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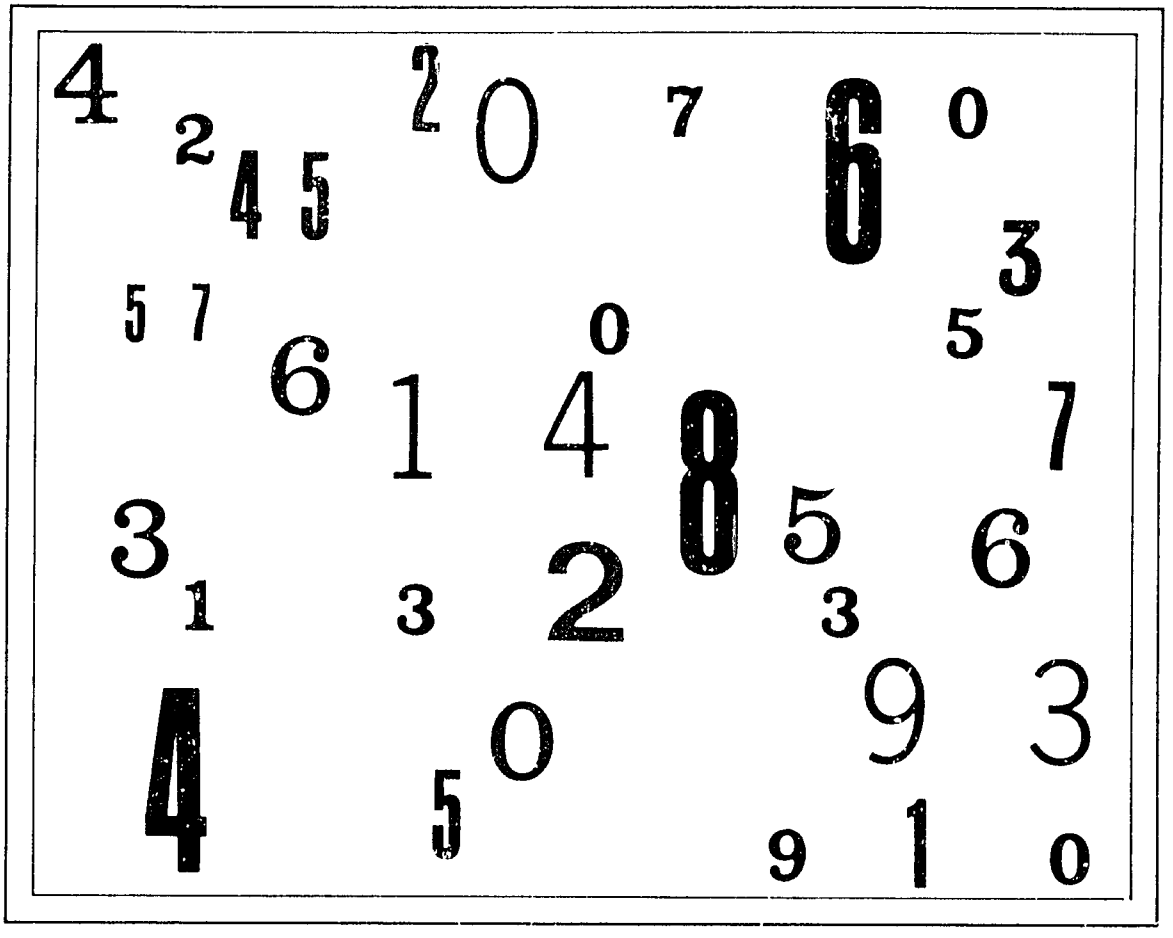

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ENTEBBE MATHEMATICS SERIES

Secondary Four Algebra
Preliminary Edition

Prepared at the 1965
Mombasa Mathematics Workshop

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TO THE STUDENT

In using this textbook you have the opportunity to continue in the great experiment which was started in the summer of 1962, when more than fifty mathematicians and mathematics teachers gathered in Entebbe, Uganda, to begin the preparation of mathematics books as up-to-date as those in use anywhere else in the world. The writers came from most countries in English-speaking Tropical Africa, from the United States and from the United Kingdom to take part in this project. In the summer of 1962, *Secondary One* of the Entebbe Mathematics Series was written, in the summer of 1963, *Secondary Two;* in the summer of 1964, *Secondary Three;* and in the summer of 1965 this book. Now you and your teachers must do your share to find out whether this book is as good as it can be made. By using the textbook critically and making the most of the opportunities it offers to make the learning of mathematics an interesting and enriching experience, you will have played a part in the new growth of mathematics education in Africa.

Since you studied the earlier Entebbe Mathematics Series Secondary books, you know that you can expect to meet many familiar ideas, sometimes expressed in unfamiliar ways, and there will be some new ideas. Careful reading will be important because this book treats mathematics as a language designed to express certain kinds of ideas and you must learn to read and speak this language.

You will continue to learn why numbers behave as they do and find that algebra helps us more and more to understand arithmetic ideas. You will also explore some geometric ideas, some of which are familiar, some of which will be new, deducing these geometric facts from a small number of simple assumptions. You will also be using the idea of rigid motion in deducing geometric facts both familiar and new.

Do not hesitate to ask questions. Talk about your work with your teacher and with your fellow pupils. Above all, try to look upon mathematics as something that can be exciting and rewarding, for if you give it half a chance, that is exactly what it is.

CONTENTS

			Student Page
Chapter	**1**	**Relations and Functions**	1
	1-1	Revision	1
	1-2	Relations	6
	1-3	Domain and Range	11
	1-4	Pictorial Representations	13
	1-5	The Inverse of a Relation	18
	1-6	Graphing the Inverse of a Relation	22
	1-7	Functions	27
	1-8	One-to-One Functions, Inverses	33
	1-9	Functions of Two Variables	38
Chapter	**2**	**Rational Functions**	42
	2-1	Rational Functions	42
	2-2	Operations with Rational Functions	47
	2-3	Factorization in Simplifying Rational Expressions	51
	2-4	Rational Functions of Two or More Variables	55
	2-5	Equations Involving Rational Expressions	57
	2-6	Graphs of Rational Functions	65
	2-7	Asymptotes	72
	2-8	Equations and Inequalities by Graphs	89
Chapter	**3**	**Exponential and Logarithmic Functions**	96
	3-1	Introduction	96
	3-2	The Exponential Function Defined by 2^x	98
	3-3	The Exponential Function Defined by 2^{-x}	101
	3-4	The Exponential Function Defined by a^x for Any Positive Number a	102
	3-5	Applications of Exponential Functions	107
	3-6	Logarithmic Functions	110
	3-7	Further Properties of Logarithmic Functions	114
Chapter	**4**	**Sequences and Series**	117
	4-1	Introduction	117
	4-2	Sequences	120
	4-3	Arithmetic Progressions	125
	4-4	Geometric Progressions	132
	4-5	Compound Interest	137
	4-6	Geometric Sequences and Infinite Series	140
	4-7	The Number e	144

			Student Page
Chapter	**5**	**Descriptive Statistics**	147
	5-1	Populations and Functions on Populations	147
	5-2	Distributions and Histograms	150
	5-3	The Mean, Median, and Mode of a Population Function	154
	5-4	Standard Deviation and Range	159
	5-5	Scatter Diagrams	163

Chapter 1
RELATIONS AND FUNCTIONS

At the end of *Secondary Three* we discussed functions of various types, including linear functions, quadratic functions, and general polynomial functions. The concept of a function is one of the most important in all of mathematics, and in this chapter we shall take a much closer look at it. As we proceed you will find yourself learning a great deal more about what a function really is, and how functions arise in mathematics and in everyday life.

1-1 REVISION.

In our earlier work we described a function as a rule or correspondence by means of which numbers are paired with one another in a perfectly definite way. For instance, the formula $y = x^2$ defines the quadratic function which associates each real number x with its square, so that 1 is paired with 1, 2 with 4, -3 with 9, $\frac{1}{2}$ with $\frac{1}{4}$, and so on. As you know, this function (or any other for that matter) can be represented in several different ways.

By a table

$y = x^2$	x	-3	-2	-1	0	1	2	3
	y	9	4	1	0	1	4	9

By a graph

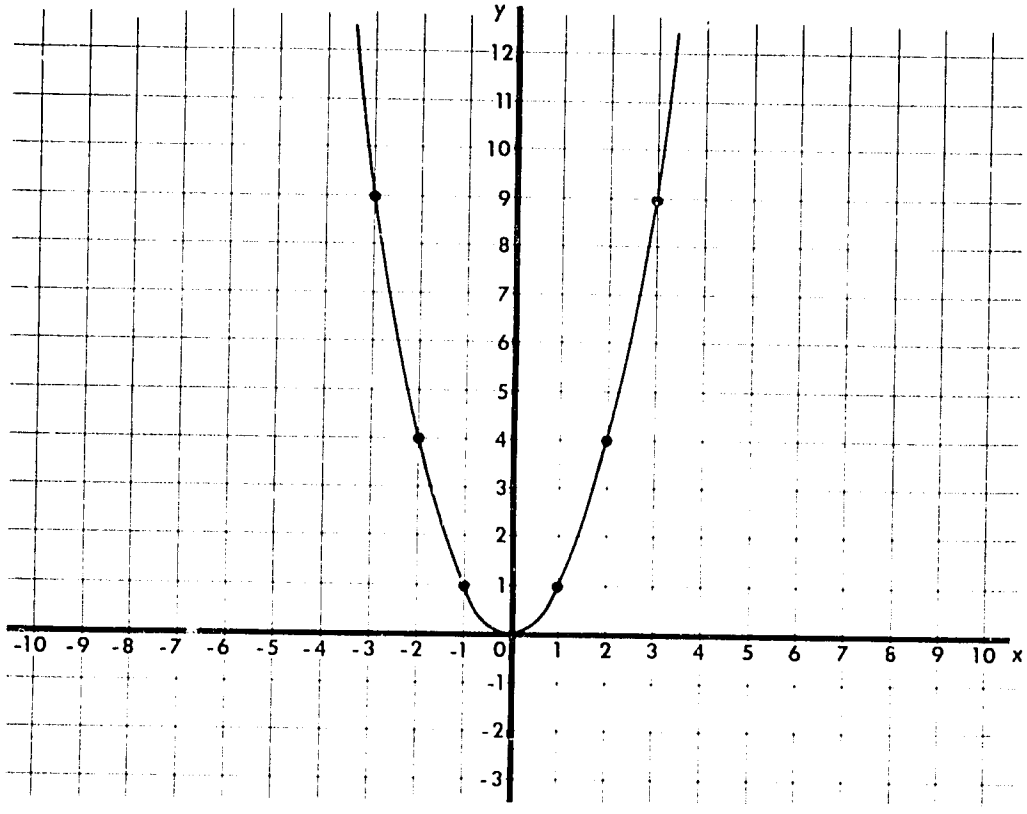


Fig. 1.

By pictures of the following types.

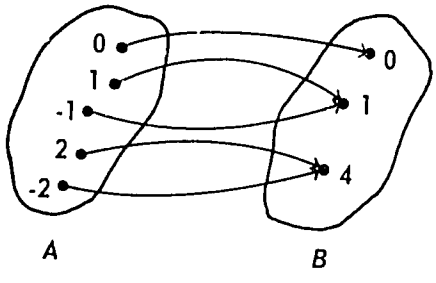


Fig. 2.

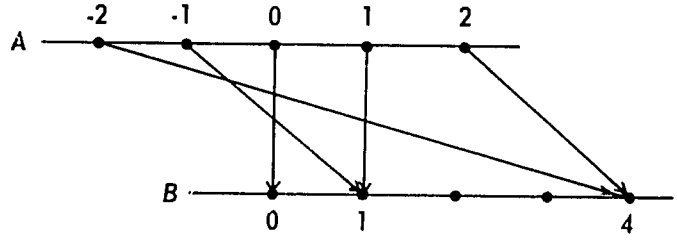


Fig. 3.

All of these representations have one thing in common. Do you know what it is? Does the word "pairing" suggest an answer? If you said that the representations show which numbers are paired with one another by the function you were right. Notice, however, that none of them lists *all* the pairs in question. (Do you think such a list is possible for this function defined by $y = x^2$? Why?) Instead they merely suggest that a pairing is present, and show a sufficient number of pairs to indicate the rule by which they are

formed. As we shall see, this idea of a pairing is the key to understanding functions.

PROBLEMS 1-1A

Represent the function defined by each of the following formulas by a table, a graph, and pictures of the type given above.

1. $y = 2x - 1$
2. $y = x^2 - 2x + 1$
3. $y = 1 - x^2$
4. $y = |x|$. [Recall that $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$, and is known as the absolute value of x.]

The sets appearing in the last two pictures given above were labeled A and B. These sets have special names. Do you remember what they are? The set A is called the *domain* of the function, and the set B the *range*. Can you describe in words the domain of the function defined by $y = x^2$? The range? To answer these questions you should recall that the domain of a function f is the set of all numbers x which can be substituted in the formula defining f, and that the range is the set of all values obtained in this way. Thus when $f(x) = x^2$, the domain is the set of *all* real numbers because every real number has a square. On the other hand, the range is the set of all *non-negative* real numbers. Why?

PROBLEMS 1-1B

1. Answer each of the following questions for the functions in Problems 1 to 4 above.

a. What is the domain of the function?

b. What is the range of the function?

c. How many values in the range of the function are paired with each value of x in its domain?

d. How many values in the domain of the function are paired with each value of y in its range? [Be careful!]

2. What are the domain and range of the functions defined by the following formulas?

a. $f(x) = \sqrt{x}$

b. $f(x) = \frac{1}{x}$

c. $f(x) = \sqrt{-x}$

d. $f(x) = 2^x$

3. Let f be the function defined by the formula

$$f(x) = x^2, \ x \geqslant 0.$$

a. What is the domain of f?

b. What is the range of f?

c. Draw the graph of f.

d. Explain how this function differs from the function defined by the formula $y = x^2$.

We have observed that a function f with domain A and range B can be described as a pairing of numbers in A with numbers in B such that each x in A is paired *exactly one* y in B, and each y in B is paired with *at least one* x in A. You may recall that in talking about such pairings we used the term *ordered pair*. Do you remember what an ordered pair of real numbers is? Plot the points in the plane represented by the ordered pairs (1, 3) and (3, 1). Are they the same? When is it true that the ordered pairs of real numbers (x, y) and (y, x) represent the same point in the plane? When do they represent different points? If you answered the last two questions correctly you know that the ordered pairs (x, y) and (y, x) are different whenever $x \neq y$. Thus an ordered pair of real numbers consists of two real numbers one of which is designated the *first entry* in the pair and the other the *second entry*. When x comes first and y second we denote the resulting ordered pair (x, y). In particular, when f is a function with domain A and range B, and a value of x in A is paired with a value of y in B, we say that the ordered pair (x, y) *belongs to* f. Note that the first entry in the pair comes from the domain of f, and the second entry from the range.

Example: Find all functions with domain A and range B when $A = \{0, 1, 2\}$, $B = \{2, 4\}$.

To solve this problem we make use of the fact that a function can be described by listing all of the ordered pairs which belong to it, or, in other words, by constructing its table of values. Since the given sets A and B are finite we can find the required functions by listing complete tables of values for each of them. There are precisely six such functions. They are

f_1	0	1	2
	2	2	4

f_3	0	1	2
	4	2	2

f_5	0	1	2
	4	2	4

f_2	0	1	2
	2	4	2

f_4	0	1	2
	4	4	2

f_6	0	1	2
	2	4	4

Questions (for class discussion): None of the following tables defines a function with domain A and range B, where A and B are as above. Why?

a.

0	1	2
2	2	2

b.

0	1	0	2
2	4	4	2

c.

0	1
2	4

If the domain and range are *not* required to be A and B respectively, which of the three tables define functions?

PROBLEMS 1-1C

1. Determine which of the following ordered pairs belong to the function defined by the formula $f(x) = x^2$.
$\left(1\frac{1}{2}, \frac{9}{4}\right)$, $(16, 4)$, $(2, -4)$, (π, π^2), $(3, \sqrt{3})$, $(2\sqrt{2}, 8)$.

2. Let f be the function defined by the formula
$$f(x) = x^2 - 5x + 6.$$
a. What is $f(0)$? $f(-1)$? $f\left(\frac{3}{2}\right)$?

b. Which of the following ordered pairs belong to f?
$(3, 0)$, $(0, 3)$, $\left(\frac{4}{5}, \frac{146}{25}\right)$, $(-2, 20)$, $(-1{\cdot}1, 12{\cdot}71)$, $(-2{\cdot}5, 17{\cdot}25)$, $(7, 20)$.

c. Complete the following table of values for this function.

x	0	1	2	$2\frac{1}{2}$	3	4	5
$f(x)$							

d. Find all ordered pairs belonging to f with the following second entries: 0, $-\frac{1}{4}$, 6.

e. Draw the graph of f.

3. Find all functions with domain A and range B when $A = \{-1, 1\}$, $B = \{3, 4\}$.

4. Find all functions with domain A and range B when $A = \{1, 2\}$, $B = \{0, 1, 2\}$. [Be careful!]

5. Which of the following tables define functions? What are their domains and ranges?

a.

0	$\frac{1}{2}$	1
-1	-1	-1

b.

2	0	4	1
0	2	0	2

c.

1	3	1	2
3	-2	1	2

d.

2	-2	1	$\frac{1}{2}$
2	-2	1	$\frac{1}{2}$

e.

3	-2	1	3
1	$\frac{1}{2}$	0	1

1-2 RELATIONS.

In order to continue our study of functions we now introduce the more general idea of a *relation*. As its name suggests, the concept of a relation is a familiar one. Everyone has relatives or relations—fathers, mothers, brothers, sisters. What may come as a surprise, however, is that this concept has an important place in mathematics. Let us start with an example

taken from everyday life—the relation of fatherhood. The problem we set for ourselves is to describe this relation *in mathematical terms*.

If you think about this problem for a few minutes you will probably find it somewhat more difficult than you expected. Although everyone knows what it means to say that y is the father of x, it is not quite so clear how to put this in the language we use in mathematics. Fortunately there is an easy way out of this difficulty, one which we have already used in our discussion of functions. There, if you will remember, we never said exactly what was meant by a rule or correspondence. Nevertheless, we felt that we knew all about a function as soon as we knew just what ordered pairs belonged to it. We now proceed to do the same thing for relations.

With this in mind, consider the relation of fatherhood among people. Whatever else it does, this relation certainly establishes a pairing between people—a pairing between all the men in the world who are fathers and all the people in the world whose fathers are living—and the set of all pairs (x, y) where y is the father of x completely describes the relation of fatherhood. Indeed, by looking at this set it is possible (at least theoretically) to determine whether or not any two people, x and y, are related to one another in such a way that y is the father of x. And this is just what we wish to do.

Before going any further we point out that here again we encounter the idea of an ordered pair, but this time for people instead of numbers. The pairs are ordered because, if y is the father of x, then x cannot possibly be the father of y, and it makes a considerable difference whether we say that the ordered pair (x, y) or (y, x) belongs to the set of ordered pairs determined by the relation of fatherhood. For this reason we define the term "relation" as follows:

Definition: A *relation* is a set of ordered pairs.

In other words, a relation is a *set* each of whose members is an *ordered pair* of objects. You should note that in contrast to a function, a relation can contain two ordered pairs of the form (x, y) and (x, z) with $y \neq z$. For example in the relation of brotherhood, it is possible that y is a brother of x and that z is also a brother of x.

Throughout this chapter we shall use the letters R, S, . . . to denote relations.

CLASS ACTIVITY

1. Make a list of all of the relations you can think of in which you appear as the *first* entry in one of the ordered pairs belonging to the relation. One such relation is the set of all ordered pairs (x, y) where x is a pupil in your school and y is one of x's teachers. Another is the set of all ordered pairs (x, y) where x lives in country y. How many more can you find? Remember that in each case you must describe a set of ordered pairs one of which has you as its first entry. Note that the second entry does not have to be a person.

2. Now do the same thing for those relations in which you appear as the *second* entry in one of the ordered pairs belonging to the relation. One such is the set of all ordered pairs (x, y) where x is a teacher in your school and y one of his pupils. If you proceed correctly you should have just as many relations of this type as you had in (1) above. How did you get them? Are these relations the same as those you found in (1)? Why? If you are given any relation R whatsoever, can you always construct a new relation in this way? Describe what would happen if you did this twice in succession to the same relation.

The examples you have just constructed should have convinced you that we live in a world of relations. We continue by giving a number of examples which illustrate how frequently relations also arise in mathematics.

Example 1: The set of all ordered pairs (x, y) of real numbers where y is greater than x.

Questions: Which of the following ordered pairs belong to this relation?

(1, 2), (2, 1), (1·1, 1·11), (1·1, -3), (-5, -3), (2, 2).

Find a number a such that

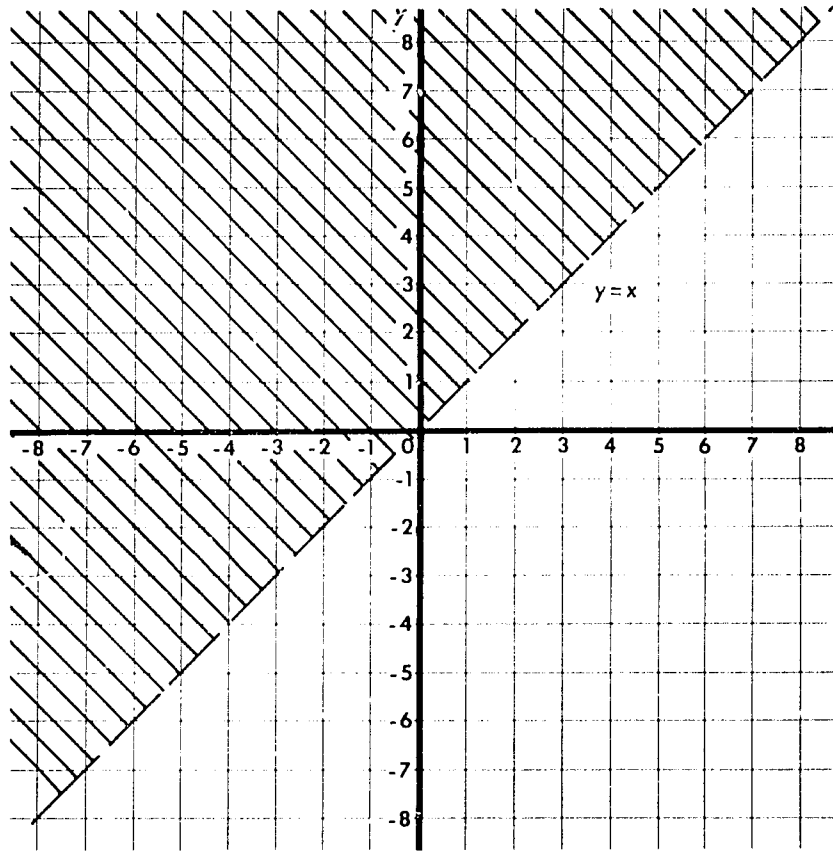
a. $(3, a)$ is in the relation;

b. $(-4, a)$ is in the relation;

c. $(3, a)$ and $(a, 4)$ are both in the relation.

Is there more than one answer to each of these questions? Since an ordered pair of real numbers can be pictured as a point on a graph, we can use graphs to represent relations. Which of the shaded regions in Figure 4 represents this relation? What relation is represented by the other picture? What relations are represented by each of these pictures if we include the line $y = x$ in the shaded regions?

(a)


y
x
y = x
-8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8
8 7 6 5 4 3 2 1 -1 -2 -3 -4 -5 -6 -7 -8

(b)

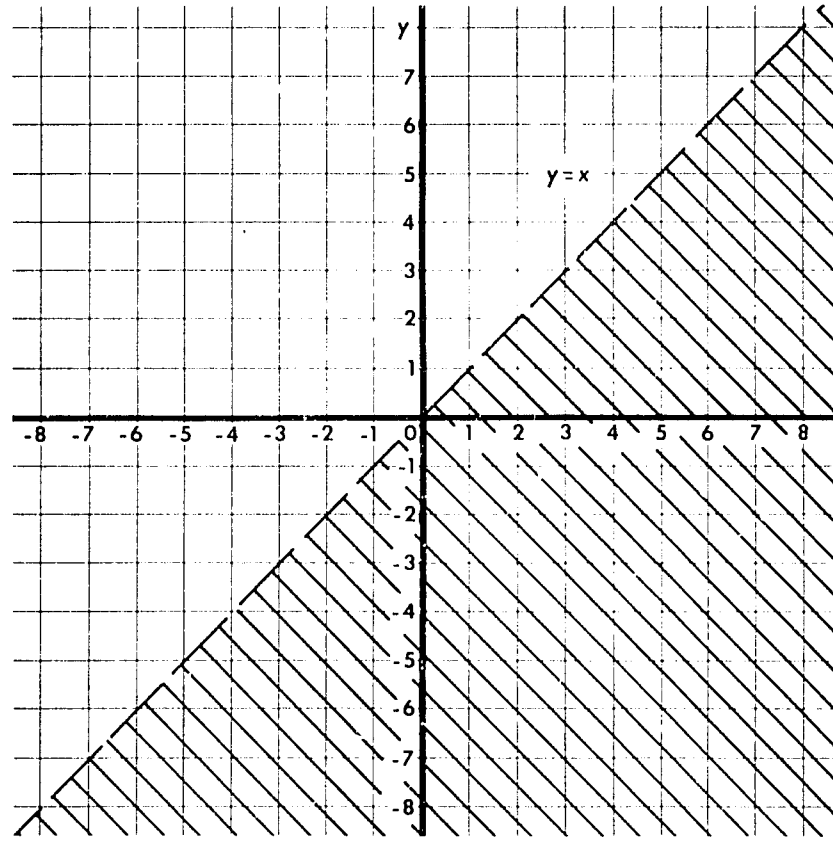

y
x
y = x
-8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8
7 6 5 4 3 2 1 -1 -2 -3 -4 -5 -6 -7 -8

Fig. 4.

Example 2: The set of all ordered pairs (Δ_1, Δ_2) of triangles where Δ_2 is congruent to Δ_1.

Questions: If the ordered pair (Δ_1, Δ_2) belongs to this relation, must (Δ_2, Δ_1) also belong to the relation? Does the ordered pair (Δ_1, Δ_1) belong to this relation for any triangle Δ_1? Why? Can you find three triangles Δ_1, Δ_2, Δ_3 for which *none* of the ordered pairs (Δ_1, Δ_2), (Δ_1, Δ_3), (Δ_2, Δ_3) belongs to this relation? For which *exactly one* of them does? For which *exactly two* of them do? For which *all three* do?

Example 3: The set of all ordered pairs (L_1, L_2) of lines in a plane, where L_2 is perpendicular to L_1.

Questions: Does an ordered pair of the form (L_1, L_1) ever belong to this relation? If (L_1, L_2) and (L_2, L_3) belong to this relation, which of the following statements is true?

a. (L_1, L_3) must belong to the relation.

b. (L_1, L_3) may or may not belong to the relation.

c. (L_1, L_3) cannot belong to the relation.

Example 4: The set of all ordered pairs (x, y) of natural numbers, where y is a multiple of x.

Questions: Which of the following ordered pairs belong to this relation?

$(2, 7)$ $(1, 5)$, $(3, 9)$, $(16, 4)$, $(\frac{1}{2}, 2)$, $(4, -8)$.

Does the ordered pair (x, x) belong to this relation for every natural number x? Is it true that the ordered pair (x, z) belongs to this relation whenever there exists a natural number y such that (x, y) and (y, z) belong to the relation? Why?

PROBLEMS 1-2

1. Let R denote the relation defined in Example 4 above. Find all ordered pairs belonging to R which have the form $(2, y)$; the form $(5, y)$; the the form $(x, 24)$; the form $(x, 17)$.

2. Let R denote the relation consisting of all ordered pairs (x, y) where y is the brother of x. Which of the following statements is true of R?
   a. R contains ordered pairs of the form (x, x).
   b. If (x, y) belongs to R, then (y, x) belongs to R.
   c. If (x, y) and (y, z) belong to R, then (x, z) belongs to R.

3. Repeat Problem 2 when R is the set of all ordered pairs (x, y) such that y is married to x.

4. Repeat Problem 2 when R is the set of all ordered pairs (x, y) of integers such that $x - y = 3$. Find five ordered pairs belonging to this relation.

5. Let R denote the set of all ordered pairs (x, y) of integers such that $x - y$ is an integral multiple of 3.

   (a is an *integral multiple* of b if $a = kb$, where k is an integer.)
   a. Which of the following ordered pairs belong to R?
   (9, 3), (3, 9), (-1, 2), (1, 5), (-7, -2), (0, 4), (-4, -7).
   b. Show that each of the statements made in Problem 2 is true for this relation.

1-3 DOMAIN AND RANGE.

Since a relation R is, by definition, a set of ordered pairs, it makes sense to speak of the set consisting of all the first entries in the pairs belonging to R. As in the case of functions, this set is call the *domain* of R. It can be described formally as follows:

> Definition: *Domain* of $R = \{x : (x, y)$ belongs to R for some $y\}$.

Similarly, the set of all the second entries in the ordered pairs belonging to R is called the *range* of R. It too can be described by a formula, as follows:

Definition: *Range* of $R = \{y : (x, y)$ belongs to R for some $x\}$.

Example 1: If R is the relation of fatherhood described in the preceding section, then the domain of R is the set of all people in the world whose fathers are living and the range of R is the set of all men in the world who have children.

Example 2: Let R be the set of all ordered pairs (x, y) of people where y is the husband of x. In this case the domain of R is the set of all married women in the world and the range of R is the set of all married men.

CLASS ACTIVITY

1. Let R denote the set of all ordered pairs (x, y) of real numbers, where $y = x^3$; that is, $R = \{(x, y) : x$ and y are real numbers and $y = x^3\}$.

   a. Find the ordered pairs belonging to R which have the following first entries:

   $0, 1, -2, -\frac{3}{2}, \sqrt[3]{-4}, \sqrt{3}, -\pi$.

   Is it true that R contains an ordered pair of the form (x, y) for every real number x? What does this tell you about the domain of R? How many such ordered pairs are there for each x? What does this tell you about R?

   b. Find all ordered pairs in R which have the following second entries:

   $0, -1, -27, 8, \frac{1}{125}, 3, -3$.

   Is it true that R contains an ordered pair of the form (x, y) for every number y? How many such ordered pairs are there for each y? What is the x appearing in this ordered pair called? What can you conclude about the range of R?

2. Let $R = \{(0, 2), (0, 4), (1, 2)\}$. What is the domain of R? The range of R? Is R a function? Why? Find *all* relations which have the same domain and range as R. How many are there? (Remember to count the one given above.) The correct answer is 7, and before going on you should make certain you have found all of them.

PROBLEMS 1-3

1. Find the domain and range of each of the following relations.

   a. The set of all ordered pairs (x, y) where y is a sister of x.

   b. The set of all ordered pairs (x, y) where x is a pupil in y's class.

   c. The set of all ordered pairs (x, y) where x and y are real numbers and $y = \sqrt{x}$.

   d. The set of all ordered pairs (x, y) where x and y are real numbers and $x \neq y$.

   e. The set of all ordered pairs (x, y) where x and y are natural numbers and y is a multiple of x.

   f. The set of all ordered pairs (x, y) where x lives in town y.

2. Let $R = \{(x, y) : x^2 + y^2 = 1\}$; that is, R is the set of all ordered pairs of real numbers the sum of whose squares is one.

   a. Determine which of the following ordered pairs belong to R:

   $(1, 1), (0, 1), (0, -1), (2, -\sqrt{3}), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), (0, 0),$
   $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

   b. Find the domain of R.

   c. Find the range of R.

1-4 PICTORIAL REPRESENTATIONS.

In the beginning of this chapter we recalled several different pictorial representations which are used in the study of functions. Similar representations can be constructed for relations and as we go on you will find these pictures extremely helpful both in understanding new ideas and in solving problems.

A general picture which can be drawn to illustrate a relation R with domain A and range B is one where A and B are shown as plane regions and the ordered pairs belonging to R are points joined by arrows as in the follow-

ing figure. In studying such a picture you should imagine that each pair of points connected by an arrow is an ordered pair belonging to R. The first entry in the pair is the point in the domain where the arrow begins, the second entry is the point in the range where the arrow ends. You will notice that the only difference between this picture and those of the same type which

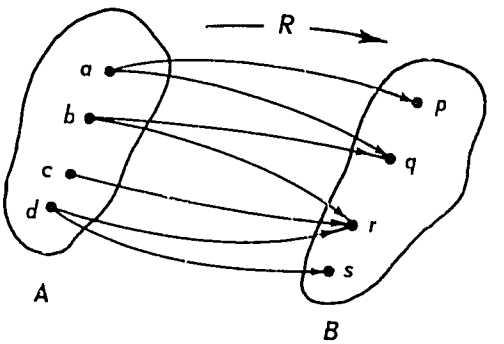


Fig. 5.

you have already drawn for functions is that now there can be more than one arrow starting from a point in the domain. This, as you will recall, *never* happens for functions.

As an example, let us draw such a picture to illustrate the relation of fatherhood. In this case we imagine that each point in the domain of R represents a person whose father is living and that each point in the range represents a man who is a father. We then connect points representing fathers with points representing their children as in Figure 6. Note that in drawing this figure we have not connected any point in the domain of R with two different points in the range. Can you explain why this was done? Also note that we have represented the domain and range of R by non-overlapping sets even though they have a certain number of people in common; namely those people whose fathers are living and who themselves have children.

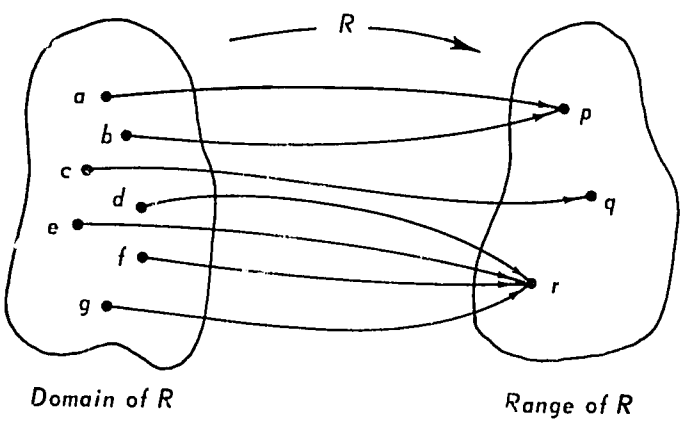


Fig. 6.

Draw a picture of the above type to illustrate each of the following relations. Be sure to include enough information in your pictures to show all of the important features of the relations.

1. R is the set of all ordered pairs (x, y) where x is a citizen of country y. (Assume that no one is a citizen of more than one country.)

2. R is the set of all ordered pairs (x, y) where y is the brother of x.

3. R is the set of all ordered pairs (x, y) where x is employed by y. (Assume that no one has more than one employer.)

4. R is the set of all ordered pairs (x, y) of integers where $x = 2y$.

When a relation R consists of a set of ordered pairs of real numbers we can also represent R by a *graph* much as we did for functions. Here, however, the resulting picture will often be a *region* of the plane rather than a curve, as the following example illustrates.

Let R be the set of all ordered pairs (x, y) of real numbers such that $y < x$. To find the graph of R we first graph the line $y = x$ as in Figure 7, and then observe that if P is any point in the plane lying *below* this line the y-coordinate of P is less than its x-coordinate. Can you explain why this is so? Thus the graph of R consists of the region in the plane lying below the line $y = x$; that is, the shaded region in Figure 7.

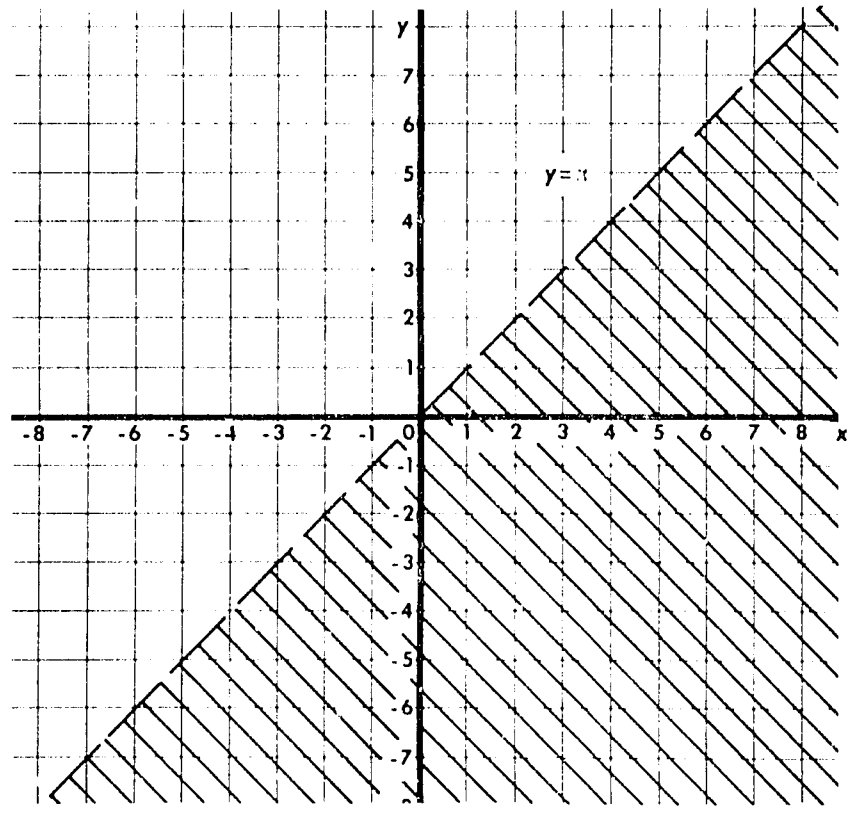


Fig. 7.

CLASS ACTIVITY

1. Use the graph in Figure 7 to help describe the set of all ordered pairs belonging to R which have 1 as their first entry. Now do the same for all ordered pairs having -2 as their second entry.

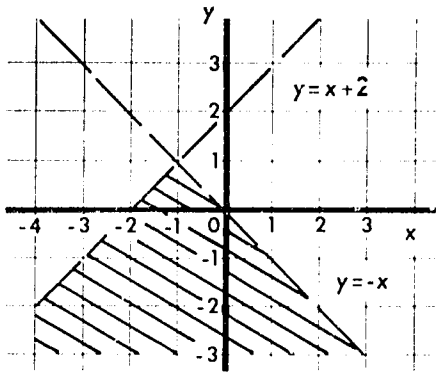
2. Let R denote the set of all ordered pairs (x, y) of real numbers such that $x + y < 2$ *and* $y < x$.

   Without drawing the graph try to find the domain and range of R. You probably found this problem somewhat difficult to solve. Let us see if it becomes any easier after you have drawn the graph of R. Take a sheet of paper and sketch the graphs of the lines $x + y = 2$ and $y = x$. Find the coordinates of their point of intersection. The lines you have drawn divide the plane into four regions, one of which represents the relation R. Determine which it is and shade it to obtain the graph of R. Now use your graph to find the domain and range of R.

PROBLEMS 1-4B

Draw the graphs of the relations given in Problems 1 to 5 below and find the domain and range of each.

1. $R = \{(x, y) : x - y < 1 \text{ and } y < -2\}$.
2. $R = \{(x, y) : x + y > 1 \text{ and } x > y + 2\}$.
3. $R = \{(x, y) : y > x^2 - 1\}$.
4. $R = \{(x, y) : y > x^2 - 1 \text{ and } y < 3\}$.
5. $R = \{(x, y) : x < y,\ x + y > 0, \text{ and } y < 2\}$.
6. Find formulas of the type given in Problems 1 to 5 which describe the relations graphed below, and determine the domain and range of each.



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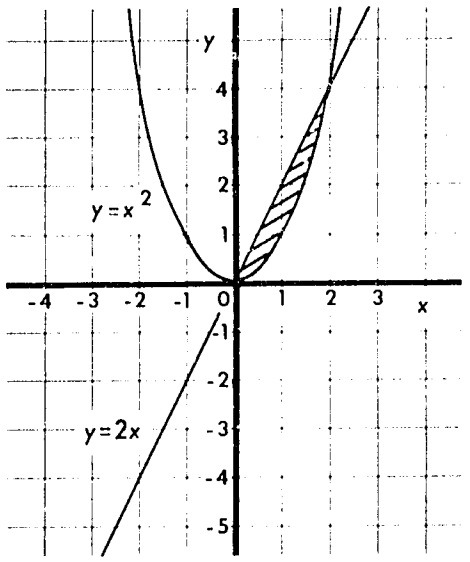

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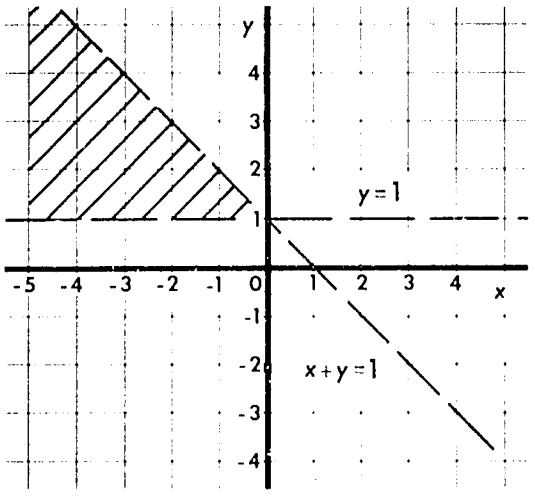

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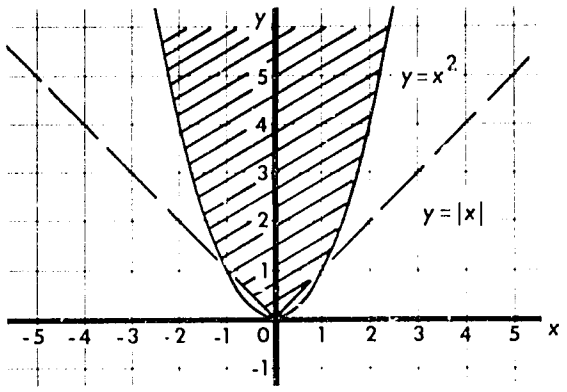

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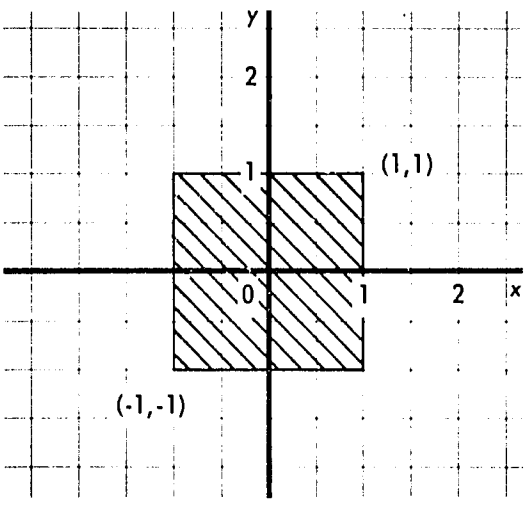

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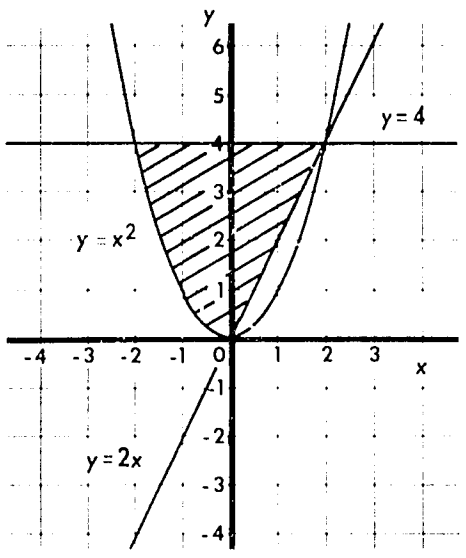


(f)

Fig. 8.

1-5 THE INVERSE OF A RELATION.

In Section 1-2 we introduced the idea of a relation by considering the relation R of fatherhood, which we described as the set of ordered pairs (x, y) of people where y is the father of x. Now suppose that instead of starting with R we had formed the set S consisting of all ordered pairs (y, x) where x is the son or daughter of the man y. By definition, S is also a relation, since it is a set of ordered pairs. Can you describe the connection between the ordered pairs belonging to R and those belonging to S? If an ordered pair (x, y) belongs to R does the pair (y, x) belong to S? Conversely, if (y, x) belongs to S does (x, y) belong to R? Clearly the answer to both of these questions is yes, and from this it follows that either of these relations can be obtained from the other by reversing the entries in each of its ordered pairs. In other words, S can be described as the set of all ordered pairs (y, x) with the property that (x, y) belongs to R, while R can be described as the set of all ordered pairs (x, y) with the property that (y, x) belongs to S.

CLASS ACTIVITY

List the names of all of the pupils in your class and after each write the name of his mother. You now have a table for a relation R. What is this relation? Starting with the table you have just constructed, write out the table for a relation S of the type described above. How did you form this table? Describe the domain and range of S in terms of the domain and range of R. Suppose you repeated this process starting with the table for S. What would you obtain?

Pairs of relations of the sort described above are encountered throughout mathematics, and without realizing it you have been using them for years. For instance, if R is the relation "greater than" for real numbers, that is $R = \{(x, y) : y > x\}$, then by reversing the entries in each of the ordered pairs belonging to R we obtain the relation "less than." Similarly, if R is the set of all ordered pairs (x, y) of real numbers with $y = x^2$, reversing the entries in these pairs yields the relation S consisting of all ordered pairs (y, x) with $y \geq 0$ and $x = \sqrt{y}$ or $x = -\sqrt{y}$. We might call R the "squaring" relation and S the "square root" relation. Question: Which of the following ordered pairs belong to the square root relation?

$$(0, 0), (2, 4), (4, 2), (4, -2), (-2, 4), (8, 2\sqrt{2}), (9, -3).$$

By now you undoubtedly realize that whenever we are given a relation R we can always produce another relation from it simply by reversing the entries in all of the ordered pairs belonging to R. The relation obtained from R in this way is known as the *inverse* of R and is denoted by R^{-1}. (Despite appearances, the -1 used here is *not* an exponent. Rather, it is just a mark to indicate that we are thinking about the inverse of R. Thus the symbol R^{-1} should always be read "R inverse", not "R to the power negative one.")

