating the y variable. After that use whichever of these methods seems simpler.

1. $\begin{cases} 3x - 2y = 7 \\ 4x + 5y = 40 \end{cases}$

2. $\begin{cases} 2x - 7y = 1 \\ 5x + 17y = 37 \end{cases}$

3. $\begin{cases} 8x + 2y = 13 \\ 16x + y = 14 \end{cases}$

4. $\begin{cases} 6x - 5y = 3 \\ 3x + 11y = -39 \end{cases}$

5. $\begin{cases} \frac{2}{5}x + \frac{1}{3}y = 1 \\ \frac{3}{5}x - \frac{1}{9}y = 7 \end{cases}$

6. $\begin{cases} -7x + 12y = -11 \\ 50x + 9y = 47 \end{cases}$

7. $\begin{cases} 12x + 3y = 3 \\ 5 + y = -2 \end{cases}$

8. CHALLENGE PROBLEM. Explain why each of the following steps is justified in finding the truth set of the system.

$$\begin{cases} 3x - 5y = 21 \\ x - 4y = 3 \end{cases}$$

$$\begin{cases} 3x - 5y = 21 \\ x = 3 + 4y \end{cases}$$

$$\begin{cases} 3(3 + 4y) - 5y = 21 \\ x = 3 + 4y \end{cases}$$

Finish finding the truth set of the system.

9. CHALLENGE PROBLEM. Use the method suggested in problem 8 to solve the following systems of simultaneous equations.

(a) $\begin{cases} 2x + 6y = 13 \\ 8x + y = 6 \end{cases}$ (b) $\begin{cases} x + 3y = 5 \\ 7x - 6y = 44 \end{cases}$

10. CHALLENGE PROBLEM. Solve the system of simultaneous equations

$$\begin{cases} 3x + 1 = 0 \\ xy + 2 = 0 \end{cases}$$

10-5 Word Problems Using Two Variables

One person bought 2 pounds of tea and 10 pounds of sugar and paid 24 shillings. At the same prices another person paid 27 shillings for 1 pound of tea and 20 pounds of sugar. What was the price per pound for tea and for sugar?

If tea cost t shillings per pound and sugar cost s shillings per pound, then

$$\begin{cases} 2t + 10s = 24 \\ t + 20s = 27 \end{cases}$$

$$\begin{aligned} (2t + 10s) - 2(t + 20s) &= 24 - (2 \times 27) \\ 2t + 10s - 2t - 40s &= 24 - 54 \\ -30s &= -30 \\ s &= 1 \end{aligned}$$

$$\begin{cases} 2t + 10s = 24 \\ s = 1 \end{cases}$$

$$\begin{aligned} 2t + (10 \times 1) &= 24 \\ 2t &= 14 \\ t &= 7 \end{aligned}$$

$$\begin{cases} t = 7 \\ s = 1 \end{cases}$$

Rather than check in our equations it is safer to check in the words of the problem. If we have made an error in setting up the equations, checking in these incorrect equations will not catch our error.

If tea cost 7 shillings per pound and sugar cost 1 shilling per pound, then

the first person paid 14 shillings for tea and 10 shillings for sugar, or a total of 24 shillings;

the second person paid 7 shillings for tea and 20 shillings for sugar, or a total of 27 shillings.

This is correct. Price of tea: 7 shillings per pound.

Price of sugar: 1 shilling per pound.

Problems 10-5

Solve the following problems, using systems of equations in two variables:

1. I am thinking of two numbers. The first number plus three times the second number is 1. The first minus three times the second is 19. Find the numbers.

2. The sum of two numbers is $\frac{53}{24}$ and the difference of the same two numbers is $\frac{13}{24}$. What are the numbers?

3. In a class there are 10 more boys than there are girls. If there were one more girl, there would be twice as many boys as girls. How many boys and how many girls are in the class?

4. I was able to buy 12 pencils and 10 rulers for 21 shillings. At the same price each I could buy 20 pencils and 4 rulers for 16 shillings. What is the price of one pencil and the price of one ruler?

5. In triangle ABC, $\hat{A}$ is an angle of 20°. The number of degrees in $\hat{B}$ is 10 more than 9 times the number of degrees in $\hat{C}$. Find the size of $\hat{B}$ and $\hat{C}$.

6. A rope was cut into two pieces so that one piece was 18 ft. longer than the other. It was also true that one piece was three times as long as the other. How long was each piece and how long was the original rope?

7. An angle is three times its supplement. Find the size of the angle.

8. Twice the length of a rectangle is three times the width. The perimeter is 320 in. Find the dimensions.

9. A boy walked for 4 hours and bicycled for 3 hours covering a total distance of 45 miles. Later he walked for 2 hours and bicycled for 4 hours covering 50 miles. What was his average speed of walking and what was his average speed of riding a bicycle if these did not change in the two cases?

10. A carpenter can build a cabinet in 30 hours. If he works for 21 hours at the normal rate of pay, and 9 hours at the overtime rate, his working time costs 56 shillings, while if he works for 27 hours at the normal rate and 3 hours at the overtime rate, his working time costs 52 shillings. Find his normal and overtime rates of pay.

11. CHALLENGE PROBLEM. In decimal numerals a two digit number has the property that it is smaller by 27 than the number with the digits reversed. The sum of the digits of the number is 13. Find the number. (Hint: If the tens' digit is x and the units' digit is y, then the number is $10x + y$. What is the number with the digits reversed?)

12. CHALLENGE PROBLEM. In decimal numerals, if 36 is added to a certain two digit number the result in the number with the digits reversed. The units' digit is the one more than twice the tens' digit. What is the number?

13. CHALLENGE PROBLEM. Two trains leave stations 200 miles apart, the first at noon, the second at 12:15 P.M. They approach each other on parallel tracks and pass at 3 P.M. Each train travels at a constant speed, the one leaving at 12:15 P.M. moving at 10 m.p.h. faster than the other. What are their speeds?

14. CHALLENGE PROBLEM. Two trains leave stations 200 miles apart, the first at noon and the second at 12:15 P.M., each train travelling at a constant speed. They approach each other on parallel tracks and pass at 3 P.M. If

the times of departure had been reversed, they would still have met at 3:00 P.M. What are their speeds?

15. CHALLENGE PROBLEM. The total weight of a mixture of two liquids is 24 lbs. wt. and the total volume is 10 cu. ft. If one cubic foot of one of the liquids weighs 2 lbs. wt. and one cubic foot of the other liquid weighs 3 lbs. wt., what volume of each liquid is present? What weight of each liquid is present?

10-6 One and Two Variable Solutions

Some word problems can be solved more naturally using one variable. In other cases it is easier and more natural to use two variables.

For example we might have the following problem.

The sum of two consecutive even numbers is 126. Find each of the numbers.

Here we can let n be the smaller number and $n + 2$ the larger. Then

$$n + (n + 2) = 126$$

which is equivalent to $n = 62$. Therefore the two numbers are 62 and 64.

We have used the single variable n. We could of course let n be the smaller number and m the larger and write the system

$$\begin{cases} m - n = 2 \\ m + n = 126 \end{cases}$$

which is equivalent to

$$\begin{cases} m = 64 \\ n = 62 \end{cases}$$

The first solution is simpler and more natural.

Consider the following problem.

Two years ago a man was 7 times as old as his son, but in 3 years he will be only 4 times as old as the boy. How old are each of them?

In this case it is easier to use two variables, say m years for the present age of the man and s years for the present age of his son. The statement of the problem gives the two equations

$$\begin{cases} m - 2 = 7(s - 2) \\ m + 3 = 4(s + 3) \end{cases}$$

which are equivalent to

$$\begin{cases} m = 7s - 12 \\ m = 4s + 9 \end{cases}$$

The solution is $m = 37$, $s = 7$ so that the man is now 37 years old and his son is 7 years old. It is easy to check this result.

In this problem it would be hard to find a solution using only one variable because there is no obvious way of expressing the man's age in terms of the son's age or the son's age in terms of his father's age.

Problems 10-6

1. The sum of two numbers is 23. Twice the larger number is 4 more than 4 times the smaller. What are the numbers?

2. A rectangle has a perimeter of 80 feet. Twice its length is three times its width. What are the dimensions of the rectangle?

3. A two digit integer is divisible by 5 but not by 10. If the digits are reversed, the new number is 18 more than the original one. What is the integer?

4. One of the acute angles of a right triangle is 16° larger than the other. How many degrees are there in each of the acute angles?

5. Euclid is supposed to have given the following problem (about 280 B.C.): A mule and a donkey were going to market laden with wheat. The mule said "If you give me one measure, I should carry twice as much as you, but if I give you one, we should have equal burdens." What were their burdens?

6. A bottle and its contents cost 60 cents and the contents cost 18 cents more than the bottle. What did the bottle cost?

7. If 5 is added to both the numerator and the denominator of a certain fraction, the result is $\frac{4}{7}$. If 1 is subtracted from both numerator and denominator, the result is $\frac{2}{5}$. Find the fraction.

8. A motor boat can run 45 miles downstream in 2 hours and 30 minutes, and 39 miles upstream in 3 hours and 15 minutes. What is the speed of the boat in still water? What is the speed of the current?

CHAPTER 11

DECIMAL NUMERALS

11-1 Revision of Base Ten Numeration

A number like 238 can be written as $2(10^2) + 3(10) + 8$. When we are using the base ten, we say that 238 is written in decimal notation (or decimal form).

When we write it all out, showing the powers of ten, as in the example above, we say that it is written in expanded form.

Problems 11-1

1. Write in expanded form:

   (a) 356 (c) 682,376

   (b) 5,009 (d) 260

2. Write in decimal notation:

   (a) $3(10^2) + 7(10) + 5$ (c) $5(10^4) + 5(10^3) + 3(10^2) + 2(10) + 5$

   (b) $4(10^4) + 3(10^2) + 9$ (d) $8(10^4) + 7$

11-2 Addition and Subtraction with Decimal Numerals

The rules for adding and subtracting integers are easy to understand if we use the expanded form. Notice how the commutative, associative and distributive properties are used.

For example:

$$\begin{aligned} 28 + 37 &= [2(10) + 8] + [3(10) + 7] \\ &= [2(10) + 3(10)] + [8 + 7] \\ &= 5(10) + 15 \\ &= 5(10) + 1(10) + 5 \\ &= 6(10) + 5 = 65 \end{aligned}$$

Show how the steps of the expanded form explain what is happening in our usual shc . form of writing the addition.

(a)
$$\begin{array}{r} 28 \\ \underline{37} \\ 65 \end{array}$$

(b)
$$\begin{array}{r} 28 \\ \underline{37} \\ 15 \\ \underline{50} \\ 65 \end{array}$$

Similarly the expanded form explains subtraction.

$$38 - 26 = [3(10) + 8] - [2(10) + 6]$$

$$\begin{array}{r} 38 \\ \underline{26} \\ 12 \end{array}$$

Continue to simplify this as in the example above, giving a reason for each step. Can you explain why the normal subtraction method works?

Consider this example where you must regroup. The first two lines are given. Complete the simplification.

$$\begin{aligned} 65 - 47 &= [6(10) + 5] - [4(10) + 7] \\ &= [5(10) + 15] - [4(10) + 7] \end{aligned}$$

Can you pick out the step where the 1(10) was taken from the 6(10) and given to the 5? The associative property of addition permits us to do this. Notice that this step is needed because 7 is larger than 5. We need a number from which we can subtract 7.

Problems 11-2

Use the expanded form to add or subtract. Then check by computing as usual.

1. (a) $23 + 12$ (c) $87 - 35$

   (b) $6304 + 5487$ (d) $1000 - 912$

11-3 Multiplication with Decimal Numerals

Multiplication rules also can be explained using expanded notation.

$$34 \times 7 = [3(10) + 4] \times 7 = [3(10) \times 7] + (4 \times 7) \qquad \text{(Why?)}$$

Continue to simplify this as in the previous examples.

You may be used to writing out your multiplication in one of the ways (a), (b), or (c), given below.

Can you explain each part of (b) and (c)?

(a)	(b)	(c)
34	34	34
7	7	7
238	28	28
	21	210
	238	238

Problems 11-3

1. Use the expanded form to find a decimal numeral for each of the following products and then check by computing in the usual way. Explain what is happening in each part of the usual method.

(a) 37×4 (b) 36×11 (c) 123×49

11-4 Division with Decimal Numerals

To understand the rule for division you must not only know how to multiply and subtract, but you must understand the definition of division.

$a \div b = c$ means that $a = bc$

Example 1. We wish to find a decimal numeral for $736 \div 23$.

You are familiar with the usual long division method shown below.

$$\begin{array}{r} 32 \\ 23\overline{)736} \\ \underline{69} \\ 46 \\ \underline{46} \end{array}$$

We wish to understand why this method works. The object is to find a number q such that

$$736 = 23\,q.$$

We easily see that q must be more than 30 since $23 \times 30 = 690$, and 690 is too small. Also q must be less than 40 since $23 \times 40 = 920$, and 920 is too large.

The desired result is 30 plus a number between 0 and 10. Let us show this by writing the division as shown below.

```
      30 +
23 )736
    690
     46
```

23×30 is too small by 46. Since $46 = 23 \times 2$, we see that we must add 2 to 30 to obtain q.

```
      30 + 2
23 )736
    690
     46
     46
```

Thus we have found that

$$736 = (23 \times 30) + (23 \times 2)$$

and so $$736 = 23 \times 32$$

or $$736 \div 23 = 32$$

The usual method omits the 0 after 69 and compresses 30 + 2 into 32. Actually we write the 3 and leave the space behind it open until we decide that the next digit is 2.

<u>Example 2</u>. To find $6376 \div 27$ we give below on the right the usual method and on the left a <u>full</u> method of writing which shows more clearly what is going on.

```
      200 + 30 + 6 + 4/27
27) 6376
    5400
     976
     810
     166
     162
       4
```

```
      236 4/27
27) 6376
    54
     97
     81
     166
     162
       4
```

$$\text{In this case } 6376 = (200 \times 27) + (30 \times 27) + (6 \times 27) + \left(\frac{4}{27} \times 27\right)$$
$$= \left(236\frac{4}{27}\right) \times 27$$
$$6376 \div 27 = 236\frac{4}{27}$$

Problems 11-4

1. Locate the following quotients between multiples of 10, 100, or 1,000 as directed. For example to locate 547 ÷ 26 between multiples of 10 would be to write "547 ÷ 26 is between 20 and 30".

(a) 736 ÷ 72 between multiples of 10

(b) 4,876 ÷ 13 between multiples of 100

(c) 87,777 ÷ 29 between muitiples of 1,000

2. Find the following quotients by the full method. Then find them by the long division method. In one example explain where the numbers of the long division method occur in the full method.

(a) 98 ÷ 7 (b) 612 ÷ 17 (c) 612 ÷ 36 (d) 2,556 ÷ 35

11-5 Rational Numbers in Decimal Notation

Consider 2345, a number written in decimal notation. In expanded form $2345 = 2(10^3) + 3(10^2) + 4(10) + 5(1)$.

We notice that the place value of each digit is 10 times the place value of the digit immediately on the right. Let us now extend the decimal notation to include all rational numbers.

The number twenty-five, three tenths and seven hundredths may be written as

$$2(10) + 5(1) + 3\left(\frac{1}{10}\right) + 7\left(\frac{1}{10^2}\right)$$

We write this in decimal notation as 25·37, using the decimal point "·" to separate the digits representing whole numbers from the digits representing numbers less than one. We say this as "twenty-five point three seven".

Similarly 2345·678 can be written in an expanded form. We say "two thousand three hundred forty-five point six seven eight".

Problems 11-5

1. Write each of the following in decimal notation.

   (a) $4(10^2) + 3\ (10) + 2\ (1) + 5\left(\frac{1}{10}\right) + 6\left(\frac{1}{10^2}\right)$

   (b) $4\left(\frac{1}{10}\right) + 5\left(\frac{1}{10^2}\right) + 6\left(\frac{1}{10^3}\right)$

2. Write each of the following in expanded form.

   (a) 391.345 (b) 600.0102

11-6 Addition and Subtraction of Rational Numbers

When we add two numbers in decimal notation, our work in expanded form would look as follows:

Example 1. 5·64 + 4·62

	tens	ones	tenths	hundredths	
$5\cdot 64 = 5(1) + 6\left(\frac{1}{10}\right) + 4\left(\frac{1}{10^2}\right) =$		5	6	4	= 5·64
$4\cdot 62 = 4(1) + 6\left(\frac{1}{10}\right) + 2\left(\frac{1}{10^2}\right) =$		4	6	2	= 4·62
	1	0	2	6	10·26

Notice how the $12\left(\frac{1}{10}\right)$ becomes $10\left(\frac{1}{10}\right) + 2\left(\frac{1}{10}\right)$ or $1 + 2\left(\frac{1}{10}\right)$ and the 10(1) becomes 1(10).

Subtracting may be performed in a similar way.

Example 2. $9 \cdot 4218 - 6 \cdot 3456$

$$\begin{array}{r} 9 \cdot 4218 \\ -\ \underline{6 \cdot 3456} \\ 3 \cdot 0762 \end{array}$$

How can you justify this result? Use the expanded form to justify the subtraction of hundredths in the example, $4 \cdot 90 - 2 \cdot 43$.

Problems 11-6

1. Add (a) $3 \cdot 12 + 6 \cdot 31$ (b) $6 \cdot 950 + 8 \cdot 432 + 1 \cdot 100$

2. Subtract (a) $7 \cdot 62 - 6 \cdot 51$ (b) $3 \cdot 1200 - 3 \cdot 0104$

11-7 Multiplication and Division by Ten

Multiplication by Ten

You will need to recall that

$$10^2 \times 10 = 100 \times 10 = 10^3$$

and $$\frac{1}{10^2} \times 10 = \frac{1}{100} \times 10 = \frac{1}{10}$$

Consider $8 \cdot 4 \times 10$:

$$\begin{aligned} 8 \cdot 4 \times 10 &= [8(1) + 4\left(\frac{1}{10}\right)] \times 10 \\ &= 80 + 4 \qquad \text{(The distributive property)} \\ &= 84 \end{aligned}$$

Examining this result you will notice that the value of each digit has been multiplied by 10 to give a number 10 times as large and thus the digits in the result have moved one place to the left with reference to the decimal point. Check this observation by finding the product of $1 \cdot 31 \times 10$ with work in the expanded form.

When a number is multiplied by 10^2 or 100 the digits move two places to the left since each digit now has 10^2 times its former value.

Can you make a similar statement for multiplying by 10^3?

Can you make a similar statement for multiplying by 10^n, where n is a positive integer?

Division by Ten

Consider $13 \cdot 15 \div 10$

Find this quotient using the expanded form and the distributive property.

Can you make a statement regarding moving a digit when you are dividing by 10?

Find the quotient $1 \cdot 13 \div 10$ to check your statement.

The result of dividing by 10 is that of moving the digits one place to the right, since each digit now has one tenth its former place value.

Can you make similar statements for dividing by 10^2, 10^3, 10^n where n is a positive integer?

Problems 11-7

1. Multiply each of the following numbers by 10, 100, 1000 and explain one of the answers using expanded notation.

   (a) $0 \cdot 11$ (b) $0 \cdot 012$ (c) $8 \cdot 001$

2. Divide each of the following numbers by 10, 100, 1000 and explain one of your answers using expanded notation.

   (a) 71 (b) $100 \cdot 51$ (c) $0 \cdot 017$

3. If $57 \cdot 6 \times 0 \cdot 675 = 38 \cdot 88$, then $0 \cdot 576 \times 0 \cdot 0675$ is equal to

   (a) $388 \cdot 8$ (b) $38 \cdot 88$ (c) $3 \cdot 888$ (d) $0 \cdot 3888$ (e) $0 \cdot 03888$

4. If $0 \cdot 0107x = 10 \cdot 7$ then x is

   (a) 1 (b) 10 (c) 100 (d) 1000 (e) 10,000

11-8 Decimal Places, Multiplication and Division

Multiplication

The positions of the digits after the decimal point are called decimal places. Thus there are three decimal places in the numeral $41 \cdot 378$ and two decimal places in $0 \cdot 13$.

Consider $1 \cdot 31 \times 2 \cdot 2$.

$$1 \cdot 31 \times 2 \cdot 2 = \left(131 \times \frac{1}{100}\right) \times \left(22 \times \frac{1}{10}\right)$$

$$= (131 \times 22) \times \left(\frac{1}{100} \times \frac{1}{10}\right) \qquad \text{(Why?)}$$

$$= 2882 \times \frac{1}{1000}$$

$$= 2 \cdot 882$$

Notice that the number of decimal places in the answer is the sum of the number of decimal places in the factors. When we multiply numbers in decimal notation, we find the product as with whole numbers and then fix the decimal point according to the sum of the number of decimal places in the factors.

Expand the product $0 \cdot 013 \times 0 \cdot 041$ as in the worked example above.

Division

We first consider division by whole numbers.

Consider $0 \cdot 0555 \div 15$

```
      0.0037
15)0.0555
   0.045
   0.0105
   0.0105
```

It is easy to divide by a whole number. We therefore try to obtain whole number divisors for division.

Example

$$5 \cdot 12 \div 6 \cdot 4 = \frac{5 \cdot 12}{6 \cdot 4}$$

$$= \frac{5 \cdot 12}{6 \cdot 4} \times \frac{10}{10} \qquad \text{(Why?)}$$

$$= \frac{51 \cdot 2}{64}$$

Do you see why we chose to multiply by $\frac{10}{10}$ and not $\frac{100}{100}$ or $\frac{1000}{1000}$ or any other form of the number one? Our main purpose was to make the divisor a whole number.

We can therefore write:

$$\begin{array}{r} 0\cdot8 \\ 64\overline{)\,51\cdot2} \\ \underline{51\cdot2} \end{array}$$

Notice the process used to make the divisor a whole number. Use a similar process to find the quotient $4\cdot608 \div 0\cdot128$.

In the worked examples in this section the decimal point has been shown in the work. Many people who have to find such quotients frequently have the decimal point showing only in the dividend and the quotient.

Problems 11-8

1. Find these products:

   (a) $1\cdot3124 \times 0.02$ (b) $813 \times 0\cdot0001$

2. Find these quotients:

   (a) $1\cdot3124 \div 0\cdot02$ (b) $1\cdot69 \div 1\cdot3$

3. Find $\dfrac{0\cdot3124 \times 0\cdot025}{0\cdot8 \times 0\cdot05}$ as a decimal numeral.

11-9 Rough Approximations in Multiplication and Division

Sometimes it is useful to make a rough approximation of a product or a quotient without multiplying or dividing completely. Often this approximation is a useful check on the accuracy of the completed product or quotient.

Consider the following product:

$$97 \times 5\cdot652$$

97 is roughly 100 and $5\cdot652$ is roughly 5. A rough approximation of the above product therefore, would be 500. The exact product is $548\cdot244$.

Consider again the following division:

$$645 \cdot 575 \div 36 \cdot 89$$

A rough approximation of $645 \cdot 575$ is 600. A rough approximation of $36 \cdot 89$ is 40. Thus $(600 \div 40)$ or 15 would be a rough approximation of the above quotient. The exact quotient is $17 \cdot 5$.

Notice that, in these two examples, our rough approximations are not too far from the exact result. In the first case, if we multiplied out $97 \times 5 \cdot 652$ and got $54 \cdot 8244$ for the product we would compare this with our rough approximation and suspect that we had put the decimal point at the wrong place. Or suppose our result for the product was $748 \cdot 244$. Again, we would mentally compare this result with our rough approximation and suspect that something was wrong with our multiplication.

Notice also that, in both examples, we have approximated one of the numbers by a larger number and the other by a smaller number. In multiplication this is usually advisable. In division however, it is frequently better to approximate both numbers the same way - either both larger or both smaller. Can you think why this should be so? Consider $635 \div 125$. Which is the better method of approximating in this case?

Notice also that we did not approximate 97 by 95 or $5 \cdot 652$ by $5 \cdot 6$. We approximate the numbers so that the product or quotient can be done quickly and mentally.

Sometimes we are not interested in the results of a product or a quotient to the last digit and a rough approximation is sufficient for our purpose.

Let us consider the area of the island of Zanzibar. From a map, using the appropriate scale, we might say that the area of the island is approximately the area of a rectangle 55 miles by 15 miles. Using our approximation methods, we find the approximate area of the island in square miles is 60×10 or 600.

Here are some rough approximations:

(a) $0 \cdot 097 \times 3 \cdot 41$

$0 \cdot 097$ is roughly $0 \cdot 1$ and $3 \cdot 41$ is roughly 3 and so the product should be roughly $0 \cdot 3$.

The exact product is 0·33077.

(b) $0 \cdot 27 \times 4 \cdot 816$

$0 \cdot 27$ is roughly $0 \cdot 25$ or $\frac{1}{4}$; $4 \cdot 816$ is roughly $4 \cdot 8$ and $\frac{1}{4} \times 4 \cdot 8$ is $1 \cdot 2$.

The exact product is 1·30032.

Notice that if we had approximated 4·816 by 5 a useful approximation would still be obtained.

(c) $6 \cdot 4 \div 0 \cdot 15$

$6 \cdot 4$ is roughly 6. $\frac{6}{0 \cdot 15} = \frac{600}{15} = 40$

Thus, the quotient is roughly 40.

Problems 11-9

(Do not use paper and pencil)

1. Without calculating say whether the following quotients and products are approximately right or not:

(a) $4 \cdot 8 \times 3 \cdot 016 = 144 \cdot 768$

(b) $4 \cdot 8 \times 3 \cdot 016 = 24 \cdot 4768$

(c) $4 \cdot 8 \times 3 \cdot 016 = 14 \cdot 4768$

(d) $456 \cdot 3225 \times 0 \cdot 51 = 89 \cdot 475$

(e) $4 \cdot 433 \times 0 \cdot 97 = 4 \cdot 3$

(f) $4 \cdot 433 \times 0 \cdot 97 = 43$

2. It costs five dollars per day to live in a Y.M.C.A. in New York. 20 shillings is equal to about 2·80 dollars. Estimate roughly how many shillings it would cost for one week in the hotel.

3. Estimate from a map the distance from Accra to Lagos. How many hours would it take to travel this distance at an average speed of 45 miles per hour?

4. Look at a map of East Africa and estimate the area of Lake Victoria.

5. Look at a map of Africa and estimate the area of Ghana. Compare this area with the area of Kenya.

6. The truth set of the equation $\frac{x}{\cdot 004} = 2\cdot 84$ is (a) 12.1 (b) 0·00121 (c) 0·01136 (d) 113·6 (e) 2·836. Find, without multiplying, which answer is likely to be correct.

11-10 Rounding Off

The East African Statistical Bulletin gives the population of Uganda, at mid-year 1961, as 6,845,000 correct to the nearest thousand persons. What does this information tell us? It tells us, for instance, that the population figure was more nearly 6,845,000 persons, than 6,844,000 persons or 6,846,000 persons. The figure may have been 6,844,989 or 6,845,252, but it has been rounded off to 6,845,000, and this gives us an approximation of the population.

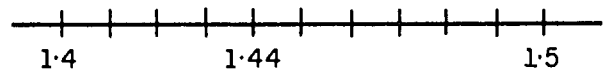
Figures are rounded off to the nearest million or thousand or tenth or thousandth etc., according to the degree of accuracy required. If a comparison is being made between the population of Uganda and that of another country whose population is given as 2,134,000, correct to the nearest thousand persons, it may be sufficient to say Uganda has 7 million people, correct to the nearest million persons. If, however, the population of Uganda is compared with that of a country whose population is given as 6,841,000, correct to the nearest thousand persons, it may not be sufficient to round off in millions, since rounding off in millions will not show the difference. On the other hand, for some purposes that difference may not be significant, and it may be sufficient to say that the two countries are the same size to the nearest million, or even to the nearest ten thousand.

If you are asked to find the length of a given line segment, you will probably use your ruler. This may give you its length to the nearest tenth of an inch. By using more refined methods, it may be possible to find the length to the nearest hundredth

of an inch. These results are approximations of the length of the line segment. It is impossible to measure the length of a line segment exactly. What is important is that the measurement should be reliable as far as it goes.

Rounding Off Decimal Numerals

Suppose we wish to round off 1·44 to the nearest tenth.



In the figure we have divided the interval of 0·1 from 1·4 to 1·5 into ten equal parts. Each part is a tenth of 0·1, or 0·01.

Hence the division marks are at 1·41, 1·42, 1·43, 1·44, 1·45, 1·46, 1·47, 1·48, 1·49.

It can easily be seen that 1·44 is nearer 1·4 than 1·5. Hence we say that 1·44 is 1·4 to the nearest tenth, or correct to one decimal place.

