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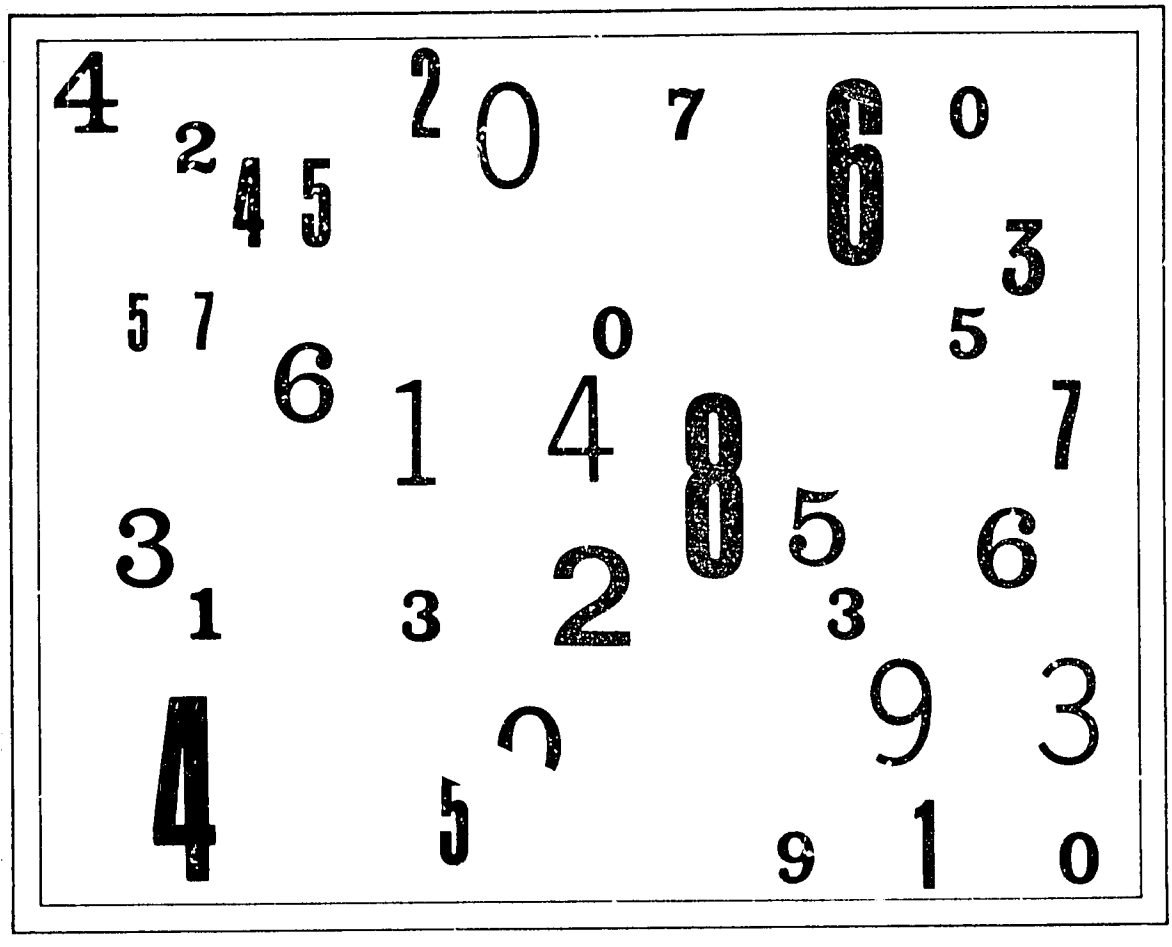

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Preliminary Edition

Prepared at the 1965
Mombasa Mathematics Workshop

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TO THE STUDENT

In using this textbook you have the opportunity to continue in the great experiment which was started in the summer of 1962, when more than fifty mathematicians and mathematics teachers gathered in Entebbe, Uganda, to begin the preparation of mathematics books as up-to-date as those in use anywhere else in the world. The writers came from most countries in English-speaking Tropical Africa, from the United States and from the United Kingdom to take part in this project. In the summer of 1962, *Secondary One* of the Entebbe Mathematics Series was written, in the summer of 1963, *Secondary Two;* in the summer of 1964, *Secondary Three;* and in the summer of 1965 this book. Now you and your teachers must do your share to find out whether this book is as good as it can be made. By using the textbook critically and making the most of the opportunities it offers to make the learning of mathematics an interesting and enriching experience, you will have played a part in the new growth of mathematics education in Africa.

Since you studied the earlier Entebbe Mathematics Series Secondary books, you know that you can expect to meet many familiar ideas, sometimes expressed in unfamiliar ways, and there will be some new ideas. Careful reading will be important because this book treats mathematics as a language designed to express certain kinds of ideas and you must learn to read and speak this language.

You will continue to learn why numbers behave as they do and find that algebra helps us more and more to understand arithmetic ideas. You will also explore some geometric ideas, some of which are familiar, some of which will be new, deducing these geometric facts from a small number of simple assumptions. You will also be using the idea of rigid motion in deducing geometric facts both familiar and new.

Do not hesitate to ask questions. Talk about your work with your teacher and with your fellow pupils. Above all, try to look upon mathematics as something that can be exciting and rewarding, for if you give it half a chance, that is exactly what it is.

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Chapter 1

ANOTHER LOOK AT THE DEDUCTIVE METHOD

1-1 WHY DO WE PROVE THEOREMS?

In your study of geometry in *Secondary Two* you learned many facts about geometric figures. Some of these facts were very easy to believe while others were rather surprising. In Secondary Three you saw how deductive arguments and the postulational method can be used to organize these facts and to discover new ones. Now that you have had more experience with deductive arguments, let us remind ourselves once again of the reasons for doing geometry in this way.

If your teacher tells you that the length of any side of a triangle is less than the sum of the lengths of the other two sides, most of you believe this at once. You believe it because many times in your life you have walked from one point A to another point C, and your experience tells you that it is nearly always faster to go from A to C in as nearly a straight line as possible than to follow a more roundabout path. If all of the facts of geometry were as simple and easy to believe as this one, it would not be necessary to study it in school.

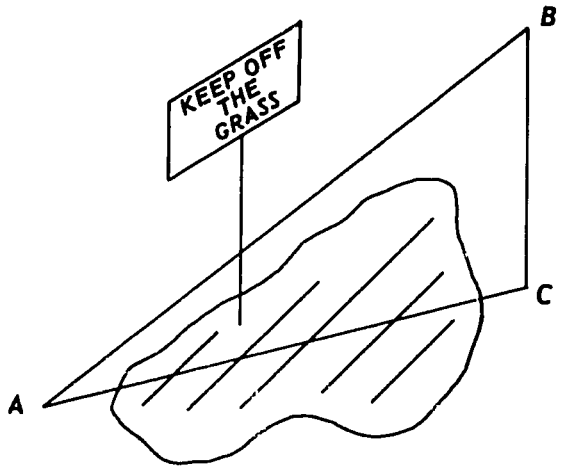


Fig. 1.

Suppose, on the other hand, that you take your compass and ruler and draw a figure like the one shown here. The centre of the circle is the point O, and the points A, B, and C are on the circle. Which of the angles $B\hat{O}A$ and $B\hat{C}A$ is smaller? Can you tell how much smaller? If your teacher would tell you that $B\hat{C}A$ is *exactly half* as large as $B\hat{O}A$ would you be convinced of this at once? Some of you may, and that would be very loyal but not at all scientific. Others of you will want to draw several more figures. Measure with your protractor the angles in each of the following figures.

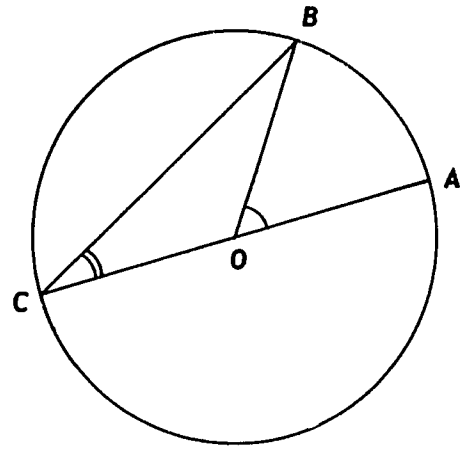


Fig. 2.

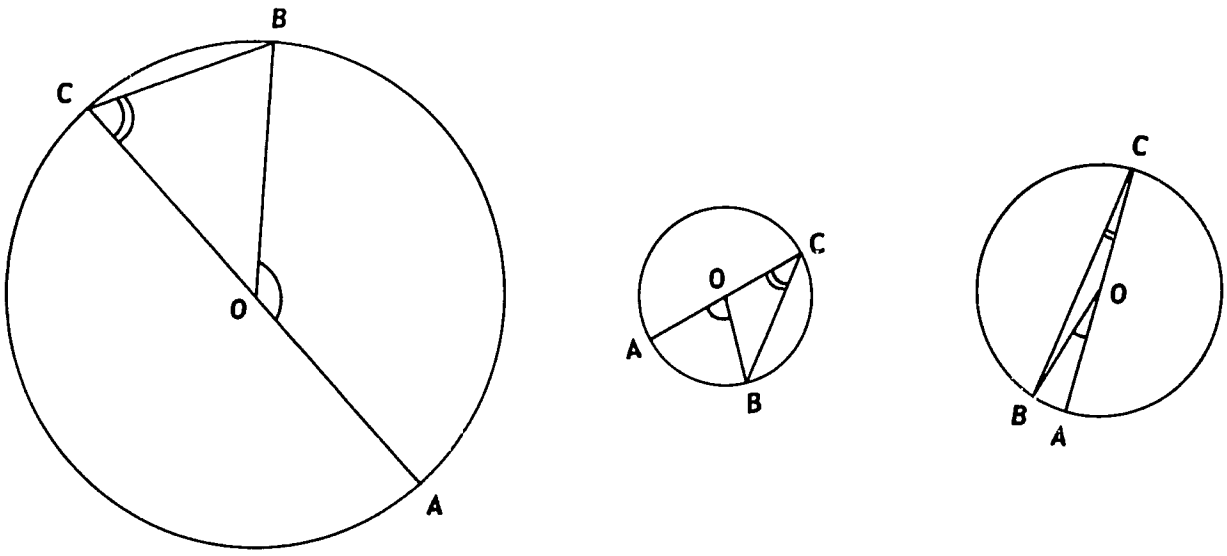


Fig. 3.

Do you find in each case that $B\hat{C}A$ is half as large as $B\hat{O}A$? Are your measurements accurate enough to be sure that it is *exactly* half as large?

You have learned, of course, that it is not possible to settle this question by measuring a million figures or even a million million figures. This would take your whole lifetime, and when you are very very old your friend might still say to you "I'm sorry, but I just don't believe that $B\hat{C}A$ is exactly half as large as $B\hat{O}A$. My measurements only show that it is

approximately half as large, and for very small circles I can't even make the measurements at all." How can you convince your friend? Surely you can't threaten to feed him to a crocodile.

Last year you learned that a much better way to settle such disputes is to give a deductive proof. You can say to your friend,

"Now look here. Angles $B\hat{C}A$ and $C\hat{B}O$ are congruent because the triangle ΔCBO is isosceles. Let x be the measure of each of these two angles; look at my figure. Then angle $B\hat{O}C$ has measure $180 - 2x$. Therefore angle $B\hat{O}A$ has measure $2x$ because it is supplementary to $B\hat{O}C$. This proves that $B\hat{O}A$ is twice as large as $B\hat{C}A$."

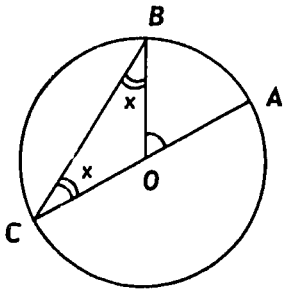


Fig. 4.

It is likely that your friend will now be convinced. For you have shown him *why* $B\hat{O}A$ must be exactly twice as large as $B\hat{C}A$. Rather it is more accurate to say that he will be convinced *if* he accepts the two statements which you used in your proof:

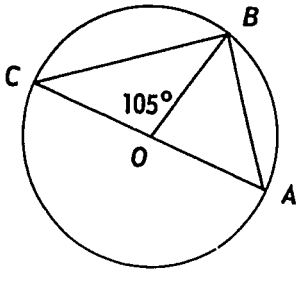
1) The base angles of an isosceles triangle are congruent.
2) The sum of the measures of the angles in any triangle is equal to the measure of a straight angle.

Since you and your friend have studied *Secondary Three* geometry together, he will probably believe both of these statements.

Every deductive argument has the two features illustrated by the proof given above. There are certain facts which are accepted to start with; we usually call them *assumptions* or *hypotheses*. And there is a clear and orderly argument which shows *why* the conclusion must follow from these hypotheses.

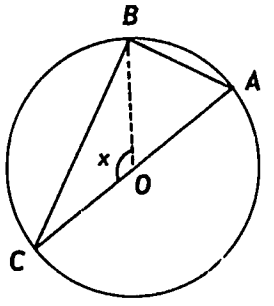
PROBLEMS 1-1

1. Use the figure given here to compute the measure of each of the following angles.



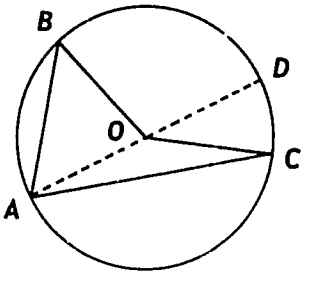
a) $B\hat{A}O$ d) $C\hat{B}O$
b) $B\hat{O}A$ e) $O\hat{B}A$
c) $B\hat{C}O$ f) $C\hat{B}A$

2. The angle CBA in Problem 1 is said to be *inscribed in the semicircle.* What did you discover about such an angle in Problem 1? Prove that this must always be the case by computing the measure of each of the following angles.
(Let the measure of $B\hat{O}C$ be x and fill in the blanks.)

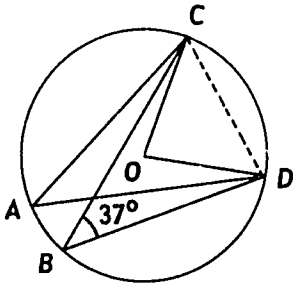


(a) $m(B\hat{O}C) = x$ (d) $m(B\hat{C}A) =$ ____
(b) $m(B\hat{A}O) =$ ____ (e) $m(C\hat{B}A) =$ ____
(c) $m(B\hat{O}A) = 180 - x$

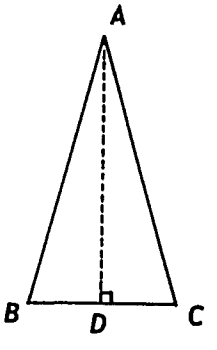
3. From the accompanying figure deduce that $m(B\hat{A}C) = \frac{1}{2}m(B\hat{O}C)$.
Hint: First show that $m(B\hat{A}D) = \frac{1}{2}m(B\hat{O}D)$ and $m(D\hat{A}C) = \frac{1}{2}m(D\hat{O}C)$.



4. Compute the measure of each of the following angles, using the result of Problem 3.
a) $C\hat{A}D$
b) $C\hat{O}D$
c) $O\hat{C}D$
Are $A\hat{C}B$ and $B\hat{D}A$ congruent? Why?

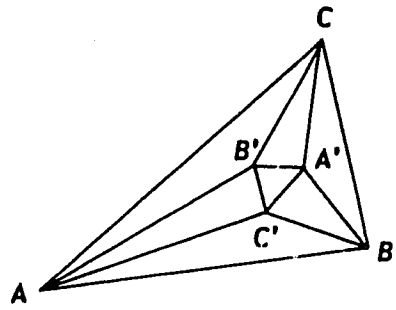


5. Your friend has drawn an isosceles triangle and has constructed the segment $\overline{AD}$ perpendicular to the base $\overline{BC}$. He then tells you, "The segment $\overline{AD}$ bisects the vertex angle $\hat{A}$ because the two right triangles, ΔADB and ΔADC, are congruent." What are the assumptions (hypotheses) which are made in his argument?
Hint: It may be easier for you to list the assumptions if you first fill in the reasons in the following expanded version of your friend's proof.



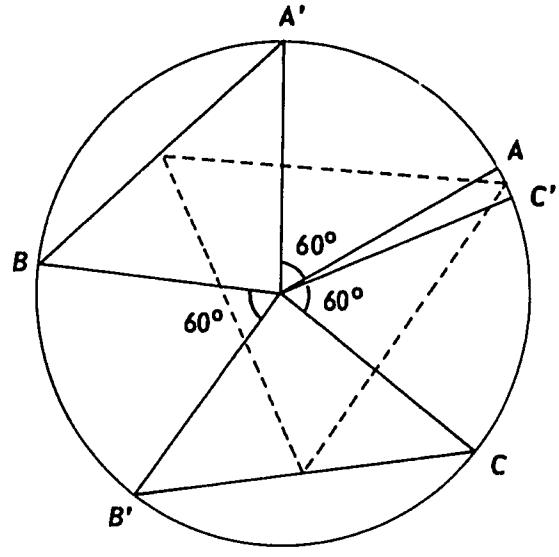
	Statements	*Reasons*
1.	$A\hat{B}C \equiv A\hat{C}B$	1.
2.	$\overline{AD} \equiv \overline{AD}$	2.
3.	$\Delta ADB \equiv \Delta ADC$	3.
4.	$B\hat{A}D \equiv C\hat{A}D$	4.
5.	$\overline{AD}$ bisects $B\hat{A}C$	5.

6. Draw any triangle you wish. Then, with your protractor, trisect each of the angles $\hat{A}$, $\hat{B}$, and $\hat{C}$ and mark the points A', B', and C' as shown in the figure. Measure the sides of $\Delta A'B'C'$. Repeat this construction starting with triangles of different sizes and shapes. Now state a theorem which you believe to be true on the basis of your experiments.



This theorem can be proved by methods familiar to you but the proof is very difficult. However next year you will learn about powerful new tools in geometry with which giving a proof becomes almost routine, though it can be quite long.

7. Draw a circle and select any three points A, B, and C on this circle. Mark the points A', B', and C' by measuring 60° angles as shown in the figure. Finally, draw the triangle which connects the midpoints of the segments $\overline{A'B}$, $\overline{B'C}$, and $\overline{C'A}$. What do you discover about this triangle? Is it obvious to you that this must happen? [Next year you will be able to prove this fact as well as the one in Problem 6.]



1-2 WHAT IS A CORRECT PROOF?

Pupils taking geometry for the first time often ask, "How do I know whether I have written a correct proof?" Unfortunately, this is a difficult question to answer. Making correct proofs is something that each of us learns by experience, and your study of geometry last year has probably given you a pretty good idea of the way in which proofs are given. It will help you a great deal if you remember that a deductive proof consists

essentially of clear and orderly thinking. What is most important is that the proof be *convincing*, and it matters very little whether you present the proof on the blackboard, or whether you write a little paragraph as was done in Section 1-1, or whether you arrange your argument in two columns giving statements in one column and reasons in the second column as you did many times in *Secondary Three.*

When given in two-column style, the proof given in Section 1-1 looks as follows (see Figure 4.).

Statements	*Reasons*
1. $CO = BO$	1. Construction. They are both radii of the same circle.
2. ΔBOC is isosceles.	2. Definition
3. $m(B\hat{C}A) = m(C\hat{B}O)$	3. The base angles of an isosceles triangle have the same measure.
4. $m(B\hat{O}C) = 180 - 2 \times m(B\hat{C}A)$	4. The sum of the angles in any triangle is 180°.
5. $m(B\hat{O}A) = 2 \times m(B\hat{C}A)$	5. $B\hat{O}C$ and $B\hat{O}A$ are supplementary angles by construction.

Writing the proof in this form reminds us that we must be able to give a reason for every statement that we make. And it also makes it easier to see the hypotheses which we accept to start with. Do you see the two hypotheses among the reasons in the second column?

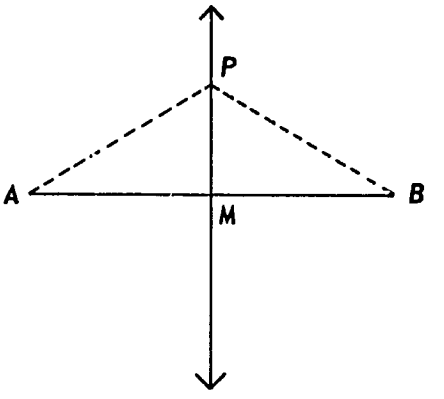
When you gave a proof to your friend in Section 1-1, you didn't bother to give all of the reasons because you felt that they were so easy that your friend would know them without being told. If you were trying to convince a pupil who has had much less practice, however, you would have had to give all of the reasons very carefully. This same pupil might not even believe your two hypotheses. Did you believe them one year ago? If he were asked to give a proof of the same fact, he would start with simpler hypotheses and give a longer proof. He might write, for example:

Statements	*Reasons*
1. $CO = BO$	1. Construction. They are both radii of the same circle.
2. $\Delta COB \equiv \Delta BOC$	2. SAS congruence postulate.
3. $B\hat{C}O \equiv C\hat{B}O$	3. They are corresponding parts of the congruent triangles in 2.
4. $m(B\hat{C}O) = m(C\hat{B}O)$	4. Congruent angles have the same measure.
5. $m(B\hat{C}O) + m(C\hat{B}O) + m(B\hat{O}C) = 180$	5. The sum of the measures of the angles in any triangle is 180°.
6. $m(B\hat{O}C) = 180 - 2 \times m(B\hat{C}O)$	6. Follows from 4 and 5.
7. $m(B\hat{O}A) = 2 \times m(B\hat{C}O)$	7. $B\hat{O}A$ and $B\hat{O}C$ are supplementary angles by construction.