Before going on let us interpret this definition in terms of pictures of the type introduced in Figure 2 on page 2. Since we have agreed that the pictorial representation of an ordered pair (x, y) belonging to R is an arrow from x to y, it follows that the picture for R^{-1} can be obtained from the picture for R by reversing the direction of all of the arrows. Hence if R is represented by the figure

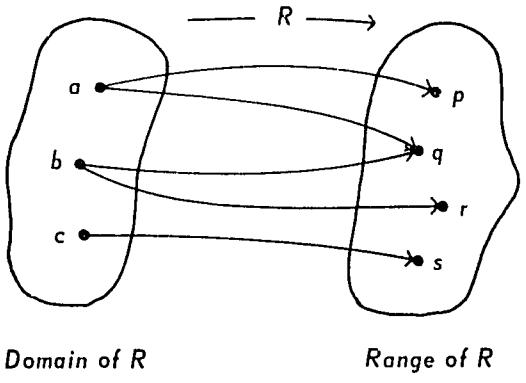


Fig. 9.

then R^{-1} is represented by the figure

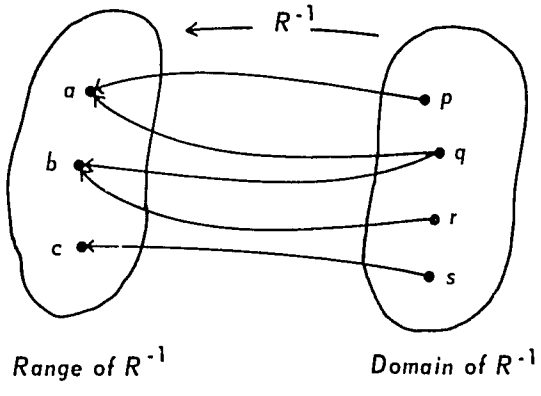


Fig. 10.

If you keep this pair of figures in mind as you read on, you should have no difficulty in understanding everything which follows.

PROBLEMS 1-5A

Describe the inverses of each of the following relations.

1. R is the set of all ordered pairs (x, y) of pupils in your class such that y is taller than x.
2. R is the set of all ordered pairs (x, y) where x is a pupil in you school and y is one of his teachers.
3. R is the set of all ordered pairs (x, y) of people in Africa where y is older than x.
4. R is the set of all ordered pairs (x, y) of real numbers where y is the cube root of x.
5. R is the set of all pairs of coordinates of points in the plane.
6. R is the set of all pairs of coordinates of points in the plane lying in the second quadrant. (Be careful! Remember that points in the plane are always described by giving their x-coordinate first and then their y-coordinate.)
7. R is the set of all pairs of coordinates of points in the plane which lie below the line $y = 4$. (See the warning in Problem 6.)

Before continuing with our study of inverses we give a formal definition in the language of sets.

> Definition: The *inverse* of a relation R is the relation defined by the formula $R^{-1} = \{(y, x): (x, y) \text{ belongs to } R\}$.

Keeping this definition in mind we now answer two simple but important questions.

Question 1: If R is a relation, what is the inverse of R^{-1}?

Answer: By definition, the inverse of R^{-1} is obtained by reversing the entries in the ordered pairs belonging to R^{-1}. But these pairs were themselves formed by reversing the entries in the ordered pairs belonging to R. The result of these two reversals is that we obtain the original relation R, and it follows that the inverse of R^{-1} is R itself. We can express this fact symbolically by the equation

$$(R^{-1})^{-1} = R.$$

CLASS ACTIVITY

Illustrate this last result by pictures of the type given in Figures 9 and 10 above. Use a similar sequence of figures to obtain an equation for the inverse of the inverse of R^{-1}. What did you find?

Question 2: If R is a relation with domain A and range B, what are the domain and range of R^{-1}?

Answer: Again we can settle the question by watching what happens when the entries in the ordered pairs belonging to R are reversed. The result is that the set of second entries in the ordered pairs belonging to R becomes the set of first entries in the ordered pairs belonging to R^{-1}, and we conclude that the domain of R^{-1} coincides with the range of R. Similarly, the range of R^{-1} coincides with the domain of R, and we have the following formulas:

Domain of R^{-1} = Range of R

Range of R^{-1} = Domain of R.

PROBLEMS 1-5B

Find the domain and range of the inverse of each of the following relations.

1. $R = \{(x, y) : y < 0\}$.

2. $R = \{(x, y) : y = x^2\}$.

3. $R = \{(x, y) : x > y\}$.

4. $R = \{(x, y) : y = x^3\}$.

5. $R = \{(x, y) : x < y + 2 \text{ and } x > 0\}$.

6. $R = \{(x, y) : y > x^2 \text{ and } y < 4\}$.

7. $R = \{(x, y) : x \text{ lies between } -2 \text{ and } 2, \text{ and } y \text{ between } -1 \text{ and } 1\}$.

1-6 GRAPHING THE INVERSE OF A RELATION.

In this section we consider the problem of drawing the *graph* of R^{-1} when R is a set of ordered pairs of real numbers. We begin by considering a simple example.

Let R be the relation "greater than" for real numbers; that is,

$$R = \{(x, y) : y > x\}$$

Then, as you know, R^{-1} is the relation "less than" for real numbers, and we have

$$R^{-1} = \{(x, y) : y < x\}$$

You should observe that in denoting the ordered pairs belonging to R^{-1} we have used the symbol (x, y) and not, as you might have expected, the symbol (y, x). Our reason for doing this is that one *always* describes a point in the plane by listing its x-coordinate first and its y-coordinate second. As we shall see, this simple agreement is the basis for graphing inverse relations.

Let us return to the relations R and R^{-1} described above. Their graphs are shown as shaded regions in Figure 11 below. If you study this figure carefully you will notice that the graph of R^{-1} can be obtained from the graph of R by holding the line $y = x$ fixed and rotating the plane about this line through an angle of 180°. To understand this process better, open your mathematics book and align the binding with the line $y = x$, as shown in Figure 12. In this position the upper page can be thought of as representing the

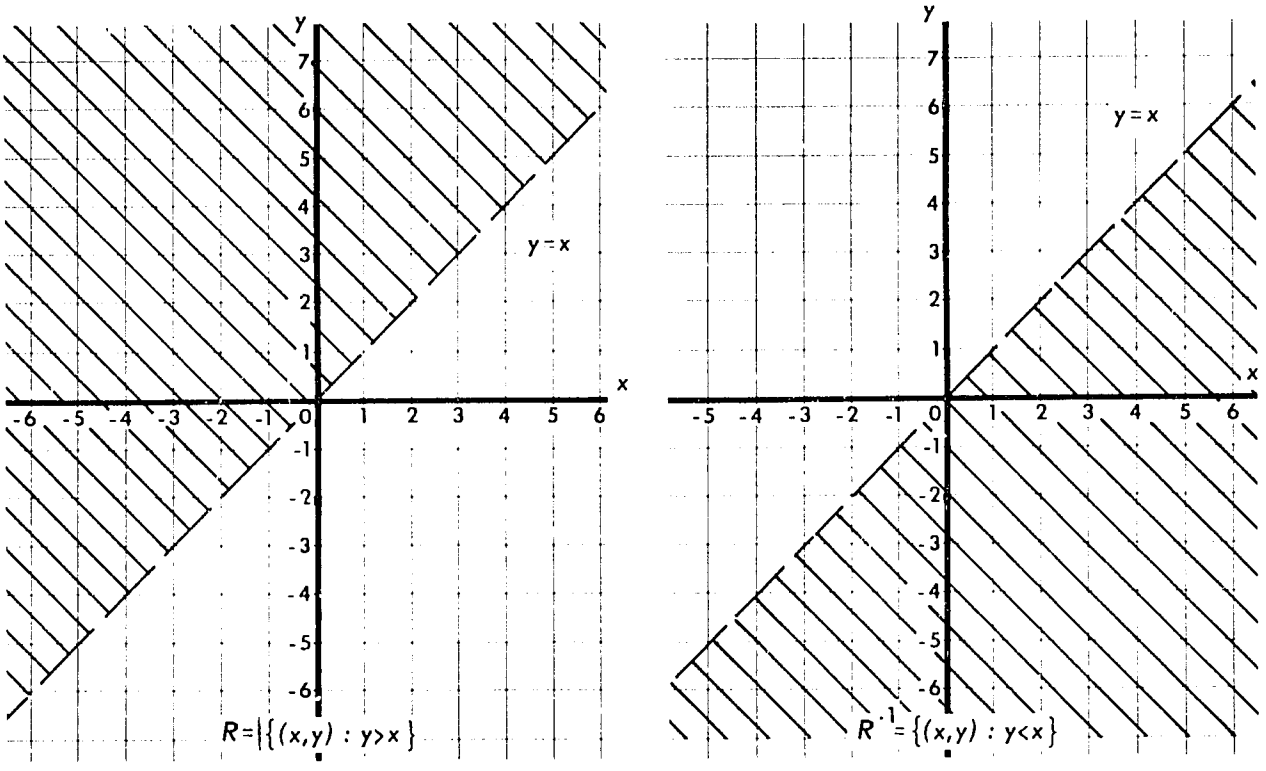


Fig. 11

graph of R, the lower page the graph of R^{-1}. Turn the upper page without moving the book. You will find that in its new position the page now represents the graph of R^{-1}.

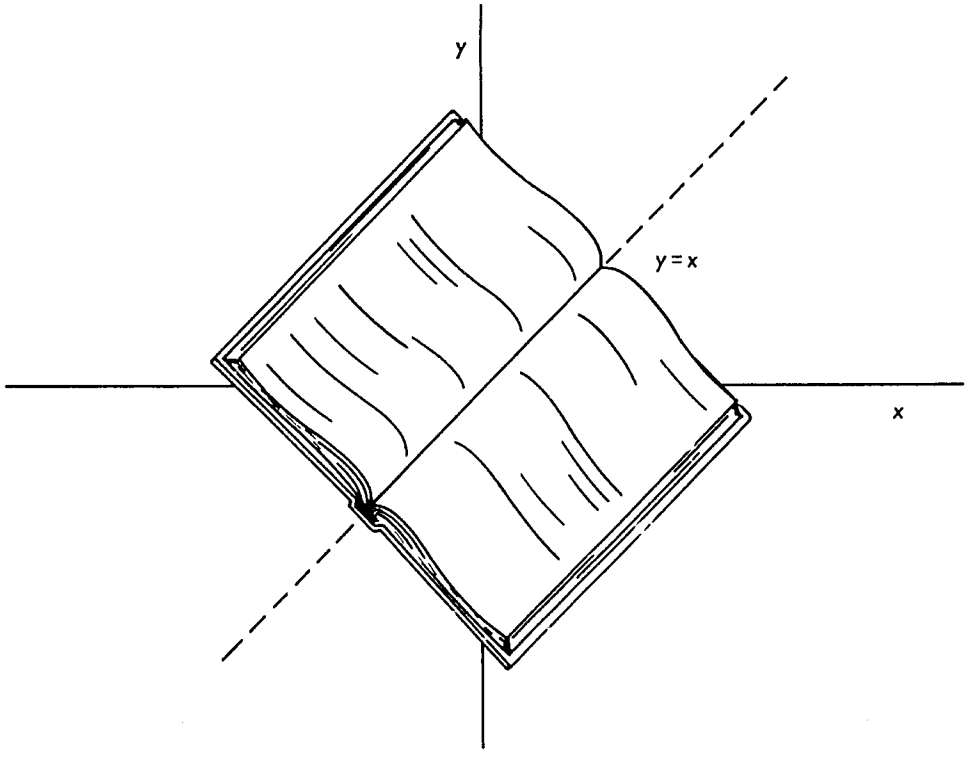


Fig. 12

CLASS ACTIVITY

Take a sheet of paper and carefully draw a pair of coordinate axes using the same scale on each. Draw the graph of the line $y = x$, and then plot the following pairs of points, marking each heavily enough so that you can see them on the opposite side of your paper.

(1, 0) and (0, 1)
(2, 0) and (0, 2)
(-1, 0) and (0, -1)
(2, 1) and (1, 2)
(3, -1) and (-1, 3)

Now fold your paper along the line $y = x$. What happened to each of these pairs of points when your paper was folded? If your drawing was accurate you should find that the points of each pair now lie on top of one another. If they don't you have made a mistake and should try again.

Using this picture as a model, answer each of the following questions.

a. What happens to points on the positive x-axis and positive y-axis when their coordinates are reversed?

b. What happens to points on the negative x-axis and negative y-axis when their coordinates are reversed?

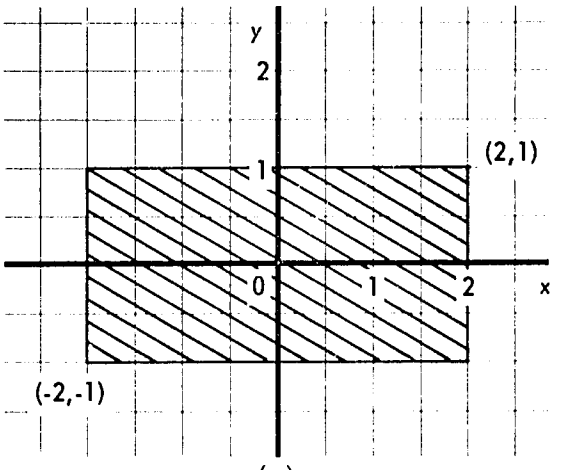
c. What happens to points on the line $y = x$ when their coordinates are reversed?

By now you should see that the effect of reversing the coordinates of any point in the plane is to move the point in question across the line $y = x$ just as though that line were a mirror. For this reason we call such a motion a *reflection* across the line $y = x$. As you have seen, we can produce such a reflection by rotating the plane through an angle of 180° about the line $y = x$.

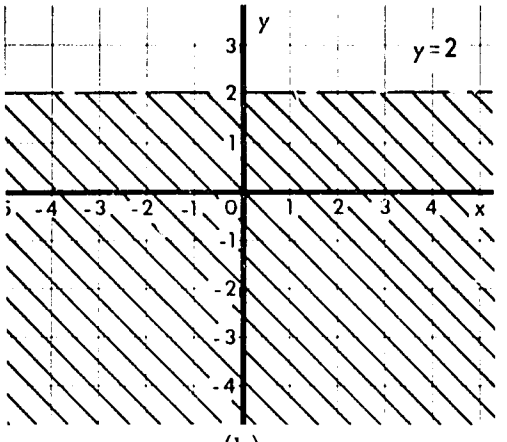
It remains to apply what we have just discovered to graphing inverses of relations. Do you see how it is done? The answer, of course, is that *the graph of R^{-1} can be obtained by reflecting the graph of R across the line $y = x$.* Thus to graph R^{-1} when R is given, first graph R and then reflect the result across the line $y = x$.

CLASS ACTIVITY

Draw the graph of the inverse of each of the relations shown below. Which of these relations is identical with its inverse?



(a)



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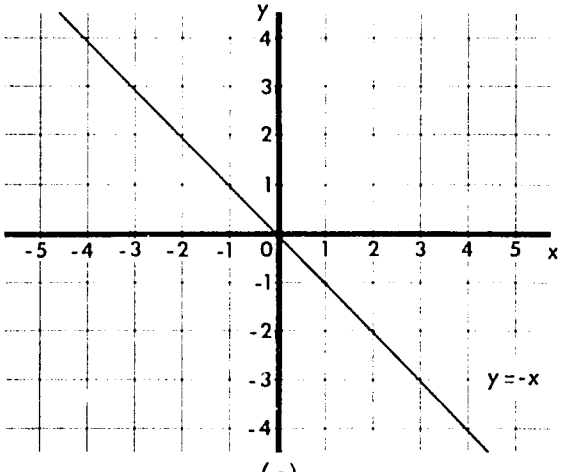

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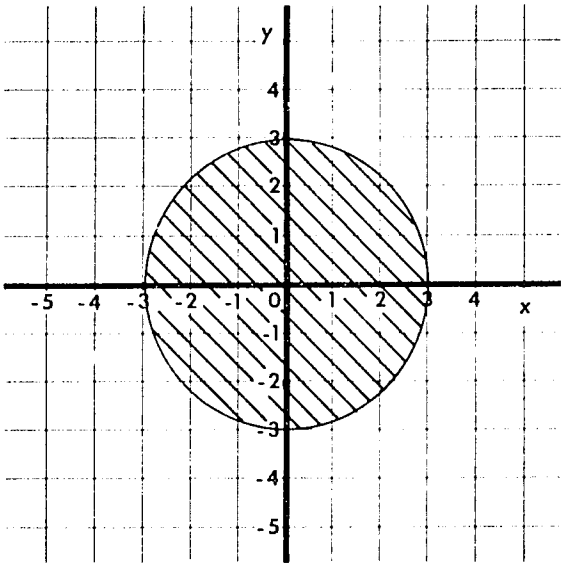

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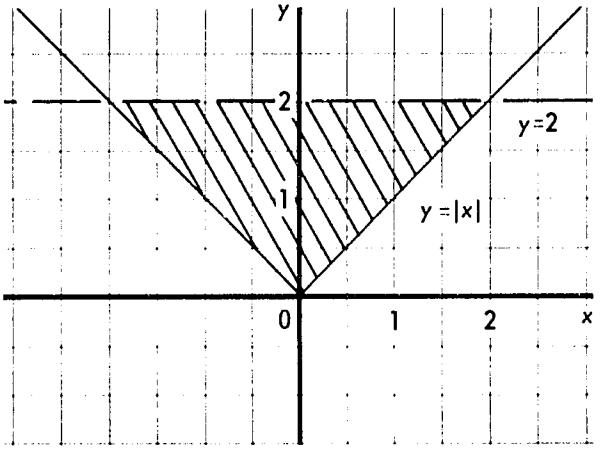


(e)

Fig. 13.

Before going on, let us summarize what we now know about relations and their inverses.

Summary

1. A relation R is a set of ordered pairs.
2. Domain of $R = \{x : (x, y)$ belongs to R for some $y\}$.
   Range of $R = \{y: (x, y)$ belongs to R for some $x\}$.
3. If R is a relation, $R^{-1} = \{(y, x) : (x, y)$ belongs to $R\}$.
4. $(R^{-1})^{-1} = R$, for any R.
5. Domain of R^{-1} = Range of R.
   Range of R^{-1} = Domain of R.
6. If R is a set of ordered pairs of real numbers the graph of R^{-1} can be obtained by reflecting the graph of R across the line $y = x$.

PROBLEMS 1-6

Draw the graph of the inverse of each of the following relations.

1. $R = \{(x, y) : y = 2x\}$.
2. $R = \{(x, y) : y > x^2\}$.
3. $R = \{(x, y) : y = x^3\}$.
4. $R = \{(x, y) : x > 0\}$.
5. $R = \{(x, y) : x < 0$ and $x > y\}$.
6. $R = \{(x, y) : x + y < 1$ and $y < x - 1\}$.
7. $R = \{(x, y) : y > 0, x < y\}$.
8. a. Let $R = \{(x, y) : y = x^2\}$. Draw the graphs of R and R^{-1}. Is R^{-1} a function? Why?
   b. Let $S = \{(x, y) : x \geq 0$ and $y = x^2\}$. Draw the graphs of S and S^{-1}. Is S^{-1} a function? Why?
9. *Challenge Problems.*
   a. Let P be a point in the plane with coordinates (x, y), let Q be the point (y, x), and suppose that $x \neq y$. Let A denote the point of intersection

of the line segment $\overline{PQ}$ and the line $y = x$. Prove that $\overline{PQ}$ is perpendicular to the line $y = x$ and that $\overline{AP} \equiv \overline{AQ}$. [Hint. Make the construction shown in Figure 14, and prove that ΔABP and ΔABQ are congruent right angled triangles.]

b. What does this result allow you to conclude about the points P and Q when the plane is reflected across the line $y = x$?

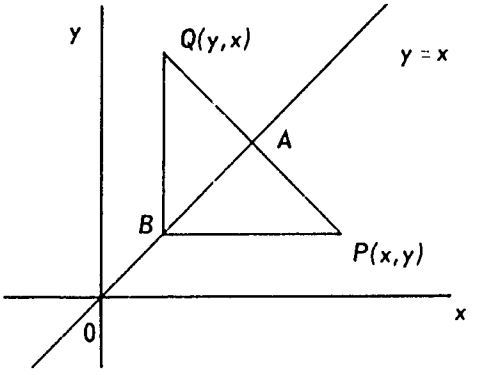


Fig. 14.

1-7 FUNCTIONS.

At the beginning of this chapter we described a function as a rule or correspondence by means of which numbers are paired with one another in a perfectly definite way. Although this description is quite adequate for most purposes, it is still desirable to have a precise definition which does not rely upon such vague and undefined words as "rule" or "correspondence." The main purpose of this section is to give such a definition.

As the first step in this direction we note that there is really no need for insisting that the entries in the ordered pairs belonging to a function be real numbers. Indeed, from our study of relations we know that there are countless examples of pairings involving objects other than real numbers, and any reasonable definition of the term "function" must take this fact into account. Thus, although we shall retain the basic idea of an ordered pair, we now drop all restrictions on the nature of the entries appearing in these pairs.

What then distinguishes a function from a relation? The answer, as you probably know, is that a function is a relation with the special property that each member of its domain is paired with *exactly one* member in its range. Hence, to complete our definition we need only phrase this restriction in mathematical terms. Do you see how this can be done? Suppose that f is a function, and that the ordered pairs (x, y) and (x, z) both belong to f. What can you say about y and z? If you see that they must be identical with

one another you have found the required definition, which we now state as follows.

> Definition: A *function* f is a set of ordered pairs with the property that whenever (x, y) and (x, z) belong to f, then $y = z$.

CLASS ACTIVITY

Give an interpretation of this definition in terms of pictures of the type shown below. Which of these pictures represents a function, and which a relation which is not a function?

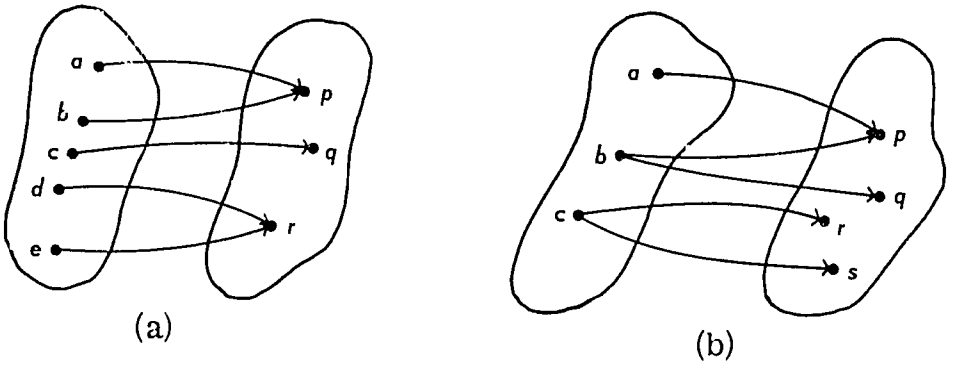


Fig. 15.

Before we continue, let us recall some of the terminology used in referring to functions. First, if (x, y) is an ordered pair belonging to a function f, we say that y is the *value* of f at x, and write $y = f(x)$. Furthermore, since x is allowed to assume arbitrary values in the domain of f, we call x a *variable* and say that y *depends on* x. By this we simply mean that once x has been chosen the corresponding value of y is uniquely determined. Finally, you should note that we have used the symbols f and $f(x)$ to stand for entirely different things. The first represents the function itself, and hence should be thought of as a set of ordered pairs, while the second denotes the value of the function at x. For this reason it is both incorrect and misleading to say that an expression such as $x^2 + 3x + 2$ is a function. Rather, it is a formula for computing the value of a function at each point in its domain, and should be referred to as such. This distinction between a formula and the function which it defines should become clear if you consider the

formulas $x^2 + 3x + 2$ and $(x + 2)(x + 1)$. As formulas, these expressions are obviously different, and yet they define the same set of ordered pairs. In other words, *different* formulas can define the *same* function, a fact which shows that the function itself is something quite different from the formula or formulas which are used to define it. This is an important point, and you should be sure to understand it.

Example 1: Let f be the set of all ordered pairs (x, y) where x is a telephone subscriber in Kampala and y is his telephone number. Then, if no one in Kampala has more than one telephone, f is a function. Moreover, the telephone directory for the city of Kampala is a listing of the ordered pairs belonging to f, and when we look up a number in this directory we are finding the value of f for a certain x.

You should observe that in this case it is altogether impossible to give a formula for computing the value of f for each x in its domain. In fact, it is clear that a telephone company assigns its numbers more or less at random as subscribers apply and vacancies occur, and not by formula. Hence, the only way of determining the value of f for a given x is to consult the telephone book which, as we have said, is a table of values for the function. In short, *the telephone book is the function*! Do you see that this conclusion agrees with our definition of a function as a set of ordered pairs?

CLASS ACTIVITY

The above example gives a second reason why it is misleading to say that a function is a formula. Explain, in your own words, what this reason is. Give two more examples of functions which cannot be defined by formulas.

Example 2: Let f be the set of all ordered pairs of the form (Δ, A), where Δ is a triangle and A its area. Then, since every triangle has a unique area, A is determined as soon as Δ is given, and f is a function. If we wished, we might call f the *area* function for triangles.

In your study of geometry you have discovered a formula for computing the value of f in terms of the lengths of the base and altitude of Δ. Do you remember what this formula is? If b is the length of one side of Δ and h the length of the altitude erected on that side, what is $f(\Delta)$? Did you answer $f(\Delta) = \frac{1}{2}bh$? Do you know any other formulas for evaluating $f(\Delta)$? Perhaps not, but,

in fact, there are several. One, which you will encounter later in geometry, expresses the area of Δ in terms of the lengths a, b, c of its sides, as follows:

$$A = \sqrt{S(S-a)(S-b)(S-c)}$$

where $S = \frac{1}{2}(a + b + c)$. Thus we have two entirely different formulas for finding $f(\Delta)$, namely

$$f(\Delta) = \tfrac{1}{2}bh \text{ and } f(\Delta) = \sqrt{S(S-a)(S-b)(S-c)},$$

and we see another example which shows why it is incorrect to call a formula a function. Instead, one should say, as we have, that the function in question is a certain set of ordered pairs, and that either of the above formulas serves to define this function by allowing us to compute its values for any triangle Δ.

PROBLEMS 1-7A

Determine which of the following relations are functions and which are not.

1. $R = \{(x, y) : y$ is the father of $x\}$.
2. $R = \{(x, y) : y$ is one of the cities in Ghana and x is a resident of $y\}$.
3. $R = \{(x, y) : y$ is the husband of $x\}$.
4. $R = \{(x, y) : x$ is a river in Africa and y is its length in miles$\}$.
5. $R = \{(x, y) : x$ and y are lines in the plane and y is perpendicular to $x\}$.
6. $R = \{(x, y) : x$ and y are congruent triangles$\}$.
7. $R = \{(x, y) : y = x^2\}$.
8. $R = \{(x, y) : x^2 + y^2 = 1\}$.
9. $R = \{(x, y) : x$ is a natural number greater than 1 and y is a prime divisor of $x\}$.
10. $R = \{(x, y) : -1 < x < 1$ and $y = 2x\}$.

11. Determine which of the following pairs of formulas define the same function.

a. $y = x^2$; $y = x^2$, $x \geq 0$.

b. $y = (x + 1)^2 (x - 2)$; $y = x^3 - 3x - 2$.

c. $y = \frac{1}{x}$; $y = \frac{x + 1}{x^2 + x}$ (Be careful!)

d. $y = 2^x$; $y = \log_2 x$.

e. $y = x + 1$; $y = \frac{x^2 - 1}{x - 1}$. (Be careful!)

f. $y = x$; $y = \sqrt{x^2}$. (Be careful!)

g. $y = x$; $y = \sqrt[3]{x^3}$.

h. $y = \frac{2x + 1}{x}$, $x > 0$; $y = \frac{2x^2 + 7x + 3}{x^2 + 3x}$, $x > 0$.

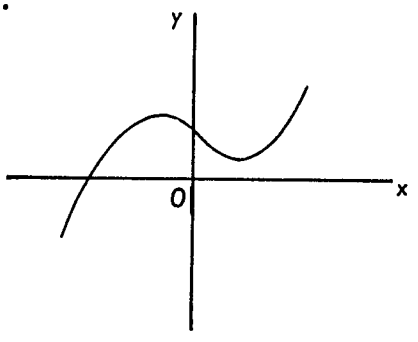
To complete this discussion it remains to obtain a graphical interpretation of the definition of function when f is a *numerical* function, that is, when f is a set of ordered pairs of real numbers. Can you guess what this interpretation is? Draw the graphs of several functions, and compare them with the graphs of a number of relations which are not functions. What is the difference between these two types of graphs?

To answer this question, let f be a numerical function and consider the vertical line in the plane passing through a point on the x-axis which belongs to the domain of f. What can you say about the x-coordinate of every point on this line? Do you see that it doesn't change? Can you use this fact, together with the definition of function, to conclude that the line in question intersects the graph of f in *exactly one* point? The argument needed to establish this fact is extremely simple. By definition, there is *one and only one point* on the graph of f whose x-coordinate is the same as the x-coordinate of the given line; hence there is one and only one point of intersection, as asserted. This observation is extremely useful, particularly in studying the inverses of functions, and for that reason we restate it, as follows.

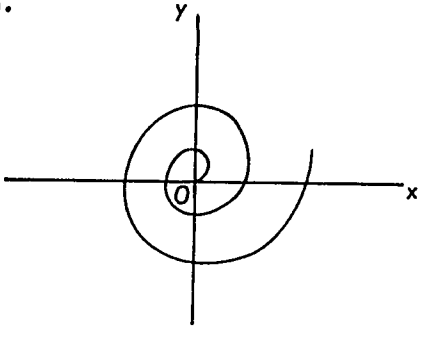
A numerical relation R is a function if and only if no vertical line in the plane intersects the graph of R in more than one point.

Determine which of the following graphs represent functions, and which do not.

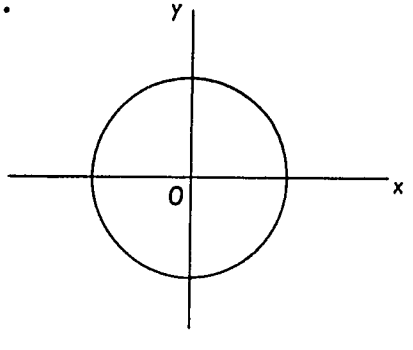
1.



2.



3.



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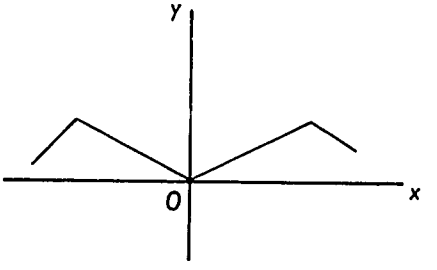

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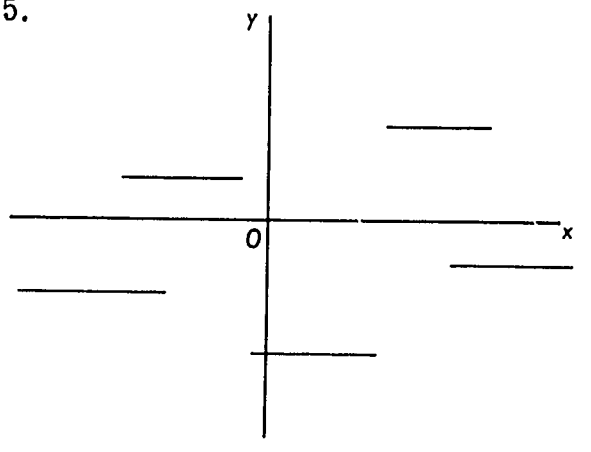

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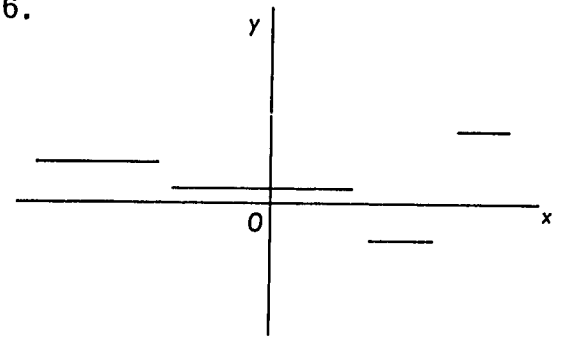


7.

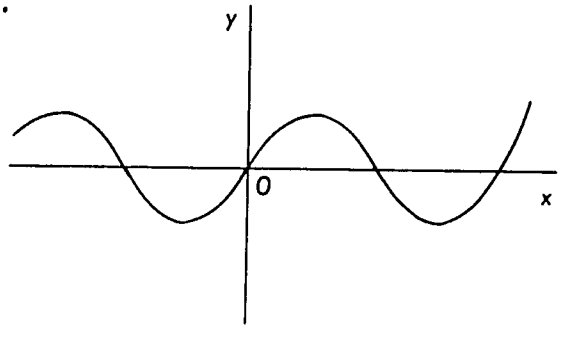


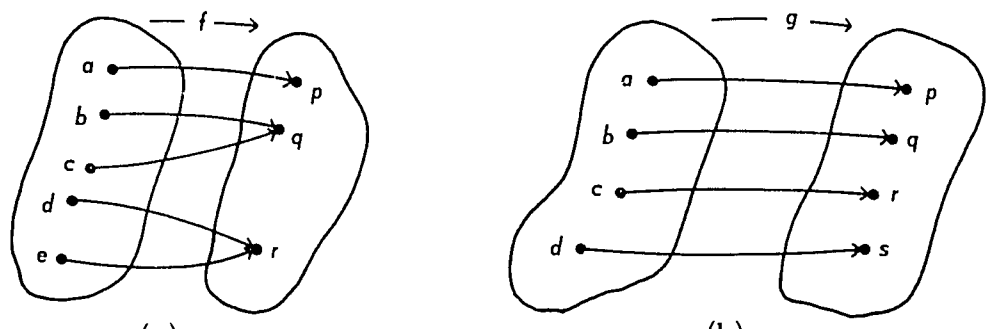
Fig. 16.

1-8 ONE-TO-ONE FUNCTIONS; INVERSES.

In this section we shall apply what we have learned about inverse relations to the study of functions. In particular we should like to determine what restrictions, if any, can be placed on a function f in order to guarantee that its inverse is also a function.

CLASS ACTIVITY

Consider the functions shown in Figures 17(a) and (b) below. On a separate sheet of paper draw a picture to illustrate the inverse of the function shown in Figure 17(a). Is the result the picture of a function? Why? Now do the same thing, for the function shown in Figure 17(b). What can you say this time? What is the difference between f and g which explains the fact that g^{-1} is a function while f^{-1} is not?



(a) (b)

Fig. 17

If you answered the above questions carefully you have discovered that the inverse of a function f is again a function whenever each point in the *range* of f is paired with *exactly one* point in its domain. Indeed, since f^{-1} is formed by interchanging the domain and range of f, it is clear that f^{-1} will be a function only if f does not contain two ordered pairs of the form (x_1, y) and (x_2, y) with $x_1 \neq x_2$. Otherwise, when the entries in the ordered pairs belonging to f were reversed we would find that f^{-1} contained the ordered pairs (y, x_1) and (y, x_2) with $x_1 \neq x_2$, and hence would not be a function.

Functions with the special property that their inverses are also functions are said to be *one-to-one*. After the above remarks the reason for this name should be obvious. If f is one-to-one with domain A and range B, then each x in A is paired with *exactly one* y in B, and each y in B is paired with

exactly one x in A. In short, we have one x to one y, one y to one x, or one-to-one. In terms of pictures of the type shown in Figure 17 a one-to-one function is characterized by the property that there is precisely one arrow terminating at each point in its range. More formally, we have the following definition.

> Definition: A function f is said to be *one-to-one* if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$.

You should convince yourself that this definition really says what it is supposed to say, namely that f does not pair two different points in its domain with the same point in its range.

Example 1: Let f be the function defined by the formula $f(x) = x^3$. In Secondary Three you learned that whenever x_1 and x_2 are distinct real numbers their cubes are also distinct. Hence $x_1^3 \neq x_2^3$ whenever $x_1 \neq x_2$, and it follows that f is one-to-one and that its inverse is a function.

CLASS ACTIVITY

1. Determine which of the following ordered pairs belongs to the inverse of the function in Example 1 above.

   $(-2, -8)$, $(-8, -2)$, $(2, \sqrt[3]{2})$, $(-\frac{1}{8}, \frac{1}{2})$, $(-\frac{1}{8}, -\frac{1}{2})$, $(0, 0)$, $(-1, 1)$.

2. Draw the graph of f, and use the technique introduced in Section 1-6, page 24, to obtain the graph of f^{-1}.

3. What is the name of f^{-1}?

Example 2: a. Let f be the function defined by $f(x) = x^2$. What is the value of f when $x = 1$? When $x = -1$? Is f one-to-one? Why? Draw the graphs of f and f^{-1} and use your graph to show that f^{-1} is not a function.

b. Now let g be the function defined by $g(x) = x^2$, $x \geq 0$. How does this function differ from the function defined in part a? Is g one-to-one? Why? Draw the graphs of g and g^{-1} and use your graph to show that g^{-1} is a function. What is the name of g^{-1}?

PROBLEMS 1-8

Determine which of the following functions are one-to-one, and which are not.

1. $f = \{(x, y) : x$ is a pupil in your school and y is his full name$\}$.
2. $f = \{(x, y) : y$ is the father of $x\}$.
3. $f = \{(x, y) : x$ is a triangle and y is its area$\}$.
4. $f = \{(x, y) : x$ is the length of the radius of a circle and y is the circumference of the same circle$\}$.
5. $f = \{(x, y) : y = x + 4\}$.
6. $f = \{(x, y) : x \leq 0,\ y = x^2 + 4\}$.
7. $f = \{(x, y) : x \geq 0,\ y = x^2 - 3x + 2\}$.
8. $f = \{(x, y) : y = x^3 - 2x^2 - x + 2\}$.
9. Draw the graphs of the function defined in Problems 5 to 8 and use the technique introduced in Section 1-6 to obtain the graphs of their inverses.

In the preceding section we learned how to use the graph of a numerical relation to determine whether or not the relation was a function. We now show how essentially the same technique can be used to decide when a function is one-to-one.

Consider the functions whose graphs are shown in Figure 18. (You should imagine that these graphs extend indefinitely in the usual fashion.) In Figure 18(a) imagine a line drawn parallel to the x-axis which intersects the graph of f in two places, and let the coordinates of these points of intersection be (x_1, y) and (x_2, y). Why do these points have the same y-coordinate? Do the ordered pairs (x_1, y) and (x_2, y) belong to f? What does this tell you about f? About f^{-1}?

Now consider the graph shown in Figure 18(b). Can you find a line parallel to the x-axis which intersects this graph in two points? What does this tell you about the function g? About g^{-1}? Do you think that you would have arrived at the same conclusion for any function whose graph had this property?

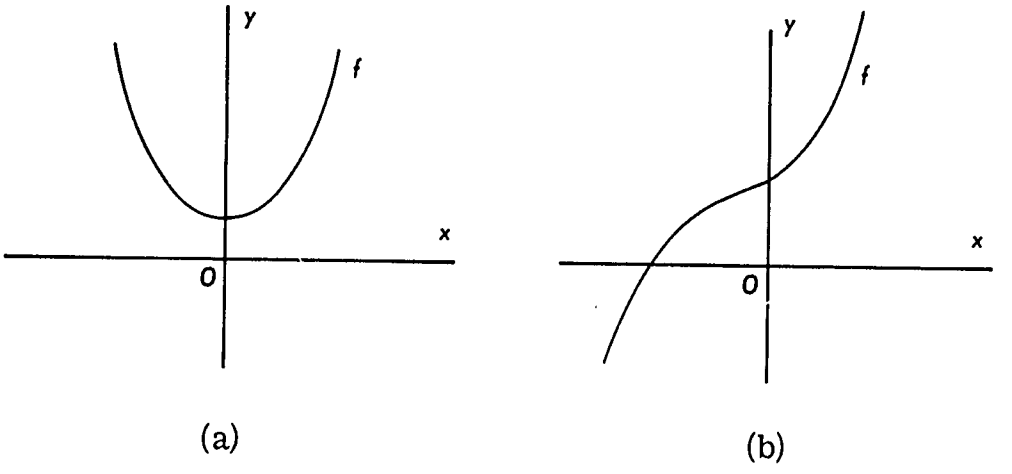


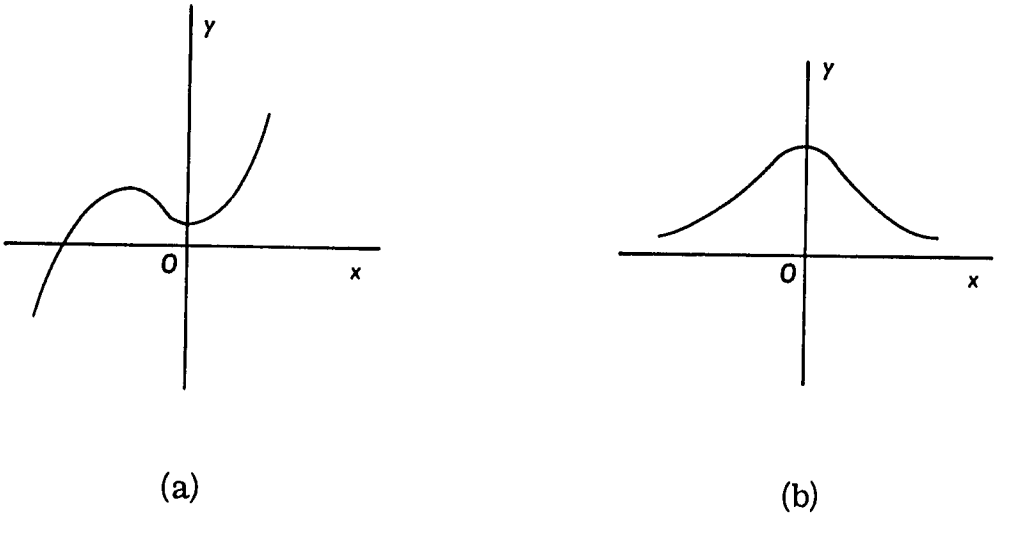
Fig. 18.

These results yield the following valuable test for determining when a function is one-to-one.

If f is a numerical function with the property that no line parallel to or coinciding with the x-axis intersects its graph in more than one point, then f is one-to-one and its inverse is a function.

CLASS ACTIVITY

Use the above test to determine which of the following graphs represent one-to-one functions.



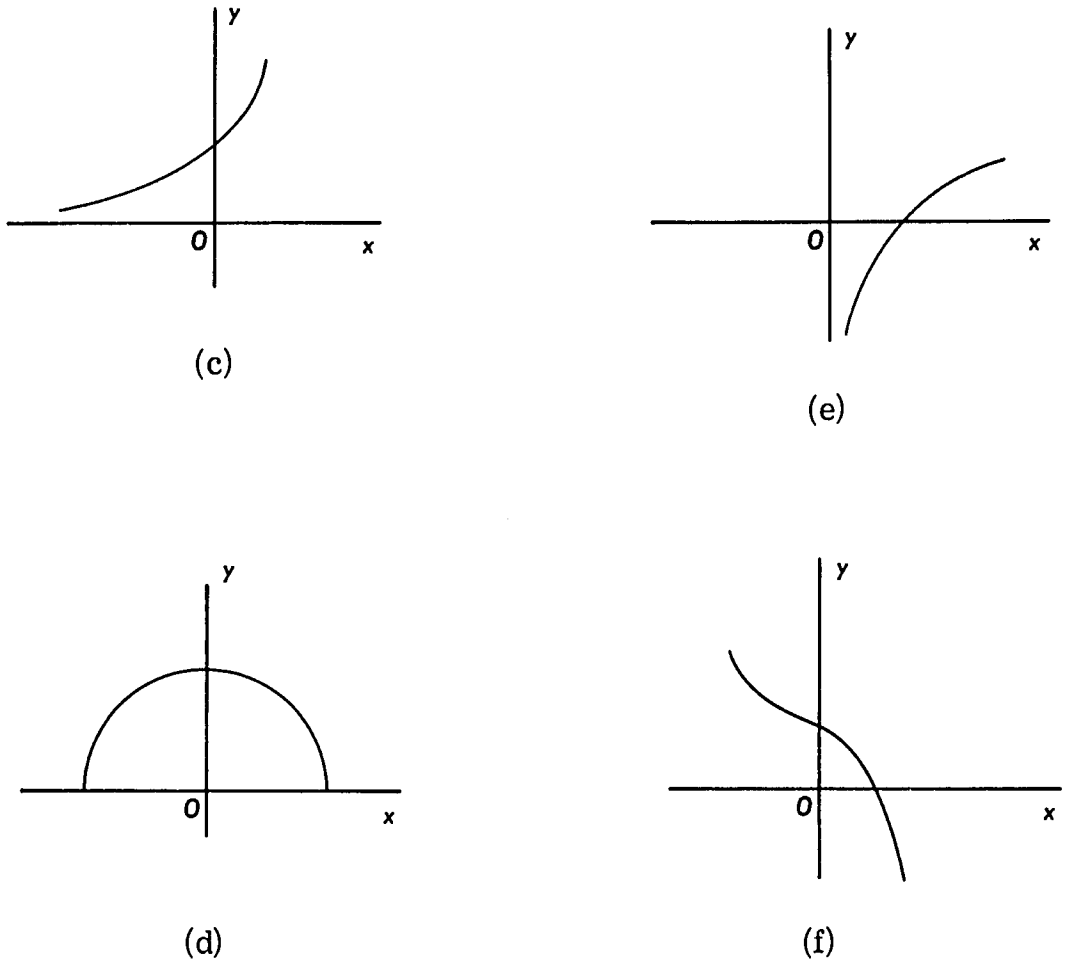


Fig. 19.

We have seen that a function is a one-to-one function if every line parallel to or coinciding with the x-axis intersects the graph in at most one point. In particular we can observe that if the graph of a function either always *rises* or always *falls* as we move along the x-axis from left to right, then this test will be satisfied and the function is one-to-one.

For a function in which the values of f either always *increase* or always *decrease* as x increases, one of the following inequalities will be satisfied:

a. $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, or

b. $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

Functions of this sort are sometimes said to be *increasing* or *decreasing* depending on whether they satisfy (a) or (b), and we shall use these terms from time to time in talking about one-to-one numerical functions.

CLASS ACTIVITY

1. Which of the functions graphed in Figure 19 above are increasing and which are decreasing?

2. Is it true that if a function is increasing or decreasing, then it is one-to-one? Does such a function have a corresponding inverse function?

3. Is the converse true? If a function is one-to-one, then is it increasing or decreasing?

4. Can you give an example of a graph of a function which is one-to-one and is neither increasing nor decreasing?

5. If a function is one-to-one *and continuous*, then must it be either increasing or decreasing?

1-9 FUNCTIONS OF TWO VARIABLES.

In our definition of a function as a set of ordered pairs no two of which have the same first entry and different second entries we placed no restriction whatever on the nature of these entries. They may be people and their telephone numbers, geometric figures and their areas, or real numbers. You will recall that functions whose ordered pairs are made up of real numbers are known as *numerical functions*. They are easily the most important class of functions in mathematics.

Of almost equal importance, however, are functions the *first* entries of whose ordered pairs are themselves ordered pairs of real numbers. As the following examples illustrate, such functions are very common indeed in mathematics.

Example 1: Consider the rule which associates with each ordered pair (x, y) of real numbers their difference $x - y$.

Question: Does this rule define a function in the sense of the definition given in Section 1-7?

Answer: Yes.

Reason: The given rule associates *precisely one* real number, namely $x - y$, with each ordered pair (x, y) and thus establishes a pairing of the type required.

