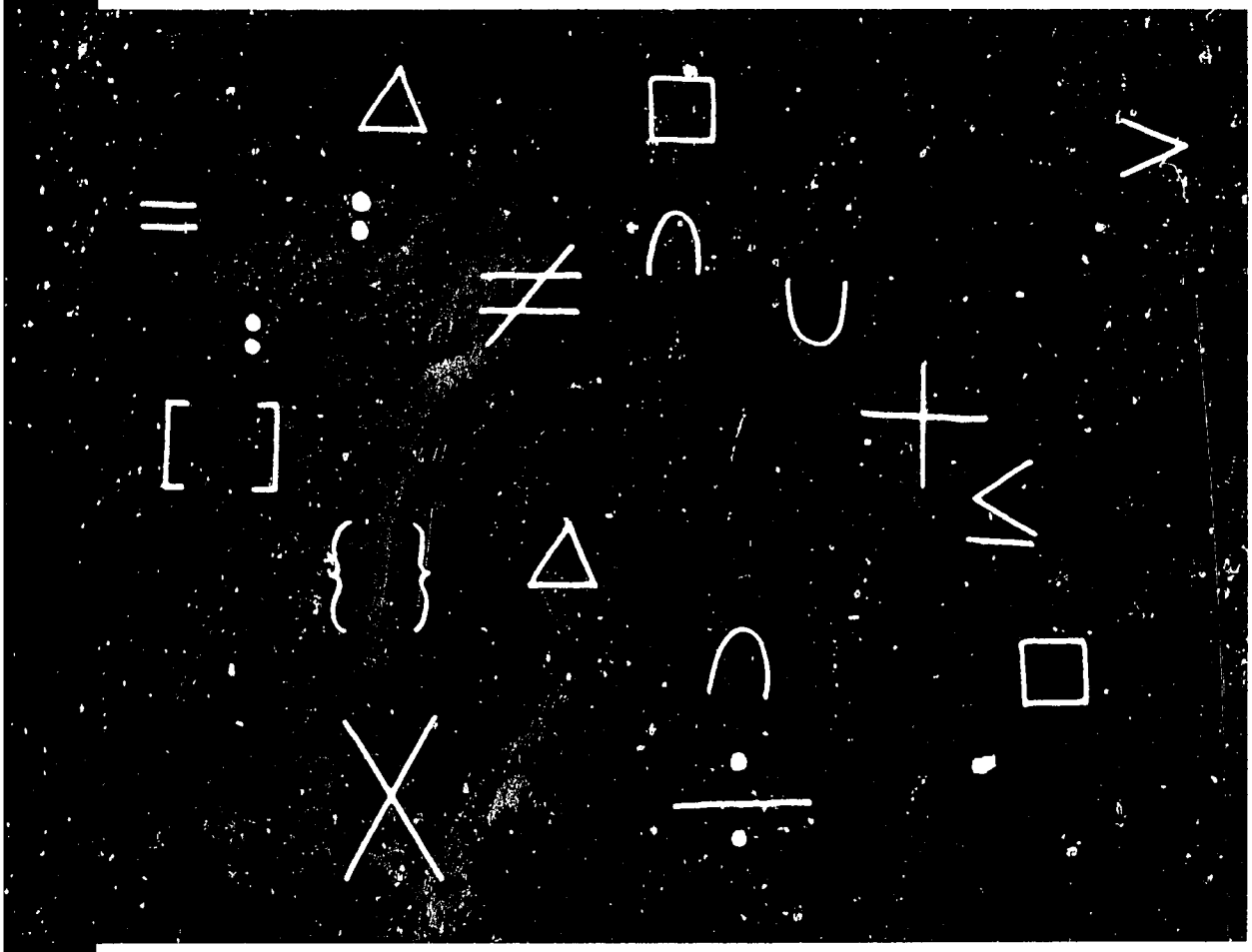
PN-AAT-088
40642

Algebra

AFK
375.12
E61a
G.2

SECONDARY
C ONE
teachers' guide



Entebbe Mathematics Series

preliminary edition

A.I.D.
Reference Center
Room 1656 NS

S R A

Educational Services Incorporated
Watertown, Massachusetts, U.S.A.

ENTEBBE MATHEMATICS SERIES

• Primary Materials •

Primary One
Revised Preliminary Edition
Pupil Book: One volume
Teachers' Guide: Two volumes

Primary Two
Preliminary Edition
Pupil Book: Two volumes
Teachers' Guide: Two volumes

Primary Three
Preliminary Edition
Pupil Book: Two volumes
Teachers' Guide: Two volumes

Primary Four
Preliminary Edition
Pupil Book: One volume
Teachers' Guide: One volume

Primary Five
Preliminary Edition
To be written at 1966 Workshop

Entebbe Mathematics Teachers' Handbook, Primary I-III, Preliminary Edition

• Secondary Materials •

FIVE YEAR COURSE

Secondary One
Preliminary Edition
Student Text: One volume
Teachers' Guide: Three volumes

Secondary Two
Preliminary Edition
Student Text: Three volumes
Teachers' Guide: Three volumes

Secondary Three
Preliminary Edition
Student Text: Algebra—One volume
Geometry—One volume
Teachers' Guide: Algebra—One volume
Geometry—One volume

Secondary Four
Preliminary Edition
Student Text: Algebra—One volume
Geometry—One volume
Teachers' Guide: Algebra—One volume
Geometry—One volume

Secondary Five
Preliminary Edition
To be written at 1966 Workshop

FOUR YEAR COURSE

Secondary C One
Preliminary Edition
Student Text: Algebra—One volume
Geometry—One volume
Teachers' Guide: Algebra—One volume
Geometry—One volume

Secondary C Two
Preliminary Edition
To be written at 1966 Workshop

• Teacher Training Materials •

Basic Concepts of Mathematics
An Introductory Text for Teachers
Preliminary Edition

Volume I —*Structure of Arithmetic*—Chapters 1-21
Volume II —*Structure of Arithmetic*—Chapters 22-46
Volume III—*Foundations of Geometry*—Chapters 47-57
Volume IV—*Measurement, Functions and Probability*—Chapters 58-68

Algebra

PN-AAT-088

teachers' guide

SECONDARY C ONE

THE FIRST YEAR OF A FOUR YEAR SECONDARY COURSE

Prepared at the

1965 Mombasa Mathematics Workshop

preliminary edition

Entebbe Mathematics Series

Published for the
AFRICAN EDUCATION PROGRAM of EDUCATIONAL SERVICES INCORPORATED
Watertown, Massachusetts
by
SCIENCE RESEARCH ASSOCIATES, INC.
(A subsidiary of International Business Machines Corporation)
Chicago, Illinois

ACKNOWLEDGEMENTS

This portion of a Student Text for Secondary C One Mathematics and its associated Teachers' Guide was prepared during the summer of 1965 at Mombasa, Kenya, as part of a program of curriculum study being conducted by the African Education Program of Educational Services Incorporated.

Funds for the Programme have been provided by the United States Government through the International Cooperation Administration, the Agency for International Development, and the Department of State; by the Ford Foundation; and by the Louis and Pauline Cowan Foundation, Inc.

The Mombasa Mathematics Workshop was directed by Professor W. T. Martin of Massachusetts Institute of Technology, Cambridge, Massachusetts and Mr. John Oyelese of the University of Ibadan, Ibadan, Western Nigeria.

Secondary C One manuscripts were written at the 1965 Mombasa Workshop by the following:

Chairman: Dr. Ross L. Finney

Princeton University, Princeton, New Jersey

Professor Jean M. Calloway

Kalamazoo College, Kalamazoo, Michigan

Professor Roy Dubisch

University of Washington, Seattle, Washington

Mr. Isaac Hunja

Kagumo School, Kiganjo, Kenya

Mr. Edward Kizza

University College, Nairobi, Kenya

Mr. Robert L. N. Offurum

King's College, Lagos, Nigeria

Mr. Joseph P. Pavlovich

Comprehensive High School, Abeokuta, Western Nigeria

Mr. Richard S. Presser

Elston Senior High School, Michigan City, Indiana

Dean Jerome M. Sachs

Chicago Teachers College North, Chicago, Illinois

Mr. Henry Swain

New Trier Township High School, Winnetka, Illinois

Mr. Robert P. Taylor

Tororo Girls' School, Tororo, Uganda

Mr. J. E. Modupe Taylor-Pierce

Secondary Technical School, Freetown, Sierra Leone

The editorial committee at Mombasa consisted of Mr. Henry Swain and Professor Jean M. Calloway; and in the United States, of Professor Donald L. Kreider, Dartmouth College, Hanover, New Hampshire, Mr. Richard S. Pieters, Phillips Academy, Andover, Massachusetts, and Professor Donald E. Richmond, Williams College, Williamstown, Massachusetts.

PREFACE

This Teachers' Guide accompanies the text which was prepared at a study and writing workshop held during the summer of 1965 at Mombasa, Kenya, in which mathematicians and teachers from most English-speaking countries of Tropical Africa, the United States, and the United Kingdom participated.

The material is intended to be used for the Algebra part of a first-year text for a four-year programme in Secondary Mathematics. Much of the material was adapted from texts prepared at Entebbe, Uganda, during the summers of 1962 and 1963. However, alterations have been made as a result of experience gained in the use of the earlier texts.

I. Preliminary Nature of This Material

The material must be considered to be preliminary until it has been tried out in classrooms and modified in accordance with the experience gained from this experimentation. The Student Text and the Teachers' Guide for MATHEMATICS SECONDARY C ONE will be subject to correction and improvement in the light of suggestions from the teachers who use them.

This means that the teacher who uses this material has the responsibility, in addition to helping interpret the new material to the students, of helping identify the areas where improvement is needed. In partial reward for this burden, the teacher will have the satisfaction of knowing that he or she is taking part in an experiment which is of great potential value.

II. Emphasis on Mathematical Ideas

In recent years there has been accumulated much evidence that young students are far more interested in <u>mathematical</u> <u>ideas</u> than they have usually been given credit for, and that they are far more competent to deal with such ideas than the current curricula would suggest. A presentation of mathematics which puts its emphasis on concepts rather than the rules of manipulation is likely to lead to far greater satisfaction on the part of the student, and will also lead to greater mathematical competence.

III. Content of the Text

The subjects of algebra and geometry are treated in two volumes; one is devoted to algebra and the other to geometry. The geometry part of the text will treat:

1. First Notions
2. Plane and Space Figures
3. Measurement
4. Symmetry

Algebra is developed as a unified structure. Techniques and concepts are all related to a few basic ideas with emphasis on <u>why</u> the numbers behave as they do. We try to be mathematically sound but at the same time the development is gradual with an effort to involve the students actively. Instead of just telling the ideas to the students outright, the teacher should use problems and careful class discussion to help the students recognize the ideas for themselves.

IV. The General Approach

The best education is the education that the student creates from his own direct efforts. The teacher should resist the temptation to tell the class exactly what to do and how to do it. It is indeed a great temptation, for by such means a class will appear to be proceeding at a rapid pace. If the teacher takes the time and effort to lead the student to think through the idea himself without telling him outright, there is considerably more assurance that the idea will be mastered and retained by the student and will become truly his own.

Here again the demands on the teacher are greater. The teacher, like the student, must be thinking at every moment, for it takes far more insight to lead than to tell. This Guide is intended primarily to assist the teacher in the actual conduct of the class. Methods are suggested which will encourage student

imagination and generate student interest. In addition the Guide contains mathematical background and explanatory material beyond that given in the Text. Answers to the problems are provided.

It should be emphasized that this modified curriculum representa a great opportunity for teacher and student alike. In making use of it, the school participates in a great experiment to help develop a strong African educational system of which we can be proud. It is a massive joint undertaking, in which students and teachers work side by side with mathematicians of international eminence, from their own country and from a dozen others, to create within Africa something that will be of major significance for Africa itself, and in a large measure for all the world.

SECONDARY _____ REPORT, CHAPTER _____

Would you please fill out one of these forms for each chapter of Secondary material as you complete the chapter. Send it by air mail to Educational Services Incorporated, African Education Program, P. O. Box 417, Watertown, Massachusetts 02172, U. S. A. Your comments will help us to improve and to rewrite the material for further use.

Please respond to *all* parts of this report.

Name of Teacher _____

Name of School _____

School Address _____

Date _____

Chapter Number _____ in Secondary Book _____ Volume _____

Chapter Title _____

Number of classes in which you teach the Entebbe Mathematics Workshop Secondary Materials _____

_____

Number of pupils using material _____ . Number of periods each class meets per week _____ .

Length of class period _____ minutes. Class periods used teaching the chapter _____.

The pupils of my class are of the following ability level (tick one of the following):

fast _____ , slow _____ , medium _____ , mixed ability _____ .

My pupils have had, on the average, _____ years of schooling before this present year.

To the best of my judgment this chapter is (tick one of the following): excellent _____ ,

good _____ , fair _____ , poor _____ , or extremely poor _____.

Why did you rate the chapter in this way? (If you need more space than is provided here, please use an additional sheet.) _____

_____

_____

If you used any supplementary materials in teaching this unit, please describe them. _____

_____

_____

Please indicate the specific sections which need improvement and how *you* would recommend improving them.

Questions	Comments
1. Should this whole chapter be omitted? Why?	
2. Name specifically any ideas which you think ought to be in this chapter but which are not there. Why should they be included?	
3. Name specifically any topics in this chapter which you think should be omitted. Why should they be omitted?	
4. Give the section and page number of any part of this chapter where you think the explanation is not clear enough. What suggestions do you have on how to make it more clear?	
5. Indicate which sets of problems need more problems to give adequate practice. What kind of problems are needed?	
6. Indicate by page and number any problems which you think are too difficult and problems which you think are too easy.	

STUDENT TEXT

(continued)

Questions	Comments
7. Indicate any problems or illustrations which are ineffective because they use words or ideas which are unfamiliar to your students. For instance, a child who had never been to sea in a schooner would not much appreciate a problem about the area of the jib and the foresail and the mainsail. For the examples which are inappropriate for your children, can you suggest local situations which would be more appropriate in the example.	

TEACHERS' GUIDE

Questions	Comments
1. In what parts of the chapter were the suggestions on how to teach the material not sufficient to help you teach it effectively? What suggestions have you for improving those parts?	
2. In what parts of the chapter were the suggestions on teaching the material trivial and unnecessary?	
3. In some places in the Teachers' Guide there are discussions for the teacher of the mathematical background, going more deeply into the	

Questions	Comments
3. *(continued)* ideas than one would with the students. Indicate by section and page number: (a) which of these discussions were not clear enough to be helpful, (b) which were too brief and need to be expanded, (c) which were unnecessary for you and might be omitted, (d) where no such discussion was given where it was needed.	
4. Indicate any answers to problems which were not sufficiently clear or complete.	

Would you please write any other comments which you believe might be helpful in rewriting this chapter.

EDUCATIONAL SERVICES INCORPORATED
African Education Program
P. O. Box 417
Watertown, Massachusetts 02172
U. S. A.

CONTENTS

Chapter 1 The Language of Sets

1-1 Sets 1

1-2 Ways of Describing Sets 1

1-3 The Empty Set 3

1-4 Subsets 4

1-5 Matching of Sets 5

1-6 Operations with Sets 7

Chapter 2 Number

2-1 Comparison of Sets by Matching 11

2-2 Sets in Their Natural Order 11

2-3 Number as a Property of Sets 12

2-5 Order in the Set of Whole Numbers 13

2-7 Finding the Number of Members in a Set 13

2-8 Equivalence Using Counting Sets 13

2-9 Ways of Representing Numbers 14

Chapter 3 Numeration Systems

3-1 Base Ten Numeration 15

3-2 Base Five Numeration 16

3-3 Addition in Base Five 16

3-4 Subtraction in Base Five 18

3-5 Base Two Numeration 18

3-6 Numeration in Other Bases 19

3-7 The Decimal System of Numeration 19

3-8 Expanded Numerals in Any Base 21

3-9 Changing Numerals Written in Any Base to Base Ten 21

3-10 Rewriting a Numeral from Base Ten to Any Base 22

3-11 Revision Problems 23

Chapter 4 Properties of Natural Numbers

4-1 Natural Numbers and Whole Numbers 27

4-2 Factors of Numbers 28

4-3 Prime Numbers 29

4-4 Prime Factorization of Natural Numbers 31

4-5 The Fundamental Theorem of Arithmetic 32

4-6 Greatest Common Factor 33

4-7 Least Common Multiple 36

Chapter 5 Rational Numbers and the Number Line

5-1 The Number Line 41

5-2 Equivalent Rational Numbers 44

5-3 Negative Rational Numbers 45

5-4 Inequalities 47

Chapter 6 Clock Arithmetic

6-1 Introduction 51

6-2 Addition 52

6-3 Multiplication 56

6-4 Summary of Basic Properties 63

Chapter 7 The Arithmetic of Rational Numbers

7-1 Addition of Rational Numbers 66

7-2 Properties of Addition of Rational Numbers 69

7-3 Applications of the Properties of Addition 73

7-4 Multiplication of Rational Numbers 76

7-5 Other Fractions for Rational Numbers 77

7-6 Properties of Multiplication of Rational Numbers 78

7-7 The Distributive Property 81

7-8 Applications of the Basic Properties 82

7-9 Revision of Properties of the Rational Number System 84

7-10 Subtraction of Rational Numbers 85

7-11 Division of Rational Numbers 86

7-12 Revision Problems 88

Chapter 8 Open Sentences

8-1 Statements 89

8-2 Variable 90

8-3 Domain of the Variable 91

8-4 Open Sentences and Truth Sets 92

8-5 Graphs of Solution Sets 94

8-6 Compound Sentences 96

8-7 Equivalent Equations 97

8-8 Truth Sets Found by Using Equivalent Equations 99

8-9 Equivalent Inequalities 100

8-10 Solution Sets of Inequalities 102

Chapter 9 Word Problems

9-1 Problems and Open Sentences 105

9-2 Solving Word Problems 106

9-3 Formulas 106

9-4 Writing Open Sentences 109

9-5 Problems Involving Inequalities 111

9-6 Solving Word Problems 111

Chapter 10 Equations in Two Variables

10-1 An Equation in Two Variables 113
10-2 A Problem in Two Variables 119
10-3 Systems of Simultaneous Equations 120
10-4 More Systems of Simultaneous Equations 123
10-5 Word Problems Using Two Variables 127
10-6 One and Two Variable Solutions 131

Chapter 11 Decimal Numerals

11-1 Revision of Base Ten Numeration 135
11-2 Addition and Subtraction with Decimal Numerals 135
11-3 Multiplication with Decimal Numerals 137
11-4 Division with Decimal Numerals 139
11-5 Rational Numbers in Decimal Notation 140
11-6 Addition and Subtraction of Rational Numbers 141
11-7 Multiplication and Division by Ten 143
11-8 Decimal Places, Multiplication and Division 144
11-9 Rough Approximations in Multiplication and Division 146
11-10 Rounding Off 148
11-11 Percentages 150
11-12 Percentages and Decimal Numerals 152
11-13 Application of Percentages 153

Chapter 12 Real Numbers

12-1 Decimal Numerals for Rational Numbers 157
12-2 Plotting Repeating Decimals on the Number Line 162
12-3 Numbers Which Are Not Rational 162
12-4 The Square Root of 2 163
12-5 Proof that $\sqrt{2}$ is an Irrational Number 164

12-6 The Real Numbers 166
12-7 Negative Real Numbers 167
12-8 Addition of Real Numbers 167
12-9 Order of Real Numbers 170
12-10 Multiplying Real Numbers 171
12-11 Absolute Value 174
12-12 More About Absolute Value 176

Chapter 13 Introduction to Graphing in the Number Plane

13-1 The Number Plane 179
13-2 Graphs as Pictures of Truth Sets 185
13-3 Guessing Rules 191
13-4 Equivalent Equations in Two Variables 194
13-5 Graphs of Linear Equations in Two Variables 194
13-6 The Slope of a Straight Line 196
13-7 The Slope of a Line from Its Equation 199
13-8 The Intercept on the y-axis 202

Chapter 14 Using Straight Line Graphs

14-1 Systems of Simultaneous Equations 207
14-2 Proportionality 211
14-3 A Practical Application 213

Chapter 15 Other Graphs

15-1 Non-linear Graphs 221
15-2 The Graph of $y = x^2$ and Tables of Squares and Square Roots . . . 236
15-3 Other Graphs 240
15-4 More Ways of Picturing Numerical Information 243

CHAPTER 1

THE LANGUAGE OF SETS

1-1 Sets

It is not only in mathematics that one forms sets. The large number of collective nouns in the English language such as class, herd, flock, fleet, tribe, bunch, and family attest to the fact that this is a common activity. Even the word "set" itself is sometimes used, for example, when we speak of the younger generation as the "younger set". One has formed a set whenever he gathers together a number of different objects and then views them collectively as one object. This "gathering together" is usually a mental activity as is the case, for example, when we speak of the Kikuyu tribe. We do not literally gather together all individuals from this tribe into one great pile. Rather, we are just mentally associating all individual Kikuyus with one another, and we use the collective term "Kikuyu tribe" to show that we are viewing all of these individuals as members of one tribal or national group. The tribe then becomes the object of interest rather than its individual members.

1-2 Ways of Describing Sets

A set may be described by listing its members or by giving a word description of its members. Thus a certain set A of African countries is described either by the phrase "the set of East African countries" or by giving the list

$$A = \{\text{Kenya, Tanzania, Uganda}\}$$

Sometimes it is very difficult to list the members of a set. This is certainly true for the set M of all mosquitoes in Africa. And sometimes it is impossible to list the members, as is the case for example with the set C of all counting numbers.

Even in such cases, however, we sometimes indicate the set by giving a partial list of its members. We might write, for example,

$$D = \{1, 4, 9, 16, 25, 36, \ldots\}$$

to describe a certain set of counting numbers, and it is hoped that the pattern for writing down still more numbers of D is clear. A word description must be given if there is any doubt. For example, D can be described as the set of counting numbers that are squares of counting numbers.

Answers to Problems 1-2

Student Text Page 3

1. (a) {Africa, Asia, Europe, North America, South America}

   (c) {January, June, July}

   (d) {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

   (f) This set contains no members. It is called the empty set.

2. (a) The set of vowels of the English alphabet.

   (b) The set of mountains in East Africa which are more than 14,000 feet high with the exception of Mount Meru in Tanzania. It can also be described as the set of mountains in Kenya and Uganda which are more than 14,000 feet high together with Mount Kilimanjaro. Still another description is the set of mountains in East Africa whose name contains more than four letters and which are more than 14,000 feet high.

   (c) The set consisting of the number 1 and the even counting numbers.

   (d) Difficult to describe. The set consists of four kinds of members of the cat family. It is best described as "the set consisting of lion, cat, tiger, and cheetah".

   (e) The set consisting of the last three letters of the English alphabet.

3. (a) Not difficult.

   (b) Impossible.

   (c) This will not be difficult unless the difference between villages and cities is not clear.

   (d) Not difficult.

   (e) Difficult.

   (f) Not difficult.

4. (a) (iii)

   (b) (ii)

   (c) (iii)

5. (a) Yes

   (b) No

   (c) No

   (d) Yes

   (e) No

6. (a) Pacific, Atlantic, Indian, Arctic

   (b) Nkrumah

   (c) This set contains <u>no</u> members.

   (d) This set contains <u>no</u> members.

1-3 The Empty Set

It is convenient to allow a set to have no members. Otherwise when one is given a word description, he would have to know ahead of time whether anything satisfies the description in order just to say that it

describes a set. Certainly one would like to be able to speak, for example, about the set of snakes in Ireland without first determining whether or not there are any snakes there (there are not).

Answers to Problems 1-3 | Student Text Page 6

1. (b) and (f) are certainly empty. (c) is probably empty. And (d) or (e) may be empty, depending on the particular class.

2. The set of elephants that live in trees. The set of Africans who like to live at the South Pole. The set of pupils in your class who were born on the moon.

1-4 Subsets

It is important to notice that a set A is a subset of itself and that the empty set is a subset of every set. In symbols this statement becomes $A \subseteq A$ and $\emptyset \subseteq A$. Thus every set in List 2 in the students' text is a subset of the corresponding set in List 1, including (f) where the sets are equal (have exactly the same members). In every case except (f), the set in List 2 is a proper subset of the corresponding set in List 1.

Answers to Problems 1-4 | Student Text Page 8

1. (a) The blank may be filled with either the word "subset" or the words "proper subset".

   (b) E is a subset of D.

2. Y is a subset of V.

3. {a, b, c, d, e} is not a subset of {a, b, c} because the first contains members (for example, d) which are not members of the second.

   However {a, b, c, d, e} is a subset of {a, b, c, d, e} because every member of the first set is also a member of the second (it is not a proper subset).

4. $\emptyset$, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}.

5. There are sixteen subsets: $\emptyset$, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}, {a, b, c, d}.

6. (a) There are 32 subsets.

   (b) 64 subsets.

   (c) There are 2^n subsets. This can be seen by noticing that adding one element to a set A doubles the number of subsets.

1-5 Matching of Sets

The notion of an exact match or one-to-one correspondence between two sets is basic to the process of counting. One can show that two sets have the same number of elements by showing that they are equivalent, that is that there is a one-to-one correspondence between them. Thus, although no one knows the exact number of people in the world, it is certain that the number of people is exactly the same as the number of heads of people.

Two sets are said to be equal if they have exactly the same members. Thus to say that two sets A and B are equal is the same as saying that $A \subseteq B$ and $B \subseteq A$. Equal sets are, of course, equivalent, but equivalent sets need not be equal (see Figure 2 in the students text).

Answers to Problems 1-5A

Student Text Page 11

1. (a) A and B can be matched exactly. For example

1 $\longleftrightarrow$ Ghana	6 $\longleftrightarrow$ Malawi
2 $\longleftrightarrow$ Tanzania	7 $\longleftrightarrow$ Zambia
3 $\longleftrightarrow$ Kenya	8 $\longleftrightarrow$ Sierra Leone
4 $\longleftrightarrow$ Liberia	9 $\longleftrightarrow$ Uganda
5 $\longleftrightarrow$ Nigeria	10 $\longleftrightarrow$ Ethiopia

1. (b) X and Y can be matched exactly, for example by letting $2 \longleftrightarrow 42$, $4 \longleftrightarrow 44$, ..., $18 \longleftrightarrow 58$.

   (c) No match is possible.

   (d) K and L can be matched exactly, for example by letting $1 \longleftrightarrow 3$, $2 \longleftrightarrow 6$, $3 \longleftrightarrow 9$, ...

   (e) No match is possible.

   (f) M and N can be matched exactly by letting each student correspond to his own nose.

2. (a) Take any three pairs of sets where the two sets in each pair have the same number of elements. Note also that $A \longleftrightarrow B$ if A and B are both the empty set.

   (b) Take any three pairs of sets where the two sets in each pair have different numbers of elements.

3. $\{a, b\} \longleftrightarrow \{p, q\}$ $\quad$ $\{a, b\} \longleftrightarrow \{q, r\}$

   $\{a, b\} \longleftrightarrow \{p, r\}$ $\quad$ $\{a, b\} \longleftrightarrow \{q, s\}$

   $\{a, b\} \longleftrightarrow \{p, s\}$ $\quad$ $\{a, b\} \longleftrightarrow \{r, s\}$

   The set $\{a, b\}$ is <u>not</u> equivalent to any of the remaining subsets which are $\emptyset$, $\{p\}$, $\{q\}$, $\{r\}$, $\{s\}$, $\{p, q, r\}$, $\{p, q, s\}$, $\{p, r, s\}$, $\{q, r, s\}$, $\{p, q, r, s\}$.

<u>Answers to Problems 1-5B</u>

Student Text Page 13

1. Yes

2. (a), (f). The sets in (b) are equivalent but are <u>not</u> equal.

3. (a), (b) and (f) contain equivalent pairs. Not all equivalent sets are equal sets, but equal sets are always equivalent sets.

4. They are all equivalent because their members can be matched exactly (they all have exactly four members). The sets C, D, and E are not equal to A. But the three sets A, B, and F are equal to A (notice that A is equal to itself according to the definition).

1-6 Operations with Sets

When taking the union of two sets A and B which are not disjoint, for example, $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$, pupils will frequently write

$$A \cup B = \{1, 2, 3, 2, 3, 4\},$$

thus writing the list of members of A followed by the list of members of B. This is perfectly correct. However, the pupils should be encouraged to observe that

$$\{1, 2, 3, 2, 3, 4\} = \{1, 2, 3, 4\}$$

because these two sets do indeed have exactly the same elements. In short, it is not incorrect but may be confusing to list some of the elements several times when listing a set.

A gifted pupil may observe that the statement in the text just before Problem set 1-6A, namely that the number of elements in $A \cup B$ is less than the sum of the numbers of elements in A and in B whenever A and B are disjoint sets, is referring to finite sets. For only in such cases have the pupils learned to add the numbers of elements of the two sets.

Answers to Problems 1-6A

Student Text Page 16

1. (a) $R \cup T = \{5, 10, 15, 20\}$

   (b) $B \cup D = \{O \ \square \ \Delta\}$

   (c) $A \cup B = \{\Delta \ \square \ O \ \star\}$

1. (d) $E \cup F = \{a, b, c, d, e, f, g, h\}$

   (e) $J \cup K = \{a, b, c, d, e, f\}$

   $= J$

   (f) $S \cup Y$ is the set of pupils in your class who are girls or are in the school choir or both.

   (g) $I \cup J$ is the set of all pupils in the class.

   (h) $M \cup N = M = N$

2. (a) Supply the missing member ◯ in the second set.

   (b) Supply the missing members g, o, and n in the second set.

   (c) Supply the missing members y, l, and i in the two sets. This can be done in six ways.

   (d) Supply the missing number 15 and any one of the numbers 1, 3, 5, 7, 9 in the second set. Thus there are five ways to fill the blanks.

   (e) Supply any two of the letters b, c, d in the blanks. This can be done in three ways.

3. (a) $A \cup B = B \cup A = \{1, 2, 3, 4, 5, 8, 16\}$

   (b) $B \cup C = C \cup B = \{1, 3, 5, a, b, c, d, e\}$

   (c) $(A \cup B) \cup C = A \cup (B \cup C)$

   $= \{1, 2, 3, 4, 5, 8, 16, a, b, c, d, e\}$

<u>Answers to Problems 1-6B</u>

Student Text Page 18

1. (a) $R \cap S = \{15\}$

   (b) $B \cap D = \{\ \}$

   (c) $A \cap B = \{\triangle\ \square\ \bigcirc\}$

   (d) $E \cap F = \{c, d\}$

   (e) $J \cap K = \{\ \}$

1. (f) $S \cap Y$ is the set of girls in your class who are in the school choir.

   (g) $I \cap J = \{\ \}$

   (h) $M \cap N = \{a, b, c, d, e\}$

2. $P \cap M$ = the set of people in Malawi.

3. $A \cap B = \{\ \}$

   $B \cap C = \{2, 6, 8\} = B$

   $A \cap D = \{4, 7\}$

   $D \cap E = \{\ \}$

4. (a) $F \cap G = \{\ \} = G \cap F$

   $G \cap H = \{4\} = H \cap G$

   $F \cap H = \{1, 3, 5, 7\} = H \cap F$

   (b) $(F \cap G) \cap H = \{\ \} \cap H = \{\ \}$

   $F \cap (G \cap H) = F \cap \{4\} = \{\ \}$

   Therefore these sets are equal.

   (c) $F \cap (G \cup H) = F \cap \{1, 2, 3, 4, 5, 6, 7, 8, 14, 24\}$

   $= \{1, 3, 5, 7\}$

   $(F \cap G) \cup (F \cap H) = \{\ \} \cup \{1, 3, 5, 7\} = \{1, 3, 5, 7\}$

   Therefore these sets are equal.

   (d) $F \cup (G \cap H) = F \cup \{4\} = \{1, 3, 4, 5, 7\}$

   $(F \cup G) \cap (F \cup H) = \{1, 3, 4, 5, 7, 14, 24\} \cap \{1, 2, 3, 4, 5, 6, 7, 8\}$

   $= \{1, 3, 4, 5, 7\}$

   Therefore these sets are equal.

   (e) $F \cap (F \cup G) = F \cap \{1, 3, 4, 5, 7, 14, 24\} = \{1, 3, 5, 7\}$

   $= F$

   $F \cup (F \cap G) = F \cup \{\ \} = F$

   Therefore these sets are equal to each other and to F.

4. (f) $F \cap F = \{1, 3, 5, 7\} = F$

$F \cup F = \{1, 3, 5, 7\} = F$

(g) $F \cap \{\ \} = \{1, 3, 5, 7\} \cap \{\ \} = \{\ \}$

(h) $F \cup \{\ \} = \{1, 3, 5, 7\} \cup \{\ \} = \{1, 3, 5, 7\} = F$

5. (a) $\{3, 7, 12\}$

(b) $\{2, 3, 6, 7, 8, 12, 13\}$

(c) Yes. Every number inside the circle is also inside the triangle.

(d) $\{3, 4, 5, 6, 7, 8, 9, 10, 12\}$

(e) $\{2, 3, 6, 7, 8, 12, 13\}$

(f) Yes. Every number inside the square is also inside the circle.

(g) $\{5, 6, 7, 8, 9\}$

(h) $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13\}$

(i) Yes.

(j) $\{7\}$

(k) $\{7\}$

(l) Yes.

(m) $\{3, 4, 5, 6, 7, 8, 9, 10, 12\}$

(n) $\{3, 4, 5, 6, 7, 8, 9, 10, 12\} = \square \cup \square$

(o) Yes.

(p) $\{3, 7, 12\} = \square$

(q) $\{3, 7, 12\}$

(r) Yes.

CHAPTER 2

NUMBER

2-1 Comparison of Sets by Matching

In the first figure of section 2-1 in the text, there are six different ways of matching the set of three pawpaws with the set of three bananas. And in the second example, there are 120 ways of matching the sets $A = \{p, q, r, s, t\}$ and $B = \{k, l, m, n, o\}$. This is because the letter p in the set A can be matched with any of the 5 elements of B, the letter q can then be matched with any of the 4 remaining elements of B, and so on. Thus there are $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways to perform the matching.

The main point of this section is to show that the idea of equivalence of sets can be used to introduce the idea of sets having the same number of elements. The idea of equivalence or of one-to-one correspondence is therefore fundamental to the process of counting.

In the remainder of the chapter, the counting numbers are introduced, and it is seen that the familiar process of "counting" the number of elements in a set is in fact just a process for setting up a certain one-to-one correspondence.

Answers to Problems 2-1 | Student Text Page 23

1. (c) The set of presidents in Africa has fewer members than the set of independent African states.

2-2 Sets in Their Natural Order

Two sets which are not equivalent can be placed in their natural order without "counting" the number of elements in each set. We simply attempt to set up a one-to-one correspondence between the two sets and observe which set has elements left over. Again, this is part of the plan of this chapter to show that the counting numbers can be introduced by using only the idea of one-to-one correspondence.

Answer to Problem 2-2 | Student Text Page 25

1. The natural order of the given sets is: c, a, d, g, f, e, b.

2-3 Number as a Property of Sets

If we consider all sets which are equivalent to a given set, these sets have only one thing in common - the number of elements in each of them. We introduce names for this number and call such names numerals. Thus the number of elements in the set {a, b, c, d, e, f, g} can be expressed by any of the numerals "seven", 7, VII, and so forth. The first of these numerals is the English name for the number, the second is the Arabic numeral, the third is the Roman numeral, etc. Other numerals for seven are $4 + 3$, $105 - 98$ and $(3 \times 3) - 2$. In this chapter the word "numeral" is used mainly to mean "Arabic numeral".

Answers to Problems 2-3A | Student Text Page 26

1. (a) {f, g, h, i, j}, {△ □ O ◇ ▭}

   (b) {1, 2, 3}, {O △ □}

   (c) {w, x, y, z}, {2, 4, 6, 8}

Answers to Problems 2-3B | Student Text Page 27

1. {17, 8, □} etc.

2. four

3. two. {1, 2}, {O □}, etc.

2-5 Order in the Set of Whole Numbers

Answers to Problems 2-5A

Student Text Page 29

1. 4, 2, 3, 1

2. {e}, {b, c}, {a, b, d}, {a, b, c, e}

Answers to Problems 2-5B

Student Text Page 30

1. 4, 3, 0, 4

2. $4 > 3$, $4 > 0$, $3 > 0$, $0 < 3$, $0 < 4$, $3 < 4$, $0 = 0$, $3 = 3$, $4 = 4$.

2-7 Finding the Number of Members in a Set

The familiar process of "counting" the members of a set A is now seen to be nothing more than setting up a one-to-one correspondence between A and a set of counting numbers. If the counting numbers are used for this purpose in their natural order, the last one used is the number of elements in A.

2-8 Equivalence Using Counting Sets

The fact that two given sets are equivalent may be established by counting the number of members in each set and showing that they have the same number of members. At first glance it appears that we are going in circles. But it must be remembered that the idea of equivalence is more fundamental than the idea of counting. In practice, to show that two sets are equivalent, we either set up a one-to-one correspondence between the two sets or else we count the number of members in each of the two sets. We do whichever of these is more convenient. Thus to show that the sets {△ O □ ▽} and {3, 7, 11, 22} are equivalent, we simply show (by counting) that each contains 4 elements. On the

other hand the set of fish in the world is equivalent to the set of fish tails although no one could possibly count the number of fishes.

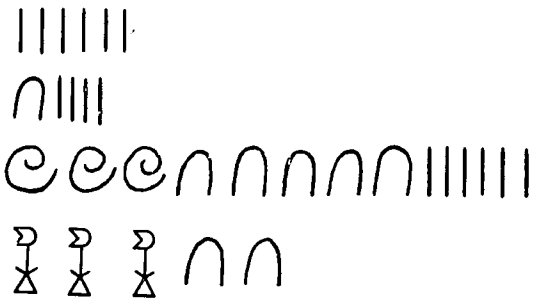
Answers to Problems 2-8 | Student Text Page 33

1. Set A contains 6 elements whereas set B contains only 5 elements. Therefore they are not equivalent.

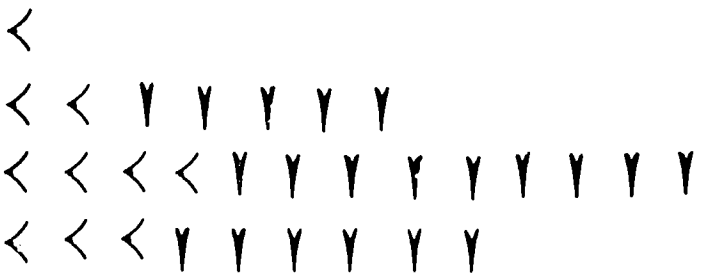
2. Each of these sets contains 4 elements, hence they are equivalent.

2-9 Ways of Representing Numbers

Answers to Problems 2-9 | Student Text Page 35

1. 

2. 3, 30, 22, 402, 1113, and 20,211

3. 

4. (a) L, C, D, M, MMMDLVI
   (b) 60, 39, 43, 264, 1198, 1650

CHAPTER 3

NUMERATION SYSTEMS

3-1 Base Ten Numeration

Perhaps the most important point of this chapter is the fact that the choice of the number ten as the basis for our system of numeration is purely an historical accident, probably related to the fact that a man has ten fingers. Any other counting number greater than one will serve just as well, and for some purposes may even be better than the base ten.

Answers to Problems 3-1

Student Text Page 38

1. thousand

2. (a) 3 tens and 5 ones

   (b) 6 hundreds and 4 tens and 9 ones

   (c) 4 hundreds and 3 tens and 8 ones

   (d) 1 million and 3 hundred thousands and 2 ten thousands and 5 thousands and 6 hundreds and 7 tens and 8 ones

   (e) 5 hundreds and 0 tens and 7 ones

   (f) 3 thousands and 5 hundreds and 0 tens and 0 ones

   (g) 4 ten thousands and 8 thousands and 0 hundreds and 1 ten and 0 ones

   (h) 8 tens and 7 ones

   (i) 1 ten thousand and 3 thousands and 0 hundreds and 0 tens and 2 ones

   (j) 5 hundreds and 0 tens and 0 ones

3-2 Base Five Numeration

Answers to Problems 3-2

Student Text Page 40

1.

1	2	3	4	10	11	12	13	14	20
21	22	23	24	30	31	32	33	34	40
41	42	43	44	100	101	102	103	104	110
111	112	113	114	120	121	122	123	124	130
131	132	133	134	140	141	142	143	144	200
201	202	203	204	210	211	212	213	214	220
221	222	223	224	230	231	232	233	234	240
241	242	243	244	300	301	302	303	304	310
311	312	313	314	320	321	322	323	324	330
331	332	333	334	340	341	342	343	344	400

2. 4 is the largest one-digit number listed.

44 is the largest two-digit number listed.

400 is the largest three-digit number listed.

Note that $(4)_{five} = (4)_{ten}$, $(44)_{five} = (24)_{ten}$, and $(400)_{five} = (100)_{ten}$.

3-3 Addition in Base Five

Learning to add and subtract numbers expressed in the base five helps us to understand the more familiar addition and subtraction processes in the base ten.

Answers to Problems 3-3

Student Text Page 42

1.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	10
2	2	3	4	10	11
3	3	4	10	11	12
4	4	10	11	12	13

2. (a) 24

   (b) 44

   (c) 34

   (d) 102

   (e) 112

3. (a) 40

   (b) 121

   (c) 32

   (d) 110

   (e) 112

4. (a) 2

   (b) 2

   (c) 34

   (d) 41

   (e) 40

3-4 Subtraction in Base Five

Answers to Problems 3-4A

Student Text Page 43

1. 1
2. 4
3. 11
4. 21
5. 12

Answers to Problems 3-4B

Student Text Page 44

1. 114
2. 203
3. 10
4. 323
5. 334

3-5 Base Two Numeration

Answers to Problems 3-5

Student Text Page 45

1. 11
2. 101
3. 1110
4. 1000
5. 1010010
6. 10

7. 100

8. 10000

9. 11

10. 1

3-6 Numeration in Other Bases

Answers to Problems 3-6

Student Text Page 47

1. (a) $(8)_{ten} = (20)_{four}$

   (b) $(8)_{ten} = (22)_{three}$

   (c) $(8)_{ten} = (11)_{seven}$

2. (a) $(19)_{ten} = (18)_{eleven}$

   (b) $(19)_{ten} = (16)_{thirteen}$

3-7 The Decimal System of Numeration

Answers to Problems 3-7A

Student Text Page 47

1. $94 = (9 \times 10) + (4 \times 1)$

2. $303 = (3 \times 10 \times 10) + (0 \times 10) + (3 \times 1)$

3. $999 = (9 \times 10 \times 10) + (9 \times 10) + (9 \times 1)$

4. $7019 = (7 \times 10 \times 10 \times 10) + (0 \times 10 \times 10) + (1 \times 10) + (9 \times 1)$

5. $5003 = (5 \times 10 \times 10 \times 10) + (0 \times 10 \times 10) + (0 \times 10) + (3 \times 1)$

6. $5 = 5 \times 1$

7. $50 = (5 \times 10) + (0 \times 1)$

8. $500 = (5 \times 10 \times 10) + (0 \times 10) + (0 \times 1)$

9. $5000 = (5 \times 10 \times 10 \times 10) + (0 \times 10 \times 10) + (0 \times 10) + (0 \times 1)$

10. $0 = 0 \times 1$

Answers to Problems 3-7B

Student Text Page 49

1. $513 = (5 \times 10^2) + (1 \times 10) + (3 \times 1)$

2. $50{,}149 = (5 \times 10^4) + (0 \times 10^3) + (1 \times 10^2) + (4 \times 10) + (9 \times 1)$

3. $222 = (2 \times 10^2) + (2 \times 10) + (2 \times 1)$

4. $357{,}000 = (3 \times 10^5) + (5 \times 10^4) + (7 \times 10^3) + (0 \times 10^2) + (0 \times 10) + (0 \times 1)$

5. $8665 = (8 \times 10^3) + (6 \times 10^2) + (6 \times 10) + (5 \times 1)$

6. $900{,}412 = (9 \times 10^5) + (0 \times 10^4) + (0 \times 10^3) + (4 \times 10^2) + (1 \times 10) + (2 \times 1)$

7. $81{,}871{,}354 = (8 \times 10^7) + (1 \times 10^6) + (8 \times 10^5) + (7 \times 10^4) + (1 \times 10^3) + (3 \times 10^2) + (5 \times 10) + (4 \times 1)$

8. $1{,}000{,}000 = (1 \times 10^6) + (0 \times 10^5) + (0 \times 10^4) + (0 \times 10^3) + (0 \times 10^2) + (0 \times 10) + (0 \times 1)$

9. $10{,}010{,}101 = (1 \times 10^7) + (0 \times 10^6) + (0 \times 10^5) + (1 \times 10^4) + (0 \times 10^3) + (1 \times 10^2) + (0 \times 10) + (1 \times 1)$

10. $10{,}000{,}000{,}001 = (1 \times 10^{10}) + (0 \times 10^{9}) + (0 \times 10^{8}) + (0 \times 10^{7}) + (0 \times 10^{6}) + (0 \times 10^{5}) + (0 \times 10^{4}) + (0 \times 10^{3}) + (0 \times 10^{2}) + (0 \times 10) + (1 \times 1)$

11. 901

12. 55,555

13. 680,170

14. 501

3-8 Expanded Numerals in Any Base

Answers to Problems 3-8 — Student Text Page 50

1. (a) $(12)_{five} = (1 \times 10) + (2 \times 1)$

   (b) $(412)_{five} = (4 \times 10^2) + (1 \times 10) + (2 \times 1)$

   (c) $(13{,}421)_{five} = (1 \times 10^4) + (3 \times 10^3) + (4 \times 10^2) + (2 \times 10) + (1 \times 1)$

2. $(16)_{seven} = (1 \times 10) + (6 \times 1)$

3. $(1101)_{two} = (1 \times 10^3) + (1 \times 10^2) + (0 \times 10) + (1 \times 1)$

3-9 Changing Numerals Written in Any Base to Base Ten

Answers to Problems 3-9 — Student Text Page 51

1. $(321)_{five} = (3 \times 10^2) + (2 \times 10) + (1 \times 1)$ in base 5

   $= (3 \times 5^2) + (2 \times 5) + (1 \times 1)$ in base 10

   $= 75 + 10 + 1 = 86$ in base 10

2. $(13)_{five} = (1 \times 10) + (3 \times 1)$ in base 5

$= (1 \times 5) + (3 \times 1)$ in base 10

$= 5 + 3 = 8$ in base 10

3. $(103)_{five} = (28)_{ten}$

4. $(57)_{eight} = (5 \times 10) + (7 \times 1)$ in base eight

$= (5 \times 8) + (7 \times 1)$ in base ten

$= 40 + 7 = 47$ in base ten

5. $(1011)_{two} = (1 \times 10^3) + (0 \times 10^2) + (1 \times 10) + (1 \times 1)$ in base 2

$= (1 \times 2^3) + (0 \times 2^2) + (1 \times 2) + (1 \times 1)$ in base 10

$= 8 + 0 + 2 + 1 = 11$ in base 10

6. $(321)_{nine} = (3 \times 10^2) + (2 \times 10) + (1 \times 1)$ in base nine

$= (3 \times 9^2) + (2 \times 9) + (1 \times 1)$ in base 10

$= 243 + 18 + 1 = 262$ in base 10

7. $(321)_{four} = (57)_{ten}$

8. $(11022)_{three} = (116)_{ten}$

3-10 <u>Rewriting a Numeral from Base Ten to Another Base</u>

<u>Answers to Problems 3-10</u>

Student Text Page 53

1. 5 | 15

5 | 3 fives and 0 ones left over

0 twenty-fives and 3 fives left over

Therefore $(15)_{ten} = (30)_{five}$.

2. $(134)_{ten} = (1014)_{five}$

3. 8 | 746
   8 | 93 eights and 2 ones left over
   8 | 11 sixty-fours and 5 eights left over
   8 | 1 five hundred twelves and 3 sixty-fours left over
   0 four thousand ninety-six and 1 five hundred twelve left over

   Therefore $(1746)_{10} = (1352)_{eight}$

4. $(26,495)_{ten} = (140150)_{seven}$

5. $(2341)_{ten} = (33331)_{five}$

6. $(645)_{ten} = (2553)_{six}$

7. $(5555)_{ten} = (134210)_{five}$

8. $(11)_{ten} = (1011)_{two}$

9. $(123,456)_{ten} = (207313)_{nine}$

3-11 Revision Problems (Answers)

Student Text Page 53

1. (a) ten
   (b) five
   (c) six
   (d) seven
   (e) two
   (f) b

2. (a) 1
   (b) 10^2
   (c) 10^7

3. (a) $12{,}312 = (1 \times 10^4) + (2 \times 10^3) + (3 \times 10^2) + (1 \times 10) + (2 \times 1)$

(b) $(321)_{five} = (3 \times 10^2) + (2 \times 10) + (1 \times 1)$ in base 5

(c) $(11010)_{two} = (1 \times 10^4) + (1 \times 10^3) + (0 \times 10^2) + (1 \times 10) + (0 \times 1)$ in base two

4. (a) $(341)_{five}$

(b) $(133)_{five}$

5.

×	0	1	2	3	4	10
0	0	0	0	0	0	0
1	0	1	2	3	4	10
2	0	2	4	11	13	20
3	0	3	11	14	22	30
4	0	4	13	22	31	40
10	0	10	20	30	40	100

6. $(1333)_{seven}$ $(256)_{seven}$

7. (a) 211

(b) 160

(c) 188

8. (a) $(443)_{five}$

(b) $(32)_{five} = (17)_{ten} = (10001)_{two}$

(c) $(40)_{five} = (10100)_{two}$

(d) $(35)_{six} = (43)_{five}$

9. (a) fifteen million, three hundred forty-eight thousand, seven hundred twenty-one

$(1 \times 10^7) + (5 \times 10^6) + (3 \times 10^5) + (4 \times 10^4) + (8 \times 10^3) + (7 \times 10^2) + (2 \times 10) + (1 \times 1)$

(b) five hundred sixty-seven million, eight hundred forty-three thousand, two hundred eighty-seven

$(5 \times 10^8) + (6 \times 10^7) + (7 \times 10^6) + (8 \times 10^5) + (4 \times 10^4) + (3 \times 10^3) + (2 \times 10^2) + (8 \times 10) + (7 \times 1)$

(c) one hundred two million, thirty thousand, four hundred five

$(1 \times 10^8) + (0 \times 10^7) + (2 \times 10^6) + (0 \times 10^5) + (3 \times 10^4) + (0 \times 10^3) + (4 \times 10^2) + (0 \times 10) + (5 \times 1)$

10. Yes

CHAPTER 4

THE NATURAL NUMBERS

The main object of this chapter is to study the factorization properties of the natural numbers. When we have found the factors of two different numbers we are then able to find the Greatest Common Factor (GCF) and the Least Common Multiple (LCM) of the two numbers. This knowledge is necessary in order to compute with fractions later on in the year. We are thus laying the groundwork here for a skill which every student will have to master.

4-1 Natural Numbers and Whole Numbers

This very brief section is simply a revision of the ideas of natural numbers and whole numbers. It is important to realize that in this chapter we deal only with natural numbers. If we allowed rational numbers to be considered as factors we would find that any rational number (except 0) is a factor of any other rational number. Thus $\frac{2}{3}$ would be a factor of $\frac{4}{5}$ since $\frac{2}{3} \times \frac{6}{5} = \frac{4}{5}$, and 4 would be a factor of $\frac{7}{13}$ since $4 \times \frac{7}{52} = \frac{7}{13}$. But this would be uninteresting and we rule it out.

Revise briefly the ideas of set and subset to help students see the relationship between the natural numbers and the whole numbers.

Answers to Problems 4-1 | Student Text Page 57

1. $N \cap W = N$

2. $N \cup W = W$

3. 1. There is no largest.

4. 0. There is no largest.

5. Yes. Since every natural number is a whole number.

6. No. Since 0 is a whole number but is <u>not</u> a natural number.

Previous Page Blank

4-2 Factors of Numbers

To find the factors of any number n you can always proceed by trial, i.e. divide n by every number x such that $1 \leq x \leq n$. Those numbers which do divide n exactly are its factors. For small numbers n this is easy but for large numbers it soon becomes impractical. Students will soon see, or you should guide them to see, that they really need to try only those numbers x which are less than $\sqrt{n}$. Also they may be guided to see that if they have found one factor of 36, say 4, they have really found another one, 9. Also some students will see that if 2 is a factor of 36, then since $2 \times 18 = 36$, any factor of 18 will also be a factor of 36. But $18 = 9 \times 2$ so $36 = (9 \times 2) \times 2$ and any factor of 9 will be a factor of 36. Students should be encouraged to proceed in this fashion if they want to, since it leads to the Prime Factorization Theorem of Section 4-4. The danger is that they may get 2 and 18 as factors of 36, also 9 and 4, 3 and 12 but may miss 6. Great care is needed to be sure that <u>all</u> factors have been listed.

Answers to Problems 4-2

Student Text Page 58

1. 6: {1, 2, 3, 6}
   12: {1, 2, 3, 4, 6, 12}
   18: {1, 2, 3, 6, 9, 18}
   24: {1, 2, 3, 4, 6, 8, 12, 24}
   36: {1, 2, 3, 4, 6, 9, 12, 18, 36}
   48: {1, 2, 3, 4, 6, 8, 12, 16, 24, 48}
   54: {1, 2, 3, 6, 9, 18, 27, 54}
   64: {1, 2, 4, 8, 16, 32, 64}
   72: {1, 2, 3, 4, 6, 8, 9, 12, 18, 36, 72}
   81: {1, 3, 9, 27, 81}
   96: {1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96}
   100: {1, 2, 4, 5, 10, 20, 25, 50, 100}

2. Yes. 1 is a factor of every number.

3. (a) The factors of $2 \times 3 \times 5$: $\{1, 2, 3, 5, 6, 10, 15, 30\}$

(b) $2 \times 3 \times 7$: $\{1, 2, 3, 6, 7, 14, 21, 42\}$

(c) $2 \times 2 \times 3$: $\{1, 2, 3, 4, 6, 12\}$

(d) $2 \times 2 \times 2$: $\{1, 2, 4, 8\}$

(e) $3 \times 3 \times 3$: $\{1, 3, 9, 27\}$

(f) $3 \times 5 \times 5$: $\{1, 3, 5, 15, 25, 75\}$

(g) $5 \times 5 \times 7$: $\{1, 5, 7, 25, 35, 175\}$

(h) $3 \times 3 \times 5 \times 7$: $\{1, 3, 5, 7, 9, 15, 21, 35, 45, 63, 105, 315\}$

(i) $3 \times 5 \times 5 \times 7$: $\{1, 3, 5, 7, 15, 21, 25, 35, 75, 105, 175, 525\}$

(j) $3 \times 5 \times 7 \times 7$: $\{1, 3, 5, 7, 15, 21, 35, 49, 105, 147, 245, 735\}$

4-3 Prime Numbers

The most important idea here is the definition of prime number. Notice that the definition excludes 1 from the set of primes. The reason for wording the definition this way is purely one of convenience. It makes certain theorems we want later on easier to state. It is incorrect to say that composite numbers are those numbers which are not prime. The natural numbers are the union of three nonintersecting sets.

$$N = \{1\} \cup \{\text{prime numbers}\} \cup \{\text{composite numbers}\}.$$

Do not drill on the sieve of Eratosthenes. If the students practice it a few times they will get the idea. Again try to get them to see that in using the sieve they only need to consider multiples of those primes which are less than $\sqrt{n}$. Thus in Problem 1 of this section when they have crossed out the multiples of 7 they are finished, since $\sqrt{100}$ is 10 and the largest prime less than 10 is 7. If we tried to cross out multiples of the next prime, which is 11, the first one we would reach which is not a multiple of some number less than 11 would be 11×11 or 121, but this is greater than 100.