We notice that this pupil has begun with different hypotheses (numbers 2, 4, and 5 in the second column) and has taken smaller steps in his proof. Is this a better proof than the first one? Is either one of these proofs better than the paragraph proof in Section 1-1?

The answer is that they are all good proofs. Each of them is clear and orderly. And each of them makes it clear what the starting hypotheses are. Which one you choose depends entirely on *whom* you want to convince. The more experienced that person is, the shorter your proof may be. Probably your teacher will want you to give proofs which are convincing to him and to everyone in your class. You will probably also be allowed to write your proofs in paragraph form. But you should be able to fill in the reasons whenever anyone asks for them.

1. "Every point on the perpendicular bisector of the segment $\overline{AB}$ is equidistant from the end points of the segment." The preceding statement is true, because M itself is equidistant from A and B and because any other point P on the bisector determines two congruent right triangles, $\triangle PMA$ and $\triangle PMB$, of which $\overline{AP}$ and $\overline{BP}$ are corresponding parts. Rewrite the proof in two-column style.

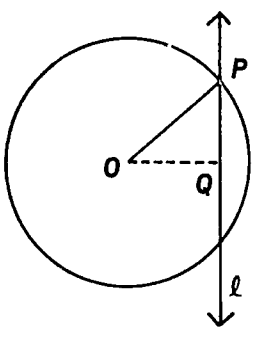


Hint: Fill in the blanks.

Statements	*Reasons*
1. $AM = MB$	1.
2. $m(P\hat{M}A) = m(P\hat{M}B)$	2.
3. $PM = PM$	3.
4. $\triangle PMA \equiv \triangle PMB$	4.
5. $AP = PB$	5.
6. Every point of the perpendicular bisector is equidistant from A and B.	6. M is equidistant by 1, and any other point is equidistant by 5.

What are the hypotheses used in this proof?

2. Take a point P on a circle and let ℓ be any line passing through P which is not perpendicular to the radius $\overline{OP}$. Here is a proof that ℓ must intersect the circle *twice*. There is a unique segment $\overline{OQ}$ which is perpendicular to ℓ, and is such that the point Q lies on ℓ. Q is different from P because $O\hat{P}Q$ is not a right angle. Since the sum of the measures of $Q\hat{P}O$ and $P\hat{O}Q$ is 90°, $m(O\hat{P}Q)$ is less than 90°. Therefore $\overline{OQ}$ is shorter than $\overline{OP}$ because it is opposite a smaller angle in $\triangle OPQ$. Therefore the point



Q is inside the circle. But then ℓ must intersect the circle twice because it passes through a point inside the circle. Try to rewrite this proof in two-column form. What are the hypotheses used in the proof?

1-3 AN EXAMPLE OF AN INCORRECT PROOF.

The deductive method teaches us to be critical in our thinking and careful in our choice of starting hypotheses. We will understand this method even better if we see what may happen when we are *not* careful. Let ΔABC be any triangle, and draw the bisector of angle $\hat{B}$ and the perpendicular bisector of $\overline{AC}$ as shown in the figure. Mark the intersection point as D.

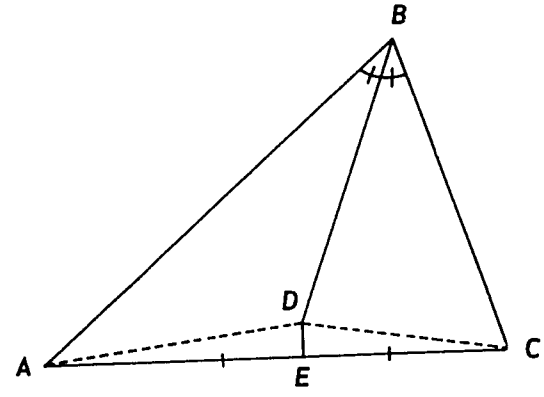


Fig. 5.

Now $\Delta DEA \equiv \Delta DEC$ (why?). Thus $\overline{AD} \equiv \overline{CD}$ because they are corresponding parts of these congruent triangles. Draw the perpendiculars from D to the sides $\overline{AB}$ and $\overline{CB}$ as shown in Figure 6. Then the right triangles ΔBDG and ΔBDF are congruent (why?) and so $\overline{GD} \equiv \overline{FD}$ and $\overline{GB} \equiv \overline{FB}$. But then the right triangles ΔAGD and ΔCFD are also congruent (why?) so $\overline{AG} \equiv \overline{CF}$. We have shown, therefore, that $GB = FB$ and $AG = CF$, and from these two equations we get $AB = CB$. Thus we conclude that the ΔABC is isosceles. But we started with *any* triangle, and hence we have proved that any triangle is isosceles.

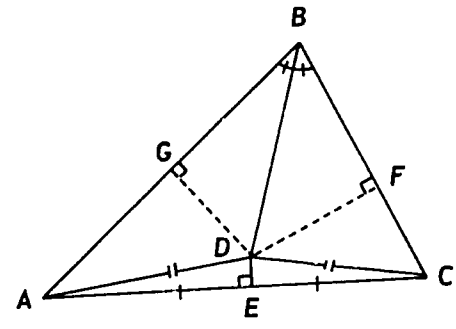


Fig. 6.

Do you believe the conclusion of this deductive argument? Probably not. But then what do you think is wrong? You will find that each step is clear and accurate, being in each case one of the facts which you learned last year about congruent triangles. But let us examine the hypotheses used in the proof.

(1) We assumed that we can construct the bisector of $\hat{B}$ and the perpendicular bisector of side $\overline{AC}$.

(2) We assumed, in drawing our figure, that these two bisectors are not parallel and that their point of intersection D lies *inside* the triangle ΔABC.

(3) We assumed that the perpendiculars $\overline{DG}$ and $\overline{DF}$ to the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$ could be constructed and that G was between A and B and F was between C and B.

(4) We assumed the facts about congruent triangles which you learned and proved last year.

Do you believe all of these hypotheses? Which do you *not* believe? You will be quite right if you point out number (2). If you draw the figure carefully you will find that the point D falls outside of the triangle (unless the triangle is isosceles). In fact you will find in Problem 2 that the third hypothesis is also false. It is hardly surprising, then, that we did not believe the conclusion of the proof.

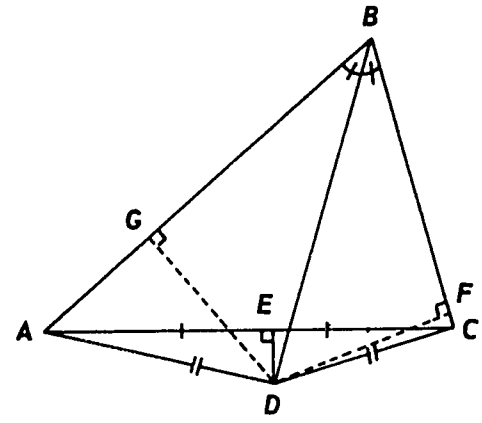
This example should convince us that one must be very careful to know *all* of the hypotheses used in a proof. Particularly when we use figures, we must make sure that no false hypotheses are hidden in the figure.

We might feel less disappointed in ourselves if we point out that even the greatest mathematicians who have ever lived have sometimes fallen into the trap of assuming incorrect hypotheses without realizing it. But you will not fall into this same trap if you keep your eyes and mind wide open at all times. This is perhaps the most important lesson to be learned from a study of the deductive method.

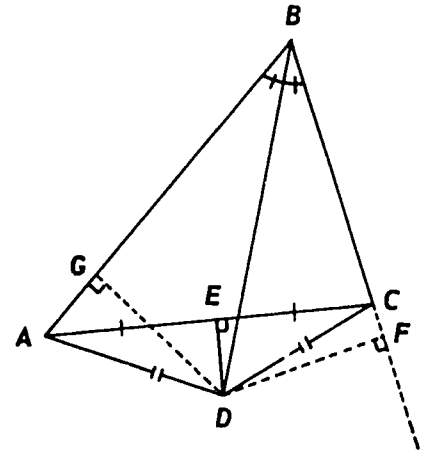
PROBLEMS 1-3.

1. Draw several triangles of different sizes and shapes and very carefully construct the angle bisector and the perpendicular bisector of the opposite side as was done in this section. Where does the point of intersection of these two bisectors lie—inside or outside the triangle?

2. Let us start with a different figure and repeat the proof of this section. Follow the proof line by line and verify that one still comes to the conclusion that the triangle is isosceles. What false assumption is made by the figure this time?



3. Let us now begin with as accurate a figure as possible. This time the figure does not hide any false assumptions. Follow the proof once again, this time referring to the new figure. Which *step* of the proof is not true?



1-4 ANOTHER EXAMPLE.

Your experience in geometry last year was more than just an introduction to the deductive method. You also discovered the rather surprising fact that most of the interesting theorems of geometry can be proved from a very small number of assumptions which you called *postulates*. Do you remember how many postulates you had?

This year we shall pay not quite as much attention to postulates and detailed proofs as we shall to the geometric facts themselves. We shall still use deductive proofs to convince ourselves about new facts, of course, but we shall not always take the time to show that all of our hypotheses can be proved from the original postulates. As long as the hypotheses are ones which we accept as true, they will suit our purpose.

If your friend insists upon giving proofs for all of these hypotheses, you may tell him that it is possible to do so and that it is indeed an interesting exercise. But while he is busy proving these facts of which we are already convinced anyway, we shall go on to new and interesting topics.

We shall give one more example in this section of how to discover and prove a new geometrical fact. And in the next section we will discover a theorem of a quite different and exciting kind. Pay particular attention while studying these examples to notice how inductive and deductive reasoning work together. And try very hard *not* to be so involved in the deduction that you stop using your imagination and common sense.

CLASS ACTIVITY

Draw any triangle ΔABC, bisect each of the three sides, and draw the line segments $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ as shown in the figure. Perhaps you remember that these segments are called the *medians* of the triangle ΔABC.

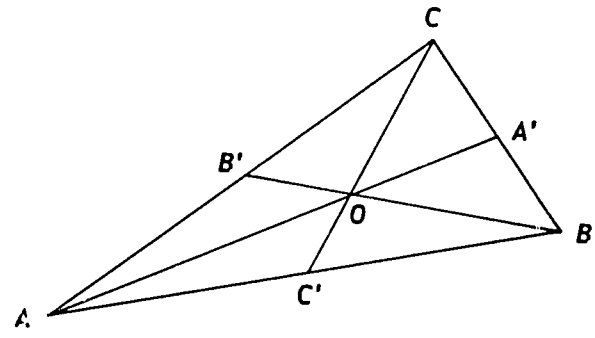


Fig. 7.

If you carry out the construction very carefully, you discover that the medians intersect in a single point O. Do you think that this must always be true? Try it out for several different triangles. Let's try to find a deductive proof.

Sometimes, when trying to prove a theorem, it is easier to prove an even stronger theorem. With your ruler, measure the segments

$$\overline{OA'}, \overline{AA'};\ \overline{OB'}, \overline{BB'};\ \overline{OC'}, \overline{CC'}.$$

What do you discover about the lengths of these segments? Do you find in each case that the longer segment is three times as long as the shorter one? Let us prove that it is.

In the triangle ΔABC with medians $\overline{AA'}$ and $\overline{BB'}$, draw the segment $\overline{A'B'}$. Then draw $\overline{A'D}$ parallel to $\overline{B'B}$ with D on the line $\overleftrightarrow{AB}$. Likewise draw $\overline{A'E}$ parallel to $\overline{AC}$ with E on the side $\overline{AB}$.

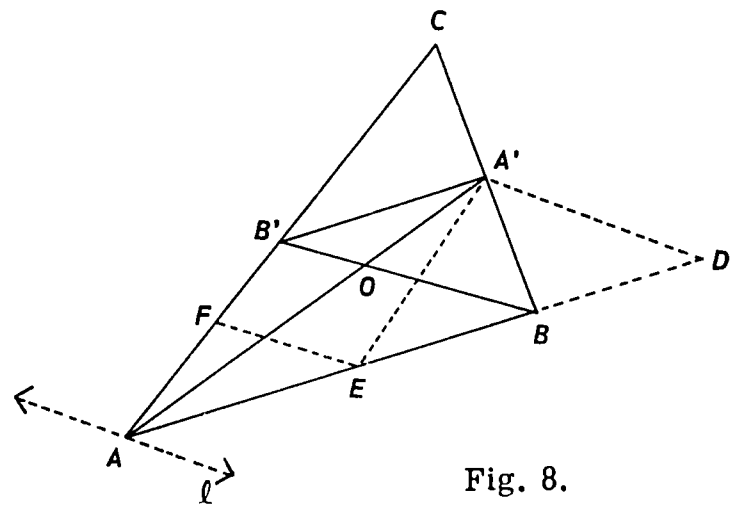


Fig. 8.

And finally draw $\overline{EF}$ and the line ℓ both parallel to $\overline{B'B}$ with F on the side $\overline{AC}$ and ℓ passing through the point A. All of these constructions are shown in the figure.

Tell why each of the following statements is true:

(1) $B'A'EA$ is a parallelogram.
(2) $B'A'DB$ is a parallelogram.
(3) $B'A' = BD$ and $B'A' = AE$.
(4) $EB = AE = BD$.

We see, therefore, that the segment $\overline{AD}$ is divided into three congruent parts by the parallel lines ℓ, $\overleftrightarrow{FE}$, $\overleftrightarrow{B'B}$, and $\overleftrightarrow{A'D}$. But these same parallel lines divide the segment $\overline{AA'}$ into three segments, and we know therefore that these three segments are congruent. Thus $OA' = \frac{1}{3} AA'$ which is what we wished to show.

What have we proved so far? We have proved that the median $\overline{BB'}$ intersects the median $\overline{AA'}$ in a point O whose distance from A' is $\frac{1}{3}$ of the distance AA'.

Suppose that we now draw the medians $\overline{AA'}$ and $\overline{CC'}$. Where do *they* intersect? You see at once that the same argument may be used to show that $\overline{CC'}$ intersects $\overline{AA'}$ in a point O' whose distance from A' is $\frac{1}{3}$ of the distance AA'. But this means that $OA' = O'A'$, and thus $O = O'$. Therefore all three medians pass through the single point O. Let us state this result as a theorem.

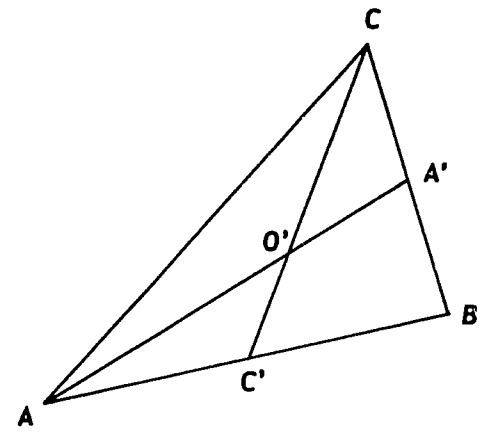


Fig. 9.

> The medians of a triangle meet in a single point O. For each median this point is located $\frac{2}{3}$ of the way from the vertex to the opposite side.

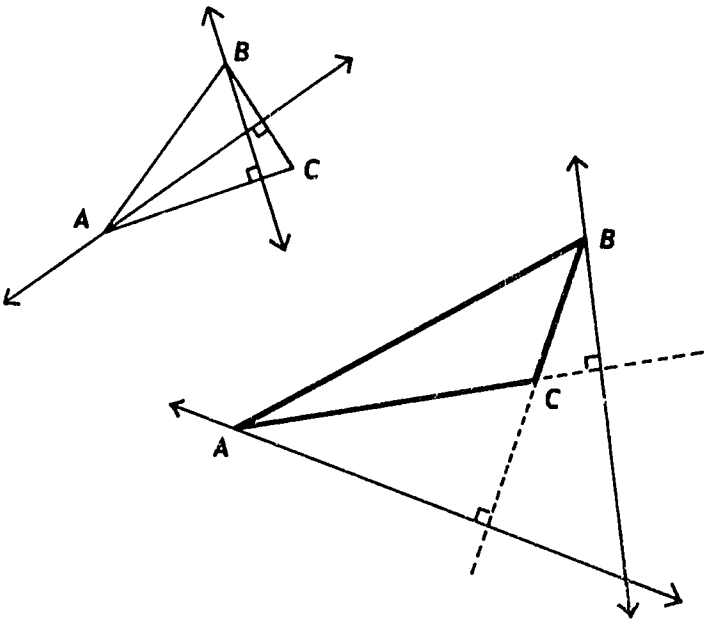
The point O where the medians intersect is called the *centroid* of the triangle. It is the point at which the triangular region could be made to "balance" on the point of a pin.

PROBLEMS 1-4

1. Make a list of facts which are assumed in the proof given in this section.

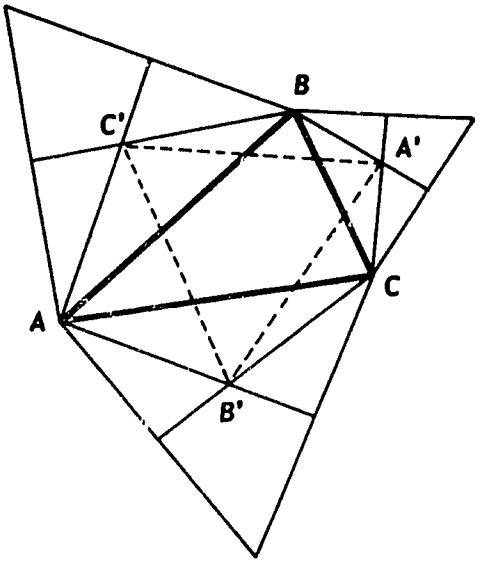
2. Draw several triangles and construct the perpendicular bisectors of the three sides. Choose at least one triangle which has an angle greater than 90°. What do you discover? State a theorem about the perpendicular bisectors which you think likely to be true.

3. The *altitudes* of a triangle ΔABC are the lines through the vertices which are perpendicular to the opposite sides. The altitudes through A and B are shown for each of the triangles in the figure. Draw several different triangles, at least one of which has an angle greater than 90°, and construct the three altitudes in each case very carefully. What do you discover? State a theorem about the altitudes which you think likely to be true.



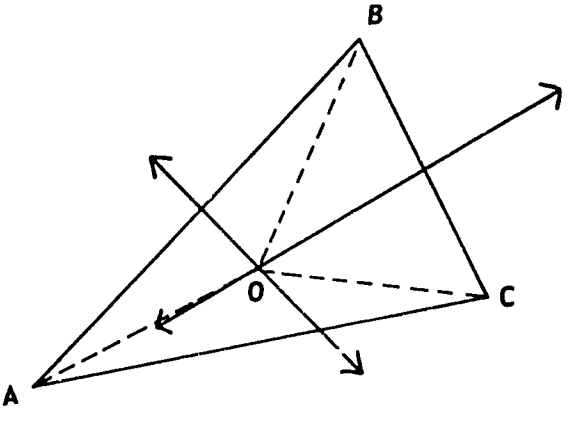
4. Draw any triangle ΔABC and construct the three angle bisectors. What do you discover? State a theorem about the angle bisectors which you think likely to be true.

5. Draw any triangle ΔABC and construct an equilateral triangle on each of the three sides. For each of these equilateral triangles, find the point where the medians intersect. These are marked in the figure as A', B', and C'. What do you discover about the triangle $\Delta A'B'C'$? Do you think it is likely that this *must* be true?



Each of problems 2-5 was an example of the discovery of a theorem by means of *inductive reasoning*. Deductive proofs for the facts discovered in problems 2-4 are easily constructed if you follow the hints given in the next three problems. A proof for Problem 5 is most easily given using the methods of Chapter 6.

6. Let ΔABC be given and construct any two of the perpendicular bisectors of the sides. For example the perpendicular bisectors of $\overline{AB}$ and $\overline{CB}$ are shown in the figure.
Answer each of the following questions (in the order given):



(a) Why do these perpendicular bisectors intersect; that is, why can't they be parallel? (Call their point of intersections O.)

(b) Why is the point O equidistant from A and B?

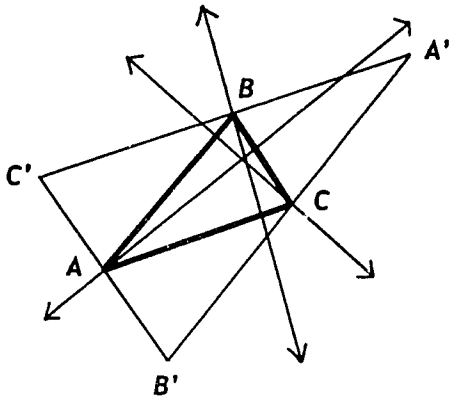
(c) Why is the point O equidistant from B and C?

(d) Why is the point O equidistant from A and C?

(e) Why must the perpendicular bisector of the remaining side $\overline{AC}$ pass through the point O?

(You see by examining your arguments that the point of intersection of the three perpendicular bisectors is equidistant from the three vertices of ΔABC. Thus the *circumscribed* circle of ΔABC, that is the circle passing through the three vertices, can be drawn by taking its centre at the point O.)

7. To show that the three altitudes of ΔABC meet in a single point, construct the triangle $\Delta A'B'C'$ as shown in the figure so that the three sides of $\Delta A'B'C'$ are parallel respectively to the three sides of ΔABC.