It is clear that the domain of this function is the set of all ordered pairs (x, y) of real numbers, and that its range is the set of all real numbers. Moreover, if f denotes the function in question the value of f at the ordered pair (x, y) is $x - y$, and we therefore write

$$f(x, y) = x - y.$$

Thus if (x, y) is the ordered pair $(2, -3)$ then

$$f(2, -3) = 2 - (-3) = 5.$$

Similarly $f(2, 3) = 2 - 3 = -1$, and the value of f at $(2, 3)$ is -1.

Questions: a. What is the value of f at each of the following ordered pairs?

$(-3, 2)$, $(1, -\frac{2}{3})$, $(3, 4)$, $(-\frac{3}{4}, \frac{1}{2})$.

b. For which ordered pairs (x, y), if any, is it true that $f(x, y) = f(y, x)$?

Example 2: Let f be the rule which associates with each ordered pair (x, y) of real numbers their product xy. Then, just as in the preceding example, f is a function whose domain is the set of all ordered pairs of real numbers, and whose range is the set of all real numbers. This time f is defined by the formula

$$f(x, y) = xy.$$

Questions: a. What is the value of f at each of the following ordered pairs?

$(0, 0)$, $(-\frac{1}{2}, \frac{2}{3})$, $(2, -1)$, $(-\frac{2}{3}, -\frac{3}{8})$, $(-\pi, -\pi)$.

b. For what ordered pairs (x, y) is it true that $f(x, y) = f(y, x)$?

Example 3: Let f be defined by the formula

$$f(x, y) = \frac{x}{y}.$$

Then $f(1, 2) = \frac{1}{2}$, $f(3, -6) = -\frac{1}{2}$, and, in general, f associates the quotient $\frac{x}{y}$ with the ordered pair (x, y). Here again f is a function whose domain is a set of ordered pairs of real numbers, and whose range is a set of real numbers.

Question: What are the domain and range of this function?

Example 4: Consider the set A of all ordered pairs (b, h) of real numbers with $b > 0$, $h > 0$, and let f be defined by the formula

$$f(b, h) = \tfrac{1}{2}bh$$

for all (b, h) in A. Then f is a function with domain A and range the set of all *positive* real numbers. (Can you prove this last statement?) If we interpret b and h respectively as the lengths of the base and altitude of a triangle Δ, then f recalls the area function for triangles discussed earlier in the chapter. The two functions differ, however, in that the domain of the first function is the set of all triangles (geometric figures) while the domain of the second function is the set of ordered pairs of numbers (b, h) where $b > 0$ and $h > 0$.

Each of the above examples illustrates a familiar mathematical situation which can be described by a function whose domain is a set of ordered pairs of real numbers. Functions of this type are known as *functions of two variables*, or, when we wish to call particular attention to the fact that the entries in the ordered pairs are real numbers, as *functions of two real variables*. When, in addition, the range of such a function is a set of real numbers; i.e., when $f(x, y)$ is a real number for each (x, y) in the domain of f, we say that f is a *numerical function* of two (real) variables. As we have said, such functions are extremely common in mathematics, and we shall have more to say about them in the chapter which follows.

PROBLEMS 1-9

In Problems 1 to 6 determine the domain and range of the functions defined by the given functions and compute their values at the given points.

1. $f(x, y) = x^2 - y^2$; $(1, 2)$, $(0, 0)$, $(-\frac{2}{3}, 1)$, $(\frac{1}{2}, -\frac{3}{4})$, $(-3, -4)$.
2. $f(x, y) = \dfrac{x + y}{y}$; $(-3, 2)$, $(0, -3)$, $(-\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{3}, \frac{2}{3})$, $(-3, -1)$.
3. $f(x, y) = x^2 + 2xy + y^2$; $(0, 0)$, $(-1, 2)$, $(\frac{1}{2}, -\frac{1}{2})$, $(-2, -2)$, $(4, 0)$.
4. $f(x, y) = |xy|$; $(0, 0)$, $(-1, -3)$, $(\frac{2}{3}, -\frac{3}{2})$, $(2, 4)$, $(-\frac{1}{2}, \frac{5}{2})$.
5. $f(x, y) = \sqrt{x^2 + y^2}$; $(0, 3)$ $(-3, 4)$, $(-12, -5)$, $(1, 1)$, $(\frac{1}{2}, -\frac{3}{2})$.
6. $f(x, y) = \sqrt{xy}$; $(-2, -2)$, $(0, 0)$, $(\frac{2}{3}, \frac{4}{3})$, $(-5, -4)$, $(-1, -1)$.

7. Let f be the function defined by the expression

$$f(x, y) = xy.$$

Show that

a. $f(x, -y) = -f(x, y)$.

b. $f(-x, y) = -f(x, y)$.

c. $f(-x, -y) = f(x, y)$.

8. Let f be the function defined by the expression

$$f(x, y) = x^2 - y^2.$$

Show that

$f(-x, y) = f(x, -y) = f(-x, -y) = f(x, y)$.

9. Describe a numerical function of two real variables as a set of ordered pairs.

Chapter 2
RATIONAL FUNCTIONS

2-1 RATIONAL FUNCTIONS.

You have been learning about functions and have seen that there are various kinds of functions. In the following set of problems let us recall some of the types of functions we can recognize.

PROBLEMS 2-1A

1. Each of the following expressions defines a function. For each one indicate the greatest possible domain and tell which of these names can correctly be applied to the function.

constant function
polynomial function
linear function
quadratic function
absolute value function
zero function
square root function
exponential function
logarithmic function
none of these

a. $2x + 6x^2$
b. $2x + 6$
c. $2x^4 + 6x^2 - 8$
d. 2
e. $2x - 2x$
f. $\sqrt{2x}$
g. $\sqrt{2}\,x$
h. $\sqrt{2}$
i. x^2
j. 2^x

k. $\frac{3}{x}$

l. $|x|$

m. $\frac{x}{x+15}$

n. $\frac{3}{2}$

o. $\frac{x+4}{x}$

p. $\log_2 x$

q. $\frac{x^2+6}{2x+3}$

r. $\sqrt[3]{\frac{x-2}{x}}$

s. $(x-3)(x-2)$

t. $\frac{(x-3)(x-2)}{x-4}$

u. $\frac{2}{3}x^3 + \frac{1}{2}x^2$

v. $\frac{x^2+2}{x^2+2}$

w. $\frac{x^2+2}{2}$

x. $\frac{x}{2x+25}$

y. $x^{\frac{1}{2}}$

z. $\sqrt{x^2}$

2. Which of the following sets are functions? For those which are functions indicate the domain and tell which of the names in Problem 1 can be applied correctly.

a. $R = \{(x, y) : x + y = 1\}$

b. $S = \{(x, y) : x^2 + y^2 = 1\}$

c. $T = \{(x, y) : y = x^2\}$

d. $U = \{(x, y) : y^2 = x\}$

e. $V = \left\{(x, y) : y = \frac{2}{x}\right\}$

f. $W = \{(x, y) : y = 2x\}$

g. $R = \{(x, y) : y = x^2 \text{ or } y = 2x\}$

h. $S = \{(x, y) : y > x\}$

i. $T = \left\{(x, y) : y = \frac{2}{x+2}\right\}$

j. $U = \left\{(r, s) : s = \frac{2}{r+2}\right\}$

k. $V = \{(r, s) : r^2 = s\}$

l. $W = \left\{(p, q) : q = 2 \text{ or } q = \frac{2p^2+2}{p^2+1}\right\}$

m. $Z = \{(p, q) : q \geq p \text{ and } q \leq p\}$

Some of these expressions for which we do not yet have a name arise in solving certain problems. Consider the following examples.

Example 1. The ratio of tenors to basses in a male chorus was $\frac{2}{5}$ and there were 15 more basses than tenors. How many tenors were there?

One way to solve this is to suppose that there were x tenors and then there were $x + 15$ basses, where of course x must represent a non-negative integer. This leads to the equation

$$\frac{x}{x+15} = \frac{2}{5}$$

The expression on the left appeared in Problem 1m above. Note, however, that it does not define the same function because in Problem 1m the domain was the set of all real numbers except -15, whereas here the domain is only the set of non-negative integers.

Example 2. A boy who can run x miles per hour carried a message for 3 miles from A to B and passed it to another boy who can run 1 mile per hour faster than the first boy. The second boy carried the message 5 miles from B to C. It took 1 hour and 45 minutes to deliver the message from A to C. How fast did each boy run?

The first boy ran for $\frac{3}{x}$ hours and since the second boy ran at $(x + 1)$ miles per hour, he ran for $\frac{5}{x+1}$ hours. We are led to the equation

$$\frac{3}{x} + \frac{5}{x+1} = \frac{7}{4}.$$

The expressions in the equation eliminate 0 and -1 from the domain, but the fact that x measures the speed of the boy reduces the domain to $\{x : x > 0\}$.

The expressions $\frac{x}{x+15}$, $\frac{3}{x}$, and $\frac{5}{x+1}$ which appear in these equations define what we call rational functions. In order to solve such equations readily we need to learn about rational functions and operations with them.

PROBLEMS 2-1B

Set up equations for solving the following problems but do not yet solve the equations. Indicate any restrictions which are imposed on the domain of the variable.

1. In a class where there are 4 more boys than girls the ratio of boys to girls is $\frac{6}{5}$. How many boys and how many girls are in the class?
2. In a box of red and black pencils there are 28 more black pencils than there are red pencils. The ratio of red pencils to all the pencils is $\frac{1}{9}$. How many red pencils are there?
3. A car travels 25 miles at an average speed of x miles per hour.

If its speed were increased by 5 m.p.h. it would complete the same journey in 15 minutes less time. What is the original speed of the car?

4. A man paid 60 shillings for some books which cost x shillings each, and 72 shillings for some more books which cost twice as much apiece. He bought 8 books altogether. What was the cost per book of each kind?

5. The area of a rectangle is 12 sq. yds. and the length of the base is x yds. Express the height of the rectangle in terms of x. Express the perimeter of the rectangle in terms of x. If the perimeter is 49 yds., what are the dimensions of the rectangle?

Let us now define exactly what we mean by a rational function of one variable.

> DEFINITION: A *rational function* of x is a function which can be defined by an expression of the form $\frac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial. The *domain* of the function is the set of all real numbers x for which $D(x) \neq 0$.

Remarks: The restriction on the domain is necessary because the symbol $\frac{N(x)}{D(x)}$ would be meaningless for values of x which make the denominator zero. Many rational functions will therefore be undefined for certain values of x. Their graphs will be *discontinuous* at those points, that is, there will be no point of the graph corresponding to these values of x.

We do not have to use the letter x for the variable in the definition. Any letter can be used because

$$\{x : x \text{ is a real number and } D(x) \neq 0\} = \{r : r \text{ is a real number and } D(r) \neq 0\}$$

Thus $\frac{x-1}{2x-1}$ and $\frac{y-1}{2y-1}$ and $\frac{r-1}{2r-1}$ define the same function, in each case the domain being the set of all real numbers except $\frac{1}{2}$.

Example: If $f(x) = \frac{3x-1}{x^2-9}$, then f is a *rational function*. In set notation this says that if $f = \left\{\left(x, f(x)\right) : f(x) = \frac{3x-1}{x^2-9}\right\}$ then f is a rational

function. The *domain* of f is the set of all real numbers except 3 and -3 since $x^2 - 9$ is zero for $x = 3$ and $x = -3$. The symbol $\frac{3x - 1}{x^2 - 9}$ is called a *rational expression*. When we assign to x a number from the domain, you will recall that the expression gives a *value of the function*. For example, if x is 5, we have $f(5) = \frac{3 \times 5 - 1}{5^2 - 9} = \frac{14}{16} = \frac{7}{8}$. Thus $\frac{7}{8}$ is the value function when x is 5. The ordered pair $\left(5, \frac{7}{8}\right)$ is a member of the set of ordered pairs which is the function.

PROBLEMS 2-1C

1. Which of the following expressions define rational functions? In each case indicate the largest possible domain.

a. $\frac{6x - 8}{5x + 10}$

b. $\frac{a^2}{3a - 1}$

c. $y^2 + 6y + 1$

d. $\frac{2a^3 - 5a + 8}{3a}$

e. $\log_2 x$

f. 2^x

g. $\frac{1}{\log_{10} x}$

h. $y + 3$

i. $\frac{1}{y + 3}$

j. $\frac{x^2 + 8x + 3}{5}$

k. $\frac{3}{a} + \frac{7}{a}$

l. $\sqrt{\frac{x - 2}{x + 2}}$

m. $3x$

n. $\frac{(a - 1)(a + 1)}{2a - 3}$

2. In PROBLEMS 2-1A, page 42, which ones give rational functions?

3. a. Is every polynomial function a rational function? Why?
   b. Is every rational function a polynomial function? Why?

4. If a rational function f is defined by $f(x) = \frac{x^2 + 8}{x - 4}$ $(x \neq 4)$, find the value of the function
   a. when x is 6.
   b. when x is 1.
   c. when x is 4.
   d. when x is 0.

5. Give some ordered pairs for the function of Problem 4.

6. Find a value of x which gives 6 for the value of the function defined by $y = \frac{x}{x - 1}$.

7. If $f(x) = \frac{x}{ax + 2}$, find a such that $f(3) = 7$.

8. $f(x) = \frac{bx + 2}{x^2 + b}$. Find b if $(-2,1)$ is an ordered pair of the function.

9. $f(x) = \frac{ax + b}{x - 1}$. Find a and b if $f(-1) = 2{\cdot}5$ and $f(2) = 1$.

10. a. The area of a triangle is 40 sq. miles and the base of the triangle is x miles long. Write an expression for the altitude of the triangle.

    b. The perimeter of a rectangle is 20 ft. and the length is x ft. Write an expression for the width. Write an expression for the area.

    c. A man can mow a lawn in x hours. His son would take two hours longer to do it. Write an expression for the fraction of the lawn mowed by the man in one hour; by the son in one hour.

    d. A train travels at an average rate of x miles per hour. Write an expression for the number of hours the train will take to go 340 miles.

    e. The area of a square is x square inches. Write an expression for the length in inches of a side of the square.

    f. Which of the expressions in parts a to e define functions? Which define rational functions? What is the domain in each case?

2-2 OPERATIONS WITH RATIONAL FUNCTIONS.

Since we shall need to operate with rational functions in solving certain problems, let us consider how we shall do so.

We saw earlier that when we multiply or divide or add or subtract any two *functions*, f and g, we find the *values* of the new function by performing the same operation on the *real numbers* which are the values of the given functions, f and g. A compact way of saying this for the sum is:

$$f + g = \{(x, f(x) + g(x))\}.$$

As we do this we must give the necessary attention to the domain. It is usually understood that if no restriction on the domain of a function is

specifically stated, the domain is the largest possible set of real numbers. Thus if we say "$f(x) = \frac{10}{x+5}$", then the domain of f is understood to be the set of all real numbers except -5.

If $f(x) = \frac{x}{3}$ and $g(x) = \frac{4x}{5}$, then the product function fg, is defined by the expression $\frac{x}{3} \times \frac{4x}{5}$ or $\frac{4x^2}{15}$. Since f and g both have the set of all real numbers for their domain, the domain of fg is also the set of all real numbers. We used the usual elementary properties of real numbers to simplify the expression.

Similarly the equation

$$\frac{3}{x+5} + \frac{7}{x+5} = \frac{10}{x+5} \qquad (x \neq -5)$$

shows that if two rational functions are defined by $\frac{3}{x+5}$ and $\frac{7}{x+5}$, then the sum function, $f+g$, is defined by $\frac{10}{x+5}$ and its domain is the set of all real numbers except -5.

In general if A, B, C, D are polynomials used in defining certain rational functions, the following familiar properties will be useful in performing operations with those functions.

$$\left.\begin{aligned}
\frac{A}{B} \times \frac{C}{D} &= \frac{AC}{BD} \\
\frac{B}{B} &= 1 \\
\frac{C}{D} \times 1 &= \frac{C}{D} \\
\frac{A}{B} + \frac{C}{B} &= \frac{A+C}{B} \\
\frac{A}{B} - \frac{C}{B} &= \frac{A-C}{B} \\
\frac{A}{B} \div \frac{C}{D} &= \frac{A}{B} \times \frac{D}{C}
\end{aligned}\right\}$$

where in all cases the domain is limited so that $B(x) \neq 0$ and $D(x) \neq 0$ and in the last one $C(x) \neq 0$.

Notice that the domain of $\frac{A}{B} \times \frac{C}{D}$ is the intersection of the domains of $\frac{A}{B}$ and $\frac{C}{D}$.

PROBLEMS 2-2A

Perform the indicated operations and indicate the restrictions on the domain of the resulting function.

1. $\frac{3}{8x} \times \frac{5}{7x}$
2. $\frac{3}{8x} \times \frac{8}{3x}$
3. $\frac{3}{8x} \times \frac{x}{3}$
4. $\frac{3}{8x} \div \frac{5}{7x}$
5. $\frac{3}{8x} \div \frac{x}{3}$
6. $\frac{3}{8x} + \frac{5}{8x}$
7. $\frac{x^2}{3} - \frac{x}{3}$
8. $\frac{3}{8x} + \frac{3}{4x}$ (Careful.)
9. $\frac{3}{8x} + \frac{x}{8}$
10. $\frac{3}{8x} - \frac{8x}{3}$
11. $\frac{6x}{x-2} \times \frac{3(x-2)}{9}$
12. $\frac{3}{x+1} \div \frac{x}{x+1}$

In these problems you probably used some of the properties of addition and multiplication listed above to help simplify the expressions. For example in Problem 2,

$$\frac{3}{8x} \times \frac{8}{3x} = \frac{3 \times 8}{8x \times 3x} = \frac{3 \times 8}{3 \times 8} \times \frac{1}{x^2} = \frac{1}{x^2} \qquad (x \neq 0)$$

you used, among others, the properties $\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}$, $\frac{B}{B} = 1$, and $\frac{C}{D} \times 1 = \frac{C}{D}$. In Problem 5, $\frac{3}{8x} \div \frac{x}{3} = \frac{3}{8x} \times \frac{3}{x} = \frac{9}{8x^2}$ $(x \neq 0)$, you used the definition of division. $\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C}$ because $\frac{D}{C}$ is the reciprocal of $\frac{C}{D}$. In Problem 8,

$$\frac{3}{8x} + \frac{3}{4x} = \frac{3}{8x} + \frac{3}{4x} \times \frac{2}{2} = \frac{3}{8x} + \frac{6}{8x} = \frac{9}{8x} \qquad (x \neq 0)$$

you used $\frac{B}{B} = 1$ again in order to prepare the fractions so you could use $\frac{A}{B} + \frac{C}{B} = \frac{A + C}{B}$.

These simple properties are at the root of all our work with rational functions.

For our present purposes we shall say that a rational expression is *simplified* if it is replaced by an equivalent rational expression which is a single fraction with no common factors in the numerator and the denominator. The simplified expression may sometimes be a polynomial.

PROBLEMS 2-2B

Point out the basic properties you use in simplifying the following expressions. In each case indicate in what domain the two expressions are equivalent.

Example: $\dfrac{2}{x+5} + \dfrac{3}{7(x+5)}$

$$= \frac{2}{x+5} \times \frac{7}{7} + \frac{3}{7(x+5)} \qquad \left(\frac{B}{B} = 1 \text{ and } \frac{C}{D} \times 1 = \frac{C}{D}\right)$$

$$= \frac{14}{7(x+5)} + \frac{3}{7(x+5)} \qquad \left(\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}\right)$$

$$= \frac{17}{7(x+5)} \qquad \left(\frac{A}{B} + \frac{C}{B} = \frac{A+C}{B}\right)$$

Domain: $\{x : x \neq -5\}$

1. $\dfrac{2y}{7} \times \dfrac{14}{3y}$
2. $\dfrac{3(a+9)}{a(a+9)}$
3. $\dfrac{4m^2+1}{3} \div \dfrac{4m^2+1}{m}$
4. $\dfrac{a+2}{6} + \dfrac{4-a}{6}$
5. $\dfrac{3y-4}{y-4} - \dfrac{y+2}{y-4}$
6. $\dfrac{x}{x^2+3} + \dfrac{1}{2(x^2+3)}$
7. $\dfrac{x}{2(x+1)} + \dfrac{1}{3(x+1)}$
8. a. Is the sum of two rational functions always a rational function? Explain why, using $\dfrac{N(x)}{D(x)}$ and $\dfrac{P(x)}{Q(x)}$ to represent the values of the functions, where $N(x)$, $D(x)$, $P(x)$, $Q(x)$ are polynomials.
   b. Answer the same question for the difference.
   c. Product
   d. Quotient
   e. What word do we use to name the property suggested here?

9. a. If f and g are rational functions, is $f+g$ always the same function as $g+f$?

b. Demonstrate using $f(x) = \dfrac{x+3}{x}$ and $g(x) = \dfrac{x}{x-2}$.

c. Demonstrate using $f(x) = \dfrac{N(x)}{D(x)}$ and $g(x) = \dfrac{P(x)}{Q(x)}$.

d. What name do we have for this property?

10. a. Is there a unique rational function g such that for any rational function f, it is true that $f+g=f$?

b. Answer the same question for $f \times g = f$.

2-3 FACTORIZATION IN SIMPLIFYING RATIONAL EXPRESSIONS.

In simplifying the expressions in 2-2B you recognized factors such as $(a+9)$, (x^2+3), and simpler factors like 2 and a. The trouble is that sometimes it is not so easy to see the factors we need. For example, how would you simplify $\dfrac{3a+12}{a^2-16} \times \dfrac{a^2-8a+16}{3a}$? You see that you need to recall your methods of factorization.

$$\frac{3a+12}{a^2-16} \times \frac{a^2-8a+16}{3a} = \frac{3(a+4)}{(a+4)(a-4)} \times \frac{(a-4)(a-4)}{3a}$$

$$= \frac{3}{3} \times \frac{a+4}{a+4} \times \frac{a-4}{a-4} \times \frac{a-4}{a}$$

$$= \frac{a-4}{a} \qquad (a \neq 0,\ a \neq 4,\ a \neq -4)$$

Usually it is not necessary to write any more steps than those shown above, but perhaps this time we should show some detailed steps which you will do mentally without writing them.

$$\frac{3a+12}{a^2-16} \times \frac{a^2-8a+16}{3a} \qquad (a \neq 0,\ a \neq 4,\ a \neq -4)$$

$$= \frac{3(a+4)}{(a+4)(a-4)} \times \frac{(a-4)(a-4)}{3a}$$ [Factorize numerators and denominators wherever possible.]

$$= \frac{3(a+4)(a-4)(a-4)}{(a+4)(a-4)3a} \qquad \left[\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}\right]$$

$$= \frac{3(a+4)(a-4)(a-4)}{3(a+4)(a-4)a}$$ [Commutative and associative properties of multiplication.]

$$= \frac{3}{3} \times \frac{a+4}{a+4} \times \frac{a-4}{a-4} \times \frac{a-4}{a} \qquad \left[\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}\right]$$

$$= 1 \times 1 \times 1 \times \frac{a-4}{a} \qquad \left[\frac{B}{B} = 1\right]$$

$$= \frac{a-4}{a} \qquad (a \neq 0,\ a \neq 4,\ a \neq -4) \qquad \left[\frac{C}{D} \times 1 = \frac{C}{D}\right]$$

Hereafter let us agree that $(a \neq 0, 4, -4)$ means $(a \neq 0,\ a \neq 4,\ a \neq -4)$. It is understood, of course, that this domain restriction is in force at every step. This insures that $B(a) \neq 0$ and $D(a) \neq 0$. Only by showing the complete restrictions on the domain at the end can we say that the rational function given by the final expression is the same function as the one we started with.

To prepare for this work with rational functions you may want to use the following set of problems as a reminder about factorization of polynomials. Practice as much as you need.

PROBLEMS 2-3A

Factorize the following polynomials.

1. $6y^2 + 7y$
2. $6y^2 + 10y$
3. $y^2 + 10y + 16$
4. $y^2 + 10y + 25$
5. $y^2 + 24y - 25$
6. $y^2 - 25$
7. $9y^2 - 25$
8. $9a^3 - 15a^2 - 12a$
9. $a^2 - 4a - 12$
10. $a^2 - 7a + 12$
11. $3a^2 - 14a + 8$
12. $7a^2 + a - 6$
13. $3a^3 - 12a$
14. $a^4 - 16$

Now we can proceed with a greater variety of rational functions. Observe how the factors help.

Example 1. $$\frac{a^2 - 7a + 12}{a^3 - 9a} = \frac{(a-3)(a-4)}{a(a+3)(a-3)}$$

$$= \frac{a-3}{a-3} \times \frac{a-4}{a(a+3)}$$

$$= \frac{a-4}{a(a+3)} \qquad (a \neq 0, 3, -3)$$

Example 2. $\dfrac{6y^2 + 12y}{y^2 + y} \times \dfrac{y^2 - 1}{y^2 + 10y + 16} = \dfrac{6y(y + 2)}{y(y + 1)} \times \dfrac{(y + 1)(y - 1)}{(y + 2)(y + 8)}$

$$= \frac{y}{y} \times \frac{y + 2}{y + 2} \times \frac{y + 1}{y + 1} \times \frac{6(y - 1)}{y + 8}$$

$$= \frac{6y - 6}{y + 8} \qquad (y \neq 0, -1, -2, -8)$$

Example 3. $\dfrac{a}{a^2 - 4} + \dfrac{a}{a^2 - 2a} = \dfrac{a}{(a - 2)(a + 2)} \times \dfrac{a}{a} + \dfrac{2}{a(a - 2)} \times \dfrac{a + 2}{a + 2}$

$$= \frac{a^2}{a(a - 2)(a + 2)} + \frac{2a + 4}{a(a - 2)(a + 2)}$$

$$= \frac{a^2 + 2a + 4}{a(a - 2)(a + 2)} \qquad (a \neq 0, 2, -2)$$

PROBLEMS 2-3B

Simplify the following expressions. Remember to indicate the domain which is necessary in the final expression if it is to be equivalent to the original expression.

1. $\dfrac{x^2 - 49}{x^2 - 7x}$

2. $\dfrac{a}{a^2 - 36} \times \dfrac{3a + 18}{3a}$

3. $\dfrac{y^2 - 4y + 4}{y^2 - 2y} \div \dfrac{y^2 - 4}{2y + 4}$

4. $\dfrac{m}{m^2 + 6m + 5} + \dfrac{5}{m + 5}$

5. $\dfrac{x^2 - 12x + 35}{2x - 14} \times \dfrac{x - 2}{x^2 - 5x}$

6. $\dfrac{x - 2}{x} - \dfrac{2}{x^2 - 7x}$

7. $\dfrac{5a^2 - 4a - 1}{a^2 - 3a + 2} \times \dfrac{8a + 24}{20a + 4}$

8. $\dfrac{x^2 + 9}{x^2 + 6x + 9} + \dfrac{3}{x + 3}$

In the following expression how shall we find the least common multiple (L.C.M.) of the denominators in order to be able to perform the addition?

$$\frac{x}{x^2 - 5x + 6} + \frac{2}{x^2 - 3x} + \frac{4}{x^2 - 2x}$$

If we factorize the denominators we find that:

$$x^2 - 5x + 6 = (x - 3)(x - 2)$$
$$x^2 - 3x = x(x - 3)$$
$$x^2 - 2x = x(x - 2)$$

A multiple of the first of these must have the factors $(x - 3)$ and $(x - 2)$.
A multiple of the second must have the factors x and $(x - 3)$.
A multiple of the third must have the factors x and $(x - 2)$.

The L.C.M. of all three of these must then contain the factors $(x - 3)$ and $(x - 2)$ and x.

$$\text{L.C.M.} = x(x - 3)(x - 2)$$

$$\frac{x}{(x-3)(x-2)} \times \frac{x}{x} + \frac{2}{x(x-3)} \times \frac{x-2}{x-2} + \frac{4}{x(x-2)} \times \frac{x-3}{x-3}$$

$$= \frac{x^2 + 2x - 4 + 4x - 12}{x(x-3)(x-2)}$$

$$= \frac{x^2 + 6x - 16}{x(x-3)(x-2)}$$

$$= \frac{(x+8)(x-2)}{x(x-3)(x-2)}$$

$$= \frac{x+8}{x(x-3)} \qquad (x \neq 3, 2, 0)$$

In general the L.C.M. of the denominators can be found as the product of the different prime polynomial factors in the various denominators. If the same factor occurs more than once in any one denominator, the L.C.M. must contain it the greatest number of times it occurs in any *one* denominator. For example:

$$\frac{1}{2x^3} + \frac{3}{x^2 + 6x} + \frac{4}{x^2 + 12x + 36}$$

$$= \frac{1}{2x^3} + \frac{3}{x(x+6)} + \frac{4}{(x+6)^2}$$

The L.C.M. of the denominators is $2x^3(x + 6)^2$.

$$\frac{1}{2x^3} \times \frac{(x+6)^2}{(x+6)^2} + \frac{3}{x(x+6)} \times \frac{2x^2(x+6)}{2x^2(x+6)} + \frac{4}{(x+6)^2} \times \frac{2x^3}{2x^3}$$

$$= \frac{x^3 + 12x + 36 + 6x^3 + 36x^2 + 8x^3}{2x^3(x+6)^2}$$

$$= \frac{14x^3 + 37x^2 + 12x + 36}{2x^3(x+6)^2} \qquad (x \neq 0, -6)$$

PROBLEMS 2-3C

Simplify the following expressions and indicate the domain in each case.

1. $\dfrac{x}{x^2 - 1} + \dfrac{3}{x^2 - 6x + 5}$

2. $\dfrac{a + 3}{a} + \dfrac{5}{a^2 - 6a} + \dfrac{a}{6a - 36}$

3. $\dfrac{y}{y^2 - 4y + 4} + \dfrac{3}{y^2 - 7y + 10}$

4. $\dfrac{a}{5a^2 - 20a - 60} - \dfrac{2}{a^3 - 4a^2 - 12a}$

5. $\dfrac{\dfrac{a^2 + 3a - 10}{5a - 10}}{\dfrac{a^2 - 25}{a + 5}}$ $\left(\text{Use } \dfrac{A}{B} \div \dfrac{C}{D} = \dfrac{A}{B} \times \dfrac{D}{C}\right)$

6. $\dfrac{\dfrac{a}{5} + \dfrac{5}{a}}{\dfrac{2a}{25} - \dfrac{25}{a^2}}$ (The L.C.M. of all the denominators is $25a^2$. Multiply by $\dfrac{25a^2}{25a^2}$.)

7. $\dfrac{\dfrac{1}{x - 3} + \dfrac{3}{x - 1}}{\dfrac{2}{x^2 - 4x + 3} + \dfrac{4}{x - 3}}$ (Use method of Problem 6.)

2-4 RATIONAL FUNCTIONS OF TWO OR MORE VARIABLES.

Rational functions of more than one variable do not present any new difficulties. For example,

$$f(a, b) = \frac{a^2 - b^2}{a^2 - 2ab + b^2}$$

represents the values of the function f for all ordered pairs of real numbers (a, b) except those where $a = b$. Thus

$$f(2,5) = \frac{4 - 25}{4 - 20 + 25} = \frac{-21}{9} = -\frac{7}{3}.$$

If we try to write $f(5,5)$, we obtain $\dfrac{25 - 25}{25 - 50 + 25} = \dfrac{0}{0}$ which is a meaningless symbol. Division by zero is undefined. Therefore $f(5,5)$ is undefined and $(5,5)$ is not in the domain of the function.

Of course if we are smart we notice that

$$f(a,b) = \frac{(a-b)(a+b)}{(a-b)(a-b)} = \frac{a+b}{a-b} \qquad (a \neq b)$$

Then $f(2,5)$ is $\frac{2+5}{2-5}$ or $-\frac{7}{3}$.

Operations with functions of two or more variables are just like operations with functions of one variable.

PROBLEMS 2-4

1. $f(x,y) = \frac{x^2 - 7xy + 12y^2}{x^2 - 9y^2}$

   a. Find $f(3,2)$ and $f(-2,1)$.

   b. Simplify $f(x,y)$ and then find $f(3,2)$ and $f(-2,1)$ from the new expression.

   c. Why is $f(3,1)$ undefined?

   d. Give the domain of f.

2. Simplify the following and give the domain.

   a. $\frac{x-y}{y} \times \frac{2y}{x^2 - y^2}$

   b. $\frac{a+b}{a-b} + \frac{a-b}{a+b}$

   c. $\frac{\frac{1}{a} - \frac{1}{b}}{\frac{1}{a} + \frac{1}{b}}$

   d. $\frac{ab - b^2}{ab} \times \frac{a^2 - 3ab + 2b^2}{a^2 - 2ab + b^2}$

   e. $\frac{x+3y}{2x} - \frac{y+2x}{3x}$

   f. $\frac{a}{a+b+c} + \frac{b}{2a+2b+2c} + \frac{c}{3a+3b+3c}$

3. In the following familiar formulas, if the variable on the left represents numbers in the range, which ones define rational functions of two or more variables?

   a. $A = bh$

   b. $A = \pi r^2$

   c. $P = 2\ell + 2w$

   d. $A = \frac{1}{2}bh$

   e. $V = abc$

   f. $V = Bh$

g. $V = s^3$

h. $A = \frac{1}{2}h(b_1 + b_2)$

i. $C = 2\pi r$

j. $D = RT$

k. $I = Prt$

l. $C = \frac{5}{9}(F - 32)$

2-5 EQUATIONS INVOLVING RATIONAL EXPRESSIONS.

Now let us return to the question of solving certain problems by means of equations which involve rational expressions. On page 44 we wanted to solve the equation

$$\frac{3}{x} + \frac{5}{x+1} = \frac{7}{4} \qquad (x > 0)$$

The L.C.M. of all the denominators is $4x(x + 1)$. It would seem helpful to make all the fractions have the same denominator.

$$\frac{3}{x} \times \frac{4(x+1)}{4(x+1)} + \frac{5}{x+1} \times \frac{4x}{4x} = \frac{7}{4} \times \frac{x(x+1)}{x(x+1)}$$

$$\frac{12x + 12}{4x(x+1)} + \frac{20x}{4x(x+1)} = \frac{7x^2 + 7x}{4x(x+1)}$$

Now notice that since the numbers 0 and -1 are not in the domain, the number $4x(x + 1)$ *cannot be zero*. We therefore are permitted to use one of our properties of equality which says:

Multiplying both sides of an equation by a *non-zero* real number gives an *equivalent* equation.

Can you see that multiplying by $4x(x + 1)$ will immediately give the equation

$$12x + 12 + 20x = 7x^2 + 7x \quad ?$$

To complete the job we now have to solve a quadratic equation. Do you remember how to solve quadratic equations? See whether you can explain each step in the following chain of equivalent equations. We recall at the end the restriction on domain which has been in force right from the start of this problem.

$$12x + 12 + 20x = 7x^2 + 7x$$

$$0 = 7x^2 + 7x - 12x - 12 - 20x$$

$$7x^2 - 25x - 12 = 0$$

$$(7x + 3)(x - 4) = 0$$

$$7x + 3 = 0 \quad \text{or} \quad x - 4 = 0$$

$$x = -\frac{3}{7} \quad \text{or} \quad x = 4 \qquad (x > 0)$$

The number $-\frac{3}{7}$ is not in the domain, of course, because we are talking about the speed at which a boy runs.

We conclude, then, that the first boy ran at 4 miles per hour and the second boy at 5 miles per hour.

Let us check to see that our solution really does satisfy the conditions of the problem. If the first boy runs at 4 miles per hour and the second boy at 5 miles per hour, then (a) it is true that the second boy runs one mile per hour faster than the first boy, and (b) the first boy runs for $\frac{3}{4}$ hours and the second boy runs for $\frac{5}{5}$ hours or 1 hour, so the total time is $1\frac{3}{4}$ hours as was required.

The other problem we set up on page 43 led to a simpler equation which we can solve as follows:

$$\frac{x}{x + 15} = \frac{2}{5} \qquad (x \text{ is a non-negative integer})$$

$$\frac{x}{x + 15} \times \frac{5}{5} = \frac{2}{5} \times \frac{x + 15}{x + 15}$$

$$5x = 2x + 30$$

$$3x = 30$$

$$x = 10 \qquad (x \text{ is a non-negative integer})$$

Therefore there were 10 tenors.

To check we observe that if there were 10 tenors, there were 10 + 15 or 25 basses. The ratio of tenors to basses was $\frac{10}{25}$ or $\frac{2}{5}$, as was required.

Since some of the equations in rational expressions lead to quadratic equations, a brief revision of quadratic equations may be helpful. Do what you need of the following problem set.

PROBLEMS 2-5A

1. In the equation $(x - 3)(x + 8) = 0$ what important principle is used to obtain the equivalent sentence

$$x - 3 = 0 \quad \text{or} \quad x + 8 = 0?$$

In Problems 2 to 15 solve by the method of factoring.

Example:

$$x^2 = 6x + 7$$
$$x^2 - 6x - 7 = 0$$
$$(x - 7)(x + 1) = 0$$
$$x - 7 = 0 \text{ or } x + 1 = 0$$
$$x = 7 \text{ or } x = -1$$

The truth set is $\{7, -1\}$.

2. $(2x - 5)(x - 2) = 0$
3. $6x(x + 4) = 0$
4. $y^2 - 9y = 0$
5. $y^2 - 9 = 0$
6. $a^2 - 10a - 11 = 0$
7. $x^2 - 14x + 24 = 0$
8. $4a^2 + 5a + 1 = 0$
9. $3x^2 + 13x - 10 = 0$
10. $y^2 + 12y + 36 = 0$
11. $4a^2 - 12a + 9 = 0$
12. $25x^2 - 49 = 0$
13. $x^2 = 2x + 15$
14. $9x = 2x^2 + 9$
15. $a^2 + a - 21 = 100 + a$
16. If $x^2 = 13$, an equivalent sentence is $x = \sqrt{13}$ or $x = -\sqrt{13}$. *Why?*
17. If $(x - 6)^2 = 5$, an equivalent sentence is $x - 6 = \sqrt{5}$ or $x - 6 = -\sqrt{5}$. What is the truth set?
18. If $x^2 - 6x = 4$, what number must be added to both sides to complete the square on the left? What is the truth set of this equation? In Problems 19-28 solve by completing the square.

Example:

$$x^2 + 8x + 4 = 0$$
$$x^2 + 8x = -4$$
$$x^2 + 8x + 16 = -4 + 16$$
$$(x + 4)^2 = 12$$
$$x + 4 = \sqrt{12} \text{ or } x + 4 = -\sqrt{12}$$
$$x = -4 + 2\sqrt{3} \text{ or } x = -4 - 2\sqrt{3}$$

The truth set is $\{-4 + 2\sqrt{3}, -4 - 2\sqrt{3}\}$

19. $a^2 - 10a - 5 = 0$
20. $y^2 + 4y + 1 = 0$
21. $x^2 - 12x + 4 = 0$
22. $x^2 + 3x - 5 = 0$
23. $a^2 - 7 = 2a$
24. $x^2 = 18x - 80$
25. $y^2 + 6y + 12 = 0$
26. $2x^2 + 16x - 8 = 0$
27. $2y^2 + 12y + 5 = 0$
28. $2y^2 - 14y - 1 = 0$

Now we are prepared to solve some equations involving rational expressions. Do it by making all the denominators alike and then multiplying by the common denominator. Watch the domain carefully.

Example:

$$\frac{x^2 + x + 2}{2x - 2} = \frac{2x}{x - 1} \quad (x \neq 1)$$

$$\frac{x^2 + x + 2}{2(x - 1)} = \frac{2x}{x - 1} \times \frac{2}{2}$$

$$x^2 + x + 2 = 4x$$

$$x^2 - 3x + 2 = 0$$

$$(x - 2)(x - 1) = 0$$

$$x = 2 \text{ or } x = 1 \quad (x \neq 1)$$

Since 1 is not in the domain, the truth set is $\{2\}$.

PROBLEMS 2-5B

1. a. Find the truth set of $\frac{x}{x - 4} = \frac{6}{5}$. ($x$ is an integer greater than 4.)

   b. Give the answers to Problem 1 on page 44.

2. a. Find the truth set of $\frac{x}{2x + 28} = \frac{1}{9}$. ($x$ is a positive integer.)

   b. Give the answer to Problem 2 on page 44.

3. a. Find the truth set of $\frac{25}{x} - \frac{25}{x + 5} = \frac{1}{4}$. ($x > 0$)

   b. Give the answer to Problem 3 on page 44.

4. a. Find the truth set of $\frac{60}{x} + \frac{72}{2x} = 8$. ($x$ is positive and a multiple of the smallest legal unit of money.)

   b. Give the answers to Problem 4 on page 45.

5. a. Find the truth set of $2\left(x + \frac{12}{x}\right) = 49$. ($x > 0$)

   b. Give the answer to Problem 5 on page 45.

6. Find the truth set of

$$\frac{1}{a - 4} + \frac{1}{a^2 - 16} = \frac{10}{a + 4}.$$

7. Find the truth set of

$$\frac{6}{x^2 + 3x - 4} = \frac{3}{5x - 5} + \frac{2}{5}.$$

8. Find the truth set of

$$\frac{2a - 9}{2a - 14} = \frac{3a}{a^2 - 7a} - \frac{1}{2a - 14} \quad \text{(Be careful.)}$$

9. Find the truth set of $\frac{x - 3}{x} = 0$.

10. *Challenge Problem*

Consider the inequality $\frac{x-3}{x} > 0$.

a. Which of the following numbers are in the truth set?
-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5

b. We would like to obtain a simpler equivalent inequality by multiplying by x, but it makes a difference whether x is positive or negative. Let us take it by cases. Suppose x is positive. Then what equivalent inequality results from multiplying both sides by x? What is the truth set of this inequality? Remember that this is only for $x > 0$.

c. Now suppose that x is negative. What equivalent inequality results now from multiplying both sides by x? What numbers does this give for the truth set? Remember that this is only for $x < 0$.

d. We could combine the results of b and c to say that the truth set of the inequality $\frac{x-3}{x} > 0$ is $\{x \mid x > 3 \text{ or } x < 0\}$. Verify that your answers in part a are in agreement with this statement.

11. *Challenge Problem*

Use the method suggested in Problem 10 to solve the following inequalities.

Example: $\frac{2x-1}{x} > 3$

$x > 0$ and $2x - 1 > 3x$	or	$x < 0$ and $2x - 1 < 3x$
$x > 0$ and $-1 > x$		$x < 0$ and $-1 < x$
There is no number for which this is true.		This can be written $-1 < x < 0$

The truth set is $\{x : -1 < x < 0\}$

a. $\frac{x+4}{x} < 0$

b. $\frac{x-1}{x} > 2$

c. $\frac{x-2}{3x} < 1$

d. $\frac{x}{x-1} < 3$

(When $x - 1 > 0$, what values has x? When $x - 1 < 0$, what values has x?)

e. $\frac{1}{x+2} > 4$

In solving equations involving rational expressions we have seen that to obtain a simpler but equivalent equation without fractions we at one point multiply both sides of the equation by the L.C.M. of the denominators. We can somewhat shorten the process by multiplying by this L.C.M. *immediately* instead of first modifying the form of the fractions. For example:

$$\frac{3}{x} + \frac{5}{x+1} = \frac{7}{4} \qquad (x \neq 0, -1)$$

The L.C.M. is $4x(x+1)$

$$4x(x+1)\left(\frac{3}{x} + \frac{5}{x+1}\right) = 4x(x+1)\left(\frac{7}{4}\right)$$

$$4x(x+1)\left(\frac{3}{x}\right) + 4x(x+1)\left(\frac{5}{x+1}\right) = 4x(x+1)\left(\frac{7}{4}\right)$$

$$4(x+1)(3) + 4x(5) = x(x+1)(7)$$

$$12x + 12 + 20x = 7x^2 + 7x \qquad (x \neq 0, -1)$$

The remainder of the solution is as before. Notice carefully how the distributive property was used on the left and how we used the fact that $\frac{x}{x} = 1$, $\frac{x+1}{x+1} = 1$, $\frac{4}{4} = 1$ for values of x in the domain. It is usually not necessary to write all of these steps. We might write only

$$\frac{3}{x} + \frac{5}{x+1} = \frac{7}{4} \qquad (x \neq 0, -1)$$

$$4x(x+1)\left(\frac{3}{x} + \frac{5}{x+1}\right) = 4x(x+1)\left(\frac{7}{4}\right)$$

$$12x + 12 + 20x = 7x^2 + 7x \qquad (x \neq 0, -1)$$

PROBLEMS 2-5C

Solve the following equations and inequalities using, if you wish, the shorter form of solution. Watch the domain.

1. $\frac{a}{5-a} = \frac{4}{5}$

2. $\frac{3}{x} + \frac{14}{x^2} = \frac{2}{3} + \frac{1}{3x}$

3. $\frac{9}{x^2 - 4} = \frac{5}{x} - \frac{4}{x+2}$

4. $2 + \frac{6}{x^2 - 11x + 10} = \frac{-13}{2x - 2}$

5. $\frac{1}{3} = \frac{7}{3x + 9} - \frac{x}{x^2 + 6x + 9}$

6. $\frac{3}{x} + \frac{1}{3} < 0$

7. $\frac{x + 7}{6x} - \frac{3}{2} > 1$

8. $\frac{1}{1 + \frac{1}{x}} = \frac{4}{5}$

9. $\frac{\frac{10}{x} + 3}{\frac{5}{4} + 4} = \frac{1}{6 - x}$

10. a. Solve the equation $\frac{2}{x} + \frac{5}{3} = \frac{7}{3x}$.

b. Simplify the expression $\frac{2}{x} + \frac{5}{3} - \frac{7}{3x}$.

c. In Part a should you multiply by $3x$? *Why*?

d. In Part b should you multiply by $3x$? *Why*?

e. Discuss the difference between an *equation* as in Part a and an *expression* as in Part b.

11. a. Solve: $\frac{3}{x} = \frac{5}{x - 2}$

b. Simplify: $\frac{3}{x} - \frac{5}{x - 2}$

12. a. Simplify: $\frac{12}{x^2} - \frac{5}{x - 5} + \frac{3}{x^2 - 5x}$

b. Solve: $\frac{5}{x - 5} = \frac{12}{x^2} + \frac{3}{x^2 - 5x}$

13. a. Solve: $\frac{4}{a^2 - a - 6} = \frac{2}{3a - 9} - \frac{1}{3a + 6}$

b. Simplify: $\frac{4}{a^2 - a - 6} - \frac{2}{3a - 9} + \frac{1}{3a + 6}$

14. Here are two equations:

(a) $\frac{x^3 - 1}{x - 1} = 7$ (b) $x^3 - 1 = 7(x - 1)$

Which of the numbers -3, -1, 1, 2 satisfies (a)?

Which of the numbers -3, -1, 1, 2 satisfies (b)?

Are equations (a) and (b) equivalent equations? Why?

15. A positive number of the form $\frac{1}{x}$ is added to the number $\frac{1}{x+2}$. The sum is 1. Find the number.

In each of the next five problems be sure to state the restrictions on the domain of the variable you use.

16. A contractor has to move 72 tons of sugar cane from the field to the factory. If he uses his small lorry he has to make a certain number of trips, but if he uses a larger lorry which can carry 2 more tons per trip, he would have to make 3 fewer trips. How many tons can the smaller lorry carry?