Answers to Problems 4-3

Student Text Page 61

1.

~~1~~	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

The primes are {2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97}

2. 293, 499

3. 2 and 7. One number must always be 2, because 2 is the only even prime and the product of two odd numbers will always be odd.

4. 7 and 5. 2 may not be used but any two other prime numbers may be, since they will always be odd and the product of any two odd numbers is always odd.

5.

16 = 5 + 11	24 = 11 + 13	32 = 3 + 29	40 = 3 + 37	48 = 5 + 43
18 = 7 + 11	26 = 13 + 13	34 = 17 + 17	42 = 11 + 31	50 = 7 + 43
20 = 7 + 13	28 = 5 + 23	36 = 5 + 31	44 = 13 + 31	52 = 5 + 47
22 = 11 + 11	30 = 7 + 23	38 = 19 + 19	46 = 5 + 41	54 = 7 + 47

In some cases there are other possibilities. Thus

26 = 3 + 23 = 7 + 19 as well as 13 + 13.

It certainly looks as if it is true that any even number greater than 4 can be expressed as the sum of two primes, but we surely

have not proved it. No number of examples can ever be enough to make us sure that the result is always true. As a matter of fact this is an example of a theorem, or rather a conjecture, which is simple to state but which is extremely difficult either to prove or to disprove. No one has yet (1965) been able to do so.

6. 29 and 31; 41 and 43; 59 and 61; 71 and 73.

   Again this leads to an unsolved (1965) problem. "Are there infinitely many prime numbers which differ by 2?" No one knows the answer.

7. You certainly could, even though it would take a very long time. There are prime numbers greater than 1,000,000. There is no last prime number. A proof of this was given by Euclid, the famous Greek mathematician who is perhaps better known as a geometer. Since there is no last prime, there is an infinite number of primes.

4-4 Prime Factorization of Natural Numbers

In factorizing numbers some students will naturally start one way and some another. Thus some will say $108 = 9 \times 12$, others will say $108 = 4 \times 27$. Be sure they continue and find the prime factors.

$$108 = 9 \times 12 = (3 \times 3) \times (4 \times 3) = 3 \times 3 \times 2 \times 2 \times 3 \quad \text{or}$$
$$108 = 4 \times 27 = (2 \times 2) \times (3 \times 9) = 2 \times 2 \times 3 \times 3 \times 3$$

Ask them to compare their final results and help them to recognize that by a simple rearrangement of their factors it is easy to see that their answers are the same. This leads naturally into the next section.

Answers to Problems 4-4 | Student Text Page 62

1. $10 = 2 \times 5$;
   $15 = 3 \times 5$;
   $16 = 2 \times 2 \times 2 \times 2$;
   $18 = 2 \times 3 \times 3$;
   $20 = 2 \times 2 \times 5$;
   $36 = 2 \times 2 \times 3 \times 3$;
   $48 = 2 \times 2 \times 2 \times 2 \times 3$;
   $72 = 2 \times 2 \times 2 \times 3 \times 3$;
   $144 = 2 \times 2 \times 2 \times 2 \times 3 \times 3$;
   $256 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$.

2. $9 = 3 \times 3$;
$12 = 2 \times 2 \times 3$;
$21 = 3 \times 7$;
$24 = 2 \times 2 \times 2 \times 3$;
$30 = 2 \times 3 \times 5$;
$42 = 2 \times 3 \times 7$;
$108 = 2 \times 2 \times 3 \times 3 \times 3$;
$125 = 5 \times 5 \times 5$;
$216 = 2 \times 2 \times 2 \times 3 \times 3 \times 3$;
$648 = 2 \times 2 \times 2 \times 3 \times 3 \times 3 \times 3$.

4-5 The Fundamental Theorem of Arithmetic

Now encourage your students to be as systematic as possible. The Fundamental Theorem of Arithmetic (which is not proved here, but assumed) says it does not matter how you factorize a number. If you factorize all the way down to the prime factors, the factors are always the same though perhaps arranged in a different order. Therefore have your students start with the smallest prime, 2, and see if it is a factor. After factorizing by 2 as many times as possible start with 3, then 5 etc. Doing this they will be led to the simplest expression for the factorization.

Answers to Problems 4-5

Student Text Page 63

$30 = 6 \times 5 = (2 \times 3) \times 5$;
$98 = 2 \times 49 = 2 \times (7 \times 7)$;
$81 = 9 \times 9 = (3 \times 3) \times (3 \times 3)$;
$80 = 4 \times 20 = (2 \times 2) \times (2 \times 2 \times 5)$;
$45 = 3 \times 15 = 3 \times (3 \times 5)$;
$100 = 25 \times 4 = (5 \times 5) \times (2 \times 2)$;
$330 = 30 \times 11 = (2 \times 3 \times 5) \times 11$;
$92 = 4 \times 23 = (2 \times 2) \times 23$;
$126 = 6 \times 21 = (2 \times 3) \times (3 \times 7)$;
$372 = 4 \times 93 = (2 \times 2) \times (3 \times 31)$;
$390 = 30 \times 13 = (2 \times 3 \times 5) \times 13$;
$258 = 6 \times 43 = (2 \times 3) \times 43$;
$231 = 7 \times 33 = 7 \times (3 \times 11)$;
$910 = 10 \times 91 = (2 \times 5) \times (7 \times 13)$;

$30 = 10 \times 3 = (2 \times 5) \times 3$;
$98 = 14 \times 7 = (2 \times 7) \times 7$;
$81 = 3 \times 27 = 3 \times (3 \times 3 \times 3)$;
$80 = 8 \times 10 = (2 \times 2 \times 2) \times (2 \times 5)$;
$45 = 5 \times 9 = 5 \times (3 \times 3)$;
$100 = 10 \times 10 = (5 \times 2) \times (5 \times 2)$;
$330 = 10 \times 33 = (2 \times 5) \times (3 \times 11)$;
$92 = 2 \times 46 = 2 \times (2 \times 23)$;
$126 = 7 \times 18 = 7 \times (2 \times 3 \times 3)$;
$372 = 12 \times 31 = (2 \times 2 \times 3) \times 31$;
$390 = 15 \times 26 = (5 \times 3) \times (2 \times 13)$;
$258 = 3 \times 86 = 3 \times (2 \times 43)$;
$231 = 11 \times 21 = 11 \times (7 \times 3)$;
$910 = 5 \times 182 = 5 \times (2 \times 91) = 5 \times (2 \times 7 \times 13)$.

The systematic way of factorizing would be

$$30 = 2 \times 15 = 2 \times 3 \times 5$$
$$98 = 2 \times 49 = 2 \times 7^2$$
$$81 = 3 \times 27 = 3^2 \times 9 = 3^4$$
$$80 = 2 \times 40 = 2^2 \times 20 = 2^3 \times 10 = 2^4 \times 5$$
$$45 = 3 \times 15 = 3^2 \times 5$$
$$100 = 2 \times 50 = 2^2 \times 25 = 2^2 \times 5^2$$
$$330 = 2 \times 165 = 2 \times 3 \times 55 = 2 \times 3 \times 5 \times 11$$
$$92 = 2 \times 46 = 2^2 \times 23$$
$$126 = 2 \times 63 = 2 \times 3 \times 21 = 2 \times 3^2 \times 7$$
$$372 = 2 \times 186 = 2^2 \times 93 = 2^2 \times 3 \times 31$$
$$390 = 2 \times 195 = 2 \times 3 \times 65 = 2 \times 3 \times 5 \times 13$$
$$258 = 2 \times 129 = 2 \times 3 \times 43$$
$$231 = 3 \times 77 = 3 \times 7 \times 11$$
$$910 = 2 \times 455 = 2 \times 5 \times 91 = 2 \times 5 \times 7 \times 13$$

4-6 Greatest Common Factor

There are two different methods given for finding the GCF of two numbers a and b. The first is simply to list the set of <u>all</u> the factors of each number and determine the intersection of these sets. This will then be the set of common factors and the GCF will be the largest number in this set. This is straightforward but long and tedious, so we give a second method which involves finding the prime factorization of each number and examining these factors to find the GCF. If there is no common factor except 1, the numbers are relatively prime. Note carefully that numbers such as 8 and 9 are relatively prime even though neither number is itself a prime.

<u>Answers to Problems 4-6</u> | Student Text Page 66 |

1. If a and b are the given numbers let A and B respectively be the sets of all factors of a and b.

   (a) 18: $\{1, 2, 3, 6, 9, 18\} = A$ $\{1, 3, 9\} = A \cap B$

   45: $\{1, 3, 5, 9, 15, 45\} = B$ $9 = \text{GCF}$

(b) 15: {1, 3, 5, 15} {1, 3, 5, 15} = A ∩ B
60: {1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60} 15 = GCF

(c) 15: {1, 3, 5, 15} {1, 3} = A ∩ B
27: {1, 3, 9, 27} 3 = GCF

(d) 12: {1, 2, 3, 4, 6, 12} {1, 2, 3, 6} = A ∩ B
42: {1, 2, 3, 6, 7, 14, 21, 42} 6 = GCF

(e) 3: {1, 3} {1} = A ∩ B
5: {1, 5} 1 = GCF

(f) 5: {1, 5} {1} = A ∩ B
7: {1, 7} 1 = GCF

(g) 96: {1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96}
36: {1, 2, 3, 4, 6, 9, 12, 18, 36}
{1, 2, 3, 4, 6, 12} = A ∩ B 12 = GCF

(h) 165: {1, 3, 5, 11, 15, 33, 55, 165} {1, 3, 11, 33} = A ∩ B
99: {1, 3, 9, 11, 33, 99} 33 = GCF

(i) 8: {1, 2, 4, 8} {1} = A ∩ B
9: {1, 3, 9} 1 = GCF

(j) 15: {1, 3, 5, 15} {1} = A ∩ B
16: {1, 2, 4, 8, 16} 1 = GCF

(k) 24: {1, 2, 3, 4, 6, 8, 12, 24} {1, 2, 3, 4, 6, 12} = A ∩ B
108: {1, 2, 3, 4, 6, 9, 12, 18, 27, 54, 108} 12 = GCF

(l) 425: {1, 5, 17, 25, 85, 425} {1, 5, 25} = A ∩ B
200: {1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 200} 25 = GCF

(m) 210: {1, 2, 3, 5, 6, 7, 10, 14, 15, 21, 30, 42, 70, 105, 210}
56: {1, 2, 4, 7, 8, 14, 28, 56}
{1, 2, 7, 14} = A ∩ B 14 = GCF

(n) 11: {1, 11} {1} = A ∩ B
13: {1, 13} 1 = GCF

1. (o) 7: $\{1, 7\}$ $\{1\} = A \cap B$

19: $\{1, 19\}$ $1 = GCF$

(p) 546: $\{1, 2, 3, 6, 7, 13, 14, 21, 26, 39, 42, 78, 91, 182, 273, 546\}$

1521: $\{1, 3, 9, 13, 39, 117, 169, 507, 1521\}$

$\{1, 3, 13, 39\} = A \cap B$ $39 = GCF$

(q) 36: $\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ $\{1, 3, 9\} = A \cap B$

243: $\{1, 3, 9, 27, 81, 243\}$ $9 = GCF$

(r) 48: $\{1, 2, 3, 4, 6, 8, 16, 24, 48\}$ $\{1, 2, 4, 8, 16\} = A \cap B$

256: $\{1, 2, 4, 8, 16, 32, 64, 128, 256\}$ $16 = GCF$

2. (a) $18 = 2 \times 3^2$; $45 = 5 \times 3^2$; $GCF = 3^2 = 9$

(b) $15 = 3 \times 5$; $60 = 2^2 \times 3 \times 5$; $GCF = 3 \times 5 = 15$

(c) $15 = 3 \times 5$; $27 = 3^3$; $GCF = 3$

(d) $12 = 2^2 \times 3$; $42 = 2 \times 3 \times 7$; $GCF = 2 \times 3 = 6$

(e) $3 = 3$; $5 = 5$; $GCF = 1$

(f) $5 = 5$; $7 = 7$; $GCF = 1$

(g) $96 = 2^5 \times 3$; $36 = 2^2 \times 3^2$; $GCF = 2^2 \times 3 = 12$

(h) $165 = 3 \times 5 \times 11$; $99 = 3^2 \times 11$; $GCF = 3 \times 11 = 33$

(i) $8 = 2^3$; $9 = 3^2$; $GCF = 1$

(j) $15 = 3 \times 5$; $16 = 2^4$ $GCF = 1$

(k) $24 = 2^3 \times 3$; $108 = 2^2 \times 3^3$ $GCF = 2^2 \times 3 = 12$

(l) $425 = 5^2 \times 17$; $200 = 2^3 \times 5^2$; $GCF = 25$

(m) $210 = 2 \times 3 \times 5 \times 7$; $56 = 2^3 \times 7$; $GCF = 2 \times 7 = 14$

(n) $11 = 1 \times 11$; $13 = 13 \times 1$; $GCF = 1$

(o) $7 = 1 \times 17$; $19 = 1 \times 19$; $GCF = 1$

(p) $546 = 2 \times 3 \times 7 \times 13$; $1521 = 3^2 \times 13^2$; $GCF = 3 \times 13 = 39$

2.	(q)	$36 = 2^2 \times 3^2$;	$243 = 3^5$;	GCF $= 3^2 = 9$
	(r)	$48 = 2^4 \times 3$;	$256 = 2^8$;	GCF $= 2^4 = 16$
3.	(a)	$28 = 2^2 \times 7$;	$54 = 2 \times 3^3$;	GCF $= 2$
	(b)	$16 = 2^4$;	$56 = 2^3 \times 7$;	GCF $= 2^3 = 8$
	(c)	$36 = 2^2 \times 3^2$;	$162 = 2 \times 3^4$;	GCF $= 2 \times 3^2 = 18$
	(d)	$4 = 2^2$;	$7 = 1 \times 7$;	GCF $= 1$
	(e)	$48 = 2^4 \times 3$;	$108 = 2^2 \times 3^3$;	GCF $= 2^2 \times 3 = 12$
	(f)	$20 = 2^2 \times 5$;	$21 = 3 \times 7$;	GCF $= 1$
	(g)	$42 = 2 \times 3 \times 7$;	$273 = 3 \times 7 \times 13$;	GCF $= 3 \times 7 = 21$
	(h)	$12 = 2^2 \times 3$; $42 = 2 \times 3 \times 7$;	$72 = 2^3 \times 3^2$;	GCF $= 2 \times 3 = 6$
	(i)	$16 = 2^4$; $72 = 2^3 \times 3^2$;	$104 = 2^3 \times 13$;	GCF $= 2^3 = 8$
	(j)	$4 = 2^2$; $5 = 1 \times 5$;	$9 = 3^2$;	GCF $= 1$
	(k)	$35 = 5 \times 7$; $420 = 2^2 \times 5 \times 3 \times 7$;	$245 = 5 \times 7^2$;	GCF $= 5 \times 7 = 35$
	(l)	$24 = 2^3 \times 3$;	$156 = 2^2 \times 3 \times 13$;	$180 = 2^2 \times 3^2 \times 5$;
			$204 = 2^2 \times 3 \times 17$;	GCF $= 2^2 \times 3 = 12$
	(m)	$4 = 2^2$; $5 = 5$; $7 = 7$;	$9 = 3^2$;	GCF $= 1$
	(n)	$30 = 2 \times 3 \times 5$;	$126 = 2 \times 3^2 \times 7$;	$105 = 3 \times 5 \times 7$;
			$255 = 3 \times 5 \times 17$;	GCF $= 3$
	(o)	$4 = 2^2$; $7 = 7$; $9 = 3^2$;	$25 = 5^2$;	GCF $= 1$

4-7 Least Common Multiple

Again we give two methods, of which the first is straightforward but tedious and long. The second method involves prime factorization just as did the second method of finding the GCF. Be careful that students do not confuse

the two ideas LCM and GCF. In finding either, they use a method involving intersection of sets or a method involving prime factorizations and selection of certain prime factors. Be sure that they understand the differences between the two methods in each case.

Answers to Problems 4-7 | Student Text Page 67

1. In order to use the first method for finding the LCM of two numbers we need the set of multiples of each of them. Let us use M_n for the set of multiples of n. We write down these sets for all the numbers we need in this problem.

$M_3 = \{3, 6, 9, 12, 15, \ldots\}$

$M_4 = \{4, 8, 12, 16, 20, 24, 28, 32, 36, \ldots\}$

$M_5 = \{5, 10, 15, 20, 25, 30, 35, \ldots\}$

$M_6 = \{6, 12, 18, 24, 30, \ldots\}$

$M_7 = \{7, 14, 21, 28, 35, \ldots\}$

$M_8 = \{8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88, 96, 104, 112, 120, \ldots\}$

$M_9 = \{9, 18, 27, 36, 45, 54, \ldots\}$

$M_{10} = \{10, 20, 30, 40, 50, \ldots\}$

$M_{11} = \{11, 22, 33, 44, 55, 66, 77, 88, 99, 110, 121, 132, 143, \ldots\}$

$M_{12} = \{12, 24, 36, 48 \ldots\}$

$M_{13} = \{13, 26, 39, 52, 65, 78, 91, 104, 117, 130, 143, \ldots\}$

$M_{15} = \{15, 30, 45, 60, 75, 90, 105, 120, 135, 150, 165, 180, 195, 210, 225, 240, \ldots\}$

$M_{16} = \{16, 32, 48, 64, 80, 96, 112, 128, 144, 160, 176, 192, 208, 224, 240, \ldots\}$

$M_{18} = \{18, 36, 54, 72, 90, \ldots\}$

$M_{20} = \{20, 40, 60, 80, \ldots\}$

$M_{21} = \{21, 42, 63, 84, \ldots\}$

$M_{24} = \{24, 48, 72, \ldots\}$

$M_{25} = \{25, 50, 75, \ldots\}$

(a) $M_4 \cap M_6 = \{12, 24, \ldots\}$ LCM(4, 6) = 12

(b) $M_4 \cap M_9 = \{36, 72, \ldots\}$ LCM(4, 9) = 36

(c) $M_6 \cap M_9 = \{18, 36, \ldots\}$ LCM(6, 9) = 18

(d) $M_8 \cap M_{15} = \{120, \ldots\}$ LCM(8, 15) = 120

(e) $M_3 \cap M_5 = \{15, 30, \ldots\}$ LCM(3, 5) = 15

(f) $M_5 \cap M_7 = \{35, 70, \ldots\}$ LCM(5, 7) = 35

(g) $M_9 \cap M_{15} = \{45, 90, \ldots\}$ LCM(9, 15) = 45

(h) $M_8 \cap M_{18} = \{72, \ldots\}$ LCM(8, 18) = 72

(i) $M_8 \cap M_9 = \{72, \ldots\}$ LCM(8, 9) = 72

(j) $M_{15} \cap M_{16} = \{240, \ldots\}$ LCM(15, 16) = 240

(k) $M_6 \cap M_{15} = \{30, 60, \ldots\}$ LCM(6, 15) = 30

(l) $M_9 \cap M_{12} = \{36, 72, \ldots\}$ LCM(9, 12) = 36

(m) $M_{15} \cap M_{20} = \{60, 120, \ldots\}$ LCM(15, 20) = 60

(n) $M_{20} \cap M_{25} = \{100, 200, \ldots\}$ LCM(20, 25) = 100

(o) $M_{10} \cap M_{15} = \{30, 60, \ldots\}$ LCM(10, 15) = 30

(p) $M_{11} \cap M_{13} = \{143, \ldots\}$ LCM(11, 13) = 143

(q) $M_{16} \cap M_{24} = \{48, 96, \ldots\}$ LCM(16, 24) = 48

(r) $M_{18} \cap M_{24} = \{72, \ldots\}$ LCM(18, 24) = 72

2. (a) $4 = 2^2$ $6 = 2 \times 3$ LCM $= 2^2 \times 3 = 12$

(b) $4 = 2^2$ $9 = 3^2$ LCM $= 2^2 \times 3^2 = 36$

(c) $6 = 2 \times 3$ $9 = 3^2$ LCM $= 2 \times 3^2 = 18$

(d) $8 = 2^3$ $15 = 3 \times 5$ LCM $= 2^3 \times 3 \times 5 = 120$

(e) $3 = 3$ $5 = 5$ LCM $= 3 \times 5 = 15$

(f) $5 = 5$ $7 = 7$ LCM $= 5 \times 7 = 35$

(g)	$9 = 3^2$	$15 = 3 \times 5$	LCM $= 3^2 \times 5 = 45$
(h)	$8 = 2^3$	$18 = 2 \times 3^2$	LCM $= 2^3 \times 3^2 = 72$
(i)	$8 = 2^3$	$9 = 3^2$	LCM $= 2^3 \times 3^2 = 72$
(j)	$15 = 3 \times 5$	$16 = 2^4$	LCM $= 2^4 \times 3 \times 5 = 240$
(k)	$6 = 2 \times 3$	$15 = 3 \times 5$	LCM $= 2 \times 3 \times 5 = 30$
(l)	$9 = 3^2$	$12 = 2^2 \times 3$	LCM $= 2^2 \times 3^2 = 36$
(m)	$15 = 3 \times 5$	$20 = 2^2 \times 5$	LCM $= 2^2 \times 3 \times 5 = 60$
(n)	$20 = 2^2 \times 5$	$25 = 5^2$	LCM $= 2^2 \times 5^2 = 100$
(o)	$10 = 2 \times 5$	$15 = 3 \times 5$	LCM $= 2 \times 3 \times 5 = 30$
(p)	$11 = 11$	$13 = 13$	LCM $= 11 \times 13 = 143$
(q)	$16 = 2^4$	$24 = 2^3 \times 3$	LCM $= 2^4 \times 3 = 48$
(r)	$18 = 2 \times 3^2$	$24 = 2^3 \times 3$	LCM $= 2^3 \times 3^2 = 72$

3.

(a) 60	(g) 144	(m) 30
(b) 42	(h) 75	(n) 72
(c) 24	(i) 63	(o) 1155
(d) 36	(j) 60	(p) 80
(e) 15	(k) 72	(q) 60
(f) 24	(l) 180	(r) 180

4.

(a)	12, 18	GCF = 6	LCM = 36
	$12 \times 18 = 216$		$6 \times 36 = 216$
(b)	7, 28	GCF = 7	LCM = 28
	$7 \times 28 = 196$		$7 \times 28 = 196$
(c)	12, 35	GCF = 1	LCM = 420
	$12 \times 35 = 420$		$1 \times 420 = 420$
(d)	96, 120	GCF = 24	LCM = 480
	$96 \times 120 = 11,520$		$24 \times 480 = 11,520$

In each case the product of the GCF and the LCM for a pair of numbers is exactly the same as the product of the numbers. This is <u>not</u> accidental and is always true. The proof, while easy enough, needs more knowledge of exponents than your pupils have at this stage. We give it here in case some of your better students might be interested. Suppose m and n are the two numbers we are considering. We first write the prime factorizations of m and n. Then $m = 2^a \times 3^b \times 5^c \times \ldots\ldots$ where we write each prime in turn with the power to which it occurs in m. If any prime is not in the factorization the corresponding exponent is zero. Thus if $m = 45$ and $n = 12$ we would write $m = 2^0 \times 3^2 \times 5^1$

$$n = 2^2 \times 3^1 \times 5^0$$

We do not write 7 or 11 or any of the other primes as factors since they would all have exponent zero anyway.

Now consider any particular prime p. m and n will each have p as a factor a certain number of times say s and t respectively. Then $m \times n$ will have as a factor p^{s+t}. Suppose $s > t$. Then the GCF of m and n has as one of its factors p^t while their LCM has as one of its factors p^s, so the product has as one of its factors p^{s+t} just as $m \times n$ did. If $s < t$ then p^s is a factor of the GCF and p^t is a factor of the LCM. Once again their product has as one of its factors p^{s+t}. If $s = t$, p^s must be a factor of both the GCF and the LCM and p^{s+s} is a factor of their product just as it is of $m \times n$. The same argument holds for any other prime q. Therefore, since $m \times n$ and (GCF) $\times$ (LCM) have the same factors, they are equal.

CHAPTER 5

RATIONAL NUMBERS AND THE NUMBER LINE

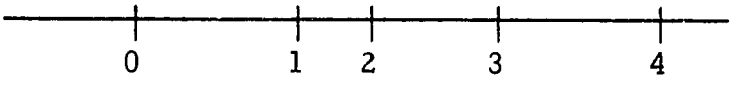
The main purpose of this chapter is to introduce the negative rational numbers. But first we want to give a practical and easy-to-grasp method of picturing or making a model for the numbers we already know about, that is the whole numbers $\{0, 1, 2, \ldots\}$ and the rational numbers such as $\frac{1}{2}, \frac{2}{3}, \frac{5}{17}$. This we do by matching these numbers with certain points on a straight line and using the numbers as names for the points. When we do this however, we find that there are many points on the number line which are not matched with numbers. This leads us to introduce the negative numbers as names for certain of these points.

We next develop some properties of both positive and negative numbers by thinking about the properties of the points which picture them. We are laying the groundwork here for the later learning of how to operate with these numbers, that is how to add, subtract, multiply and divide them. In order for these rules of operation not to be just rote memorization, it is necessary for students to develop a deeper understanding than they have usually had in the past about the relationships between numbers and between numbers and their numerals. The object of this chapter is to begin to develop this understanding of both positive and negative numbers.

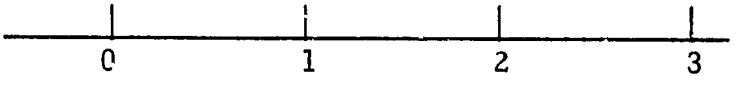
5-1 The Number Line

To use the points on a line as pictures for numbers, the students must realize that they have to start by picking two particular points which they are to call the pictures of 0 and 1. At this stage you should encourage them to call the point to the left 0 and the point to the right 1. The distance between the points does not matter, and if you let the students pick their own points for 0 and 1 it is unlikely that many will have them exactly the same distance apart.

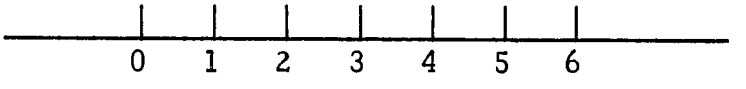
But once a student has picked his first two points then the points for the other numbers will be fixed, since the distance between the points marked 1 and 2, the distance between 2 and 3, etc. must be the same as the distance between the original points marked 0 and 1. Thus a number line which looked like this



would not be allowed but if Aya's line looked like this



and Modupe's like this



they would both be all right.

Once a student has understood how he can picture the whole numbers on a number line he should have little trouble picturing the rational numbers such as $\frac{1}{2}$ or $\frac{2}{3}$. He may not actually be able to divide the unit segment into 3 equal parts but he can easily imagine it being done by some process, so that a point can be found to represent $\frac{1}{3}$. By starting at zero and laying off two segments equal to the distance from 0 to $\frac{1}{3}$ he finds the point representing $\frac{2}{3}$.

In finding the point which represents $\frac{a}{b}$ we divided the unit segment into b equal parts. Clearly we can't divide a segment into 0 equal parts. There is therefore no point to represent $\frac{a}{0}$. This should strengthen our conviction that there is no number which can be represented as the fraction $\frac{a}{0}$. You see, we want a point to correspond to each number and also we want a number to correspond to each point. This is the reason we note here that there must be other kinds of numbers. There are all those points to the left of the point labelled 0 which do not as yet represent any numbers. Many of them will be used to represent the negative rational numbers which will be introduced

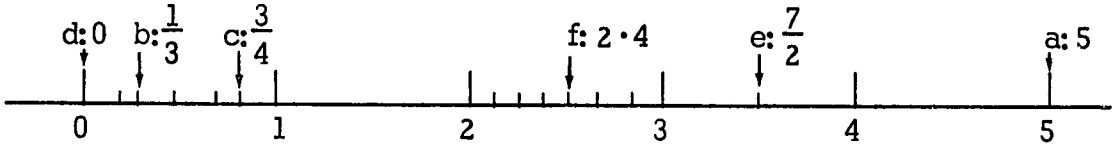
shortly. Will we have used up all the points on the number line when we have all the rationals matched up? The answer is "No" but we will have to wait until we learn about the real numbers before we can finish matching <u>all</u> the points on the number line with numbers.

You may have noticed that we are often using the words "rational number" instead of the word "fraction". There is a distinction between the meanings we want to be attached to these words. "Fraction" is used to signify a certain form in which a numeral may be written. Thus $\frac{\pi}{2}$ is a fraction whose numerator is the irrational number π and whose denominator is 2. 2·25 is one way of writing a numeral for a certain number. However, another numeral for this same number is the fraction $\frac{9}{4}$. The number represented by this fraction is called a "rational number" since one numeral for it is in the form of a fraction with numerator a whole number and denominator a natural number. Fractions such as $\frac{\sqrt{3}}{2}$ or $\frac{5}{\pi}$ are numerals for irrational numbers. A fraction may or may not represent a rational number. Different fractions such as $\frac{4}{6}$, $\frac{6}{9}$, and $\frac{\sqrt{8}}{\sqrt{18}}$ all represent the same number. This number is a rational number since <u>one</u> of its numerals is a fraction whose numerator is a whole number and whose denominator is a natural number. Be careful about these words when the distinction is important. But do not be pedantic. We all use the word "fraction" many times when we really mean "rational number represented by the fraction" but we should do so only when the meaning is clear from the way in which the word is used.

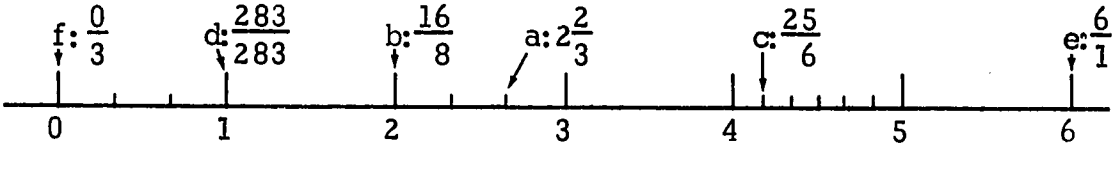
<u>Answers to Problems 5-1</u>

Student Text Page 73

1.



2.



3. In the symbol $\frac{a}{b}$, b represents the number of equal parts into which the unit segment is divided. Since it is impossible to divide a segment into zero parts, $\frac{a}{0}$ is meaningless.

4. (a) $6\frac{1}{2} = \frac{13}{2} = \frac{26}{4}$ (d) $4 \cdot 3 = \frac{43}{10} = \frac{430}{100}$ (g) $0 \cdot 83 = \frac{83}{100}$

(b) $1\frac{7}{8} = \frac{15}{8} = \frac{45}{24}$ (e) $9\frac{3}{11} = \frac{102}{11}$ (h) $8763 \cdot 1 = \frac{87631}{10}$

(c) $23 = \frac{23}{1} = \frac{46}{2}$ (f) $0 = \frac{0}{2} = \frac{0}{3}$ (i) $4\frac{6}{8} = \frac{38}{8} = \frac{19}{4}$

5-2 Equivalent Rational Numbers

In this section the student should see that the point representing $\frac{2}{3}$ and the point representing $\frac{4}{6}$ are the same point. If the matching of points and numbers is really one-to-one then these two different fractions should represent the same number. At this stage we want the students to accept the fact that $\frac{a}{b} = \frac{a \times m}{b \times m}$ on the basis of his seeing that when we represent these two fractions on the number line we arrive at the same point. Since we want each point to represent one and only one number, $\frac{a}{b}$ and $\frac{a \times m}{b \times m}$ must be different numerals for the same number.

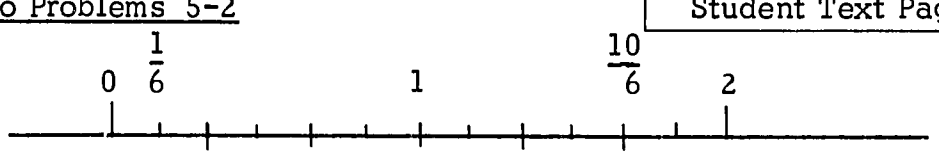
Note how use is made here of the GCF of numerator and denominator when we try to change a fraction to its lowest terms.

Note that we say two fractions are equivalent if the numbers they represent are equal. Remember that we always use "equal" to mean identical. $\frac{9}{12}$ and $\frac{6}{8}$ are equivalent fractions since they both are numerals for the number $\frac{3}{4}$.

Answers to Problems 5-2

Student Text Page 77

1. (a)



(b)

(c) The picture demonstrates that $\frac{10}{6} = \frac{5}{3}$.

2. (a) $\frac{3}{2} = \frac{3 \times 3}{2 \times 3} = \frac{9}{6}$

(b) $\frac{2}{5} = \frac{2 \times 7}{5 \times 7} = \frac{14}{35}$

(c) $\frac{2}{8} = \frac{2 \times 1}{2 \times 4} = \frac{1}{4}$

(d) $\frac{12}{10} = \frac{12 \times 3}{10 \times 3} = \frac{36}{30}$

(e) $\frac{5}{4} = \frac{5 \times 13}{4 \times 13} = \frac{65}{52}$

(f) $\frac{11}{9} = \frac{11 \times 15}{9 \times 15} = \frac{165}{135}$

3. (a) $\frac{12}{15} = \frac{4}{5}$ (b) $\frac{1}{6}$ (c) $\frac{3}{5}$ (d) $\frac{3}{2}$ (e) $\frac{3}{2}$ (f) $\frac{5}{1}$

4. (a) $\frac{15}{20}$ (b) $\frac{28}{8}$ (c) $\frac{500}{600}$ (d) $\frac{2}{7}$ (e) $\frac{36}{4}$ (f) $\frac{52}{169}$

5. $\frac{5}{2} = \frac{10}{4} = \frac{15}{6} = \frac{25}{10} = \frac{30}{12} = \frac{35}{14} = \frac{50}{20} = \frac{100}{40} = \frac{45}{18}$

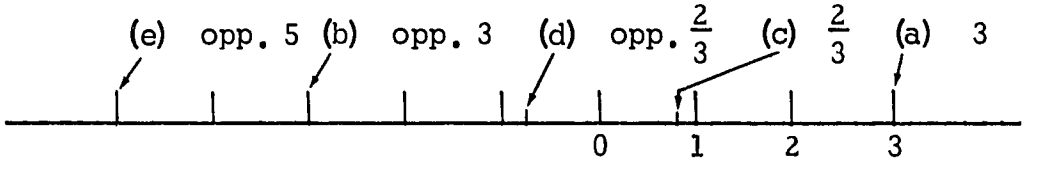
5-3 Negative Rational Numbers

If this is the first time your students have heard of negative numbers you will want to use many illustrations to get across the idea of "opposites". Several are given here. What number name shall we give to a point as far to the left of 0 as 1 is to the right of it? If 5 indicates a credit of 5 shillings how shall we record a debit of 5 shillings? John and Sam play several games of tic-tac-toe. If John wins the first three games and then loses five times, how shall we record the score at the end of each game and the total score for the afternoon? In each case we might say "opposite 1", "opposite 5" or "opposite 2" but a neater way is to say "negative 1", "negative 5" or "negative 2" and write "-1", "-5" or "-2". Students need to be reminded frequently that the symbol "-" is used in two different senses: as a symbol for "negative"; and as a symbol for "minus" or "subtract". You may find it convenient for a while to have your students write $^{-}2$ and $^{+}2$ for "negative two" and "positive

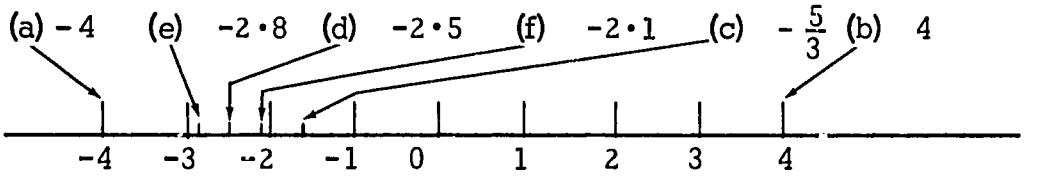
two" but "-2" for "subtract 2". Usually "positive two" is written just as "2". This may help in a later chapter where students have to learn to add and subtract negative numbers from positive or negative numbers.

Answers to Problems 5-3 | Student Text Page 80

1.



2.



3.

(a)	negative	(d)	negative
(b)	minus	(e)	minus
(c)	negative	(f)	minus

4.

(a)	$\frac{2}{5}$	positive rational number
(b)	-7	negative integer, negative rational number
(c)	-7·5	negative rational number
(d)	43	positive integer, positive rational number, whole number, counting number
(e)	0	zero, whole number
(f)	-58	negative integer, negative rational number
(g)	$-\frac{5}{9}$	negative rational number
(h)	$\frac{43}{5}$	positive rational number
(i)	$2\frac{7}{8}$	positive rational number
(j)	-700	negative integer, negative rational number
(k)	48·862	positive rational number
(l)	$-\frac{11}{2}$	negative rational number

5. Many answers are possible.

(a) The expenses of a store are 100 shillings per day. It sells items for 12 shillings each which cost it 8 shillings each. What are the profits per day if it sells 20 items? if it sells 32 items? 25 items? The profits would be -20 shillings, +28 shillings and 0 shillings in the three cases.

(b) If Mombasa is 4° south of the equator its latitude is -4°.
If Lusaka is 15° south of the equator its latitude is -15°.
If Lagos is 6° north of the equator its latitude is $+6^{\circ}$.

(c) John's score if he and Sam play many games of tic-tac-toe.

6. Each student's answer should be different.

5-4 Inequalities

In learning to think about numbers one of the first ideas a student gets is some idea of "more" or "less". One student is taller than another; he may weigh more. If John and Sam stand next to each other, it is quickly obvious which one is taller. But if each student is measured and John is 62 inches tall while Sam is 64 it may not be quite so obvious at first. But we know that 64 is larger than 62, or in symbols $64 > 62$, since $64 - 62$ is the positive number 2. How do we see on the number line that $64 > 62$? We look at the points which represent them on the number line. We put successive points to the right to represent successive numbers. We put 63 to the right of 62 and 64 to the right of 63. Thus 64 is to the right of 62. We know $64 > 62$ and 64 on the number line is to the right of 62. We do not know yet how to add and subtract negative numbers such as -7 and -5 but just as for positive numbers we can say which one of the two numbers is larger by seeing which point is to the right. It is -5 so we know $-5 > -7$.

Students should practise using the symbols $>$, $<$, $\geq$, $\leq$, $\neq$ for inequalities. They may find it difficult to see that $5 \geq 3$ and $5 \geq 5$ are both true statements. They should practise with these until they realize that what $5 \geq 5$ says is that "either $5 > 5$ or $5 = 5$" and, since one of the statements, namely $5 = 5$, is true, the whole sentence is true. Since every rational number is represented by a point on the number line we can always compare two rational numbers if by looking at the points which represent them we can see which one is to the right. But how would we compare $\frac{2}{3}$ and $\frac{5}{7}$? It might be hard to tell which point is to the right. The best thing to do is use the idea of equivalent fractions and express each number as a fraction whose denominator is the LCM of the two denominators. Thus $\frac{2}{3} = \frac{14}{21}$ and $\frac{5}{7}$ is $\frac{15}{21}$. Now it is easy to see that $\frac{15}{21}$ is to the right of $\frac{14}{21}$ so $\frac{5}{7} > \frac{2}{3}$.

The two properties of order given at the end of the chapter follow easily from looking at the points representing the numbers on the number line. They are important, and we will use them many times.

Answers to Problems 5-4 | Student Text Page 87

1.	(a)	T	(g)	T	(m)	F
	(b)	F	(h)	T	(n)	T
	(c)	T	(i)	F	(o)	T
	(d)	T	(j)	F	(p)	T
	(e)	T	(k)	T	(q)	T
	(f)	T	(l)	F	(r)	F
2.	(a)	$<$	(f)	$=$		
	(b)	$<$	(g)	$<$		
	(c)	$>$	(h)	$>$		
	(d)	$<$	(i)	$=$		
	(e)	$>$	(j)	$<$		

3. John is taller than James by the second property of order.

4. Mary's age is the same as Margaret's. One and only one of the three $a > b$, $a = b$ and $a < b$ must be true. If $a > b$ and $a < b$ are both false then $a = b$ must be true.

5. Many other examples are possible. For instance:

   If $a \geq b$ and $b = c$ then $a \geq c$.

   If $a \geq b$, $b > c$, $c \geq d$, then $a > d$.

   If Isaac has more money than John but less than Sam who has more money, John or Sam?

   If it is warmer in Lagos than in Accra, if it is colder in Cairo than in Entebbe and if Entebbe and Accra have the same temperature, how would you compare Lagos and Entebbe? How about Cairo and Lagos?

CHAPTER 6

CLOCK ARITHMETIC

In arithmetic and algebra we work with numbers, learning how to combine them by the operations of addition and multiplication. We study the properties of these operations separately and in combination, discovering their so-called commutative and associative properties as well as the existence of identities, inverses, order properties, etc. But the set of rational numbers is an infinite set and students may find it hard to study these properties in such a large set. Could we study them in a smaller set?

The main purpose of this chapter is to consider a very small set of numbers. In fact, we will be concerned mainly with a set A which has only five elements: $A = \{0, 1, 2, 3, 4\}$. We define a special kind of addition and multiplication which we call "clock addition" and "clock multiplication". We use new symbols, $\oplus$ and $\otimes$, for these operations so that students will not confuse them with the ordinary operations of arithmetic. Having defined these operations we can study their properties easily because there are only a finite number of cases to investigate. We can check to see that $a \oplus b = b \oplus a$ for every a and b because there are only five possibilities for a and five for b.

The whole chapter should be fun for the student because he can think of it as a game played on a clock, his own special kind of a clock, of which he can make a copy and on which he can experiment.

6-1 Introduction

In this section we simply introduce the special clock face with only five numbers represented. You should have each student draw such a clock face and you yourself should draw a big one on the chalkboard, so that they can check their own results with those on the board. If some students ask if they can make clocks with 4 or 7 or 12 numbers on the faces the answer is "Yes, of course, but it will be simpler if we all start with the same kind of a clock."

Previous Page Blank

6-2 Addition

Have the students determine $2 \oplus 2$ and $2 \oplus 3$ by starting at 2 and counting 2 or 3 places around the face. They should see what is happening very quickly. Then draw the addition table on the board and fill it in as the students give you the answers.

$\oplus$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

The right position for $2 \oplus 3$ is the square where the row labelled 2 crosses the column labelled 3. We put 0 there since this is the label of the point we landed on when we moved 3 places from the point labelled 2.

The first property we notice is that of closure. Note of course that if we considered the set $\{0, 1, 2, 3, 4\}$ and ordinary addition we would not have closure, since $2 + 3 = 5$ and 5 does not belong to the given set.

The second property, that of commutativity, can be easily checked by noticing that the addition table is symmetric with respect to the diagonal.

$\oplus$					
				a	
		a			

There will be 25 cases to check if a student checks every single one.

To check the associative property completely would require 125 cases and you certainly should not ask anyone to do them all. If each student checks 3 or 4 different cases they should be satisfied. The reason there are 125 cases is that in the equation

$$(a \oplus b) \oplus c = a \oplus (b \oplus c)$$

there are 5 choices for a, 5 for b and 5 for c. It might be interesting to ask your class how many cases they think there might be before you tell them how many there are.

Answers to Problems 6-2A

Student Text Page 92

1. (a) $3 \oplus (4 \oplus 4) = 3 \oplus 3 = 1$; $(3 \oplus 4) \oplus 4 = 2 \oplus 4 = 1$

   (b) $(2 \oplus 3) \oplus 2 = 0 \oplus 2 = 2$; $2 \oplus (2 \oplus 3) = 2 \oplus 0 = 2$; $(2 \oplus 2) \oplus 3 = 4 \oplus 3 = 2$

   (c) $(4 \oplus 2) \oplus (3 \oplus 1) = 1 \oplus 4 = 0$; $(3 \oplus 2) \oplus (1 \oplus 4) = 0 \oplus 0 = 0$; $1 \oplus (3 \oplus 4) \oplus 2 = 1 \oplus 2 \oplus 2 = 3 \oplus 2 = 0$

   (d) $1 \oplus 1 \oplus 1 \oplus 1 = 4$; $4 \oplus 4 \oplus 4 = 2$; $2 \oplus 2 \oplus 2 = 1$; $3 \oplus 3 \oplus 3 \oplus 3 = 2$

   (e) $0 \oplus 1 \oplus 2 \oplus 3 \oplus 4 = 0$

   (f) $0 \oplus (1 \oplus 4) \oplus (2 \oplus 3) = 0 \oplus 0 \oplus 0 = 0$

2. (a) Testing each number on our clock we find

   $0 \oplus 1 = 1$ is true.
   but $1 \oplus 1 = 1$ is false since $1 \oplus 1 = 2$
   and $2 \oplus 1 = 1$ is false since $2 \oplus 1 = 3$
   and $3 \oplus 1 = 1$ is false since $3 \oplus 1 = 4$
   and $4 \oplus 1 = 1$ is false since $4 \oplus 1 = 0$.

   Thus the only number which makes $x \oplus 1 = 1$ a true statement is 0.

In each equation the pupils should test all five of the numbers 0, 1, 2, 3, 4. They will find that only one number makes the equation into a true statement. That number is

(b) 2	(e) 3	(h) 3	(k) 3	(n) 4
(c) 4	(f) 0	(i) 0	(l) 1	(o) 1
(d) 2	(g) 2	(j) 0	(m) 2	(p) 0

3. (a) 3 (f) 1

(b) 2 (g) 0

(c) 1 (h) 4

(d) 0 (i) 3

(e) 4 (j) 2

The set of numbers is closed under subtraction because subtraction is defined in terms of addition. Also we could set up a subtraction table. It would look like this.

$\ominus$	0	1	2	3	4
0	0	4	3	2	1
1	1	0	4	3	2
2	2	1	0	4	3
3	3	2	1	0	4
4	4	3	2	1	0

We see immediately that subtraction is not commutative since the table is not symmetric with respect to the diagonal. Also we can see $3 \ominus 1 = 2$ but $1 \ominus 3 = 3$.

By actual checking in each possible case we find that $0 \oplus a = a$ for every number in the set. Then since $0 \oplus a = a \oplus 0$ we do have enough evidence to support the statement that 0 is an identity element for clock addition.

Again by checking each case we find that there always exists some number which, added to a given number, gives 0. We can also see this by noting that there is a 0 in each row of the $\oplus$ addition table. The number b such that $a \oplus b = 0$ is called the additive inverse or opposite of a for clock arithmetic. Students may find it difficult to think of 3 as the <u>opposite</u> of 2 so soon after they have learned that -2 is the opposite of 2. But you should remind them that clock arithmetic is different from ordinary arithmetic and the opposite of 2 in any arithmetic is that number which added to 2 gives 0. We have not yet shown that in ordinary arithmetic $2 + (-2) = 0$, but it is true, while in our clock arithmetic $2 \oplus 3 = 0$, as the table shows.

In Problems 6-2B we ask the students to set up clocks with different numbers on their faces. They should do these problems by actually making clock faces and counting out the results.

<u>Answers to Problems 6-2B</u> | Student Text Page 94

1.	(a) 7	(d) 3	(g) 1
	(b) 12	(e) 2	(h) 11
	(c) 1	(f) 10	(i) 12

The identity element in this clock addition is 12. This is true since if we add 12 hours to any time on the clock face we get back to the same position of the hands. Thus 12 hours added to 3 o'clock in the afternoon gives us 3 in the morning. $12 \oplus 3 = 3$.

There is an inverse for each clock number.

$1 \oplus 11 = 12$, so 11 is the inverse of 1 and 1 is the inverse of 11.

$2 \oplus 10 = 12$, so 10 is the inverse of 2 and 2 is the inverse of 10.

$3 \oplus 9 = 12$, so 3 and 9 are inverses of each other.

$4 \oplus 8 = 12$, so 4 and 8 are inverses of each other.

$5 \oplus 7 = 12$, so 5 and 7 are inverses of each other.

$6 \oplus 6 = 12$, and 6 is its own inverse.

$12 \oplus 12 = 12$, and so 12 is also its own inverse.

2. (a)

$\oplus$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

(b)

$\oplus$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

(c)

$\oplus$	0	1
0	0	1
1	1	0

It is easy to check to see that all the properties we have been discussing are satisfied.

6-3 Multiplication

Have each student make a multiplication table. Then you should fill out one on the chalkboard as they give you the entries.

$\otimes$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

The property of closure holds since the only numbers in the table are those in the set $\{0, 1, 2, 3, 4\}$. We can check commutativity exhaustively (25 cases) or we can see that it is true from the symmetry of the table with respect to the diagonal. The associative property also holds, but once again it would take 125 cases to show it completely, so we only ask each student to check a few cases.

Answers to Problems 6-3A

Student Text Page 98

1. (a) $2 \otimes 1 = 1 \otimes 2 = 2$ (d) $4 \otimes 3 = 3 \otimes 4 = 2$

   (b) $3 \otimes 0 = 0 \otimes 3 = 0$ (e) $0 \otimes 2 = 2 \otimes 0 = 0$

   (c) $1 \otimes 0 = 0 \otimes 1 = 0$ (f) $0 \otimes 0 = 0$

2. $2 \otimes \left[2 \otimes (2 \otimes 2)\right] = 2 \times \left[2 \otimes 4\right] = 2 \otimes 3 = 1$

3. $(4 \otimes 2) \otimes 3 = 3 \otimes 3 = 4$; $4 \otimes (3 \otimes 2) = 4 \otimes 1 = 4$;
   $(3 \otimes 4) \otimes 2 = 2 \otimes 2 = 4$

4. $3 \otimes 1 \otimes 3 = 4$; $4 \otimes 0 \otimes 2 = 0$; $4 \otimes 4 \otimes 4 = 4$

5. (a) $3 \otimes (4 \oplus 1) = 3 \otimes 0 = 0$ (e) $(3 \otimes 4) \oplus (3 \otimes 1) = 2 \oplus 3 = 0$

   (b) $3 \otimes (2 \oplus 3) = 3 \otimes 0 = 0$ (f) $(3 \otimes 2) \oplus (3 \otimes 3) = 1 \oplus 4 = 0$

   (c) $4 \otimes (3 \oplus 0) = 4 \otimes 3 = 2$ (g) $(4 \otimes 3) \oplus (4 \otimes 0) = 2 \oplus 0 = 2$

   (d) $1 \otimes (1 \oplus 1) = 1 \otimes 2 = 2$ (h) $(1 \otimes 1) \oplus (1 \otimes 1) = 1 \oplus 1 = 2$

In Problem 5 above the student is asked to check a few cases of the distributive property. This also is a property of ordinary multiplication and addition in the set of all rational numbers.

Answers to Problems 6-3B

Student Text Page 100

1. (a) Testing each of the numbers 0, 1, 2, 3, 4 in turn we find

$1 \otimes 0 = 3$ is false, since $1 \otimes 0 = 0$

$1 \otimes 1 = 3$ is false, since $1 \otimes 1 = 1$

$1 \otimes 2 = 3$ is false, since $1 \otimes 2 = 2$

$1 \otimes 3 = 3$ is true

$1 \otimes 4 = 3$ is false since $1 \otimes 4 = 4$

Thus, the only number to make this equation a true statement is 3. Testing in each of the equations we find that for each equation except (h) there is only one number that makes the equation a true statement. This number is:

(b) 4	(d) 2	(f) 1	(h) See below
(c) 1	(e) 1	(g) 1	(i) 0

For (h), any number in the set makes the equation a true statement, since $0 \otimes a = 0$ for any number a.

2. (a) There is no answer, since $0 \otimes a = 0$ for any a.

(b) 1	(d) 2
(c) 3	(e) 4

3. (a) $4 \otimes (1 \oplus 3) = (4 \otimes 1) \oplus (4 \otimes 3)$

$4 \otimes 4 = 1 \qquad 4 \oplus 2 = 1$

(b) $2 \otimes (4 \oplus 4) = (2 \otimes 4) \oplus (2 \otimes 4)$

$2 \otimes 3 = 1 \qquad 3 \oplus 3 = 1$

(c) $1 \otimes (3 \oplus 2) = (1 \otimes 3) \oplus (1 \otimes 2)$

$1 \otimes 0 = 0 \qquad 3 \oplus 2 = 0$

(d) $3 \otimes (1 \oplus 1) = (3 \otimes 1) \oplus (3 \otimes 1)$

$3 \otimes 2 = 1 \qquad 3 \oplus 3 = 1$

4. (a) $(4 \otimes 3) \oplus (4 \otimes 4) = 4 \otimes (3 \oplus 4)$

$2 \oplus 1 = 3 \qquad 4 \otimes 2 = 3$

(b) $(3 \otimes 2) \oplus (3 \otimes 0) = 3 \otimes (2 \oplus 0)$

$1 \oplus 0 = 1 \qquad 3 \otimes 2 = 1$

(c) $(3 \otimes 3) \oplus (3 \otimes 3) = 3 \otimes (3 \oplus 3)$

$4 \oplus 4 = 3 \qquad 3 \otimes 1 = 3$

(d) $(2 \otimes 3) \oplus (2 \otimes 4) = 2 \otimes (3 \oplus 4)$

$1 \oplus 3 = 4 \qquad 2 \otimes 2 = 4$

5. (a) $(3 \otimes 2) \oplus (3 \otimes a) = 3 \otimes (2 \oplus a)$

(b) $(y \otimes 4) \oplus (y \otimes 2) = y \otimes (4 \oplus 2) = y \otimes 1$

(c) $2 \otimes (1 \oplus a) = (2 \otimes 1) \oplus (2 \otimes a) = 2 \oplus (2 \otimes a)$

(d) $4 \otimes (a \oplus b) = (4 \otimes a) \oplus (4 \otimes b)$

(e) $(3 \otimes x) \oplus (3 \otimes x) = 3 \otimes (x \oplus x)$

(f) $(a \otimes 3) \oplus (a \otimes 2) = a \otimes (3 \oplus 2) = a \otimes 0$

6. (a) $(x \otimes 3) \oplus (x \otimes 3) = 2$

$x \otimes (3 \oplus 3) = 2$

$x \otimes 1 = 2$

$x = 2$ since $2 \otimes 1 = 2$

(b) $(x \otimes 2) \oplus (x \otimes 1) = 4$

$x \otimes (2 \oplus 1) = 4$

$x \otimes 3 = 4$

$x = 3$ since $3 \otimes 3 = 4$

(c) $(x \otimes 3) \oplus (x \otimes 2) = 2$

$x \otimes (3 \oplus 2) = 2$

$x \otimes 0 = 2$

There is no solution since $x \otimes 0 = 0$ for any x.

