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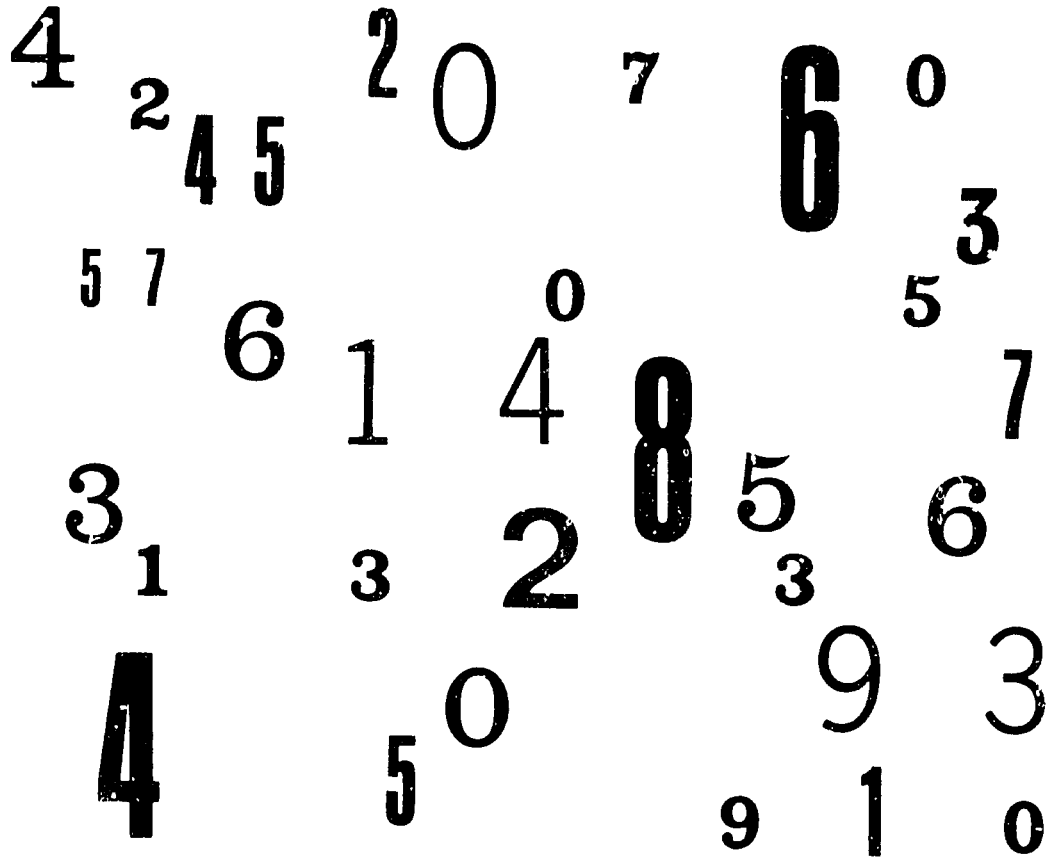
SECONDARY FOUR ALGEBRA

teachers'



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Teachers'
Guide

ENTEBBE MATHEMATICS SERIES

Secondary Four Algebra
Preliminary Edition

Prepared at the 1965
Mombasa Mathematics Workshop

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PREFACE

This Teachers' Guide accompanies the text which was prepared at a study and writing workshop held during the summer of 1965 at Mombasa, Kenya, in which mathematicians and teachers from most English-speaking countries of Tropical Africa, the United States and the United Kingdom participated.

This material is intended to follow *Secondary Three*, *Secondary Two*, and *Secondary One* of the Entebbe Mathematics Series which were prepared at similar workshops during the summers of 1964, 1963, and 1962 at Entebbe, Uganda.

I. Preliminary Nature of This Material

The material must be considered to be preliminary until it has been tried out in class-rooms and modified in accordance with the experience gained from this experimentation. The Student Text and Teachers' Guide for SECONDARY FOUR ALGEBRA and GEOMETRY of the Entebbe Mathematics Series will be subject to correction and improvement in the light of suggestions from the teachers who use them.

This means that the teacher who uses this material has the responsibility, in addition to helping interpret the new material to students, of helping identify the areas where improvement is needed. To guide us in future revisions, we hope that teachers will fill out Secondary Chapter Reports. (See sample Report on pages ix, x, xi, xii.) In partial reward for this burden, the teacher will have the satisfaction of knowing that he or she is taking part in an experiment which is of great potential value.

II. Emphasis on Mathematical Ideas

In recent years there has been accumulated much evidence that young students are far more interested in mathematical *ideas* than they have usually been given credit for, and that they are far more competent to deal with such ideas than the current curricula would suggest. A presentation of mathematics which puts its emphasis on concepts rather than the rules of manipulation is likely to lead to far greater satisfaction on the part of the student, and will also lead to greater mathematical competence.

III. Content of the Text

The subjects of algebra and geometry are treated in two volumes; one is devoted to algebra and the other to geometry, as in *Secondary Three*. The material is arranged so that there is considerable flexibility in the order in which the topics can be studied, although Chapter 1 of ALGEBRA should be studied before beginning Chapter 6 of GEOMETRY.

The treatment of algebra continues the start made in *Secondary One*, *Two*, and *Three* to develop algebra as a unified structure. All five chapters are about functions. Chapter 1 treats the general concept of function. In Chapters 2 and 3 the pupil studies some of the most important types of functions; the rational functions and the exponential and logarithmic functions. As a result of having studied the polynomial functions in *Secondary Three* and the functions in Chapters 2 and 3 of this volume, the pupil will be familiar with all the basic types of elementary functions except the trigonometric functions which will be studied in *Secondary Five*. In Chapter 4 the domain is narrowed to the set of positive integers, with emphasis on two of the most important types, arithmetic progressions which are really just linear functions with the domain restricted to the positive integers, and geometric progressions, which are really just exponential functions with the domain restricted to the positive integers. Finally the central concept in Chapter 5 is that of a function whose domain is a finite set. These functions are studied in the context of descriptive statistics. The finite set is called a population and the functions are called population functions. Chapter 1 should be done first, so that the student will see the other four chapters as the study of special types of functions. The remaining four chapters are independent of each other and may be studied in any order.

As the pupils will have studied formal, deductive geometry in *Secondary Three*, there is now no need to be so formal in SECONDARY FOUR GEOMETRY. He will have already been exposed to formal proofs and will by this time have an idea about rigour in these proofs. Therefore, Chapter 1 of SECONDARY FOUR GEOMETRY indicates how much loosening up from *Secondary Three* occurs in this volume. The pupils however, are now equipped to be able to take any of the theorems or examples in the text and write out as formal a proof as they wish. Chapters 2 to 5 follow the pattern of *Secondary Three* except that not so much time is spent on the formal aspect of deductive reasoning. Chapter 6 is quite different both in content and spirit from the other chapters. The treatment of rigid motion is informal and intuitive. This approach avoids the complications of establishing a set of axioms from which the basic facts about rigid motion can be deduced. It should be mentioned that it is quite possible to develop all our geometric ideas by starting off with rigid motions. Except for Chapter 6 which could be studied any time after Chapter 1 of ALGEBRA, the chapters are best studied in the order in which they appear. Teachers who prefer the informal, intuitive treatment of geometry may wish to spread out the study of Chapter 6 throughout the school year.

IV. The General Approach

As in previous texts we try to be mathematically sound but at the same time the development is gradual with an effort to involve the pupils actively. Instead of just telling the ideas to the pupils outright, the teacher should use problems and careful class discussion to help the pupils recognize the ideas for themselves.

The best education is the education that the pupil creates from his own direct efforts. The teacher should resist the temptation to tell the class exactly what to do and how to do it. It is indeed a great temptation, for by such means a class will appear to be proceeding at a rapid pace. If the teacher takes the time and effort to lead the pupil to think through the ideas himself without telling him outright, there is considerably more assurance that the idea will be mastered and retained by the pupil and will become truly his own.

Here again the demands on the teacher are greater. The teacher, like the student, must be thinking at every moment, for it takes far more insight to lead than to tell. This Guide is intended primarily to assist the teacher in the actual conduct of the class. Methods are suggested which will encourage student imagination and generate student interest. In addition the Guide contains mathematical background and explanatory materials beyond that given in the Text. Answers to the problems are provided.

V. Relationship to Other Materials

This Text is based on the assumption that the pupil has studied from the previous Texts in the Entebbe Mathematics Series. If a pupil has not studied all of the earlier texts it may be necessary to go back and study portions of *Secondary Two* and *Secondary Three*. However, the pupil can make good progress in the present Text without having a complete mastery of these previous Texts.

This Text is designed to prepare the pupil for further mathematics in either the conventional school curriculum or in curricula evolving from this and similar experimental programmes. It is also designed to conform to the requirements of the present examinations.

It should be emphasized that this modified curriculum represents a great opportunity for teacher and pupil alike. In making use of it the school participates in a great experiment to help develop a strong African educational system of which we can be proud. It is a massive joint undertaking, in which pupils and teachers work side by side with mathematicians of international eminence, from their own country and from a dozen others, to create within Africa something that will be of major significance for Africa itself, and in a large measure for all the world.

ENTEBBE MATHEMATICS SERIES

SECONDARY _____ REPORT, CHAPTER _____

Would you please fill out one of these forms for each chapter of Secondary material as you complete the chapter. Send it by air mail to educational Services Incorporated, African Education Program, 55 Chapel Street, Newton, Massachusetts 02158, U. S. A. Your comments will help us to improve and to rewrite the material for further use.

Please respond to *all* parts of this report.

Name of Teacher _____

Name of School _____

School Address _____

Date _____

Chapter Number _____ in Secondary Book _____ Volume _____

Chapter Title _____

Number of classes in which you teach the Entebbe Mathematics Workshop Secondary Materials _____

_____

Number of pupils using material _____ . Number of periods each class meets per week _____ .

Length of class period _____ minutes. Class periods used teaching the chapter _____.

The pupils of my class are of the following ability level (tick one of the following):

fast _____ , slow _____ , medium _____ , mixed ability _____ .

My pupils have had, on the average, _____ years of schooling before this present year.

To the best of my judgment this chapter is (tick one of the following): excellent _____,

good _____ , fair _____, poor _____ , or extremely poor _____.

Why did you rate the chapter in this way? (If you need more space than is provided here, please use an additional sheet.) _____

_____

_____

If you used any supplementary materials in teaching this unit, please describe them. _____

_____

_____

Please indicate the specific sections which need improvement and how *you* would recommend improving them.

Questions	Comments
1. Should this whole chapter be omitted? Why?	
2. Name specifically any ideas which you think ought to be in this chapter but which are not there. Why should they be included?	
3. Name specifically any topics in this chapter which you think should be omitted. Why should they be omitted?	
4. Give the section and page number of any part of this chapter where you think the explanation is not clear enough. What suggestions do you have on how to make it more clear?	
5. Indicate which sets of problems need more problems to give adequate practice. What kind of problems are needed?	
6. Indicate by page and number any problems which you think are too difficult and problems which you think are too easy.	

STUDENT TEXT

(continued)

Questions	Comments
7. Indicate any problems or illustrations which are ineffective because they use words or ideas which are unfamiliar to your students. For instance, a child who had never been to sea in a schooner would not much appreciate a problem about the area of the jib and the foresail and the mainsail. For the examples which are inappropriate for your children, can you suggest local situations which would be more appropriate in the example.	

TEACHERS' GUIDE

Questions	Comments
1. In what parts of the chapter were the suggestions on how to teach the material not sufficient to help you teach it effectively? What suggestions have you for improving those parts?	
2. In what parts of the chapter were the suggestions on teaching the material trivial and unnecessary?	

Questions	Comments
3. In some places in the Teachers' Guide there are discussions for the teacher of the mathematical background, going more deeply into the ideas than one would with the students. Indicate by section and page number: (a) which of these discussions were not clear enough to be helpful, (b) which were too brief and need to be expanded, (c) which were unnecessary for you and might be omitted, (d) where no such discussion was given where it was needed.	
4. Indicate any answers to problems which were not sufficiently clear or complete.	

Would you please write any other comments which you believe might be helpful in rewriting this chapter.

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Chapter 1
RELATIONS AND FUNCTIONS

This chapter continues the study of functions begun in *Secondary Three*, and introduces the concept of a relation as a natural generalization of the notion of a function. The elementary theory of relations is developed and then applied to the study of functions and their inverses. The unifying idea throughout is that of an ordered pair, and, without undue exaggeration, this chapter could be entitled "The Mathematical Theory of Ordered Pairs." As usual, we proceed with the aid of examples and pictures which have been designed to enable the teacher to guide pupils to the point where they can discover most of the stated results for themselves.

1-1 REVISION.

This section is a brief revision of the material on functions introduced in *Secondary Three*. The amount of time which will have to be devoted to this revision will of course depend upon the preparation of the pupils. In the event that your class has not covered the last unit in the *Secondary Three Algebra Text* (Chapters 12 and 13) you may wish to start the year's work at that point. If this is done, the present section may be omitted.

We begin by recalling our earlier description of a function as a rule or correspondence by means of which numbers are paired together in a perfectly definite way. Your pupils should understand that the phrase "in a perfectly definite way" means that the function associates a *unique* number in its range with each number in its domain. (For the moment we restrict our attention to numerical functions, but, if you wish, you may observe that functions can also be formed by pairing other types of objects. This point will be treated explicitly later in the chapter.)

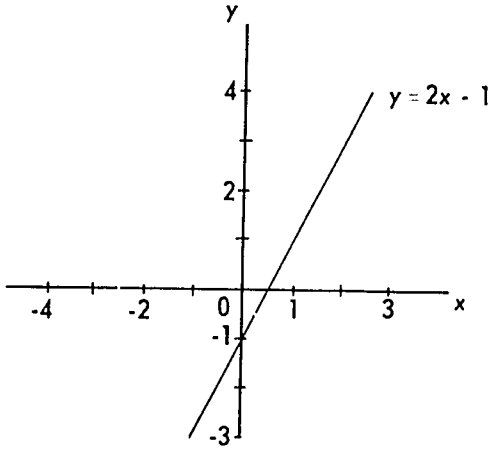
The main purpose of this section—besides recalling the terminology used in speaking about functions—is to focus attention on the notion of an ordered pair. Without being excessively formal you should attempt to bring your pupils to the point where they see that a set of ordered pairs is a precise and often convenient way of expressing the correspondence determined by a rule or formula which was their earlier viewpoint of function. To this end, the example given on p. 5 should be helpful, and your pupils should be encouraged to work it out for themselves. Once they understand how these functions are expressed you will be able to point out that functions cannot always be described by a simple formula such as $y = x^2$. Finally, they should also be made to appreciate that two functions may be defined by the same formula and yet be different because their domains are different. This point should be familiar since it was discussed in *Secondary Three* where examples were given. It is briefly covered in the third problem in Problems 1-1B, page 4.

Answers to
PROBLEMS 1-1A

Student Text Page 3

The pupil's tables should include enough values to allow them to draw the graphs of these functions. The fourth problem is the only one which may cause difficulty, and you may find it necessary to discuss the absolute value function before having the pupils attempt the graph. The graphs are as follows:

1.



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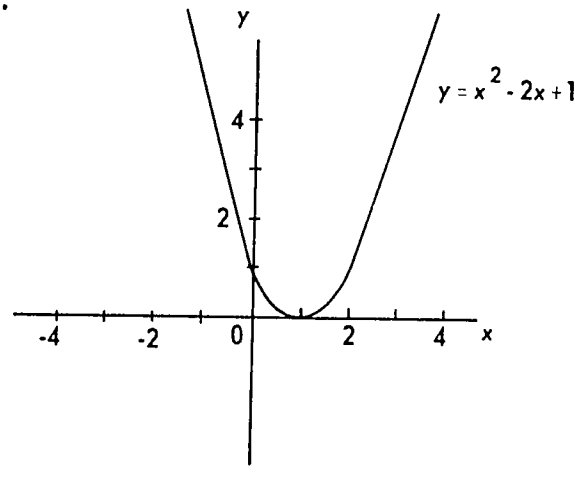

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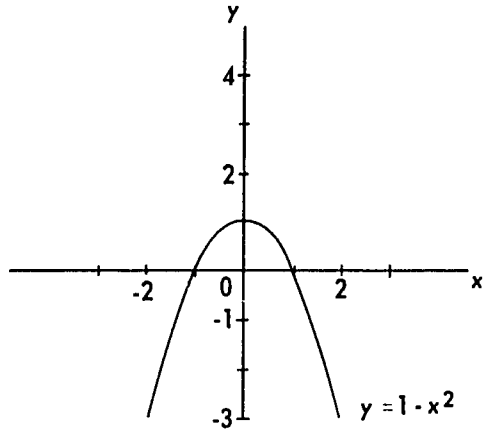

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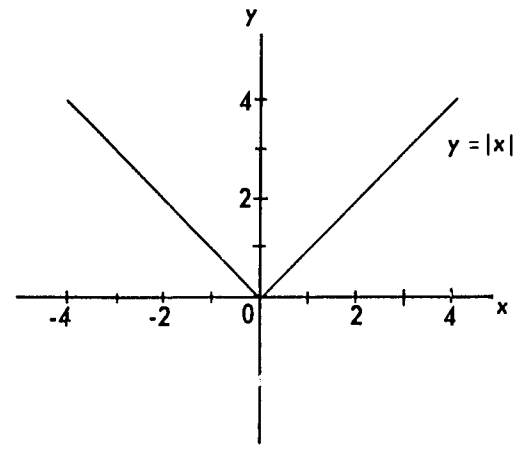


Answers to
PROBLEMS 1-1B

Student Text Page 3

1. Pupils should answer this problem by reading the required information from graphs they constructed in PROBLEMS 1-1A. In particular, they should realize that part d can be answered by counting the number of points of intersection of the graph of the function with lines parallel to the x-axis. This point will be important later when we consider one-to-one functions.

   a. The domain of each function is the set of all real numbers.

   b. The range for $y = 2x - 1$ is the set of all real numbers. The range for $y = x^2 - 2x + 1$ and $y = |x|$ is the set of all nonnegative real numbers.
   The range for $y = 1 - x^2$ is the set of all real numbers ≤ 1.

   c. In every case one y is paired with each x in the domain of the function. (Remind your pupils that, *by definition*, this is true of *every* function.)

   d. For $y = 2x - 1$ one value in the domain is paired with each value in the range.
   For $y = x^2 - 2x + 1$ and $y = |x|$ two values in the domain are paired with each value in the range, except when $y = 0$ where there is only one value of x in the domain.
   For $y = 1 - x^2$ two values in the domain are paired with each value in the range, except when $y = 1$ where there is only one value of x in the domain.

2. a. Domain = $\{x: x \geq 0\}$. Since the symbol $\sqrt{x}$ is understood to be the *positive* square root, the range is the set of all nonnegative real numbers, that is, Range = $\{y: y \geq 0\}$.

b. Domain = $\{x: x \neq 0\}$. Range = $\{y: y \neq 0\}$.

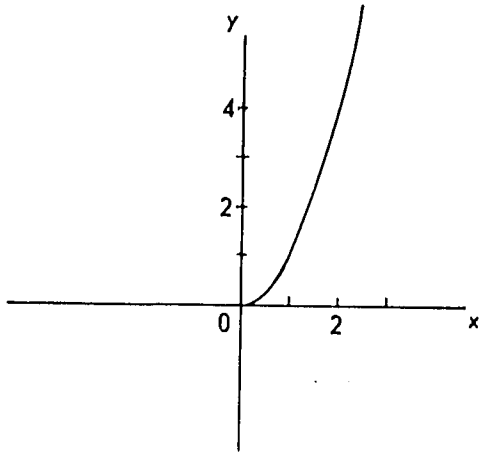
c. Domain = $\{x: x \leq 0\}$. Range = $\{y: y \geq 0\}$.

d. Domain is the set of all real numbers. Range = $\{y: y > 0\}$.

3. a. Domain = $\{x: x \geq 0\}$

b. Range = $\{y: y \geq 0\}$

c.



Note that the graph of this function is half of the graph of $y = x^2$. The origin is included in the graph.

d. The function differs from the function defined by the expression $y = x^2$ in that its domain is the set of *nonnegative* real numbers.

Answers to Questions following the example: **Student Text Page 5**

Table a has only part of the range B. Table b has two different values in the range for the same value, 0, in the domain. Table c has only part of the domain A. If the domain and range are not specified, a and c define functions.

Answers to
PROBLEMS 1-1C

Student Text Page 5

1. The pairs $\left(1\frac{1}{2}, \frac{9}{4}\right)$, (π, π^2), $(2\sqrt{2}, 8)$ belong to the function, the rest do not.

2. a. $f(0) = 0^2 - 5(0) + 6 = 6$

$f(-1) = (-1)^2 - 5(-1) + 6 = 12$

$f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^2 - 5\left(\frac{3}{2}\right) + 6 = \frac{3}{4}$

b. The pairs (3,0), (-2,20), (-1·1, 12·71), (7,20) belong to f, the rest do not.

c. The completed table should read

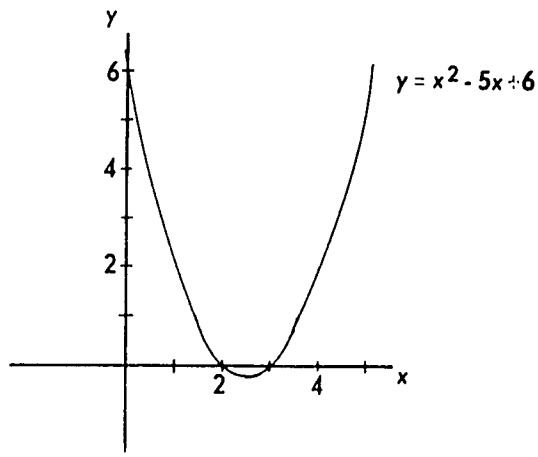
x	0	1	2	$2\frac{1}{2}$	3	4	5
$f(x)$	6	2	0	$-\frac{1}{4}$	0	2	6

d. When $f(x) = 0$, $x = 2$ or $x = 3$.

When $f(x) = -\frac{1}{4}$, $x = 2\frac{1}{2}$

When $f(x) = 6$, $x = 0$ or $x = 5$.

e.



3.

f_1	-1	1
	3	4

f_2	-1	1
	4	3

4. There are no such functions.

5. Tables a, b, d, e define functions, c does not.

a. Domain = $\{0, \frac{1}{2}, 1\}$. Range = $\{-1\}$.

b. Domain = $\{0, 1, 2, 4\}$. Range = $\{0, 2\}$.

d. Domain = $\{-2, \frac{1}{2}, 1, 2\}$. Range = $\{-2, \frac{1}{2}, 1, 2\}$.

e. Domain = $\{-2, 1, 3\}$. Range = $\{0, \frac{1}{2}, 1\}$.

1-2 RELATIONS.

In this section we introduce the mathematical definition of a relation as a set of ordered pairs. Despite the simplicity of this concept pupils find it strange at first, and you will have to proceed slowly until they are ready to think in terms of sets of ordered pairs. Thus, in discussing the relation of fatherhood, which is eventually defined to be the set of all ordered pairs (x, y) where y is the father of x, you should attempt to get your class to suggest this definition for themselves. Emphasize that we seek a *workable, mathematical* definition; that is, one which allows us to determine when two people x and y are related to one another in such a way that y is the father of x, and which, at the same time, avoids all complications arising from the vague feelings usually associated with this term. As you proceed you should stress that regardless of these feelings (where, of course, no two people will completely agree) we can still give a definition which allows us to study the relation of fatherhood in a systematic way. The definition finally chosen does just this, and hence from a mathematical point of view is entirely satisfactory.

Once your class has grasped the idea of thinking about relations as sets of ordered pairs have them go on to the CLASS ACTIVITY (Page 8) and make their lists. If they have difficulty you might suggest that they consider relations such as the set of ordered pairs (x, y) of pupils in their class where y is taller than x or where y is older than x, and the set of all ordered pairs (x, y) where x is a citizen of country y. To stimulate class discussion you may find it helpful to list the relations they find on the chalkboard.

In Question 2 of the CLASS ACTIVITY they should, after some experimentation, hit upon the idea of reversing the entries in the ordered pairs belonging to the relations found in Question 1. This operation of reversing the order in the ordered pairs should be emphasized since it will be introduced formally when inverses are studied. For the moment, however, you need only bring your pupils to the point where they realize that this can be done with any relation whatever, and that two successive reversals of this type will take them back to the original relation.

Answers to Questions following the examples: **Student Text Page 8**

Example 1: The ordered pairs (1,2), (1·1, 1·11), (-5, -3) belong to the relation, the rest do not.

a. Any number a greater than 3 will do.

b. Any number a greater than -4 will do.

c. Any number a between 3 and 4 will do.

(Make sure your pupils understand that a, b, and c have many different answers, all of them correct.)

Before studying Figure 4 remind the class that we use a broken line to indicate that the points on that line do *not* belong to the region in question; a solid line when they do. The correct answer to the first question concerning the figure is Figure 4 (a). Figure 4 (b) represents the relation consisting of the set of all ordered pairs (x, y) with $y < x$. When the line $y = x$ is included in the region we must allow the possibility that $y = x$ in both relations.

Example 2: If Δ_2 is congruent to Δ_1 (i.e., if the ordered pair (Δ_1, Δ_2) belongs to the relation) then Δ_1 is certainly congruent to Δ_2, and it follows that (Δ_2, Δ_1) also belongs to the relation. Since a triangle is always congruent to itself, (Δ_1, Δ_1) belongs to the relation for every Δ_1. Finally, if Δ_1, Δ_2, and Δ_3 are triangles, no two of which are congruent, none of the given pairs belongs to the relation. If two of the triangles are congruent to each other but not to the third, only one of the ordered pairs belongs to the relation. It is impossible to have *exactly* two of these pairs in the relation since all of the triangles would then be congruent and all three ordered pairs would belong to the relation.

Example 3: The questions here should not cause any difficulty if the pupils phrase them following the model given above for Example 2.

Example 4: The ordered pairs (1,5) and (3,9) belong to the relation, the rest do not. Since $x = 1 \times x$, the ordered pair (x, x) belongs to the relation for every natural number x. If y is a multiple of x we have $y = mx$ for some integer $m > 0$. Similarly, if z is a multiple of y we have $z = ny$ for some integer $n > 0$. Substituting mx for y in the second equation we get $z = (nm)x$, and z is a multiple of x. Hence (x, z) belongs to the relation.

1. In pairs of the form $(2, y)$, y must be even; i.e., $y = 2, 4, 6, \ldots$. In pairs of the form $(5, y)$, y must be a multiple of 5; i.e., $y = 5, 10, 15, \ldots$. In pairs of the form $(x, 24)$, x must be one of the numbers 1, 2, 3, 4, 6, 8, 12, 24. In pairs of the form $(x, 17)$, x must be one of the numbers 1, 17.

2. a. False
   b. False; x may be the *sister* of y.
   c. True.

3. a. False.
   b. True
   c. False

4. a. False; $x - x = 0$.
   b. False; if $x - y = 3$, then $y - x = -3$
   c. False; if $x - y = 3$ and $y - z = 3$, then $x - z = 6$.

5. a. The pairs $(9,3)$, $(3,9)$, $(-1,2)$, $(-4,-7)$ belong to the relation, the rest do not.
   b. The pair (x, x) belongs since $x - x = 0 = 3 \times 0$. If (x, y) belongs, then $x - y = 3k$ for some integer k. Hence $y - x = -3k = 3(-k)$, and $y - x$ is also an integral multiple of 3, the multiplier now being $-k$. If $x - y = 3k$ and $y - z = 3m$ for integers k and m, then

$$\begin{aligned} x - z &= (x - y) + (y - z) \\ &= 3k + 3m \\ &= 3(k + m). \end{aligned}$$

   Since $(k + m)$ is an integer (by the closure property of addition for integers), $x - z$ is an integral multiple of 3, and (x, z) belongs to the relation.

1-3 DOMAIN AND RANGE.

This section will not cause any difficulty if pupils remember the meanings of the terms "domain" and "range" from their study of functions since the definitions here are exactly the same. You may find it necessary, however, to recall the meaning of the set notation used in the formal definition.

Answers to
CLASS ACTIVITY

Student Text Page 12

1. Here for the first time we use set notation to define a relation. Make sure your pupils know how to read and interpret such definitions because they will be used constantly from now on.

   a. The ordered pairs are $(0,0)$, $(1,1)$, $(-2,-8)$, $\left(-\frac{3}{2}, -\frac{27}{8}\right)$, $(\sqrt[3]{-4}, -4)$, $(\sqrt{3}, 3\sqrt{3})$, $(-\pi, -\pi^3)$. Since every real number has a cube, R contains an ordered pair of the form (x, y) for every real number x. This implies that the domain of R is the set of all real numbers, and since y is uniquely determined by x, R is a *function.*

   b. The ordered pairs are $(0,0)$, $(-1,-1)$, $(-3,-27)$, $(2,8)$, $\left(\frac{1}{5}, \frac{1}{125}\right)$, $(\sqrt[3]{3}, 3)$, $(\sqrt[3]{-3}, -3)$. Since R contains an ordered pair of the form (x, y) for every real number y, the range of R is the set of all real numbers. The x appearing in this pair is called the cube root of y.

2. Domain of $R = \{0,1\}$. Range of $R = \{2,4\}$. R is not a function since 0 is paired with more than one member in the range of R. The remaining relations are

$$\{(0,2), (0,4), (1,2), (1,4)\}$$
$$\{(0,2), (0,4), (1,4)\}$$
$$\{(0,2), (1,2), (1,4)\}$$
$$\{(0,4), (1,2), (1,4)\}$$
$$\{(0,2), (1,4)\}$$
$$\{(0,4), (1,2)\}$$

Answers to
PROBLEMS 1-3

Student Text Page 13

1. a. Domain = $\{x\colon\ x$ has a sister$\}$.
   Range = $\{y\colon\ y$ is a sister$\}$.

   b. Domain = $\{x\colon\ x$ is a pupil in a class$\}$.
   Range = $\{y\colon\ y$ has a class$\}$.

   c. Domain = $\{x\colon\ x \geq 0\}$.
   Range:= $\{y\colon\ y \geq 0\}$.

d. Domain = the set of all real numbers.
Range = the set of all real numbers.

e. Domain = $\{x\colon\ x$ is a natural number$\}$.
Range = $\{y\colon\ y$ is a natural number$\}$.

f. Domain = $\{x\colon\ x$ is a person living in a town$\}$
Range = $\{y\colon\ y$ is a town$\}$.

2. a. The ordered pairs $(0,1)$, $(0,-1)$, $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ belong to R, the rest do not.

b. Domain of $R = \{x\colon\ -1 \le x \le 1\}$.

c. Range of $R = \{y\colon\ -1 \le y \le 1\}$.

1-4 PICTORIAL REPRESENTATIONS.

The objective of this section is to introduce some of the pictorial methods of representing relations. Two such are given: general pictures of the type shown in Figure 5 of the Student Text, and graphs. Your pupils should be reasonably familiar with each from their earlier work with functions, and although the pictures are now a little different from those they have seen heretofore, they should have no difficulty drawing them. You should point out that whereas the first type of picture is perfectly general and can be used to represent any relation whatever, graphs can be used only to represent numerical relations; that is, relations whose domain and range are sets of real numbers. Make sure your pupils understand that the graph of a numerical relation is a set of points in the plane. Often the graph is a *region* of the plane, although in special cases it may turn out to be a curve or collection of curves or even a set of isolated points. They should realize also that the converse of this statement is true, and that *every* (nonempty) subset of the plane is the graph of a numerical relation, namely, the relation whose ordered pairs are the pairs of coordinates of the points in the given subset. In short, (nonempty) subsets of the plane and numerical relations are two different ways of looking at exactly the same thing.

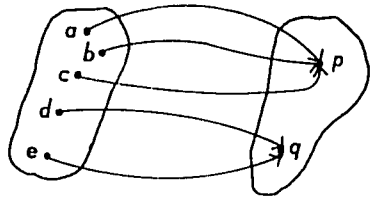
After your pupils have studied this section they should be encouraged to draw pictures (and graphs) of relations both as an aid in solving problems and to illustrate new concepts as they are introduced. A number of the problems at the end of the section have been designed with this in mind, and will be quite difficult unless graphs are drawn.

Answers to
PROBLEMS 1-4A

Student Text Page 15

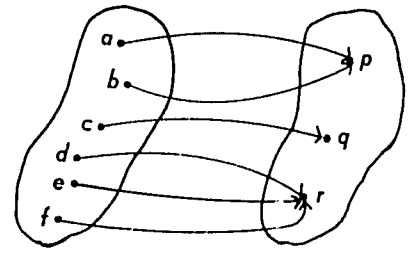
The pupils' pictures should look something like this.

1.



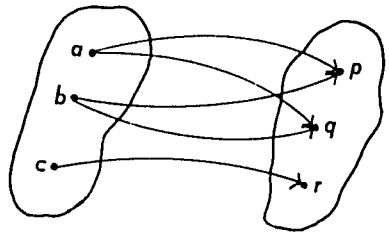
Domain Range

3.



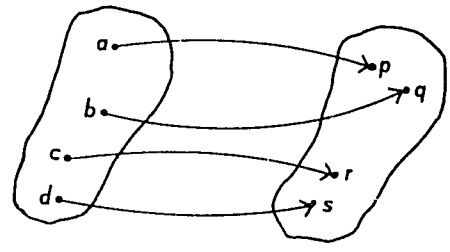
Domain Range

2.



Domain Range

4.



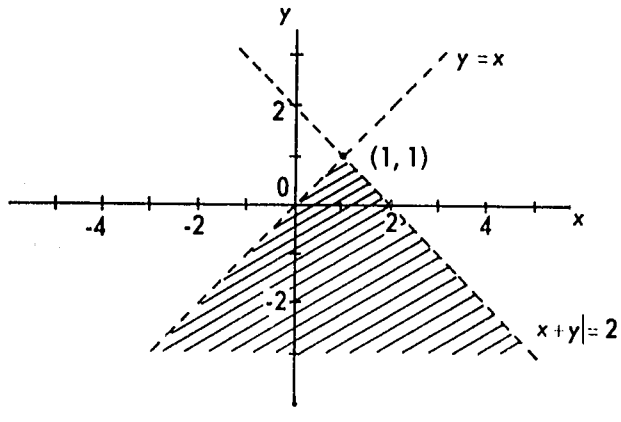
Domain Range

Answers to
CLASS ACTIVITY

Student Text Page 16

1. The set of ordered pairs having 1 as their first entry is represented by the portion of the vertical line $x = 1$ below the line $y = x$. Similarly the set of ordered pairs having -2 as their second entry is represented by the portion of the horizontal line $y = -2$ to the right of the line $y = x$.

2. The picture should look like this:



Once the pupils have sketched this graph they can determine the domain and range of R by considering the values of x and y which lie in the shaded region. The correct answers, of course, are that the domain of R is the set of all real numbers, the range the set of all real numbers less than 1. If your class has difficulty in deciding which of the four regions to choose you can point out that the graph of R is the *intersection* of the graphs of the relations

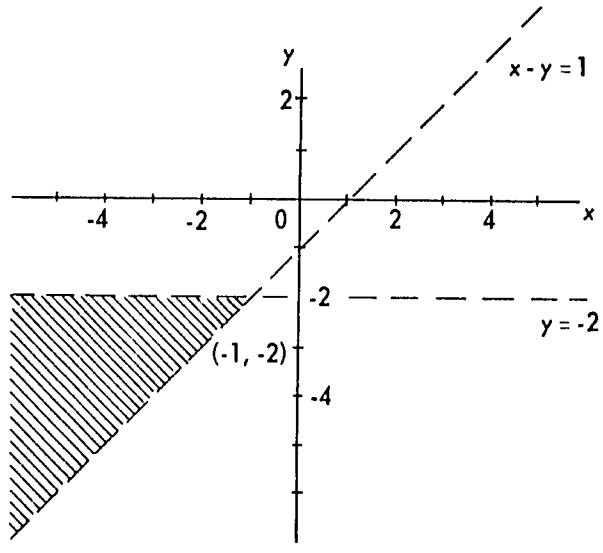
$$R_1 = \{(x, y): \ x > y\} \text{ and } R_2 = \{(x, y): \ x + y < 2\}.$$

This will allow them to find the graph of R by graphing R_1 and R_2 and then taking the region common to the two graphs.

Answers to
PROBLEMS 1-4B

Student Text Page 16

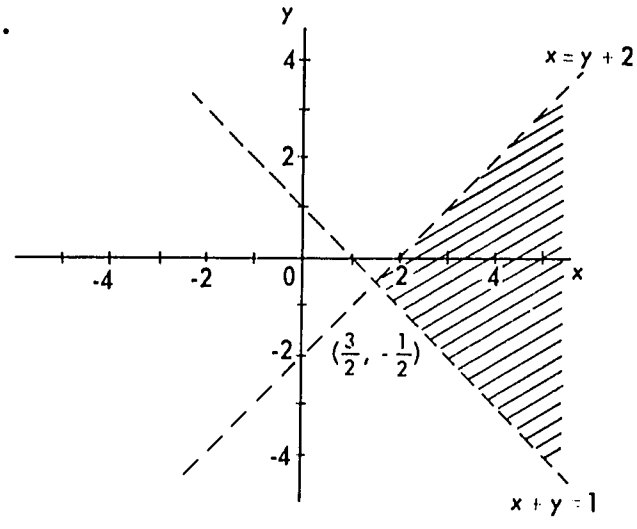
1.



Domain of $R = \{x: \ x < -1\}$

Range of $R = \{y: \ y < -2\}$

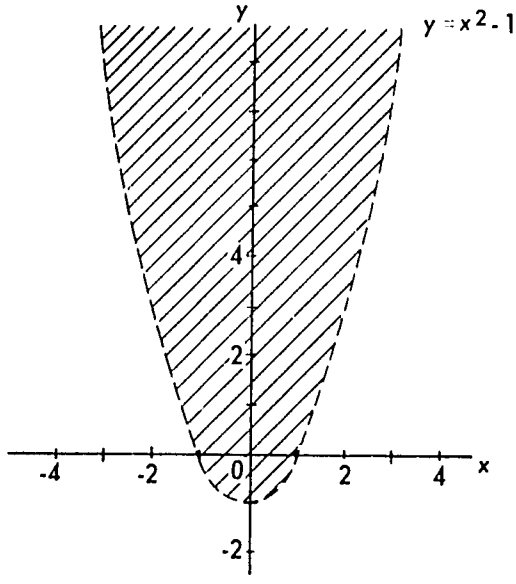
2.



Domain of $R = \{x: \ x > \frac{3}{2}\}$.

Range of R is the set of all real numbers.

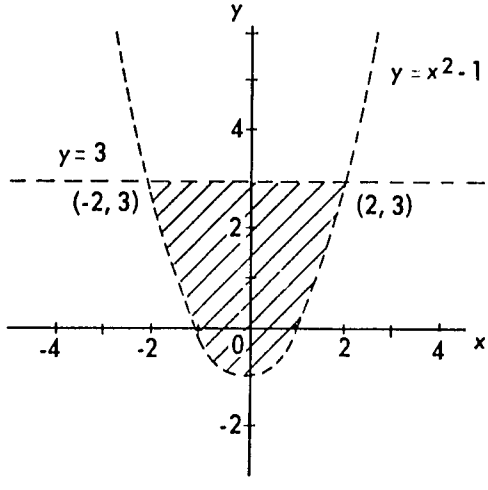
3.



Domain of R is the set of all real numbers.

Range of $R = \{y\colon\ y > -1\}$

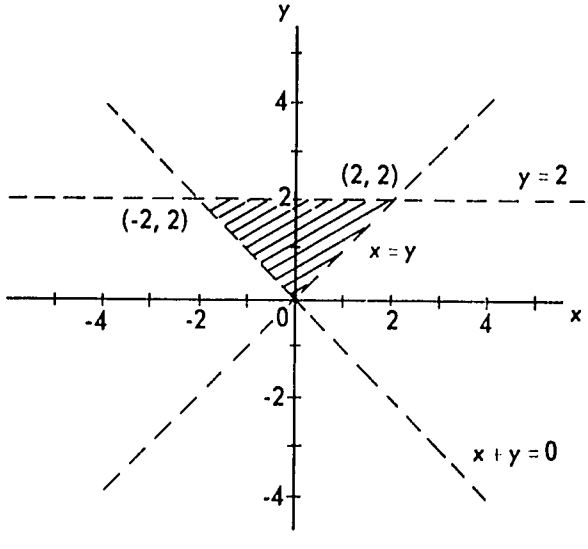
4.



Domain of $R = \{x\colon\ -2 < x < 2\}$

Range of $R = \{y\colon\ -1 < y < 3\}$

5.



Domain of $R = \{x\colon\ -2 < x < 2\}$

Range of $R = \{y\colon\ 0 < y < 2\}$

6. a. $R = \{(x, y): \ y < -x \text{ and } y < x + 2\}$
Domain of R is the set of all real numbers.
Range of $R = \{y: \ y < 1\}$.

b. $R = \{(x, y): \ y > 1 \text{ and } x + y < 1\}$
Domain of $R = \{x: \ x < 0\}$
Range of $R = \{y: \ y > 1\}$.

c. $R = \{(x,y): \ -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1\}$
Domain of $R = \{x: \ -1 \leq x \leq 1\}$.
Range of $R = \{y: \ -1 \leq y \leq 1\}$.

d. $R = \{(x, y): \ y \geq x^2 \text{ and } y \leq 2x\}$
Domain of $R = \{x: \ 0 \leq x \leq 2\}$.
Range of $R = \{y: \ 0 \leq y \leq 4\}$.

e. $R = \{(x, y): \ y > |x| \text{ and } y \geq x^2\}$
Domain of R is the set of all real numbers.
Range of $R = \{y: \ y > 0\}$.

f. $R = \{(x, y): \ y \geq 2x, \ y \geq x^2, \text{ and } y \leq 4\}$.
Domain of $R = \{x: \ -2 \leq x \leq 2\}$.
Range of $R = \{y: \ 0 \leq y \leq 4\}$.

1-5 THE INVERSE OF A RELATION.

In this section we discuss the important notion of the inverse of a relation. To form R^{-1} once R is given, simply reverse the entries in each of the ordered pairs belonging to R. This operation was introduced informally as a class activity in Section 1-2 and your pupils might profit from a revision of the results they obtained at that time before you introduce the formal definition.

Before you answer the questions following the definition encourage your pupils to draw pictures such as the one given in Figure 9 of their text and attempt to discover the answers for themselves. Those who understand how inverses are formed will be able to answer them without difficulty.

The questions in the CLASS ACTIVITY on page 18 besides illustrating the operation of forming inverses, point out that if the operation is performed twice in succession on any relation whatever, the result is necessarily the original relation.

You will notice that we have not discussed the problem of graphing

the inverse of a numerical relation in this section. Most pupils find this difficult at first, and for that reason we have postponed the problem to the next section where it is treated separately.

Answers to Question:

Student Text Page 18

The ordered pairs (0, 0), (4, 2), (4, -2), (8, $2\sqrt{2}$), (9, -3) belong to the square root relation S.

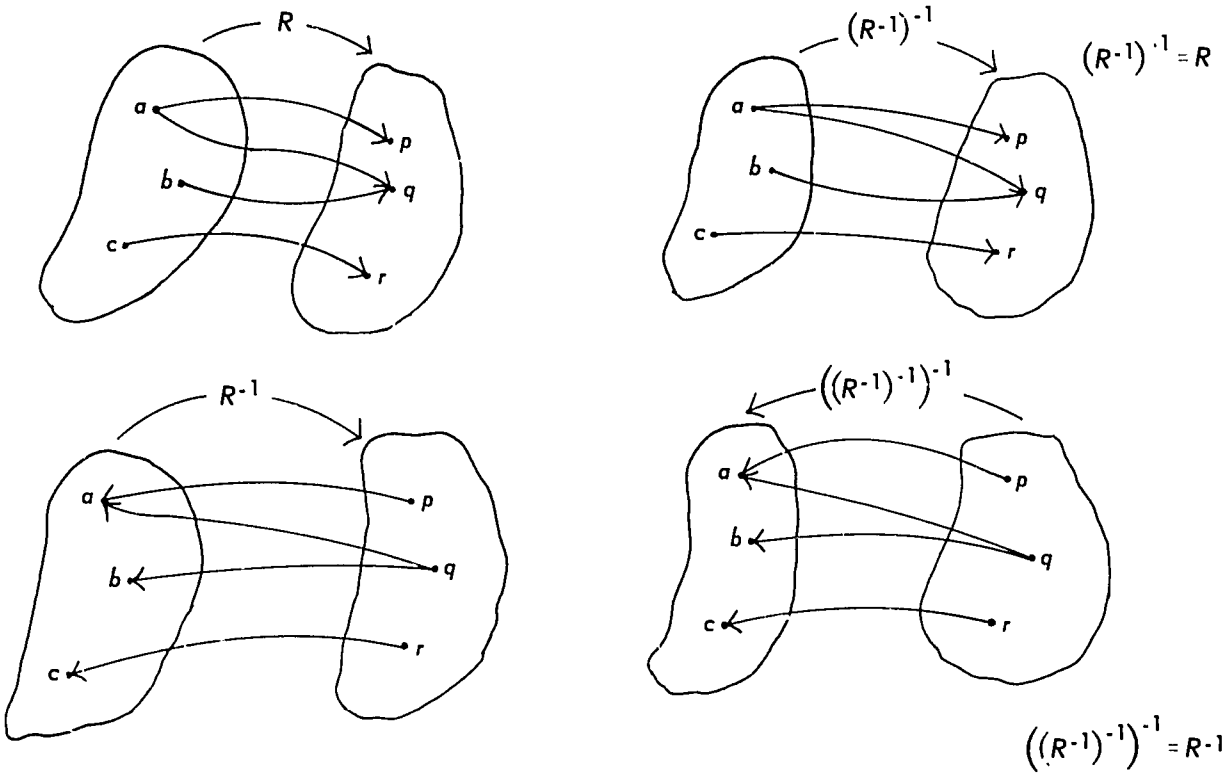
Answers to
PROBLEMS 1-5A

Student Text Page 20

1. R^{-1} is the set of all ordered pairs (y, x) of pupils in your class where x is shorter than y.
2. R^{-1} is the set of all ordered pairs (y, x) where y is a teacher in your school and x is one of y's pupils.
3. R^{-1} is the set of all ordered pairs (y, x) of people in Africa where x is younger than y.
4. R^{-1} is the set of all ordered pairs (y, x) of real numbers where x is the cube root of y.
5. R^{-1} is the set of all pairs of coordinates of points in the plane.
6. R^{-1} is the set of all pairs of coordinates of points in the plane lying in the fourth quadrant; that is, points whose first coordinate is positive and second coordinate negative.
7. R^{-1} is the set of all pairs of coordinates of points in the plane to the left of the line $x = 4$.

Answers to
CLASS ACTIVITY

Student Text Page 21



Answers to
PROBLEMS 1-5B

Student Text Page 21

1. Domain of $R^{-1} = \{x\colon\ x < 0\}$.
   Range of R^{-1} is the set of all real numbers.

2. Domain of $R^{-1} = \{x\colon\ x \geq 0\}$
   Range of R^{-1} is the set of all real numbers.

3. Domain and range of R^{-1} are the set of all real numbers.

4. Domain and range of R^{-1} are the set of all real numbers.

5. Domain of R^{-1} is the set of all real numbers.
   Range of $R^{-1} = \{y\colon\ y > 0\}$.

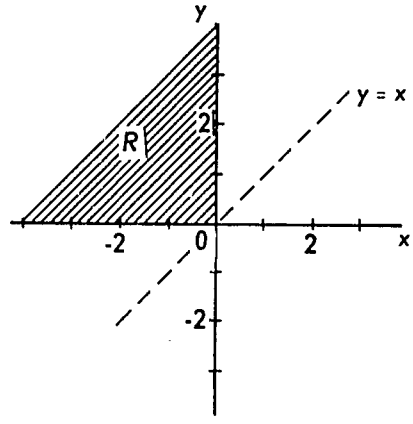
6. Domain of $R^{-1} = \{x\colon\ 0 < x < 4\}$.
   Range of $R^{-1} = \{y\colon\ 0 < y < 2\}$.

7. Domain of $R^{-1} = \{x\colon\ -1 < x < 1\}$.
   Range of $R^{-1} = \{y\colon\ -2 < y < 2\}$.

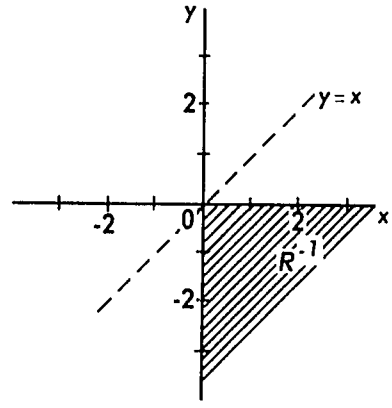
1-6 GRAPHING THE INVERSE OF A RELATION.

We now turn our attention to the problem of graphing the inverse of a numerical relation. You will probably find that your pupils have more trouble with this section than with any other in the chapter, and you will have to proceed slowly and carefully. In the last analysis the point at issue can be made only by drawing a large number of figures on the chalkboard, even though it can be described simply. Indeed, since R^{-1} is formed by reversing the entries in the ordered pairs belonging to R, the graph of R^{-1} consists of the set of points obtained by reversing the x and y coordinates of the points belonging to the graph of R. To describe the effect of this reversal observe that points on the line $y = x$ remain fixed since their x and y coordinates are the same; that a point $(a,0)$ on the x-axis goes into the point $(0,a)$ on the y-axis, and conversely; and that in general a point (a, b) goes into the point (b, a) located symmetrically across the line $y = x$. (A proof of this fact has been assigned as Challenge Problem 9 at the end of the section.) Thus the result of interchanging the x and y coordinates of the points in the plane is to reflect the plane across the line $y = x$. In the Student Text we have attempted to lead up to this result in such a way that the pupil can discover it for himself. However, the success of this effort will depend in large measure on pictures drawn on the chalkboard in the classroom.

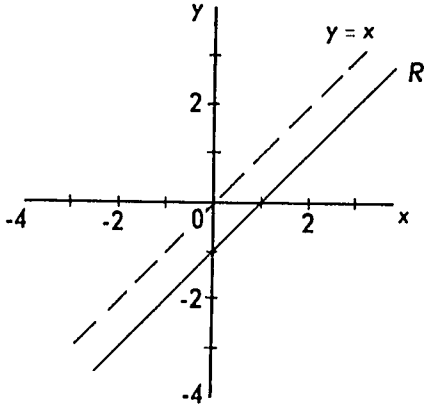
Once your pupils understand this description and realize that it can be interpreted as a rotation of the plane through an angle of 180° about the line $y = x$ all should be well, for then they will be able to obtain the graph of R^{-1} by visualizing what happens to the graph of R under this rotation. Here again pictures will be necessary, and you may find it helpful to display a sequence such as the one shown below, encouraging your pupils to draw the graph of R^{-1} before you construct it for them. In addition, you may wish to draw a number of graphs on square sheets of thin paper in such a way that the graph can be seen through the paper when viewed from the opposite side. Then, by holding the paper so that the graph is facing away from the pupils, and rotating it about the diagonal running from the upper right to lower left corners, you will be able to make the graph of R^{-1} appear before them.