Answer each of the following questions:

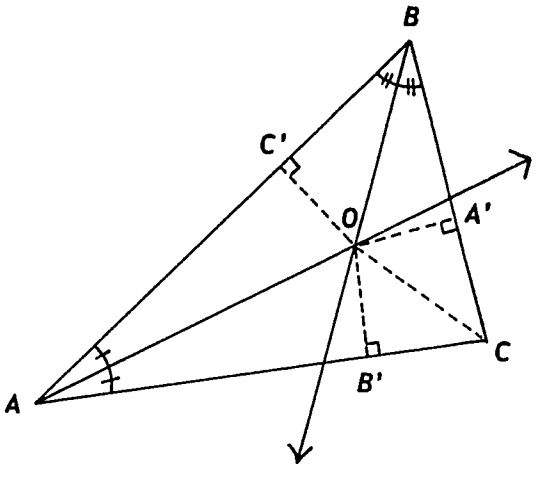
(a) Can you find three parallelograms in the figure? Name them.

(b) Why is $A'C = B'C$? (Hint: Compare them with AB)

(c) Why is the altitude of ΔABC through the vertex C the same as the perpendicular bisector of $\overline{A'B'}$?

(d) Do you see that the altitudes of ΔABC are the same as the perpendicular bisectors of the sides of $\Delta A'B'C'$? Now use the result of Problem 6.

8. To show that the angle bisectors of ΔABC meet in a single point, answer the following questions.



(a) Why must the bisectors of angles $\hat{A}$ and $\hat{B}$ intersect? (Why can't they be parallel?)

(b) Construct the perpendiculars $\overline{OA'}$, $\overline{OB'}$, and $\overline{OC'}$ as shown in the figure. Why do these segments have the same length?

(c) Does the bisector of $\hat{C}$ pass through the point O? Hint: Show that $\Delta OCA' \equiv \Delta OCB'$.

(While giving the proof that the angle bisectors meet in a single point, you also discovered that their point of intersection is equidistant from the three sides of ΔABC. The circle with centre at O which passes through A', B', and C' is called the *inscribed* circle of ΔABC. This circle touches all the three sides of the triangle ΔABC—at the points A', B', C'.)

1-5* EULER'S FORMULA.

Examine the figures shown below. Each figure represents a hollow three-dimensional object that is constructed from plane polygons by fastening them together along their edges. You can make models of these objects by cutting triangles, squares, pentagons, etc., out of paper and gluing them together in appropriate fashion.

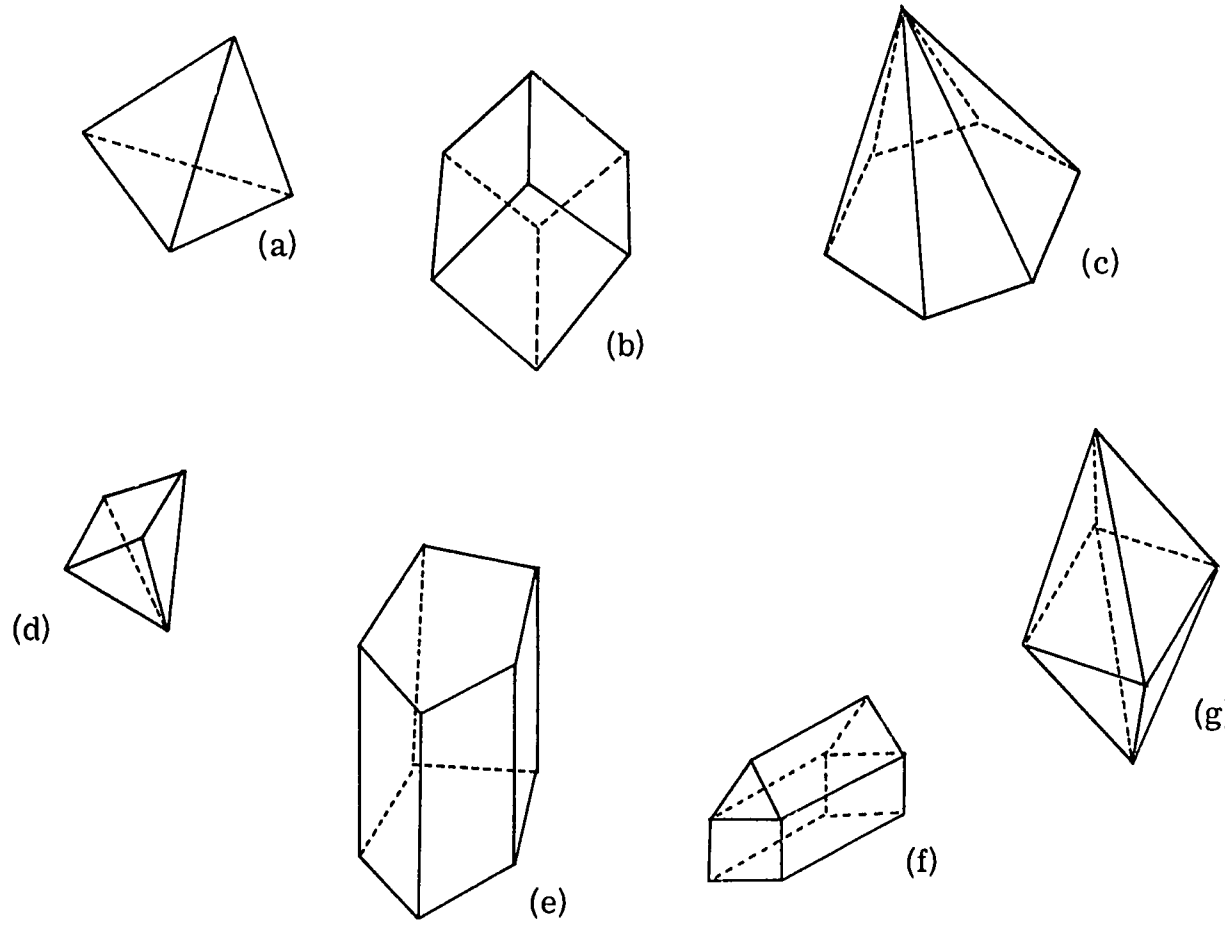


Fig. 10.

Such objects are called *polyhedra*, although you will recognize some of them as having special names. Can you find a cube, a tetrahedron, a square pyramid, a pentagonal prism? Which of these objects is constructed from eight triangles? Do you know what this object is called? What name might you apply to (c)?

The line segments along which the polygons are joined are called the *edges* of the polyhedron. The corners are called its *vertices*, and the interiors of the polygons are called the *faces* of the polyhedron. Thus the cube has eight vertices, twelve edges, and six faces (count them). Fill in the following table giving the number of vertices, edges, and faces of each of the other polyhedra in Figure 10. Also fill in the last column of this table.

	NAME	V	E	F	$V - E + F$
(a)					
(b)	cube	8	12	6	2
(c)					
(d)					
(e)					
(f)	no name				
(g)	octahedron				

What do you discover by filling in this table? If you have counted correctly, the last column will contain the number 2 in each case. Is it possible that the value of $V - E + F$ *must* be 2 for any polyhedron no matter how complicated? Let's try another example. Count the numbers of vertices, edges, and faces of the star-shaped polyhedron shown in the next figure. You can construct this polyhedron, if you wish, by gluing together 16 triangles and 2 squares.

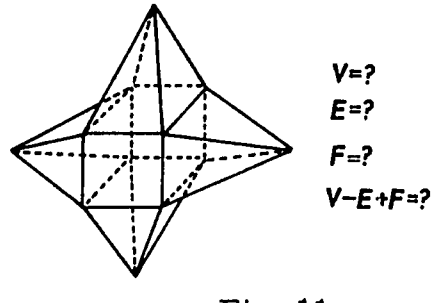


Fig. 11.

You are probably beginning to believe that we have found a very general relationship between the number of vertices, edges, and faces of any polyhedron. In all cases that have been considered so far, we have found that $V - E + F = 2$. But how can we convince ourselves that this must be true?

One of the ways that complicated problems can be solved in mathematics is to reduce the problem to a simpler situation. In the case of our polyhedra, it is difficult to draw these objects because they are three-dimensional, and it is difficult to count the number of vertices, edges, and faces even if we succeed in getting a good picture. Let us replace each polyhedron, therefore, by a *plane* figure obtained by removing one face and spreading the polyhedron out flat. For example, if we remove the face of the cube which is shaded in the figure below, then we can spread the remaining five faces out flat as shown.

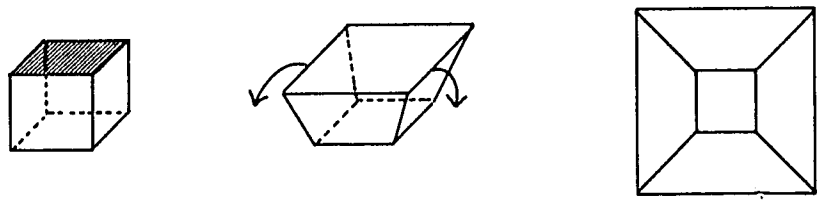
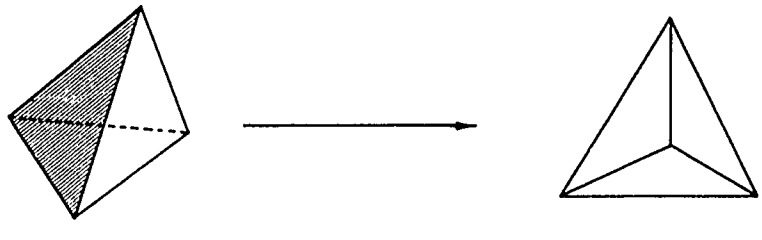


Fig. 12.

It is necessary to imagine that the cube is made of rubber or of some other elastic material in order that the face may be stretched out into the plane. Each of the other polyhedra shown in Figures 10 and 11 can be spread out in a similar way after a face has been removed. Some of the resulting plane figures are shown below in Figure 13. In each case it is the shaded face of the polyhedron which has been cut away.

(a)



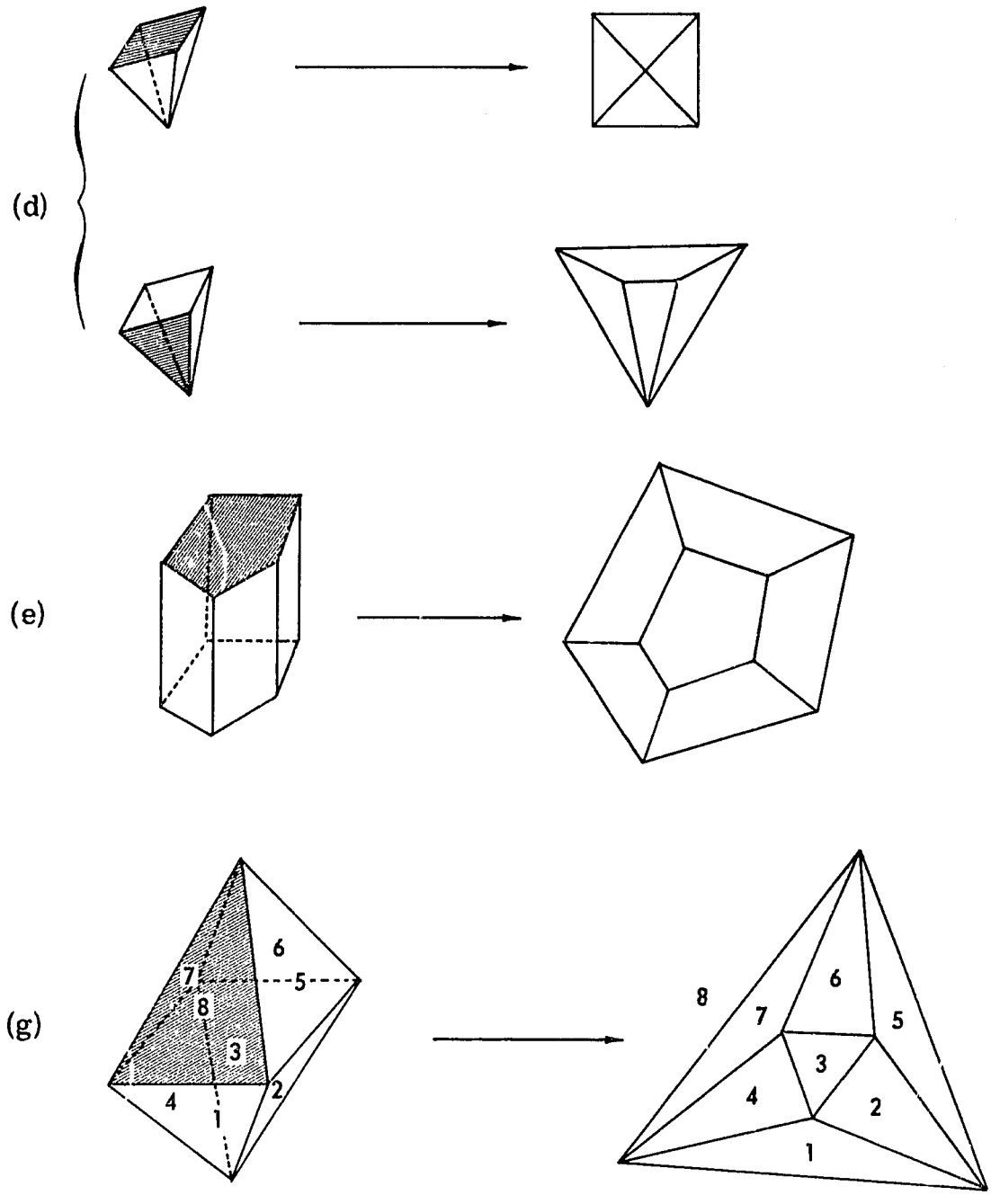


Fig. 13.

In each of these examples, the vertices and edges of the polyhedron are converted into points in the plane and straight line segments connecting these points. Let us continue to call them *vertices* and *edges*, and let us call the whole plane figure a *network*. (Some people are reminded of a network of roads connecting several towns.)

Into how many regions do each of the networks in Figure 13 divide the plane? Do you see that these regions correspond to the faces of the original polyhedron? Which region corresponds to the face that was removed?

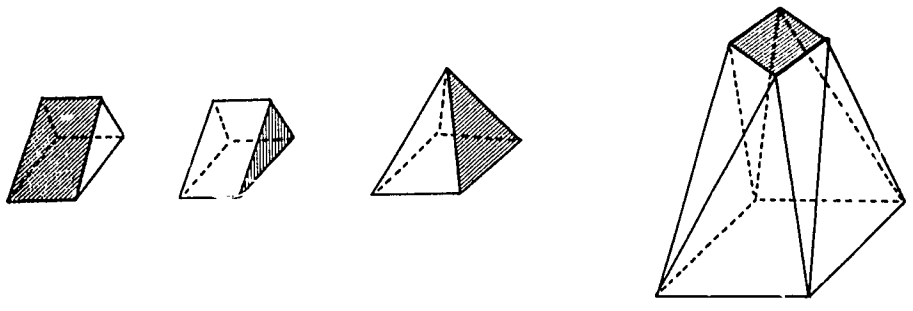
CLASS ACTIVITY

In Figure 13(a), point out the regions corresponding to each of the four faces of the tetrahedron. Which vertex of the network corresponds to the vertex of the tetrahedron opposite the shaded triangle? Which edges of the network correspond to the edges of the shaded triangle?

In Figures 13(d), the square pyramid is spread out in two different ways. Point to the vertex of the network in each case which corresponds to the bottommost vertex of the pyramid. Point to the four edges in the network in each case which correspond to the edges of the square base of the pyramid.

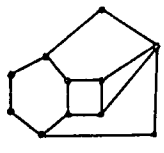
PROBLEMS 1-5A

1. Draw figures showing what each of the following polyhedra look like when the shaded face is removed and the polyhedron is spread out into a plane network.

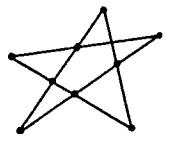


2. For each of the following networks, count the number V of vertices, the number E of edges, and the number F of regions, and fill in the table which is given.

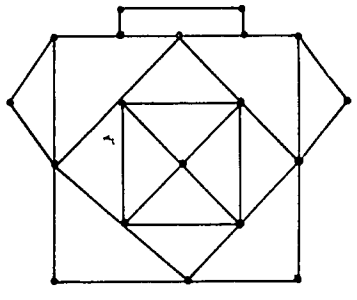
(a)



(b)



(c)



	V	E	F	$V - E + F$
(a)				
(b)				
(c)				

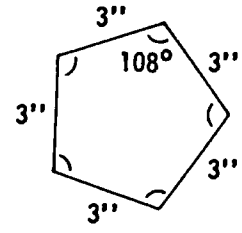
3. Construct several of the following models.

(a) A tetrahedron by gluing together 4 congruent equilateral triangles.

(b) A cube, by gluing together 6 congruent squares.

(c) An octahedron, by gluing together 8 congruent equilateral triangles.

(d) Cut out 12 pentagons as shown, and glue them together to make a polyhedron. The resulting object is called a dodecahedron (meaning 12 faces).



Keep for later use any of these models which you make.

4. Try to construct a polyhedron from 20 congruent equilateral triangles. Hint: Five triangles come together at each vertex.

You have seen how the value of $V - E + F$ may be computed for any polyhedron by counting the numbers of vertices, edges and regions of a certain plane network which is obtained from the polyhedron. Let us now perform several experiments with networks.

Draw a plane network such as the one shown in the figure. For this figure we find that

$V = 13$

$E = 18$

$F = 7$

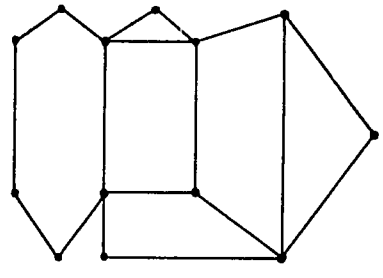


Fig. 14.

What is the value of $V - E + F$? Now select any one of the polygons in the figure which is *not* a triangle, and draw an additional edge which connects two of its vertices. Does this change the number of vertices? How many regions does it add? What is the new value of $V - E + F$? You will have discovered that adding an edge in this way does not change the value of V, while both E and F increase

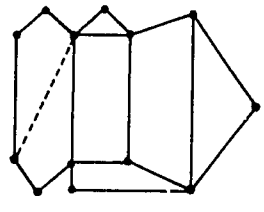


Fig. 15.

by 1. Since E and F occur in the expression $V - E + F$ with opposite signs, the value of $V - E + F$ is not changed. If you are given any network, therefore, you may draw as many additional edges in the above fashion as you wish. In fact you may do this until the only polygonal regions which are left are triangular.

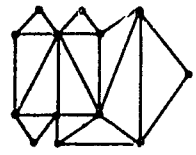


Fig. 16.

Now perform a second experiment. Begin with any network which has only triangular regions, and consider what happens if we "remove" one of these regions which lies along the outside edge of the network. There are two cases that we must consider. The region numbered 1 in Figure 17 has *two* edges in common with the network. This region may be "removed" by taking away the edge $\overline{AB}$ but *not* the vertices A and B. This results in the network shown in Figure 18(a). On the other hand, the region numbered 2 in Figure 17 has only *one* edge in common with the network. We may "remove" this region, therefore, by taking away edges $\overline{BC}$ and $\overline{CD}$ together with vertex C, but *leaving* vertices B and D. This results in the network shown in Figure 18(b).

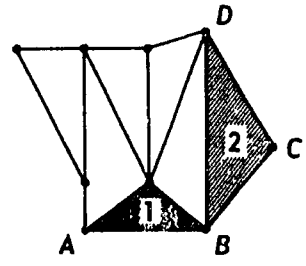


Fig. 17.

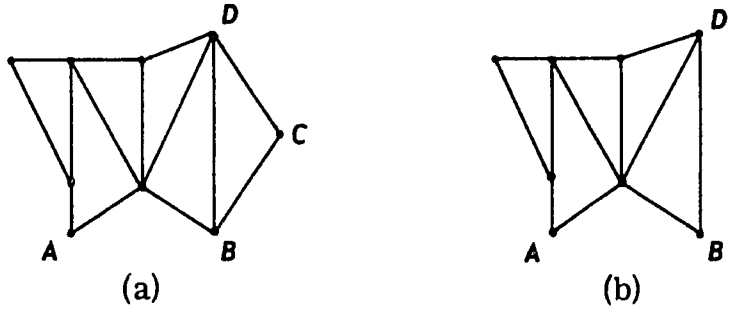


Fig. 18.

As we remove these triangles in the way described, let us see how the numbers V, E, and F change.

	Number of Vertices Removed	Number of Edges Removed	Number of Regions Removed
No. 1	0	1	1
No. 2	1	2	1

In each case the sum of the number of vertices and regions removed is equal to the number of edges removed. Do you see, therefore, that the value of $V - E + F$ does not change?

Let us now put together all that we have learned. Starting with a polyhedron, we can first spread it out into a plane network. Then we can draw additional edges until only triangular regions remain. Finally, we can "peel away" the triangles one-by-one until only one triangle is left. We have seen that none of these activities changes the value of $V - E + F$. Therefore, the value of $V - E + F$ is the same for the original polyhedron as it is for the network consisting of the single triangle. But here we can easily compute its value as

$$V - E + F = 3 - 3 + 2 = 2$$

Let us summarize our results as a theorem.

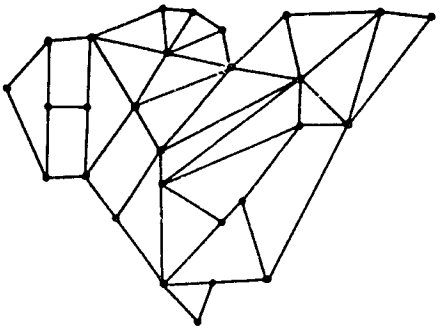
> For any simple polyhedron, the number of vertices, edges, and faces satisfies the equation $V - E + F = 2$.