(d) $(x \otimes 4) \oplus (x \otimes 4) = 0$

$x \otimes (4 \oplus 4) = 0$

$x \otimes 3 = 0$

$x = 0$ since $0 \otimes 3 = 0$

(e) $(x \otimes 4) \oplus (x \otimes 1) = 0$

$x \otimes (4 \oplus 1) = 0$

$x \otimes 0 = 0$ Any number is a solution of this equation since $x \otimes 0 = 0$ for any x.

7. (a) $1 \div 2 = 3$, since it is true that $1 = 2 \otimes 3$.

(b) $2 \div 3 = 4$, since it is true that $2 = 3 \otimes 4$.

(c) $1 \div 3 = 2$, since it is true that $1 = 3 \otimes 2$.

(d) $3 \div 4 = 2$, since $3 = 4 \otimes 2$ is true.

(e) $4 \div 4 = 1$, since $4 = 4 \otimes 1$ is true.

(f) $1 \div 4 = 4$, since $1 = 4 \otimes 4$ is true.

(g) $4 \div 3 = 3$, since $4 = 3 \otimes 3$ is true.

(h) $3 \div 2 = 4$, since $3 = 2 \otimes 4$.

(i) $1 \div 1 = 1$, since $1 = 1 \otimes 1$.

Clock division is not in general commutative since

$$3 \div 4 = 2 \text{ but } 4 \div 3 = 3.$$

In ordinary arithmetic the number 1 is an identity for multiplication, since $1 \times a = a \times 1 = a$ for any number a. In our clock arithmetic 1 is an identity for $\otimes$, since $1 \otimes a = a \otimes 1 = a$ for any number a in the set $\{0, 1, 2, 3, 4\}$. In ordinary arithmetic if we consider any number a, except 0, there always exists another number which when multiplied by a gives 1. For instance if $a = \frac{2}{3}$ we can multiply it by $\frac{3}{2}$ and get 1. Such a number is called the reciprocal. That the same situation holds in our clock arithmetic will seem surprising to many students. But looking at the multiplication table we see a 1 in every row except the row labelled 0. So $3 \otimes 2 = 1$ and 2 is the reciprocal or multiplicative inverse of 3.

In Problems 6-3C the Challenge Problems 2 and 3 should not be assigned to any except the best students. They are intended to show operations which have none of the properties we have been discussing in this chapter. You might want to discuss these problems briefly in your class.

Probably if you discuss the fact that subtraction and division are not commutative or associative in either ordinary arithmetic or in clock arithmetic your students will realize that the properties we have been discussing really are important for addition and multiplication.

Answers to Problems 6-3C

Student Text Page 103

1. (a) Yes

(b) Yes

(c) Yes in the set $\{0, 1, 2\}$, since in this arithmetic $2 \otimes 2 = 1$.

No in the set $\{0, 1, 2, 3\}$, since in this arithmetic $2 \otimes 0 = 0$, $2 \otimes 1 = 2$, $2 \otimes 2 = 0$ and $2 \otimes 3 = 2$ and there is no number such that $2 \otimes x = 1$.

(d) Yes in the first case, since $1 \otimes 1 = 1$ and $2 \otimes 2 = 1$.
No in the second case, since in (c) we saw there is no reciprocal for 2.

(e) Set up the multiplication table and check each case.

$\otimes$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

$\otimes$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

2. Challenge Problem

$\circ$	0	1	2	3	4
0	0	1	2	3	4
1	2	3	4	5	6
2	4	5	6	7	8
3	6	7	8	9	10
4	8	9	10	11	12

(a) The operation $\circ$ is <u>not</u> commutative since $2 \circ 1 = 5$ while $1 \circ 2 = 4$.

(b) The set $\{0, 1, 2, 3, 4\}$ is <u>not</u> closed with respect to $\circ$ since $1 \circ 4 = 6$ and 6 is not in the set.

(c) In the set of all whole numbers $\{0, 1, 2, 3, \ldots\}$ the operation is closed, since $2 \times a$ is always a whole number and therefore $(2 \times a) + b$ is also a whole number.

(d) The operation $\circ$ is <u>not</u> associative since by the table

$$(1 \circ 1) \circ 1 = 3 \circ 1 = 7$$

while $1 \circ (1 \circ 1) = 1 \circ 3 = 5$. Although one such example is enough, others are easily found.

Thus $(1 \circ 2) \circ 3 = 4 \circ 3 = 11$,

while $1 \circ (2 \circ 3) = 1 \circ 7$, which is not even defined.

(e) There is no identity element for $\circ$ in this set. Although $0 \circ x = x$ for each number in the set, $1 \circ 0 = 2$ and $2 \circ 0 = 4$, so $x \circ 0 = x$ is not always true. For an element a in the set to be an identity for $\circ$ it must be true that $a \circ x = x \circ a = x$ for every x in the set.

(f) No. If there is no identity it is impossible to have an inverse.

(g) This operation differs in every respect from clock addition. Not one of the properties, closure, commutativity, associativity, identity, or inverse, holds.

3. Challenge Problem

$$a * b = (3 \otimes a) \oplus b$$

*	0	1	2	3	4
0	0	1	2	3	4
1	3	4	0	1	2
2	1	2	3	4	0
3	4	0	1	2	3
4	2	3	4	0	1

(a) * is <u>not</u> commutative since $2*1 = 2$, but $1*2 = 0$.

(b) * is closed in this set, since, for every pair of numbers a and b in $\{0, 1, 2, 3, 4\}$, $a*b$ is a number in the set.

(c) * is closed in the whole numbers for the same reason.

(d) * is not associative, since, for example, $(1*2) * 3 = 0*3 = 3$, while $1* (2*3) = 1*4 = 2$.

(e) * has no identity element, since $0*x = x$ for every x, but $2*0 = 1$, not 2.

(f) No. Without an identity, inverses are impossible.

(g) The only resemblance between this operation and clock addition is that both are closed in the set $\{0, 1, 2, 3, 4\}$.

6-4 Summary of Basic Properties

As we said at the beginning of this chapter the main purpose has been to provide a small finite arithmetic which displays these eleven properties and which makes it possible to check the properties easily. Any mathematical

system composed of a set of elements, finite or infinite in number, and two operations that have these eleven properties is called a "field". The set $\{0, 1, 2, 3, 4\}$, with $\oplus$ and $\otimes$ defined as we have defined them, is a field. The set of rational numbers with the operations of ordinary addition and multiplication is a field. Fields are of great importance in more advanced mathematics but for the present all we wish is that the student become acquainted with the field properties. He will meet them again and study them in greater detail later on.

CHAPTER 7

THE ARITHMETIC OF RATIONAL NUMBERS

Introductory Remarks

There are many ways to handle rational numbers. We have chosen an informal and intuitive treatment. We build on the student's previous knowledge of adding and multiplying fractions in arithmetic to suggest definitions of addition and multiplication for all rational numbers. The motivation for these definitions comes from various sources - observation of what happens in a number of instances, the desire to keep the commutative property, the interpretation of addition on the number line, and the interpretation of multiplication as the way to find the area of a rectangle. The teacher should make clear that these are not proofs of the "rules" for adding and multiplying but only suggestions of what agreements it seems desirable to make about the way in which rational numbers should be added or multiplied.

We also try to make it plausible that when we do multiply in accordance with the usual methods the rational numbers have the "field" properties listed in 7-9. While all of these properties can be proved from the definitions

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

and

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

there is no intention of doing so here. We observe that the properties hold in some specific cases and then <u>assume</u> that these properties are true for all rational numbers. We have given some geometrical arguments and other lines of thought which we believe will make the conclusions rather convincing.

It is felt that at this stage it is too early to introduce the student to strict formal definitions and proofs. However, we have given some arguments in this Teacher's Guide which would help to satisfy a bright student who asks for more in the way of proof.

7-1 Addition of Rational Numbers

We begin with the addition of positive rational numbers (fractions). We believe that the student will readily accept the principle that

$$\frac{a}{c} + \frac{b}{c} = \frac{a + b}{c}$$

This amounts to the fact that if a unit is divided into c equal parts, a of these parts plus b of these parts is, of course, (a + b) of these parts. This may be made clear by choosing particular whole number values for a, b and c. Thus, 2 sevenths + 3 sevenths = (2 + 3) sevenths. The use of the word "seventh" for the numeral $\frac{1}{7}$ may help to bring out the principle.

To add fractions with <u>different</u> denominators we replace the given fractions by equivalent fractions with the same denominator and then proceed as in the previous special case. Here we rely on the method of Chapter 4 for finding the LCM of the denominators.

There is of course an automatic method of adding fractions by using the scheme

$$\frac{a}{b} + \frac{c}{d} = \frac{(a \times d) + (b \times c)}{b \times d}$$

This will always work, but it often requires an unnecessary amount of arithmetic and often gives a result with a large numerator and denominator. Ordinarily we should like to simplify this result. For example, by this method

$$\frac{5}{12} + \frac{7}{24} = \frac{(5 \times 24) + (7 \times 12)}{12 \times 24} = \frac{120 + 84}{288} = \frac{204}{288},$$

which finally reduces to $\frac{17}{24}$. It requires much less effort to write

$$\frac{5}{12} + \frac{7}{24} = \frac{10}{24} + \frac{7}{24} = \frac{17}{24}$$

It is also better if the student uses his understanding without relying too much on rules which he may be tempted to memorize.

In the second part of this section, we turn to the addition of rational numbers which are not necessarily positive. We illustrate the ideas by first adding integers on the number line, using arrows pointing to the right for positive integers and arrows pointing to the left for negative integers. It is then easy to apply the same ideas with rational numbers. It is important for the student to picture clearly what is going on. Logically we can reduce everything to the addition of integers (positive, negative and 0) by choosing a new unit on the number line which is $\frac{1}{\text{LCD}}$ of the original one.

Answers to Problems 7-1A

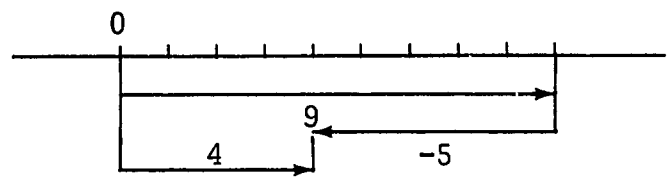
Student Text Page 110

1.	$\frac{13}{15}$	5.	$\frac{5}{12}$	9.	$\frac{13}{6}$
2.	$\frac{17}{18}$	6.	$\frac{21}{20}$	10.	$\frac{15}{8}$
3.	$\frac{41}{14}$	7.	$\frac{7}{6}$	11.	$\frac{139}{42}$
4.	$\frac{38}{9}$	8.	$\frac{31}{6}$	12.	$\frac{9}{8}$

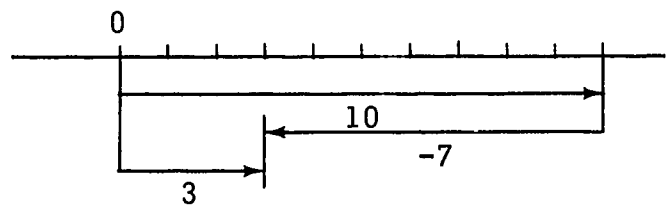
Answers to Problems 7-1B

Student Text Page 114

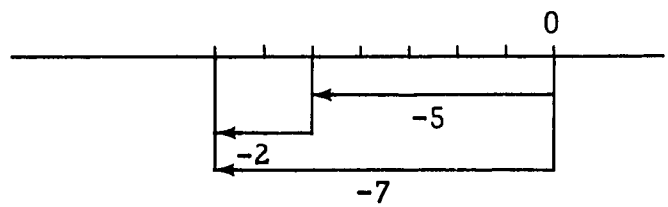
1. (a) 4



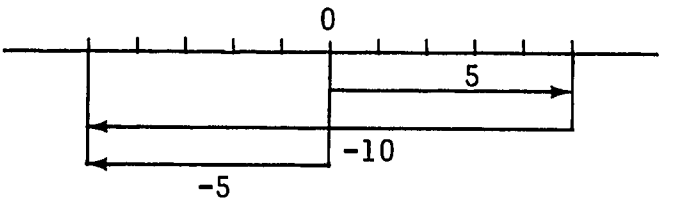
(b) 3



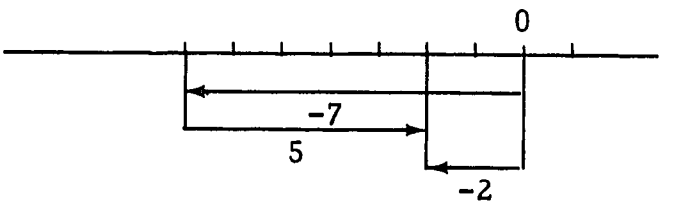
(c) -7



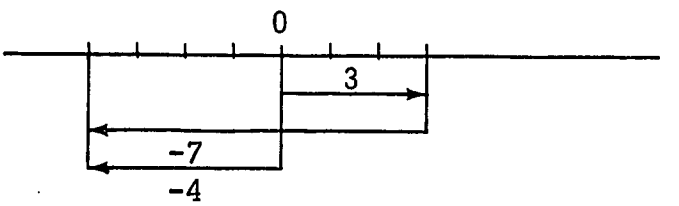
1. (d) -5



(e) -2



(f) -4



2. (a) 9
   (b) -12
   (c) -4
   (d) -5
   (e) -25
   (f) 0

3. (a) $-\frac{1}{6}$
   (b) $-\frac{5}{4}$
   (c) $\frac{7}{15}$
   (d) $\frac{1}{2}$
   (e) 0
   (f) $-\frac{5}{12}$

4. (a) Use positive numbers to represent profits and negative numbers to represent losses.

   (b) 5000 shs. loss

   (c) 1000 shs. profit

   (d) 12,000 shs. loss

5. Call his still water speed 4 miles per hour (positive) and the speed of the current -2 miles per hour. His actual speed upstream is 2 miles per hour.

6. (a) 13,000 feet + 5,000 feet + (-3,000 feet)

(b) 15,000 feet.

7-2 Properties of Addition of Rational Numbers

The purpose of this section is to convince the student of the four basic properties of addition summarized at the end of the section. The method is to appeal to examples and to the picture of addition on the number line. No attempt has been made to give a strict proof. If the better students require more convincing arguments something of the following sort can be given.

To establish the commutative property for addition of positive fractions we can assume that the fractions have been replaced by equivalent ones with a common denominator d. Suppose that they are $\frac{p}{d}$ and $\frac{q}{d}$. Let the unit on the number line be subdivided into d equal parts. To add p of these parts to q of these parts will give the same result as if we add q parts to p parts (we know that the commutative property of addition is true for the whole numbers). This means that

$$\frac{p}{d} + \frac{q}{d} = \frac{q}{d} + \frac{p}{d}$$

Similarly, since the whole numbers have the associative property of addition

$$\frac{p}{d} + \left(\frac{q}{d} + \frac{r}{d}\right) = \left(\frac{p}{d} + \frac{q}{d}\right) + \frac{r}{d}$$

because p parts + (q + r) parts = (p + q) parts + r parts.

Answers to Problems 7-2 | Student Text Page 120

We shall use the abbreviations AA for the associative property of addition and CA for the commutative property of addition.

1. $a = h$ AA

   $b = k$ CA

   $c = i$ AA

   $d = l$ CA

   $e = m$ AA

   $f = j$ AA

   $g = n$ CA

2. (a) 6

   (b) -4

   (c) -1

   (d) 4

   (e) 1

   (f) 8

   (g) 3

3. (a) $5 + (6 + 7) = (6 + 7) + 5$ AA

   $= (7 + 6) + 5$ CA

   (b) $\left[(-8) + 3\right] + (-4) = (-8) + \left[3 + (-4)\right]$ AA

   $= (-8) + \left[(-4) + 3\right]$ CA

   (c) $\frac{3}{5} + \left[(-\frac{2}{5}) + \frac{1}{5}\right] = \left[\frac{3}{5} + (-\frac{2}{5})\right] + \frac{1}{5}$ AA

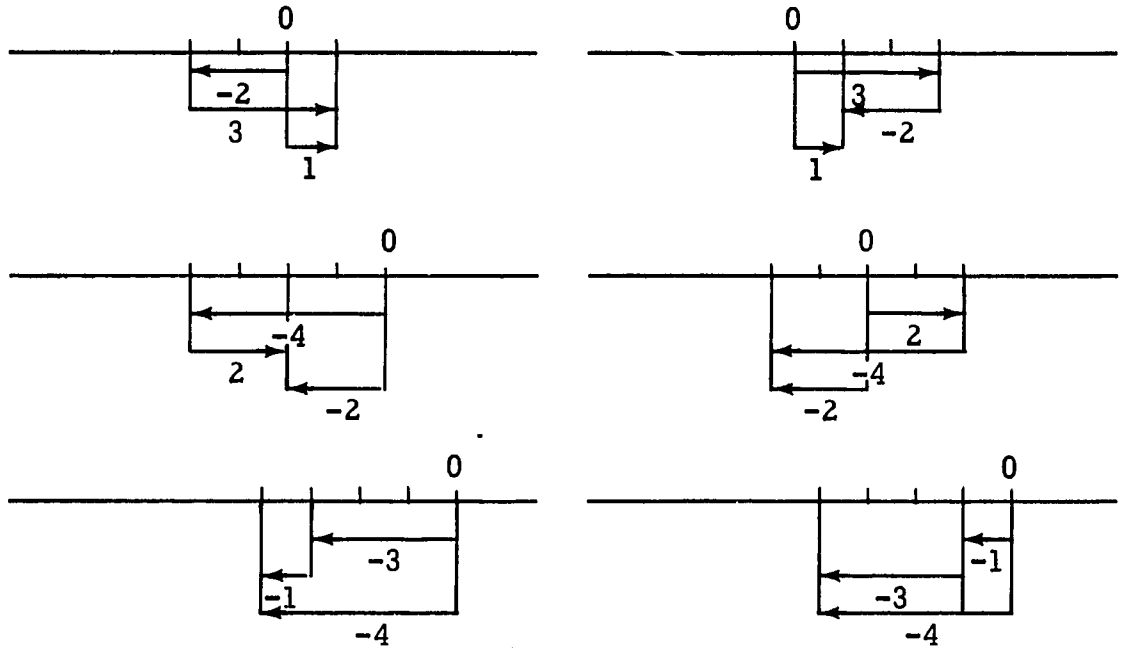
   $= \frac{1}{5} + \left[\frac{3}{5} + (-\frac{2}{5})\right]$ CA

   (d) $\left[(-\frac{5}{8}) + (-\frac{3}{5})\right] + (-2) = \left[(-\frac{3}{5}) + (-\frac{5}{8})\right] + (-2)$ CA

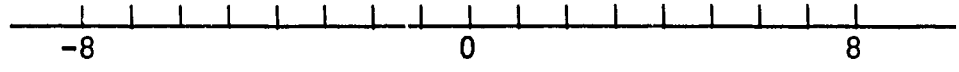
   $= (-\frac{3}{5}) + \left[(-\frac{5}{8}) + (-2)\right]$ AA

   $= (-\frac{3}{5}) + \left[(-2) + (-\frac{5}{8})\right]$ CA

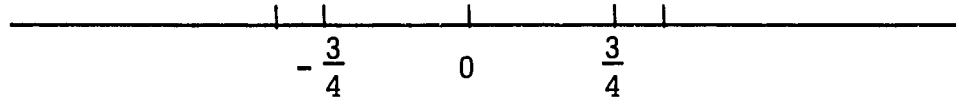
4.



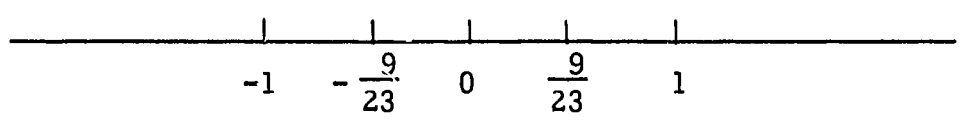
5. (a) -8



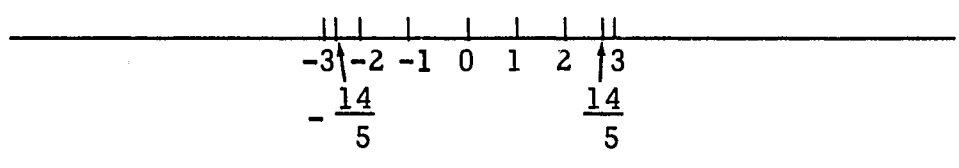
(b) $\frac{3}{4}$



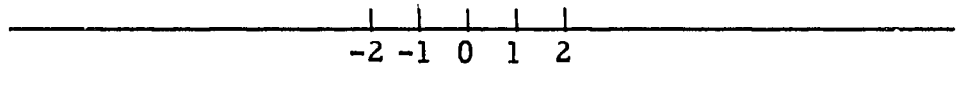
(c) $-\frac{9}{23}$



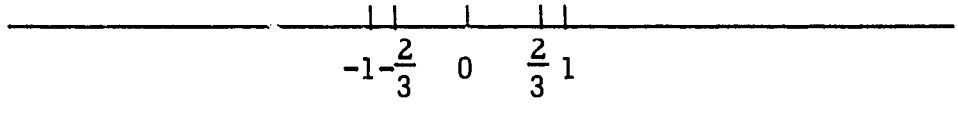
(d) $\frac{14}{5}$



(e) 0



(f) $-\frac{2}{3}$



6.	(a) 0	7.	Property of zero
	(b) $-\frac{5}{9}$		Additive inverse
	(c) $-\frac{5}{11}$		Property of zero
	(d) 94		Additive inverse
	(e) 0		Additive inverse
	(f) 0		Property of zero
	(g) 16.8		Additive inverse

7-3 Applications of the Properties of Addition

Application 1. We give the reasons for the steps indicated by "Why?" in the example, with the abbreviations AA and CA.

$$(20 + 4) + (30 + 7)$$

$$= \left[(20 + 4) + 30\right] + 7 \qquad \text{AA}$$

$$= \left[20 + (4 + 30)\right] + 7 \qquad \text{AA}$$

$$= \left[20 + (30 + 4)\right] + 7 \qquad \text{CA}$$

$$= \left[(20 + 30) + 4\right] + 7 \qquad \text{AA}$$

$$= (20 + 30) + (4 + 7) \qquad \text{AA}$$

$$50 + (10 + 1) = (50 + 10) + 1 \qquad \text{AA}$$

Application 3.

$$(2 + \tfrac{3}{4}) + (7 + \tfrac{5}{8})$$

$$= (2 + 7) + (\tfrac{3}{4} + \tfrac{5}{8})$$ by four applications of AA and one of CA as in Application 1.

The next step replaces $\frac{3}{4}$ by the equivalent fraction $\frac{6}{8}$.

$$9 + (1 + \tfrac{3}{8}) = (9 + 1) + \tfrac{3}{8}$$ by AA.

Application 4.

$$7 + \left[(-4) + (-3)\right] = (4 + 3) + \left[(-4) + (-3)\right] \quad \text{because } -7 = (-4) + (-3)$$

$$= \left[4 + (-4)\right] + \left[3 + (-3)\right] \quad \text{AA and CA}$$

$$= 0 + 0 \quad \text{Additive inverses}$$

$$= 0 \quad \text{Property of zero}$$

$$(1 + 3) + (-3)$$

$$= 1 + \left[3 + (-3)\right] \quad \text{AA}$$

$$= 1 + 0 \quad \text{Additive inverses}$$

$$= 1 \quad \text{Property of zero}$$

$$4 + (-7) = 4 + \left[(-4) + (-3)\right] \quad \text{because } -7 = (-4) + (-3)$$

$$= \left[4 + (-4)\right] + (-3) \quad \text{AA}$$

$$= 0 + (-3) \quad \text{Additive inverses}$$

$$= 3 \quad \text{Property of zero}$$

Application 5. $-(-a) = a$

Answers to Problems 7-3

Student Text Page 126

1. $29 + 34 = (20 + 9) + (30 + 4)$

$= 20 + [9 + (30 + 4)]$

$= 20 + [(30 + 4) + 9]$

$= 20 + [30 + (4 + 9)]$

$= (20 + 30) + (4 + 9)$

$= 50 + 13$

$= 50 + (10 + 3)$

$= (50 + 10) + 3$

$= 60 + 3$

$= 63$

$125 + 397 = (100 + 25) + (300 + 97)$

$= 100 + [25 + (300 + 97)]$

$= 100 + [(300 + 97) + 25]$

$= 100 + [300 + (97 + 25)]$

$= (100 + 300) + (97 + 25)$

$= 400 + [97 + (3 + 22)]$

$= 400 + [(97 + 3) + 22]$

$= 400 + (100 + 22)$

$= (400 + 100) + 22$

$= 500 + 22$

$= 522$

2. (a) $164 + (35 + 65) = 164 + 100 = 264$

(b) $(7 + 3) + (9 + 1) + (6 + 4) + 18 = 10 + 10 + 10 + 18 = 48$

(c) $(9{,}998 + 1{,}002) + 2{,}344 = 11{,}000 + 2{,}344 = 13{,}344$

(d) $(\frac{5}{9} + \frac{4}{9}) + (\frac{11}{2} + \frac{1}{2}) = 1 + 6 = 7$

(e) $(-\frac{3}{4} + \frac{11}{4}) + (\frac{7}{8} + \frac{73}{8}) = 2 + 10 = 12$

(f) $\left[(-3\cdot 7) + (-9\cdot 3)\right] + \left[-12\cdot 9 + 3\cdot 9\right] + 8\cdot 0 = (-13) + (-9) + 8 = -14$

3. (a) $3\frac{5}{8} + 5\frac{7}{12} = (3 + 5) + (\frac{5}{8} + \frac{7}{12}) = 8\frac{29}{24} = 9\frac{5}{24}$

(b) $7\frac{2}{3} + 6\frac{3}{5} = (7 + 6) + (\frac{2}{3} + \frac{3}{5}) = 13 + \frac{19}{15} = 14\frac{4}{15}$

(c) $8\frac{2}{3} + 9\frac{3}{4} + 2\frac{1}{3} = (8 + 9 + 2) + (\frac{2}{3} + \frac{3}{4} + \frac{1}{3})$

$= 19 + (1 + \frac{3}{4}) = 20\frac{3}{4}$

(d) $5\frac{1}{4} + 8\frac{2}{3} + 3\frac{3}{4} + 7\frac{2}{5} = (5 + 8 + 3 + 7) + (\frac{1}{4} + \frac{2}{3} + \frac{3}{4} + \frac{2}{5})$

$= 23 + (1 + \frac{16}{15}) = 25\frac{1}{15}$

4. (a) $5 + (-5) = 0$ Now $(-3) + (-2) = -5$ if

$5 + \left[(-3) + (-2)\right] = 0$

This is true since

$5 + \left[(-3) + (-2)\right] = (3 + 2) + \left[(-3) + (-2)\right]$

$= \left[3 + (-3)\right] + \left[2 + (-2)\right] = 0 + 0 = 0$

(b) $9 + (-9) = 0$

$(-4) + (-5) = -9$ if

$9 + \left[(-4) + (-5)\right] = 0$

or $(4 + 5) + \left[(-4) + (-5)\right] = 0$

or $\left[4 + (-4)\right] + \left[5 + (-5)\right] = 0$

True, since $\left[4 + (-4)\right] + \left[5 + (-5)\right] = 0 + 0 = 0.$

4. (c) $6 + (-3) = (3 + 3) + (-3)$
$= 3 + \left[3 + (-3)\right] = 3 + 0 = 3.$

(d) $5 + (-7) = 5 + \left[(-5) + (-2)\right]$
$= \left[5 + (-5)\right] + (-2)$
$= 0 + (-2) = -2.$

(e) $(-5) + 3 = \left[(-2) + (-3)\right] + 3$
$= -2 + \left[(-3) + 3\right] = -2 + 0 = -2.$

(f) $(-9) + 14 = (-9) + (9 + 5)$
$= \left[(-9) + 9\right] + 5 = 0 + 5 = 5.$

5. $9;\ \frac{15}{16}$

6. $-\left[-(-9)\right] = -9$
$-\left[-(-a)\right] = -a$

7-4 Multiplication of Rational Numbers

To multiply two fractions $\frac{a}{b}$ and $\frac{c}{d}$ we use the principle

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}.$$

We have tried to make this principle reasonable by pointing out that if this method of multiplication is used, the area of a rectangle is found by multiplying the length of the base by the height, just as is the case if the base and height are whole numbers of units.

Then we turn to the case where one or both of the rational numbers is negative. In working out what sign the product should have, we are guided by experience with the integers.

Answers to Problems 7-4

Student Text Page 131

1. (a) True

   (b) True

   (c) False. The product is zero.

   (d) True

   (e) False

   (f) False. Both factors may be negative.

2. (a) $\frac{1}{2}$

   (b) $\frac{4}{5}$

   (c) $\frac{10}{3}$

   (d) $-\frac{4}{5}$

   (e) $\frac{4}{5}$

   (f) -1.9

   (g) $-\frac{4}{5}$

   (h) $\frac{4}{5}$

   (i) $-\frac{187}{9}$

   (j) $\frac{25}{9}$

   (k) 0

   (l) 125

7-5 Other Fractions for Rational Numbers

The important result is that we may write $-\frac{2}{3}$ as $\frac{-2}{3}$ or $\frac{2}{-3}$, for example, and that $\frac{-2}{-3} = \frac{2}{3}$. More generally, if a and b are integers,

$$\frac{-a}{b} = \frac{a}{-b} = -\frac{a}{b} \text{ and } \frac{-a}{-b} = \frac{a}{b}.$$

Answers to Problems 7-5

Student Text Page 134

1.	(a) $\frac{-2}{3}$	(b) $\frac{-4}{5}$	(c) $\frac{-3}{7}$	(d) $\frac{-5}{16}$
2.	(a) $\frac{3}{-7}$	(b) $\frac{4}{-9}$	(c) $\frac{5}{-24}$	(d) $\frac{7}{-20}$
3.	(a) $\frac{3}{4}$	(b) $\frac{5}{7}$	(c) $\frac{9}{10}$	(d) $\frac{8}{15}$
4.	(a) $-\frac{5}{19}$	(b) $-\frac{3}{25}$	(c) $\frac{84}{11}$	(d) $-\frac{678}{5}$

7-6 Properties of Multiplication of Rational Numbers

In justifying the commutative and associative properties of multiplication of rational numbers we have used geometrical examples. We believe that this approach appeals to the student's imagination. However, if desired, such an approach can be replaced by the following formal proofs.

A formal proof for the commutative property of multiplication of positive rational numbers would go as follows:

$\frac{p}{q} \times \frac{r}{s} = \frac{p \times r}{q \times s}$ [Definition of multiplication of rational numbers]

$\frac{r}{s} \times \frac{p}{q} = \frac{r \times p}{s \times q}$ [Same]

Since whole numbers have the commutative property, $r \times p = p \times r$ and $s \times q = q \times s$.

Therefore

$$\frac{p}{q} \times \frac{r}{s} = \frac{r}{s} \times \frac{p}{q}.$$

The corresponding proof of the associative property of multiplication of rational numbers is the following:

$$\left(\frac{p}{q} \times \frac{r}{s}\right) \times \frac{t}{u} = \left(\frac{p \times r}{q \times s}\right) \times \frac{t}{u} = \frac{(p \times r) \times t}{(q \times s) \times u}$$

$$\frac{p}{q} \times \left(\frac{r}{s} \times \frac{t}{u}\right) = \frac{p}{q} \times \left(\frac{r \times t}{s \times u}\right) = \frac{p \times (r \times t)}{q \times (s \times u)}$$

Since whole numbers have the associative property of multiplication, $p \times (r \times t) = (p \times r) \times t$ and $q \times (s \times u) = (q \times s) \times u$. Therefore

$$\left(\frac{p}{q} \times \frac{r}{s}\right) \times \frac{t}{u} = \frac{p}{q} \times \left(\frac{r}{s} \times \frac{t}{u}\right).$$

<u>Answers to Problems 7-6</u> Student Text Page 139

We use the abbreviations AM for the associative property of multiplication and CM for the commutative property of multiplication.

1. $a = g$ AM
   $b = k$ AM
   $c = h$ CM
   $d = l$ CM
   $e = i$ CM
   $f = j$ CM

2. (a) -30
   (b) 60
   (c) $-\frac{28}{75}$
   (d) $-\frac{12}{35}$
   (e) $-\frac{4}{45}$
   (f) $\frac{1}{20}$

3. (a) $(5 \times 7) \times 2 = (7 \times 5) \times 2$ CM
   $= 7 \times (5 \times 2)$ AM

   (b) $\left[(-3) \times 9\right] \times (-4) = (-4) \times \left[(-3) \times 9\right]$ CM
   $= (-4) \times \left[9 \times (-3)\right]$ CM

   (c) $\frac{3}{5} \times \left(\frac{2}{3} \times \frac{5}{3}\right) = \frac{3}{5} \times \left(\frac{5}{3} \times \frac{2}{3}\right)$ CM
   $= \left(\frac{3}{5} \times \frac{5}{3}\right) \times \frac{2}{3}$ AM

3. (d) $\left[(-\frac{1}{2}) \times 20\right] \times (-35) = \left[20 \times (-\frac{1}{2})\right] \times (-35)$ CM

$= 20 \times \left[(-\frac{1}{2}) \times (-35)\right]$ AM

(e) $(-\frac{3}{4}) \times \left[(-\frac{1}{5}) \times (-\frac{7}{8})\right] = \left[(-\frac{1}{5}) \times (-\frac{7}{8})\right] \times (-\frac{3}{4})$ CM

$= \left[(-\frac{7}{8}) \times (-\frac{1}{5})\right] \times (-\frac{3}{4})$ CM

$= (-\frac{7}{8}) \times \left[(-\frac{1}{5} \times (-\frac{3}{4})\right]$ AM

4.

(a) $\frac{9}{4}$	(d) $\frac{1}{3}$	(g) 2	(j) -1
(b) $\frac{-7}{-13} = -\frac{7}{13}$	(e) $-\frac{8}{17}$	(h) -3	(k) $\frac{2}{5}$
(c) $\frac{12}{5}$	(f) $-\frac{1}{2}$	(i) 1	(l) $\frac{1}{5000}$

5. Repeated use of AM

6. $3 \times \frac{1}{3} = 1$ and $5 \times \frac{1}{5} = 1$

Multiplicative inverses

$(1 \times 1) \times 9 = 1 \times 9 = 9$ Property of one.

7. (a) $(\frac{3}{4} \times \frac{4}{3}) \times (2 \times \frac{1}{2}) \times 13 = 1 \times 1 \times 13 = 13$

(b) $(-2) \times (\frac{7}{8} \times 8) \times (-5) = (-2) \times 7 \times (-5) = 70$

(c) $\left[\frac{9}{7} \times (-\frac{7}{9})\right] \times \left[(-\frac{5}{4}) \times 4\right] \times 2 = (-1) \times (-5) \times 2 = 10$

(d) $\left[\frac{15}{4} \times (-4)\right] \times \left[(-3) \times \frac{1}{3}\right] \times \frac{7}{15} = (-15) \times (-1) \times \frac{7}{15} =$

$15 \times \frac{7}{15} = 7$

(e) $\left[(0.125) \times 8\right] \times 17 = 1 \times 17 = 17$

8. and 9.

(a) 1 Property of one

(b) $\frac{17}{5}$ Mult. Inv.

(c) $-\frac{2}{5}$ Prop. of one

(d) -13 Mult. Inv.

(e) 1 Property of one

(f) 1 Mult. Inv.

(g) $\frac{1}{23}$ Mult. Inv.

(h) 1 Prop. of one

7-7 The Distributive Property

We have given a geometrical argument for the distributive property when all three rational numbers are positive. It is, of course, possible to give an algebraic argument to prove that

$$\frac{p}{q} \times \left(\frac{r}{s} + \frac{t}{u}\right) = \left(\frac{p}{q} \times \frac{r}{s}\right) + \left(\frac{p}{q} \times \frac{t}{u}\right)$$

We apply the definitions of addition and multiplication of positive rational numbers and rely upon the properties of whole numbers. We sketch the proof. On the left we have $\frac{p}{q} \times \left(\frac{ru + st}{s\,u}\right) = \frac{p \times (ru + st)}{q \times s\,u} = \frac{p\,r\,u + p\,t\,s}{q\,s\,u}$

On the right we have

$$\left(\frac{p}{q} \times \frac{r}{s}\right) + \left(\frac{p}{q} \times \frac{t}{u}\right) = \frac{pr}{qs} + \frac{pt}{qu} = \frac{pru}{qsu} + \frac{pts}{qsu} = \frac{pru + pts}{q\,s\,u}$$

The results are the same.

We do not advise using this algebraic proof at the present stage.

Answers to Problems 7-7

Student Text Page 144

1. (a) $\left[2 \times (-3)\right] + \left[2 \times 5\right]$

(b) $\left(\frac{4}{9} \times \frac{3}{5}\right) + \left(\frac{4}{9} \times \frac{10}{7}\right)$

(c) $\left[\frac{2}{3} \times \left(-\frac{3}{4}\right)\right] + \left[\frac{2}{3} \times \frac{7}{8}\right]$

(d) $\left[\left(-\frac{4}{9}\right) \times \frac{3}{5}\right] + \left[\left(-\frac{4}{9}\right) \times \left(-\frac{1}{3}\right)\right]$

2. (a) $2 \times \left[(-5) + (-3)\right]$

(b) $\frac{1}{2} \times \left[\frac{3}{5} + \frac{7}{10}\right]$

(c) $(\frac{-2}{3}) \times \left[\frac{1}{5} + \frac{3}{7}\right]$

(d) $\frac{7}{5} \times \left[(-\frac{3}{2}) + (-\frac{1}{2})\right]$

3. In 1. (c), both results $= \frac{1}{12}$

In 2. (c), both results $= -\frac{44}{105}$

7-8 Applications of the Basic Properties

These applications of the multiplication properties of the rational numbers are analogous to the applications of the addition properties in 7-3.

In Application 1, the change from $7 \times (20 + 4)$ to $(7 \times 20) + (7 \times 4)$ is, of course, an application of the distributive property. The last steps in detail are

$$7 \times 20 = 7 \times (2 \times 10) = (7 \times 2) \times 10 = 14 \times 10 = 140$$

and

$$140 + 28 = 140 + (20 + 8) = (140 + 20) + 8$$
$$= 160 + 8 = 168.$$

Answers to Problems 7-8

Student Text Page 146

1. $6 \times 32 = 6 \times \left[30 + 2\right]$

$= (6 \times 30) + (6 \times 2)$

$= 180 + 12 = 192$

$9 \times 54 = 9 \times \left[50 + 4\right]$

$= (9 \times 50) + (9 \times 4)$

$= 450 + 36 = 486.$

2. (a) $124 \times (12 + 88) = 124 \times 100 = 12{,}400$

(b) $\frac{5}{7} \times (\frac{3}{4} + \frac{1}{4}) = \frac{5}{7} \times 1 = \frac{5}{7}$

(c) $(-25) \times \left[298 + 102\right] = (-25) \times 400 = -10{,}000$

(d) $27 \times \left[13 + (-3)\right] = 27 \times 10 = 270$

(e) $(12 \times \frac{5}{6}) + (12 \times \frac{7}{4}) = 10 + 21 = 31$

(f) $8 \times \left[(-\frac{4}{5}) + (-\frac{6}{5})\right] = 8 \times (-\frac{10}{5}) = 8 \times (-2) = -16$

(g) $\left[(-\frac{14}{3}) \times \frac{9}{7}\right] + \left[(-\frac{14}{3}) \times \frac{3}{2}\right] = (-6) + (-7) = -13$

(h) $(-39) \times \left[(-\frac{17}{15}) + (-\frac{8}{15})\right] = -39 \times -\frac{25}{15} = 39 \times \frac{5}{3}$

$= 13 \times 5 = 65.$

3. (a) $(-4) \times 5 = 5 \times (-4)$

$(-4) + 4 = 0$

$\left[5 \times (-4)\right] + \left[5 \times 4\right] = 5 \times 0 = 0$

Hence $\left[5 \times (-4)\right] + 20 = 0$

Hence $5 \times (-4) = -20.$ Unique additive inverse

(b) $7 + (-7) = 0$

$(6 \times 7) + \left[6 \times (-7)\right] = 6 \times 0 = 0$

$42 + \left[6 \times (-7)\right] = 0$

Hence $6 \times (-7) = -42$ Unique additive inverse

(c) $5 + (-5) = 0$

$\left[(-4) \times (5)\right] + \left[(-4) \times (-5)\right] = (-4) \times 0 = 0$

Hence $-20 + \left[(-4) \times (-5)\right] = 0$

$(-4) \times (-5) = 20$ Unique additive inverse

3. (d) $(-7) + 7 = 0$

$$\left[(-6) \times (-7)\right] + \left[(-6) \times (7)\right] = 0$$

$$\left[(-6) \times (-7)\right] + (-42) = 0$$

Hence $(-6) \times (-7) = 42$

7-9 Revision of the Properties of the Rational Number System

The 11 "field" properties are summarized in convenient form for reference. The examples in the text and the problems illustrate the fact that there are systems in which some of these properties do not hold.

For the set of whole numbers with ordinary addition and multiplication.

1. The system is closed for addition and multiplication. (The sum of two whole numbers is a whole number and the product of two whole numbers is a whole number.)

2. $a + b = b + a$ and $a \times b = b \times a$ for all whole numbers a and b.

3. $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$ for all whole numbers a, b and c.

4. $a \times (b + c) = (a \times b) + (a \times c)$ for all whole numbers a, b and c.

5. $a + 0 = 0 + a = a$ and $a \times 1 = 1 \times a = a$ for all whole numbers a. Hence 0 is the identity element for addition and 1 is the identity element for multiplication.

6. No element except 0 has an additive inverse. No element except 1 has a multiplicative inverse. For example, there is no whole number b for which

$$b \times 2 = 1$$

Answers to Problems 7-9 | Student Text Page 150

1. Not a field. There is no additive identity. Also 2, for example, has no additive inverse and no multiplicative inverse.

2. Not a field: 2, for example, has no multiplicative inverse.

3. Not a field. No additive or multiplicative identity. No additive or multiplicative inverses.

4. Not a field. Lack of additive inverses.

5. Field.

6. Not a field. 2, 3, 4, and 6 have no multiplicative inverses.

7. Field. Additive identity = a. Multiplicative identity = c. Every element except a has both kinds of inverse.

7-10 Subtraction of Rational Numbers

The important point is to see that subtracting one rational number b from another a is the same as adding a and -b, the additive inverse of b. That is, $a - b = a + (-b)$. The operation of subtraction can be eliminated in favor of addition.

Answers to Problems 7-10 | Student Text Page 154

1. (a) 4; $4 + 3 = 7$

   (b) -3; $(-3) + 8 = 5$

   (c) -13; $(-13) + 8 = -5$

   (d) -8; $(-8) + 3 = -5$

   (e) 11; $11 + (-3) = 8$

   (f) 12; $12 + (-7) = 5$

   (g) 2; $2 + (-8) = -6$

   (h) -4; $-4 + (-4) = -8$

1. (i) $-\frac{10}{4} = -\frac{5}{2}$; $\frac{-10}{4} + \frac{7}{4} = \frac{-3}{4}$

   (j) 0; $0 + (-\frac{4}{5}) = -\frac{4}{5}$

   (k) $\frac{13}{6}$; $\frac{13}{6} + (-\frac{2}{3}) = \frac{3}{2}$

   (l) $-\frac{7}{9}$; $-\frac{7}{9} + \frac{12}{9} = \frac{5}{9}$

   (m) $-\frac{41}{24}$; $-\frac{41}{24} + \frac{20}{24} = -\frac{21}{24} = -\frac{7}{8}$

   (n) $1\frac{19}{24}$; $1\frac{19}{24} + 1\frac{21}{24} = 3\frac{16}{24} = 3\frac{2}{3}$

   (o) $4\frac{9}{14}$; $4\frac{9}{14} + (-\frac{5}{14}) = 4\frac{2}{7}$

   (p) $-0{\cdot}38$; $-0{\cdot}38 + (-0{\cdot}46) = -0{\cdot}84$

   (q) 1000; $1000 + (-136) = 864$

2. Yes.

3. No. For example, $2 - 1 \neq 1 - 2$

   No. For example, $3 - (2 - 1) \neq (3 - 2) - 1$

4. Yes, because $a \times (b - c) = a \times \left[b + (-c)\right]$

   $= (a \times b) + \left[a \times (-c)\right] = (a \times b) - (a \times c)$

7-11 Division of Rational Numbers

The important point is that division of one rational number a by another b ($b \neq 0$) is the same as multiplication of a by $\frac{1}{b}$, the multiplicative inverse of b. That is, $\frac{a}{b} = a \times \frac{1}{b}$. The operation of division can be eliminated in favor of multiplication.

Notice the analogy with the situation in 7-10.

There is however one important difference. Every rational number has an additive inverse. Not every rational number has a multiplicative inverse. The number 0 has no multiplicative inverse, since there is no rational number whose product with 0 is equal to 1. The product of any rational number and 0 is 0, never 1. See Section 7-6.

The equation

$$\frac{a}{b} = a \times \frac{1}{b}$$

is an expression of the well-known rule for dividing fractions. Let $a = \frac{p}{q}$ and $b = \frac{r}{s}$. Since $\frac{1}{b} = \frac{s}{r}$, the reciprocal of $\frac{r}{s}$, to divide $\frac{p}{q}$ by $\frac{r}{s}$ is equivalent to multiplying $\frac{p}{q}$ by $\frac{s}{r}$. This is often expressed by the rule "To divide fractions, invert and multiply." Study of this section should remove the mystery from this rule.

Answers to Problems 7-11

Student Text Page 158

1. (a) $\frac{2}{3} \times \frac{7}{4} = \frac{7}{6}$

(b) $(-\frac{3}{5}) \times \frac{5}{9} = -\frac{1}{3}$

(c) $(-\frac{8}{7}) \times (-\frac{3}{4}) = \frac{6}{7}$

(d) $\frac{12}{7}$

(e) $(-\frac{18}{5}) \times \frac{10}{3} = -12$

(f) $(-5) \times (-\frac{4}{3}) = \frac{20}{3}$

(g) $(\frac{9}{2}) \times (-\frac{1}{2}) = -\frac{9}{4}$

(h) $-\frac{3}{4} \times \frac{1}{5} = -\frac{3}{20}$

(i) $\frac{1}{7}$

(j) $27 \times (-\frac{7}{15}) = -\frac{63}{5}$

2. Division of rational numbers is not commutative. For example, $2 \div 3 \neq 3 \div 2$.

Division of rational numbers is not associative. For example, $6 \div (3 \div 2) \neq (6 \div 3) \div 2$, because $4 \neq 1$.

3. It is not true that $a \div (b + c) = (a \div b) + (a \div c)$. For example, $1 \div (1 + 1) \neq (1 \div 1) + (1 \div 1)$, $(\frac{1}{2} \neq 2)$.

Answers to Revision Problems

Student Text Page 159

1. (a) $-\frac{44}{21}$

(b) $\frac{26}{21}$

(c) $-\frac{2}{3}$

(d) $2\frac{2}{3}$

(e) $\frac{11}{30}$

(f) $-\frac{45}{22}$

(g) $-\frac{2}{3}$

(h) 1

(i) $\frac{2}{3}$

(j) $-\frac{1}{5}$

(k) $\frac{7}{15}$

(l) $\frac{28}{15}$

(m) 0

(n) $\frac{9}{16}$

(o) $\frac{9}{17}$

(p) $\frac{9}{17}$

(q) 0

(r) 1

2. Yes. $\frac{a \times b}{c} = a \times \frac{b}{c}$ $(c \neq 0)$

3. No. $\frac{a}{b} \times c \neq \frac{a}{b \times c}$ unless $a = 0$, or $c = 1$ or -1

For example $\frac{2}{3} \times 3 = 2$, but $\frac{2}{3 \times 3} = \frac{2}{9}$.

4. $\frac{a + b}{c} = (a + b) \times \frac{1}{c}$ Definition of multiplication of rational numbers: $(\frac{a + b}{1}) \times \frac{1}{c}$

$= \frac{1}{c} \times (a + b)$ CM

$= (\frac{1}{c} \times a) + (\frac{1}{c} \times b)$ Distributive property

$= \frac{a}{c} + \frac{b}{c}$ Definition of multiplication: $(\frac{1}{c} \times \frac{a}{1}$ and $\frac{1}{c} \times \frac{b}{1})$

CHAPTER 8

OPEN SENTENCES

8-1 Statements

The students are familiar with the idea of a sentence in their language-study classes. A statement is a particular kind of sentence, the kind that states that something is the case like "The sky is blue" or "It is raining." A statement must be either true or false. There are of course English sentences that are questions, commands or requests. These are not statements. For example, "Is it raining?" and "Go to the blackboard" are not statements. They are neither true nor false.

We shall be concerned with mathematical statements, usually statements about numbers. These are written in mathematical language. That is, they use mathematical symbols like $=$, $\neq$, $>$, $<$, $\geq$ and $\leq$. For example $3 + 2 = 5$ is a mathematical statement, a true one. The corresponding English sentence is "The sum of three and two is five." Encourage the students to write mathematical statements on the board.

They may write something like $12 \div 4$, or $\frac{12}{4}$. This of course is not a statement.

$$12 \div 4 = 3$$

<u>is</u> a statement.

Some mathematical statements are true and some are false, just as some English statements are true and some are false. The problems should bring this out and at the same time recall the meanings of the symbols. If they give difficulty, encourage the students to translate the mathematical statements into English statements.

<u>Answers to Problems 8-1</u> | Student Text Page 161 |

1. c and d are true.
   a, b, e, f, g, h, i, j are false.

2. a, d, g, and h are true.
   b, c, e, f and i are false.

8-2 Variable

A sentence in which there is at least one symbol like □ or x is neither true nor false. It becomes true or false only when we replace the box or x by a numeral. We shall eventually abandon boxes and use letters. Thus $2 + \square = 7$ becomes $2 + x = 7$ or $2 + y = 7$. But it appears that boxes are easier for a student to understand at first.

We have treated □ and x in exactly the same way. We call either of them a <u>variable</u>. However, the box is often thought of as a physical object within which we can write various numerals. This is not quite the same as saying that we substitute the numeral for □ but this fine logical distinction will probably not bother the student.

This topic might be introduced as follows: Write on the board a statement like "Today is his birthday" and ask "Is this true or false?" The class will say that they cannot tell because they do not know to whom "his" refers. Rewrite the sentence replacing "his" by the names (in the possessive) of different boys in the class, for example "John's". Ask "Now is the sentence true or false?" In the English sentence "Today is his birthday" the word "his" acts like " □" or "x" in mathematical sentences.

Answers to Problems 8-2

Student Text Page 163

1.	True	6.	True, False
2.	False	7.	True, False
3.	False	8.	False, True
4.	True	9.	False, True
5.	False	10.	True, False

8-3 Domain of the Variable

The important idea here is that of the domain of a variable like □ or x. This is the set of numbers from which we may choose replacements for the variable. The variable represents any number of this set. Ordinarily the mathematical sentence is true for certain members of the domain and false for other members of the domain. In special cases the sentences may be true for all members of the domain or true for no members of the domain.

The term open sentence is introduced to describe sentences like those in the previous section. The truth or falsity remains open, or undecided, until the variable is replaced by the name of a number of the domain.

It is useful to return to "Today is his birthday", which we now call an open sentence. The domain might be the set of boys in the class. Note that it is very likely that the statement is false for all members of the class. If the domain is extended to include all of the boys in the school it is more likely that the open sentence is true for at least one boy that day.

Answers to Problems 8-3 | Student Text Page 165

In each case we list the members of D for which the statement is true, and omit the members for which it is false.

1. 1
2. 0
3. 2, 3, 4, 5
4. All natural numbers
5. All integers less than 2
6. All rational numbers greater than or equal to 1
7. All positive rational numbers less than or equal to 1
8. $\frac{11}{2}$
9. 4, $\frac{9}{2}$, 5
10. 6, $\frac{13}{2}$, 7, $\frac{15}{2}$, 8
11. All rational numbers greater than or equal to $\frac{11}{2}$
12. All positive rational numbers less than or equal to $\frac{11}{2}$
13. All rational numbers less than or equal to $\frac{11}{2}$
14. All negative rational numbers
15. 1, 2, 3, 4, 5

8-4 Open Sentences and Truth Sets

The important idea here is that of truth set (or solution set). The idea has already been introduced in the previous section. We are merely learning how to express it. The truth set is the subset of the domain for which the open sentence

becomes a true statement. The truth set may be empty or it may be the whole domain.

Open mathematical sentences may be equations or inequalities.

It is well to use the summary at the end of the section to revise the ideas and the language. Try to have the students explain in their own words the terms open sentence, equation, inequality, truth set. If possible get them to invent new examples.

Answers to Problems 8-4

Student Text Page 167

1. (a) $\{1\}$
   (b) $\{2, 3, 4\}$
   (c) $\{1, 2, 3, 4, 5\}$
   (d) $\{0\}$
   (e) $\{0, 1\}$
   (f) $\{1\}$
   (g) $\{1, 2, 3, \ldots\}$
   (h) $\{2, 3, 4, \ldots\}$
   (i) $\{0, -1, -2, -3, \ldots\}$
   (j) $\{1, 0, -1, -2, \ldots\}$
   (k) $\{1\}$
   (l) {All rational numbers greater than 1}
   (m) {All rational numbers greater than or equal to 1}
   (n) {All rational numbers less than 1}
   (o) {All rational numbers less than or equal to 1}

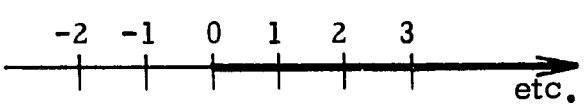
2. (a) $\{4, 6, 8\}$
   (b) $\{30\}$
   (c) $\{1, 2, 3, 4, 5, 6, 7\}$
   (d) $\{\ \}$
   (e) $\{-1, -2, -3, \ldots\}$
   (f) {All negative rational numbers}
   (g) $\{\ \}$
   (h) $\{1, 2, 3, 4, 5, 6, 8, 9, 10\}$
   (i) $\{\frac{10}{3}\}$
   (j) $\{13, 15\}$

3. (a) $\{9, -9\}$
   (b) $\{\ \}$
   (c) $\{8, -10\}$
   (d) {All rational numbers}
   (e) $\{0\}$
   (f) $\{\ \}$
   (g) {All rational numbers}
   (h) $\{0\}$
   (i) $\{\ \}$
   (j) {All whole numbers}
   (k) $\{10, -10\}$
   (l) $\{2, 3\}$
   (m) $\{4, 5\}$
   (n) $\{\ \}$
   (o) {All integers except 2 and 3}

8-5 Graphs of Solution Sets

The student will learn how to show certain truth sets (or solution sets) as sets of points on the number line. There are some solution sets which the student is not yet ready to handle. For example, consider the open sentence $x > 0$ where the domain is the set of all rational numbers.