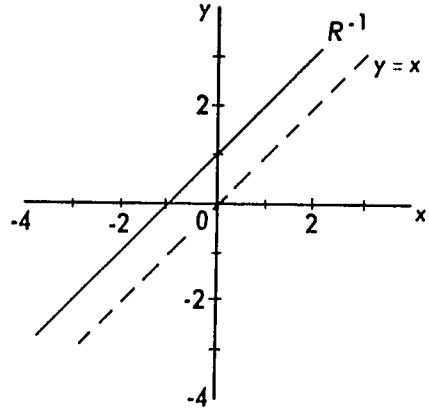
(a_1)



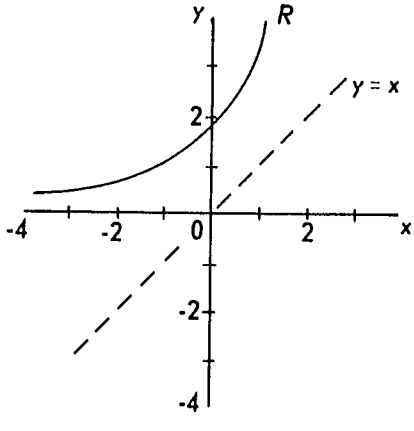
(a_2)



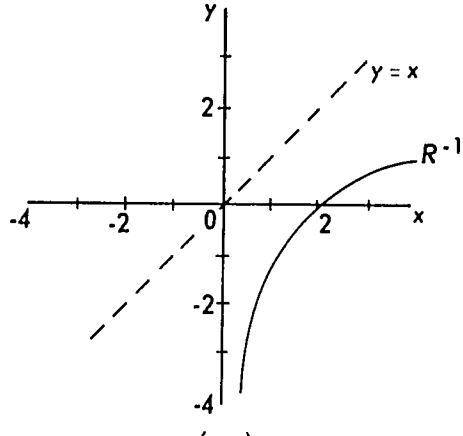
(b_1)



(b_2)



(c_1)



(c_2)

Answers to
CLASS ACTIVITY

Student Text Page 24

a. Points on the positive x-axis change places with the corresponding points on the positive y-axis.

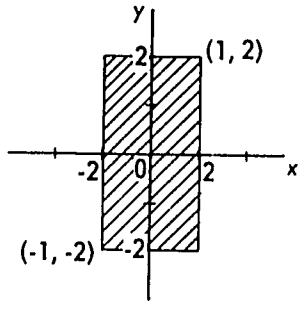
b. Points on the negative x-axis change places with the corresponding points on the negative y-axis.

c. Points on the line $y = x$ remain unchanged.

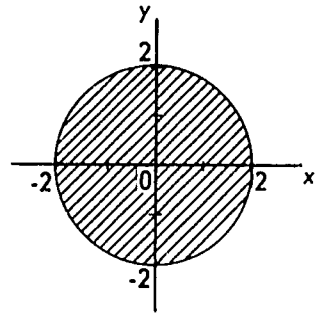
Answers to
CLASS ACTIVITY

Student Text Page 25

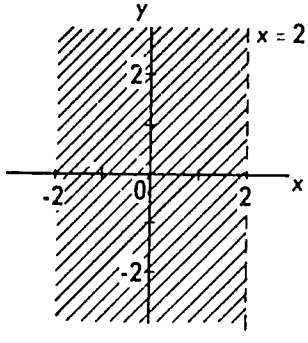
a.



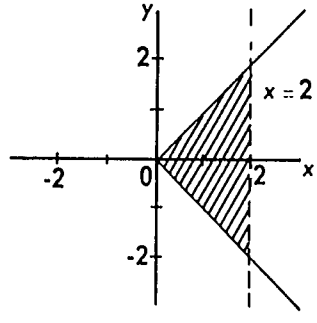
d.



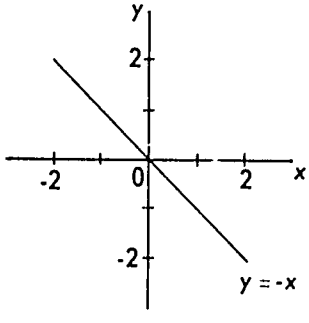
b.



e.



c.

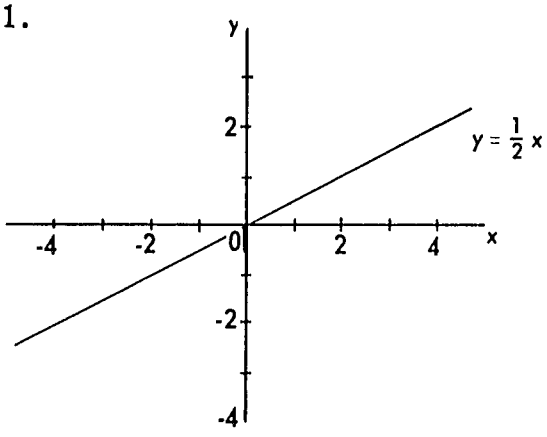


In parts c and d the relation is identical with its inverse.

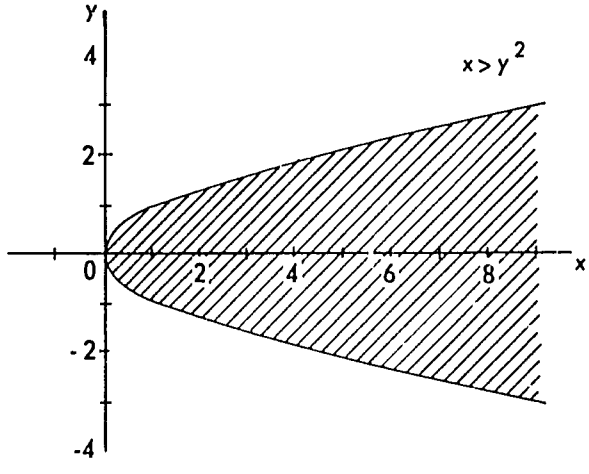
Answers to
PROBLEMS 1-6

Student Text Page 26

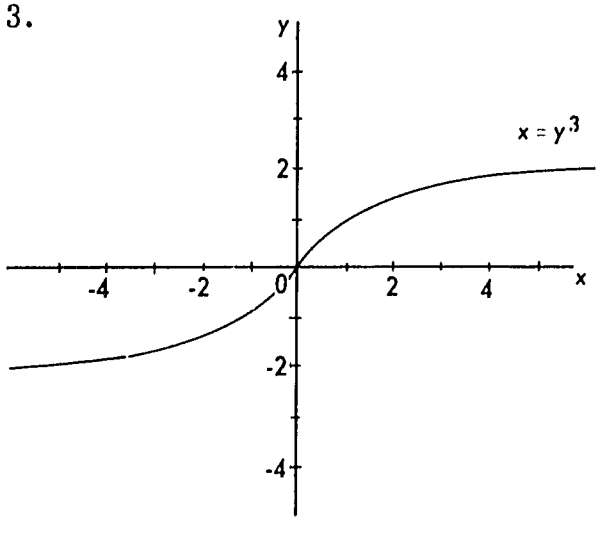
1.



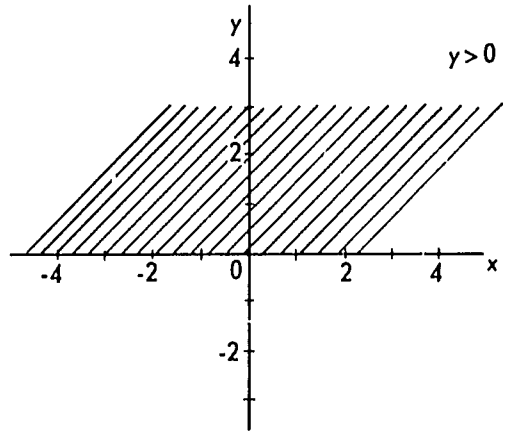
2.



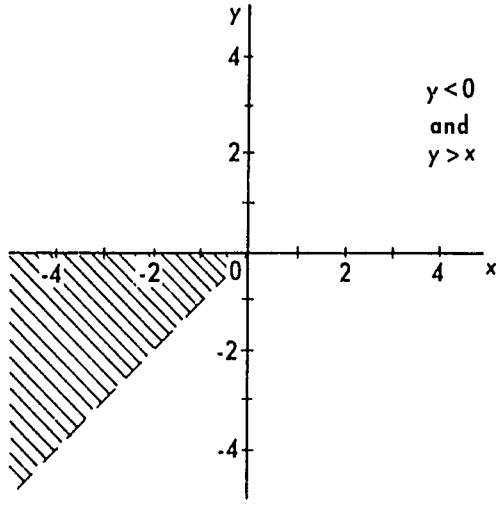
3.



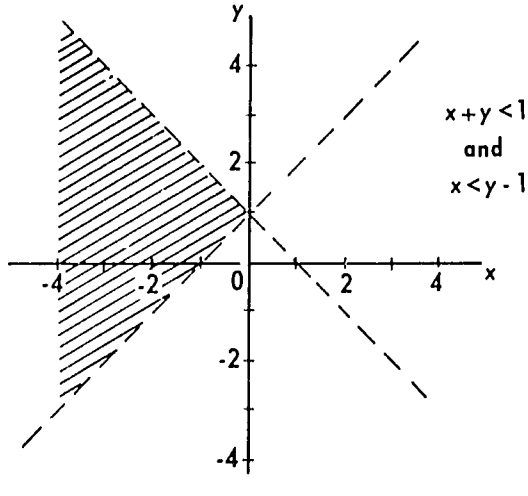
4.



5.



6.



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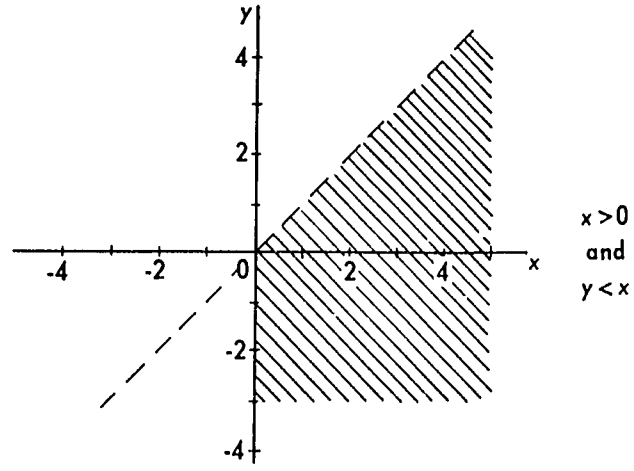
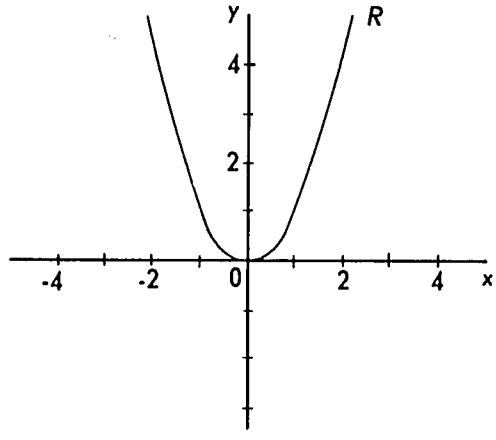
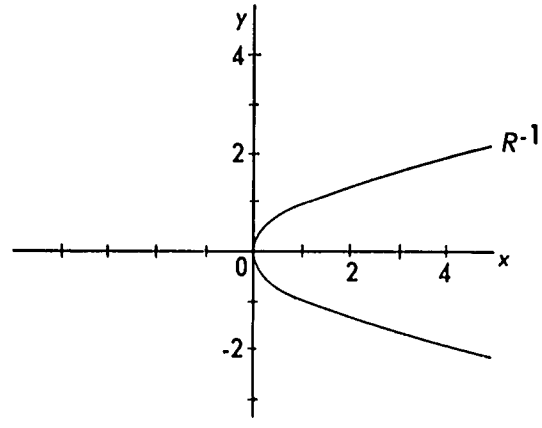


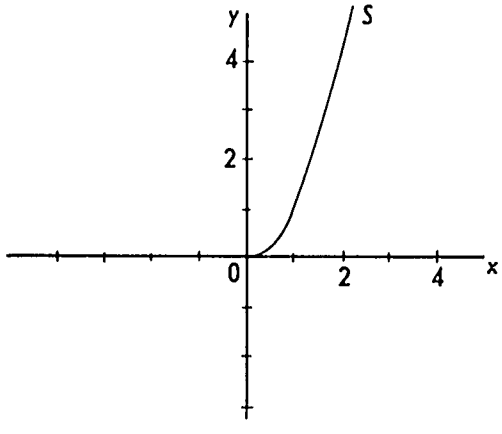
8. a.

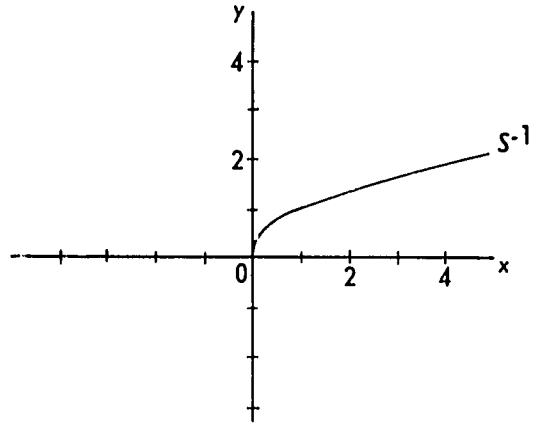




R^{-1} is not a function since each $x \neq 0$ in its domain is paired with two distinct y's in its range.

b.





(Note: The origin is included in each graph.)

S^{-1} is a function since each x in its domain is paired with exactly one y in its range.

9. *Challenge Problem*

a. The coordinates of B are (y, y), so the distances BP and BQ are both $|x - y|$, that is, $\overline{BP} \equiv \overline{BQ}$.
Of course, $\overline{BA} \equiv \overline{BA}$.
Since lines $\overleftrightarrow{BP}$ and $\overleftrightarrow{OX}$ are parallel, $A\hat{B}P \equiv A\hat{O}X$.
Similarly $A\hat{B}Q \equiv A\hat{O}Y$. But $A\hat{O}X \equiv A\hat{O}Y$ because $\overrightarrow{OA}$ bisects $Y\hat{O}X$.
It follows that $A\hat{B}P \equiv A\hat{B}Q$.
We now have $\Delta ABP \equiv \Delta ABQ$ by SAS. Therefore $\overline{AP} \equiv \overline{AQ}$ and $B\hat{A}P \equiv B\hat{A}Q$, so $\overline{PQ}$ is perpendicular to the line $x = y$.

b. This result shows that points P and Q are mirror images of each other in the line $y = x$.

1-7 FUNCTIONS.

In this section we formally define a function as a relation with the additional property that no two of its ordered pairs have the same first entry and different second entries. After recalling the terminology used in talking about functions we give two examples which are designed to show that in certain circumstances describing a function by means of a rule or formula is neither natural nor convenient. At this point you should encourage your pupils to appreciate the fact that although most functions in mathematics *are* described by formulas, the ordered pair definition has the advantage of being sufficiently general to include *all* functions which they will ever encounter. This is precisely the reason for choosing it as our basic definition. At the same time however, you should insist that they be flexible in their approach, and use both points of view interchangeably, as convenient.

The telephone book example furnishes an illustration of a function which cannot be described by a formula, and you should have your pupils propose additional examples of this type. To help them you might suggest the set of ordered pairs consisting of mountain peaks in Africa and their heights in feet, or the set of ordered pairs consisting of countries and their populations as given in an atlas. If this last example is used you might point out that countries take censuses periodically precisely because they cannot determine their populations by a formula.

In the example involving areas we have a function which can be described by two entirely different formulas. Nevertheless, *as a set of ordered pairs*, the function in question is the same regardless of which formula is used to compute its values.

Answers to
PROBLEMS 1-7A

Student Text Page 30

1-10. The relations given in 1, 3, 4, 7, and 10 are functions; the rest are not.

11. The pairs of formulas in b, d, g, and h define the same function, the rest do not.

The problem of graphing functions was discussed in *Secondary Three*, and by now your pupils undoubtedly know that a curve in the plane is the graph of a function if and only if it satisfies the geometric condition stated in the Student Text. This point should still be emphasized, however, since it will recur in the next section when we discuss inverse functions.

Answers to
PROBLEMS 1-7B

Student Text Page 32

The graphs in 1, 4, 6, and 7 represent functions, the rest do not.

1-8 ONE-TO-ONE FUNCTIONS; INVERSES.

The notion of a one-to-one correspondence was introduced in Section 2-4 of the *Secondary Three Algebra Text*. We now combine this idea with the notion of the inverse of a relation to show that the inverse of a function f is again a function if and only if f is one-to-one. By now your pupils will have had sufficient experience both with inverses and with functions to see the point immediately upon drawing the figures suggested in the text.

The first example following the definition should be perfectly straightforward since all of the ideas were discussed in detail in *Secondary Three*.

Answers to
CLASS ACTIVITY

Student Text Page 33

1. The ordered pairs $(-8, -2)$, $(2, \sqrt[3]{2})$, $\left(-\frac{1}{8}, -\frac{1}{2}\right)$, $(0, 0)$ belong to the inverse function.

2. See Problem 3 of Problems 1-6, page 26 of Student Text, page 20 of Teachers' Guide.

3. The inverse is known as the "cube root" function and is defined by the formula $y = \sqrt[3]{x}$.

The second example is designed to show that if the domain of a function is changed the resulting function may be one-to-one even though the original function was not. This will be important in *Secondary Five* when we study the inverse trigonometric functions.

After the discussion in Section 1-7 the graphical criterion for determining when a function is one-to-one should be all but obvious. The same should be true of the concepts of increasing and decreasing functions, although the terminology may seem strange at first. Have your pupils draw graphs of a number of functions of each type until they know the meaning of these terms.

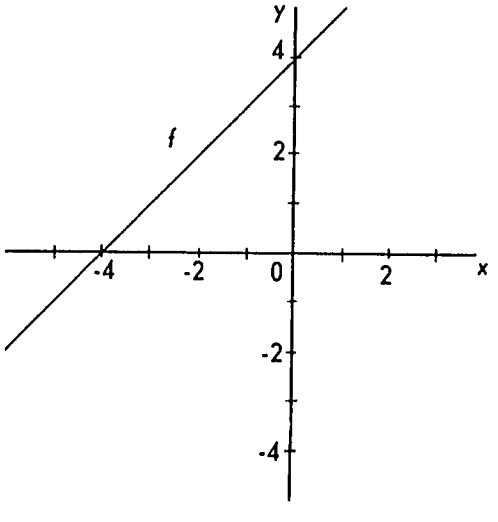
Answers to
PROBLEMS 1-8

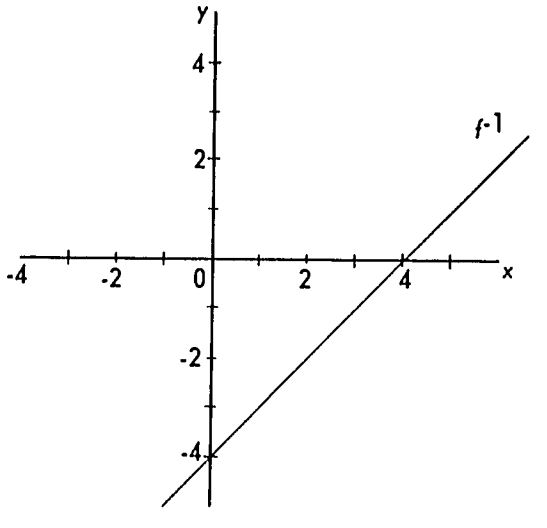
Student Text Page 35

1-8 The functions defined in 1, 4, 5, and 6 are one-to-one; the rest are not.

9.

From 5.





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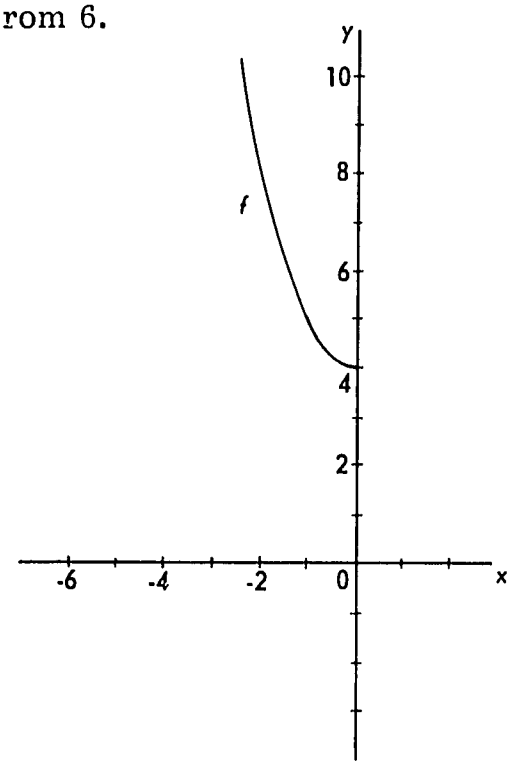

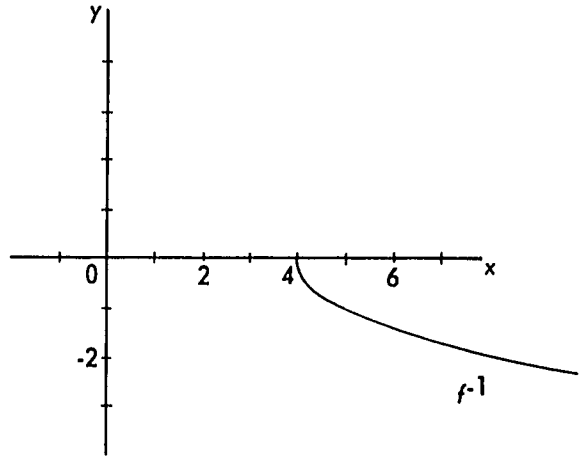

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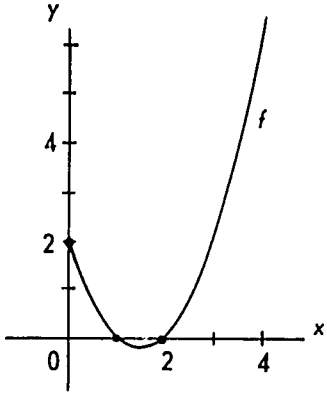
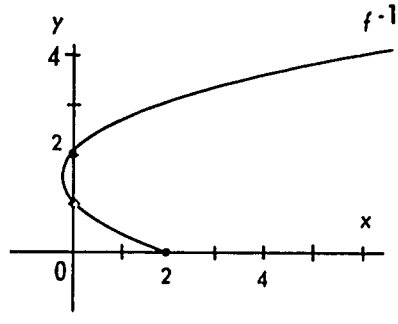
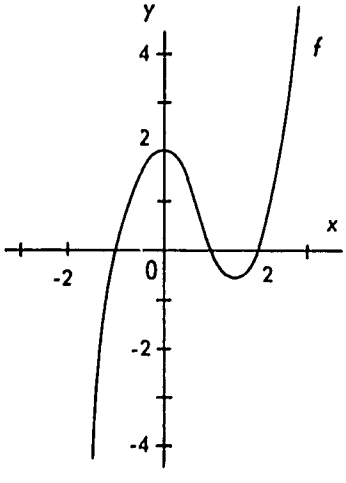
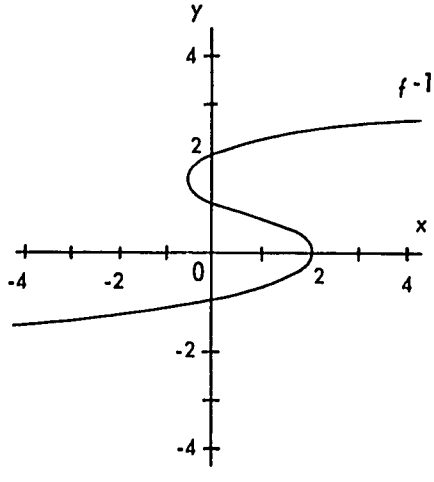




From 8.





Answers to
CLASS ACTIVITY

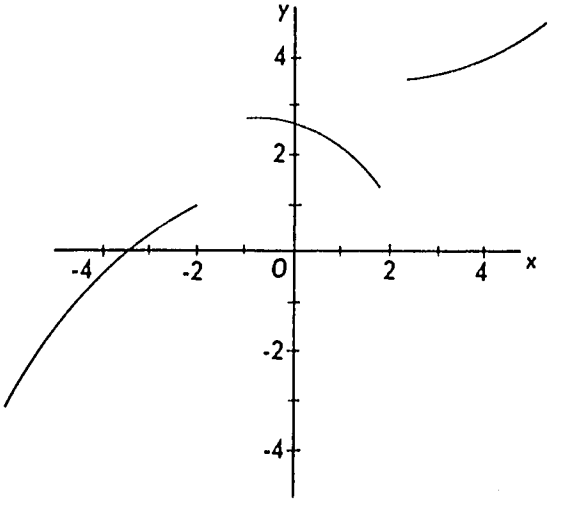
Student Text Page 36

The graphs in c, e, and f represent one-to-one functions.

Answers to
CLASS ACTIVITY

Student Text Page 38

1. The functions represented in c and e are increasing and the one in f is decreasing.
2. Yes. Yes.
3. No.
4. The pupils may be able to invent other examples.



5. Yes.

1-9 FUNCTIONS OF TWO VARIABLES.

This is an optional section devoted to a brief introduction to functions of two real variables. Rather than start with a formal definition, we proceed by means of examples chosen with an eye to illustrating that such functions are perfectly natural and must inevitably be encountered. After discussing these examples you should ask your pupils to produce further illustrations of the same type, particularly from elementary geometry. You might

suggest that they consider the formulas for the area of a rectangle and a parallelogram, and those for the volume of a cylinder and a right circular cone. Each, of course, is a function of two real variables. Finally, as the first three examples in the student text suggest, all of the elementary operations of arithmetic—addition, subtraction, multiplication, and division—can also be viewed as functions of this type.

We have deliberately avoided discussing graphs in this setting since this is a complicated subject which requires a knowledge of three-dimensional geometry. If any of your pupils are curious about this point you should tell them that such graphs are studied in more advanced courses, but encourage them to do some experimenting on their own.

Student Text Page 38

Answers to Questions following the examples:

Example 1. a. $f(-3,2) = -5$, $f\left(1, -\frac{2}{3}\right) = \frac{5}{3}$, $f(3,4) = -1$, $f\left(-\frac{3}{4}, \frac{1}{2}\right) = -\frac{5}{4}$.

b. $f(x, y) = f(y, x)$ whenever $x - y = y - x$; hence whenever $2x = 2y$, or $x = y$.

Example 2. a. $f(0,0) = 0$, $f\left(-\frac{1}{2}, \frac{2}{3}\right) = -\frac{1}{3}$, $f(2,-1) = -2$, $f\left(-\frac{2}{3}, -\frac{3}{8}\right) = \frac{1}{4}$, $f(-\pi, -\pi) = \pi^2$.

b. $f(x, y) = f(y, x)$ whenever $xy = yx$, a condition which is satisfied for *all* ordered pairs (x, y) of real numbers.

Example 3. Domain of $f = \{(x, y): y \neq 0\}$. (The restriction $y \neq 0$ is necessary because $\frac{x}{y}$ is not defined when $y = 0$.)

Range of f: the set of all real numbers. (To prove this it suffices to observe that if r is any real number whatever, then $f(r,1) = r$.)

Answers to
PROBLEMS 1-9

Student Text Page 40

1. Domain: the set of all ordered pairs of real numbers.
   Range: the set of all real numbers.

$f(1,2) = -3$, $f(0,0) = 0$, $f\left(-\frac{2}{3}, 1\right) = -\frac{5}{9}$, $f\left(\frac{1}{2}, -\frac{3}{4}\right) = -\frac{5}{16}$, $f(-3, -4) = -7$.

2. Domain: the set of all ordered pairs (x, y) of real numbers with $y \neq 0$.
Range: the set of all real numbers.

$f(-3,2) = -\frac{1}{2}$, $f(0,-3) = 1$, $f\left(-\frac{1}{2}, \frac{1}{2}\right) = 0$, $f\left(\frac{1}{3}, \frac{2}{3}\right) = \frac{3}{2}$, $f(-3, -1) = 4$.

3. Domain: the set of all ordered pairs of real numbers.
Range: the set of all nonnegative real numbers.

$f(0,0) = 0$, $f(-1,2) = 1$, $f(\frac{1}{2}, -\frac{1}{2}) = 0$, $f(-2, -2) = 16$, $f(4,0) = 16$.
[Note that $f(x, y) = (x + y)^2$]

4. Domain: the set of all ordered pairs of real numbers.
Range: the set of all nonnegative real numbers.

$f(0,0) = 0$, $f(-1, -3) = 3$, $f\left(\frac{2}{3}, -\frac{3}{2}\right) = 1$, $f(2,4) = 8$, $f\left(-\frac{1}{2}, \frac{5}{2}\right) = \frac{5}{4}$.

5. Domain: the set of all ordered pairs of real numbers.
Range: the set of all nonnegative real numbers.

$f(0,3) = 3$, $f(-3,4) = 5$, $f(-12, -5) = 13$, $f(1,1) = \sqrt{2}$, $f\left(\frac{1}{2}, -\frac{3}{2}\right) = \frac{\sqrt{10}}{2}$.

6. Domain: the set of all ordered pairs (x, y) of real numbers with $x > 0$ and $y > 0$, or $x < 0$ and $y < 0$.
Range: the set of all nonnegative real numbers.

$f(-2, -2) = 2$, $f(0,0) = 0$, $f\left(\frac{2}{3}, \frac{4}{3}\right) = \frac{2\sqrt{2}}{3}$, $f(-5, -4) = 2\sqrt{5}$, $f(-1, -1) = 1$.

7. Given $f(x, y) = xy$.
Then

$$f(x, -y) = x(-y) = -(xy) = -f(x, y).$$
$$f(-x, y) = (-x)y = -(xy) = -f(x, y).$$
$$f(-x, -y) = (-x)(-y) = xy = f(x, y).$$

8. Given $f(x, y) = x^2 - y^2$.
Then

$$f(-x, y) = (-x)^2 - y^2 = x^2 - y^2 = f(x, y).$$
$$f(x, -y) = x^2 - (-y)^2 = x^2 - y^2 = f(x, y).$$
$$f(-x, -y) = (-x)^2 - (-y)^2 = x^2 - y^2 = f(x, y).$$

9. f is a numerical function of two real variables if f is a set of ordered pairs each of whose first entries is an ordered pair of real numbers, whose second entries are real numbers, and which has the additional property that no two ordered pairs have the same first entry and different second entries.

Chapter 2
RATIONAL FUNCTIONS

2-1 RATIONAL FUNCTIONS.

Before defining rational functions we recall the fact that some functions can be classified into types according to the nature of the rule which defines them.

Answers to
PROBLEMS 2-1A

Student Text Page 42

1. a. All real numbers; polynomial, quadratic.
   b. All real numbers; polynomial, linear.
   c. All real numbers; polynomial.
   d. All real numbers; constant, polynomial.
   e. All real numbers; constant, polynomial, zero.
   f. $\{x : x \geq 0\}$; square root.
   g. All real numbers; polynomial, linear.
   h. All real numbers; constant, polynomial.
   i. All real numbers; polynomial, quadratic.
   j. All real numbers; exponential.
   k. $\{x : x \neq 0\}$; none.
   l. All real numbers; absolute value.
   m. $\{x : x \neq -15\}$; none
   n. All real numbers; constant, polynomial.
   o. $\{x : x \neq 0\}$; none
   p. $\{x : x > 0\}$; logarithmic.
   q. $\{x : x \neq -\frac{3}{2}\}$; none.
   r. $\{x : x \neq 0\}$; none

s. All real numbers; polynomial, quadratic.

t. $\{x : x \neq 4\}$; none

u. All real numbers; polynomial.

v. All real numbers; constant, polynomial.

w. All real numbers; polynomial, quadratic.

x. $\{x : x \neq -\frac{25}{2}\}$; none.

y. $\{x : x \geq 0\}$; square root.

z. All real numbers; absolute value.

2. a. All real numbers; polynomial, linear.

b. Not a function.

c. All real numbers; polynomial, quadratic.

d. Not a function.

e. $\{x : x \neq 0\}$; none.

f. All real numbers; polynomial, linear.

g. Not a function.

h. Not a function.

i. $\{x : x \neq -2\}$; none.

j. $\{r : r \neq -2\}$; none.

k. All real numbers; polynomial, quadratic.

l. All real numbers; constant, polynomial.

m. All real numbers; polynomial, linear.

We show a need for rational functions by setting up equations for some problems which lead to the use of rational expressions. Do not attempt to solve the equations now. A little later after the pupils have developed some understanding of rational expressions we shall return to these problems and complete their solution.

Answers to
PROBLEMS 2-1B

Student Text Page 44

1. If there are x boys, then there are $(x - 4)$ girls.

$$\frac{x}{x - 4} = \frac{6}{5} \qquad (x \text{ is an integer greater than } 4)$$

2. If there are x red pencils, then there are $x + 28$ black pencils and the total number of pencils is $x + (x + 28)$ or $2x + 28$.

$$\frac{x}{2x + 28} = \frac{1}{9} \qquad (x \text{ is a positive integer})$$

3. The time for the journey originally is $\frac{25}{x}$ hours. At the faster rate of $(x + 5)$ miles per hour the time is $\frac{25}{x + 5}$ hours.

$$\frac{25}{x} - \frac{25}{x + 5} = \frac{1}{4} \qquad (x > 0)$$

4. At x shillings each the man bought $\frac{60}{x}$ books. At $2x$ shillings each he bought $\frac{72}{2x}$ books.

$$\frac{60}{x} + \frac{72}{2x} = 8$$

(x is positive and a multiple of the smallest legal unit of money)

5. The rectangle is $\frac{12}{x}$ yds. in height and the perimeter is

$$\left(x + x + \frac{12}{x} + \frac{12}{x}\right) \text{ yds. or } 2\left(x + \frac{12}{x}\right) \text{ yds.}$$

$$2\left(x + \frac{12}{x}\right) = 49 \qquad (x > 0)$$

In the definition of rational function help the pupils to understand the need for restricting the domain. The notation $\left\{\left(x, f(x)\right) : f(x) = \frac{3x - 1}{x^2 - 9}\right\}$ merely gives the set of ordered pairs of numbers which is the function as we saw in the preceding chapter. The pupils should be aware of the difference in meaning of *rational function*, *rational expression*, and *value of the function*.

Answers to
PROBLEMS 2-1C

Student Text Page 46

1. Rational functions: $a, b, c, d, h, i, j, k, m, n$

a. $\{x : x \neq -2\}$ b. $\{a : a \neq \frac{1}{3}\}$

c. All real numbers

d. $\{a : a \neq 0\}$

e. $\{x : x > 0\}$

f. All real numbers

g. $\{x : x > 0 \text{ and } x \neq 1\}$.

h. All real numbers.

i. $\{y : y \neq -3\}$

j. All real numbers

k. $\{a : a \neq 0\}$

l. $\{x : x \geq 2 \text{ or } x < -2\}$

m. All real numbers

n. $\{a : a \neq \frac{3}{2}\}$

2. All except Problem 1, parts f, j, l, p, r, y, z, and Problem 2, parts b, d, g, h.

3. a. Yes, because a polynomial $P(x)$ can be written $\frac{P(x)}{1}$ and since 1 is a polynomial, $\frac{P(x)}{1}$ defines a rational function.

   b. No, because $\frac{x}{x+1}$, for example, defines a rational function but not a polynomial function.

4. a. 22

   b. -3

   c. undefined

   d. -2

5. (6, 22), (1, -3), (0, -2), $\left(-5, -\frac{11}{3}\right)$, $\left(\frac{1}{2}, -\frac{33}{14}\right)$, $\left(\pi, \frac{\pi^2 + 8}{\pi - 4}\right)$ and many others.

6. $6 = \frac{x}{x - 1} \qquad (x \neq 1)$

   $6x - 6 = x$

   $5x = 6$

   $x = \frac{6}{5} \qquad (x \neq 1)$

   This problem anticipates some of the work later in this chapter on solution of equations involving rational expressions. If the pupils have difficulty with it now, omit it until they are more nearly ready for it. The same remark applies to Problems 7 and 8.

7. $a = -\frac{11}{21}$

8. $b = -\frac{2}{3}$

9. $\begin{cases} 2 \cdot 5 = \frac{-a + b}{-2} \\ 1 = \frac{2a + b}{1} \end{cases}$ is equivalent to $\begin{cases} a = 2 \\ b = -3 \end{cases}$

10. a. $\frac{80}{x}$ d. $\frac{340}{x}$

b. Width: $10 - x$ e. $\sqrt{x}$

Area: $x(10 - x)$ or $10x - x^2$

c. $\frac{1}{x}, \frac{1}{x + 2}$

f. All of the expressions in parts a to e define functions.

All but part e define rational functions.

Domains: a. $x > 0$ d. $x > 0$

b. $0 < x < 10$ e. $x > 0$

c. $x > 0$

2-2 OPERATIONS WITH RATIONAL FUNCTIONS.

We observe that addition and multiplication of rational *functions* look very much like addition and multiplication of rational numbers and we have the same properties to guide us in manipulating them.

There are three different things involved in what we are doing.

1.	The *rational function*	A set of ordered pairs of numbers.
2.	The *rational expression*	Marks on the paper.
3.	*Numbers* represented by the expression for various values of x in the domain.	Values of the function, numbers in the range.

Actually what happens in practice is that we manipulate the polynomials and rational *expressions* which appear on the paper (2 above). We hope that the pupils are thinking of the properties of *numbers* (3 above) as they manipulate the symbols. The *function* (1 above) is defined by the expression and the domain.

Answers to
PROBLEMS 2-2A

Student Text Page 49

1. $\frac{15}{56x^2}$ $(x \neq 0)$

2. $\frac{1}{x^2}$ $(x \neq 0)$

3. $\frac{1}{8}$ $(x \neq 0)$

4. $\frac{21}{40}$ $(x \neq 0)$

5. $\frac{9}{8x^2}$ $(x \neq 0)$

6. $\frac{1}{x}$ $(x \neq 0)$

7. $\frac{x^2 - x}{3}$

8. $\frac{9}{8x}$ $(x \neq 0)$

9. $\frac{3 + x^2}{8x}$ $(x \neq 0)$

10. $\frac{9 - 64x^2}{8x}$ $(x \neq 0)$

11. $2x$ $(x \neq 2)$

12. $\frac{3}{x}$ $(x \neq 0,\ x \neq -1)$

In the next problem set the purpose of having the pupils write down the properties they are using is not that we expect them always to write the properties but we hope that they will, by writing them a few times, become more aware of how much we depend on these properties continuously and perhaps will think of them more frequently and more clearly as they work.

Answers to
PROBLEMS 2-2B

Student Text Page 50

1. $\frac{2y}{7} \times \frac{14}{3y} = \frac{2y \times 14}{7 \times 3y}$ $\left(\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}\right)$

$= \frac{y \times 7 \times 2 \times 2}{y \times 7 \times 3}$ (Associative and commutative properties of multiplication.)

$= \frac{y}{y} \times \frac{7}{7} \times \frac{4}{3}$ $\left(\frac{AC}{BD} = \frac{A}{B} \times \frac{C}{D}\right)$

$= 1 \times \frac{4}{3}$ $\left(\frac{B}{B} = 1\right)$

$= \frac{4}{3}$ $\left(\frac{B}{D} \times 1 = \frac{C}{D}\right)$

Domain is $\{y : y \neq 0\}$.

The properties for Problems 2 to 7 should be shown in a similar manner.

2. $\frac{3}{a}$ $(a \neq 0,\ a \neq -9)$

3. $\frac{m}{3}$ $(m \neq 0)$

4. 1

5. $\frac{2y - 6}{y - 4}$ $(y \neq 4)$

6. $\frac{2x + 1}{2(x^2 + 3)}$

7. $\frac{3x + 2}{6(x + 1)}$ $(x \neq -1)$

8. Yes. $\frac{N(x)}{D(x)} + \frac{P(x)}{Q(x)} = \frac{N(x) \times Q(x) + P(x) \times D(x)}{D(x) \times Q(x)}$ $(D(x) \neq 0,\ Q(x) \neq 0)$

Since the set of polynomials is closed under addition and multiplication, the numerator and the denominator of this fraction are both polynomials. The fraction therefore defines a rational function.

b. Yes, by similar reasoning.

c. Yes, by similar reasoning.

d. No. For example $(x + 3)$ defines a rational function and 0 defines a rational function, but $\frac{x+3}{0}$ does not.

e. The rational functions have the property of *closure* for addition, subtraction, and multiplication.

9. a. Yes.

b.

$$\frac{x+3}{x} + \frac{x}{x-2}$$
$$= \frac{(x+3)(x-2)}{x(x-2)} + \frac{x^2}{x(x-2)}$$
$$= \frac{(x^2 + x - 6) + x^2}{x(x-2)}$$
$$= \frac{2x^2 + x - 6}{x(x-2)} \quad (x \neq 0,\ x \neq 2)$$

$$\frac{x}{x-2} + \frac{x+3}{x}$$
$$= \frac{x^2}{x(x-2)} + \frac{(x-2)(x+3)}{x(x-2)}$$
$$= \frac{x^2 + (x^2 + x - 6)}{x(x-2)}$$
$$= \frac{2x^2 + x - 6}{x(x-2)} \quad (x \neq 0,\ x \neq 2)$$

c. Similarly by using properties of *polynomial* functions we see that for rational functions f and g, $f(x) + g(x) = g(x) + f(x)$.

d. Commutative property of addition.

10. a. $g(x) = 0$ b. $g(x) = 1$

In problems 8, 9, and 10 we have mentioned some properties which you recognize as properties of a field. While it is of interest and instructive to observe these properties for rational functions, we do *not* want to pursue them very far here because we would run into a complication which is probably too difficult and too subtle to be presented to the pupils at this time. The discussion below is for the benefit of the teacher and not to be given to the pupils unless you have one or two particularly able pupils who ask questions and could understand it.

First, let us list the properties of a field which *can* be proved for rational functions by using the corresponding properties for the polynomial functions in terms of which the rational functions are defined.

If f, g, and h are rational functions, then

$$\left.\begin{array}{l} f + g \text{ is a rational function} \\ fg \text{ is a rational function} \end{array}\right\} \quad \text{Closure properties}$$

$f + g = g + f$ $fg = gf$	Commutative properties
$(f + g) + h = f + (g + h)$ $(fg)h = f(gh)$	Associative properties
$f(g + h) = fg + fh$	Distributive property
$f + 0 = f$ $f \times 1 = f$	Identity properties
$f + (-f) = 0$	Inverse property

The remaining field property, the inverse property of multiplication, strictly speaking is not true for rational functions. Fully stated this property says:

> Corresponding to each rational function f (except $f = 0$) there is a unique rational function $\frac{1}{f}$ such that

$$f \times \frac{1}{f} = \frac{1}{f} \times f = 1$$

It is understood in this statement that the function 1 is the *unique* identity function for multiplication, which has as its domain the set of all real numbers. Consider, however, the following examples.

$$(x - 3) \times \frac{1}{(x - 3)} = 1 \qquad (x \neq 3)$$

$$(x + 8) \times \frac{1}{(x + 8)} = 1 \qquad (x \neq -8)$$

Since the domains are different, the 1 in the first case does not define the same function as the 1 in the second case, and neither of them is the same function as the *unique* identity function 1 which has as its domain the set of *all* real numbers. Since there is not a unique identity function which will serve to determine all multiplicative inverses, this inverse property is not strictly true.

While this prevents the set of rational functions from being a field, it does not hamper our use of the multiplicative inverse idea because in ordinary circumstances we do not insist on there being a unique function for 1.

2-3 FACTORIZATION IN SIMPLIFYING RATIONAL EXPRESSIONS.

Use your judgment about whether to use PROBLEMS 2-3A on revision of factorization. If the class has retained its proficiency in factorization,

these problems can be omitted. If they do need help, you should recall for them the types they have studied.

Quadratic trinomial: $3a^2 - 14a + 8$

We look for two numbers whose product is 3×8 or 24 and whose sum is -14. We can test factors of 24 systematically.

$$\left.\begin{array}{ll} 1 \times 24 & (-1) \times (-24) \\ 2 \times 12 & (-2) \times (-12) \\ 3 \times 8 & (-3) \times (-8) \\ 4 \times 6 & (-4 \times (-6) \end{array}\right\} \quad \begin{array}{l} -2 \text{ and } -12 \\ \text{will work.} \end{array}$$

$$\begin{aligned} 3a^2 - 2a - 12a + 8 &= a(3a - 2) - 4(3a - 2) \\ &= (a - 4)(3a - 2) \end{aligned}$$

Common monomial: $6y^2 + 10y = 2y(3y + 5)$

Difference of two squares: $9y^2 - 25 = (3y + 5)(3y - 5)$

Perfect square trinomial: $y^2 + 10y + 25 = (y + 5)^2$

Combinations of these: $3a^3 - 12a = 3a(a^2 - 4) = 3a(a + 2)(a - 2)$

Answers to
PROBLEMS 2-3A

Student Text Page 52

1. $y(6y + 7)$
2. $2y(3y + 5)$
3. $(y + 8)(y + 2)$
4. $(y + 5)^2$
5. $(y + 25)(y - 1)$
6. $(y + 5)(y - 5)$
7. $(3y + 5)(3y - 5)$
8. $3a(a^2 - 5a - 4)$
9. $(a - 6)(a + 2)$
10. $(a - 4)(a - 3)$
11. $(a - 4)(3a - 2)$
12. $(7a - 6)(a + 1)$
13. $3a(a + 2)(a - 2)$
14. $(a^2 + 4)(a + 2)(a - 2)$

We must continue to emphasize awareness of the domain. It is particularly important here because in some of the final simplified expressions the domain is not evident just from the expression.

Answers to
PROBLEMS 2-3B

Student Text Page 53

1. $\frac{x+7}{x}$ $(x \neq 0, 7)$
2. $\frac{1}{a+6}$ $(a \neq 0, 6, -6)$
3. $\frac{2}{y}$ $(y \neq 0, 2, -2)$
4. $\frac{6m+5}{(m+5)(m+1)}$ $(m \neq -5, -1)$
5. $\frac{x-2}{2x}$ $(x \neq 0, 7, 5)$
6. $\frac{x^2-9x+12}{x(x-7)}$ $(x \neq 0, 7)$
7. $\frac{2a+6}{a-2}$ $(a \neq 2, 1, -\frac{1}{5})$
8. $\frac{x^2+3x+18}{(x+3)^2}$ $(x \neq -3)$

Answers to
PROBLEMS 2-3C

Student Text Page 54

1. $\frac{x^2-2x+3}{(x+1)(x-1)(x+5)}$ $(x \neq 1, -1, -5)$
2. $\frac{7a^2-18a-78}{6a(a-b)}$ $(a \neq 0, 6)$
3. $\frac{y^2-2y-6}{(y-2)^2(y-5)}$ $(y \neq 2, 5)$
4. $\frac{a^2-10}{5a(a^2-4a-12)}$ $(a \neq 0, 6, -2)$
5. $\frac{a+5}{5(a-5)}$ $(a \neq 2, 5, -5)$
6. $\frac{5a^3+125a}{2a^2-625}$ $(a \neq 0, \frac{25}{2}\sqrt{2}, -\frac{25}{2}\sqrt{2})$
7. $\frac{2x-5}{2x-1}$ $(x \neq \frac{1}{2}, 3, 1)$

2-4 RATIONAL FUNCTIONS OF TWO OR MORE VARIABLES.

For functions of two variables the main thing to bear in mind is that the elements of the domain are ordered pairs of numbers, one for the first variable and one for the second variable.

1. a. $f(3,2) = -\frac{5}{9}$, $f(-2,1) = -6$

   b. $f(x,y) = \frac{x - 4y}{x + 3y}$ $(x \neq 3y,\ x \neq -3y)$

   c. (3, 1) is not in the domain because it makes the original denominator zero.

   d. $\{(x, y) : x \neq 3y \text{ and } x \neq -3y\}$

2. a. $\frac{2}{x + y}$ $(x \neq y,\ x \neq -y)$

   b. $\frac{2a^2 + 2b^2}{a^2 - b^2}$ $(a \neq b,\ a \neq -b)$

   c. $\frac{b - a}{b + a}$ $(a \neq 0,\ b \neq 0,\ a \neq -b)$

   d. $\frac{a - 2b}{a}$ $(a \neq 0,\ b \neq 0,\ a \neq b)$

   e. $\frac{-x + 7y}{6x}$ $(x \neq 0)$

   f. $\frac{6a + 3b + 2c}{6a + 6b + 6c}$ $(a + b + c \neq 0)$

3. All except b, g, i, and l, which define rational functions of one variable.

2-5 EQUATIONS INVOLVING RATIONAL EXPRESSIONS.

We use two slightly different methods of simplifying equations involving rational expressions. In both we use the least common multiple (L.C.M.) of all the denominators.

In the first method we use the properties of one repeatedly on the various terms until all the terms in the equation have the same denominator. Then multiplying both sides of the equation by this same denominator gives a simpler and equivalent equation with no denominators.

After using this basic method for a while we observe that we can obtain the same result with fewer steps if we multiply by the L.C.M. of the denominators at once. Simplification by the properties of one produces the same equation with no denominators which we had by the first method.

An advantage of doing the longer method for a while first is that it may help the pupils to see more clearly what they are doing, so when they come to the shorter method, it is more meaningful than it might be otherwise.

Whatever method we use, it is important not to forget the restrictions

on the domain. We do not bother to write them at every step, but they are constantly present and should always be written clearly by the last equation in the chain of equivalent equations.

On page 57 be sure that the pupils *can* fill in the following details when they multiply by $4x(x + 1)$.

$$4x(x + 1)\left[\frac{12x + 12}{4x(x + 1)} + \frac{20x}{4x(x + 1)}\right] = 4x(x + 1)\left[\frac{7x^2 + 7x}{4x(x + 1)}\right]$$

$$4x(x + 1)\frac{12x + 12}{4x(x + 1)} + 4x(x + 1)\frac{20x}{4x(x + 1)} = 4x(x + 1)\frac{7x^2 + 7x}{4x(x + 1)}$$

(by the distributive property)

$$\frac{4x(x + 1)}{4x(x + 1)}(12x + 12) + \frac{4x(x + 1)}{4x(x + 1)}(20x) = \frac{4x(x + 1)}{4x(x + 1)}(7x^2 + 7x)$$

$$\left(A \times \frac{C}{D} = A \times C \times \frac{1}{D} = A \times \frac{1}{D} \times C = \frac{A}{D} \times C\right)$$

$$12x + 12 + 20x = 7x^2 + 7x$$

$$\left(\text{by } \frac{B}{B} = 1 \text{ and } 1 \times A = A\right)$$

Do not ask the pupils to write out all of this, but now and then in class discussion ask them some questions about how this works to be sure that they are doing it with understanding and not just manipulating mechanically.

In problems such as the one about the running boys be sure that the pupils clearly answer the question, as it is shown in the text. Do not let them stop at $x = 4$. A check in the words of the problem should also be done both to verify correctness and to bring out the full meaning.