We can make a table of approximations as follows:

1·41	is	nearer	1·4	than	1·5.	Hence	1·41	is	1·4	correct	to	1	decimal	place.
1·42	"	"	1·4	"	1·5.	"	1·42	"	1·4	"	"	"	"	"
1·43	"	"	1·4	"	1·5.	"	1·43	"	1·4	"	"	"	"	"
1·44	"	"	1·4	"	1·5.	"	1·44	"	1·4	"	"	"	"	"
1·46	"	"	1·5	"	1·4.	"	1·46	"	1·5	"	"	"	"	"
1·47	"	"	1·5	"	1·4.	"	1·47	"	1·5	"	"	"	"	"
1·48	"	"	1·5	"	1·4	"	1·48	"	1·5	"	"	"	"	"
1·49	"	"	1·5	"	1·4.	"	1·49	"	1·5	"	"	"	"	"

1·45 is exactly halfway between 1·4 and 1·5. By convention we associate it with the higher of the two numbers. Hence 1·45 is 1·5 correct to 1 decimal place.

Example 1.

Write the following, correct to (a) 2 decimal places, (b) 3 decimal places.

(i) 5·8761 (ii) 68·2459 (iii) 3·1415 (iv) 146·9992.

For a result correct to 2 decimal places, we consider the digit in the third place and round off.

(i) 5·88 (ii) 68·25 (iii) 3·14 (iv) 147·00

For a result correct to 3 decimal places, we consider the digit in the fourth place and round off.

(i) 5·876 (ii) 68·246 (iii) 3·142 (iv) 146·999.

Example 2.

Write $\frac{1}{13}$ as a decimal numeral, correct to 3 decimal places. To do this we need to find $\frac{1}{13}$ as a decimal numeral to 4 places.

$$\begin{array}{r} 0\cdot0769 \\ 13\overline{)\,1\cdot0000} \end{array}$$

Here, considering the fourth figure, we have $\frac{1}{13}$ = 0·077 correct to 3 decimal places.

Problems 11-10

1. Write the following correct to the nearest thousand.

   (a) 6,123,456 (b) 5,432,109 (c) 1,235,678

2. Write the following correct to the nearest whole number.

   (a) 625·1 (b) 96·5 (c) 7·9 (d) 8·45131782

3. Write the following correct to 1 decimal place.

   (a) 25·234 (b) 6·519 (c) 619·991 (d) 8·512

4. Write the following correct to:

   (a) 2 decimal places (b) 3 decimal places

   (i) 1·23456 (ii) 9652·85491

   (iii) 2615·2112 (iv) 8·651397

5. Write the following numbers in decimal notation correct to:

(a) 2 decimal places (b) 3 decimal places

(i) $\frac{1}{3}$ (ii) $\frac{1}{6}$ (iii) $\frac{1}{7}$ (iv) $\frac{3}{11}$ (v) $\frac{5}{9}$

(vi) $\frac{13}{65}$ (vii) $\frac{17}{9}$ (viii) $\frac{81}{7}$ (ix) $\frac{98}{13}$ (x) $\frac{101}{19}$

11-11 Percentages

When you take an examination, the teacher often reports your grades in per cent. He may, for instance, say that Ngozi got 75 per cent. Do you know what this means? The expression "per cent" comes from a Latin phrase "per centum", meaning "for each hundred".

Suppose that in a particular examination in mathematics the teacher gave 10 problems and one pupil had 6 of the problems correct. If all the problems have equal marks, can you write this score in terms of per cent? You will notice that the ratio of the number of correct problems to the total number of problems is $\frac{6}{10}$. From our knowledge of equivalent fractions we know that this ratio can be written as $\frac{12}{20}$, $\frac{18}{30}$, $\frac{30}{50}$, $\frac{60}{100}$, etc. The fraction which indicates the ratio in terms of one hundred is $\frac{60}{100}$. Therefore, we say that the student had 60 percent of the problems right. He had 4 problems wrong. The ratio of the number 4 to the total number of problems given is $\frac{4}{10}$. This again can be written as $\frac{8}{20}$, $\frac{12}{30}$, $\frac{20}{50}$, $\frac{40}{100}$, etc. Similarly we can say that the pupil had 40 per cent of the problems wrong. Thus if a fraction is expressed with 100 as its denominator, it is said to be expressed as a <u>percentage</u>. The numerator of the fraction is called the <u>rate</u>.

For example, $\frac{35}{100}$ is a percentage, denoting 35 out of a hundred. This may be written as 35 per cent or 35%. The symbol % is a short way of writing per cent. It is convenient to have a symbol for $\frac{1}{100}$ and we use % to represent it.

$$\% = \frac{1}{100}$$

Thus we have the following relations:

$$35 \text{ per cent} = \frac{35}{100} = 35 \times \frac{1}{100} = 35\%$$

$$50 \text{ per cent} = \frac{50}{100} = 50 \times \frac{1}{100} = 50\%$$

$$6 \text{ per cent} = \frac{6}{100} = 6 \times \frac{1}{100} = 6\%$$

Problems 11-11A

Express the following in three equivalent forms:

1. 90%
2. 30 per cent
3. $\frac{5}{100}$
4. $7 \times \frac{1}{100}$
5. 55%
6. 100%
7. 45 per cent
8. $\frac{67}{100}$
9. $\frac{15}{100}$
10. $37 \times \frac{1}{100}$

A class of 30 pupils is made up of 18 boys and 12 girls. The ratio of the number of boys to the number of pupils in the class is $\frac{18}{30}$. This ratio can be expressed in various ways as $\frac{18}{30}, \frac{6}{10}, \frac{12}{20}, \frac{30}{50}, \frac{60}{100}, \frac{72}{120}$. Here, boys constitute 60 per cent of the class. Why? We can now state the following:

Any number $\frac{x}{y}$ can be expressed as a percentage by finding a number a such that $\frac{x}{y} = \frac{a}{100} = a \times \frac{1}{100} = a\%$ Here a is called the <u>rate</u>.

Example

Express $\frac{7}{8}$ as a percentage.

Let a denote the rate. Then we have,

$$\frac{7}{8} = \frac{a}{100}$$

$$a = \frac{7}{8} \times 100 = 87\frac{1}{2}$$

$$\frac{7}{8} = 87\frac{1}{2}\%$$

Problems 11-11B

1. Write each of the following numbers as a percentage.

(a) $\frac{2}{3}$	(f) 4	(k) $\frac{3}{5}$
(b) $\frac{1}{2}$	(g) $\frac{1}{3}$	(l) $\frac{4}{5}$
(c) $\frac{3}{4}$	(h) $\frac{5}{3}$	(m) $\frac{5}{5}$
(d) $\frac{4}{3}$	(i) $\frac{1}{5}$	(n) $\frac{8}{15}$
(e) $\frac{7}{5}$	(j) $\frac{2}{5}$	(o) $\frac{6}{5}$

2. Express the following as fractions in simplest form:

(a) $3\frac{1}{2}\%$	(c) 65 per cent
(b) $33\frac{1}{3}\%$	(d) 5.5%

3. Find the value of

(a) $5\frac{1}{2}\%$ of £75	(d) 10% of £230 10s
(b) $3\frac{1}{2}\%$ of £120	(e) 25% of $750.28
(c) 7% of £720	(f) 4% of $312.00

4. Express 33 as a per cent of 300.

5. What per cent of $27\frac{1}{2}$ tons is 4 tons 8 cwt.?

6. A room is 10 ft. 6 in. wide and 15 ft. 6 in. long. What per cent of the length is the width?

7. In an examination 70 problems were given and all problems had equal marks. A boy had 55 right. What percentage is this? In order to pass the examination it is necessary to answer at least 40% correctly. What is the lowest number of correct answers needed to pass?

8. In an election there were two candidates. One of them received 65% of the votes cast and secured a majority of 1500 votes, that is, 1500 votes more than his competitor. How many people voted?

9. In the year 1952 the population of a town was 72,000; in 1961 the population had risen to 85,200. Find the per cent increase of the population.

10. The enrollment in a certain secondary school was 450 in 1950. By 1960 the enrollment had increased by 16%. What was the enrollment in 1960?

11. A primary school had an enrollment of 850 pupils in January 1951. In January 1960 the enrollment was 1120. What was the percentage increase in enrollment?

12. During 1950 a family spent £250 on food. In 1960 the same family spent spent £375 on food. What was the percentage increase in the money spent on food?

11-12 Percentages and Decimal Numerals

In the preceding section you learned that $\frac{65}{100}$ can be written as 65% or as $65 \times \frac{1}{100}$. From your study of decimal numerals you know that $\frac{65}{100}$ can also be written as 0·65. Hence we can write $\frac{65}{100} = 65\% = 0{\cdot}65$. Similarly, $\frac{1}{2} = \frac{50}{100} = 50\% = 0{\cdot}50 = 0{\cdot}5$.

Suppose that we wish to express $\frac{1}{8}$ as a percentage. We know that $\frac{1}{8}$ as a decimal numeral is 0·125 and this can be written as $\frac{12{\cdot}5}{100}$. Hence, $\frac{1}{8}$ as a percentage becomes 12·5%. Similarly we have,

$$0{\cdot}01 = \frac{1}{100} = 1\%$$

$$0{\cdot}375 = \frac{37{\cdot}5}{100} = 37{\cdot}5\%$$

$$1{\cdot}75 = \frac{175}{100} = 175\%$$

Problems 11-12

1. Express the following as percentages:

(a) 0·325

(b) 0·05

(c) 2·3

(d) $\frac{1}{125}$

(e) $\frac{6}{7}$

2. Express the following as decimal numerals:

(a) 33%

(b) $\frac{1}{2}\%$

(c) 1·5%

(d) $12\frac{1}{2}\%$

(e) 250%

(f) 75%

11-13 Application of Percentages

Percentages have a very wide application in everyday transactions. One such application is in the comparison of business transactions in terms of per cent profit or loss.

Profit and Loss

When goods are bought for a sum of money and sold for another, there is a gain or loss on the transaction according as the selling price is more or less than the cost price. In business transactions the actual gains or losses do not give enough comparative information about the transactions. For instance, consider the following:

1. A set of books which cost £25 was sold at £30; and
2. Another set of books which cost £75 was sold at £80.

In each case the actual profit is £5, but one of the transactions is better business than the other. To find out which, we have to express the profit as a percentage of the cost.

In the first case this percentage is 20%.

$$\frac{5}{25} = \frac{20}{100} = 20 \times \frac{1}{100}$$

In the second case the percentage is $6\frac{2}{3}$ %.

$$\frac{5}{75} = \frac{5}{75} \times \frac{100}{100} = \frac{500}{75} \times \frac{1}{100} = 6\frac{2}{3} \times \frac{1}{100}$$

Hence the first transaction is better business. Notice that the profit has been calculated with reference to the cost price. This is the general practice. It is also a general and convenient practice to abbreviate the terms <u>cost price</u> and <u>selling price</u> by the letters <u>c.p.</u> and <u>s.p.</u>, respectively.

<u>Example 1</u>.

A trader bought 20 bicycles for £220 and sold them at £13 10s. each. What was his gain or loss per cent?

$$\text{s.p. of 20 bicycles} = 13\frac{1}{2} \times 20 = 270$$

$$\text{gain} = 270 - 220 = 50$$

$$\text{gain per cent} = \frac{50}{220} \times \frac{100}{100} = \frac{5,000}{220} \times \frac{1}{100} = 22\frac{8}{11} \times \frac{1}{100}$$

$$= 22\frac{8}{11}\%$$

<u>Example 2</u>.

An article which cost £150 was sold at a loss of 5%. What was the selling price?

$$\begin{aligned} \text{s.p.} &= \text{c.p.} - \text{loss} \\ &= \text{c.p.} - 5\% \text{ of c.p.} \\ &= 150 - \left(\frac{5}{100} \times 150\right) \\ &= 150 - 7\frac{1}{2} \\ &= 142\frac{1}{2} \end{aligned}$$

The selling price is £142 10s.

Example 3.

By selling goods for £168, a profit of 12% was made by a merchant. How much did the goods cost him?

If the cost of the goods was x pounds, then the profit was $168 - x$ pounds. The profit can be expressed also as $\frac{12}{100}x$ or $\frac{3x}{25}$.

$$168 - x = \frac{3x}{25}$$

$$25(168-x) = 25 \times \frac{3x}{25}$$

$$25(168-x) = 3x$$

$$25 \times 168 - 25x = 3x$$

$$25 \times 168 = 28x$$

$$\frac{25 \times 168}{28} = x$$

$$x = 25 \times 6$$

$$x = 150$$

The goods cost £150.

Problems 11-13A

1. A man bought a car for £875 and sold it for £750. Find his loss per cent.

2. A trader bought eggs at 4d. each and sold them at $6\frac{1}{2}$ d. each. Find his per cent profit.

3. A shopkeeper bought eggs at 5s. 6d. a dozen. He found that 10% of the eggs were rotten and sold the rest at 7s. 6d. a dozen. Find his per cent profit or loss.

4. An increase in wages of 15% makes the monthly wage bill of a company £5,980. What is the amount of the increase?

5. A woman bought eggs at 4s. 6d. per dozen. At what price per score must she sell them in order to gain 15%?

6. By selling goods for £175 10s., a profit of 17% was made by a merchant. How much did the goods cost him?

7. By selling goods for £250, a merchant lost 25% on his outlay. For how much should he have sold the goods in order to gain 10%?

8. At a clearance sale an article was reduced in price from 5s. to 4s. 9d. If the first price represents 25% profit, how much is the per cent of profit if the article is sold at the reduced price?

9. How much would I have to pay for a book which cost 3s. to produce, if the printing company sold it to a book seller at 20% profit and the book seller sold it to me at a profit of 25%?

10. A reduction of 25% in the price of rice will enable a trader to buy 50 pounds more for £1 5s. than he could have before the price reduction. What was the price per pound before reduction?

Commissions and Discounts

The business of buying and selling may be conducted by an agent on behalf of another person. For his services the agent is paid a fee called commission, which is a sum calculated as a percentage of the sum received or spent by the agent for his employer. For example: A book salesman is paid a commission of 15% of the selling price of a set of books. If he sells a set of books for £240, his commission is 15% of £240, or £36.

Stores often allow some reduction on cash payment for goods purchased. Such a reduction is normally calculated as a percentage of the original price and is called discount. During a sale, goods may be sold at a discount of, say, 10%. In such a case, goods originally marked £25 would then sell at £22 10s. Discount is also a sum of money deducted from a debt in consideration of its being paid before it is legally due.

Problems 11-13B

1. A real estate agent receives a commission of $7\frac{1}{2}$% of the selling price for any sale that he makes. What would be his commission on the sale of a house for £7,200?

2. A certain store gives 20% discount for cash. A woman paid shs. 110 cash for a dress. How much was the marked price of the dress?

3. A bookseller receives $12\frac{1}{2}$% commission on the sale of books for a publisher. In a certain month he received a commission of £75 from the company for his services for that month. How much was his sale for that month?

4. If a car dealer allows 5% cash discount and I paid £855 cash for a car, how much was its marked price?

5. How much cash must a man pay to settle an account of shs. 766.70 at 5% discount?

6. A bookseller received £90 commission for selling a set of books for £640. What was his rate of commission?

Simple Interest

When money is lent, particularly for business, the borrower is expected to pay for the use of the money. The amount of money lent is called the principal and the charge made for the use of the money is called interest.

Interest on money borrowed is paid at definite intervals (monthly, quarterly, half-yearly or yearly.) It is usually reckoned as a percentage of the principal for the period stated until the loan is repaid. If the interest is eckoned on the principal only, throughout the whole term of the loan, it is called simple interest.

The principal together with its interest for a stated period is called the amount for that time. Per annum means in one year.

Example

A sum of £750 was lent for 5 years at 3% per annum. Find the simple interest and the amount.

Interest in pounds on £750 for 1 year at 3% is $\frac{3}{100} \times 750$.

Interest in pounds on £750 for 5 years at 3% is $\frac{3}{100} \times 750 \times 5$

or $112\frac{1}{2}$.

$$\begin{aligned} \text{Amount} &= \text{Principal} + \text{Interest} \\ &= 750 + 112\tfrac{1}{2} \\ &= 862\tfrac{1}{2} \end{aligned}$$

The interest is £112 10s. and the amount is £862 10s.

From this example it is easy to see that the simple interest is equal to the principal multiplied by the rate per cent and the number of years. Thus if a principal of P pounds earns interest of I pounds at the rate of r per cent per year for a period of t years, then

$$I = \frac{Prt}{100}$$

Problems 11-13C

1. Find the simple interest on

   (a) £300 for 3 years at 5%.
   (b) £525 10s. for 4 years at $3\frac{1}{2}$%.
   (c) £750 for 3 years 4 months at $2\frac{1}{2}$%.
   (d) 750 dollars for $2\frac{1}{2}$ years at 3%.
   (e) 500 francs for 4 years at 5%.

2. If the simple interest on £500 at 4% per annum is £35, find the time.

3. If the simple interest on £250 for 5 years is £43 15s., find the rate per cent per annum.

4. If £350 amounts to £475 at $2\frac{1}{2}$% per annum, find the time.

5. If the simple interest on a sum of money invested at $3\frac{1}{2}\%$ per annum for 4 years is shs. 420, find the principal.

6. Find the principal which amounts, at simple interest, to £ 729 in 4 years at 2% per annum.

7. Find the quarterly interest on sh.16,800 at $2\frac{1}{2}$ per cent per annum.

8. A man borrows £ 500 on the condition that he repays £ 750 at the end of 4 years. What is the rate of simple interest?

9. A man received shs. 110 as one year's interest from a Post Office Savings Bank which pays 3% interest. How much had he in the bank?

Revision Problems

1. Write in decimal form each of the following:

(a) $30 + 5 + \frac{6}{10} + \frac{3}{100}$

(b) $400 + 3 + \frac{1}{10} + \frac{7}{1000}$

(c) $1 + \frac{1}{100} + \frac{3}{10000}$

(d) $\frac{7}{100} + \frac{7}{1000} + \frac{5}{100000}$

(e) $9000 + 90 + \frac{9}{10} + \frac{9}{1000}$

(f) $58\ \frac{87}{1000}$

(g) $105\ \frac{77}{10000}$

(h) $3\ \frac{1003}{1000000}$

2. Write each of the following decimal numerals in the expanded form:

(a) 1·0249 (b) 5·03 (c) 37035·449

(d) 0·0075 (e) 0·00821 (f) 1000·0001

3. In each of the following equations find x.

(a) $1924 = 1924x$

(b) $25{\cdot}01x = 2501$

(c) $0{\cdot}00157 = 1{\cdot}57x$

(d) $63004x = 6300{\cdot}4$

(e) ${\cdot}000051 = 510x$

(f) $78{\cdot}5x = {\cdot}00785$

(g) $0{\cdot}00107x = 10{\cdot}7$

(h) $0{\cdot}50501x = 50{\cdot}501$

4. The following is a record of rainfall in a certain town in West Africa for the first week of June, 1960:

Monday	3.2 inches
Tuesday	1·0 inches
Wednesday	2·3 inches
Thursday	1·5 inches
Friday	3·0 inches
Saturday	2·5 inches
Sunday	2·0 inches

(a) What is the average rainfall per day for that week? (Give the answer to 1 decimal place.)

(b) By how many inches was the total rainfall for Thursday to Sunday heavier than that for Monday to Wednesday?

(c) By how many inches was the rainfall for Monday heavier than the average rainfall for the week?

5. A clerk gets an increase of 2.5 per cent on his salary. What is his present monthly salary if he was receiving £25 a month before his increase?

6. How many yards are there in 0·125 of a mile?

7. Express 400 metres in yards and inches if 1 metre is equal to 39·37 inches.

8. It a car travelled at an average speed of 45·6 miles per hour, what distance did it cover in 12 hours?

9. On June 1st, 1961, my car had travelled 37477·3 miles. On June 1st, 1962, it had travelled 46285·9 miles.

(a) How many miles did the car cover between June 1, 1961 and June 1, 1962?

(b) How many miles will the car cover after June 1, 1962 before it has travelled 48888·8 miles?

(c) How many miles will the car cover after June 1, 1962 before it has travelled 50000·0 miles?

10. What decimal numerals could you put for a in each of the following equations to make it true?

(a) $a \times 485 = 45\%$ of 485 (b) $a \times 1 = 12\frac{1}{2}\%$ of 1

(c) $2a \times 80 = 15\%$ of 80

11. Express the following percentages as decimal numerals:

(a) 25% (b) 50% (c) 85% (d) $2\frac{1}{2}\%$ (e) 305% (f) $112\frac{1}{2}\%$

12. How may cwt. are there in 5·25 per cent of 48 tons?

13. Ama bought fish at 100 for £1 12s., and sold them at 6d. each; how much profit did she make by buying and selling 1000 fish?

14. A country with a population of 7,250,000 had 54·8% literates. How many were illiterate?

CHAPTER 12

REAL NUMBERS

12-1 Decimal Numerals for Rational Numbers

A rational number can be expressed in decimal notation by considering it as a quotient.

Thus: $\frac{1}{5} = 1 \div 5 = 0 \cdot 2$

and $\frac{3}{8} = 3 \div 8 = 0 \cdot 375$

Problems 12-1A

1. Express as decimal numerals:

(a) $\frac{1}{2}$ (c) $\frac{3}{4}$ (e) $\frac{3}{40}$

(b) $\frac{1}{4}$ (d) $\frac{3}{8}$ (f) $\frac{13}{80}$

2. Express the following as fractions in lowest terms:

(a) $0 \cdot 25$ (b) $1 \cdot 75$ (c) $2 \cdot 8$ (d) $1 \cdot 414$

Repeating Decimal Numerals

Consider the rational number $\frac{3}{22}$. This may be expressed in decimal notation by dividing $3 \cdot 000\ldots$ by 22 as follows:

```
      0·13636...
22 )3·00000...
    2 2
      80
      66
      140
      132
        80
        66
        140
        132
          8
```

where the three dots indicate that you can go on in this way forever. It is clear that the set of digits 36 is repeated over and over again. Why? We call such a numeral a <u>repeating decimal</u>. The answer can be abbreviated by placing dots over the first and the last of the set of digits that repeat.

Hence $\frac{3}{22} = 0{\cdot}13636\ldots = 0{\cdot}1\dot{3}\dot{6}.$

Consider also the decimal numeral for $\frac{6}{37}$.

```
      0·162162...
37 ) 6·000000
     3 7
     2 30
     2 22
        80
        74
         60
         37
         230
         222
           80
           74
            6
```

Do you see why the digits in the quotient are repeating?

If the division does not terminate, there is always a remainder. After a few steps of the division we must come to the same remainder or to the original numerator. Why? In the above division 6 appears as a remainder in the third step and again in the sixth step. The digits in the quotient, then, must repeat in sets of three.

Hence $\frac{6}{37} = 0{\cdot}\dot{1}6\dot{2}.$

The set of digits 162 repeats. We indicate this by placing a dot over the first and last in the set of repeating digits.

Problems 12-1B

Express as decimal numerals:

(1) $\frac{1}{9}$ (4) $\frac{2}{3}$ (7) $\frac{1}{11}$ (9) $\frac{1}{37}$

(2) $\frac{1}{99}$ (5) $\frac{1}{7}$ (8) $\frac{3}{11}$ (10) $\frac{3}{41}$

(3) $\frac{1}{999}$ (6) $\frac{5}{7}$

Terminating Decimals and Repeating Decimals

In expressing fractions in decimal notation, you may have noticed that some fractions come out exactly; that is, they end or terminate, e.g. $\frac{1}{2} = 0\cdot5$ or $\frac{3}{8} = 0\cdot375$. Such decimal numerals are called terminating decimals.

If you try some examples, you will notice that, if a rational number can be expressed as a terminating decimal, then when written in simplest fractional form its denominator will have no prime factors other than 2 and 5. Can you connect this statement with the fact that 10 equals 2×5?

All other rational numbers can be expressed as repeating decimals. Can you show why this should be so?

We shall be able to show also that all repeating decimals represent rational numbers. The following set of problems will help prepare us to do this.

Problems 12-1C

1. By what number has each number in the first column been multiplied to obtain the number in the second column?

(a)	$0\cdot\dot{8}$	$8\cdot\dot{8}$
(b)	$0\cdot\dot{7}\dot{6}$	$76\cdot\dot{7}\dot{6}$
(c)	$0\cdot\dot{6}9\dot{7}$	$697\cdot\dot{6}9\dot{7}$
(d)	$8\cdot\dot{9}3476$	$893476\cdot\dot{9}3476$
(e)	$5\cdot5\dot{2}0\dot{1}$	$5520\cdot1\dot{2}0\dot{1}$

2. If $x = 0\cdot\dot{3}4\dot{5}$, find:

(a) $10x$ (b) $100x$ (c) $1000x$

3. Simplify:

(a) $10x - x$ (b) $100x - x$ (c) $1000x - x$

4. Simplify:

(a) $8\cdot\dot{8} - 0\cdot\dot{8}$ (c) $7245\cdot\dot{2}4\dot{5} - 0\cdot\dot{2}4\dot{5}$

(b) $93476\cdot\dot{9}3476\dot{} - 0\cdot\dot{9}3476\dot{}$ (d) $86\cdot4\dot{7}\dot{5} - 0\cdot0\dot{7}\dot{5}$

Let us consider the question:

Does $0\cdot\dot{1}\dot{5}$ represent a rational number?

We can write

(1) $x = 0\cdot\dot{1}\dot{5}$ and

(2) $100x = 15\cdot\dot{1}\dot{5}$

Subtracting (1) from (2) we obtain:

$$(100 - 1)x = 15 \qquad \text{(Distributive Property)}$$
$$99x = 15$$
$$x = \frac{15}{99} \qquad \text{(Why?)}$$

Hence $0\cdot\dot{1}\dot{5} = \frac{15}{99} = \frac{5}{33}$. This can be checked by doing the division $5 \div 33$. Since $0\cdot\dot{1}\dot{5}$ can be represented as the quotient of two integers, it is a rational number.

Do you see why we chose to multiply by 100 in the first step and not by 10 or 1000?

Consider $3\cdot\dot{2}16\dot{3}$. Does this repeating decimal represent a rational number? We proceed in a similar way:

$$x = 3\cdot\dot{2}16\dot{3}$$
$$10000x = 32163\cdot\dot{2}16\dot{3} \qquad \text{(Why?)}$$
$$9999x = 32160 \qquad \text{(Why?)}$$
$$x = \frac{32160}{9999}$$
$$x = \frac{10720}{3333}$$

Hence $3\cdot\dot{2}16\dot{3} = \dfrac{10720}{3333}$

Problems 12-1D

1. Find fractional numerals which indicate that the following decimals represent rational numbers: (Check your answer by dividing.)

(a) $4\cdot\dot{6}$

(b) $0\cdot875$

(c) $0\cdot\dot{1}\dot{4}$

(d) $0\cdot\dot{1}4285\dot{7}$

(e) $3\cdot\dot{2}\dot{7}$

(f) $1\cdot\dot{1}\dot{6}$

12-2 Plotting Repeating Decimals on the Number Line

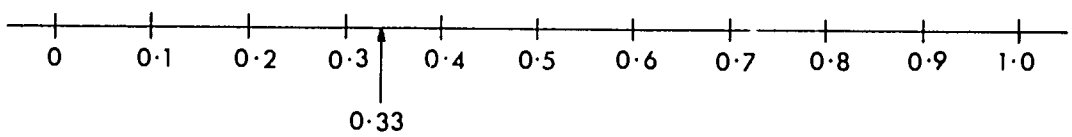
Consider $0\cdot333...$

$0\cdot33$ is another name for $0\cdot3 + 0\cdot03$

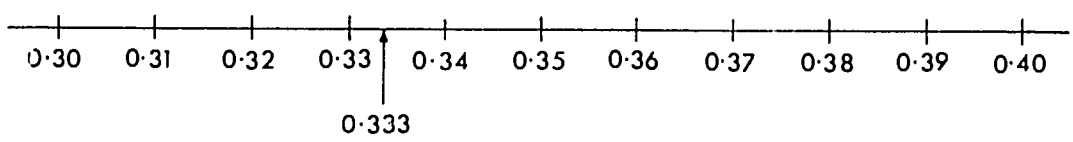
$0\cdot333$ is another name for $0\cdot3 + 0\cdot03 + 0\cdot003$

$0\cdot3333$ is another name for $0\cdot3 + 0\cdot03 + 0\cdot003 + 0\cdot0003$

We divide the segment between 0 and 1 into ten equal parts, each part being of length $0\cdot1$. The point at the end of the third part corresponds to $3 \times 0\cdot1$ or $0\cdot3$.