One might be tempted to draw a figure like this,



making the number line heavy to the right of 0.

The trouble is that it is not possible to tell the difference between this picture and that for $x > 0$ with the domain the set of all real numbers. The set of positive rationals is full of "holes" but the graph cannot show this. Discussion of such cases must be postponed until real numbers have been studied.

Answers to Problems 8-5

Student Text Page 171

1. (a) 0 1 2 3 4 5

(b) 0 1 2 3 4 5 6 7 8 9 etc.

(c) etc. -4 -3 -2 -1 0

(d) -4 -3 -2 -1 0 1 2 3 4

(e) etc. -4 -3 -2 -1 0 1 2 3 4 etc.

(f) -5 -4 -3 -2 -1 0 1 2 3 4 5

(g) -2 -1 0 1 2

(h) -1 0 1 2 3 4 5 6 7 8 9 10

2. (a) {1, 3}

(b) { }

(c) {natural numbers}

(d) $\{-3, -\frac{3}{2}, 4\}$

(e) {integers less than 3}

3. (a) { } -3 -2 -1 0 1 2 3

(b) {whole numbers} 0 1 2 3 etc.

(c) Same as b.

(d) { } -2 -1 0 1 2 3

(e) Same as d.

(f) {whole numbers} 0 1 2 3 etc.

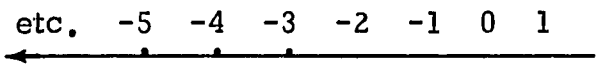
(g) {-2} -3 -2 -1 0 1 2

(h) {integers greater than -2} -3 -2 -1 0 1 2 etc.

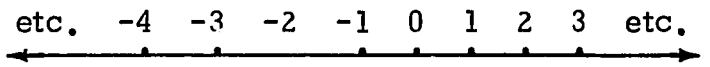
(i) {integers greater than -3} -3 -2 -1 0 1 2 etc.

(j) {..., -4, -3, -2} etc. -4 -3 -2 -1 0 1

3. (k) {..., -5, -4, -3}



(l) {integers except -2}



8-6 Compound Sentences

The important idea is to learn to handle the results of connecting two statements by "and" and "or". The word "or" is used in a precise way that does not correspond exactly to general English usage. It has been found convenient in mathematics to say that if S_1 and S_2 are two statements, the new statement, S_1 or S_2, is true in any one of the following cases:

if S_1 is true and S_2 is false
if S_1 is false and S_2 is true
if S_1 is true and S_2 is true

In English we sometimes exclude the third possibility. For example when we say "You must pass this examination or you will fail the course" it is usually understood that you cannot pass the examination and fail the course.

Answers to Problems 8-6A

Student Text Page 174

We shall omit the graphs on the number line.

1. {1, 2, 3, 4, 5, 6, 7}

2. {51, 52, 53, 54, 55, 56, 57, 58, 59}

3. {-3, -2, -1, 0, 1}

4. {7}

5. {-99, -98, ..., -1, 0, 1, ..., 98, 99}

6. { }

7. { }

8. { }

9. {..., -1, 0, 1, 2, 3, 4, 5}

10. {6}

11. { }

12. {-1, 0, 1, 2, 3, ...}

Answers to Problems 8-6B

Student Text Page 175

(We shall omit the graphs on the number line)

1. {integers except 10, 11, 12}

2. {... , -6, -5, -4}

3. {0, 3, 4, 5, ...}

4. {3, 4, 5, 6, ...}

5. {... , -5, -4, 0, 1, 2, ...}

6. {integers except -1}

7. {integers}

8. {... , -1, 0, 1, 2} or {integers less than 3}

9. {-3, -2, -1, 0, 1, ...}

10. {1, 6}

8-7 Equivalent Equations

Equations with the same truth set are said to be equivalent. The important principles are these:

If the same number is added to or subtracted from both sides of an equation, the new equation is equivalent to the old one.

If both sides of an equation are multiplied or divided by the same non-zero number, the new equation is equivalent to the old one.

To prove that two statements are equivalent, it is necessary to show that every solution of the first is a solution of the second and that every solution of the second is a solution of the first. In establishing the principles above it is necessary to emphasize this fact.

Answers to Problems 8-7

Student Text Page 179

1. (a) Equivalent. Adding -4 to both sides.

   (b) Equivalent. Adding $\frac{3}{8}$ to both sides.

   (c) Not equivalent.

   (d) Equivalent. Multiplying both sides by 5.

   (e) Equivalent. Adding $-\frac{5}{8}$ to both sides and multiplying both sides by $\frac{4}{3}$.

   (f) Equivalent. Adding 3 to both sides.

   (g) Equivalent. Adding $\frac{3}{4}$ to both sides.

   (h) Not equivalent.

   (i) Equivalent. Both have the same truth set, $\{8\}$.

   (j) Not equivalent.

   (k) Equivalent. Multiplying both sides by $\frac{1}{3}$.

2. (a) $x = 82$. By adding 9 to both sides.

   (b) $y = \frac{5}{6} - \frac{2}{3}$. By adding $-\frac{2}{3}$ to both sides.

   (c) $a = \frac{13}{6}$. By multiplying by $\frac{1}{6}$ on both sides.

   (d) $x = 16$. By multiplying both sides by 4.

   (e) $y = \frac{2}{7} \times \frac{5}{3}$. By multiplying both sides by $\frac{5}{3}$.

   (f) $x = 19(-\frac{1}{8})$ By multiplying both sides by $(-\frac{1}{8})$.

8-8 Truth Sets Found by Using Equivalent Equations

As we know, when we are given an equation, we can obtain an equivalent equation by adding (or subtracting) the same number on both sides, or by multiplying (or dividing) both sides by a non-zero number. By repeated use of these principles, a given equation can often be reduced to the form

$$x = \text{a number.}$$

The truth set of an equation of this form is evident by inspection.

The student should solve enough problems so that he can himself see what steps to take to reduce a given equation to the form

$$x = \text{a number.}$$

It will be useful to write equations on the board and ask for suggestions as to what to do. There are often alternative solutions.

For example, given the equation

$$2x + 3 = \frac{x}{2} + 5$$

we might write the following equivalent equations in order.

$2x = \frac{x}{2} + 2$	(Subtracting 3)
$\frac{3x}{2} = 2$	(Subtracting $\frac{x}{2}$)
$x = \frac{2}{3} \times 2 = \frac{4}{3}$	(Multiplying by $\frac{2}{3}$)

An alternative method is this:

$2x + 3 = \frac{x}{2} + 5$	
$4x + 6 = x + 10$	(Multiplying by 2)
$4x = x + 4$	(Subtracting 6)
$3x = 4$	(Subtracting x)
$x = \frac{4}{3}$	(Dividing by 3)

This method makes less use of fractions than the other. When we reach $4x + 6 = x + 10$ we might subtract x first and then 6 obtaining

$$\begin{aligned} 3x + 6 &= 10 \\ 3x &= 4 \\ x &= \frac{4}{3} \end{aligned}$$

It is desirable to encourage flexibility. It is important that the student shall take such steps as seem natural to him.

Answers to Problems 8-8 | Student Text Page 184

1.	$\{13\}$	9.	$\{-\frac{5}{3}\}$
2.	$\{13\}$	10.	$\{56\}$
3.	$\{1\}$	11.	$\{-\frac{45}{2}\}$
4.	$\{-1\}$	12.	$\{-\frac{7}{2}\}$
5.	$\{-3\}$	13.	$\{\frac{21}{2}\}$
6.	$\{12\}$	14.	$\{2\}$
7.	$\{1\}$	15.	$\{5\}$
8.	{rational numbers}	16.	$\{7\}$

8-9 Equivalent Inequalities

The following operations on an inequality give an equivalent inequality.

Adding or subtracting the same number on both sides

Multiplying or dividing on both sides by the same positive number

Multiplying or dividing on both sides by the same negative number and reversing the direction of the inequality sign.

It is important to emphasize the reversal of the inequality sign in the last case.

If desired the order properties for inequalities may be proved algebraically.

To say that $a < b$ means that $b = a + p$ where p is some positive number.

1. To prove that if $a < b$, $c + a < c + b$ we say

$$c + b = c + (a + p)$$
$$= (c + a) + p$$

Since $c + b$ is $c + a$ plus a positive number

$$c + a < c + b.$$

2. Given $a < b$ and $c > 0$, to prove $ca < cb$.

$$b = a + p$$
$$cb = c(a + p) = ca + cp$$

Since c and p are positive, cp is positive.

Therefore $ca < cb$.

3. Given $a < b$ and $c < 0$, to prove $ca > cb$.

In this case cp is <u>negative</u> in the equation above and therefore

$$ca > cb.$$

<u>Answers to Problems 8-9A</u>

Student Text Page 188

1.	Equivalent	6.	Equivalent
2.	Equivalent	7.	Not equivalent
3.	Not equivalent	8.	Equivalent
4.	Not equivalent	9.	Equivalent
5.	Equivalent	10.	Equivalent

Answers to Problems 8-9B

Student Text Page 189

1.

(a)	True	(h)	False
(b)	False	(i)	False
(c)	True	(j)	True
(d)	True	(k)	True
(e)	False	(l)	False
(f)	True	(m)	False
(g)	True	(n)	True

2. All the pairs are equivalent inequalities.

8-10 Solution Sets of Inequalities

We use the principles of 8-9 to reduce a given inequality to one of the simple forms

$x >$ a number

$x <$ a number

$x \geq$ a number

$x \leq$ a number

The student should learn to reduce a given inequality to one of the simpler forms by a succession of steps which he can take for himself. Board work is helpful here. As in the case of equalities, there are often many different ways to obtain the result.

For example

$$4 - 2x < 5x + 8$$

is equivalent to:

$$-2x < 5x + 4$$

which is equivalent to:

$$-7x < 4$$

which is equivalent to:

$$x > -\frac{4}{7}.$$

(Note the change from < to >)

Alternatively,

$$-2x < 5x + 4$$

is equivalent to:

$$0 < 7x + 4$$

which is equivalent to:

$$-4 < 7x$$

which is finally equivalent to:

$$-\frac{4}{7} < x.$$

Notice that this method does not require a reversal of the inequality sign.

<u>Answers to Problems 8-10</u> Student Text Page 191

We abbreviate $\left\{\text{rational numbers } x \text{ such that } x \geq \frac{11}{2}\right\}$ to $\left\{x \geq \frac{11}{2}\right\}$ and so on.

1. (a) $\{x \geq \frac{11}{2}\}$ (f) $\{x < \frac{17}{15}\}$

(b) $\{x \leq 5\}$ (g) $\{x \geq -\frac{4}{5}\}$

(c) $\{x > 5\}$ (h) $\{x \geq 2\}$

(d) $\{x < 0\}$ (i) $\{y < 14\frac{1}{3}\}$

(e) $\{x \geq -2\}$ (j) $\{x \geq 3\}$

2. (a) $\{x > -\frac{9}{4}\}$
   (b) $\{x \leq -\frac{28}{17}\}$
   (c) $\{x > -2\}$
   (d) $\{x \leq \frac{18}{7}\}$
   (e) $\{x < \frac{4}{25}\}$
   (f) $\{x \leq -\frac{1}{5}\}$
   (g) $\{x \leq 1\}$
   (h) $\{x < -4\}$
   (i) $\{x \geq \frac{4}{3}\}$
   (j) $\{x < 0\}$

3. (a) $\{x > 13\}$
   (b) $\{x < 13\}$
   (c) $\{x \geq 1\}$
   (d) $\{x \geq -3\}$
   (e) $\{x < \frac{5}{2}\}$
   (f) $\{x > \frac{3}{4}\}$
   (g) $\{y < \frac{4}{5}\}$
   (h) $\{x \geq -1\}$
   (i) $\{x \leq 5\}$
   (j) $\{x < \frac{51}{7}\}$
   (k) $\{x > \frac{15}{4}\}$
   (l) $\{x \leq \frac{28}{3}\}$
   (m) {rational numbers}
   (n) {rational numbers}

CHAPTER 9

WORD PROBLEMS

One of the most important skills that a student may acquire is the ability to solve simple problems stated in words. Success in doing so gives him a feeling of confidence and is a strong motivating factor for further study of mathematics and science.

9-1 Problem and Open Sentences

A problem about numbers can often be stated mathematically as an open sentence. The solution of the problem reduces to the determination of the truth set for the open sentence. Since we know how to do this for many open sentences we have a method for solving problems.

It is first necessary to choose a variable to represent the number which we wish to find and then, after careful reading of the problem, make a statement concerning it. This statement will usually be an equation.

Sometimes there will be two numbers which we wish to find, and it may be necessary to write two equations. However, in this chapter we shall meet problems in which one variable will be sufficient.

In this section the student learns how to state a problem as an open sentence and is asked to invent problems which lead to given open sentences.

Answers to Problems 9-1 | Student Text Page 195

There are of course, many possible answers. The following are possibilities.

1. A five gallon tank is partly full. If you add two gallons the tank is full. How many gallons did the tank contain?

2. By adding 1 to a certain even number the odd number 9 is obtained. What is the even number?

3. Twelve feet is how many yards?

4. Forty-three bananas are to be divided equally among five children. How many bananas should each of them receive?

5. If a boy spends 6 shillings and has 2 shillings left, how many shillings did he have to begin with?

6. The sum of the length and width of a rectangle is 10 feet. What are the dimensions? (There is insufficient information to answer the question.)

9-2 Solving Word Problems

Here we take a problem and show exactly how to translate it into an open sentence. There are in the example two numbers to be found, the number of geography books and the number of history books. At first we use two variables p and q and set up two open sentences. To see how to do this, it is helpful to guess the number of geography books and test the guess.

Later we see how we can use one variable and set up a single open sentence of the sort that we know how to solve.

This problem should be studied very carefully and thoroughly understood. An important step is to express the number of history books in terms of x, the number of geography books. This is done by the formula, $11 - x$.

It is also important to check the answers to see that they satisfy the conditions of the problem.

9-3 Formulas

The important skill to be developed in this section is the ability to express one of the numbers which occurs in a problem in terms of another number. There are numerous examples in the text which should prepare the student to write formulas for himself.

Answers to Problems 9-3

Student Text Page 200

1. $100 - x$

2. $\frac{20}{n}$ or $20 \div n$

3. $y - 7$ if y is the larger number

   $y + 7$ if y is the smaller number

4. $30 - p$

5. $12 + x$

6. (a) $12s$
   (b) $36m$
   (c) $100q$
   (d) $5280d$
   (e) $16p$
   (f) $12s + t$
   (g) $16p + q$

7. If n is the number chosen, the number obtained is $2n - 5$.

8. If Olayunka has n pineapples, Olu has $20 - n$ pineapples.

9. If n is the larger number and s the smaller number, $n = 2s$.

10. If n is the larger number and s the smaller number, $s = \frac{n}{3}$.

11. If n is the given number, the number obtained is $3(n + 10) - n$.

12. (a) $x - 5$
    (b) -2

13. (a) $k + 2$
    (b) -12

14. (a) If n is the other number, the first number is $n - \frac{3}{4}$.
    (b) $-8\frac{1}{4}$

15. (a) If n is the larger number and s the smaller number, $n = 2s + 11$.
    (b) $10\frac{1}{8}$

16. (a) If f is the first number and n the other number, $f = \frac{n}{2} + 15$.

    (b) $18\frac{1}{2}$

    (c) $11\frac{1}{2}$

17. If A is Alfred's age and K is Kasule's age, $A = 5 + K$.

18. If b is the boy's age and s his sister's age, $b = s - 8$.

19. If f is the father's age and s the son's age, $f = 4s$.

20. If f is the father's age and s the son's age, $f = 7s$.

21. If g is the number of girls and b the number of boys, $g = \frac{2}{3}b$.

22. If n is the number of tickets bought, the total number of shillings is $4n$.

23. $\frac{1}{x}$

24. $\frac{y}{6}$ of the pool

25. If f is the length of the first side and s the length of the second side, $s = \frac{f}{2} - 2$.

26. If the plant grows n inches per week, in five weeks its height will be $10 + 5n$.

27. If d miles is the distance travelled in h hours, $d = 8h$.

28. $\frac{2000}{h}$ miles per hour.

29. If t hours is the time he walks at a speed of s miles per hour, $t = \frac{23}{s}$.

30. If n feet is the length of the rectangle and w feet its width, $n = 10 + w$.

31. $(50 + 2w)$ feet.

32. $\frac{600}{w}$ feet.

33. If w feet is the width of the rectangle, $(2w - 5)$ feet is the length.

34. $(6w - 10)$ feet, or $2w + 2(2w - 5)$ feet

35. $(2w - 5)w$ square feet is the area.

36. $n + 2$

37. $m + 2$

9-4 Writing Open Sentences

Now that the student has learned to express one number which occurs in a problem in terms of another he is ready to translate problems about two numbers into open sentences. Here one learns by doing, after studying several examples. The sample problems should be studied with care. In this section the emphasis is in writing the open sentence, not in solving it. That comes later.

Answers to Problems 9-4 | Student Text Page 207

1. If x feet is the length $25x = 6125$.

2. If w inches is the width, $14w = 144$.

3. If w inches is the width, $96 + 4w = 100$.

4. If w inches is the width, $20 + 4w = 140$.

5. If w feet is the width, $w(3 + w) = 70$.

6. If n is the number, $n + \frac{3}{8}n + \frac{5}{16}n = 7\frac{1}{2}$.

7. If n is the number of which I am thinking, $\frac{1}{3}(6n - 18) = -2$.

8. If n is the smaller of the two numbers, $n + (n + 2) = 54$.

9. If n is the smaller of the two numbers, $n + (n + 2) = 208$.

10. If n is the smaller of the two numbers, $(10 - n) - n = 2$.

11. If n is the number of 3 shilling books purchased, $3n + 8(20 - n) = 110$.

12. If s is the number of shilling pieces in the box, $\frac{s}{2} + \frac{s}{20} = 154$.

13. If a is the number of shillings A gets, $a + \frac{a}{2} + \frac{3}{2}a = 102$.

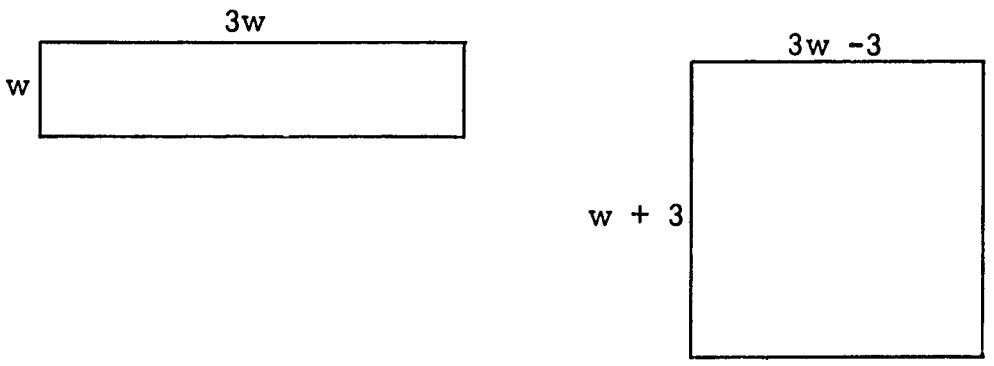
14. If m is the cost of each mug, $3m + 3(3 + m) = 24$.

15. If s is the son's age now, his father's present age is $4s$. Then $(4s - 10) + (s - 10) = 60$.

16. If s is the sister's present age, $s - 3$ is the boy's present age. Then $(s - 3) - 3 = \frac{2}{3}(s - 3)$.

17. If w feet is the width, the length is $3w$ feet. The dimensions of the square are: length $3w - 3$ feet and width $w + 3$ feet. Then

$$3w - 3 = w + 3.$$



18. If s is the smaller number, $s + 15$ is the larger number. $5(s + 15) - 2s = 3$

19. If n is the number, $\frac{n - 7}{5} = \frac{n - 5}{7} + 4$.

20. If x years is the age of the oldest son, $x + (x - 2) + (x - 4) + (x - 6) = 52$.

21. Many possible answers. For example:

(a) In another year I shall be 18. How old am I now?

(b) If you pay out 5 shillings and want to have 20 shillings left how much money must you have?

(c) A father is three times as old as his son. In 8 years he will be 35. How old is the son?

(d) A rectangle has an area of 70 square inches. Its length is 3 inches more than its width. How long is it?

22. If b is my brother's age, $b + (b + 3) = 185$.
A boy's father has three more pounds than the boy's uncle. Together they have 185 pounds. How many pounds does each one have?

9-5 Problems Involving Inequalities

Certain problems lead to open sentences that are not equalities because they contain phrases like "greater than" or "less than". The solution set in such a case includes a number of possibilities.

Answers to Problems 9-5 | Student Text Page 210

1. If x is the length of the third side, $4 < x < 24$.
2. If x is any other member of the set of integers, $512 < x < 999$.
3. If x is any other even integer in the set, $512 < x < 800$.
4. If x is any rational number of the set different from $-\frac{5}{16}$ and $\frac{5}{16}$, $-\frac{5}{16} < x < \frac{5}{16}$.
5. If c cents is the amount the boy lost, $0 < c \leq 10$.
6. If t minutes is his time, $t < 4 \cdot 1$.
7. If t minutes is his time, $t \geq 4 \cdot 1$.
8. If x marks are scored in the third test, $\frac{70 + 76 + x}{3} \geq 80$.
Solution: $x \geq 94$

9-6 Solving Word Problems

In this section we find the truth set of the open sentence which translates the given problem. The method is the familiar one of writing equivalent statements until the truth set is evident.

At the same time the steps in solving a word problem are summarized. Encourage the students to write their answers clearly and fully and to check their correctness against the problem itself.

Answers to Problems 9-6

Student Text Page 212

1. 245 feet
2. $10\frac{2}{7}$ inches
3. 1 inch
4. 30 inches
5. 7 feet (By trial)
6. $\frac{40}{9}$
7. 2
8. 26 and 28
9. 103 and 105
10. 4 and 6
11. 10
12. 280
13. 34 shillings for A, 17 shillings for B, and 51 shillings for C
14. $2\frac{1}{2}$ shillings
15. 16 years for the son, 64 years for the father
16. 12 years for the sister, 9 years for the boy
17. 9 feet long and 3 feet wide
18. The smaller number is -24 and the larger number is -9.
19. 82
20. 16 years
22. 91 pounds

CHAPTER 10

EQUATIONS IN TWO VARIABLES

In this chapter we first examine a single equation in two variables and observe that members of its truth set are ordered pairs of numbers, one for the first variable and one for the second variable. Usually there are many such ordered pairs of numbers in the truth set.

By considering a problem which leads to two equations we find that there is usually only one ordered pair which will satisfy both of two equations. This means that if we have a method of finding such an ordered pair, we have a means of answering a great variety of problems. We proceed to develop a method of solving systems of two linear (first degree) equations in two variables. Later (in Chapter 14) a graphical method of solving such systems is presented. Questions of higher degree equations or of equations in three or more variables will be postponed to later years. It will be helpful to the teacher, however, if he has some understanding of these extensions of the idea as background for the present work.

The chapter is concluded with some problems illustrating the convenience of using two variables, and with problems in which a choice can be made between a one variable solution and a two variable solution.

10-1 An Equation in Two Variables

The main objective of this section is to help the students get used to thinking of two numbers at once, a first one and a second one. In particular we are interested in such ordered pairs of numbers as will make a given sentence true.

The table of pairs of numbers satisfying the equation $x + y = 8$ might be extended as shown. Since x and y stand for numbers of children, they must be whole numbers and these are the only possibilities.

x	y
2	6
7	1
0	8
6	2
1	7
3	5
4	4
5	3
8	0

If possible, lead the students to observe for themselves that 2 in the first column and 6 in the second column does not describe the same situation as 6 in the first column and 2 in the second. Observe it for other pairs of numbers. Then as they grow familiar with the notation for ordered pairs observe the difference in meaning between (7, 1) and (1, 7), etc.

There are 9 different ordered pairs of numbers which express the number of children of Mr. X and Mr. Y, as can be seen from the table.

We shall usually use x and y as the two variables. If other letters, such as p and q, are used, either an arbitary choice must be made as to which is the "first" variable, or frequently they are taken in the order in which they occur in the alphabet.

In the example $2x + y = 10$, nothing has been said to restrict the variables to whole numbers. If possible, let the students discover this themselves when they are asked to find more than six ordered pairs satisfying the equation. Once they start using other rational numbers, they will be able to produce many solutions. Help them conclude that there is an unlimited number of solutions in this case. Also help them realize that (2, 6) is a solution but (6, 2) is not.

Answers to Problems 10-1 | Student Text Page 214

1. Most of the present problems are simple enough so that the following form is sufficient for the student to write in verifying a solution of an equation.

   $(2 \times 1) + 10 = 12$ is true.

   $(2 \times 5) + 2 = 12$ is true.

   $(2 \times 0) + 12 = 12$ is true.

   $(2 \times 8) + (-4) = 12$ is true.

2. $(2 \times 6) + 0 = 12$ is true.

   $(2 \times 2) + 8 = 12$ is true.

   $(2 \times \frac{1}{2}) + 11 = 12$ is true.

3. $(2 \times 4) + 6 = 12$ is false.

   $(2 \times 2) + 5 = 12$ is false.

   $(2 \times 6) + 1 = 12$ is false.

4. If some students find fancy solutions, such as $(\frac{57}{8}, -\frac{9}{4})$, they will need a more complete form to show their verification. The following is recommended.

   $(2 \times \frac{57}{8}) + (-\frac{9}{4}) = 12$ is true because

   $(2 \times \frac{57}{8}) + (-\frac{9}{4}) = \frac{57}{4} + (-\frac{9}{4}) = \frac{48}{4} = 12$

   This form will later make clear the verifications of solutions of more involved equations. For instance to show that $(\frac{1}{24}, \frac{5}{9})$ is a solution of the equation $x + 7y = 3 + 9x + y$ we would write:

$$\frac{1}{24} + (7 \times \frac{5}{9}) = 3 + (9 \times \frac{1}{24}) + \frac{5}{9} \text{ is true because}$$

$$\begin{aligned}
\frac{1}{24} + (7 \times \frac{5}{9}) &= \frac{1}{24} + \frac{35}{9} \\
&= (\frac{1}{24} \times \frac{3}{3}) + (\frac{35}{9} \times \frac{8}{8}) \\
&= \frac{3}{72} + \frac{280}{72} \\
&= \frac{283}{72}
\end{aligned}$$

and

$$\begin{aligned}
3 + (9 \times \frac{1}{24}) + \frac{5}{9} &= 3 + \frac{3}{8} + \frac{5}{9} \\
&= (3 \times \frac{72}{72}) + (\frac{3}{8} \times \frac{9}{9}) + (\frac{5}{9} \times \frac{8}{8}) \\
&= \frac{216}{72} + \frac{27}{72} + \frac{40}{72} \\
&= \frac{283}{72}
\end{aligned}$$

5. $(1, 1), (5, -2)$

6. $(11, 2), (2, 5), (-4, 7), (-283, 100), (23, -2), (13, \frac{4}{3}), (24, -\frac{7}{3}), (\frac{31}{2}, \frac{1}{2}), (6 \cdot 5, 3 \cdot 5)$

7. Previous to this we have not suggested any systematic way to find pairs of numbers which will satisfy an equation. The student may just be making good guesses or they may be working out methods of their own. This is fine. It is time, though, to help them see that for any given value of y, they can methodically find the corresponding value of x. At first they may substitute the value of y and solve the resulting equation for x each time.

For example, if $y = -\frac{37}{18}$, then

$$x + [3 \times (-\frac{37}{18})] = 17$$

$$x + (-\frac{37}{6}) = 17$$

$$x = 17 + \frac{37}{6}$$

$$x = \frac{102}{6} + \frac{37}{6}$$

$$x = \frac{139}{6}$$

Hence, $(\frac{139}{6}, -\frac{37}{18})$ is a solution.

If however, they are trying to find several solutions, help them to discover that they can save themselves work by solving the equation for x in terms of y before they substitute any of the given values of y.

$$x + 3y = 17$$

$$x = 17 - 3y$$

Then they have a formula which will give them more directly the value of x for any value of y.

8. It is hoped that the students will use the method suggested in Problem 7. Since values of x are given, the equation should be solved for y in terms of x.

$$2x - 3y = 8$$

$$2x - 8 = 3y$$

$$\frac{1}{3}(2x - 8) = y$$

$(1, -2)$, $(7, 2)$, $(-2, -4)$, $(5, \frac{2}{3})$, $(\frac{1}{2}, -\frac{7}{3})$

9. See Problem 8. If x is $\frac{57}{23}$,

$$y = \frac{1}{3}\left[\left(2 \times \frac{57}{23}\right) - 8\right]$$
$$= \frac{1}{3}\left(\frac{114}{23} - \frac{184}{23}\right)$$
$$= \frac{1}{3}\left(-\frac{70}{23}\right)$$
$$= -\frac{70}{69}$$

Hence, $\left(\frac{57}{23}, -\frac{70}{69}\right)$ is a solution.

Some of the solutions should be verified by direct substitution into the original equation.

10. Each of the given ordered pairs satisfies the equation. Any value of x less than zero would not be a whole number so it is not allowed.

Any value of x greater than 5 makes y negative so is not allowed. Hence this is the complete truth set.

11. There is an unlimited set of solutions of $x-y = 2$. Some of them are $(2,0)$, $(3,1)$, $(4,2)$, $(5,3)$. It is not possible to write all of them, but any one of them is of the form $(x,x-2)$, where x is a whole number greater than 1.

12. <u>CHALLENGE PROBLEM</u>. If the larger whole number is x and the smaller whole number is y, then the equation is $2x + 3y = 24$ and the truth set is $\{(12, 0), (9, 2), (6, 4)\}$. Notice that $(3, 6)$ cannot be used because we chose the first number to be the larger number. If a pupil chooses x as the <u>smaller</u> whole number, the equation is $2y + 3x = 24$ and the truth set is $\{(0, 12), (2, 9), (4, 6)\}$. In either case the required numbers are 12 and 0, 9 and 2, or 6 and 4.

10-2 A Problem in Two Variables

A number problem quite naturally leads us into considering two equations connected by the conjunction "and", for which we are seeking a pair of numbers which will make both equations true. It is not enough to make only one of them true.

Our first method of trying to find a solution is not very efficient but hopefully it helps the pupil to understand exactly what is meant by a solution of the system of simultaneous equations. Separate lists are made of members of the truth sets of the two equations. A solution of the system must be in both lists. With good fortune and a little cleverness one may find a common solution this way. Even if a solution is found, there is nothing about the method which assures us that there are not more solutions of the system. Nevertheless, a little experimenting with this viewpoint is of value for understanding.

Answers to Problems 10-2

Student Text Page 218

1. (3,-2)

2. (a) $\{(0,9), (2,6), (4,3), (6,0)\}$
   (b) $\{(0,30), (1,25), (2,20), (3,15), (4,10), (5,5), (6,0)\}$
   (c) (6,0)

3. (a) $\{(0,7), (5,5), (10,3), (15,1)\}$
   (b) $\{(0,23), (1,17), (2,11), (3,5)\}$
   (c) There are no pairs in both sets, so the truth set in the domain of whole numbers of the system of two equations is the empty set.

10-3 Systems of Simultaneous Equations

The systematic method which we use here for finding the truth set of a system of two simultaneous equations in two variables is built on the idea of removing one of the variables to give simpler equations with only one variable which we know how to solve. Of course, the new systems of equations must be equivalent to (that is, have the same truth set as) the original system.

Instead of trying to develop a complete theory of equivalent systems of equations, it seems better at this stage -- simpler for the students to understand -- to work in one direction from the original equation and then check. Thus we reason as follows:

If there is a pair of numbers (x, y) which satisfies the system

$$\begin{cases} 5x + 3y = 9 \\ -5x + 2y = 1 \end{cases}$$

then we can show that this same pair of numbers satisfies the system

$$\begin{cases} 5x + 3y = 9 \\ y = 2 \end{cases}$$

and in turn satisfies the system

$$\begin{cases} x = \frac{3}{5} \\ y = 2 \end{cases}$$

In other words, the truth set of the first system is a subset of the truth set of the third system.

In order to know that the truth sets are the same we must show that the truth set of the third system is a subset of the truth set of the first one. We can show that by checking the only solution of the third system, $(\frac{3}{5}, 2)$, in the first system. We do not say all this to the students, but the teacher should be aware of this logical basis.

The addition idea which is used to remove a variable could be formally stated:

If a, b, c, d are numbers, and
if $a = b$ and $c = d$, then
$a + c = b + d$.

The reasoning which explains this, however, is little more than the reasoning given in the text in an informal way. It depends on the closure property of addition. If two numbers are added, the sum is a unique number regardless of what names have been used for the numbers. If a and b are names for a number and c and d are names for another number (or the same one), then the unique sum of the two numbers can be named in various ways, one of which is $a + c$, another is $b + d$. Of course one could also choose the names $a + d$ and $b + c$, but we prefer to choose the ones which serve our purpose.

In this case we have planned ahead. Since the sum of $5x$ and $-5x$ is zero, we can see that addition will give an equation without the variable x.

Since any numbers which make both equations in the original system true will also make the simple equation $y = 2$ true, we can use the new system.

$$\begin{cases} 5x + 3y = 9 \\ y = 2 \end{cases}$$

or equally well the system

$$\begin{cases} y = 2 \\ -5x + 2y = 1 \end{cases}$$

Encourage flexibility in replacing either the first or the second equation by the equation in one variable. Having found y, we substitute this value in whichever of the original equations is then easier to solve for x.

When we reach the system

$$\begin{cases} x = \frac{3}{5} \\ y = 2 \end{cases}$$

we have showed that if an ordered pair satisfies the original system, <u>then</u> it satisfies this final very simple system whose truth set we can see by inspection. It remains to be proved that if an ordered pair of numbers satisfies the final system, <u>then</u> it satisfies the original system. This is why we must always check the ordered pair we obtain by substituting into both of the equations we started with.

In the second example we observe $4y$ in both equations. It is this which suggests using subtraction because $4y - 4y = 0$. Encourage the students to watch the terms in the given equations and to be able to predict whether addition or subtraction will give an equation with only one variable. Write several systems of equations on the board and have them tell which variable can easily be eliminated and whether to use addition or subtraction. Then have them make up some systems and tell the same thing. Possibly they will give some systems where neither addition nor subtraction will work as they stand. This will show nicely the need for an extension of the method, which comes in the next section.

We follow our policy of trying to keep the reasoning alive by asking occasionally for the reasons which justify the detailed steps. The reasons in the second example are as follows:

$(2x + 4y)-(5x + 4y) = (-12)-(-33)$	(Subtraction using different names for the same numbers)
$-3x = 21$	(Simplification using definition of subtraction and the associative, commutative and distributive properties)
$x = -7$	(Multiplication by $-\frac{1}{3}$)

$$\begin{cases} 2x + 4y = -12 \\ x = -7 \end{cases}$$

(Can replace $5x + 4y = -33$ by $x = -7$ because any pairs which make both equations true also make $x = -7$ true)

$2(-7) + 4y = -12$ (x must be -7 in both equations)

$4y = 2$ (Addition of 14)

$y = \frac{1}{2}$ (Multiplication by $\frac{1}{4}$)

$$\begin{cases} x = -7 \\ y = \frac{1}{2} \end{cases}$$

(Can replace $2x + 4y = -12$ by $y = \frac{1}{2}$ because any pairs which make both equations true also make $y = \frac{1}{2}$ true)

Notice that we have rearranged the equations, writing $x = -7$ first. This is because of our habit of calling x the first variable. In writing the ordered pairs $(-7, \frac{1}{2})$ we are careful to put the value of x first.

Answers to Problems 10-3

Student Text Page 220

1. $\{(5,3)\}$
2. $\{(3,-2)\}$
3. $\{(5, \frac{3}{2})\}$
4. $\{(-3, -\frac{1}{3})\}$

10-4 More Systems of Simultaneous Equations

We use the word "coefficient" in this discussion. The pupils have met it before but they may need to be reminded. In $3x + 2y = 12$, the coefficient of y is 2. In $7x - 4y = 2$, the coefficent of y is -4.

We hope that the students will discover that (1) it would be convenient to multiply $2y$ by 2 in the first equation to make it $4y$, (2) we are

permitted to multiply $2y$ by 2 if we also multiply $3x$ and 12 by 2 in the first equation, because the result is an equivalent equation. Since the coefficient of y is now 4 in one equation, and -4 in the other, addition will eliminate the variable y.

Completion of the first example:

By addition,

$$(6x + 4y) + (7x - 4y) = 24 + 2$$
$$13x = 26$$
$$x = 2$$

$$\begin{cases} x = 2 \\ 7x - 4y = 2 \end{cases}$$

$$(7 \times 2) - 4y = 2$$
$$-4y = -12$$
$$y = 3$$

$$\begin{cases} x = 2 \\ y = 3 \end{cases}$$

Check: $(3 \times 2) + (2 \times 3) = 12$ is true.

$(7 \times 2) - (4 \times 3) = 2$ is true.

The truth set is $\{(2,3)\}$.

Completion of the second example:

By subtraction,

$$(21x + 12y) - (10x + 12y) = 129 - 52$$
$$11x = 77$$
$$x = 7$$

$$\begin{cases} x = 7 \\ 7x + 4y = 43 \end{cases}$$

$$(7 \times 7) + 4y = 43$$
$$4y = -6$$
$$y = -\frac{3}{2}$$

$$\begin{cases} x = 7 \\ y = -\frac{3}{2} \end{cases}$$

Check: $(7 \times 7) + [4 \times (-\frac{3}{2})] = 43$ is true.

$(5 \times 7) + [6 \times (-\frac{3}{2})] = 26$ is true.

The truth set is $\left\{ (7, -\frac{3}{2}) \right\}$.

To find the solution of the second example by eliminating the variable x, the children should multiply in the first equation by 5, in the second equation by 7, and subtract.

If in a system such as

$$\begin{cases} 13x + 18y = 5 \\ 17x + 24y = 7 \end{cases}$$

the pupils have any difficulty in deciding what numbers to multiply by in the two equations to make the coefficients of y the same ask them how they would get equivalent fractions with the same denominator in the sum $\frac{13}{18} + \frac{17}{24}$.

Both problems involve finding the least common multiple of 18 and 24.

Answers to Problems 10-4 | Student Text Page 222

1. $\{ (5,4) \}$
2. $\{ (4,1) \}$
3. $\{ (\frac{5}{8}, 4) \}$
4. $\{ (-2, -3) \}$
5. $\{ (10,-9) \}$
6. $\{ (1, -\frac{1}{3}) \}$
7. $\{ (2, -7) \}$
8. CHALLENGE PROBLEM The equations

$$x - 4y = 3$$

and

$$x = 3 + 4y$$

are equivalent ($4y$ has been added to each side of the first equation to obtain the second), so one can be replaced by the other in the system without changing the truth set.

Since for any ordered pair in the truth set, x and (3 + 4y) are names for the same number, we may replace x by (3 + 4y) in the first equation without changing the truth set. The resulting equation has only one variable.

$$3(3 + 4y) - 5y = 21$$
$$9 + 12y - 5y = 21$$
$$7y = 12$$
$$y = \frac{12}{7}$$
$$\begin{cases} y = \frac{12}{7} \\ x = 3 + 4y \end{cases}$$
$$x = 3 + (4 \times \frac{12}{7})$$
$$x = \frac{21}{7} + \frac{48}{7}$$
$$x = \frac{69}{7}$$
$$\begin{cases} x = \frac{69}{7} \\ y = \frac{12}{7} \end{cases}$$

Check: $(3 \times \frac{69}{7}) - (5 \times \frac{12}{7}) = 21$ is true because

$(3 \times \frac{69}{7}) - (5 \times \frac{12}{7}) = \frac{207}{7} - \frac{60}{7} = \frac{147}{7} = 21.$

$\frac{69}{7} - (4 \times \frac{12}{7}) = 3$ is true because

$\frac{69}{7} - (4 \times \frac{12}{7}) = \frac{69}{7} - \frac{48}{7} = \frac{21}{7} = 3.$

The truth set is $\left\{(\frac{69}{7}, \frac{12}{7})\right\}$.

The above method, sometimes called the <u>method</u> of <u>substitution</u>, is often convenient when it is easy to solve one of the equations for one variable in terms of the other. We make the method available to those students who are interested, but we do not consider it a necessity at present.

9. CHALLENGE PROBLEM (a) Solve the second equation for y in terms of x, giving $y = 6 - 8x$. Then use $(6 - 8x)$ for y in the first equation. The truth set is $\left\{\left(\frac{1}{2}, 2\right)\right\}$.

(b) Solve the first equation for x in terms of y, giving $x = 5 - 3y$. Then use $(5 - 3y)$ for x in the second equation. The truth set is $\left\{\left(6, -\frac{1}{3}\right)\right\}$.

10. CHALLENGE PROBLEM From the first equation, $x = -\frac{1}{3}$. Use this value for x in the second equation to show that $y = 6$. The truth set is $\left\{\left(-\frac{1}{3}, 6\right)\right\}$.

10-5 Word Problems Using Two Variables

In solving word problems we have an opportunity to practice clear self expression and good form in written work. Help the students to develop good habits in such things as

1. Describing the variables clearly and showing the units of measure at the start. Not "Let x = butter", which is nonsense, but "If he bought x pounds of butter".

2. Showing sufficient steps in solving the system of equations so the method is clear.

3. Checking in the words of the original problem.

4. Showing the answers to the questions clearly. Not "$x = 3$", but "He bought 3 pounds of butter". Steps 3 and 4 should also help the student eliminate nonsense answers to problems where the domains of the variables are restricted. For example, in a problem involving "He bought x pounds of butter", a solution of "$x = -3$" might be correct for the system of equations, but is not for the word problem.

Answers to Problems 10-5

Student Text Page 224

1. The numbers are 10 and -3.

2. The numbers are $\frac{11}{8}$ and $\frac{5}{6}$.

3. If there are b boys and g girls,

then

$$\begin{cases} b = g + 10 \\ b = 2(g + 1). \end{cases}$$

There are 18 boys and 8 girls.

4. If one pencil costs p shillings and one ruler costs r shillings, then

$$\begin{cases} 12p + 10r = 21 \\ 20p + 4r = 16. \end{cases}$$

Pencils cost $\frac{1}{2}$ shilling each and rulers cost $1\frac{1}{2}$ shillings each.

5. If the size of $\hat{B}$ is x° and the size of $\hat{C}$ is y°, then

$$\begin{cases} x + y + 20 = 180 \\ x = 9y + 10 \end{cases}$$

$\hat{B}$ is an angle of 145° and $\hat{C}$ is an angle of 15°.

6. If the longer piece was x feet long and the other piece was y feet long, then

$$\begin{cases} x = y + 18 \\ x = 3y \end{cases}$$

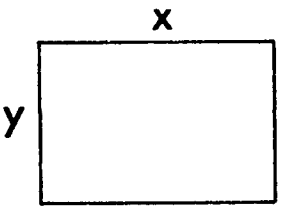
One piece was 27 ft. long and the other was 9 ft. long. The original rope was 36 ft. long.

7. If the size of the angle is x° and the size of its supplement is y°, then

$$\begin{cases} x + y = 180 \\ x = 3y \end{cases}$$

The size of the angle is 135°.

8. If the rectangle is x inches long and y inches wide, then



$$\begin{cases} 2x = 3y \\ 2x + 2y = 320 \end{cases}$$

The rectangle is 96 inches long and 64 inches wide.

9. If he walked at w miles per hour and bicycled at b miles per hour, then

$$\begin{cases} 4w + 3b = 45 \\ 2w + 4b = 50 \end{cases}$$

He walked at 3 m.p.h. and bicycled at 11 m.p.h.

10. If his normal rate of pay is x shillings per hour and his overtime rate is y shillings per hour, then

$$\begin{cases} 21x + 9y = 56 \\ 27x + 3y = 52 \end{cases}$$

His normal rate of pay is $1\frac{2}{3}$ shillings per hour and his overtime rate is $2\frac{1}{3}$ shillings per hour.

11. <u>CHALLENGE PROBLEM</u> If the tens' digit is x and the units digit is y, the number is $10x + y$ and the number with the digits reversed is $10y + x$.

$$\begin{cases} 10x + y = (10y + x) - 27 \\ x + y = 13 \end{cases}$$

$$\begin{cases} 9x - 9y = -27 \\ x + y = 13 \end{cases}$$

The number is 58.

12. <u>CHALLENGE PROBLEM</u> If the tens' digit is x and the units' digit is y, then

$$\begin{cases} (10x + y) + 36 = 10y + x \\ y = 2x + 1 \end{cases}$$

The number is 37.

13. <u>CHALLENGE PROBLEM</u> If the speed of the first train is x m.p.h. and the speed of the second train is y m.p.h. then

$$\begin{cases} 3x + \frac{11}{4}y = 200 \\ y = x + 10 \end{cases}$$

The speed of the first train is 30 m.p.h. and the speed of the second train is 40 m.p.h.

14. <u>CHALLENGE PROBLEM</u> If the speed of the first train is x m.p.h. and the speed of the second train is y m.p.h. then

$$\begin{cases} 3x + \frac{11}{4}y = 200 \\ \frac{11}{4}x + 3y = 200 \end{cases}$$

Both trains travel at the speed of $34\frac{18}{23}$ m.p.h. or about 34·8 m.p.h.

15. <u>CHALLENGE PROBLEM</u> If there are x cu. ft. of one liquid and y cu. ft. of the other, then

$$\begin{cases} x + y = 10 \\ 2x + 3y = 24 \end{cases}$$

There are 6 cu. ft. on one liquid and 4 cu. ft. of the other.
The weights are both 12 lbs. weight.

<u>Answers to Problems 10-6</u> | Student Text Page 227

1. <u>Two variable solution.</u>

Let n be the larger of the two numbers and s the smaller. Then

$$\begin{cases} n + s = 23 \\ 2n - 4s = 4 \end{cases}$$

or

$$\begin{cases} n + s = 23 \\ n - 2s = 2 \end{cases}$$

The solution set is $\{16, 7\}$.

<u>One variable solution.</u>

Let n be the larger number. Then $23 - n$ is the smaller number. Therefore

$$2 - 4(23 - n) = 4$$

which is equivalent to

$$n = 16$$

Then $$23 - n = 7.$$

There is not much difference in the difficulty of these two solutions.

2. <u>Two variable solution.</u>

Let n feet be the length and w feet the width of the rectangle. Then

$$\begin{cases} 2n + 2w = 80 \\ 2n = 3w \end{cases}$$

The rectangle is 24 feet long and 16 feet wide.

<u>One variable solution.</u>

Let n feet be the length and $\frac{2}{3}n$ feet the width of the rectangle. Then

$$n + \frac{2}{3}n = 40$$

<u>Another one variable solution.</u>

Let n feet be the length and $40 - n$ feet the width of the rectangle. Then

$$2n = 3(40 - n)$$

3. There is only one variable. The given integer must be of the form $10x + 5$. If the digits are reversed the new number is $50 + x$.

Then

$$50 + x = (10x + 5) + 18$$

and $$x = 3.$$

4. Let x° be the size of the smaller angle and $(x + 16)^{\circ}$ the size of the larger angle. Then $x + (x + 16) = 90$.

The angles have sizes 37° and 53°.

This is a little simpler than a two variable solution in which $x + 16$ is replaced by y, and we have the two equations

$$y = x + 16$$

$$x + y = 90$$

5. <u>Two variable solution.</u>

Let m be the number of measures in the mule's load and d the number of measures in the donkey's load. Then

$$\begin{cases} m + 1 = 2(d - 1) \\ m - 1 = d + 1 \end{cases}$$

or $$\begin{cases} m - 2d = -3 \\ m - d = 2 \end{cases}$$

The mule's load is 7 measures, the donkey's 5 measures.

6. Let the cost of the bottle be b cents and the cost of the contents $(b + 18)$ cents.

$$b + (b + 18) = 60$$

$$b = 21$$

$$b + 18 = 39$$

This is somewhat simpler than the two variable solution in which $b + 18$ is replaced by c and we have

$$\begin{cases} c + b = 60 \\ c - b = 18 \end{cases}$$

7. <u>Two variable solution</u>.

Let $\frac{x}{y}$ be the given fraction. Then

$$\begin{cases} \dfrac{x + 5}{y + 5} = \dfrac{4}{7} \\ \dfrac{x - 1}{y - 1} = \dfrac{2}{5} \end{cases}$$

These equations may be replaced by

$$\begin{cases} 7x + 35 = 4y + 20 \\ 5x - 5 = 2y - 2 \end{cases}$$

or

$$\begin{cases} 7x - 4y = -15 \\ 5x - 2y = 3 \end{cases}$$

The solution set is $\{7, 16\}$. Hence the given fraction is $\frac{7}{16}$.

8. <u>Two variable solution</u>.

Let x miles per hour be the speed of the boat in still water and y miles per hour the speed of the current. Then the speed of the boat downstream is $(x + y)$ miles per hour and the speed upstream is $(x - y)$ miles per hour. So

$$\begin{cases} (x + y) \times \dfrac{5}{2} = 45 \\ (x - y) \times \dfrac{13}{4} = 39 \end{cases}$$

These equations are equivalent to

$$\begin{cases} x + y = 18 \\ x - y = 12 \end{cases}$$

whose solution set is $\{15, 3\}$

CHAPTER 11

DECIMAL NUMERALS

This chapter is concerned with a particular way of representing numbers. Much of the work in earlier chapters has been concentrated on properties of the numbers, properties which are not dependent on the way the numbers are written. For example, the commutative property remains valid however we represent numbers. The statements $7 + 9 = 9 + 7$ and VII + IX = IX + VII are true regardless of what notation is used.

Here we study one important representation of numbers. We study decimal numerals, or more briefly "decimals".

The main feature of the chapter is the clarification, by means of expanded notation, of the four fundamental operations of addition, subtraction, multiplication, and division. With this clarification, the work on percentage follows readily.

11-1 Revision of Base Ten Numeration

Do this section in class as rapidly as possible. Begin by asking the class what several decimal numerals represent. Be sure that the index notation is clear to all. Problems 1 and 2 can probably be done orally, or with students putting the work on the board.

11-2 Addition and Subtraction with Decimal Numerals

This material, too, should be revised briefly.

Addition

Begin the discussion without reference to the text, by asking the pupils to explain simple sums. For example:

$$\begin{array}{r} 53 \\ \underline{42} \\ 95 \end{array} \qquad \begin{array}{r} 57 \\ \underline{64} \\ 121 \end{array} \qquad \begin{array}{r} 637 \\ \underline{795} \\ 1432 \end{array}$$

Keep asking questions until pupils, on their own, make use of expanded notation to explain their work. For example, in $57 + 64 = 121$, you might ask a pupil how he got the "2". If he talks about "carrying one", get him to explain why the "one" is really "ten". Perhaps he may argue that 57 plus 60 is 117 and then 4 more gives 121. Or that 50 plus 60 is 110, 7 plus 4 is 11, and their sum is 121, i.e.,

$$\begin{array}{r} 57 \\ 60 \\ \underline{4} \\ 117 \\ \underline{4} \\ 121 \end{array} \qquad\qquad \begin{array}{r} 50 \\ 7 \\ 60 \\ \underline{4} \\ 110 \\ \underline{11} \\ 121 \end{array}$$

While you would like everyone to discover for himself that

$$\begin{aligned} 57 + 64 &= [5(10) + 7] + [6(10) + 4] \\ &= 11(10) + 11 \\ &= 1(10^2) + 1(10) + 1(10) + 1 \\ &= 121 \end{aligned}$$

not everyone will think of doing it this way. However, when all are aware of how the notation might be used, then you can go to the text. Have pupils explain the steps in the text and how the basic principles are used. Also, show how these steps occur in the usual shorthand computational algorithm.

Subtraction

You may treat as a class activity the completion of the two examples given in the text. Have the pupils explain and discuss the steps in the simplification. Make certain that they really understand the process which is often called "borrowing", but which is really regrouping by the associative property of addition.

Example 1.

$$\begin{aligned} 38 - 26 &= [3(10) + 8] - [2(10) + 6] \\ &= [3(10) - 2(10)] + [8 - 6] \\ &= 1(10) + 2 = 12 \end{aligned}$$

Example 2.

$$\begin{aligned} 65 - 47 &= [6(10) + 5] - [4(10) + 7] \\ &= [5(10) + 15] - [4(10) + 7] \\ &= [5(10) - 4(10)] + [15 - 7] \\ &= 1(10) + 8 = 18 \end{aligned}$$

Answers to Problems 11-2

Student Text Page 230

1. (a) $3(10) + 5 = 35$
   (b) $1(10^4) + 1(10^3) + 7(10^2) + 9(10) + 1 = 11{,}791$
   (c) $5(10) + 2 = 52$
   (d) $8(10) + 8 = 88$

11-3 Multiplication with Decimal Numerals

By this time, a pupil should by himself be able to use expanded notation to explain the usual computational algorithm. It is of great importance that the pupil see how the distributive property is applied.

A good way to begin the discussion is to write a problem on the board such as is shown here. Then ask pupils to explain where each digit comes from in the computational algorithm. As soon as the class sees the idea, you can proceed to a study of the text, but do not stop class discussion just to go to the text. It is quite possible that some pupil may suggest something like this:

Algorithm for 42×27

$$\begin{array}{r} 27 \\ \underline{42} \\ 54 \\ \underline{108} \\ 1134 \end{array}$$

$$\begin{aligned} 42 \times 27 &= (40 + 2)(20 + 7) \\ &= [40 \times (20 + 7)] + [2 \times (20 + 7)] \\ &= [(40 \times 20) + (40 \times 7)] + [(2 \times 20) + (2 \times 7)] \\ &= [800 + 280] + [40 + 14] \\ &= 1080 + 54 \end{aligned}$$

Now the problem is reduced to a problem in addition, a problem which has all ready been discussed in Section 11-2. At this point you can once more ask the class where the different digits in the computational algorithm come from.

Observe that the explanation given by the pupil does not fully use the expanded form. This is quite all right. The purpose of this section is to make perfectly clear to each pupil the reasons underlying the algorithm which he may have learned by rote. Once this has been achieved, there is no necessity for insisting that the pupil write out everything in expanded form.

For example, the multiplication shown above may also be written as follows:

$$\begin{aligned} 42 \times 27 &= [4(10) + 2] \times [2(10) + 7] \\ &= 8(10^2) + 4(10) + 28(10) + 14 \\ &= 8(10^2) + 4(10) + 2(10^2) + 8(10) + 1(10) + 4 \\ &= 10(10^2) + 13(10) + 4 \\ &= 1(10^3) + 1(10^2) + 3(10) + 4 \end{aligned}$$

Remember: <u>Use of expanded notation is not a skill to be learned</u>. Its purpose is to clarify. The example in the text is simplified thus:

$$\begin{aligned} 34 \times 7 = [3(10) + 4] \times 7 &= [3(10) \times 7] + (4 \times 7) \\ &= 21(10) + 28 \\ &= 20(10) + 1(10) + 2(10) + 8 \\ &= 2(10^2) + 1(10) + 2(10) + 8 \\ &= 2(10^2) + 3(10) + 8 \\ &= 238 \end{aligned}$$

Some pupils may need help with the fourth line.