17. Triangle ABC is similar to triangle DEF and $AC = x + 3$, $AB = x + 6$, $DF = 2x + 4$, $DE = 3x + 6$. Find the measure of AC, AB, DF, and DE. Remember that because the triangles are similar,

$$\frac{AC}{DF} = \frac{AB}{DE}.$$

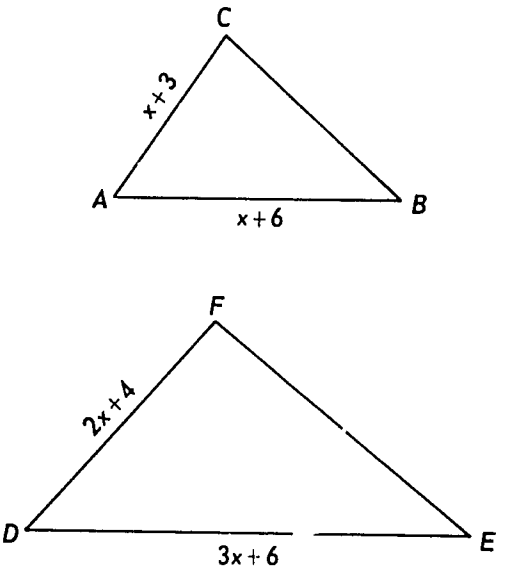


Fig. 1.

18. A building casts a shadow 40 ft. long. A flagpole 8 ft. tall is on top of the building. The end of its shadow at the same time is 50 ft. from the building. Find the height of the building. (Fig. 2 on page 65)

19. A boy would take x hours to do a certain job if he worked alone, and another boy who is a slower worker would need 4 more hours to do it. If they work together on it for 6 hours they can finish the job. How many hours would the first boy need to do it alone?

20. *Challenge Problem.*

A man and a boy pick bananas. The man can pick twice as many pounds per hour as the boy if they both pick steadily. They work for one hour but the boy takes a rest in the meantime. At the end of the hour the man has picked $\frac{3}{4}$ of the bananas gathered. How long did the boy rest?

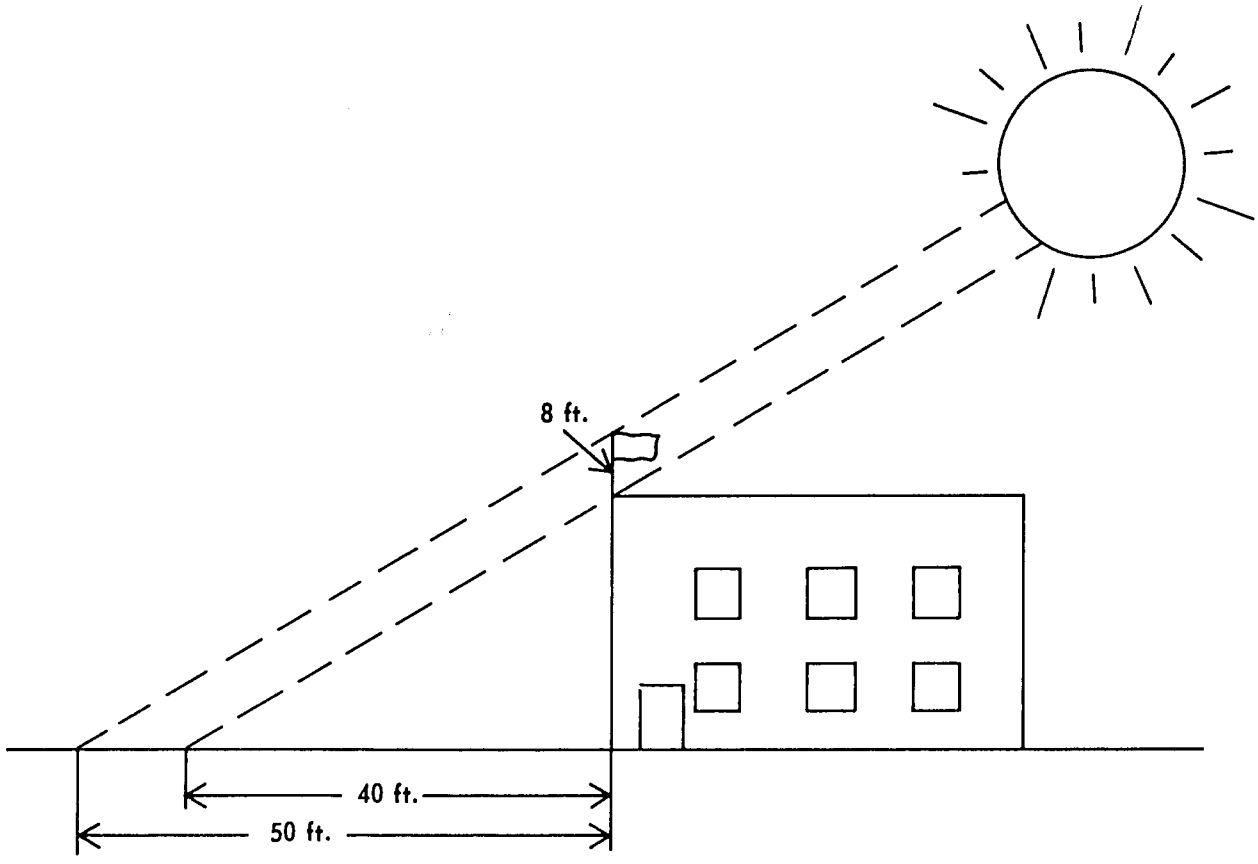


Fig. 2.

2-6 GRAPHS OF RATIONAL FUNCTIONS.

Recall that the graph of a function is the set of points corresponding to the set of ordered pairs which is the function. We have seen graphs of linear and quadratic functions and other polynomial functions and even some exponential and logarithmic functions. The graphs of rational functions do some interesting things which we should now like to explore.

Suppose we want the graph of $y = \frac{3}{x}$. A partial table of values might be as follows.

$y = \frac{3}{x}$	x	-5	-4	-3	-2	-1	0	1	2	3	4	5
	y	$-\frac{3}{5}$	$-\frac{3}{4}$	-1	$-\frac{3}{2}$	-3		3	$\frac{3}{2}$	1	$\frac{3}{4}$	$\frac{3}{5}$

The points corresponding to these ordered pairs would look like this.

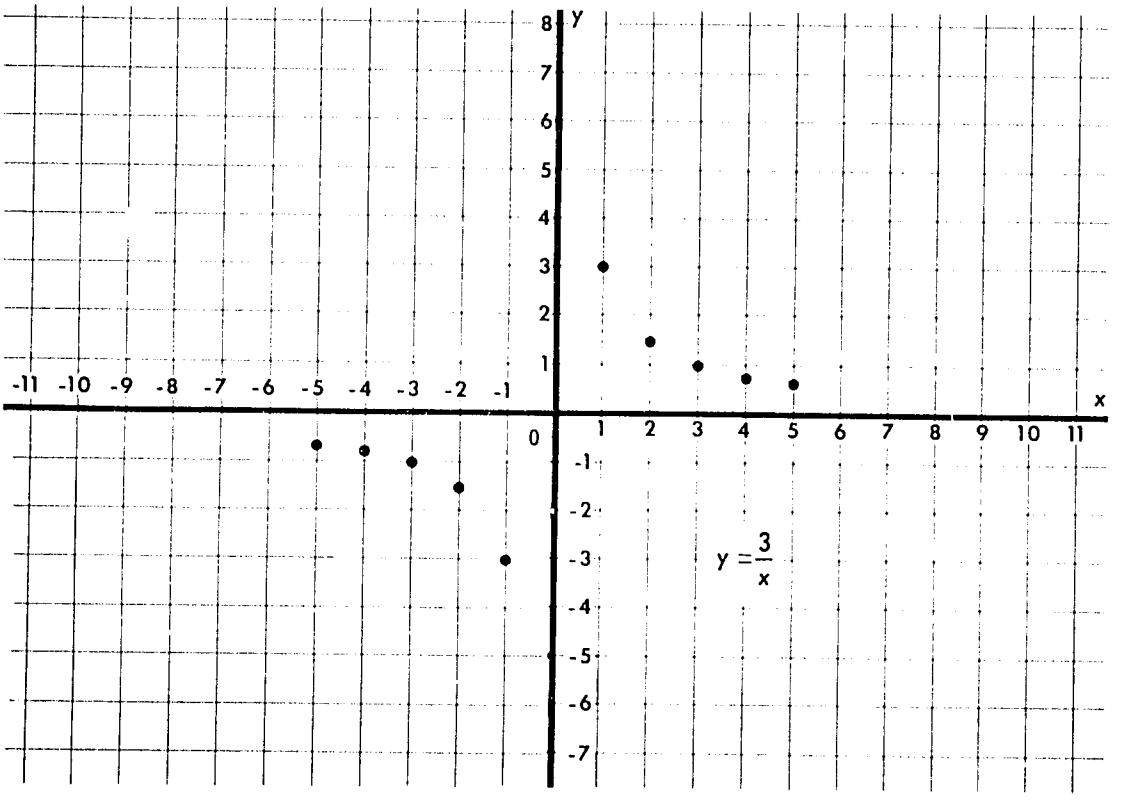


Fig. 3.

You noticed that no value of y was filled in for $x = 0$. Why? Since there is no division by zero and y is therefore undefined when $x = 0$, there can be no point on the graph when $x = 0$. As we noted in Section 2-1, the graph is said to be discontinuous at $x = 0$. This is a peculiar situation and we naturally wonder what happens to the graph near this peculiarity. To find out we should take some values of x *near* zero. Intelligent choice of these values helps keep the computation easy.

$y = \frac{3}{x}$	x	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
	y	-4	-6	-12	-24	24	12	6	4

Now our graph looks like that shown in Fig. 4. Since zero is the only value of x for which this function is undefined, we would expect the graph elsewhere to be a continuous curve, so we now join the points we have plotted with a smooth curve as in Fig. 5. Note that the graph never touches the vertical line $x = 0$.

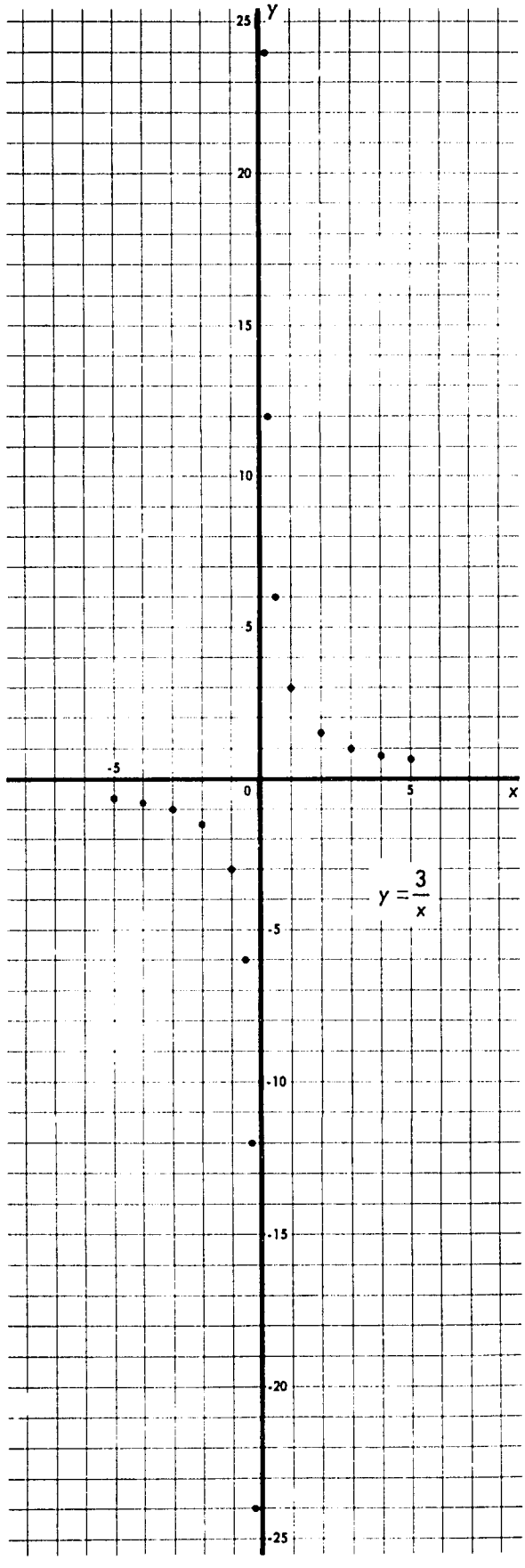

y
25
20
15
10
5
-5
0
5
x
-5
-10
-15
-20
-25
$y = \frac{3}{x}$

Fig. 4.

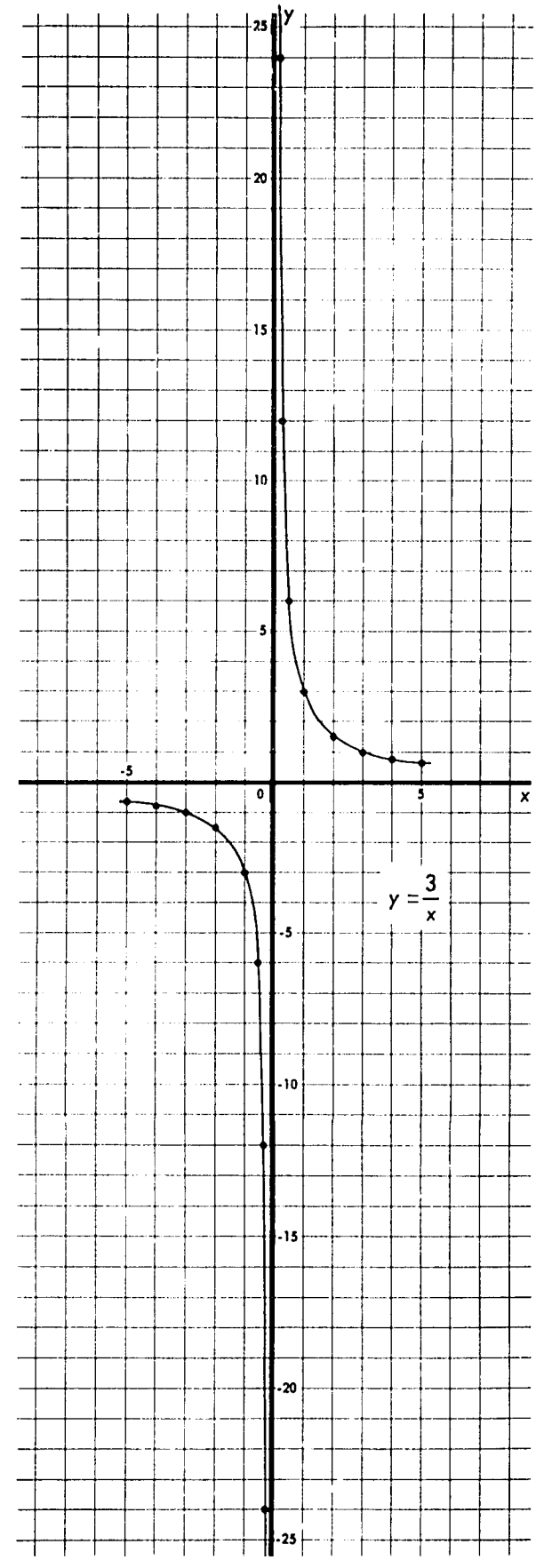

y
25
20
15
10
5
-5
0
5
x
-5
-10
-15
-20
-25
$y = \frac{3}{x}$

Fig. 5.

PROBLEMS 2-6A

1. Which of the following points lie on the graph of $y = \frac{1}{x - 2}$?

$\left(4, \frac{1}{2}\right)$, $\left(-4, -\frac{1}{2}\right)$, $\left(0, -\frac{1}{2}\right)$, $(8, 1)$, $\left(-1, -\frac{1}{3}\right)$

$(2, 0)$, $\left(\frac{1}{2}, -\frac{2}{3}\right)$, $\left(\frac{3}{2}, -2\right)$, $\left(7, \frac{1}{9}\right)$, $(1, -1)$, $(\sqrt{5}, \sqrt{5} + 2)$

2. Which of the following points lie on the graph of $y = \frac{2x - 3}{x + 4}$?

$\left(0, -\frac{3}{4}\right)$, $(1, 5)$, $(-3, -9)$, $(-5, -13)$, $\left(2, \frac{1}{6}\right)$

$\left(\frac{1}{6}, 2\right)$, $(-4, 11)$, $\left(\frac{1}{2}, -\frac{4}{9}\right)$, $\left(\frac{3}{2}, 0\right)$.

3. Draw the graph of $y = \frac{1}{x - 2}$ following the procedure illustrated above for $y = \frac{3}{x}$. Note carefully where the function is undefined and find enough ordered pairs near that point to help see the behaviour of the graph.

4. Suppose we were able to extend indefinitely the paper on which we have drawn the graph of $y = \frac{1}{x - 2}$. Find values to complete the following table for this function and describe where the resulting points will be.

$y = \frac{1}{x - 2}$	x	12	50	88	102	1002	10,002	1,000,002
	y							

$y = \frac{1}{x - 2}$	x	-12	-50	-88	-98	-998	-9998	-999,998
	y							

5. Do as in Problem 4 for this table.

$y = \frac{1}{x - 2}$	x	1·5	1·9	1·99	1·999	1·99999
	y					

$y = \frac{1}{x - 2}$	x	2·5	2·1	2·01	2·001	2·00001
	y					

In Problems 6, 7, 8, 9 use the procedure in Problems 3, 4, 5 to find and discuss the graph, that is,

(a) Make a table of values for some integral values of x.

(b) Observe for what value of x the function is not defined.

(c) Make a table of values for x near that value for which the function is undefined.

(d) Consider values of x off the paper such as 1,000,000 and -1,000,000 and observe the behaviour of y for these extreme values.

(e) Similarly consider what will happen to y as x takes values closer and closer to the value for which the function is undefined.

(f) Draw a sketch of the graph from the information you have obtained.

6. $y = \frac{1}{x}$

7. $y = \frac{8}{x - 4}$

8. $y = \frac{2}{x + 3}$

9. $y = \frac{36}{x^2}$

Symmetry

In Problem 9 above you may have observed that the graph of $y = \frac{36}{x^2}$ is *symmetrical with respect to the y-axis*, that is, for every point on the graph, such as (2, 9), there is another point on the graph, (-2, 9), such that the y-axis is the perpendicular bisector of the segment joining the two points. In general, if the point (a, b) is on the graph, then the point $(-a, b)$ also is on the graph. This phenomenon will always happen when x occurs only to even

powers in the expression. Can you explain why? Thus the graphs of $\frac{4x^2}{x^2 - 4}$, $6 + x^4$, $\frac{x^2 - 5}{3 + x^6}$ will all be symmetrical with respect to the y-axis.

Perhaps you also observed that in Problem 6 the graph of $y = \frac{1}{x}$ is *symmetrical with respect to the origin.* For example, the points $\left(5, \frac{1}{5}\right)$ and $\left(-5, -\frac{1}{5}\right)$ are both on the graph and the origin bisects the segment joining them. In general, if the point (a, b) is on the graph, then the point $(-a, -b)$ also is on the graph. This will happen if in the rational expression either

(1) every term in the numerator is of odd degree and every term in the denominator is of even degree, or

(2) every term in the numerator is of even degree and every term in the denominator is of odd degree.

Can you explain why this is so? The graphs of $\frac{2x^3}{x^4 + 8}$, $\frac{3x^2 - 5}{x^3 + 6x}$, $x^5 - x$ are all symmetrical with respect to the origin.

PROBLEMS 2-6B

1. Which of the following expressions have graphs which are symmetrical with respect to the y-axis? Which with respect to the origin?

   a. $y = \frac{x^2}{x^3 + x}$

   b. $y = \frac{6}{x^2 - 9}$

   c. $y = x^2$

   d. $y = \frac{x}{x^4 + 1}$

   e. $y = \frac{x^2 + 5x + 2}{x^3 + 1}$

   f. $y = \frac{x^4 + 2}{x^6 - 2}$

2. a. Is there any function which has a graph symmetrical with respect to the x-axis?

   b. Can a relation have a graph symmetrical with respect to the x-axis?

3. *Challenge Problem.*

   If f is a function such that for all numbers a in the domain $f(a) = f(-a)$, the function is sometimes said to be an *even* function. If, on the other hand, $f(a) = -f(-a)$, the function is called an *odd* function.

a. What symmetry will the graph of an even function have?

b. What symmetry will the graph of an odd function have?

c. How can you tell by the degrees of the terms whether a given polynomial defines an even function or an odd function?

d. If $P(x)$ and $Q(x)$ are polynomials and P and Q are both even functions, is the rational function $\frac{P}{Q}$ even or odd? What symmetry has the graph of the rational function $\frac{P}{Q}$?

e. Answer the questions of part d if P is even and Q is odd.

f. Answer the questions of part d if P is odd and Q is even.

g. Answer the questions of part d if P is odd and Q is odd.

Intercepts.

In Problem 2 on page 68 you found that the points with coordinates $\left(0, -\frac{3}{4}\right)$ and $\left(\frac{3}{2}, 0\right)$ are points of the graph. The first of these points is on the y-axis and the second is on the x-axis. (Why?)

It often happens that such points which are on the axes are easy to find and are useful bits of information about the graph. A *y-intercept* is the value of y at a point where the graph intersects the y-axis, so of course the value of x at that point is zero. (Why?) The graph of a function never has more than one y-intercept. (Why?) To find the y-intercept we let x be zero and find the corresponding value of y. Similarly to find the *x-intercepts,* we let y be zero and find the corresponding values of x.

For example if the function is defined by $y = \frac{3x + 4}{x - 2}$, then when $x = 0$, $y = \frac{3 \times 0 + 4}{0 - 2}$, so the point $(0, -2)$ is a point on the graph. The y-intercept is -2. When $y = 0$, $0 = \frac{3x + 4}{x - 2}$. This is true when $x = -\frac{4}{3}$ (Why?), so the point $\left(-\frac{4}{3}, 0\right)$ is on the graph and the x-intercept is $-\frac{4}{3}$.

While this has given only two points of the graph, it gives them quickly and easily and we shall find that, combined with other available information, these intercepts are a definite help.

PROBLEMS 2-6C

Find the x- intercept and the y-intercept for each of the following equations.

1. $y = \dfrac{x - 8}{x + 2}$
2. $y = \dfrac{3x}{x^2 + 4}$
3. $y = \dfrac{2x + 5}{3x^2 + x + 7}$
4. $y = \dfrac{6x - 8}{2x^2 - 3x}$
5. $y = \dfrac{x^2 - 7x - 18}{x - 18}$
6. $y = \dfrac{3x^2 + 12x}{x + 1}$
7. $y = \dfrac{8}{x + 2}$
8. $y = \dfrac{x^2 + 8}{2x - 1}$
9. $y = \dfrac{x^2 - 16}{4x - 16}$
10. a. Explain why in Problems 7 and 8 there is no x-intercept.
    b. Explain why in Problem 4 there is no y-intercept.

2-7 ASYMPTOTES

In PROBLEMS 2-6A on pages 68-69 you found for each function a value of x for which the function is undefined and for values of x near that number you found that the graph goes far off the paper either above or below. For example, if $y = \dfrac{7}{x - 9}$, you can predict at once that the graph will behave in this peculiar way at $x = 9$. (Why?) If we draw a vertical line at $x = 9$, it is clear that the graph never touches that line (Why?), but that for values of x a *little less* than 9 the graph is far down (Why?) and for values of x a *little greater* than 9 the graph is far up. Also, the closer to 9 we choose the value of x, the farther up or down the corresponding point will be, giving a picture as in Fig. 6.

We sometimes call such a line an *asymptote* of the graph. The line is not part of the graph but it has the property that the graph comes closer and closer to the line as we go farther up or down. A more precise statement is that if the asymptote is the line with equation $x = 9$, then as x takes values closer and closer to 9, y becomes larger and larger without limit. At $x = 9$ the function is undefined so there is no point of the graph at $x = 9$. We say that the graph is discontinuous there.

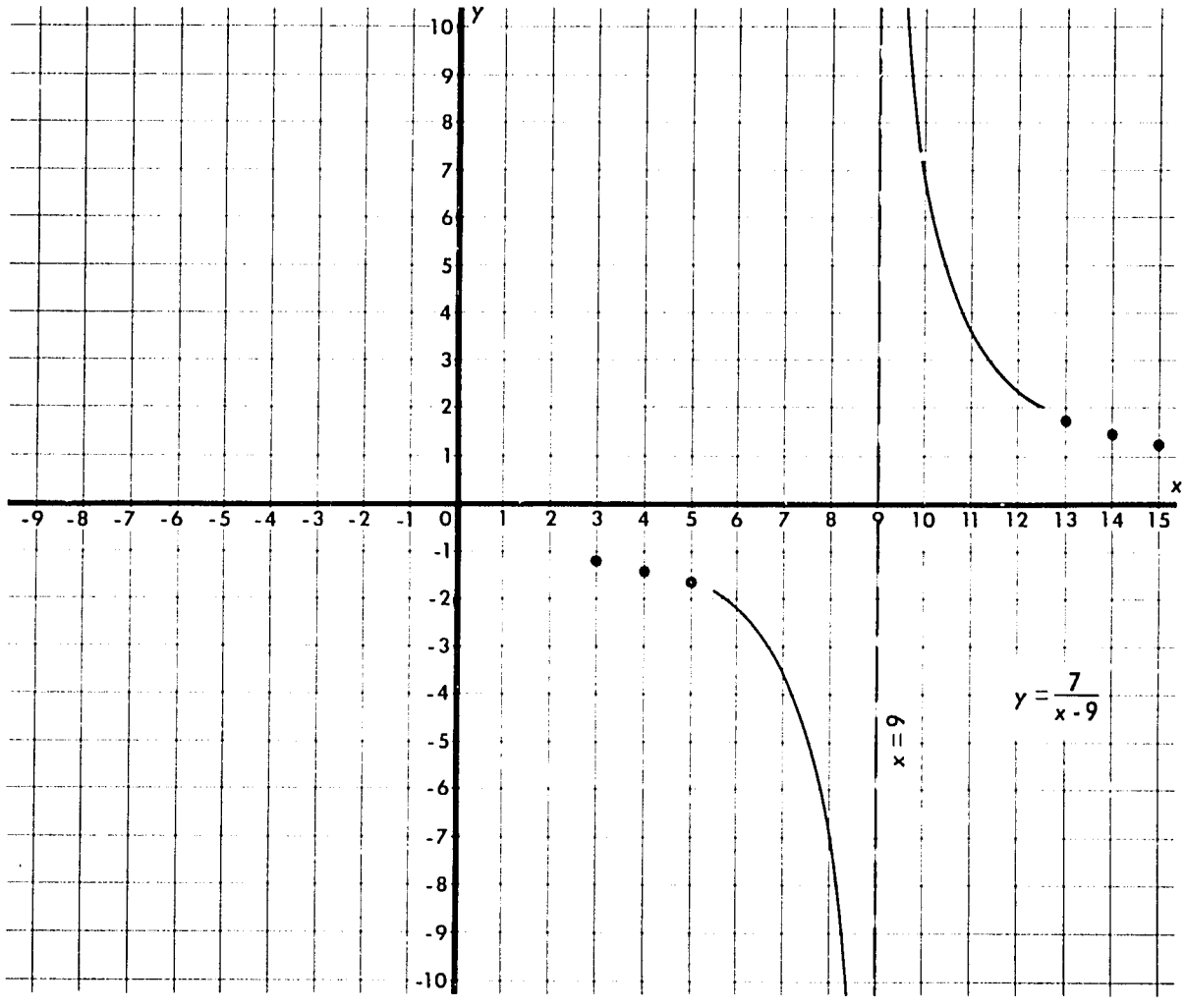


Fig. 6.

PROBLEMS 2-7A

1. a. In the graph of $y = \frac{5}{x - 7}$, at what value of x will there be an asymptote?

   b. As x takes values near 7, the values of $(x - 7)$ will be near zero. What, then, will happen to the values of $\left|\frac{5}{x - 7}\right|$?

   c. For values of x a *little less* than 7 (e.g., 6·8, 6·99) will $(x - 7)$ be positive or will it be negative? Will $\frac{5}{x - 7}$ be positive or will it be negative?

   d. For values of x a *little greater* than 7 (e.g., 7·2, 7·01) will $(x - 7)$ be positive or will it be negative? Will $\frac{5}{x - 7}$ be positive or will it be negative?

Without plotting points you should be able to see from your answers that this graph has a vertical asymptote at $x = 7$ and the graph is far down (y negative and $|y|$ very large) close to the asymptote on the left, and far up (y positive and $|y|$ very large) close to the asymptote on the right. We can sketch a part of the graph as shown in Fig. 7. The dotted part means we are not yet sure of that part. The fact that the y-intercept is $-\frac{5}{7}$ helps a little.

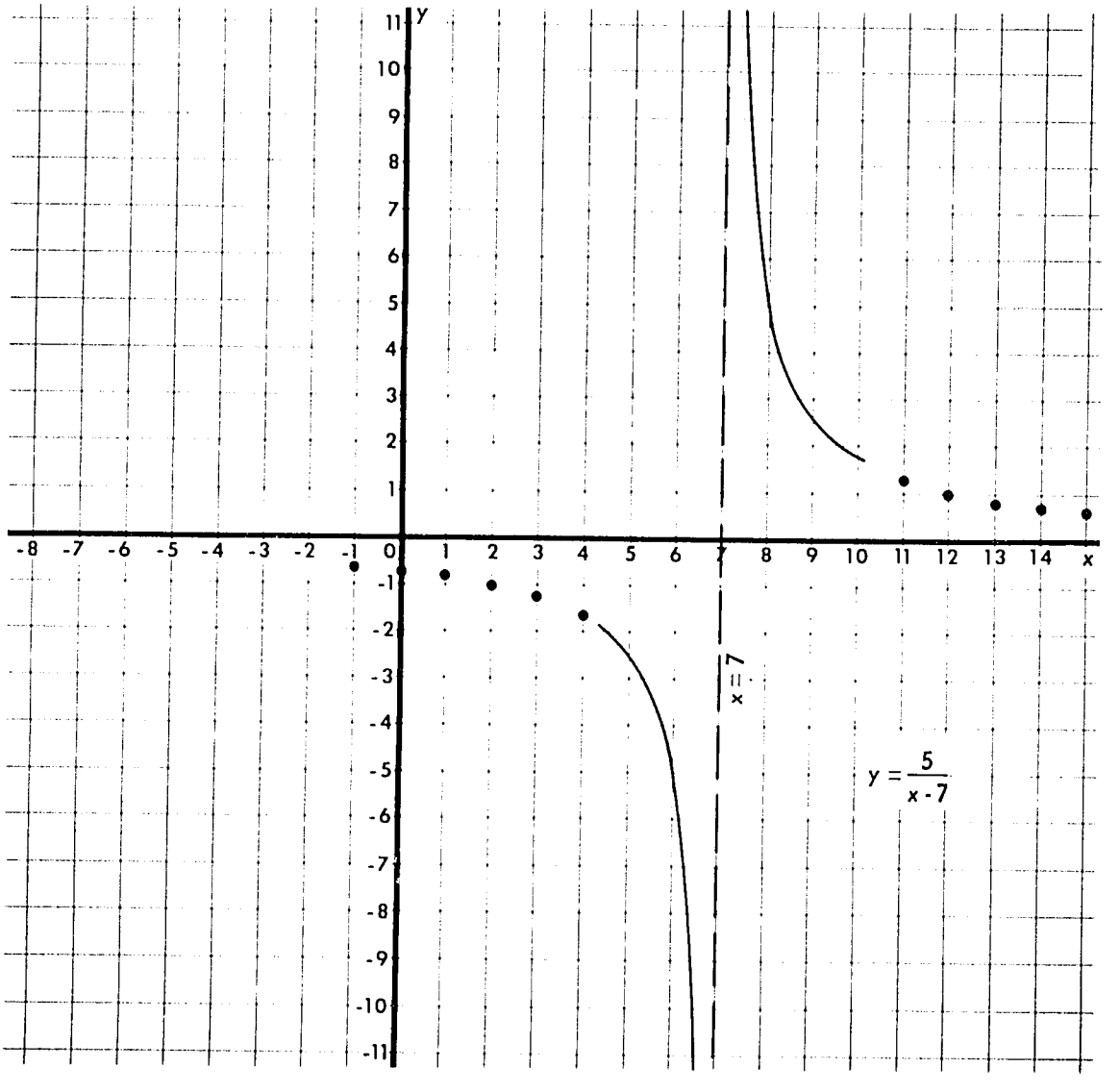


Fig. 7.

2. Following the procedure in Problem 1 find the vertical asymptotes, determine the behavior of the graph near the asymptotes, find the y-intercept, and sketch what you can of the graph.

a. $y = \dfrac{6}{x + 5}$

b. $y = \dfrac{-2}{x - 4}$

c. $y = \dfrac{11}{2x - 3}$

d. $y = \dfrac{-1}{x + 17}$

e. $y = \dfrac{2}{3 - x}$

f. $y = \dfrac{-7}{4 - 3x}$

3. Some graphs have more than one vertical asymptote.

Example:

$$y = \frac{2}{x^2 - 8x + 12}$$

$$y = \frac{2}{(x - 6)(x - 2)}$$

This function will be undefined at $x = 6$ and at $x = 2$. (Why?) Let us examine values of y for x *near* these numbers 6 and 2. If $x = 1{\cdot}9$, then $(x - 6)$ is negative and $(x - 2)$ is negative and near 0. Show why this tells you that at $x = 1{\cdot}9$, $|y|$ is large and y is positive.
If $x = 2{\cdot}1$, then $(x - 6)$ is negative and $(x - 2)$ is positive and near 0. Show why this tells you that at $x = 2{\cdot}1$, $|y|$ is large and y is negative.
Similarly what happens to y when x is $5{\cdot}9$ and $6{\cdot}1$?
We see there are asymptotes at $x = 6$ and $x = 2$ and can sketch the graph.

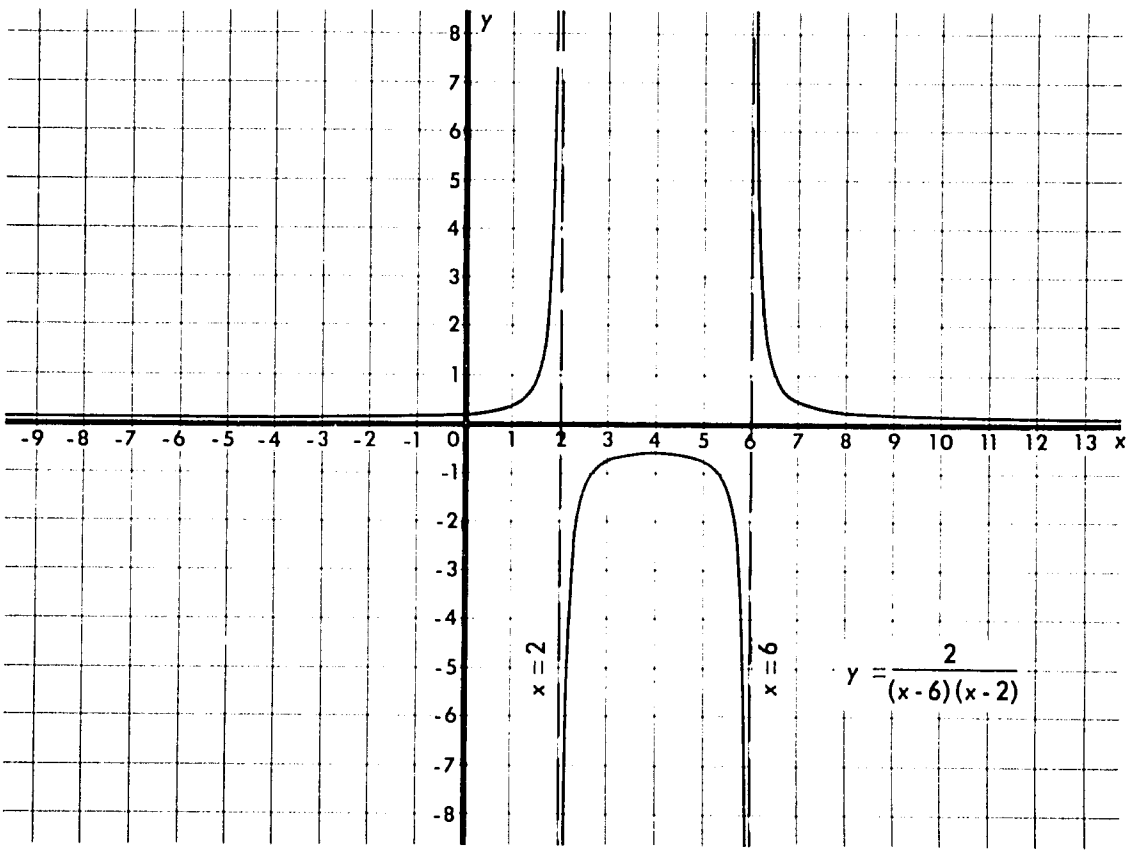


Fig. 8.

See whether you can determine the asymptotes and sketch part of the graph for each of the following equations.

a. $y = \dfrac{1}{(x - 3)(x - 5)}$

b. $y = \dfrac{-3}{(x + 2)(x - 6)}$

c. $y = \dfrac{5}{x^2 - 6x + 5}$

d. $y = \dfrac{x - 2}{(x + 3)(x - 5)}$

Very large $|x|$

In studying the graph of $y = \dfrac{1}{x - 2}$ we found it helpful to consider values of x such as 1,000,000 and -1,000,000, in general values for which $|x|$ *is very large*. In fact, in that example we found what can be called a *horizontal asymptote* of the graph because for $|x|$ very large we found that values of y were very close to zero. This meant that the line $y = 0$ (the x-axis) was behaving like an asymptote.

It will not be difficult to see that for any rational expression of the form $\dfrac{N(x)}{D(x)}$, if the degree of $N(x)$ is less than the degree of $D(x)$, the graph will have a horizontal asymptote at $y = 0$. (Remember that the degree of a polynomial is the exponent of the highest power of the variable in the polynomial.)

Consider the example

$$y = \frac{x^2 + 8x + 2}{2x^3 - 5x^2 + 9}$$

We shall find that it serves our purpose to multiply this rational expression by $\dfrac{\frac{1}{x^3}}{\frac{1}{x^3}}$, where the x^3 is the highest power of x in the denominator. The result is that for all values of x in the domain except zero the given equation is equivalent to the equation

$$y = \frac{\frac{1}{x} - \frac{8}{x^2} + \frac{2}{x^2}}{2 - \frac{5}{x} + \frac{9}{x^3}}$$

When $|x|$ is very large, the numbers $\frac{1}{x}$, $\frac{8}{x^2}$, $\frac{2}{x^3}$, $-\frac{5}{x}$, and $\frac{9}{x^3}$ are all very close to zero. (Why?) Therefore when $|x|$ is very large, $\dfrac{\frac{1}{x} + \frac{8}{x^2} + \frac{2}{x^3}}{2 - \frac{5}{x} + \frac{9}{x^3}}$ must be very close to $\dfrac{0 + 0 + 0}{2 - 0 + 0}$ or 0. We say then that for very large $|x|$, y *approaches zero*. Since y *does* approach zero when $|x|$ is very large, the graph has a horizontal asymptote at $y = 0$.

Notice that our exclusion of the value 0 for x at the start causes no trouble because we are using the method only to study large values of $|x|$.

PROBLEMS 2-7B

1. a. Multiply $\dfrac{10x}{x^2 + 6x + 8}$ by $\dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}$ $(x \neq 0)$.

   b. If $y = \dfrac{10x}{x^2 + 6x + 8}$, what does your result in part a tell you about the value of y when $|x|$ is very large?

   c. Why does the graph of $y = \dfrac{10x}{x^2 + 6x + 8}$ have a horizontal asymptote at $y = 0$?

   d. From the equivalent equation $y = \dfrac{10x}{(x + 4)(x + 2)}$ answer the following questions.
   When $|x|$ is large and x is positive, which is y, positive or negative?
   When $|x|$ is large and x is negative, which is y, positive or negative?

   e. What vertical asymptotes are there in the graph of $y = \dfrac{10x}{x^2 + 6x + 8}$?

   f. Determine the behaviour of the graph near the vertical asymptotes.

   g. What are the intercepts of $y = \dfrac{10x}{x^2 + 6x + 8}$?

   h. Use the information in b, d, e, f, and g to help sketch the graph of this function.

2. a. Multiply $\frac{6}{3x - 5}$ by $\frac{\frac{1}{x}}{\frac{1}{x}}$ $(x \neq 0)$.

b.-h. Carry out for this expression the instructions of parts b.-h. in Problem 1.

3. a. Multiply $\frac{3}{x^2 - 4}$ by $\frac{\frac{1}{x^2}}{\frac{1}{x^2}}$ $(x \neq 0)$.

b.-h. Carry out for this expression the instructions of parts b.-h. in Problem 1.

4. In each of the following expressions what would you multiply by to do part a of the procedure in Problem 1?

a. $\frac{2x - 9}{x^3 + 8x + 5}$

b. $\frac{2}{x^2 + 10}$

c. $\frac{9x + 8}{3x^2 + 4x + 5}$

d. $\frac{2}{4x + 3}$

e. $\frac{4x}{x^4 - 3x + 1}$

5. In which of the following expressions does the graph have a horizontal asymptote at $y = 0$?

a. $\frac{x}{x^2 - 6x + 9}$

b. $\frac{3x - 5}{x + 2}$

c. $\frac{9x^2 + 8x + 2}{x - 4}$

d. $\frac{x^3}{2x^4 + x + 9}$

e. $\frac{7x^2 + 4}{4x^2 + 7}$

f. $12x^2 - 8x + 2$

g. $\frac{x^4 + 1}{x^2 + 1}$

The method used in Problems 1-3 above can be applied to any rational expression $\frac{N(x)}{D(x)}$ if the degree of $N(x)$ is less than the degree of $D(x)$. If the degree of $D(x)$ is k, then multiplying by $\frac{\frac{1}{x^k}}{\frac{1}{x^k}}$ $(x \neq 0)$ will always give the same kind of result as you saw in the problems. We are justified in

using this test of degree of $N(x)$ and $D(x)$ to show when the graph of a rational function has a horizontal asymptote at $y = 0$.

What will happen, however, if the degree of $N(x)$ is the same as or higher than the degree of $D(x)$?

Consider
$$y = \frac{3x - 8}{x + 4}$$

By a long division this expression can be rewritten as
$$y = 3 - \frac{20}{x + 4}$$

[Remember long division of polynomials?

$$\begin{array}{r} 3 \\ x + 4\overline{)3x - 8} \\ \underline{3x + 12} \\ -20 \end{array} \qquad \text{so} \qquad \frac{3x - 8}{x + 4} = 3 + \frac{-20}{x + 4}$$
]

In this new expression the term $-\frac{20}{x + 4}$ approaches 0 when $|x|$ becomes very large. (Why?) In that case $3 - \frac{20}{x + 4}$ must be very near 3 when $|x|$ is very large, that is, y approaches 3. This tells us that the graph has a *horizontal asymptote at* $y = 3$.

How does the graph behave near the asymptote? Far to the right is it above the asymptote or below it? Decide by seeing whether $3 - \frac{20}{x + 4}$ is greater than 3 or less than 3 when $|x|$ is large and x is positive. Make a similar decision on the left when x is negative.

If we use the vertical asymptote at $x = -4$, we can now sketch the graph. (See page 80)

In the following problems you will need to do some long division of polynomials. If you have forgotten how to do this, you would be wise to practice a few with your teacher's help.

PROBLEMS 2-7C

1. a. Use long division to rewrite the rational expression $\frac{8x + 1}{2x - 6}$ as a sum, as illustrated in the brackets above.

   b. Find the horizontal asymptote of the graph of $y = \frac{8x + 1}{2x - 6}$ by finding what y approaches as $|x|$ becomes very large.

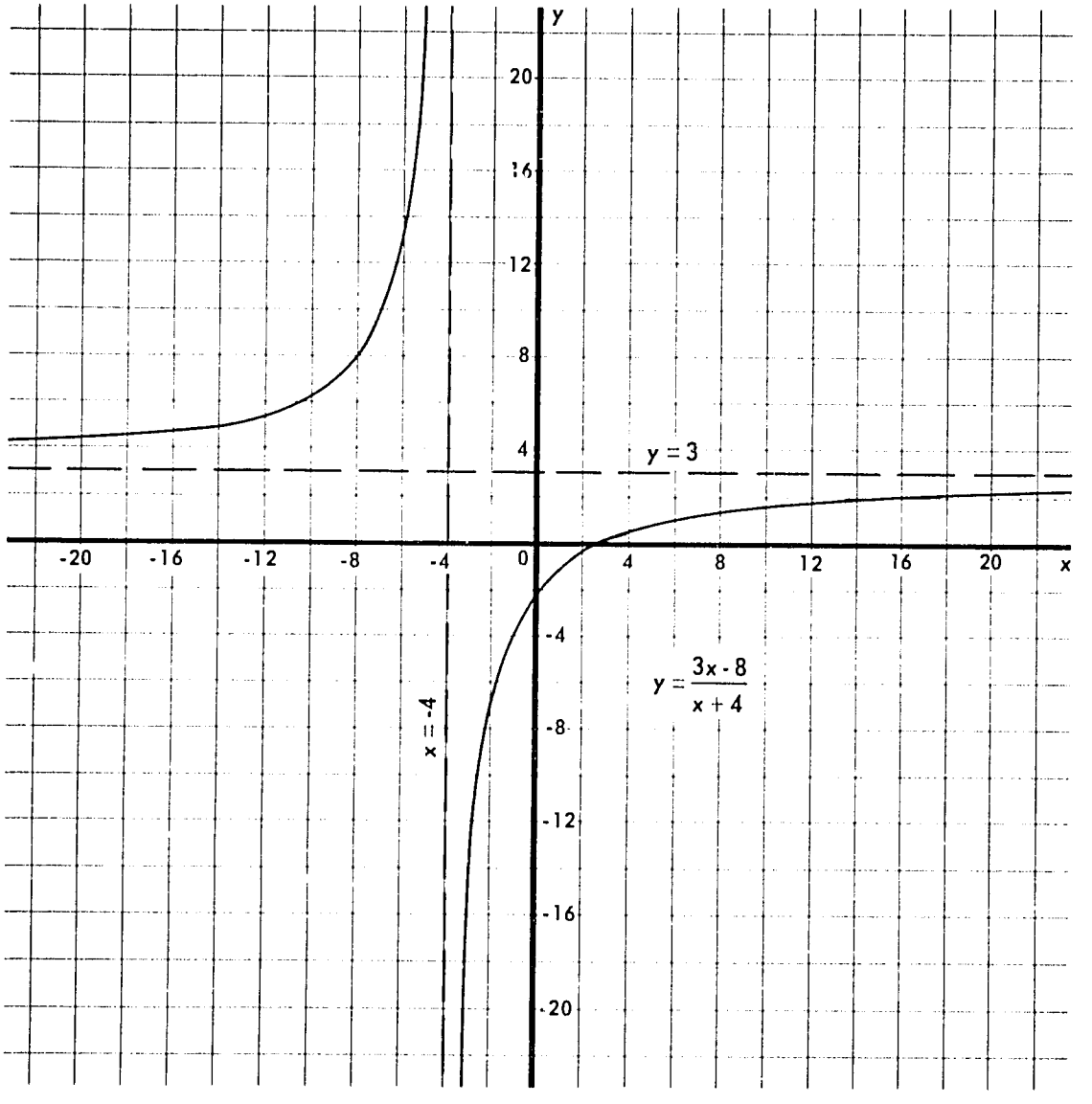


Fig. 9.

c. Find the behaviour of the graph near this asymptote by considering what happens when $|x|$ is large and $x > 0$, and what happens when $|x|$ is large and $x < 0$.

d. Find the vertical asymptote and find the behaviour of the graph near this asymptote.

e. Find the intercepts.

f. Test for symmetry.

g. Use the information obtained in b, c, d, e, and f to help sketch the graph.