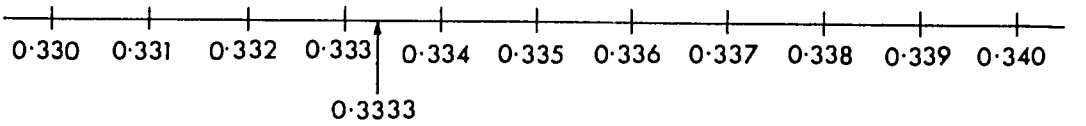


We divide the segment between $0\cdot3$ and $0\cdot4$ into ten equal parts, each part being of length $0\cdot01$. Three of these parts give $3 \times 0\cdot01$ or $0\cdot03$. Thus we find the point that corresponds to $0\cdot3 + 0\cdot03$ or $0\cdot33$.



We continue by dividing the segment between $0\cdot33$ and $0\cdot34$ into ten equal parts and take three of these. The new point will correspond to $0\cdot333$. The process can be continued on and on.

Every time we divide the segment into ten equal parts and take three of them we are getting closer and closer to the point that corresponds to $\frac{1}{3}$ or $0.\dot{3}$.



Problems 12-2

1. Plot each of the following successively on the number line as was done in the example.

(a) 0·6, 0·66, 0·666

(b) 3·6, 3·65, 3·656, 3·6565

12-3 Numbers Which are Not Rational

From the above discussion we can conclude that:

> Every terminating or repeating decimal numeral represents a rational number and every rational number may be represented by a terminating or repeating decimal numeral.

We are, however, left with the question, "Can every number be represented by a terminating or repeating decimal numeral?"

Consider the following decimal numerals:

(1) 1·01

(2) 1·01001

(3) 1·010010001

(4) 1·01001000100001

(5)

Can you find the fifth numeral? Can you find the twentieth numeral? Let us consider the fourth numeral, 1·01001000100001. The digits after the decimal point consist of a 0 followed by a 1, then two 0's followed by a 1, then three 0's followed by a 1, then four 0's followed by a 1. The fifth numeral will have an additional five 0's followed by a 1.

Each of these numerals will represent a rational number, since each is a terminating decimal.

Consider, however, 1·01001000100001000001... where the three dots indicate that the numeral does not end, but continues according to the pattern described above. This is neither a terminating nor a repeating decimal. Hence it does not represent a rational number.

What about 0·1234567891011... ?

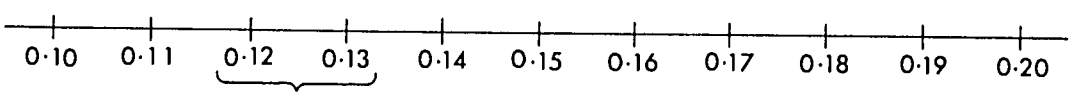
How would you continue to write it?

Is it a repeating decimal?

Does it represent a rational number?

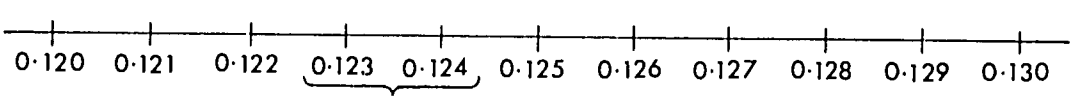
Let us locate on the number line the point which corresponds to 0·1234567891011...

Here is a segment of the line between 0·1 and 0·2 subdivided into hundredths.



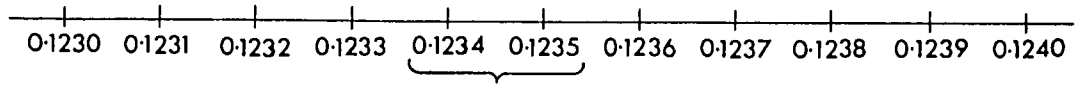
Our required point lies within the segment 0·1 to 0·2, in fact, between 0·12 and 0·13.

We can divide the segment between 0·12 and 0·13 into ten equal parts and plot 0·121, 0·122, 0·123, etc. On an enlarged scale, it will look like this:



Our required point lies between the points marked 0·123 and 0·124.

If we divide this interval (0·123 to 0·124) into ten equal parts and enlarge the scale again we obtain a figure like this:



Our required point lies in the interval from 0·1234 to 0·1235.

We can continue subdividing each segment into ten parts each one-tenth as long as before and find points that correspond to 0·12345 and 0·12346, to 0·123456 and 0·123457, and so on.

In this way we can locate the required point within intervals of length 0·1, 0·01, 0·001, 0·0001, and so on. Since the given decimal does not terminate we shall never locate it exactly in this way. However, we can locate it within an interval as short as we please.

It should be clear that there is a definite point on the number line which corresponds to each decimal.

We conclude

> There are decimal numerals which neither terminate nor repeat. They do not correspond to rational numbers. Every decimal numeral locates a definite point on the number line.

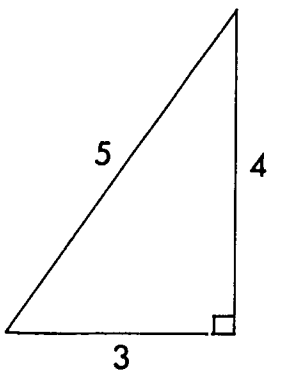
<u>Problems 12-3</u>

1. Use the method of this section to locate within an interval of length 0·0001, the point which corresponds to each of the following decimals:

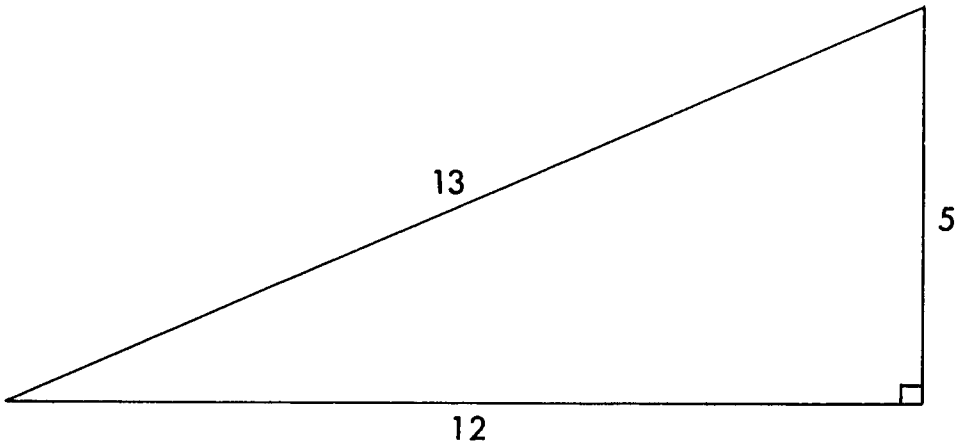
(a) 0·101001... (b) 2·124816

12-4 The Square Root of 2

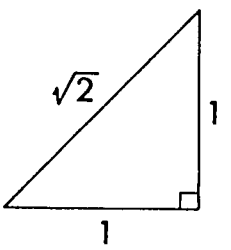
You remember the Pythagorean Theorem about right-angled triangles. The following figures illustrate this property.



$3^2 + 4^2 = 5^2$

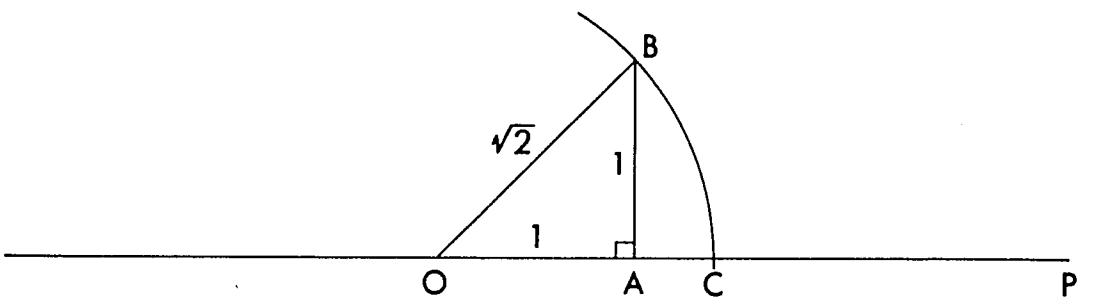


$5^2 + 12^2 = 13^2$



$1^2 + 1^2 = (\sqrt{2})^2$

Let us find a point that corresponds to $\sqrt{2}$ on the number line.



Draw a segment $\overline{OA}$ that is 1 unit long. At A we construct another segment $\overline{AB}$ that is perpendicular to $\overline{OA}$ and is 1 unit long. Join B to O. The segment $\overline{OB}$, the hypotenuse of $\triangle OAB$, is of length $\sqrt{2}$. (Why?) With O as centre and $\overline{OB}$ as radius draw part of a circle cutting the number line at the point C. OC $= \sqrt{2}$.

$\sqrt{2}$ corresponds to a point on the number line.

How can we find a rational number which is close to $\sqrt{2}$?

Since $(\sqrt{2})^2$ is the number 2, by the definition of square root, we can test some numbers which we think might be close to $\sqrt{2}$ by squaring them. Only if the square of a number is exactly 2 can we say that the number is $\sqrt{2}$.

We might try $\frac{7}{5}$ as our first approximation to $\sqrt{2}$.

Since $\left(\frac{7}{5}\right)^2 = \frac{49}{25}$ and $\frac{49}{25} < 2$ we know that $\frac{7}{5}$ is too small.

Compare the following sentences, both of which are true.

$$(\sqrt{2})(\sqrt{2}) = 2, \qquad \left(\frac{7}{5}\right)\left(\frac{10}{7}\right) = 2.$$

How was the number $\frac{10}{7}$ determined?

Is $\frac{7}{5} \times \frac{10}{7} = (\sqrt{2})(\sqrt{2})$?

If $\frac{7}{5}$ is less than $\sqrt{2}$, what is true of the other factor, $\frac{10}{7}$?

Since $\frac{7}{5}$ is less than $\sqrt{2}$ and $\frac{10}{7}$ is greater than $\sqrt{2}$, the number $\sqrt{2}$ must be between $\frac{7}{5}$ and $\frac{10}{7}$. Therefore, the <u>average</u> of $\frac{7}{5}$ and $\frac{10}{7}$ might be a good next approximation.

$$\frac{1}{2}\left(\frac{7}{5} + \frac{10}{7}\right) = \frac{99}{70}.$$

Now use this second approximation, $\frac{99}{70}$, to write the following equation.

$$\frac{99}{70} \times \frac{140}{99} = 2.$$

How was the second factor $\frac{140}{99}$ determined?

If we reason as before, we see that of the two factors $\frac{99}{70}$ and $\frac{140}{99}$, if one of them is less than $\sqrt{2}$, the other must be greater than $\sqrt{2}$, since their product is $(\sqrt{2})^2$. A third approximation is then found by taking the average of $\frac{99}{70}$ and $\frac{140}{99}$.

$$\frac{1}{2}\left(\frac{99}{70} + \frac{140}{99}\right) = \frac{19601}{13860}.$$

Hence, $\frac{19601}{13860}$ or $1\frac{5741}{13860}$ or about $1 \cdot 4142$ is a third approximation.

If still better approximations are desired, this process of finding the other factor of 2 and averaging the two factors can be continued.

As a test of how close $\frac{19601}{13860}$ is to $\sqrt{2}$ we can square it.

$$\left(\frac{19601}{13860}\right)^2 = \frac{384199201}{192099600} = 2\frac{1}{192099600}$$

Hence, $\frac{19601}{13860}$ is a little larger than $\sqrt{2}$ but not much.

Actually, the square root of 2 lies somewhere between 1·414214 1·414215, or

$$1 \cdot 414214 < \sqrt{2} < 1 \cdot 414215$$

We say that 1·414214 is an approximation to $\sqrt{2}$ to six decimal places.

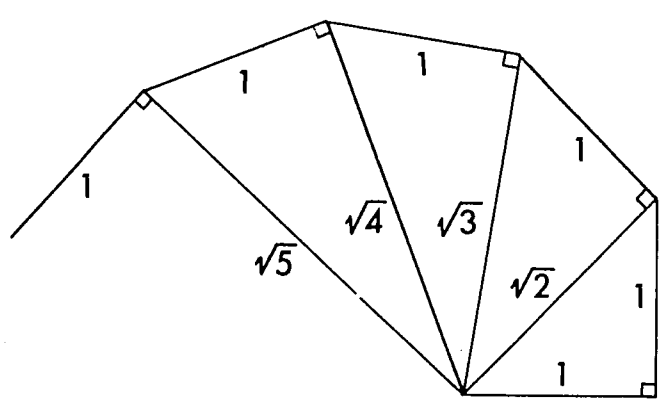
It will be proved in the next section that $\sqrt{2}$ is not a rational number.

We call numbers like $\sqrt{2}$ irrational numbers. They are irrational because they cannot be written as ratios of integers.

We can also find approximate values of square roots by geometry. The following problem illustrates this fact.

Problems 12-4

1. Continue drawing the following diagram as accurately as you can, and by measurement get approximate values of the square roots of the first few whole numbers.



2. Using the arithmetic method described above, find rational approximations to the following square roots: (Make a good first approximation to start with.)

(a) $\sqrt{7}$ (b) $\sqrt{11}$ (c) $\sqrt{13}$ (d) $\sqrt{17}$

12-5 Proof that $\sqrt{2}$ is an Irrational Number

Is $\sqrt{2}$ a rational number? That is, can $\sqrt{2}$ be written as a quotient of two integers?

We have approximated $\sqrt{2}$ by the rational numbers $\frac{7}{5}$, $\frac{10}{7}$, $\frac{99}{70}$, $\frac{140}{99}$. The squares of these numbers are very close to 2, but none of them is exactly equal to 2.

We could go on, trying other possible rational numbers, but no matter how many times we might try without succeeding, we would never be sure that some other choice of rational number might not work. We might begin to suspect that we could never succeed in finding a rational number which is exactly the square root of 2. But how could we be sure?

Let us look at our approximations more closely. We might choose $\frac{99}{70}$. We can see of course that $\left(\frac{99}{70}\right)^2 = \frac{99 \times 99}{70 \times 70} = \frac{9801}{4900}$ is a little more than 2. Can we see without squaring and dividing why $\left(\frac{99}{70}\right)^2$ cannot be exactly equal to 2?

Suppose that $\left(\frac{99}{70}\right)^2 = 2.$

Then $\frac{99 \times 99}{70 \times 70} = 2$

and $99 \times 99 = 2 \times 70 \times 70.$

If we multiply out we get

$$9801 = 9800$$

which is not true.

Can we see, without multiplying, that 99×99 and $2 \times 70 \times 70$ cannot possibly be equal?

Yes, we can. You remember that a whole number can be written as the product of prime factors in only one way.

Suppose that 99×99 were equal to $2 \times 70 \times 70$.

Let us look at the factor 2. This factor does not appear at all on the left-hand side, because 99 is not divisible by 2. But 2 does occur as a factor on the right-hand side. Since $2 \times 70 \times 70 = 2 \times 2 \times 35 \times 2 \times 35$, the factor 2 occurs three times on the right-hand side. If 99×99 and $2 \times 70 \times 70$ did represent the same number, 2 would have to occur the same number of times as a factor in both. We see, then, without multiplying, that since 99×99 and $2 \times 70 \times 70$ cannot be the same number, $\frac{99}{70}$ can not possibly be exactly equal to the square root of 2.

Let us generalize the above argument.

Let $\frac{a}{b}$ be any fraction where a and b are <u>whole numbers</u> and $b \neq 0$.

Suppose that $\frac{a}{b} = \sqrt{2}$.

Then $\left(\frac{a}{b}\right)^2 = 2$

or $\frac{a^2}{b^2} = 2$.

This means that

$$\frac{a \times a}{b \times b} = 2$$

and finally

$$a \times a = 2 \times b \times b.$$

Is this possible? Can $a \times a$ and $2 \times b \times b$ be two numerals for the same number? To answer this question, we ask whether the number of factors 2 on the left and right-hand sides could conceivably be the same.

On the left-hand side $(a \times a)$ there might be no 2's at all. On the right-hand side, $(2 \times b \times b)$, 2 occurs at least once as a factor. In this case, we surely could not have an agreement in the number of 2's.

Suppose then, that 2 does occur as a factor on the left-hand side. Any factor 2 which occurs in the first a occurs also in the second a. Hence there must be an <u>even</u> number of 2's on the left-hand side, (two, four, six,...).

Let us turn to the right-hand side, $(2 \times b \times b)$. What are the possibilities? There is at least one factor 2. There may be more. Every factor 2 that occurs in the first b occurs also in the second b. Hence there must be an <u>odd</u> number of 2's on the right-hand side, (one, three, five,...).

No matter what whole numbers we choose for a and b, the sentence $a \times a = 2 \times b \times b$ cannot be true. (Why?) This means that $\left(\frac{a}{b}\right)^2 = 2$ cannot be true and therefore there is no rational number exactly equal to $\sqrt{2}$. Then $\sqrt{2}$ must be an <u>irrational</u> number.

<u>Problems 12-5</u>

1. Prove that the following are irrational.

   (a) $\sqrt{3}$ (b) $\sqrt{5}$

2. What happens if you attempt to prove that $\sqrt{4}$ is irrational by the procedure described above?

<u>12-6</u> <u>The Real Numbers</u>

We have learned that decimals which terminate or repeat correspond to rational numbers, and that decimals which do not terminate and do not repeat correspond to irrational numbers.

The set of decimals of all kinds corresponds to the union of the set of rational numbers and the set of irrational numbers.

The union of these two sets is called the set of <u>real numbers</u>.

Then every decimal corresponds to a real number.

Every real number locates a definite point on the number line. We have illustrated this for both rational and irrational numbers.

Are there any points on the number line which have not been accounted for? The answer is "No". We can see this as follows:

Imagine any point P on the number line. To be definite, let us choose P between 0 and 1. When we mark the points for $0\cdot1$, $0\cdot2$,...$0\cdot9$, it could happen that P is one of these points. In this case, P corresponds to a terminating decimal, with one decimal place. Otherwise P lies between points marked with successive tenths. Then we know the first place of its decimal representation and we must continue. Now we divide this interval into ten equal parts. If our point P falls on one of the new points of division, we can locate it by a terminating decimal with two decimal places. If not, we have found the first two decimal places and must continue. Going on in this way we see that our given point P corresponds either to a terminating decimal or a non-terminating one. In any case, P corresponds to some decimal and therefore to some real number.

> To every real number there corresponds a single point on the number line, and to every point on the number line there corresponds a real number.

Problems 12-6

Fill in the table with "Yes" or "No".

	Real	Rational	Irrational	Positive	Negative
$\sqrt{3}$					
3					
-3					
$\frac{4}{7}$					
$-0\cdot\dot{8}$					
$0\cdot\dot{1}\dot{7}$					

12-7 Negative Real Numbers

The number line extends indefinitely to the right and to the left. The points to the right of 0 correspond to numbers that are positive.

$$1,\ 2,\ 3,\ \frac{3}{2},\ \frac{19}{3},\ \sqrt{17},\ 1\cdot 67,\ 1\cdot 010010001\ldots$$

(Whenever we write $\sqrt{\ }$ we mean the positive square root.)

The points to the left of 0 correspond to numbers that are negative.

$$-1,\ -2,\ -3,\ -\frac{3}{2},\ -\frac{19}{3},\ -\sqrt{17},\ -1\cdot 67,\ -1\cdot 010010001\ldots$$

(By $-\sqrt{17}$ we mean the negative number which squared gives 17.)

Just as the points to the right of 0 correspond to positive rational and irrational numbers, the points to the left of 0 will correspond to negative rational and irrational numbers.

We remember that if the sum of two numbers a and b is zero, he is called the additive inverse of the other or the opposite of the other. That is:

if $a + b = 0$

a is the opposite of b

and b is the opposite of a.

Thus, since $2 + (-2) = 0$

-2 is the opposite of 2

and 2 is the opposite of -2.

What is the opposite of $\frac{19}{3}$? of $-\frac{13}{5}$?

On the number line, opposites are at the same distance from 0 and on opposite sides.

What should be the opposite of $\sqrt{17}$? In other words, what must we add to $\sqrt{17}$ to obtain 0? Clearly $-\sqrt{17}$.

What real number must be added to $1\cdot 67$ to give 0? What real number must be added to $-1\cdot 010010001$ to give 0?

Problems 12-7

1. Find the opposites of:

(a) $3 \cdot 205$ (b) $-1 \cdot 23$ (c) $\sqrt{3}$ (d) $0 \cdot 2020020002\ldots$

2. Find the opposites of:

(a) $\frac{3}{2}$ (b) $-0 \cdot 1111\ldots$ (c) $\sqrt{6}$ (d) $-0 \cdot 12345\ldots$

12-8 Addition of Real Numbers

As you know, repeating and terminating decimals represent rational numbers. You also know that we can add these numbers.

Problems 12-8A

1. Find the following sums:

(a) $4 \cdot 13 + 5 \cdot 42$ (c) $3 \cdot 86 + 4 \cdot 57 + 0 \cdot 05$

(b) $4 \cdot 13 + 5 \cdot 423$ (d) $4 \cdot 37 + 7 \cdot 321 + 2 \cdot 2$

2. Verify that $0 \cdot \dot{2}7\dot{5} + 0 \cdot \dot{7}1\dot{4}$ equals $0 \cdot \dot{9}8\dot{9}$ by converting the numerals to fractional form (quotients of whole numbers) and adding in that form.

Can irrational numbers be added?

(a) Is $4 + 1 \cdot 030030003\ldots$ a real number?

(b) Is $3 \cdot 1010010001\ldots + 1 \cdot 3030030003\ldots$ a real number?

(c) What can we say about $(\sqrt{2} + \sqrt{3})$?

The answer to (a) is clearly $5 \cdot 030030003\ldots$, since $4 + 1 \cdot 030030003\ldots$ means moving on the number line four more units to the right of $1 \cdot 030030003\ldots$

To perform the addition in (b), we can form successive sums:

$$3\cdot 1 + 1\cdot 3 = 4\cdot 4$$
$$3\cdot 10 + 1\cdot 30 = 4\cdot 40$$
$$3\cdot 101 + 1\cdot 303 = 4\cdot 404 \qquad \text{and so on.}$$

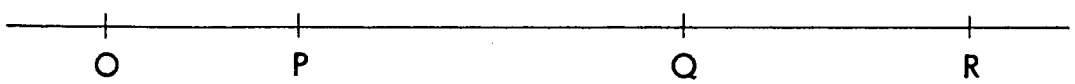
Until we have the sum:

$$\begin{array}{r} 3\cdot 1010010001\ldots \\ +\ \underline{1\cdot 3030030003\ldots} \\ 4\cdot 4040040004\ldots \end{array}$$

We see that in both (a) and (b), the sums are represented by non-terminating and non-repeating decimals. They correspond to irrational numbers. Like all real numbers they can be represented on the number line.

Let us use the number line to help us understand these additions.

Let the point P on the number line correspond to $1\cdot 3030030003\ldots$, and let Q correspond to $3\cdot 1010010001\ldots$



The point R, such that $OR = OP + OQ$, represents $4\cdot 4040040004\ldots$ There clearly is a definite point R corresponding to a definite real number which is the sum of the given numbers.

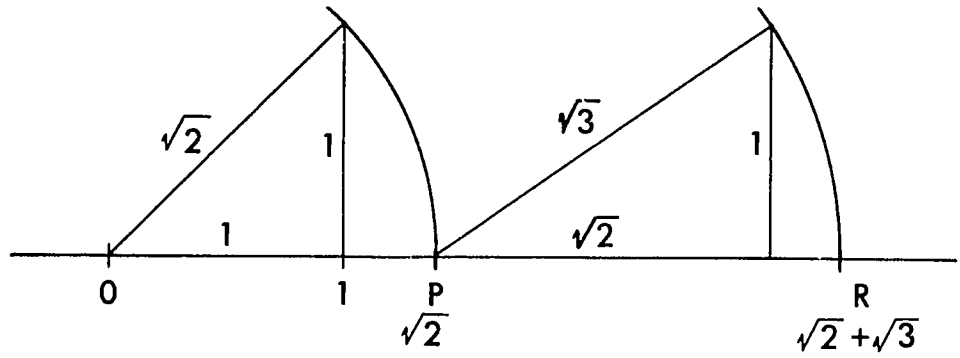
Problems 12-8B

Continue each of the decimals below to eight decimal places assuming that the unwritten digits follow some pattern suggested by the first few and perform the indicated addition:

1. $4\cdot 12345\ldots + 5\cdot 891011\ldots$
2. $3\cdot 101001\ \ldots + 6\cdot 152535\ldots$
3. $3\cdot 010110111\ldots + 4\cdot 102003\ldots$

Addition of Square Roots on the Number Line

As a special case of addition on the number line, we can construct the sum $\sqrt{2} + \sqrt{3}$. We already know how to construct $\sqrt{2}$ and $\sqrt{3}$, as in the following figures:



On the number line, mark P to the right of O such that OP = $\sqrt{2}$ and mark R to the right of P such that PR = $\sqrt{3}$. The coordinate of R, then is $\sqrt{2}+\sqrt{3}$. There clearly is a real number corresponding to $\sqrt{2}+\sqrt{3}$.

Problems 12-8C

Write the first four decimal places for each of the following numbers:

1. $\sqrt{2}$
2. $\sqrt{3}$
3. $\sqrt{2}+\sqrt{3}$
4. $\sqrt{5}$
5. Does $\sqrt{(2+3)} = \sqrt{2}+\sqrt{3}$?
6. Locate $\sqrt{5}$ on the number line by using a geometrical construction.

In question 5 above, notice that

$$\sqrt{(2+3)} = \sqrt{5} \qquad \text{but} \qquad \sqrt{2}+\sqrt{3} \neq \sqrt{5}$$

The Sum of a Rational Number and an Irrational Number is an Irrational Number

Consider $3+\sqrt{2}$. Can this be a rational number?

Let us assume that $3+\sqrt{2}$ is a rational number.

Then $3+\sqrt{2} = \frac{a}{b}$ where a and b are integers and $b \neq 0$

$$\sqrt{2} = \frac{a}{b} - 3$$

$$\sqrt{2} = \frac{a-3b}{b}$$

Since $a-3b$ and b are integers (Why?) $\frac{a-3b}{b}$ is a rational number.

The original assumption leads us to say that $\sqrt{2}$ is a rational number. We know, however, that $\sqrt{2}$ is not rational. Therefore our assumption that $3 + \sqrt{2}$ is rational is false.

In the same way we can prove that the sum of any rational number and any irrational number is an irrational number.

What about the sum of two irrational numbers?

Consider the following addition of two irrational numbers:

$$\begin{array}{r} 0{\cdot}01001000100001\ldots \\ +\ \underline{0{\cdot}10110111011110\ldots} \\ 0{\cdot}11111111111111\ldots \end{array}$$

The resulting decimal represents $\frac{1}{9}$ and therefore the sum is rational.

The sum of two irrational numbers <u>may</u> be a rational number; that is, the set of irrational numbers is not closed under addition.

<u>Problems 12-8D</u>

1. What can you add to each of the following to make each a rational number? (There are many possible answers. In each case give two answers.)

(a) $5 - \sqrt{3}$ (d) $6 + \sqrt{5}$

(b) $4{\cdot}000100001000001\ldots$ (e) $6{\cdot}123456\ldots$

(c) $7{\cdot}030030003\ldots$ (f) $7{\cdot}789101112\ldots$

2. Add:

(a) $3{\cdot}6 + 4{\cdot}21$ (d) $3{\cdot}\dot{4}5\dot{1} + 2{\cdot}\dot{3}2\dot{7}$

(b) $4{\cdot}71 + 0{\cdot}06$ (e) $(7 + \sqrt{3}) + (3 - \sqrt{5})$

(c) $6{\cdot}\dot{2} + 3{\cdot}\dot{5}$ (f) $1{\cdot}04004\ldots + 4{\cdot}40440444\ldots$

3. Compare the following using the symbols $=$ or $\neq$

(a) $\sqrt{(3 + 5)}$, $\sqrt{3} + \sqrt{5}$ and $\sqrt{8}$

(b) $\sqrt{(2 + 5)}$, $\sqrt{2} + \sqrt{5}$ and $\sqrt{7}$

4. Show that $\sqrt{\sqrt{2} + 1}$ is irrational.

12-9 Order of Real Numbers

Suppose that we have two different positive real numbers written as decimals. For example,

$$a = 0 \cdot 32842\ldots$$
$$b = 0 \cdot 32719\ldots$$

How can we compare them so that we can say that one is less than the other? In this example, the two numbers agree in the tenth's place and in the hundredth's place. In the thousandth's place they differ. Let us locate a and b on the number line.

The point that corresponds to a lies to the right of the point that corresponds to $0 \cdot 328$, and the point that corresponds to b is to the left of the point that corresponds to $0 \cdot 328$. Therefore if we go along the number line from left to right we would meet b first. We say, then, that b is less than a and write $b < a$.