<u>Answers to Problems 11-3</u>

Student Text Page 231

1. (a) $1(10^2) + 4(10) + 8 = 148$

   (b) $3(10^2) + 9(10) + 6 = 396$

   (c) $6(10^3) + 0\ (10^2) + 2(10) + 7 = 6027$

11-4 Division with Decimal Numerals

Begin the discussion by reminding the class of the definition of division in the text. Then start doing a long division problem such as is shown on the right. Ask the class what the 2 means, and what the 94 means. Ask why one selects 2 rather than some other digit. These questions should promote a fair amount of discussion and argument. It is likely that pupils will not keep in mind or tell you the aim of the whole process. That is, they will lose sight of the fact that they are trying to find a decimal numeral for the number q such that

$$\begin{array}{r} 2 \\ 47\,\overline{)1081} \\ 94 \end{array}$$

$$1081 = 47\,q.$$

Keep asking questions until it happens (it may not happen so you will have to tell the class, but try to be patient) that some student remarks that q must be less than 30 because $47 \times 30 = 1410$, and $1410 > 1081$. Also q must be more than 20 because $47 \times 20 = 940$, and $940 < 1081$. Now the class should be able to answer two of your earlier questions. The 2 in the method stands for 20 and the 94 in the method stands for 940.

At this point you can either abandon your example of $1081 \div 47$ and go to the text, or you can continue on with the example. What you should do will depend on how the class feels. Hopefully, the class will want to continue. If so, ask a pupil to explain where the 141 comes from in the long division algorithm shown here. He should be able to tell you that 141 is the difference between 1081 and 940. In other words, he should know that he is looking for a number p such that

$$\begin{array}{r} 23 \\ 47\,\overline{)1081} \\ \underline{94} \\ 141 \\ \underline{141} \end{array}$$

$$1081 = 47(20 + p), \text{ or}$$
$$1081 = (47 \times 20) + (47 \times p), \text{ or}$$
$$141 = 47 \times p$$

The problem now is a simpler one. Obviously, $p = 3$, so

$$1081 = (47 \times 20) + (47 \times 3) = 47 \times 23, \quad \text{and}$$
$$1081 \div 47 = 23$$

Now turn to the text and ask the pupils to follow the two examples and answer the questions in the text.

<u>Answers to Problems 11-4</u>

Student Text Page 233

1. (a) Between 10 and 20
   (b) Between 300 and 400
   (c) Between 3,000 and 4,000

2. (a) $98 = 7x$
   $x = 10 + p$
   where p is between 0 and 10
   $98 = 7(10 + p)$
   $98 = 70 + 7p$
   $28 = 7p$
   $4 = p$
   $x = 10 + 4$

$$\begin{array}{r} 10 + \\ 7\overline{)98} \\ \underline{70} \\ 28 \end{array} \qquad \begin{array}{r} 10 + 4 \\ 7\overline{)98} \\ \underline{70} \\ 28 \\ \underline{28} \end{array} \qquad \begin{array}{r} 14 \\ 7\overline{)98} \\ \underline{7} \\ 28 \\ \underline{28} \end{array}$$

   (b) 36
   (c) 17
   (d) $73\frac{1}{35}$

11-5 Rational Numbers in Decimal Notation

The pupils already know how to write whole numbers in decimal notation. You might take them quickly through a few exercises of writing whole numbers in the expanded form.

You could then start a discussion on the place value of 3 in each of the following whole numbers: (a) 4300, (b) 430, (c) 43, (d) 43,000 and (e) 430,000. Because of the work they have already done on decimal notation they

should have no difficulty in explaining, for example, that the 3 in (a) is 10 times the 3 in (b) and also that the 3 in (b) is 10 times the 3 in (c).

Ask the pupils to write $\frac{43}{10}$ in the expanded form. If they are able to do so after some effort, the exercise may be extended to $\frac{43}{100}$.

They should then be asked to write the number $28\frac{43}{100}$ in the expanded form. If they are able to state that it is

$$2(10) + 8(1) + 4\left(\frac{1}{10}\right) + 3\left(\frac{1}{10^2}\right)$$

ask them whether they can say something about the place values of the digits 2, 8, 4, and 3 in the above expanded number. They should notice that the place value of each digit is 10 times the place value of the digit immediately on the right. Now tell them that we write the number $2(10) + 8(1) + 4\left(\frac{1}{10}\right) + 3\left(\frac{1}{10^2}\right)$ in decimal form as 28·43 and use the decimal point to separate the digits representing a whole number from the digits representing a number less than 1. It is read as "twenty-eight point four three".

<u>Answers to Problems 11-5</u> **Student Text Page 234**

1. (a) 432·56
   (b) 0·456
2. (a) $3(10^2) + 9(10) + 1 + 3\left(\frac{1}{10}\right) + 4\left(\frac{1}{10^2}\right) + 5\left(\frac{1}{10^3}\right)$
   (b) $6(10^2) + 0(10) + 0(1) + 0\left(\frac{1}{10}\right) + 1\left(\frac{1}{10^2}\right) + 0\left(\frac{1}{10^3}\right) + 2\left(\frac{1}{10^4}\right)$
   or $6(10^2) + 1\left(\frac{1}{10^2}\right) + 2\left(\frac{1}{10^4}\right)$

11-6 Addition and Subtraction of Rational Numbers

The work which the pupils have already done in addition and subtraction with whole numbers should simplify the work of this section.

Begin by considering a sum such as the following:

$$3 \cdot 64 + 4 \cdot 12$$

Ask the pupils to suggest how this should be done. They may be led to recall what was done with whole numbers and this will suggest the expanded form.

Thus

$$\begin{aligned} 3 \cdot 64 &= 3(1) + 6\left(\frac{1}{10}\right) + 4\left(\frac{1}{10^2}\right) \\ 4 \cdot 12 &= 4(1) + 1\left(\frac{1}{10}\right) + 2\left(\frac{1}{10^2}\right) \\ \hline 3 \cdot 64 + 4 \cdot 12 &= 7(1) + 7\left(\frac{1}{10}\right) + 6\left(\frac{1}{10^2}\right) \end{aligned}$$

Hence, the sum is 7·76. The use of the distributive law should be noted here as the teacher leads the pupils to see how each digit is obtained.

The pupils should soon reach the stage where the process of adding may be understood without using the expanded form.

Subtraction can be studied in a similar manner. Encourage the pupils to check subtraction by addition.

Stress the importance of keeping the decimal points directly under one another when writing decimal numerals to be added or subtracted.

$$\begin{array}{r} 3 \cdot 64 \\ \underline{4 \cdot 12} \\ 7 \cdot 76 \end{array} \qquad \begin{array}{r} 8 \cdot 73 \\ -\ \underline{3 \cdot 45} \\ 5 \cdot 28 \end{array}$$

Answers to Problems 11-6 | Student Text Page 235

1. (a) 9·43 (b) 16·482
2. (a) 1·11 (b) 0·1096

11-7 Multiplication and Division by Ten

The pupils have seen the distributive property applied to multiplication with decimal numerals (Section 11-3). A good way to start will, therefore, be to set a problem like $3 \cdot 2 \times 10$ for the children to do on the board in the expanded form.

The problem is done as follows:

$$\begin{aligned} 3 \cdot 2 \times 10 &= [3(1) + 2\left(\frac{1}{10}\right)] \times 10 \\ &= 3(10) + 2(1) \\ &= 32 \end{aligned}$$

At this stage the teacher could start a discussion with the pupils on place values of the digits in $3 \cdot 2$ and 32. In particular, the teacher could ask them whether they notice what has happened to the place values of the digits in $3 \cdot 2$ after it is multiplied by 10.

They may say that the place values of the digits have changed. Ask them to tell how. Some may be able to say that the value of each digit is now 10 times what it was at first. That is a correct answer, but in order to draw attention to the behavior of the digits in the decimal numeral, the teacher should lead them on to what they may have observed, namely that each digit has moved one place to the left with reference to the decimal point.

Let them see what happens in this example which may be worked on the board:

$72 \cdot 65 \times 10$	There is, as it were,
Result $= 726 \cdot 5$	"promotion" for each digit

Now ask them to tell what happens when a number is multiplied by 100 or 10^2. Lead them to make a general statement about multiplying a number by 10^n where n is a positive integer.

Division by Ten

This should, of course, follow from multiplication by 10. One example may be put on the board for the students:

$$24 \cdot 38 \div 10 \qquad \text{or} \qquad 24 \cdot 38 \times \frac{1}{10}$$

They proceed as follows:

$$24\cdot38 \div 10 = [2(10) + 4(1) + 3\left(\frac{1}{10}\right) + 8\left(\frac{1}{10^2}\right)] \times \frac{1}{10}$$

$$= 2(1) + 4\left(\frac{1}{10}\right) + 3\left(\frac{1}{10^2}\right) + 8\left(\frac{1}{10^3}\right) \qquad \text{(The distributive property)}$$

$$= 2\cdot438$$

Now ask them to tell what has happened to the digits. It is easy for them to see that the digits have been moved one place to the right with reference to the decimal point.

We may then find out from them whether there has been another "promotion" in this case. This, of course, will lead to their stating that there has been a demotion, i.e., the reverse process.

Then follow up by asking for a statement for dividing by 10^2, 10^3, and 10^n where n is a positive integer.

Answers to Problems 11-7 | Student Text Page 236

1. (a) 1·1; 11; 110
   (b) 0·12; 1·2; 12
   (c) 80·01; 800·1; 8001

2. (a) 7·1; 0·71; 0·071
   (b) 10·051; 1·0051; 0·10051
   (c) 0·0017; 0·00017; 0·000017

3. (e) 0·03888

4. (d) 1000

11-8 Decimal Places, Multiplication and Division

The purpose of this section is to show that multiplication and division involving rational numbers in decimal notation are the same as multiplication and division of whole numbers in decimal notation, except that decimal places come into the picture.

Start off by telling the pupils what is meant by decimal places and then have them give orally the number of decimal places of some numerals such as the following: 37·5, 0·24, 40·105, 0·0056, 149·35043.

The next step would be to pose the following problem: What is the truth set for the following equation?

$$2 \cdot 54 = 254 \times a$$

They should be able to give the answer $\left\{ \frac{1}{100} \right\}$. From the above, they are able to see that we can write $2 \cdot 54$ as $254 \times \frac{1}{100}$. Proceed to work $4 \times 2 \cdot 54$ on the board as follows:

$$\begin{aligned} 4 \times 2 \cdot 54 &= 4 \times \left(254 \times \frac{1}{100}\right) \\ &= 1016 \times \frac{1}{100} && \text{(Associative property of multiplication)} \\ &= 10 \cdot 16 \end{aligned}$$

The pupils may now do $1 \cdot 24 \times 3 \cdot 2$ in a similar way. The work will take this form:

$$\begin{aligned} 1 \cdot 24 \times 3 \cdot 2 &= \left(124 \times \frac{1}{100}\right) \times \left(32 \times \frac{1}{10}\right) \\ &= (124 \times 32) \times \left(\frac{1}{100} \times \frac{1}{10}\right) && \text{(Why?)} \\ &= 3968 \times \frac{1}{1000} \\ &= 3 \cdot 968 \end{aligned}$$

Ask the pupils what they notice about the decimal places in the answers of both examples and lead them on through inductive reasoning to the general statement: "When we multiply numbers in decimal notation, we find the product as with whole numbers and then fix the decimal point so that the number of decimal places in the product is the sum of the number of decimal places in the factors."

Division

We go about division first by doing one or two division examples on the blackboard with the pupils. The following will do:

$$\begin{array}{r} 0 \cdot 23 \\ 47 \overline{)10 \cdot 81} \\ \underline{9 \cdot 4} \\ 1 \cdot 41 \\ \underline{1 \cdot 41} \end{array} \qquad\qquad \begin{array}{r} 0 \cdot 0058 \\ 7 \overline{)0 \cdot 0406} \end{array}$$

They see that division is easy with whole number divisors.

The teacher may now ask them: "What should be done in a division like $10 \cdot 81 \div 4 \cdot 7$?" After some effort they might come out with the suggestion that if $4 \cdot 7$ were 47, the problem could be done easily. They may then be asked to put the problem in a form such that the divisor is a whole number.

If $108 \cdot 1 \div 47$ is suggested, ask for the explanation. This may lead to:

$$\frac{10 \cdot 81}{4 \cdot 7} = \frac{10 \cdot 81}{4 \cdot 7} \times \frac{10}{10}$$

We could now generalize by telling them that since it is easy to divide by whole numbers, we try to obtain whole numbers for divisors as was done above.

Answers to Problems 11-8 | Student Text Page 238

1. (a) $0 \cdot 026248$ (b) $0 \cdot 0813$

2. (a) $65 \cdot 62$ (b) $1 \cdot 3$

3. $0 \cdot 19525$

11-9 Rough Approximations in Multiplication and Division

This section may be introduced by a game in which the pupils make snap guesses of products. After a number of examples a discussion may be started by having the pupils whose approximations were good describe how they obtained their answers. This could lead to a further discussion of the value of such approximation.

The text does not attempt to give rules for approximations. A few examples are given but each pupil should feel free to use the method easiest for him in the particular example being tackled.

You are aware that almost always when we deal with decimal numerals we are concerned with measurement. For example $2 \cdot 5$ is likely to have appeared as a measure of length, weight, or capacity. In dealing with such measures we know that there is no point in keeping decimal places beyond those which can be

measured in the given situation or beyond those which are justified by the given measurements. The number of decimal places which have meaning depends on the accuracy with which the measurements have been and can be made.

In this section, we have not concerned ourselves with this aspect of decimal numerals. You will find that some of the products have been taken to five or more places. We have assumed that the given numerals are not the result of measurement but that they are exact representations of numbers, from which products can be found exactly. We are concerned now with the computation process with decimal numerals and the "exact products" are given to help demonstrate the usefulness of the rough approximation as a check on the computation.

Discuss with the class the difference between computation with exact numbers which we are studying here, and application to situations involving measurement. Make sure that the pupils understand that in the latter we must frequently throw away some of the last decimal places because they are meaningless.

Answers to Problems 11-9 | Student Text Page 240

1. (a) Wrong (d) Wrong
   (b) Wrong (e) Right
   (c) Right (f) Wrong

2. Approximately 250 shillings. If we estimate 5 dollars as about 40 shillings, we get a satisfactory estimate of 280 shillings.

3. 6 hrs., approximately

4. Between 35,000 and 25,000 sq. miles.

5. Ghana has approximately 150,000 sq. miles. Kenya has twice the area of Ghana.

6. (c) 0·01136

11-10 Rounding Off

Pupils may be asked to say whether the population of a country at 8 a.m. on a certain day will be the same as the population of the country at 8 p.m. on the same day. This should lead to the fact that the population may change at every instant of the day, and that it is practically impossible to obtain the exact number of people in a country at any one time, since difficulties may arise by some being in the process of being born and others in the process of dying. Discussion on the practical means of estimating the population of a country should reveal that an approximation to the population is all that is really necessary or practicable.

Records of weights and measures are liable to error because of the nature of the instruments. The measurement that we obtain for the length of a line segment depends on the accuracy of the instrument used, which in turn depends on physical conditions. Thus, the measurement we obtain for the length of a line segment can be only an approximation.

Suppose that we wish to find the area of a rectangle whose dimensions are measured to be 12·5 inches and 9·8 inches, carried to one decimal place, respectively. The statement that the length is 12·5 inches means that it is anything between 12·45 inches and 12·55 inches. In the same way, the breadth is between 9·75 inches and 9·85 inches. The area, then, is between $(12{\cdot}45 \times 9{\cdot}75)$ and $(12{\cdot}55 \times 9{\cdot}85)$ square inches, or between 121·5 square inches and 123·5 square inches. This demonstrates how the approximate nature of the measurements affects the product. We cannot even say what the area will be to the nearest square inch and certainly not to the nearest tenth of a square inch. The best we can say is that the area is about 122 or 123 square inches.

We do not now go into this matter of computation with numbers which are approximations, but it is a matter which should be studied later, and of which the teacher should be aware now.

In talking about mathematics, most of us have acquired language habits which do not always describe accurately the mathematical situation they deal with. An example of this appears in the text in this section when we talk of measuring a line segment. Strictly speaking, a line segment is an abstraction and

cannot be seen and, therefore, cannot be measured. An alternative, rather long-winded, but more accurate, description could be "measure the mark which represents the line segment". Of course, a line segment has associated with it a number which is its measure, but there is no physical measuring process for obtaining it.

A question may arise as to how we know at which level a number like 85,000 has been rounded off. Has it been rounded off at thousands, hundreds, tens or units? In our ordinary decimal numeral, there is no way of saying.

85,000 could be 85,251 or 84,763 rounded off at thousands
or 85,046 or 84,963 rounded off at hundreds
or 85,004 or 84,996 rounded off at tens
or 85,000·4 or 84,999·8 rounded off at units

If, however, we use standard notation to express the number, the accuracy of the number can be clearly indicated. For instance, with standard notation we know that $8 \cdot 50 \times 10^4$ has been rounded off at hundreds.

Not everyone uses the convention we have given for rounding off a number such as 1·45 to one decimal place. Instead the agreement is often to keep the nearest <u>even</u> digit in the last place retained. Thus

1·45 would be rounded off to 1·4
1·75 would be rounded off to 1·8
285,000 would be rounded off to 280,000
0·3615 would be rounded off to 0·362

This procedure is used only when the digit to be dropped is 5 and digits after that digit are zero or not known. An advantage of this convention is that, in a large number of computations with approximations, the errors caused by rounding off tend to balance out instead of accumulating, as they can when the next higher digit is always used.

Answers to Problems 11-10

Student Text Page 243

1. (a) 6,123,000 (b) 5,432,000 (c) 1,236,000

2. (a) 625 (b) 97 (c) 8 (d) 8

3. (a) 25·2 (b) 6·5 (c) 620·0 (c) 8·5

4. Correct to 2 decimal places

(i) 1·23 (ii) 9652·85 (iii) 2615·21 (iv) 8·65

Correct to 3 decimal places

(i) 1·235 (ii) 9652·855 (iii) 2615·211 (iv) 8·651

5. Correct to 2 decimal places

(i) 0·33 (ii) 0·17 (iii) 0·14 (iv) 0·27

(v) 0·56 (vi) 0·20 (vii) 1·89 (viii) 11·57

(ix) 7·54 (x) 5·32

Correct to 3 decimal places

(i) 0·333 (ii) 0·167 (iii) 0·143 (iv) 0·273

(v) 0·556 (vi) 0·200 (vii) 1·889 (viii) 11·571

(ix) 7·538 (x) 5·316

11-11 Percentages

The aim of this section is to introduce that aspect of rational numbers which we call percentages. The definition of the term percentage can be easily understood by the pupils if they recall what they have learnt earlier about fractions. They must have learnt, in particular, that $\frac{7}{10}$ means that a whole thing is divided into ten equal parts and 7 of those parts are represented by the fraction $\frac{7}{10}$. Every part is a tenth part of the whole.

Explain that a "per cent" means a hundredth part. You might at this stage ask a pupil to express "one per cent" as a fraction on the blackboard. He should

write $\frac{1}{100}$. Then ask for some others such as 8 per cent, 30 per cent, 89 per cent, etc. to be written as fractions.

To introduce the use of per cent in expressing a given ratio, you might talk about the boy who got 7 sums out of 10 correct. They know that the ratio of correct sums to total sums is represented by the fraction $\frac{7}{10}$. To show that $\frac{7}{10}$ is the same as $\frac{70}{100}$ they multiply by one in the form $\frac{10}{10}$. Then explain that if a fraction is transformed into one with 100 as its denominator, it is said to be expressed as a <u>percentage;</u> the numerator of the fraction is called the <u>rate</u>. In the percentage $\frac{75}{100}$, 75 denotes the rate. The symbol ($\% = \frac{1}{100}$) can now be introduced. Point out clearly that $\frac{75}{100}$, $75 \times \frac{1}{100}$, 75%, and 75 per cent are four ways of writing the same number.

Some people think of the symbol % as representing $\frac{\quad}{100}$, in other words, part of the fraction symbol. Thus, 35% can be written as $\frac{35}{100}$, the number in front of the symbol % being the numerator of the fraction. This is quite acceptable and may be taught. Whatever concept is taught, however, it is desirable that consistency be maintained throughout. If this second meaning is taught you would want to change the wording in some of the examples, e.g., "$35 \times \frac{1}{100}$" would be omitted in the first example.

<u>Answers to Problems 11-11A</u> | Student Text Page 245

1. $90\% = \frac{90}{100} = 90 \times \frac{1}{100} = 90$ per cent. Similarly for the others.

<u>Answers to Problems 11-11B</u> | Student Text Page 246

1.	(a)	$66\frac{2}{3}\%$	(f)	400%	(k)	60%
	(b)	50%	(g)	$33\frac{1}{3}\%$	(l)	80%
	(c)	75%	(h)	$166\frac{2}{3}\%$	(m)	100%
	(d)	$133\frac{1}{3}\%$	(i)	20%	(n)	$53\frac{1}{3}\%$
	(e)	140%	(j)	40%	(o)	120%

2. (a) $\frac{7}{200}$ (c) $\frac{13}{20}$

(b) $\frac{1}{3}$ (d) $\frac{11}{200}$

3. (a) £4 2s.6d. (d) £23 1s.0d.

(b) £4 4s.0d. (e) $187.57

(c) £50 8s.0d. (f) $12.48

4. $33 = \frac{11}{100} \times 300 = 11$ per cent of 300

5. 16%

6. $67\frac{23}{31}\%$

7. $78\frac{4}{7}\%$; 28.

8. 5,000

9. $18\frac{1}{3}\%$

10. 522

11. $31\frac{13}{17}\%$

12. 50%

11-12 Percentages and Decimal Numerals

The pupils already know how to convert fractions into percentages and vice versa. The aim of this section is to show how we can convert percentages into decimal numerals and vice versa.

The conversion is easily done by using the fraction form as an intermediate step. Thus it is a short step from 45% to $\frac{45}{100}$ and another short step from $\frac{45}{100}$ to 0·45. Point out the following pattern of converting back and forth between the different forms.

Percentage ⟷ Fraction with denominator 100 ⟷ Decimal numeral

Answers to Problems 11-12 Student Text Page 248

1. (a) 32·5% (d) 0·8%

(b) 5% (e) $85\frac{5}{7}\%$

(c) 230%

2. (a) 0·33 (d) 0·125
(b) 0·005 (e) 2·5
(c) 0·015 (f) 0·75

11-13 Application of Percentages

You might begin by discussing why anyone goes into business. Many reasons may be given for this, but it is generally true that a businessman wishes to make a profit. He buys and sells, and hopes to sell for more than his cost.

You may wish to give some oral exercises in profit and loss:

(a) If I buy a dozen eggs for 1s. 9d. and sell them at 2d. each, how much do I gain? Since I sell the eggs for 12 × 2d. or 2s., I gain the difference between 2s. and 1s. 9d. I gain 3d.

(b) If I buy six exercise books for 3s.6d., how much do I gain or lose if I sell them at 5d. each? I lose 1s. since I sell them for 6 × 5d. or 2s.6d. and 3s.6d. - 2s.6d. is 1s.

Lead the class to the next topic by asking the class which is the better transaction, buying a book for 10s. and selling it for 10s.6d., or buying a book for 5s. and selling it for 5s.6d?

The person who invested in the first book could have used the money to buy two of the second books and in that way gain 1s. Thus, the second transaction is better than the first.

Not every set of transactions may be so clear. Which is the best transaction, buying a book for 9s. and selling it for 12s., or buying a book for 8s. and selling it for 10s., or buying a book for 10s. and selling it for 13s.6d.? We shall be able to find the best transaction if we compare the ratios $\frac{3}{9}$, $\frac{2}{8}$, and $\frac{3\frac{1}{2}}{10}$.

Hence, the ratio of the profit to the cost price gives us an adequate basis of comparison. We express this ratio as a percentage, so that comparison is easy.

Answers to Problems 11-13A

Student Text Page 250

1. $14\frac{2}{7}\%$
2. $62\frac{1}{2}\%$
3. $22\frac{8}{11}\%$ profit
4. £780
5. 8s.$7\frac{1}{2}$d.
6. £150
7. £366 13s.4d.
8. $18\frac{3}{4}\%$
9. 4s.6d.
10. 2d.

Commissions and Discounts

The teacher may prepare for this lesson by obtaining local material on commission and discounts. Shops or newspaper offices may supply this. It will be of value if the students could have a practical notion at first hand of commissions and discounts.

Answers to Problems 11-13B

Student Text Page 252

1. £540
2. shs.137.50
3. £600
4. £900
5. shs. 728.37
6. $14\frac{1}{16}\%$

Simple Interest

Many boys and girls use the Post Office bank system. The Post Office uses compound interest to compute amounts. This interest system should be explained to the pupils, and also the reason that simple interest is studied as a prelude to compound interest.

Answers to Problems 11-13C

Student Text Page 253

1. (a) £45
   (b) £73 11s.$4\frac{4}{5}$d.
   (c) £62 10s.
   (d) 56 dollars 25 cents = \$56.25
   (e) 100 francs

2. 1 year 9 months
3. $3\frac{1}{2}$%
4. $14\frac{2}{7}$ years
5. shs.3,000
6. £675
7. shs.105
8. $12\frac{1}{2}$%
9. shs.3,666.67

Answers to Revision Problems

Student Text Page 254

1. (a) 35·63
   (b) 403·107
   (c) 1·0103
   (d) 0·07705
   (e) 9090·909
   (f) 58·087
   (g) 105·0077
   (h) 3·001003

2. (a) $1 + 2\left(\frac{1}{10^2}\right) + 4\left(\frac{1}{10^3}\right) + 9\left(\frac{1}{10^4}\right)$
   (b) $5 + 3\left(\frac{1}{10^2}\right)$
   (c) $3(10^4) + 7(10^3) + 3(10) + 5(1) + 4\left(\frac{1}{10}\right) + 4\left(\frac{1}{10^2}\right) + 9\left(\frac{1}{10^3}\right)$
   (d) $7\left(\frac{1}{10^3}\right) + 5\left(\frac{1}{10^4}\right)$
   (e) $8\left(\frac{1}{10^3}\right) + 2\left(\frac{1}{10^4}\right) + 1\left(\frac{1}{10^5}\right)$
   (f) $1(10^3) + 1\left(\frac{1}{10^4}\right)$

3. (a) $x = 1$

(b) $x = 10^2 = 100$

(c) $x = \frac{1}{10^3} = \frac{1}{1000}$

(d) $x = \frac{1}{10}$

(e) $x = \frac{1}{10^7} = \frac{1}{10000000}$

(f) $x = \frac{1}{10^4} = \frac{1}{10000}$

(g) $x = 10^4 = 10000$

(h) $x = 10^2 = 100$

4. (a) 2·2 inches (b) 2·5 inches (c) 1·0 inch

5. £25 12s.6d.

6. 220 yards

7. 437 yards 16 inches

8. 547 miles

9. (a) 8808·6 miles (b) 2602·9 miles (c) 3714·1 miles

10. (a) 0·45 (b) 0·125 (c) 0·075

11. (a) 0·25 (c) 0·85 (e) 3·05
(b) 0·50 (d) 0·025 (f) 1·125

12. 50·4 cwt.

13. £9

14. 3,280,000

CHAPTER 12

REAL NUMBERS

This chapter has a rather different emphasis from the others in the text. There are fewer problems than usual and emphasis is more on ideas. The problems are chosen to illustrate the ideas rather than to teach procedures which the pupils will often use.

The pupil knows about the rational numbers. He has learned that these correspond to points on the number line. But there are points on the number line which do not correspond to any rational number. A non-terminating, non-repeating decimal numeral like 0·1010010001... locates a point but it does not represent a rational number. The numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$,... locate points on the number line but they are not rational numbers. Ideas like these increase the pupils' understanding. They are particularly important if one goes on in mathematics because they form a good background for the idea of limit.

12-1 Decimal Numerals for Rational Numbers

In this section, the pupil will learn that a rational number, $\frac{a}{b}$ (a, b integers and $b \neq 0$) corresponds to a decimal numeral (or decimal, for short) which either terminates (for example $\frac{1}{4} = 0{\cdot}25$) or repeats without end (for example $\frac{1}{3} = 0{\cdot}333...$).

Conversely, he will learn that a decimal that terminates or repeats corresponds to a rational number. He will enjoy the method by which the rational fraction actually can be found.

You should remind the pupils that the numeral 1 may be written as 1·000... etc., and in general a fraction can be represented as a decimal numeral by dividing the numerator by the denominator.

Answers to Problems 12-1A

Student Text Page 257

1. (a) 0·5
   (b) 0·25
   (c) 0·75
   (d) 0·375
   (e) 0·075
   (f) 0·1625

2. (a) $\frac{1}{4}$
   (b) $\frac{7}{4}$
   (c) $\frac{14}{5}$
   (d) $\frac{707}{500}$

Repeating Decimal Numerals

Ask the class to find the decimal numerals for $\frac{1}{3}$ and $\frac{1}{9}$. Ask the pupils to give the digit in the 10th or the 20th position. Point out to the class that a decimal is not completely known until it is possible to state the digit in any position.

Now ask the class how they could indicate that a digit, or a set of digits, repeats indefinitely, that is, how one might abbreviate the following:

0·333...	(all the digits are 3's)
0·111...	(all the digits are 1's)
0·232323...	(all digits are alternately 2's and 3's)

The pupils may wish to invent their own ways of doing this. You may then use the pupils' notation for a time before leading the class to the standard notation:

$$0{\cdot}333\ldots = 0{\cdot}\dot{3}$$
$$0{\cdot}111\ldots = 0{\cdot}\dot{1}$$
$$0{\cdot}2323\ldots = 0{\cdot}\dot{2}\dot{3}$$

In the division for $\frac{6}{37}$ we must come to the same remainder sooner or later, because the remainder must be less than 37. There are only 36 possible different remainders.

Of course it is possible that there will be some digits at the start of the decimal numeral which are not part of the repeating pattern. For example $\frac{478}{130} = 3.6\dot{7}6923\dot{0}$.

Answers to Problems 12-1B

Student Text Page 259

1. $0.\dot{1}$
2. $0.\dot{0}\dot{1}$
3. $0.\dot{0}0\dot{1}$
4. $0.\dot{6}$
5. $0.\dot{1}4285\dot{7}$
6. $0.\dot{7}1428\dot{5}$
7. $0.\dot{0}\dot{9}$
8. $0.\dot{2}\dot{7}$
9. $0.\dot{0}2\dot{7}$
10. $0.\dot{0}731\dot{7}$

Terminating Decimals and Repeating Decimals

Lead the pupils to see, by means of some examples, that terminating decimals are formed only from rational numbers whose denominators have no prime factors other than 2 or 5. Give them examples of terminating decimal numerals like 0·58, 0·96, and 0·43217. If these decimal numerals are expressed as fractions, their denominators will be powers of 10 and hence have only 2 and 5 as prime factors. When reduce to lowest terms, one or the other of these factors may be missing. For example, $\frac{25}{100} = \frac{5 \times 5}{2 \times 5 \times 2 \times 5} = \frac{1}{2 \times 2}$. In any case there will be no factors in the denominator other than 2 and 5.

The converse also is true. If a fraction has a denominator with no prime factors other than 2 or 5, this denominator can easily be converted into a power of 10 so that the fraction is represented as a terminating decimal.

For example, $\frac{17}{2^2 \times 5^4} = \frac{2^2 \times 17}{2^4 \times 5^4} = \frac{4 \times 17}{10^4} = 0{\cdot}0068$

Incidentally, if a fraction has a denominator in which neither 2 nor 5 occurs as a factor, and numerator 1, the decimal repeats from the beginning.

For example, $\frac{1}{3} = 0\cdot\dot{3}$ and $\frac{1}{7} = 0\cdot\dot{1}4285\dot{7}$

If the denominator has 2 or 5 as a factor and at least one factor other than 2 or 5, the decimal repeats but not from the beginning.

For example, $\frac{1}{6} = \frac{1}{2 \times 3} = 0\cdot1\dot{6}$

The fact that all rational numbers which cannot be expressed as <u>terminating</u> decimal numerals can be expressed as <u>repeating</u> decimal numerals has been explained briefly above for $\frac{6}{37}$. Each remainder in the division process must be less than the divisor, and since the divisor is finite, there is a finite number of different remainders. For a non-terminating decimal the remainder cannot be zero. If the divisor is n, there can be at most (n - 1) different remainders.

Illustrate the discussion by expressing $\frac{1}{7}$ as a decimal numeral. The remainders, in order, are 3, 2, 6, 4, 5, 1 and the next remainder must be one of these numbers; it turns out to be 3.

Discuss carefully the ideas in Problems 16-1c, so the pupils will be ready to go on to the following material easily.

<u>Answers to Problems 12-1C</u> **Student Text Page 259**

1. (a) 10 (d) 100000
   (b) 100 (e) 1000
   (c) 1000

2. (a) $3\cdot45\dot{3}4\dot{5}$ (b) $34\cdot5\dot{3}4\dot{5}$ (c) $345\cdot\dot{3}4\dot{5}$

3. (a) 9x (b) 99x (c) 999x

4. (a) 8 (b) 93476 (c) 7245 (d) 86.4

The process for finding the rational fraction which corresponds to a repeating decimal, strictly speaking, requires some justification.

Let us illustrate by an example.

In setting $x = 0{\cdot}11111\ldots$ and multiplying by 10 to obtain

$$10x = 1{\cdot}1111\ldots$$

we assume that even though the 1's go on forever, we can multiply by 10 by moving the decimal point one place to the right. This is surely true when the decimal terminates as we saw in the previous chapter. The justification can be given in the case of a non-terminating decimal, but this requires some discussion of infinite series. The pupil is unlikely to worry about this. If he does he can be told that if he goes on with mathematics this matter will be treated fully; or you can observe that

$$\begin{array}{ll} \text{if} & 0{\cdot}11111 < x < 0{\cdot}111111 \\ \text{then} & 1{\cdot}1111 < 10x < 1{\cdot}11111 \end{array}$$

which should convince him.

Any rational fraction found by the method of the text can be divided out to give the original repeating decimal so that in any particular case the pupil can verify the correctness of his result.

The example $0{\cdot}\dot{9} = 0{\cdot}99999\ldots$ is often puzzling to the pupil. He may have difficulty in believing that this decimal represents 1. Of course, the method of the text gives this result.

Thus

$$\begin{aligned} x &= 0{\cdot}\dot{9} \\ 10x &= 9{\cdot}\dot{9} \\ 9x &= 9 \\ x &= 1 \end{aligned}$$

It is worth pointing out that if

$$\frac{1}{3} = 0{\cdot}333\ldots$$

$$\frac{1}{3} \times 3 = 1 = 0{\cdot}999\ldots$$

After studying Section 16-2, the pupil should be able to see that $0{\cdot}\dot{9}$ would not correspond to any point on the number line to the left of the point 1.

Answers to Problems 12-1D

Student Text Page 261

1. (a) $\frac{14}{3}$ (b) $\frac{7}{8}$ (c) $\frac{14}{99}$ (d) $\frac{1}{7}$ (e) $\frac{36}{11}$ (f) $\frac{115}{99}$

12-2 Plotting Repeating Decimals on the Number Line

This section shows how a decimal numeral, even if unending, specifies a definite point on the number line. It should stretch the pupils' imaginations.

12-3 Numbers which are not Rational

In this section, the pupil will learn that there are unending decimals which do not repeat. Since they do not repeat, they cannot represent rational numbers. They represent irrational numbers. They are called irrational because they cannot be written as ratios of integers. These irrational numbers also correspond to points on the number line.

12-4 The Square Root of 2

There is a definite point on the number line that corresponds to $\sqrt{2}$. If there were not, the circle of radius $\sqrt{2}$ and centre O would not intersect the line at C.

We could draw the entire circle. However, we would then have to explain the other point of intersection at $-\sqrt{2}$. It seems best to postpone negative irrational numbers to a later section.

The method explained here of obtaining fractions which approximate $\sqrt{2}$ is widely used in programming computing machines. It rests, first of all, on the fact that if a and b are positive numbers such that

$$a \times b = 2 = \sqrt{2} \times \sqrt{2}$$

and a is less than $\sqrt{2}$, then b must be greater than $\sqrt{2}$. Then with a too small and b too large, when we take $\frac{a+b}{2}$ we find a number halfway between a and b which is our next approximation to $\sqrt{2}$. Repeated use of this procedure narrows down on $\sqrt{2}$ very rapidly.

This method will be used in the next section.

You may be familiar with another method of extracting square roots which has been used in the schools. Very few pupils understand why this method works. The present method is very effective and at the same time is easy to understand.

Answers to Problems 12-4

Student Text Page 267

1. The pupils will probably be unable to obtain the following degree of accuracy, unless they use a very large scale. Accept the nearest tenth as a satisfactory approximation.

$\sqrt{2} \approx 1\cdot41$	$\sqrt{3} \approx 1\cdot73$	$\sqrt{4} = 2$	$\sqrt{5} \approx 2\cdot24$
$\sqrt{6} \approx 2\cdot45$	$\sqrt{7} \approx 2\cdot65$	$\sqrt{8} \approx 2\cdot83$	$\sqrt{9} = 3$

This is a rather amusing geometric construction of $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$... in succession. Notice that $\sqrt{4}$, $\sqrt{9}$, $\sqrt{16}$... are whole numbers. As we shall see in Section 16-5, the others are irrational.

2. The particular fractions found will vary with choice of a first approximation. In any case, the third approximation, when written as a decimal numeral, may be correct as far as the fourth digit.

(a)	$\sqrt{7}$:	$\frac{13}{5}, \frac{35}{13}, \frac{172}{65}$	2·646
(b)	$\sqrt{11}$:	$\frac{33}{10}, \frac{110}{33}, \frac{2189}{600}$	3·317
(c)	$\sqrt{13}$:	$\frac{7}{2}, \frac{26}{7}, \frac{101}{28}$	3·606
(d)	$\sqrt{17}$:	$\frac{41}{10}, \frac{170}{41}, \frac{3381}{820}$	4·123

12-5 Proof that $\sqrt{2}$ is an Irrational Number

We have given a famous proof. This is real mathematics. The good pupil should get a stimulus from this proof. It is worth spending some time on it.

The actual proof begins with $\frac{a}{b} = \sqrt{2}$ and is fairly short. A good pupil could perhaps begin there. It seems desirable however to begin with numerical illustrations. If necessary the teacher can illustrate the argument by using other fractions, for example if $\sqrt{2}$ were $\frac{10}{7}$ we would have

$$\frac{10 \times 10}{7 \times 7} = 2$$

$$\text{and} \quad 10 \times 10 = 2 \times 7 \times 7$$

This is impossible since there are two factors 2 on the left-hand side and only one factor 2 on the right-hand side.

The proof is based on the Fundamental Theorem of Arithmetic (see Section 4-5) which says that a given whole number which is not prime can be written as the product of prime factors in only one way, apart from the order in which the factors are written.

Example: $75 = 5 \times 15 = 5 \times 5 \times 3$

$75 = 3 \times 25 = 3 \times 5 \times 5$

<u>Answers to Problems 12-5</u> **Student Text Page 270**

1. (a) If $\frac{a}{b} = \sqrt{3}$, $3 = \frac{a^2}{b^2}$, $a^2 = 3b^2$, $a \times a = 3 \times b \times b$.

On the left-hand side there would be an even number of factors 3 (zero, two, four, six, ...). On the right-hand side, there would be an odd number of factors 3 (one, three, five,...). Thus $a \times a$ has an even number of 3's and $3 \times b \times b$ has an odd number of 3's.

Since, by the Fundamental Theorem of Arithmetic, a whole number can be expressed as a product of prime factors in only one way, $a \times a = 3 \times b \times b$ cannot be true for any whole numbers a and b. The equation $\frac{a}{b} = \sqrt{3}$ therefore cannot be satisfied by any whole numbers a and b.

(b) If $\frac{a}{b} = \sqrt{5}$, $\frac{a^2}{b^2} = 5$, $a^2 = 5b^2$ and $a \times a = 5 \times b \times b$.

Consider how many factors 5 there could be on the left and right-hand sides. On the left-hand side the possibilities are: zero, two, four...; on the right-hand side: one, three, five... The equation $\frac{a}{b} = \sqrt{5}$ cannot be true for any whole numbers a and b.

2. As before, if we assume that $\frac{a}{b} = \sqrt{4}$ we get

$$a \times a = 4 \times b \times b$$

We do not count the number of factors 4 because 4 is not a prime number. We write

$$a \times a = 2 \times 2 \times b \times b$$

and consider the number of factors 2. This number must be even on both sides. If we take $a = 2$ and $b = 1$ everything works out. Hence we run into no contradiction and cannot prove that $\sqrt{4}$ is irrational.

Another Example.

Prove that $\sqrt{6}$ is irrational.

From $\frac{a}{b} = \sqrt{6}$ we are led to

$$a \times a = 6 \times b \times b = 2 \times 3 \times b \times b.$$

We can consider either the number of factors 2 on both sides of the equation, or the number of factors 3, and carry through the usual argument.

12-6 The Real Numbers

Every decimal corresponds to a definite point on the number line.

The terminating and repeating decimals represent the rational numbers. The other decimals (non-repeating and non-terminating) represent irrational numbers. The union of the set of rational numbers and the set of irrational numbers is the set of real numbers.

Does every point on the number line correspond to a single decimal? There is a slight difficulty here. As we have seen $0{\cdot}999\ldots$ (or $0{\cdot}\dot{9}$) and 1 locate exactly the same point. Similarly $0{\cdot}499\ldots$ (or $0{\cdot}4\dot{9}$) and $0{\cdot}5$ locate the same point. If we decide not to use unending decimals which ultimately have only 9's, then every point on the number line has exactly one corresponding decimal. Decimals of this type will not arise if we follow the procedure described in the text.

Answers to Problems 12-6

Student Text Page 271

	Real	Rational	Irrational	Positive	Negative
$\sqrt{3}$	yes	no	yes	yes	no
3	yes	yes	no	yes	no
-3	yes	yes	no	no	yes
$\frac{4}{7}$	yes	yes	no	yes	no
$-0{\cdot}\dot{8}$	yes	yes	no	no	yes
$0{\cdot}\dot{1}\dot{7}$	yes	yes	no	yes	no

12-7 Negative Real Numbers

We suggest that not much time be spent on this section. The problems can be done orally in class.

We have used "opposite" in preference to "additive inverse" but you should gradually accustom the pupils to the term "additive inverse" which they will meet in later work.

Answers to Problems 12-7 Student Text Page 273

1. (a) $-3 \cdot 205$ (c) $-\sqrt{3}$
   (b) $1 \cdot 23$ (d) $-0 \cdot 2020020002\ldots$
2. (a) $-\frac{3}{2}$ (c) $-\sqrt{6}$
   (b) $0 \cdot 111\ldots$ (d) $0 \cdot 12345\ldots$

Note: When we write a number a with $\ldots$, it is understood that in writing its opposite $-a$, the dots stand for the same digits as before.

12-8 Addition of Real Numbers

Addition of real numbers is looked at from the point of view of:

1. Adding two rational numbers in decimal notation,
2. Adding numbers in decimal notation when one or both of them are irrational,
3. Interpreting these sums on the number line,
4. Adding square roots on the number line,
5. Considering whether a sum of real numbers will be rational or irrational.

Answers to Problems 12-8A

Student Text Page 273

1. (a) 9·55 (c) 8·48
   (b) 9·553 (d) 13·891

2. $\frac{275}{999} + \frac{714}{999} = \frac{989}{999} = 0\cdot\dot{9}\dot{8}\dot{9}$

Answers to Problems 12-8B

Student Text Page 274

1.
```
 4·1234567891...
 5·8910111213...
10·0144679104...
```

2.
```
3·1010010001...
6·1525354555...
9·2535364556...
```

3.
```
3·0101101110...
4·1020030004...
7·1121131114...
```

Note: If the pupil finds some other pattern than the one intended here, his results should be accepted.

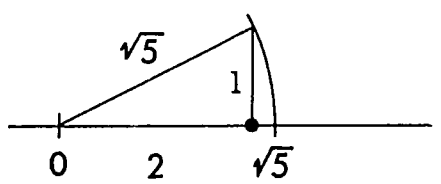
Addition of Square Roots on the Number Line

We point out incidentally the distinction between $\sqrt{(2 + 3)}$ and $\sqrt{2} + \sqrt{3}$. In $\sqrt{(2 + 3)}$ we add first and then take the square root. In $\sqrt{2} + \sqrt{3}$ we take the square roots first and then add. The results are quite different.

Answers to Problems 12-8C

Student Text Page 275

1. 1·4142
2. 1·7320
3. 3·1462
4. 2·2360
5. No. $2.2360\ldots \neq 3.1462\ldots$
6.



The Sum of a Rational Number and an Irrational Number is an Irrational Number

We write $\sqrt{2} = \frac{a - 3b}{b}$, where a and b are integers and $b \neq 0$. Since the integers are closed under the operation of subtraction, we know $a - 3b$ stands for an integer. Hence, $\frac{a - 3b}{b}$ is a rational number, being the ratio of two integers.

In general the sum of any rational number and any irrational number is irrational.

The formal proof is as follows:

Let R be any rational number and I any irrational number.
Suppose $I + R = R'$
where R' is a rational number.
Then $I = R' - R$.
But, the rational numbers are closed under the operation of subtraction.

That is, the difference of two rational numbers is a rational number. We have a contradiction so $I + R$ cannot be rational. It must be irrational.

Answers to Problems 12-8D **Student Text Page 276**

1. (a) $\sqrt{3}$, $5 + \sqrt{3}$
   (b) $-0\cdot000100001000001\ldots$, $-1\cdot000100001000001\ldots$
   (c) $-0\cdot030030003\ldots$, $-1\cdot030030003\ldots$
   (d) $-\sqrt{5}$, $1 - \sqrt{5}$
   (e) $-0\cdot123456\ldots$, $-1\cdot123456\ldots$
   (f) $-0\cdot789101112\ldots$, $-1\cdot789101112\ldots$

2. (a) $7\cdot81$
   (b) $4\cdot77$
   (c) $9\cdot\dot{7}$
   (d) $5\cdot\dot{7}\dot{8}$
   (e) $10 + \sqrt{3} - \sqrt{5}$
   (f) $5\cdot4444\ldots = 5\cdot\dot{4}$

3. (a) $\sqrt{3 + 5} \neq \sqrt{3} + \sqrt{5}$, $\sqrt{3 + 5} = \sqrt{8}$
   (b) $\sqrt{2 + 5} \neq \sqrt{2} + \sqrt{5}$, $\sqrt{2 + 5} = \sqrt{7}$

In general, $\sqrt{a} + \sqrt{b} \neq \sqrt{a+b}$ unless $a = 0$ or $b = 0$

For example

$$\sqrt{9} + \sqrt{16} \neq \sqrt{25}$$
$$3 + 4 \neq 5$$

If we assume that

$$\sqrt{a} + \sqrt{b} = \sqrt{a+b}$$

and square both sides of the equation, we obtain

$$a + 2\sqrt{a} \times \sqrt{b} + b = a + b$$

from which it follows that $2 \times \sqrt{a} \times \sqrt{b} = 0$.

This is impossible unless $a = 0$ or $b = 0$ (or both).

4. If $\sqrt{\sqrt{2} + 1}$ were rational, its square $(\sqrt{2} + 1)$ would have to be rational. This is impossible because we know that the sum of a rational number and an irrational number is irrational.

12-9 Order of Real Numbers

The decimal representation of irrational numbers is infinitely long. To compare two irrational numbers, and tell which is larger, we need not ordinarily go far. Certainly 4·10100100001... is less than 5·090090009... simply because a number between 4 and 5 is less than a number greater than 5; and 0·4321... is less than 0·435... because a number between 0·432 and 0·433 is less than a number greater than 0·435 whatever digit comes in the fourth decimal place.

It is suggested in the problems that Property 3 can be used to help compare two real numbers. If you so wish you can incorporate this idea in the text discussion. Thus

$$a = 0{\cdot}32842\ldots = 0{\cdot}32 + 0{\cdot}00842\ldots$$
$$b = 0{\cdot}32719\ldots = 0{\cdot}32 + 0{\cdot}00719\ldots$$

We know $0{\cdot}00719\ldots < 0{\cdot}00842\ldots$

therefore $0{\cdot}32 + 0{\cdot}00719\ldots < 0{\cdot}32 + 0{\cdot}00842\ldots$ by Property 3.

Answers to Problems 12-9

Student Text Page 278

1. (a) $0{\cdot}432\ldots < 0{\cdot}437\ldots$ (c) $-0{\cdot}128\ldots < -0{\cdot}123\ldots$

   (b) $3{\cdot}824\ldots > 3{\cdot}816\ldots$ (d) $-0{\cdot}128\ldots < 0{\cdot}001\ldots$

2. Since $0{\cdot}002\ldots < 0{\cdot}007\ldots$

   $0{\cdot}43 + 0{\cdot}002\ldots < 0{\cdot}43 + 0{\cdot}007\ldots$ by Property 3.

   That is $0{\cdot}432\ldots < 0{\cdot}437\ldots$

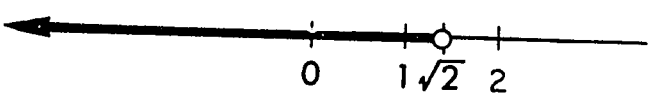
3. $3{\cdot}824\ldots = 3{\cdot}8 + 0{\cdot}024\ldots$

   $3{\cdot}816\ldots = 3{\cdot}8 + 0{\cdot}016\ldots$

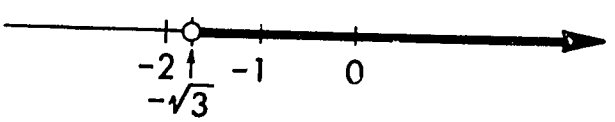
   Since $0{\cdot}024\ldots > 0{\cdot}016$

   $3{\cdot}824\ldots > 3{\cdot}816\ldots$

4. (a)



(b)



12-10 Multiplying Real Numbers

We observe the basic property of square roots that, for non-negative numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$. We observe also that for real numbers, just as it was for rational numbers, the product of a positive

real number and a negative real number is a negative real number, and the product of two negative real numbers is a positive real number.

Proof that $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$

$$\begin{aligned}
(\sqrt{a} \times \sqrt{b})^2 &= (\sqrt{a} \times \sqrt{b}) \times (\sqrt{a} \times \sqrt{b}) && \text{Definition of Square} \\
&= [(\sqrt{a} \times \sqrt{b}) \times \sqrt{a}] \times \sqrt{b} \\
&= [\sqrt{a} \times (\sqrt{b} \times \sqrt{a})] \times \sqrt{b} \\
&= [\sqrt{a} \times (\sqrt{a} \times \sqrt{b})] \times \sqrt{b} && \text{Associative and commutative properties of multiplication.} \\
&= [(\sqrt{a} \times \sqrt{a}) \times \sqrt{b}] \times \sqrt{b} \\
&= (\sqrt{a} \times \sqrt{a}) \times (\sqrt{b} \times \sqrt{b}) \\
&= (\sqrt{a})^2 \times (\sqrt{b})^2 \\
&= a \times b && \text{Definition of square root.}
\end{aligned}$$

Since $(\sqrt{a} \times \sqrt{b})^2 = a \times b$

$\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$

by the definition of square root.

Proof that $\sqrt{2} \times (-\sqrt{2}) = -2$

$$\sqrt{2} \times [\sqrt{2} + (-\sqrt{2})] = \sqrt{2} \times 0 = 0$$ Addition property of opposites and multiplication property of zero.

$$[\sqrt{2} \times \sqrt{2}] + [(\sqrt{2}) \times (-\sqrt{2})] = 0$$ Distributive property.

$$2 + [(\sqrt{2}) \times (-\sqrt{2})] = 0$$ Definition of square root.

Therefore

$$(\sqrt{2}) \times (-\sqrt{2}) = -2$$ Uniqueness of the additive inverse. Any number which, added to 2, gives zero, must be the opposite of 2.

Proof that $(-\sqrt{2}) \times (-\sqrt{2}) = 2$

$$(-\sqrt{2}) \times [\sqrt{2} + (-\sqrt{2})] = 0$$

$$[(-\sqrt{2}) \times (\sqrt{2}) + [(-\sqrt{2}) \times (-\sqrt{2})] = 0$$

$$-2 + [(-\sqrt{2}) \times (-\sqrt{2}) = 0$$

Therefore

$$(-\sqrt{2}) \times (-\sqrt{2}) = 2$$

Answers to Problems 12-10

Student Text Page 281

1. $a \times b > (0\cdot1) \times (0\cdot3) = 0\cdot03$

$> (0\cdot11) \times (0\cdot33) = 0\cdot0363$

$> (0\cdot111) \times (0\cdot333) = 0\cdot036963$

$a \times b < (0\cdot2) \times (0\cdot4) = 0\cdot08$

$< (0\cdot12) \times (0\cdot34) = 0\cdot0408$

$< (0\cdot112) \times 0.334 = 0\cdot037408$

2. $\frac{1}{9} \times \frac{1}{3} = \frac{1}{27} = 0\cdot037037\ldots$

3. $\sqrt{2} \times \sqrt{3} > (1\cdot414) \times (1\cdot732) = 2\cdot449048$

$< (1\cdot415) \times (1\cdot733) = 2\cdot452195$

$\sqrt{6} = 2\cdot44948\ldots$

4. $\sqrt{3}(\sqrt{2} + 1) = [\sqrt{3} \times \sqrt{2}] + \sqrt{3}$

$= \sqrt{6} + \sqrt{3}$

5. $(\sqrt{2} - 1) \times (\sqrt{2} + 1) = [\sqrt{2} \times (\sqrt{2} + 1)] - [1 \times (\sqrt{2} + 1)]$

$= 2 + \sqrt{2} - \sqrt{2} - 1 = 1$

6. $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2}) = [\sqrt{5} \times (\sqrt{5} - \sqrt{2})] + [\sqrt{2} \times (\sqrt{5} - \sqrt{2})]$

$= 5 - \sqrt{10} + \sqrt{10} - 2 = 3$

7. CHALLENGE PROBLEM

$$\left(\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}\right)^2 = 3 + 2\sqrt{2} - 2\left(\sqrt{3 + 2\sqrt{2}}\right)\left(\sqrt{3 - 2\sqrt{2}}\right) + 3 - 2\sqrt{2}$$

$$= 6 - 2\sqrt{9 - 8}$$

$$= 6 - 2\sqrt{1} = 6 - 2 = 4$$

Since $\left(\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}\right)^2 = 4$,

$\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}}$ must be 2 or -2.

It must be the positive number because

$3 + 2\sqrt{2} > 3 - 2\sqrt{2}$. Hence $\sqrt{3 + 2\sqrt{2}} - \sqrt{3 - 2\sqrt{2}} = 2$.