2. Do all the parts of Problem 1 for each of the following expressions.

a. $\dfrac{5x + 35}{x + 9}$

b. $\dfrac{x + 8}{4 - x}$

c. $\dfrac{x^2}{x^2 - 9}$

d. *Challenge Problem*

$\dfrac{2x^2 - 18x + 36}{x^2 - 9x + 14}$

3. a. In the expression $\dfrac{2x^3}{x^2 - 16}$, $N(x)$ is of *higher* degree than $D(x)$.

If you perform the long division you obtain

$$y = 2x + \frac{32x}{x^2 - 16}.$$

Is $\dfrac{32x}{x^2 - 16}$ close to zero for large values of $|x|$? Why?

b. If $y = 2x + \dfrac{32x}{x^2 - 16}$, what is y close to when $|x|$ is very large?

Does y approach a fixed number? Does the graph have a horizontal asymptote?

c. If y approaches $2x$ when $|x|$ becomes large, it might help to draw the graph of $y = 2x$ as a guide to the graph of $y = 2x + \dfrac{32x}{x^2 - 16}$. Do so. Is the graph of $y = 2x$ a straight line? We could could call this line an *oblique asymptote*.

d. Find the behaviour of the graph near this new asymptote by considering when $\dfrac{32x}{x^2 - 16}$ is positive and when it is negative for large $|x|$. When it is positive, the graph is above the line $y = 2x$; when it is negative, the graph is below.

e. Does the graph of $y = 2x + \dfrac{32x}{x^2 - 16}$ have any vertical asymptotes?

If so, find them and find the behaviour of the graph near them.

f. Find the intercepts.

g. Test for symmetry.

h. The graph can now be sketched using the information you have obtained.

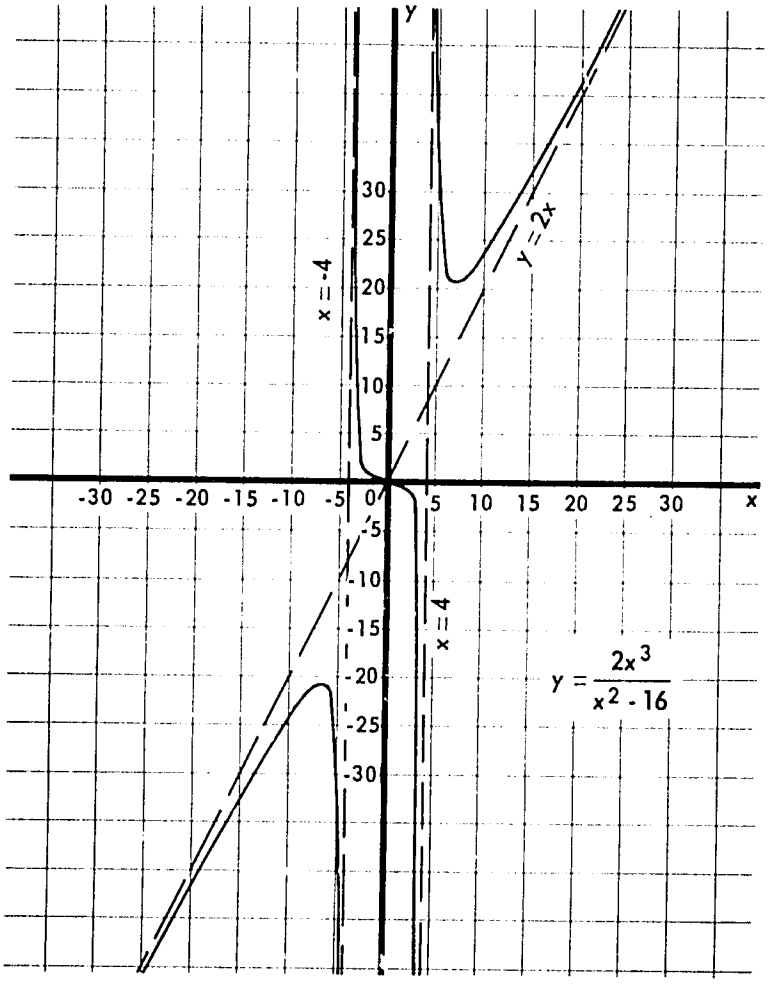


Fig. 10.

4. Analyze the graphs of the following expressions as was done in Problem 3, that is:

   Perform the long division.

   Find the oblique asymptote if there is one.

   Find the behaviour of the graph near the oblique asymptote.

   Find vertical asymptotes and behaviour of the graph near them.

   Find the intercepts.

   Test for symmetry.

   Sketch the graph.

a. $\dfrac{x^2 + 2x - 3}{x + 2}$

b. $\dfrac{x^2 - 5x + 7}{x - 2}$

c. $\dfrac{x^2 - 4x}{3 - x}$

We have seen now that in the rational expression $\frac{N(x)}{D(x)}$, by studying the effect of large values of $|x|$ we observe the following:

1. If the degree of $N(x)$ is *less* than the degree of $D(x)$, the graph has a horizontal asymptote at $y = 0$.
2. If the degree of $N(x)$ is *equal* to the degree of $D(x)$, the graph has a horizontal asymptote which can be found by performing the long division.
3. If the degree of $N(x)$ is *greater* than the degree of $D(x)$, the graph does not have a horizontal asymptote, but if the degree of $N(x)$ is *one* higher than the degree of $D(x)$, the graph will have an oblique asymptote which can be found by performing the long division.

In the second and third cases we used the fact that if the degree of $N(x)$ is greater than or equal to the degree of $D(x)$, then $\frac{N(x)}{D(x)}$ can always be expressed by long division as the sum of a polynomial and a fraction in which the degree of the numerator is *less* than the degree of the denominator.

We can combine the information now available about the graphs of rational functions to help us draw a fairly good approximation to the graph without plotting very many points. In such an analysis of the graph we usually expect that you find as much of the following information as is easily obtainable.

1. The y-intercept and the x-intercepts.
2. Symmetry.
3. Values of x for which the graph is discontinuous.
4. Vertical asymptotes and behaviour of the graph near them.
5. Behaviour of the graph for large $|x|$.
6. A few additional points where they are most needed.

Example 1: $$y = \frac{x}{(x+4)^2(x-1)}$$

x-intercept and y-intercept: (0,0)

Discontinuous at $x = -4$ and $x = 1$.

Vertical asymptote at $x = -4$: y positive on both sides

Vertical asymptote at $x = 1$: y negative on left, positive on right

Horizontal asymptote at $y = 0$; graph above on extreme left, above on extreme right.

Preliminary sketch:

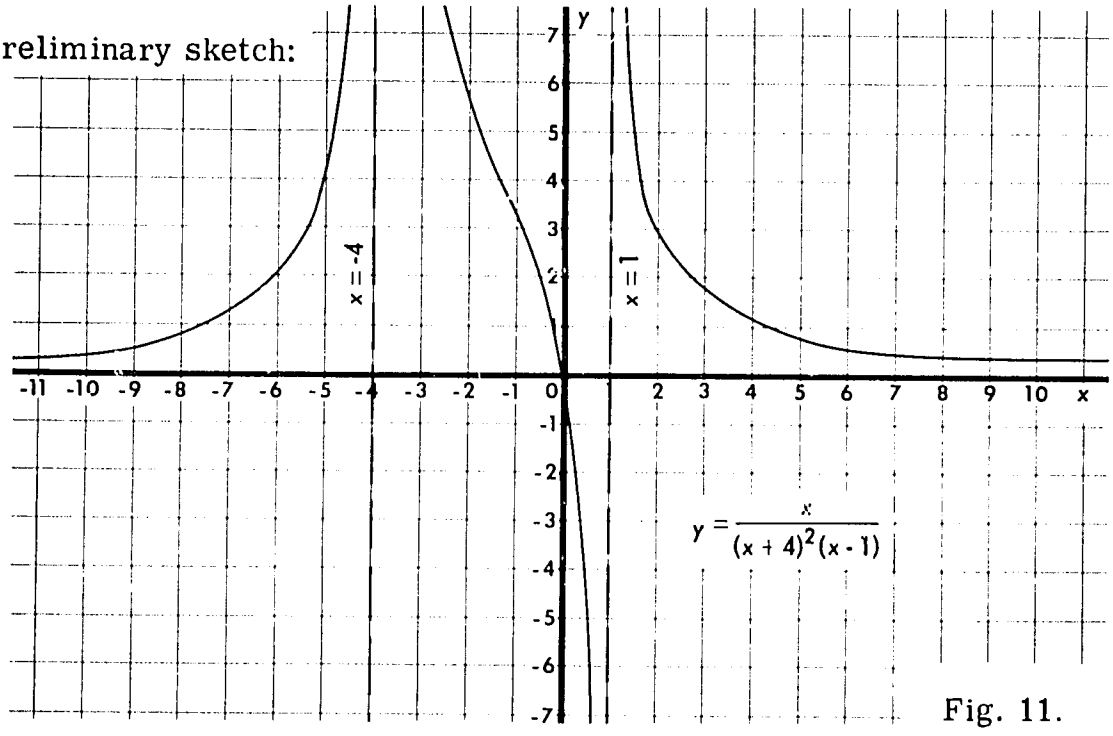


Fig. 11.

Additional points for further guidance:

$y = \frac{x}{(x+4)^2(x-1)}$	x	-5	-3	-2	$\frac{1}{2}$	2
	y	$\frac{5}{6}$	$\frac{3}{4}$	$\frac{1}{6}$	$-\frac{4}{81}$	$\frac{1}{18}$

Improved sketch.

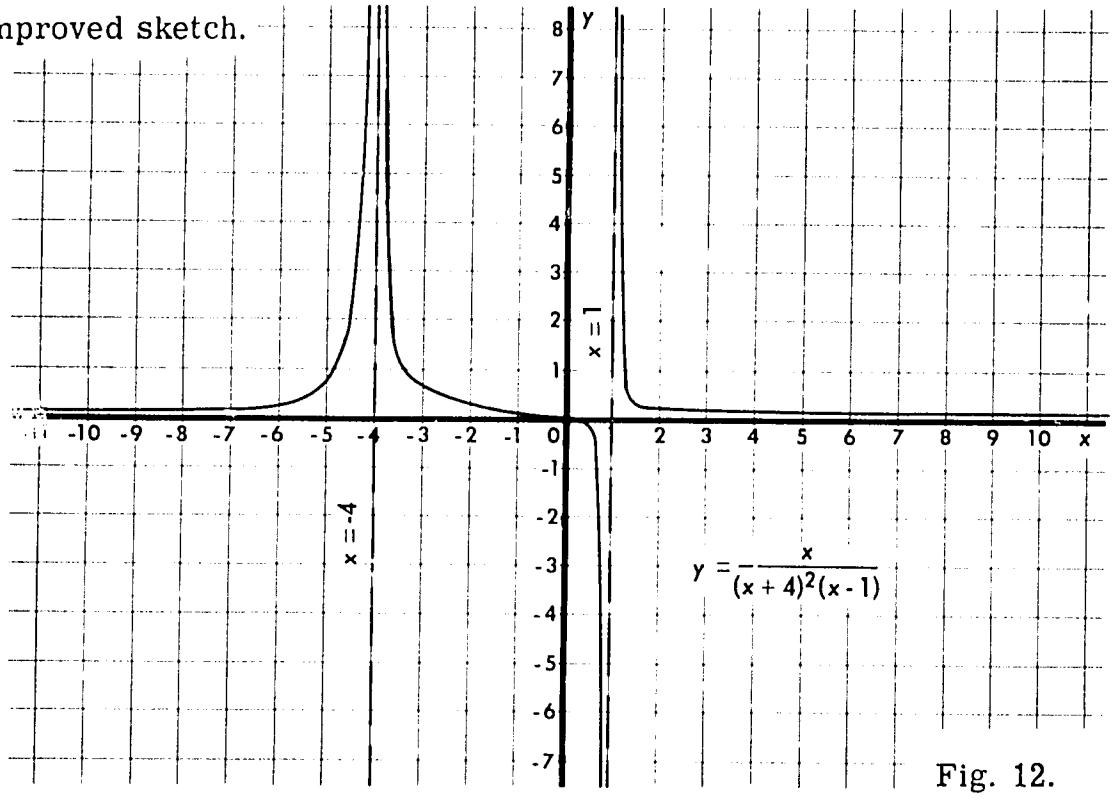


Fig. 12.

Example 2.

$$y = \frac{4x^2}{x^2 - 4}$$

$$y = \frac{4x^2}{(x + 2)(x - 2)}$$

$$y = 4 + \frac{16}{x^2 - 4}$$

x-intercept and y-intercept: (0,0)

Symmetrical with respect to the y-axis.

Discontinuous at $x = -2$ and $x = 2$.

Vertical asymptote at $x = -2$; y positive on left, negative on right.

Vertical asymptote at $x = 2$; y negative on left, positive on right.

Horizontal asymptote at $y = 4$; graph above on extreme left, above on extreme right.

Preliminary sketch:

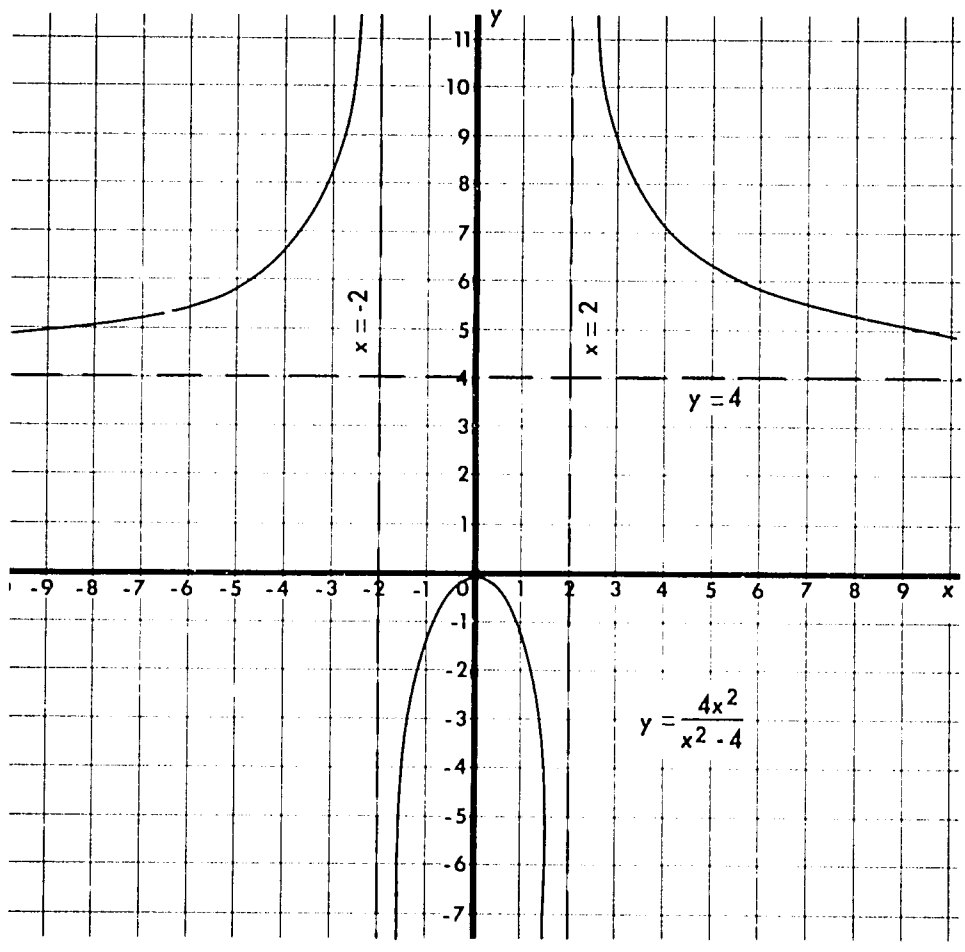


Fig. 13.

Additional points for further guidance:

$y = \dfrac{4x^2}{x^2 - 4}$	x	-4	-3	-1	1	3	4
	y	$5\frac{1}{3}$	$7\frac{1}{5}$	$-1\frac{1}{3}$	$-1\frac{1}{3}$	$7\frac{1}{5}$	$5\frac{1}{3}$

Improved sketch:

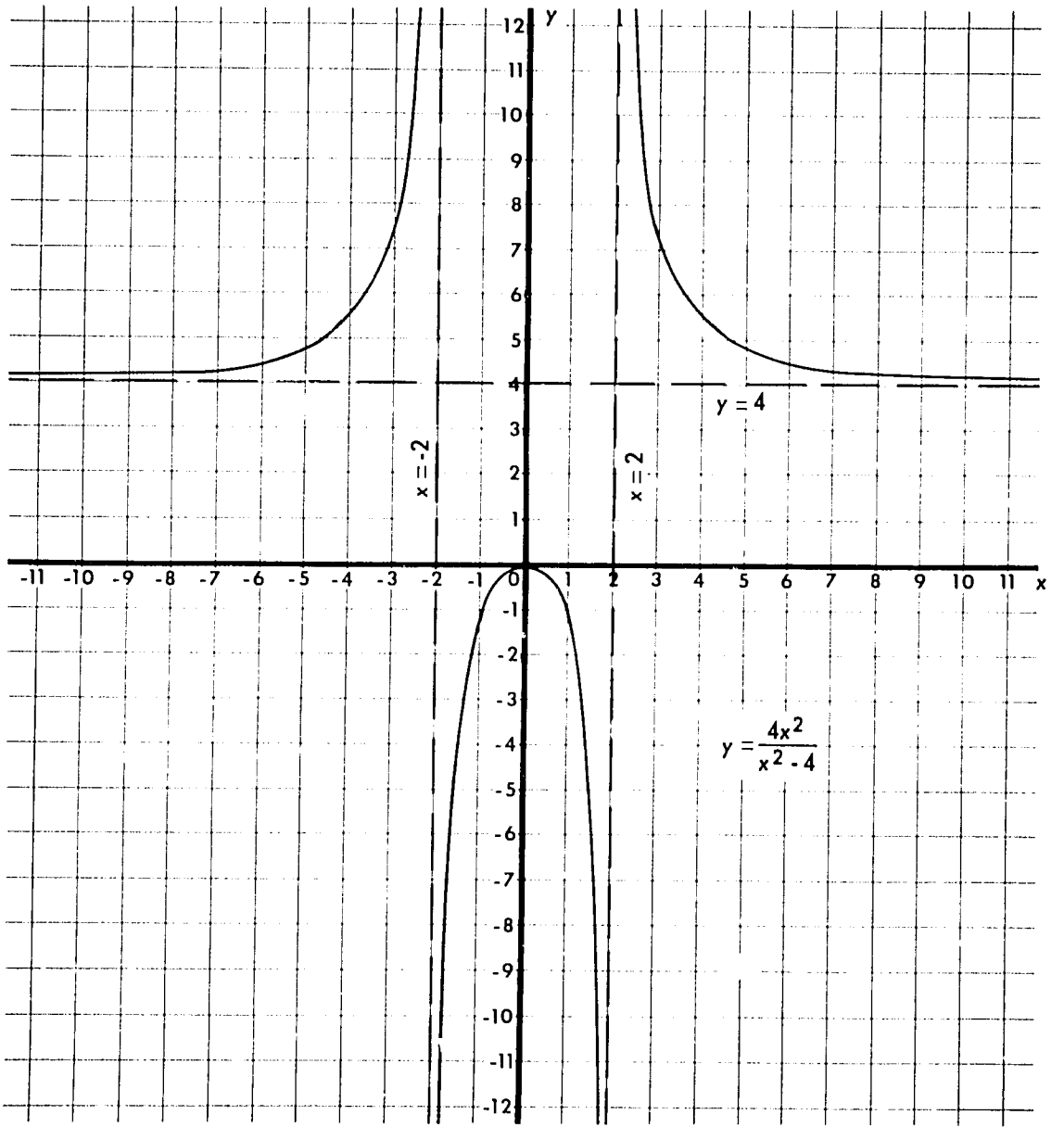


Fig. 14.

Example 3.

$$y = \frac{x^2 + 5x}{3x + 6}$$

$$y = \frac{x(x + 5)}{3(x + 2)}$$

$$y = \frac{1}{3}x + 1 - \frac{2}{x + 2}$$

y-intercept: (0,0)

x-intercepts: (0,0), (-5, 0)

Discontinuous at $x = -2$

Vertical asymptote at $x = -2$; y positive on left, negative on right.

Oblique asymptote $y = \frac{1}{3}x + 1$; graph above on extreme left, below on extreme right.

Preliminary sketch.

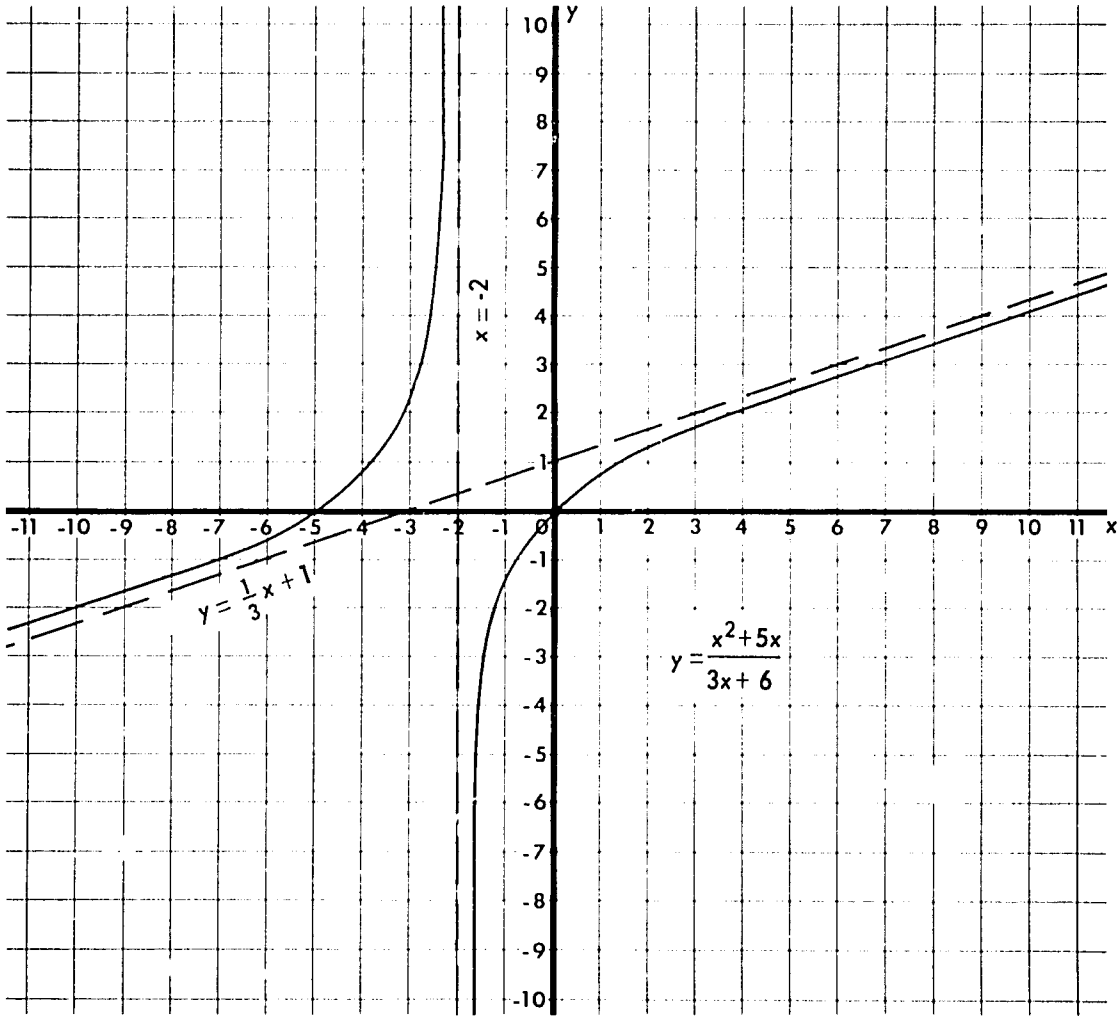


Fig. 15.

Additional points for further guidance:

$y = \dfrac{x^2 - 5x}{3x + 6}$	x	-4	-1	1
	y	$\frac{2}{3}$	$-\frac{4}{3}$	$\frac{2}{3}$

Improved sketch:

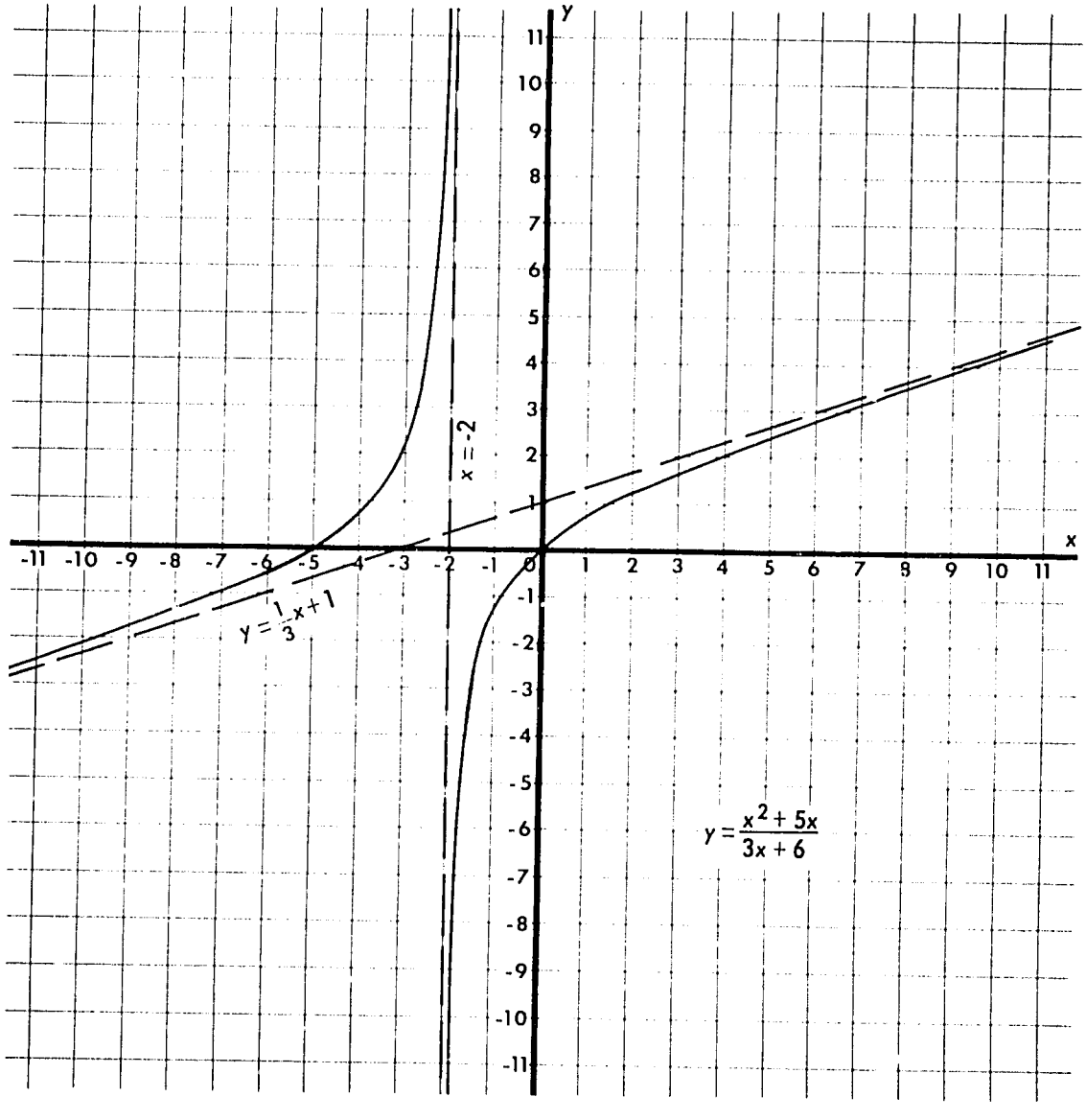


Fig. 16.

Notice that this time the intercepts gave enough help so that the additional points did not make much improvement.

PROBLEMS 2-7D

Analyze the graph of each of the following expressions as illustrated above, draw a preliminary sketch, select and compute additional helpful points, and make an improved sketch.

1. $y = \dfrac{x + 2}{x - 3}$
2. $y = \dfrac{x^2}{x^2 + 4}$
3. $y = \dfrac{x - 5}{x^2 - 25}$
4. $y = \dfrac{2x^2 - 4x}{x^2 - 2x - 3}$
5. $y = \dfrac{x - 2}{x^2 - 16}$
6. $y = \dfrac{5x - 15}{x^2 - 2x}$
7. $y = \dfrac{x^2 + 4}{x}$
8. Draw the graph of $y = \dfrac{|x|}{x + 1}$. (This does not define a rational function.) What are its horizontal and vertical asymptotes? Can a rational function have more than one horizontal asymptote? More than one vertical asymptote?

2-8 EQUATIONS AND INEQUALITIES BY GRAPHS.

We found last year that some polynomial equations which were not easily solved by factorization could be solved approximately by a graph. For example, to solve the equation

$$2x^3 - 3x^2 - 12x + 10 = 0$$

we draw the graph of $y = 2x^3 - 3x^2 - 12x + 10$ and observe the values of x where $y = 0$ on the graph. (See top of page 90)
There are three numbers in the truth set. One is between -3 and -2, one is between 0 and 1, and one is between 2 and 3. We might estimate them as -2·4, 0·8, and 2·9. By plotting more points of the graph in the vicinity of these values we could obtain better approximation to the solutions.

To solve the rational equation

$$\frac{x^2}{x^2 - 9} + \frac{2}{x^2 - 3x} = \frac{6}{x} \qquad (x \neq 0, 3, -3)$$

we could first obtain the equivalent polynomial equation

$$x^3 + 2(x + 3) = 6(x^2 - 9)$$

$$x^3 - 6x^2 + 2x + 60 = 0 \qquad (x \neq 0, 3, -3)$$

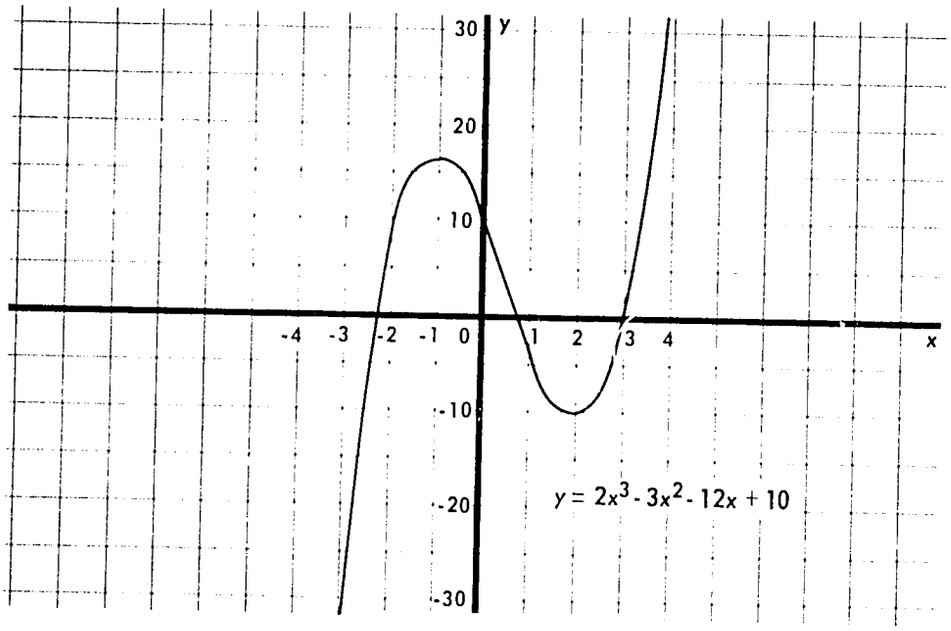


Fig. 17.

The truth set of this equation can be found approximately from the graph of $y = x^3 - 6x^2 + 2x + 60$ which is shown in Fig. 18. The graph shows that y is zero somewhere between $x = -2$ and $x = -3$, approximately at -2·5. Therefore we can say that the solution of the equation

$$\frac{x^2}{x^2 - 9} + \frac{2}{x^2 - 3x} = \frac{6}{x}$$

is about -2·5 and it has only one real solution.

Notice that in this method it is unnecessary to draw the graph of the rational function. The graph of the equivalent polynomial function serves the purpose.

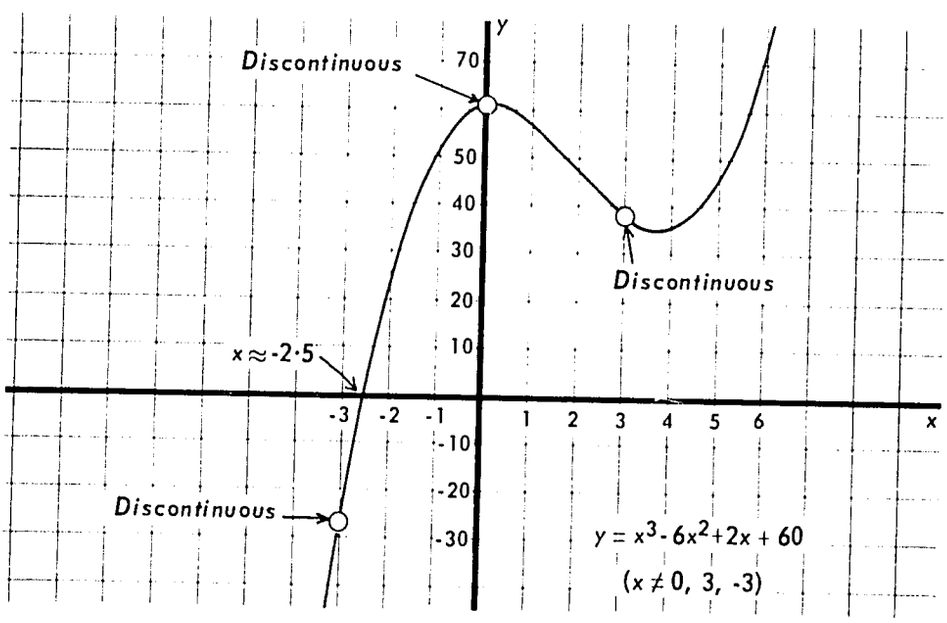


Fig. 18.

PROBLEMS 2-8A

Use the method illustrated above to solve graphically the following equations.

1. $\dfrac{x^2 + 1}{x^2 + 2x - 8} + \dfrac{x^2}{2x - 4} = 0$

2. $\dfrac{x^2}{x^2 - 4} - \dfrac{8x}{x^2 - 2x} = \dfrac{1}{x}$

In some equations where there are comparatively simple rational expressions a quite different use of graphical solution is possible in which the rational functions themselves are graphed. For example,

$$\frac{x}{x - 1} = \frac{2}{x^2}$$

can be solved by writing the system

$$y = \frac{x}{x - 1}$$

$$y = \frac{2}{x^2}$$

and asking for values of x which make the values of y equal. We draw the graphs of $y = \dfrac{x}{x - 1}$ and $y = \dfrac{2}{x^2}$ as shown in Fig. 19. At approximately $x = -1{\cdot}7$ the two graphs intersect. For that value of x, then, $\dfrac{x}{x - 1} = \dfrac{2}{x^2}$, so $-1{\cdot}7$ is approximately a solution of the equation.

PROBLEMS 2-8B

Solve the following equations by the method illustrated above. In Problem 1 check your answers by solving the equation algebraically.

1. $\dfrac{1}{x} = \dfrac{x}{x + 3}$

2. $\dfrac{1}{x} - \dfrac{x^2}{x - 4} = 0$

3. $\dfrac{2x}{x + 3} = \dfrac{10}{x^2 - 4}$

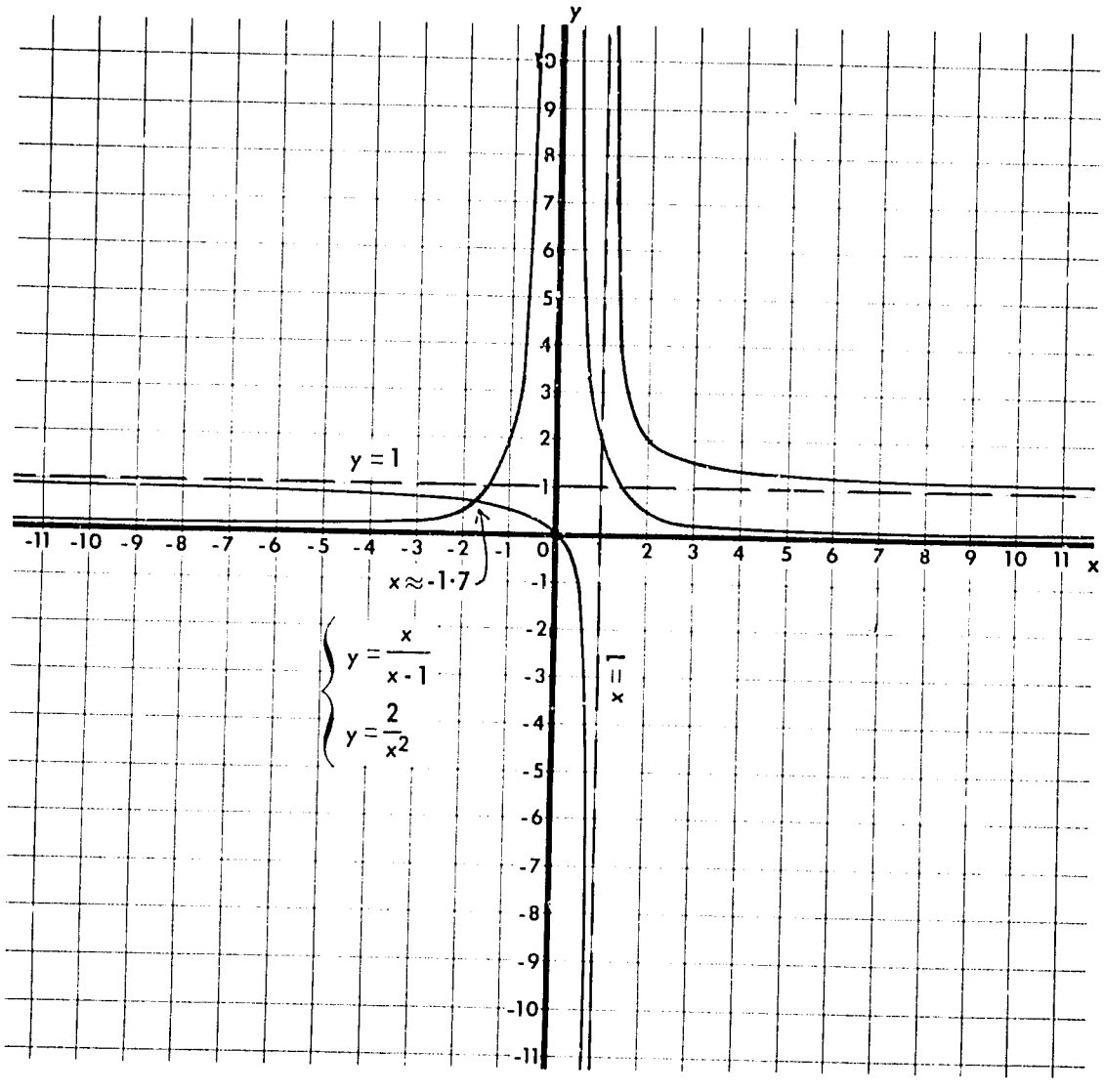


Fig. 19.

The graph is often a help in solving certain inequalities. Consider the inequality

$$\frac{x - 3}{x} > 0$$

which we solved earlier. All we need to know is for what values of x the graph of

$$y = \frac{x - 3}{x}$$

stays above the x-axis. (Why?)

CLASS ACTIVITY

1. What is the behaviour of the graph of $y = \frac{x - 3}{x}$ at $x = 3$?
2. What is the behaviour of the graph at $x = 0$?
3. Does the graph cross the x-axis or have a vertical asymptote anywhere else except at $x = 3$ and $x = 0$?
4. Why can we say that the graph stays on one side of the x-axis to the left of $x = 0$?
5. Between $x = 0$ and $x = 3$?
6. To the right of $x = 3$?
7. How can we determine on *which* side of the x-axis the graph is to the left of $x = 0$?
8. Between $x = 0$ and $x = 3$?
9. To the right of $x = 3$?

As a result of your answers it should be clear now that $\frac{x - 3}{x}$ is positive for x less than 0 and for x greater than 3, and it is negative for x between 0 and 3. The truth set of $\frac{x - 3}{x} > 0$ is

$$\{x : x < 0 \text{ or } x > 3\}.$$

Actually we did not draw the graph at all but just observed for what values of x it would be above the x-axis and for what values of x below. If we did draw the graph, it would look like Fig. 20. The only places where the graph can change from above to below or below to above are at x-intercepts (there it can cross the axis) or at vertical asymptotes (there it can jump from one side to the other). To find these points of change it is sufficient to find the values of x which make either $N(x) = 0$ or $D(x) = 0$ in the rational expression $\frac{N(x)}{D(x)}$. These values, which we might call *boundary values*, divide the x-axis into regions which then can be studied for positiveness or negativeness of the rational expression.

Let us use another example to illustrate how the necessary information can be quickly exhibited.

$$\frac{6 - x}{x + 2} > 0$$

The boundary values are $x = 6$ and $x = -2$. We can make the following table.

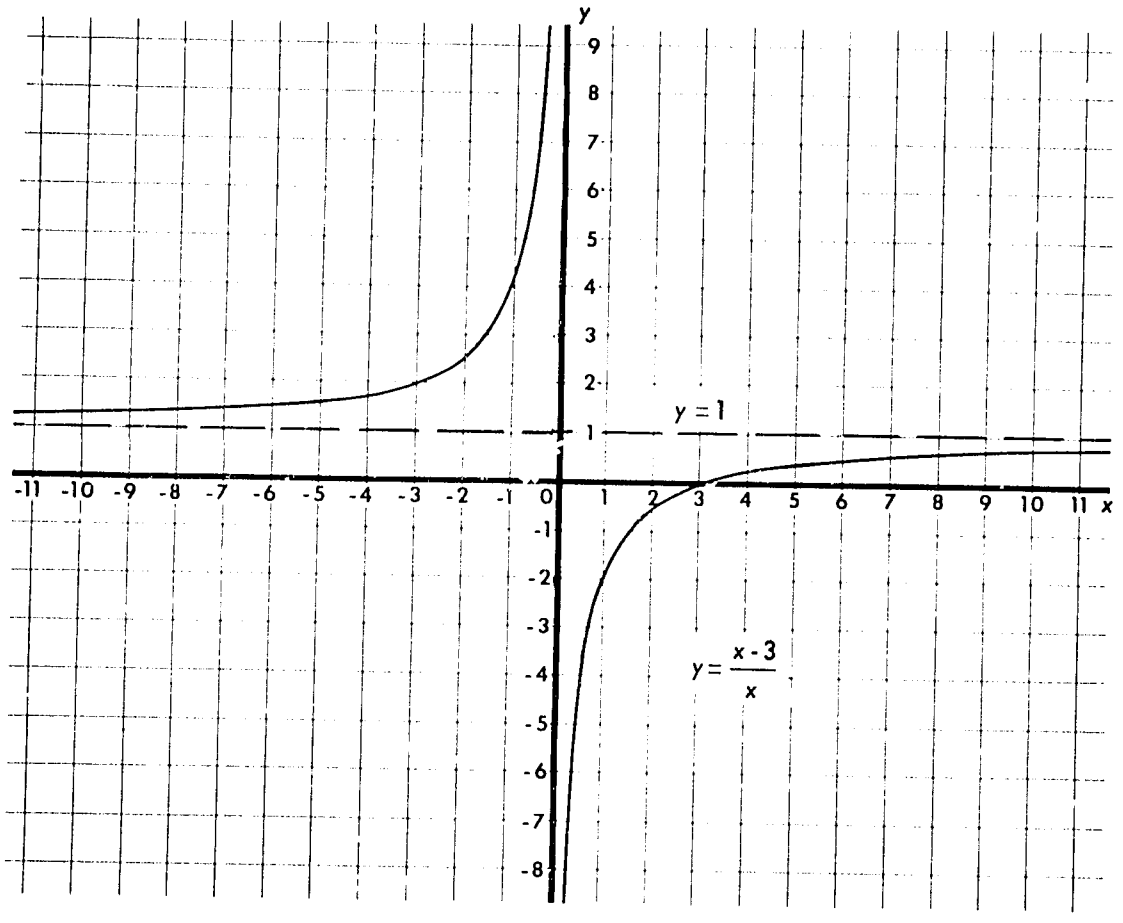


Fig. 20.

x	$x < -2$	2	$-2 < x < 6$	6	$x > 6$
$6 - x$	pos.	pos.	pos.	0	neg.
$x + 2$	neg.	0	pos.	pos.	pos.
$y = \frac{6 - x}{x + 2}$	neg.	undefined	pos.	0	neg.

The region where $\frac{6 - x}{x + 2}$ is positive is between -2 and 6, so the truth set of $\frac{6 - x}{x + 2} > 0$ is

$$\{x: -2 < x < 6\}.$$

There are times in the construction of graphs of rational functions when a table such as this would be a helpful item of information to add to the six items listed on page 83.

PROBLEMS 2-8C

Solve the following inequalities by the method described above.

1. $\dfrac{x}{4 + x} < 0$

2. $\dfrac{x - 1}{x^2 - 9} > 0$

3. $\dfrac{x^2 - 8x + 7}{x - 5} > 0$

4. $\dfrac{x^2 + 4x}{x^2 + 7x + 12} < 0$

5. $\dfrac{x^2}{8 - x} \geq 0$

6. $\dfrac{x + 3}{x^2 + 8x + 16} \leq 0$

7. $\dfrac{7 - x}{x^3 - 7x^2} > 0$

8. $\dfrac{7 - x}{x^3 - 7x^2} < 0$

9. $\dfrac{x^2 + x - 2}{x^2 - x - 30} \leq 0$

Chapter 3

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

3-1 INTRODUCTION.