In the same way if:

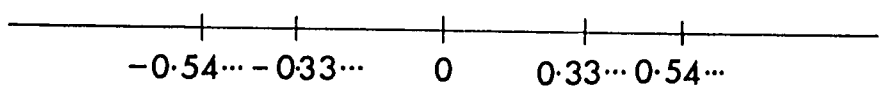
$$a = 0 \cdot 5321\ldots$$
$$b = 0 \cdot 5324\ldots$$

we would come to a before we come to b, regardless of what the digits are after the fourth decimal place.

If we wish to compare two negative real numbers, say:

$$-0 \cdot 54\ldots \qquad \text{and} \quad -0 \cdot 33\ldots$$

we notice that since $0 \cdot 33\ldots$ is to the left of $0 \cdot 54\ldots$



their opposites are in the reverse order. $-0 \cdot 54\ldots$ is to the left of $-0 \cdot 33\ldots$ That is, $-0 \cdot 54\ldots < -0 \cdot 33\ldots$

Of course, any negative number is located to the left of 0 and any positive number is to the right of 0. Therefore, any negative real number is less than any positive real number.

The order of real numbers has the following properties:

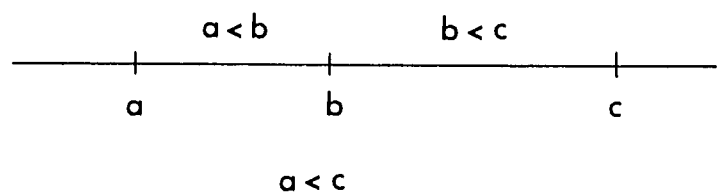
<u>Property 1</u>. If a and b are any real numbers, exactly one of the following is true.

$$a = b \quad \text{or} \quad a < b \quad \text{or} \quad b < a$$

<u>Property 2</u>. If a, b and c are any real numbers

and if $a < b$ and $b < c$

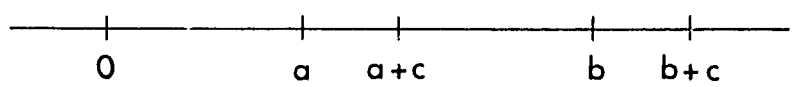
then $a < c$



<u>Property 3</u>. If a, b and c are any real numbers

and if $a < b$

then $a + c < b + c$



<u>Problems 12-9</u>

1. Compare

(a) 0·432... and 0·437...

(b) 3·824... and 3·816...

(c) -0·128... and -0·123...

(d) -0·128... and 0·001...

2. Think of $0 \cdot 432...$ and $0 \cdot 437...$ as

$$0 \cdot 43 + 0 \cdot 002...$$
$$\text{and } 0 \cdot 43 + 0 \cdot 007...$$

Then use Property 3 to conclude that

$$0 \cdot 432... < 0 \cdot 437...$$

3. Use the idea in problem 2 above to compare the numbers in Problem 1(b).

4. Indicate on the number line the truth set of the inequalities.

(a) $x < \sqrt{2}$ (b) $x > -\sqrt{3}$

12-10 Multiplying Real Numbers

As we know, a real number can be written in decimal form. Suppose that we have two real numbers,

For example

$$a = 0 \cdot 1213...$$
$$b = 0 \cdot 2145...$$

How shall we multiply them?

Since $a > 0 \cdot 1$ and $b > 0 \cdot 2$

$$a \times b > (0 \cdot 1)(0 \cdot 2) = 0 \cdot 02$$

Again $a > 0 \cdot 12$ and $b > 0 \cdot 21$

so that $a \times b > (0 \cdot 12)(0 \cdot 21) = 0 \cdot 0252$

Similarly $a \times b > (0 \cdot 121)(0 \cdot 214) = 0 \cdot 025894$

Each time we are closer to the value of the desired product. In each case we have left something off, so that the products we found are always too small.

We can get approximations which are too large by noticing that

$$a < 0\cdot 2 \quad \text{and} \quad b < 0\cdot 3$$
$$a < 0\cdot 13 \quad \text{and} \quad b < 0\cdot 22$$
$$a < 0\cdot 122 \quad \text{and} \quad b < 0\cdot 215 \quad \text{and so on.}$$

From the last of these statements we see that

$$a \times b < (0\cdot 122)(0\cdot 215) = 0\cdot 026230$$

By going on in this way we come closer and closer to the value $a \times b$, but each time the result will be too large.

We know definitely that $a \times b$ is between

$$0\cdot 025894 \quad \text{and} \quad 0\cdot 026230$$

If we continue we can obtain as close an approximation as we please.

In the case of real numbers which are square roots of whole numbers, we can write the products in another form.

For example,

$$\sqrt{2} \times \sqrt{2} = 2$$

since $\sqrt{2}$ is a number whose square is 2.

What is $\sqrt{2}(\sqrt{2} - 1)$?

Using the distributive property we obtain the result

$$\sqrt{2}(\sqrt{2} - 1) = \sqrt{2} \times \sqrt{2} - \sqrt{2} \times 1 = 2 - \sqrt{2}.$$

What is $\sqrt{2} \times \sqrt{3}$?

If we consider the result of squaring the product, we obtain

$$(\sqrt{2} \times \sqrt{3})^2 = (\sqrt{2})^2 \times (\sqrt{3})^2$$ (Use the commutative and associative properties to show that this is true.)

$$(\sqrt{2})^2 \times (\sqrt{3})^2 = 2 \times 3$$ (Definition of square root)

Since $(\sqrt{2} \times \sqrt{3})^2 = 6$

$\sqrt{2} \times \sqrt{3} = \sqrt{6}$ (Why?)

Show in the same way that if a and b are any two positive real numbers,

$$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$$

What is $\sqrt{2} \times \sqrt{8}$?

So far we have considered only multiplication of two positive real numbers. What happens when we multiply a positive real number by a negative real number?

To find $\sqrt{2} \times (-\sqrt{2})$ consider the following questions.

Why is $\sqrt{2}\left[\sqrt{2} + (-\sqrt{2})\right] = 0$?

Why, then, is $2 + \left[\sqrt{2} \times (-\sqrt{2})\right] = 0$?

And so, why is $\sqrt{2} \times (-\sqrt{2}) = -2$?

What is the result of multiplying two negative real numbers? Is it a positive number or a negative number?

What, for example, is $(-\sqrt{2}) \times (-\sqrt{2})$?

The answer is 2. Can you prove this?

<u>Problems 12-10</u>

1. We wish to multiply a by b where

$$a = 0 \cdot 111\ldots$$
$$b = 0 \cdot 333\ldots$$

Multiply $0 \cdot 1$ by $0 \cdot 3$, $0 \cdot 11$ by $0 \cdot 33$ and so on, to get better approximations to $a \times b$. These results are all too small. Get approximations which are too large by multiplying $0 \cdot 2$ by $0 \cdot 4$, $0 \cdot 12$ by $0 \cdot 34$ and so on.

2. Check your result in Problem 1 from the fact that $a = \frac{1}{9}$ and $b = \frac{1}{3}$, so $a \times b = \frac{1}{27}$.

3. $\sqrt{2} = 1\cdot414...$

$\sqrt{3} = 1\cdot732...$

$\sqrt{6} = 2\cdot449...$

Use the process of Problem 1 to make it plausible that

$$\sqrt{2} \times \sqrt{3} = \sqrt{6}$$

4. Using the distributive property, find $\sqrt{3}\ (\sqrt{2} + 1)$ in terms of square roots.

5. Show that $(\sqrt{2} - 1)\ (\sqrt{2} + 1)$ is a whole number. (Use the distributive property.)

6. Show that $(\sqrt{5} + \sqrt{2}) \times (\sqrt{5} - \sqrt{2})$ is a whole number.

7. <u>CHALLENGE PROBLEM</u>

Show by squaring that

$$\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}} \text{ is rational.}$$

8. <u>CHALLENGE PROBLEM</u>

Show by squaring that

$$\sqrt{2 + \sqrt{3}} + \sqrt{2 - \sqrt{3}} = \sqrt{6}$$

9. <u>CHALLENGE PROBLEM</u>

Find a simple numeral for

$$\sqrt{2 + \sqrt{3}} - \sqrt{2 - \sqrt{3}}$$

10. Find two irrational numbers whose sum is irrational and whose product is rational.

You have learnt that the set of rational numbers under addition and multiplication forms a <u>field</u>.

The set of real numbers under addition and multiplication also forms a field.

That is,

The set of real numbers is closed under addition and multiplication. (The sum and product of two real numbers are real numbers.)

Addition and multiplication of real numbers are commutative and associative.

$x + y = y + x$ $\qquad$ $x \times y = y \times x$

$(x + y) + z = x + (y + z)$ $\qquad$ $(x \times y) \times z = x \times (y \times z)$

There are additive and multiplicative identities (0 and 1) such that

$x + 0 = x$ $\qquad$ $x \times 1 = x$

Every real number x has an additive inverse (opposite) $-x$ such that

$$x + (-x) = 0$$

Every real number x $(x \neq 0)$ has a multiplicative inverse (reciprocal) $\frac{1}{x}$ such that

$$x \times \frac{1}{x} = 1$$

12-11 Absolute Value

We can measure distance along the number line using the segment from 0 to 1 as a unit of length. Thus, the point that corresponds to 2 is at a distance of 2 units from 0. The point that corresponds to -2 is also at a distance 2 from 0. We say that both 2 and -2 have the absolute value 2.

The absolute value of a number a will be written $|a|$.

Thus $|2| = 2$ and $|-2| = 2$.

Similarly, $|0 \cdot 132...| = 0 \cdot 132...$

and $|-0 \cdot 132...| = 0 \cdot 132...$

Any two opposites, a and $-a$ have the same absolute value:

$$|a| = |-a|$$

What is $|0|$? How far is 0 from 0?

Do you see that $|0| = 0$?

We have explained absolute value in terms of the distance from 0 along the number line. This distance is always non-negative. That is, distance can be only positive or 0.

We can say that $|a|$ is that one of the pair of numbers, a and $-a$, which is non-negative.

If a is positive, then $|a| = a$.
For example, since 2 is positive, $|2| = 2$.

If a is negative, then $|a| = -a$. (Notice that when a is negative, its opposite, $-a$, is positive.)

For example, since -2 is negative $|-2| = -(-2) = 2$.

<u>Problems 12-11</u>

1. Find the absolute value of each of the following.

   (a) $-0\cdot1234\ldots$ (c) $(-0\cdot111\ldots) - 0\cdot01001\ldots$
   (b) $\frac{4}{3} - \frac{3}{2}$ (d) $-3\cdot68 + 3\cdot89$

2. What is the truth set of $|x| = 2$?

3. Indicate on the number line the truth set of:

   (a) $|x| < 2$ (b) $|x| > 2$

4. Indicate on the number line the truth set of:

   (a) $|x| \leq 2$ (b) $|x| \geq 2$

Note: $\leq$ means "is less than or equal to".

5. Find the truth set of $|6 - x| = 1$.

6. We know that on the number line the distance between the points with coordinates 9 and 12 is 3. We found the distance by taking the difference, 12-9. If we use 9-12, we would have a negative number, but by using absolute value, we can write $|9-12|$ for the distance between the points. Either $|12-9|$ or $|9-12|$

gives the distance. In general $|a-b|$ is the distance between the points with coordinates a and b.

Using this viewpoint, we can say that $|6-x|$ is the distance between the points with coordinates 6 and x. Use this viewpoint to find the truth sets of the following sentences:

(a) $|6-x| = 1$ (c) $|2 - x| = 1$

(b) $|6-x| = 3$ (d) $|2 - x| = 5$

7. Show the truth sets of the following sentences on the number line.

(a) $|5 - x| = 3$ (c) $|(-3) - x| = 2$ (e) $|(-2) - x| = 4$

(b) $|3 - x| = 5$ (d) $|x - 4| = 5$ (f) $|x - (-2)| = 2$

8. Find the truth set and show it on the number line.

(a) $|x - 5| \leq 4$ (b) $|x - 5| \geq 4$

12-12 More About Absolute Value

The Absolute Value of a Product

How is $|a \times b|$ related to $|a|$ and $|b|$?

Let us take a few examples:

If a = 3 and b = 2, then

$|a| = 3, \quad |b| = 2, \quad |a \times b| = |6| = 6$

In this case $|a| \times |b| = |a \times b|$

If a = -3 and b = -2, then $|a| = 3$, $|b| = 2$, $|a \times b| = |6| = 6$. In this case, $|a| \times |b| = |a \times b|$.

If a = 3 and b = -2, then $|a| = 3$, $|b| = 2$, $|a \times b| = |-6| = 6$. In this case, $|a| \times |b| = |a \times b|$.

If a = 3 and b = 0, then $|a| = 3$, $|b| = 0$, $|a \times b| = |0| = 0$. In this case, $|a| \times |b| = |a \times b|$.

We suspect that $|a| \times |b| = |a \times b|$ for all real numbers a and b. This is true because in every case, $|a| \times |b|$ and $|a \times b|$ are both equal to the <u>non-negative</u> number of the two numbers $(a \times b)$ and $-(a \times b)$.

It is true, then, that the absolute value of the product of two real numbers is equal to the product of their absolute values.

The Absolute Value of a Sum

How is $|a + b|$ related to $|a|$ and $|b|$?

You might guess that $|a + b|$ is the sum of $|a|$ and $|b|$.

A few examples should convince you that this is not always true.

Example 1. If $a = 3$ and $b = -1$, $a + b = 2$.

Then $|a| = 3$, $|b| = 1$ and $|a + b| = 2$.

In this case, $|a + b|$ is less than $|a| + |b|$.

Example 2. If $a = -3$ and $b = -1$, $a + b = -4$

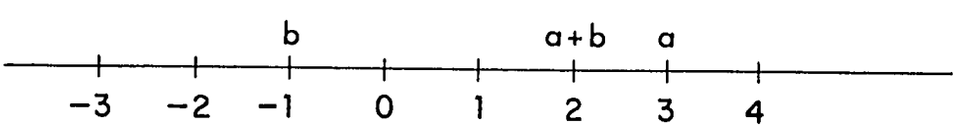
Then $|a| = 3$, $|b| = 1$ and $|a + b| = 4$.

In this case, $|a + b| = |a| + |b|$.

You might try $a = 3$ and $b = 1$, also $a = -3$ and $b = 1$, to see if you can draw a general conclusion.

It will be helpful if we show the results of Examples 1 and 2 on the number line.

In Example 1, a and b are on opposite sides of 0, and a is the farthest away from 0.



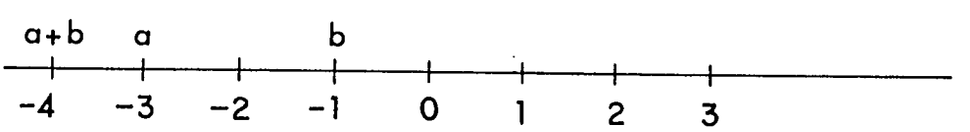
Therefore, $a + b$ must be nearer to 0 than a is.

That is,

$$|a + b| < |a|$$

and surely $$|a + b| < |a| + |b|.$$

In Example 2, a and b are on the same side of 0 at distances 3 and 1.



Their sum is on the same side at distance 4.

Here $$|a + b| = |a| + |b|$$

If we consider the general situation, we can see that one of two things can occur:

(1) If a and b are on the same side of 0 or if one of them is 0

$$|a + b| = |a| + |b|$$

(2) If a and b are on opposite sides of 0,

$$|a + b| < |a| + |b|$$

We can put these two results together in a single open sentence which applies to both cases.

$$|a + b| \leq |a| + |b|$$

We can summarize our results as follows:

For all a and b
$$\lvert a \times b\rvert = \lvert a\rvert \times \lvert b\rvert$$
$$\lvert a + b\rvert \leq \lvert a\rvert + \lvert b\rvert$$

<u>Problems 12-12</u>

1. Verify that $|a \times b| = |a| \times |b|$ when:
   (a) $a = -\frac{1}{2}$, $b = -\frac{1}{4}$
   (b) $a = \sqrt{3}$, $b = -\sqrt{2}$
   (c) $a = 1 + \sqrt{3}$, $b = 1 - \sqrt{3}$

2. Verify that $|a + b| \leq |a| + |b|$ when:
   (a) $a = -2$, $b = -4$
   (b) $a = 2$, $b = -2$
   (c) $a = \sqrt{2} - 1$, $b = \sqrt{2} + 1$

3. What is a simple name for $|2|^2$? For $|-2|^2$?
   Show that, $|a|^2 = a^2$ whether a is positive, negative or 0.
   Then show that $|a| = \sqrt{a^2}$.

(Remember that when we write $\sqrt{}$, it is understood that we take the non-negative square root. If we wanted the negative square root we would write $-\sqrt{}$.)

4. Verify that $|a| = \sqrt{a^2}$
for $a = -5$, $a = \frac{1}{4}$, $a = -\frac{1}{9}$, $a = -\sqrt{3}$

5. What is the truth set of $\sqrt{x^2} = |x|$? of $\sqrt{x^2} = x$? of $\sqrt{x^2} = -x$?

6. <u>CHALLENGE PROBLEM.</u> Use $|x| = \sqrt{x^2}$ to prove that $|a \times b| = |a| \times |b|$.
Hint: Is $\sqrt{(a \times b)^2} = \sqrt{a^2} \times \sqrt{b^2}$?

7. <u>CHALLENGE PROBLEM.</u>
   (a) Show that $|a - b| \leq |a| + |b|$
   (b) Use this result to show that $|a - b| \geq |b| - |a|$

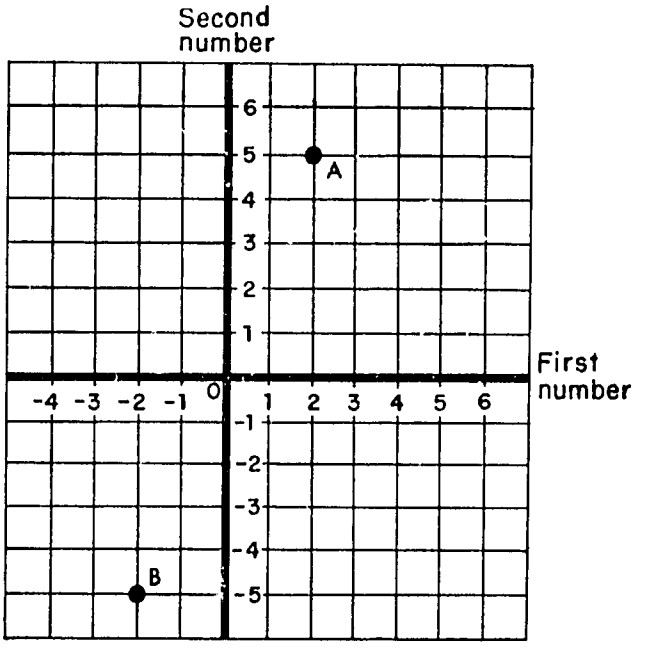
   Hint: Start (b) with $|(a - b) - a| \leq |a - b| + |a|$

CHAPTER 13

INTRODUCTION TO GRAPHING IN THE NUMBER PLANE

13-1 The Number Plane

To the right is a picture of two number lines intersecting at right angles. What first number goes with A ? If you look at the picture, you should see that it is 2. What second number goes with A ? Do you see why the pair of numbers that corresponds to B is (-2, -5)?

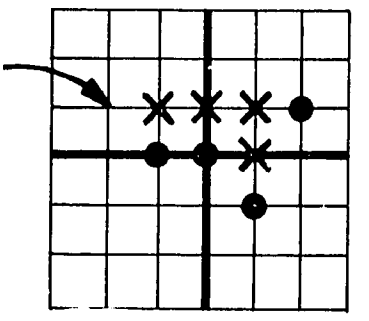


Problems 13-1

1. On a piece of graph paper, draw the diagram that appears above; then mark and label the points that go with each of the following pairs of numbers:

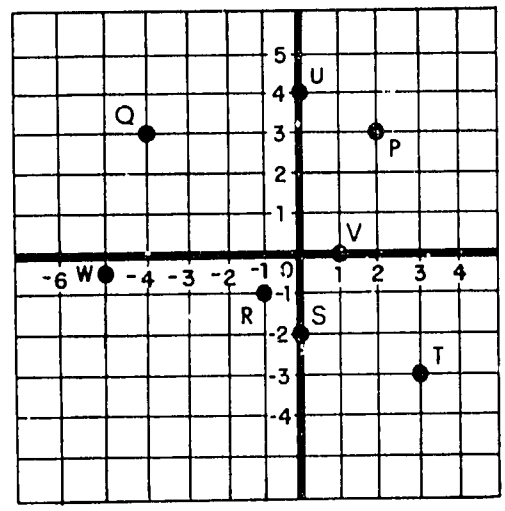
C: (0, 3)	E: (-2, 2)	G: (-2, 5)	I: (10, -4)
D: (3, 0)	F: (6, 2)	H: (0, 0)	J: $(2\frac{1}{2}, 3)$

2. John is on the X-team and wishes to win for his team by placing an X where the arrow is pointing. (Four in a line wins.) He says, "One, negative two." Does he win for his team?



As you have probably already discovered, and as John just learnt, the point that corresponds to (1, -2) is different from the point that corresponds to (-2, 1). The numbers are the same, but the <u>order</u> is different. We call the two numbers an <u>ordered pair</u>, and use brackets when we write them to show that the order is important.

3. Find the ordered pair of numbers that corresponds to each point marked in the diagram to the right. The first one is correctly answered.



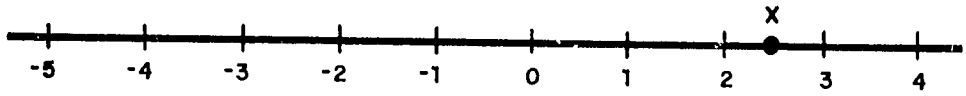
Point	Ordered Pair	Point	Ordered Pair
P	(2, 3)	T	
Q		U	
R		V	
S		W	

4. (a) For any ordered pair of numbers, how many corresponding points in the number plane are there?

(b) For any point in the number plane, how many corresponding ordered pairs of numbers are there?

<u>Coordinates</u>

When you learnt about the number line, you spoke of the <u>coordinate of a point</u>. For example, in the following picture, we call $2\frac{1}{2}$ the coordinate of the point X.



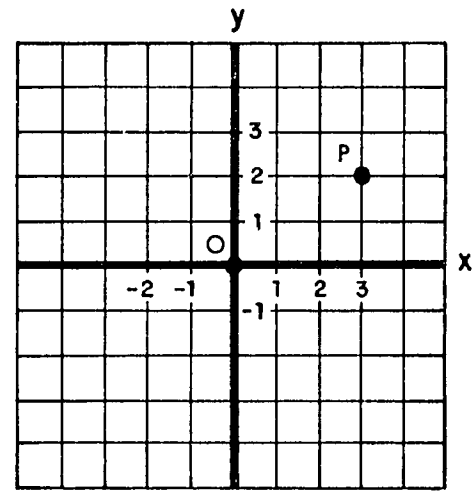
When we use two intersecting number lines, we speak of the coordinates of a point, since we now have a pair of numbers that goes with each point: A first coordinate, and a second coordinate. The special point whose coordinates are both zero, i.e. the point that corresponds to (0, 0), is the point where we start counting when we are marking, or plotting, a point. This point has the special name origin. The two intersecting number-lines are called axes (plural); if you are speaking of one of the axes, for example, the horizontal one, it is called the horizontal axis (singular).

1. The axis marked with x is called the horizontal axis, the first coordinate axis, or the x-axis.

2. The axis marked with y is called the vertical axis, the second coordinate axis, or the y-axis.

3. The x and y axes intersect at the point O, called the origin.

4. The point P has first coordinate 3 and second coordinate 2.



13-2 Graphs as Pictures of Truth Sets

Class Activity

1. What is the truth set of the sentence $-3 < x < 5$ if the domain of x is the set of integers? Draw a picture of the truth set on the number line.

2. What is the truth set of $-3 < x < 5$ if the domain of x is the set of real numbers? Draw a picture of the truth set on the number line.

3. Let us see if we can find the truth set of an open sentence in <u>two</u> variables, say the equation $y = x + 3$. If x is 2, what should y be to make the equation true? If we agree that the first coordinate is the value of x, and the second coordinate the value of y, which of these ordered pairs belong to the truth set of $y = x + 3$?

(a)	(5, 8)	(c)	(4, 1)	(e)	(-1, 4)	(g)	$(\frac{1}{2}, 4)$
(b)	(0, 3)	(d)	(2, 5)	(f)	(-2, 1)		

4. Fill in the table to the right with any of the ordered pairs from above that belong to the truth set of $y = x + 3$. Then find two more ordered pairs that belong to the truth set.

x	y
5	8

How many ordered pairs are there in the truth set?

5. For $-3 < x < 5$, you drew a picture of the truth set on the number line. For $y = x + 3$, the truth set contains ordered pairs of numbers, so we shall need a number plane to draw a picture of the truth set. Draw a set of axes at right angles on your graph paper; let us agree to call the horizontal axis the x-axis, and to use the same scale on both axes. Plot the points that correspond to the ordered pairs in your table.

6. Pretend for a moment that you do not know any arithmetic, or how to count, but you do have good eyesight. Mark another point that looks as though it should belong to the truth set of $y = x + 3$. What pair of numbers goes with this last point that you marked? Check to see whether this ordered pair really does make the equation true. If it does not, erase the point and guess again. Mark 3 or 4 more points on the graph paper that look as though they belong to the truth set. Find the ordered pair of numbers that goes with each point after you have marked it, and check to see whether it really does satisfy $y = x + 3$.

7. If you allow the domain of each of the variables in $y = x + 3$ to be the set of real numbers, what picture do you get for the truth set? Can you draw a picture which shows all the ordered pairs in the truth set?

8. For the equation $y = 2x + 3$, what ordered pair with x-coordinate 1 belongs to the truth set? What ordered pair with x-coordinate 2 belongs to the truth set?

x	y
1	?
2	?

On the same graph paper used for exercise 5, plot the points that go with these ordered pairs. (If you marked the points representing the truth set of $y = x + 3$ with $\cdot$, mark these points with $\times$ so that you will be able to tell them apart.) Without doing any more arithmetic or counting, mark another point that you think belongs to the truth set of $y = 2x + 3$. What ordered pair corresponds to the point you marked? Check to see whether this pair of numbers really does make $y = 2x + 3$ true. If it does not, erase it and guess again. Mark 3 or 4 more points on the graph paper that look as though they belong to the truth set of $y = 2x + 3$. Find the ordered pair that goes with each point after you have marked it, and check to see whether it really does satisfy $y = 2x + 3$. Let the domain of both x and y be the set of real numbers, and draw a picture of as much of the truth set of $y = 2x + 3$ as will fit on the paper. The set of points representing the truth set of an open sentence is called a graph of the open sentence.

9. On the same picture of the number plane, graph $y = 3x + 3$.

x	y
1	?

Mark the point that goes with the ordered pair in your table above. Without doing any more arithmetic or counting, mark another point that you think belongs to the graph of $y = 3x + 3$. (You may wish to mark the point with $\square$ to tell it apart from points on the other two graphs.) Find the ordered pair of numbers that goes with the point you marked. Check to see whether it really satisfies $y = 3x + 3$. Mark other points that look as though they belong to the graph. Check each one after you have marked it. Complete as much of the graph as you can on the paper.

Problems 13-2A

Assume that the domain of any variable in the rest of this section is the set of real numbers.

1. Graph these equations on the same set of axes:

   (a) $y = 5x + 3$ (c) $y = -2x + 3$

   (b) $y = 10x + 3$ (d) $y = \frac{1}{2}x + 3$

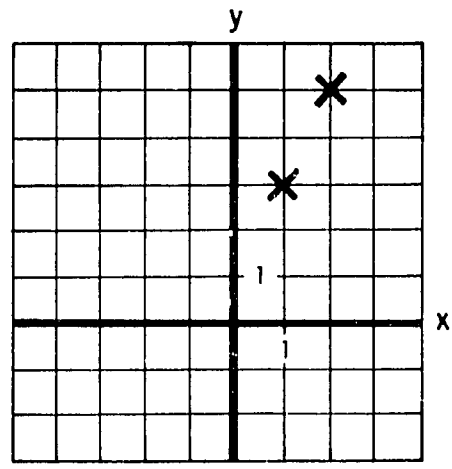
2. Graph these equations on the same set of axes:

   (a) $y = 2x + 1$ (c) $y = 2x + (-2)$

   (b) $y = 2x + 5$ (d) $y = 2x$

Class Activity

1. If we want an equation whose graph contains the two points marked in the picture, what value should you give to a in the equation $y = ax + 1$?