Again, use your judgment about how much of PROBLEMS 2-5A on quadratic equations needs to be done by your class. If they are proficient, the set of problems may be omitted entirely. Possibly some pupils in the class will need more practice than others to recall what they knew about solving quadratic equations.

Answers to
PROBLEMS 2-5A

Student Text Page 58

1. For all real numbers a and b, $ab = 0$ if and only if $a = 0$ or $b = 0$.
2. $\left\{\frac{5}{2}, 2\right\}$
3. $\{0, -4\}$
4. $\{0, 9\}$
5. $\{3, -3\}$

6. $\{11, -1\}$
7. $\{12, 2\}$
8. $\left\{-1, -\frac{1}{4}\right\}$
9. $\left\{\frac{2}{3}, -5\right\}$
10. $\{6\}$
11. $\left\{\frac{3}{2}\right\}$
12. $\left\{\frac{7}{5}, -\frac{7}{5}\right\}$
13. $\{-3, 5\}$
14. $\left\{3, \frac{3}{2}\right\}$
15. $\{11, -11\}$
16. By the definition of square root, if $x^2 = 13$, then $x = \sqrt{13}$ or $x = -\sqrt{13}$.
17. $\{6 + \sqrt{5}, 6 - \sqrt{5}\}$
18. $9, \{3 + \sqrt{13}, 3 - \sqrt{13}$
19. $\{5 + \sqrt{30}, 5 - \sqrt{30}\}$
20. $\{-2 + \sqrt{3}, -2 - \sqrt{3}\}$
21. $\{6 + 4\sqrt{2}, 6 - 4\sqrt{2}\}$
22. $\left\{\frac{-3 + \sqrt{29}}{2}, \frac{-3 - \sqrt{29}}{2}\right\}$
23. $\{1 + 2\sqrt{2}, 1 - 2\sqrt{2}\}$
24. $\{10, 8\}$
25. $\{\ \}$
26. $\{-4 + 2\sqrt{5}, -4 - 2\sqrt{5}\}$
27. $\left\{\frac{-6 + \sqrt{26}}{2}, \frac{-6 - \sqrt{26}}{2}\right\}$
28. $\left\{\frac{7 + \sqrt{51}}{2}, \frac{7 - \sqrt{51}}{2}\right\}$

Answers to
PROBLEMS 2-5B

Student Text Page 60

1. a. $\{24\}$ b. 24 boys and 20 girls.
2. a. $\{4\}$ b. 4 red pencils and 32 black pencils.
3. a. $\{20\}$ b. 20 miles per hour
4. a. $\{12\}$ b. 12 shillings and 24 shillings.
5. a. $\left\{24, \frac{1}{2}\right\}$ c. 24 yds. by $\frac{1}{2}$ yd.

Notice that in Problem 3 the negative solution, -25, of the quadratic equation is eliminated because it is not in the domain.

In Problem 5 both solutions are in the domain, but if $x = 24$, the height, $\frac{12}{x}$ is $\frac{1}{2}$ and if $x = \frac{1}{2}$, the height is 24. Thus we obtain the same two dimensions whichever value we choose for the base.

6. $\{5\}$ 7. $\left\{2, -\frac{13}{2}\right\}$ 8. $\{\ \}$ 9. $\{3\}$

In Problems 10 and 11 the method depends heavily on the order property:

If $a < b$ and $c > 0$, then $ac < bc$.

If $a < b$ and $c < 0$, then $ac > bc$.

We take the two cases, one where the multiplier is positive and the other where the multiplier is negative.

Using the number line may help the pupils to see that, for example, there is no number which is both greater than 0 and less than -4. Have a pupil try to put his finger on the number line to the right of 0 and at the same time to the left of -4. Similarly the number line will help the pupils to see that the statement "$x > 0$ and $x > 3$" can be replaced with the simpler statement "$x > 3$" because if a number is greater than 3, it will necessarily also be greater than 0. This is just a use of the transitivity property: If $x > 3$ and $3 > 0$, then $x > 0$.

These two problems are marked "Challenge Problems" and are optional, but if the class is reasonably able and has time it would be a profitable experience to do these.

10. *Challenge Problem*

a. -5, -4, -3, -2, -1, 4, 5

b. $x - 3 > 0$; Since $x > 0$ and $x > 3$, the truth set is $\{x : x > 3\}$.

c. $x - 3 < 0$; Since $x < 0$ and $x < 3$, the truth set is $\{x : x < 0\}$.

11. *Challenge Problem*

a.

$$\frac{x+4}{x} < 0$$

$x > 0$ and $x + 4 < 0$	or	$x < 0$ and $x + 4 > 0$
$x > 0$ and $x < -4$		$x < 0$ and $x > -4$
There is no number for which this is true.		This can be written $-4 < x < 0$

The truth set is $\{x : -4 < x < 0\}$

b. $\{x : -1 < x < 0\}$

c. $\{x : x > 0 \text{ or } x < -1\}$

d.

$$\frac{x}{x-1} < 3$$

d. Cont.

$x - 1 > 0$ and $x < 3x - 3$	or	$x - 1 < 0$ and $x > 3x - 3$
$x > 1$ and $3 < 2x$		$x < 1$ and $3 > 2x$
$x > 1$ and $x > \frac{3}{2}$		$x < 1$ and $x < \frac{3}{2}$
This is equivalent to		This is equivalent to
$x > \frac{3}{2}$		$x < 1$

The truth set is $\left\{x : x < 1 \text{ or } x > \frac{3}{2}\right\}$

e. $\left\{x : -2 < x < -\frac{7}{4}\right\}$

In the example on page 62 if the pupils do not immediately see that $4x(x + 1)\left(\frac{3}{x}\right) = 4(x + 1)3$, give them a little practice for a few times on the intervening steps.

$$4x(x + 1)\left(\frac{3}{x}\right) = 4x(x + 1)\left(3 \times \frac{1}{x}\right) = \left(x \times \frac{1}{x}\right)4(x + 1)3 = 4(x + 1)3$$

$$4x(x + 1)\left(\frac{5}{x + 1}\right) = 4x(x + 1)\left(5 \times \frac{1}{x + 1}\right) = \left[(x + 1) \times \frac{1}{x + 1}\right]4x(5) = 4x(5)$$

$$4x(x + 1)\left(\frac{7}{4}\right) = 4x(x + 1)\left(7 \times \frac{1}{4}\right) = \left(4 \times \frac{1}{4}\right)x(x + 1)7 = x(x + 1)7$$

Answers to
PROBLEMS 2-5C

Student Text Page 62

1. $\left\{\frac{20}{9}\right\}$
2. $\{7, -3\}$
3. $\{5, -4\}$
4. $\left\{-2, \frac{39}{4}\right\}$
5. $\{-1 + \sqrt{13}, -1 - \sqrt{13}\}$
6. $\{x : -9 < x < 0\}$
7. $\{x : 0 < x < \frac{1}{2}\}$
8. $\{4\}$ (First simplify the left expression to $\frac{x}{x + 1}$)
9. $\left\{5, -\frac{11}{3}\right\}$

Problems 10 to 13 are intended to emphasize the need for care in distinguishing between equations and expressions.

In *solving equations* the truth set must remain unchanged as we go from equation to equation. This permits multiplying both sides of the equation by a non-zero number.

In *simplifying expressions* the value of the expression must remain unchanged. This does not permit multiplying by any number other than 1.

10. a. $\left\{\frac{1}{5}\right\}$ b. $\frac{5x - 1}{3x}$

    c. Yes. Since the domain is $\{x : x \neq 0\}$, $3x$ is not 0 and multiplying both sides of the equation by $3x$ will give an equivalent equation (same truth set).

    d. No. Since $3x$ might not be 1, multiplying the expression by $3x$ would, for most values of x, change the value of the expression. We want the value of the expression to remain unchanged.

    e. In an *equation* we are interested in the truth set. It will remain unchanged if we multiply both sides by a non-zero real number. In an *expression* we are interested in finding another expression for the same number for every value of x in the domain. This does not permit multiplying the expression by any number other than 1.

11. a. $\{-3\}$ b. $\frac{-2x - 6}{x(x - 2)}$

12. a. $\frac{-5x^2 + 15x - 60}{x^2(x - 5)}$ b. $\{\ \}$

13. a. $\{5\}$ b. $\frac{-a + 5}{3(a - 3)(a + 2)}$

14. -3 and 2 satisfy (a).
    -3 and 2 and 1 satisfy (b).
    The equations are not equivalent because they do not have the same truth set. The number 1 is not in the domain of (a), but it is a solution of (b).

15. The positive number is $\frac{1}{x}$.

$$\frac{1}{x} + \frac{1}{x + 2} = 1 \qquad (x > 0)$$

    The truth set of this is $\{\sqrt{2}\}$

    The positive number $\frac{1}{x}$ is $\frac{\sqrt{2}}{2}$.

16. $\frac{72}{x} - \frac{72}{x + 2} = 3 \qquad (x > 0)$

16. Cont.

The truth set is $\{6\}$.

The smaller lorry can carry 6 tons.

17. $\frac{x+3}{2x+4} = \frac{x+6}{3x+6} \qquad (x > -2)$

The truth set is $\{3\}$.

$AC = 6$, $AB = 9$, $DF = 10$, $DE = 15$.

17. If the building is x ft. high, the top of the flagpole is $(x + 8)$ ft. above the ground. By similar triangles

$$\frac{x}{x+8} = \frac{40}{50} \qquad (x > 0)$$

The truth set is $\{32\}$.

The building is 32 ft. high.

19. The first boy does $\frac{1}{x}$ of the job in 1 hour, $\frac{6}{x}$ of the job in 6 hours.

The second boy does $\frac{1}{x+4}$ of the job in 1 hour, $\frac{6}{x+4}$ of the job in 6 hours.

$$\frac{6}{x} + \frac{6}{x+4} = 1 \qquad (x > 0)$$

The truth set is $\{4 + 2\sqrt{10}\}$. (Notice that $4 - 2\sqrt{10}$ is not in the domain.)

$$4 + 2\sqrt{10} \approx 4 + 2(3 \cdot 16) \approx 10 \cdot 32$$

The first boy would need about 10 hours and 20 minutes to do it alone.

20. *Challenge Problem*

This problem can be set up in a variety of ways. Encourage the more able pupils to try to do it in more than one way.

If the boy can pick p pounds per hour, the man can pick $2p$ pounds per hour. If the boy rests for x minutes, he works for $(60 - x)$ minutes or $\frac{60-x}{60}$ hours.

$$2p = \frac{3}{4}\left(2p + \frac{60-x}{60}p\right)$$

The boy rests for 20 minutes.

(There are two unknown quantities involved, but one of the variables drops out leaving an equation with only one variable.)

2-6 GRAPHS OF RATIONAL FUNCTIONS.

The development at the beginning of this section (pages 65-67) should be done in class with much discussion and participation by the pupils. Do it for another simple expression such as $\frac{2}{x}$ or $\frac{1}{2x}$ or $\frac{1}{x-1}$. Have them help to compute the table of values, not just read it from the book.

For working with graphs the pupils should be trained right from the start to do the following:

1. In making tables of values, arrange the numbers selected in an orderly way.
2. Use judgment in how far the table of values is carried. Stop if the graph seems to be going way off the paper but continue far enough to show the main characteristics. Take points closer together at turning points or near peculiar points of interest.
3. Use good judgment about selecting the unit on the two axes. The resulting graph should be as large as possible consistent with keeping the interesting part on the paper.
   Occasionally it is desirable to use a larger unit on the x-axis than on the y-axis, for instance, in order to show clearly the parts of interest. Since this distorts the shape of the graph, it is better not to do so unless it is really needed.
4. Use judgment in placing the axes on the page so that there is enough room above and below, to the right and to the left, for the particular graph you are drawing. This means not drawing the axes until after at least some of the information about the graph has been obtained.
5. Show the units clearly on the axes by writing numerals on the axes at least part way out.
6. Write the labels "x" and "y" (or whatever letters are being used for the variables) on the positive ends of their respective axes.
7. Preferably use pencil instead of ink so that errors in plotting points and drawing the graph may be more easily corrected.

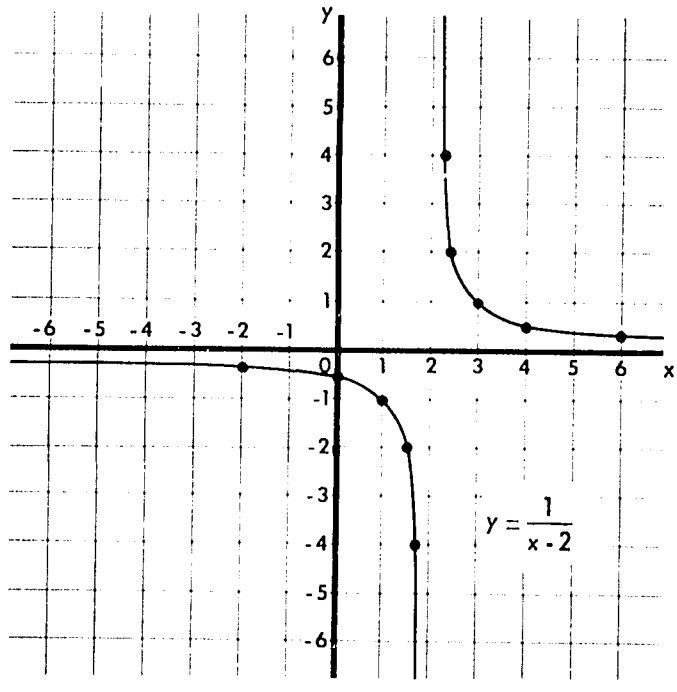
1. $\left(4, \frac{1}{2}\right)$, $\left(0, -\frac{1}{2}\right)$, $(3, 1)$, $\left(-1, -\frac{1}{3}\right)$, $\left(\frac{1}{2}, -\frac{2}{3}\right)$, $(1, -1)$, $(\sqrt{5}, \sqrt{5} + 2)$

2. $\left(0, -\frac{3}{4}\right)$, $(-3, -9)$, $\left(2, \frac{1}{6}\right)$, $\left(\frac{1}{2}, -\frac{4}{9}\right)$, $\left(\frac{3}{2}, 0\right)$

3.

$y = \frac{1}{x-2}$	x	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
	y	$-\frac{1}{7}$	$-\frac{1}{6}$	$-\frac{1}{5}$	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1		1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

$y = \frac{1}{x-2}$	x	$1\frac{1}{2}$	$1\frac{3}{4}$	$2\frac{1}{4}$	$2\frac{1}{2}$
	y	-2	-4	4	2



4.

$y = \frac{1}{x-2}$	x	12	50	88	102	1002	10,002	1,000,002
	y	$\frac{1}{10}$	$\frac{1}{48}$	$\frac{1}{86}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10,000}$	$\frac{1}{1,000,000}$

$y = \frac{1}{x-2}$	x	-12	-50	-88	-98	-998	-9998	-999,998
	y	$-\frac{1}{14}$	$-\frac{1}{52}$	$-\frac{1}{90}$	$-\frac{1}{100}$	$-\frac{1}{1000}$	$-\frac{1}{10,000}$	$-\frac{1}{1,000,000}$

Points of the graph very far to the right will be above and very near the x-axis.

Points of the graph very far to the left will be below and very near the x-axis.

5.

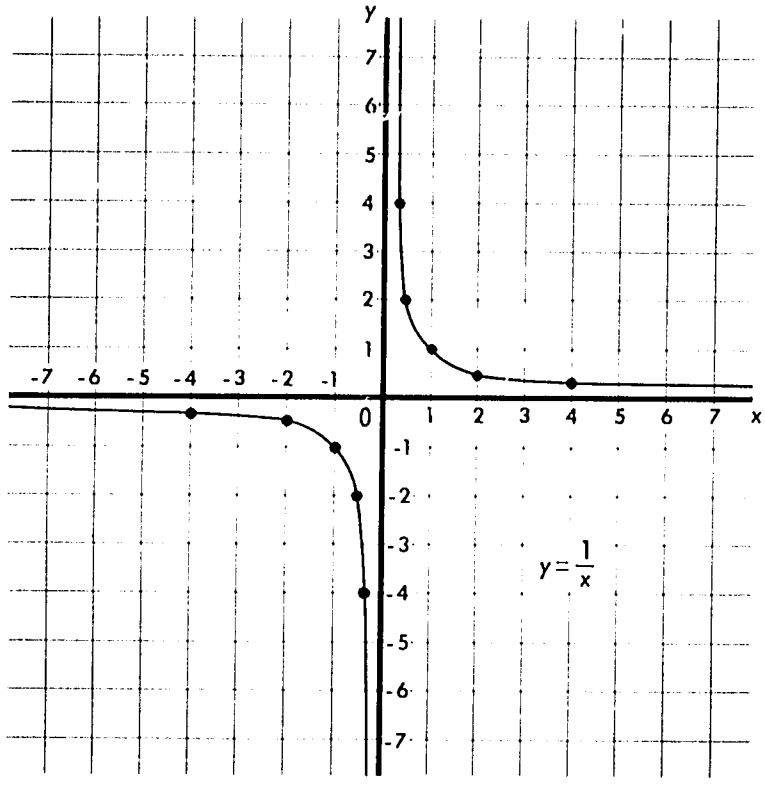
$y = \frac{1}{x - 2}$	x	1·5	1·9	1·99	1·999	1·99999
	y	-2	-10	-100	-1000	-100,000

$y = \frac{1}{x - 2}$	x	2·5	2·1	2·01	2·001	2·00001
	y	2	10	100	1000	100,000

For x very near to 2 on the left, points of the graph will be "far down", way off the paper in the negative y direction.

For x very near to 2 on the right, points of the graph will be "far up" way off the paper in the positive y direction.

6.



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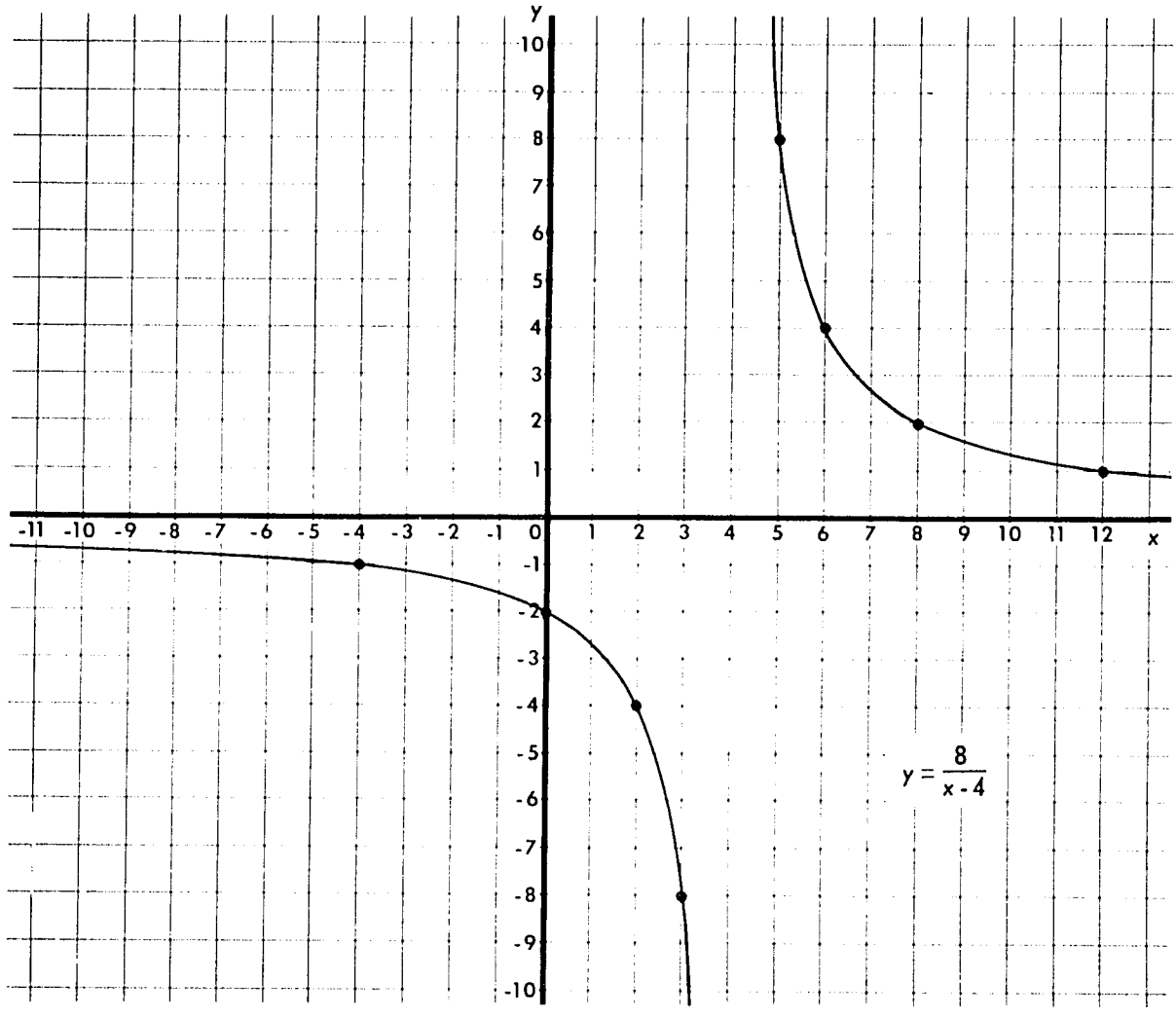

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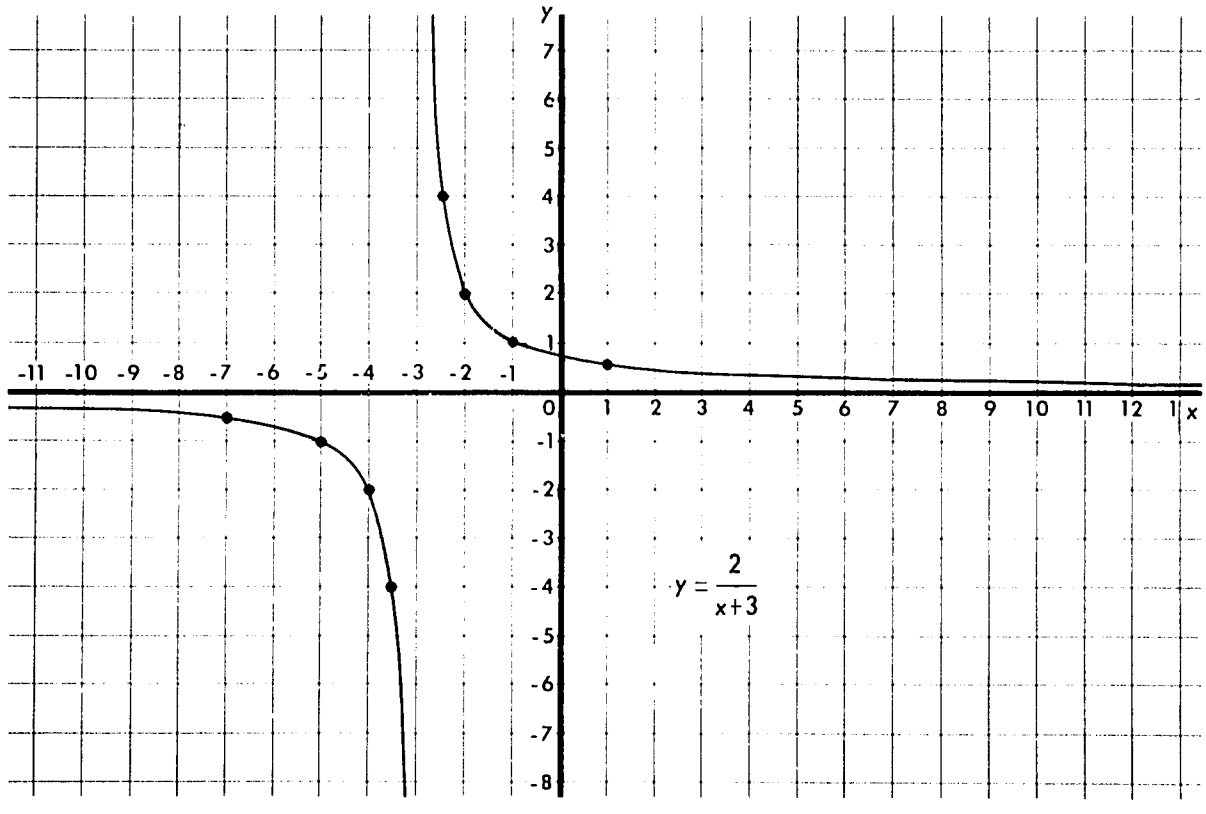
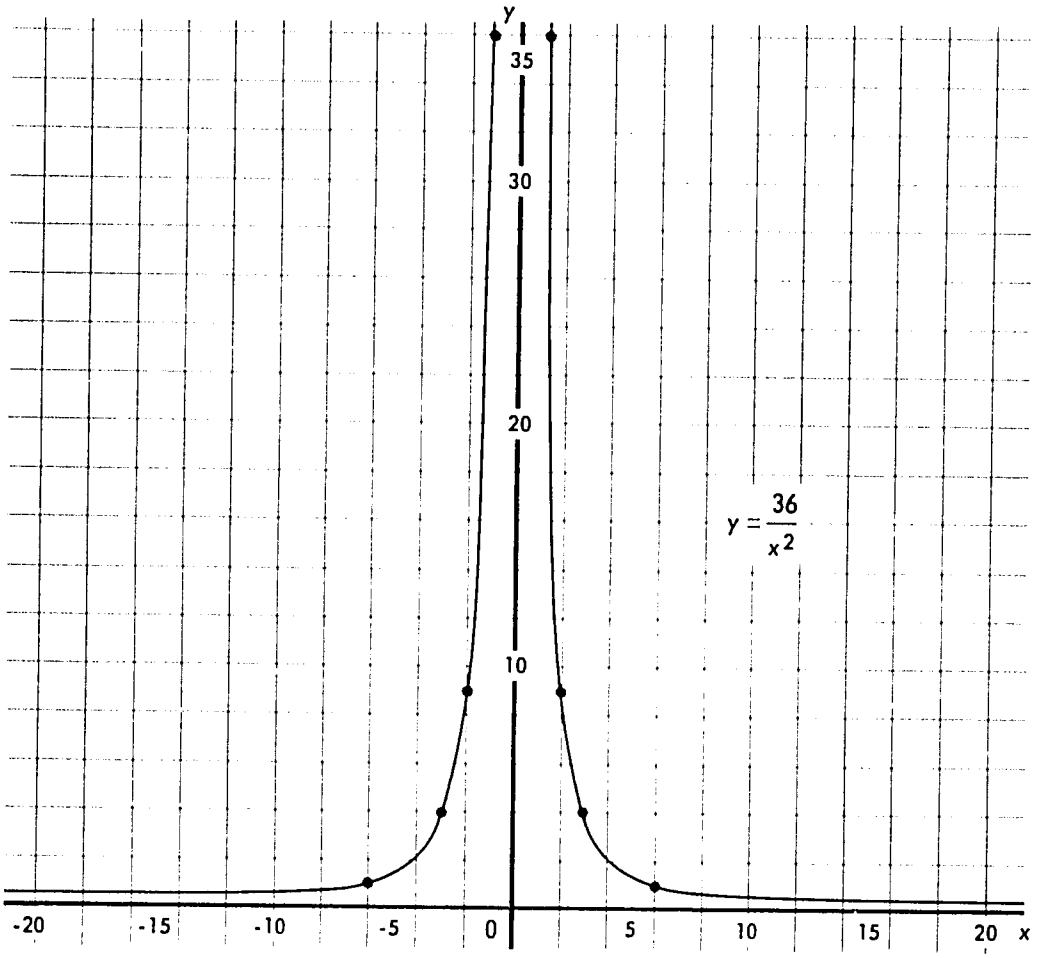


9.



The idea of symmetry was studied earlier in Secondary Two. Since the value of x^2, x^4, and any even power of x is the same for $x = a$ that it is for $x = -a$, we see that the graph of an expression involving only even powers of x has its points in pairs symmetrically placed with respect to the y-axis. For every point of the graph to the right of the y-axis there is a corresponding point of the graph the same distance away on the left.

The value of x, x^3, and any odd power of x when $x = a$ is the opposite of its value when $x = -a$. Thus $2^3 = 8$ and $(-2)^3 = -8$. If *all* the terms of a polynomial $P(x)$ are of odd power, then $P(a) = -P(-a)$, that is, using the opposite value of the variable gives the opposite value of the polynomial. In a rational function we are concerned with the quotient of two polynomials. If the value of one of these polynomials changes to its opposite and the other stays the same, the value of the quotient changes to its opposite. This explains why the two cases stated in the text will give pairs of points on the graph of the form (a,b) and $(-a, -b)$. Observation of such points on the graph quickly shows that they are symmetrical with respect to the origin.

Remember that a constant term is of degree zero and is therefore an even power of x, not an odd power.

Answers to
PROBLEMS 2-6B

Student Text Page 70

1. Symmetrical with respect to the y-axis: b, c, f.
   Symmetrical with respect to the origin: a, d.

2. a. A function defined by $y = 0$ is symmetrical with respect to the x-axis. All its points are on the x-axis and each one is symmetrical to itself with respect to the x-axis. Points such as (5,3) and (5, -3), however, cannot both be on the graph of a function (for any value of x there must be only one value of y). There are no other functions with symmetry with respect to the x-axis.
   b. Yes.

3. *Challenge Problem*
   a. Symmetry with respect to the y-axis.
   b. Symmetry with respect to the origin.
   c. If all terms are of even degree, the polynomial defines an even function.
      If all terms are of odd degree, the polynomial defines an odd function.
   d. Even. Symmetry with respect to the y-axis.
   e. Odd. Symmetry with respect to the origin.
   f. Odd. Symmetry with respect to the origin.
   g. Even. Symmetry with respect to the y-axis.

Answers to
PROBLEMS 2-6C

Student Text Page 72

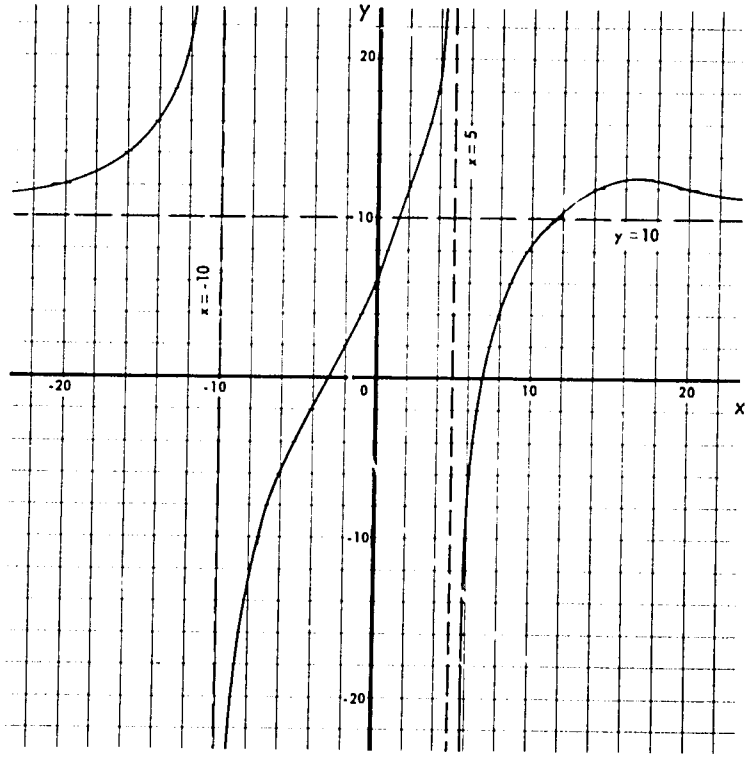
1. (8,0), (0, -4)
2. (0,0)
3. $\left(-\frac{5}{2}, 0\right)$, $\left(0, \frac{5}{7}\right)$
4. $\left(\frac{4}{3}, 0\right)$
5. (9,0), (-2, 0), (0,1)
6. (0, 0), (-4, 0)
7. (0,4)
8. (0, -8)
9. (4,0), (-4, 0), (0,1)

10. a. The solution set of $0 = \frac{8}{x + 2}$ and of $0 = \frac{x^2 + 8}{2x - 1}$ is { } because there is no real number for which $8 = 0$ or $x^2 + 8 = 0$. Hence there is no x-intercept.

b. When $x = 0$, $\frac{6x - 8}{2x^2 - 3x}$ is undefined because the denominator is 0. Hence there is no y-intercept.

2-7 ASYMPTOTES.

The word "asymptote" comes from Greek words meaning "does not touch". Actually it is possible for a graph of a rational function to cross a horizontal asymptote at some other part of the graph, as is shown here, but where the graph is "approaching" its asymptote for very large values of $|x|$ it will never quite reach it.



Try to stimulate an active class discussion in introducing this idea. Use an example similar to that on page 72. In that example the peculiarity at $x = 9$ is that the graph is discontinuous because the denominator of $\frac{7}{x - 9}$ becomes zero. The graph could not touch the line $x = 9$ because, as we have said, there is no point of the graph when $x = 9$.

To show what happens for values of x a *little less* than 9, take numbers such as 8·8, 8·9, 8·999 and have the pupils compute the corresponding values of y. They have just been doing this sort of thing and should see rather quickly that y will then be negative and $|y|$ very large, which we can describe informally as "far down". A similar treatment gives "far up" for values of x a *little greater* than 9.

As you observe, we draw asymptotes with dashed lines to indicate that they are not part of the graph.

Answers to
PROBLEMS 2-7A

Student Text Page 73

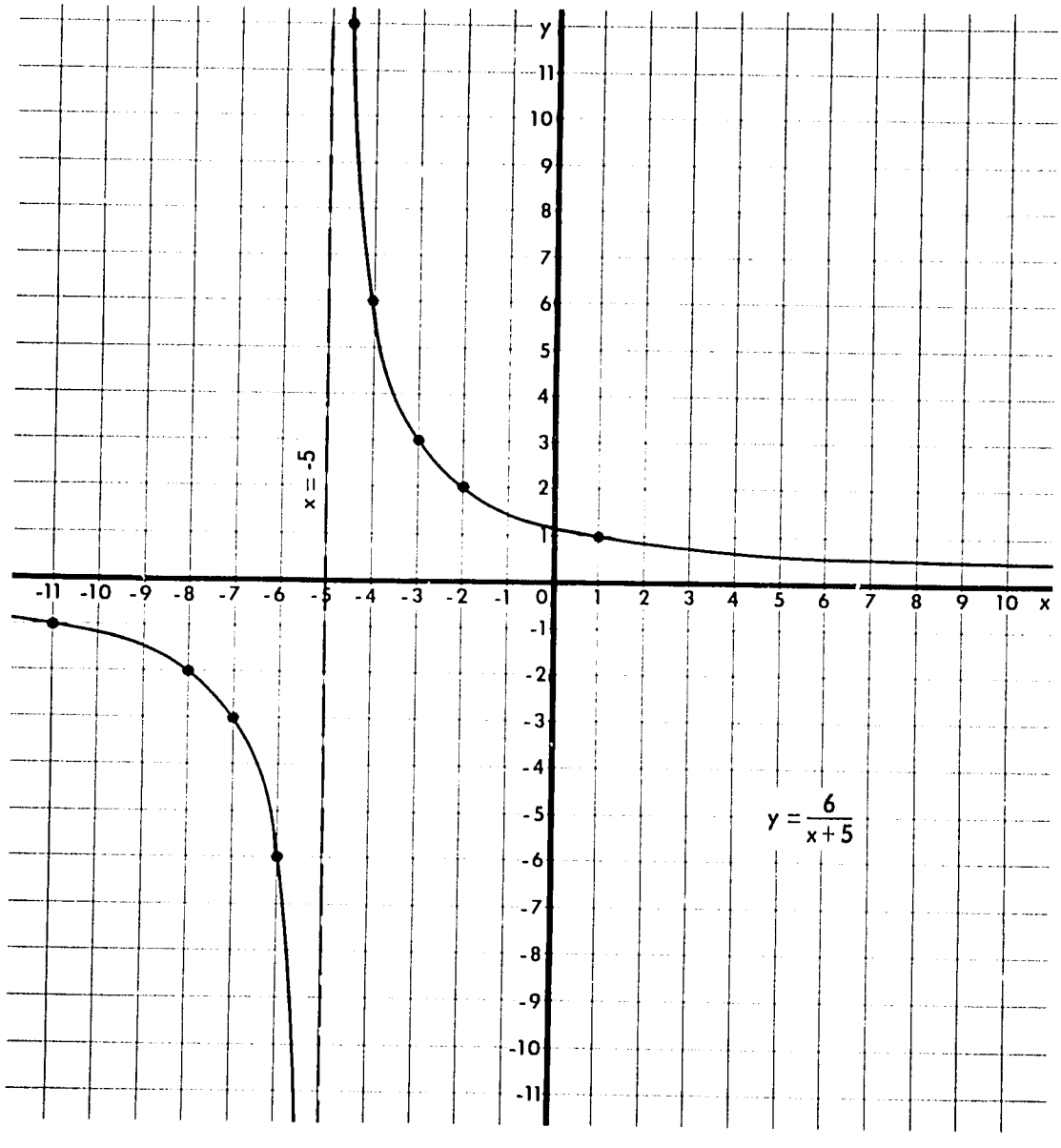
1. a. $x = 7$

   b. $\left|\frac{5}{x - 7}\right|$ becomes very large.

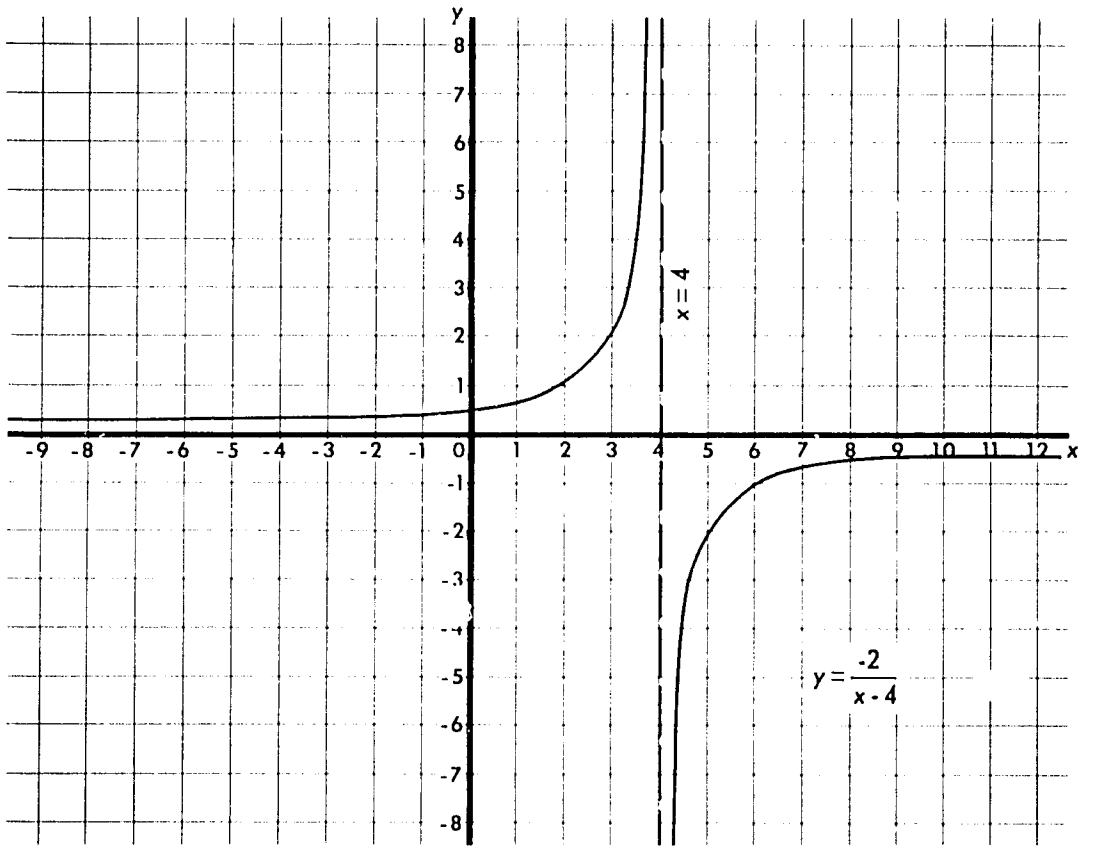
   c. Negative, negative.

   d. Positive, positive.

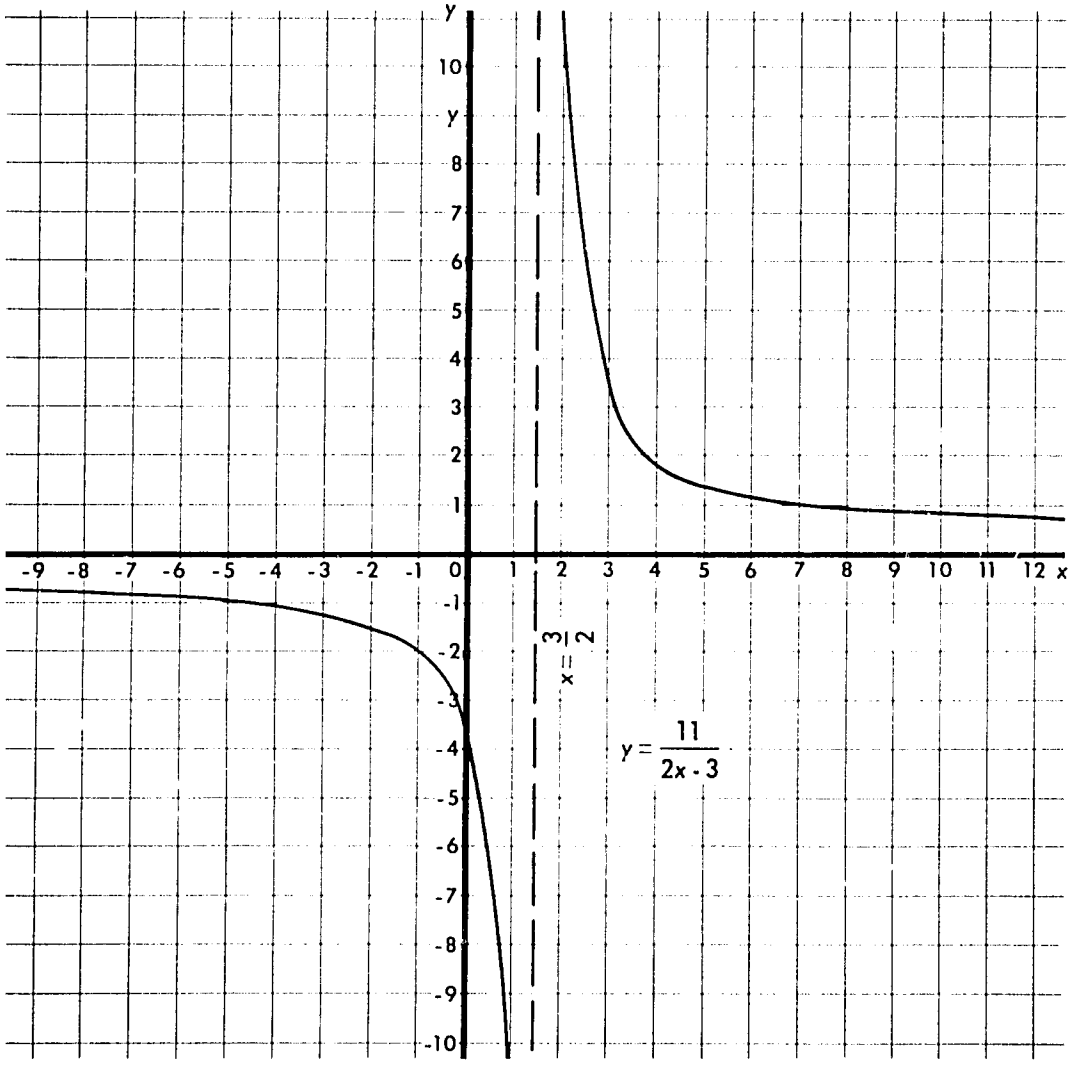
2. a.



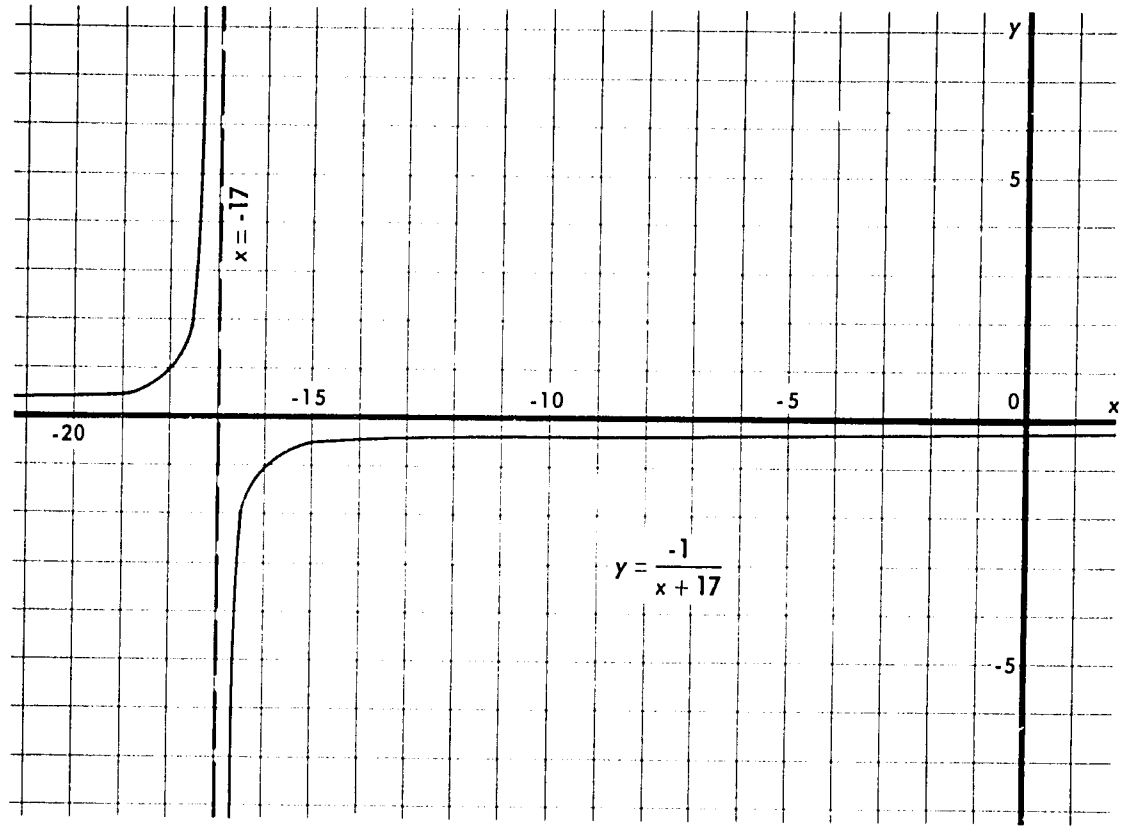
b.



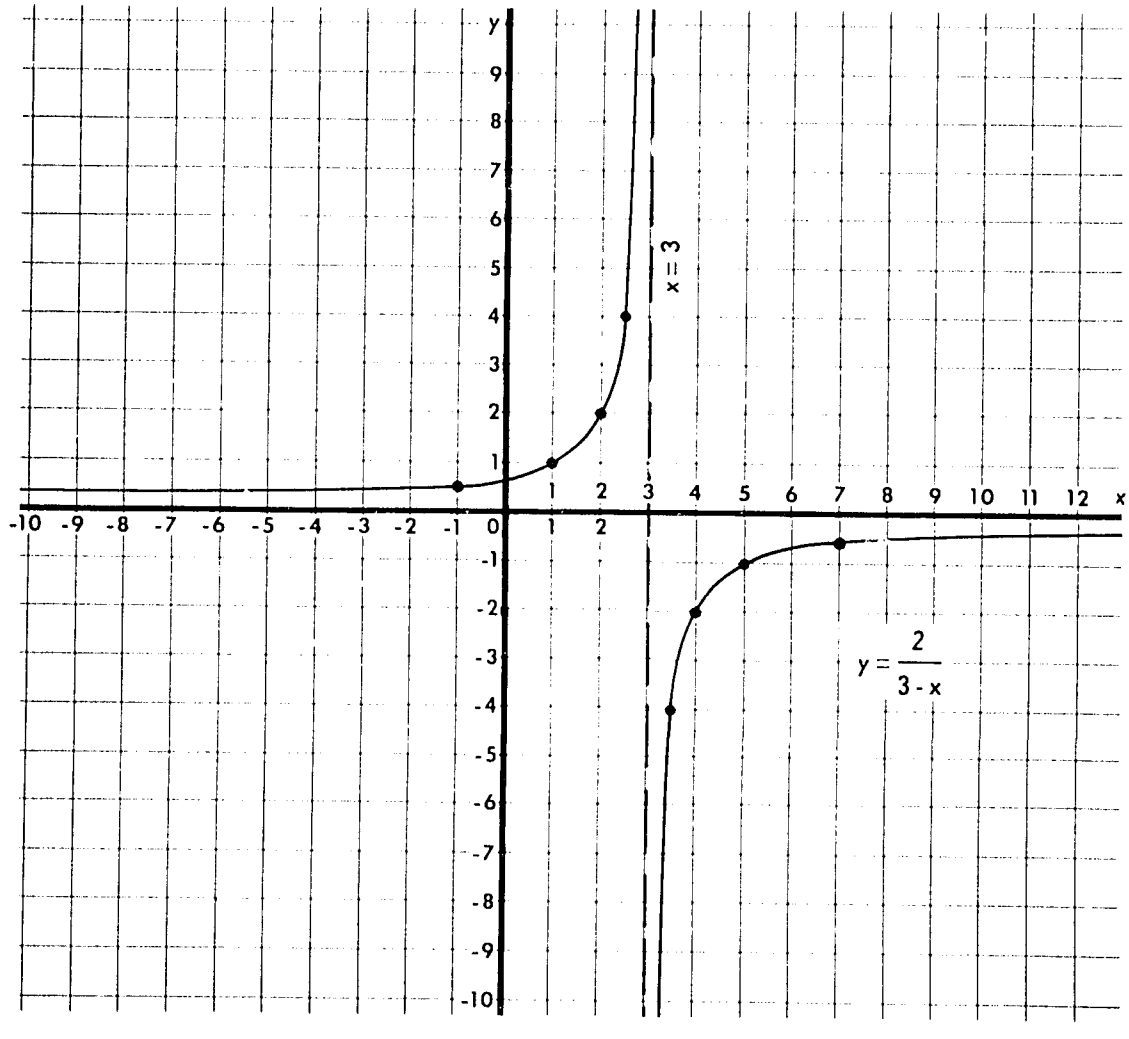
c.