8. CHALLENGE PROBLEM

$$\left(\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}}\right)^2 = 2+\sqrt{3}+2\sqrt{(2+\sqrt{3})\times(2-\sqrt{3})}+2-\sqrt{3}$$
$$= 4+2\sqrt{4-3} = 4+2 = 6$$

Reasoning as in Problem 7 we see that $\sqrt{2+\sqrt{3}}+\sqrt{2-\sqrt{3}} = \sqrt{6}.$

9. CHALLENGE PROBLEM

$$\left(\sqrt{2+\sqrt{3}}-\sqrt{2-\sqrt{3}}\right)^2 = 2+\sqrt{3}-2\sqrt{(2+\sqrt{3})\times(2-\sqrt{3})}+2-\sqrt{3}$$
$$= 4-2\sqrt{4-3} = 2$$

Reasoning as in Problem 7 we see that $\sqrt{2+\sqrt{3}}-\sqrt{2-\sqrt{3}} = \sqrt{2}.$

10. $\sqrt{2}$ and $2\sqrt{2}$ for example.
Their sum is $3\sqrt{2}$
Their product is 4.

12-11 Absolute Value

Answers to Problems 12-11

Student Text Page 284

1. (a) $0{\cdot}1234\ldots$

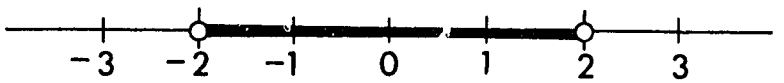
(b) $\left|\frac{4}{3}-\frac{3}{2}\right| = \frac{3}{2}-\frac{4}{3} = \frac{1}{6}$

(c) $\left|(-0{\cdot}111\ldots)-0{\cdot}01001\ldots\right| = \left|-0{\cdot}12112\ldots\right| = 0{\cdot}12112\ldots$

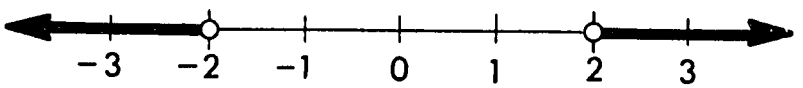
(d) $\left|-3{\cdot}68+3{\cdot}89\right| = \left|0{\cdot}21\right| = 0{\cdot}21$

2. $\{2, -2\}$

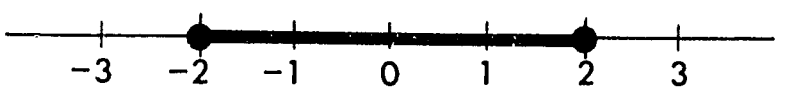
3. (a) $-2 < x < 2$



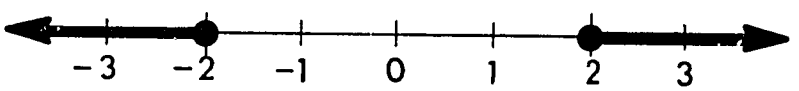
(b) $x < -2$ or $x > 2$



4. (a) $-2 \leq x \leq 2$



(b) $x \leq -2$ or $x \geq 2$



5. For $|6 - x| = 1$ to be true, $6 - x$ has to be 1 or -1.

If $6 - x = 1$, x is 5; if $6 - x = -1$, x is 7.

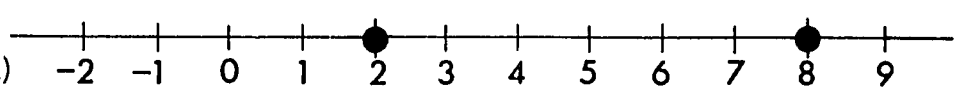
Hence the truth set is $\{5, 7\}$.

6. (a) $\{5, 7\}$ (c) $\{1, 3\}$

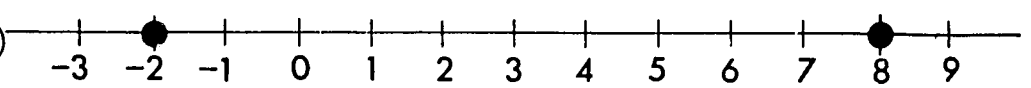
(b) $\{3, 9\}$ (d) $\{7, -3\}$

7.

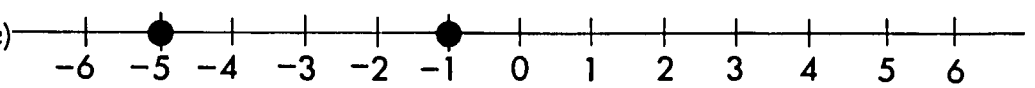
(a)



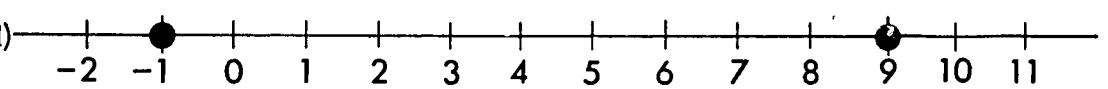
(b)



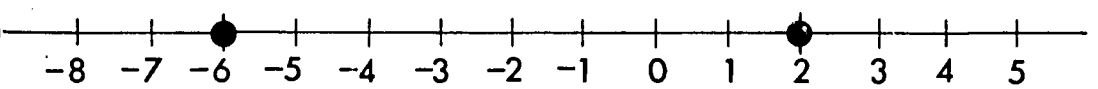
(c)



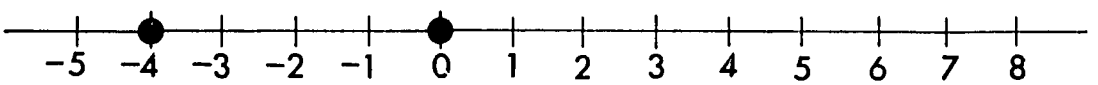
(d)



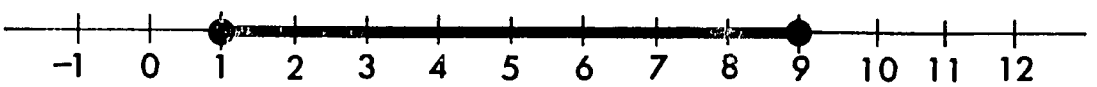
(e)



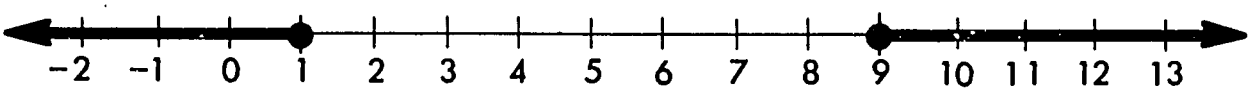
(f)



8. (a) $1 \leq x \leq 9$



(b) $x \leq 1$ or $x \geq 9$



12-12 More About Absolute Value

If you are pressed for time, this section can be omitted.

Answers to Problems 12-12

Student Text Page 287

1. (a) $a \times b = \frac{1}{8}$, $|a \times b| = \frac{1}{8}$, $|a| = \frac{1}{2}$, $|b| = \frac{1}{4}$
   (b) $a \times b = -\sqrt{6}$, $|a \times b| = \sqrt{6}$, $|a| = \sqrt{3}$, $|b| = \sqrt{2}$
   (c) $a \times b = -2$, $|a \times b| = 2$, $|a| = 1 + \sqrt{3}$, $|b| = \sqrt{3} - 1$

2. (a) $a = -2$, $b = -4$, $a + b = -6$
   $|a| = 2$, $|b| = 4$, $|a + b| = 6$

   (b) $a = 2$, $b = -2$, $a + b = 0$
   $|a| = 2$, $|b| = 2$, $|a + b| = 0$

   (c) $a = \sqrt{2} - 1$, $b = \sqrt{2} + 1$, $a + b = 2\sqrt{2}$
   $|a| = \sqrt{2} - 1$, $|b| = \sqrt{2} + 1$, $|a + b| = 2\sqrt{2}$
   $|a| + |b| = 2\sqrt{2}$

3. $|2|^2 = 4$ $|-2|^2 = 4$

   $|a|^2 = a^2$ because both of them are the non-negative number $a \times a$.
   If $|a|^2 = a^2$
   then $|a| = \sqrt{a^2}$ by the definition of square root.

4. $a = -5, \quad a^2 = 25, \quad \sqrt{a^2} = \sqrt{25} = 5 = |a|$

$a = \frac{1}{4}, \quad a^2 = \frac{1}{16}, \quad \sqrt{a^2} = \sqrt{\frac{1}{16}} = \frac{1}{4} = |a|$

$a = -\frac{1}{9}, \quad a^2 = \frac{1}{81}, \quad \sqrt{a^2} = \sqrt{\frac{1}{81}} = \frac{1}{9} = |a|$

$a = -\sqrt{3}, \quad a^2 = 3, \quad \sqrt{a^2} = \sqrt{3} = |a|$

5. $\sqrt{x^2} = |x|$ for all values of x.

$\sqrt{x^2} = x$ for $x \geq 0$.

$\sqrt{x^2} = -x$ for $x \leq 0$.

6. <u>CHALLENGE PROBLEM</u>

$$|a \times b| = \sqrt{(a \times b)^2} = \sqrt{a^2 \times b^2} = \sqrt{a^2} \times \sqrt{b^2} = |a| \times |b|$$

7. <u>CHALLENGE PROBLEM</u>

$$|a - b| = |a + (-b)| \leq |a| + |-b| = |a| + |b|$$

$$|(a - b) - a| \leq |a - b| + |a|$$

$$|-b| \leq |a - b| + |a|$$

$$|b| \leq |a - b| + |a|$$

$$|a - b| \geq |b| - |a|$$

CHAPTER 13

INTRODUCTION TO GRAPHING IN THE NUMBER PLANE

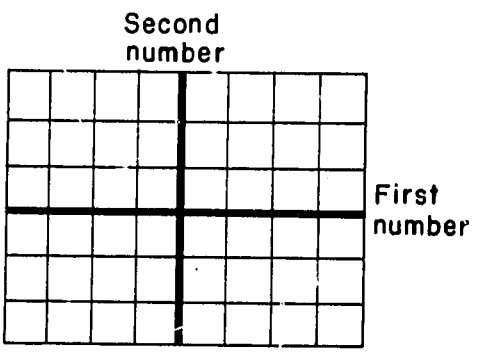
There is provision for much student activity in this chapter. While such activity is time-consuming and you will want to be careful not to spend too long on it, the ideas being presented are significant and deserve careful attention.

13-1 The Number Plane

In the first part of this section the pupils can learn simply and quickly to plot points on the number plane without any formal directions and without learning any new language. The language follows later. This part will take only a few minutes of class time.

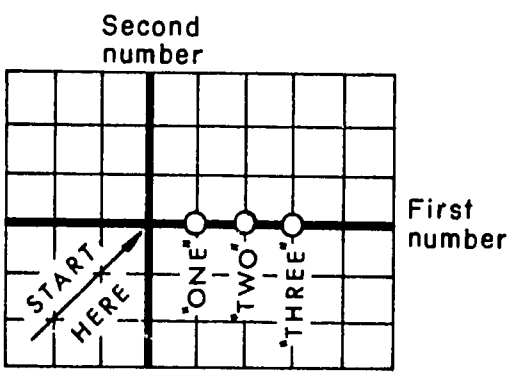
For some classes a brief discussion such as that on the first page of this section in the text will be enough. For classes which need more help you may want to proceed somewhat as follows:

1. Put this diagram on the board (or use a graph board, if you have one).



2. Ask a pupil to name two integers, not too large.

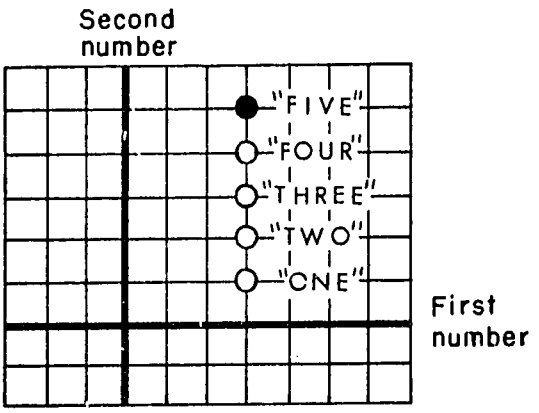
3. Starting at the point where the axes intersect, count over on the first-number axis, saying "One" at the first intersection, "Two" at the second intersection, etc. For example, if the student said "Three and five", the o's in the picture on the right indicate where you put



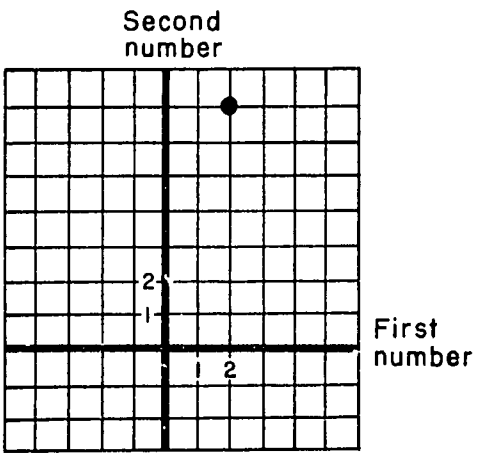
Previous Page Blank

your chalk (but not making a mark on the board), and what you say is written below.

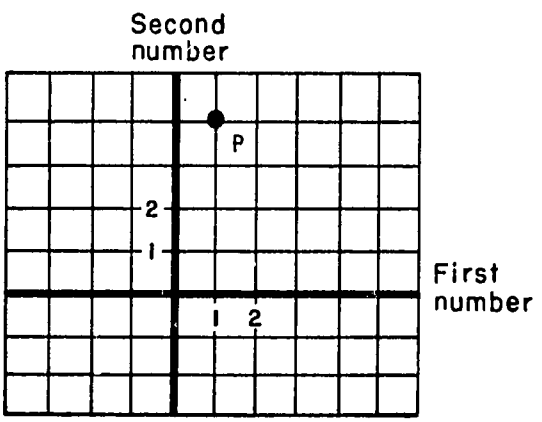
4. Holding your chalk where you said "Three", you might ask the student what the second number he named was. (You may remember, but some of the students may not.) After he has repeated "Five", count aloud up the intersections. When you reach the fifth intersection, mark it with a solid dot. If the student had said "Seven" or "Eight" as one of the numbers, you can add a line or two so that the point can be marked, but if he said "Twenty", you would probably want to ask him for a smaller number.



5. Ask another student to name another pair of integers. Repeat the same process, starting where the axes intersect, counting over for the first number, and up for the second. If the student said "Two and seven", the picture would now look like this:

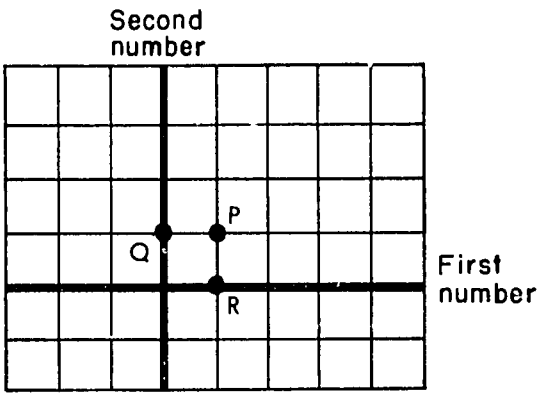


6. Then you can mark a point somewhere at an intersection, say at P, and ask, "What numbers go with this point?" Check the two numbers for the class by counting over and up as you did for the other pairs of numbers.



7. Mark a few more points and have someone in the class give you the pair of numbers that goes with each point, checking each one for them by counting over and up. Have one student name a pair of numbers and ask another to come to the board and mark the corresponding point.

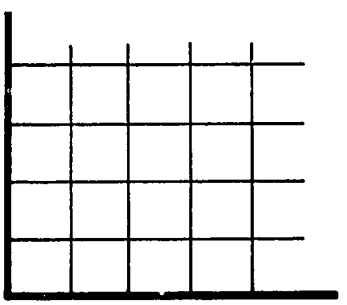
Note: The question of a point being marked on one of the axes, or one with either one or both coordinates negative, probably will not arise. This is all right because it can be introduced later. If it does arise at this time, it is probably best to take care of it here. For example, suppose a student wants to know what numbers correspond to the point Q. Ask someone what numbers he thinks correspond to Q. You may get an answer of "One, one". Mark "One, one" correctly at P, and ask for another guess. Suppose someone says "One, zero". Mark that correctly at R. The next number would probably be the one for Q: "Zero, one."



Dots and Crosses

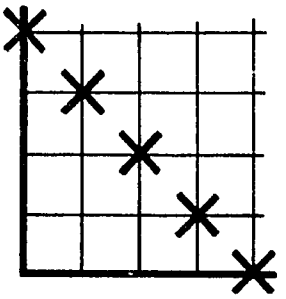
We describe here a game which you may wish to use with your class if they need practice in associating points of the number plane with their ordered pairs of numbers.

Game 1: Put this diagram on the board:

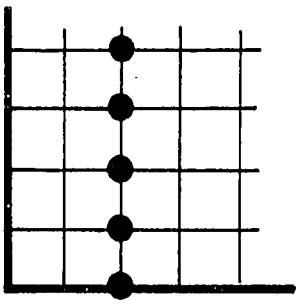


Split the class up into two teams--one can be the cross-team, the other the dot-team. The easiest way is to divide the class down the middle; those on the left are the

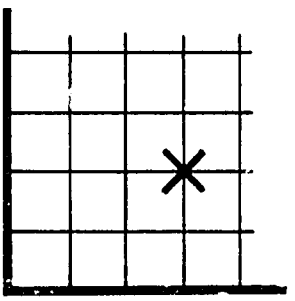
cross-team, on the right the dot-team. Tell the class that a team wins when it gets five of its marks in a line, playing on the intersections in the picture (the picture cannot be made larger). For example, the cross-team would win this game:



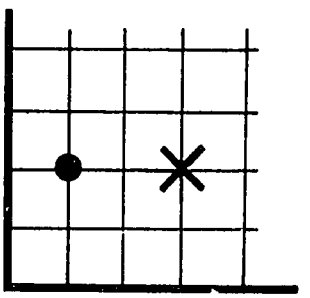
The dot-team wins this one:



Have a person on one team, say the cross-team, name two integers. Suppose he says "Three, two". Count over three, up two, and mark the intersection with ×.



It is now the dot-team's turn. Suppose someone on the team says "One, two". Again, count over and up, and mark this point with a big, heavy dot like this:

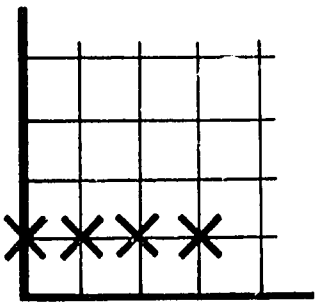


The game continues until one team sets five in a line, or neither side can get five in a line. A team loses its turn if either number named is too large (for example, the point that goes with "Seven, two" will not fit on the picture), <u>or</u> the choice of numbers results in an intersection that already has a mark on it.

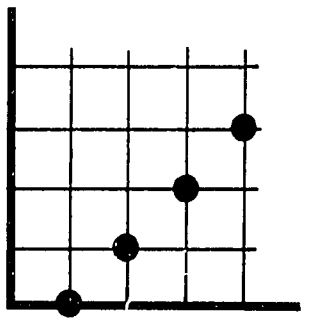
<u>Comments</u>: Having to get five in a line does not make for a very interesting game after the children have had a little practice. Since one team needs four turns to get four in a line, and the other team needs only one turn to stop them, neither team will be able to win. Therefore, after one or two games, you should change the

rule and say that <u>four</u> in an <u>uninterrupted</u> line wins.

Like this:

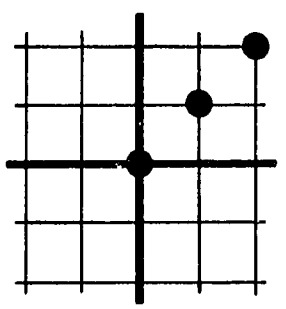


Or perhaps this:



This does make an interesting game, and one team is likely to win if it uses some good strategy. The reason for playing at least one game with five in a line is to force the children to figure out how to get a mark on one of the axes. If they have to figure out for themselves that they need to name zero as one of the numbers, they are likely to remember it.

There is another variation that can be introduced after four or five games, or perhaps earlier. Since the children have learnt about negative numbers previously, the following diagram can be used:



Here, the dot-team has to figure out that "negative one, negative one" is the play that will win for them.

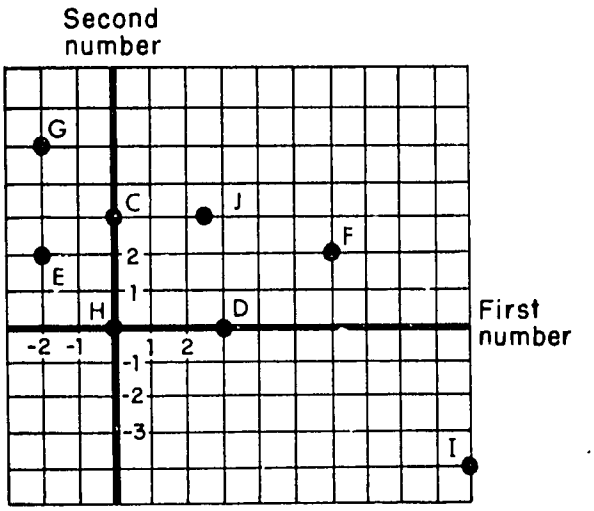
If the teacher does not say anything about negative numbers before the new diagram is put on the board, the class will probably fill up all the intersections which do not use any negative numbers, look a bit puzzled, and perhaps try to tell you "No, I meant you to count to the <u>left</u> one place." You can refuse this by saying, "The only instructions I can accept are the names for two numbers." Eventually, some child will probably announce "negative something, . . ." and get a point in another quadrant. If they start to get discouraged by not being able to do it, you may have to give them a hint.

At this point, the children should be able to handle **Section 13-1 in their** text, either in class or on their own.

Answers to Problems 13-1

Student Text Page 289

1. The point I, (10, -4) is included to remind the children that the picture of a number plane can be enlarged. The point J, $(2\frac{1}{2}, 3)$, is included to remind them that the numbers are not restricted to the integers.



2. This question emphasizes the importance of the order in the ordered pair of numbers.

3.

P:	(2, 3)	T:	(3, -3)
Q:	(-4, 3)	U:	(0, 4)
R:	(-1, -1)	V:	(1, 0)
S:	(0, -2)	W:	$(-5, -\frac{1}{2})$

4. (a) One

(b) One

You might let the children describe roughly the location of the points that correspond to these ordered pairs:

(a) (-100, -100) (c) $(\sqrt{2}, \sqrt{2})$

(b) (1000, -2) (d) $(-\frac{1}{2}, 50)$

Coordinates

You might wish to point out to the children that it is a convention to call the horizontal axis the x-axis; it could be called the y-axis, but it usually is not.

13-2 Graphs as Pictures of Truth Sets

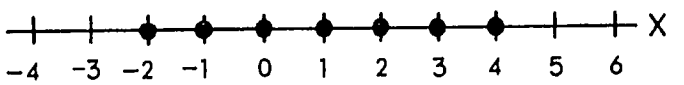
Student Text Page 291

Much of the problem work in this section is referred to and used for further developments in later sections of the chapter. Be sure that the pupils keep their written work available.

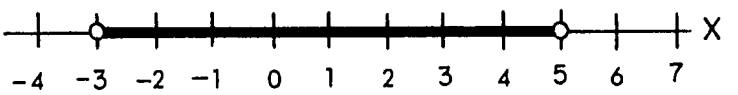
Class Activity

The class activity can be done either on the board with the whole class, or individually on graph paper. Perhaps a combination of the two would be best.

1.



2.



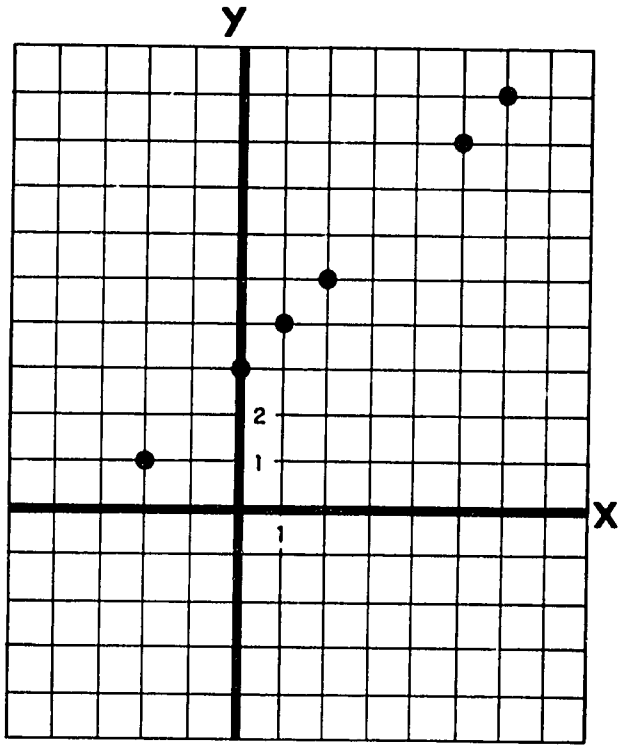
The hollow dots at -3 and 5 indicate that these numbers are not in the truth set. You may need to remind the pupils that the domain of a variable is the set of numbers from which values of the variable may be chosen. For each of these numbers the sentence is either true or false. The truth set of of the open sentence contains those numbers which make it true.

3. For $y = x + 3$, if x is 2, then y is 5. The ordered pairs in (a), (b), (d), and (f) belong to the truth set.

4. The table can be filled in on the blackboard with the pupils filling in two or three more ordered pairs. The number of ordered pairs in the truth set is unlimited.

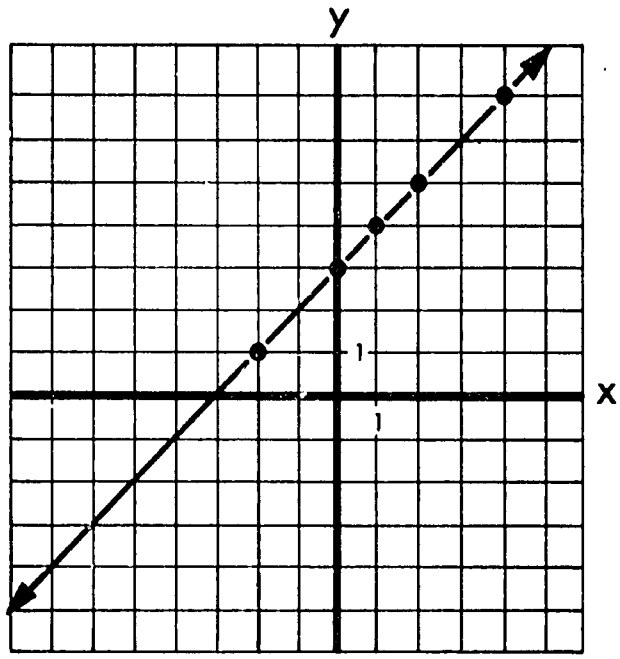
5. For activities 5, 6, 7, and 8, some of the work should probably be done on the blackboard with the whole class before they complete their own graphs individually. For example, the axes can be drawn first on the board, and several children can come up to the board to mark two or three points from

the table, and then the rest of the points can be marked individually on their own papers. Then the remaining points can be marked on the blackboard so that the children can check their graphs for mistakes. The graphs may now look something like this:



6. It is probably a good idea to have several points marked on the board before the children go back to their own graphs so that you can be sure that they understand what they are to do. Ask a pupil to come up and mark a point that he thinks belongs to the truth set <u>without</u> doing any counting. Do <u>not</u> be surprised if he marks one that is <u>not</u> on the line suggested by the other points already marked. Many children do not realize right away that there is a pattern to see. Whatever point is marked, ask for the numbers that go with it, and then go back to the equation $y = x + 3$ and check whether the equation is true for these numbers. This last step is important because a child should realize that he has his own way, independent of the teacher's help, of finding out whether a point belongs to the graph. When several points that belong to the graph have been marked, then the children can mark more on their own graphs. As soon as a child has marked a point on his own graph and checked that it belongs to the graph, he can come up and mark the point on the blackboard graph.

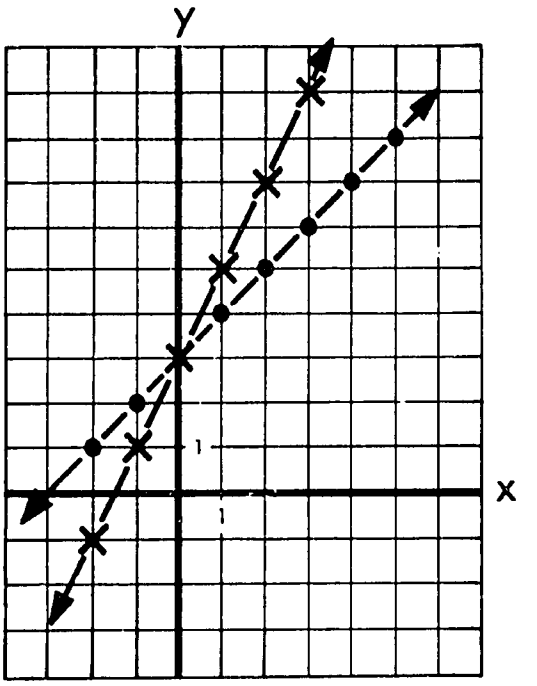
7. Not all the ordered pairs in the truth set can be shown; the line runs off the graph paper.



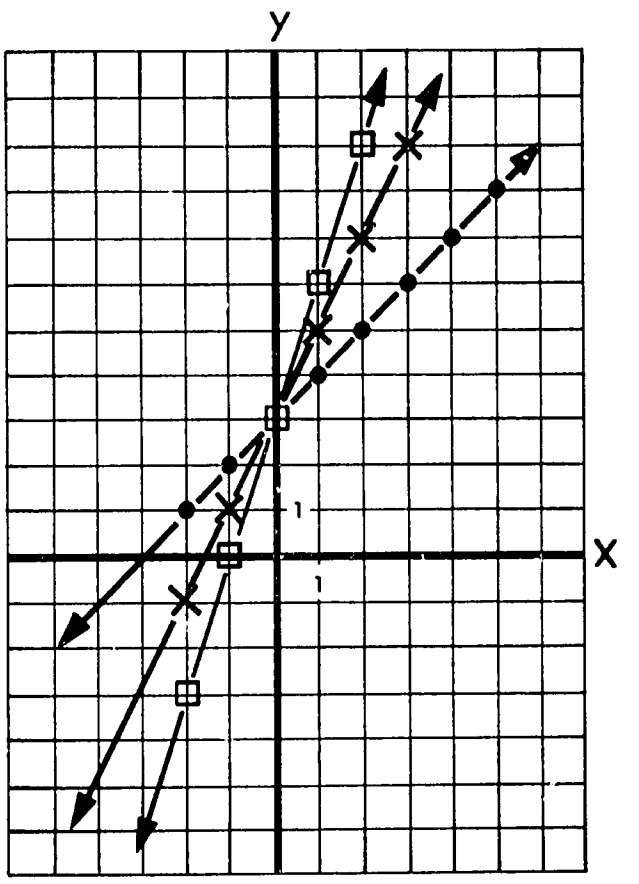
8. There are two things that we wish to give the children an opportunity to discover in the class activity. The first is that the graph of the truth set for some equations is a straight line. This the children found when they finished the graph in 7. The second thing is that the coefficient of the x term for equations of the form $y = ax + b$ determines the "up and over" pattern in going from one point of the graph to another. Many children can discover this for themselves, and for those who do not, it is not necessary to tell them at this point; the subject is dealt with more formally later in this chapter in the section on slope. It is often valuable for children to gain informal experience with a concept before they even know the name attached to the concept.

For the equation $y = 2x + 3$, (1, 5) and (2, 7) belong to the truth set. Have these points marked on the board (perhaps in a different colour if you have coloured chalk). We ask the children to guess where the third point goes even though they may put it in the wrong place. If they do, let them find out for themselves by testing whether the coordinates make the sentence true. Many of them will have discovered by now that the points may be expected to be on a straight line. Some of the children will notice the relation of "up and

over" to the coefficient of x. Again, a few more correct points for $y = 2x + 3$ can be marked, and the children can complete the graph on their individual papers. Their papers should look like this:



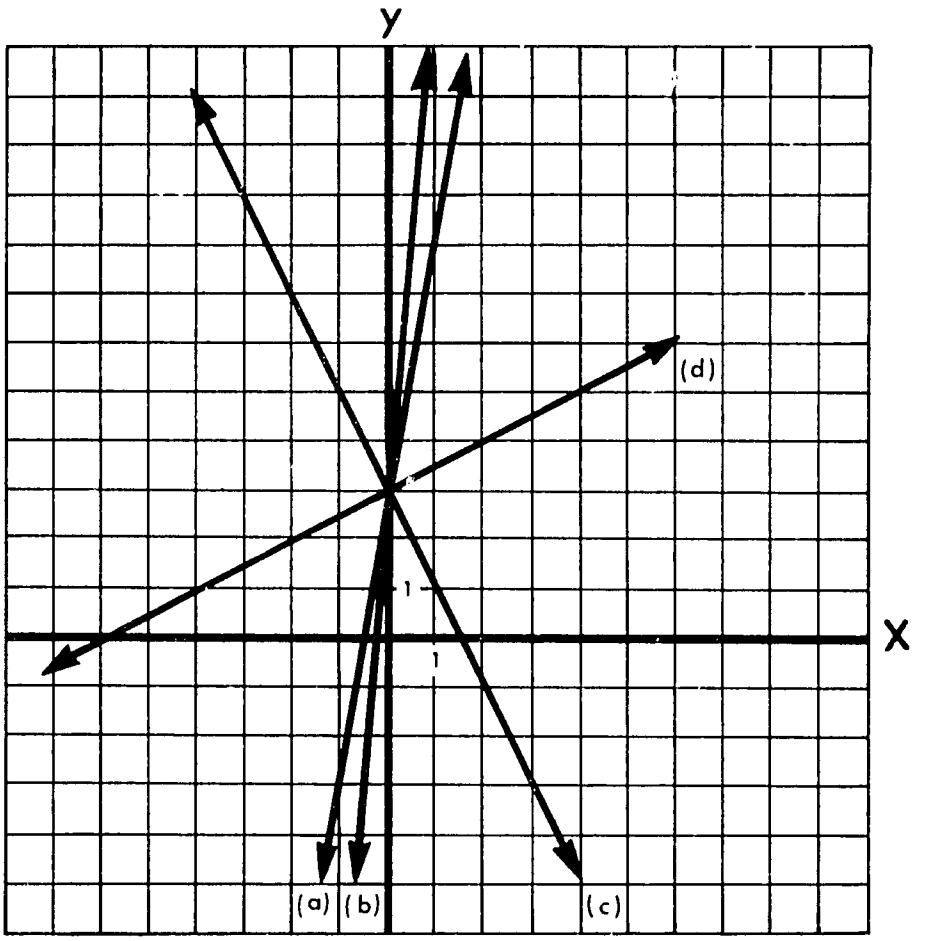
9. This time we let them try to guess a second point after having located only the single point (1,6). Some will recognize the pattern of behaviour quickly and will know that because of the $3x$ in $y = 3x + 3$, the line will climb three units for an increase of one in x. Others may have to do more experimenting before they see the pattern. Their graph paper when $y = 3x + 3$ has been completed will look like this:



Answers to Problems 13-2A

Student Text Page 294

1.

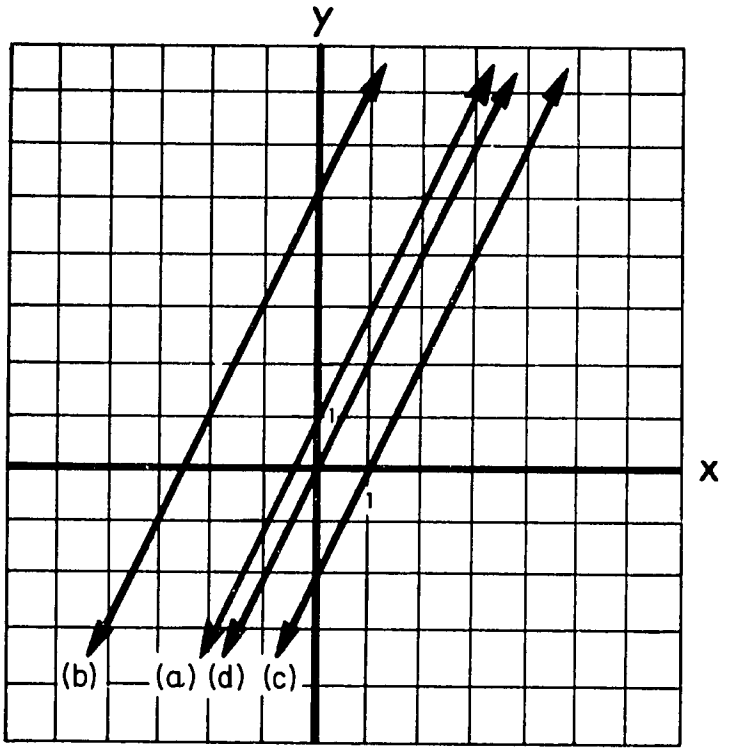


You may want to ask the pupils after they have finished the above graphs to point to where they think the graphs of the following would roughly be:

(a) $y = -10x + 3$ (c) $y = 100x + 3$

(b) $y = -\frac{1}{2}x + 3$ (d) $y = \frac{1}{3}x + 3$

2. Until this point, the children have graphed only straight lines which intersect the y-axis at (0, 3). This exercise is designed so that the children have an opportunity to discover what role the constant term of a linear equation plays in the graph. This idea is introduced formally later in the chapter, so there is no need to use the definition of y-intercept here.



Class Activity

1. This exercise may be a guessing game for the children who have not yet noticed the relationship between the coefficient of x and the "up and over" pattern from one point to another. Accept whatever answers the children give, and have them check the two points to find out if their value of a is correct. They will get the correct value 2 for a sooner or later, and it is good practice for them to test ordered pairs to see whether they make the equation true. If they have a good deal of trouble obtaining the correct value **for a, you may wish to do the first two or three of Problems 13-2B in class.**

Some pupils may be clever enough to see that, since the point (1, 3) is on the line, the equation $3 = (a \times 1) + 1$ must be true, and this will be true only if a is 2.

2. Yes, all the points on the line have coordinates that make $y = 2x + 1$ true. Have the pupils check two or three of them.

Answers to Problems 13-2B

Student Text Page 295

For the first two problems, the children are asked to find only the coefficient of the x term; for the rest, they are asked to find the constant term, also.

1. $a = 1$
2. $a = \frac{1}{2}$
3. $a = 3;\ b = 1$
4. $a = -1;\ b = 0$
5. $a = -2;\ b = 2$
6. $a = -1;\ b = -1$
7. $a = 5;\ b = -3$
8. $a = 0;\ b = 1$

13-3 Guessing Rules

Class Activity

Student Text Page 297

The purpose of this section is to approach linear equations and straight line graphs from another point of view. The children have already had some experience with graphing a linear equation of the form $y = ax + b$; they have also reversed this process, i.e., starting with two points marked in the number plane, they have written the equation that goes with the line through the points. Two points are determined, however, when you know the ordered pairs that go with them. This section is aimed at having the children find the linear equation determined by two ordered pairs of numbers.

1. John's rule is: Multiply by 5 and add one.

0	1
4	21
10	51

If other rules are suggested first, have the children decide if it works by checking several of the pairs in the table.

2. Tunde's rule is: Multiply by seven.

1	7
3	21
4	28

3. "Divide by four" does work; the rule Edison was thinking of is "multiply by one-fourth" (or perhaps it was "multiply by two and divide by eight").

0	0
2	$\frac{1}{2}$
3	$\frac{3}{4}$
12	3

4. The variable x represents the number which you tell John and y represents the number John tells you after using his rule. Tunde's rule can be written as $q = 7p$, if we assume that p is the number you tell Tunde, and q is the number Tunde tells you. Edison's rule is $y = \frac{1}{4}x$, or $q = \frac{1}{4}p$, etc., depending on what two variables are chosen. John would write the rule as $y = \frac{x}{4}$. The two equations are equivalent, of course.

Before going on to Problems 13-3, the children may enjoy making up and guessing their own rules. For example, you can have two children go outside the room, agree upon a rule, and then let the rest of the children give them numbers on which to work the rule. (You can write the table on the board.) The first child who can guess a rule that works for any number (it may <u>not</u> be the rule the two children were actually using) and write it correctly on the board is allowed to pick an assistant and make up the next rule to be guessed by the class. (The reason for an assistant is that he can act as a checker for the arithmetic involved in using the rule.)

<u>Answers to Problems 13-3</u>

Student Text Page 298

1.

x	y
5	12
10	17

$y = x + 7$

2.

w	z
0	-4
1	-3
7	3

$z = w - 4$

3.

s	t
-1	-4
0	-1
5	14
7	20

$t = 3s - 1$

4.

p	q
1	$-\frac{1}{2}$
101	$49\frac{1}{2}$

$q = \frac{1}{2}p - 1$

(There are other possibilities for the form of the rules that the children may find that might also be correct.)

5. The point of this problem is to have some of the children figure out by themselves the following:

a. If the rule is of the form "multiply by a number and then add a number", the rule is an equation of the form $y = ax + b$.

b. Only two ordered pairs are needed to determine the equation. (They may even associate the equation with its graph and the ordered pairs with the two points needed to determine the line.)

c. Knowing y when x is 0 tells me the constant term in the equation. The coefficient of the x term can be obtained by subtracting the second coordinate for 0 from the second coordinate for 1. (This actually determines algebraically the slope and y-intercept of the linear equation.) Some of the children may discover these two things by examining the tables and rules they have already guessed; some of them perhaps have already realized it when they were guessing rules made up in class.

The four rules Sonko guessed:

(a) $y = 2x + 3$ (c) $y = \frac{1}{3}x$

(b) $y = x - 1$ (d) $y = 4x + 5$

6. The purpose of this question is to indicate to the children that knowing only two ordered pairs determines exactly one equation of the form $y = ax + b$ but there are other equations of different forms which will also have the two ordered pairs in the truth set. In Problem 7, the children may begin to realize that equations of the form $y = ax + b$ have straight line graphs, but equations of different forms may not. Edward's rule is: $y = x^2 + 1$.

7. How the children decide whether the graphs of the rules are straight lines or not does not matter at this point; they will probably decide in different ways. Some may just "know", others may actually draw each graph to decide. All the rules except the last one, $y = x^2 + 1$, have straight line graphs. It is not necessary to complete the graph of $y = x^2 + 1$ (plotting three points will indicate that it is not in a straight line) unless the children are curious

to find out what its graph looks like. It is perhaps helpful to graph in class one or two of the rules from Problem 5 by plotting the two points given so that the children have another opportunity to associate the algebra they did to guess the rule with the geometry they did when they were finding a and b for $y = ax + b$ from the two points marked in the number plane.

13-4 Equivalent Equations in Two Variables

Thus far the children have met linear equations of the form $y = ax + b$. The purpose of this section is to introduce them to linear equations in different forms.

The pupils should find that any ordered pair that makes $y = \frac{1}{2}x - 1$ true also makes $2y - x + 2 = 0$ true and the converse also is true.

Answers to Problems 13-4 | Student Text Page 301

1. Equivalent
2. Not equivalent
3. Equivalent

There is a variety of ways in which the equations in 1 and 3 can be proved equivalent. You might ask the children if it matters with which equation you begin the proof.

13-5 Graphs of Linear Equations in Two Variables

We are turning now to consideration of more general equations of the first degree in two variables - equations of the form

$$ax + by + c = 0$$

In this section we observe informally that the graphs are straight lines. We mention the name "linear equations".

Answers for Class Activity

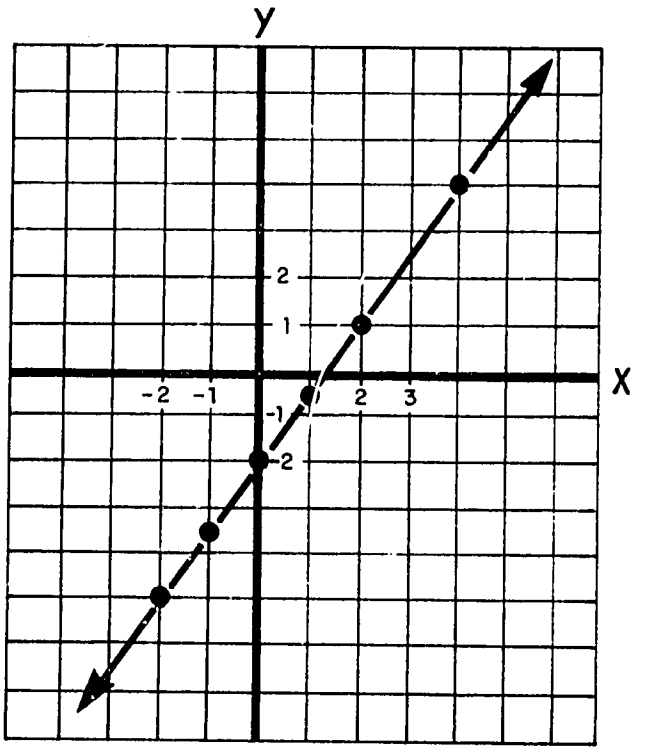
Student Text Page 302

1. (4, 4), (−2, −5), and (0, −2) belong to the truth set of $2y - 3x + 4 = 0$.

2.

x	y
−2	−5
−1	$-3\frac{1}{2}$
0	−2
1	$-\frac{1}{2}$
2	1
4	4

3.

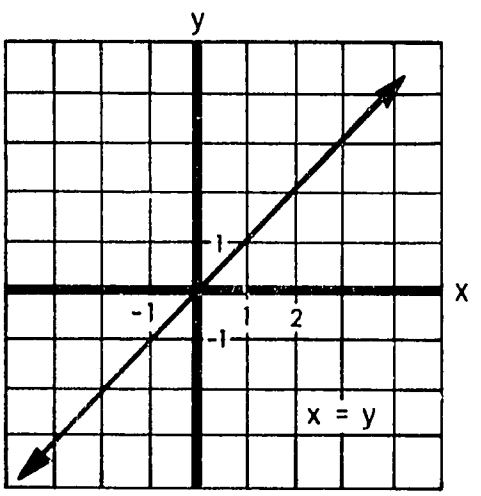


4. $\left(3, 2\frac{1}{2}\right)$, $\left(5, 5\frac{1}{2}\right)$, (−4, −8), (−6, −11)

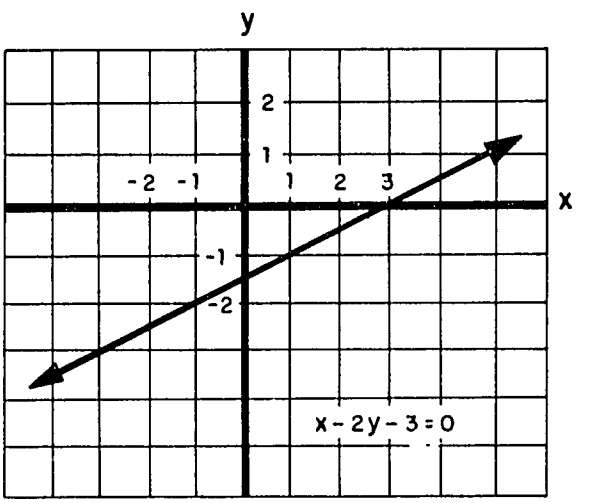
Answers to Problems 13-5

Student Text Page 303

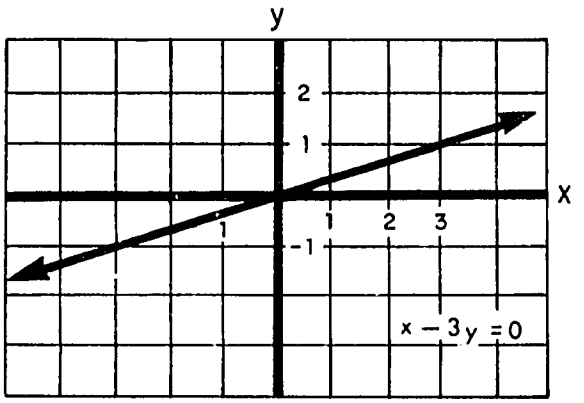
1.



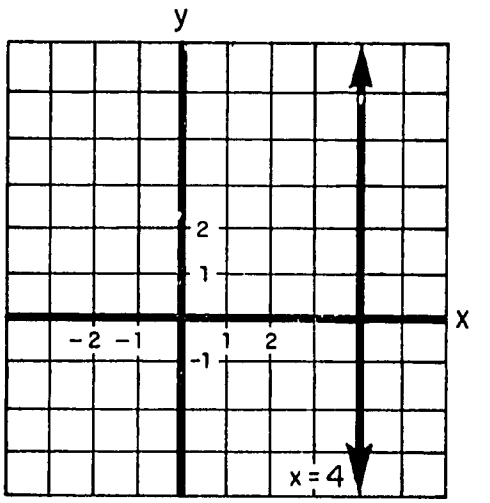
2.



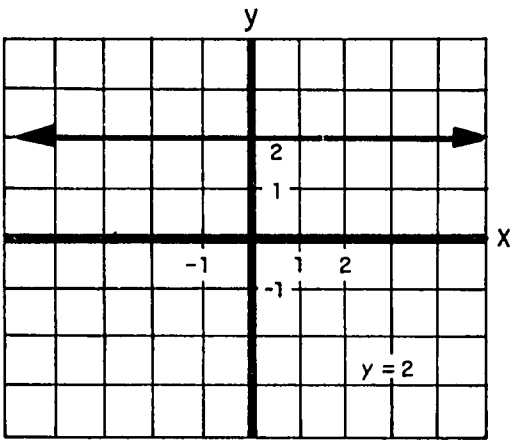
3.



4.



5.



13-6 The Slope of a Straight Line.

Again we are very informal in our approach to the idea of slope. It may be helpful to speak of slope as a convenient measure of the direction of a line. This number is convenient because, as we see later in Section 13-7, it appears in one form of the equation of the line.

The idea that the slope is the same number, no matter what two points on the line are chosen to determine it, is basic. The pupils discover this idea inductively here. No proof is given at this time. Proof of it comes in a later year when the pupils return to the idea of slope from a more formal viewpoint.

Other possible words to use in defining slope are

$$\frac{\text{change in } y}{\text{change in } x} \quad \text{or} \quad \frac{\text{rise}}{\text{run}} .$$

<u>Answers for Class Activity</u> **Student Text Page 303**

3. Yes, the children should finish at point F. The ratio is $\frac{6}{4}$ or $\frac{3}{2}$. The ratio $\frac{3}{2}$ should be obtained for all the pairs of points in (a) - (f).

4. Count down 3 units; count to the left 2 units; vertical increase is -3, horizontal increase is -2; ratio is $\frac{-3}{-2}$ or $\frac{3}{2}$.

5. The vertical increase is $-1\frac{1}{2}$; the horizontal increase is -1; the ratio is $\frac{-1\frac{1}{2}}{-1}$ or $\frac{3}{2}$.

6. Vertical increase: $(-3\frac{1}{2}) - (-2) = -1\frac{1}{2}$

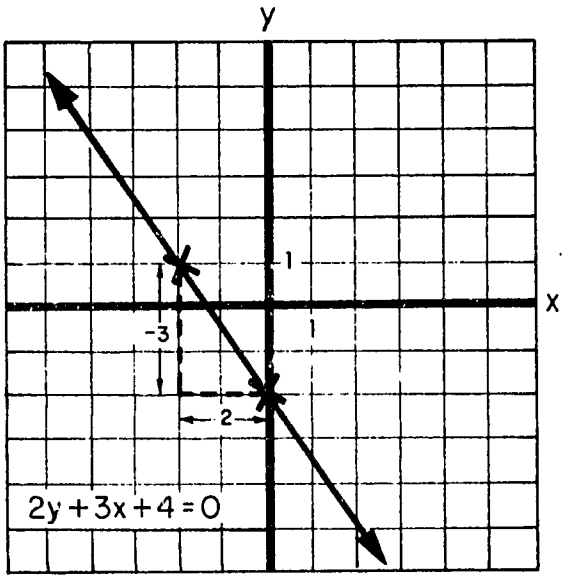
horizontal increase: $(-1) - 0 = -1$

ratio: $\frac{-1\frac{1}{2}}{-1} = \frac{3}{2}$

Answers to Problems 13-6

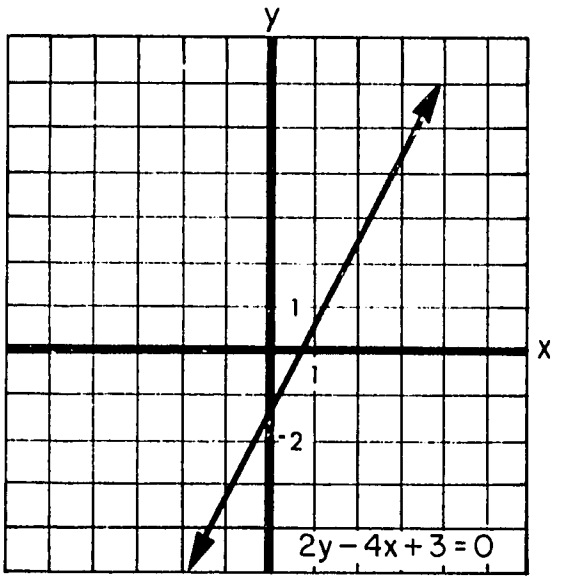
Student Text Page 305

1. (a)



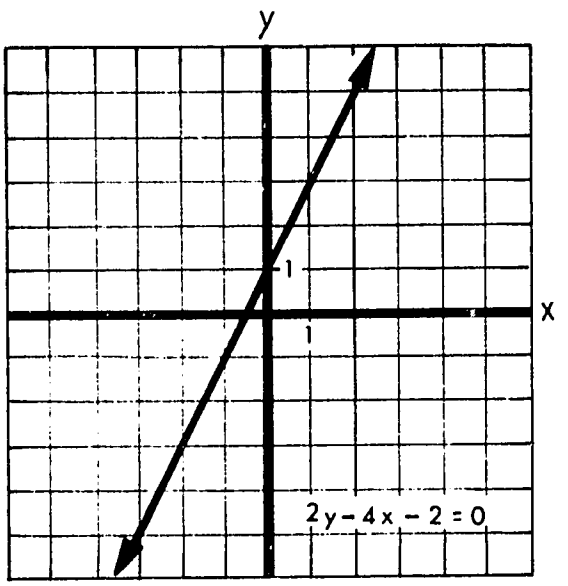
Slope is $-\frac{3}{2}$.

(b)



Slope is 2.

(c)



Slope is 2.

(d)

Same as (c)

2.

Line	Slope
$y = x + 3$	1
$y = 2x + 3$	2
$y = 3x + 3$	3
$y = 5x + 3$	5
$y = 10x + 3$	10
$y = -2x + 3$	-2
$y = \frac{1}{2}x + 3$	$\frac{1}{2}$

3.