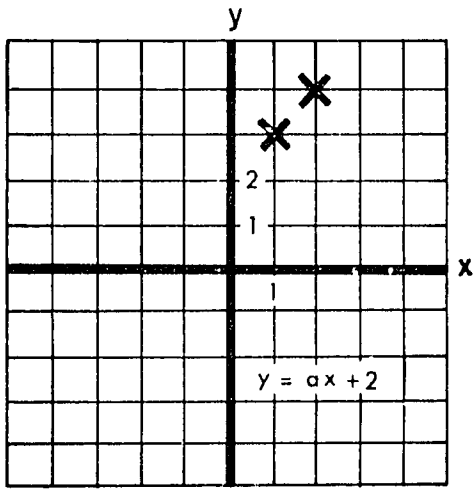
Find the coordinates of the two points marked and check to see whether they both make the equation true.

2. From your previous work in geometry, you know that two points are contained in exactly one line. If you select any of the other points on the line, will they have coordinates that satisfy your equation in Exercise 1?

Problems 13-2B

Find the values of a and b in each of the following equations so that the points marked are on the graph of the equation. For each, check your equation by finding the coordinates of the points and seeing whether they belong to the truth set of your equation.

1.



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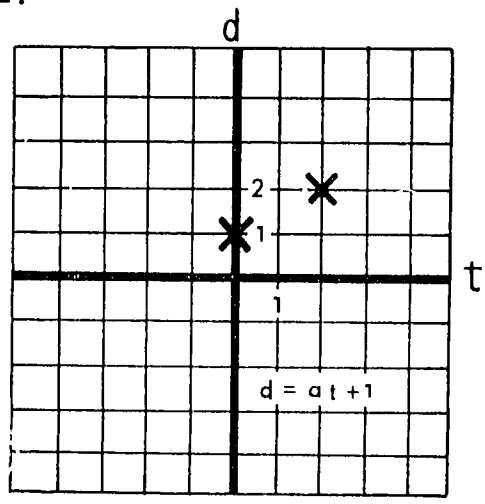

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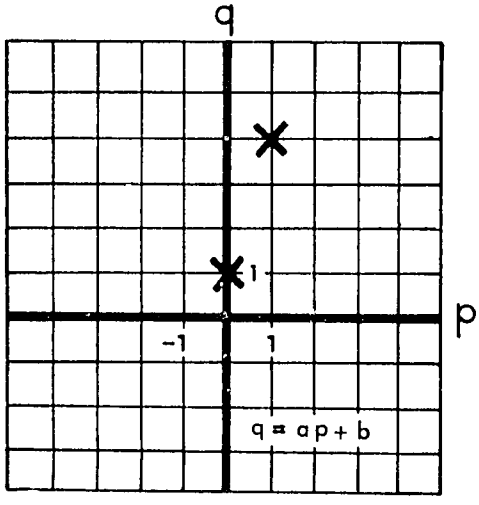

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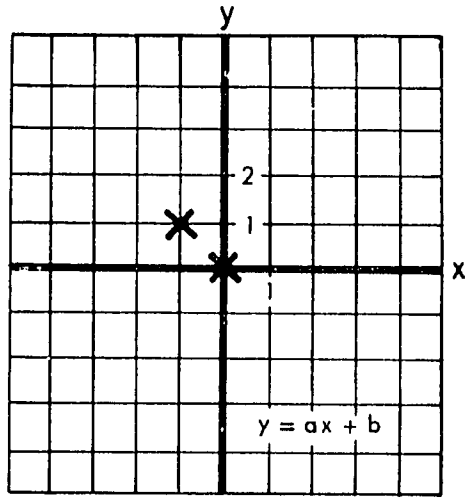

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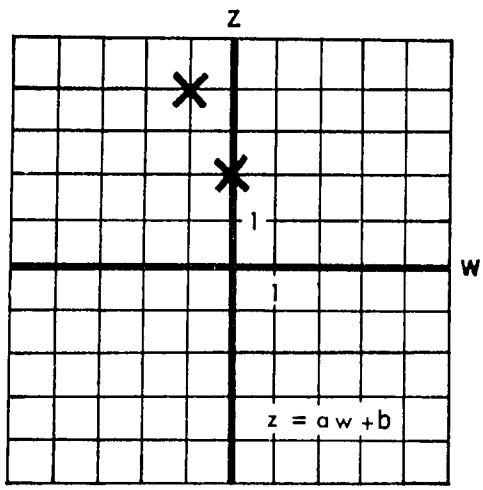

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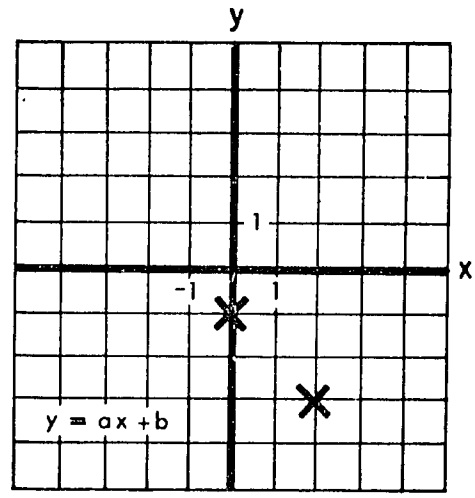

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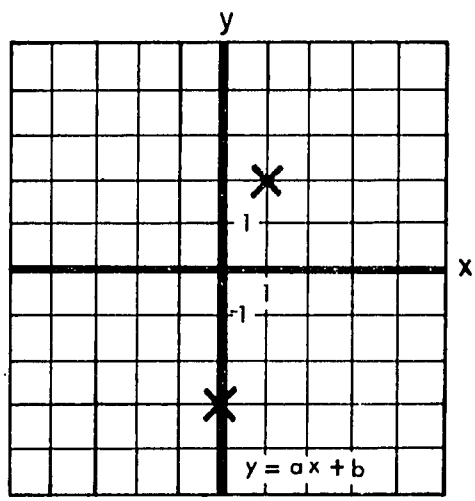

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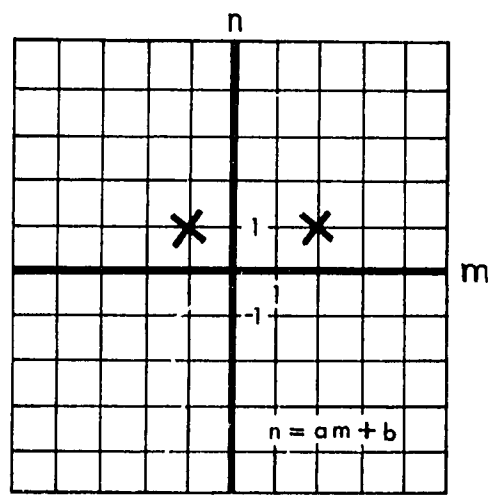


13-3 Guessing Rules

Class Activity

1. John has a rule that he is using on numbers. If you tell him "5" he uses his rule and says "26". If you tell him "3", he says "16". If you tell him "1", he says "6". If you tell him "2", he says "11". If we put this information in a table with the numbers you tell John in the first column, and the numbers John tells you after he has used his rule in the second column, it looks like this:

0	?
1	6
2	11
3	16
4	?
5	26
10	?

What would John say if you told him "0"? "4"? "10"?

Mary says that John's rule is "adding five", because when he used it on "1", he got "6". But Sonko says that "adding five" does not work for the other numbers, so John's rule must be something else. Can you guess John's rule?

2. Here is a table showing how Tunde's rule works for some numbers.

0	0
1	?
2	14
3	?
4	?
7	49
100	700

Try to fill in the question marks and guess Tunde's rule.

3. Here is a table showing how Edison's rule works for some numbers. Try to fill in the question marks and guess Edison's rule.

-1	$-\frac{1}{4}$
0	?
1	$\frac{1}{4}$
2	?
3	?
4	1
8	2
100	25

John says he thinks Edison's rule is "Divide by Four". Does this rule work? Edison says this is not the rule he was using. Can you guess what his rule is?

4. John says he has a short way of writing his rule using two variables, like this: $y = 5x + 1$. Which variable has as its domain all the numbers you might tell John? Which variable has as its domain all the numbers John would tell you after he has used his rule?

Tunde says he has written his rule using p and q. What do you think he wrote?

Edison is not sure how to write his rule using two variables. Can you do it for him?

Problems 13-3

Fill in the question marks and write a rule for each of the following:

1.

x	y
0	7
1	8
2	9
5	?
?	17

2.

w	z
-4	-8
0	?
5	1
?	3
8	4
9	5

3.

s	t
-1	?
0	?
1	2
2	5
3	8
?	14
6	17
7	?

4.

p	q
0	-1
1	?
20	9
100	49
101	?
1000	499

5. Sonko and Edward had nothing to do one afternoon because it was raining, so they decided to play a game of guessing rules. To make the game easier, they agreed that the rules they made up had to be: Multiply by some real number, and then add some real number. The winner was the person who could determine the other's rule from the fewest pairs of numbers. After a while Sonko was winning each time because he needed only to know how Edward's rule worked on two numbers, and he usually asked about 0 and 1. Here are four of the rules Sonko guessed correctly.

(a)

x	y
0	3
1	5

(b)

x	y
0	-1
1	0

(c)

x	y
0	0
1	$\frac{1}{3}$

(d)

x	y
0	5
1	9

What rules did Sonko guess for each of these? Can you write each one as an equation? Why did he ask how the rule worked on zero and one? (You may want to look at the tables and rules you have already guessed to answer this.)

6. Edward got tired of always losing (He had not yet figured out how to guess the rules knowing only how they worked on two numbers.), so he tried to make up a hard rule that Sonko would not be able to guess so easily. Sonko asked about 0 and 1, and the table looked like this:

x	y
0	1
1	2

Sonko said, "That's easy -- it's multiplying by one and adding one," and he wrote it like this: $y = 1x + 1$, but Edward said, "No, that is <u>not</u> my rule!" Sonko looked puzzled, and finally asked about some more numbers. The table looked like this before Sonko finally guessed the right rule:

x	y
0	1
1	2
2	5
8	65
10	101
12	145

Can you guess Edward's rule? Sonko finally decided that his method for guessing the rule knowing about only two numbers did not work this time because Edward's rule was not of the form they had agreed upon: Multiply by a number and then add a number.

7. For each of the rules in this section, decide whether or not its graph would be a straight line.

13-4 Equivalent Equations in Two Variables

You will recall that equations with the same truth set are called equivalent equations.

Look at this pair of equations:

$$y = \frac{1}{2}x - 1$$

$$2y - x + 2 = 0$$

Find three pairs of numbers that make the first equation true. Do they also make the second equation true? Let us try to prove that the two equations are equivalent.

$y = \frac{1}{2}x - 1$	
is equivalent to	If the number represented on each side of an equation is multiplied by the same non-zero number, the truth set remains unchanged.
$2y = 2(\frac{1}{2}x - 1)$	
which is equivalent to	The numeral on the right has been changed by the distributive property to another name for the same number.
$2y = x - 2$	
which is equivalent to	If the same number is added to (or subtracted from) the number represented on each side of an equation, the truth set remains unchanged.
$2y - x + 2 = x - 2 - x + 2$	
which is equivalent to	The numeral on the right has been changed by basic properties of addition and subtraction to another name for the same number.
$2y - x + 2 = 0$	

In practice, while we must think of the reasons why the equations are equivalent, we usually write more briefly as follows:

$$y = \frac{1}{2}x - 1$$

$$2y = 2(\frac{1}{2}x - 1) \qquad \text{Multiplied by 2.}$$

$$2y = x - 2$$

$$2y - x + 2 = x - 2 - x + 2 \qquad \text{Added 2 and subtracted } x.$$

$$2y - x + 2 = 0$$

At each step either we use one of the four rules for equivalent equations or we change the name of a number by using properties of real numbers.

Explain why the following equations are equivalent:

$$2x + 6y - 7 = 0$$

$$2x + 6y - 7 + 7 = 0 + 7$$

$$2x + 6y = 7$$

$$2x + 6y - 2x = 7 - 2x$$

$$6y = 7 - 2x$$

$$\frac{1}{6}(6y) = \frac{1}{6}(7 - 2x)$$

$$y = -\frac{1}{3}x + \frac{7}{6}$$

Problems 13-4

Which of the following are pairs of equivalent equations? If the equations of a pair are not equivalent, prove it by finding an ordered pair of numbers that belongs to the truth set of one, but does not belong to the truth set of the other. If the equations of a pair are equivalent, prove it by using the appropriate rules for equivalent equations and properties of real numbers.

1. $y = \frac{2}{3}x + 2$

   $3y - 2x - 6 = 0$

2. $4y - 5x + 1 = 0$

   $y = \frac{4}{5}x - 4$

3. $10x + 5y - 3 = 0$

   $y = -2x + \frac{3}{5}$

13-5 Graphs of Linear Equations in Two Variables

You have seen that the graphs of equations such as

$$y = 3x + 2$$
$$y = -\frac{1}{2}x + 8$$
$$y = x - \frac{9}{2}$$

are straight lines and you have observed some relations between the positions of those straight lines and the numbers in the equations.

What can we learn about the following equations?

$$5x + 2y - 3 = 0$$
$$7x - y = 0$$
$$2y - 3x + 4 = 0$$

Let us experiment with some.

Class Activity

1. Consider the equation $2y - 3x + 4 = 0$. Which of these ordered pairs belong to the truth set? (4, 4), (1, 2), (-2, -5), (0, -2), (-2, 0).

2. Find numbers for the question marks in the table so that each pair of numbers makes the equation $2y - 3x + 4 = 0$ true.

x	y
-2	?
-1	?
0	?
1	?
2	?
?	4

3. Plot the points represented in the table. Do they appear to lie on a straight line? How could we test other points of the line to see whether they belong to the graph of $2y - 3x + 4 = 0$?

4. Draw the line through the points you have plotted and from the line find numbers for the question marks.

(3, ?), (5, ?), (?, -8), (?, -11)

Do the resulting ordered pairs of numbers satisfy the equation? Since equations such as $2y - 3x + 4 = 0$ do have straight line graphs, they are often referred to as linear equations.

Problems 13-5

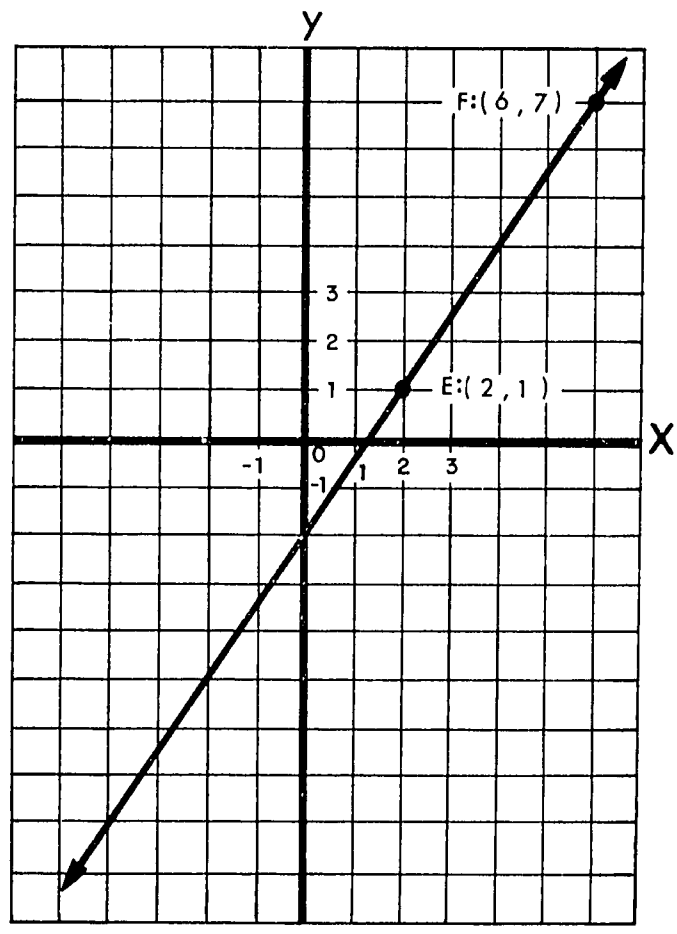
Graph each of the following equations by making a table of five ordered pairs of the truth set, plotting the points, and drawing the straight line through the points.

1. $x = y$
2. $x - 2y - 3 = 0$
3. $x - 3y = 0$
4. $x = 4$ (Suggestion: You may think of this as $x + 0y = 4$ if you wish.)
5. $y = 2$

13-6 The Slope of a Straight Line

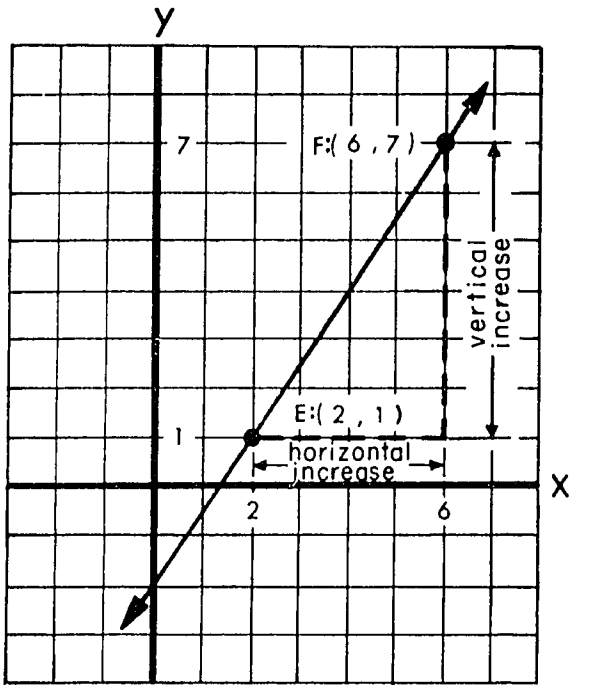
Class Activity

1. On your graph of $2y - 3x + 4 = 0$ that you made for the last section, label the point which corresponds to (2, 1) with E and (6, 7) with F as in the picture.



2. Label the following points on your graph: A: (-4, -8); B: (-2, -5); C: $(-1, -3\frac{1}{2})$; D: (0, -2).

3. Put your pencil point on E. Count over four units to the right, and up six units. Did you finish at F ? By counting over four units, you increased the x-coordinate by four; by counting up six units you increased the y-coordinate by six. This can be shown in a diagram.



What is the ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$?

The same result can be obtained by using arithmetic. In going from E: (2, 1) to F: (6, 7), the vertical (or y-coordinate) increase is: 7 - 1 = 6; the horizontal (or x-coordinate) increase is: 6 - 2 = 4. The ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$ is $\frac{6}{4}$, or $\frac{3}{2}$. From your graph, find the ratio $\frac{\text{vertical increase}}{\text{horizontal increase}}$, for the following pairs of points:

(a) D to E
(b) D to F
(c) C to D
(d) A to E
(e) A to D
(f) A to F

4. Put your pencil point on the point E. If you wanted to reach D, would you count up or down? How many? Would you count over to the left or right? How many? In going from E to D the vertical increase is <u>negative</u>; the horizontal increase is <u>negative</u>. What is the ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$?

5. What is the vertical increase in going from D to C ? What is the horizontal increase? What is the ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$?

6. The ratio $\frac{\text{vertical increase}}{\text{horizontal increase}}$ in going from E: (2, 1) to D: (0, -2) which you obtained in Exercise 4 can be found by arithmetic:

$$\text{vertical increase: } -2 - 1 = -3$$

$$\text{horizontal increase: } 0 - 2 = -2$$

$$\frac{\text{vertical increase}}{\text{horizontal increase}}: \frac{-3}{-2} = \frac{3}{2}$$

Show how you can obtain your answer to Exercise 5 by using arithmetic.

In your work with the equation $2y - 3x + 4 = 0$ and its graph, you found that the ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$ was the same for any two points considered.

The constant ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$ for any two points on a line is called the <u>slope</u> of the line.

How many points of a line do you need to find the slope of that line?

<u>Problems 13-6</u>

1. Draw the graph of each of the following equations. Choose two points on the line in your completed graph and determine the slope of the line.

(a) $2y + 3x + 4 = 0$ (c) $2y - 4x - 2 = 0$

(b) $2y - 4x + 3 = 0$ (d) $y = 2x + 1$

2. In Section 13-2 you graphed $y = x + 3$, $y = 2x + 3$, and $y = 3x + 3$ on the same set of axes. Look at these graphs you made and find the slope of each line. Do the same for the graphs you made for Problem 1 of Problems 13-2A.

3. Refer to Problems 13-2B and for each, find the slope of the line that contains the two points pictured.

4. For each problem in Problems 13-2B, show how the slope can be obtained by computation. For example, for Problem 1, the coordinates of the two points are (1, 3) and (2, 4). If we go from (1, 3) to (2, 4), we have:

$$\text{vertical increase: } 4 - 3 = 1$$
$$\text{horizontal increase: } 2 - 1 = 1$$

Therefore, the slope is $\frac{1}{1}$, or 1.

13-7 The Slope of a Line from its Equation

In Problem 1 of Problems 13-6 you may have noticed that the graphs of (c) $2y - 4x - 2 = 0$ and (d) $y = 2x + 1$ were the same. Can you prove that the two equations are equivalent by the method you used in Section 13-4?

You may also have noticed that the coefficient of the x-term in $y = 2x + 1$ is the same as the slope of the line. You should have found this true also of the equations referred to in Problem 2 of Problems 13-6. The coefficient of the x-term in $2y - 4x - 2 = 0$, however, is -4. Thus, in this form of the equation, the coefficient of the x-term is <u>not</u> the slope.

Problems 13-7A

1. In the graph of the equation $y = 5x + 3$, how much does y increase if the x-increase is 1? How much does y increase if the x-increase is 3? What is the slope of the line? What is the coefficient of x?

2. Answer the same questions for $y = 7x + 9$.
3. Answer the same questions for $y = \frac{3}{2}x - 2$.
4. Answer the same questions for $y = x + 8$.
5. Answer the same questions for $y = -4x + 2$.
6. Answer the same questions for $y = 1000x + 4$.

We have mentioned that an equation whose graph is a straight line is called a linear equation. There can be several forms for the equation of a line. We have observed that a linear equation in the special form $y = ax + b$ tells us the slope of the line. For the equation $y = 5x + 3$ the slope of the graph is 5; for the equation $y = -4x + 2$ the slope is -4. We make the following generalization:

If a linear equation is in the form $y = ax + b$, then a is the slope of the line.

If a linear equation is <u>not</u> in the form $y = ax + b$, we can try to find an <u>equivalent equation</u> which is in that form. Thus, to find the slope of the graph of $3x + 4y - 5 = 0$, we could proceed as follows:

$$3x + 4y - 5 = 0$$

is equivalent to $\quad 4y = -3x + 5 \quad$ (Why?)

which is equivalent to $\quad y = -\frac{3}{4}x + \frac{5}{4} \quad$ (Why?)

What, then, is the slope of the graph?

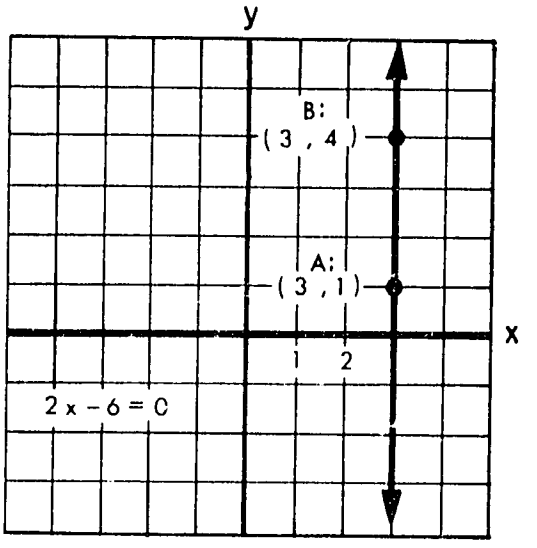
<u>Problems 13-7B</u>

For each of the following equations, obtain the slope, where possible, from an equivalent equation in the $y = ax + b$ form. Check 1 and 2 by drawing a graph and finding the ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$ for two points; check 3 and 4 by finding two ordered pairs in the truth set and computing the slope; use either method for 5.

1. $5y - 3x + 1 = 0$
2. $3y + 4x - 5 = 0$
3. $x + 3y - 5 = 0$
4. $3y - 7 = 0$
5. $2x - 6 = 0$

A Special Case

Were you able to find a slope of the line for $2x - 6 = 0$ in Problem 5? Examine the graph of $2x - 6 = 0$ to the right, and find the change in y and in x from A: (3, 1) to B: (3, 4).



You should find that the vertical increase is 3 and the horizontal increase is 0. When we consider the slope, that is, the ratio, $\frac{\text{vertical increase}}{\text{horizontal increase}}$ we find that $\frac{3}{0}$ is <u>not</u> a real number. We obtain the same result by computing:

$$y \text{ - increase: } 4 - 1 = 3$$

$$x \text{ - increase: } 3 - 3 = 0$$

Since $\frac{3}{0}$ does not make sense, slope does not exist for this line.

If we attempt to find an equivalent equation in the form $y = ax + b$ for $2x - 6 = 0$, we find that we cannot. Even if we introduce a y-term into the equation obtaining

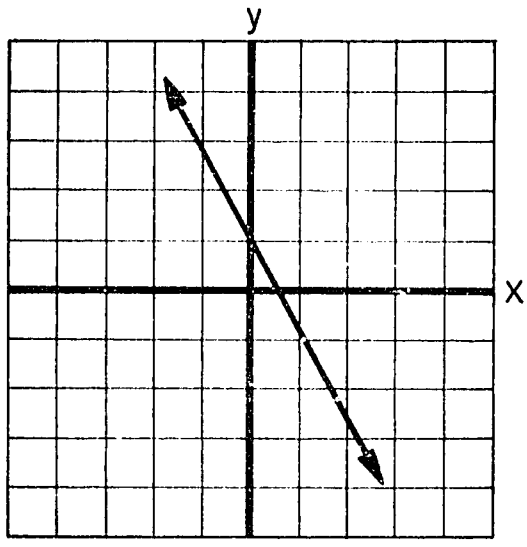
$$0y + 2x - 6 = 0$$

we cannot solve for y because we cannot divide by 0. The linear equation $2x - 6 = 0$ does not have a slope. Can you name another equation whose graph is a straight line that does not have a slope? We might have expected that the slope would never exist for a line in which the horizontal increase between any two points is 0. How can you describe all such lines?

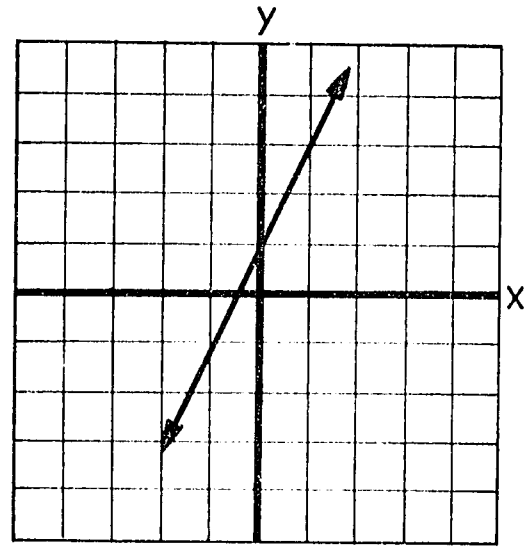
Problems 13-7C

For which of the following lines is the slope positive, for which is the slope negative, for which is the slope zero, for which does the slope not exist?

1.



2.