The equation $V - E + F = 2$ is often called Euler's formula after the Swiss mathematician Leonard Euler (1707 - 1783) who first discovered it. In the next section we shall see how this formula may be used to obtain a very interesting result about polyhedra.

The use of the term *simple polyhedron* instead of just *polyhedron* is intended to emphasize that we are thinking of objects of the kind illustrated in Figures 10 and 11. If we were to imagine any one of these polyhedra to be made of rubber, we could inflate it into the shape of a sphere. These are exactly the polyhedra which can be flattened into a plane network in the manner described. Another way to describe a *simple* polyhedron is to say that its interior is one solid piece without any holes through it. Problem 4 gives an example of a polyhedron whose interior *does* have a hole through it and hence is *not* a simple polyhedron. You will see that in such cases Euler's formula does not hold.

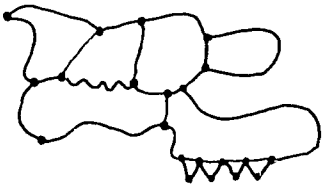
PROBLEMS 1-5B

1. Count the number of vertices and regions in the given network and *compute* the number of edges from the formula $V - E + F = 2$. Now *count* the number of edges and compare the results.

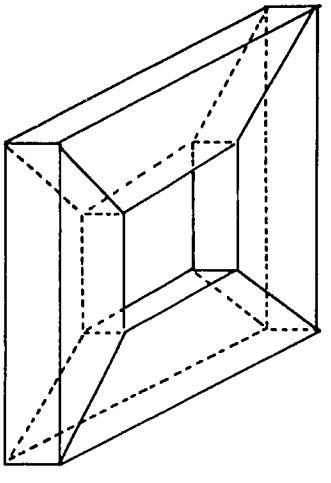


2. Your friend has just constructed the model suggested in Problem 4 of Problems 1-5A. He tells you that this model has 12 vertices and 28 edges. Do you believe him? Which piece of information is incorrect?

3. There is no special reason why the edges of a network must be straight line segments. The numbers V, E, and F remain the same if curved edges are used instead. Count the number of vertices, edges, and regions for the adjoining figure and verify that $V - E + F = 2$.



4. The figure shows a polyhedron which is *not* a simple polyhedron. Count the number of vertices, edges, and faces and show that Euler's formula does not hold. What value does $V - E + F$ have?



1-6* REGULAR POLYHEDRA.

When we are studying polygons, we sometimes like to distinguish between ordinary polygons such as the ones shown in Figure 19 and the regular polygons of Figure 20 for which all of the edges

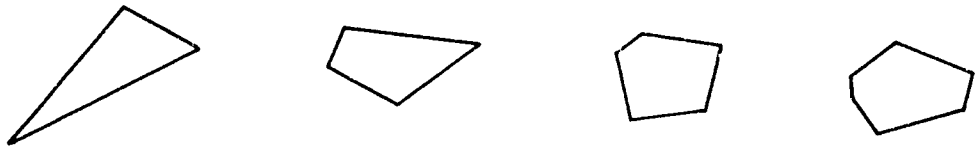


Fig. 19.

are congruent and all of the angles are congruent.

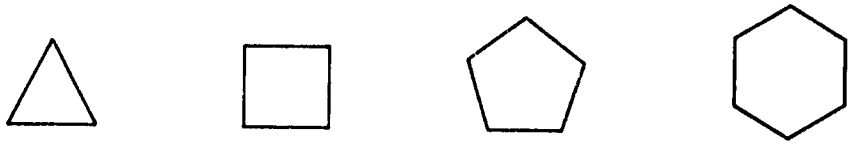


Fig. 20.

Regular polygons appeal to us as especially beautiful because of the many symmetries which they display.

When studying polyhedra we are likewise attracted by those which show particular simplicity and symmetry such as the tetrahedron and the cube shown in Figure 21.

Fig. 21.

The tetrahedron is constructed from four congruent equilateral triangles with three triangles coming together at each vertex, and the cube is constructed from six congruent squares with three squares being joined at each vertex. In general, a *regular polyhedron* is constructed from congruent regular polygons in such a way that the *same number* of polygons are joined at each vertex.

Do you know examples of regular polyhedra besides those shown in Figure 21? What object do you obtain by gluing together eight congruent equilateral triangles in such a way that four triangles come together at each vertex? Perhaps you have already constructed a model of this object.

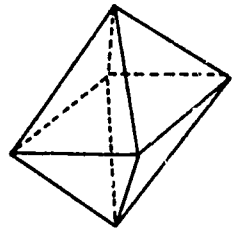


Fig. 22.

QUESTIONS FOR CLASS DISCUSSION.

Why is it not possible to construct a regular polyhedron from congruent squares by joining *four* squares at each vertex? Is it possible to construct a regular polyhedron from congruent regular hexagons? (Look at Figure 23).

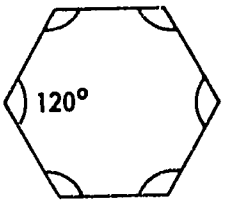


Fig. 23.

Let us try to find other examples of regular polyhedra. In particular, let's try to construct such a polyhedron from congruent regular polygons each of which has e edges. Let n be the number of polygons to be joined at each vertex. Do you see that the number n also represents the number of edges which are joined at each vertex?

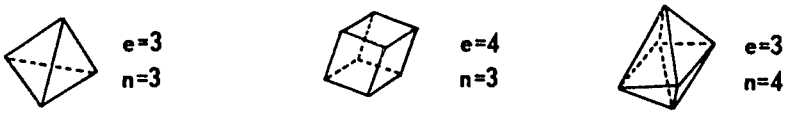


Fig. 24.

Suppose that you are told the number of vertices and faces of a certain regular polyhedron. Can you compute the number of edges? There are two ways to do this. First of all, there are F faces and each face has e edges. Thus the polygons have eF edges in all. But since each edge of the polyhedron is obtained by joining *two* edges of polygons, the total number of edges of the polyhedron must be $\frac{1}{2}\,eF$. Secondly, we can begin with the number V of vertices of the polyhedron and remember that n edges are joined at each vertex. Does the number nV represent the total number of edges of the polyhedron? Why must we take $\frac{1}{2}\,nV$ instead?

You have seen from this discussion that if either V or F is given to you, the value of E can be computed from the equation $E = \frac{1}{2}\,eF$ or $E = \frac{1}{2}\,nV$. If you multiply each of these equations by 2 and remember that the numbers V, E, and F must also satisfy Euler's formula, the following three equations result.

$$eF = 2E$$
$$nV = 2E$$
$$V - E + F = 2 .$$

Let us see what can be deduced from these equations.

Multiply the third equation by en. This gives $e(nV) - enE + n(eF) = 2en$. Then, using the first two equations, replace both nV and eF by $2E$. This results in the equation

$$2eE - enE + 2nE = 2en,$$

and you can now solve for E in terms of e and n. You should get the formula

$$E = \frac{2en}{2e - en + 2n} .$$

Verify that this formula works for each of the objects shown in Figure 24.

Now the number E must be a positive number since it represents the number of edges of a polyhedron. But this means that the expression $2e - en + 2n$ must also be positive; that is, $2e - en + 2n > 0$. Writing this inequality in the form $2e - n(e - 2) > 0$ and adding $n(e - 2)$ to both sides, we obtain $n(e - 2) < 2e$. Finally, since $e - 2$ is a positive number (why?), we may divide both sides of the last inequality by $e - 2$ and obtain

$$n < \frac{2e}{e - 2} .$$

What new facts do we learn about e and n from this inequality? Fill in the blanks below and then see if you can answer the question.

When $e = 3$, n is less than 6.
When $e = 4$, n is less than ____.
When $e = 5$, n is less than ____.
When $e = 6$, n is less than ____.
When $e = 7$, n is less than ____.

What happens when e is greater than 5? Do you see that the only possible values for e are 3, 4, and 5? When $e = 3$, you will see that $n < 6$, and hence n must have one of the values 3, 4, or 5. When $e = 4$, you know that $n < 4$, therefore the only possible value for n is 3. And when $e = 5$, you know that $n < 10/3$, so again the only possible value for n is 3.

If we put all of these facts together we see that there can be at most five kinds of regular polyhedra corresponding to the cases:

(1) $e = 3$ and $n = 3$
(2) $e = 3$ and $n = 4$
(3) $e = 3$ and $n = 5$
(4) $e = 4$ and $n = 3$
(5) $e = 5$ and $n = 3$

For each of these cases we can compute the numbers E, V, and F from the formulas:

$$E = \frac{2en}{2e - en + 2n}$$

$$V = \frac{2E}{n}$$

$$F = \frac{2E}{e}$$

For example in case (2), we use the values $e = 3$, $n = 4$, to obtain

$$E = \frac{2 \times 3 \times 4}{2\times3-3\times4+2\times4} = \frac{24}{6 - 12 + 8} = \frac{24}{2} = 12$$

$$V = \frac{2\times 12}{4} = \frac{24}{4} = 6$$

$$F = \frac{2\times 12}{3} = \frac{24}{3} = 8$$

Thus the polyhedron arising from this case is constructed from eight triangles ($F = 8$ and $e = 3$) with four triangles being joined at each vertex ($n = 4$). Do you recognize this polyhedron? Write its name in the proper space in the following table, and then fill in the remainder of the table.

	e	n	E	V	F	name
Case 1	3	3				
Case 2	3	4	12	6	8	
Case 3	3	5	30			icosahedron
Case 4	4	3				
Case 5	5	3				dodecahedron

Three of the polyhedra named in this table are familiar to you. The other two may be new, so we have filled in their names. The word *dodecahedron* means polyhedron with twelve faces, and the word *icosahedron* means polyhedron with twenty faces. Is this in agreement with the numbers you computed for F?

The discussion given above shows that there are at most five kinds of regular polyhedra. There is a picture of each of them in Figure 25, but you will be able to imagine them much better if you and your classmates construct models.

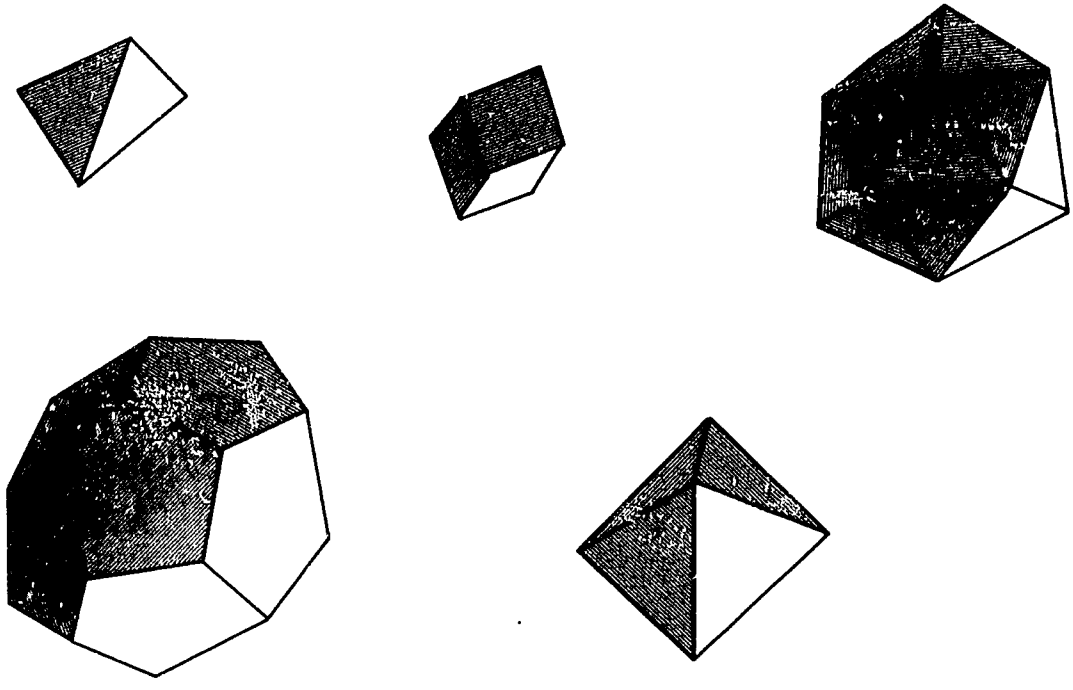
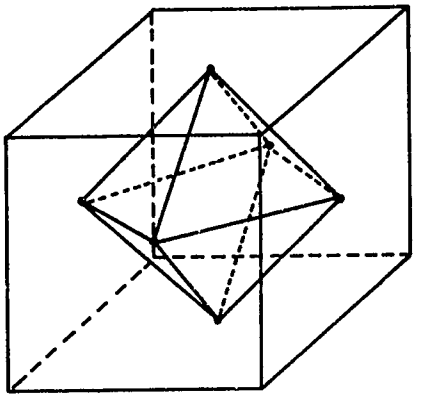


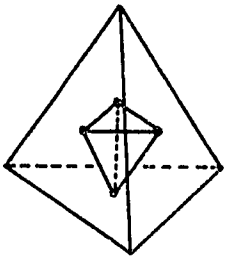
Fig. 25.

PROBLEMS 1-6

1. Do you notice an interesting relationship between the number of vertices, edges, and faces of the cube and the octahedron? Examine the following figure carefully. If the outside figure is a cube and the points which are marked are the midpoints of its faces, what can you say about the figure inside?



2. What figure results if one connects the midpoints of the four faces of a regular tetrahedron?



3. What relationship would you expect to find between a dodecahedron and an icosahedron? This will be difficult to imagine unless you have already constructed models of these two figures.

Chapter 2
AREAS OF POLYGONAL REGIONS

2-1 INTRODUCTION.

You are already familiar with the notion of area and have calculated areas of figures such as triangles, parallelograms and circles. In this chapter, we will consider the concept of area in a more precise manner. We will study area as a measure of polygonal regions just as we studied length as a measure of segments.

Let us recall our previous notion of area by considering the following problem:

A seamstress went into a shop to buy material to make handkerchiefs. For each handkerchief, she would require a piece of material one foot square. The shopkeeper offered her two rectangular pieces of the same material for exactly the same price. The first piece measured 9 feet by 6 feet; the length of the second was 1 foot shorter but its width was 1 foot longer. Which piece would you choose if you were the seamstress, and why?

2-2 UNIT OF AREA.

Before we make a careful study of the concept of area, let us revise some of our previous ideas. Just as we chose a unit segment to measure the lengths of segments and a unit angle relative to which we could find the measure of angles, so we choose a unit region in order to measure the areas of other regions. We would remark, to begin with, that there is no natural unit for measuring areas of regions, just as there is no natural unit for measuring lengths of segments. You know, of course, that the straight angle is a natural unit for finding the measure of an angle.

For our preliminary discussion, we take the following statement as a definition:

DEFINITION 2-1. The area of a square region of side 1 (unit of length) is 1 (unit of area). We will call a square of side 1 a unit square.

If length is measured in some standard unit such as inch, centimetre, mile, etc. the corresponding unit of area is the square inch (abbreviated to sq. in.), square centimetre (sq. cm.), square mile (sq. ml.), etc.

2-3 AREAS BY SUMMING UP SQUARES.

You will recall that "congruent" means, roughly, "duplicate" or "copy." It is reasonable then to assume that congruent regions have equal areas. We make this assumption in the following informal discussion, and introduce it later as one of our postulates about area.

We shall now try to find the areas of some plane regions; that is, regions all of whose points are in one plane. Let us consider a square of side 2 units. We find that if we divide the square as shown in the figure, we obtain four smaller squares. These squares are all congruent to each other (each is in fact a unit square) and they are non-overlapping. It is natural to conclude that the original square has area 4.

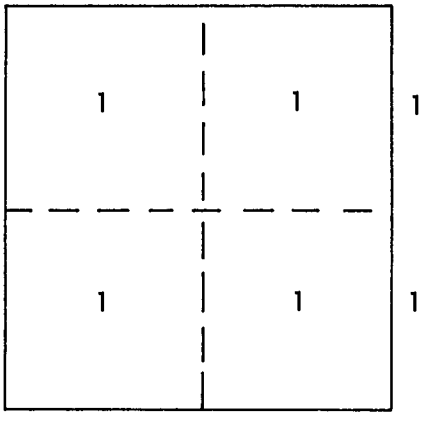


Fig. 1.

You will have observed that we have here assumed that we can find the area of the large square by summing up the areas of the four non-overlapping small squares. Later we will formulate this notion precisely, when we will call it "The Area Addition Postulate".

PROBLEMS 2-3A

Show by diagram how you could find the area of a square whose side is:

(a) 3 cm., (b) 8 in., (c) 14 ft., (d) s (where s is a positive integer).

You will have found from Problem (d) that if a square has side s (s a positive integer), then its area is s^2.

Do you think we can use this method of dividing a region into unit squares to find the area of some other polygons? You will find that we can do it for a rectangle whose sides are integral multiples of the chosen unit of length. Show how this could be done for a rectangle of sides 5 by 3.

Use this method to determine the area of a rectangle whose sides are a by b, where a and b are positive integers.

We now consider the areas of rectangles the lengths of whose sides are rational numbers, which are not necessarily integers. We shall illustrate the method we use in such cases by considering a square of side $\frac{1}{2}$. By dividing a unit square as shown in Figure 2, we find that we need four non-overlapping squares, each of side $\frac{1}{2}$ to cover exactly the unit square. These squares are all congruent to each other and so should have equal areas. Let us call each area A. Then

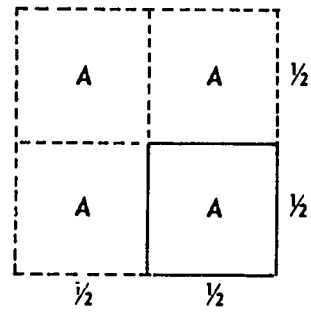


Fig. 2.

$$A + A + A + A = 1$$

which means $\quad A = \frac{1}{4}$.

PROBLEMS 2-3B

Find, by drawing suitable figures, the areas of the following:

(i) a square of side $\frac{1}{3}$ ft.,

(ii) a square of side $3\frac{1}{3}$ cm.,

(iii) a rectangle of sides $\frac{1}{3}$ ft. by $\frac{1}{2}$ yd.,

(iv) a rectangle of sides $\frac{3}{5}$ ml. by $\frac{2}{7}$ ml.,

(v) a rectangle of sides $4\frac{1}{2}$ km. by $2\frac{2}{5}$ km.,

(vi) a rectangle of sides $\frac{p}{q}$ by $\frac{r}{s}$,

where p, q, r, s are positive integers.

You will have found from the above problems that, if a rectangle has sides a by b where a and b are positive rational numbers, then we can find its area by dividing it into suitable squares and summing up the areas of these squares. The area we obtain by this procedure is ab. There are, however, rectangles which we cannot subdivide into small squares such that we already know the areas of these squares or can easily compute them by our previous methods. Let us try to find the area of a rectangle R whose sides are $\sqrt{2}$ by 1. You will observe that this rectangle has a side which is a non-rational multiple of the unit of length.

You will recall that, in *Secondary Two*, we obtained the following sets of inequalities for $\sqrt{2}$:

$$1{\cdot}4 < \sqrt{2} < 1{\cdot}5$$
$$1{\cdot}41 < \sqrt{2} < 1{\cdot}42$$
$$1{\cdot}414 < \sqrt{2} < 1{\cdot}415$$
$$1{\cdot}4142 < \sqrt{2} < 1{\cdot}4143$$

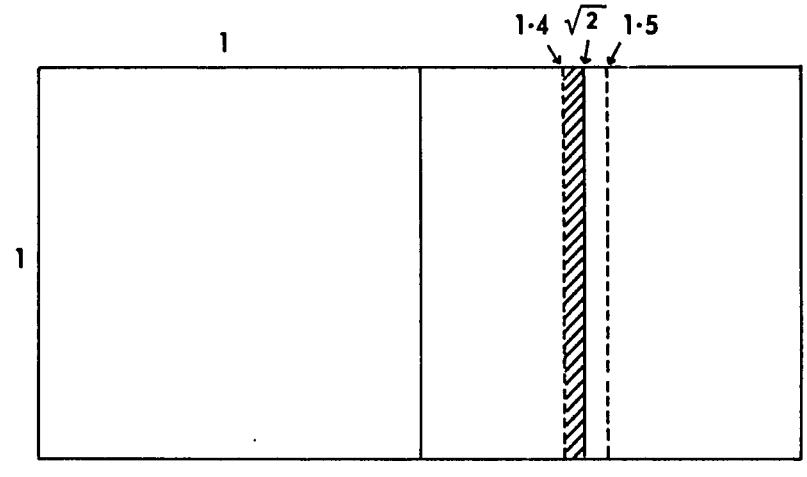


Fig. 3.

We consider a rectangle R_1 whose sides are 1·4 by 1. If you look at the figure, you will see that such a rectangle can be made to fit into the given rectangle R with the shaded region left over. You will agree then that the area of the rectangle R_1 should be smaller than the area we are trying to find. By our method of dividing a rectangle into suitable squares and

summing up their areas, we find that the area of rectangle R_1 is $1{\cdot}4 \times 1 = 1{\cdot}4$. This means that the number 1·4 is less than the number we are trying to find. On the other hand, a rectangle of sides 1·5 by 1 will be larger than the rectangle R. This means that $1{\cdot}5 \times 1 = 1{\cdot}5$ is greater than the number we want. Arguing in this way, we obtain an increasing sequence of numbers $1{\cdot}4 \times 1$, $1{\cdot}41 \times 1$, $1{\cdot}414 \times 1$, . . . , each of which is less than the required number and also a decreasing sequence of numbers $1{\cdot}5 \times 1$, $1{\cdot}42 \times 1$, $1{\cdot}415 \times 1$, . . . , each of which is greater than the required number.