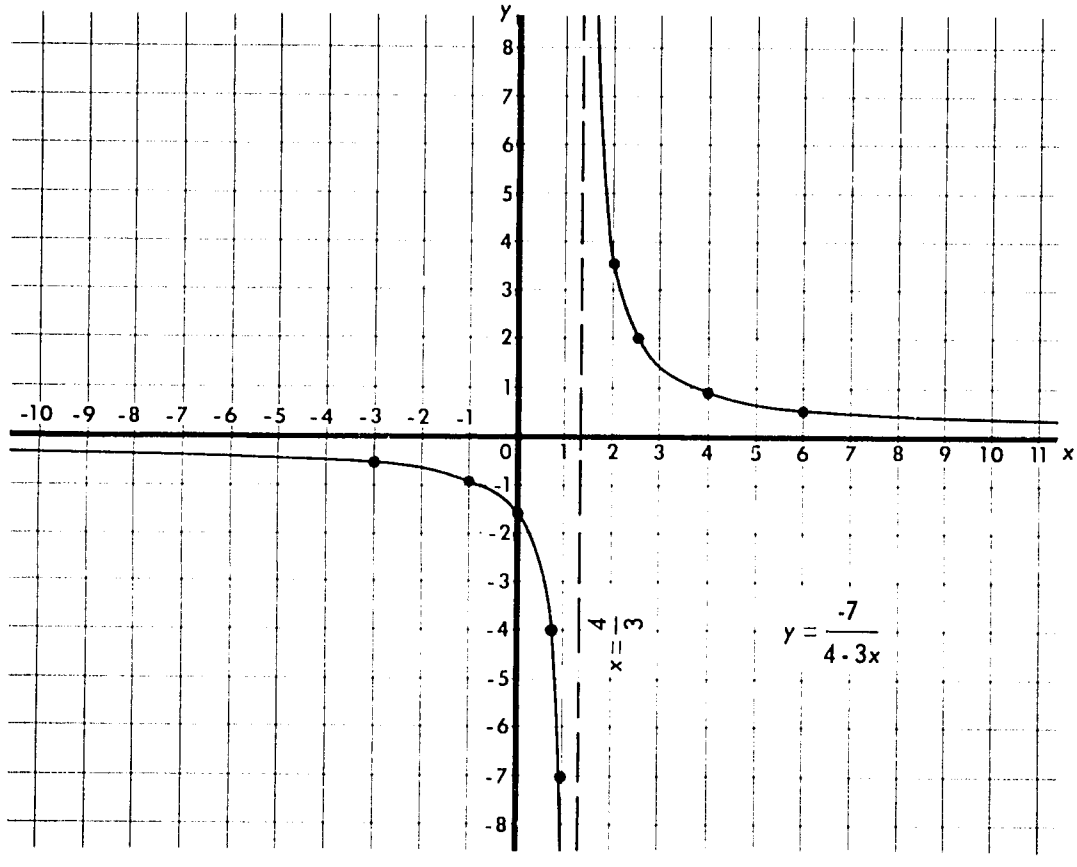
d.


y
5
x = -17
-15
-10
-5
0
x
-20
$y = \frac{-1}{x + 17}$
-5

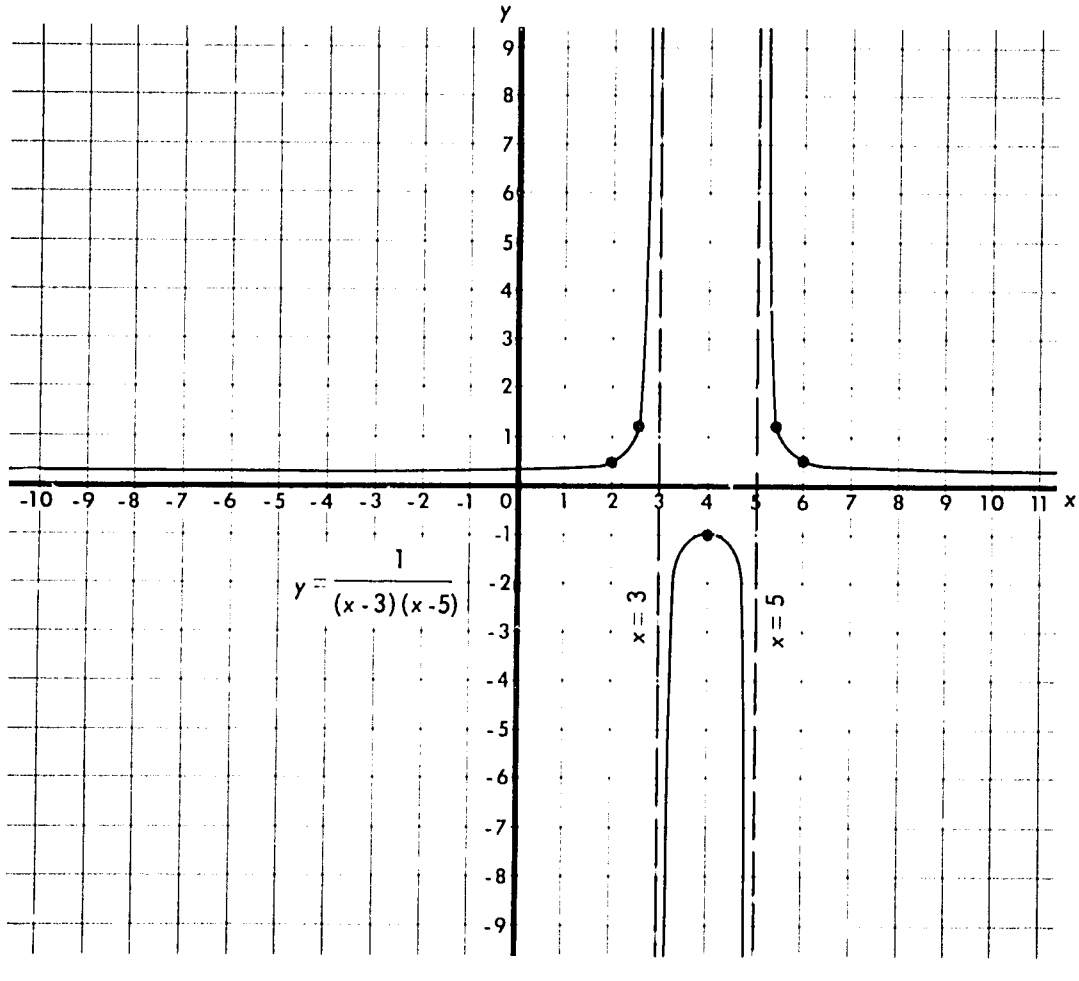
e.


y
9
8
7
6
5
4
3
2
1
x = 3
1 2 3 4 5 6 7 8 9 10 11 12 x
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2
-1
-2
-3
-4
-5
-6
-7
-8
-9
-10
$y = \frac{2}{3 - x}$

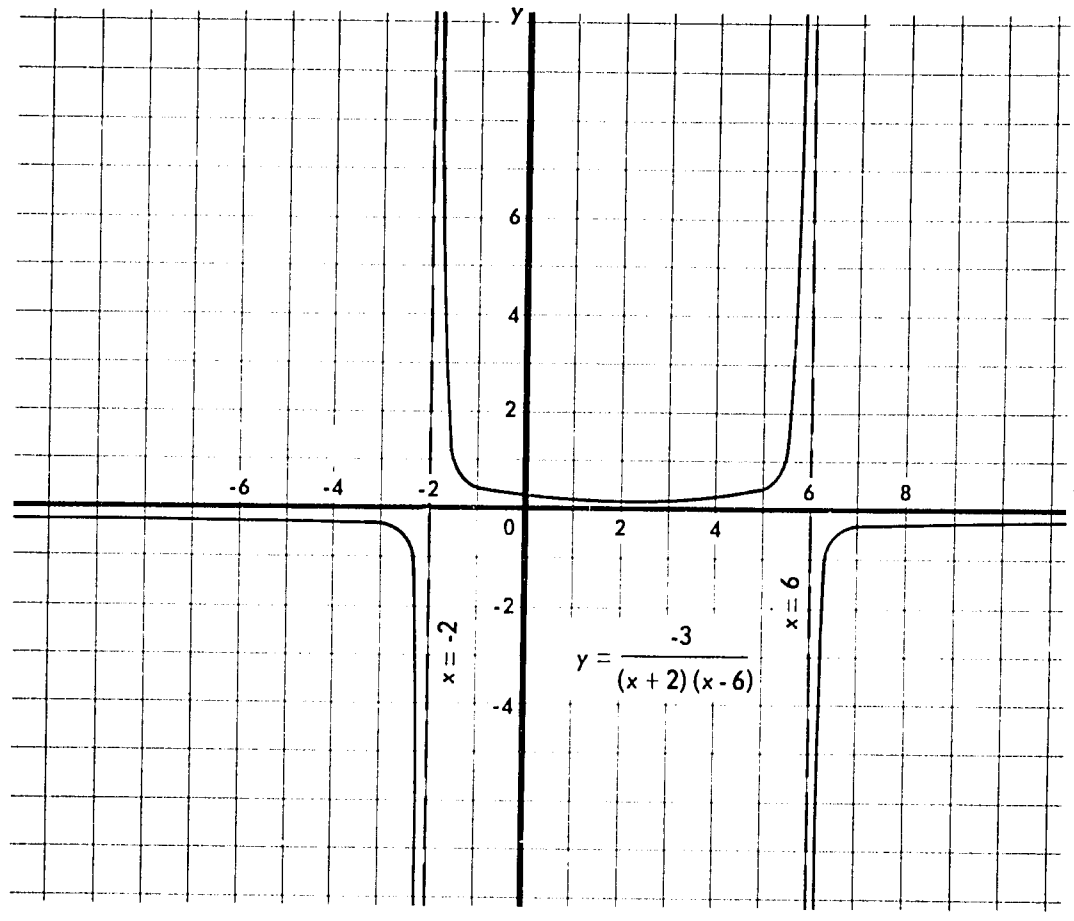
f.



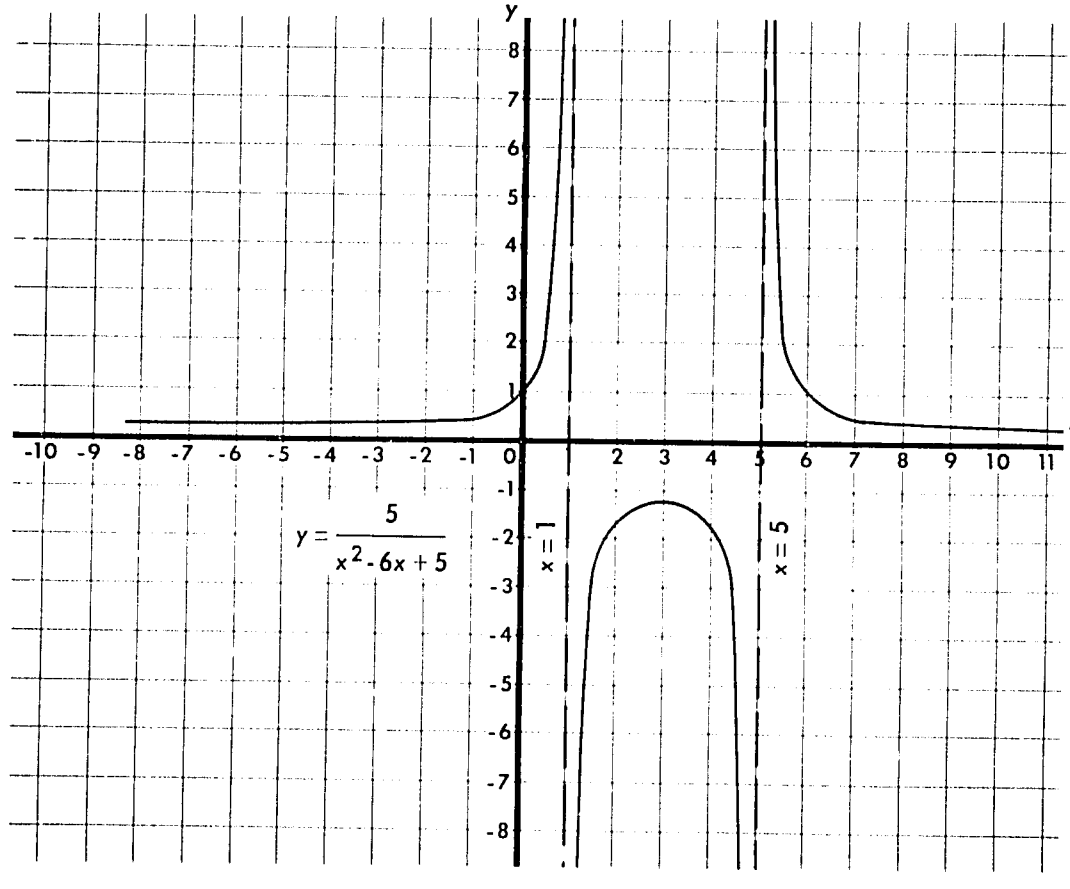
3. a.



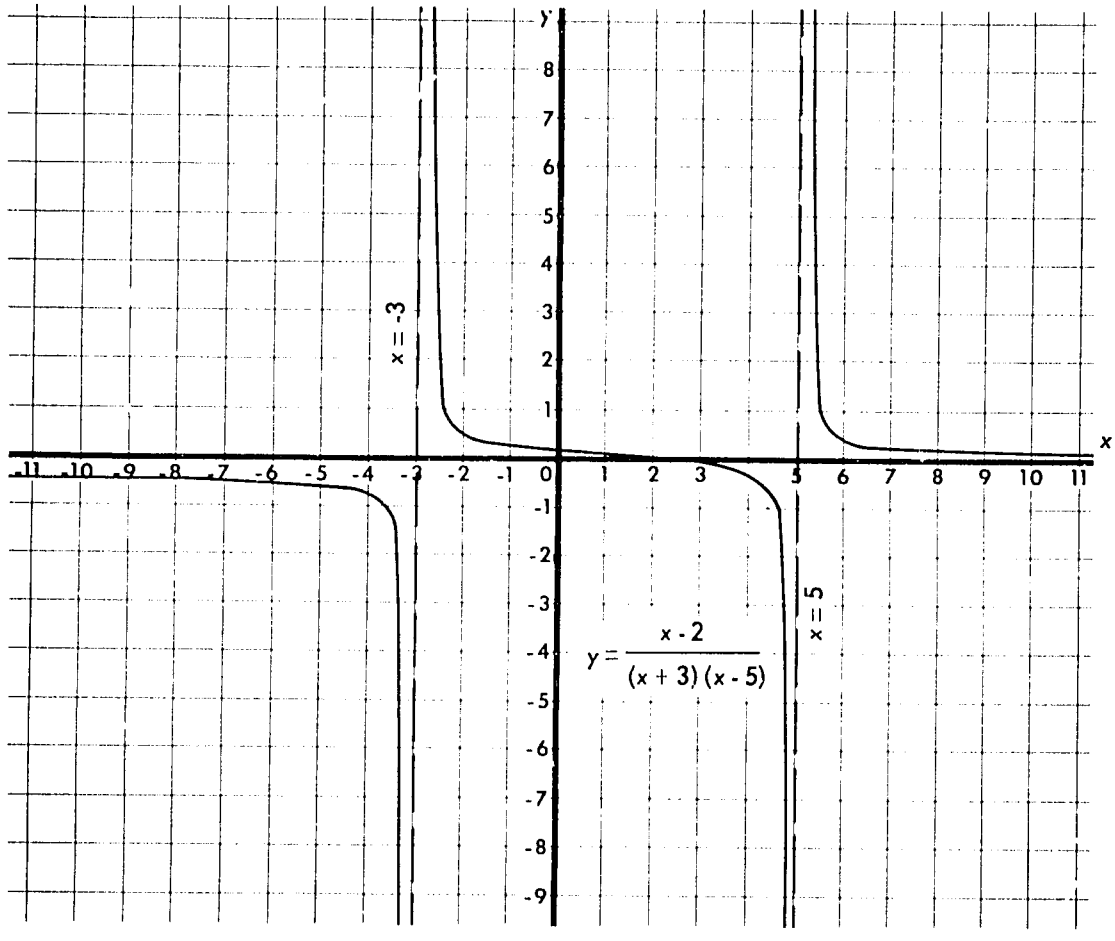
b.


y
x
6
4
2
-6
-4
-2
0
2
4
6
8
-2
-4
x = -2
x = 6
$y = \frac{-3}{(x+2)(x-6)}$

c.


y
x
8
7
6
5
4
3
2
1
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 2 3 4 5 6 7 8 9 10 11
-1
-2
-3
-4
-5
-6
-7
-8
x = 1
x = 5
$y = \frac{5}{x^2 - 6x + 5}$

d.



Try to stimulate a good class discussion in introducing horizontal asymptotes. We examine expressions which approach zero as $|x|$ becomes very large. We use the idea that expressions such as $\frac{1}{x}, \frac{8}{x^2}, \frac{2}{x^3}$ all approach zero as $|x|$ increases to large values. Have the pupils demonstrate this fact with specific values of x such as 100,000 and -1,000,000,000. Every expression $\frac{a}{x^n}$ (where $a \neq 0$ and n is a positive integer) has this property of approaching zero as $|x|$ becomes large.

When we multiply by $\frac{\frac{1}{x^3}}{\frac{1}{x^3}}$ in the example on page 76 our purpose is to put all but one of the terms into the form $\frac{a}{x^n}$. As you see, one term in the denominator does not approach zero and all the other terms in the numerator and denominator do approach zero as $|x|$ becomes large. As long as the degree of the denominator is greater than the degree of the numerator, this result can always be obtained.

1. a. $\dfrac{\frac{10}{x}}{1 + \frac{6}{x} + \frac{8}{x^2}}$

   b. y approaches zero.

   c. There is a horizontal asymptote at $y = 0$ because y approaches zero as $|x|$ becomes large.

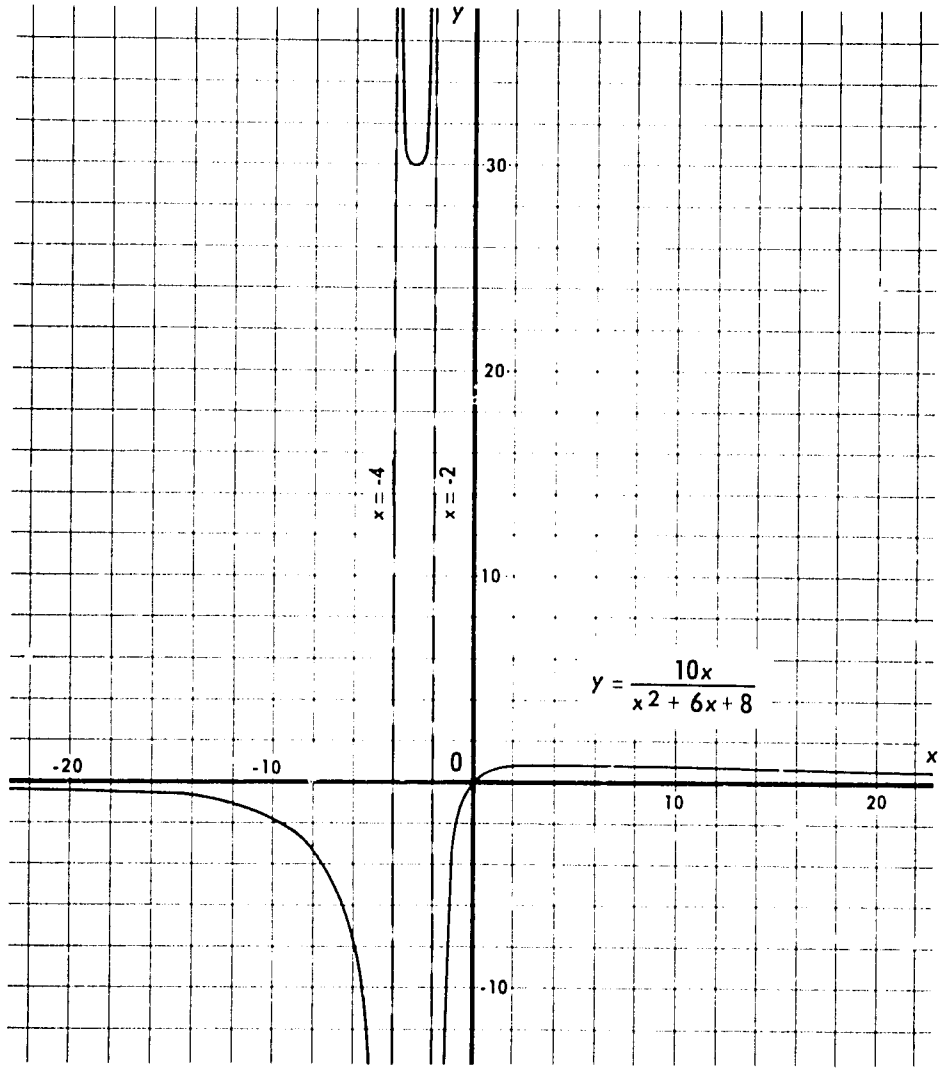
   d. Positive, negative.

   e. $x = -4$, $x = -2$

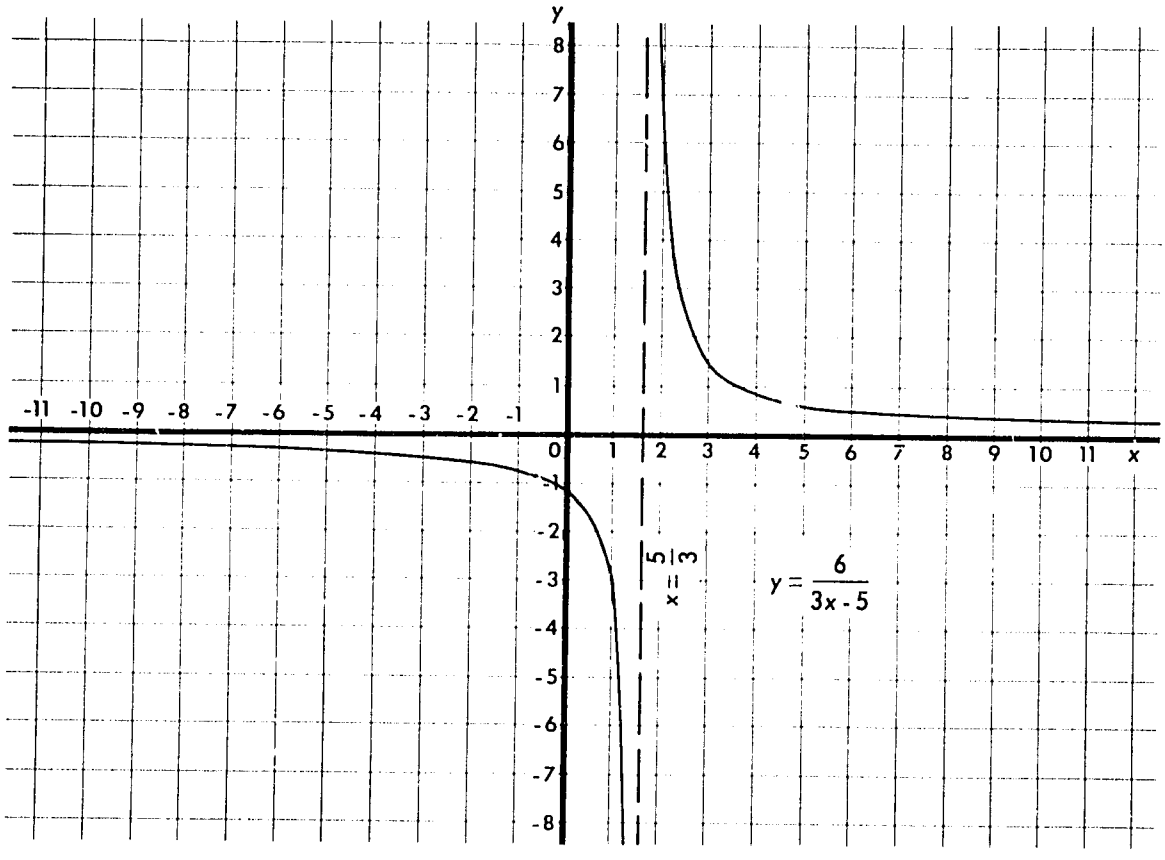
   f. Left of $x = -4$; far down.
   Right of $x = -4$; far up.
   Left of $x = -2$; far up.
   Right of $x = -2$; far down.

   g. (0,0)

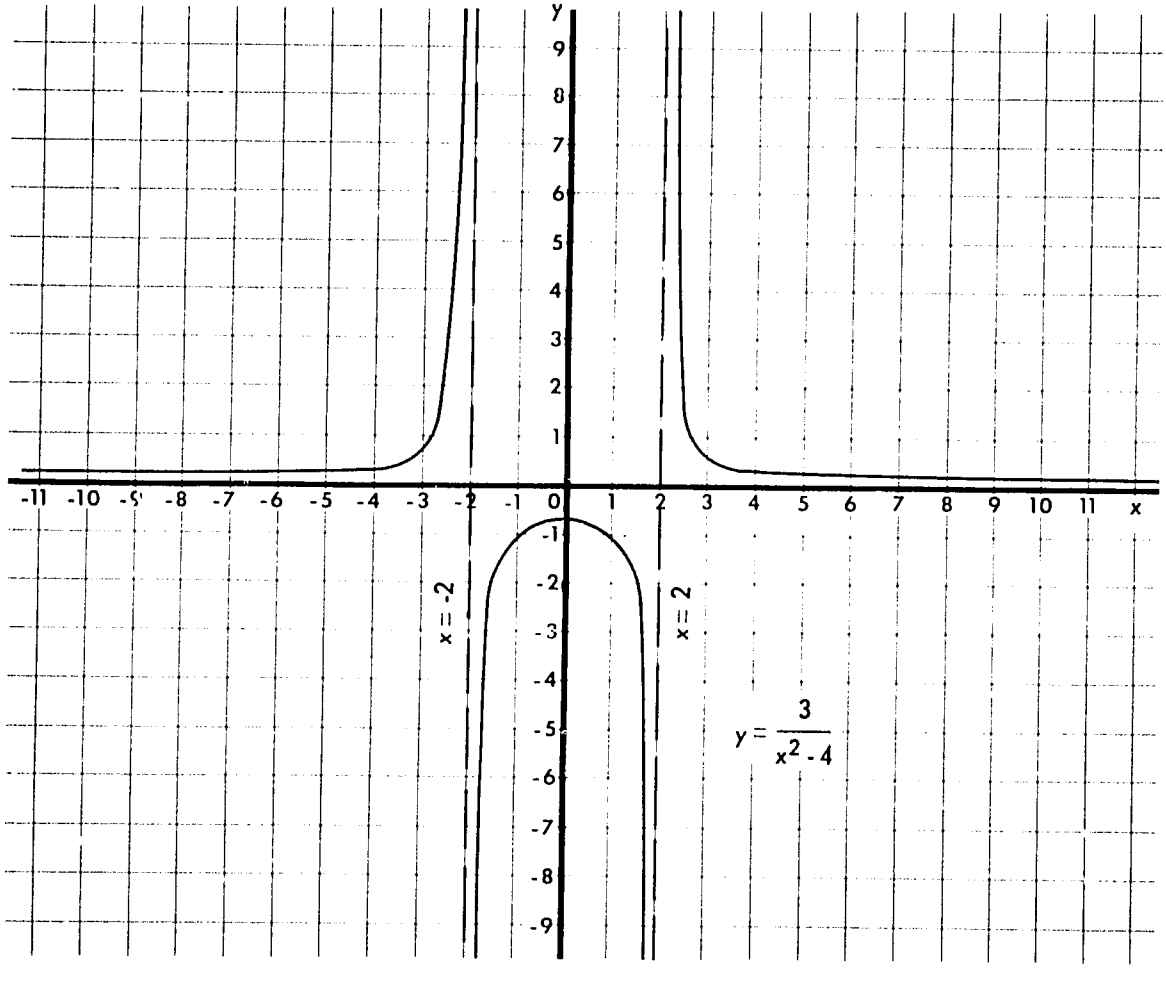
   h.



2.



3.



4. a. $\dfrac{\frac{1}{x^3}}{\frac{1}{x^3}}$ b. $\dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}$ c. $\dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}$ d. $\dfrac{\frac{1}{x}}{\frac{1}{x}}$ e. $\dfrac{\frac{1}{x^4}}{\frac{1}{x^4}}$

5. a, d.

Once the pupils have seen *why* $\dfrac{N(x)}{D(x)}$ has a horizontal asymptote at $y = 0$ when the degree of $N(x)$ is *less than* the degree of $D(x)$, let them use this fact to recognize immediately when such an asymptote occurs.

The other cases where the degree of $N(x)$ is *greater than* or *equal to* the degree of $D(x)$ can be studied by using long division and observing what happens in the resulting form for large $|x|$. Again encourage active participation by the pupils in the introductory discussion of these ideas. Help them to see how the given expression is rewritten as a sum of two parts. The first part is a simple polynomial (constant or linear in the examples we consider) whose behaviour we can quickly recognize. The second part is a fraction whose value approaches zero as $|x|$ becomes large. Therefore far out to the right and left the graph is almost like the graph of the polynomial.

Use your judgment about how far you take your class in this section. If the pupils are finding this difficult or if you are pressed for time, the discussion and problems on oblique asymptotes (Problems 3 and 4 in 2-7C, Example 3 before 2-7D, and Problem 7 in 2-7D) can be omitted entirely.

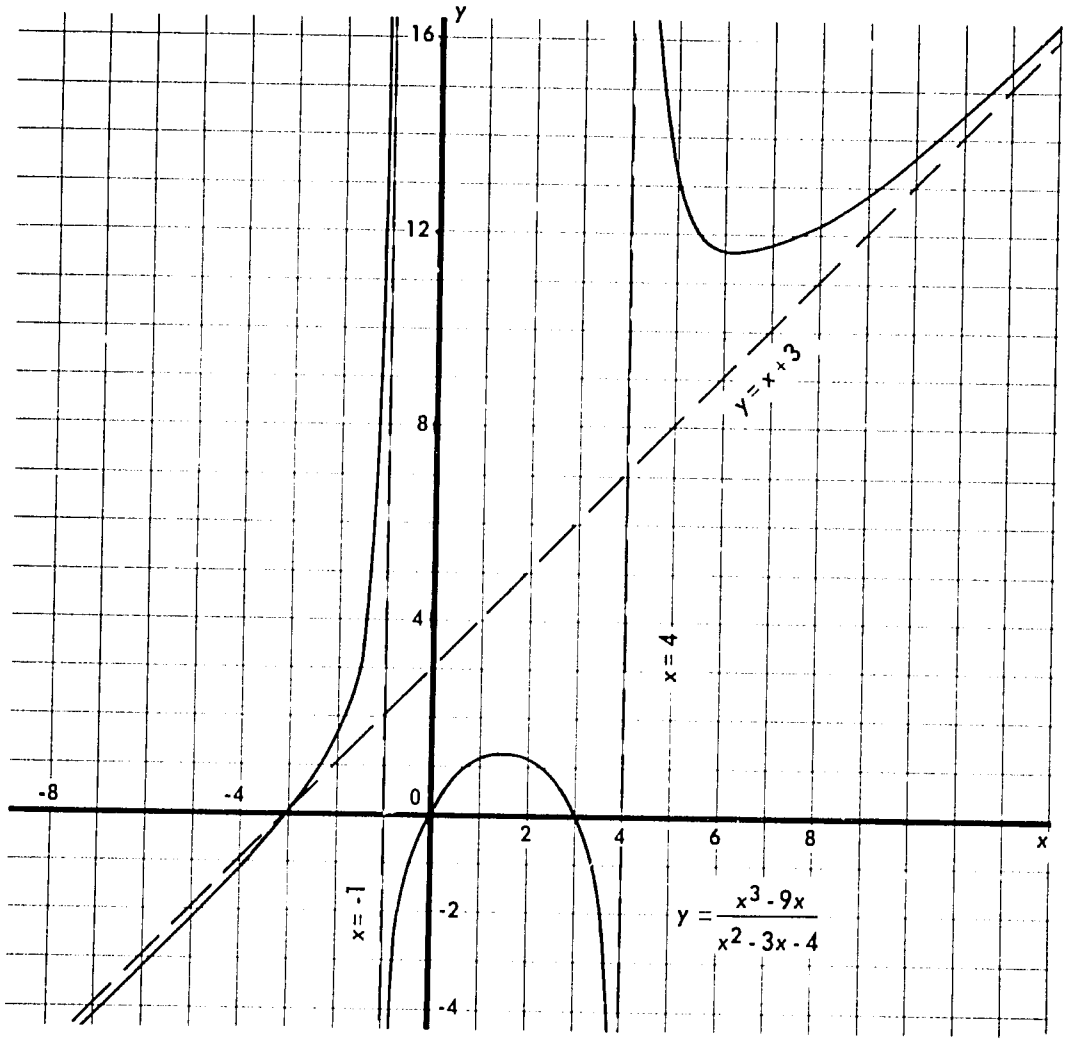
On the other hand, if you have some pupils who are able and eager, they may enjoy and profit from working on some more complicated graphs. Here are two examples.

1. $\dfrac{x^3 - 9x}{x^2 - 3x - 4}$ 2. $\dfrac{x^3 - 5x^2 + 8}{x - 5}$

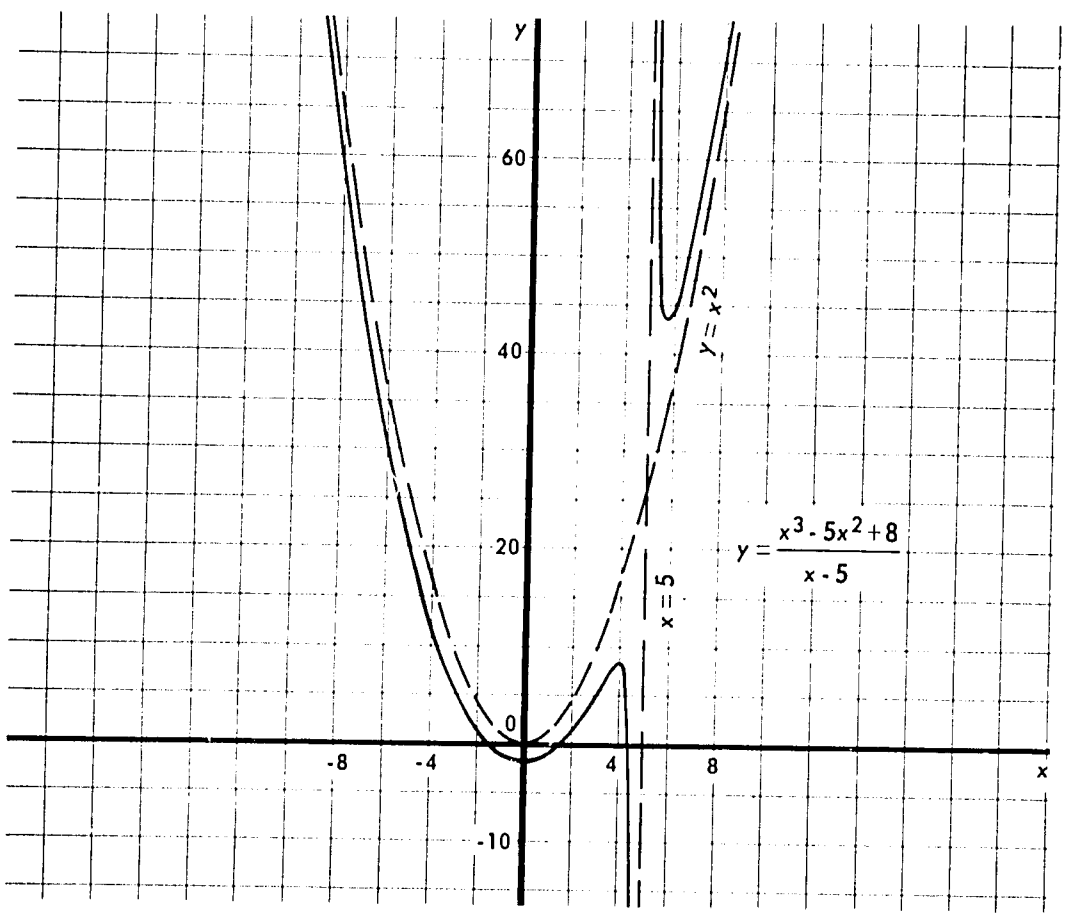
In 1. the graph crosses its own oblique asymptote.

In 2. the result of the long division, $x^2 + \dfrac{8}{x - 5}$, shows that for large $|x|$ the graph approaches the parabola $y = x^2$, so we have a curve instead of a straight line acting as an asymptote.

1.


y
16
12
8
4
0
-4
-8
-4
2
4
6
8
x
-2
y = x + 3
x = 4
x = -1
$y = \frac{x^3 - 9x}{x^2 - 3x - 4}$

2.


y
60
40
20
0
-10
-8
-4
4
8
x
$y = x^2$
x = 5
$y = \frac{x^3 - 5x^2 + 8}{x - 5}$

Some pupils may have forgotten how to do long division of polynomials. Here are a few problems for them to practice on if you need them.

1. $\dfrac{8x - 2}{x + 6}$

2. $\dfrac{15x + 7}{5x - 1}$

3. $\dfrac{x^2 - 8x + 2}{x + 7}$

4. $\dfrac{6x^2 - x - 4}{3x - 5}$

5. $\dfrac{-7x^2 + x - 9}{x^2 - 8x + 1}$

6. $\dfrac{10x^2 + 3x - 4}{2x^2 + 3}$

7. $\dfrac{3x - 7}{5 - x}$

8. $\dfrac{9x^2 + 4x - 2}{1 - 3x}$

9. $\dfrac{x^3 - 5x^2 + 8x - 2}{x - 3}$

10. $\dfrac{3x^4 + x^2 - 7x + 19}{x + 2}$

11. $\dfrac{6x^3 + 5x^2 - 9x - 11}{2x - 1}$

12. $\dfrac{5x^3 + 8x - 1}{x^2 + 5x + 2}$

13. $\dfrac{10x^3 - x^2 + 13x - 5}{2x^2 - x + 3}$

14. $\dfrac{12x^3 - 8x^2 - 28x + 3}{4x^2 - 9}$

Answers to
PROBLEMS 2-7C

Student Text Page 79

1. a. $4 + \dfrac{25}{2x - 6}$ b. $y = 4$

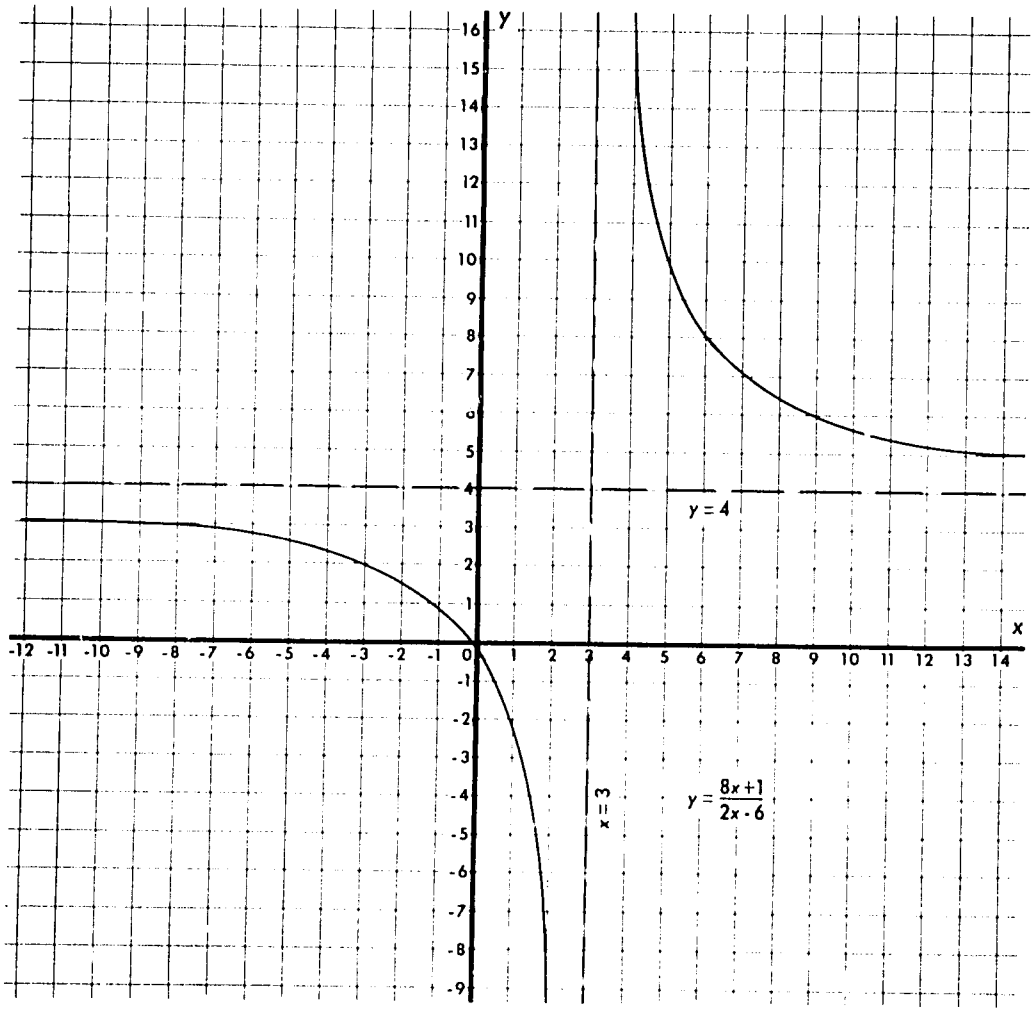
   c. Graph above on far right, below on far left.

   d. $x = 3$ Left of $x = 3$, far down; right of $x = 3$, far up.

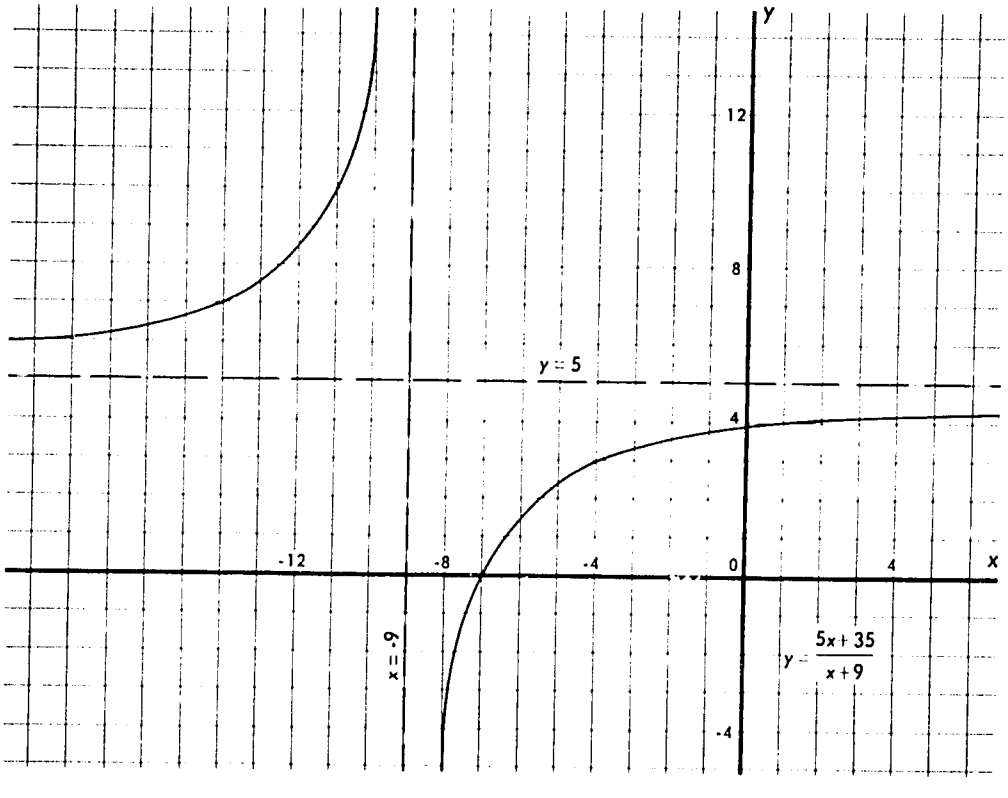
   e. $\left(0, -\frac{1}{6}\right)$, $\left(-\frac{1}{8}, 0\right)$

   f. No symmetry with respect to the y-axis or the origin.

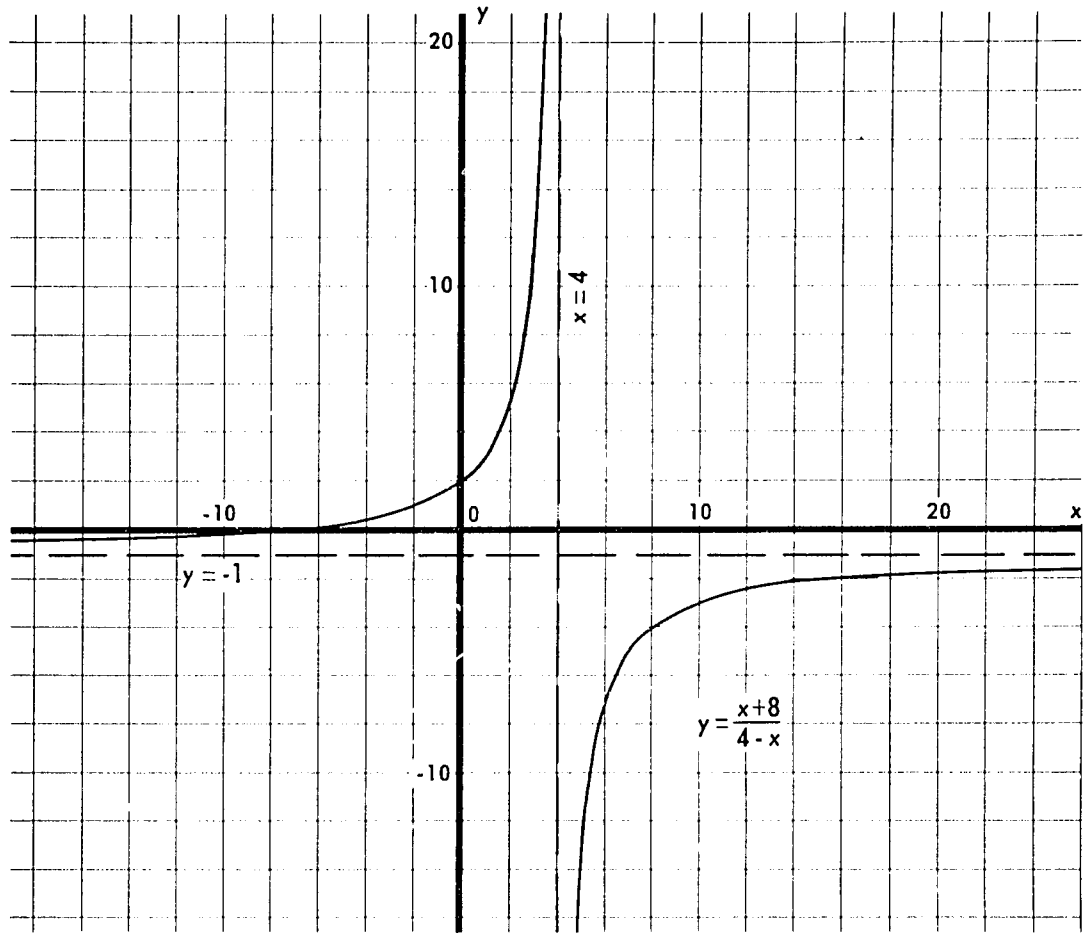
g.



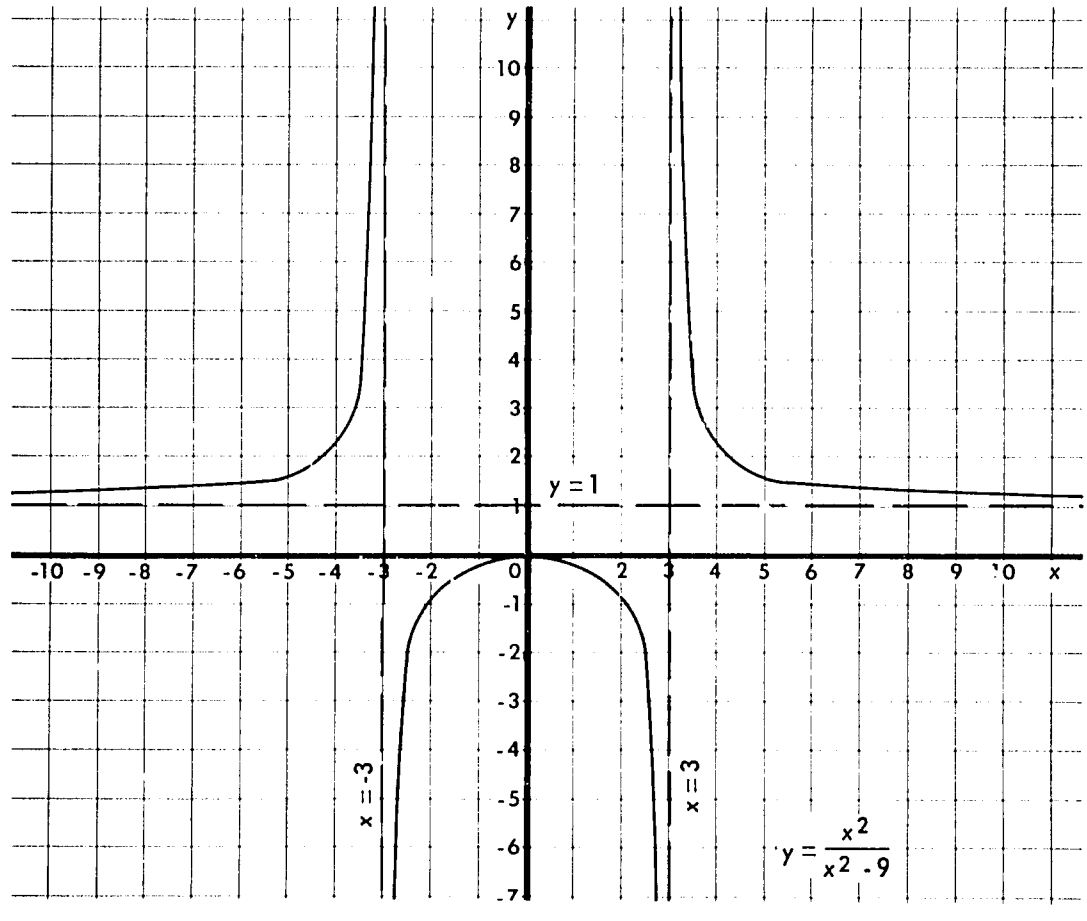
2. a.



b.