In *Secondary Three* you learnt the meaning of a^x when a is a positive number and x is any real number. You spoke of a^x as a *power of* a, you called a the *base*, and you called x the *exponent*. Expressions like this lead to an important new type of functions called *exponential functions*. Before we explore these new functions we should recall some of the properties of exponents which you have learnt.

CLASS ACTIVITY

Simplify the following expressions and state the law of exponents you have used in each case.

1. $2^3 \times 2^4$	6. $(6^{\frac{1}{2}})^2$	11. $\dfrac{64^{\frac{3}{2}}}{8^{\frac{3}{2}}}$	16. $\left(3^{-\sqrt{2}}\right)^{\sqrt{8}}$
2. $2^{\frac{1}{2}} \times 2^{\frac{3}{2}}$	7. $(\sqrt[3]{5})^6$	12. $\left(4^{\sqrt{3}}\right)^{\sqrt{3}}$	17. $\dfrac{12^{-2}}{4^{-2}}$
3. $2^{\sqrt{2}} \times 2^{2\sqrt{2}}$	8. $4^2 \times 5^2$	13. $5^{-2} \times 5^{-4}$	18. $2^9 \times 2^{-9}$
4. $\dfrac{3^5}{3^2}$	9. $2^3 \times 3^3$	14. $\dfrac{\pi^{-3}}{\pi^2}$	19. $\dfrac{3^{4\cdot 6}}{3^{2\cdot 6}}$
5. $\dfrac{5^{\frac{5}{2}}}{5^{\frac{3}{2}}}$	10. $\dfrac{49^3}{7^3}$	15. $(8^{-2})^{-\frac{1}{2}}$	20. $(\frac{3}{4})^{0\cdot 69} \times (\frac{4}{3})^{0\cdot 69}$

Let us list the laws of exponents which you have been using. If x and y are real numbers and a and b are positive numbers, then

1. $a^x \times a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = a^{xy}$
4. $a^x \times b^x = (ab)^x$
5. $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

MORE CLASS ACTIVITY

1. Do you remember how the five laws of exponents stated above were justified when x and y were *positive integers*? Give an example for each of the five laws which makes it seem plausible.
   For example: $3^5 \times 3^2 = (3 \times 3 \times 3 \times 3 \times 3) \times (3 \times 3)$
   $= 3 \times 3 \times 3 \times 3 \times 3 \times 3 \times 3$
   $= 3^7$

2. Do you remember how we defined such expressions as a^0, a^{-3}, $a^{\frac{1}{4}}$, $a^{\frac{3}{4}}$, $a^{0.75}$? Discuss briefly the meaning of each of these expressions, pointing out how the definitions were chosen so that the above laws would still be true.

3. We extended the law $a^x \times b^x = (ab)^x$ to rational values of x, such as $\frac{1}{3}$, by proving that $\left(a^{\frac{1}{3}} \times b^{\frac{1}{3}}\right)^3 = a \times b$.
   a. How did we prove that $\left(a^{\frac{1}{3}} \times b^{\frac{1}{3}}\right)^3 = a \times b$?
   b. Why does this result prove that $a^{\frac{1}{3}} \times b^{\frac{1}{3}} = (a \times b)^{\frac{1}{3}}$?

4. After we had defined a^x for all rational numbers x, we were faced with the task of defining a^x when x is irrational. Tell briefly how it was that we used the following sequences to help define $a^{\sqrt{3}}$.

$$a^1, a^{1.7}, a^{1.73}, a^{1.732}, a^{1.7321}, \ldots$$

$$a^2, a^{1.8}, a^{1.74}, a^{1.733}, a^{1.7322}, \ldots$$

3-2 THE EXPONENTIAL FUNCTION DEFINED BY 2^x.

In *Secondary Three* you studied the meaning of the expression 2^x at some length. You found that 2^x could be defined for all real values of x, even for irrational values, as you recalled in the preceding section. You examined closely the graph of 2^x.

Let us now consider the *function* which is defined by 2^x and explore some of the properties of this function, much as you have been exploring properties of rational functions in Chapter 2.

You have learnt that 2^x has a definite value for any real number x. Since for every real number x we can find exactly one corresponding value of 2^x, we conclude that 2^x *defines a function* whose *domain is the set of all real numbers* x.

To study the *range* of this function let us examine the following table and graph. The table lists some of the ordered pairs of the function defined by 2^x. The graph is the graph of $y = 2^x$.

CLASS ACTIVITY

1. In both the table and the graph are there values of x for which 2^x is
   a. negative? b. zero? c. positive?

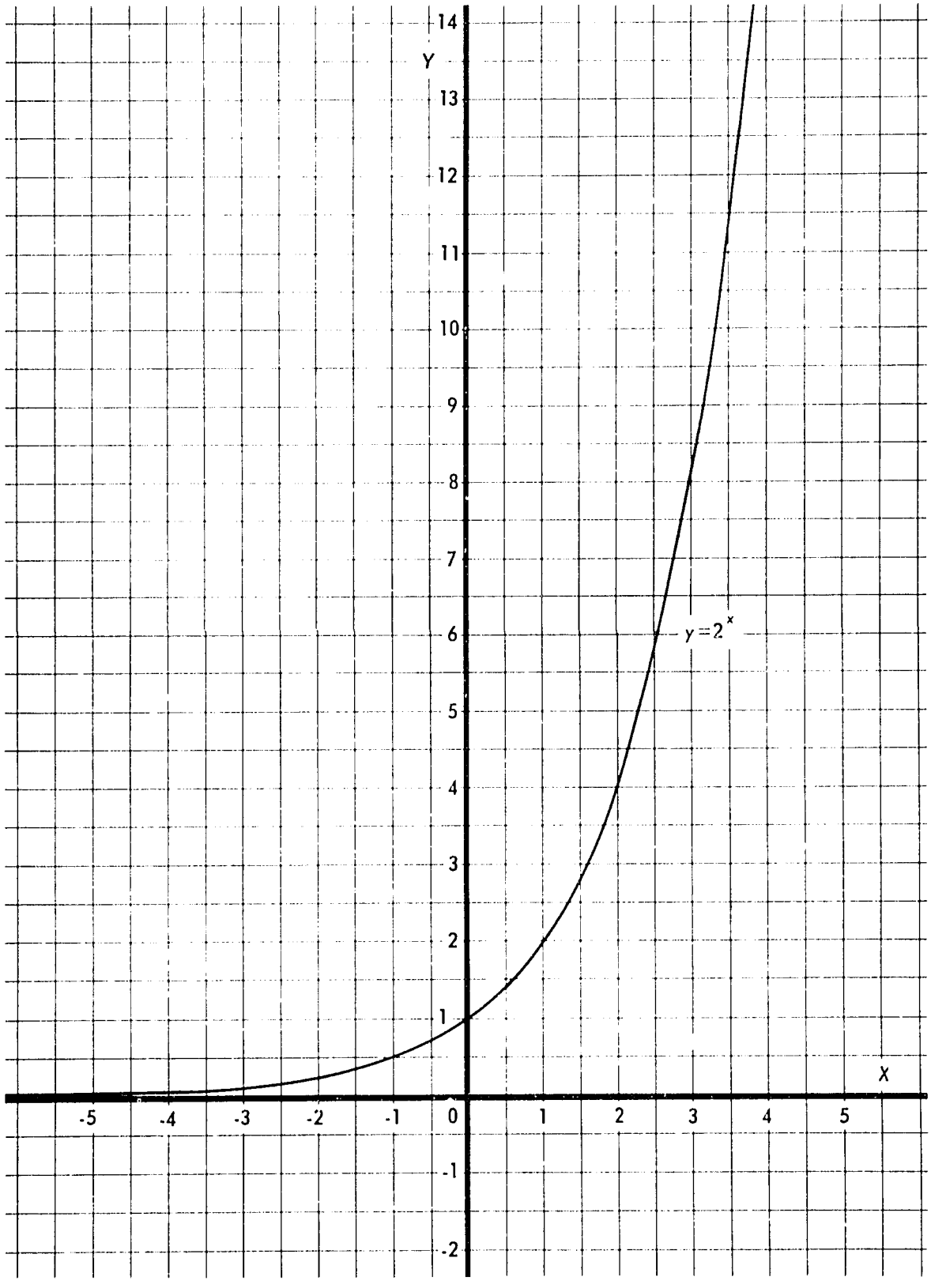
2. By the definition of 2^x can the value of 2^x ever be negative or zero
   a. if x is a positive integer?
   b. if x is zero?
   c. if x is a negative integer?
   d. if x is any rational number?
   e. if x is any irrational number?

3. From the graph does it appear that if we pick *any* positive number p, there is always a value of x for which $2^x = p$?

4. We have seen that all values of 2^x are positive and, conversely, for every positive number p there is a value of 2^x which is equal to p. What, therefore, is the *range* of the function defined by 2^x?

5. By observing the graph, answer the following questions:
   a. If, in the domain, $x < 0$, what is the corresponding subset of the range? If $x > 0$? If $x > 1$? If $x > 2$?
   b. If, in the range, $f(x) < 1$, what is the set of corresponding values of x in the domain? If $f(x) > 1$? If $f(x) > 2$?

x	-20	$-\frac{15}{2}$	3	$-\sqrt{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{7}{3}$	3	$\frac{22}{7}$	20
2^x	$\cdot 95 \times 10^{-6}$	$\cdot 55 \times 10^{-2}$	$\cdot 125$	$\cdot 38$	$\cdot 714$	1	$1\cdot 414$	2	$5\cdot 44$	8	$8\cdot 80$	$1\cdot 05 \times 10^{6}$



We have seen above that if $f(x) = 2^x$, then the *domain* of the exponential function f is the set of all real numbers, and the *range* is the set of all positive real numbers. Another property of this exponential function is that it is *continuous* for all real numbers x. You will recall that we defined 2^x for irrational exponents in such a way that the graph would be a curve without any breaks or holes in it, as we see in the graph above.

Let us examine the graph of 2^x for extreme values of x.

CLASS ACTIVITY

1. What happens to the graph of the function defined by 2^x when x is negative and $|x|$ is very large?
2. Name the asymptote to the graph of this function.
3. What happens to the graph of this function when we take larger and larger positive values of x?

The graph of 2^x appears to rise continuously from left to right, that is, 2^x appears always to increase as x increases. Let us see if we can show why this is true.

CLASS ACTIVITY

1.

a. $3 > 2$.	Is $2^3 > 2^2$?	d. $0 > -3$.	Is $2^0 > 2^{-3}$?
b. $1 > -1$.	Is $2^1 > 2^{-1}$?	e. $2\sqrt{2} > \sqrt{2}$	Is $2^{2\sqrt{2}} > 2^{\sqrt{2}}$?
c. $\frac{3}{2} > 1$.	Is $2^{\frac{3}{2}} > 2^1$?	f. $\sqrt{13} > 2\sqrt{3}$	Is $2^{\sqrt{13}} > 2^{2\sqrt{3}}$?

2. You may not have been sure in the last cases in Question 1. Look at it this way.
   a. If $\sqrt{13} > 2\sqrt{3}$, then $\sqrt{13} - 2\sqrt{3} > 0$. Why?
   b. If $\sqrt{13} - 2\sqrt{3} > 0$, then $2^{\sqrt{13} - 2\sqrt{3}} > 1$. Why?
   c. If $2^{\sqrt{13} - 2\sqrt{3}} > 1$, then $\dfrac{2^{\sqrt{13}}}{2^{2\sqrt{3}}} > 1$. Why?
   d. If $\dfrac{2^{\sqrt{13}}}{2^{2\sqrt{3}}} > 1$, then $2^{\sqrt{13}} > 2^{2\sqrt{3}}$. Why?
   e. Do you see that we have proved that if $\sqrt{13} > 2\sqrt{3}$, then $2^{\sqrt{13}} > 2^{2\sqrt{3}}$?
3. Now prove that for any real numbers a and b, if $a > b$, then $2^a > 2^b$.
   a. If $a > b$, then $a - b > 0$. Why?
   b. If $a - b > 0$, then $2^{a-b} > 1$. Why?

c. If $2^{a-b} > 1$, then $\frac{2^a}{2^b} > 1$. Why?

d. If $\frac{2^a}{2^b} > 1$, then $2^a > 2^b$. Why?

Since $2^a > 2^b$ whenever $a > b$, we say that the function defined by 2^x is *increasing* for all real numbers x. (See page 37 in Chapter 1.)

3-3 THE EXPONENTIAL FUNCTION DEFINED BY 2^{-x}.

You are familiar now with the graph of $y = 2^x$. How would the graph of $y = 2^{-x}$ be different from the graph of $y = 2^x$?

x	-4	-3	-2	-1	0	1	2	3	4
2^x	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16

x	-4	-3	-2	-1	0	1	2	3	4
2^{-x}	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

CLASS ACTIVITY

Use the tables above and your knowledge of 2^x to help answer the following questions.

1. Why is $2^{-x} = \frac{1}{2}$ when $x = 1$?
2. Why is $2^{-x} = 8$ when $x = -3$?
3. Why is 2^{-x} less than 1 when x is positive?
4. Why is 2^{-x} greater than 1 when x is negative?
5. As the value of x increases, does 2^{-x} increase or does it decrease?
6. Describe the behaviour of 2^{-x} when x has values such as 200,000 or 58,000,000.
7. Describe the behaviour of 2^{-x} when x has values such as -200,000 or -58,000,000.
8. When $2^x = 8$, what is the value of x? When $2^{-x} = 8$, what is the value of x?
9. Answer question 8 when 2^x and 2^{-x} are both equal to 16, $\frac{1}{2}$, $\frac{1}{4}$, 4, 1.

Since $5 = -(-5)$, it is true that 2^5 and $2^{-(-5)}$ have the same value, 32. That is, 2^x for $x = 5$ is equal to 2^{-x} for $x = -5$. In general, if two values of x are opposites of each other, the corresponding values of 2^x and 2^{-x} will be equal, that is, the y-values on the graph will be the same.

Do you see from the above discussion that there is a simple geometric relation between the graph of $y = 2^x$ and that of $y = 2^{-x}$? What kind of symmetry do you observe? Try sketching the graph of $y = 2^{-x}$ before you turn the page.

You will see that the functions defined by 2^x and 2^{-x} have the following properties in common:

1. The domain is the set of all real numbers.
2. The range is the set of positive real numbers.
3. The function is continuous for all values of x.
4. The graph includes the point (0, 1).
5. The graph approaches the x-axis as an asymptote in one direction and increases without limit in the other direction.

A striking difference of the two functions is that 2^x is increasing and 2^{-x} is decreasing. The symmetry of the two graphs can be seen by turning the paper over about the y-axis and seeing how each graph goes into the other. The graphs of 2^x and 2^{-x} are mirror images of each other in the y-axis.

Now observe how all these properties appear on the graph. (See page 104)

3-4 THE EXPONENTIAL FUNCTION DEFINED BY a^x FOR ANY POSITIVE NUMBER a.

You have a graph of $y = 2^x$. If you wanted a graph of $y = 4^x$, one way to get it would be to write 4^x as 2^{2x}.

CLASS ACTIVITY

1. Why is $4^x = 2^{2x}$?

2. Copy and complete the following table, using your graph of $y = 2^x$ on page 99 to determine some of the values.

x	-3	-2	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{3}{2}$	2	3
$2x$					-1			$\frac{1}{2}$						6
2^{2x}					0·5			1·4						64
4^x					0·5			1·4						64

3. Use the table you have completed to help draw carefully the graph of $y = 4^x$. Make your graph as large as your paper allows.

4. On the same set of axes draw carefully the graph of $y = 2^x$. Use the graph given on page 99 to help find enough points for a good graph.

5. Compare the two graphs you have drawn. What properties do they have which are alike and how are they different?

6. From the graph of $y = 2^x$ verify the fact that $2^{1\cdot6} \approx 3$.
Why is $3^x \approx (2^{1\cdot6})^x = 2^{1\cdot6x}$?
How could we use this fact to find values of 3^x?

7. Copy and complete the following table:

x	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$
$1\cdot6x$		$-3\cdot2$			$-0\cdot8$				$2\cdot4$		
$2^{1\cdot6x}$		$0\cdot1$			$0\cdot6$				$5\cdot3$		
3^x									$5\cdot3$		

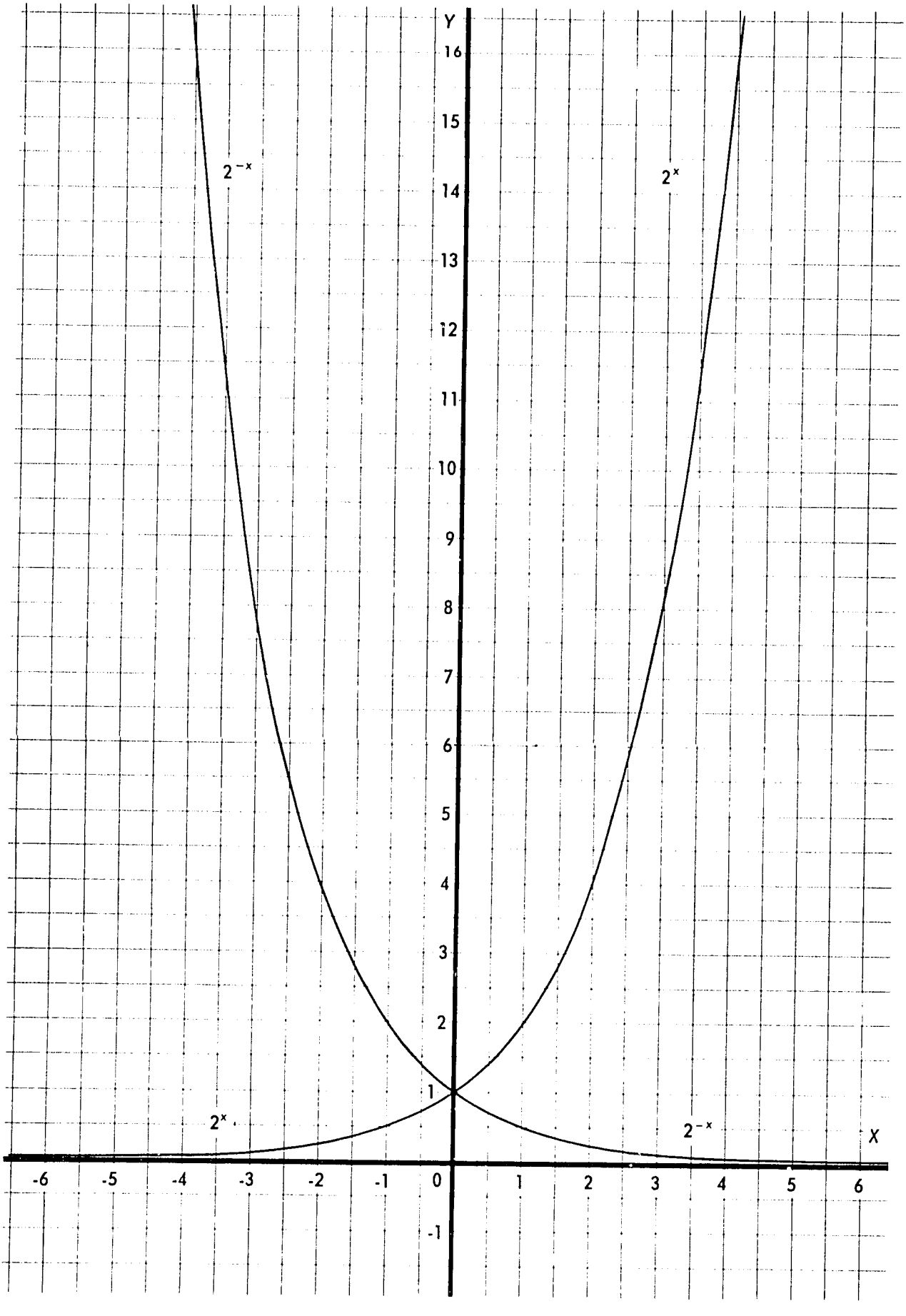
8. Use the values you have found in the table to draw carefully the graph of $y = 3^x$ on the same set of axes on which you graphed 2^x and 4^x.

9. What do you observe about the properties of 3^x?

10. Could we similarly find values of 5^x? 10^x? $(\frac{7}{2})^x$? $(\sqrt{2})^x$?

From your experiments above you probably see now that for any positive number a greater than 1 there is an exponential function defined by a^x and these functions have the following properties in common.

1. The domain is the set of all real numbers.
2. The range is the set of positive real numbers.
3. The function is continuous for all values of x.
4. The graph includes the point (0, 1).
5. The function is increasing for all values of x.
6. Values of the function are greater than 1 for $x > 0$ and less than 1 for $x < 0$.
7. The graph approaches the x-axis as an asymptote on the left and increases without limit on the right.


Y
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-2
-1
0
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4
5
6

The graphs differ in that for x greater than 0 those with larger values of a rise more rapidly than those with smaller values of a. For x less than 0, those graphs with larger values of a are closer to the x-axis.

PROBLEMS 3-4A

1. You have drawn the graphs of the functions defined by 2^x, 3^x, and 4^x. For each function estimate from the graph the values of x for which the function has the values 0·75, 1·5, and 2·5.
2. What is the truth set of the equation $a^x = b^x$ for $a > b > 1$?
3. Describe the graph of $y = 1^x$. Is the function defined by 1^x increasing or is it decreasing? What kind of function is it?
4. *CHALLENGE PROBLEM*
   Prove that for all real numbers $x (x \neq 0)$ if $a > b > 1$, then $a^x > b^x$ for $x > 0$ and $a^x < b^x$ for $x < 0$.
   Hint: Consider $\frac{a^x}{b^x} = \left(\frac{a}{b}\right)^x$.

In Section 3-3 you studied the graph of $y = 2^{-x}$. We can now learn more from that graph.

CLASS ACTIVITY

1. Why is 2^{-x} equal to $(\frac{1}{2})^x$?
2. Why is $(\frac{1}{4})^x$ equal to $(\frac{1}{2})^{2x}$ and also to 2^{-2x}?
3. How could we use the graph of $y = 2^{-x}$ to help obtain the graph of $y = (\frac{1}{4})^x$?
4. Copy and complete the following table:

x	-3	-2	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	2	3
$2x$	-6			$-\frac{3}{2}$						$\frac{3}{2}$			
$-2x$	6			$\frac{3}{2}$						$-\frac{3}{2}$			
2^{-2x}	64			2·8						0·4			
$(\frac{1}{4})^x$	64												

5. Draw on the same set of axes the graphs of $y = (\frac{1}{2})^x$ (Use the graph of $y = 2^{-x}$ on page 104) and $y = (\frac{1}{4})^x$. (Use the table above.)

6. Verify from the graph of $y = 2^x$ on page 104 that $\frac{3}{4} \approx 2^{-0\cdot42}$.

7. Describe how you could use the fact that $(\frac{3}{4})^x \approx (2^{-0\cdot42})^x = 2^{-0\cdot42x}$ to help find the graph of $y = (\frac{3}{4})^x$.

8. Without actually drawing the graph can you visualize how the graph of $y = (\frac{3}{4})^x$ would differ from the graphs of $y = (\frac{1}{4})^x$ and $y = (\frac{1}{2})^x$? What do you think the graphs would be like for $y = (\frac{1}{10})^x$? For $y = (0\ 6)^x$?

From this discussion you may see that for any base less than 1 $(0 < a < 1)$, there is an exponential function defined by a^x. The properties dealing with domain, range, continuity, and including the point (0, 1) are all the same as those for a^x when the base is greater than 1. (See page 103, properties 1-4.) We see, however, that when the base is less than 1,

5. The function is *decreasing* for all values of x.

6. Values of the function are greater than 1 for $x < 0$ and less than 1 for $x > 0$.

7. The graph approaches the x-axis as an asymptote on the *right* and increases without limit on the *left*.

Also, for x less than 0, the curve will descend more rapidly for smaller values of a (nearer 0) than with larger values of a (nearer 1), and for x greater than 0, those graphs with smaller values of a are closer to the x-axis.

We observe also that exponential functions with base greater than 1 and those with base less than 1 are related by the fact that the graphs of $y = a^x$ and $y = \left(\frac{1}{a}\right)^x$ are mirror images of each other in the y-axis.

PROBLEMS 3-4B

1. From your graphs of the functions defined by $(\frac{1}{2})^x$ and $(\frac{1}{4})^x$, estimate the values of $(\frac{1}{2})^x$ and $(\frac{1}{4})^x$ when x is -1·4, 0·6, 2·5.

2. For what values of x will 2^x and 4^x have the same values that you have estimated in Problem 1?

3. *CHALLENGE PROBLEMS*
   Prove that if $0 < a < b < 1$, then
   a. $a^x > b^x$ for $x < 0$
   b. $a^x < b^x$ for $x > 0$

4. *CHALLENGE PROBLEM*
   Do you think that the function defined by a^x for $a \neq 1$ is a polynomial function?

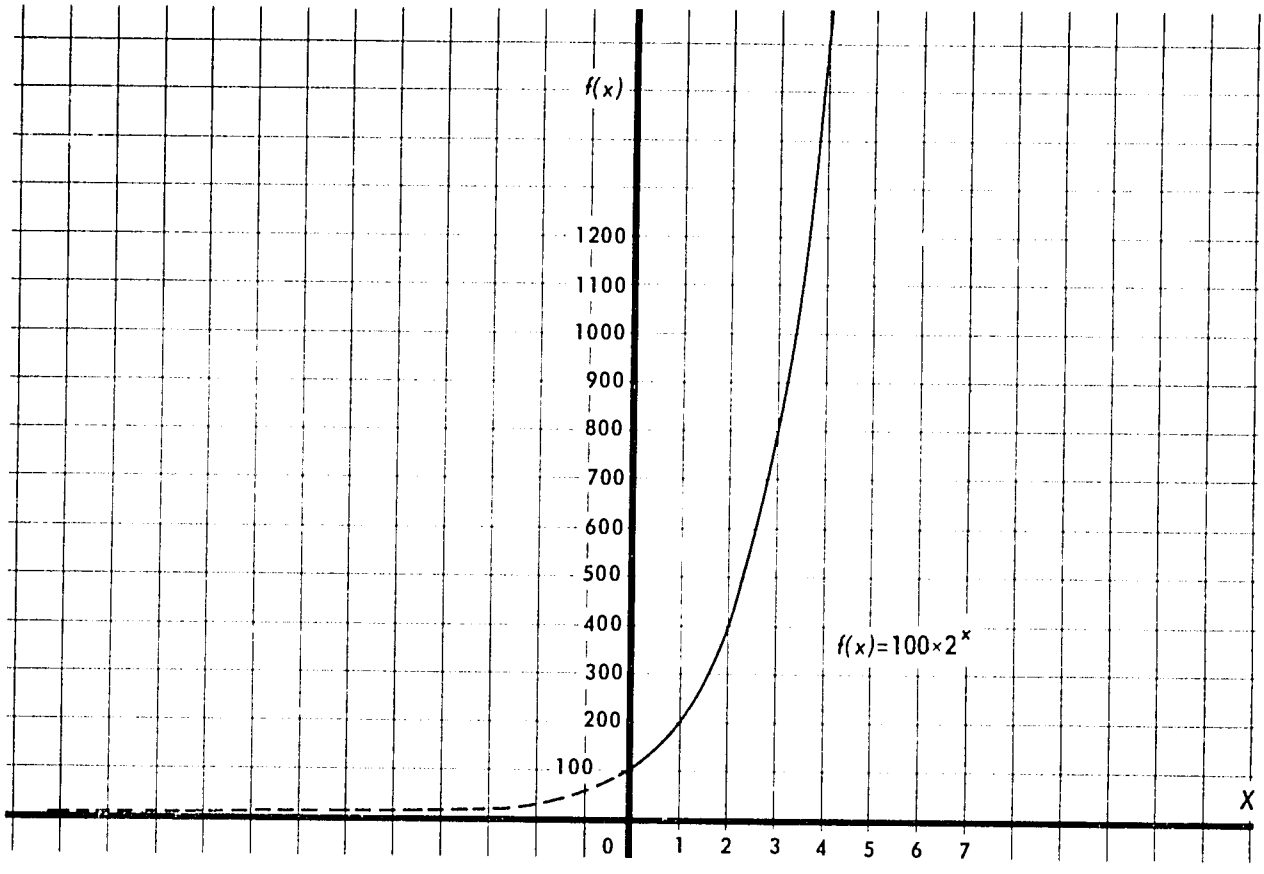
   Let us write $f(x) = a^x$. Then since $a^x \times a^x = a^{2x}$ (Why?) we have $f(x) \times f(x) = f(2x)$. Suppose, now, that $f(x)$ is a polynomial function of degree n; then the term of highest degree on the left-hand side of $f(x) \times f(x) = f(2x)$ is of degree $2n$, while that on the right-hand side is still of degree n. (Why?) This is a contradiction since two polynomials of different degrees cannot be identical. We say, then, that *the function defined by a^x $(a \neq 1)$ is not a polynomial function.*

3-5 APPLICATIONS OF EXPONENTIAL FUNCTIONS.

1. *GROWTH OF BACTERIA*

Suppose in a bacteria colony at a certain time of the day a biologist observes that there are 100 bacteria in the colony. Then if the biologist knows that each bacterium breaks into two after one hour he may wish to find the size of the colony at any other time.

After the first hour there will be 100×2 bacteria. After the second hour there will be $100 \times 2 \times 2$ bacteria. What will be the size of the colony after the 3rd hour, 5th hour, 6th hour, 10th hour? Can you see that after the xth hour the size of the colony will be 100×2^x? You may see that the size of the bacteria colony after the xth hour is the value of an exponential function defined by 100×2^x.



Here we have the graph of the function for the size of the bacteria colony. On the left of the y-axis the graph is dotted in to indicate that there must have been some bacteria alive before the biologist took his first measurement. It is of interest to realize that here we have a mathematical model to describe the growth in size of the bacteria colony. With the model one can answer questions such as: At what time is the size of the bacteria colony 7500? What was the size of the colony $4\frac{1}{2}$ hours before the biologist took the original measurement?

2. *COMPOUND INTEREST*

Suppose you invest shs. 150/-- in a business at an annual rate of $2\frac{1}{2}\%$ and at the end of each year interest is compounded or added to the amount which you have in the investment at that time.

Let us say the amount after the first year is A_1 shillings. Then

$$A_1 = 150 + 150 \times \frac{2\frac{1}{2}}{100}$$

$$= 150 + 150 \times \frac{5}{200}$$

$$= 150 + 150 \times \frac{1}{40}$$

$$= 150\left(1 + \frac{1}{40}\right)$$

The amount A_2 shillings after the 2nd year would be given by

$$\begin{aligned} A_2 &= A_1 + A_1 \times \frac{2\frac{1}{2}}{100} \\ &= A_1\left(1 + \frac{5}{200}\right) \\ &= A_1\left(1 + \frac{1}{40}\right) \\ &= 150\left(1 + \frac{1}{40}\right)\left(1 + \frac{1}{40}\right) \\ &= 150\left(1 + \frac{1}{40}\right)^2 \end{aligned}$$

The amount A_3 shillings after the 3rd year would be given by

$$\begin{aligned} A_3 &= A_2 + A_2 \times \frac{2\frac{1}{2}}{100} \\ &= A_2\left(1 + \frac{1}{40}\right) \\ &= 150\left(1 + \frac{1}{40}\right)^2\left(1 + \frac{1}{40}\right) \\ &= 150\left(1 + \frac{1}{40}\right)^3 \end{aligned}$$

This suggests that the amount A_t shillings after t years would be given by

$$\begin{aligned} A_t &= 150\left(1 + \frac{1}{40}\right)^t \\ &= 150\left(\frac{41}{40}\right)^t \end{aligned}$$

You may see that the expression which gives the amount you have in the business after t years defines an exponential function. Again we should recognize that this is a mathematical model we have found which can enable us to calculate the amounts we have in a business every year when we know the annual rate of compound interest. In general if the annual rate of interest is $r\%$ and our principal is P shillings, then the amount after t years is given by

$$A_t = P\left(1 + \frac{r}{100}\right)^t.$$

The domain of the function is restricted to the set of whole numbers because the interest is added only once a year. Similar functions can be described which give the amount in the investment if the interest is calculated and added on a half yearly or quarter yearly basis.

PROBLEMS 3-5

1. A man invests shs. 100/- in a business at an annual rate of 2%, with interest compounded half yearly. Set up the exponential function which will give the number of shillings in the investment at the end of t half-yearly periods. Be careful to observe that if the interest rate is 2% per year, then it is only 1% per half year. Find the amount in the investment after 2 years. After $2\frac{1}{2}$ years.

2. The population of a village is 600 people. Given that the population of the village increases at the annual rate of 5%, set up the exponential function which can be used to find the population at the end of t years. What will be the approximate size of the population after 5 years? Does this population increase behave as compound interest does?

3-6 LOGARITHMIC FUNCTIONS.

CLASS ACTIVITY

1. You have seen from your experience with exponential functions that, given a positive real number x in an equation such as $2^y = x$, a unique real number y can be found to satisfy the equation. Find the value of y which satisfies each of the following equations:

   a. $2^y = 32$ c. $5^y = 125$ e. $10^y = 10{,}000$

   b. $2^y = \frac{1}{4}$ d. $(\frac{1}{3})^y = 27$ f. $(\frac{1}{10})^y = (\sqrt[3]{10})^2$

2. In *Secondary Three* you saw that the unique real number y which satisfies the expression $2^y = x$, for any positive real number x, is defined to be the logarithm of x to the base 2; and y is denoted by $\log_2 x$. Thus since $5^3 = 125$ then $3 = \log_5 125$ and since $10^2 = 100$, then $2 = \log_{10} 100$. Find the values of the following expressions showing the exponential equation which you used to find your answer in each case:

   a. $\log_2 64$ d. $\log_7 \sqrt[3]{49}$

   b. $\log_3 \frac{1}{27}$ e. $\log_{\frac{1}{2}} \frac{1}{16}$

   c. $\log_{10} 10$

We have seen that the exponential function defined by $a^x (a > 1)$ has as its domain the set of all real numbers and as its range the set of positive real numbers. We also saw that this function has a one-to-one correspondence between the elements of its domain (x) and the elements of its range (a^x). Therefore the function defined by a^x for $a > 1$ has *an inverse function* whose domain is the set of positive real numbers and whose range is the set of all real numbers. This inverse function is called the *logarithm with base* a and its value at x is written as $\log_a x$. As with all inverse functions, the ordered pairs of the logarithmic function defined by $y = \log_a x$ are the same as those of the exponential function defined by $y = a^x$ except that the order has been reversed.

CLASS ACTIVITY

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	$y = \log_2 x$
$y = 2^x$	$\frac{1}{64}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64	x

Using this table and the same set of axes sketch the graphs of the functions defined by 2^x and $\log_2 x$. Read the table from the left for $y = 2^x$ and from the right for $y = \log_2 x$. Then answer the following questions *for the functions defined by* $\log_2 x$.

1. For what subset of the domain are the values of the function positive?

2. For what subset of the domain are the values of the function negative?

3. Does the graph of $y = \log_2 x$ have any jumps or breaks for any value of x? Is the function continuous?

4. From the graph of $y = \log_2 x$ do you think that $\log_2 4 > \log_2 3$? Do you think that $\log_2 a > \log_2 b$ whenever $a > b$? Is the function defined by $\log_2 x$ decreasing? Increasing?

5. Can you see what happens to the graph of the function defined by $\log_2 x$ when x is given positive values closer and closer to zero? What line is an asymptote of the graph of the function?

6. From your graphs of $y = 2^x$, $y = 3^x$, and $y = 4^x$ can you visualize the graphs of $y = \log_2 x$, $y = \log_3 x$, and $\log_4 x$? (Remember the relation between graphs of inverse functions.) Would the functions defined by

$\log_3 x$ and $\log_4 x$ have the same properties as the function defined by $\log_2 x$?

7. State as many as you can of the properties of the function defined by $\log_a x$ for any a greater than 1.

We can summarize some of the properties of the logarithmic function defined by $\log_a x$ for any a greater than 1.

1. The domain is the set of all positive numbers.
2. The range is the set of all real numbers.
3. The function is continuous for all values of x in the domain.
4. The graph includes the point (1, 0).
5. The function is increasing for all values of x in the domain.
6. The y-axis is an asymptote of the graph.
7. The values of the function are negative when $x < 1$ and they are positive when $x > 1$.

PROBLEMS 3-6A

1. Using the definition of the logarithm of a number, verify the following relations:

a. $\log_2 32 > \log_2 16$

b. $\log_3 81 > \log_3(\sqrt{3})^4$

c. $\log_3(\frac{1}{27}) < \log_3(\frac{1}{9})$

d. $\log_5(\frac{1}{25}) < \log_5 125$

e. $\log_9(\sqrt[5]{81}) > \log_9\left(\frac{1}{\sqrt[3]{9}}\right)$

f. $\log_{10}(\frac{1}{1000}) < \log_{10} 1000$

2. *CHALLENGE PROBLEM*

Given that $\log_a x > \log_a y$ and $a > 1$, $x > 0$, $y > 0$, use the definition of the logarithm to prove that $x > y$. Hint: Let $\log_a x = p$ and $\log_a y = q$, and then compare a^p and a^q.

CLASS ACTIVITY

1. $y = 2^{-x}$ defines a familiar exponential function. Show why the inverse function is defined by $y = \log_{\frac{1}{2}} x$.

2. Make a table similar to the one on page 111 using x and $y = 2^{-x}$ on the left and $y = \log_{\frac{1}{2}} x$ and x on the right.

3. With the help of the table in Question 2 and the graph of $y = 2^{-x}$ (which is really the graph of $y = (\frac{1}{2})^x$) draw carefully the graph of $y = \log_{\frac{1}{2}} x$.

4. Answer the following questions about the function defined by $\log_{\frac{1}{2}} x$:
   a. Is the function decreasing or increasing?
   b. For what values of x are values of the function positive?
   c. For what values of x are values of the function negative?

5. If $y = \log_2 32$, then $2^y = 32$ (Why?), so $y = 5$. If $y = \log_{\frac{1}{2}} 32$, then $(\frac{1}{2})^y = 32$ (Why?) and $2^{-y} = 2^5$ (Why?) so $y = -5$.
   a. Find the values of $\log_2 16$, $\log_2 64$, $\log_2 \frac{1}{32}$.
   b. Then find the corresponding values of $\log_{\frac{1}{2}} 16$, $\log_{\frac{1}{2}} 64$, $\log_{\frac{1}{2}} \frac{1}{32}$.

6. a. Find the values of $\log_3 9$, $\log_5 625$, $\log_7 49$, $\log_4 16$.
   b. Then find the corresponding values of $\log_{\frac{1}{3}} 9$, $\log_{\frac{1}{5}} 625$, $\log_{\frac{1}{7}} 49$, $\log_{\frac{1}{4}} 16$.

7. What do you think is the relation between the graphs of the functions defined by $\log_2 x$ and $\log_{\frac{1}{2}} x$?

You should see now that the graphs of the functions defined by $\log_a x$ and $\log_{\frac{1}{a}} x$, for $a > 1$, are mirror images of each other with respect to the x-axis. In other words for a given value of x the values of the functions are opposites, that is $\log_a x = -\left(\log_{\frac{1}{a}} x\right)$. Also whereas $\log_a x$ is increasing, $\log_{\frac{1}{a}} x$ is decreasing. Both functions are continuous for all values of x in the domain and their graphs have the y-axis as an asymptote.

If we write $b = \frac{1}{a}$, we see that when $a > 1$, then $0 < b < 1$. Thus all that we have said about $\log_{\frac{1}{a}} x (a > 1)$ can be repeated for $\log_b x \, (0 < b < 1)$. We see, then, that the base of a logarithmic function can be any positive number except 1, but bases greater than 1 give increasing functions while bases less than 1 give decreasing functions.

PROBLEMS 3-6B

1. Using the definition of the logarithm of a number verify the following relations:

   a. $\log_{\frac{1}{2}} 32 < \log_{\frac{1}{2}} 16$

   b. $\log_{\frac{1}{8}} 64 < \log_{\frac{1}{8}} (\frac{1}{64})$

   c. $\log_{\frac{1}{3}} (\sqrt[4]{9}) > \log_{\frac{1}{3}} 27$

   d. $\log_{\frac{1}{10}} (\frac{1}{1000}) < \log_{\frac{1}{10}} (\frac{1}{10,000})$

2. *CHALLENGE PROBLEM*

   Using the definition of the logarithm, prove that, for $a > 1$, $x > 0$ and $y > 0$, $\log_{\frac{1}{a}} x < \log_{\frac{1}{a}} y$ whenever $x > y$. Hint: Let $\log_{\frac{1}{a}} x = p$ and $\log_{\frac{1}{a}} y = q$ and compare $\left(\frac{1}{a}\right)^p$ and $\left(\frac{1}{a}\right)^q$.

3-7 FURTHER PROPERTIES OF LOGARITHMIC FUNCTIONS.

In *Secondary Three* you studied some properties of logarithmic functions which were particularly useful in doing computations with the help of logarithms. We do not intend at this time to dwell on the computational use of logarithms, but let us list the properties to recall them to our minds. In a more complete treatment of logarithmic functions these properties would be examined more closely.

For $x > 0$, $y > 0$, $a > 0$, and $a \neq 1$,

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a (x)^k = k \times \log_a x$$

In this section we are going to study some further properties of the function defined by $\log_a x$ which, besides being interesting in themselves, will be useful when more than one base in involved.

First we observe a direct consequence of the inverse relation between exponential and logarithmic functions.

When $a^y = x$ ($x > 0$, $a > 0$, $a \neq 1$), then $y = \log_a x$. (Why?) Therefore, by direct substitution,

$$a^{\log_a x} = x$$

For example, $5^{\log_5 29} = 29$, and $3^{\log_3 92} = 92$.

We now derive a relation by which $\log_a x$, for any base a, can be written in terms of logarithms with base 10.

$$a^{\log_a x} = x$$

implies that $\log_{10}\left(a^{\log_a x}\right) = \log_{10} x$ (Why?)

which implies that $(\log_a x)(\log_{10} a) = \log_{10} x$ (Why?)

and this means that

$$\log_a x = \frac{\log_{10} x}{\log_{10} a} .$$

This result is useful for finding logarithms of numbers to any base if only tables of logarithms to the base 10 are available.

EXAMPLES

1. $\log_{2.5} 7{\cdot}6 = \dfrac{\log_{10} 7{\cdot}6}{\log_{10} 2{\cdot}5} = \dfrac{{\cdot}8808}{{\cdot}3979}$

$= 2{\cdot}202$

2. $\dfrac{\log_5 7}{\log_5 6} = \dfrac{\log_{10} 7}{\log_{10} 5} \div \dfrac{\log_{10} 6}{\log_{10} 5}$ (Why?)

$= \dfrac{\log_{10} 7}{\log_{10} 5} \times \dfrac{\log_{10} 5}{\log_{10} 6} = \dfrac{\log_{10} 7}{\log_{10} 6} = \log_6 7$ (Why?)

The second example above illustrates the more general formula:

$$\log_a x = \frac{\log_b x}{\log_b a} \qquad (x > 0,\ a > 0,\ b > 0,\ a \neq 1,\ b \neq 1)$$

This general formula is useful in converting from any base to any other base. It can be proved just as we proved the relation involving base 10.

Two interesting properties of the logarithmic function can be derived using the relation $\log_a x = \dfrac{\log_{10} x}{\log_{10} a}$.

1. We have been excluding 1 as a base. Let us examine the expression $\log_1 x$. If the above relation were to hold for base 1, we would have

$$\log_1 x = \frac{\log_{10} x}{\log_{10} 1}$$

We know, however, that $\log_{10} 1 = 0$. (Why?) This would give

$$\log_1 x = \frac{\log_{10} x}{0}$$

and the right hand side of this equation is, of course, undefined. This shows why *$\log_1 x$ is not defined for any value of x.*

2. $$\log_a b \times \log_b a = \left(\frac{\log_{10} b}{\log_{10} a}\right) \times \left(\frac{\log_{10} a}{\log_{10} b}\right) \qquad \text{(Why?)}$$

$$= 1 \qquad \text{(Why?)}$$

We therefore have the reciprocal of a logarithm expressed in terms of another logarithm.

$$\frac{1}{\log_a b} = \log_b a$$

Chapter 4
SEQUENCES AND SERIES

4-1 INTRODUCTION.

Most of us remember at one time or another having the feeling that a certain experience seems so familiar that we feel sure it must have happened before. When this feeling occurs in mathematics we try to figure out precisely what is so familiar and if possible to determine the pattern that seems to be repeating. This recognition of patterns plays an important part in all learning. If we can discover the pattern we often understand the situation in a way which was impossible before.

We shall look at some simple patterns which emerge from counting problems and try to use these insights to solve other problems.

When we first learn to count we repeat the names for the numbers 1, 2, 3, ... If our teacher asks us to count by 2's beginning with 1, we say 1, 3, 5, 7, ... If we are asked to count by 2's beginning with 2, we say 2, 4, 6, 8, ... In all three cases we are actually repeating in order a first number, a second number, a third number and so on.

This same process may be used to choose two teams to play a game. If we count off, we may then let the first team consist of the persons whose numbers were 1, 3, 5, 7, . . . , while the second team will consist of those persons whose numbers were 2, 4, 6, 8, ...

CLASSROOM ACTIVITY

1. (a) What number would the third person on the first team have?
   (b) the fifth person on the first team?
   (c) the tenth person on the first team?

(d) the nth person on the first team?

2. (a) What number would the third person on the second team have?
   (b) the eighth person?
   (c) the twelfth person?
   (d) the nth person?

3. (a) If John's number was 15, which team was he on?
   (b) If Hamisi's number was 20, which team was he on?
   (c) If Ali's number was 23, how many persons were already selected for his team before he was chosen?

4. Suppose a very rich man agrees to raise the wages of a favorite worker by 2 shillings each week for a year. If the worker received 10 shillings the first week, how much did he receive
   (a) the fourth week?
   (b) the eighth week?
   (c) the twentieth week?
   (d) the fifty-second week?

5. In Problem 4 above, for what week would the worker's salary be
   (a) 14 shillings?
   (b) 20 shillings?
   (c) 40 shillings?

6. In Problem 4 above, what would be the worker's total income for
   (a) the first four weeks?
   (b) the first eight weeks?

   CHALLENGE PROBLEMS
   (c) the first 20 weeks?
   (d) the entire year?

To answer questions like these and many others, it would be good to have dependable formulas which would give any particular number and sum of numbers in lists such as the ones we have been considering. In this chapter we shall develop such formulas.

A good example of the kind of result we shall obtain is contained in the following story told about Carl Friedrich Gauss as a young schoolboy. Gauss (1777-1855) is considered to be one of the greatest mathematicians of all time.

One morning Gauss's teacher had some important reports to write and to keep his pupils busy for a while he asked them to find the sum of the first 100 natural numbers, $1 + 2 + 3 + \ldots + 100$. When the teacher looked up from his desk a few minutes later he was surprised to find that Gauss was not working. When the teacher asked why he was not doing his work, Gauss showed the teacher his answer, 5050. The teacher was very much surprised and asked how he had found the answer so quickly. Here is Gauss's method:

To find the sum	$1 + 2 + 3 + \ldots + 99 + 100,$
write the numbers in reverse order	$100 + 99 + 98 + \ldots + 2 + 1.$
Add two by two	$101 + 101 + 101 + \ldots + 101 + 101$
and we have	$100 \times 101.$

Since this is twice the required sum, the answer is $\frac{10100}{2}$ or 5050.