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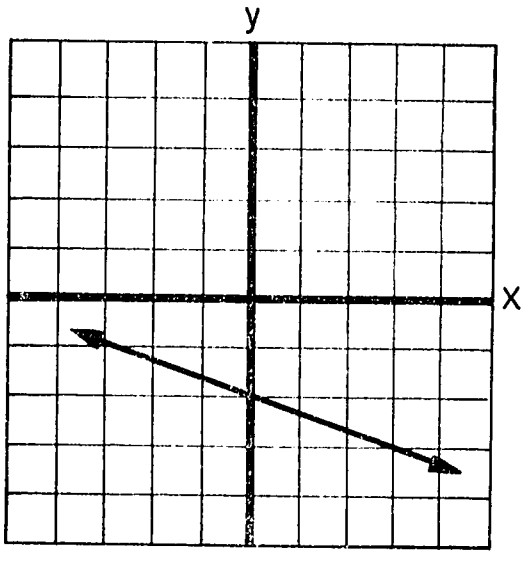

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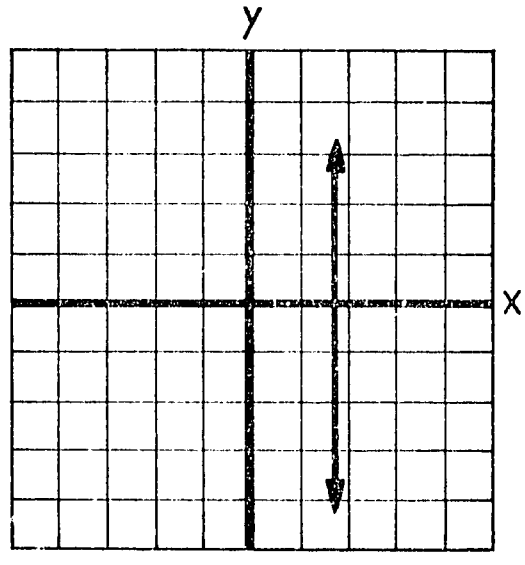
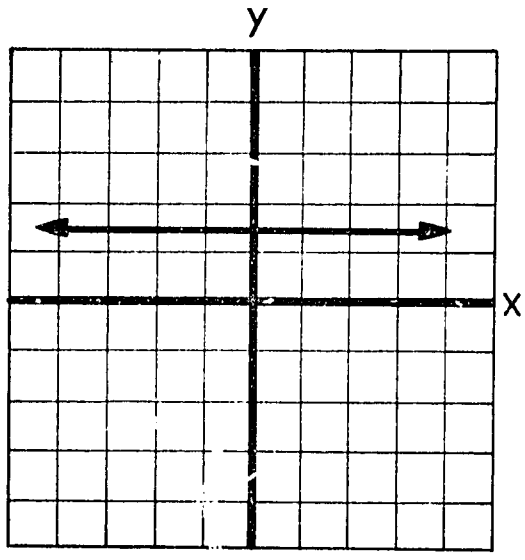
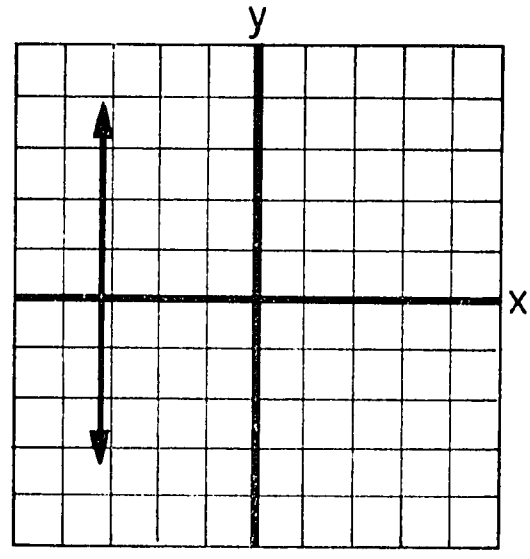


5.

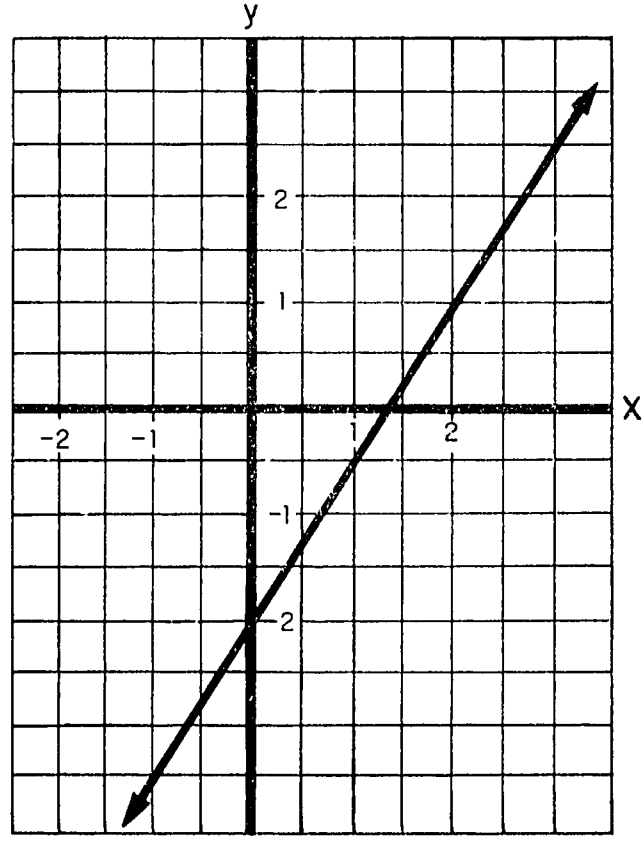


6.



13-8 The Intercept on the y-axis

To the right is an enlarged picture of a portion of the graph of $2y - 3x + 4 = 0$ which we considered in Section 13-5.



If x is 0, what is y? What are the coordinates of the point where the line intersects the y-axis? Refer back to your graphs for Problem 2 of Problems 13-2A. For each line, find the coordinates of the point where the line intersects the y-axis. Compare the second coordinate of this point with the constant term in the equation of the line. Do the same for the problems in Problems 13-2B.

The second coordinate of the point where a line intersects the y-axis is called the <u>intercept</u> on the y-axis.

What is the first coordinate of the point where a line intersects the y-axis?

You may have noticed that if an equation is in the form $y = ax + b$, then b is the intercept of the line. The proof of this is simple. Since any point on the y-axis has 0 for its x-coordinate, the equation $y = ax + b$ becomes $y = a \times 0 + b$ for a point on the y-axis; therefore, y is b, and $(0, b)$ belongs to the truth set of $y = ax + b$. Since a nonvertical line can intersect the y-axis in at most one point, the intercept is b. For any equation $y = ax + b$, the intercept on the y-axis is b.

<u>Problems 13-8</u>

1. Graph each of the following and read the y-intercept from the graph; check by writing the equation in $y = ax + b$ form and reading the y-intercept from the equation.

   (a) $2y + 3x - 6 = 0$ (b) $3x - y - 3 = 0$

2. Graph each of the following and read the y-intercept n from the graph; check by seeing whether $(0, n)$ makes the equation true.

   (a) $4x - 3y - 9 = 0$ (b) $2y + 6x - 5 = 0$

3. Graph the equation $7y - 14x + 6 = 0$. When you try to read the intercept on the y-axis from the graph, you will find it difficult to read exactly. Estimate it as closely as you can from your graph and test your estimate by seeing whether your coordinates make the equation true. If the two sides of

the equation differ by very little, your estimate is close. You may then want to find the exact intercept by some other method which does not depend on the graph.

4. If we combine what we know about the slope and the intercept of a line, we obtain the following:

For any equation $y = ax + b$,
a is the slope of the line and
b is the intercept on the y-axis.

For the equations in (a) and (b), name the slope and y-intercept; for (c) and (d), find an equation which has the slope and y-intercept as given.

(a) $y = -\frac{2}{3}x + 5$

(b) $y = 100x - 3$

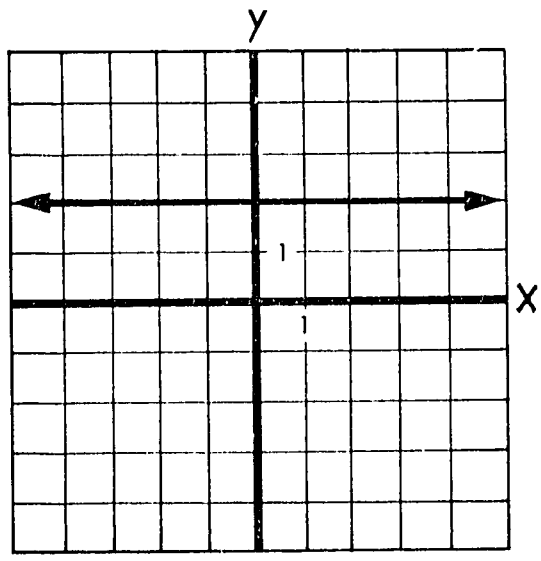
(c) Slope is 5; y-intercept is 4.

(d) Slope is 0; y-intercept is -2.

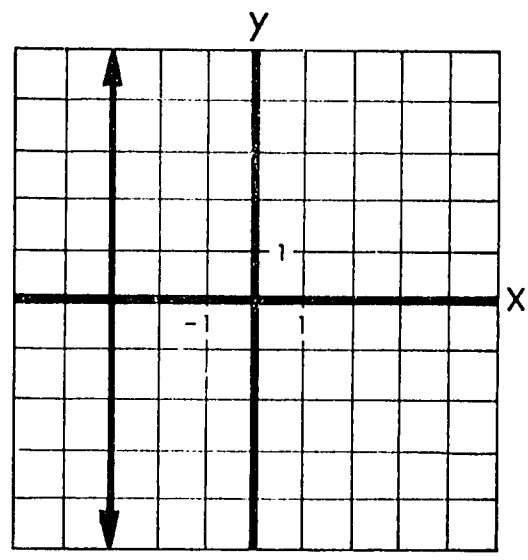
5. In the last section you found that not every linear equation has a slope. Can you find a linear equation that does not have an intercept on the y-axis?

6. For each of the following lines, determine the slope and y-intercept, and write an equation whose truth set is pictured.

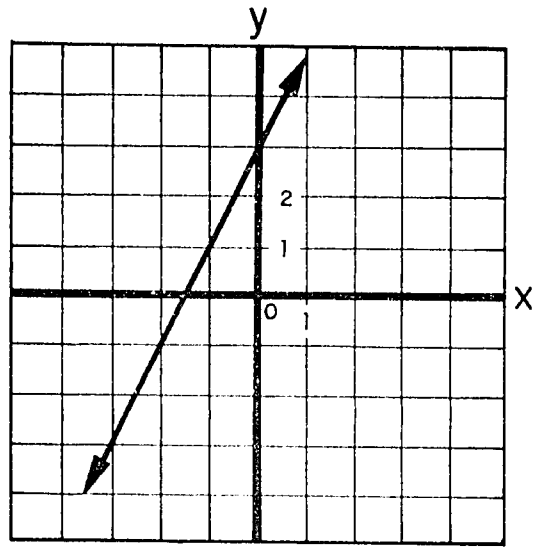
(a)



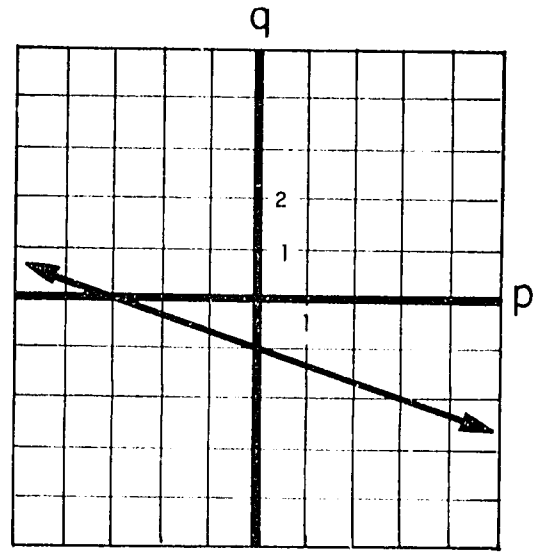
(b)



(c)



(d)



7. For each of the following, determine an equation, the graph of which contains the two points named.

(a) (0, 1) and (1, 7)

(b) (0, -3) and (3, -3)

(c) (0, 0) and (1, -2)

(d) (0, 4) and (2, 7)

8. Can you find the intercept on the y-axis for the equation $5y - 13x + 35 = 0$ without putting the equation in the form $y = ax + b$?

9. Find the intercept on the y-axis without doing any writing:

(a) $x + 3y = 12$

(b) $3x - y + 8 = 0$

(c) $7y - 8x - 4 = 0$

(d) $x + y + 15 = 97x$

(e) $\frac{3}{5}x + \frac{2}{3}y = 1$

(f) $2x + 3y = 8 + 5x + y$

10. <u>CHALLENGE PROBLEM</u>. What meaning might appropriately be given to "the intercept on the x-axis"? Find this number for each part of Problem 9.

11. <u>CHALLENGE PROBLEM</u>. Find an equation of the line which contains the points (1, 4) and (2, 7).

CHAPTER 14

USING STRAIGHT LINE GRAPHS

14-1 Systems of Simultaneous Equations

In Chapter 10 you solved the following pair of simultaneous equations:

$$\begin{cases} 2x + 4y = -12 \\ 5x + 4y = -33 \end{cases}$$

By subtracting, you obtained another equation, $-3x = 21$, and you then considered this pair of equations

$$\begin{cases} 2x + 4y = -12 \\ -3x = 21 \end{cases}$$

which is equivalent to the pair above.

Class Activity

1. On the same set of axes, graph the three equations $2x + 4y = -12$, $5x + 4y = -33$, and $-3x = 21$.

2. Read from the graph the coordinates of the point of intersection of the lines whose equations are $2x + 4y = -12$ and $5x + 4y = -33$.

3. Read from the graph the coordinates of the point of intersection of the lines whose equations are $2x + 4y = -12$ and $-3x = 21$.

4. You should have found that the graphs of the two pairs of equivalent equations intersect at the same point. Do you think this will occur for the graphs of any two pairs of equivalent equations?

5. Without doing any more algebra do you know the solution of

$$\begin{cases} 2x + 4y = -12 \\ 5x + 4y = -33 \end{cases} ?$$

Previous Page Blank

Problems 14-1A

For each of the following pairs of equations, obtain the solution by graphing; for problems 1 and 2, check by seeing whether your solution makes both equations true; for problem 3, check by finding the solution using equivalent systems of equations as you did in Chapter 10.

1. $\begin{cases} 2x + 3y = 5 \\ x + y = 1 \end{cases}$

2. $\begin{cases} 3x + 5y = 11 \\ y = \frac{1}{7}x - 3 \end{cases}$

3. $\begin{cases} 7x + 4y = 11 \\ 14x + 3y = 12 \end{cases}$

Suppose we wish to find the solution for this system of equations:

$$\begin{cases} 3x - y + 1 = 0 \\ 2y - 6x - 7 = 0 \end{cases}$$

We might proceed by using algebra to find equivalent pairs of equations. Give a reason why each of the following pairs is equivalent to the pair above.

$$\begin{cases} 3x - y + 1 = 0 \\ -6x + 2y - 7 = 0 \end{cases}$$

$$\begin{cases} 6x - 2y + 2 = 0 \\ -6x + 2y - 7 = 0 \end{cases}$$

If we add the last pair of equations, we obtain: $0 + 0 - 5 = 0$, or $-5 = 0$. Our attempt to find the solution of this system of equations algebraically resulted in a sentence which is false.

Class Activity

Graph the pair of equations on the same set of axes and see whether you can explain why the attempt to solve them algebraically resulted in a

false statement. What is the truth set for this system of simultaneous equations?

Let us examine another system of equations:

$$\begin{cases} y = \frac{1}{3}x - 2 \\ 6y - 2x + 12 = 0. \end{cases}$$

Is (3, -1) a solution?

Is (0, -2) a solution?

Is (9, 1) a solution?

The graph of each of the above equations is a straight line. Can two lines intersect in more than one point?

Class Activity

1. Graph the pair of equations above and see whether you can explain why there is more than one solution for this pair of equations.

2. Try to solve the same pair of equations algebraically. What happens?

3. What is the truth set of this system of equations?

Problems 14-1B

Using any method you wish, find the truth set for each of the following:

1. $\begin{cases} 3x + 4y - 11 = 0 \\ 2x + 3y = 9 \end{cases}$

2. $\begin{cases} 2x - y + 8 = 0 \\ 4x - 2y + 17 = 0 \end{cases}$

3. $\begin{cases} 2x + 8y = 19 \\ x - 16y + 3 = 0 \end{cases}$

4. $\begin{cases} \frac{1}{3}x - 5y + 2 = 0 \\ 30y - 2x = 12 \end{cases}$

14-2 Proportionality

You may recall that we say y is proportional to x if $y = kx$, and k is any non-zero real number called the constant of proportionality. Let us examine the tables of some of the rules you met in Chapter 13.

(a)

x	y
0	7
1	8
2	9
10	17

(b)

a	b
0	0
2	14
7	49
100	700

(c)

x	y
-1	$-\frac{1}{4}$
0	0
2	$\frac{1}{2}$
4	1
8	2

(d)

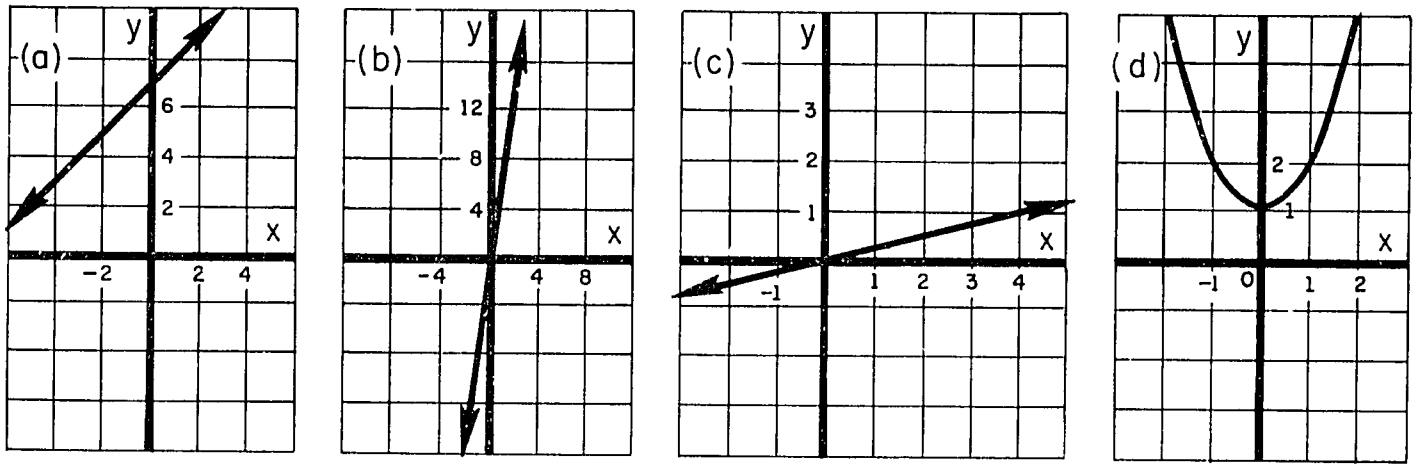
x	y
0	1
1	2
2	5
8	65
10	101
12	145

1. For which of these rules is the second variable proportional to the first variable?

2. For those that are proportional, what is the constant of proportionality?

3. Write an equation for each of the rules (a) - (d). Describe how to tell if y is proportional to x by examining the equation.

4. Here are graphs of the four rules.



From examining the equations you should have been able to know that (a) and (d) do not have proportionality, while (b) and (c) do. Can you describe how to tell whether two variables are proportional from examining the graph?

5. Let us examine the equation $y = kx$ again. Is the equation linear, that is, can it be put in the form of $y = ax + b$? What is the slope of $y = kx$? What is its intercept on the y-axis? What point must the graph of $y = kx$ contain, for <u>any</u> value of k?

Your answers to the previous questions should lead you to the following conclusion:

> y is proportional to x if there is a linear equation relating y and x, the graph of which goes through the origin and has a non-zero slope. The constant of proportionality is the slope of the line.

<u>Problems 14-2</u>

1. If y is proportional to x with a constant of proportionality k, is x proportional to y? If your answer is "yes", then what is the constant of proportionality?

2. For which of the following is y proportional to x ? What is the constant of proportionality?

(a) $y = 3(x + 2)$

(b) $7x - 2y = 0$

(c) $y = 0$

(d) $y = 2x^2$

3. For each of the following, assume that a rule can be written which is a linear equation. Which relations show that y is proportional to x ?

(a)

x	y
-1	0
5	2

(b)

x	y
0	0
1	100

(c)

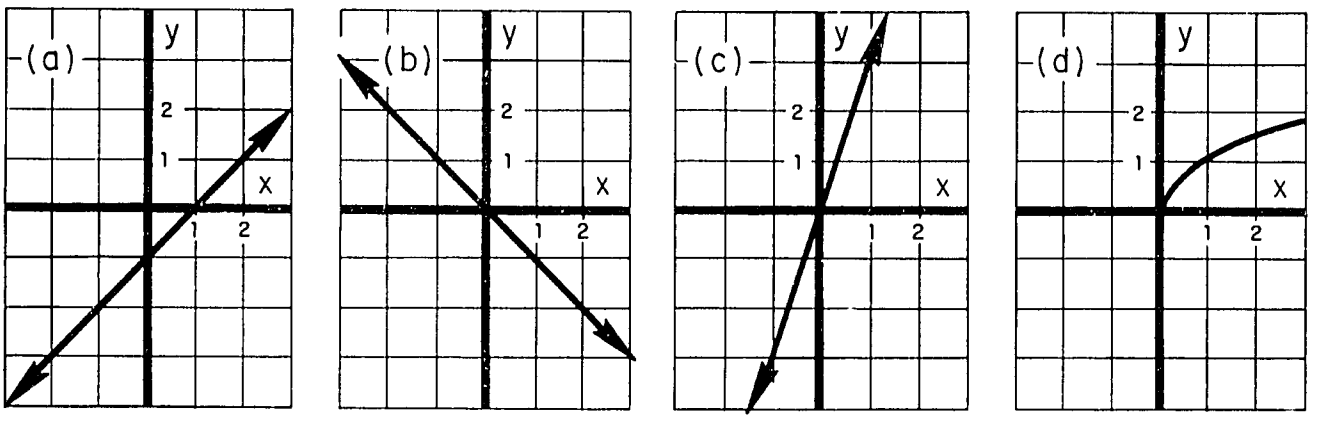
x	y
$\frac{1}{2}$	0
0	3

(d)

x	y
0	0
-2	4

For those that show proportionality, can you tell what the constant of proportionality is?

4. For which of the following is y proportional to x ? What, if any, is the constant of proportionality?



5. (a) A class of 45 children took an examination which had 35 questions. The teacher scored each paper and wrote the number of correct answers at the top of the paper. Then he gave the papers to Sonko, and asked him to figure out what grade out of 100 each paper should have if the allowable grades were numerals ending in either 0 or 5. For example, if Sonko had a grade of 71 out of 100 for a paper, then he was to round it off to 70. The

first paper Sonko looked at had 28 out of 35 correct, so he set up the following proportion: $\frac{28}{35} = \frac{x}{100}$. What value should Sonko get for x? What grade should he put on the paper?

"That was easy," thought Sonko. The next paper had a score of 23. So Sonko set up this proportion: $\frac{23}{35} = \frac{x}{100}$. What value should he get for x? What grade should he put on the paper?

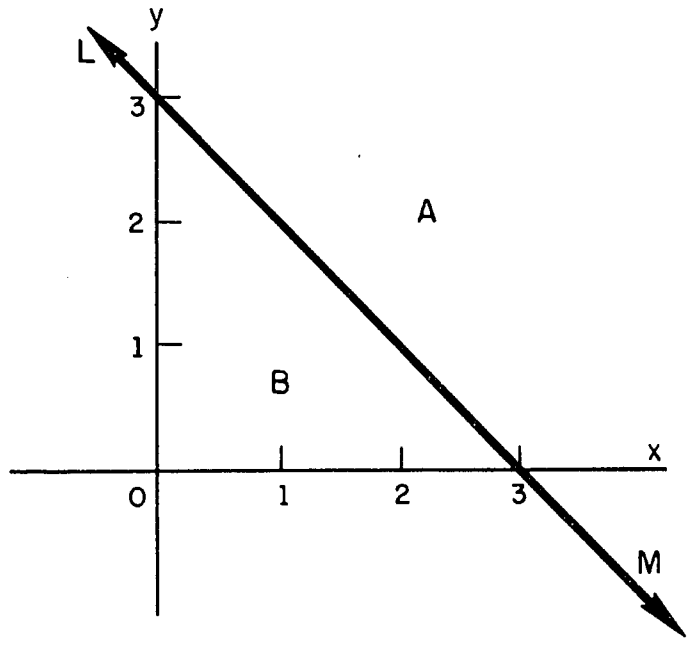
(b) Sonko decided at this point that this was too much work - it would take all day to do 43 more papers. After thinking for a minute, he decided that the grade out of 100 <u>before</u> rounding off was proportional to the number correct out of 35, so he could save time by finding the constant of proportionality. Since 100 had to be $k \times 35$, Sonko found that k must be $\frac{100}{35}$, or $\frac{20}{7}$. The next paper had a score of 14, so Sonko wrote: $y = \frac{20}{7} \times 14$. What value should he obtain for y? What grade should he put on the paper? The next paper had a score of 18. What value for y should Sonko obtain? What grade should he put on the paper?

(c) Sonko still had 41 papers to go when he began to think, "Since the exact number out of 100 is not needed, maybe I could use a graph... Let me see, I need 35 units on the x-axis and 100 units on the y-axis." So he drew the axes on a large piece of graph paper and marked the scales. Then he thought, "(0, 0) and (35, 100) have to be points on the graph," so he marked them, drew the line, and finished putting grades on the remaining 41 papers in 7 minutes. He even had time to check the grades for the first four papers. Use Sonko's last method to find the grades for the next 10 papers with scores of 10, 34, 20, 17, 30, 26, 4, 12, 31, and 24.

14-3 A Practical Application

1. Graphs of Inequalities

Look at the graph shown below of the equation, $x + y = 3$. Let us call the line $\overleftrightarrow{LM}$.



The line divides the rest of the plane into two half-planes. Let us try to find the open sentences whose truth sets give us these half-planes.

Name the coordinates of some points in the half-plane A.

Find the values of $x + y$ at these points.

Compare these values with the values of $x + y$ at points on the line $\overleftrightarrow{LM}$.

What do you notice?

The point (0, 3) lies on the line $\overleftrightarrow{LM}$. What can you say about y when the point $(0, y)$ lies in the half-plane A ? What can you say about the value of $x + y$ at this point. Is it greater or less than 3?

Can you find any point in the half-plane A at which the value of $x + y$ is not greater than 3?

You can now see that any member of the truth set of

$$x + y > 3,$$

gives a point in the half-plane A, and the coordinates of any point in the half-plane A is a member of the truth set of $x + y > 3$.

Make similar statements about points in the half-plane B.

2. <u>An Application</u>

Ideas like those above can be used in the solution of many problems. Let us look at an example:

A teacher has 60/- to spend on text books for his class.

At Nazarali's they cost 5/- each.

At Shariff's they cost 6/- each.

But Nazarali has only 6 of the books. How must the teacher buy the books in order to obtain the greatest number?

This is an easy problem and you will probably be able to give the answer using only your common sense. But let us see how we could solve the problem using graphical methods and then you will be able to use this method to solve harder problems.

Let x be the number of books bought from Nazarali.

Let y be the number of books bought from Shariff.

The cost of the books is then $5x + 6y$ shillings and so $5x + 6y \leq 60$.

Since Nazarali has only 6 books $x \leq 6$.

We know that $x \geq 0$ and $y \geq 0$.

We shall therefore need to draw the lines.

$5x + 6y = 60$, $x = 6$, $x = 0$, $y = 0$ on the same axes.

When we graph an inequality such as $5x + 6y \leq 60$, it is customary to shade the half-plane whose coordinates satisfy $5x + 6y \leq 60$. In this work however this sometimes gives a rather untidy diagram. We shall instead, shade the region for which $5x + 6y > 60$, and then the clear part of the plane is the part in which we are interested.