If A is the area of the rectangle R, we have shown that A satisfies the following inequalities:

$$\begin{aligned} 1{\cdot}4 \times 1 &< A < 1{\cdot}5 \times 1 \\ 1{\cdot}41 \times 1 &< A < 1{\cdot}42 \times 1 \\ 1{\cdot}414 \times 1 &< A < 1{\cdot}415 \times 1 \\ 1{\cdot}4142 \times 1 &< A < 1{\cdot}4143 \times 1 \end{aligned}$$

Or, $1{\cdot}4 \times 1 < 1{\cdot}41 \times 1 < 1{\cdot}414 \times 1 < \ldots < A < \ldots < 1{\cdot}415 \times 1 < 1{\cdot}42 \times 1 < 1{\cdot}5 \times 1$.

It would seem that the area (A) of the rectangle R is $\sqrt{2} \times 1$. Do you agree?

PROBLEMS 2-3C

1. Using the above method, suggest what should be the area of a rectangle whose sides are:

   (a) $\sqrt{3}$ by 2; (b) $\sqrt{3}$ by $\sqrt{2}$.

2. Can we use this method to obtain the area of a square whose side is $\sqrt{3}$? If your answer is yes, what would be its area?

2-4 APPROXIMATIONS TO AREAS OF SOME PLANE REGIONS.

CLASS ACTIVITY

The following exercises are to be done on squared paper.

1. Draw an equilateral triangle of side 8 cm.

(a) Count the number of complete squares contained in the triangular region.
(b) Count the number of complete squares which together just contain the triangular region.
(c) What can you say about the area of the triangle?

2. As in Problem 1, but with a rhombus of side 2·5 in. and one diagonal of length 3 in.

[Make a note of your answers to these two problems; we shall refer to them later.]

3. Trace out an outline map of your country from an atlas onto a piece of squared paper. (Note the scale given for this map.)

(a) Count the number of complete squares enclosed by your outline map.
(b) Count the number of complete squares which together just enclose your outline map.
(c) What area of land does one square on the graph represent?
(d) What, very roughly, is the area of your country?

[Note: You will already know that the surface of the Earth is curved, so that a map is a representation of a curved surface on a plane. There are a number of different ways of doing this, and a map preserves or accurately represents some features of the original surface such as angles or distances, while it distorts others. Your teacher will discuss the special features of the map you use. Let us hope that it represents areas reasonably accurately.]

We could improve on the estimates we have obtained in the three problems above. Here is one way of doing it: In the (a) part of each problem, you include as a complete square every square which is such that the greater part of it is inside the region. (You will, of course, have to estimate this.)

A second method of improving your results is this: You divide each square which is cut by the boundary into four non-overlapping congruent squares, as in Figure 2 above. You now include in the various counts the

number of these smaller squares, which are either inside the region or are cut by the boundary of the region, as the case may be.

Which of the two methods do you think is the better and why?

You may have realised that the second method we have suggested for improving your estimates of areas may be repeated a number of times to give closer and closer approximations. The idea underlying this method for finding the areas of a region will be used again when you study the areas of other regions, such as circular regions.

For our present study, which is that of plane regions bounded by segments, we find that a slightly different approach is more suitable. The first two exercises given at the beginning of this section may have shown you that it is not always possible to subdivide such regions into a *finite* number of square regions. It can be shown, however, that such regions can be subdivided into a finite number of triangular regions.

Our definition of a polygonal region will reflect this last fact. Our basic unit still remains the unit square. From it, we shall first derive the area formula for a rectangular region. Then, by relating a triangular region to a rectangular region, we shall derive the area formula for the former.

2-5 POLYGONAL REGIONS.

We now begin our formal study of area. Before we state any postulates and use them to prove some theorems, we shall define what we mean by a polygonal region.

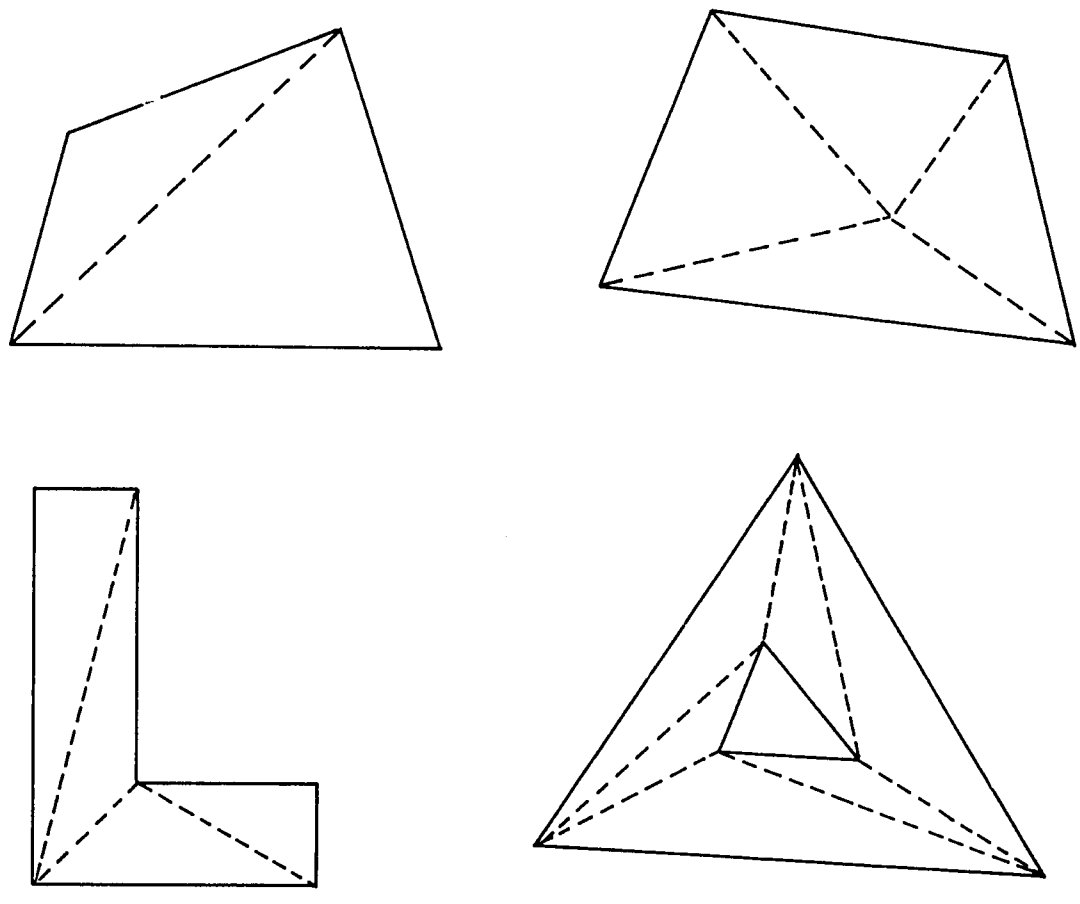
You have already met the term "triangular region." We re-state the definition here:

> DEFINITION 2-2. *A triangular region* is the union of a triangle and its interior. The sides and vertices of the triangle will also be called the sides and vertices of the triangular region.

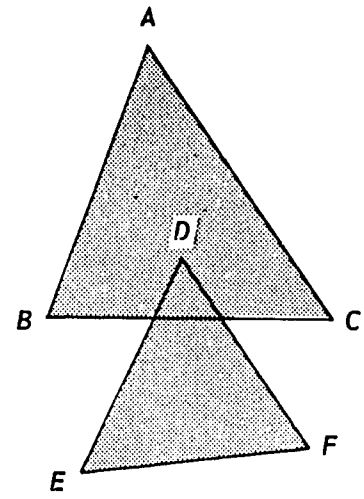
As already remarked, the regions we wish to study are such that they can always be subdivided into a finite number of non-overlapping triangular regions. This suggests our next definition.

DEFINITION 2-3. *A polygonal region* is the union of a finite number of coplanar triangular regions such that, if any two of the regions intersect, they do so only in vertices and sides of these regions.

Triangular regions, which are related as in this last definition, are said to be *non-overlapping*. It is true, of course, that the regions have edges in common. We use the term *non-overlapping* here in the sense that the regions have *no interior points in common*. Here are some examples of regions cut up such that they satisfy the definition of polygonal region. (The continuous line segments are the boundaries of the regions.)

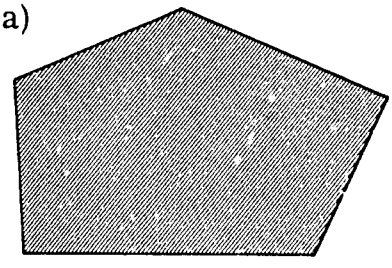


1. Does the region shaded in the figure satisfy the definition of a polygonal region if it is considered as the union of the triangular regions *ABC* and *DEF*? If it does not, could you divide it in such a way that it does satisfy the definition?

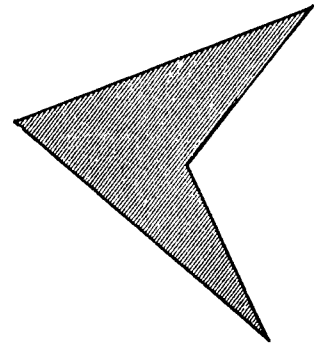


2. Divide each of the following regions into triangular regions in a way which will make it a polygonal region:

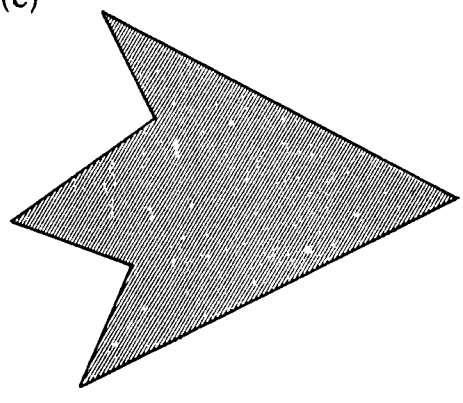
(a)



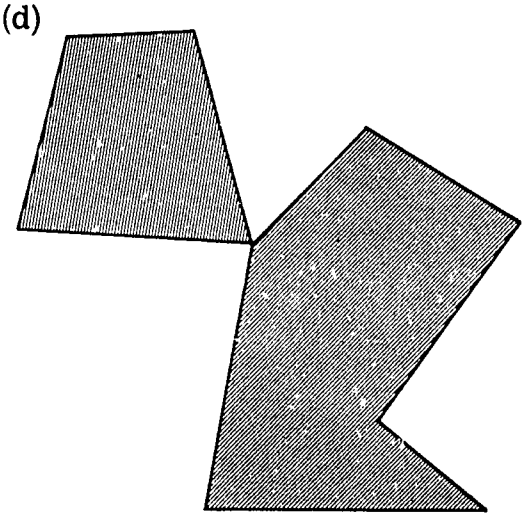
(b)



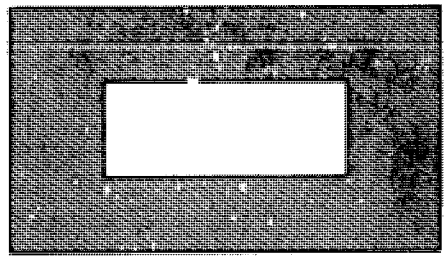
(c)



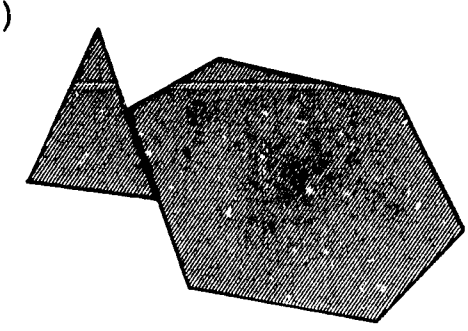
(d)



(e)



(f)



3. Draw a few more regions bounded by segments and cut them up as in Problem 2.

4. Suppose a region R can be subdivided as in Problem 2. In how many different ways do you think this can be done?

DEFINITION 2-4. A *square region* is the union of a square and its interior.

Can you give a similar definition for a rectangular region?

CONVENTION: You may have observed that, in the above discussions, we have used the expressions "area of a triangle," "area of a square," "area of a rectangle" when we should have said "area of a triangular region," "area of a square region," "area of a rectangular region" respectively. This is an accepted and useful convention, and we shall continue to adopt it. The convention will also apply to other polygonal regions whose boundaries are easily named.

2-6 AREAS OF POLYGONAL REGIONS.

We now state a few postulates which will enable us to prove theorems about areas of polygonal regions. Actually, we could have used our previous

postulates to prove these new ones as theorems. But this would be rather difficult to do, and so we take the statements as postulates.

Area Postulate 1:

To every polygonal region, there corresponds, in terms of a chosen unit, a unique positive real number, which is its area.

Note: In stating our other postulates and when we prove theorems later, we shall always assume that the chosen unit of area is fixed.

You will recall that our choice of the unit square as the unit of area enabled us to show that a square of side s has area s^2, where s is a rational number. Though we did not prove it, we found it reasonable to accept this result as holding for all positive real numbers s. We now state this as our next postulate.

Area Postulate 2:

The area of a square of side s is s^2.

Area Postulate 3:

If two triangles are congruent, they have the same area.

DEFINITION 2-5. Let the polygonal region R_1 be the union of the finite set S_1 of non-overlapping coplanar triangular regions, and similarly R_2 the union of the finite set S_2. We say that R_1 and R_2 are two *non-overlapping polygonal regions* if they intersect in at most the vertices and sides of the triangular regions belonging to S_1 and S_2.

Area Postulate 4 (Area Addition Postulate):

Let the polygonal region R be the union of two non-overlapping polygonal regions R_1 and R_2. Then the area of R is the sum of the areas of R_1 and R_2.

Remarks:

1. Postulate 4 gives us a way of finding the area of a polygonal region R in terms of the areas of two other regions: R_1 and R_2. The definition preceding this postulate can be extended to give a

meaning to n non-overlapping polygonal regions $R_1, R_2, \ldots, R_n$, for any finite n. We can show that, if polygonal region R is the union of n non-overlapping regions $R_1, R_2, \ldots, R_n$, the area of R is the sum of the areas of $R_1, R_2, \ldots, R_n$. We shall use this result very often in the work that follows.

2. You may have realised that if a region R can be subdivided so that it satisfied the definition of polygonal region, then the subdividing can be done in many ways (in fact, in infinitely many ways). Area Postulate 4 and our earlier remark tell us that if R is subdivided into a finite number of non-overlapping triangular regions, then the area of R is the sum of the areas of these triangles. If our area postulates are to be consistent, then we must be able to prove that all the infinitely many ways of subdividing R lead to the same area, which is that assigned to R by Area Postulate 1. This can actually be done, though we shall not do so here.

> *Notation*:
> When a polygonal region can be completely described by naming the vertices of the polygon which forms its boundary, we may denote its area by just naming the vertices (without any symbol or descriptive expression). For instance, the area of a triangular region ABC will be denoted by ABC, and that of a quadrilateral region $PQRS$ by $PQRS$.

PROBLEMS 2-6

1. Is a triangular region a polygonal region?

2. Is a square a polygonal region?

3. Can we meaningfully use the expression "area of a quadrilateral"?

4. Suppose we are given a positive real number r. Can we conclude from Area Postulate 1 alone that there is a polygonal region whose area is r?

5. "If two triangles have the same area, then they are congruent." Is this statement true or false?

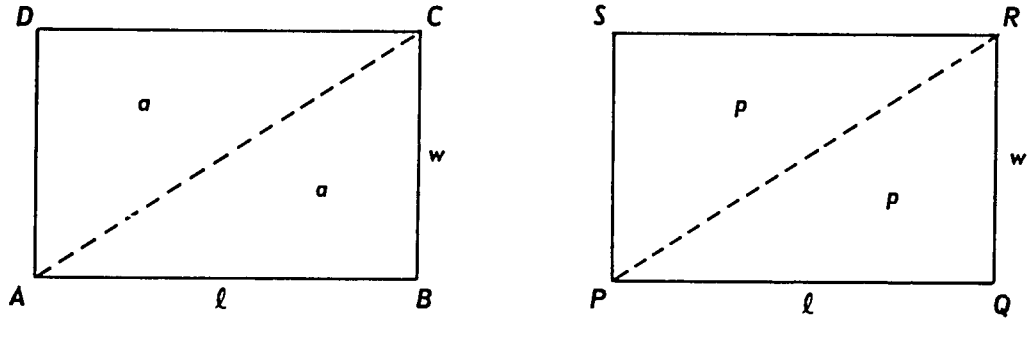
2-7 SOME AREA FORMULAE.

In this section, we shall state and prove theorems, which will give us expressions for the areas of certain polygons in terms of their sides and the lengths of certain segments associated with these polygons.

> *NOTE*: In Chapter 1 we considered the altitudes of triangles as *lines* in order to show that these lines (altitudes) meet in a single point for a given triangle. In this chapter, we will use the term *altitude* to mean the *perpendicular distance* between any vertex of a triangle and the side opposite that vertex, or the perpendicular distance between the parallel sides of a parallelogram, a rectangle, a square, a rhombus, or a trapezium.

We now prove a lemma, which we need for the proof of our first theorem. (A lemma is a "helping theorem," which we prove as a step towards proving our main theorem.)

> *Lemma*: Two rectangles which have equal bases and altitudes have equal areas.



Given: Two rectangles, $ABCD$ and $PQRS$, each with sides of ℓ and w.

Prove: $ABCD = PQRS$.

Proof:

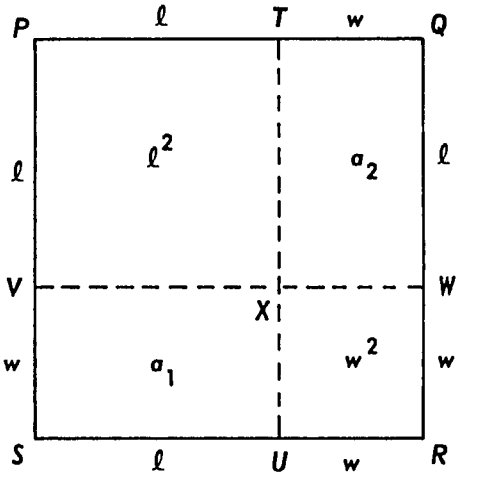
1.	$\Delta ABC \equiv \Delta CDA$	1.	SSS
2.	$ABC = CDA = a$	2.	Area Postulate 3.
3.	$ABCD = 2a$	3.	Area Postulate 4.
4.	$PQRS = 2p$	4.	Why?
5.	$\Delta ABC \equiv \Delta PQR$	5.	SAS
6.	$ABC = PQR$ (or $a = p$)	6.	Area Postulate 3.
7.	$ABCD = PQRS$	7.	From 3, 4 and 6.

> THEOREM 2-1. The area of a rectangle with sides ℓ by w is ℓw.

Given: Rectangle $TQWX$ with sides ℓ by w.

Prove: The area of the rectangle is ℓw.

Proof: From the lemma, we deduce that the rectangles $TQWX$ and $SVXU$ have equal areas. Let the area of each rectangle be a.



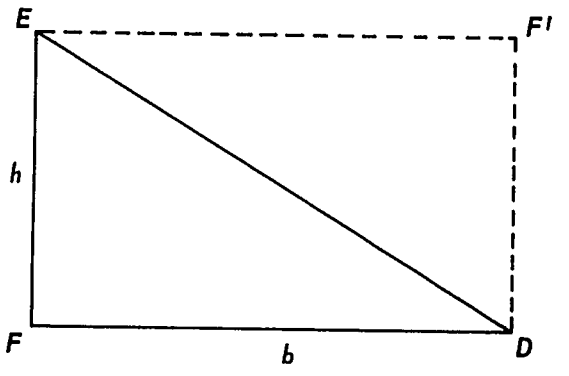
1.	$a_1 = a_2 = a$	1.	Lemma
2.	$PQRS = (\ell + w)^2$	2.	Area Postulate 2.
3.	$PQRS = \ell^2 + a_1 + a_2 + w^2$	3.	Area Postulates 4 and 2.
4.	$PQRS = \ell^2 + 2a + w^2$	4.	Statement 1.
5.	$\ell^2 + 2a + w^2 = (\ell + w)^2$	5.	Statements 2 and 4 and uniqueness of area (Area Postulate 1).

6.	$(\ell + w)^2 = \ell^2 + 2\ell w + w^2$	6.	Using algebraic properties of numbers.
7.	So $a = \ell w$	7.	Statements 5 and 6 and algebra.

> THEOREM 2-2. The area of a right triangle is half the product of its legs.

Given: A right triangle ΔDEF with the right angle at F, $DF = b$ and $EF = h$.

Prove: $DEF = \frac{1}{2} bh$.



Proof: Let F' be the point of intersection of the line through E parallel to $\overleftrightarrow{FD}$ and the line through D parallel to $\overleftrightarrow{EF}$. (Why must these two lines intersect?)