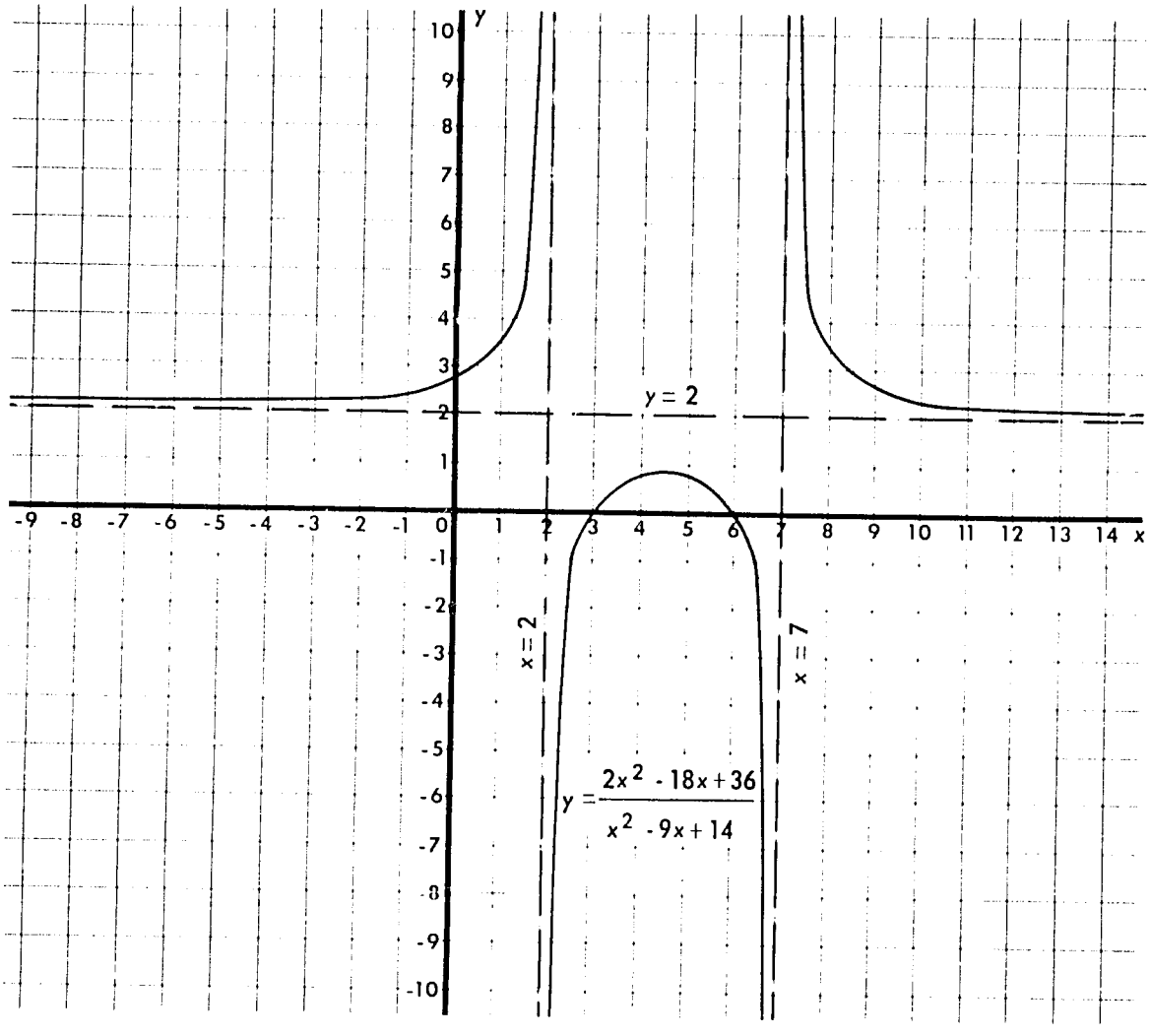


c.



Observe the symmetry with respect to the y-axis.

d. *Challenge Problem*



3. See remarks above about possibility of omitting oblique asymptotes.

a. Yes, because the degree of the numerator is less than the degree of the denominator.

b. y is close to $2x$; no; no.

c. The oblique asymptote $y = 2x$ is a straight line through the origin with slope 2.

d. Graph above $y = 2x$ on far right; below on far left.

e. Yes, at $x = -4$ and $x = 4$.

Left of $x = -4$, far down.

Right of $x = -4$, far up.

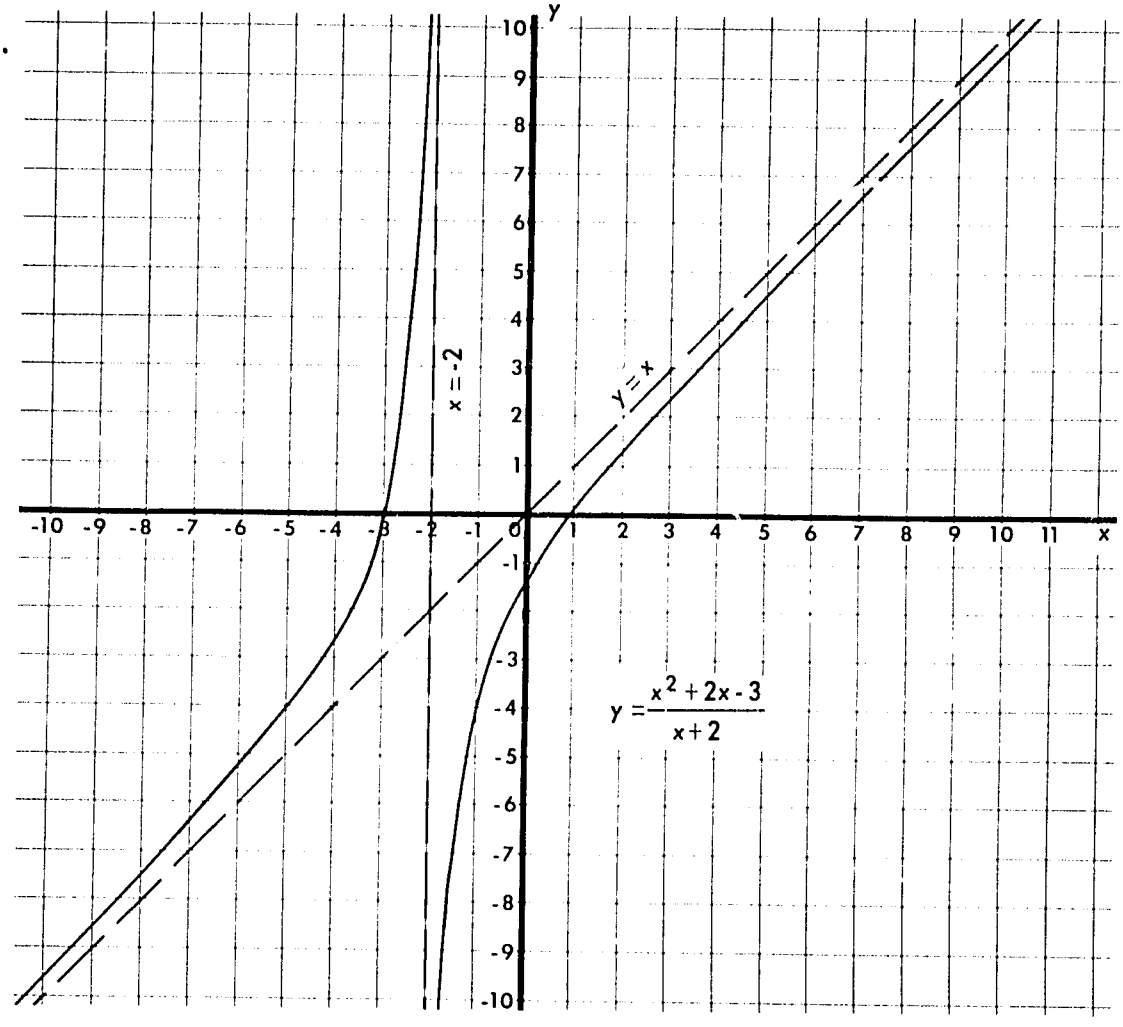
Left of $x = 4$, far down.

Right of $x = 4$, far up.

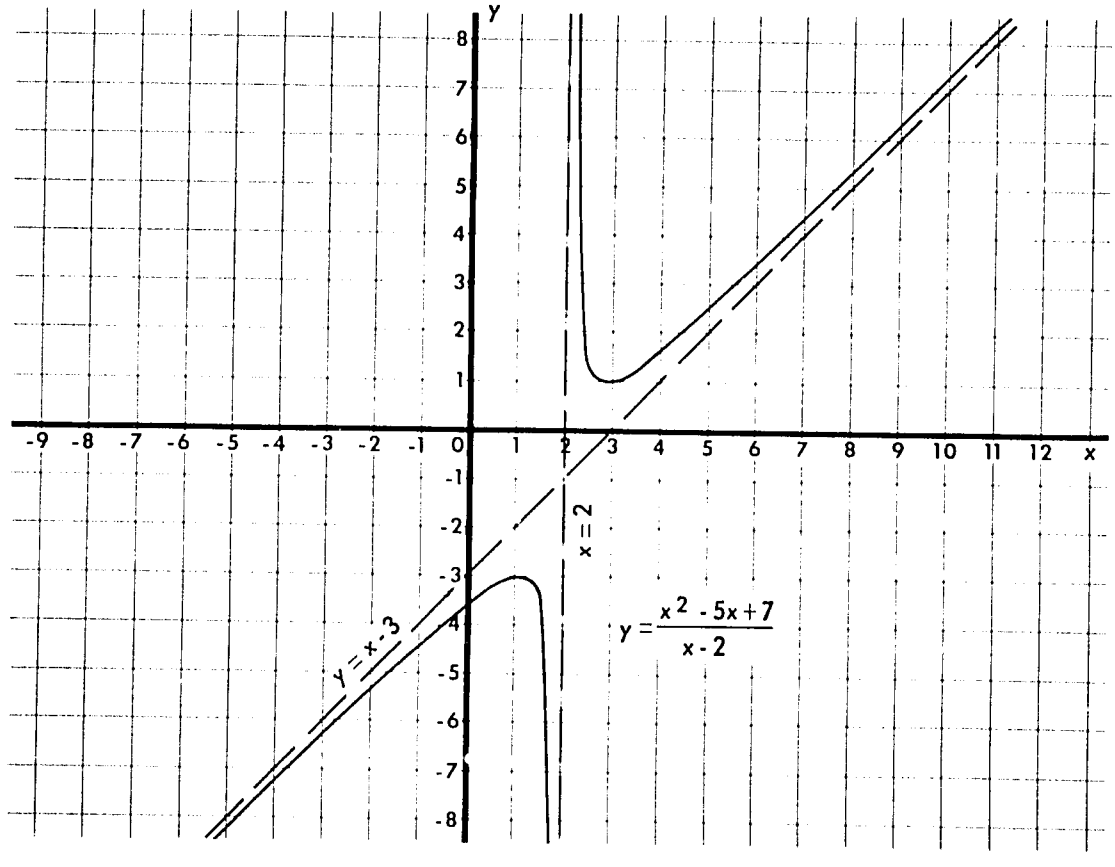
f. (0,0)

g. Symmetry with respect to the origin.

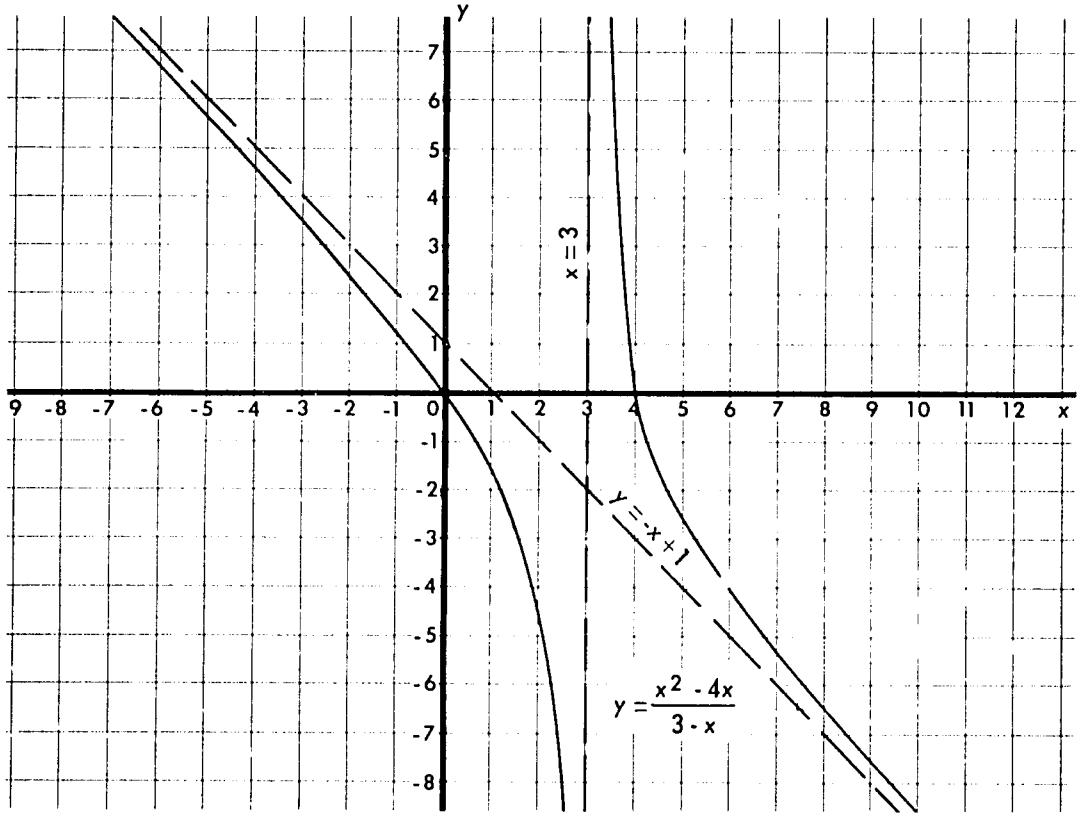
4. a.



b.



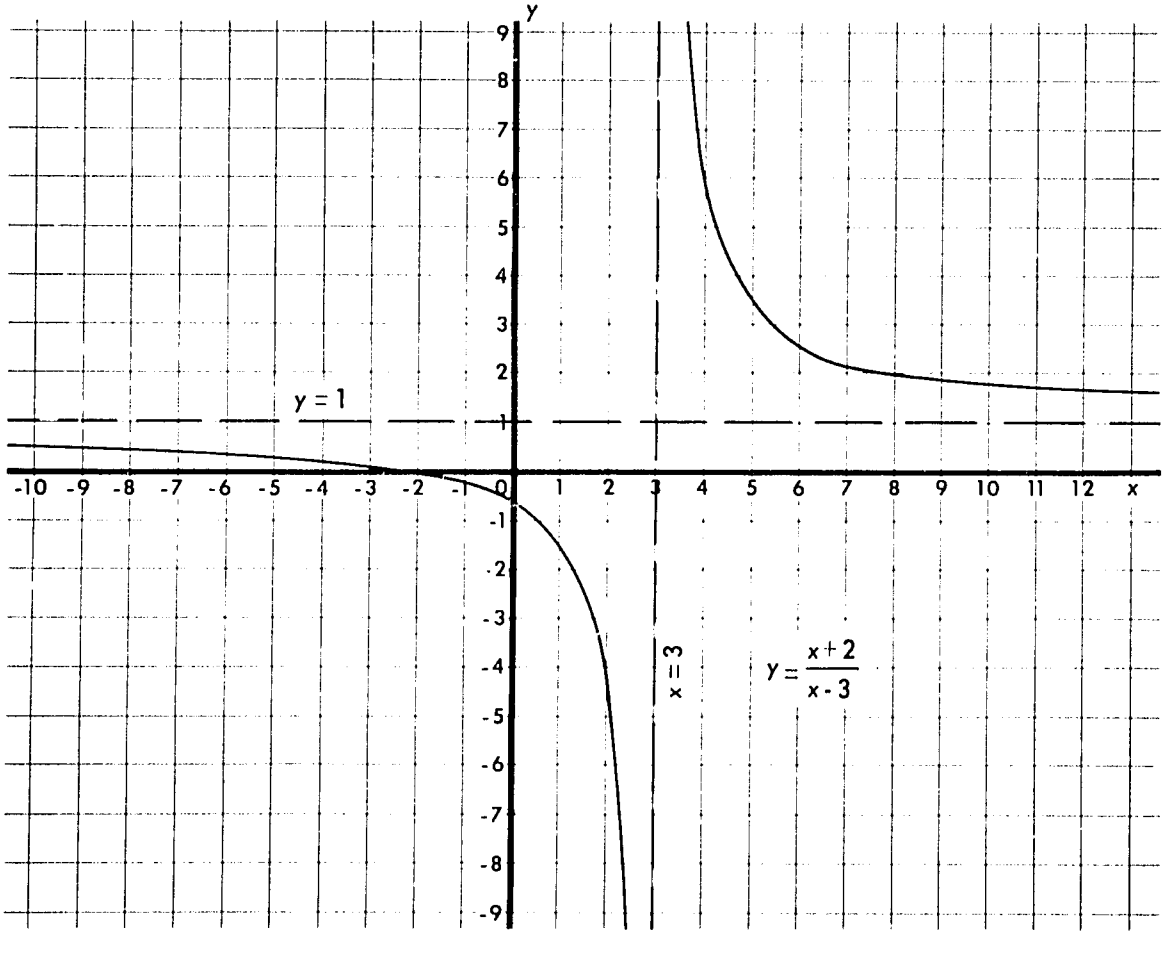
c.



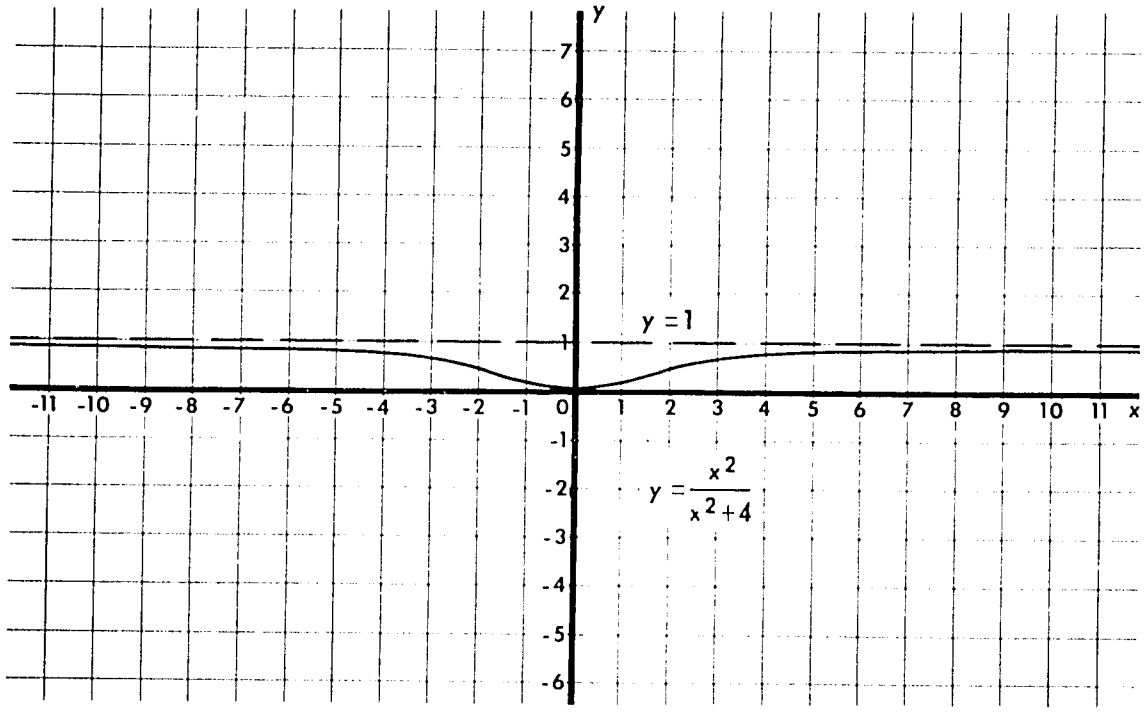
Answers to
PROBLEMS 2-7D

Student Text Page 89

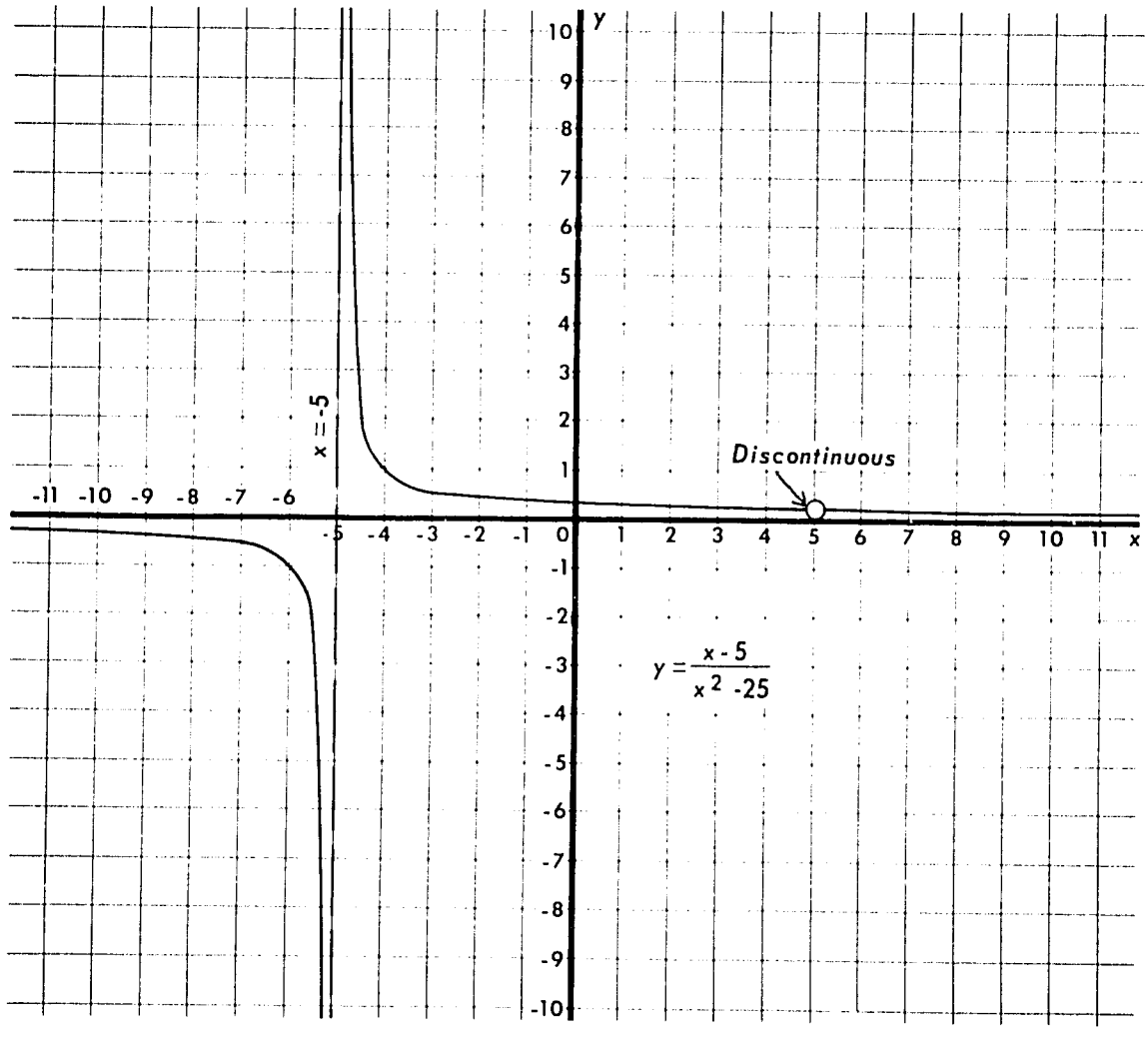
1.



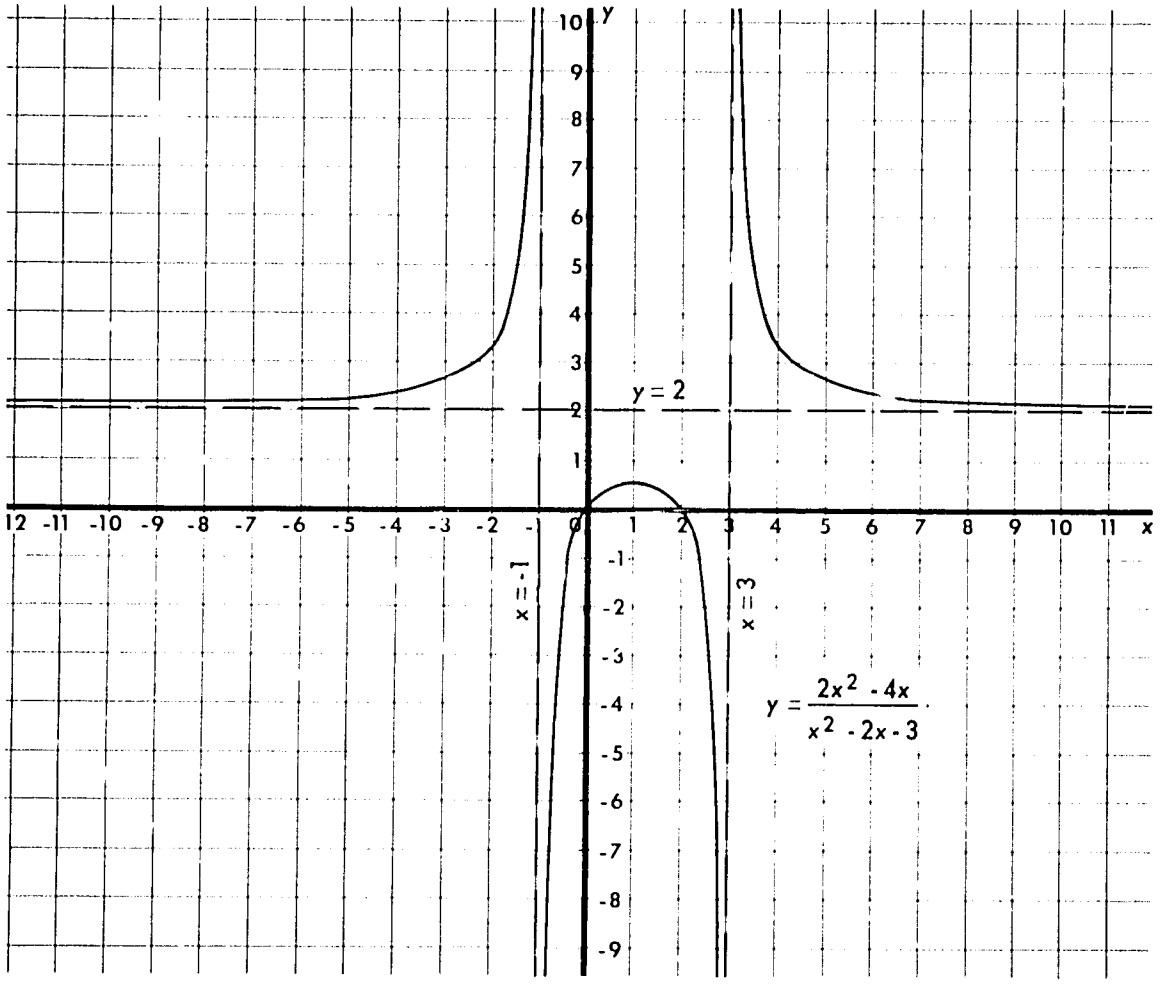
2.



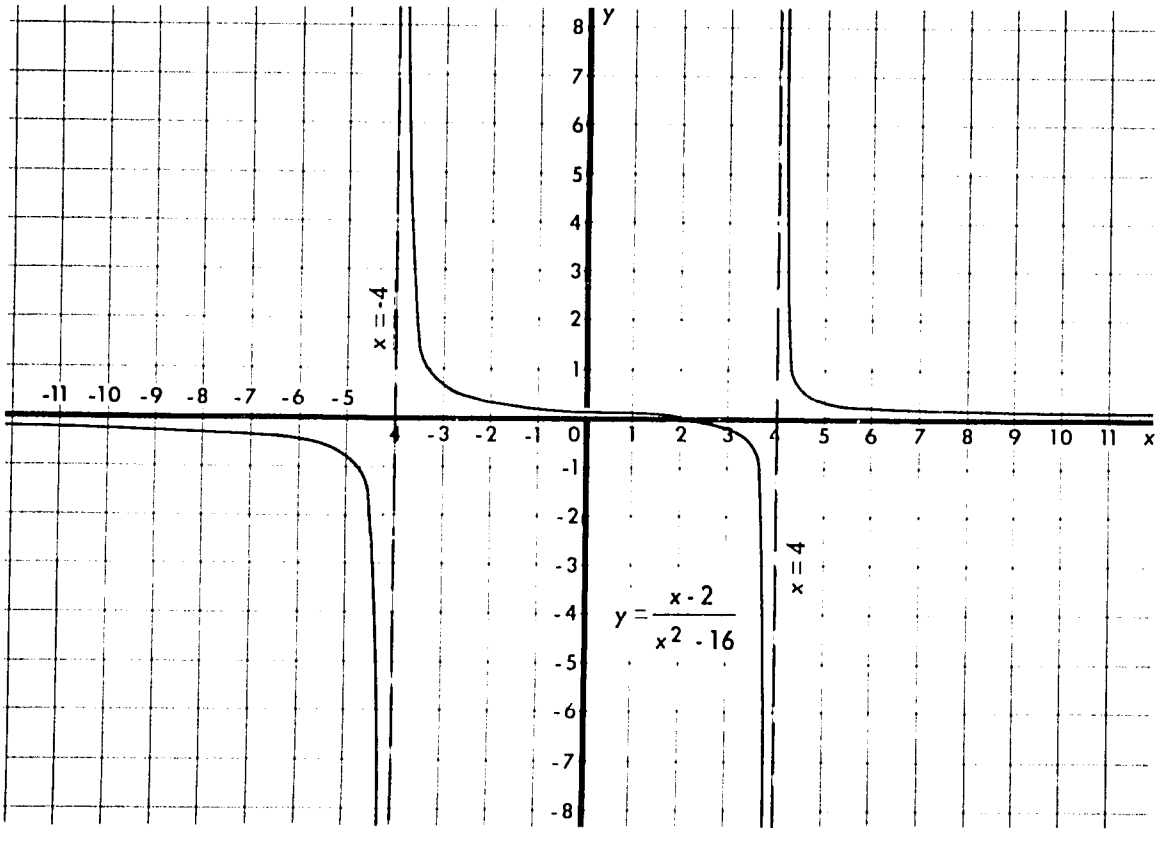
3.



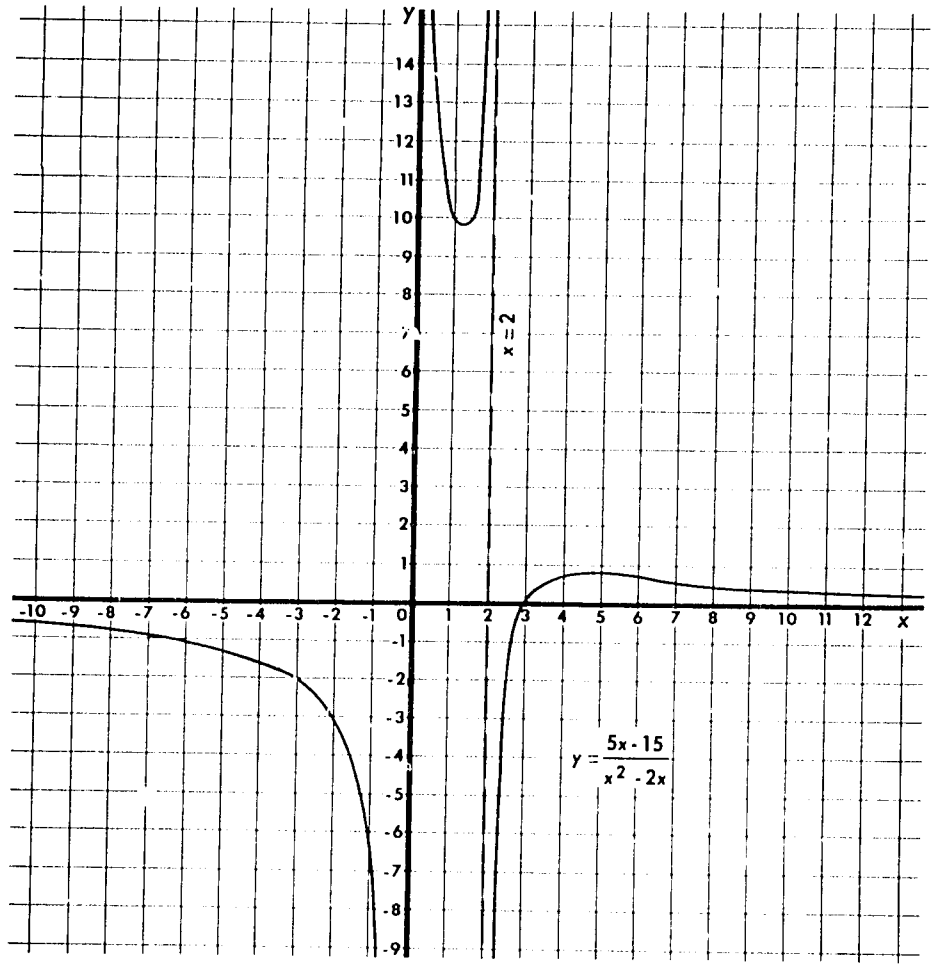
4.



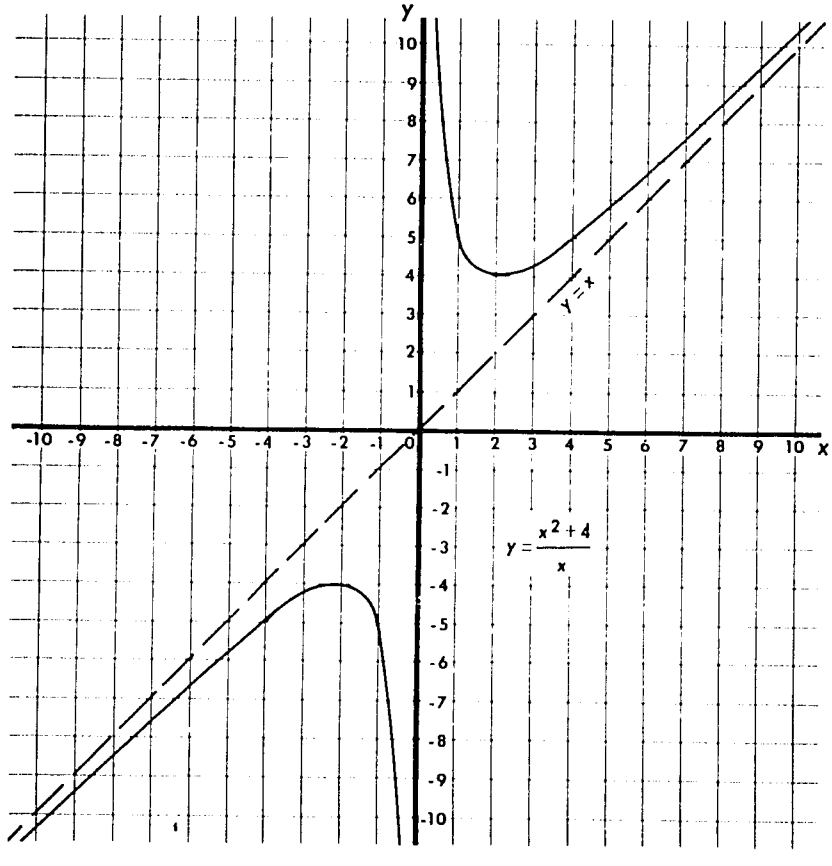
5.



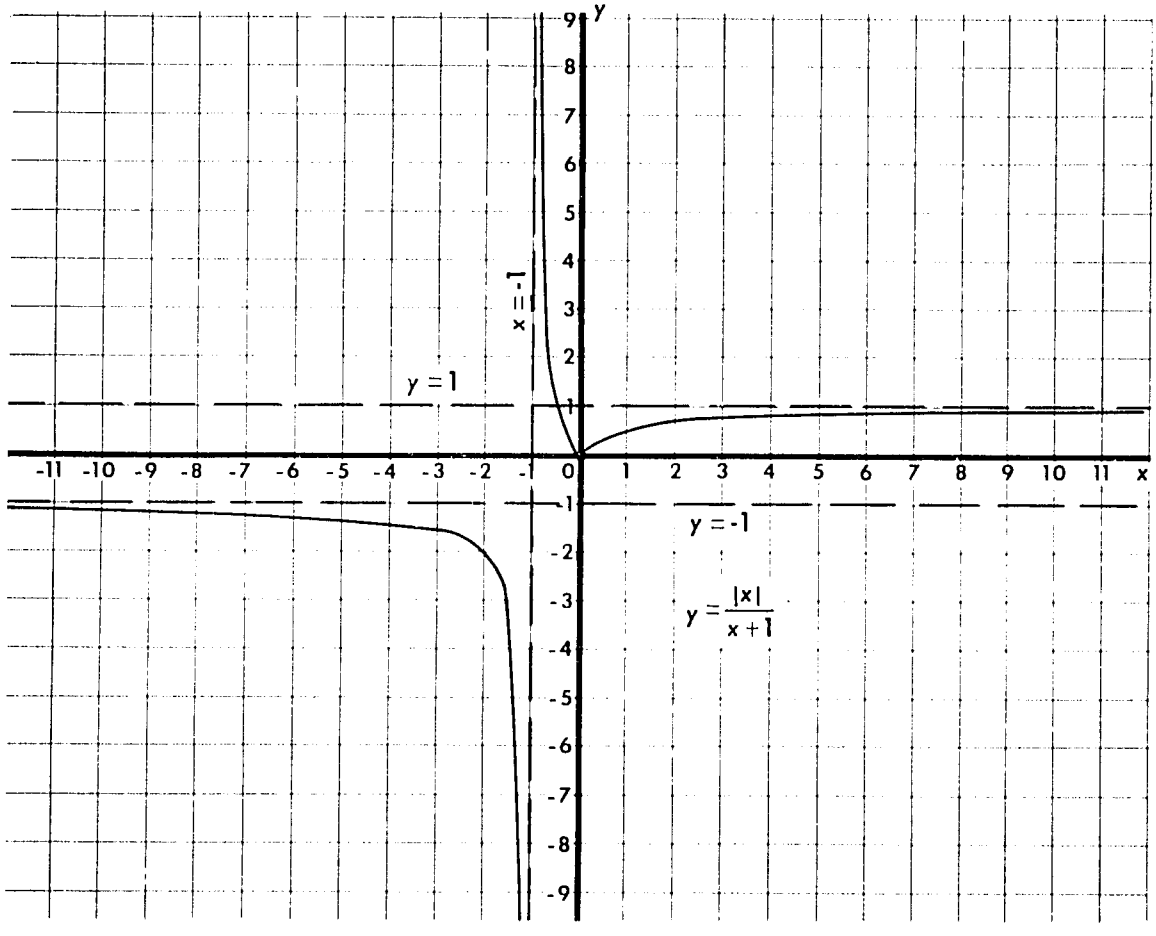
6.



7.



8.



A rational function will have at most one horizontal asymptote but it may have many vertical asymptotes.

2-8 EQUATIONS AND INEQUALITIES BY GRAPHS.

Try to help the pupils to see the difference between the equation

$$2x^3 - 3x^2 - 12x + 10 = 0$$

which is true for some values of x and false for others, and the equation

$$y = 2x^3 - 3x^2 - 12x + 10$$

which defines a function by pairing values of y with values of x. Of the ordered pairs in this function we are interested in those of the form $(a,0)$ because the number a in such a pair makes the first equation true. Those pairs $(a,0)$, however, all lie on the x-axis, so we can use the graph of the function to help find, at least approximately, the numbers a which make the first equation true.

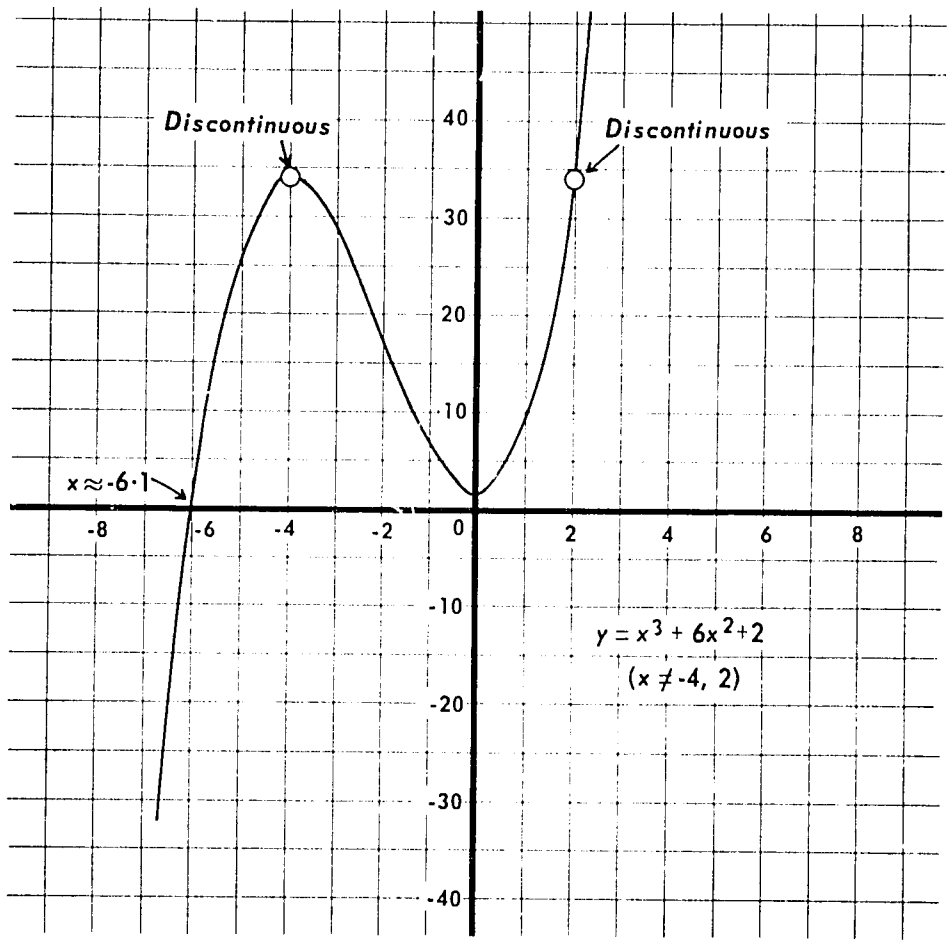
If the first equation involves rational expressions, not just polynomials, we could still find approximate solutions from the graph of the corresponding

rational function. It is usually easier, though, to work with an equivalent polynomial equation as is shown in the example. We do have to be careful to watch the domain which may not be evident in the polynomial by itself.

Answers to
PROBLEMS 2-8A

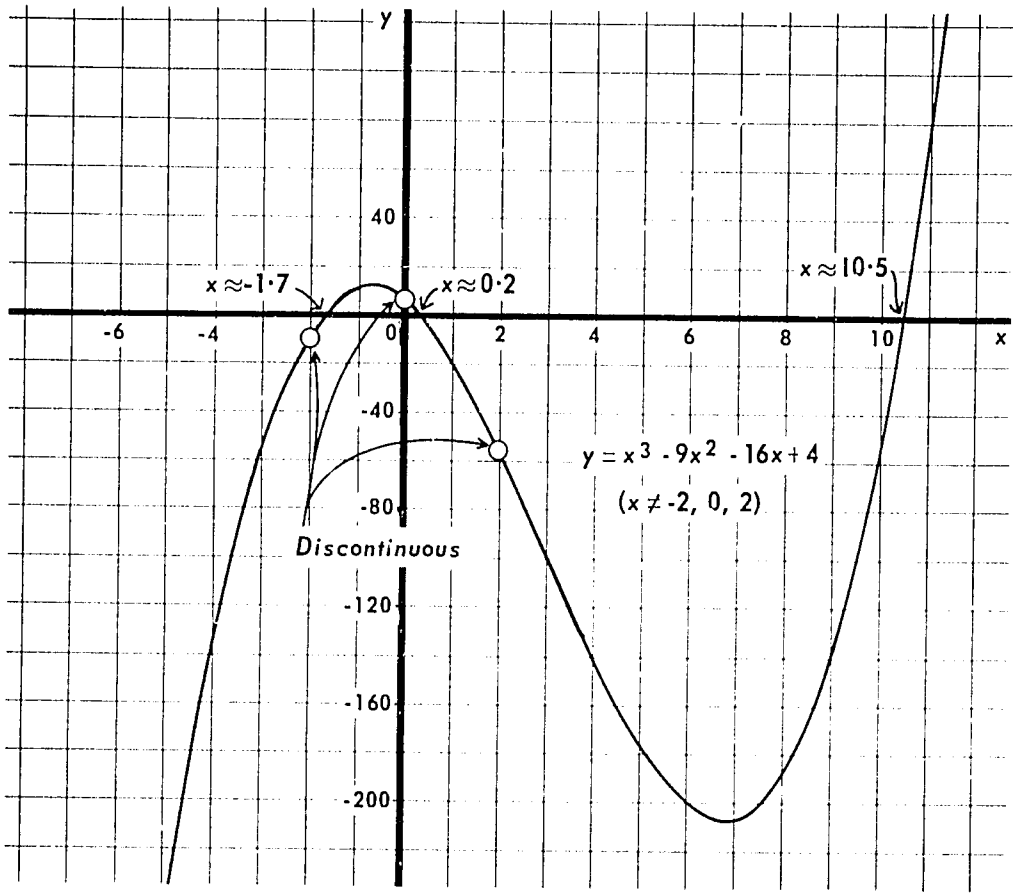
Student Text Page 91

1.



The truth set is approximately $\{-6{\cdot}1\}$

2.

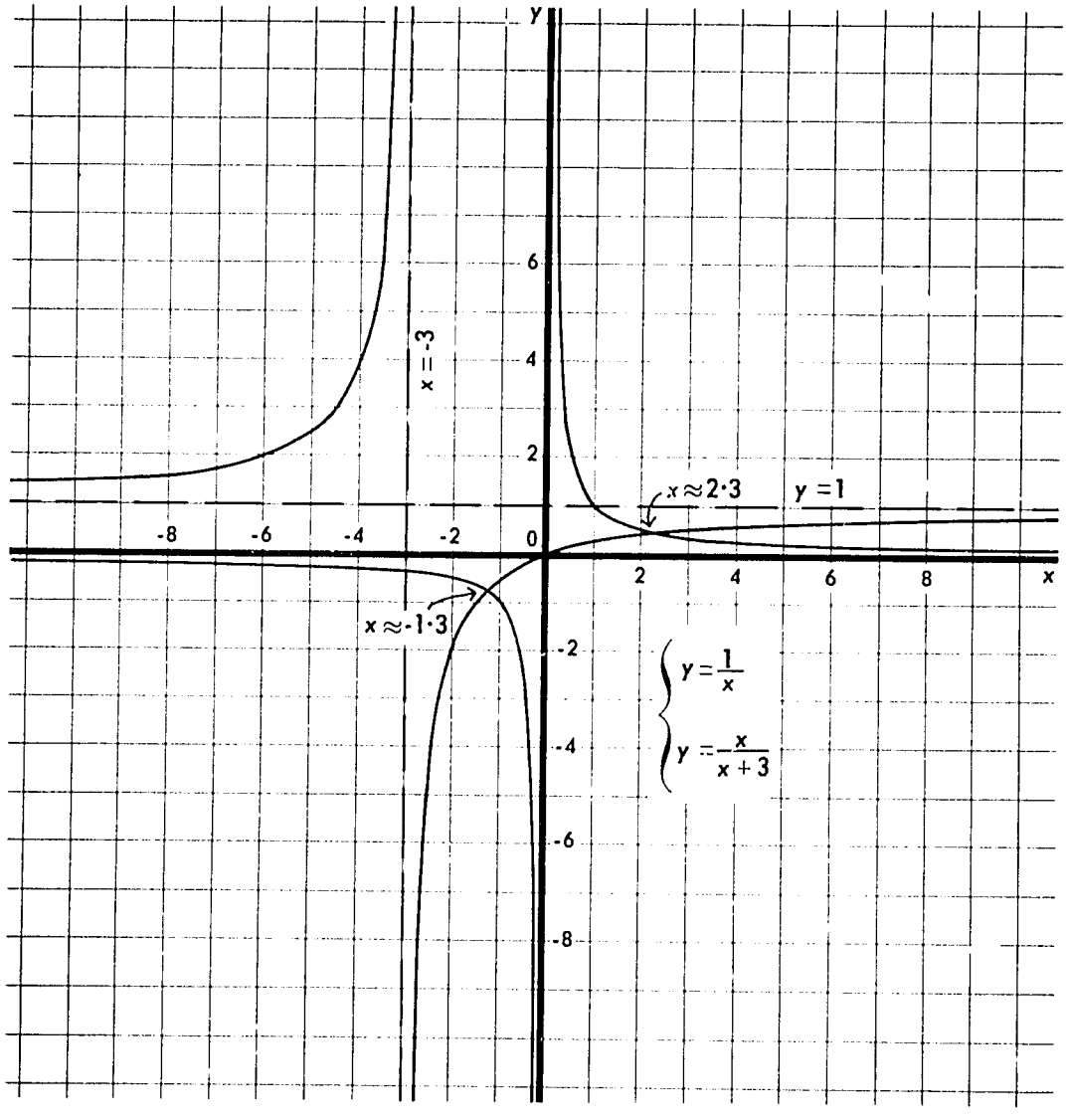


The truth set is approximately $\{-1{\cdot}7,\ 0{\cdot}2,\ 10{\cdot}5\}$

Sometimes a complicated expression can be broken into two or more simpler expressions which are easier to graph and we can consider whether there is a simple relation between the simpler graphs which will serve our purpose. In the case of $\frac{x}{x-1} = \frac{2}{x^2}$ we want a value of x for which the two simple graphs of $y = \frac{x}{x-1}$ and $y = \frac{2}{x^2}$ have the same y-value. This means that we are interested in the point or points of intersection of the two graphs.