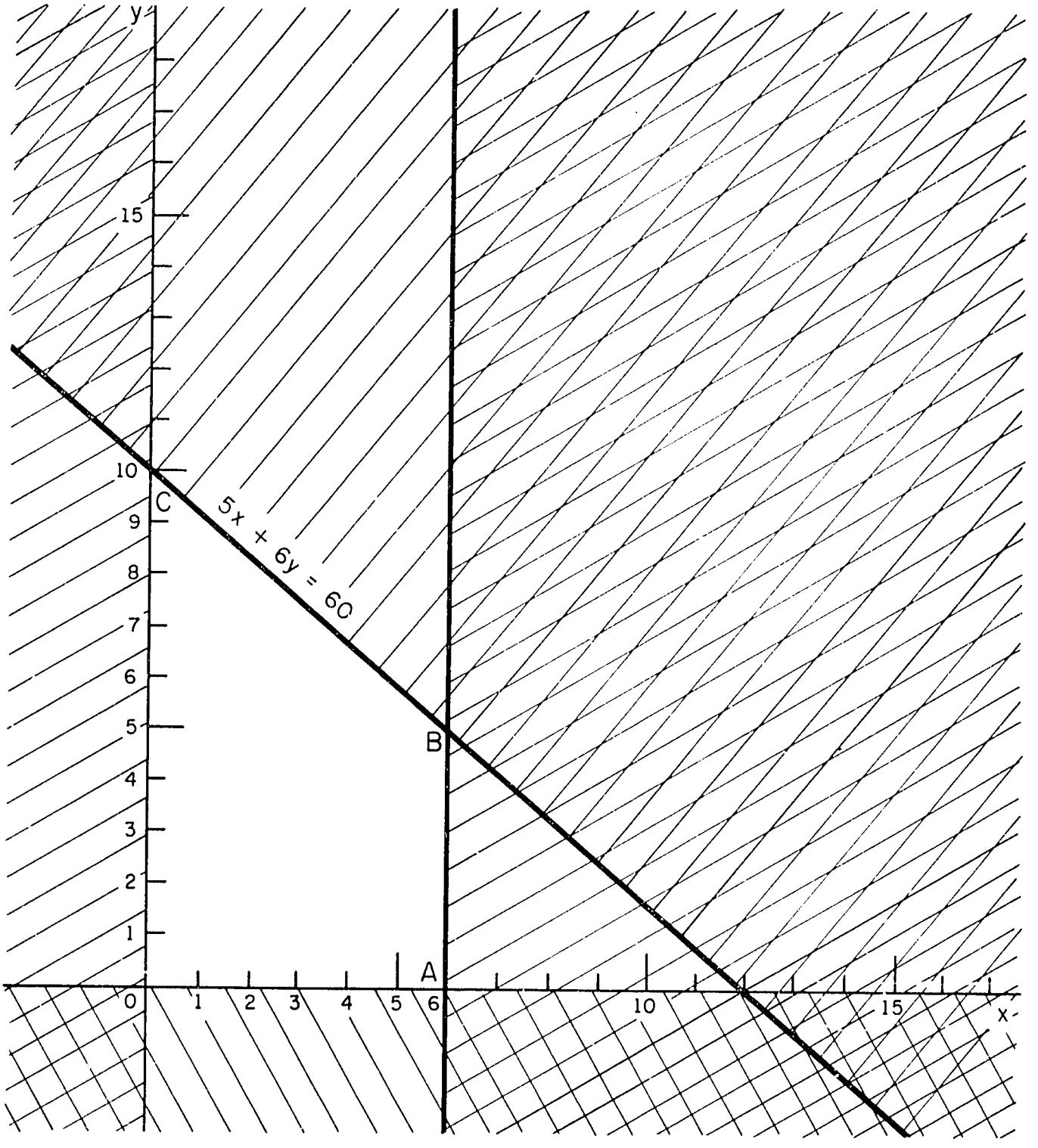


Fig. 1

The truth set of $5x + 6y \leq 60$ gives us the set of points in the region below the line $5x + 6y = 60$, so we shade the region above the line (a coloured pencil might make it more clear). Similarly, we shade the regions:-

to the right of $x = 6$

to the left of $x = 0$

below $y = 0$.

We are now left with an unshaded region within the quadrilateral OABC as seen in Fig. 1. The coordinates of points in this region are members of the truth set of the system of simultaneous inequalities.

$$\left.\begin{array}{r} 5x + 6y \leq 60 \\ 0 \leq x \leq 6 \\ 0 \leq y \end{array}\right\} \quad \text{---------} \ (1)$$

We now want the member of this set with whole number coordinates (we cannot buy $7\frac{1}{2}$ books) which makes $x + y$ as large as possible.

The lines containing the truth sets for $x + y = 5$, 10, and 15 are drawn on the graph. (See Fig. 2) It is clear that there are members of the truth set of (1) above for which $x + y = 5$ and for which $x + y = 10$ but not for which $x + y = 15$.

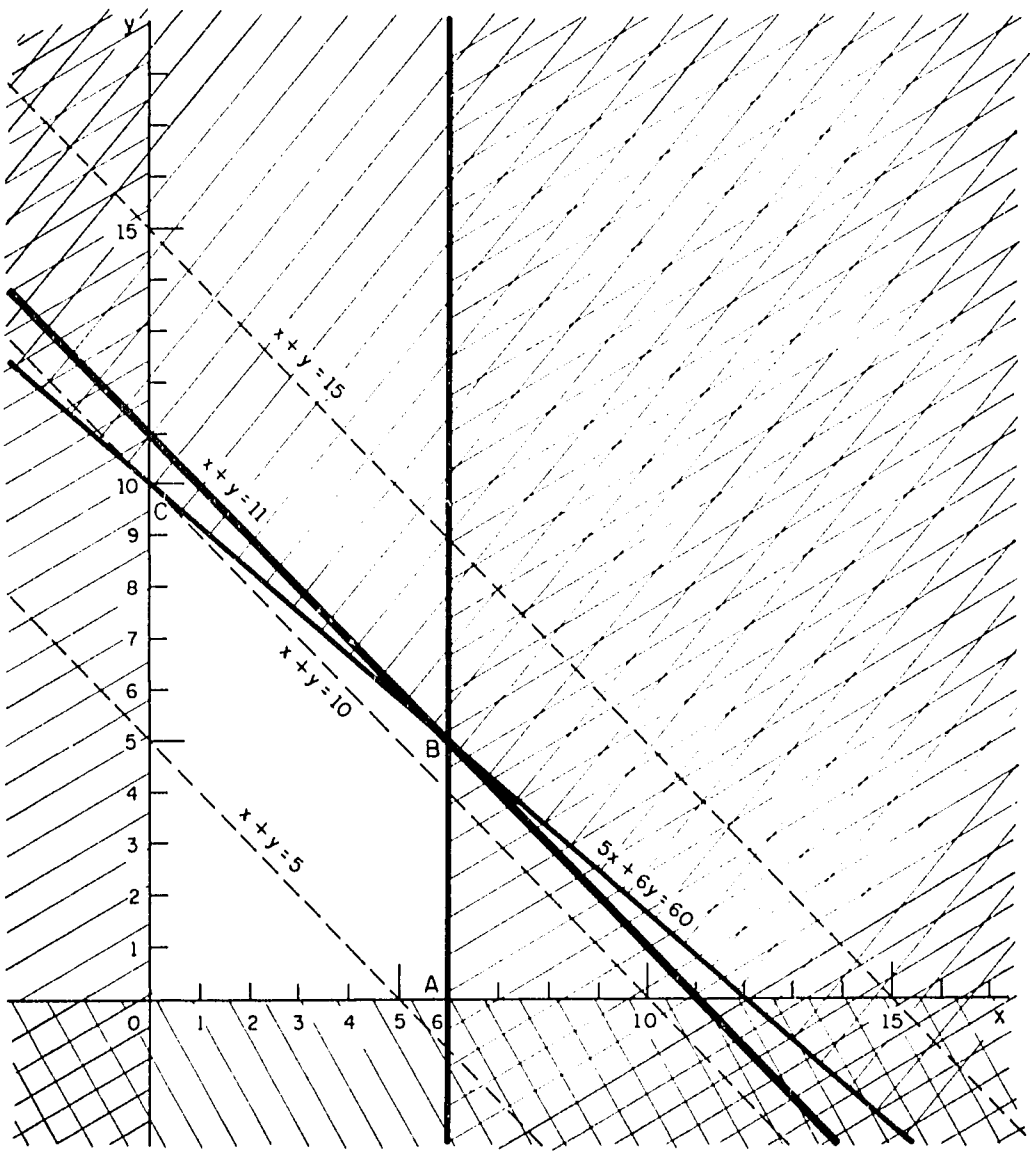


Fig. 2.

The lines $x + y = 5$, $x + y = 10$, and $x + y = 15$ are members of a set of parallel lines. There are other members of the set along which the value of $x + y$ is different from 5, 10, and 15. Which member of the set of parallel lines cuts the unshaded region, and has the largest value of $x + y$?

What is this value of $x + y$?

What are the coordinates of the point where it cuts the quadrilateral OABC ?

You may find it easier to use your ruler and set square to answer these questions.

Where, then, does the teacher buy his books?

What would the solution be if Nazarali had

(1) 9 books?

(2) 5 books?

(3) 4 books?

3. <u>Another Application</u>

Here is a longer example. This is a kind of problem which a manufacturer might meet.

A manufacturer makes 2 products A and B. For every ton of A he makes, he can make from 1 to 2 tons of B.

Not more than 10 tons of A can be made in 1 day.

The product A can be sold at a town 30 miles away at a profit of £ 2 per ton.

The product B can be sold at a town 20 miles away at a profit of £ 1 per ton.

But there is only enough transport for 600 ton-miles per day. (This means that in 1 day 600 tons can be carried 1 mile or 30 tons carried 20 miles or 10 tons carried 60 miles, etc.)

How many tons of each product does the manufacturer make each day to obtain the greatest profit?

If the manufacturer makes x tons per day of A and y tons per day of B, we have

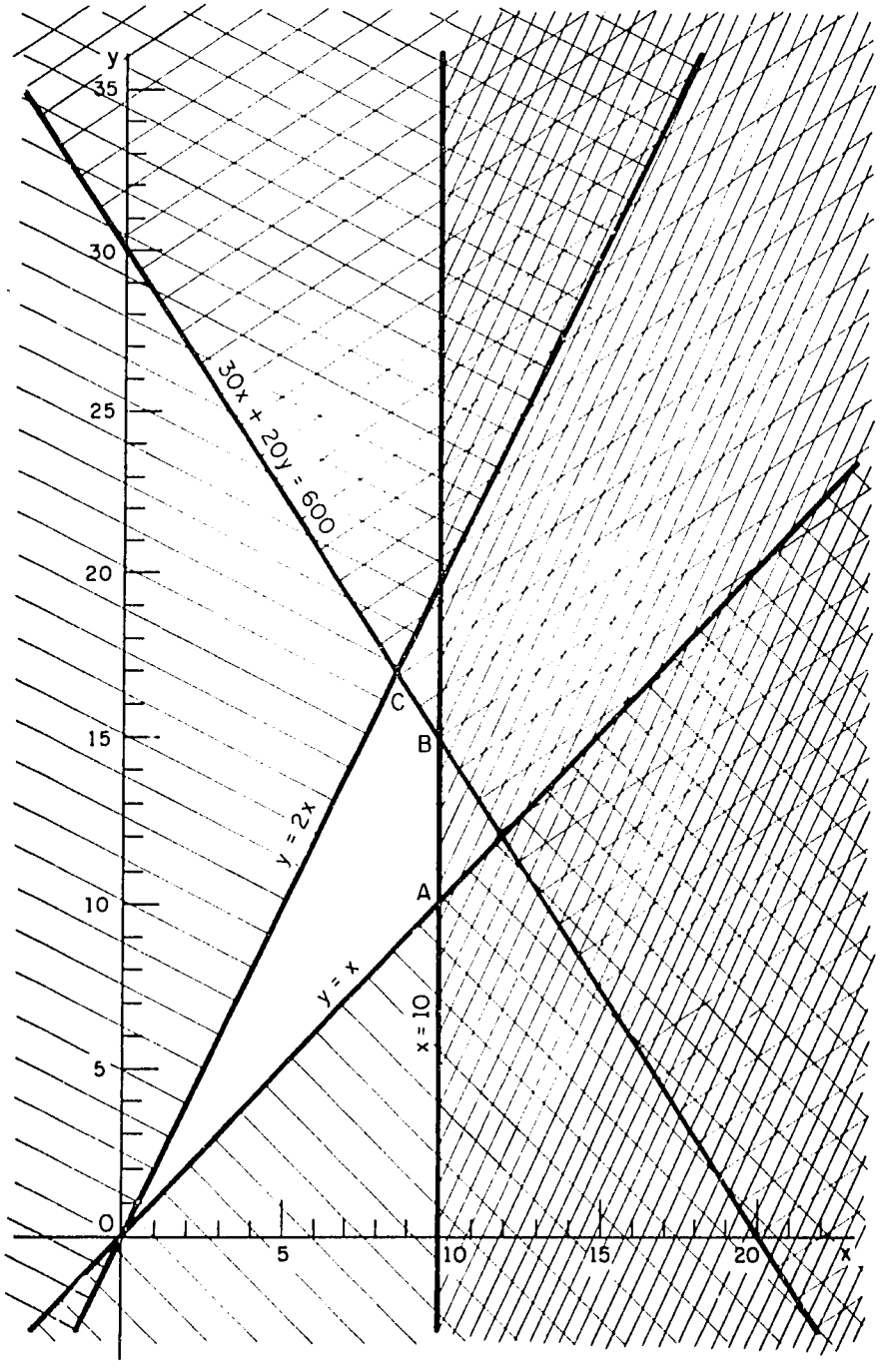
$y \leq 2x$	Why?
$y \geq x$	Why?
$x \leq 10$	Why?

and, from looking at the transport information, we have

$$30x + 20y \leq 600.$$

How do we get this?

Therefore we need to draw the graphs of $y = x$, $y = 2x$, $x = 10$, and $30x + 20y = 600$ on the same axes.



Having done this we now have the unshaded region OABC. This is the set of points whose coordinates satisfy the system

$$\left.\begin{aligned} y &\geq x \\ y &\leq 2x \\ x &\leq 10 \\ 30x + 20y &\leq 600 \end{aligned}\right\}$$

We want the point in this quadrilateral which makes the profit as large as possible. The profit is £ $(2x + y)$. Why?

We must therefore find the member of the set of parallel lines given by $2x + y = d$, which cuts the quadrilateral OABC and for which d is as large as possible.

Use your ruler and set square to find this line.

Where does it cut the quadrilateral?

How many tons of A and B should the manufacturer make per day?

What would the solution be if the profit on both A and B were £2 per ton?

Problems 14-3

Here are more examples you can try. The first is not a word problem:

1. Construct the polygon which contains the points whose coordinates are in the truth set of the simultaneous inequalities:

$$\left.\begin{aligned} y &\leq 2x \\ x &\leq 6 \\ y &\geq 2 \\ 2x + 3y &\leq 30 \end{aligned}\right\}$$

Find the members of this truth set which make:

(a) $y - x$ a minimum

(b) $y - x$ a maximum

(c) $5y + 6x$ a minimum

(d) $4x + 6y$ a maximum

2. A manufacturer makes two compounds "X" and "Y" which contain

the following quantities of Vitamins A and B:

	"X"	"Y"
Vitamin A	5 units per oz.	5 units per oz.
Vitamin B	15 units per oz.	5 units per oz.

"X" costs 2/- per oz. and "Y" costs 1/- per oz. The manufacturer wants a mixture of "X" and "Y" to contain at least 50 units of Vitamin A and at least 60 units of Vitamin B. How many ounces of each does he use to make the mixture as cheaply as possible? (Hint: Let the manufacturer mix x oz. of "X" with y oz. of "Y".)

3. A manufacturer has two warehouses in Kenya. One is in Nairobi and contains 25 units of his product. The other is in Nakuru and contains 20 units of his product. He has to supply a shop in Nanyuki with 20 units and a shop in Thompson Falls with 15 units. The costs of transport for each unit are as follows:

From \ To	Thompson Falls	Nanyuki
Nairobi	36/-	45/-
Nakuru	30/-	42/-

How does the manufacturer send his products as cheaply as possible? (Hint: If x units are sent from Nakuru to Thompson Falls, how many are sent from Nairobi to Thompson Falls? If y units are sent from Nakuru to Nanyuki, how many are sent from Nairobi to Nanyuki?)

4. A builder has 2 stores at S_1 and S_2. He is building houses at 3 places, P_1, P_2, P_3. He needs 5 tons of bricks at P_1, 6 tons at P_2, and 4 tons at P_3. The stores contain 9 tons at S_1 and 6 tons at S_2.

The transport costs per ton are as follows:

From \ To	P_1	P_2	P_3
S_1	6/-	3/-	4/-
S_2	4/-	2/-	6/-

How does the builder send the bricks at minimum cost? (Hint: If x tons are sent from S_1 to P_1 and y tons sent from S_1 to P_2, complete the following table:

From \ To	P_1	P_2	P_3
S_1	x	y	
S_2	$5 - x$		

5. A mining company owns 2 small copper mines, each of which produces 3 grades of copper ore, grade 1, grade 2, and grade 3. The first mine at A costs £ 40 per day to run and produces 3 tons of grade 1, 1 ton of grade 2, and 2 tons of grade 3 ore per day. The second mine costs £ 30 per day to run and produces 1 ton of grade 1, 1 ton of grade 2, and 6 tons of grade 3 ore per day.

The company has orders for 6 tons of grade 1, 4 tons of grade 2, and 12 tons of grade 3 ore per week.

For how many days each week should each mine work, to fill the order as cheaply as possible?

Hint: Arrange the information in a table like that shown below:-

Mine \ Ore	Grade 1	Grade 2	Grade 3	Cost per day
A				
B				
Orders for				

CHAPTER 15

OTHER GRAPHS

15-1 Non-linear Graphs

Thus far we have been concerned with the straight line graphs of equations of the form $y = ax + b$, or $ax + by + c = 0$. Let us investigate the graphs of some equations which are not of these forms.

Class Activity

The graph of $y = |x|$

Recall that the absolute value of x is the same as x when x is positive or zero, and that it is the opposite of x when x is negative. One way to get an idea of what the graph will look like is to find some ordered pairs of numbers that belong to the truth set, plot the points, and see whether there is a pattern.

1. Find the values of y which would complete this table.

x	0	1	2	3	4	5
y						

2. Plot the points that correspond to these ordered pairs.

3. Thus far you should find that the graph is the same as for $y = x$. We have not yet considered any negative values for x. Find the values of y which would complete this table, and then plot the points.

x	-1	-2	-3	-4	-5
y					

4. This part of the graph is not the same as $y = x$, but it is the same as part of the graph of a straight line. Can you name the equation for it?

5. Complete the graph of $y = |x|$.

6. For what values of x is the graph of $y = |x|$ the same as for $y = x$? For what values of x is the graph the same as for $y = -x$?

7. By recalling the definition of $|x|$, could you have predicted what the graph of $y = |x|$ would be <u>without</u> having graphed it?

8. Describe the graph of each of the following open sentences in terms of the graph of $y = |x|$:

(a) $y < |x|$ (c) $y > |x|$

(b) $y \leq |x|$ (d) $y \geq |x|$

<u>The graph of $y = x^2$</u>

Again, we can begin by finding some ordered pairs in the truth set, plotting the points, and seeing whether there is a pattern.

1. For $y = x^2$, find the values of y which would complete this table.

x	-2	-1	0	1	2	3	4	5
y								

2. Plot the points on graph paper or on the blackboard. Use the same scale on both axes; unless you use small units, you may find that there are several points that do not fit on your graph.

3. You should have found that the points you plotted do <u>not</u> lie on a straight line. Decide how you think the graph should be completed, and then discuss your decision with the rest of the class. Graphs such as the one you have constructed are called <u>parabolas</u>.

<u>Problems 15-1</u>

Sketch the graph of each open sentence. Let the x-axis be horizontal, the y-axis vertical, and use the same scales on both.

1. Sketch these on the same set of axes.

(a) $y = x^2$

(b) $y = 2x^2$

(c) $y = -2x^2$

(d) $y = 10x^2$

(e) $y = \frac{1}{2}x^2$

2. Sketch these on the same set of axes.

(a) $y = x^2 + 3$

(b) $y = x^2 + 0$

(c) $y = x^2 - 4$

3. $x = y^2$

4. Use a separate set of axes for each of the following:

(a) $x + y = 10$; $x + y \leq 10$; $x + y \geq 10$

(b) $|x| + y = 10$; $|x| + y < 10$

(c) $x + |y| = 10$; $x + |y| \geq 10$

(d) $|x| + |y| = 10$; $|x| + |y| \leq 10$

(e) $|x + y| = 10$; $|x + y| > 10$

5. $x + y = y + x$

6. $xy = 12$

7. (a) $y = x^3$

(b) $|x|^3$

8. <u>CHALLENGE PROBLEM</u>

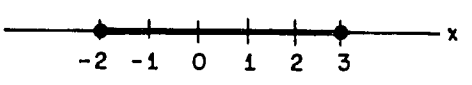
(a) Graph $x \leq 4$ on the number-<u>line</u>.

(b) Graph $x \leq 4$ on the number-<u>plane</u>.

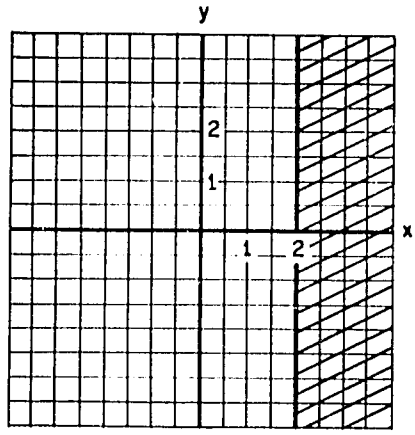
(c) What do you think the graph of $x \leq 4$ in the number-<u>space</u> would be?

9. Consider each of the following as a graph of a truth set, and write an open sentence for each. (The shaded portions indicate the points of the graph. A solid boundary of a region belongs to the graph; a dashed boundary does not.)

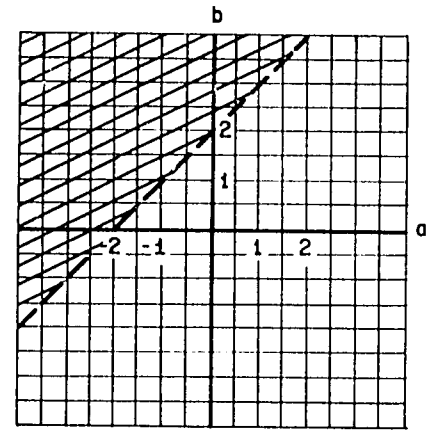
(a)



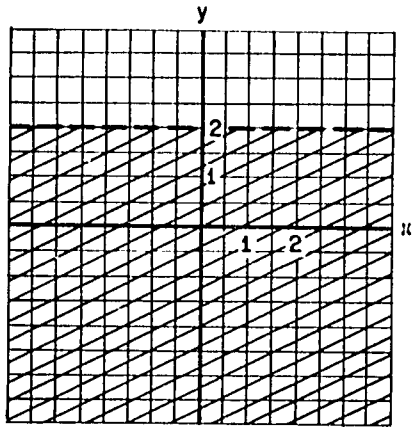
(b)



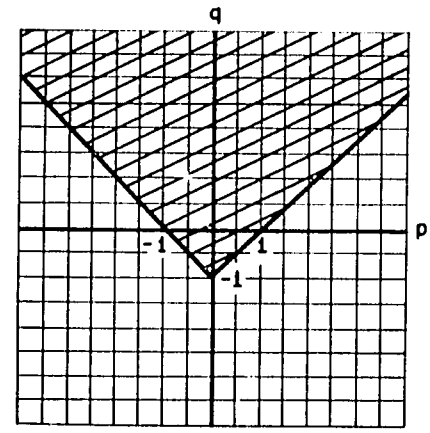
(d)



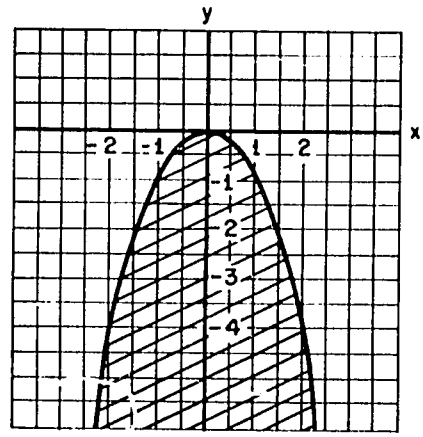
(c)



(e)



(f)



10. CHALLENGE PROBLEM

In a preceding chapter symmetry was discussed in relation to geometric figures. Look at your graphs for Problems 1-7 and see if you can answer the following questions:

(a) Which graphs have the x-axis as a line of symmetry?

(b) Which graphs have the y-axis as a line of symmetry?

(c) Which graphs have other lines of symmetry?

(d) Where you can, give the equation of each line of symmetry.

15-2 The Graph of $y = x^2$ and Tables of Squares and Square Roots

Section 15-1 dealt with the graphs of a variety of open sentences. This section deals in more detail with one of these, the graph of $y = x^2$, and its use in producing tables of squares and square roots.

You are already aware of the general shape of the graph of $y = x^2$ from working the exercise in section 15-1. You are aware also of its symmetry about the y-axis and the fact that it passes through the origin and exists only for non-negative values of y.

Class Activity

Let us draw the graph of $y = x^2$ for values of x from -5 to 5. When you draw your graph, take great care for you will need to take readings from the graph with considerable accuracy.

1. If $y = x^2$, find values of y for this table.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y											

What do you notice about the values of y for those values of x which are opposites of each other?

2. Decide on the position of each axis and the scales to be used so as to make the greatest use of your graph paper. Then draw the axes and plot the ordered pairs of numbers you have calculated.

3. Do the points you have plotted suggest a line or a curve? Would it be sensible to join the points by line segments? Additional points will be needed for a more accurate graph. Calculate the y-values when x is 0·5, 1·5, 2·5, 3·5, and 4·5, plot the points, and use symmetry to plot the points when x is -4·5, -3·5, ..., -0·5. Complete the graph by drawing a smooth curve through the points you have plotted.

A Table of Squares

One way of producing a table of squares is to use the multiplication process throughout. (You did this in obtaining the ordered pairs used for graphing $y = x^2$.) Another way of producing a table of squares is to use the graph of $y = x^2$ itself. Look at your graph of $y = x^2$ for values of x from -5 to 5 and see if you can find the value of $(3{\cdot}2)^2$. The exact square of 3·2 is 10·24. Were you able to find this answer from the graph? Again, see if you can find the value of $(1{\cdot}2)^2$ and $(3{\cdot}7)^2$ from the graph. Check by arithmetic to see how accurate your graph readings are. How could greater accuracy be obtained?

Class Activity

Draw a large graph of $y = x^2$ for values of x from 0 to 1. (Calculate at intervals of 0·1.) Use the graph to prepare a table of values of x^2 when x is 0·00, 0·05, 0·10, 0·15, 0·20, ..., 0·95, 1·00, estimating the values of x^2 to 3 decimal places.

A Printed Table of Squares

In practice we do not have to refer to a table of squares which we have made ourselves. Tables can be obtained giving an accuracy far greater than any we are likely to achieve by the graphical method. Compare the tables of squares you have prepared with the one in the back of this book.

At first glance it would appear that the table of squares in the book gives only the squares of numbers between 1 and 99. Decimal points are absent. The reason for this is simply explained.

When numbers such as 3·2, 32, 0·32, and 320 are squared, the numerals for the answers are all the same if one ignores the position of the decimal point.

$$\begin{aligned} 3{\cdot}2^2 &= 10{\cdot}24 \\ 32^2 &= 1024{\cdot} \\ 0{\cdot}32^2 &= 0{\cdot}1024 \\ 320^2 &= 102{,}400{\cdot} \end{aligned}$$

The printed table makes use of this property and leaves the placement of the decimal point to the reader.

Example 1. Find the square of 780.

We first note that

$$780 = 78 \times 10$$

Then

$$\begin{aligned} (780)^2 &= (78 \times 10)^2 \\ &= (78)^2 \times (10)^2 \end{aligned}$$

The table lists the square of 78 as 6084.

Thus

$$\begin{aligned} (780)^2 &= 6084 \times 10^2 \\ &= 608{,}400 \end{aligned}$$

Example 2. Find the square of 0·98.

We note that

$$\begin{aligned} 0\cdot98 &= 98 \times 0\cdot01 \\ (0\cdot98)^2 &= (98 \times 0\cdot01)^2 \\ &= (98)^2 \times (0\cdot01)^2 \end{aligned}$$

The table lists the square of 98 as 9604.

Thus

$$\begin{aligned} (0\cdot98)^2 &= 9604 \times 0\cdot0001 \\ &= 0\cdot9604 \end{aligned}$$

Problems 15-2A

Using the table of squares in the back of the book, find the squares of the following:

(a) 2·4

(b) 240

(c) 7100

(d) 0·071

(e) 9·3

(f) 0·00093

A Table of Square Roots

You will recall that if we have a number x such that $x^2 = y$, then x is said to be a square root of y. Since $7 \times 7 = 49$, 7 is a square root of 49. Since $(-7) \times (-7) = 49$, (-7) also is a square root of 49. We indicate the positive square root of y as $\sqrt{y}$ and the negative square root of y as $-\sqrt{y}$. Thus $7 = \sqrt{49}$ and $-7 = -\sqrt{49}$.

If $y = x^2$, is it true that $\sqrt{y} = x$? What if x is negative? In that case is $\sqrt{y} = -x$? Since $\sqrt{y} = x$ if $x \geq 0$ and $\sqrt{y} = -x$ if $x < 0$, is it correct to say that $y = x^2$ and $\sqrt{y} = |x|$ are equivalent equations?

Can we use the graph of $y = x^2$ to find square roots, that is, for every value of y on the graph, will the corresponding positive value of x be $\sqrt{y}$? What will the corresponding negative value of x be ?

Turn to the graph of $y = x^2$ that you used for finding the squares of numbers from -5 to 5. You know that a positive number has two square roots. Look at the graph and see how it shows that the square roots of 9 are 3 and -3, that is, when y is 9, x is 3 or -3. In the same way, find the square roots of 4, 16, and 2 from your graph.