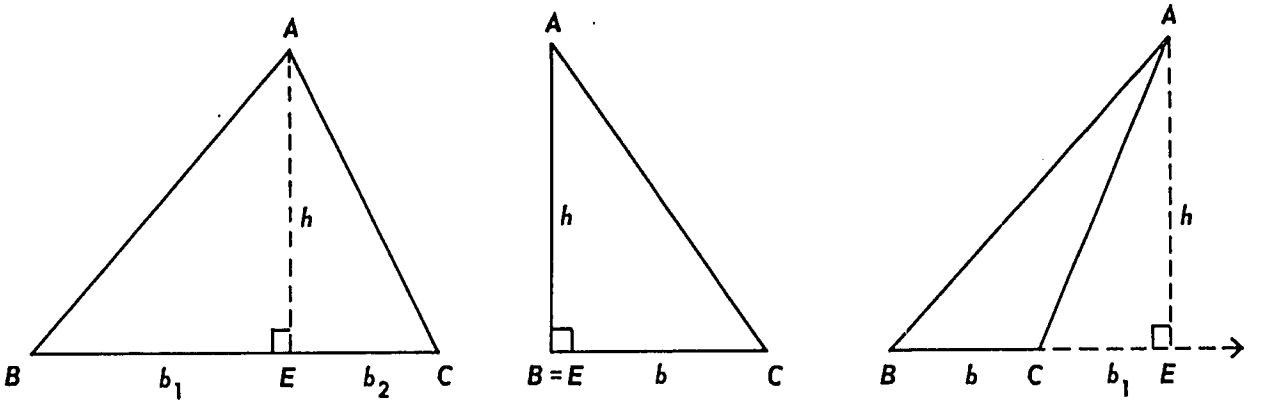
1. The quadrilateral $EFDF'$ is a rectangle.
2. $\Delta EFD \equiv \Delta DF'E$
3. $EFD = DF'E$ (Denote each area by a.)
4. $2a = EFDF'$
5. $EFDF' = bh$
6. $a = \frac{1}{2} bh$

Give reasons for the statements in the above proof.

> THEOREM 2-3. The area of a triangle is half the product of any base and the corresponding altitude.

Given: Triangle ΔABC with base $BC = b$ and corresponding altitude $AE = h$.

Prove: $ABC = \frac{1}{2} bh$.



(1) (2) (3)

Proof: As seen from the figures, there are three cases to consider:

(1) The point E is between the end-points B and C. Let $BE = b_1$, and $EC = b_2$, so that $b_1 + b_2 = b$. By Theorem 2-2,

$$ABE = \tfrac{1}{2} b_1 h, \qquad AEC = \tfrac{1}{2} b_2 h.$$

By Area Postulate 4,

$$ABC = ABE + AEC .$$

Hence
$$\begin{aligned} ABC &= \tfrac{1}{2} b_1 h + \tfrac{1}{2} b_2 h \\ &= \tfrac{1}{2}(b_1 + b_2) h \\ &= \tfrac{1}{2} bh. \end{aligned}$$

(2) The point E is one of the end-points of $\overline{BC}$ ($E = B$). Triangle ABC is then a right triangle and the result is given immediately by Theorem 2-2.

(3) The point E falls outside the segment $\overline{BC}$. We take C to be between B and E. Let $CE = b_1$. By Theorem 2-2,

$$ABE = \tfrac{1}{2}(b + b_1) h , \qquad ACE = \tfrac{1}{2} b_1 h .$$

By Area Postulate 4,

$$ABC + ACE = ABE .$$

Hence $ABC + \frac{1}{2} b_1 h = \frac{1}{2}(b + b_1)h$,
from which we deduce that

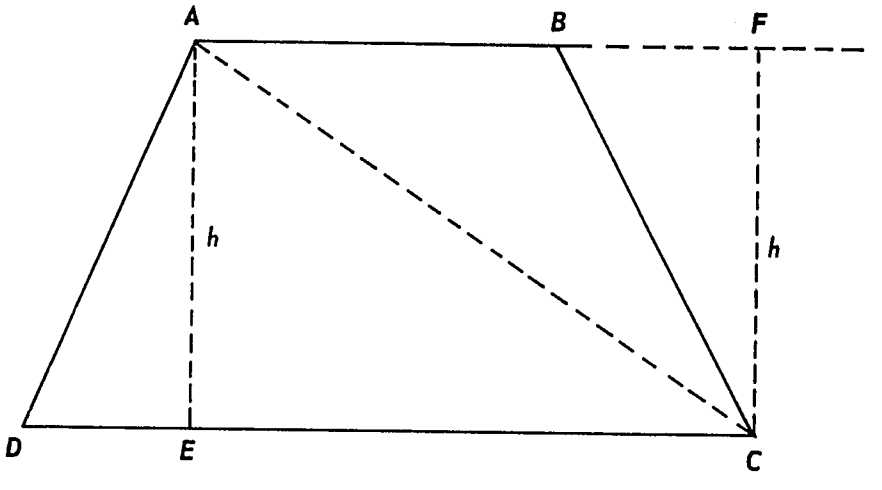
$$ABC = \frac{1}{2} bh .$$

COROLLARY 2-3-1. If two triangles have equal altitudes, then the ratio of their areas is equal to the ratio of the corresponding bases.

THEOREM 2-4. The area of a trapezium is half the product of the sum of the parallel sides and its altitude.

Given: A trapezium $ABCD$ with parallel sides $\overline{AB}$ and $\overline{CD}$, whose respective lengths are $AB = b_1$ and $CD = b_2$. The height of the trapezium $ABCD$ is h.

Prove: $ABCD = \frac{1}{2}(b_1 + b_2)h$.



Proof: We draw one diagonal, $\overline{AC}$, of the trapezium $ABCD$. This divides the trapezium into two triangles ABC and ΔADC. By Theorem 2-3,

$$ABC = \tfrac{1}{2} b_1 h \quad \text{and} \quad ADC = \tfrac{1}{2} b_2 h \ .$$

$$\begin{aligned} ABCD &= ABC + ADC \quad \text{(Area Postulate 4)} \\ &= \tfrac{1}{2} b_1 h + \tfrac{1}{2} b_2 h \\ &= \tfrac{1}{2}(b_1 + b_2) h \ . \end{aligned}$$

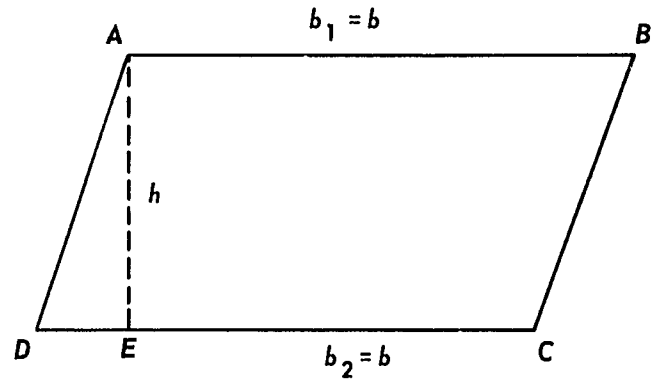
You will recall that a quadrilateral with a pair of opposite sides parallel and congruent is a parallelogram. What must be the relationship between b_1 and b_2 (in Theorem 2-4) for the trapezium to be a parallelogram?

You will have found the answer to the last question to be $b_1 = b_2$. We use this fact to prove our next theorem.

> THEOREM 2-5. The area of a parallelogram is the product of any base and the corresponding altitude.

Given: A parallelogram $ABCD$ with base $DC = b$ and altitude $AE = h$.

Prove: $ABCD = bh$.



Proof: The parallelogram may be considered as a trapezium $ABCD$ with $b_1 = b_2 = b$. Theorem 2-4 then gives us immediately,

$$\begin{aligned} ABCD &= \tfrac{1}{2}(b_1 + b_2) \\ &= bh \ . \end{aligned}$$

PROBLEMS 2-7

1. Prove Corollary 2-3-1.

2. Write a true statement about the ratio of the areas of two triangles which have equal bases. Prove that your statement is true.
   [Hint: Look at Corollary 2-3-1.]

3. Triangles ΔABC and ΔDEF are such that the vertices B, C, E, F are collinear and the lines $\overleftrightarrow{AD}$ and $\overleftrightarrow{BC}$ are parallel.
   (a) What can you say about the ratio of the areas of the two triangles?
   (b) What can you deduce if $\overline{BC} \equiv \overline{EF}$?

4. $\overline{BC}$ is a fixed segment. P is a point such that the area of the triangle ΔPBC is always equal to a given fixed positive number. What can you say about the point P if
   (i) all the triangles ΔPBC lie in one plane?
   (ii) the triangles ΔPBC do not have to lie in one plane?

5. A right triangle has legs 5 and 12, and hypotenuse 13.
   (i) Find the area of the triangle.
   (ii) Find the altitude to the hypotenuse.

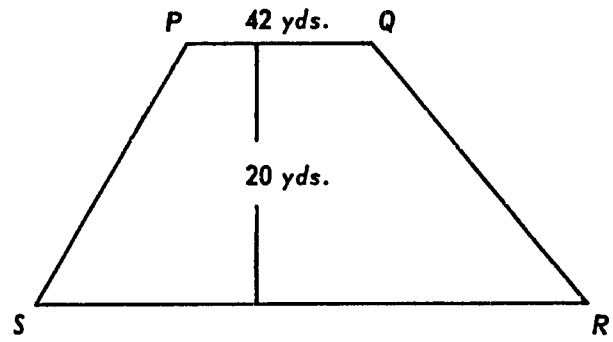
6. The hypotenuse of a right triangle is 25, one leg is 24, and its area is 84.
   (a) Find the altitude to the hypotenuse.
   (b) Find the other leg.
   (c) What is the altitude to this second leg?

7. A triangle has area 35 and base 7. Determine the altitude to this base.

8. A triangle has area 48. One altitude is 10. What is the corresponding base?

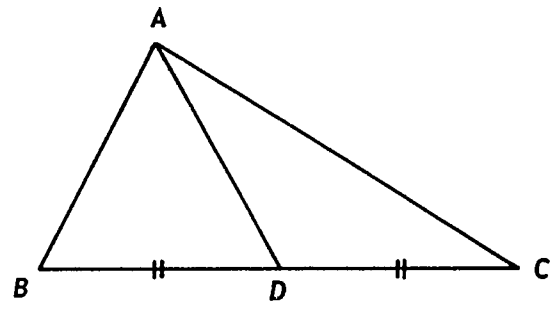
9. The lengths of the parallel sides of a trapezium are 10 and 16. The altitude is 12. Calculate its area.

10. A plot of land is in the form of a trapezium $PQRS$ (see figure). $PQ = 42$ yds. The altitude of the trapezium is 20 yds., and the area of the plot is 1000 sq. yds. Determine SR.

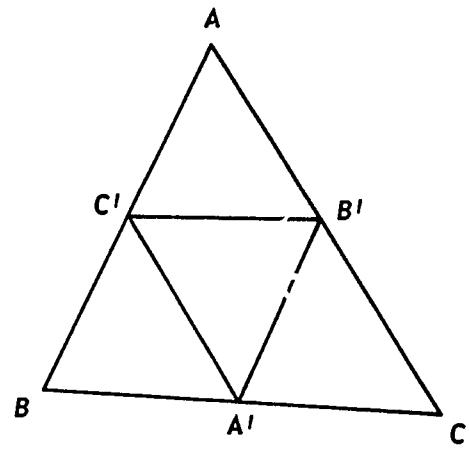


11. A parallelogram has area 40 sq. in. If one side is 1 ft., find the corresponding altitude. Give your answer in feet.

12. Prove that a median of a triangle divides the triangle into two triangles of equal areas. Will the two smaller triangles be congruent in every case? In any case?

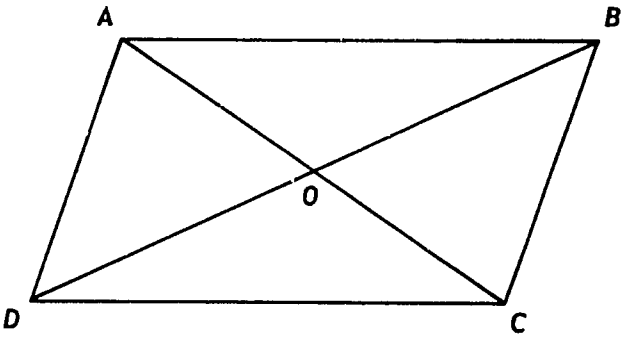


13. If A', B', C' are respectively the midpoints of the segments $\overline{BC}$, $\overline{CA}$, $\overline{AB}$, prove that the area of the triangle $A'B'C'$ is $\frac{1}{4}$ of that of triangle ΔABC.



14. The points E, F are the mid-points of the sides $\overline{AB}$, $\overline{AD}$ of the parallelogram $ABCD$. Prove that $AECF = \frac{1}{2} ABCD$.

15. Prove that the diagonals of a parallelogram divide it into four triangles which have equal areas.



16. A quadrilateral has perpendicular diagonals. Prove that its area is half the product of the lengths of the diagonals. Can you give an area formula for a rhombus from this theorem? If so, what is the formula? Look at Problem 2 of Problems 2-4. How good was your area estimation?

17. A square has side s and diagonal d. Find its area
    (i) in terms of s;
    (ii) in terms of d.
    What relationship between s and d do you deduce from this?

18. An isosceles right triangle has legs of length a. Find
    (i) the length of the hypotenuse;
    (ii) the area of the triangle;
    (iii) the length of the altitude to the hypotenuse.

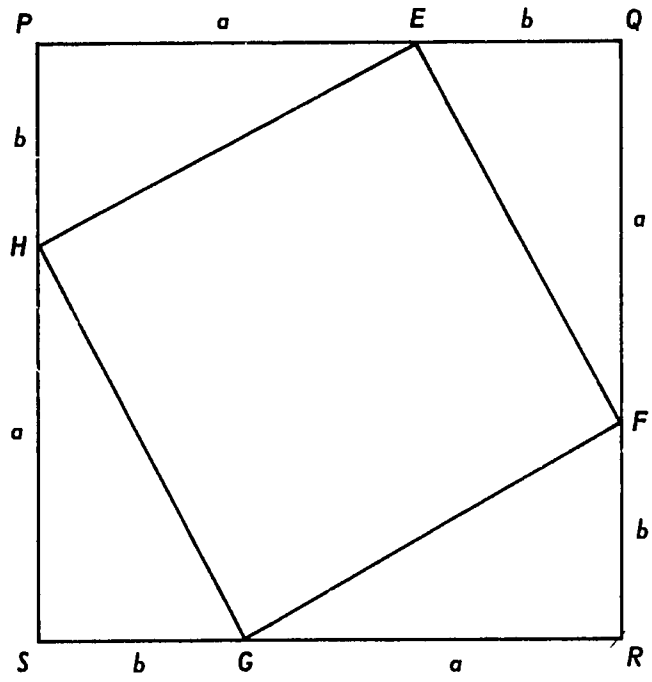
    [Hint: Use the relationship you found in Problem 17.]

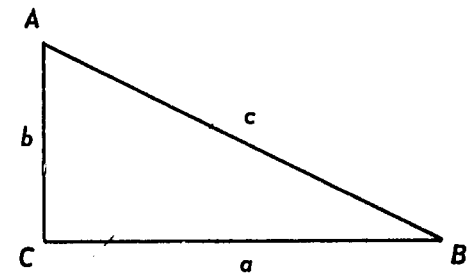
2-8 PYTHAGORAS' THEOREM.

In this section we shall use some of our previous results on area to prove a very important relationship between the lengths of the legs and the length of the hypotenuse of a right triangle. You have already met this result in your work in Secondary Two, when it was called "The Pythagorean Property."

THEOREM 2-6. If a right triangle has legs of lengths a, b and hypotenuse of length c, then $c^2 = a^2 + b^2$.

Given: A right triangle ΔABC with the right angle at C.
Prove: $c^2 = a^2 + b^2$.





Proof: We construct a square $PQRS$ of side $a + b$, and subdivide the square region determined by $PQRS$ as shown in the figure.

1. The quadrilateral $EFGH$ is a square of side c.
2. $PQRS = (a + b)^2$
3. $PQRS = HEP + EFQ + FGR + GHS + EFGH$
4. $HEP = EFQ = FGR = GHS$
5. $HEP = \frac{1}{2}ab$
6. $(a + b)^2 = 4(\frac{1}{2}ab) + c^2$ — Statements 1-5, and Area Postulate 1.
7. $a^2 + 2ab + b^2 = 2ab + c^2$ — Statement 6 and algebra
8. $a^2 + b^2 = c^2$ — Statement 7 and algebra

Give reasons for the first five statements of the above proof.

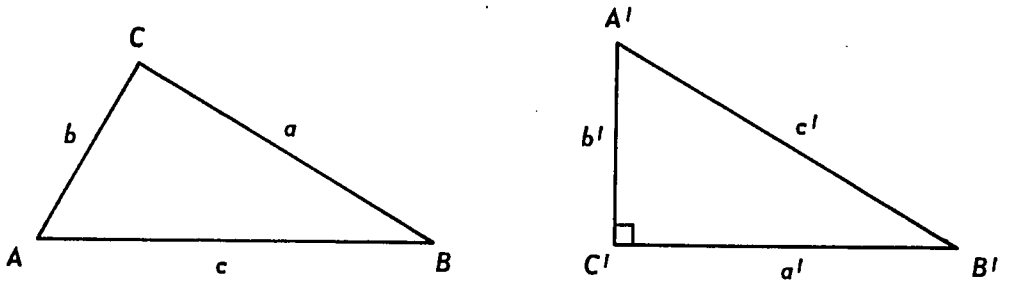
The theorem just proved is named after the Greek mathematician Pythagoras, who lived in the 6th century B. C. It is not known for certain whether he himself discovered this theorem. It is reputed that there are more proofs of this theorem than there are of any other theorem in the whole field of mathematics. In Problems 9-11 below, we give other methods for proving the theorem. In particular, we give in Problem 11 the proof which has come down to us from Euclid. You will observe that this proof is purely geometric in character. This is because the ancient Greeks thought of area, not entirely as a number (as we have done), but more as a region.

The converse of Pythagoras' Theorem is also true. We prove this in our next theorem.

> THEOREM 2-7. If the square of one side of a triangle is equal to the sum of the squares of the two remaining sides, then the triangle is right-angled, the right angle being opposite the first side.

Given: A triangle ΔABC with $AB = c$, $BC = a$, $CA = b$ such that $c^2 = a^2 + b^2$.

Prove: $\widehat{C}$ is a right angle.



Proof: We construct a right triangle $\Delta A'B'C'$ with legs $B'C' = a' = a$, and $C'A' = b' = b$. Then

(1) $c'^2 = a'^2 + b'^2$	(1) Pythagoras' Theorem
(2) $c'^2 = a^2 + b^2$	(2) Statement (1) and $a' = a$, $b' = b$ by construction

(3)	$c'^2 = c^2$	(3)	Statement (2) and $a^2 + b^2 = c^2$ by hypothesis.
(4)	$c' = c$	(4)	
(5)	$\Delta ABC \equiv \Delta A'B'C'$	(5)	SSS
(6)	$\widehat{C} = \widehat{C'}$	(6)	
(7)	$\widehat{C}$ is a right angle	(7)	Statement (6) and $\widehat{C'}$ is a right angle (by construction).

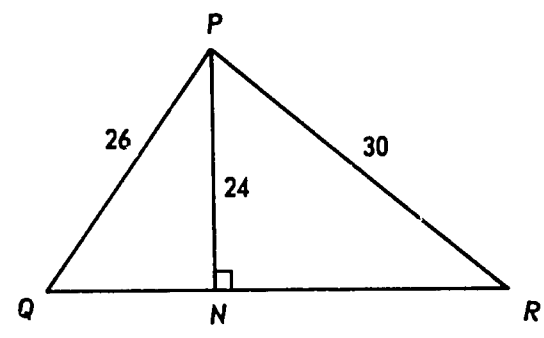
Give reasons for statements (4) and (6) in the above proof.

PROBLEMS 2-8

1. If a, b, c are respectively the lengths of the sides and hypotenuse of a right triangle, complete the following table:

	a	b	c
(i)	12		13
(ii)		24	25
(iii)	16	20	
(iv)	9	9	
(v)	$\frac{15}{4}$		$\frac{25}{4}$
(vi)		15	17
(vii)	6		8
(viii)	0·9		4·1

2. Calculate the length of the side $\overline{QR}$ of the triangle ΔPQR.



3. A window-cleaner has a ladder which is 25 feet long. He places it so that it just reaches a window sill 20 feet from the ground. How far from the house is the foot of the ladder?

4. Which of the following sets of numbers could be the lengths of a right triangle?

(a)	7,	40,	41	(e)	6,	12,	14
(b)	15,	36,	39	(f)	$\frac{12}{3}$,	$2\frac{2}{3}$,	$3\frac{1}{3}$
(c)	8,	15,	19	(g)	$\frac{5}{12}$,	$\frac{5\sqrt{3}}{2}$,	5
(d)	0·75,	1,	1·25	(h)	10,	20,	25

What theorem(s) have you used here?

5. If p and q are positive integers such that $p > q$, prove that $p^2 - q^2$, $2pq$, and $p^2 + q^2$ can be taken as the lengths of the sides of a right triangle.
Which integer gives the length of the hypotenuse?