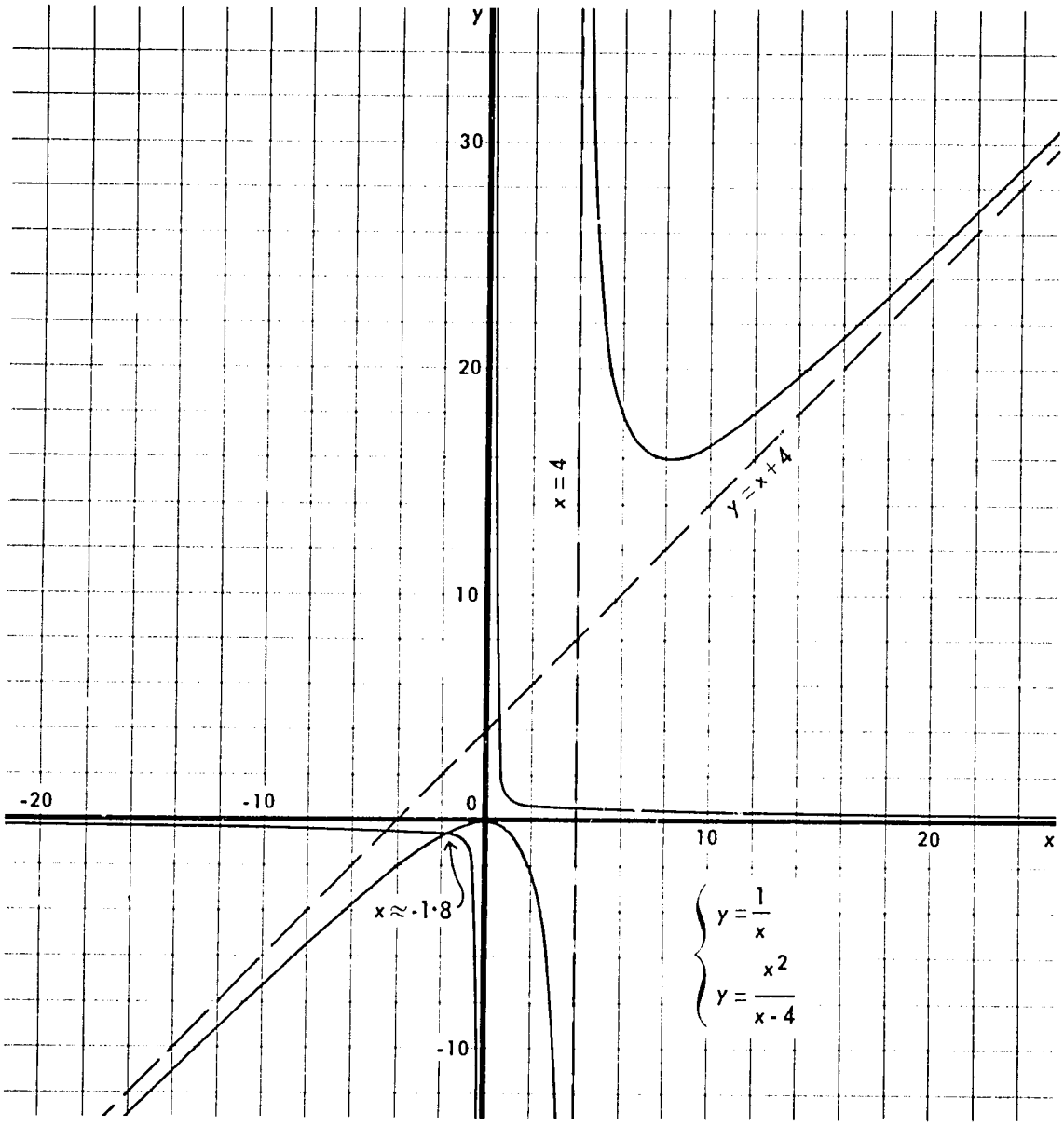
This device is a convenient one to know about. There are problems which would be very difficult to solve without it.

1.



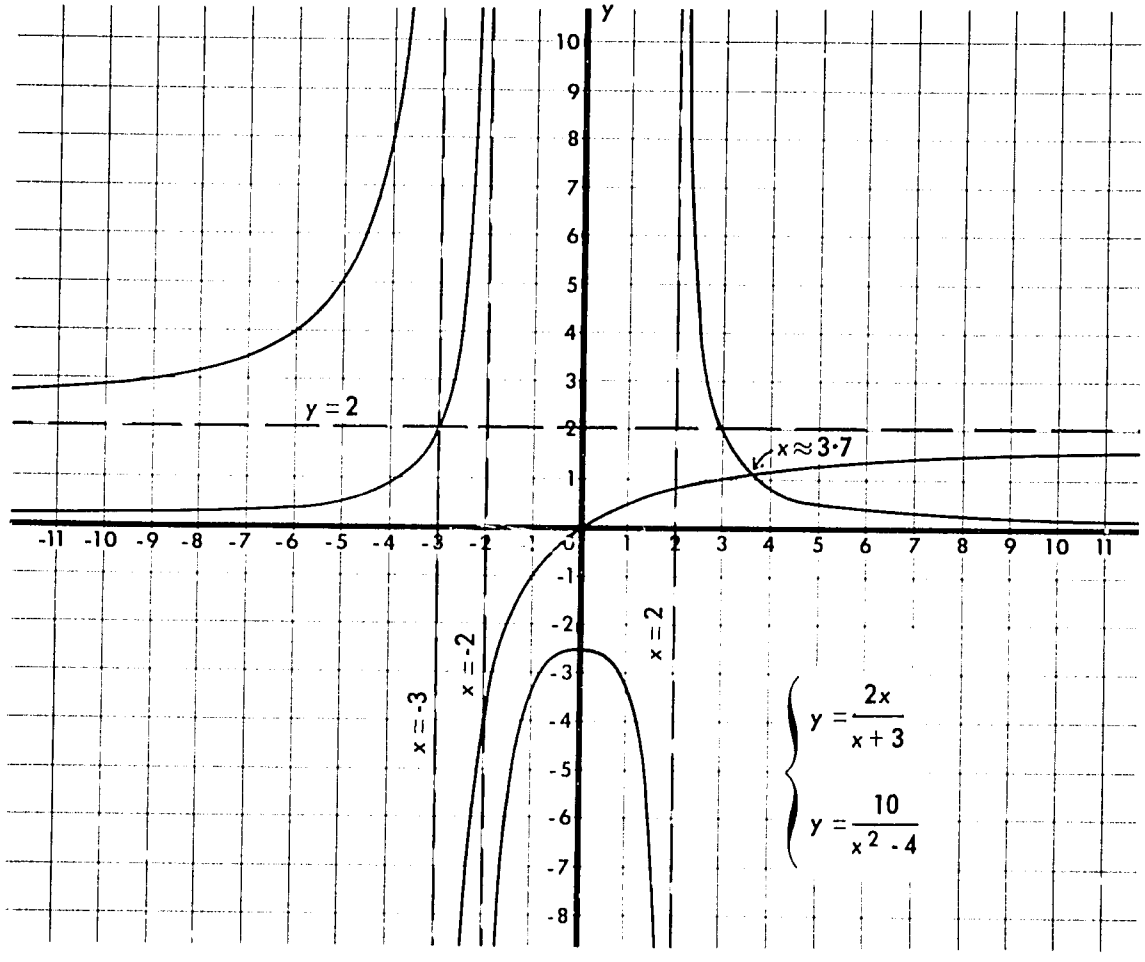
Truth set: $\left\{\frac{1+\sqrt{13}}{2}, \frac{1-\sqrt{13}}{2}\right\}$ or approximately $\{-1{\cdot}3, 2{\cdot}3\}$

2.



Truth set approximately $\{-1{\cdot}8\}$

3.



Truth set approximately $\{3{\cdot}7\}$

To solve inequalities of the form $f(x) > 0$, we can use the function defined by $y = f(x)$. In this function we are interested in ordered pairs (x, y) for which the second number, y, is greater than zero. This just means that we are interested in points of the graph which are *above* the x-axis. Similarly for $f(x) < 0$ we are interested in points *below* the x-axis.

Lead the class carefully through the Class Activity and at the end accumulate your results in a table similar to that on page 94.

Answers to
CLASS ACTIVITY

Student Text Page 93

1. The graph crosses the x-axis.

2. There is a vertical asymptote.

3. No.

4, 5, 6. Except at a vertical asymptote the graph of a rational function is continuous and y cannot change from positive to negative or from negative to positive without going through the value zero. Therefore a change of sign of y can occur only at a vertical asymptote or an x-intercept, in this case at $x = 0$ and $x = 3$. It follows that within each of the intervals $x < 0$, $0 < x < 3$, and $x > 3$, the value of y does not change sign, that is, the graph stays on one side of the x-axis.

7, 8, 9. It is sufficient to test one value of y in each interval. If y is positive for that value of x, it is positive for every value of x in that interval. In this case, for example, when $x = -1$ (which is in the interval $x < 0$), then y is 4, a positive number. Hence y is positive throughout the interval $x < 0$.

Another way to determine the positiveness and negativeness of y in the intervals is merely to observe the positiveness and negativeness of the factors of the numerator and denominator of the expression. A table such as the following helps.

x	$x < 0$	0	$0 < x < 3$	3	$x > 3$
$x - 3$	neg.	neg.	neg.	0	pos.
x	neg.	0	pos.	pos.	pos.
$y = \frac{x-3}{x}$	pos.	undefined	neg.	0	pos.

Answers to
PROBLEMS 2-8C

Student Text Page 95

1. $\{x: -4 < x < 0\}$
2. $\{x: -3 < x < 1 \text{ or } x > 3\}$
3. $\{x: 1 < x < 5 \text{ or } x > 7\}$
4. $\{x: -3 < x < 0\}$
5. $\{x: x < 8\}$
6. $\{x: x \leq -3 \text{ and } x \neq -4\}$
7. $\{\ \}$
8. $\{x: x \neq 7, x \neq 0\}$
9. $\{x: -5 < x \leq -2 \text{ or } 1 \leq x < 6\}$

Chapter 3

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

3-1 INTRODUCTION.

In this section we help the pupils recall the meaning and some of the properties of exponents. The CLASS ACTIVITY (page 96) gives practice material to stimulate discussion of the basic laws of exponents which are then stated. As you go through these, try to help the pupils see that the laws are plausible. Illustrate with the cases having positive integral exponents (Problems 1, 4, 8, 10 and others like them) with emphasis on how the exponents are *counting* the factors. Then remind them how the other kinds of exponents were defined in such a way that the laws continue to be true.

Answers to
CLASS ACTIVITY

Student Text Page 96

(We refer to the laws by number here, but in class continue to speak of them in terms of variables, as, for example, "the law that $a^x b^x = (ab)^x$.")

1. 2^7 Law 1.
2. 2^2 Law 1.
3. $2^{3\sqrt{2}}$ Law 1.
4. 3^3 Law 2.
5. 5 Law 2.
6. 6 Law 3.
7. 5^2 Law 3.
8. 20^2 Law 4.
9. 6^3 Law 4.
10. 7^3 Law 5.
11. $8^{\frac{3}{2}}$ Law 5.
12. 4^3 Law 3.
13. 5^{-6} Law 1.
14. π^{-5} Law 2.
15. 8 Law 3.
16. 3^{-4} Law 3.
17. 3^{-2} Law 5.
18. 2^0 Law 1.
19. 3^2 Law 2.
20. $1^{0.69}$ Law 4.

Many of these can, of course, be further simplified, but the point to be made is the law used to change to the form shown above.

Answers to
MORE CLASS ACTIVITY

Student Text Page 97

1. Examples for the other four laws might be:

$$\frac{3^5}{3^2} = \frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3} = \frac{3 \times 3}{3 \times 3} \times 3 \times 3 \times 3 = 3^3 = 3^{5-2}$$

$$(3^2)^5 = (3 \times 3)(3 \times 3)(3 \times 3)(3 \times 3)(3 \times 3) = 3^{10} = 3^{2 \times 5}$$

$$3^4 \times 2^4 = (3 \times 3 \times 3 \times 3)(2 \times 2 \times 2 \times 2)$$

$$= (3 \times 2)(3 \times 2)(3 \times 2)(3 \times 2) = (3 \times 2)^4$$

$$\frac{3^5}{2^5} = \frac{3 \times 3 \times 3 \times 3 \times 3}{2 \times 2 \times 2 \times 2 \times 2} = \left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^5$$

Emphasize the way the exponents are *counting* the factors and discuss the properties of multiplication and of fractions which are used along the way. (For example, the associative and commutative properties of multiplication and $\frac{a \times b}{c \times d} = \frac{a}{c} \times \frac{b}{d}$ and $\frac{a}{a} = 1$.)

2. $a^0 = 1$. The definition was chosen so that $\frac{a^5}{a^5} = a^{5-5}$ will be true.

$a^{-3} = \frac{1}{a^3}$. The definition was chosen so that $\frac{a^5}{a^8} = a^{5-8}$ will be true.

$a^{\frac{1}{4}} = \sqrt[4]{a}$. The definition was chosen so that $\left(a^{\frac{1}{4}}\right)^1 = a^1$ will be true.

$a^{\frac{3}{4}} = \sqrt[4]{a^3} = \left(\sqrt[4]{a}\right)^3$. The definition was chosen so that $\left(a^3\right)^{\frac{1}{4}} = \left(a^{\frac{1}{4}}\right)^3 = a^{\frac{3}{4}}$ will be true.

$a^{0.75} = a^{\frac{3}{4}}$.

3. a. $\left(a^{\frac{1}{3}} \times b^{\frac{1}{3}}\right)^3 = \left(a^{\frac{1}{3}} \times b^{\frac{1}{3}}\right)\left(a^{\frac{1}{3}} \times b^{\frac{1}{3}}\right)\left(a^{\frac{1}{3}} \times b^{\frac{1}{3}}\right)$ Definition of exponent 3.

$= \left(a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}}\right)\left(b^{\frac{1}{3}} \times b^{\frac{1}{3}} \times b^{\frac{1}{3}}\right)$ Associative and commutative properties of multiplication.

$= a \times b$ Definition of exponent $\frac{1}{3}$.

b. By the definition of cube root, since $\left(a^{\frac{1}{3}} \times b^{\frac{1}{3}}\right)^3 = a \times b$ then $a^{\frac{1}{3}} \times b^{\frac{1}{3}}$ must be $\sqrt[3]{a \times b}$.

4. We showed that $\sqrt{3}$ is squeezed between the increasing sequence

$$1 < 1{\cdot}7 < 1{\cdot}73 < 1{\cdot}732 < 1{\cdot}7321 < \ldots$$

and the decreasing sequence

$$2 > 1{\cdot}8 > 1{\cdot}74 > 1{\cdot}733 > 1{\cdot}7322 > \ldots$$

We showed that as x increases, a^x increases. We saw that powers with rational exponents are defined and that the sequence

$2^1 < 2^{1{\cdot}7} < 2^{1{\cdot}73} < 2^{1{\cdot}732} < 2^{1{\cdot}7321} < \ldots$ is increasing and the sequence

$2^2 > 2^{1{\cdot}8} > 2^{1{\cdot}74} > 2^{1{\cdot}733} > 2^{1{\cdot}7322} > \ldots$ is decreasing and their values are squeezing down on some number. Since the values of x are squeezing down on $\sqrt{3}$, we give the name $2^{\sqrt{3}}$ to the number which the corresponding values of 2^x are squeezing down on.

3-2 THE EXPONENTIAL FUNCTION DEFINED BY 2^x.

We use what we have learnt about exponents to help define an *exponential function* and we explore its properties.

Answers to
CLASS ACTIVITY

Student Text Page 98

1. a. No b. No c. Yes

2. a. No b. No c. No d. No e. No

3. Yes

4. Range = $\{y: y > 0\}$

5. a. If $x < 0$, $\{y: 0 < y < 1\}$.
   If $x > 0$, $\{y: y > 1\}$.
   If $x > 1$, $\{y: y > 2\}$.
   If $x > 2$, $\{y: y > 4\}$.

   b. If $f(x) < 1$, $\{x: x < 0\}$.
   If $f(x) > 1$, $\{x: x > 0\}$.
   If $f(x) > 2$, $\{x: x > 1\}$.

Point out what we are doing. We have observed informally the domain and range of the function and the fact that it is continuous. We next explore how the graph of 2^x behaves for extreme values of x.

Answers to
CLASS ACTIVITY

Student Text Page 100

1. The graph approaches the x-axis as an asymptote.
2. $y = 0$.
3. The graph rises to the right without limit.

Finally we prove that the function defined by 2^x is *increasing*.

Answers to
CLASS ACTIVITY

Student Text Page 100

1. Yes in all cases.
2. a. Addition property of order. (Added $-2\sqrt{3}$ to both sides.)
   b. If $x > 0$, then $2^x > 1$. (See Problem 5a in *CLASS ACTIVITY*, page 98)
   c. A law of exponents. $2^{a-b} = \frac{2^a}{2^b}$
   d. Multiplication property of order.
3. See Problem 2 above.

3-3 THE EXPONENTIAL FUNCTION DEFINED BY 2^{-x}.

Answers to
CLASS ACTIVITY

Student Text Page 101

1. By definition, $2^{-1} = \frac{1}{2^1} = \frac{1}{2}$

2. $2^{-(-3)} = 2^3 = 8$

3. When x is positive, $(-x)$ is negative, so 2^{-x} has a negative exponent and is therefore less than 1, as we saw in Section 3-2.

4. When x is negative, $(-x)$ is positive, so 2^{-x} has a positive exponent and is therefore greater than 1.

5. Decreases.

6. 2^{-x} approaches zero.

7. 2^{-x} increases without limit.

8. 3, -3

9. 4, -4; -1, 1; -2, 2; 2, -2; 0, 0.

The fact that the graphs of $y = 2^x$ and $y = 2^{-x}$ are symmetrical to each other with respect to the y-axis is observed. For example, the point (5, 32) is on the graph of y^x and (-5, 32) is on the graph of y^{-x}.

Encourage the pupils to draw the graph of $y = 2^{-x}$, making use of the various properties which have been developed in this section, before he looks at the graph on page 104.

3-4 THE EXPONENTIAL FUNCTION DEFINED BY a^x FOR ANY POSITIVE NUMBER a.

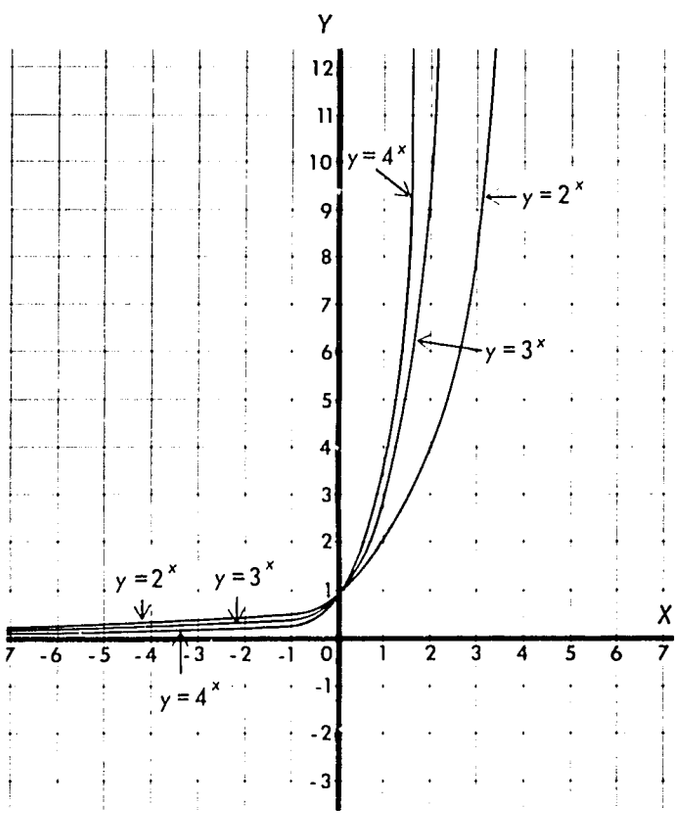
The discussion in the preceding sections has been limited to exponential functions with the base 2. Exponential functions with other bases can be developed *from* our base two functions and this is what we shall do now.

1. $4^x = (2^2)^x = 2^{2x}$

2.

x	-3	-2	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{3}{2}$	2	3
$2x$	-6	-4	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	3	4	6
2^{2x}	0·02	0·06	0·25	0·35	0·5	0·7	1·0	1·4	2·0	2·8	4	8	16	64
4^x	0·02	0·06	0·25	0·35	0·5	0·7	1·0	1·4	2·0	2·8	4	8	16	64

3.
4.
8.



5. Alike: The graphs are both increasing, both go through the point (0,1), both are continuous, both have for domain the set of all real numbers and for range the set of positive real numbers, both have the x-axis as an asymptote to the left and increase without limit to the right.

Different: The graph of $y = 4^x$ is above the graph of $y = 2^x$ to the right and below to the left. On the right the graph of $y = 4^x$ rises more rapidly than the graph of $y = 2^x$.

6. If $2^{1\cdot6} \approx 3$, then $(2^{1\cdot6})^x \approx 3^x$. A law of exponents tells us that $(2^{1\cdot6})^x = 2^{1\cdot6x}$. Therefore $3^x \approx 2^{1\cdot6x}$ and we can find information about 3^x from what we know about the base-two exponential function.

7.

x	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$
$1\cdot6x$	-4·0	-3·2	-2·4	-1·6	-0·8	0	0·8	1·6	2·4	3·2	4·0
$2^{1\cdot6x}$	0·06	0·11	0·19	0·33	0·57	1·00	1·74	3·0	5·3	9·2	16
3^x	0·06	0·11	0·19	0·33	0·57	1·00	1·74	3·0	5·3	9·2	16

8. See graph above.

9. See Problem 5. The likenesses are the same. To the right and to the left of $x = 0$ the graph of $y = 3^x$ is *between* the graphs of $y = 2^x$ and $y = 4^x$.

10. Yes.

Answers to
PROBLEMS 3-4A

Student Text Page 105

1. If $2^x = 0\cdot75$, then $x \approx -0\cdot4$ If $2^x = 1\cdot5$, then $x \approx 0\cdot6$
If $3^x = 0\cdot75$, then $x \approx -0\cdot3$ If $3^x = 1\cdot5$, then $x \approx 0\cdot4$
If $4^x = 0\cdot75$, then $x \approx -0\cdot2$ If $4^x = 1\cdot5$, then $x \approx 0\cdot3$
If $2^x = 2\cdot5$, then $x \approx 1\cdot3$
If $3^x = 2\cdot5$, then $x \approx 0\cdot8$
If $4^x = 2\cdot5$, then $x \approx 0\cdot7$

2. If $a^x = b^x$, then $\frac{a^x}{b^x} = 1$ and $\left(\frac{a}{b}\right)^x = 1$. Hence the truth set is $\{0\}$.

3. If $y = 1^x$, then $y = 1$ for all values of x and the graph is the straight line $y = 1$. The function is a constant function and is neither increasing nor decreasing.

4. *CHALLENGE PROBLEM*

If $a > b > 1$ and $x > 0$, then

$\frac{a}{b} > 1$	Multiplication property of order.
$\left(\frac{a}{b}\right)^x > 1$	Property 6 on page 103.
$\frac{a^x}{b^x} > 1$	A property of exponents.
$a^x > b^x$	Multiplication property of order.

Similarly, when $x < 0$, we obtain by Property 6 that $a^x < b^x$.

Answers to
CLASS ACTIVITY

Student Text Page 105

1. $2^{-x} = \frac{1}{2^x} = \frac{1^x}{2^x} = \left(\frac{1}{2}\right)^x$

   or $2^{-x} = 2^{(-1)x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x$

2. $\left(\frac{1}{4}\right)^x = \left[\left(\frac{1}{2}\right)^2\right]^x = \left(\frac{1}{2}\right)^{2x}$

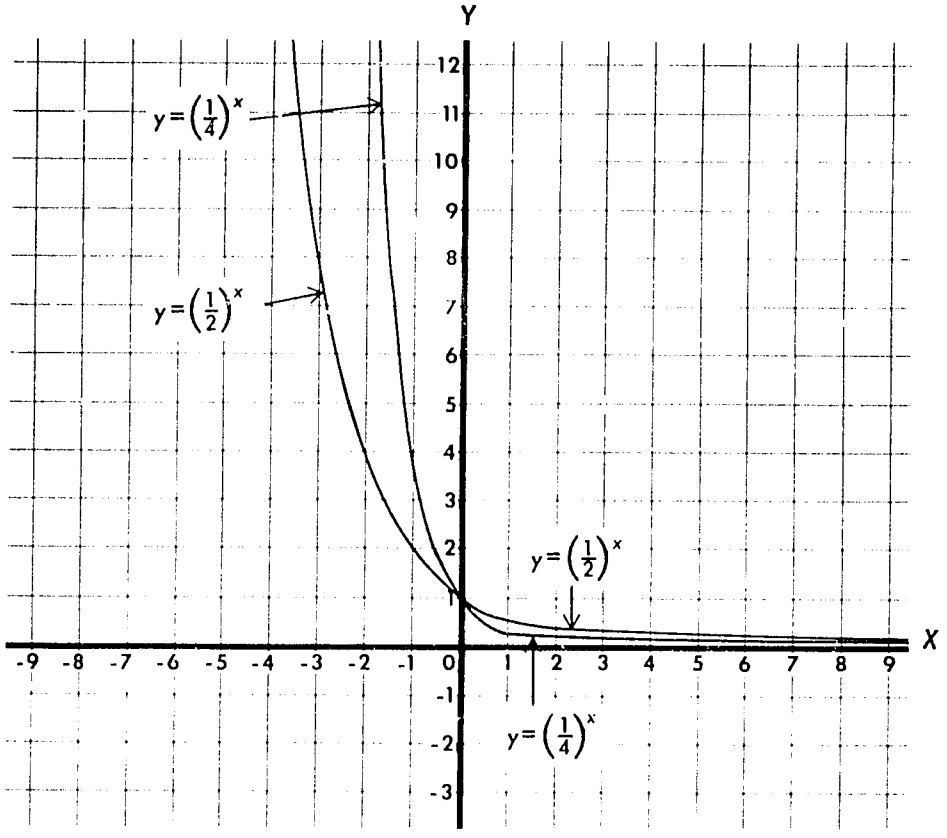
   $\left(\frac{1}{4}\right)^x = \left(\frac{1}{2^2}\right)^x = (2^{-2})^x = 2^{-2x}$

3. Corresponding to any real number x, we can find the value of 2^{-2x} from the graph of $y = 2^{-x}$. (For example if $x = 0{\cdot}8$, we could find the value of $2^{-2(0{\cdot}8)}$ by looking for $2^{-1{\cdot}6}$ on the graph of $y = 2^{-x}$.) Since $2^{-2x} = \left(\frac{1}{4}\right)^x$, we then have a value of $\left(\frac{1}{4}\right)^x$ corresponding to the given number x.

4.

x	-3	-2	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	2	3
$2x$	-6	-4	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	4	6
$-2x$	6	4	2	$\frac{3}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2	-4	-6
2^{-2x}	64	16	4	2·8	2·0	1·4	1·0	0·7	0·5	0·35	0·25	0·06	0·02
$\left(\frac{1}{4}\right)^x$	64	16	4	2·8	2·0	1·4	1·0	0·7	0·5	0·35	0·25	0·06	0·02

5.



7. For any real number x we can find $-0{\cdot}42x$ and from that can find $2^{-0{\cdot}42x}$ from the graph of $y = 2^{-x}$.

8. The graph of $y = \left(\frac{3}{4}\right)^x$ will be below the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = \left(\frac{1}{4}\right)^x$ on the left and above them on the right.

The graph of $y = \left(\frac{1}{10}\right)^x$ will be above all three previous graphs on the left and below them on the right.

The graph of $y = (0{\cdot}6)^x$ will be between the graphs of $y = \left(\frac{1}{2}\right)^x$ and $y = \left(\frac{3}{4}\right)^x$.

Answers to
PROBLEMS 3-4B

Student Text Page 107

1. $\frac{1}{2}^{-1{\cdot}4} \approx 2{\cdot}6$ $\quad \frac{1}{2}^{0{\cdot}6} \approx 0{\cdot}66$ $\quad \frac{1}{2}^{2{\cdot}5} \approx 0{\cdot}18$

$\frac{1}{4}^{-1{\cdot}4} \approx 7{\cdot}0$ $\quad \frac{1}{4}^{0{\cdot}6} \approx 0{\cdot}44$ $\quad \frac{1}{4}^{2{\cdot}5} \approx 0{\cdot}03$

2. $2^{1\cdot4} \approx 2\cdot6$ $\quad 2^{-0\cdot6} \approx 0\cdot66$ $\quad 2^{-2\cdot5} \approx 0\cdot18$

$4^{1\cdot4} \approx 7\cdot0$ $\quad 4^{-0\cdot6} \approx 0\cdot44$ $\quad 4^{-2\cdot5} \approx 0\cdot03$

3. *CHALLENGE PROBLEM*

If $0 < a < b < 1$ and

a. $x < 0$, then

$\frac{a}{b} < 1$	Multiplication property of order.
$\left(\frac{a}{b}\right)^x > 1$	Property 6 on page 106.
$\frac{a^x}{b^x} > 1$	A property of exponents.
$a^x > b^x$	Multiplication property of order.

b. Similarly, when $x > 0$, we obtain by Property 6 that $a^x < b^x$.

4. *CHALLENGE PROBLEM*

An exponential function is not a polynomial function.

If $f(x) = a^x$, then $f(x) \times f(x) = f(2x)$ since $a^x \times a^x = a^{2x}$ by a law of exponents (Law 1, page 97).

If, for example, $f(x)$ were the polynomial $5x^3 + 3x^2 + \ldots$, then

$f(x) \times f(x) = 25x^6 + 30x^5 + \ldots$ (of degree 6), whereas

$f(2x) = 5(2x)^3 + 3(2x)^2 + \ldots$

$= 40x^3 + 12x^2 + \ldots$ (of degree 3).

Therefore $f(x) \times f(x)$ could *not* be equal to $f(2x)$ (their degrees are different), so our assumption that $f(x)$ is a polynomial is false.

3-5 APPLICATIONS OF EXPONENTIAL FUNCTIONS.

We discuss briefly two of the many real life situations in which an exponential function is a mathematical model which gives man the ability to analyse and predict in such situations.

The bacteria colony is an example of population growth in general; compound interest is heavily used in business.

1. If there are A_t shillings in the investment at the end of t *half-yearly* periods, then $A_t = 100\left(1 + \frac{1}{100}\right)^t$.

   At the end of 2 years $t = 4$, $A_4 = 100(1{\cdot}01)^4 = 104{\cdot}06$
   At the end of $2\frac{1}{2}$ years $t = 5$. $A_5 = 100(1{\cdot}01)^5 = 105{\cdot}10$

2. At the end of 1 year the population will be about $600\left(1 + \frac{5}{100}\right)$.

   At the end of 2 years it will be about

$$600\left(1 + \frac{5}{100}\right) + 600\left(1 + \frac{5}{100}\right)\left(\frac{5}{100}\right) = 600\left(1 + \frac{5}{100}\right)^2.$$

   If the population at the end of t years is N_t, then

$$N_t = 600\left(1 + \frac{5}{100}\right)^t$$

   At the end of 5 years,

$$N_5 = 600\left(1 + \frac{5}{100}\right)^5 \approx 600(1{\cdot}276) \approx 765{\cdot}6$$

   The population is about 760 or 770 at the end of 5 years. This population increase behaves as compound interest does, except that in compound interest the total amount changes only at regular intervals, once a year or half year or quarter year, when the interest is added to the investment. In the population, changes in the total occur irregularly and more frequently as people are born or die or move into or out of the village.

3-6 LOGARITHMIC FUNCTIONS.

In *Secondary Three* we learned about logarithms. In Chapter 1 of *Secondary Four* we learned about inverse functions. We now combine these ideas to bring out the fact that exponential functions and logarithmic functions are inverses of each other. We emphasize the idea that the ordered pairs defined by $y = a^x$ and those defined by $y = \log_a x$ are the same except that the order has been reversed. The domain and range have changed places. This means that we can draw a great deal of information about the logarithmic functions from what we have already learned about the exponential functions.

Answers to
CLASS ACTIVITY

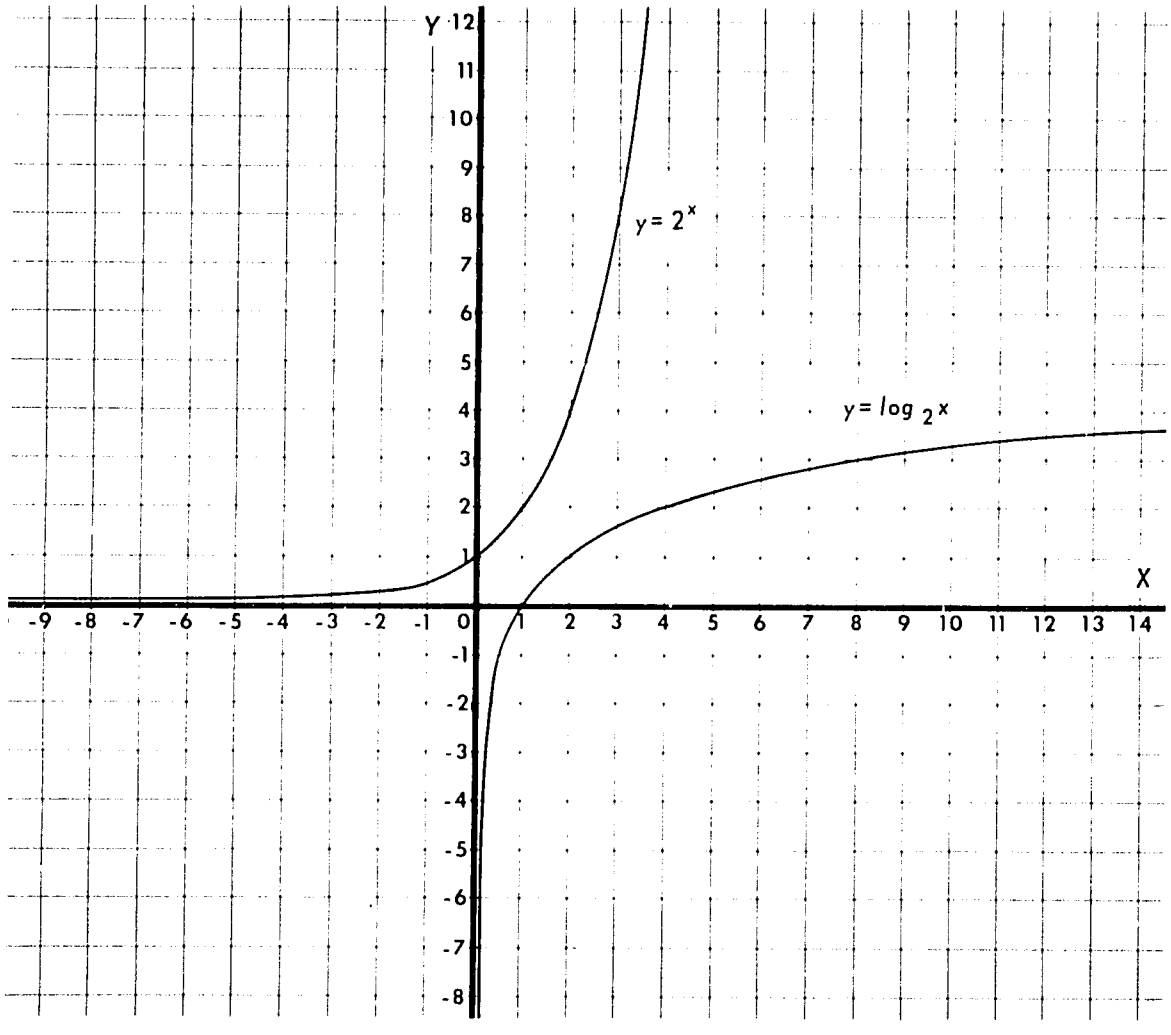
Student Text Page 110

1. a. 5 c. 3 e. 4
   b. -2 d. -3 f. $-\frac{2}{3}$

2. a. $2^y = 64$, $\log_2 64 = 6$
   b. $3^y = \frac{1}{27}$, $\log_3 \frac{1}{27} = -3$
   c. $10^y = 10$, $\log_{10} 10 = 1$
   d. $7^y = \sqrt[3]{49} = 7^{\frac{2}{3}}$, $\log_7 \sqrt[3]{49} = \frac{2}{3}$
   e. $\left(\frac{1}{2}\right)^y = \frac{1}{16}$, $\log_{\frac{1}{2}} \frac{1}{16} = 4$

Answers to
CLASS ACTIVITY

Student Text Page 111



1. $\{x : x > 1\}$

2. $\{x : 0 < x < 1\}$

3. No. The function is continuous.

4. $\log_2 a > \log_2 b$ whenever $a > b$. The function is increasing.

5. The y-axis is an asymptote of the graph of $y = \log_2 x$.

6. The graph of $y = \log_3 x$ can be obtained by reflecting the graph of $y = 3^x$ in the line $y = x$. The functions defined by $\log_3 x$ and $\log_4 x$ and $\log_2 x$ have many common properties.

7. See the summary on page 112 of the student text.

Answers to
PROBLEMS 3-6A

Student Text Page 112

1. a. $2^5 = 32$, $2^4 = 16$, $5 > 4$.

   b. $3^4 = 81$, $3^{\frac{4}{2}} = (\sqrt{3})^4$, $4 > \frac{4}{2}$

   c. $3^{-3} = \frac{1}{27}$, $3^{-2} = \frac{1}{9}$, $-3 < -2$

   d. $5^{-2} = \frac{1}{25}$, $5^3 = 125$, $-2 < 3$

   e. $9^{\frac{2}{3}} = \sqrt[5]{81}$, $9^{-\frac{1}{3}} = \frac{1}{\sqrt[3]{9}}$, $\frac{2}{5} < -\frac{1}{3}$

   f. $10^{-3} = \frac{1}{1000}$, $10^3 = 1000$, $-3 < 3$

2. *CHALLENGE PROBLEM*

   Let $\log_a x = p$ and $\log_a y = q$.

   Then $a^p = x$ and $a^q = y$.

   Since $\log_a x > \log_a y$, $p > q$.

   Since a^x is *increasing* when $a > 1$, $a^p > a^q$.

   Therefore $x > y$ as was to be proved.

Answers to
CLASS ACTIVITY

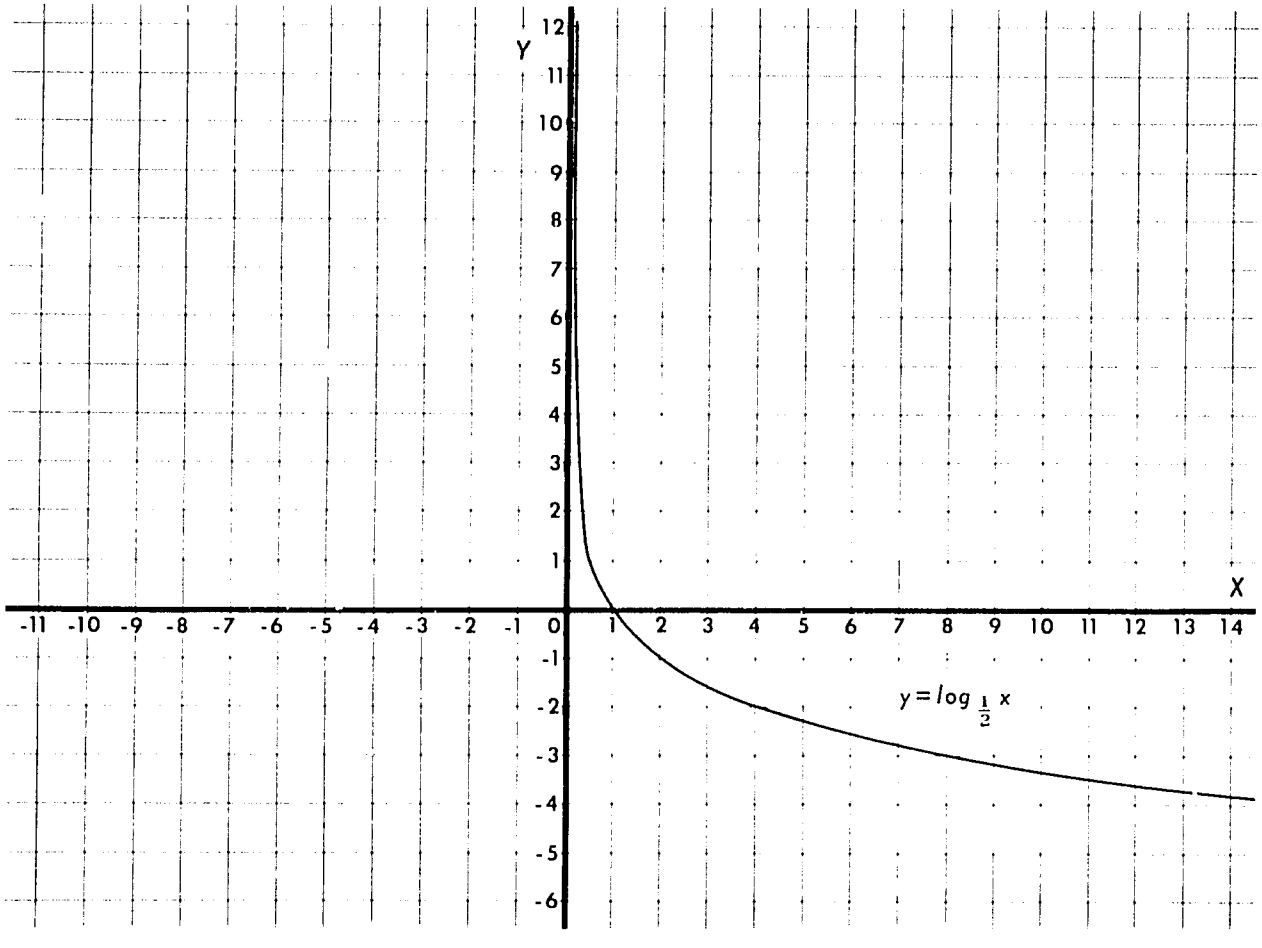
Student Text Page 113

1. $y = 2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x$. The inverse function therefore is defined by $y = \log_{\frac{1}{2}} x$.

2.

x	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	$y = \log_{\frac{1}{2}} x$
$y = 2^{-x}$	64	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	x

3.



4. a. Decreasing.
   b. $\{x : 0 < x < 1\}$
   c. $\{x : x > 1\}$

5. a. 4, 6, -5. b. -4, -6, 5.

6. a. 2, 4, 2, 2. b. -2, -4, -2, -2.

7. The graphs of the functions defined by $\log_{\frac{1}{2}} x$ and $\log_2 x$ are mirror images of each other in the x-axis.

Answers to
PROBLEMS 3-6B

Student Text Page 114

1. a. $\left(\frac{1}{2}\right)^{-5} = 32$, $\left(\frac{1}{2}\right)^{-4} = 16$, $-5 < -4$.

   b. $\left(\frac{1}{8}\right)^{-2} = 64$, $\left(\frac{1}{8}\right)^{2} = \frac{1}{64}$, $-2 < 2$.

   c. $\left(\frac{1}{3}\right)^{-\frac{2}{4}} = \left(\frac{1}{3}\right)^{-\frac{1}{2}} = \sqrt[4]{9}$, $\left(\frac{1}{3}\right)^{-3} = 27$, $-\frac{1}{2} > -3$.

   d. $\left(\frac{1}{10}\right)^{3} = \frac{1}{1000}$, $\left(\frac{1}{10}\right)^{4} = \frac{1}{10{,}000}$, $3 < 4$.

2. *CHALLENGE PROBLEM*

   Let $\log_{\frac{1}{a}} x = p$ and $\log_{\frac{1}{a}} y = q$.

   Then $\left(\frac{1}{a}\right)^{p} = x$ and $\left(\frac{1}{a}\right)^{q} = y$

   Since $x > y$, $\left(\frac{1}{a}\right)^{p} > \left(\frac{1}{a}\right)^{q}$.

   Since $a > 1$, $\frac{1}{a} < 1$ and $\left(\frac{1}{a}\right)^{x}$ is *decreasing*, that is, a decrease in $\left(\frac{1}{a}\right)^{x}$ corresponds to an increase in x.

   Therefore $p < q$, that is, $\log_{\frac{1}{a}} x < \log_{\frac{1}{a}} y$, as was to be proved.

3-7 FURTHER PROPERTIES OF LOGARITHMIC FUNCTIONS.

Very briefly we remind the pupils of the properties of logarithmic functions which make them useful in computational work, and we point out a few further interesting properties having to do with changes of base in logarithmic functions. This is only a hint of further study of logarithmic functions which can be done in a more advanced course. The hint is intended here to give the pupils a little appreciation of the nature of mathematics as a rich expanding body of ideas.

If you feel that it is important at this time to spend time on revision of computation by means of logarithms, you can give the class some work similar to that in Chapter 11, pages 198 to 220, in *Secondary Three*. Otherwise this section is intended just for class discussion.

Chapter 4
SEQUENCES AND SERIES

4-1 INTRODUCTION.

This chapter is essentially another chapter on functions. The distinctive feature of the functions studied in this chapter is that the domain is the set of *natural numbers*, that is, the set of positive integers.

While it is normally assumed that the domain is the set of all natural numbers, in some problems it is clear that the domain for all practical purposes is a finite subset of this set; for example, the sequence for which the general term is the number of ships entering the harbour at Mombasa on the nth day counting from January 1, 1966. No mention is made of this difficulty in the pupil's text. However if some pupil should raise the question, we might have distinguished between finite sequences with domain $\{1, 2, 3, \ldots, n\}$ in which there is a last term, and infinite sequences with domain $\{1, 2, 3, \ldots, n, \ldots\}$ in which there is no last term. This is a minor point and should not be mentioned unless the question arises in class.

This chapter is concerned with three principal topics:

(1) the general concept of a sequence

(2) the special cases of arithmetic and geometric progressions with the accompanying formulas for the nth term and the sum of its first n terms and their applications, and

(3) the notion of the limit of a sequence treated very informally in Section 4-6, and appearing again in the optional material of Section 4-7.

The discussion about counting and trying to find a function which gives the correct number for the general term offers the teacher an opportunity to point out the similarity between mathematics and the experimental sciences in which one gathers information and then tries to formulate a general theory which will include all the data collected. One can also point out the impor-

tance of furnishing a convincing argument for the conjectures (guesses) one proposes. The discussion is intended to serve as an introduction for sequences in general, but should be especially helpful in leading to the results obtained in the section on arithmetic progressions. In particular the story about Gauss is an attempt to show by an example the method for deriving the formula for the sum of the terms of an arithmetic progression.

Answers to
CLASSROOM ACTIVITY

Student Text Pages 117-118

1. (a) 5 (b) 9 (c) 19 (d) $2n - 1$
2. (a) 6 (b) 16 (c) 24 (d) $2n$
3. (a) first team (b) second team (c) 11
4. (a) 16 shillings (b) 24 shillings (c) 48 shillings (d) 112 shillings
5. (a) 3rd week (b) 6th week (c) 16th week
6. (a) 52 shillings (b) 136 shillings

Challenge Problems

(c) 580 shillings (d) 3172 shillings

All of these questions except 1(d) and 2(d) can be answered by simply counting or adding. However we would hope that some pupils would see patterns emerging and discover for themselves for instance that the nth term when counting by 2's beginning with 1 is $2n - 1$ or that in Problem 4 the worker's wages are obtained by counting by 2's beginning with 10 and that the nth term is $10 + (n-1)2$ or $8 + 2n$. These are of course special cases of the formula for the nth term of an arithmetic progression developed in Section 4-3. If some pupil discovers the formula, fine; if not, simply let the class count, add, and guess.

In Problem 6, parts (c) and (d) are too time consuming unless some ambitious pupil loves to do long sums or unless some one discovers his own formula for computing the sum. For example, he might, like Gauss, anticipate the method used in deriving the formula for the sum of the first n terms of an arithmetic progression.

1. (a) 10 (b) 19 (c) 28 (d) 58 (e) $1 + (n-1)3$ or $3n - 2$
   (f) 13 and 16. They are the fifth and sixth terms.

2. (a) 22 (b) 70 (c) 145

3. (a) 8 (b) 14 (c) 23 (d) 47 (e) $2 + (n-1)3$ or $3n - 1$

4. (a) 15 (b) 40 (c) 155
   CHALLENGE PROBLEM
   (d) $\frac{n}{2}\left[4 + (n-1)3\right]$ or $\frac{n}{2}(3n + 1)$

In Problem 4 probably only parts (a) and (b) should be assigned. If some pupils are interested and discover the pattern themselves, then encourage them to do (c) and (d) and to share their results with the class.

5. (a) 10 (b) 13 (c) 17 (d) $7 + (n-1)$ or $6 + n$ (e) 34 (f) 70
   (g) 132 *CHALLENGE PROBLEM* (h) $\frac{n}{2}\left[14 + (n-1)\right]$ or $\frac{n}{2}(n + 13)$

Parts (g) and (h) are probably too difficult unless someone thinks of using Gauss's method to solve them.

6. (a) 5 (b) 1 (c) 0 (d) -4

7. 4. The sixth term will be 19.

8. We could be counting by 2's, 3's, or 6's, but not by 5's. The difference between 17 and 5 is 12 and since this must be a multiple of the common difference d, we can be counting only by numbers which are divisors of 12; that is, by 1's, 2's, 3's, 4's, 6's, or 12's.

4-2 SEQUENCES.

The formal definition of a sequence should be followed immediately in class with many examples of sequences. Let pupils select any function (perhaps ones which they have encountered in previous chapters) and ask them to calculate in order $f(1), f(2), f(3), \ldots$. Write the rule defining the function on the board and list the values of the function in order. Give the pupils a great deal of practice using the new notation.

For example if the function is $f = \{(x, x^2)\}$ write

$$f(x) = x^2$$
$$f_1 = 1^2 = 1$$
$$f_2 = 2^2 = 4$$
$$f_3 = 3^2 = 9$$
$$f_4 = 4^2 = 16$$
$$f_5 = 5^2 = 25$$
$$\vdots$$
$$f_n = n^2$$

Notice that we may write the sequence in the form $\{1, 4, 9, 16, 25, \ldots, n^2, \ldots\}$.

Other functions which might be used now with the domain restricted to the natural numbers, are:

1) linear functions, defined by $f(x) = mx + b$
2) polynomial functions
3) rational functions
4) the exponential and logarithm functions.

You might also encourage pupils to think of non-mathematical sequences such as the nth person standing in line to enter a movie, or the temperature in the schoolroom at 8 a.m. on the nth day of the year, or the nth name in the telephone directory, etc.

The point of the discussion up to PROBLEMS 4-2A is to show that all the problems we considered in Section 4-1 can be solved if we can find a function whose values are those we used in making the various lists.

The function w requested for the worker's wages is given by $w(n) = 10 + (n - 1)\,2 = 8 + 2n$.

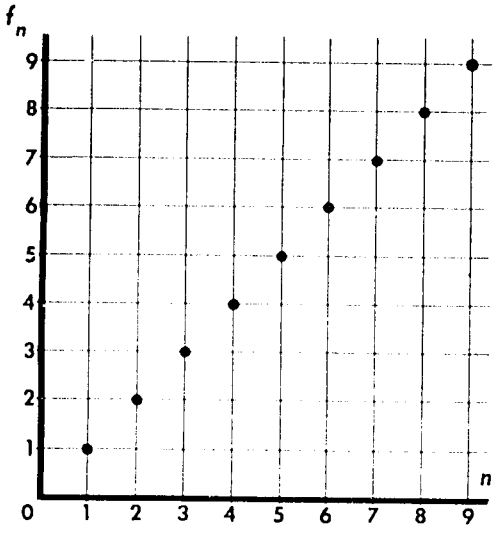
There may be more than one way to state a rule defining the same function; however the function $w(n)$ was devised to show that although the first few values of a formula may give correct results we need something more than a guess to be sure that the function we propose really solves the problem for *all* values of n.

The graphs requested in PROBLEMS 4-2A are to give practice in drawing graphs and to see how graphs of sequences differ from the graphs of the corresponding functions with the real numbers as their domain. The point is that the graph of a sequence is always a subset of the graph of the corresponding function with domain the set of real numbers.