While getting correct answers is an important part of mathematics, it is even more important to be able, as Gauss did, to give a convincing argument to show that our answers are correct.

PROBLEMS 4-1

1. If we count by 3's beginning with 1
   (a) What will the fourth number be?
   (b) the seventh?
   (c) the tenth?
   (d) the twentieth?
   (e) the nth?
   (f) Which of the numbers 12, 13, 14, 15, 16 occur in this counting process? Which terms in the list are they?

2. In Problem 1, what is the sum of
   (a) the first 4 terms?
   (b) the first 7 terms?
   (c) the first 10 terms?

3. If we count by 3's beginning with 2, what is
   (a) the third term?

(b) the fifth term?
(c) the eighth term?
(d) the sixteenth term?
(e) the nth term?

4. In Problem 3, what is the sum of
(a) the first 3 terms?
(b) the first 5 terms?
(c) the first 10 terms?

CHALLENGE PROBLEM
(d) the first n terms?

5. If we begin with the number 7 and count by 1's, what is
(a) the fourth term?
(b) the seventh term?
(c) the eleventh term?
(d) the nth term?
(e) the sum of the first 4 terms?
(f) the sum of the first 7 terms?
(g) the sum of the first 11 terms?

CHALLENGE PROBLEM
(h) the sum of the first n terms?

6. If we count backward by 1's starting with 10, what is
(a) the sixth term?
(b) the tenth term?
(c) the eleventh term?
(d) the fifteenth term?

7. A pupil started counting by 3's. The third term was 10. Where did he begin? What will the sixth term be?

8. In a certain counting, 5 and 17 occur. Could we be counting by 2's? by 3's? by 5's? by 6's?

4-2 SEQUENCES.

In problems which we considered in Section 4-1 we actually used a very special kind of function called a *sequence*.

Definition: A *sequence* is a function whose domain is the set of natural numbers. The values of the function are called *terms* of the sequence and are always listed in order corresponding to the natural order of the natural numbers.

The value of the function corresponding to the number n of the domain of the function is called the nth *term* or the *general term* of the sequence.

For these special functions called sequences it is customary to write f_n instead of $f(n)$ for the value of the function corresponding to the natural number n. This change of notation serves as a reminder that the domain of the function is the set of natural numbers and that we list the values in order, $f_1, f_2, f_3, \ldots, f_n, \ldots$. We sometimes write $\{f_n\}$ for the sequence instead of $\{(n, f(n))\}$.

The problems which we considered in Section 4-1 may now be stated as finding the rule which defines a certain function.

Counting by 1's beginning with 1 is the same as listing the values of the sequence defined by the formula $f_n = n$.

$$f_1 = 1 \quad f_2 = 2 \quad f_3 = 3 \quad \ldots \quad f_n = n$$

We may write the sequence as $\{1, 2, 3, \ldots n, \ldots\}$.

Counting by 2's beginning with the number 1 gives 1, 3, 5, 7, ... What kind of formula would define a function g for which $g(1) = 1$, $g(2) = 3$, $g(3) = 5$, $g(4) = 7$ and so on? Since $g(1)$ must be 1 and we are counting by 2's,

$g(2)$ should be $g(1) + 2$ or $1 + 2$.

$g(3)$ should be $g(2) + 2$ or $(1 + 2) + 2 = 1 + 2 \times 2$.

Similarly $g(4)$ should be $g(3) + 2$ or $(1 + 2 \times 2) + 2 = 1 + 3 \times 2$,

and $g(5)$ should be $g(4) + 2$ or $(1 + 3 \times 2) + 2 = 1 + 4 \times 2$.

If we try to see a pattern emerging, it seems that $g(n)$ should be $1 + (n - 1) \times 2$. If we check values of the function defined by $g(n) = 1 + (n - 1) \times 2 = 2n - 1$, then, in the notation for a sequence, $g_1 = 1$, $g_2 = 3$, $g_3 = 5$, $g_4 = 7$, ... These however are the numbers used in counting by 2's beginning with 1.

If we try to find a sequence h with values $h_1 = 2$, $h_2 = 4$, $h_3 = 6$, $h_4 = 8$ we again see that to get h_n we add 2 to h_{n-1};

i.e.,

$$h_2 = h_1 + 2$$
$$h_3 = h_2 + 2 = (h_1 + 2) + 2 = h_1 + 2 \times 2$$
$$h_4 = h_3 + 2 = (h_1 + 2 \times 2) + 2 = h_1 + 3 \times 2.$$

Since h_1 is 2, as before the pattern seems to be

$$h_n = 2 + (n - 1)\,2 = 2n.$$

The values of this sequence $\{h_n\}$ are just the numbers we used in counting by 2's beginning with 2.

Can you find a function w such that $w(n)$ will give the worker's wages during the nth week? (Problem 4, CLASSROOM ACTIVITY, Page 118.)

Check your answer for $n = 1, 2, 3, 4, 5$.

It is an easy matter to check your answer for a particular value of n, but how do we know that the function you propose will give the correct answer for all values of n?

Find the values of the function w defined by $w(n) = 2n + 8 + \frac{(n-1)(n-2)(n-3)}{6}$ for $n = 1, 2, 3$. Now find $w(n)$ for $n = 4, 5, 6$.

We see that for $n = 1, 2, 3$ the answers check with our previous results for the worker's wages but for $n > 3$ the function does not give the correct values.

How does one go about finding such a function w, and how does one show that it gives correct answers for all values of n?

The answer to the first question lies in collecting information and then making an intelligent guess.

The second question needs more thought and we shall consider it in some detail in later sections.

Let us now look at the graphs of these special functions which we have called sequences. It is interesting to see how they differ from the graphs of functions whose domain is the set of real numbers.

PROBLEMS 4-2A

1. Draw the graph of the functions f, g, h, and of your function w in the discussion above. (Remember that the domain is the set of natural numbers.)

2. If the domain is the set of real numbers instead of the set of natural numbers, draw the graphs of the functions f, g, h, and w defined by the same formulas.

You may notice in the problems above that all of these graphs consist of points which lie on straight lines. Such sequences are a very special kind and the terms of the sequence are said to be *in arithmetic progression.* The list of values of such a sequence (in the natural order $f_1, f_2, f_3, f_4, \ldots$) is called *an arithmetic progression.* These special functions are of considerable importance and will be studied in some detail in the next section.

PROBLEMS 4-2B

1. Find the sequence which corresponds to counting by 5's beginning with the number 1. With the number 3.

2. If $S(n)$ is the sum of the first n natural numbers, find $S(n)$ for $n = 1, 2, 3, 4, 5$.

CHALLENGE PROBLEM

3. Try to find an expression for the nth term of the sequence S in Problem 2 above.

Of course not all sequences have graphs which lie on straight lines.

Draw the graph of $S(n)$ in Problem 2 (Problems 4-2B) for $n = 1, 2, 3, 4, 5$. Do the points lie on a straight line?

Some sequences are given by very complicated formulas, while others are given by rules which cannot be reduced to a formula at all. (See Problem 1 (n) and 1 (o) below.)

PROBLEMS 4-2C

1. Examine each of the following rules. Does the rule define a sequence? If possible find the first 5 terms of the sequence.

(a) $f_n = 2 + 3n$

(b) $f_n = 2 \times 3^n$

(c) $f_n = 2 + 3^n$

(d) $2f_n^2 = 6n + 12$

(e) $f_n = 2 + (-1)^n$

(f) $f_n = 3$

(g) $f_n^2 = 4$

(h) $f_n = 2^{n-1}$

(i) $f_n = f_n + n, f_1 = 1$

(j) $f_{n+1} = f_n + n, f_1 = 1$

(k) f_n is the nth prime number

(l) $f_n = p_n - 1$ where p_n is the nth prime number

(m) $f_{n+2} = f_{n+1} + f_n + 1$ where $f_1 = 1$ and $f_2 = 2$

(n) f_n is the number of ships arriving in Mombasa on the nth day of the year 1966

(o) f_n is the number of lions born on the nth day of 1966 in Nairobi National Park

CHALLENGE PROBLEM

(p) $f_n = \dfrac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n\sqrt{5}}$

2. Try to find a rule for a sequence whose first 4 terms are:

(a) 1, 3, 5, 7

(b) 1, 4, 9, 16

(c) 1, -1, 1, -1

(d) 10, 40, 160, 640

(e) $\frac{1}{1 \times 2}, \frac{1}{2 \times 3}, \frac{1}{3 \times 4}, \frac{1}{4 \times 5}$

(f) $1, \frac{1}{1 \times 2}, \frac{1}{1 \times 2 \times 3}, \frac{1}{1 \times 2 \times 3 \times 4}$

(g) 0, 1, 3, 7

(h) 2, 1, $\frac{1}{2}$, $\frac{1}{4}$

(i) $\frac{3}{4}, \frac{6}{7}, \frac{9}{10}, \frac{12}{13}$

(j) 2, 3, 5, 7

(k) 1, 1, 2, 3

(l) $-\frac{2}{3}, -\frac{5}{9}, -\frac{17}{27}, -\frac{65}{81}$

(m) 1, 3, 6, 10

(n) $\frac{5}{2}, \frac{3}{2}, \frac{7}{3}, \frac{5}{3}$

3. Using the rules which you gave in Problem 2, find the first seven terms of each of the sequences in that problem.

4. *CHALLENGE PROBLEM*

The solution to the following problem is said to have been responsible for a sequence called the sequence of Fibonacci. "How many pairs of rabbits can be produced from a single pair in a year if (i) each pair begets a new pair every month, which from the second month on becomes productive; (ii) no deaths occur."

(a) The number of pairs of rabbits in each month will define a sequence f. Find the first 12 terms of this sequence. (f_{12} is the answer to the problem above.)

(b) Show that f_n is given by $\dfrac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$

for $n = 1, 2, 3$.

4-3 ARITHMETIC PROGRESSIONS.

We mentioned in Section 4-2 that the list of values of the special kind of sequence whose graph lies on a straight line is called an *arithmetic progression.* In this section we want to consider these sequences in more detail since they occur in many applications.

In the counting problems we considered, when we count by 1's or 2's any term of the sequence is obtained by adding 1 or 2 respectively to the preceding term. This property of adding a fixed number to any term to obtain the following term is the particular property of the sequence which makes its terms form an arithmetic progression.

> Definition: An *arithmetic progression* is a sequence in which each term after the first is obtained by adding a fixed number to the preceding term. If we denote the sequence by $\{A_n\}$ and the fixed number by d, $A_{n+1} = A_n + d$ for all natural numbers n. The number d is called the *common difference.*

The problem of counting by 2's beginning with 1 can be written in the notation of this definition as

$$A_1 = 1,\ d = 2$$
$$A_2 = 1 + 2$$
$$A_3 = (1 + 2) + 2 = 1 + 2 \times 2$$
$$A_4 = (1 + 2 \times 2) + 2 = 1 + 3 \times 2$$
$$A_5 = (1 + 3 \times 2) + 2 = 1 + 4 \times 2 \text{ etc.}$$

To get the 23rd term we add 22 times the common difference 2 to the first term 1; i.e.,

$$A_{23} = 1 + 22 \times 2 = 45$$

In general the nth term is obtained by adding $(n - 1)$ times the common difference to the first term:

$$A_n = 1 + (n - 1)\ 2 = 2n - 1.$$

PROBLEMS 4-3A

1. Write the first 5 terms and find a formula for the nth term obtained in counting by 5's beginning with 3.
2. Write the first 5 terms and find a formula for the nth term obtained in counting by 7's beginning with 5.
3. The first term of an arithmetic progression is 3 and the common difference is 4. Find the tenth term, the nth term.
4. The third term of an arithmetic progression is 9 and the common difference is 2. Find
   (a) the second term
   (b) the first term
   (c) the 200th term
   (d) the nth term
5. The fourth and fifth terms of an arithmetic progression are $A_4 = 47$ and $A_5 = 52$. What is d? Find A_1. Find A_{50}.

If we try to find A_n for any arithmetic progression in which the difference between successive terms is d, we can proceed in the same way.

$$A_1 = A_1$$
$$A_2 = A_1 + d$$
$$A_3 = A_2 + d = (A_1 + d) + d = A_1 + 2d$$
$$A_4 = A_3 + d = (A_1 + 2d) + d = A_1 + 3d$$
$$A_5 = A_4 + d = (A_1 + 3d) + d = A_1 + 4d$$

We see the nth term is obtained by adding $(n - 1)$ times the common difference to the first term.

> The nth term of an arithmetic progression with first term A_1 and common difference d is
> $$A_n = A_1 + (n - 1)\, d$$

If we consider again the wages of the rich man's worker, we recall that he was to receive 10 shillings the first week and an increase of 2 shillings per week for the rest of the year. The arithmetic progression $\{A_n\}$ with $A_1 = 10$ and $d = 2$ would give the worker's wages for any given week. By the formula which we developed for A_n above, his wages for the nth week are $A_n = A_1 + (n - 1)\, d = 10 + (n - 1)\, 2 = 8 + 2n$.

A second interesting problem is to find how much the worker received in all by the end of the year.

The amount is $\quad S = 10 + 12 + 14 + \ldots + (10 + 51 \times 2)$

$$S = 10 + 12 + 14 + \ldots + 110 + 112$$

If we use Gauss's idea and write $\quad S = 112 + 110 + 108 + \ldots + 12 + 10$

Then add $\quad 2S = 122 + 122 + 122 + \ldots + 122 + 122$

$$2S = 52 \times 122$$
$$S = \frac{52 \times 122}{2} = 3172$$

The worker received 3172 shillings in the year.

Here is a picture which represents the situation:

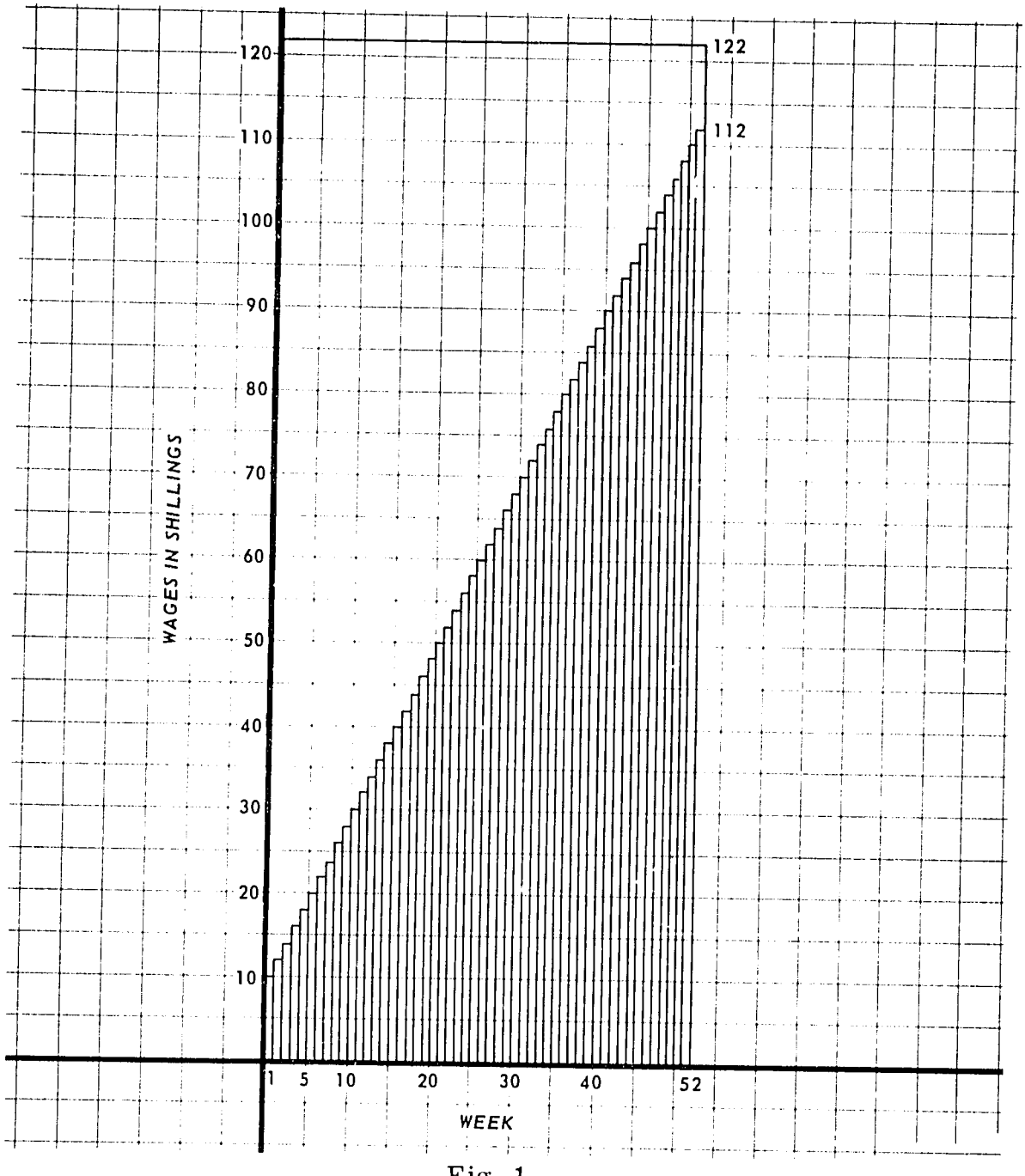


Fig. 1

The areas of the rectangles represent the worker's wages for the 1st, 2nd, . . ., 52nd week, so that the total wages for the year are represented by the area of the entire shaded region. A congruent region fits neatly together with the original region to form a rectangle 52×122. Therefore twice the shaded area is 52×122; that is $2S = 52 \times 122$ and $S = 3172$.

For any arithmetic progression with first term A_1 and common difference d, the sum S_n of the first n terms is

$$S_n = A_1 + (A_1 + d) + (A_1 + 2d) + \ldots + (A_n - d) + A_n$$

We may also write

$$S_n = A_n + (A_n - d) + (A_n - 2d) + \ldots + (A_1 + d) + A_1$$

Therefore

$$2S_n = (A_1 + A_n) + (A_1 + A_n) + (A_1 + A_n) + \ldots + (A_1 + A_n) + (A_1 + A_n)$$

$$2S_n = n(A_1 + A_n)$$

and

$$S_n = n\left(\frac{A_1 + A_n}{2}\right)$$

> The sum of the first n terms of an arithmetic progression with first term A_1 and n^{th} term A_n is
>
> $$S_n = n\left(\frac{A_1 + A_n}{2}\right);$$
>
> that is, the sum equals the number of terms times the average of the first and last terms.

If we now replace A_n by $A_1 + (n - 1)d$ we obtain a second useful formula for S_n.

$$S_n = n\left(\frac{A_1 + A_1 + (n - 1)d}{2}\right)$$

or

$$\boxed{S_n = \frac{n}{2}\left[2A_1 + (n - 1)d\right]}$$

We use whichever of those formulas is most convenient in a given problem.

If we apply the second formula to the problem of finding the total amount received by the end of the nth week by the rich man's worker, we get

$$\begin{aligned} S_n &= \frac{n}{2}\left[2(10) + (n - 1)2\right] \\ &= 10n + n(n - 1) \\ &= n^2 + 9n \end{aligned}$$

It is now an easy matter to find his total wages for the first four weeks: $S_4 = 4^2 + 9 \times 4 = 52$ shillings. For the first 8 weeks we have $S_8 = 8^2 + 9 \times 8 = 64 + 72 = 136$ shillings.

On the other hand if we had already found out that during the fourth week he earned $A_4 = 10 + 3 \times 2 = 16$ shillings, we might have found S_4 by using the first formula:

$$S_4 = \frac{4}{2}(10 + 16) = 2(26) = 52 \text{ shillings}$$

Similarly if we know that $A_8 = 10 + 7 \times 2 = 24$, then

$$S_8 = \frac{8}{2}(10 + 24) = 4(34) = 136 \text{ shillings.}$$

Using the second formula we find that

$$S_{20} = \frac{20}{2}(2(10) + 19(2)) = 10(20 + 38) = 580 \text{ shillings.}$$

And finally $S_{52} = \frac{52}{2}\big(2(10) + 51(2)\big) = 26(20 + 102) = 26\,(122) = 3172$ shillings.

The formulas

$$A_n = A_1 + (n - 1)d$$

$$S_n = n\left(\frac{A_1 + A_n}{2}\right)$$

$$S_n = \frac{n}{2}\left[2A_1 + (n - 1)d\right]$$

may now be used to solve many different problems involving numbers which are in arithmetic progression.

PROBLEMS 4-3B

1. A rich man dies and leaves a distant relative a fortune of 100,000 shillings. The relative invests the money at 6% simple interest. If he lets the interest accumulate, how much does his inheritance amount to after 5 years? After 10 years? How many years will it take his money to double?

2. If the third term of an arithmetic progression is 13 and the tenth term is 27, what is the first term? the 20th term? the 100th term? the sum of the first ten terms?

3. Find a general formula for the sum of the first n natural numbers.

4. What is the sum of the first n odd natural numbers? the first n even natural numbers?

5. The count down for a launching of a spacecraft has proceeded to launch minus 100 minutes. If from now on the count down can be interrupted only at 15 minute intervals, what is the last possible time that the count down can be halted before blast off?

6. How many integers are there between 20 and 100 which are divisible by 3?

7. How many integers are there between 10 and 1000 which are divisible by 17?

8. The *arithmetic mean* between two numbers a and b is the number M such that a, M, b are successive terms in an arithmetic progression. Show that a, $\frac{a+b}{2}$, b are in arithmetic progression and thus that $\frac{a+b}{2}$ is the arithmetic mean between a and b.

9. Find the arithmetic mean between

(a) 2 and 28
(b) -3 and 3
(c) 17 and 20
(d) $2-\sqrt{2}$ and $2+\sqrt{2}$
(e) 5 and 65
(f) -7 and 3
(g) $(m+n)^2$ and (m^2-n^2)

10. Given some information about an arithmetic progression, other facts can be found. In each of the following, from the facts given find the information requested:

Given	Find
(a) $A_1 = 2$, $d = 3$	A_5, S_5
(b) $A_2 = 10$, $A_4 = 12$	d, S_{10}
(c) $A_9 = 40$, $d = 4$	A_1, S_9
(d) $A_1 = 2$, $A_2 = 2$	d, S_{10}
(e) $A_1 = 5$, $A_2 = 2$	d, S_1, S_2, S_3, S_4
CHALLENGE PROBLEMS	
(f) $A_1 = 2$, $S_{10} = 100$	d
(g) $A_1 = 19$, $A_2 = 17$	the largest value of S_n

11. Each year a tree produces 3 more coconuts than it did the previous year. It produced 10 coconuts in 1965. How many will it produce in

1980? How many coconuts did it produce altogether during the years 1965 through 1980? In what year will it produce 40 coconuts? Will there be a year in which it produces exactly 60 coconuts?

12. A body falling from rest in a vacuum falls approximately 16 feet the first second and 32 feet farther in each succeeding second. How far will the body fall during the third second? the fifth second? the tenth second? How far does it fall in the first 10 seconds? the first n seconds?

13. If weights $W_1, W_2, \ldots, W_n$ are placed at positions $X_1, X_2, \ldots, X_n$ on a rod made of a homogeneous material, the centre of gravity of the rod is at position

$$\frac{W_1 X_1 + W_2 X_2 + \ldots + W_n X_n}{W_1 + W_2 + \ldots + W_n}$$

If unit weights are placed at position 1, 2, . . ., n, find the centre of gravity of the rod in two ways

(a) by just thinking about it and

(b) by using the formula just given.

Do you recognize the formula you obtain by equating the two answers?

4-4 GEOMETRIC PROGRESSIONS.

In the last section we considered special sequences called *arithmetic progressions* which had the property that each term after the first was obtained by adding a fixed number to the preceding term. Another special kind of sequence is obtained by multiplying each term by a fixed number in order to get the next term. Such sequences are called *geometric progressions.*

> Definition: A *geometric progression* is a sequence in which each term after the first is obtained by multiplying the preceding term by a fixed number. If we denote the sequence by $\{G_n\}$ and the fixed number by r, $G_{n+1} = r\,G_n$ for all natural numbers n. The number r is called the *common ratio.*

The following story shows how a knowledge of geometric progressions might be useful even for kings.

A court wizard had just pleased his king with an especially dazzling performance. The king was so delighted with his clever wizard that he said, "Make one request and it shall be granted." The clever wizard pointed to the royal chessboard and said, "O mighty king, give me 1 grain of wheat for the first square of the chessboard, 2 for the second, 4 for the third, and so on." This sounded reasonable enough to the king and he said, "Let it be done."

How many grains of wheat did the clever wizard receive?

The chessboard has 8 squares on a side or 64 squares in all. The wizard was to receive 1 grain of wheat for the first square, 2 for the second, 4 for the third, 8 for the fourth, and so on. In other words the number of grains of wheat for each square of the board is a term of the geometric progression $\{G_n\}$ with $G_1 = 1$ and $r = 2$. The first few terms of this progression are:

$$G_1 = 1$$
$$G_2 = 1 \times 2 = 2$$
$$G_3 = 2 \times 2 = 2^2 = 4$$
$$G_4 = 2^2 \times 2 = 2^3 = 8$$
$$G_5 = 2^3 \times 2 = 2^4 = 16$$

If we continue to make this kind of calculation $(n - 1)$ times we get $G_n = 2^{n-2} \times 2 = 2^{n-1}$, so the 64th term is $G_{64} = 2^{63}$.

In order to calculate how many grains of wheat the king promised the wizard we would need to find

$$S_{64} = 1 + 2 + 2^2 + \ldots + 2^{62} + 2^{63}$$

If we observe that $2S_{64}$ has many terms in common with S_{64} and then subtract S_{64} from $2S_{64}$, we obtain the following result:

$$2S_{64} = 2 + 2^2 + 2^3 + \ldots + 2^{63} + 2^{64}$$

Subtract $$S_{64} = 1 + 2 + 2^2 + \ldots + 2^{62} + 2^{63}$$

$$S_{64} = 2^{64} - 1 = 18{,}446{,}744{,}073{,}709{,}551{,}615 \text{ grains of wheat.}$$

If we count 1,000,000 grains to the bushel, it would take 41 years to grow this much wheat if every acre of Africa—mountains, deserts, lakes, and all—could be made to yield 60 bushels of wheat a year.

The wizard should not have been so greedy and the king should have understood about geometric progressions!

PROBLEMS 4-4A

1. A geometric progression has first term 16 and common ratio $\frac{1}{2}$.
   (a) What is the third term?
   (b) What is the fifth term?
   (c) What is the tenth term?

2. The second term of a geometric progression is 5 and the third term is 1; what is the first term?

3. If the fourth term of a geometric progression is 9 and the sixth term is 81, what is the common ratio? What is the first term?

4. The first three terms of a geometric progression are -4, 1, $-\frac{1}{4}$. What is the fifth term?

5. If a geometric progression is given by $G_n = 2^n$, find the sum of the first five terms.

In the general case the terms of a geometric progression $\{G_n\}$ with the common ratio r will be

$$G_1 = G_1$$
$$G_2 = r\,G_1$$
$$G_3 = r\,G_2 = r^2 G_1$$
$$G_4 = r\,G_3 = r^3 G_1$$

And if we continue the process $(n - 1)$ times we see that we get

$$\boxed{G_n = r^{n-1} G_1}$$

To find the sum of the first n terms we use the same method which allowed us to find the number of grains of wheat that the king promised the court wizard.

If we call S_n the sum of the first n terms of the geometric progression $\{G_n\}$ with common ratio r, we have

$$S_n = G_1 + G_2 + \ldots + G_n$$
$$S_n = G_1 + r\,G_1 + r^2 G_1 + \ldots + r^{n-1} G_1$$
$$r\,S_n = \qquad r\,G_1 + r^2 G_1 + \ldots + r^{n-1} G_1 + r^n G_1$$
$$r\,S_n - S_n = r^n G_1 - G_1$$

$$S_n (r - 1) = G_1 (r^n - 1)$$

$$S_n = \frac{G_1 (r^n - 1)}{r - 1} \quad , r \neq 1$$

If $r = 1$, then $S_n = G_1 + G_1 + \ldots + G_1 = n G_1$

$$\boxed{S_n = \left\{ \begin{array}{ll} \dfrac{G_1 (r^n - 1)}{r - 1} & , r \neq 1 \\ n G_1 & , r = 1 \end{array} \right\}}$$

PROBLEMS 4-4B

1. The first 2 terms of a geometric progression are 3 and 12.
   (a) What is the fifth term?
   (b) What is the sum of the first 3 terms?
   (c) What is the sum of the first 10 terms?

2. Find a geometric progression with $G_1 = -1$ and $G_3 = -4$. How many such geometric progressions are there? How many geometric progressions are there with $G_1 = -1$ and $G_3 = 4$?

3. The first and third terms of a geometric progression are 2 and 8. If r is positive, find the second term. This term is called the *geometric mean* between 2 and 8. Find the arithmetic mean between 2 and 8. Which is larger, the arithmetic mean or the geometric mean?

4. For the geometric progression with $G_1 = 1$ and $G_2 = -1$, find r, S_2, S_3, S_4, S_{100}, and S_{273}. Make a general statement about S_n.

5. If M is the foot of a perpendicular drawn from a point P of a semicircle to the diameter $\overline{AB}$, show that lengths MB, MP, and MA are in geometric progression.

CHALLENGE PROBLEMS

6. Find the arithmetic mean and the geometric mean between two positive numbers a and b. Can you say which is larger? Try to prove your answer.

7. A geometric progression has $G_1 = 2$, $G_2 = 4$. Find the product $G_1 G_2 \ldots G_{100}$ of the first 100 terms.

8. A man owns a lot bounded by an isosceles right-angled triangle with equal sides 100 ft. and hypotenuse $100\sqrt{2}$ ft. He divides the lot into triangular lots 1, 2, 3, . . . by starting at C, building the shortest fence to $\overline{AB}$ (the fence will be perpendicular to $\overline{AB}$), then the shortest fence from there to $\overline{AC}$, then the shortest fence to $\overline{AB}$, etc.

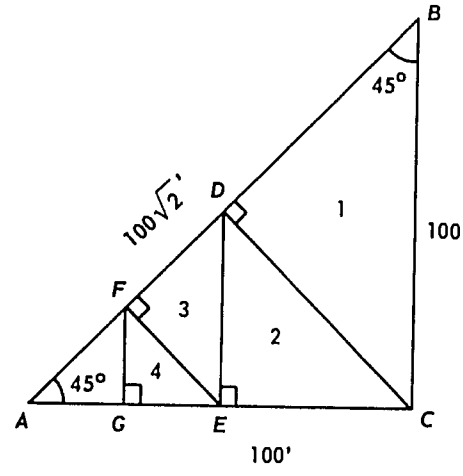


Fig. 2

Figure 2 shows the picture after the first four fences $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FG}$ are built.

(a) Find the lengths of the first four fences $\overline{CD}$, $\overline{DE}$, $\overline{EF}$, $\overline{FG}$.

(b) Denoting the length of the nth fence by L_n, find L_9 and L_{10}. Why do the numbers $L_1, L_2, \ldots$ form a geometric progression? What is the common ratio?

(c) If the man starts with enough fencing material to build 250 ft. of fence, will he ever run out of material? What is the smallest amount of fencing material he can start with so that he will never run out?

(d) Find the areas of the triangular lots 1, 2, 3, 4, 9, 10. If we denote the area of triangular lot n by A_n, why do the numbers $A_1, A_2, \ldots$ form a geometric progression? What is the ratio?

(e) Denote by R_n the area of the part of the original triangular lot not included in lots 1, 2, . . ., n. Find R_1, R_2, R_3. Why do the numbers R_1, R_2, . . . form a geometric progression? What is the ratio? What is R_{100}?

(f) Use the fact that $A_1 + A_2 + \ldots + A_{100} + R_{100}$ is the area of the original lot and the value of R_{100} to find $A_1 + A_2 + \ldots + A_{100}$. Check your answer by using the formula for the sum of a geometric progression.

9. Answer the questions of Problem 8, if the original lot is bounded by a 30°-60° right-angled triangle with hypotenuse 100 ft. and sides 50 ft. and $50\sqrt{3}$ ft.

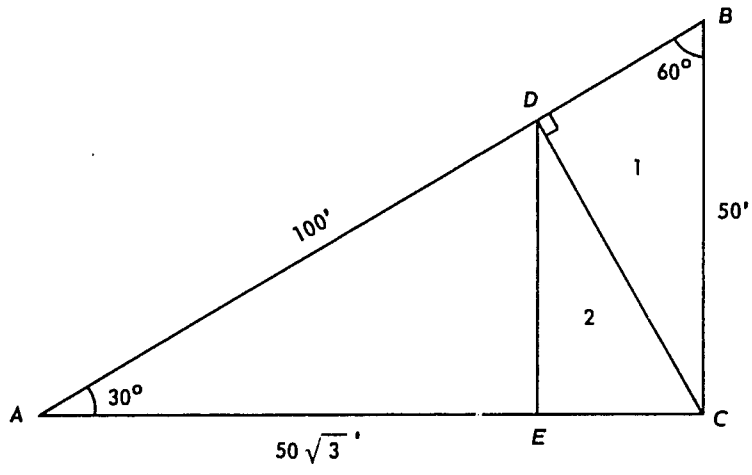


Fig. 3

4-5 COMPOUND INTEREST.

In *Secondary Three* we studied problems involving *simple interest.* In these problems the borrower agrees to pay a fixed percent of the amount borrowed in interest during each interest period. The amount borrowed is called the *principal* and the fixed percent the *rate of interest.*

For example if one borrows 1000 shillings at 6% simple interest, the interest each year is 60 shillings ($\cdot 06 \times 1000$). 6% is the *rate,* 1000 shillings is the *principal,* and 60 shillings is the *interest.* The total interest paid under this arrangement over a period of n years is given by 60 n shillings ($n \times \cdot 06 \times 1000$). For example 600 shillings (60×10) is the amount paid in interest over a 10 year period.

In general if the amount borrowed is P shillings and the rate of interest is $i\%$, then the total interest paid during n interest periods is $n \times \frac{i}{100} \times P$ shillings

If one has money to invest, a more profitable arrangement than simple interest is one called *compound interest*. Under this plan after each interest period the interest is added to the principal and interest is then received on the sum of the principal and the interest accumulated in previous interest periods.

For example if one invests 1000 shillings at 4% interest compounded annually, at the beginning of the second year the interest earned during the first year, 40 shillings ($\cdot 04 \times 1000$), is added to the principal. During the second year one receives interest on 1040 shillings. Thus the interest earned during the second year is 41·60 shillings ($\cdot 04 \times 1040$). At the beginning of the third year the principal on which interest will be paid has increased to 1081·60 shillings (1040 + 41·60). The interest received during the third year is $\cdot 04 \times 1081 \cdot 60$ shillings.

The amount at the beginning of each year may be written as follows:

first year 1000

second year $1000 + 1000 \times \cdot 04 = 1000\,(1 + \cdot 04) = 1000 \times 1 \cdot 04$

third year $1000 \times 1 \cdot 04 + \big((1000 \times 1 \cdot 04) \times \cdot 04\big) = (1000 \times 1 \cdot 04)\,(1 + \cdot 04) = 1000 \times (1 \cdot 04)^2$

fourth year $\big(1000 \times (1 \cdot 04)^2\big) + \big(\big(1000 \times (1 \cdot 04)^2\big) \times \cdot 04\big) = \big(1000 \times (1 \cdot 04)^2\big)\,(1 + \cdot 04) = 1000 \times (1 \cdot 04)^3$

In general at the beginning of the nth year the accumulated amount would be $1000 \times (1 \cdot 04)^{n-1}$. We note that these amounts are the terms of the geometric progression $\{G_n\}$ with $G_1 = 1000$ and $r = 1 \cdot 04$.

More generally if P shillings is invested at a rate of i% compounded annually, the terms of the geometric progression $\{G_n\}$ with $G_1 = P$ and $r = 1 + \frac{i}{100}$ gives the amount accumulated at the beginning of the nth year.

PROBLEMS 4-5A

1. Find the amount at the beginning of the third and sixth years of
   (a) 1200 shillings invested at 6% compounded annually
   (b) 500 shillings invested at 5% compounded annually
   (c) 2000 shillings invested at 4% compounded annually
   (d) 1500 shillings invested at 4·5% compounded annually.

2. Find the amount at the beginning of the nth year for [illegible]h of the parts of Problem 1.

Sometimes interest is paid semi-annually or even quarterly. If 1000 shillings is invested at 6% compounded semi-annually, then 30 shillings ($1000 \times \cdot 06 \times \frac{1}{2}$) is received at the end of the first six months and interest is received on 1030 shillings for the second half of the year; the interest for the second half year would be 30·90 shillings ($1030 \times \cdot 06 \times \frac{1}{2}$).

In general if P shillings is invested at a rate of i% compounded n times a year, the amount at the end of *one year* (at the beginning of the $(n+1)^{\text{st}}$ interest period) is

$$A = P\left(1 + \frac{i}{100n}\right)^{n}.$$

After t years interest will have been received for nt interest periods and the amount accumulated at the end of t *years* (at the beginning of the $(nt+1)^{\text{st}}$ interest period) will be

$$A = P\left(1 + \frac{i}{100n}\right)^{nt},$$

which is the $(nt+1)^{\text{st}}$ term of the geometric progression with first term P and ratio $\left(1 + \frac{i}{100n}\right)$.

PROBLEMS 4-5B

1. Find the amount of

(a) 1000 shillings at 6% compounded semi-annually, at the end of 3 years. 5 years.

(b) 500 shillings at 4% compounded quarterly, at the end of 1 year. 3 years. 5 years.

(c) 2000 shillings at $4\frac{1}{2}$% compounded semi-annually at the end of 1 year. 3 years. 5 years.

2. *CHALLENGE PROBLEM*

In how many years would one double one's investment if

(a) 1000 shillings is invested at 4% compounded quarterly?

(b) 200 shillings at 6% compounded semi-annually?

(c) 500 shillings at 5% compounded annually?

4-6 GEOMETRIC SEQUENCES AND INFINITE SERIES.

In Sections 4-3 and 4-4 as we studied arithmetic and geometric progressions, we found, in each case, a new sequence by taking the sum of the first n terms of these progressions.

For example in the arithmetic progression with $A_1 = 1$ and $d = 2$, the sum of the first n terms is

$$S_n = \frac{n}{2}[2 + (n - 1)\,2] = n^2.$$

The sums form a new sequence; namely, the sequence of squares of the natural numbers.

In the geometric progression with $G_1 = 2$ and $r = \frac{1}{2}$,

$$S_n = \frac{2\,(\frac{1}{2})^n - 2}{\frac{1}{2} - 1}$$

$$= 4 - 4\,(\tfrac{1}{2})^n$$

CLASS ACTIVITY

1. (a) Find the first 10 terms of the sequence defined by

$$S_n = 4 - 4\,(\tfrac{1}{2})^n.$$

(b) Draw the graph of this sequence.

(c) What is S_{20}? S_{100}?

(d) What happens to S_n for large values of n?

(e) What happens to $G_n = 2(\frac{1}{2})^{n-1}$ for large values of n?

2. (a) Find the first 10 terms of the sequence $\{S_n\}$ defined by $S_n = G_1 + G_2 + \ldots + G_n$ for the geometric progression $\{G_n\}$ if $G_1 = 2$ and $r = 1$.

(b) Draw the graph of the sequence $\{S_n\}$.

(c) What is S_{20}? S_{100}?

(d) What happens to S_n for large values of n?

3. (a) Find the first 10 terms of the sequence $\{S_n\}$ defined by $S_n = G_1 + G_2 + \ldots + G_n$ for the geometric progression $\{G_n\}$ if $G_1 = 1$ and $r = -1$.
(b) Draw the graph of the sequence $\{S_n\}$.
(c) What happens to S_n for large even values of n? For large odd values of n?

4. (a) Find the first 10 terms of the sequence $\{S_n\}$ defined by $S_n = G_1 + G_2 + \ldots + G_n$ for the geometric progression $\{G_n\}$ if $G_1 = 2$ and $r = 2$.
(b) Draw the graph of this sequence.
(c) What is S_{20}? S_{100}?
(d) What happens to S_n for large values of n?

From the examples we have just considered, we see that if S_n is the sum of the first n terms of a geometric progression the behaviour of S_n for large values of n differs according to the value of the ratio r.

If r is 1, as in Problem 2 above, $S_n = n\,G_1$ and for large values of n, $|S_n|$ becomes very large.

If r is -1, as in Problem 3 above, $S_n = \begin{cases} G_1 & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$ and the values of S_n continue to oscillate back and forth between G_1 and 0.

If $|r| > 1$, as in Problem 4, then for large values of n, $G_1 r^n$ is very large in absolute value and accordingly $S_n = \frac{G_1 r^n}{r - 1} - \frac{G_1}{r - 1}$ also becomes very large in absolute value for large values of n.

However, in the case $|r| < 1$, as in Problem 1 above, $G_1 r^n$ is very small for large values of n and $S_n = \frac{G_1}{1 - r} - \frac{G_1 r^n}{1 - r}$ is close to the number $\frac{G_1}{1 - r}$.

Such geometric sequences are especially interesting. While S_n is close to $\frac{G_1}{1 - r}$ for large values of n, S_n is not actually equal to $\frac{G_1}{1 - r}$ for any number n. On the other hand, S_n is as close to $\frac{G_1}{1 - r}$ as we please, provided we take n large enough.

The expression $G_1 + G_2 + \ldots + G_n$ is an example of a *finite series* and $G_1 + G_2 + \ldots + G_n + \ldots$ an example of an *infinite series*. Clearly we cannot add infinitely many numbers. However, when $|r| < 1$, $S_n = G_1 + G_2 + \ldots + G_n$ becomes close to $\frac{G_1}{1 - r}$ for large values of n and we say the infinite series $G_1 + G_2 + \ldots + G_n + \ldots$ *converges* to the *sum* $\frac{G_1}{1 - r}$.

We may also consider infinite series whose terms are not sums of geometric progressions, but sums of terms of any sequence. In general if we have an infinite series

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

we say that the series *converges* to a *sum* S if and only if the sequence of "partial sums"

$$S_n = u_1 + u_2 + \ldots + u_n$$

becomes close to S for large values of n. If the terms of the sequence $\{S_n\}$ go back and forth without coming close to any one fixed number, or simply become large in absolute value, as n increases, we say that the infinite series *diverges*.

PROBLEMS 4-6A

1. For the given geometric progression, find S_n. Does the infinite series converge or diverge? If it converges, find its sum.
   (a) $G_1 = 16,\ r = \frac{1}{4}$
   (b) $G_1 = 10,\ r = 2$
   (c) $G_1 = 15,\ r = \frac{6}{5}$
   (d) $G_1 = -4,\ r = -\frac{1}{2}$
   (e) $G_1 = 1,\ r = 1$
   (f) $G_1 = 0,\ r = -2$

2. For the infinite series $U_1 + U_2 + \ldots + U_n + \ldots$ find $S_n = U_1 + U_2 + \ldots + U_n$ for $n = 3, 5, 10$ if
   (a) $U_n = n + 1$
   (b) $U_n = (\frac{1}{2})^n$
   (c) $U_n = 2$
   (d) $U_n = \dfrac{1}{n(n+1)}$
   (e) $U_n = (-2)^n$
   (f) $U_n = \dfrac{1}{n}$

3. Try to guess for each series in Problem 2 above, whether the series converges or diverges. Give a reason for your answer.

4. A ball is dropped from a height of 6 feet. On each bounce it rebounds $\frac{2}{3}$ of the distance it fell. How far does it travel before coming to rest?

5. *CHALLENGE PROBLEM*
   Find an expression for S_n for the series

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{n(n+1)} + \cdots$$

Hint: Show first that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

What does S_n approach as n becomes large?

The representation of rational numbers as repeating decimals offers another example of convergent infinite series. Consider the repeating decimal ·666 . . . which we may also write as

$$\cdot 6 + \cdot 06 + \cdot 006 + \ldots .$$

In this form the infinite decimal is written as the indicated sum of the terms of the geometric progression with $G_1 = \cdot 6$ and $r = \cdot 1$. From our study of geometric progressions

$$\begin{aligned} S_n &= G_1 + G_2 + \ldots + G_n \\ &= \frac{G_1 r^n - G_1}{r - 1} \\ &= \frac{\cdot 6\left((\cdot 1)^n - 1\right)}{\cdot 1 - 1} \\ &= \tfrac{6}{9}\left(1 - (\tfrac{1}{10})^n\right) \\ &= \tfrac{2}{3} - \tfrac{2}{3}(\tfrac{1}{10})^n \end{aligned}$$

Clearly S_n is as close to $\frac{2}{3}$ as we please if we take n large enough. Another way to say this is to say that the infinite series $\cdot 6 + \cdot 06 + \cdot 006 + \ldots$ converges to the sum $\frac{2}{3}$. Notice that we could have used the expression $\frac{G_1}{1 - r}$ which we discussed on page 141. In that case we have $\frac{G_1}{1 - r} = \frac{\cdot 6}{1 - \cdot 1} = \frac{\cdot 6}{\cdot 9} = \frac{2}{3}$.

A slight modification of this procedure allows us to write any repeating decimal as the sum of a convergent infinite series whose terms are terms of a geometric progression with $0 < r < 1$.

Let us look at the repeating decimal 2·146146. . ., that is $2{\cdot}\dot{1}4\dot{6}$. (You may recall that in *Secondary Two* we indicated the repeating part of a repeating decimal by placing a dot over the first and last digits of the repeating part of the decimal.) This number can be written as

$$2 + \cdot 146 + \cdot 000146 + \cdot 000000146 + \ldots$$

that is, $2 + S$, where $S = \cdot 146 + \cdot 000146 + \ldots .$

$$S = \frac{\cdot 146}{1 - \cdot 001} = \frac{146}{999}$$

or

$$2{\cdot}\dot{1}4\dot{6} = 2 + \frac{146}{999} = \frac{2144}{999}.$$

PROBLEMS 4-6B

Write each of the following repeating decimals as the sum of an infinite series and find a representation for the number as the quotient of two integers.

1. $\cdot\dot{5}$
2. $\cdot\dot{1}\dot{4}$
3. $\cdot\dot{2}8\dot{1}$
4. $\cdot 21\dot{4}\dot{3}$
5. $1\cdot 4\dot{1}$
6. $1\cdot\dot{4}\dot{1}$
7. $1\cdot\dot{4}1\dot{4}$
8. $2\cdot\dot{1}82\dot{8}$

---

*4-7 THE NUMBER e

A very special case of the compound interest formula in Section 4-5 leads to a number which is very important in many parts of mathematics.

In the formula $A = P\left(1 + \frac{i}{100n}\right)^{nt}$ suppose we take $P = 1$, $i = 100$, and $t = 1$. (This is hardly a practical application of the formula since banks paying 100% on 1 shilling would be hard to find.) We would then have

$$A = \left(1 + \frac{1}{n}\right)^n$$

and we see that A now depends only on n. Let us indicate this dependence by writing $A(n)$ for A. $A(n)$ represents the amount at the end of 1 year of 1 shilling invested at 100% compounded n times a year.