1.	1	5.	-2
2.	$\frac{1}{2}$	6.	-1
3.	3	7.	5
4.	-1	8.	0

4. Each of these can be done in two ways, depending on the order in which the points are taken. For example, in Problem 1, one could have gone from (2,4) to (1,3) and obtained these results:

vertical increase: $3 - 4 = -1$

horizontal increase: $1 - 2 = -1$

slope is $\frac{-1}{-1}$, or 1.

13-7 The Slope of a Line from its Equation

Sections 13-7 and 13-8 may be considered optional and omitted if time is limited or if the ability of the class is such that these sections are too difficult at present.

In this section the pupil learns that if the equation of a line is in the form $y = ax + b$, the slope of the graph is a and if the equation is not in this form, it can usually be put in this form with the help of equivalent equations.

Exceptions are lines parallel to the y-axis for which the slope is undefined. Be careful that the pupils do not confuse lines with slope zero (horizontal lines) with lines which have no slope (vertical lines).

The student is led inductively to discover the ideas in this section. The proof that a is the slope in $y = ax + b$ comes later in a more formal treatment.

Proof of equivalence:

(c)	$2y - 4x - 2 = 0$	
	$2y = 4x + 2$	Added $4x + 2$
	$\frac{1}{2}(2y) = \frac{1}{2}(4x + 2)$	Multiplied by $\frac{1}{2}$
(d)	$y = 2x + 1$	

The proof can be reversed if you start with (d).

<u>Answers to Problems 13-7A</u> **Student Text Page 306**

1. y increases 5 as x increases 1. (The $5x$ tells us this.)
   y increases 15 as x increases 3. The slope is 5.
   The coefficient of x is 5.

2. The slope and the coefficient of x are both 7.

3. $\frac{3}{2}$ 4. 1 5. -4 6. 1000

You have a chance for a little review of equivalent equations.

$3x + 4y - 5 = 0$ is equivalent to $4y = -3x + 5$ because adding the real numbers $-3x$ and 5 on both sides does not change the truth set.

$4y = -3x + 5$ is equivalent to $y = -\frac{3}{4}x + \frac{5}{4}$ because multiplying on both sides by the non-zero number $\frac{1}{4}$ does not change the truth set.

The slope is $-\frac{3}{4}$.

Answers to Problems 13-7B

Student Text Page 307

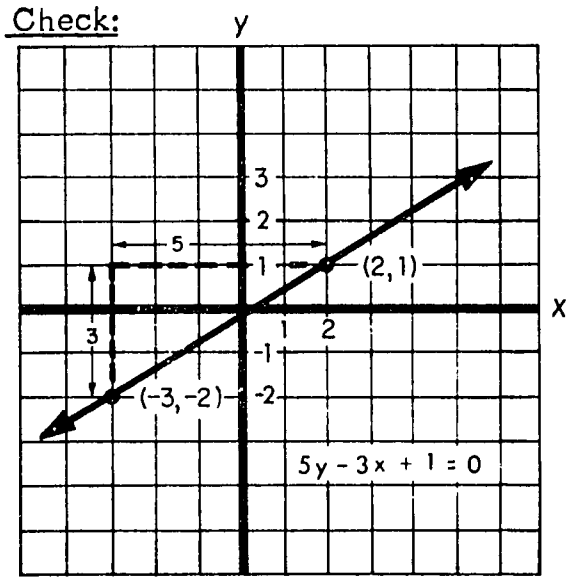
1. $5y - 3x + 1 = 0$

$5y = 3x - 1$

$y = \frac{3}{5}x - \frac{1}{5}$

Slope is $\frac{3}{5}$.

Check:



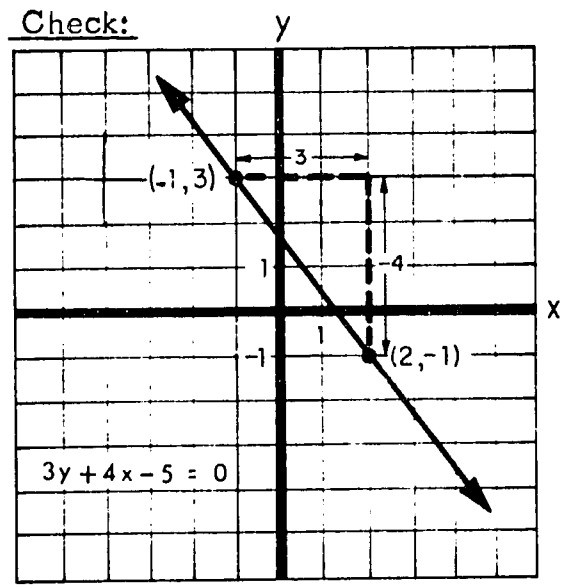
2. $3y + 4x - 5 = 0$

$3y = -4x + 5$

$y = -\frac{4}{3}x + \frac{5}{3}$

Slope is $-\frac{4}{3}$.

Check:



3. $x + 3y - 5 = 0$

$3y = -x + 5$

$y = -\frac{1}{3}x + \frac{5}{3}$

Slope is $-\frac{1}{3}$

Check: Depends on what two points the children choose. Example: From $(-1, 2)$ to $(5, 0)$:

y - increase: $0 - 2 = -2$

x - increase: $5 - (-1) = 6$

Slope is $\frac{-2}{\ }$, or $-\frac{1}{3}$

4. $3y - 7 = 0$

$3y = 7$

$y = \frac{7}{3}$

or $y = 0x + \frac{7}{3}$

Slope is 0.

Check: From $\left(0, \frac{7}{3}\right)$ to $\left(1, \frac{7}{3}\right)$:

y - increase: 0

x - increase: 1

Slope is $\frac{0}{1}$, or 0.

5. $2x - 6 = 0$.

This presents a special problem since the line does not have a slope. It is discussed in the text immediately after this problem.

A Special Case

An equation of the form $x = c$, or $ax + b = 0$, has no slope. All lines which are parallel to the y-axis have no slope. Do no confuse these lines with lines which are parallel to the x-axis, which do have slope; their slope is the number zero.

Answers to Problems 13-7C

Student Text Page 309

1. Slope is a negative number.
2. Slope is a positive number.
3. Slope is a negative number.
4. Line has no slope.
5. Slope is zero.
6. Line has no slope.

13-8 The Intercept on the y-axis

If x is 0, y -2. The coordinates of the point of intersection with the y-axis are $(0, -2)$.

In referring back to the problems in Section 13-2, we find that the second coordinate of the point where the line crosses the y-axis is the same number as the constant term in the equation of the line when the equation is written in the form $y = ax + b$. For example, when the equation of the line is $y = 2x + 5$, the coordinates of the point of intersection with the y-axis are $(0, 5)$.

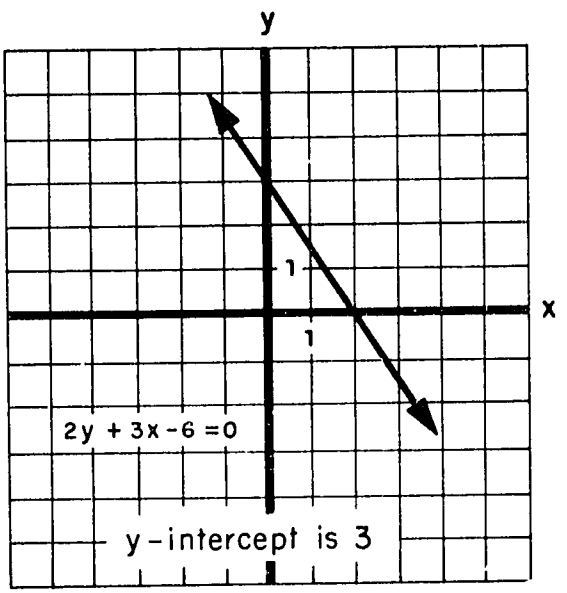
The first coordinate is 0 for every point of intersection with the y-axis.

In the proof, you may wish to discuss the number of points of intersection of a line with the y-axis. If the line is parallel to the y-axis, there is no point of intersection; if the line is the y-axis, then every point of the line is in the intersection; for any other line, there is exactly one point in the intersection.

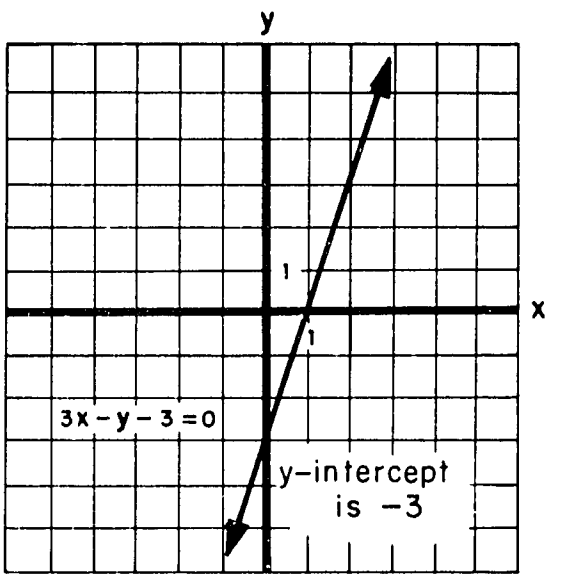
Answers to Problems 13-8

Student Text Page 311

1. (a)



(b)



Check: $2y + 3x - 6 = 0$

$2y = -3x + 6$

$y = -\frac{3}{2}x + 3$

The y-intercept is 3.

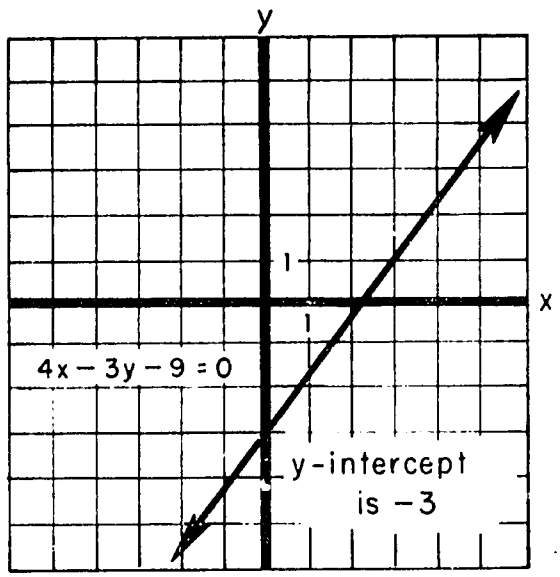
Check: $3x - y - 3 = 0$

$3x - 3 = y$

$y = 3x - 3$

The y-intercept is -3.

2. (a)



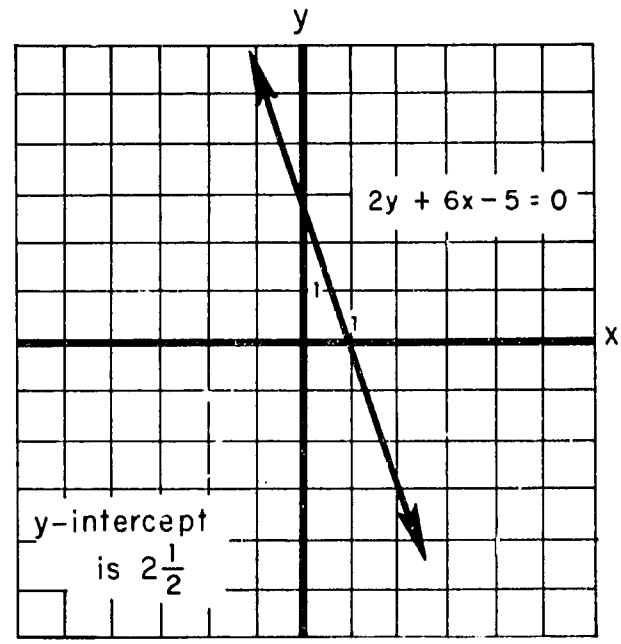
Check: If $x = 0$ and $y = -3$,

$4x - 3y - 9 = 0$ becomes

$[4 \times 0] - [3 \times (-3)] - 9 = 0$

which is true.

(b)



Check: If $x = 0$ and $y = \frac{5}{2}$,

$2y + 6x - 5 = 0$ becomes

$\left(2 \times \frac{5}{2}\right) + (6 \times 0) - 5 = 0$

which is true.

3. If a pupil estimates $\left(-\frac{3}{4}\right)$ for the y-intercept, he would try $\left(0, -\frac{3}{4}\right)$ in the equation.

$[7 \times \left(-\frac{3}{4}\right)] - [14 \times 0] + 6 = -\frac{21}{4} + \frac{24}{4} = \frac{3}{4}$

$\frac{3}{4} \neq 0$

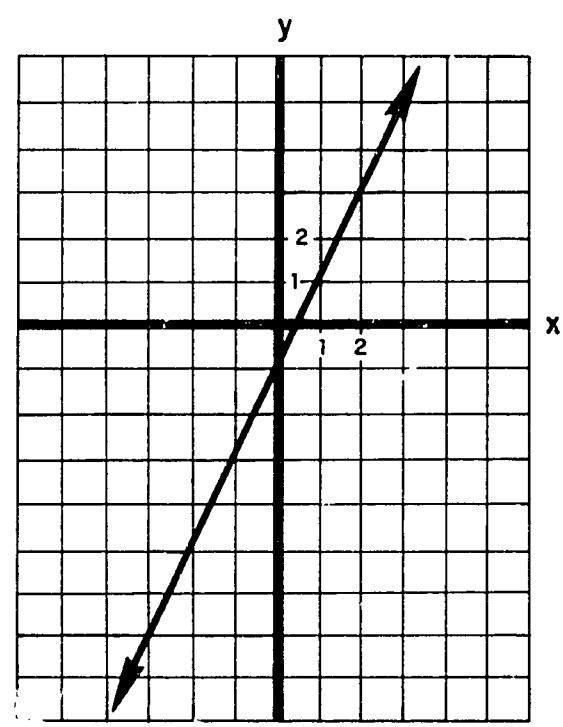
For $(0, -0{\cdot}8)$ we have

$[7 \times (-0{\cdot}8)] - [14 \times 0] + 6 = -5{\cdot}6 + 6$

$= 0{\cdot}4$

This is a closer estimate.

The exact intercept is $-\frac{6}{7}$, as can be found from the $y = ax + b$ form or by letting x be 0 in the given equation and solving for y.



4. (a) Slope is $-\frac{2}{3}$; y-intercept is 5.

(b) Slope is 100; y-intercept is -3.

(c) $y = 5x + 4$.

(d) $y = -2$.

5. Yes, any equation $x = c$ $(c \neq 0)$ gives a line which is parallel to the y-axis, and therefore has no y-intercept.

6.

	Slope	y-intercept	Equation
(a)	0	2	$y = 2$
(b)	no slope	no intercept	$x = -3$
(c)	2	3	$y = 2x + 3$
(d)	$-\frac{1}{3}$	-1	$y = -\frac{1}{3}x - 1$

7.

	Slope	y-intercept	Equation
(a)	$\frac{6}{1}$ or 6	1	$y = 6x + 1$
(b)	$\frac{0}{3}$ or 0	-3	$y = -3$
(c)	$\frac{-2}{1}$ or -2	0	$y = -2x$
(d)	$\frac{3}{2}$	4	$y = \frac{3}{2}x + 4$

8. Let x be 0 and solve for y.

$$5y + 35 = 0$$
$$5y = -35$$
$$y = -7$$

When $x = 0$, $y = -7$, so the y-intercept is -7.

9. Use the method of Problem 8, but do it mentally.

(a) 4

(b) 8

(c) $\frac{4}{7}$

(d) -15

(e) $\frac{3}{2}$

(f) 4

10. <u>CHALLENGE PROBLEM.</u> The intercept on the x-axis is the x-coordinate of the point where the graph intersects the x-axis. The y-coordinate at this point is 0.

(a) 12

(b) $-\frac{8}{3}$

(c) $-\frac{1}{2}$

(d) $\frac{5}{32}$

(e) $\frac{5}{3}$

(f) $-\frac{8}{3}$

11. <u>CHALLENGE PROBLEM.</u> Since the point with first coordinate 0 is not given, the pupils will be looking for some other way to find the intercept on the y-axis. Some may find it by drawing the graph. Some may see that y goes from 7 to 4 (a change of -3) as x goes from 2 to 1, so y must go from 4 to 1 as x goes from 1 to 0. Some may write $y = 3x + b$ (they have found that the slope is 3) and say that the coordinates (1, 4) must make this true. This gives them an equation which they can solve for b.

The equation is $y = 3x + 1$.

CHAPTER 14

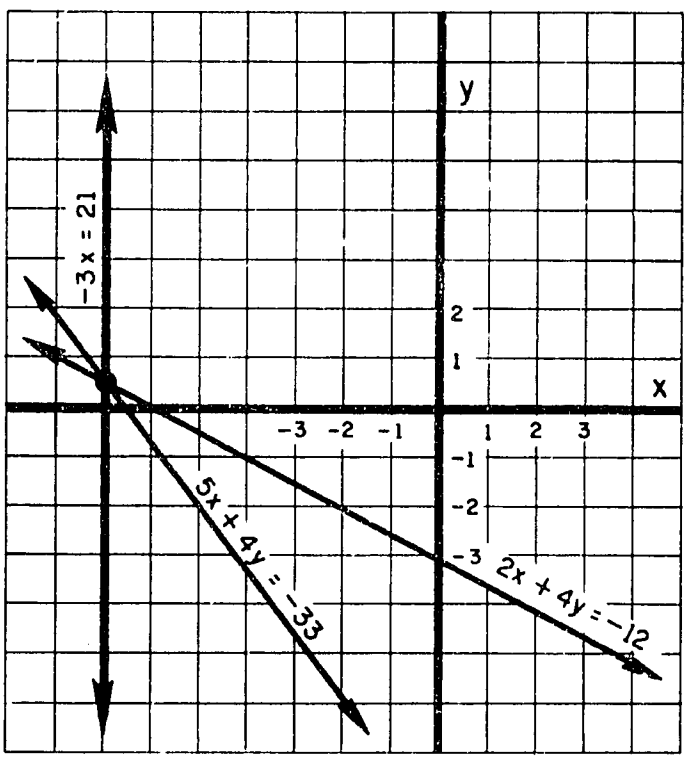
USING STRAIGHT LINE GRAPHS

14-1 Systems of Simultaneous Equations

The purpose of this section is to illustrate graphically the solution of systems of simultaneous linear equations, and to give examples of the special cases that arise.

Answers to Class Activity Student Text Page 315

1.



2. $(-7, \frac{1}{2})$

3. $(-7, \frac{1}{2})$. The three lines intersect at the same point.

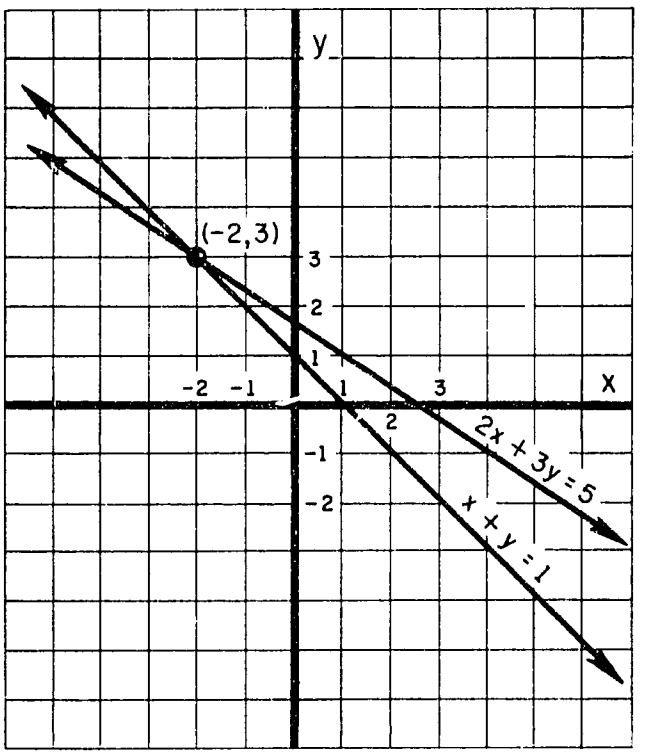
4. Yes, if the straight lines intersect, they will do so at the same point. This should not surprise the children since the coordinates of the point of intersection of two lines is the solution to the corresponding system of simultaneous equations, and if two pairs of simultaneous equations are equivalent, they have the same solution.

5. $(-7, \frac{1}{2})$. The children should realize that the solution of the system of simultaneous equations is the pair of coordinates of intersection of their graphs.

Answers to Problems 14-1A

Student Text Page 316

1.

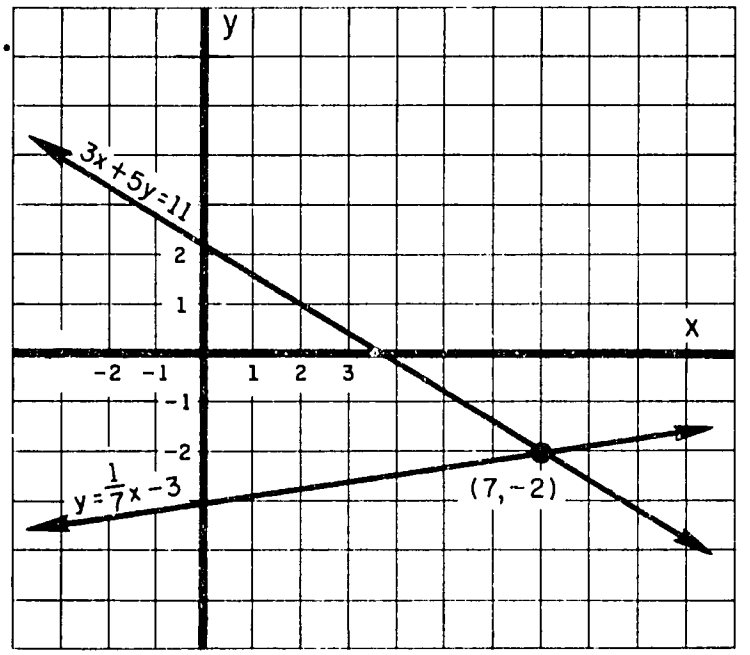


Check:

$2 \times (-2) + 3 \times 3 = 5$ is true.

$-2 + 3 = 1$ is true.

2.



Check:

$3 \times 7 + 5 \times (-2) = 11$ is true.

$-2 = \frac{1}{7} \times 7 - 3$ is true.

3. A problem where the solution involves fractions was included to illustrate to the children the difficulty in obtaining the solution from a graph; if accuracy is desired, the algebraic method should be used.

$$\begin{cases} 7x + 4y = 11 \\ 14x + 3y = 12 \end{cases}$$

$$\begin{cases} 14x + 8y = 22 \\ 14x + 3y = 12 \end{cases}$$

$$5y = 10$$

$$y = 2$$

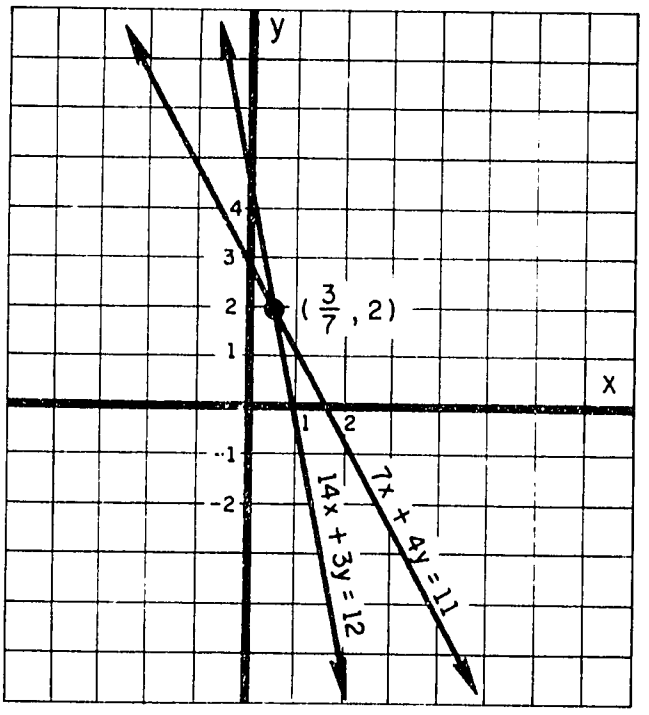
$$\begin{cases} 7x + 4y = 11 \\ y = 2 \end{cases}$$

$$7x + 4 \times 2 = 11$$

$$7x = 3$$

$$x = \frac{3}{7}$$

$$\begin{cases} x = \frac{3}{7} \\ y = 2 \end{cases}$$



There are two cases where a pair of simultaneous equations does not have a unique solution. The truth set of

$$\begin{cases} 3x - y + 1 = 0 \\ 2y - 6x - 7 = 0 \end{cases}$$

is the empty set. The children will discover that their graphs are parallel lines which, of course, have no point of intersection. This explains why the truth set is the empty set.

The system of equations

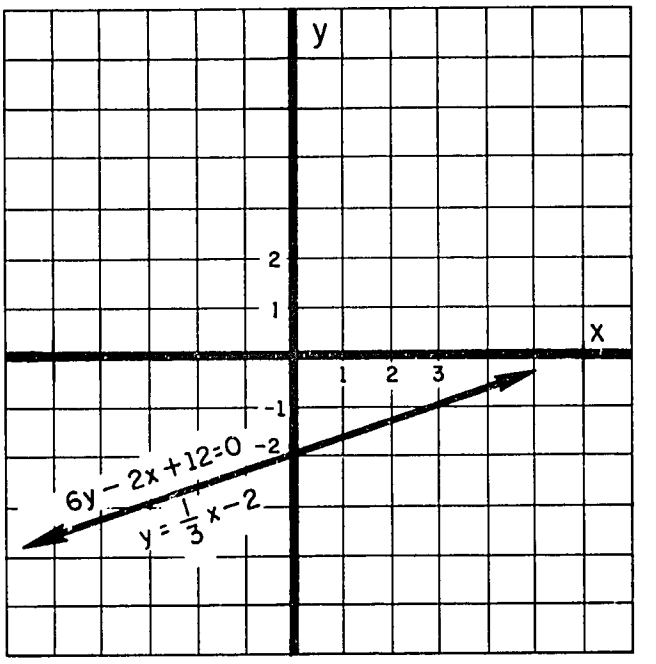
$$\begin{cases} y = \frac{1}{3}x - 2 \\ 6y - 2x + 12 = 0 \end{cases}$$

has many solutions, in particular (3, -1), (0, -2), and (9, 1). Since two distinct lines cannot intersect in more than one point, some of the children may be able to predict that the two equations have the same graph before they actually draw their graph in Exercise 1 of the Class Activity.

Answers to Class Activity

Student Text Page 317

1. You may wish to ask the pupils if they can prove the two equations equivalent without using the graph.



2. An equivalent system can be obtained in which the two equations are identical.

$$\begin{cases} \frac{1}{3}x - 2 = y \\ 6y - 2x + 12 = 0 \end{cases} \qquad \begin{cases} 3y - x + 6 = 0 \\ 6y - 2x + 12 = 0 \end{cases}$$

$$\begin{cases} x - 6 = 3y \\ 6y - 2x + 12 = 0 \end{cases} \qquad \begin{cases} 6y - 2x + 12 = 0 \\ 6y - 2x + 12 = 0 \end{cases}$$

3. The children may need to discuss this question before they decide on an answer. The truth set can be described using the graph as the set of ordered pairs that correspond to the points on the line; or algebraically as

the set of ordered pairs satisfying $y = \frac{1}{3}x - 2$ (or $6y - 2x + 12 = 0$); or as the set of ordered pairs of the form $(x, \frac{1}{3}x - 2)$.

Answers to Problems 14-1B

Student Text Page 317

1. $\{(-3, 5)\}$
2. $\{\quad\}$ (the empty set)
3. $\left\{\left(7, \frac{5}{8}\right)\right\}$
4. $\left\{\left(x, \frac{1}{15}x + \frac{2}{5}\right)\right\}$ or the set of all ordered pairs of numbers which satisfy either one of the given equations.

14-2 Proportionality

Answers to Questions in Text

Student Text Page 318

1. (b) and (c).
2. 7 for (b); $\frac{1}{4}$ for (c).
3. (a) $y = x + 7$ (c) $y = \frac{1}{4}x$

   (b) $y = 7x$ (d) $y = x^2 + 1$

(b) and (c) show proportionality because the rule can be written in the form $y = kx$; (a) and (d) cannot.

4. The pupils should realize that the graph of an equation showing proportionality must be a straight line (which (d) is not) and it must pass through the origin (which (a) does not).

5. Yes, the equation is linear with slope k and intercept 0 on the y-axis. The point (0, 0) will be on the graph of $y = kx$ for any k.

Answers to Problems 14-2 | Student Text Page 319

1. x is proportional to y with constant of proportionality $\frac{1}{k}$. The proof of this is simple:

If $y = kx, \quad k \neq 0$

Then $\frac{1}{k} y = \frac{1}{k}(kx)$, Multiply by the non-zero number $\frac{1}{k}$

$\frac{1}{k} y = \left(\frac{1}{k} \times k\right)x,$

and $\frac{1}{k} y = x$

2. (b) $k = \frac{7}{2}$

3. (b) and (d), since they both contain the point (0, 0) and do not have slope zero. (a) and (c) do not contain (0, 0). The assumption that the rule is a linear equation is important, since other wise (d) might be $y = x^2$. For (b), $k = 100$; for (d) $k = -2$.

4. (b) with $k = -1$

(c) with $k = 3$

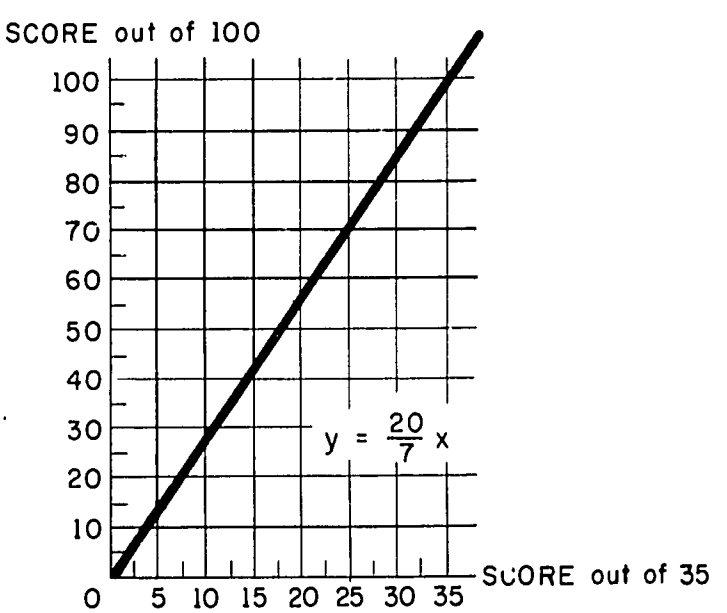
5. (a) For $\frac{28}{35} = \frac{x}{100}$, $x = 80$, and 80 is the grade put on the paper.

For $\frac{23}{35} = \frac{x}{100}$, $x = 65\frac{5}{7}$, and 65 is the paper grade.

(b) For $y = \frac{20}{7} \times 14$, y is 40, and 40 is the paper grade.

For $y = \frac{20}{7} \times 18$, y is $51\frac{3}{7}$, and 50 is the paper grade.

(c)



Number correct	10	34	20	17	30	26	4	12	31	24
Grade	30	95	55	50	85	75	10	35	90	70

14-3 A Practical Application

Introduction

This section is an optional one. Although there is no necessity to take up this material, it nevertheless provides a variety of interesting problems and these problems can provide the pupils with better understanding of linear equations.

We give here a brief introduction to what is known as linear programming. The problems are of a type which manufacturers meet when they try to make or distribute their products in the most economical way. The central mathematical topic is that of linear inequalities and the first objective is to show that a line (such as $x + y = 3$) divides the plane into two half-planes (given by $x + y > 3$ and $x + y < 3$).

Not all pupils should be able to carry out a problem on their own. This does not matter. The purpose is to develop insight into linear equations and inequalities and to show the usefulness of inequalities.

1. Graphs of Inequalities

As mentioned in the introduction, the aim here is to show that a linear equation gives a line which divides the rest of the plane into two half-planes, each given by a linear inequality.

After completing the discussion in the text of the graphs of $x + y = 3$, $x + y > 3$, and $x + y < 3$, you may wish to review the ideas with the following Class Activity.

1. Have a pupil draw the graph of $2x + y = 10$ on the blackboard.
2. Is the point (4, 2) on the graph?

3. If the point (4, y) is in the half-plane above the graph, what numbers can y be?

4. Show all points (4, y) such that $y > 2$.

5. What values can $2x + y$ have for these points (4, y) above the graph?

6. If $2x + y > 10$, where is the point (4, y)?

7. If $x = 3$, what must y be to give points (x, y) in the upper half-plane?

8. What inequality will describe all points in the upper half-plane?

9. What inequality gives all points in the lower half-plane?

Other questions at this point might be:

10. What inequality describes the half-plane

(a) to the right of the graph of $x + 4y = 12$?

(b) to the left of the graph of $x + 4y = 12$?

(Examine the point (x, 2), for instance, for values of x which put the point on, to the right of, or to the left of the given line.)

11. Draw the graphs of the inequalities (on the same axes) $x + y > 2$ and $x + 3y \leq 6$. Name four points whose coordinates satisfy both inequalities.

2. An Application (Linear Programming)

It is not necessary to make a point of introducing the name "linear programming". Some classes may be put off by the name while others may be stimulated by it.

The first task is to establish the inequalities. This does not differ in any way from the methods used in working problems earlier in the course. It should not present much difficulty if it is not rushed.

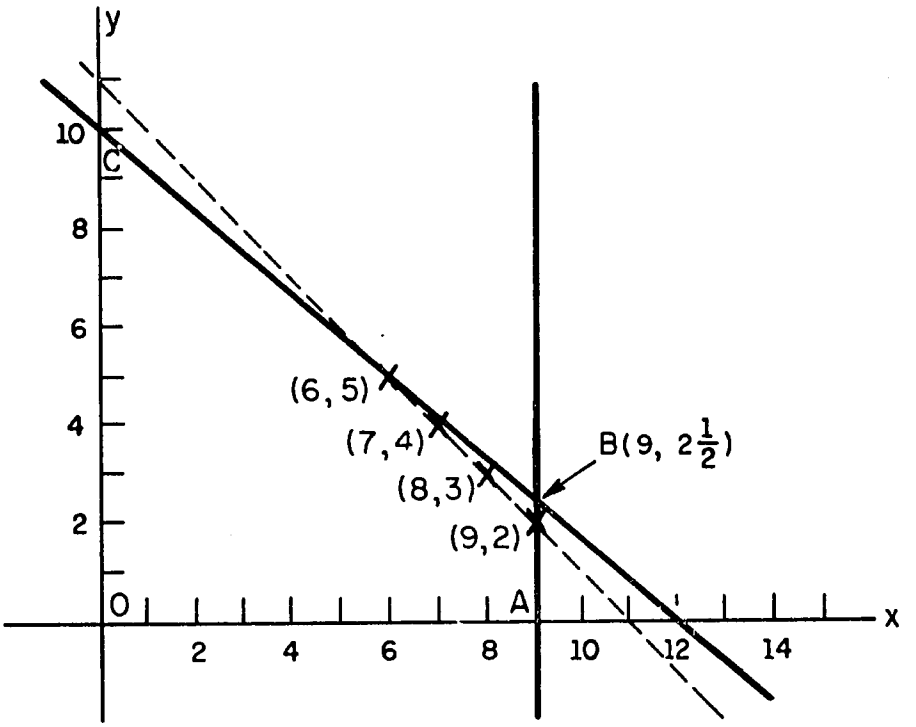
When graphing the lines which give the half-planes, the easiest way will usually be to find where the lines cut the axes. These two points should be all that is necessary to draw the lines. The important thing is to be able to draw the lines quickly.

The polygon containing the points whose coordinates satisfy all the inequalities, will be built up step by step. The order is not important.

To indicate the half-plane given, for instance, by $5x + 6y \leq 60$, we shade the region for which this is <u>not</u> satisfied. This is not the usual practice, but it gives a clearer diagram eventually. You may find that a coloured pencil makes the shading more clear. We eventually have an unshaded region which is the set of points whose coordinates satisfy all the inequalities. This is shown in the figure **on page 324 of the Student Text.**

Having drawn the polygon, we must now find the point (x, y) within it which makes $(x + y)$, the number of books, as large as possible. That is, we want the line (with equation $x + y = c$) which cuts the polygon and has c as large as possible. This is most easily found by using a set square and ruler. Arrange the instruments so that one edge of the set square gives lines parallel to $x + y = c$. It is then easy to see, by sliding the set square along the ruler, that the required line is the one which passes through B. This is $x + y = 11$. The solution is $x = 6$, $y = 5$, that is, 6 books from Nazarili and 5 books from Shariff.

It will be necessary to point out that x and y must be whole numbers. We obviously cannot buy $7\frac{1}{2}$ books. This problem was made easier by the fact that the point B itself has whole number co-ordinates. This does not always happen. Consider the case when $x \leq 9$. B is now the point $(9, 2\frac{1}{2})$ and the required line through B has the equation $x + y = 11\frac{1}{2}$. Since $x + y$ must be a whole number, we use the line $x + y = 11$. We see from the figure on the next page that there are four points on this line with integral coordinates and which are contained in the region OABC. They are $(9, 2)$, $(8, 3)$, $(7, 4)$ and $(6, 5)$.



Any of these, then would be a solution. There is one more factor to consider. Although it is not so stated in the question, we should expect the cost in this case to be kept as low as possible. That is, we want $5x + 6y$ to be as low as possible. The four points make $5x + 6y$ equal to 57, 58, 59, and 60 respectively, so the best solution is $x = 9$, $y = 2$, that is, 9 books from Nazarili and 2 from Shariff.

If Nazarili had 5 books, one should buy 5 books from Nazarili and 5 from Shariff. If Nazarili had 4 books, one should buy 4 books from Nazarili and 6 from Shariff.

3. <u>Another Application</u>

Very little additional comment is needed on this second example. The method is exactly the same. The unit "ton-miles" will need careful explanation. Reference should also be made to other compound units such as man-hours, machine-hours, etc. The manufacturer should make 10 tons per day of A and 15 tons per day of B. If the profit were £2 per ton for both A and B, he should make $8\frac{1}{2}$ tons per day of A and 17 tons per day of B.

Answers to Problems 14-3

Student Text Page 328

1. (a) (6, 2) (b) $(\frac{15}{4}, \frac{15}{2})$

(c) (6, 6)

(d) Any point on the line $2x + 3y = 30$ between the points (6, 6) and $(\frac{15}{4}, \frac{15}{2})$

2. The inequalities are

$$5x + 5y \geq 50$$
$$15x + 5y \geq 60$$

You may wish to point out that these are equivalent to

$$x + y \geq 10$$
$$3x + y \geq 12$$

but this is not necessary.

The cost, $2x + y$, is to be a minimum.

The solution is 1 oz. of "X" to 9 oz. of "Y".

3. This table shows the number of units transported.

From \ To	Thompson Falls	Nanyuki
Nairobi	$15 - x$	$20 - y$
Nakuru	x	y

Number of units sent from Nakuru is $x + y$. Therefore $x + y \leq 20$.

Number of units sent from Nairobi is $35 - (x + y)$. Therefore

$$35 - (x + y) \leq 25$$

or $$10 \leq x + y$$

Subtract 25 from both sides and add $x + y$ to both sides.

Also, 15 units go to Thompson Falls and 20 units go to Nanyuki.

Therefore $x \leq 15$

$y \leq 20$

The system of inequalities is

$$\begin{cases} 10 \leq x + y \leq 20 \\ x \leq 15 \\ y \leq 20 \end{cases}$$

The cost of transport will be

$$[36(15 - x) + 45(20 - y) + 30x + 42y] \text{ shillings}$$

or $$(1440 - 6x - 3y) \text{ shillings}$$

We want this to be as small as possible so we want $6x + 3y$ to be as large as possible. The solution is $x = 15$, $y = 5$. So numbers of units sent are:

From \ To	Thompson Falls	Nanyuki
Nairobi	0	15
Nakuru	15	5

4. The completed table is:

From \ To	P_1	P_2	P_3
S_1	x	y	$9 - (x + y)$
S_2	$5 - x$	$6 - y$	$6 - [(5 - x) + (6 - y)]$ or $(x + y) - 5$

The inequalities are

$x \leq 5$ (from $5 - x \geq 0$)

$y \leq 6$ (from $6 - y \geq 0$)

$x + y \leq 9$ (from $9 - (x + y) \geq 0$)

$x + y \geq 5$ (from $(x + y) - 5 \geq 0$)

Cost of transport in shillings is:

$$6x + 3y + 4[9 - (x + y)] + 4(5 - x) + 2(6 - y) + 6[(x + y) - 5]$$

or $$4x + 3y + 38$$

$4x + 3y$ must therefore be as small as possible.

The solution is:

$$x = 0, \quad y = 5$$

The table indicating the number of tons delivered is:

From \ To	P_1	P_2	P_3
S_1	0	5	4
S_2	5	1	0

5. Let the mine at A operate for x days per week.

Let the mine at B operate for y days per week.

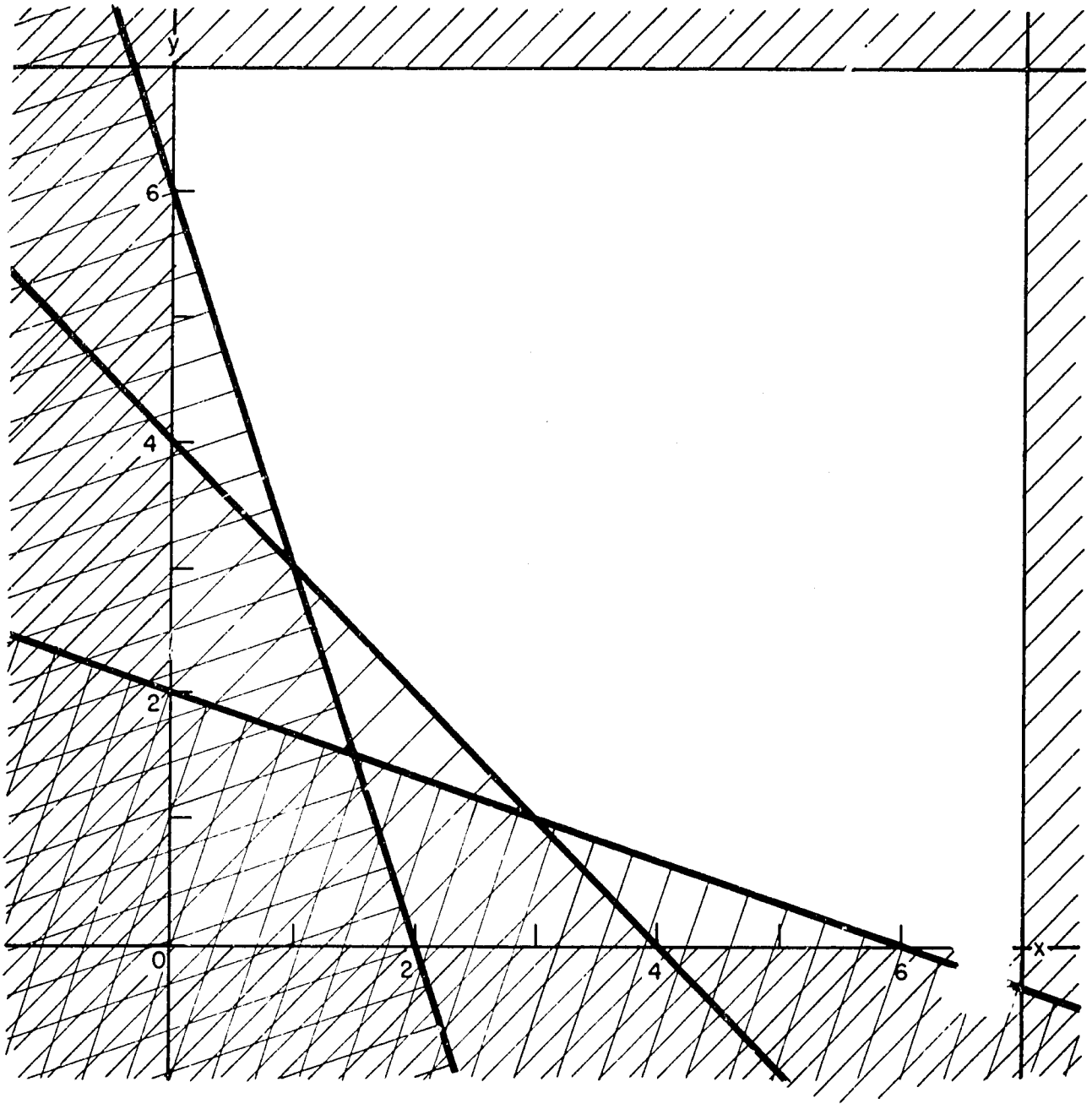
The following table helps to clarify the information:

Mine \ Ore	Grade 1	Grade 2	Grade 3	Cost per day
A	3	1	2	£ 40
B	1	1	6	£30
Orders for	6	4	12	

The system of inequalities is:

$$\begin{cases} 3x + y \geq 6 \\ x + y \geq 4 \\ 2x + 6y \geq 12 \\ 0 \leq x \leq 7 \\ 0 \leq y \leq 7 \end{cases}$$

The cost per week of running the mines is £$(40x + 30y)$, that is, $10(4x + 3y)$. We therefore want $4x + 3y$ to be as small as possible.



The solution is: $x = 1$ $y = 3$

Mine A works for 1 day per week.

Mine B works for 3 days per week.

CHAPTER 15

OTHER GRAPHS

Much of this chapter may be considered optional if it is not appropriate for your pupils at this time. Look ahead through the chapter and plan in advance what parts of the chapter you want to emphasize.

15-1 Non-linear Graphs

The purpose of this section is to acquaint the children with some graphs which are not straight lines. Since all these graphs will be considered at a more advanced level in later years, how much you do with this section should be determined by time and the children's interest. If time is short, you may wish to spend only 15 or 20 minutes in class on the two graphs in the class activity, and either omit the rest or assign the problems as long-range individual projects or perhaps as extra-credit work for interested children. If your class has much difficulty doing the graphs, or finds them uninteresting, then it is probably a good idea to omit most of the problems. You may find, however, that many children get considerable enjoyment from discovering that graphs of open sentences can be circles, squares, parabolas, etc. as well as straight lines. If the class seems to enjoy the section, it is suggested that you **spend a part of class time on Problems 15-1 for several days, as you go on into other sections.**

Teaching Suggestions for Class Activity — **Student Text Page 331**

The graph of $y = |x|$. The graph may be done either on the blackboard with the class working together, or on graph paper with each child making his own graph.

1.

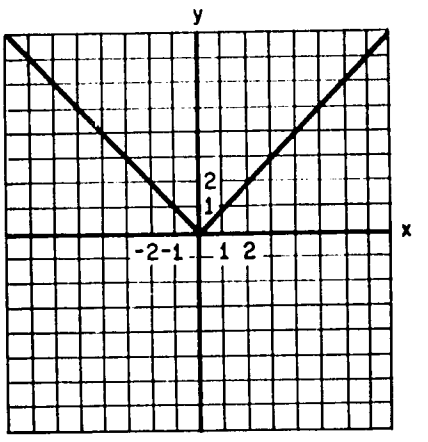
x	0	1	2	3	4	5
y	0	1	2	3	4	5

3.

x	-1	-2	-3	-4	-5
y	1	2	3	4	5

4. $y = -x$

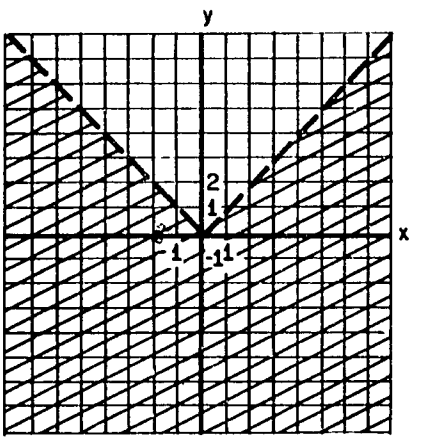
5. Completed graph of $y = |x|$.



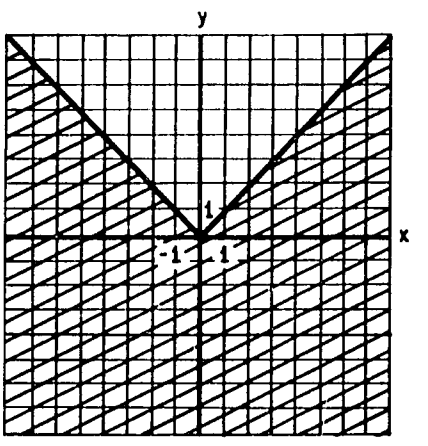
6. The graphs $y = |x|$ and $y = x$ are the same for $x \geq 0$; the graphs of $y = |x|$ and $y = -x$ are the same for $x \leq 0$.

7. This question's purpose is to have the children notice the connection between the graph and the equation. Since $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x \leq 0$, the children should not be surprised that the graph of $y = |x|$ is made up of parts of the graphs of $y = x$ and $y = -x$. Additional questions you can ask: "According to your graph, what are the possible values of y?" Answer: $y \geq 0$, i.e. y cannot be negative. "Could you have predicted this before you drew the graph?" Answer: Yes, y cannot be negative because the absolute value of any number cannot be negative.

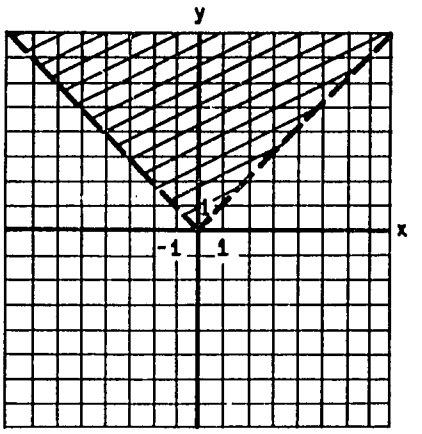
8. (a) The graph of $y < |x|$ is the region below the graph of $y = |x|$. In the picture below, the rays are dashed to show that these points are <u>not</u> part of the graph; the shading is used to show points that do belong to the graph.



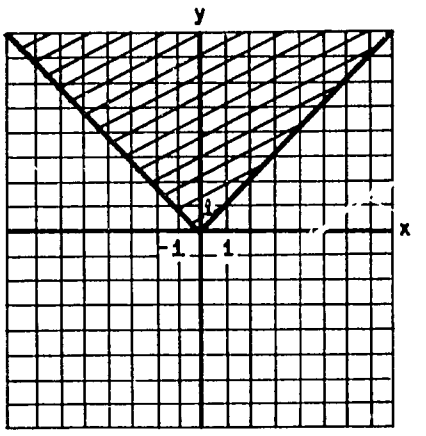
(b) The graph of $y \leq |x|$ is the region below the graph of $y = |x|$ <u>and</u> the graph of $y = |x|$.



(c) The graph of $y > |x|$ is the region above the graph of $y = |x|$.



(d) The graph of $y \geq |x|$ is the region above the graph of $y = |x|$, <u>and</u> the graph of $y = |x|$.



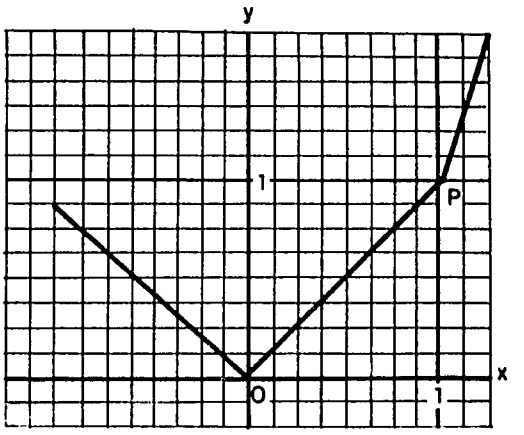
<u>The graph of $y = x^2$</u>

1.

x	-2	-1	0	1	2	3	4	5
y	4	1	0	1	4	9	16	25

3. This section needs to be handled with care. If the class is doing the graph on the blackboard, you can ask someone to come up and complete the graph as he thinks it should be. If the children are making their own graphs, you can have all the <u>different</u> ways they have completed the graph put on the board. They should realize that the graph is symmetric about the y-axis, and therefore, since (3, 9) is in the truth set, (-3, 9) is also. It is likely that many, or even all, of the children will complete the graph by connecting the points by line segments, which is not surprising since their previous experience has not included curves. If no one objects to the line segments,

you can direct their attention to the piece of their graph for x between 0 and 1, perhaps draw an enlarged picture on the blackboard,



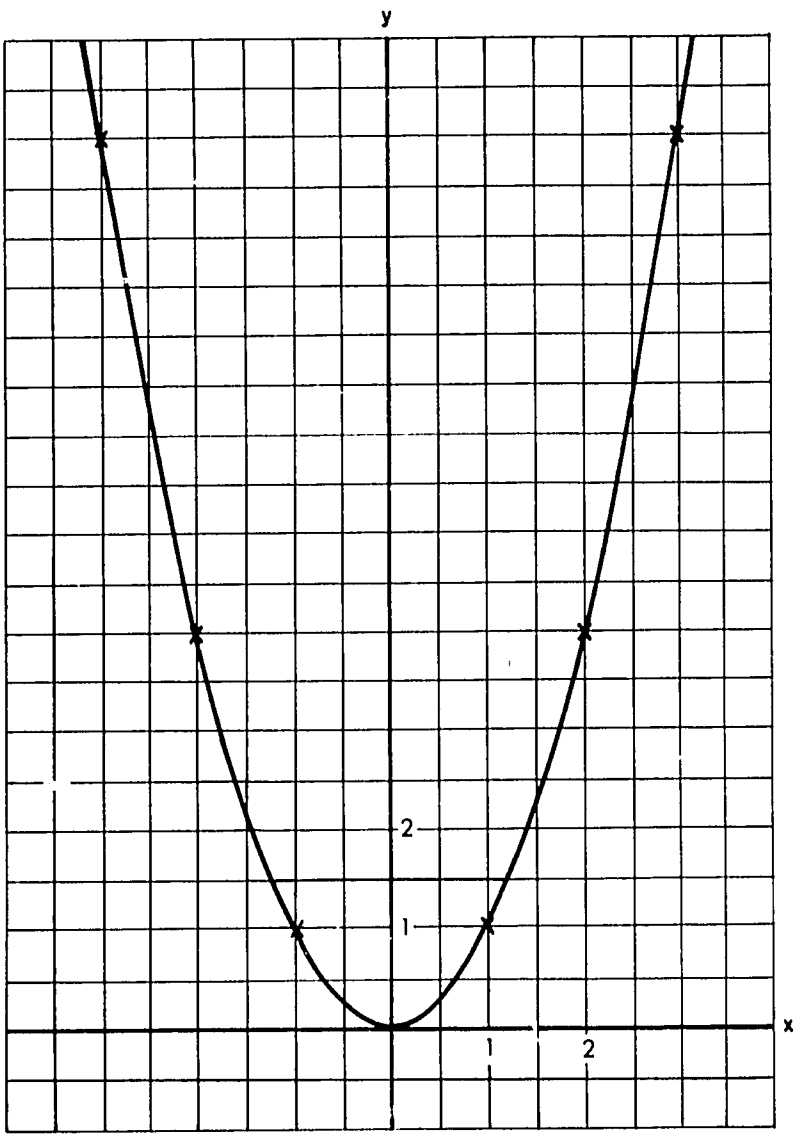
and ask the following questions:

(a) What is the equation of the line that contains $\overline{OP}$? [Ans: $y = x$].