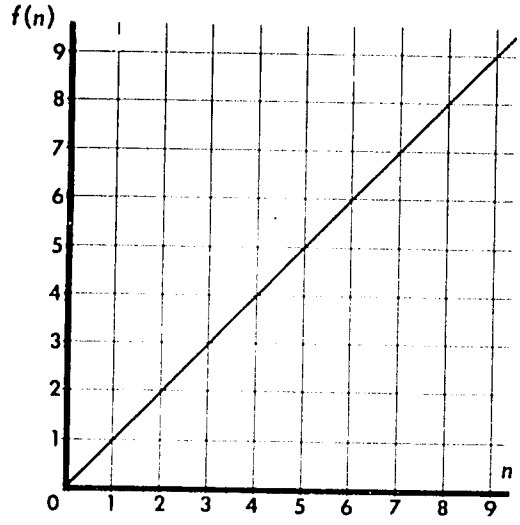
Answers to
PROBLEMS 4-2A

Student Text Pages 122-123

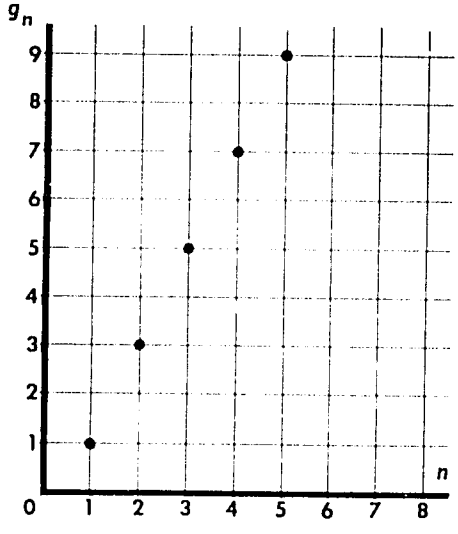
1. $f = \{(n, f_n) : f_n = n\}$

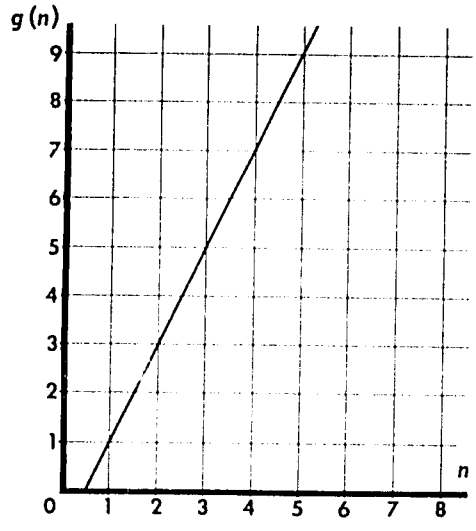


2.

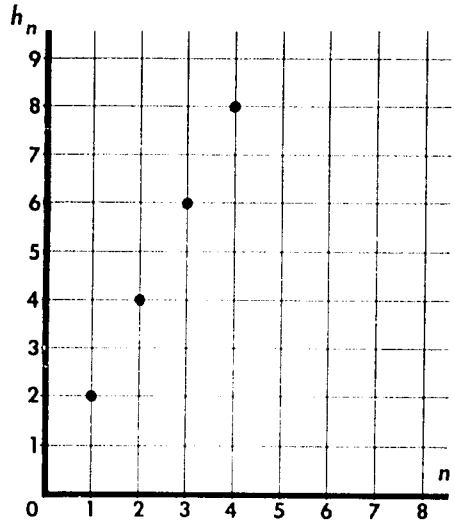


$g = \{(n, g\ ) : g_n = 2n - 1\}$

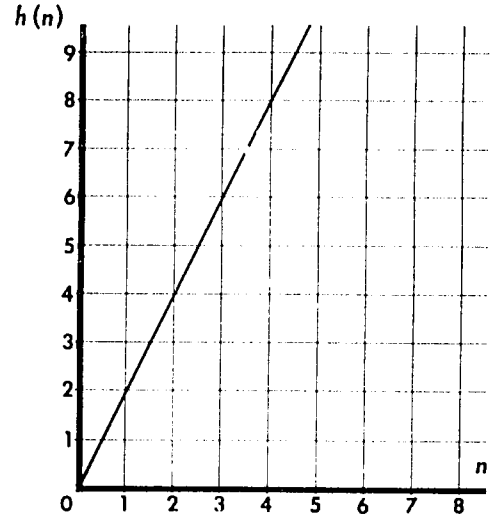




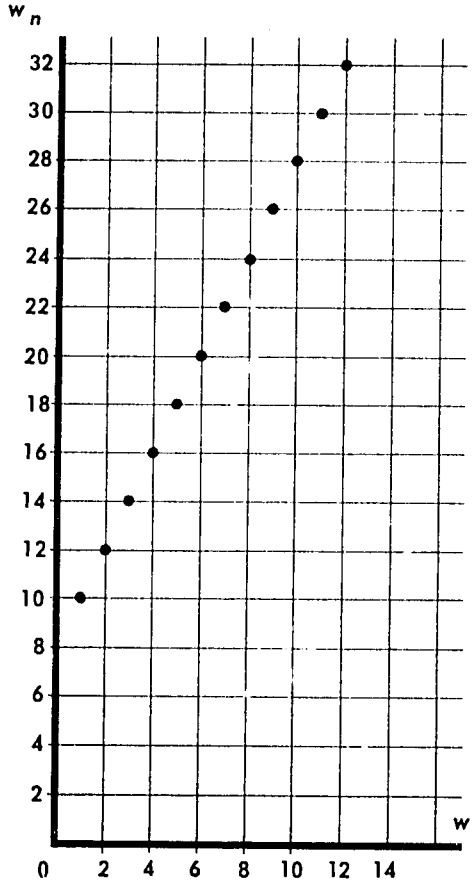
1. $h = \{(n, h_n) : h_n = 2n\}$

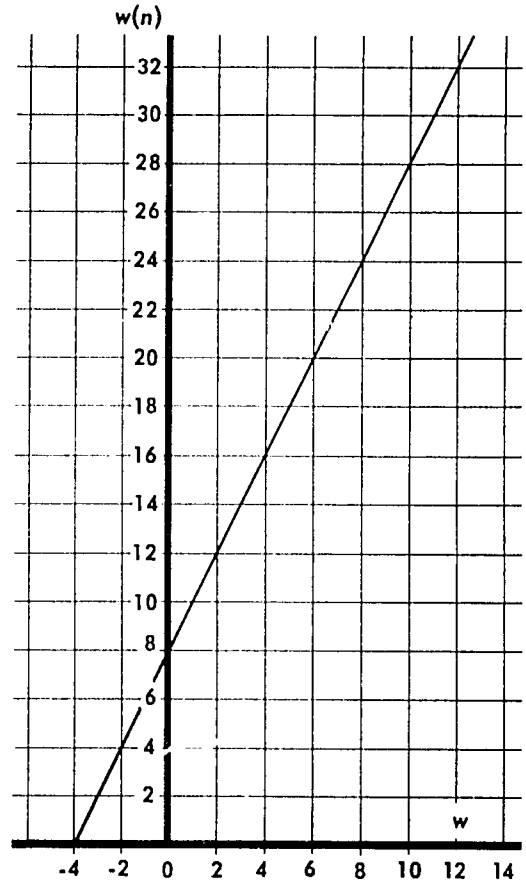


2.



$w = \{(n, 8 + 2n)\}$





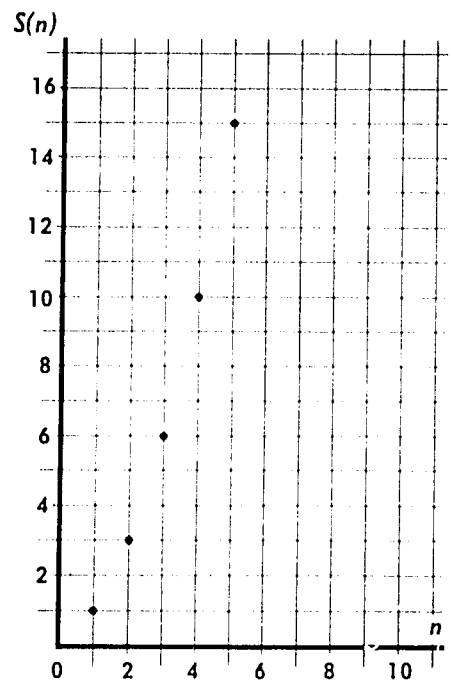
1. $\{1, 6, 11, 16, \ldots, 5n - 4, \ldots\}$ or $f(n, 5n - 4)$ or the sequence defined by $f_n = 5n - 4$. We obtained $5n - 4$ from $1 + (n - 1)5$.
   $\{3, 8, 13, 18, \ldots, 5n - 2, \ldots\}$ or $f(n, 5n - 2)$ or the sequence defined by $f_n = 5n - 2$. We obtained $5n - 2$ from $3 + (n - 1)5$.

2. $S(1) = 1$
   $S(2) = 1 + 2 = 3$
   $S(3) = 1 + 2 + 3 = 6$
   $S(4) = 1 + 2 + 3 + 4 = 10$
   $S(5) = 1 + 2 + 3 + 4 + 5 = 15$

3. *CHALLENGE PROBLEM*

$$\begin{aligned} S(n) &= 1 + 2 + 3 + \ldots + (n - 1) + n \\ S(n) &= n + (n - 1) + (n - 2) + \ldots + 2 + 1 \\ \hline 2S(n) &= (n + 1) + (n + 1) + (n + 1) + \ldots + (n + 1) + (n + 1) \\ &= n(n + 1) \\ S(n) &= \frac{n(n + 1)}{2} \end{aligned}$$

n	$S(n)$
1	1
2	3
3	6
4	10
5	15



No, the points do not lie on a straight line.

1.	Sequence?	First 5 terms
	(a) yes	5, 8, 11, 14, 17
	(b) yes	6, 18, 54, 162, 486
	(c) yes	5, 11, 29, 83, 245
	(d) no	$2f_1^2 = 18$, $f_1 = 3$ or -3 f_1 is not uniquely determined. Defines a relation but not a function.
	(e) yes	1, 3, 1, 3, 1
	(f) yes	3, 3, 3, 3, 3,
	(g) no	$f_1 = 2$ or -2. Defines a relation but not a function.
	(h) yes	1, 2, 4, 8, 16
	(i) no	Cannot find f_2 or f_n for > 1.
	(j) yes	1, 2, 4, 7, 11
	(k) yes	2, 3, 5, 7, 11
	(l) yes	1, 2, 4, 6, 10
	(m) yes	1, 2, 4, 7, 12
	(n) yes (o) yes	Although each of these defines a function, we do not have the information which would give us the terms of the sequences.
	CHALLENGE PROBLEM	
	(p) yes	1, 1, 2, 3, 5

2. In this problem there is no *one* answer. Any rule which gives the first four values correctly is acceptable. It will be instructive to find how many different sequences pupils can define which begin in the same way. At least one rule defining a function is given below for each part, but the teacher is urged to examine each proposed answer and accept any one which gives the correct first four terms. Also it does not matter how informal the statement of the rule defining the sequence may be if values for the sequence can be determined.

(a) $f_n = 2n - 1$; the sequence of odd natural numbers.

(b) $f_n = n^2$; the sequence of squares of the natural numbers.

(c) $f_n = (-1)^{n-1}$; alternately 1 and -1; odd terms are 1 and the even terms are -1.

(d) $f_n = 10 \times 4^{n-1}$: first term is 10 and any succeeding term is 4 times the preceding one.

(e) $f_n = \frac{1}{n(n+1)}$; fractions with numerator 1 and denominator the product of the number of the term and the next larger natural number.

(f) $f_n = \frac{1}{1 \times 2 \times \ldots \times n}$; fraction with numerator 1 and denominator the product of all the natural numbers up through the number of the term.

(g) $f_n = 2^{n-1} - 1$; one less than the power of two which is one less than the number of the term.

(h) $f_n = 2\left(\frac{1}{2}\right)^{n-1}$; first term is 2 and every term after that is $\frac{1}{2}$ the preceding term.

(i) $f_n = \frac{3n}{3n+1}$; fractions in which we count by 3's beginning with 3 to get the numerator, and the denominator is 1 greater than the numerator.

(j) f_n is the nth prime number.

(k) $f_1 = f_2 = 1, f_{n+2} = f_{n+1} + f_n$; first 2 terms are 1 and add two consecutive terms to get the next one.

(l) $f_n = -\frac{4^{n-1}+1}{3^n}$; negative fractions with denominators powers of 3 and numerators 1 more than powers of 4 beginning with the zeroth power of 4.

(m) $f_1 = 1$, $f_{n+1} = f_n + (n+1)$; first term 1 and after that the sum of the preceding term and the number of the term.

(n)
$$f_n = \begin{cases} \frac{2m+3}{m+1} & \text{for } n = 2m-1, \text{ i.e. } n \text{ is the } m^{\text{th}} \text{ odd number} \\ \frac{2m+1}{m+1} & \text{for } n = 2m, \text{ i.e. } n \text{ is the } m^{\text{th}} \text{ even number} \end{cases};$$

or for *odd* terms fractions with odd numbers in the numerator beginning with 5 and denominators natural numbers beginning with 2, for *even* terms fractions with odd numbers in the numerator beginning with 3 and natural numbers in the denominator beginning with 2.

3. (a) 1, 3, 5, 7, 9, 11, 13

(b) 1, 4, 9, 16, 25, 36, 49

(c) 1, -1, 1, -1, 1, -1, 1

(d) 10, 40, 160, 640, 2560, 10240, 40960

(e) $\frac{1}{1 \times 2}, \frac{1}{2 \times 3}, \frac{1}{3 \times 4}, \frac{1}{4 \times 5}, \frac{1}{5 \times 6}, \frac{1}{6 \times 7}, \frac{1}{7 \times 8}$

(f) $1, \frac{1}{1 \times 2}, \frac{1}{1 \times 2 \times 3}, \frac{1}{1 \times 2 \times 3 \times 4}, \frac{1}{1 \times 2 \times 3 \times 4 \times 5},$

$\frac{1}{1 \times 2 \times 3 \times 4 \times 5 \times 6}, \frac{1}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}$

(g) 0, 1, 3, 7, 15, 31, 63

(h) $2, 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$

(i) $\frac{3}{4}, \frac{6}{7}, \frac{9}{10}, \frac{12}{13}, \frac{15}{16}, \frac{18}{19}, \frac{21}{22}$

(j) 2, 3, 5, 7, 11, 13, 17

(k) 1, 1, 2, 3, 5, 8, 13

(l) $-\frac{2}{3}, -\frac{5}{9}, -\frac{17}{27}, -\frac{65}{81}, -\frac{257}{243}, -\frac{1025}{729}, -\frac{4097}{2187}$

(m) 1, 3, 6, 10, 15, 21, 28

(n) $\frac{5}{2}, \frac{3}{2}, \frac{7}{3}, \frac{5}{3}, \frac{9}{4}, \frac{7}{4}, \frac{11}{5}$

4. *CHALLENGE PROBLEM*

(a) 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144

(b) $n = 1 \quad \frac{1 + \sqrt{5} - (1 - \sqrt{5})}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1$

$$n = 2 \quad \frac{(1 + \sqrt{5})^2 - (1 - \sqrt{5})^2}{2^2\sqrt{5}} = \frac{1 + 2\sqrt{5} + 5 - (1 - 2\sqrt{5} + 5)}{4\sqrt{5}} =$$

$$\frac{4\sqrt{5}}{4\sqrt{5}} = 1$$

$$n = 3 \quad \frac{(1 + \sqrt{5})^3 - (1 - \sqrt{5})^3}{2^3\sqrt{5}} =$$

$$\frac{1 + 3\sqrt{5} + 3 \times 5 + 5\sqrt{5} - (1 - 3\sqrt{5} + 3 \times 5 - 5\sqrt{5})}{8\sqrt{5}} =$$

$$\frac{6\sqrt{5} + 10\sqrt{5}}{8\sqrt{5}} = \frac{16\sqrt{5}}{8\sqrt{5}} = 2$$

4-3 ARITHMETIC PROGRESSIONS.

The aim of this section is to develop and apply the formulas for the general term and the sum of the first n terms of an arithmetic progression.

The pupils should work enough problems so that they understand what makes a sequence an arithmetic progression.

PROBLEMS 4-3A are to be done from the definition of an arithmetic progression: i.e. $A_{n+1} = A_n + d$. As yet we have developed no general formula for A_n in terms of n and d. It is assumed that the pupil will be able to guess from looking at the first few terms how the sequence continues. If he does not observe a pattern, then he will have to work out for himself that the tenth term is obtained by starting with the first term and adding the common difference 9 times. If some pupils have difficulty getting the nth term don't push them. The formula will be that much more impressive after we obtain it.

Answers to
PROBLEMS 4-3A

Student Text Page 126

1. 3, 8, 13, 18, 23. $A_n = 3 + (n-1)5$ or $5n - 2$.

2. 5, 12, 19, 26, 33. $A_n = 5 + (n-1)7$ or $7n - 2$.

3. $A_1 = 3$, $d = 4$, $A_{10} = 3 + 9 \times 4 = 39$, $A_n = 3 + (n - 1)4 = 4n - 1$.

4. (a) $A_3 = 9$, $d = 2$, $A_2 = 9 - 2 = 7$ (c) $A_{200} = 5 + 199 \times 2 = 403$
   (b) $A_1 = A_2 - d = 7 - 2 = 5$ (d) $A_n = 5 + (n - 1)2 = 2n + 3$

5. $A_4 = 47$, $A_5 = 52$, $A_5 = A_4 + d$
   $52 = 47 + d$
   $5 = d$

   $A_4 = A_1 + 3d$ $\quad A_{50} = A_1 + 49 \times d$
   $47 = A_1 + 3 \times 5$ $\quad A_{50} = 32 + 49 \times 5$
   $32 = A_1$ $\quad A_{50} = 277$

The derivation given in the text of the formula $A_n = A_1 + (n - 1)d$ is not a proof but simply an informal argument to convince pupils that the formula is true or at least seems plausible. For a rigorous proof we need the method of proof called mathematical induction. Such proofs are based on a property of the natural numbers which appears as an axiom in some formulations of

arithmetic. We have chosen to omit a discussion of mathematical induction in the pupils' text at this time because the informal arguments seem convincing, while the induction argument is very sophisticated and takes a good deal of effort to explain to pupils of this age. We feel that it should not be introduced until there is a clear need which cannot be met by a more informal argument and until the pupils are more mature. However the teacher should know that there is a way to prove these formulas using only the definition of an arithmetic progression and the induction property of the natural numbers. Accordingly we give a brief account of the method and its application to the proof of the formulas of this section. (Remember that "$k \in S$" means that k is an element of set S.)

> The Principle of Mathematical Induction is as follows:
> If S is any set of natural numbers which has the properties
> i) $1 \in S$
> ii) if $k \in S$, then $(k + 1) \in S$,
> then S is the set of *all* natural numbers.

This Principle expresses a property of the natural numbers which seems very reasonable and even obvious. Any set of natural numbers S having properties i) and ii) contains 1 by property i). By property ii) since $1 \in S$, so is 2. By property ii) again, since $2 \in S$ (by the previous step), $3 \in S$. Again by the same argument, since by the previous step $3 \in S$ by property ii) $4 \in S$, etc. A repetition of this argument $(n-1)$ times would show that n also belongs to S. This is the point at which we call on the Induction Principle to say that *all* natural numbers belong to S. We are unable to repeat an argument infinitely often. Although we could show that 10,000,001 belongs to S by repeating the argument 10,000,000 times, that is not quite the same as saying that all natural numbers belong to S. However since we could show that any particular natural number belongs to S, we might as well agree that all natural numbers do. This is the content of the Induction Principle. It states an intrinsic property of the natural numbers and is taken as an axiom or postulate.

We illustrate how this property of the natural numbers is used to prove propositions which are conjectured to be true for all natural numbers by proving that the formulas of Section 4-3 hold for all natural numbers. The proofs use only the definition of an arithmetic progression and the Principle of Mathematical Induction.

The plan of all such proofs is first to consider the set, S, of all natural numbers for which the proposition holds. We show that S has properties i) $1 \in S$ and ii) if $k \in S$, then $(k + 1) \in S$. We then call on the Induction Principle to assert that S is the set of *all* natural numbers; that is, the set of natural numbers for which the proposition is true, is the set of *all* natural numbers.

Consider the proposition: If $\{A_n\}$ is an arithmetic progression, $A_n = A_1 + (n - 1)d$ for all natural numbers n.

Let S be the set of natural numbers for which the formula $A_n = A_1 + (n-1)d$ is true; i.e. $S = \{n : A_n = A_1 + (n-1)d\}$.

Step 1. $1 \in S$ since
$$\begin{aligned} A_1 &= A_1 + (1 - 1)d \\ &= A_1 + 0 \times d \\ &= A_1 \end{aligned}$$

Step 2. If $k \in S$, then $A_k = A_1 + (k - 1)d$.
We know, however, that $A_{k+1} = A_k + d$ since $\{A_n\}$ is an arithmetic progression, so if $A_k = A_1 + (k - 1)d$, then

$$A_{k+1} = \big(A_1 + (k - 1)d\big) + d = A_1 + \big((k + 1) - 1\big)d.$$

However this is just the statement that $(k + 1) \in S$; i.e. $A_{k+1} = A_1 + \big((k + 1)-1\big)d$. Hence we have shown that if $k \in S$, then so is $k + 1$.

Step 3. Since we have shown that i) $1 \in S$
and ii) if $k \in S$ then $(k + 1) \in S$,
by the Principle of Mathematical Induction, S is the set of all natural numbers.

We recall that S is the set of natural numbers for which the formula $A_n = A_1 + (n - 1)d$ is true. Therefore $A_n = A_1 + (n - 1)d$ is true for all natural numbers n.

This kind of proof is a very powerful weapon in the mathematician's arsenal, but it is somewhat sophisticated and should not be used the first time to prove statements which seem obvious to pupils without such a fancy argument. There are many formulas whose validity is not nearly so obvious and in proving some such formulas mathematical induction is a most convincing and indeed indispensable tool. Most arguments which contain "..." or "and so on" could be replaced by a proof which depends on the Principle of Mathematical Induction.

As a second illustration we shall prove that the sum of the first n terms of an arithmetic progression is given by the expression $\frac{n}{2}\left[2A_1 + (n - 1)d\right]$.

The informal argument which is used in the pupil's text to derive the formula $A_1 + A_2 + \ldots + A_n = \frac{n}{2}\left[A_1 + A_n\right]$ depends on the "obviousness" of adding two expressions for S_n both of which contain "..." and getting $n(A_1 + A_n)$. We may avoid this bit of informality by replacing the argument with one which relies on the Principle of Mathematical Induction.

We now show that: The sum of the first n terms of an arithmetic progression is given by $\frac{n}{2}\left[2A_1 + (n - 1)d\right]$ for all natural numbers n.

Let S be the set of all natural numbers for which

$A_1 + A_2 + \ldots + A_n = \frac{n}{2}\left[2A_1 + (n - 1)d\right]$; i.e.

$S = \left\{n : A_1 + A_2 + \ldots + A_n = \frac{n}{2}\left[2A_1 + (n - 1)d\right]\right\}$.

Step 1. $1 \in S$ since $A_1 = \frac{1}{2}\left[2A_1 + (1 - 1)d\right]$

$$= \frac{1}{2}\left[2A_1 + 0 \times d\right]$$

$$= \frac{1}{2}(2A_1)$$

$$= A_1$$

Step 2. If $k \in S$, $A_1 + A_2 + \ldots + A_k = \frac{k}{2}\left[2A_1 + (k - 1)d\right]$

However $A_1 + A_2 + \ldots + A_k + A_{k+1} = (A_1 + A_2 + \ldots + A_k) + (A_1 + kd)$ Why?

But if $k \in S$, $A_1 + A_2 + \ldots + A_k + A_{k+1} = \frac{k}{2}\left[2A_1 + (k - 1)d\right] + A_1 + kd$

$$= k A_1 + \frac{k}{2}(k - 1)d + A_1 + kd$$

$$= (k + 1)A_1 + \left(\frac{k^2}{2} - \frac{k}{2} + k\right)d$$

$$= (k + 1)A_1 + \frac{k^2 + k}{2}d$$

$$= (k + 1)A_1 + \frac{(k + 1)k}{2}d$$

$$= \frac{k + 1}{2}\left[2A_1 + kd\right]$$

$$= \frac{k + 1}{2}\left[2A_1 + \left((k + 1) - 1\right)d\right]$$

This is simply the statement that $(k + 1) \in S$. We have proved that if $k \in S$, then $(k + 1) \in S$.

Step 3. Since i) $1 \in S$ and ii) if $k \in S$ then $(k + 1) \in S$, by the Principle of Mathematical Induction S is the set of *all* natural numbers. However S was the set of natural numbers for which $A_1 + A_2 + \ldots + A_n = \frac{n}{2}\left[2A_1 + (n - 1)d\right]$; hence this formula is true for all natural numbers n.

The formula $A_1 + A_2 + \ldots + A_n = \frac{n}{2}\left[A_1 + A_n\right]$ is now obtained by replacing $A_1 + (n - 1)d$ by A_n in the previous result.

Answers to
PROBLEMS 4-3B

Student Text Pages 130-132

1. If $A_1 = 100{,}000$ shillings, $d = 0.06 \times 100{,}000 = 6{,}000$ shillings, then A_{n+1} shillings is the amount accumulated by the end of the nth year.

$A_6 = 100{,}000 + 5(6{,}000) = 130{,}000$, the amount after 5 years.

$A_{11} = 160{,}000$, the amount after 10 years.

$A_{n+1} = 100{,}000 + n(6{,}000)$, the amount after n years.

To find how many years it will take to double his inheritance set

$$A_{n+1} = 200{,}000 = 100{,}000 + 6000n$$

$$n = \frac{100{,}000}{6000} = 16\frac{2}{3} \text{ years}$$

At the end of 17 years he will have accumulated 202,000 shillings and his inheritance will have doubled.

2. $A_3 = 13$, $A_{10} = 27$,

$$\begin{aligned} A_{10} &= A_1 + 9d = 27 \\ A_3 &= A_1 + 2d = 13 \\ \hline A_{10} - A_3 &= \quad 7d = 14 \\ d &= 2 \end{aligned}$$

$$A_1 = A_3 - 2d = 13 - 2 \times 2 = 9$$

$$A_{20} = 9 + 19 \times 2 = 47$$

$$A_{100} = 9 + 99 \times 2 = 207$$

$$S_{10} = \frac{10}{2}\left[2 \times 9 + 9 \times 2\right] = 180$$

3.
$$\begin{array}{rcccccccccc} S_n = & 1 & + & 2 & + & 3 & + \ldots + & (n-1) & + & n \\ S_n = & n & + & (n-1) & + & (n-2) & + \ldots + & 2 & + & 1 \\ \hline \end{array}$$

$$2S_n = (n+1) + (n+1) + (n+1) + \ldots + (n+1) + (n+1)$$
$$2S_n = n(n+1)$$
$$S_n = \frac{n}{2}(n+1)$$

Or just use the formula $S_n = n\left(\frac{A_1 + A_n}{2}\right)$ with $A_1 = 1$ and $A_n = n$.

4. $S_n = 1 + 3 + 5 + \ldots + (2n - 1)$
$$= \frac{n}{2}\left[1 + (2n - 1)\right] = n^2$$
$$S_n = 2 + 4 + 6 + \ldots + 2n$$
$$= \frac{n}{2}\left[2 + 2n\right] = n(n+1)$$

5. $A_1 = 100$, $d = -15$. $A_n = 100 + (n - 1) \times (-15)$
$$= 115 - 15n$$

The last possible halt occurs the last time $A_n > 0$.
$$A_n = 115 - 15_n > 0$$
$$115 > 15_n$$
$$7\frac{2}{3} > n$$

Hence n must be 7.
$A_7 = 100 + 6(-15) = 10$. The last possible hold occurs at launch minus 10 minutes.

6. $A_1 = 21$, $d = 3$, $A_n < 100$
$$A_n = 21 + (n - 1)3 < 100$$
$$n < 27\frac{1}{3}$$

There are 27 integers between 20 and 100 which are divisible by 3.

7. $A_1 = 17$, $d = 17$, $A_n = 17 + (n - 1)17 < 1000$
$$n < 58\frac{14}{17}$$

There are 58 integers between 10 and 1000 which are divisible by 17.

8. $\frac{a+b}{2} - a = \frac{a + b - 2a}{2} = \frac{b - a}{2}$

$b - \frac{a+b}{2} = \frac{2b - a - b}{2} = \frac{b - a}{2}$

Since these differences are equal, the numbers a, $\frac{a+b}{2}$, and b are in arithmetic progression by the definition.

9. (a) 15 (b) 0 (c) $18\frac{1}{2}$ (d) 2 (e) 35 (f) -2 (g) $m^2 + mn$

10. (a) $A_5 = 14,\ S_5 = 40$
(b) $d = 1,\ S_{10} = 135$
(c) $A_1 = 8,\ S_9 = 216$
(d) $d = 0,\ S_{10} = 20$
(e) $d = -3,\ S_1 = 5,\ S_2 = 7,\ S_3 = 6,\ S_4 = 2$

CHALLENGE PROBLEMS

(f) $d = \frac{16}{9}$. $S_{10} = \frac{10}{2}\,[2 \times 2 + 9 \times d] = 100$

$$20 + 45d = 100$$

$$d = \frac{80}{45} = \frac{16}{9}$$

(g) $A_2 = 17 = A_1 + d = 19 + d.$

$$-2 = d$$

The largest value of S_n will occur for the largest n for which A_n is positive. After that A_n will be negative and S_n will start decreasing.

$$A_n = A_1 + (n - 1)d$$
$$= 19 + (n - 1)(-2) > 0$$
$$21 > 2n$$
$$10\tfrac{1}{2} > n$$

$$A_{10} = 19 + 9(-2) = 1$$

$$S_{10} = \frac{10}{2}\left[2 \times 19 + 9(-2)\right] = 100 \qquad (S_9 = S_{11} = 99)$$

11. If 1965 is the first year, 1980 will be the 16th year.
$A_1 = 10,\ d = 3,\ A_{16} = 10 + 15 \times 3 = 55$ coconuts in 1980.
$S_{16} = 520$
$A_n = 40 = 10 + (n - 1)3,\ n = 11.$
In 1975 it will produce 40 coconuts.
Does $A_n = 60$ for any n? $60 = 10 + (n - 1)^3$, $(n - 1)3 = 50$ for no natural number n. No, if we mean calendar years.

12. $A_1 = 16,\ d = 32.$

80 feet in the third second.
144 feet in the fifth second.
304 feet in the tenth second.
1600 feet in the first ten seconds.
$16n^2$ feet in the first n seconds.

13. (a) The centre of gravity should be at $\frac{1+n}{2}$ if the weights are unit weights evenly spaced, because we'd expect the rod to balance at the mid-point.

(b) The formula gives the centre of gravity to be

$$\frac{1(1) + 1(2) + 1(3) + \ldots + 1(n)}{1 + 1 + 1 + \ldots + 1} = \frac{1 + 2 + 3 + \ldots + n}{n}.$$

By Problem 3 above, $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

Hence $\frac{1 + 2 + 3 + \ldots + n}{n} = \frac{n(n+1)}{2} \times \frac{1}{n} = \frac{n+1}{2}$

and we have agreement with the intuitive answer in part (a).

4-4 GEOMETRIC PROGRESSIONS.

The developments in this section parallel those in Section 4-3. Since the definitions of the two kinds of progressions are analogous with addition of a fixed number replaced by multiplication by a fixed number, we did not think it necessary to indulge in another long build-up to the formulas for geometric progressions.

It is worth noting that the graph of a geometric progression will lie, not on a straight line as was the case for an arithmetic progression, but on an exponential curve with base r; $G_n = G_1 r^{n-1}$.

Answers to
PROBLEMS 4-4A

Student Text Pages 134-135

1. (a) 4 (b) 1 (c) $\frac{1}{32}$

2. 25

3. $r = 3$ or -3; $G_1 = \frac{1}{3}$ or $-\frac{1}{3}$

4. $-\frac{1}{64}$

5. 62

A proof by mathematical induction can be given for the formulas for G_n and S_n as in Section 4-3. However it is hoped that the pupils will find the informal arguments convincing.

Answers to
PROBLEMS 4-4B

Student Text Pages 135-137

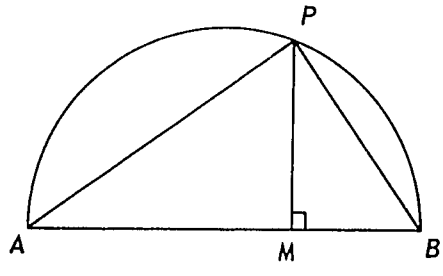
1. (a) 768 (b) 63 (c) 1,048,575

2. $G_3 = -4 = G_1 \times r^2 = -r^2$; $r^2 = 4$, $r = 2$ or -2
There are *two* such progressions: $G_n = -1(2)^{n-1}$ or
$G'_n = -1(-2)^{n-1}$
If $G_3 = 4$ and $G_1 = -1$, $4 = -r^2$ or $r^2 = -4$. There are no geometric progressions whose terms are real numbers with $G_1 = -1$ and $G_3 = 4$.

3. $G_2 = 4$; $A_2 = 5$. The arithmetic mean is larger than the geometric mean.

4. $r = -1$, $S_2 = 0$, $S_3 = 1$, $S_4 = 0$, $S_{100} = 0$, $S_{273} = 1$,
$$S_n = \begin{cases} 1 \text{ if } n \text{ is odd} \\ 0 \text{ if } n \text{ is even} \end{cases}$$

5.



ΔAPM is similar to ΔPBM.
Therefore $\frac{MA}{MP} = \frac{MP}{MB} = k$.

Thus $MA = k \times MP$ and $MP = k \times MB$. Hence MB, MP, and MA are in geometric progression by the definition.

CHALLENGE PROBLEMS

6. Arithmetic mean is $\frac{a+b}{2}$; the geometric mean is $\sqrt{ab}$.

$\sqrt{ab} \leq \frac{a+b}{2}$ will be true if each of the following is true.

$ab \leq \left(\frac{a+b}{2}\right)^2$

6. Cont.

$$\begin{aligned} ab &\le \frac{a^2 + 2ab + b^2}{4} \\ 4ab &\le a^2 + 2ab + b^2 \\ 0 &\le a^2 - 2ab + b^2 \\ 0 &\le (a - b)^2 \end{aligned}$$

Since the square of any real number is non-negative, reading the chain of steps backwards shows that the arithmetic mean of two positive numbers is always greater than or equal to the geometric mean.

7. $G_1 = 2,\ G_2 = 4,\ r = 2,\ G_n = 2^n,\ G_1 G_2 \ldots G_{100} = 2^1\, 2^2 \ldots 2^{100} =$

$2^{1+2+\ldots+100} = 2^{\frac{100}{2}(1+100)} = 2^{5050}$

8. (a) $CD = 50\sqrt{2},\quad DE = 50,\quad EF = 25\sqrt{2},\quad FG = 25.$

(b) $L_9 = \frac{25}{8}\sqrt{2},\ L_{10} = \frac{25}{8}.$ The numbers form a geometric progression $\{L_n\}$ with $L_1 = 50\sqrt{2}$ and $r = \frac{1}{\sqrt{2}}.$ Each term is $\frac{1}{\sqrt{2}}$ times the preceding term since it is the side of a 45° isosceles right-angled triangle whose hypotenuse is the preceding term.

(c) $$S_n = L_1 + L_2 + \ldots + L_n = \frac{50\sqrt{2}\left[\left(\frac{1}{\sqrt{2}}\right)^n - 1\right]}{\frac{1}{\sqrt{2}} - 1}$$

This can be simplified to $S_n = 100(1 + \sqrt{2})\left[1 - \left(\frac{1}{\sqrt{2}}\right)^n\right].$

For large values of n this is close to $100(1 + \sqrt{2})$ and for all values of n this is less than $100(1 + \sqrt{2})$ or approximately 241·4. Hence with 250 ft. of fence he will not run out of material. In fact he will never need more than 242 ft. of fencing material.

(d) $A_1 = 2500,\ A_2 = 1250,\ A_3 = 625,\ A_4 = 312{\cdot}5,\ A_9 = \frac{625}{64} \approx 9{\cdot}8,$

$A_{10} = \frac{625}{128} \approx 4{\cdot}9.$ Each area is $\frac{1}{2}$ the preceding area since the sides of the right-angled triangle are both $\frac{1}{\sqrt{2}}$ times the sides of the preceding triangle.

(e) $R_1 = 2500,\ R_2 = 1250,\ R_3 = 625,\ R_{n+1} = \frac{1}{2}R_n,\ \{R_n\}$ forms a geometric progression with $r = \frac{1}{2} \times R_{100} = 2500\left(\frac{1}{2}\right)^{99}$

(f) The area of the original lot is 5000 square feet.

$$A_1 + A_2 + \ldots + A_{100} + R_{100} = 5000$$

$$A_1 + A_2 + \ldots + A_{100} = 5000 - R_{100}$$

$$= 5000 - 2500\left(\frac{1}{2}\right)^{99}$$

$$S_{100} = A_1 + A_2 + \ldots + A_{100} = \frac{2500\left(\left(\frac{1}{2}\right)^{100} - 1\right)}{\frac{1}{2} - 1}$$

$$= 5000\left(1 - \left(\frac{1}{2}\right)^{100}\right)$$

$$= 5000 - 2500\left(\frac{1}{2}\right)^{99}$$

9. (a) $CD = 25\sqrt{3}$, $DE = \frac{75}{2}$, $EF = \frac{75}{4}\sqrt{3}$, $FG = \frac{225}{8}$

(b) $L_9 = 25\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)^8$, $L_{10} = 25\sqrt{3}\left(\frac{\sqrt{3}}{2}\right)^9$. The numbers $L_1, L_2, \ldots$ form a geometric progression with $L_1 = 25\sqrt{3}$ and $r = \frac{\sqrt{3}}{2}$.

(c) L_n approaches $100\sqrt{3} + 150$ for large n. Hence 250 feet would not do but 324 feet would.

(d) $A_1 = \frac{625}{2}\sqrt{3}$, $A_2 = \frac{1875}{8}\sqrt{3}$, $A_3 = \frac{5625}{32}\sqrt{3}$,

$A_4 = \frac{16875}{128}\sqrt{3}$, $A_9 = \frac{625}{2}\sqrt{3}\left(\frac{3}{4}\right)^8$, $A_{10} = \frac{625}{2}\sqrt{3}\left(\frac{3}{4}\right)^9$.

$A_1, A_2, \ldots$ form a geometric progression with $A_1 = \frac{625}{2}\sqrt{3}$ and $r = \frac{3}{4}$.

(e) $R_1 = \frac{1875}{2}\sqrt{3}$, $R_2 = \frac{5625}{8}\sqrt{3}$, $R_3 = \frac{16875}{32}\sqrt{3}$.

$R_1, R_2, \ldots$ form a geometric progression with $R_1 = \frac{1875}{2}\sqrt{3}$ and $r = \frac{3}{4}$. $R_{100} = \frac{1875}{2}\sqrt{3}\left(\frac{3}{4}\right)^{99}$.

(f) $A_1 + A_2 + \ldots + A_{100} + R_{100} = 1250\sqrt{3}$

$$A_1 + A_2 + \ldots + A_{100} = 1250\sqrt{3} - \frac{1875}{2}\sqrt{3}\left(\frac{3}{4}\right)^{99}$$

$$S_{100} = \frac{\frac{625}{2}\sqrt{3}\left(\left(\frac{3}{4}\right)^{100} - 1\right)}{\frac{3}{4} - 1}$$

$$= 1250\sqrt{3} - \frac{1875}{2}\sqrt{3}\left(\frac{3}{4}\right)^{99}$$

4-5 COMPOUND INTEREST.

Application to interest and annuities is a traditional topic in school mathematics. While simple interest can be computed using the formulas for arithmetic progressions, progressions are really unnecessary in simple interest. The amount of interest paid over n years is simply n times the interest for a single year. Consequently the inclusion of simple interest is intended only as an introduction to compound interest. Compound interest, on the other hand, is a very good practical application of geometric progressions.

Since we want the principal to be the first term of the geometric progression, the second term (under interest compounded annually) will be the amount *at the beginning of the second year*, and in general the nth term, $P\left(1+\frac{i}{100}\right)^{n-1}$, will be the amount at the beginning of the nth year. We shall therefore speak of the amount at the *beginning* of a year in interest compounded annually.

Answers to
PROBLEMS 4-5A

Student Text Pages 138-139

1.	Amount 3rd year	6th year	2.	nth year
(a)	$1200(1\cdot06)^2$	$1200(1\cdot06)^5$		$1200(1\cdot06)^{n-1}$
(b)	$500(1\cdot05)^2$	$500(1\cdot05)^5$		$500(1\cdot05)^{n-1}$
(c)	$2000(1\cdot04)^2$	$2000(1\cdot04)^5$		$2000(1\cdot04)^{n-1}$
(d)	$1500(1\cdot045)^2$	$1500(1\cdot045)^5$		$1500(1\cdot045)^{n-1}$

The amounts indicated as answers may be computed directly or they may be approximated by using logarithms. In many books of tables the amount of 1 shilling invested at various rates of compound interest is given. Of course this is just a table of values of $\left(1+\frac{i}{100n}\right)^n$ for different values of i and n.

Below we give the answers for Problem 1 when the amounts are computed from a four-place compound interest table.

3rd year				6th year			
(a)	1348·32	(c)	2163·20	(a)	1605·84	(c)	2433·40
(b)	551·25	(d)	1638·00	(b)	638·15	(d)	1869·30

Although the rate of interest i is always given as an annual rate, if the interest is compounded more often than annually, the terms of the geometric progression are actually the amounts at the beginning of each interest period and the ratio r of the geometric progressions is $1 + \frac{i}{100n}$ where n is the number of times per year the interest is compounded.

Answers to
PROBLEMS 4-5B

Student Text Pages 139-140

The answers given below were computed using a four-place compound interest table.

		1 year	3 years
1.	(a)		$1000(1\cdot03)^6 \approx 1194\cdot10$
	(b)	$500(1\cdot01)^4 \approx 520\cdot30$	$500(1\cdot01)^{12} \approx 555\cdot60$
	(c)	$2000(1\cdot0225)^2 \approx 209\cdot05$	$2000(1\cdot0225)^6 \approx 2285\cdot20$

5 years

(a) $1000(1\cdot03)^{10} \approx 1343\cdot90$

(b) $500(1\cdot01)^{20} \approx 610\cdot60$

(c) $2000(1\cdot0225)^{10} \approx 2498\cdot50$

2. *CHALLENGE PROBLEMS*

(a) $1000(1\cdot01)^{4t} = 2000$, $\quad t = \frac{\log 2}{4 \log(1\cdot01)} \approx 12$ years

(b) $200(1\cdot03)^{2t} = 400$, $\quad t = \frac{\log 2}{2 \log 1\cdot03} \approx 12$ years

(c) $500(1\cdot05)^t = 1000$, $\quad t = \frac{\log 2}{\log 1\cdot05} \approx 15$ years

4-6 GEOMETRIC SEQUENCES AND INFINITE SERIES.

This section is designed to use the information on geometric progressions to introduce informally the idea of the limit of a sequence. This idea is then used to describe convergence of an infinite geometric series. The attempt is a very modest one and no effort is made to introduce new symbols or precise definitions for limits. The hope is that this informal introduction

to limits of sequences and to convergence for infinite geometric series will make the study of limits and infinite series easier when it is met again in a more formal and complete presentation.

The main difficulty pupils are likely to have is that of confusing the two sequences $\{G_n\}$ and $\{S_n\}$. $\{G_n\}$ is the sequence whose terms are added to obtain the infinite series, while $\{S_n\}$ is the sequence of partial sums of the infinite series. The point must be made clear that there are *two* sequences which are involved and that it is the sequence of partial sums $\{S_n\}$ which is used to find the sum of a convergent geometric series.

While we prove no theorems, pupils may notice that whenever the infinite series converges, the terms of the sequence $\{G_n\}$ which are added to obtain the infinite series, approach a limit and that limit is zero. This is a necessary condition for the infinite series to converge. However for infinite series in general this is not sufficient. For example the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n} + \ldots$ has the property that $\frac{1}{n}$ approaches zero as n becomes large; however the sequence of partial sums becomes as large as we like. This can be seen by grouping the terms in the following way:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \ldots + \frac{1}{18}\right) \ldots$$
$$> 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots$$

The definition of convergence for an infinite series states that the series converges if and only if the sequence of partial sums $\{S_n\}$ has a limit.

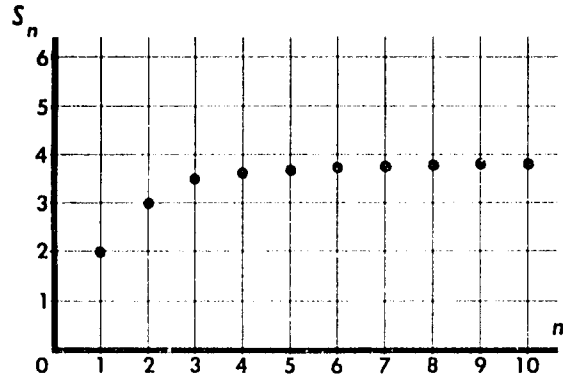
Answers to
CLASS ACTIVITY

Student Text Pages 140-141

The point of these exercises is to help the pupils to discover that when $|r| < 1$, the sequence of partial sums $S_n = G_1 + G_2 + \ldots + G_n$ of the geometric series does approach the limit $\frac{G_1}{1 - r}$, while in the other cases, $|r| \geq 1$, the series diverges.

1. (a) $2,\ 3,\ 3\frac{1}{2},\ 3\frac{3}{4},\ 3\frac{7}{8},\ 3\frac{15}{16},\ 3\frac{31}{32},\ 3\frac{63}{64},\ 3\frac{127}{128},\ 3\frac{255}{256}$

(b)



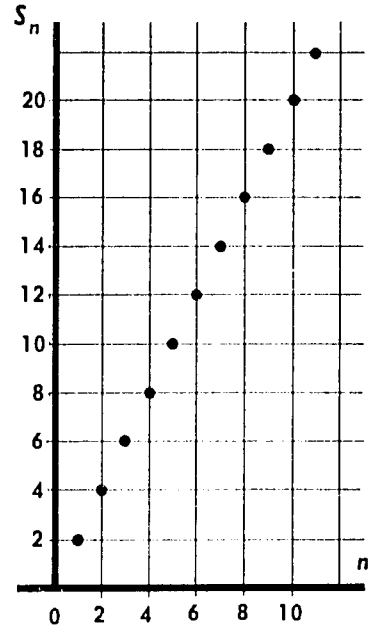
(c) $S_{20} = 4 - \frac{1}{2^{18}}$, $S_{100} = 4 - \frac{1}{2^{98}}$

(d) S_n is always less than 4, but becomes closer and closer to 4 as n becomes large.

(e) $G_n = \frac{1}{2^{n-2}}$ approaches 0 as n becomes large.

2. (a) 2, 4, 6, 8, 10, 12, 14, 16, 18, 20

(b)



(c) $S_{20} = 40$

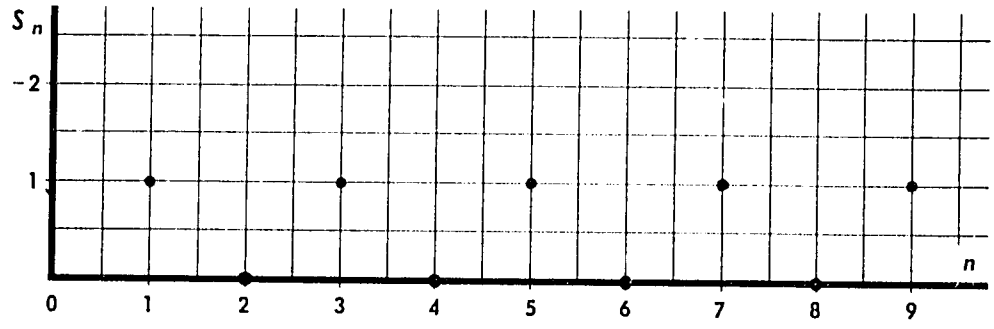
$S_{100} = 200$

(d) S_n becomes as large as we like for large enough n. ($S_n = 2n$)

In other words, n increases without limit.

3. (a) 1, 0, 1, 0, 1, 0, 1, 0, 1, 0

(b)



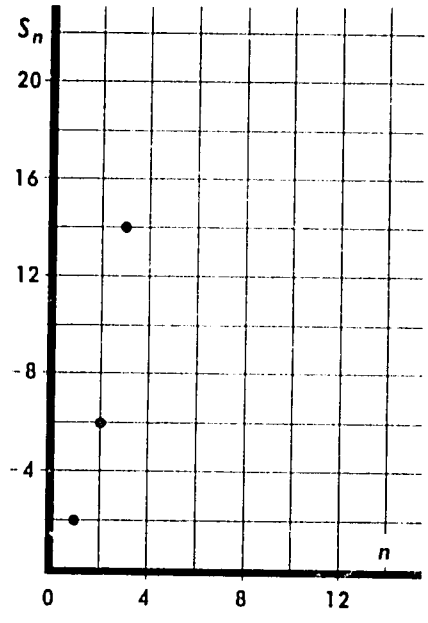
(c) $S_{2n} = 0$

$S_{2n-1} = 1$

For large even values of n, S_n is zero and for large odd values of n, S_n is 1.

4. (a) 2, 6, 14, 30, 62, 126, 254, 510, 1022, 2046

(b)



(c) $S_{20} = 2^{21} - 2$

$S_{100} = 2^{101} - 2$

(d) S_n becomes very large for large values of n.

1.

	S_n		Sum
(a)	$\dfrac{16\left(1-\left(\frac{1}{4}\right)^n\right)}{1-\frac{1}{4}}$	Converges	$\frac{64}{3}$
(b)	$\dfrac{10(2^n-1)}{2-1}$	Diverges	
(c)	$\dfrac{15\left(\left(\frac{6}{5}\right)^n-1\right)}{\frac{6}{5}-1}$	Diverges	
(d)	$\dfrac{-4\left(1-\left(-\frac{1}{2}\right)^n\right)}{1-\left(-\frac{1}{2}\right)}$		$-\frac{8}{3}$
(e)	n	Diverges	
(f)	0	Converges	0

2.

	S_3	S_5	S_{10}
(a)	9	20	65
(b)	$\frac{7}{8}$	$\frac{31}{32}$	$\frac{1023}{1024}$
(c)	6	10	20
(d)	$\frac{3}{4}$	$\frac{5}{6}$	$\frac{10}{11}$
(e)	-6	-22	682
(f)	$1\frac{5}{6}$	$2\frac{17}{60}$	$2\frac{2341}{2520}$

3.

	S_n		Reason
(a)	$n(n+3)$	Diverges	$S_n > n$ for all n
(b)	$1-\left(\frac{1}{2}\right)^n$	Converges	A geometric series with $r < 1$. The sum is 1.
(c)	$2n$	Diverges	$S_n > n$ for all n
(d)	$1-\dfrac{1}{n+1}$	Converges	The sum is 1.
(e)		Diverges	$\lvert S_n \rvert$ becomes large
(f)		Diverges	See page 117 in the Teachers' Guide.

4. $6 + 4 + \frac{8}{3} + \frac{16}{9} + \ldots$

$4 + \frac{8}{3} + \frac{16}{9} + \ldots$

The first sum gives the total distance the ball falls. The second sum gives the total distance the ball rises on the rebounds. If $S = 4 + \frac{8}{3} + \frac{16}{9} + \ldots$, then the total distance travelled by the ball is

$$6 + 2S.$$

Since S is the sum of terms of a geometric progression with first term 4 and ratio $\frac{2}{3}$,

$$S = \frac{G_1}{1 - r} = \frac{4}{1 - \frac{2}{3}} = 12$$

and $6 + 2S = 30$. The ball travelled less than 30 ft.