In an earlier chapter we found the value of $\sqrt{2}$ to be approximately 1·414, expressed to 3 decimal places. How accurately were you able to estimate $\sqrt{2}$ from your graph? How could greater accuracy be obtained?

Problems 15-2B

1. Use your graph of $y = x^2$ from -5 to 5 to find values for the following table of the positive square roots (to 1 decimal place) of the integers from 1 to 25:

y	1	2	3	...	25
$\sqrt{y} = \lvert x \rvert$					

2. Use your graph of $y = x^2$ from 0 to 1 to find values for the following table of positive square roots to two decimal places:

y	0·1	0·2	0·3	...	0·9
$\sqrt{y} = \lvert x \rvert$					

A Printed Table of Square Roots

As with the squares of numbers, tables have been produced for square roots of numbers giving an accuracy far greater than that we can obtain by the graphical method. Look at the table of square roots given in the back of this book and compare it with the tables you made from your graphs.

You will notice that for each of the numbers from 1 to 99, there are two square roots listed; again, no decimal points are indicated. Let us see how to use the table.

We know that if we squared each of the following numbers, the decimal numerals for the answers would all be the same if we ignore the position of the decimal point:

4000, 400, 40, 4, 0·4, 0:04

Does the same thing occur when we take the <u>square root</u> of each? Try it and see.

You know that the square root of 36 is 6. How many of the following values of $\sqrt{n}$ can you find without doing additional arithmetic?

n	3600	360	36	3·6	0·36	0·036	0·0036
$\sqrt{n}$			6				

Since $\sqrt{3600} = \sqrt{36 \times 100} = \sqrt{36} \times \sqrt{100} = 6 \times 10$, you could see that $\sqrt{3600} = 60$.

To write $\sqrt{360}$ in terms of $\sqrt{36}$, we have to write $\sqrt{360} = \sqrt{36 \times 10} = \sqrt{36} \times \sqrt{10} = 6 \times \sqrt{10}$. This is not simple. Perhaps you noticed, however, that in the table there is a column marked $\sqrt{10n}$. When $n = 36$, $\sqrt{10n} = \sqrt{36 \times 10} = \sqrt{360}$. The table gives the digits 1897 for $\sqrt{10n}$ when n is 36. A decimal point is needed. Where should it be?

Since 360 is between 100 and 10,000, $\sqrt{360}$ must be between 10 and 100. This indicates that the approximation to $\sqrt{360}$ given by the table is 18·97.

A useful check is to observe from the table that $19^2 = 361$. This means that $\sqrt{361} = 19$, so $\sqrt{360}$ should be a bit less than 19, as we found.

Can you now find values for $\sqrt{n}$ in the rest of the table above without doing additional arithmetic? Use the facts that

$$3{\cdot}6 = 36 \times 10 \times \frac{1}{100}$$
$$0{\cdot}36 = 36 \times \frac{1}{100}$$
$$0{\cdot}036 = 36 \times 10 \times \frac{1}{10{,}000}$$
$$0{\cdot}0036 = 36 \times \frac{1}{10{,}000}$$

If you wanted to find $\sqrt{39}$ to four figures, would you look in the $\sqrt{n}$ column, or the $\sqrt{10n}$ column? In which column would you look to find $\sqrt{390}$? $\sqrt{3900}$? $\sqrt{3{\cdot}9}$?

Since 39 is listed in the n column, $\sqrt{39}$ is listed in the $\sqrt{n}$ column. We find $\sqrt{39} \approx 6{\cdot}245$. (The symbol $\approx$ means "is approximately".) Since $3900 = 39 \times 100$, we can write

$$\sqrt{3900} = \sqrt{39} \times \sqrt{100}$$
$$\sqrt{3900} \approx 6{\cdot}245 \times 10$$
$$\sqrt{3900} \approx 62{\cdot}45$$

The square root of 390 is listed in the $\sqrt{10n}$ column since $390 = 39 \times 10$. We find $\sqrt{390} \approx 19{\cdot}75$. Since $3{\cdot}9 = 39 \times 10 \times \frac{1}{100}$,

$$\sqrt{3{\cdot}9} = \sqrt{39 \times 10} \times \sqrt{\frac{1}{100}}$$
$$\sqrt{3{\cdot}9} \approx 19{\cdot}75 \times \frac{1}{10}$$
$$\sqrt{3{\cdot}9} \approx 1{\cdot}975$$

Problems 15-2C

1. Using the square root tables in the book, find the approximate square root to 4 figures of each of the following:

(a)	15	(d)	760
(b)	150	(e)	76000
(c)	1·5	(f)	0·00076

2. CHALLENGE PROBLEM

You have seen how to produce tables of squares and square roots either arithmetically or by using the graph of $y = x^2$. How could you produce a table of cubes and cube roots?

3. CHALLENGE PROBLEM

Produce tables of cubes and cube roots for numbers from -2 to 2 at intervals of 0·5. How many real cube roots has a number?

4. CHALLENGE PROBLEM

If you were to produce a printed table of cube roots for numbers from 10 to 99 which could also be used to find the cube roots of other numbers, how many columns would you need for each of the numbers from 10 to 99? You may wish to consider the following before you decide.

Since $10 \times 10 \times 10 = 1000$, the cube root of 1000 is 10. Which of the following cube roots would have the same digits with the decimal point differently placed?

$$\sqrt[3]{32},\ \sqrt[3]{320},\ \sqrt[3]{3200},\ \sqrt[3]{3\cdot 2},\ \sqrt[3]{0\cdot 32},\ \sqrt[3]{0\cdot 032},\ \sqrt[3]{320{,}000}$$

15-3 Other Graphs

Class Activity

Let us see what we can find out about the graph of $x^2 + y^2 = 25$ before we plot any points. Answer these questions about the coordinates of the ordered pairs in the truth set by examining the equation.

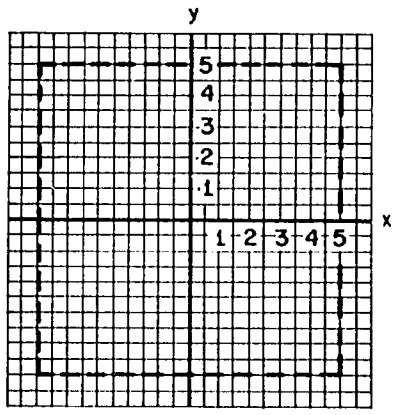
1. As x^2 becomes larger, does y^2 become larger or smaller?

2. How small can x^2 be? How small can y^2 be?

3. How large can x^2 be? Can it be 100? (If it were, then y^2 would be -75. Is this possible?)

4. If you answered the preceding questions correctly, you now know that for $x^2 + y^2 = 25$, $0 \leq x^2 \leq 25$. What, then, are the possible values of x in the truth set? Can x be negative? How small can x be? How large can x be? Can you answer the same questions about y?

5. Your answers to 4 should indicate to you that no points of $x^2 + y^2 = 25$ can be outside the 10 by 10 square with the origin as its center.



Draw axes on your graph paper with an appropriate scale. Use the same scale for both axes.

6. Find values for y and plot the points.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y											

You may wish to use your tables of square roots to estimate some of the y-values to one decimal place. For example, if x is -2, then:

$$(-2)^2 + y^2 = 25$$
$$4 + y^2 = 25$$
$$y^2 = 21$$
$$y = \sqrt{21} \text{ or } y = -\sqrt{21}$$

We can then put in the table 4·6 and -4·6 for values of y corresponding to -2 for x.

7. Can you find other ordered pairs for the graph of $x^2 + y^2 = 25$ without doing any more arithmetic? If x is 4·6 correct to one decimal place, do you know what values y can have? Mark any other points you can without doing any more arithmetic. What do you think the graph of $x^2 + y^2 = 25$ is?

8. $x^2 + y^2 = 25$ defines a locus algebraically. You have met this locus before in geometry. How was it defined there?

9. Describe the graphs of these inequalities in terms of the graph of $x^2 + y^2 = 25$.

(a) $x^2 + y^2 < 25$

(b) $x^2 + y^2 > 25$

(c) $x^2 + y^2 \leq 25$

(d) $x^2 + y^2 \geq 25$

Problems 15-3

For each of the following equations or inequalities sketch the graph. Try to find out as much as you can about the graph before you plot any points. You may wish to make use of your tables of squares and square roots.

1. (a) $x^2 + y^2 = 100$

(b) $x^2 + y^2 \geq 100$

(c) $x^2 + y^2 < 100$

2. $x^2 + 4y^2 = 100$

3. $4x^2 + y^2 = 100$

4. $x^2 - y^2 = 25$

5. $y^2 - x^2 = 25$

6. You met figures like the graphs in 1 (a), 2, 3, 4, and 5 in your work on loci. Do you remember the special name of each and how each was obtained?

7. Find lines of symmetry, if possible, for each of your graphs and write the equations of the lines of symmetry.

15-4 More Ways of Picturing Numerical Information

There are many ways of picturing numerical information. We are now going to examine two ways.

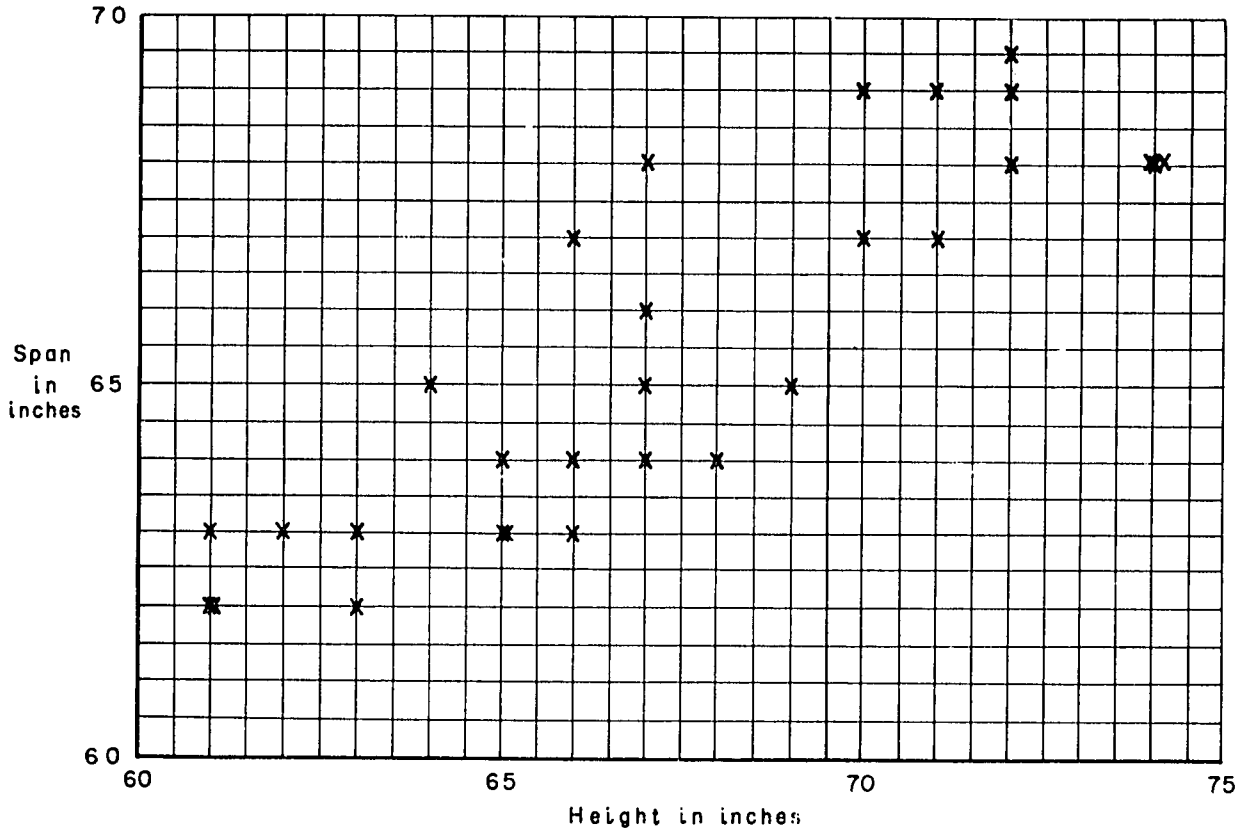
1. Scatter Diagram

Below is a chart listing the heights and spans of 30 boys. How can we picture this information?

Height	Span	Height	Span	Height	Span
66	63	67	66	66	64
65	63	69	65	65	63
62	63	71	69	61	62
63	63	70	69	61	63
72	68	72	69	61	60
66	67	61	62	65	64
71	67	61	62	74	68
70	67	63	62	74	68
67	64	72	69	67	65
67	68	68	64	64	65

All measurements are in inches.

One of the questions we may ask about this information is whether there is any connection between the heights and spans of the boys. The following picture of this information may help us to answer this.



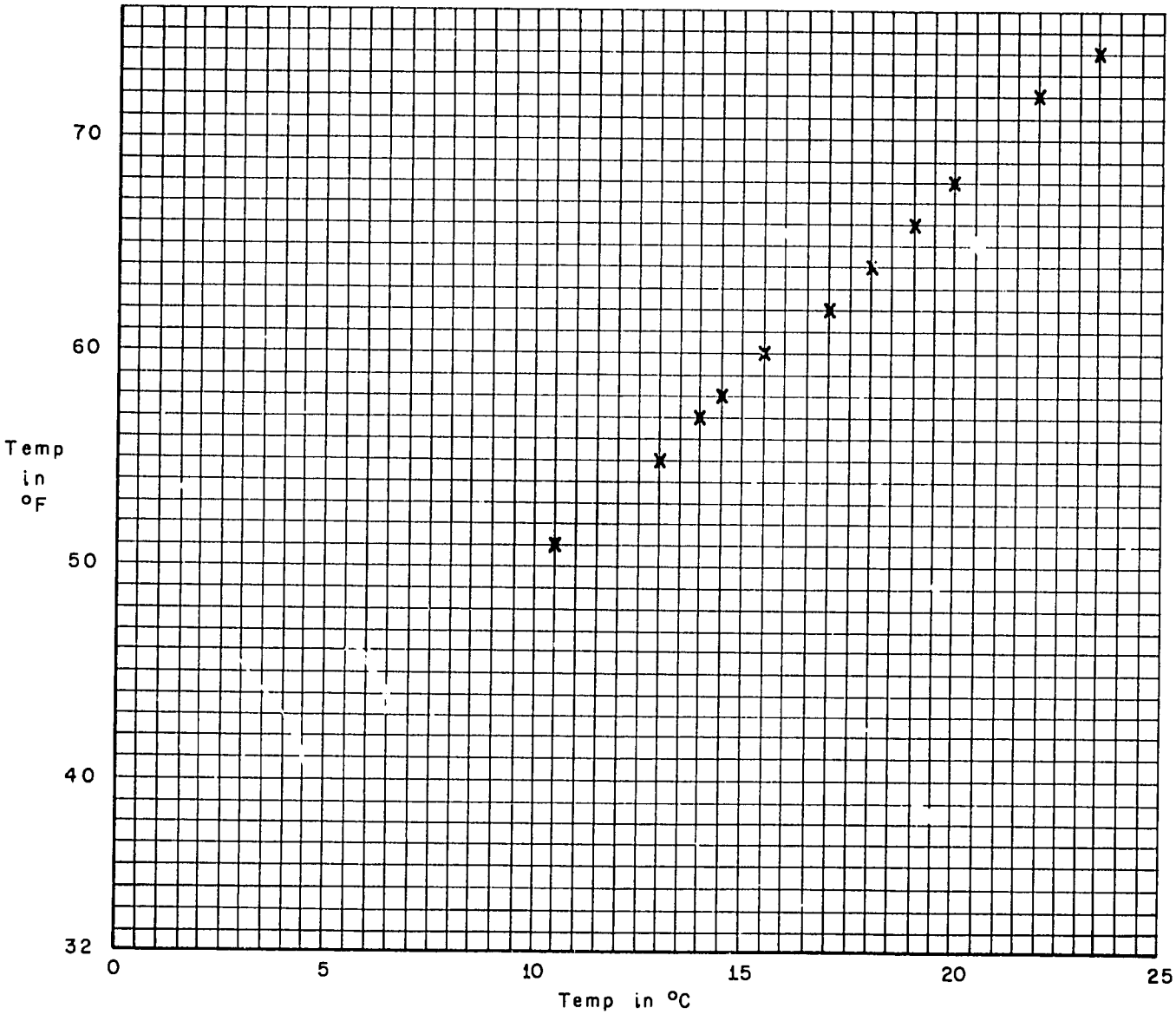
This is known as a <u>scatter diagram</u>. Why do you think it is called a scatter diagram?

How has it been constructed? What do you think the diagram tells us? What can you say about the span of a boy 66 in. tall?

Now look at the information shown below. These are temperatures taken during a day, by two boys. One used a centigrade thermometer, and the other a Fahrenheit thermometer. They obtained these temperatures.

y (temp. in $^{\circ}$F.)	51	55	57	58	60	62	64	66	68	72	74
x (temp. in $^{\circ}$C.)	10.5	13	14	14.5	15.5	17	18	19	20	22	23.5

You can see below the scatter diagram for these two sets of measurements.



How is this scatter diagram different from the previous one?

The points of this second diagram approximate a straight line so closely that we should expect to find the equation of a straight line relating x and y.

Do you know from your work in science what this equation should be? Draw this line.

Why do not all the points lie on the line?

Summary

When we have two sets S_1, S_2 of numbers which can be matched in ordered pairs, we can investigate the relation between S_1 and S_2 using a scatter diagram. In some cases, such as our second example, the scatter diagram is very nearly a set of points on a straight line. In these examples, we can usually find an equation connecting the members of the sets. This is the case in many science experiments. It is not often, however, that we get such a close approximation to a straight line as in our second example. Experimental error can produce much scattering.

Class Activity

1. Measure the diameters and circumferences of a number of circular objects. Plot their scatter diagram.

2. Measure the heights and spans of the pupils in your class, and plot their scatter diagram.

3. Plot the scatter diagram for the results of any of your science experiments which are suitable.

4. Plot a scatter diagram of the marks your class obtained in two subjects, e.g., mathematics and English.

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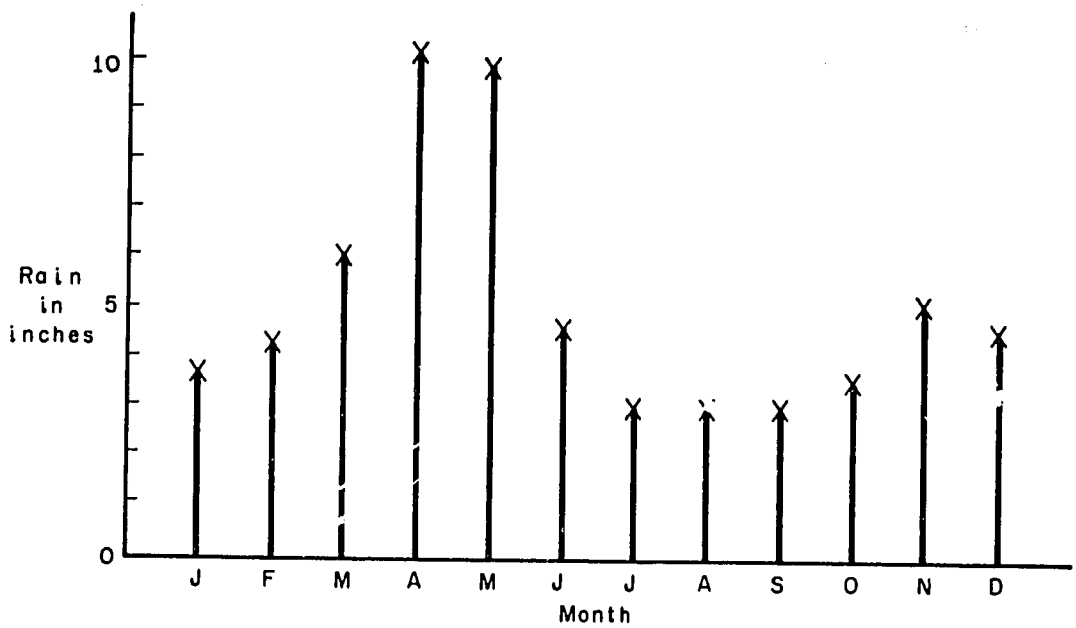
2. Continuous Line Chart

Now look at the average monthly rainfall figures for Entebbe which are given again below:

Month	Jan	Feb	Mar	Apr	May	June	July	Aug	Sept	Oct	Nov	Dec
Rain in In.	3·6	4·2	6·0	10·2	9·7	4·6	3·0	3·0	3·0	3·5	5·1	4·5

Here is a bar graph for the information:

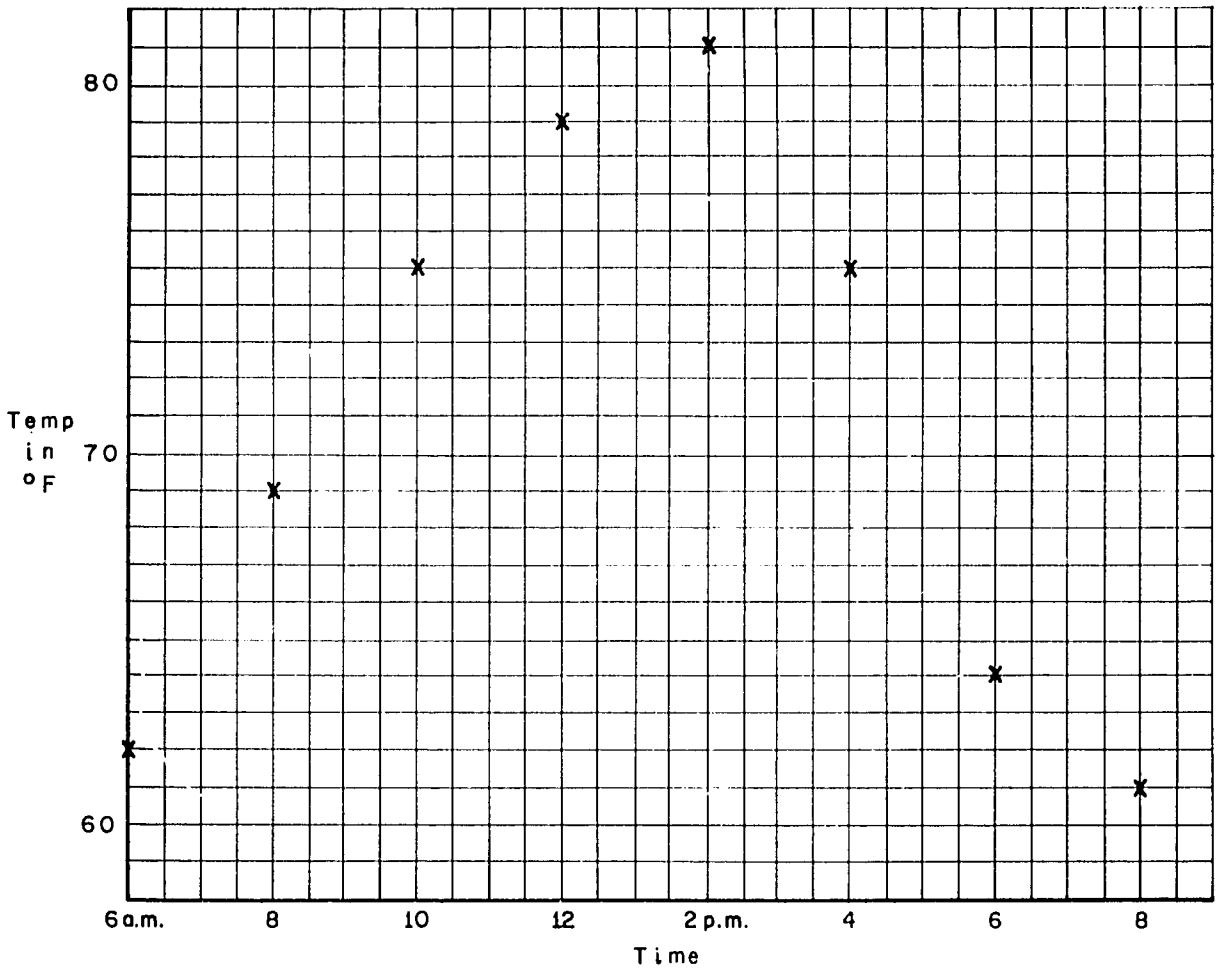


How many points are on this graph? Why can we plot no more points than this? With this sort of graph, there is no room to give additional information which might be available to us. Every drop of rain fell during one of the twelve months, and the monthly total only is recorded. We have no indication on the graph, for example, how much rain fell on 10 April, or even during the first 10 days of April.

In contrast, now look at the information below. These are temperatures as they were measured every 2 hours during a day.

Time	6 a.m.	8 a.m.	10 a.m.	12 noon	2 p.m.	4 p.m.	6 p.m.	8 p.m.
Temp. in $^{\circ}$F	62	69	75	79	81	75	64	61

We now plot these measurements:



In this example we could easily put more points on the graph by taking the temperature every hour, or even more frequently.

The main difference between this graph and the rainfall graph is that here we are able to plot the temperature at each moment of the day, provided this information is available to us. As our experience indicates that temperature does not change radically in a very short time, the graph allows us to make reasonable estimates of the temperature at different times. For example, what do you think the temperature was at 11 a.m.? The points we have plotted may be joined by a curve and, if we draw a reasonable curve, the curve should very nearly approximate the temperature at each moment throughout the day. In fact, if we could have recorded the temperature at each instant, we would have obtained a smooth curve. We call such a graph a <u>continuous curve graph</u>. Such a graph gives us a complete

picture of the behaviour of the temperature during the day. By contrast, the bar graph gives a rather incomplete picture of the rainfall during the year.

Give some other examples of continuous curve graphs.

Table of Squares and Square Roots

n	n^2	$\sqrt{n}$	$\sqrt{10n}$
1	1	1000	3162
2	4	1414	4472
3	9	1732	5477
4	16	2000	6325
5	25	2236	7071
6	36	2449	7746
7	49	2646	8367
8	64	2828	8944
9	81	3000	9487
10	100	3162	1000
11	121	3317	1049
12	144	3464	1095
13	169	3606	1140
14	196	3742	1183
15	225	3873	1225
16	256	4000	1265
17	289	4123	1304
18	324	4243	1342
19	361	4359	1378
20	400	4472	1414
21	441	4583	1449
22	484	4690	1483
23	529	4796	1517
24	576	4899	1549
25	625	5000	1581
26	676	5099	1612
27	729	5196	1643
28	784	5292	1673
29	841	5385	1703
30	900	5477	1732

n	n^2	$\sqrt{n}$	$\sqrt{10n}$
31	961	5568	1761
32	1024	5657	1789
33	1089	5745	1817
34	1156	5831	1844
35	1225	5916	1871
36	1296	6000	1897
37	1369	6083	1924
38	1444	6164	1949
39	1521	6245	1975
40	1600	6325	2000
41	1681	6403	2025
42	1764	6481	2049
43	1849	6557	2074
44	1936	6633	2098
45	2025	6708	2121
46	2116	6782	2145
47	2209	6856	2168
48	2304	6928	2191
49	2401	7000	2114
50	2500	7071	2236
51	2601	7141	2258
52	2704	7211	2280
53	2809	7280	2302
54	2916	7348	2324
55	3025	7416	2345
56	3136	7483	2366
57	3249	7550	2388
58	3364	7616	2408
59	3481	7681	2429
60	3600	7746	2449

n	n^2	$\sqrt{n}$	$\sqrt{10n}$
61	3721	7810	2470
62	3844	7874	2490
63	3969	7937	2510
64	4096	8000	2530
65	4225	8062	2550
66	4356	8124	2569
67	4489	8185	2588
68	4624	8246	2608
69	4761	8307	2627
70	4900	8367	2646
71	5041	8426	2665
72	5184	8485	2683
73	5329	8544	2702
74	5476	8602	2720
75	5625	8660	2739
76	5776	8718	2757
77	5929	8775	2775
78	6084	8832	2793
79	6241	8888	2811
80	6400	8944	2828
81	6561	9000	2846
82	6724	9055	2864
83	6889	9110	2881
84	7056	9165	2898
85	7225	9220	2915
86	7396	9274	2933
87	7569	9327	2950
88	7744	9381	2966
89	7921	9434	2983
90	8100	9487	3000

n	n^2	$\sqrt{n}$	$\sqrt{10n}$
91	8281	9539	3017
92	8464	9592	3033
93	8649	9644	3050
94	8836	9695	3066
95	9025	9747	3082
96	9216	9798	3098
97	9409	9849	3114
98	9604	9899	3130
99	9801	9950	3146

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