6. Make a table with the following column headings:

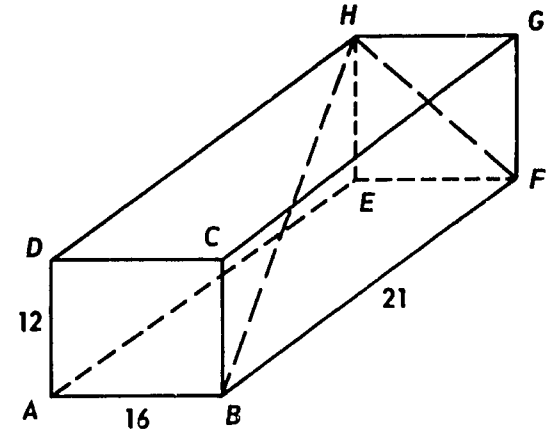
p	q	$p^2 - q^2$	$2pq$	$p^2 + q^2$

Using the results of Problem 5, list in the table the integral lengths of sides of right triangles, the hypotenuse being less than or equal to 25.
(There are six such triangles.)
Continue the table to give the lengths of the sides of such triangles where the hypotenuse is less than or equal to 41. How many such triangles are there? Can you generalize this last result?

7. A set of three positive integers, such as 3, 4, 5 which can be regarded as the lengths of the sides of a right triangle is called a *Pythagoras' Triple*. A Pythagoras' Triple is said to be *primitive* if it is not an integral multiple of some other Pythagoras' Triple. For example, the set $\{5, 12, 13\}$ is a primitive Pythagoras' Triple, but the set $\{15, 36, 39\}$ is not. Write down all the primitive Pythagoras' Triples which are included in your table in Problem 6.

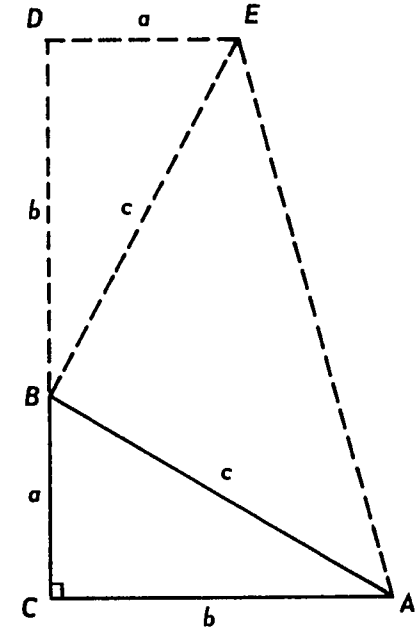
8. The rectangular solid shown in the figure has the edges as indicated. Calculate the length of the diagonal $\overline{BH}$. (Hint: First

calculate the length of the segment $\overline{HF}$. What type of triangle is ΔHFB?)

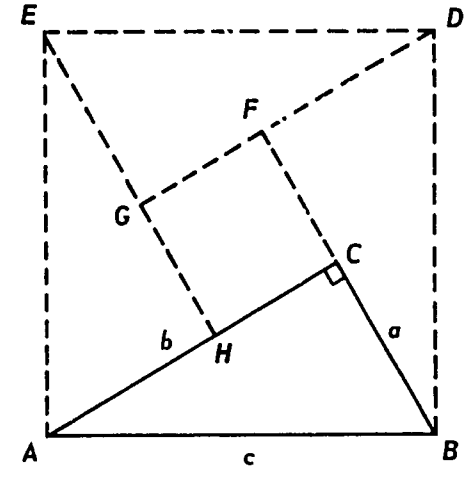


9. Starting with a right triangle ΔABC, we construct a second triangle ΔBED congruent to the first as shown in the figure. Prove that the quadrilateral $ACDE$ is a trapezium, and also that $E\hat{B}A$ is a right angle. Using the fact that the area of the trapezium $ACDE$ is the sum of the areas of the triangles ΔABC, ΔABE and ΔBED, prove that

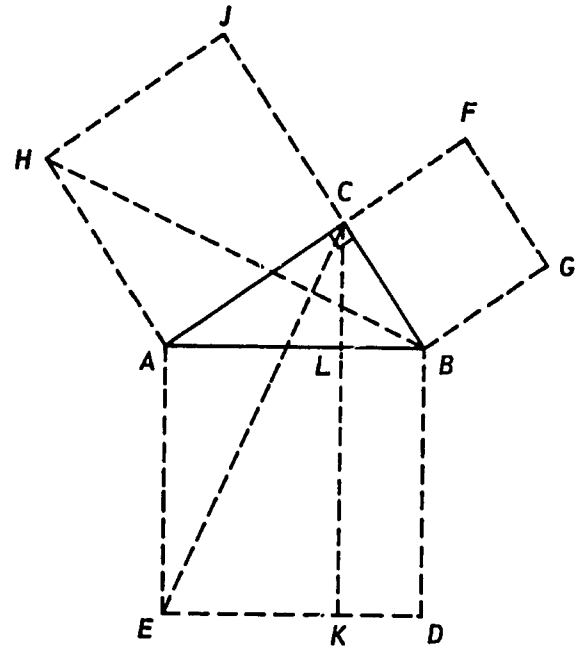
$$a^2 + b^2 = c^2.$$



10. In the figure, $ABDE$ is a square of side c, drawn on the hypotenuse $\overline{AB}$ of the right triangle ΔABC; $\overline{EH} \parallel \overline{BC}$ and $\overline{DG} \parallel \overline{AC}$. Show that the quadrilateral $CFGH$ is a square of side $b - a$, and hence prove Pythagoras' Theorem.



11. In this problem we shall consider area as a region, rather than as a number, and prove Pythagoras' Theorem in the form: If ΔABC is a right triangle, the square on the hypotenuse $\overline{AB}$ is equal to the sum of the squares on the legs $\overline{AC}$ and $\overline{BC}$. We shall give the steps in the proof and you should supply the reasons for them. To prove that $ABDE = BCFG + CAHJ$.



(1) $\Delta AEC \equiv \Delta ABH$

(2) $AEC = ABH$

(3) $CAHJ = 2ABH$

(4) $AEKL = 2AEC$

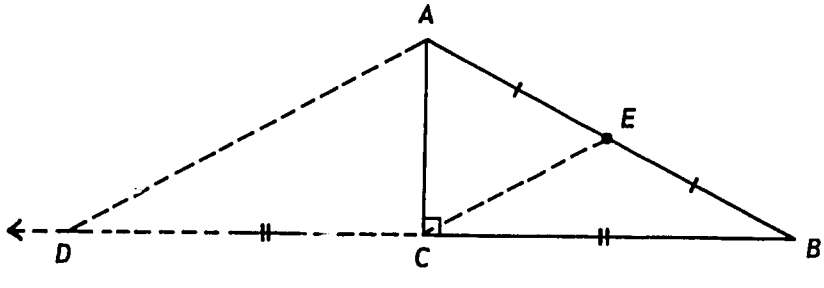
(5) $CAHJ = AEKL$

(6) $BCFG = BDKL$

(There are a number of steps involved in this.)

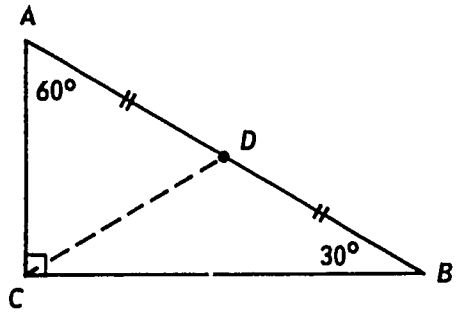
(7) $ABDE = BCFG + CAHJ$

12. Prove that the median to the hypotenuse in a right triangle is half as long as the hypotenuse. Hint: In the figure, let ΔABC be right-angled at C with $\overline{CE}$ the median to the hypotenuse. On



the ray opposite $\overrightarrow{CB}$, take the point D such that $\overline{CB} \equiv \overline{CD}$. Then $\Delta ACB \equiv \Delta ACD$ (why?), and $CE = \frac{1}{2} AD$ (why?). Complete the proof.

13. Prove that in a 30° - 60° - 90° right triangle the leg opposite the 30° angle is one half as long as the hypotenuse. Hint: In the figure, let $\overline{CD}$ be the median to the hypotenuse of the right triangle ΔABC. Use the result of Problem 12 to determine the type of triangle ΔACD is.



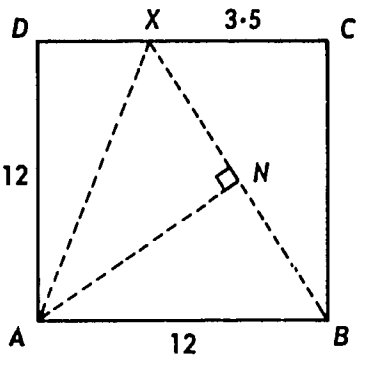
14. Find the length of the altitude of an equilateral triangle of side s. What is the area of the triangle? Now look back at Problem 1 of Problems 2-4. How good was your approximation to part (c)?

15. Find the area of an equilateral triangle whose altitude is 4 cm.

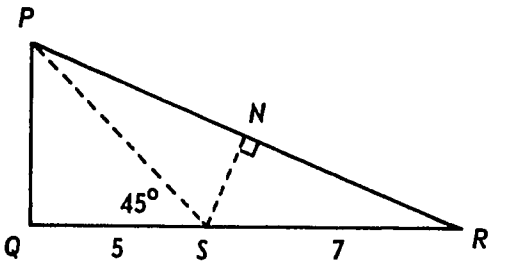
16. Show that the side of an equilateral triangle of altitude h is $\frac{2\sqrt{3}}{3} h$. What is the area of the triangle?

17. The sides of a triangle are 5, 6·5, 6·5 respectively. Find the length of the altitude drawn to the side of length 5. What is the length of the altitude drawn to a side of length 6·5?

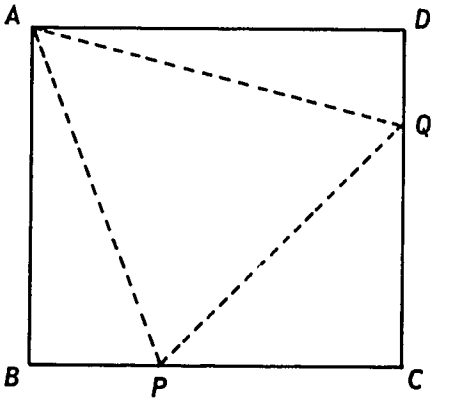
18. $ABCD$ is a square of side 12, and $XC = 3{\cdot}5$. Calculate the area of the triangle ΔABX, and also the length of the segment $\overline{BX}$. Hence find the length of the altitude AN.



19. Calculate the length of $\overline{PR}$, and the altitude SN.

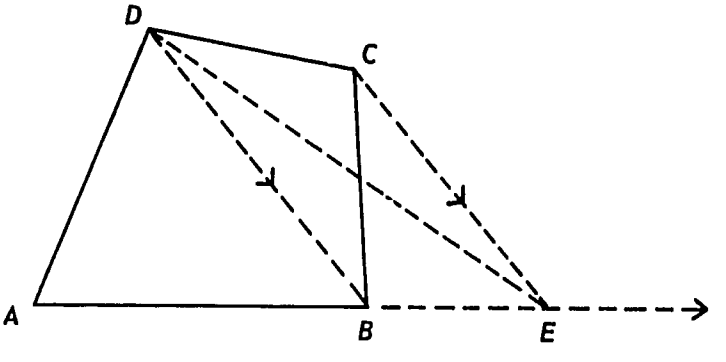


20. *Challenge Problem*
The triangle APQ is an equilateral triangle inscribed in the square $ABCD$. Find the ratio of a side of the triangle to a side of the square.



21. *Challenge Problem*
An equilateral triangle ΔXAB and a square $ABCD$ have a common side $\overline{AB}$ and are such that the points X and C are on opposite sides of the line $\overleftrightarrow{AB}$.
Prove that the length of the segment $\overline{XC}$ is $(\sqrt{6} + \sqrt{2})a$, where $AB = 2a$.

22. Let $ABCD$ be a convex quadrilateral. On the ray $\overrightarrow{AB}$, we choose the point E such that $\overleftrightarrow{CE}$ is parallel to $\overleftrightarrow{DB}$.



Prove that the area of the triangle ΔADE is equal to the area of the quadrilateral $ABCD$.

23. Look at Problem 22 again. You will see that it provides us with a ruler-and-compass method for constructing a triangle whose area is equal to that of given convex quadrilateral.
Does the construction hold if the quadrilateral is non-convex?

24. Construct, using ruler and compasses only, a triangle whose area is equal to that of a given convex pentagon.

25. *Challenge Problem*:
Some mathematicians claim that Pythagoras' Theorem is true only if the sum of the measures of the angles of a triangle is 180. Is the fact that the sum of the measures of the angles of a triangle is 180 used anywhere in our proof of the theorem (Theorem 2-6)?
(Remember that the "angle-sum" theorem is equivalent to the parallel postulate.)

Chapter 3
SIMILAR TRIANGLES AND POLYGONS

3-1 INTRODUCTION.

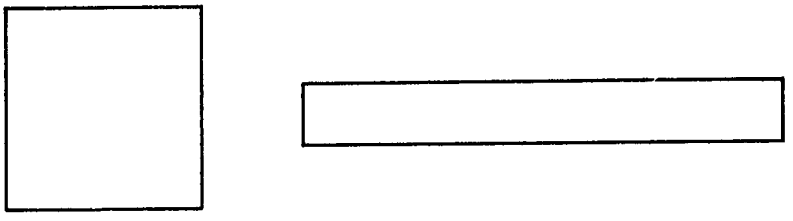
Recall that when we studied congruence of triangles we required that two triangles not only have the same shape but also the same size before we agreed to call them *congruent*. We set up a one-to-one correspondence between the angles and sides of the two triangles in a special way so that if $A \leftrightarrow D$, $B \leftrightarrow E$, $C \leftrightarrow F$ and if $\widehat{A} \equiv \widehat{D}$, $\widehat{B} \equiv \widehat{E}$, $\widehat{C} \equiv \widehat{F}$, $\overline{AB} \equiv \overline{DE}$, $\overline{BC} \equiv \overline{EF}$, and $\overline{AC} \equiv \overline{DF}$, then we defined $\triangle ABC$ to be congruent to $\triangle DEF$.

It often happens in the planning of buildings, the construction of bridges, the making of maps, etc. that we need to draw figures that are small but exact models of larger figures. Also when we work with a microscope in biology we often want to draw figures that are much larger than the objects we look at. For these and other reasons our study of geometry will not be complete if we do not consider some of the more important properties of those geometric figures that have the same shape, but not necessarily the same size.

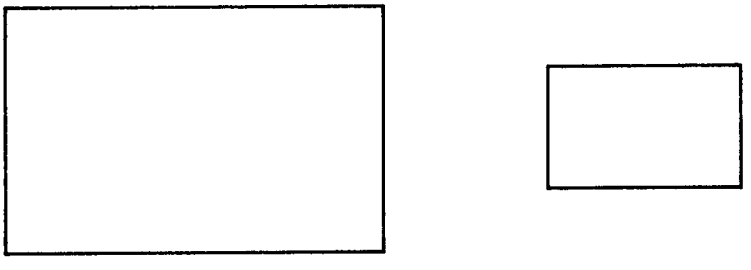
CLASS ACTIVITY

1. Draw two circles that are not the same size. Do they have the same shape? Do all circles have the same shape?
2. Draw two squares that are not the same size. Do they have the same shape? Do all squares have the same shape?
3. Draw two rectangles that are not the same size. Do they have the same shape? Do all rectangles have the same shape? Explain.

You may have discovered in Problem 3 above that the two rectangles you drew did not appear to have the same shape. For example, if one of them were short and fat and the other long and thin as in the figure below, it is evident that one is not an exact model of the other.



On the other hand if your two rectangles looked like this:



you might have concluded that all rectangles have the same shape.

This last example helps us see two things. First it can be misleading to draw conclusions from looking at only one example, and second it is necessary for us to define exactly what we mean when we say that one geometric figure is similar to (has the same shape as) another.

The following definition of similar polygons was given in *Secondary Two*: "Two polygons have the same shape if the angles of one are congruent to the corresponding angles of the other and the ratios of the measures of corresponding sides are equal." In a later section we will be more precise as to what is meant by the correspondence mentioned in the definition, but since you may not have had much experience working with ratios and proportions we proceed to study these terms first.

3-2 RATIO AND PROPORTION.

You were introduced to the terms "ratio" and "proportion" in your work in *Secondary One*. Recall that if a and b are positive numbers the *ratio* of a to b is the number $\frac{a}{b}$. We may also write a ratio in the form "$a:b$"

(this is read "the ratio of a to b"). When a number is described or thought of as a ratio it is usually described as the quotient of two other numbers. For example, the ratio "5 to 1" is usually expressed as "5 : 1" or "$\frac{5}{1}$" rather than "5".

It should be noted that if $b = 0$, the expression $\frac{a}{b}$ is not defined. We need not be concerned about this here, however, since the numbers in our geometry will represent measures of angles, segments, areas, etc. and hence will be positive numbers.

PROBLEMS 3-2A

Unless otherwise stated, the letters in the problems of this section and in later sections will represent positive numbers.

1. Which of the following ratios are equal to $\frac{3}{7}$?

   (a) $\frac{12}{28}$ (d) $\frac{6k}{14k}$

   (b) $\frac{9}{49}$ (e) $\frac{3a}{7b}$

   (c) $\frac{1\frac{1}{2}}{3\frac{1}{2}}$ (f) $\frac{3+a}{7a}$

2. Reduce the following ratios to lowest terms.

   (a) $\frac{15x}{3x}$ (c) $\frac{51\,abc}{69\,a}$

   (b) $\frac{42\,ax}{27\,az}$ (d) $\frac{x^2 - x}{x^2 + x}$

3. For what value of x does each of the following ratios have the value $\frac{4}{9}$?

   (a) $\frac{12}{3x}$ (c) $\frac{16x}{18}$

   (b) $\frac{3x}{45}$ (d) $\frac{x-3}{x+2}$

4. Determine the ratio of x to y from the following statements:

   (a) $3x = 4y$ (c) $4x - 6y = 0$

   (b) $ay = bx$ (d) $y = 5x$

5. Express each of the following as a ratio in the form $\frac{a}{b}$ and reduce to lowest terms.

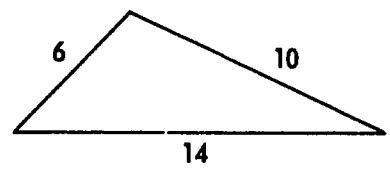
(a) 21 to 15

(b) $6x : 15x$

(c) $\frac{91}{13}$

(d) $2x + 3 : 1$

6. To say that three numbers are in the ratio $a:b:c$ is to say that the ratio of the first to the second is a to b, the second to the third is b to c, and the first to the third is a to c. For example, in the figure below we might say that the lengths of the sides of the triangle are in the ratio 3:5:7.



Find two other sets of numbers which have the ratio 3:5:7.

7. If the length of the side of a square is represented by the number s, find the ratio of its perimeter to its side.

8. A square is circumscribed about a circle of radius r. Find the ratio of the area of the square to the area of the circle.

If $a, b, c \ldots$ and $a', b', c' \ldots$ are two sequences of positive numbers, we say that the first sequence is *proportional* to the second if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \ldots = k$. The number k is called the *constant of proportionality*. Why is k positive? We note that if the first sequence is proportional to the second, then the second sequence is proportional to the first. Because if $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} = \ldots = k$, then $\frac{a'}{a} = \frac{b'}{b} = \frac{c'}{c} = \ldots = \frac{1}{k}$.

PROBLEM:

What is the constant of proportionality for the two sequences 2, 3, 7 and 6, 9, 21 if we compare the first sequence to the second? If we compare the second to the first?

> DEFINITION 3-1. If a, b, c, d are positive numbers such that $\frac{a}{b} = \frac{c}{d}$, we call this statement a *proportion.*

Note that we have restricted the terms of a proportion to positive numbers. Although this restriction is not necessary (we need only restrict b and d in the definition to non-zero numbers), we do so for the reason stated earlier—the numbers we use in geometry are positive numbers.

A *proportion* is a statement that two ratios are equal. Four numbers are said to be *in proportion* if the ratio of the first two is equal to the ratio of the last two. The numbers 2, 3, 4, 6 are in proportion. Why? Note that the order in which the numbers are given is important since the same numbers in the order 2, 6, 4, 3 are not in proportion. We sometimes use the notation 2:3 = 4:6 for a proportion. This is read "2 is to 3 as 4 is to 6" and means $\frac{2}{3} = \frac{4}{6}$.

If the second and third terms of a proportion are the same number, as in $a:b = b:c$, we call b a *mean* proportional between a and c or a *geometric mean* between a and c. *The* geometric mean between a and c is *the positive number*, $\sqrt{ac}$.

Example

To find a geometric mean between 9 and 16 we set up the equation $\frac{9}{x} = \frac{x}{16}$ and solve for x. We obtain $x^2 = 144$ which has two solutions, $x = 12$ and $x = -12$. *The* geometric mean between 9 and 16, then, is the number 12.

When more than two ratios are equal we may state this in the form of an *extended proportion.* For example, $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$ is an extended proportion. It is a short (and convenient) form for expressing the three proportions $\frac{a}{b} = \frac{c}{d}$, $\frac{c}{d} = \frac{e}{f}$, and $\frac{a}{b} = \frac{e}{f}$.

Since a proportion is a statement about numbers, the properties of proportions can be proved quite easily using the laws of algebra. The following theorem is fundamental to the work that follows.

> THEOREM 3-1. If a, b, c, and d are positive numbers, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.