5. *CHALLENGE PROBLEM*

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

Hence the sequence of partial sums is given by

$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

Therefore the series converges to the sum 1.

Rational numbers as we have seen in *Secondary Three* can be represented by terminating decimal numerals if the denominator contains only the prime factors 2 and 5; otherwise the decimal representation is a repeating decimal. A repeating decimal can be represented as the sum of a convergent geometric series in which $r < 1$. Such a series always converges to the sum $\frac{G_1}{1 - r}$.

A rational number whose decimal representation does not repeat immediately but eventually repeats, can be written as the sum of a terminating decimal numeral and a convergent infinite geometric series.

1. $\cdot\dot{5} = \cdot 5 + \cdot 05 + \cdot 005 + \ldots \quad (G_1 = \cdot 5,\ r = \cdot 1)$

$$= \frac{\cdot 5}{1 - \cdot 1} = \frac{5}{9}$$

2. $\cdot\dot{1}\dot{4} = \cdot 14 + \cdot 0014 + \cdot 000014 + \ldots \quad (G_1 = \cdot 14,\ r = \cdot 01)$

$$= \frac{\cdot 14}{1 - \cdot 01} = \frac{14}{99}$$

3. $\cdot\dot{2}8\dot{1} = \cdot 281 + \cdot 000281 + \cdot 000000281 + \ldots \quad (G_1 = \cdot 281,\ r = \cdot 001)$

$$= \frac{\cdot 281}{1 - \cdot 001} = \frac{281}{999}$$

4. $\cdot 21\dot{4}\dot{3} = \cdot 21 + \cdot 0043 + \cdot 000043 + \cdot 00000043 + \ldots \quad (G_1 = \cdot 0043,\ r = \cdot 01)$

$$= \cdot 21 + \frac{\cdot 0043}{1 - \cdot 01} = \frac{21}{100} + \frac{43}{9900} = \frac{1061}{4950}$$

5. $1\cdot 4\dot{1} = 1\cdot 4 + \cdot 01 + \cdot 001 + \cdot 0001 + \ldots \quad (G_1 = \cdot 01,\ r = \cdot 1)$

$$= \frac{14}{10} + \frac{\cdot 01}{1 - \cdot 1} = \frac{14}{10} + \frac{1}{90} = \frac{127}{90}$$

6. $1\cdot\dot{4}\dot{1} = 1 + \cdot 41 + \cdot 0041 + \cdot 000041 + \ldots \quad (G_1 = \cdot 41,\ r = \cdot 01)$

$$= 1 + \frac{\cdot 41}{1 - \cdot 01} = 1 + \frac{41}{99} = \frac{140}{99}$$

7. $1\cdot\dot{4}1\dot{4} = 1 + \cdot 414 + \cdot 000414 + \cdot 000000414 + \ldots \quad (G_1 = \cdot 414,\ r = \cdot 001)$

$$= 1 + \frac{\cdot 414}{1 - \cdot 001} = 1 + \frac{414}{999} = \frac{1413}{999}$$

8. $2\cdot\dot{1}82\dot{8} = 2 + \cdot 1828 + \cdot 00001828 + \ldots \quad (G_1 = \cdot 1828,\ r = \cdot 0001)$

$$= 2 + \frac{\cdot 1828}{1 - \cdot 0001} = 2 + \frac{1828}{9999} = \frac{21826}{9999}$$

*4-7 THE NUMBER e.

This section can be considered as optional.

We present a very interesting way of obtaining one of the important transcendental numbers of mathematics. The number e appears as the limit of the sequence $\left(1 + \frac{1}{n}\right)^n$ (that is, the number which the terms of the sequence approach for large values of n). It is the base of the system of natural logarithms. Logarithms to this base are universally used in calculus and

other parts of higher mathematics because of the convenient form which the derivative of the logarithm to this base has.

The discussion in this section is another attempt to introduce the idea of the limit of a sequence. The reader is led step by step to see that a sequence which is increasing and bounded approaches a limit. This material could be very interesting for bright pupils, but should not be attempted by below average or even average pupils. The argument is quite sophisticated and involves several rather delicate inequality relations.

The reader is led to conclude that the sequence $\{A_n\}$ is an increasing function of n by exhibiting a proof that the subsequence $A(2^n)$ is increasing. To show that no term of the sequence ever exceeds four, we examine a similar sequence $B(n) = \left(1 - \frac{1}{n}\right)^n$ and show that the subsequence $B(2^n)$ is also increasing and hence that the sequence $\frac{1}{B(2^n)}$ decreases. It is easy to show that for all n $A(n) < \frac{1}{B(n)}$ and hence the terms of the sequence $A(n)$ are all less than any term of the decreasing sequence $\frac{1}{B(n)}$. Since $\frac{1}{B(2)} = 4$, we have shown that $A(2^n) < 4$ for all n.

The details are as follows:

$$A_1 = \left(1 + \frac{1}{1}\right)^1 = 2$$

$$A_2 = \left(1 + \frac{1}{2}\right)^2 = 2{\cdot}25$$

$$A_3 = \left(1 + \frac{1}{3}\right)^3 = \frac{64}{27} \approx 2{\cdot}37$$

$$A_4 = \left(1 + \frac{1}{4}\right)^4 = \frac{625}{256} \approx 2{\cdot}44$$

The values of A_n seem to be increasing with n. Since this represents the amount at the end of a year of 1 shilling invested at rate 100% compounded n times a year, if n increases we would expect the amount to increase.

In the argument that $A(2n) > A(n)$ the reason for the step $\left[1 + \frac{1}{n} + \left(\frac{1}{2n}\right)^2\right]^n > \left(1 + \frac{1}{n}\right)^n$ is that $1 + \frac{1}{n} < 1 + \frac{1}{n} + \left(\frac{1}{2n}\right)^2$ $\left(\text{since } 0 < \left(\frac{1}{2n}\right)^2\right)$ and for positive numbers a and b if $a < b$, then $a^n < b^n$.

1. $B(1) = 0,\ B(2) = \cdot 25,\ B(3) = \frac{8}{27} \approx \cdot 296,\ B(4) = \frac{81}{256} \approx \cdot 316$

2. Yes. $B(n)$ does seem to be increasing (rather slowly however).

3. $B(2n) = \left(1 - \frac{1}{2n}\right)^{2n} = \left[\left(1 - \frac{1}{2n}\right)^2\right]^n = \left(1 - \frac{1}{n} + \frac{1}{4n^2}\right)^n > \left(1 - \frac{1}{n}\right)^n = B(n).$

4. $A(n) \times B(n) = \left(1 + \frac{1}{n}\right)^n \left(1 - \frac{1}{n}\right)^n = \left(1 - \frac{1}{n^2}\right)^n < 1^n = 1.$

5. $$\frac{1}{B(2^{20})} = \frac{1}{\left(1 - \frac{1}{2^{20}}\right)^{2^{20}}} \frac{\left(1 + \frac{1}{2^{20}}\right)^{2^{20}}}{\left(1 + \frac{1}{2^{20}}\right)^{2^{20}}} = \frac{\left(1 + \frac{1}{2^{20}}\right)^{2^{20}}}{\left(1 - \frac{1}{2^{40}}\right)^{2^{20}}} > \left(1 + \frac{1}{2^{20}}\right)^{2^{20}} = A(2^{20})$$

   The inequality is true since replacing the denominator by a larger number $1^{2^{20}}$ makes the fraction smaller.
   Since by Problem 2, $B(2n) > B(n)$ for all n,

   $$B(2) < B(2^2) < B(2^3) < \ldots < B(2^{20})$$

   or $$\frac{1}{B(2)} > \frac{1}{B(2^2)} > \frac{1}{B(2^3)} > \ldots > \frac{1}{B(2^{20})} > A(2^{20})$$

6. $\frac{1}{B(4)} = \frac{256}{81} = 3\cdot 16,\ \frac{1}{B(2)} = 4.$

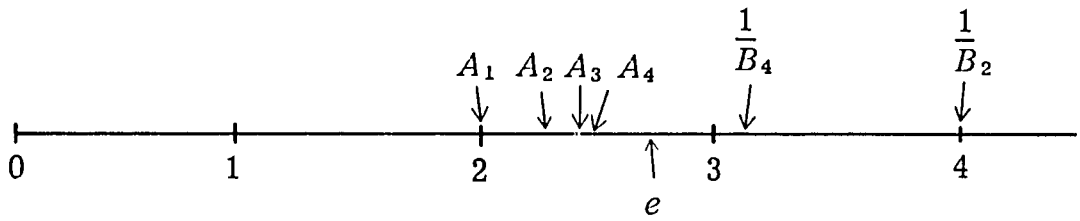
7. Since $\frac{1}{B(2)} > \frac{1}{B(4)} > \ldots > \frac{1}{B(2^{20})} > A(2^{20}),\ A(2^{20}) < 4.$

Since there was nothing special about $n = 20$ in the subsequences $A(2^n)$ and $B(2^n)$ we might as well have written

$$A(2^n) < \frac{1}{B(2^n)} \leq \frac{1}{B(2^2)} < \frac{1}{B(2)} \quad \text{for all } n \geq 2.$$

Near the bottom of page 145 of the pupil's text, there is a disguised and, hopefully, obvious theorem: any monotonic sequence which is bounded has a limit.

If we plot the values of the two sequences $\{A_n\}$ and $\left\{\frac{1}{B_n}\right\}$ on the number line we have the following picture:



Answers to
PROBLEMS 4-7

Student Text Page 146

If the bank charged $i\%$ per year and deducted the charge n times a year, then after the first deduction the amount would be $1 - \frac{i}{100n}$. After the second deduction $\left(1 - \frac{i}{100n}\right) - \left(1 - \frac{i}{100n}\right)\frac{i}{100n} = \left(1 - \frac{i}{100n}\right)^2$. After the third deduction $\left(1 - \frac{i}{100n}\right)^2 - \left(1 - \frac{i}{100n}\right)^2 \frac{i}{100n} = \left(1 - \frac{i}{100n}\right)^3$, etc. After the nth deduction one would have $B(n) = \left(1 - \frac{i}{100n}\right)^n$ shillings.

One can show that $B(2n) > B(n)$ just as before and so one would like to have the bank deduct the charge very often. The more times per year the deduction is made, the more money one would have in the bank at the end of the year.

Chapter 5
DESCRIPTIVE STATISTICS

The aim of this chapter is to familiarize the pupil with some of the more common descriptive statistics: frequency distribution, histogram, mean, median, mode, standard deviation, and range of a population function, and scatter diagram for two population functions.

5-1 POPULATIONS AND FUNCTIONS ON POPULATIONS.

There is nothing new in this section except the words "population" and "population function." Make clear that we are just dealing with a function whose domain is a finite set, called the population, and whose values are real numbers. Since the values of our functions are usually obtained not from a formula but by empirical measurement, we usually describe the function by listing the elements of the population and, opposite each, the corresponding value of the function.

Here are the possibilities for listing the pupils in order of increasing height:

	Pupil x						$H(x)$
1	A	A	C	C	G	G	60
2	C or	G or	A or	G or	A or	C	60
3	G	C	G	A	C	A	60
4	D						62
5	F						63
6	H	I					64
	or						
7	I	H					64
8	E						66
9	B	J					67
	or						
10	J	B					67

We can choose any of 6 orders for places 1, 2, 3 with either of 2 orders for places 6, 7 with either of 2 orders for places 9, 10, so there are $6 \times 2 \times 2$ or 24 different correct lists, but you should not go into this unless a pupil raises the question.

The answers to the questions "What is the height of the second person?" etc., are the same for any of the 24 correct listings.

In the temperature example, the graphical representation of the population functions H and I should again remind the pupils that these are just functions with finite domains.

Be sure the pupils list the elements (members) of the population and the values of L, S, $\frac{L+S}{2}$ so they will have a concrete picture of how $\frac{L+S}{2}$ is related to L and S.

Element x	$L(x)$	$S(x)$	$\frac{L+S}{2}(x)$
Su	88	82	85
M	85	81	83
Tu	86	81	83·5
W	91	85	88
Th	87	84	85·5
F	87	86	86·5
Sa	88	84	86

For the two digit number population, encircling the subsets for which $F + S$ has each value helps to make it clear that the values of $F + S$ are 2, 3, . . . , 8 and that the corresponding frequencies are 1, 2, 3, 4, 3, 2, 1:

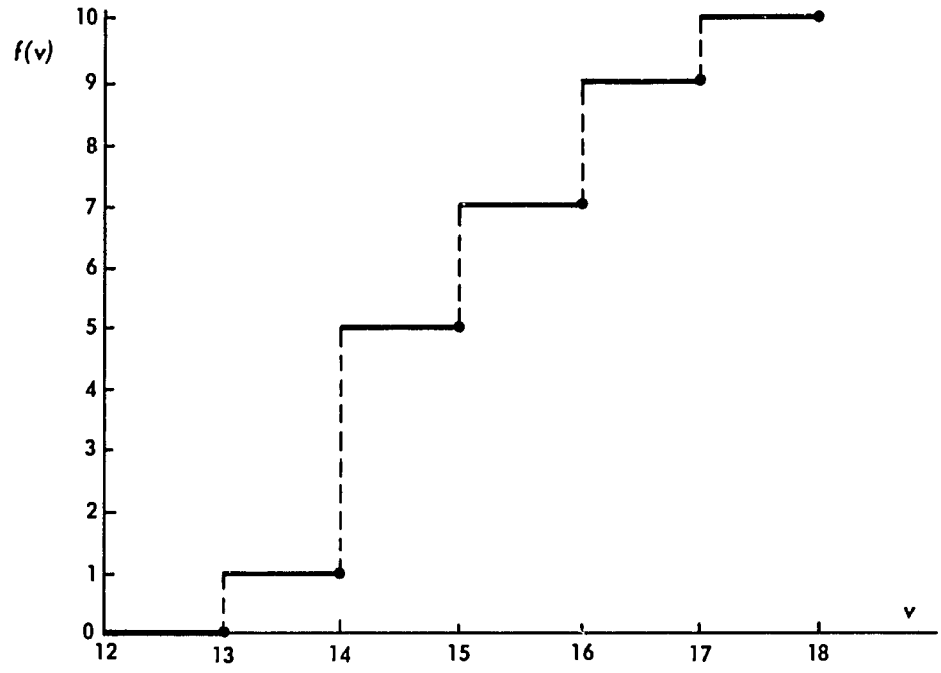
$F + S = 2$ →	11	12	13	14
$= 3$ →	21	22	23	24
$= 4$ →	31	32	33	34
$= 5$ →	41	42	43	44
		$F + S = 6$	$= 7$	$= 8$

The list of pupils in your class, and the corresponding values of several population functions, will be used to illustrate all the concepts introduced in this chapter. One way to prepare the list is: on the day before the list is to be prepared, you choose the population functions to be used, and announce that each pupil should be prepared on the following day to report his weight to the nearest 5 pounds, age at last birthday, or whatever information you may have chosen to use.

Answers to
PROBLEMS 5-1

Student Text Page 150

1. 0, 1, 7, 7, 8, 10. Here is the graph:



The dotted lines are not a part of the graph. The dot at the point (14, 1) emphasizes that (14, 1) is part of the graph, (14, 5) is not. The function is discontinuous.

2.

Element	Value of the function				
	L	E	Z	$L - E$	$E + Z$
THERE	5	2	0	3	2
IS	2	0	0	2	0
NO	2	0	0	2	0
LARGEST	7	1	0	6	1
INTEGER	7	2	0	5	2

The point of Z is that the value of a population function need not vary. Again this is nothing new, as the pupils are familiar with constant functions.

3. A has the value 0 at *bbbb*
   " 1 at *abbb, babb, bbab, bbba*
   " 2 at *aabb, abab, abba, baab, baba, bbaa*
   " 3 at *aaab, aaba, abaa, baaa*
   " 4 at *aaaa*

B has the values 0, 1, 2, 3, 4 for 1, 4, 6, 4, 1 elements respectively. This exercise is a small first step toward the binomial coefficients:

$$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

The values of $A - B$ (which, since $B = 4 - A$, is just the function $2A - 4$) are $-4, -2, 0, 2, 4$. The only value of $A + B$ is 4 (another constant function). The values of $\text{Max}(A, B)$ are 2 (for the six elements with $A = B = 2$), 3 (for the 8 elements with $A = 1$ or 3), 4 (for the 2 elements with $A = 0$ or 4).

5-2 DISTRIBUTION AND HISTOGRAMS.

The distribution of a population function is also known as the frequency distribution or frequency function of the population function.

Answers to
PROBLEMS 5-2

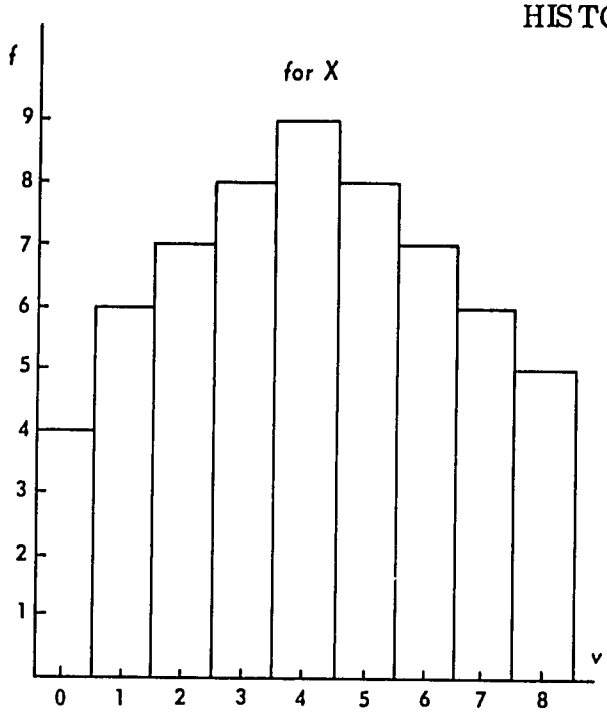
Student Text Page 153

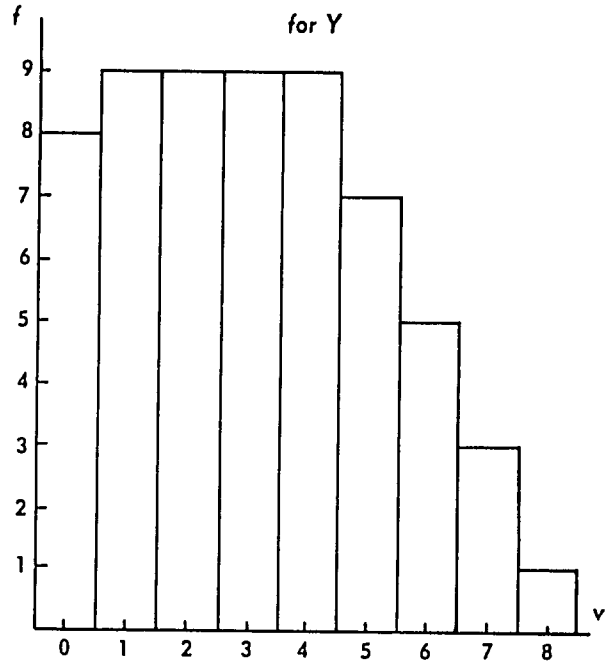
2. Distributions:

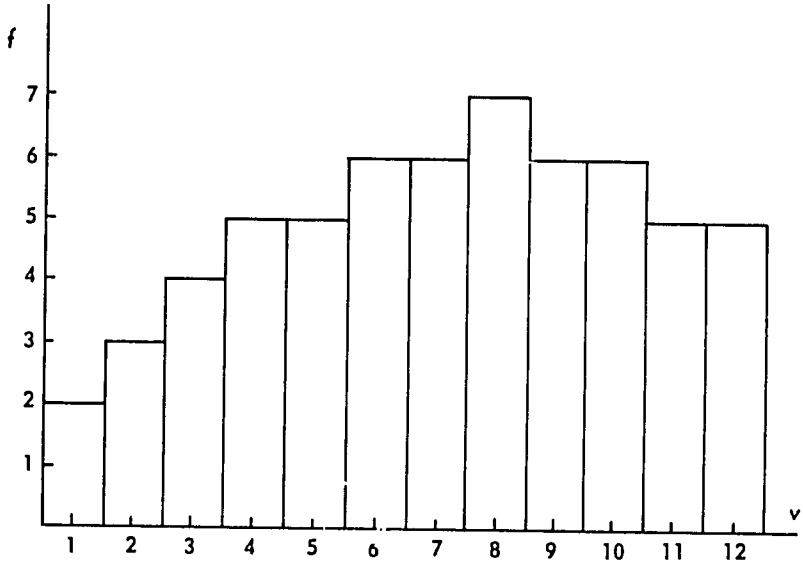
X		Y		$X + Y$		Z	
v	f	v	f	v	f	v	f
0	4	0	8	1	2	0	12
1	6	1	9	2	3	1	12
2	7	2	9	3	4	2	11
3	8	3	9	4	5	3	10
4	9	4	9	5	5	4	9
5	8	5	7	6	6	5	5
6	7	6	5	7	6	6	1
7	6	7	3	8	7		
8	5	8	1	9	6		
				10	6		
				11	5		
				12	5		

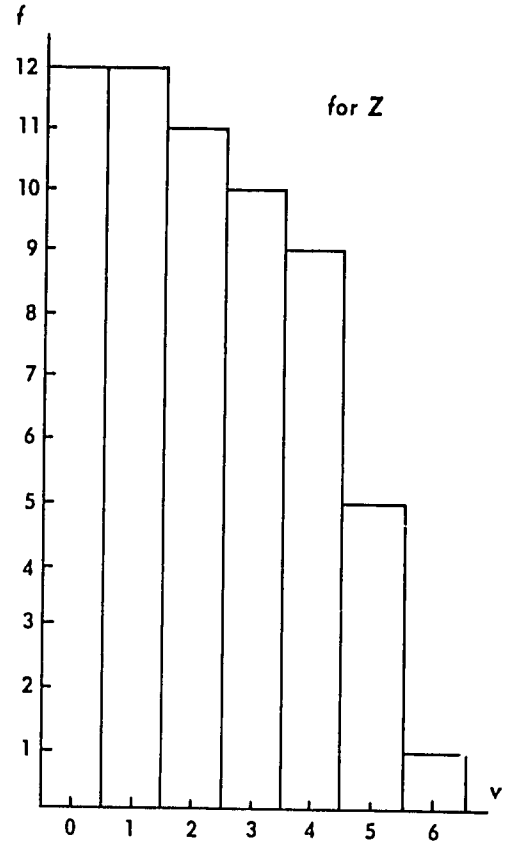
(The pupils are really doing a little coordinate geometry here; with the market as origin, the functions X, Y give just the x and y coordinates of the shop; they are observing that the locus of $x + y$ = constant is a line.)

HISTOGRAMS









5-3 THE MEAN, MEDIAN, AND MODE OF A POPULATION FUNCTION.

The title of the section and the notation $M(X)$ (read M of X or the mean of X or the mean of the function X) emphasize a small but important point: the things that have means are population functions. It is true that the mean of a population function depends only on its distribution, so statisticians often speak of the mean of a distribution, but at this stage any shift of emphasis away from population functions would be a mistake: the pupil will never understand why $M(X + Y) = M(X) + M(Y)$ if he thinks of the mean only in terms of distributions.

The point of the peculiar way of writing the values for $L - S$, etc., for example, writing 88 - 82 instead of 6, is that if we add the columns before doing the subtraction, we see at once why $M(L - S) = M(L) - M(S)$.

When the pupil finds $M(S - 80)$ without first finding $M(S)$:

Day	Value of $S - 80$
Su	2
M	1
Tu	1
W	5
Th	4
F	6
Sa	4
TOTAL	23
MEAN	$\frac{23}{7}$

then adds 80 to get $M(S) = 80 + \frac{23}{7} = \frac{583}{7}$, he will see how the subtraction simplifies the calculation of $M(S)$.

The formula (described in words in the text)

$$M(X) = \frac{v_1 f_1 + v_2 f_2 + \ldots + v_n f_n}{f_1 + f_2 + \ldots + f_n}$$

is also the formula for the centre of gravity of weights $f_1, f_2, \ldots, f_n$ at positions $v_1, v_2, \ldots, v_n$ on a line, so the mean of a population function is the point where its histogram balances. You may want to mention this if your pupils have met this idea in physics.

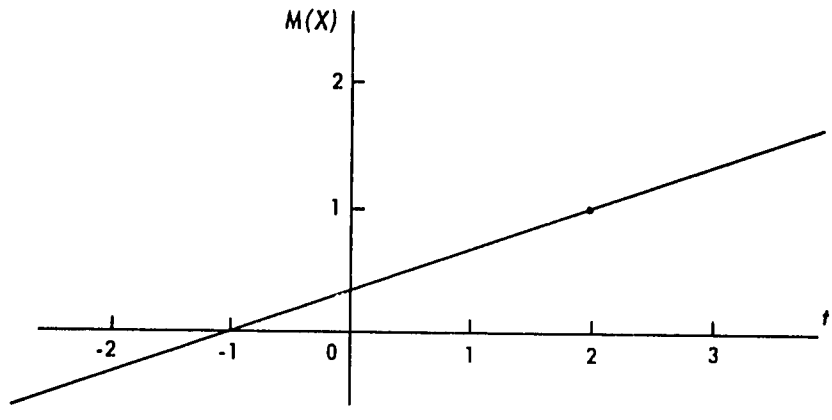
Answers to
PROBLEMS 5-3

Student Text Page 159

2.

t	$M(X)$	median	mode(s)
1	$\frac{2}{3}$	1	1
1·5	$\frac{5}{6}$	1	0, 1, 1·5
2	1	1	0, 1, 2
10	$\frac{11}{3}$	1	0, 1, 10

$M(X) = \frac{0 + 1 + t}{3} = \frac{1 + t}{3}$; its graph is

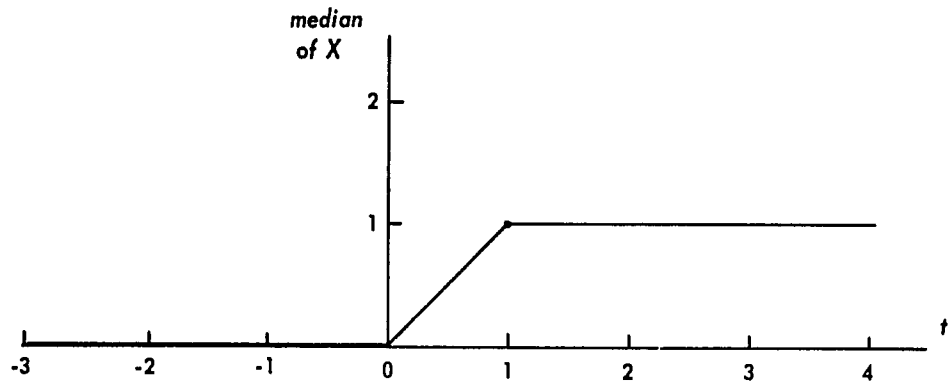


The median of X is 1 for $t \geq 1$

" t for $0 \leq t \leq 1$

" 0 for $t \leq 0$;

its graph is



The point of this problem and the next is to give the pupil some appreciation of the way the mean and median are affected by changes in a single individual. The median is affected rather little, but every change in a value of a population function is reflected by a proportional change in the mean.

3. $M(S) = \frac{182}{11} = 16\frac{6}{11}$ (a change of more than 8)

median of S = 2 (a change of 1)

mode of S is 0 (no change)

4. $$M(X) = \frac{(0\times 4)+(1\times 6)+(2\times 7)+(3\times 8)+(4\times 9)+(5\times 8)+(6\times 7)+(7\times 6)+(8\times 5)}{4+6+7+8+9+8+7+6+5}$$

$$= \frac{244}{60} \approx 4{\cdot}067$$

The median of X is 4; the mode of X is 4.

$$M(Y) = \frac{(0\times 8)+(1\times 9)+\ldots+(8\times 1)}{8+9+\ldots+1} = \frac{184}{60} \approx 3{\cdot}067$$

The median of Y is 3; 1, 2, 3, 4 are all modes of Y.

$$M(X+Y) = \frac{(1\times 2)+\ldots+(12\times 5)}{60} = \frac{428}{60} \approx 7{\cdot}133$$

Alternatively $M(X+Y) = M(X) + M(Y) = \frac{244}{60} + \frac{184}{60} = \frac{428}{60}$

The median of $X + Y$ is 7; the mode of $X + Y$ is 8.

$$M(Z) = \frac{(0 \times 12) + \ldots + (6 \times 1)}{60} = \frac{131}{60} \approx 2{\cdot}183$$

The median of Z is 2; the modes of Z are 0 and 1.

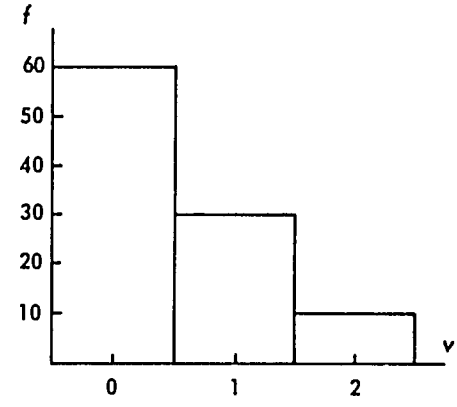
5. The pupils discover here that the relative frequencies of the values of a population function, that is the fractions of the population for which a function has each of its values, are sufficient to determine the shape of its histogram, and its mean, median, and modes.

For 100 pupils:

Distribtuion of X

v	f
0	60
1	30
2	10

Histogram of X



$$M(X) = \frac{(0 \times 60) + (1 \times 30) + (2 \times 10)}{60 + 30 + 10} = \frac{50}{100} = {\cdot}5$$

median of X = mode of X = 0

For 200 pupils:

Distribution of X

v	f
0	120
1	60
2	20

The histogram is as before except that the points labelled 10, 20, . . ., 60 on the f axis are now labelled 20, 40, . . ., 120.

$$M(X) = \frac{(0 \times 120) + (1 \times 60) + (2 \times 20)}{120 + 60 + 20} = \frac{100}{200} = {\cdot}5$$

median of X = mode of X = 0.

5-4 STANDARD DEVIATION AND RANGE.

Standard deviation is a subtle concept, and, with the brief discussion given here, the pupil cannot be expected to have the same clear intuitive

grasp of it that he has for mean. Standard deviation is a square root of a mean of a square of a difference of two population functions: four consecutive operations, and five if you count finding the mean of the original function as an operation.

Concentrate on the station example, and the easily grasped idea that the average delivery cost is some kind of measure of how spread out the stations are. Any population function can be interpreted as locations of an imagined set of filling stations along a road, and this interpretation will be helpful. For instance it shows at once that adding a constant to the values of a function doesn't change the variance, since this can be interpreted as just measuring distances from another town, without moving the stations or depots at all, which obviously will not affect delivery costs.

Where the pupils are asked to fill in the missing entries, which are

	$\frac{36}{49}$
	$\frac{169}{49}$
$-\frac{8}{7}$	
	$\frac{1}{49}$
0	
0	

point out that the 0's are not accidental: the mean is just the constant to subtract from the values of a function to make the mean of the resulting function 0.

pupil	A	$A - M(A)$	$[A - M(A)]^2$
A	14	- ·8	·64
B	15	·2	·04
C	16	1·2	1·44
D	14	- ·8	·64
E	17	2·2	4·84
F	15	·2	·04
G	16	1·2	1·44
H	13	-1·8	3·24
I	14	- ·8	·64
J	14	- ·8	·64
TOTAL	148	0	13·60
MEAN	14·8	0	1·36

The variance of A is 1·36.

Here are the calculations for the variance of A - 13:

pupil	A - 13	$(A - 13) - M(A - 13)$	$[(A - 13) - M(A - 13)]^2$
A	1	- ·8	·64
B	2	·2	·04
C	3	1·2	1·44
D	1	- ·8	·64
E	4	2·2	4·84
F	2	·2	·04
G	3	1·2	1·44
H	0	-1·8	3·24
I	1	- ·8	·64
J	1	- ·8	·64
TOTAL	18	0	13·60
MEAN	1·8	0	1·36

The variance of $A - 13$ is 1·36. Having already done the longer previous calculation, the pupil will appreciate the computational advantage of subtracting a suitable constant from the values of the function before calculating variance. Had we subtracted 15 the numbers would have been even smaller, and you may want the pupils to do this.

Answers to
PROBLEMS 5-4

Student Text Page 162

2.

X		$X - M(X)$		$[X - M(X)]^2$	
v	f	v	f	v	f
1	7	- ·4	7	·16	7
2	2	·6	2	·36	2
3	1	1·6	1	2·56	1

$$M(X) = \frac{(1 \times 7) + (2 \times 2) + 3 \times 1}{7 + 2 + 1} = \frac{14}{10} = 1{\cdot}4$$

$$M\left([X - M(X)]^2\right) = \frac{({\cdot}16 \times 7) + ({\cdot}36 \times 2) + (2{\cdot}56 \times 1)}{7 + 2 + 1} = \frac{4{\cdot}40}{10} = {\cdot}44$$

The mean of X is 1·4. The variance of X is ·44.

The largest value of X is 3, the smallest value of X is 1, the range of X is $3 - 1 = 2$. You can find the standard deviation and range of any population function from its distribution. This is the point of the problem. Yes, since the distribution can be produced from the histogram.

3. For 1000 globes manufactured, the distribution of L is

v	f
1	$1000x$
2	$1000\,(1 - 2x)$
3	$1000x$

L		$L - M(L)$		$[L - M(L)]^2$	
v	f	v	f	v	f
1	$1000\,x$	-1	$1000\,x$	1	$1000\,x$
2	$1000\,(1 - 2x)$	0	$1000\,(1 - 2x)$	0	$1000\,(1 - 2x)$
3	$1000\,x$	1	$1000\,x$	1	$1000\,x$

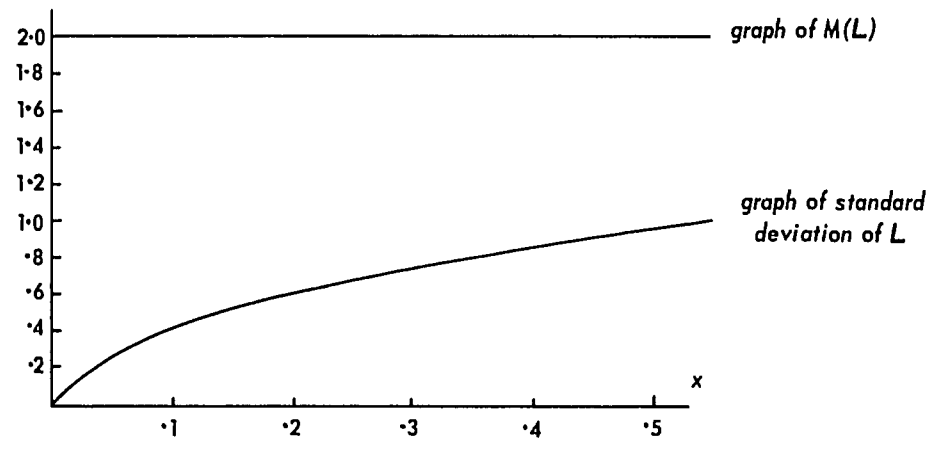
$$M(L) = \frac{1\,(1000)\,x + 2\,(1000)\,(1 - 2x) + 3\,(1000)\,x}{1000} = \frac{2000}{1000} = 2$$

$$M\left([L - M(L)]^2\right) = \frac{1\,(1000)\,x + 0\,(1000)\,(1 - 2x) + (1000)\,x}{1000}$$

$$= \frac{2000\,x}{1000} = 2x$$

The mean of L is 2. The standard deviation is $\sqrt{2x}$.

x	mean of L	standard deviation of L
0	2	0
·1	2	·45
·2	2	·63
·3	2	·78
·4	2	·90
·5	2	1·00



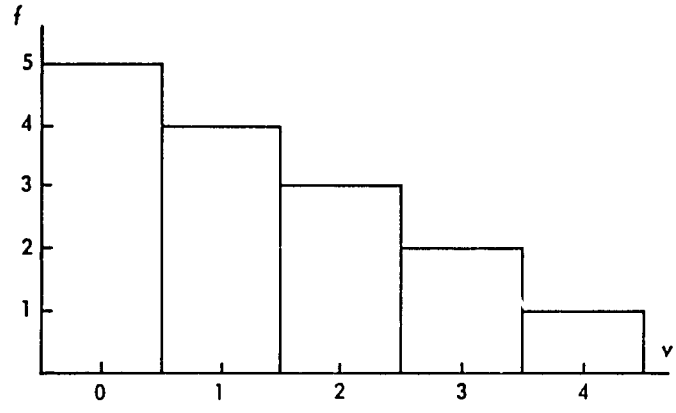
5-5 SCATTER DIAGRAMS.

Scatter diagrams have already been treated in *Secondary Two,* but their revision here is a natural continuation of the work of the first four sections.

Answers to
PROBLEMS 5-5

Student Text Page 165

2. (a) Histogram for X



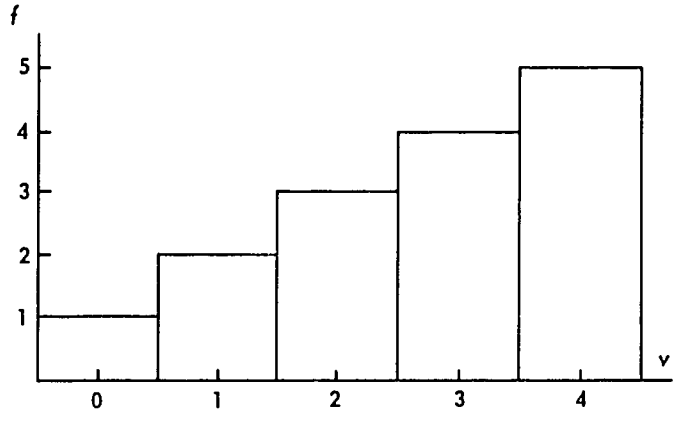
$M(X) = \frac{4}{3}$, variance of $X = \frac{14}{9}$,

standard deviation (s.d.) of $X = \sqrt{\frac{14}{9}} \approx 1{\cdot}2$,

range of X is 4.

The histogram and (therefore) mean, standard deviation, and range of Y are the same as for X.

Histogram for $X + Y$



$M(X+Y) = \frac{8}{3}$, variance of $X+Y = \frac{14}{9}$,

s. d. of $X+Y = \sqrt{\frac{14}{9}} \approx 1{\cdot}2$,

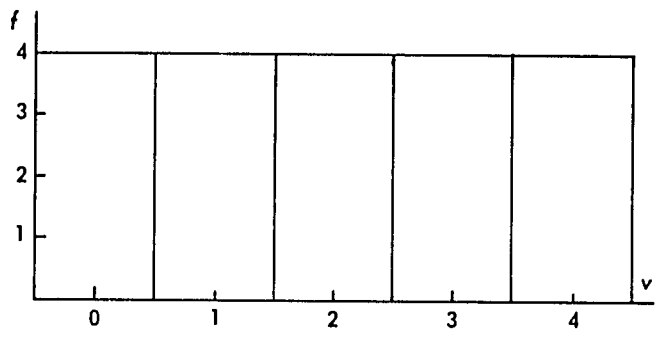
range of $X+Y$ is 4.

Point out that the histogram for $X+Y$ is just the reflection of that for X in a vertical line. The histogram for the function $4 - X$ would be identical with that for $X+Y$, so

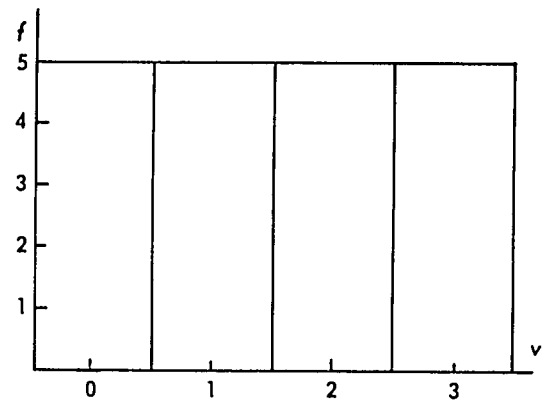
the mean of $X+Y$ = mean of $4 - X$ = 4 - mean $X = \frac{8}{3}$,

s.d. of $X+Y$ = s.d. of $4 - X$ = s.d. of $X \approx 1{\cdot}2$

(b) Histogram for X



Histogram for Y



$M(X) = 2$, variance of $X = 2$,

s.d. of $X = \sqrt{2} \approx 1{\cdot}4$,

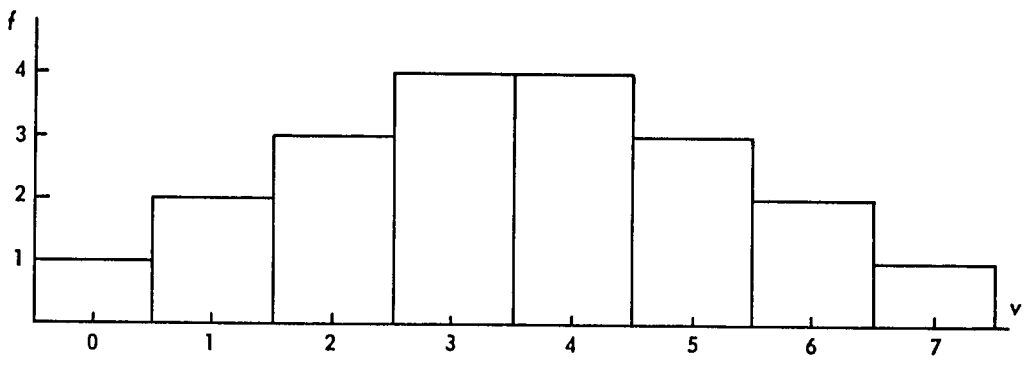
range of X is 4.

$M(Y) = \frac{3}{2}$, variance of $Y = \frac{5}{4}$,

s.d. of $Y = \sqrt{\frac{5}{4}} \approx 1{\cdot}1$,

range of Y is 3.

Histogram for $X+Y$

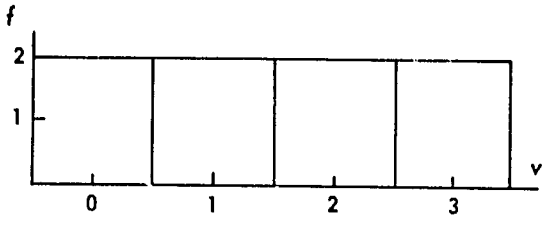


$M(X+Y) = \frac{7}{2}$, variance of $X+Y = \frac{13}{4}$,

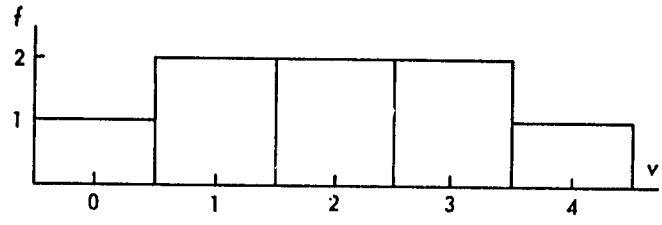
s.d. of $X+Y = \sqrt{\frac{13}{4}} \approx 1{\cdot}8$,

range of $X+Y$ is 7.

(c) Histogram for X



Histogram for Y



$M(X) = \frac{3}{2}$, variance of $X = \frac{5}{4}$,

s.d. of $X = \sqrt{\frac{5}{4}} \approx 1{\cdot}1$,

range of X is 3.

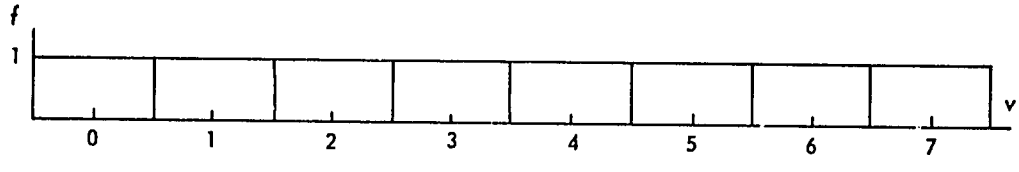
(compare with Y in (b))

$M(Y) = 2$, variance of $Y = \frac{3}{2}$,

s.d. of $Y = \sqrt{\frac{3}{2}} \approx 1{\cdot}2$,

range of Y is 4.

Histogram for $X+Y$

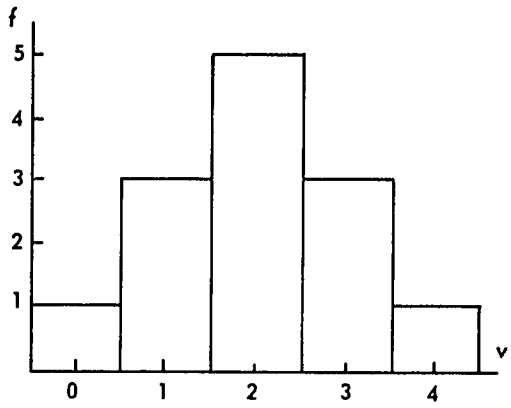


$M(X+Y) = \frac{7}{2}$, variance of $X+Y = \frac{21}{4}$,

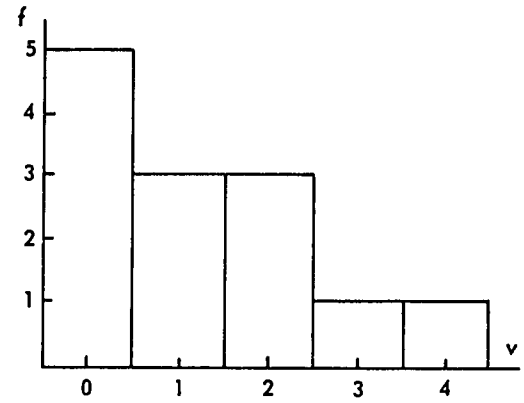
s.d. of $X+Y = \sqrt{\frac{21}{4}} \approx 2{\cdot}3$,

range of $X+Y$ is 7.

(d) Histogram for X



Histogram for Y



$M(X) = 2$, variance of $X = \frac{14}{13}$,

s.d. of $X = \sqrt{\frac{14}{13}} \approx 1{\cdot}0$,

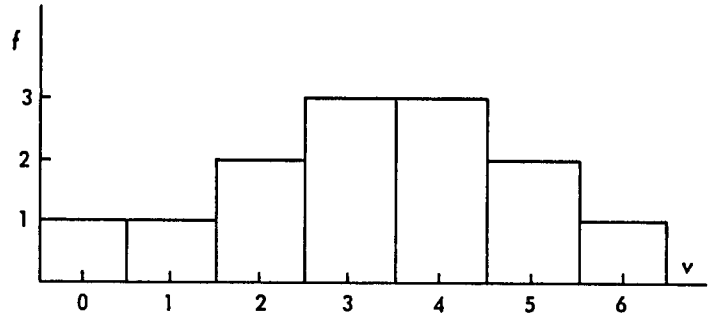
range of X is 4.

$M(Y) = \frac{16}{13}$, variance of $Y = \frac{264}{169}$,

s.d. of $Y = \sqrt{\frac{264}{169}} \approx 1{\cdot}2$,

range of Y is 4.

Histogram for $X + Y$

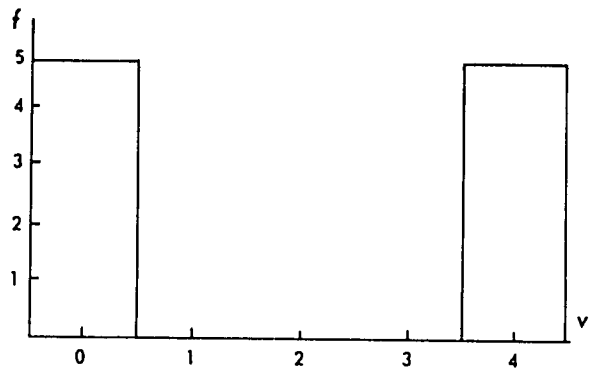


$M(X+Y) = \frac{42}{13}$, variance of $X + Y = \frac{446}{169}$,

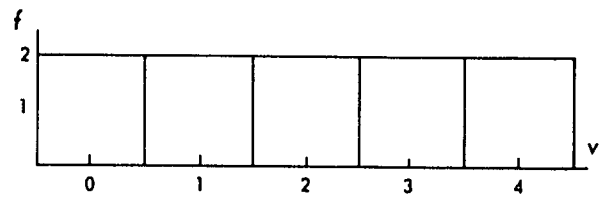
s.d. of $X + Y = \sqrt{\frac{446}{169}} \approx 1{\cdot}6$,

range of $X + Y$ is 6.

(e) Histogram for X



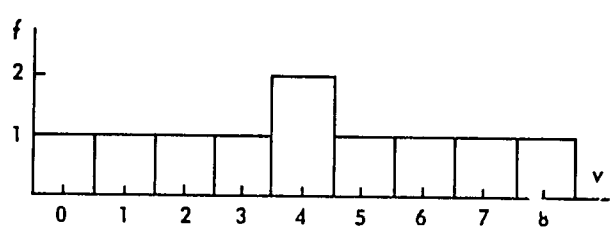
Histogram for Y



$M(X) = 2$, variance of $X = 4$,
s.d. of X is $\sqrt{4} = 2$,
range of X is 4.

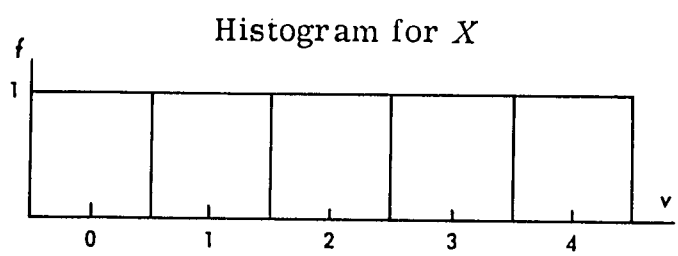
$M(Y) = 2$, variance of $X = 2$,
s.d. of $Y = \sqrt{2} \approx 1{\cdot}4$,
range of Y is 4.
(compare with X in (b))

Histogram for $X + Y$



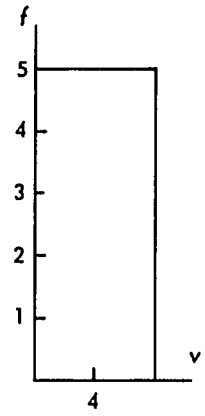
$M(X + Y) = 4$, variance of $X + Y = 6$,
s.d. of $X + Y = \sqrt{6} \approx 2{\cdot}4$,
range of $X + Y$ is 8.

(f) Histogram for X



$M(X) = 2$, variance of $X + Y = 2$,
s.d. of $X = \sqrt{2} \approx 1{\cdot}4$,
range of X is 4.
All answers for Y are the same as for X.
(compare with X in (b), Y in (e))

Histogram for $X + Y$



$M(X + Y) = 4$, variance of $X + Y = 0$,
s.d. of $X + Y = \sqrt{0} = 0$,
range of $X + Y$ is 0.

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