Calculate $A(1)$, $A(2)$, $A(3)$, and $A(4)$.

Do the values seem to be increasing with n?

Is this reasonable? (That is, would you expect that the more often interest is compounded, the more money it would amount to at the end of the year?)

It certainly ought to be the case. However a proof that the function $A(n)$ is an increasing function is not easy to give. It *is* easy to see that doubling n increases A, for

$$\begin{aligned} A(2n) = \left(1 + \frac{1}{2n}\right)^{2n} = \left[\left(1 + \frac{1}{2n}\right)^2\right]^n &= \left[1 + 2\left(\frac{1}{2n}\right) + \left(\frac{1}{2n}\right)^2\right]^n \\ &= \left[1 + \frac{1}{n} + \left(\frac{1}{2n}\right)^2\right]^n \\ &> \left(1 + \frac{1}{n}\right)^n = A_n \end{aligned}$$

The sequence $A(1), A(2), A(4), A(8), \ldots$ is increasing. However do the terms of the sequence become very very large? For instance is there a term of this sequence as large as 4? If $n = 2^{20}$ the interest would be compounded 1,048,576 times a year or about once a second and it would be quite a task to find the amount.

However the following argument will show that $A(2^n)$ is not as much as 4 for any n.

CLASS ACTIVITY

Consider the sequence $B(n) = \left(1 - \frac{1}{n}\right)^n$.

1. Calculate $B(1)$, $B(2)$, $B(3)$, $B(4)$.
2. Does $B(n)$ seem to be increasing?
3. Prove that $B(2n) > B(n)$ by the same method used to show that $A(2n) > A(n)$.
4. Prove that $A(n) \times B(n) < 1$ for all n.
5. Explain each step in the inequality $A(2^{20}) < \frac{1}{B(2^{20})} < \frac{1}{B(4)} < \frac{1}{B(2)}$.
6. What is $\frac{1}{B(4)}$? $\frac{1}{B(2)}$?
7. Does $A(2^{20})$ exceed 4?

In the above argument 20 can be replaced by $n \geq 2$ in the inequality and thus

$$A(2^n) < \frac{1}{B(2^n)} \leq \frac{1}{B(4)} < \frac{1}{B(2)} \text{ for all } n \geq 2.$$

Since the terms of the sequence $A(1), A(2), A(4), A(8), \ldots$ increase but never exceed 4, there is a smallest number which no term exceeds, and which the successive terms approximate more and more closely, just as the successive terms in the sequence $\{1 - (\frac{1}{2})^n\}$ never exceed 1 and approximate it more and more closely. This smallest number which no A exceeds is the famous one which we mentioned in the beginning of this discussion. It is denoted by e (for the great Swiss mathematician Leonhard Euler (1707-1783)). Its value is approximately

$$e = 2{\cdot}71828 \ldots$$

Therefore if the interest is calculated very very often, you will have nearly 2·72 shillings at the end of the year.

Suppose instead of paying you interest the bank charged you at the rate $i\%$ per year for guarding your money, deducting the charge from your account. Show that your initial deposit will be reduced to $B = P\left(1 - \frac{i}{100n}\right)^n$ at the end of the year, if deductions are calculated n times a year. Would you like them to calculate deductions often? Why?

Chapter 5
DESCRIPTIVE STATISTICS

5-1 POPULATIONS AND FUNCTIONS ON POPULATIONS.

Here is a list of the pupils in a certain class, with their heights to the nearest inch, weights to the nearest 5 pounds, and age at last birthday:

Pupil x	$H(x)$	$W(x)$	$A(x)$
Alice	60	90	14
Ben	67	100	15
Carol	60	95	16
Don	62	100	14
Enos	66	110	17
Frank	63	105	15
Gail	60	100	16
Henry	64	100	13
Isaac	64	105	14
Joe	67	115	14

How much does Ben weigh? How tall is Gail? Who is the oldest member of the class? How many members of the class are 13 years old? 14? 15? 16? 17? 18? What is the most frequent age?

List the pupils in order of increasing height, starting with the shortest. Note that there are several ways to do this because of ties: Alice, Carol, or Gail could come first, but all three must come before Don. What is the height of the second person? the 5th? the 6th? the 10th? If you interchanged Henry

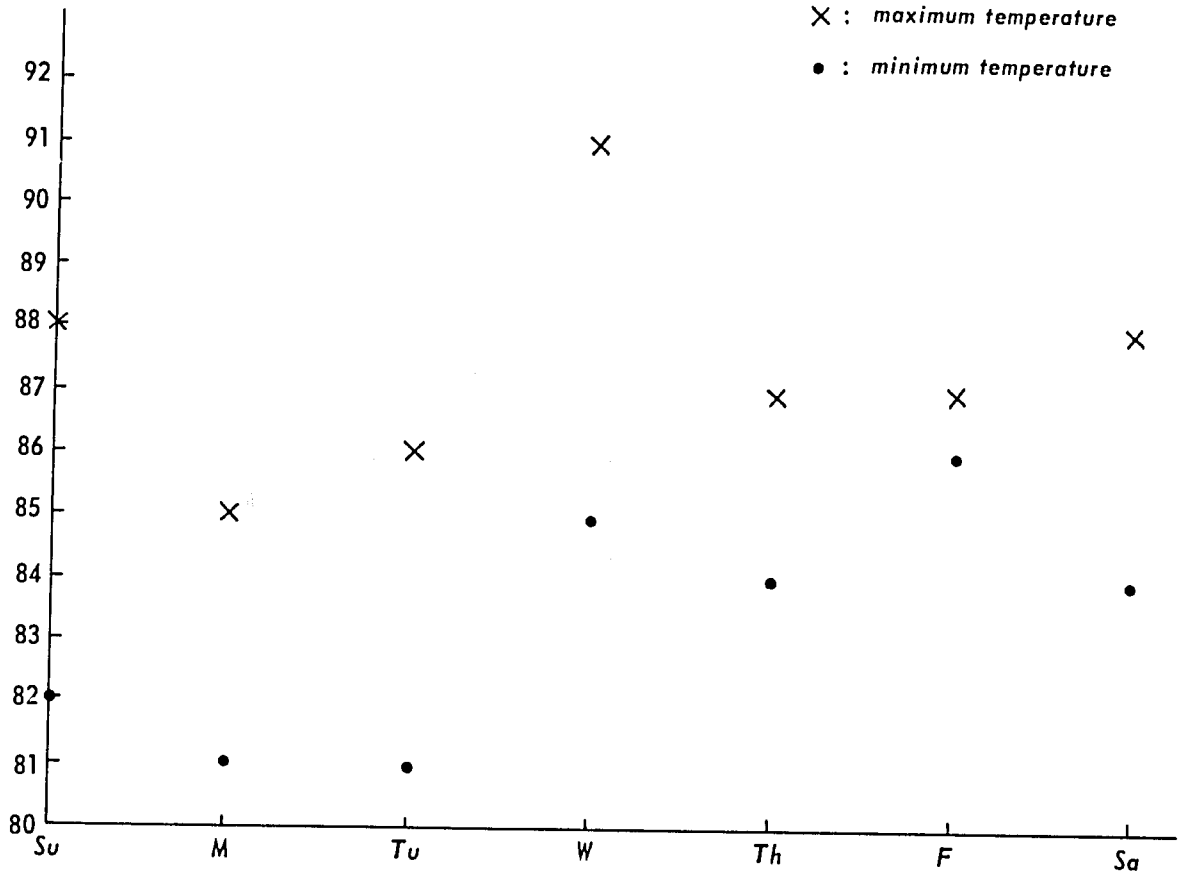
and Isaac on your list, would it still be correct? Would the answers to the earlier questions in this paragraph be the same?

The set of pupils in the class is an instance of a *population*, and the functions H, W, A are examples of *functions on this population*.

> Definition: In statistics, any finite set is called a *population*, and any numerical-valued function on a population, that is, any rule which associates with each member of the population a corresponding real number, is called a *function on the population* or a *population function*.

Functions on populations whose values are not numbers, for example, sex, occupation, tribe, are also called population functions and are also of interest but will not be discussed in this chapter.

Here is a graph of the minimum and maximum temperatures in a certain city on 7 successive days:



Think of the seven days as a population, and the functions L: maximum temperature and S: minimum temperature as functions on this population. What is the largest value of L? For what member(s) of the population does L have this value? Is 89 a value of L? What is the largest value of S? The smallest? What is the value of L on Wednesday? of S? of $L - S$? What is the largest value of the population function $L - S$? The smallest? Make a list of the elements of the population, and of the corresponding values of the population functions L, S, $\frac{L+S}{2}$.

Here are the 16 two-digit numbers that can be formed, using only 1, 2, 3, 4 as digits:

11	12	13	14
21	22	23	24
31	32	33	34
41	42	43	44

Consider this set as a population with 16 members. Let F be the function of this population having as its values the first digit of the number. Let S be the population function having as its values the second digit. For what members of the population does the function F have the value 1? For what members does the function S have the value 3? What are the possible values of the function $F + S$, the sum of the digits? For how many elements does $F + S$ have each of these values?

Make a list of the pupils in your class, and of the corresponding values of four population functions such as

H: height to the nearest inch,

W: weight to the nearest 5 pounds,

A: age at last birthday,

C: waist circumference to the nearest inch,

F: length of foot to the nearest half inch,

B: number of brothers.

Save this list.

PROBLEMS 5-1

1. For our 10 pupil class, how many of the pupils have $A(x)$, the age at last birthday, less than 12? How many have $A(x) < 14$? $A(x) < 16$? $A(x) < 16{\cdot}5$? $A(x) < 17$? $A(x) < 17{\cdot}1$? Draw a graph of the function f: the number of pupils whose age at last birthday is less than v, on the interval $12 \leq v \leq 18$.

2. For the population whose members are the 5 words in the sentence

   THERE IS NO LARGEST INTEGER,

   list the members of the population and the corresponding values of the population functions.

   (a) L: number of letters in the word
   (b) E: number of E's in the word
   (c) Z: number of Z's in the word
   (d) $L - E$
   (e) $E + Z$

3. Here is a list of the 16 4-letter "words" which can be formed, using only two symbols, a and b, as letters

aaaa	*abaa*	*baaa*	*bbaa*
aaab	*abab*	*baab*	*bbab*
aaba	*abba*	*baba*	*bbba*
aabb	*abbb*	*babb*	*bbbb*

   For what members of the population does the population function A, defined as the number of a's in the word, have the value 0? 1? 2? 3? 4? For how many members does the function B, defined as the number of b's in the word, have each of these values? What are the possible values of the function $A - B$? of the function $A + B$? What are the possible values of the function $\text{Max}(A, B)$, defined as the larger of the values of A and B, or their common value if they are equal?

5-2 DISTRIBUTIONS AND HISTOGRAMS.

For the population of 10 pupils in Section 5-1, here is a list of the different values of each of the population functions H, W, A, and the frequency with which each occurs:

H		W		A	
Value	Frequency	Value	Frequency	Value	Frequency
60	3	90	1	13	1
62	1	95	1	14	4
63	1	100	4	15	2
64	2	105	2	16	2
66	1	110	1	17	1
67	2	115	1		

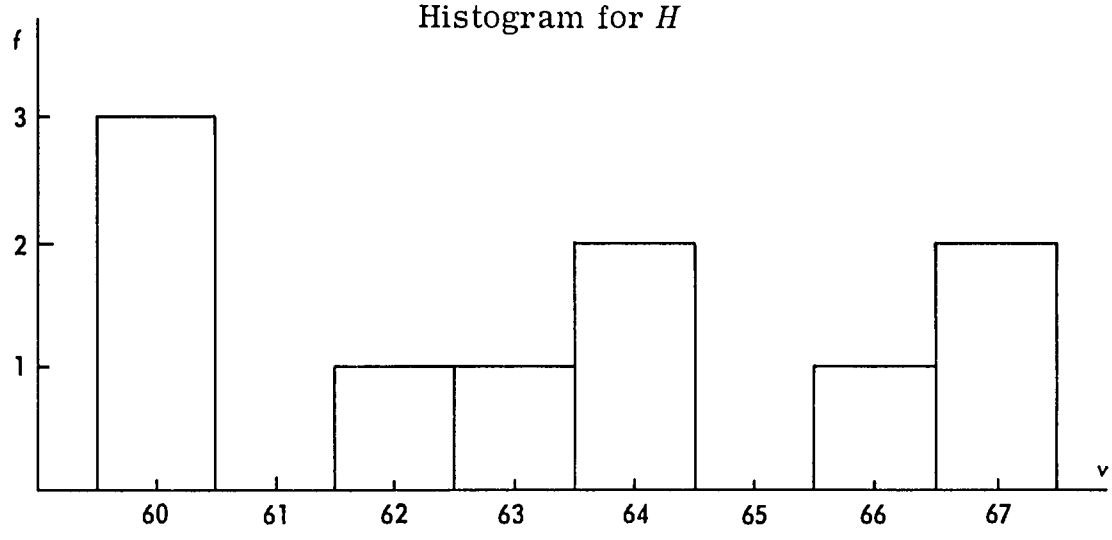
Such tables represent *distributions* of the population functions.

> Definition: The *distribution* of a population function is the function which associates with each value of the population function the number of members of the population for which the function has this value.

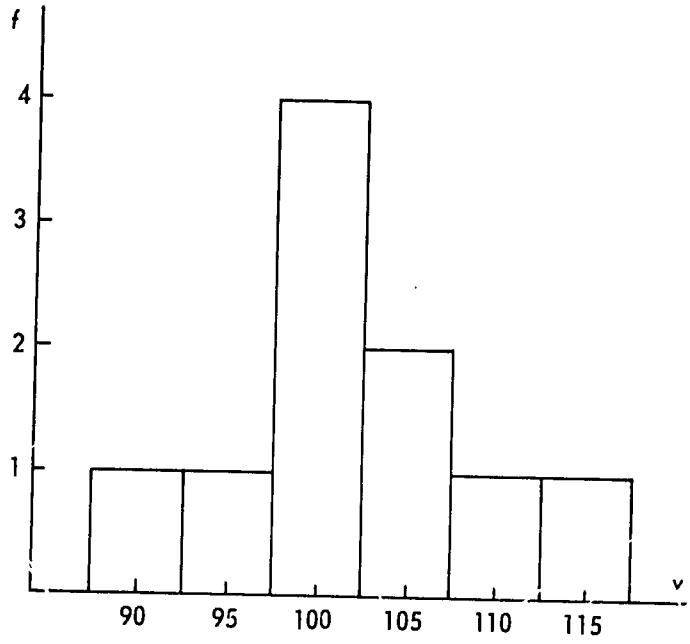
For the population of days of the week in Section 5-1, make tables for the distribution of each of these population functions: H, L, $H - L$, $\frac{H + L}{2}$.

Here are pictorial representations, called *histograms*, for the distributions of H, W, A in the 10 pupil class:

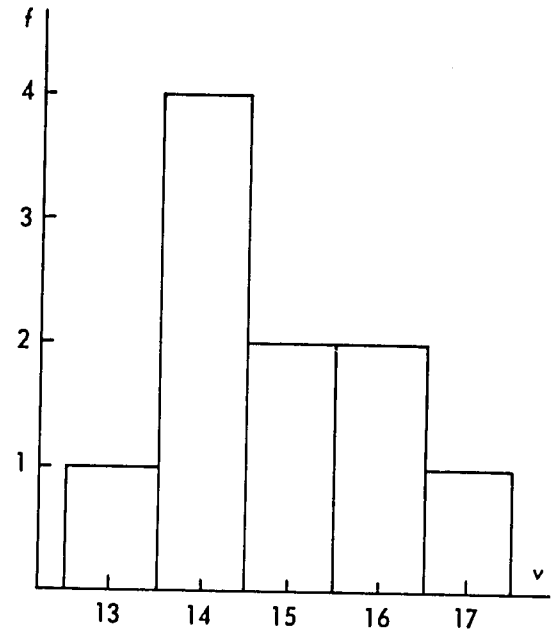
Histogram for H



Histogram for W



Histogram for A



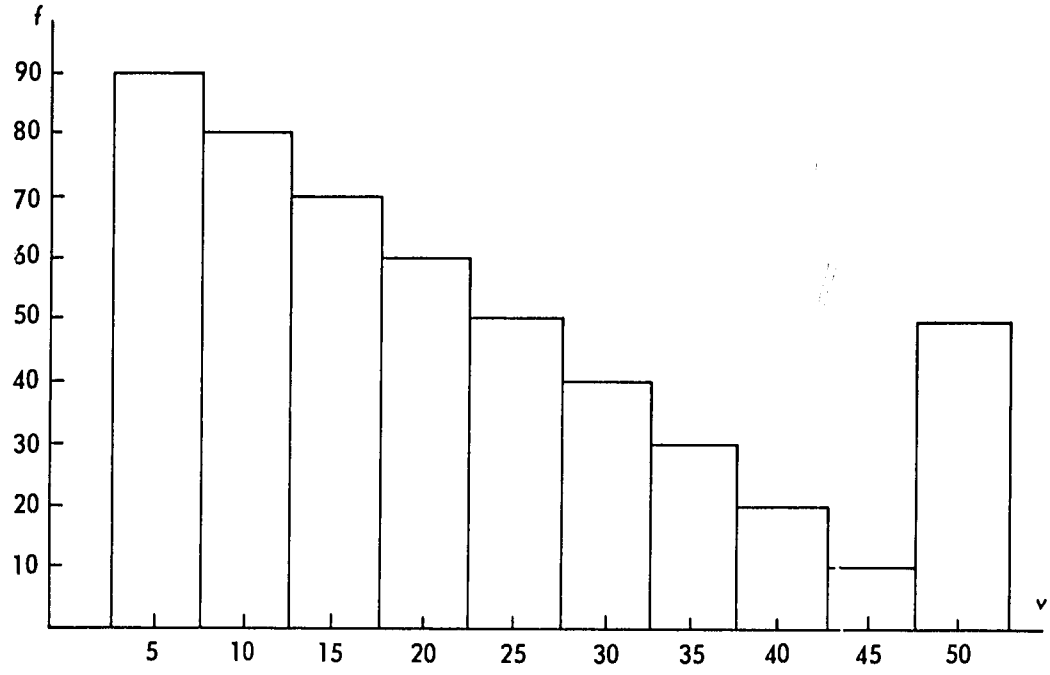
> To draw a histogram for a population function, mark the values of the function on a horizontal number line, and mark non-overlapping intervals of equal length, one centered at each value of the function. Then draw with each interval as base a rectangle whose height equals the frequency of the value of the function at the centre of the interval.

Note that the histogram of a population function is rather like a graph of its distribution.

Answer these questions by looking at the histogram for W: What is the smallest weight? How many pupils weigh 105 pounds? What is the most frequent weight? How many pupils weigh at least 100 pounds? more than 100 pounds? less than 100 pounds?

For the population of 16 two-digit numbers in Section 5-1, find the distribution of $F + S$, and draw its histogram.

Here is the histogram for a function X:

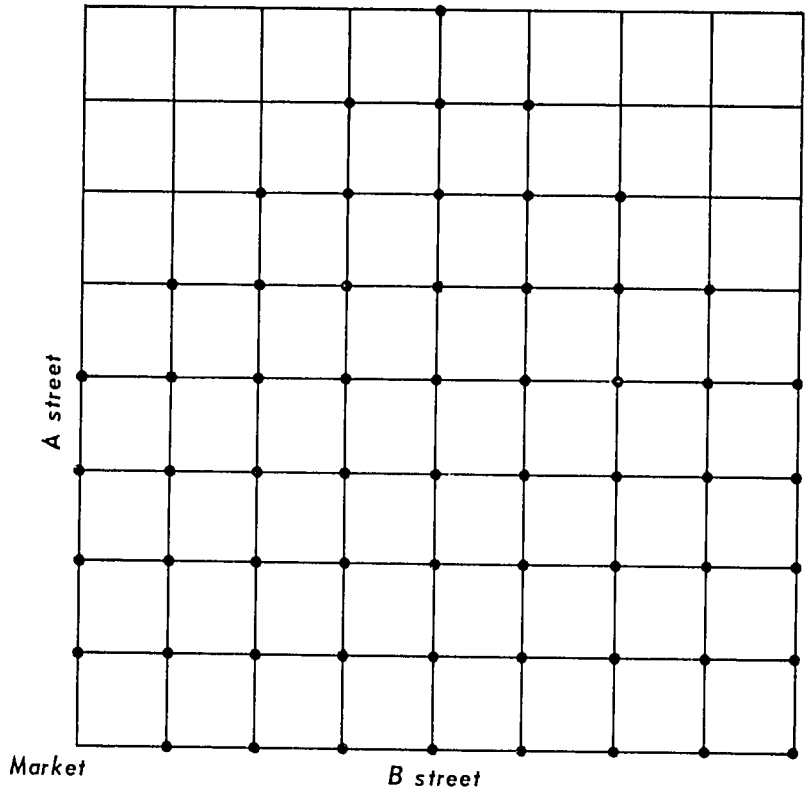


What are the values of X? For how many members is the value of X 25? What is the most frequent value of X? What is its frequency? How many members does the population have? Find the distribution of X.

PROBLEMS 5-2

1. Find the distributions and draw the histograms for four population functions for the population of pupils in your class.

2. Here is a map of a town. (See page 154.) The horizontal and vertical lines are streets 1 block apart, the dots are shops, and the market is at the lower left corner. For the population of shops, find the distribution and draw the histogram for each of these population functions:
   X: the distance of the shop from A street,
   Y: the distance of the shop from B street,
   $X + Y$: the distance of the shop from the market,
   Z: the smaller of X, Y (their common value if they are equal, i.e. the distance of the shop from the nearer of the two streets A and B).



5-3 THE MEAN, MEDIAN, AND MODE OF A POPULATION FUNCTION.

Here is the number $S(x)$ of shillings in the pocket of each member x of our 10 pupil class:

Member x	$S(x)$
Alice	0
Ben	0
Carol	2
Don	0
Enos	4
Frank	50
Gail	22
Henry	0
Isaac	0
Joe	4

If we collect all the money and divide it equally among the pupils, how much will each get? You should find that 82 shillings $(0 + 0 + 2 + \ldots + 4)$ will be collected, so that each pupil will get 8·2 shillings $\left(\frac{82}{10}\right)$. The number

8·2 is called the *average* or *mean* of the population function S, and is denoted by $M(S)$.

> Definition: The *mean* of a population function X is the number, denoted by $M(X)$, obtained by adding the values of X for all members of the population, and dividing the sum by the number of members of the population.

Look at these calculations:
for the mean of L, the maximum temperature for our 7 day population,
for the mean of S, the minimum temperature,
for the mean of $L - S$, the fluctuation in temperature,
for the mean of $S - 32$, the amount by which the minimum temperature exceeds freezing, and
for the mean of $\frac{5}{9}(S - 32)$, the minimum Centigrade temperature.

Day	Value of the Function				
	L	S	$L - S$	$S - 32$	$\frac{5}{9}(S - 32)$
Su	88	82	88 - 82	82 - 32	$\frac{5}{9}(82 - 32)$
M	85	81	85 - 81	81 - 32	$\frac{5}{9}(81 - 32)$
Tu	86	81	86 - 81	81 - 32	$\frac{5}{9}(81 - 32)$
W	91	85	91 - 85	85 - 32	$\frac{5}{9}(85 - 32)$
Th	87	84	87 - 84	84 - 32	$\frac{5}{9}(84 - 32)$
F	87	86	87 - 86	86 - 32	$\frac{5}{9}(86 - 32)$
Sa	88	84	88 - 84	84 - 32	$\frac{5}{9}(84 - 32)$
Total	612	583	612 - 583	$583 - (7 \times 32)$	$\frac{5}{9}\left(583 - (7 \times 32)\right)$
Mean	$\frac{612}{7}$	$\frac{583}{7}$	$\frac{612}{7} - \frac{583}{7}$	$\frac{583}{7} - 32$	$\frac{5}{9}\left(\frac{583}{7} - 32\right)$

Check the addition in the L and S columns. Can you see, without any further calculations, that the sums and means in each of the remaining columns are correct? How is $M(L - S)$ related to $M(L)$ and $M(S)$? How is $M(S - 32)$ related to $M(S)$? How is the mean of the function $\frac{5}{9}(S - 32)$ related to the mean of the function $S - 32$? What is $M(L + S)$? How is $M(S - 80)$ related to $M(S)$? Find $M(S - 80)$ directly without using $M(S)$. Now find $M(S)$ from $M(S - 80)$. Is this a good way to calculate $M(S)$?

Do you see that each of the following sentences is true?

> (1) The mean of a sum or difference of population functions is the sum or difference of the means.
>
> (2) The mean of a constant times a population function is that constant times the mean of the population function.

Can you find the mean of a population function from its distribution? Here is the distribution of a certain function X. (v is the value, f is the frequency.)

v	f
1	4
2	3
5	3

For how many individuals does X have the value 1? What is the sum of the values of X for these individuals? What is the sum of the values of X for all individuals for which X has the value 2? For which X has the value 5? What is the sum of the values of X for all individuals? How many individuals are there in the population? What is the mean of X? Do you see that

$$M(X) = \frac{(1 \times 4) + (2 \times 3) + (5 \times 3)}{4 \quad + \quad 3 \quad + \quad 3} = \frac{25}{10} = 2{\cdot}5?$$

> The mean of a population function can be calculated from its distribution by multiplying each value by its frequency, adding these products, and dividing their sum by the sum of the frequencies.

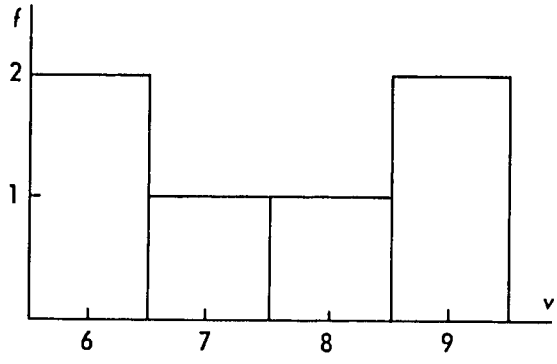
For our 10 pupil class, find $M(A)$ by two methods: (1) from the original list of the pupils and the values of A in Section 5-1 and (2) from the distribution of A in Section 5-2.

For our 10 pupil class, what is the most frequent value of A? What is the most frequent value of S, the number of shillings in the pupil's pocket?

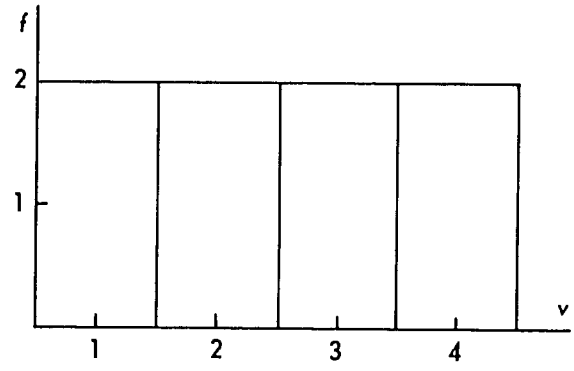
> The most frequent value of a population function is called the *mode* of the population function.

A population function may have more than one mode. Here are histograms for two functions X and Y.

Histogram for X



Histogram for Y



Both 6 and 9 are modes of X, and 1, 2, 3, 4 are all modes of Y. Find $M(X)$ and $M(Y)$. Could you have seen the answers directly, from the symmetry of the histograms?

We sometimes are interested in the value of the middle member of the population. Suppose we have the members of a class listed in the order of their scores on a test as follows.

Pupil x	Test Scores $T(x)$
Frank	58
Enos	63
Joe	64
Tunde	75
John	79
Martha	82
Mary	85
Ben	85
Isaac	91

What pupil is in the middle place in the list? What is the value of T for that pupil? This value is called the *median* of the population function T. In this case the median of T is 79.

If the population has an *even* number of members, there is no single member at the middle position. What shall we do then to find a median of the population function? List the pupils in our 10 pupil class in order of increasing values of S, starting with those with the smallest value of S. Since there are ties, there are several correct ways to do this. Who are the pupils in the two middle places (5 and 6) of your list? What is the value of S for the pupil in place 5? For the pupil in place 6? You should have found that the values of S in the two places nearest the middle are 0 and 2. We shall use the *average* of these two numbers as the *median* of S. Thus in this case the median of S is $\frac{0+2}{2}$ or 1.

> Definition: If the values of a population function are arranged in increasing order, the *median* of the function is defined as follows:
> (1) If the population has an odd number of members, the median is the value of the middle member in the list.
> (2) If the population has an even number of members, the median is the average (half the sum) of the values of the two middle members in the list.

Find the median of each of the population functions H, W, A for our 10 pupil class and the median of each of the functions L, S, $L - S$, $S - 32$, $\frac{5}{9}(S - 32)$ for our 7 day population.

The mean, median, and mode are all called *measures of location* for a population function, since they give us some idea where the values of the function are located. They tell us different things, though. For the function S on our 10 pupil population, we have the following information. The mean, $M(S)$ is 8·2, which tells us that if the total wealth were divided equally, each pupil would have 8·2 shillings. The median of S is 1. It tells us that half of the pupils have less than one shilling and half have more. The mode of S is 0 which tells us that there are more pupils with 0 shillings than with any other one number of shillings.

PROBLEMS 5-3

1. Find the mean, median, and mode(s) for each of four functions for the population of pupils in your class.

2. The values of a function X on a population with three members are:

Member	Value of X
A	0
B	1
C	t

Find the mean, median and mode(s) of X for each of these values for t : 1, 1·5, 2, 10. Graph the mean and median of X as functions of t, for $-3 \leq t \leq 4$.

3. Suppose an 11th pupil, Kombi, joins the 10 pupil class, and that he has 100 shillings in his pocket. Find the new mean, median, and mode(s) for S.

4. For the population of shops in Problem 2 of Section 5-2, find the mean, median, and mode(s) for the functions X, Y, $X + Y$, Z.

5. In a certain school, 60% of the pupils own no compass, 30% of the pupils own one compass, and 10% of the pupils own 2 compasses. Can you find the mean, median, and mode(s) for the population function X: number of compasses owned, without knowing the number of pupils in the school? Suppose there are 100 pupils in the school. Find the distribution of X, draw its histogram, and find the mean, median, and mode(s). Now suppose there are 200 pupils in the school, and find the distribution of X, etc. What do you conclude?

5-4 STANDARD DEVIATION AND RANGE.

Four petrol stations are located along a road, at distances 1, 3, 6, 10 miles east of a certain town, T. We can think of 1, 3, 6, 10 as values of a population function X.

A supply depot is to be located somewhere along the road, from which a lorry will make a separate delivery trip to each station every day. The cost of each trip, in shillings, equals the square of the distance from the depot to the station, so that for instance if the depot is on the road 4 miles east of T, a trip to D costs $(10 - 4)^2$ or 36 shillings, a trip to C costs $(6 - 4)^2$ or 4 shillings, a trip to B costs $(3 - 4)^2$ or 1 shilling, and a trip to A costs $(1 - 4)^2$ or 9 shillings and the average cost per trip is $\frac{36 + 4 + 1 + 9}{4}$ or 12·5 shillings.

Where on the road should the depot be located to make the average cost per trip as small as possible, and what is the smallest average cost?

Try locating the depot 5 miles east of T and calculate the average cost. Now try 6, 5·1, 4·9. What do you conclude?

If we locate the depot at a point t miles east of T, our average delivery cost is

$$\frac{(10 - t)^2 + (6 - t)^2 + (3 - t)^2 + (1 - t)^2}{4}$$

$$= \frac{4t^2 - 40t + 146}{4} = t^2 - 10t + 36{\cdot}5.$$

We are therefore trying to find t to make the value of the quadratic expression $t^2 - 10t + 36{\cdot}5$ as small as possible. Notice that

$$t^2 - 10t + 36{\cdot}5 = t^2 - 10t + 25 + 11{\cdot}5 = (t - 5)^2 + 11{\cdot}5.$$

Can you see at once that the function will have its smallest value when $t = 5$, and that this smallest value is 11·5? We should therefore locate the depot 5 miles east of T, and our average delivery cost is then 11·5 shillings.

Notice that 5 is the mean distance of the stations from T $\left(M(X) = \frac{1 + 3 + 6 + 10}{4} = 5\right)$ and that the number 11·5 is a kind of measure of how spread out the stations are: if they were closer together we could make the average delivery cost smaller, and if they were farther apart it would be higher. If we consider the 4 stations as a population, and the distances of the stations east of T as values of the population function X, then the number 11·5 is called the *variance* of X, and the (non-negative) square root of the variance is called the *standard deviation* of X.

> Definition: For any population function X, the mean of the function $[X - M(X)]^2$ is called the *variance* of X, and the square root of the variance of X is called the *standard deviation* of X.

Fill in the missing entries in this calculation of the variance and standard deviation of the function $X = L - S$ on our 7 day population.

	Value of the Function		
Day	X	$X - M(X)$	$[X - M(X)]^2$
Su	6	$\frac{13}{7}$	$\frac{169}{49}$
M	4	$-\frac{1}{7}$	$\frac{1}{49}$
Tu	5	$\frac{6}{7}$	$\boxed{}$
W	6	$\frac{13}{7}$	$\boxed{}$
Th	3	$\boxed{}$	$\frac{64}{49}$
F	1	$\boxed{}$	$\frac{484}{49}$
Sa	4	$-\frac{1}{7}$	$\boxed{}$
Total	29	$\boxed{}$	$\frac{924}{49}$
Mean	$\frac{29}{7}$	$\boxed{}$	$\frac{132}{49}$

Thus the variance of $L - S$ is $\frac{132}{49}$, and the standard deviation of $L - S$ is $\sqrt{\frac{132}{49}} \approx 1{\cdot}64$.

Find the variance of the population function A on our 10 pupil population.

What is the largest value of A? What is the smallest value of A? The difference between these two numbers is called the *range* of A, so the range of A is 17 - 13 or 4. Notice that this meaning of range is quite different from another common use of the term, to denote the set of values of a function.

In our 10 pupil class, find the range of each of these population functions: W, $W - 100$, $\frac{W - 100}{5}$.

The standard deviation and range of a population function are called measures of *spread* or *dispersion*, since they tell us something about how spread out or dispersed the values of the function are.

In our original filling station example, suppose we had measured distances from a town 10 miles west of the one we used, so the distances of the stations would be 11, 13, 16, 20 miles respectively. Where should the depot be located? What is the smallest average cost per trip? Do you see that adding a constant to the values of a population function does not change its variance? How could this fact be used to simplify calculation of variance? Find the variance of the function $A - 13$ in our 10 pupil class.

PROBLEMS 5-4

1. Find the standard deviation and range for four population functions for your class.

2. Here is the distribution of a population function X:

v	f
1	7
2	2
3	1

What is the mean of X? What are the possible values of $X - M(X)$? What is the frequency of each of these values? What are the values and their frequency for $[X - M(X)]^2$? What is the variance of X? What is the largest value of X? What is the smallest value of X? What is the range of X? Can you find the standard deviation and range of any function from its distribution? From its histogram?

3. Among all globes manufactured by a certain company, a fraction x last exactly one month, a fraction $1 - 2x$ last exactly 2 months, and the remaining fraction x last exactly 3 months. Find the mean and standard deviation of the function L, the life of the bulb, for these values of x: 0, ·1, ·2, ·3, ·4, ·5. Graph the mean and standard deviation as functions

of x, $0 \leq x \leq \cdot 5$. You may want to choose a definite number, say 1000, as the total number of globes manufactured; the mean and standard deviation of L will not depend on your choice.

5-5 SCATTER DIAGRAMS.

Look at the histograms for H and W in our 10 pupil class. Which of these questions can you answer just from the two histograms?

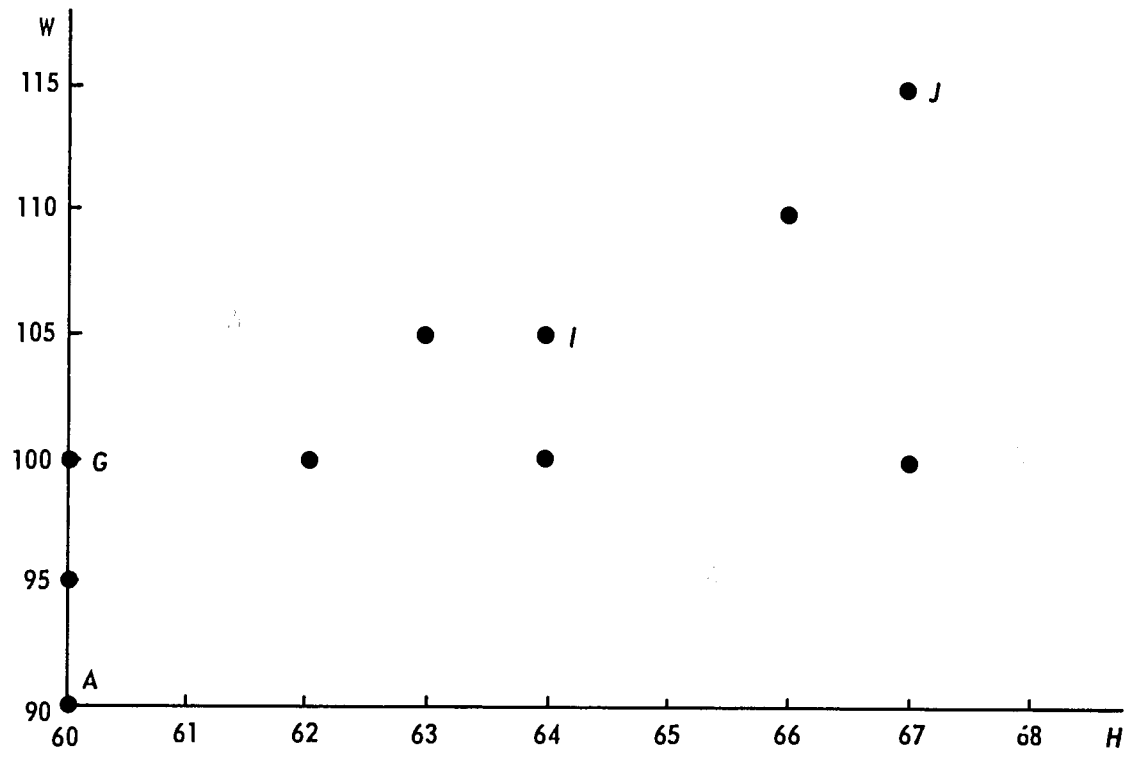
(a) How many pupils have height 64 inches?

(b) How many pupils weigh at least 110 pounds?

(c) Does the pupil who is 66 inches tall weigh more than the pupil who is 62 inches tall?

You should have been able to answer (a) and (b), but not (c). Questions like (c), involving two population functions cannot usually be answered by looking at the separate distributions of the functions. We need a picture which represents the joint behaviour of the functions.

Here is such a picture, called a *scatter diagram*, for the functions H, W.



You see 10 points in the picture, one for each pupil in the class. The points labeled A, G, I, J represent Alice, Gail, Isaac, Joe in the height-weight scatter diagram. Do you see how to tell from the picture that Isaac is 64 inches tall and weighs 105 pounds? Look at the scatter diagram and answer these questions.

(a) How tall is Alice?

(b) How tall is Gail?

(c) How much does Alice weigh?

(d) Is Isaac heavier than Gail?

(e) Is any other pupil as tall as Joe?

(f) Is any other pupil as heavy as Joe?

(g) How many pupils are 60 inches tall?

(h) How many pupils are 61 inches tall?

(i) What is the range of H?

Can you find the distribution of H from the scatter diagram? of W?

Notice the tendency for the points to rise as we go to the right. What does this say about the relationship between height and weight in this class?

Sometimes several members of the population are represented by the same point in a scatter diagram. We must indicate this, perhaps as shown in the following example.

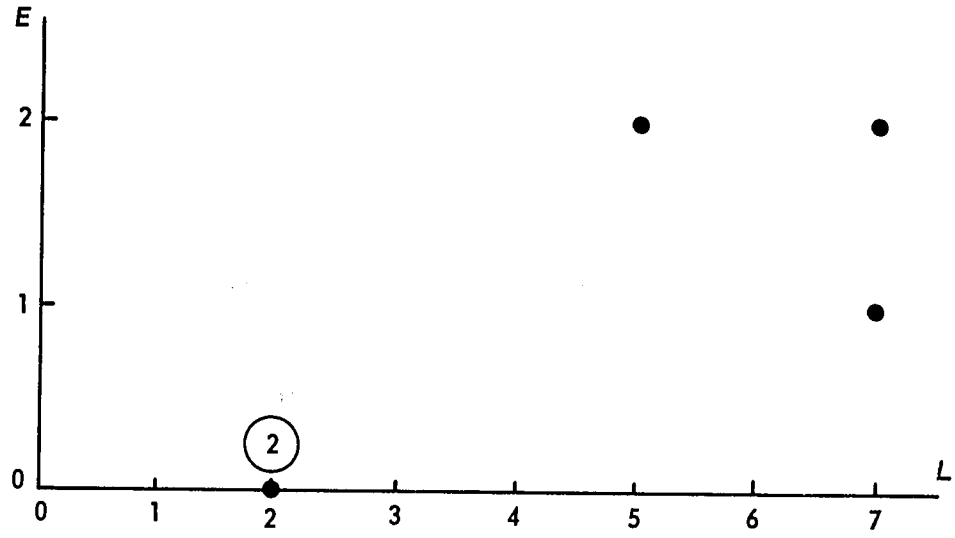
Here is a scatter diagram for the functions L and E on the population of five words.

L: number of letters

E: number of E's.

Word	Value of L	Value of E
THERE	5	2
IS	2	0
NO	2	0
LARGEST	7	1
INTEGER	7	2

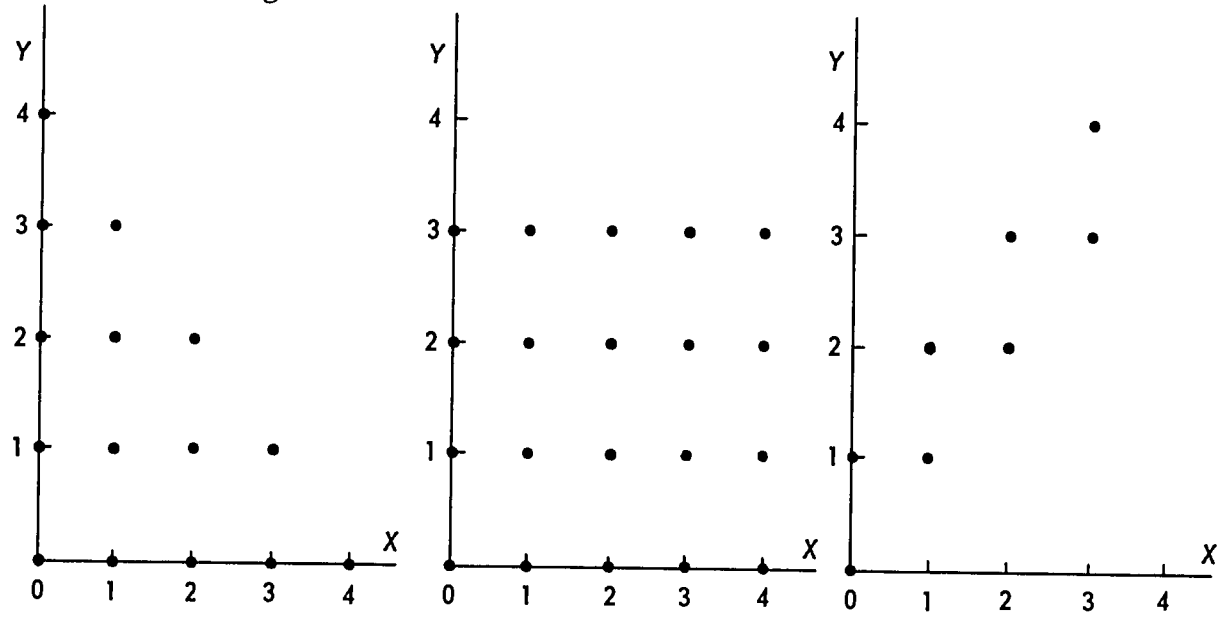
The (2) at the point (2, 0) indicates that two elements of the population have 2 and 0 as values of L and E respectively. From the scatter diagram, find the value of the function $S = L + 2E$ at each point. Find the distribution, mean, and standard deviation of S, without reference to the original list of words.



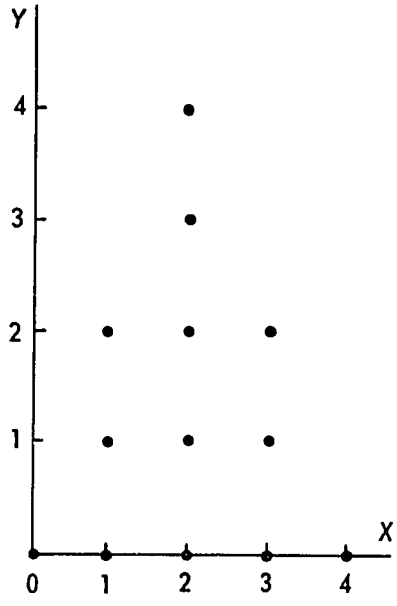
PROBLEMS 5-5

1. Draw scatter diagrams for three pairs of population functions on your class.

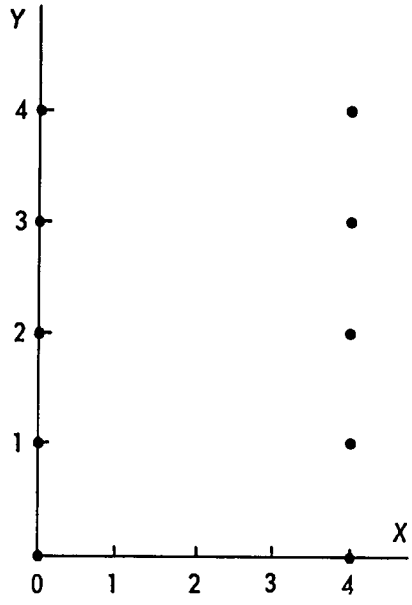
2. For each of these scatter diagrams, find the mean, standard deviation, and range for each of the functions X, Y, $X+Y$, and draw the histogram for each of the three functions. Mark the point $\big(M(X), M(Y)\big)$ in each scatter diagram.



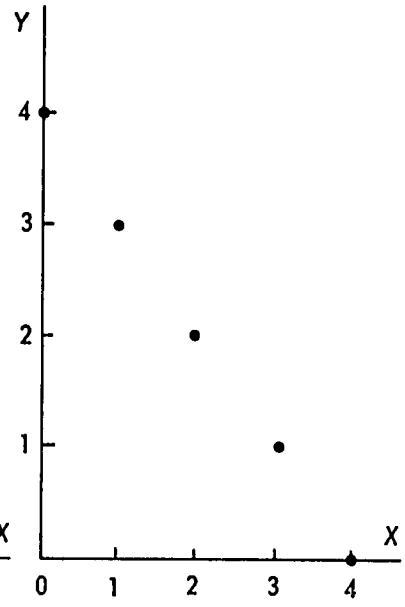
(a) (b) (c)



(d)



(e)



(f)

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