(b) If x is $\frac{1}{2}$, and $y = x$, what is y? [Ans: $\frac{1}{2}$]

(c) Is $(\frac{1}{2}, \frac{1}{2})$ in the truth set of $y = x^2$? [Ans: No, $(\frac{1}{2}, \frac{1}{4})$ is.]

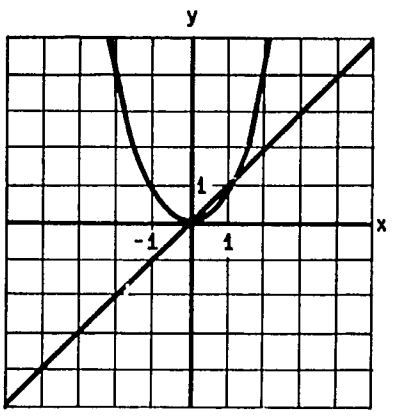
At this point in the discussion, the children should understand that connecting (0, 0) and (1, 1) by a line segment will not give a correct graph of $y = x^2$, and that they need to find more ordered pairs that belong to the truth set. It is probably a good idea to find pairs for such x-values as $\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4}$. After the points have been plotted, someone can come up and mark a point that he thinks belongs to the graph, estimate the ordered pair that goes with it, and check it in the equation to see how close it is. Points can then be marked for other parts of the graph, and finally the curve drawn.



Some questions you can ask when the graph of $y = x^2$ is completed are:

1. Looking at the graph, do you find any points with a negative y-coordinate? [No.] Could you have predicted this <u>before</u> you drew the graph? [Yes; y cannot be negative because the square of any real number cannot be negative.]

2. Have the children draw the graph of $y = x$ on the same axes as $y = x^2$,



and ask:

(a) For what values of x are the y-values of $y = x^2$ greater than the y-values of $y = x$? [For $x > 1$ and $x < 0$; from the graphs we see that $y = x^2$ is <u>above</u> $y = x$; from the algebra we know that the square of any number greater than 1 or less than 0 is greater than the number.]

(b) For what values of x are the y-values of $y = x^2$ <u>less</u> than the y-values of $y = x$? [For x between 0 and 1; the graphs show that $y = x^2$ is <u>below</u> $y = x$; also the square of any number between 0 and 1 is <u>less</u> than the number.]

(c) For which values of x are the y-values of $y = x^2$ and $y = x$

equal? [For $x = 0$ and $x = 1$; the graphs intersect at the points (0, 0) and (1, 1).]

3. For $y = x$, if x increases one, how much does y increase? [1]. For $y = x^2$, if x increases one, how much does y increase? [It depends on the value of x.]

4. Is (12, 144) in the truth set of $y = x^2$? [Yes.] What other ordered pair do you then know is in the truth set? [(-12, 144)] Is there a line of symmetry for the graph of $y = x^2$? [Yes, the y-axis.] What is its equation? [$x = 0$.]

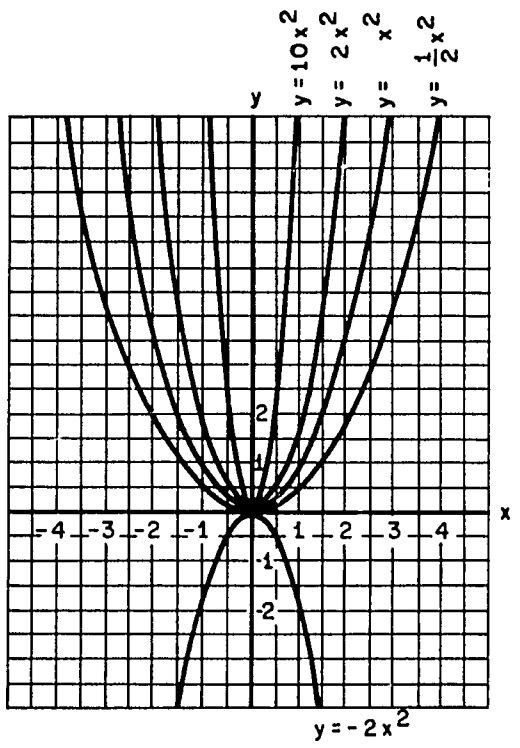
Answers to Problems 15-1

Student Text Page 332

1. This problem you may wish to do in class right after doing the graph of $y = x^2$ if there is time. Additional questions you might ask when the graphs to the right have been completed:

(a) Without drawing the graphs, how could you describe the graphs of $y = -x^2$, $y = -10x^2$, $y = -\frac{1}{2}x^2$? [Several answers are possible; for example, the graph of $y = -10x^2$ could be described as the graph of $y = 10x^2$ "flipped over" the x-axis; or the graphs of all of them can be obtained by turning the paper upside-down and re-labeling the axes.]

(b) Would $y = 100x^2$ be thinner or fatter than $y = x^2$? [Thinner]



How about $y = \frac{1}{10}x^2$? [Fatter.] Can you name an equation of a parabola that would be very fat? [$y = \frac{1}{100}x^2$, $y = \frac{1}{1000}x^2$, etc.]

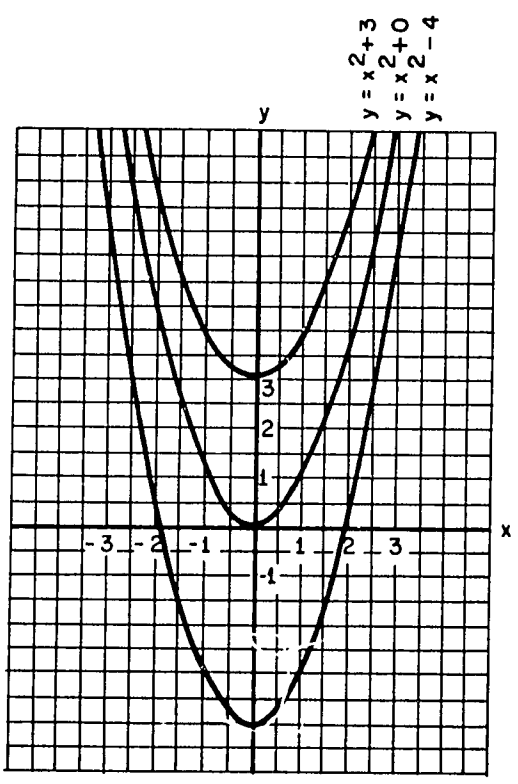
(c) When $x > 0$, for which graph does the y-value increase faster, $y = 100x^2$ or $y = x^2$? [$y = 100x^2$]; $y = \frac{1}{100}x^2$ or $y = x^2$? [$y = x^2$.]

(d) <u>If</u> the children can use variables well, you can describe <u>all</u> the parabolas whose turning points are at the origin by the equation $y = kx^2$, where k is any real number, and then ask the following questions:

(1) For what values of k does the parabola open up? [$k > 0$.] For what values of k does the parabola open down? [$k < 0$.] What happens when $k = 0$? [$y = kx^2$ becomes $y = 0$. The equation for the x-axis.]

(2) For what values of k is $y = kx^2$ thinner than $y = x^2$? [$k > 1$ or $k < -1$.] For what values of k is $y = kx^2$ fatter than $y = x^2$? [$-1 < k < 1$.]

2.



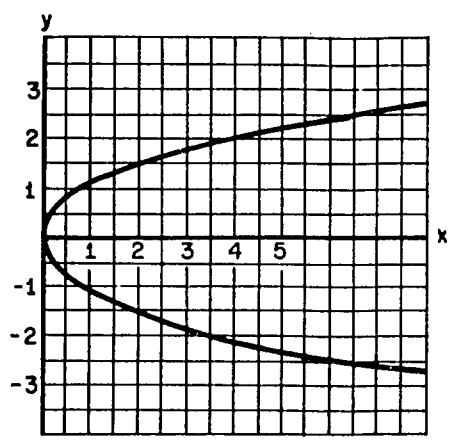
Additional questions you can ask when the above graphs are completed:

(a) Without graphing, can you describe the graph of $y = x^2 + 1$? of $y = x^2 + 100$? of $y = x^2 - 100$? of $y = -x^2 + 100$? [The first three graphs have the same shape as $y = x^2$, but have been moved up or down the y-axis so that the turning points are at 1, 100, and -100 respectively; the last one is the graph of $y = x^2$ "flipped over" the x-axis and moved up the y-axis 100 units.]

(b) Without graphing, can you describe the graph of $y = 2x^2 + 5$? [Same shape as $y = 2x^2$ (thinner than $y = x^2$) but moved up the y-axis 5 units.]

(c) Again, if the children can handle variables well, you can now discuss the graphs of $y = x^2 + p$, for all real numbers p. The ordered pair (0, p) gives the parabola's turning point. In addition, the graphs of $y = kx^2 + p$ can be discussed. The children may be able to give rough descriptions of the parabola for various values of k and p; for example, they might describe the graph of $y = -100x^2 + 10$ as: A parabola much thinner than $y = x^2$, opening down, with turning point at (0, 10). An interesting special case occurs for $k = 0$. The graphs are a family of straight lines $y = p$ parallel to the x-axis.

3.

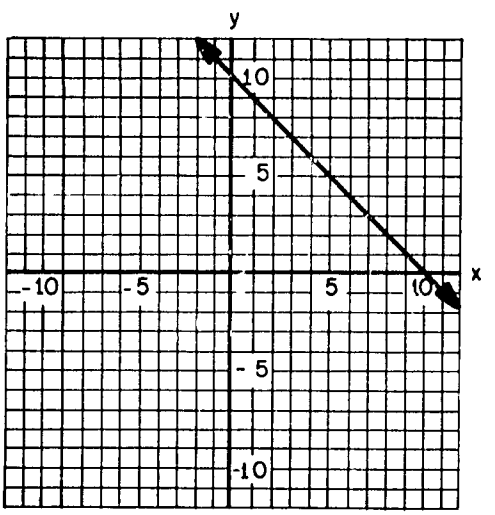


$x = y^2$

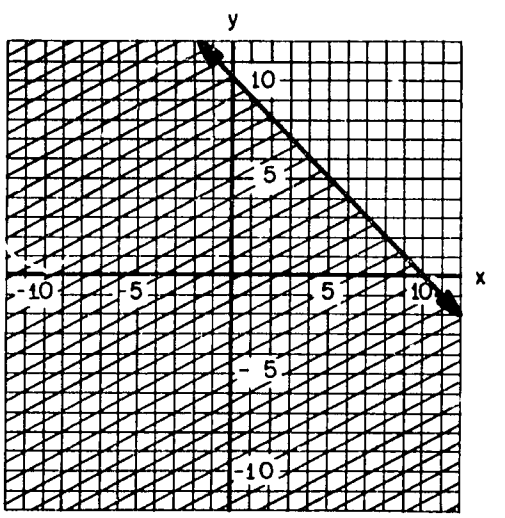
Additional questions similar to those in 1, 2, and 3 can be asked when this graph is completed. For example, describe the graphs of $x = -y^2$, $x = y^2 + 3$, $x = -10y^2 + 2$, etc. If the children happen to ask how you can get a parabola whose graph has its turning point off the axes, you can have them try something like $y = (x + 1)^2 - 2$, whose turning point is at $(-1, -2)$.

4.

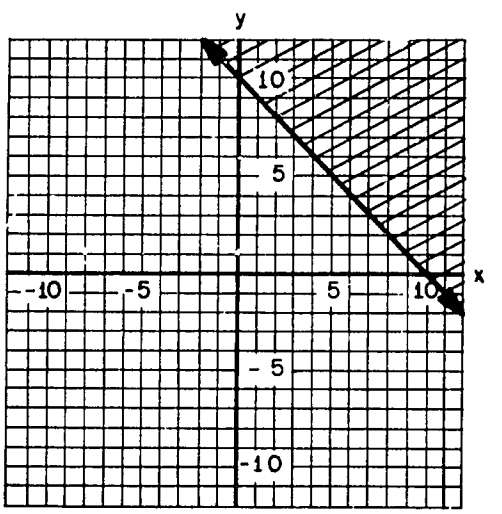
(a)



$x + y = 10$

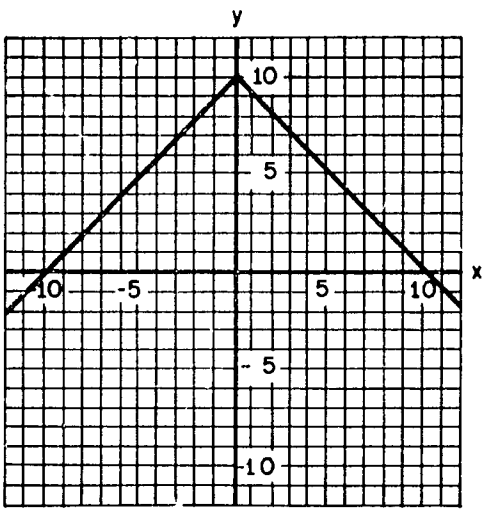


$x + y \leq 10$

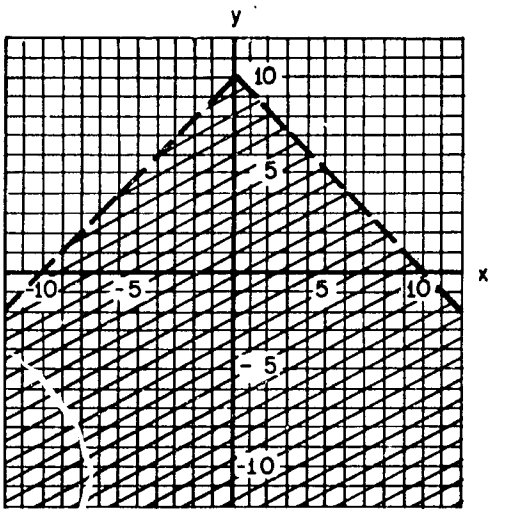


$x + y \geq 10$

(b)

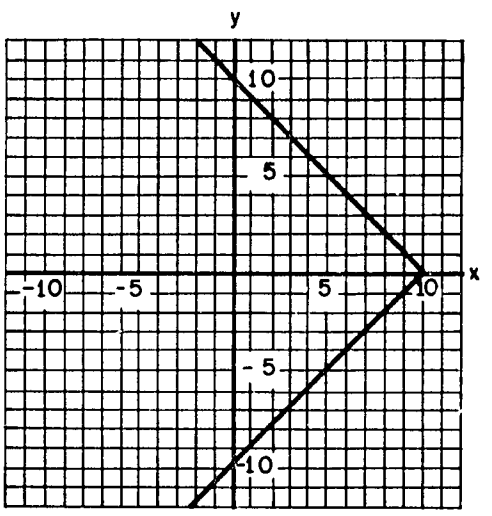


$|x| + y = 10$

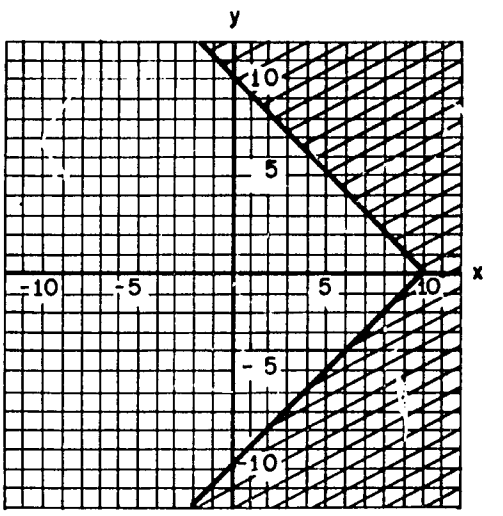


$|x| + y < 10$

(c)

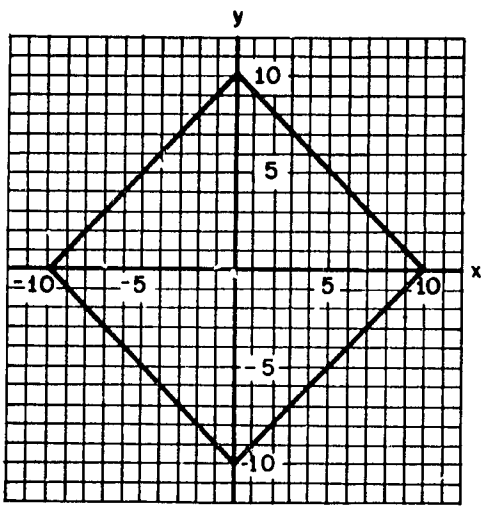


$x + |y| = 10$

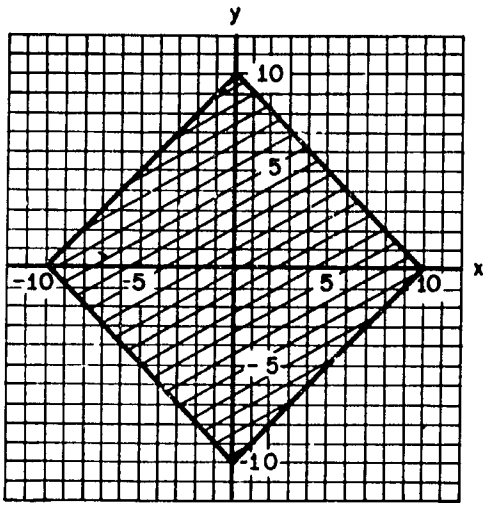


$x + |y| \geq 10$

(d)

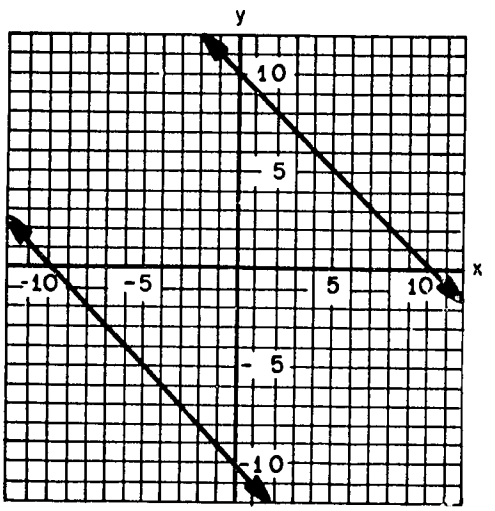


$|x|+|y|=10$

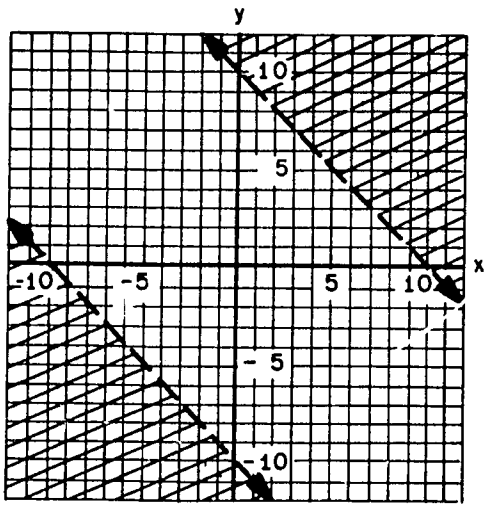


$|x|+|y|\leq 10$

(e)



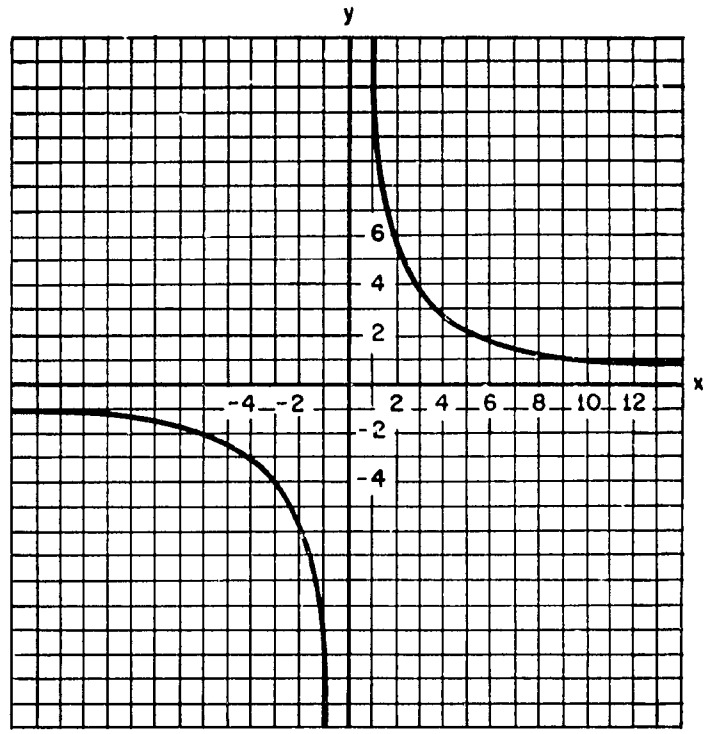
$|x+y|=10$



$|x+y|>10$

5. Since this is an identity, the graph is the number-plane; i.e., every ordered pair of numbers is in the truth set.

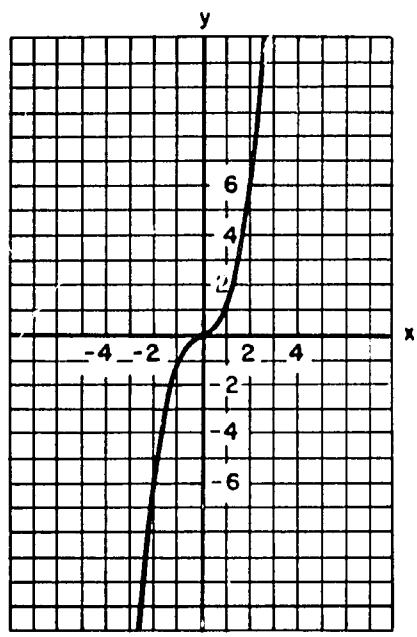
6.



$xy = 12$

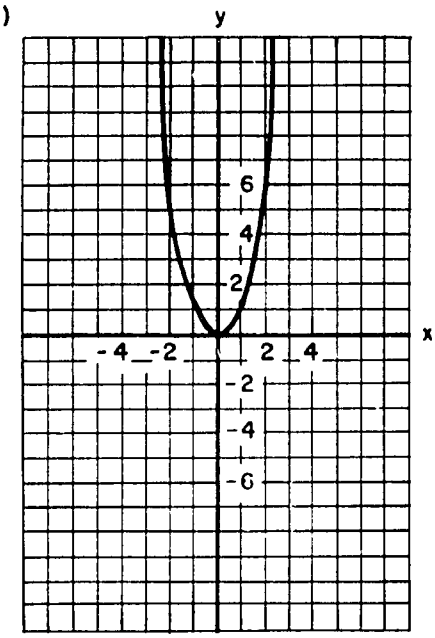
7.

(a)



$y = x^3$

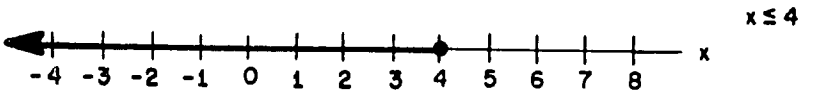
(b)



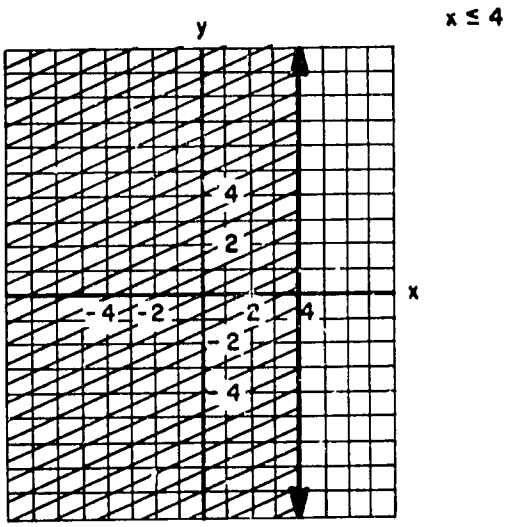
$y = |x|^3$

8. CHALLENGE PROBLEM

(a)



(b)



(c) The children have already gone from the number-line to two intersecting number-lines to get the number-plane; it is hoped that they can figure out that if they consider three mutually perpendicular number-lines intersecting at one point, they can describe any point in space by an ordered triple of numbers. They can have practice visualizing this by using a corner of the room and describing points in the room by an ordered triple of numbers. The first coordinate can be the distance of the point from the left side wall, the second coordinate its distance from the floor, and the third coordinate its distance from the front wall. (The distance can be measured in

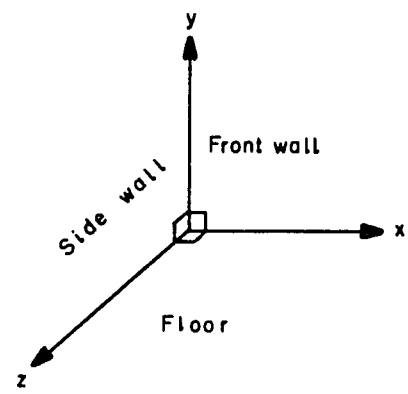
units of feet, yards, etc. as long as the same unit is used.) The children might describe the following points by ordered triples:

(1) One corner of the teacher's desk.

(2) The top of someone's head.

(3) A point marked with chalk on the blackboard. (If the board is at the front of the room, the third coordinate will be zero.)

(4) A point somewhere on the floor. (The second co-ordinate will be zero.) You can ask someone to find a point whose first co-ordinate is zero. (Any point on the left-wall.) You can ask where the origin, (0, 0, 0), is, and then where the three axes are, and which is the x-axis (first co-ordinate), which is the y-axis (second co-ordinate). They will need a third variable, say z, for the third axis. They should now have this picture in mind:



Ask the children how they could describe with an ordered triple of numbers a point which is 2 units in from the left side wall, 3 units <u>underneath</u> the floor, and 5 units in from the front wall. [(2, -3, 5)]. They should realize that the three axes extend to include the negative numbers too.

For the graph of $x \leq 4$, if they have trouble seeing what it would be, start with $x = 4$, i.e. all the points whose first co-ordinate is 4. Since y and z are not limited in any way, this graph will be the whole plane of points 4 units to the right of the plane determined by the y-axis and the z-axis. The graph of $x \leq 4$ in the number-space consists of all points to the <u>left</u> of or on this plane. The children can visualize this by thinking of putting up a partition in their room which is four units from the left side wall. The points on one side of the partition will have the first co-ordinate bigger than four, on the other side less than four.

9. (a) $-2 \leq x \leq 3$
   (b) $x \geq 2$
   (c) $y < 2$
   (d) $y > x + 2$
   (e) $y \geq |x| - 1$
   (f) $y \leq -x^2$

10. <u>CHALLENGE PROBLEM</u>.

		Lines of Symmetry
1.	All parts	$x = 0$ (y-axis)
2.	All parts	$x = 0$ (y-axis)
3.		$y = 0$ (x-axis)
4.	(a)	$y = x + k$ for any k
	(b)	$x = 0$ (y-axis)
	(c)	$y = 0$ (x-axis)
	(d)	$x = 0$ (y-axis) and $y = 0$ (x-axis) and $y = x$ and $y = -x$
	(e)	$y = x + k$ for any k and $y = -x$
5.		Any line
6.		$y = x$, $y = -x$
7.	(a)	None
8.	(b)	$x = 0$ (y-axis)

15-2 The Graph of $y = x^2$ and Tables of Squares and Square Roots

Answers for Class Activity

Student Text Page 335

1.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	25	16	9	4	1	0	1	4	9	16	25

Any value of x and its opposite correspond to the same value of y.

2. The children should realize from looking at their previous sketch of $y = x^2$ that the x-axis should be placed at the bottom of their graph paper; because of symmetry, the y-axis should be down the middle of the paper. The scale to be used on each axis will depend on the particular graph paper you have, and the class should discuss this before they put the scales on the axes. The reason for including negative values of x for the graph is that we wish to have a graphical illustration that a positive number has two square roots when we come to this section later in the chapter.

3. The purpose of these questions is to remind the children that the graph of $y = x^2$ is a smooth curve, not a series of line segments.

x	0·5	1·5	2·5	3·5	4·5
y	0.25	2.25	6.25	12.25	20.25

A Table of Squares

The children will find that reading squares from their graphs does not give great accuracy. They can estimate to only one decimal place, and this may give a relatively large error even though they have drawn their graphs carefully. The children may have a variety of answers for how to obtain greater accuracy. A few of the possibilities are listed here.

1. Use a larger scale. This will mean a larger piece of graph paper will be needed to show the values of x from -5 to 5, or only a portion of this range can be shown on the paper.
2. Use a pencil with a very sharp point.
3. Plot more points before completing the curve.
4. Compute the squares arithmetically. (This, of course, will give the exact square.)

A Printed Table of Squares

The students should be encouraged to decide where the decimal point belongs by estimating the size of the answer. The method illustrated in the examples should give the students some help in locating the position of the decimal point. In all cases, the student should look at the size of his answer to make sure the answer is reasonable.

Answers to Problems 15-2A | Student Text Page 338

(a) 5·76

(b) 57,600

(c) 50,410,000

(d) 0·005041

(e) 86·49

(f) 0·0000008649

A Table of Square Roots

The purpose of the discussion in the first paragraph is to lead the children to see that since $y = x^2$ and $\sqrt{y} = |x|$ are equivalent equations (and therefore have the same graphs), they can use their graph of $y = x^2$ to find square roots of numbers from 0 to 25.

Since $\sqrt{y}$ represents only the non-negative square root of y, $\sqrt{y} = x$ is true only when $x \geq 0$. If $x < 0$, we must take the opposite of x to obtain a positive number, so if $x < 0$, $\sqrt{y} = -x$. Both these cases are included in the statement $\sqrt{y} = |x|$.

Yes, $y = x^2$ and $\sqrt{y} = |x|$ are equivalent equations.

Answers to Problems 15-2B

Student Text Page 339

1.

y	1	2	3	4	5	6	7	8	9	10	11	12	13
$\sqrt{y} = \lvert x \rvert$	1	1·4	1·7	2	2·2	2·4	2·6	2·8	3	3·2	3·3	3·5	3·6

y	14	15	16	17	18	19	20	21	22	23	24	25
$\sqrt{y} = \lvert x \rvert$	3·7	3·9	4	4·1	4·2	4·4	4·5	4·6	4·7	4·8	4·9	5

2.

y	0·1	0·2	0·3	0·4	0·5	0·6	0·7	0·8	0·9
$\sqrt{y} = \lvert x \rvert$	0·32	0·45	0·55	0·63	0·71	0·77	0·84	0·89	0·95

A Printed Table of Square Roots

This part may be omitted, especially if the children are not yet ready to handle the necessary work with radicals.

The children should find that the square roots of 400, 4, and 0·04 are 20, 2, and 0·2, all having the same digit 2. They can see from Problem 2 of Problems 19-2b that the square root of 0·4 is about 0·63, which has entirely different digits.

The children should be able to find values for only part of the table.

n	3600	360	36	3·6	0·36	0·036	0·0036
$\sqrt{n}$	60		6		0·6		0·06

When they have found that $\sqrt{360} \approx 18{\cdot}97$, they can find values for the rest of the table.

n	3600	360	36	3·6	0·36	0·036	0·0036
$\sqrt{n}$	60	18·97	6	1·897	0·6	0·1897	0·06

Answers to Problems 15-2C

Student Text Page 341

1. (a) 3·873 (d) 27·57
   (b) 12·25 (e) 275·7
   (c) 1·225 (f) 0·02757

2. CHALLENGE PROBLEM

A table of cube roots can be constructed either graphically or arithmetically. The graphical method is much easier, the arithmetic more accurate, but very time consuming. Probably a combination of the two would work best - the graph for a first approximation, and then arithmetic to get the accuracy needed. The graphical readings would require a graph of $y = x^3$.

3. CHALLENGE PROBLEM

Any real number has exactly one real cube root. The student should be reminded that this is not the case for square roots, where no negative number has a real square root, zero has one square root, and any positive number has two square roots.

4. CHALLENGE PROBLEM

The cube root table would need three columns, namely $\sqrt[3]{n}$, $\sqrt[3]{10n}$, and $\sqrt[3]{100n}$.

$\sqrt[3]{32}$ can be read in the $\sqrt[3]{n}$ column.

$\sqrt[3]{320}$ is $\sqrt[3]{32 \times 10}$ and can be read in the $\sqrt[3]{10\,n}$ column.

$\sqrt[3]{3200}$ is $\sqrt[3]{32 \times 100}$ and can be read in the $\sqrt[3]{100n}$ column.

$$\sqrt[3]{3{\cdot}2} = \sqrt[3]{32 \times 100 \times \frac{1}{1000}} = \sqrt[3]{32 \times 100} \times \frac{1}{10}$$

$$\sqrt[3]{0{\cdot}32} = \sqrt[3]{32 \times 10 \times \frac{1}{1000}} = \sqrt[3]{32 \times 10} \times \frac{1}{10}$$

$$\sqrt[3]{0{\cdot}032} = \sqrt[3]{32 \times \frac{1}{1000}} = \sqrt[3]{32} \times \frac{1}{10}$$

$$\sqrt[3]{320{,}000} = \sqrt[3]{32 \times 10 \times 1000} = \sqrt[3]{32 \times 10} \times 10$$

$\sqrt[3]{32}$ and $\sqrt[3]{0{\cdot}032}$ have the same digits, namely 317...

$\sqrt[3]{320}$, $\sqrt[3]{0{\cdot}32}$, and $\sqrt[3]{320{,}000}$ have the same digits, namely 684...

$\sqrt[3]{3200}$ and $\sqrt[3]{3{\cdot}2}$ have the same digits, namely 1474...

15-3 Other Graphs

This section, as was 15-1, is optional and should be done only if time permits and the children seem to enjoy graphing.

Teaching Suggestions for Class Activity

Student Text Page 342

One of the purposes of this section is to have the children realize that they can find some information about a graph by first considering the equation. The graph and work can be done on the board or individually as you choose.

1. y^2 becomes smaller.

2. The smallest either x^2 or y^2 can be is zero.

3. The largest x^2 can be is 25. It cannot be 100 since y^2 cannot be negative.

4. The possible x-values are from -5 to 5; same for y.

6.

x	-5	-4	-3	-2	-1	1	2	3	4	5	0
y	0	3 -3	4 -4	4·6 -4·6	4.9 -4.9	4·9 -4·9	4·6 -4·6	4 -4	3 -3	0	5 -5

7. Yes; additional ordered pairs:

x	4·6	-4·6	4·9	-4·9
y	2 -2	2 -2	1 -1	1 -1

The children will probably realize that from one ordered pair in the truth set, they can get others without doing additional arithmetic. It should now be obvious from the points plotted that the graph is a circle.

8. A circle is the set of all points at a given distance from a given point. You might ask the children what the centre and radius are for their graphs.

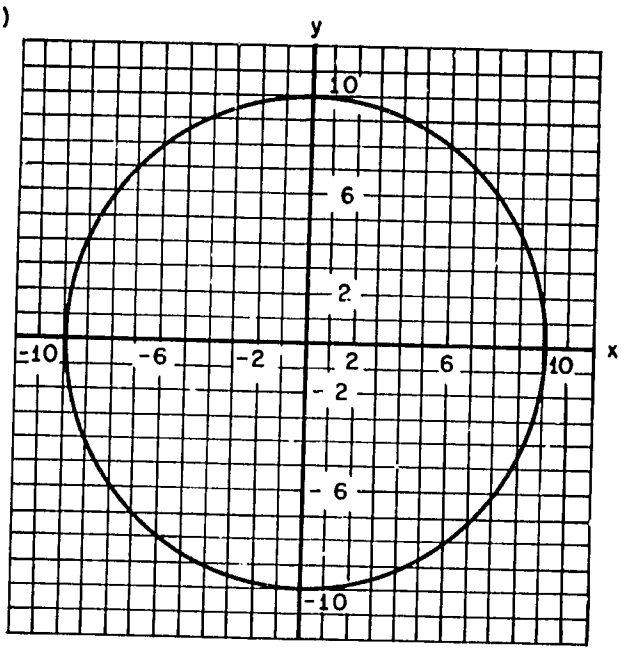
9. (a) The points in the interior of the circle.
   (b) The points in the exterior of the circle.
   (c) The circle <u>and</u> the points in the interior of the circle.
   (d) The circle <u>and</u> the points in the exterior of the circle.

Answers to Problems 15-3

Student Text Page 344

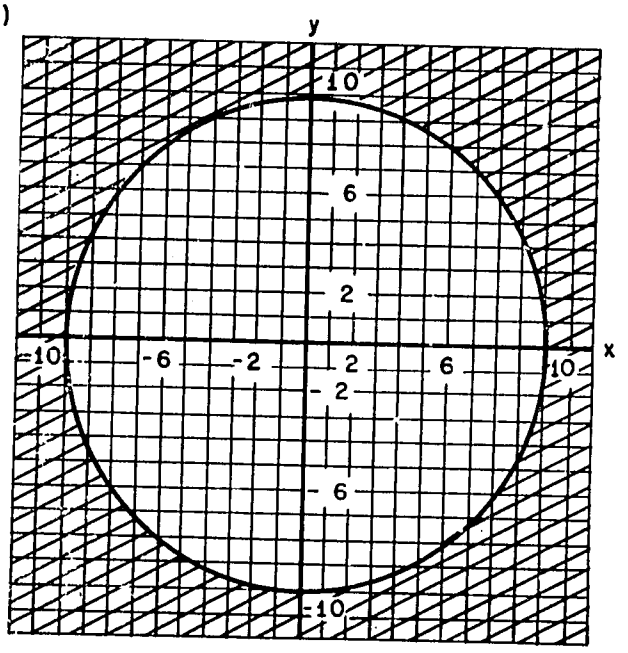
1.

(a)



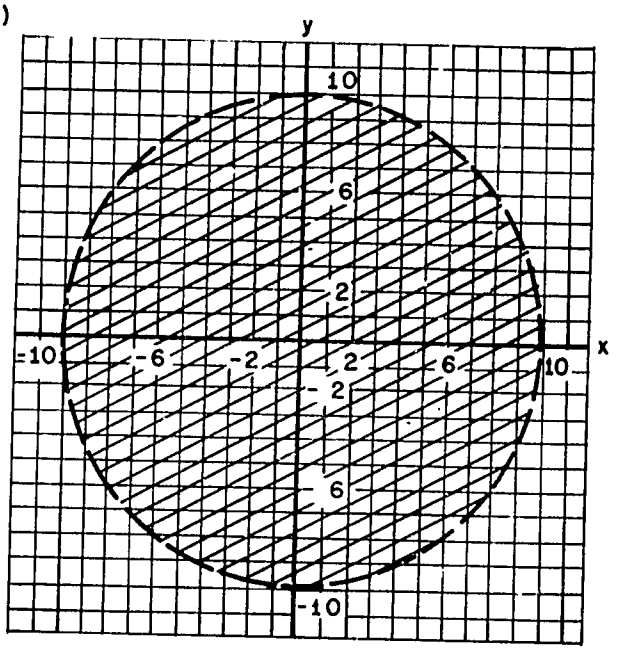
$x^2 + y^2 = 100$

(b)



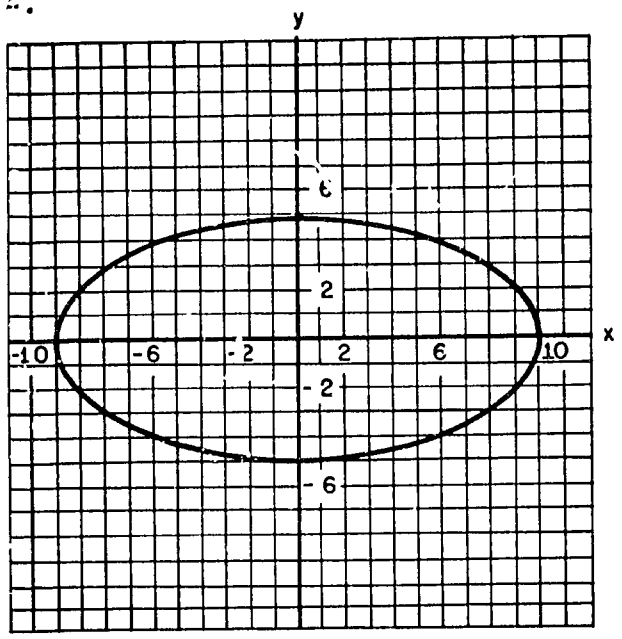
$x^2 + y^2 \geq 100$

(c)



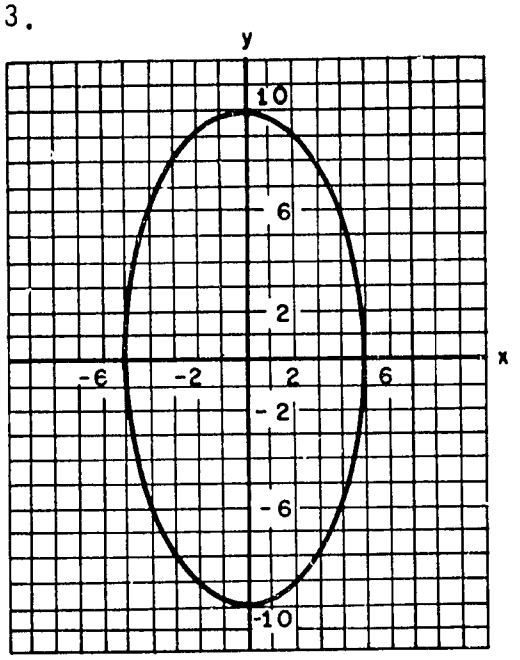
$x^2 + y^2 < 100$

2.



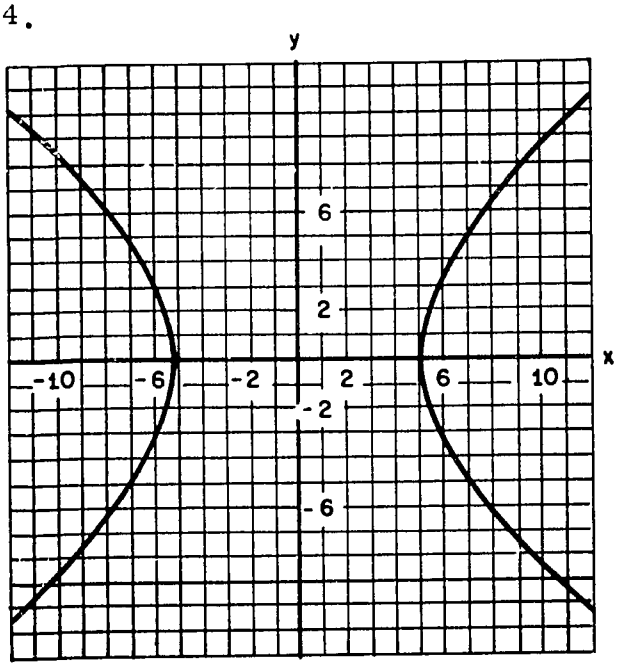
$x^2 + 4y^2 = 100$

3.



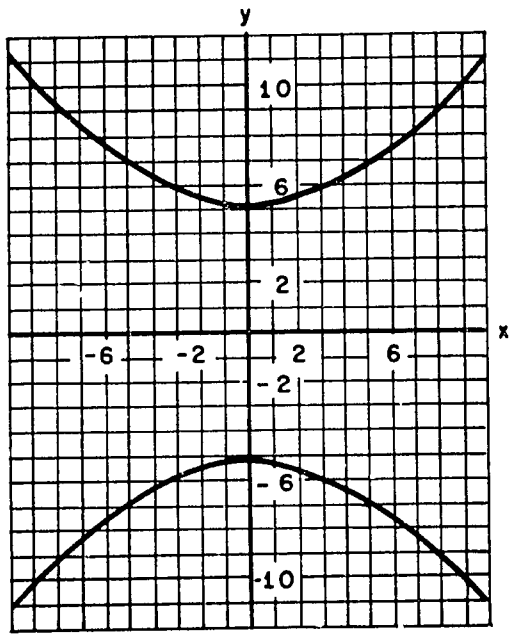
$4x^2 + y^2 = 100$

4.



$x^2 - y^2 = 25$

5.



$y^2 - x^2 = 25$

6. 1. (a) circle

2. ellipse.

3. ellipse.

4. hyperbola.

5. hyperbola.

Additional questions you may ask after graphs are completed: Suppose you have just finished the graph in Problem 2 and you feel lazy, how could you obtain the graph in Problem 3 with very little effort? [Turn the paper through 90° and relabel the axes.] How could you obtain the graph in Problem 5 with very little work? [Turn the graph in Problem 4 through 90° and relabel the axes.]

7. For the graphs in Problem 1, any line through the center of circle is a line of symmetry. Since the center is the origin, the family of lines of symmetry can be given by $y = ax$, where a is any real number.

For the graphs in Problems 2-5, the x-axis (equation $y = 0$) and the y-axis (equation $x = 0$) are lines of symmetry.

15-4 More Ways of Picturing Numerical Information

The main aim of this section is to show clearly the difference between a discrete and a continuous variable. These terms need not be mentioned to the pupils.

1. Scatter Diagram: As mentioned in the student's text, this helps us see whether there is any relationship between the two sets of data, the heights and spans of boys. The name "scatter diagram" comes from the fact that the points are scattered on the graph paper.

The construction of the scatter diagram should be explained (or discussed) fully. There is no special significance in putting the heights on the horizontal axis. Questions such as, "What is the greatest (smallest) height (span) we have?" should help the pupils see why we start labelling our axes where we do (around 60) and stop

where we do (in the seventies). This should also bring out the reason why the scales we have shown have been used. In response to the question, "What do you think the diagram tells us?" you should expect quite a number of answers. Here you should consider each answer given and in turn try to ask the child who gives the answer why he thinks the diagram tells us what he says it does. The diagram indicates that the span of a boy 66 inches tall may be anything from 63 inches to 67 inches. You might ask the pupils whether they can name some factors which contribute to such a wide scatter. Included in these might be errors in measurement and different arm lengths.

At this stage, the pupils should be familiar with the two scales of temperature from their geography and science lessons. They should know that 32° F and 0° C represent the same temperature.

The difference between this scatter diagram and the one of heights and arm spans is that the former shows much more scatter than the latter. The formula $F = \frac{9}{5} C + 32$, if not already learnt in the science lessons, should be introduced. The pupils should be able to see that it is an equation of a straight line.

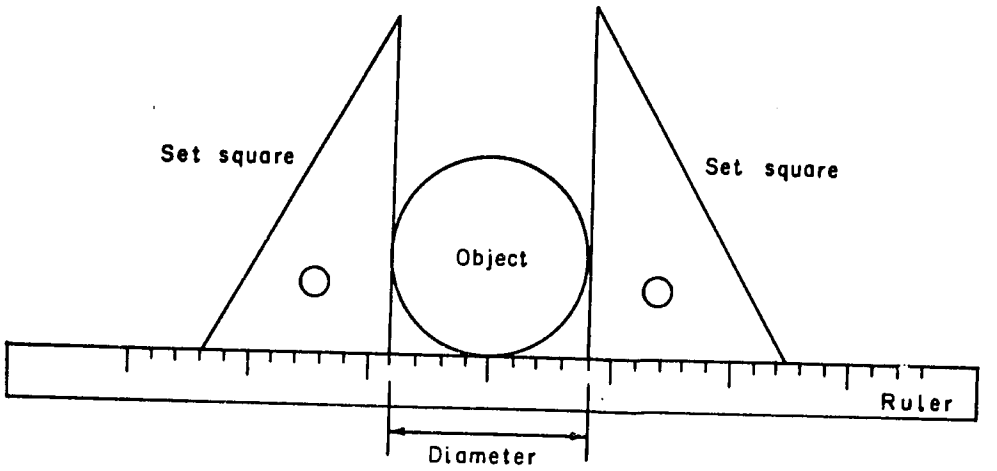
The points of the scatter diagram only approximate a straight line, again because of errors in reading our thermometer scales. Let the pupils draw what they think is the "best" straight line connecting the points, and then ask them why they think the line they have drawn is the best.

The points do not all lie exactly on a straight line because of our errors in measurement, in this case errors in reading the thermometer scales. If there were no such errors all the points would lie on a straight line. The pupils may be familiar with experiments in physics such as Hooke's law from which scatter diagrams may be drawn which approximate straight lines. You may want to seek out and refer to such experiments if they exist in your school.

<u>Class Activity</u>

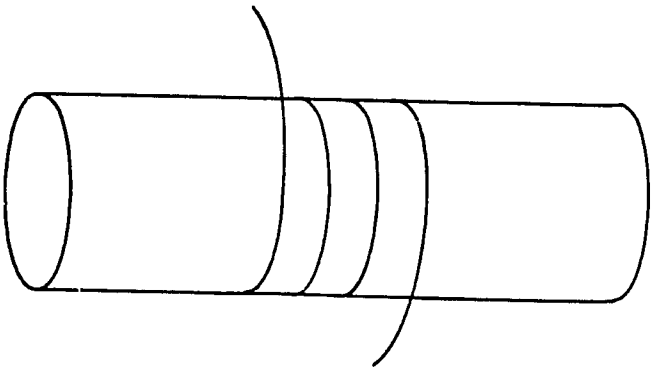
Student Text Page 348

In doing the exercises in measurement, it may be a good idea to divide the class in groups of about four each. Methods of measuring the diameters and circumferences of the circular objects should be explained. One method for measuring the diameter is shown in the figure below.



For the circumference a number of ways may be used. Some of these are illustrated below:

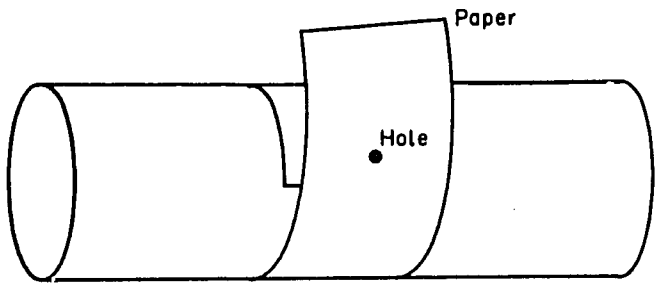
(a)



A piece of string is wrapped tightly round the object. Each complete turn represents the length of the circumference of the object. Wrapping ten

turns and dividing the total length by 10 gives a good approximation to the circumference.

(b)



A piece of paper is wrapped tightly round the object until it overlaps. A pin is then pushed into the paper so that it makes a hole right through the paper. When the paper is unwrapped, the distance between the two pin holes is the measure of the circumference.

(c) Another method is to roll the object on a ruler. A point is marked on the circular edge of the object. The marked point is placed on the ruler and the object is rolled carefully along the ruler (without slipping) until marked point is again on the ruler. The measure of the distance between the two positions is the measure of the circumference.

For the measurement of heights, use a vertical scale graduated in inches on the wall or blackboard. The pupils then stand against the wall (with their backs to the wall). A ruler on their heads helps to read the scale to the nearest inch. For the spans, use a horizontal scale on the blackboard, again graduated at one inch intervals.

2. Continuous Line Chart: The rainfall chart is an example of a quantity which is not varying continuously. The rainfall for the entire month may have fallen entirely in one, two, or ten days of the month. Some days may not have had any rain at all. Therefore we are able to say from the graph only that the accumulated rainfall for the month was so much. We can plot only twelve points on the graph, representing the twelve months of the year. The chart cannot tell us what was happening on various days of the month. With the temperature figures, we could have as many points as we please by reading the thermometer more frequently. The temperature is continuously changing. We therefore join the points we have decided to plot by a continuous free-hand line. There is no equation that we know of which connects the temperature and the time. Therefore we do not need to use a ruler to draw straight lines connecting the points to get the pattern of the changing temperature. From this kind of graph we could estimate the temperature at any given time during the day. For example, if we had read the temperature at 11:00 a.m. it is likely that it would have been about 77° F.

The pupils should be asked to draw the continuous line and then estimate from the graph what they think the temperature might have been at times other than the ones at which it was actually read and recorded.

Other examples of data suitable for this kind of graph are:

(a) the weight of a child at different ages.

(b) the temperature of a cooling body at different times.

(c) the girth of a tree at different ages.

(d) the height of a child at different ages.

The pupils should be encouraged to give a number of examples and the examples should be discussed to see whether they are suitable.

AFRICAN EDUCATION PROGRAM
MATHEMATICS STEERING COMMITTEE

Professor W. T. Martin, Chairman
Massachusetts Institute of Technology
Cambridge, Massachusetts

Dr. Iya Abubakar
Ahmadu Bello University
Zaria, Northern Nigeria

Mr. James L. Aldrich
Educational Services Incorporated
Watertown, Massachusetts

Professor Edward G. Begle
Stanford University
Stanford, California

Professor David H. Blackwell
University of California
Berkeley, California

Mr. Hugh P. Bradley
Educational Services Incorporated
Watertown, Massachusetts

Professor Jerome S. Bruner
Harvard University
Cambridge, Massachusetts

Professor Robert B. Davis
Webster College
Webster Groves, Missouri

Professor Robert P. Dilworth
California Institute of Technology
Pasadena, California

Professor Rov Dubisch
University of Washington
Seattle, Washington

Professor Vincent Haag
Franklin and Marshall College
Lancaster, Pennsylvania

Professor Clarence Hardgrove
Northern Illinois University
DeKalb, Illinois

Mr. D. K. Abbiw-Jackson
Kwame Nkrumah University of Science and Technology
Kumasi, Ghana

Mr. Christopher Modu
West African Examinations Council
Yaba, Lagos, Nigeria

Mr. Paul E. Mugambi
Makerere University College
Kampala, Uganda

Mr. John O. Oyelese
University of Ibadan
Ibadan, Western Nigeria

Professor David A. Page
Educational Services Incorporated
Watertown, Massachusetts

Professor B. J. Pettis
University of North Carolina
Chapel Hill, North Carolina

Mr. J. E. Phythian
University College
Dar es Salaam, Tanzania

Professor Walter Prenowitz
Brooklyn College
Brooklyn, New York

Professor Alfred L. Putnam
University of Chicago
Chicago, Illinois

Professor Donald E. Richmond
Williams College
Williamstown, Massachusetts

Professor Patrick Suppes
Stanford University
Stanford, California

Dr. Onyerisara Ukeje
University of Nigeria
Nsukka, Eastern Nigeria

Mr. Stephen White
Columbia Broadcasting System, Inc.
New York, New York

Dr. Awadagin Williams
University of Sierra Leone
Freetown, Sierra Leone

Dr. Grace Alele Williams
University of Lagos
Lagos, Nigeria

Professor Jerrold R. Zacharias
Massachusetts Institute of Technology
Cambridge, Massachusetts

Ex Officio:
Dr. James R. Killian, Jr.
Chairman of the Board
Educational Services Incorporated
Watertown, Massachusetts

Printed in the United States of America

248