PROBLEMS 3-2B

1. Theorem 3-1 above consists of two parts.
   (a) State the "only if" part of Theorem 3-1 and prove it. (This is sometimes referred to as the "cross-product" theorem.)
   (b) State the "if" part of Theorem 3-1 and prove it.
2. Are the numbers 2, 5, 6, 20 in proportion? Why?
3. Which of the following are true statements?
   (a) $\frac{3}{7} = \frac{12}{28}$ (c) $\frac{6x}{9} = \frac{4x^2}{6x}$
   (b) $\frac{13}{5} = \frac{27}{11}$ (d) $\frac{14xy}{26x} = \frac{21y}{39}$
4. Determine x so that the following statements are true.
   (a) $\frac{9}{2x} = \frac{5}{7}$ (c) $\frac{x-3}{5} = \frac{13}{6}$
   (b) $4 : x = x : 9$ (d) $\frac{2x-5}{3} = \frac{x-2}{2}$
5. Find the mean proportional between 4 and 12.
6. Find the ratio of x to y if $\frac{x-y}{x+y} = \frac{7}{13}$.
7. Show that if $\frac{a}{x} = \frac{a}{y}$ and $a \neq 0$, then $x = y$.
8. Find x and y if $\frac{x}{3} = \frac{2}{y} = \frac{3}{5}$.

It is often convenient to transform a known proportion into some equivalent form. This is particularly useful when we prove the Basic

Proportionality Theorem in Section 3-4 and again, later, when we prove that perimeters of similar polygons have the same ratio as any pair of corresponding sides. We list below six of the more useful of these equivalent forms. You should spend a little time studying them so that you recognise how each is obtained from the given proportion.

It is quite easy to prove that each of the statements below is equivalent to the given proportion, since an application of Theorem 3-1 and algebra reduces each to the form $ad = bc$.

THEOREM 3-2. $\frac{a}{b} = \frac{c}{d}$ if and only if

(a) $\frac{d}{b} = \frac{c}{a}$

(b) $\frac{a}{c} = \frac{b}{d}$

(c) $\frac{b}{a} = \frac{d}{c}$

(d) $\frac{a+b}{b} = \frac{c+d}{d}$

(e) $\frac{a-b}{b} = \frac{c-d}{d}$

(f) $\frac{a+c}{b+d} = \frac{a}{b}$

We can use Theorem 3-2 to generate even more proportions that are equivalent to $\frac{a}{b} = \frac{c}{d}$. For example, if we apply the theorem to the proportion $\frac{a+b}{b} = \frac{c+d}{d}$ we may conclude

$$\frac{a+b+b}{b} = \frac{c+d+d}{d} \quad \text{or} \quad \frac{a+2b}{b} = \frac{c+2d}{d}.$$

Similarly, applying the theorem to

$\frac{a+b}{c+d} = \frac{a}{b}$ we get $\frac{a+b+a}{c+d+b} = \frac{a}{b}$ etc.

PROBLEMS 3-2C

1. Tell how each of the statements of Theorem 3-2 is obtained from the given proportion, $\frac{a}{b} = \frac{c}{d}$.

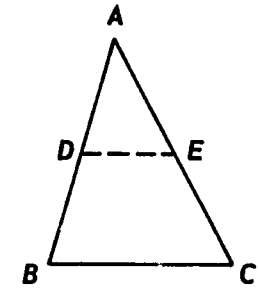
2. Complete each statement.

   (a) If $\frac{x}{y} = \frac{8}{5}$, then $\frac{x+y}{y} =$ _____ and $\frac{x-y}{y} =$ _____.

   (b) If $\frac{x+y}{y} = \frac{13}{4}$, then $\frac{x}{y} =$ _____ and $\frac{x-y}{y} =$ _____.

3. If $p = a + b + c$, $p' = a' + b' + c'$, and $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$, prove $\frac{p}{p'} = \frac{a}{a'}$.

4. In the figure on the right, if $\frac{DB}{AD} = \frac{EC}{AE}$ prove $\frac{AB}{AD} = \frac{AC}{AE}$.



3-3 RATIO AND PROPORTION IN GEOMETRY.

In geometry, and in particular in the study of similar polygons, we will use the properties of ratio and proportion as they apply to segment lengths. When we talk about the ratio of one segment to another we will mean by this the *positive number* that is the ratio of their *lengths*. Of course, such a ratio will have meaning only if the two segments under consideration have been measured in terms of the same unit.

> DEFINITION 3-2. If $\overline{AB}$ and $\overline{CD}$ are segments, the ratio of $\overline{AB}$ to $\overline{CD}$ is the number $\frac{AB}{CD}$. Two segments $\overline{AB}$ and $\overline{CD}$ are proportional to the two segments $\overline{A'B'}$ and $\overline{C'D'}$ if $\frac{AB}{A'B'} = \frac{CD}{C'D'}$.

The statement that the sides of ΔABC are *proportional* to the sides of $\Delta A'B'C'$ will mean that their corresponding sides have the *same ratio*; that is, $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$.

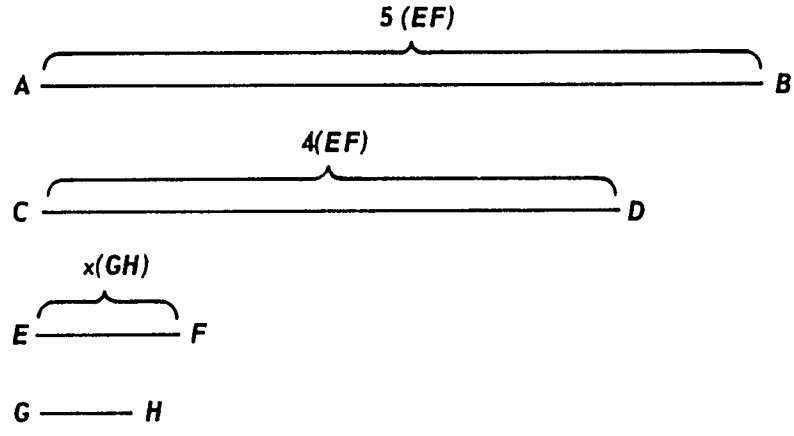
As you know from your earlier experience in measuring segments the number you get for the length of a segment depends on the unit you use in measuring it. For example, using the foot as a unit we might get the number $1\frac{1}{2}$ for the length of a certain segment whereas if we had used the inch as our unit we would get the number 18. This poses an important question. Does the number that we get for the ratio of two segments depend on the unit that we use in measuring them? How would you answer this question? Can you make up an example that would give you some insight to the answer?

Suppose the lengths of two sides of a room are 24 ft. and 18 ft. What is the ratio of the lengths of the shorter side to the longer? What would be the lengths of the two sides if they were measured in yards? in inches? Is the ratio of the lengths of the shorter side to the longer the same for these new units of measure as it was before? What does changing the unit of measure do to the lengths of each side? Do you see that it just multiplies each of them by the same positive number?

Example 1:

The lengths of two segments when measured in feet are 2 and 5. The ratio of the shorter to the longer is $\frac{2}{5}$. If they were measured in inches their lengths would be 2×12 and 5×12. If they were measured in yards their lengths would be $2 \times \frac{1}{3}$ and $5 \times \frac{1}{3}$. But $\frac{2 \times 12}{5 \times 12} = \frac{2}{5}$ and $\frac{2 \times \frac{1}{3}}{5 \times \frac{1}{3}} = \frac{2}{5}$. We see again that changing the unit of measure of two segments multiplies the length of each segment by the same positive number, but leaves their ratio unchanged.

Example 2:



Let $\overline{AB}$ and $\overline{CD}$ be any two segments whose lengths are 5 and 4 when $\overline{EF}$ is the unit of measure and let the length of $\overline{EF}$ be x when $\overline{GH}$ is the unit as in the figure.

We have $AB = 5(EF)$

and $EF = x(GH)$,

so $AB = 5[x(GH)] = 5x(GH)$.

Similarly, $CD = 4(EF)$

and $EF = x(GH)$,

so $CD = 4[x(GH)] = 4x(GH)$.

Once again we see that changing the unit of measure for the lengths of $\overline{AB}$ and $\overline{CD}$ from $\overline{EF}$ to $\overline{GH}$ only multiplied each of their lengths by the same

positive number x, while the ratio $\frac{AB}{CD} = \frac{5}{4}$ when $\overline{EF}$ is the unit and is $\frac{5x}{4x} = \frac{5}{4}$ when $\overline{GH}$ is the unit.

It appears from our examples that the answer to the question posed above is "no". Do these examples constitute a proof that the answer is no? Does a consideration of more examples make you more sure of your answer? We will state the answer in the form of a theorem.

> THEOREM 3-3. If the same unit is used to measure the lengths of two segments, their ratio is the same number no matter what unit is used to measure them.

Proof:

We see from our examples that changing the unit of measure of two segments just multiplies each of their lengths by the same positive number. Thus if ℓ and L are the lengths of two segments for one unit of measure, their lengths would be ℓx and Lx respectively for a second unit, where x is the length of the first unit when measured in terms of the second unit. Since $\frac{\ell x}{Lx} = \frac{\ell}{L}$, their ratio is the same number no matter what unit is chosen to measure them.

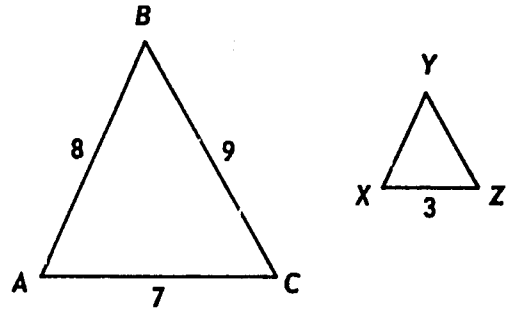
PROBLEMS 3-3

1. Write an extended proportion that shows the sides of $\triangle ABC$ are proportional to the sides of $\triangle FDE$.

2. If the sides of $\triangle ABC$ are proportional to the sides of $\triangle A'B'C'$ and if $AB = 7$, $BC = 9$, $AC = 13$, and $A'B' = 21$, find the lengths of $B'C'$ and $A'C'$.

3. Two segments have lengths of 24 inches and 36 inches. What would be their ratio if they were measured in feet? in yards? in miles?

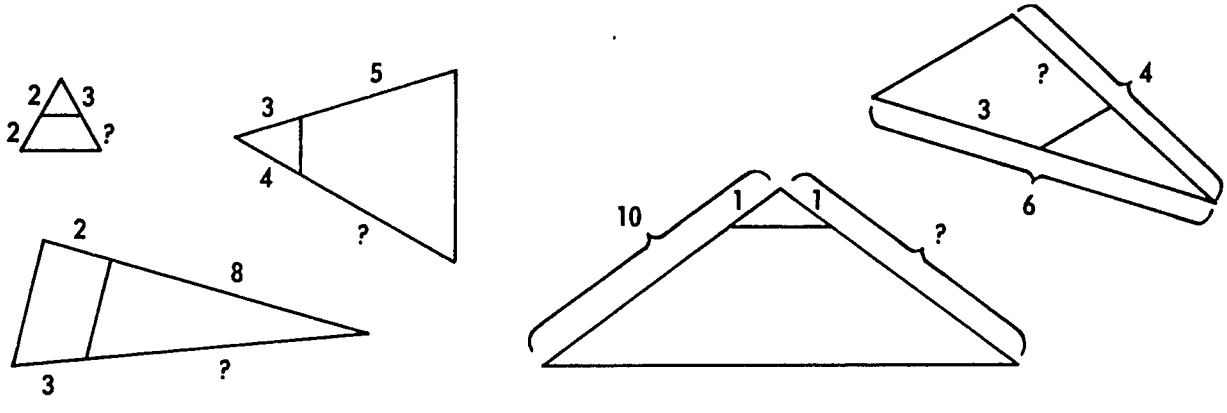
4. In the figure on the following page, the sides of $\triangle ABC$ are proportional to the sides of $\triangle XYZ$. If the lengths of the sides are as shown, what is the ratio of AB to XY? of BC to YZ?



3-4 THE BASIC PROPORTIONALITY THEOREM.

We shall find that triangles with the same shape have a number of interesting properties. A number of these properties deal with proportional segments. In particular we shall want to explore what proportions appear if, in a triangle, there is a segment parallel to one side and intersecting the other two sides.

In the following figures if the line segment intersecting the two sides of the triangle is parallel to the third side what would you guess for the measures of the segments marked "?" ?

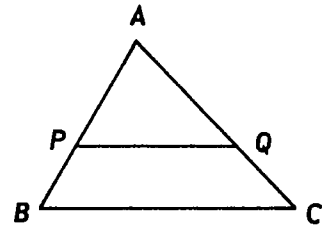


Which of the following proportions do you believe are true if $\overline{PQ} \| \overline{BC}$?

$\frac{AP}{AB} = \frac{AQ}{AC}$, $\frac{AP}{PB} = \frac{AC}{AQ}$,

$\frac{PB}{PA} = \frac{QC}{QA}$, $\frac{AB}{BP} = \frac{AC}{CQ}$,

$\frac{AP}{AQ} = \frac{PB}{QC}$, $\frac{AB}{AC} = \frac{AQ}{AP}$, $\frac{AB}{AP} = \frac{AC}{AQ}$

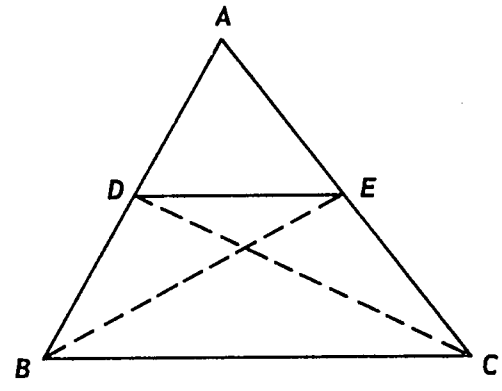


We shall be able to prove that $\frac{AB}{AP} = \frac{AC}{AQ}$ is true and from it we can easily derive a number of other useful proportions. A convenient way to prove this proportion makes use of a theorem you proved earlier about the ratio of the areas of two triangles which have equal altitudes. Do you remember it?

> THEOREM 3-4. If a line parallel to one side of a triangle intersects the other two sides in distinct points, then it cuts off segments which are proportional to these sides.

Given: $\triangle ABC$ with points D and E on $\overline{AB}$ and $\overline{AC}$ respectively such that $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$.

Prove: $\frac{AB}{AD} = \frac{AC}{AE}$.



Proof:

1. Draw $\overline{DC}$ and $\overline{BE}$.
2. $\frac{\text{area } \triangle BED}{\text{area } \triangle ADE} = \frac{BD}{AD}$. Why?
3. $\frac{\text{area } \triangle CED}{\text{area } \triangle ADE} = \frac{CE}{AE}$. Why?
4. $\triangle BED$ and $\triangle CED$ have the same base and equal altitudes. (Why?) $\therefore$ Area $\triangle BED$ = area $\triangle CED$. Why?
5. Combining the three statements (2), (3), and (4) gives $\frac{BD}{AD} = \frac{CE}{AE}$.
6. $\frac{BD + AD}{AD} = \frac{CE + AE}{AE}$. Why?
7. $\frac{AB}{AD} = \frac{AC}{AE}$.

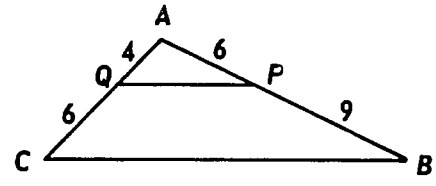
> COROLLARY 3-4-1.
> Using the above figure we also have
> $\frac{AD}{BD} = \frac{AE}{CE}$, $\frac{AB}{BD} = \frac{AC}{EC}$.

This corollary follows directly from steps in the proof of Theorem 3-4. Do you know which ones?

COROLLARY 3-4-2.

If $\overleftrightarrow{DE}$ cuts the extensions of the sides of $\triangle ABC$ so that D and E are on $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ respectively, then $\frac{AB}{AD} = \frac{AC}{AE}$.

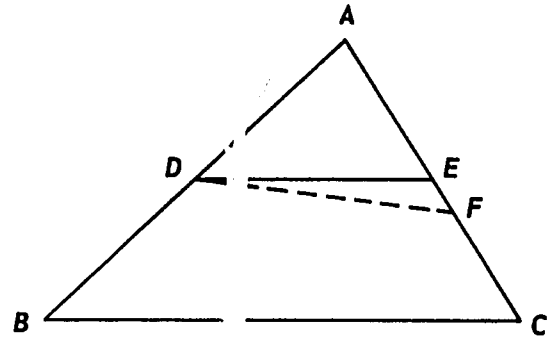
After we have proved a theorem it is always interesting to see whether the converse is true. Do you think the converse of Theorem 3-4 is true? In this figure if the indicated measures were correctly drawn do you think that $\overline{PQ} \parallel \overline{BC}$? We can prove that it is.



THEOREM 3-5. If a line intersects two sides of a triangle and cuts off segments proportional to these sides, then it is parallel to the third side.

Given: $\triangle ABC$ with points D and E on segments $\overline{AB}$ and $\overline{AC}$ respectively such that $\frac{AB}{AD} = \frac{AC}{AE}$.

Prove: $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$.



Proof:

We shall use the method of indirect proof which you studied in *Secondary Three*. Suppose $\overleftrightarrow{DE}$ is not parallel to $\overleftrightarrow{BC}$. Then let $\overleftrightarrow{DF}$ be the line through D parallel to $\overleftrightarrow{BC}$ intersecting $\overleftrightarrow{AC}$ in F, which is distinct from E. By Theorem 3-4,

$$\frac{AB}{AD} = \frac{AC}{AF}.$$

But by hypothesis,

$$\frac{AB}{AD} = \frac{AC}{AE}$$

$$\therefore \frac{AC}{AF} = \frac{AC}{AE}$$

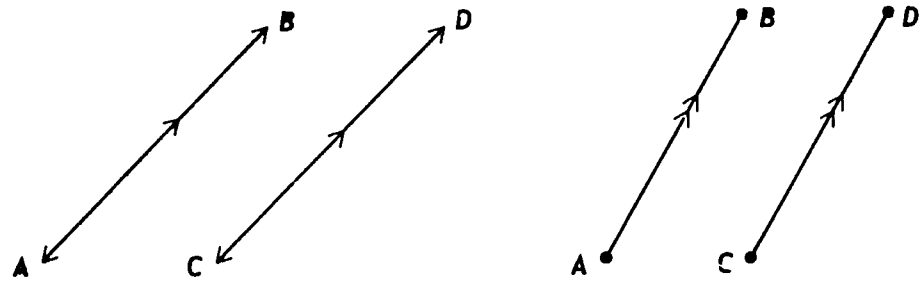
$$\therefore AF = AE.$$

Hence F is the same point as E, contradicting the previous statement that F and E are distinct points. Therefore $\overleftrightarrow{DE} \| \overleftrightarrow{BC}$.

In the proof of this theorem we have assumed that F lies between A and C. Is it possible for (a) C to be between A and F or (b) A to be between C and F? If not, why not?

NOTATION We are going to use arrows on lines or line segments to indicate they are parallel.

Thus: or



indicates that $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$ or $\overline{AB} \| \overline{CD}$. Lines with the same number of arrow heads are parallel to each other.

PROBLEMS 3-4

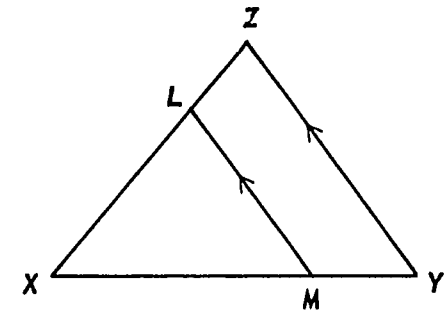
1. In the figure if $\overleftrightarrow{LM} \| \overleftrightarrow{YZ}$ complete the following blanks:

$\frac{XL}{XZ} =$ _____, $\frac{MY}{MX} =$ _____,

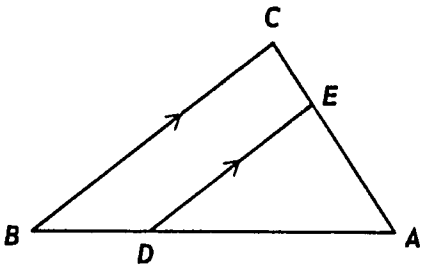
$\frac{XM}{XL} =$ _____

$\frac{XY}{XZ} =$ _____, $\frac{XZ}{LZ} =$ _____,

$\frac{MY}{LZ} =$ _____



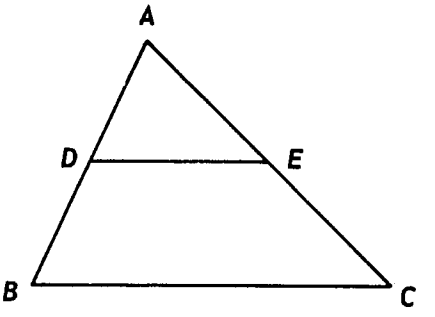
2. In the figure if $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$ and if $BD = 6$, $AD = 9$, $CE = 2$, find AC.



3. In the figure if $A\hat{D}E \equiv A\hat{B}C$, complete the following statements:

$\frac{AD}{BD} =$ _____, $\frac{AC}{AE} =$ _____

$\frac{AD}{AE} =$ _____, $\frac{AC}{AB} =$ _____

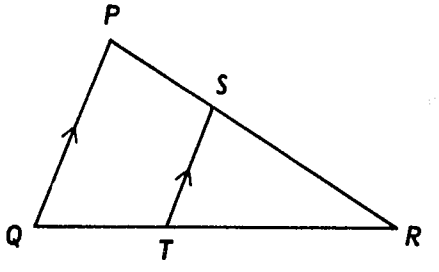


4. In the figure if

(a) $PS = 3$, $SR = 6$, $TR = 8$, find QT.

(b) $QT = 5$, $QR = 15$, $PS = 6$, find PR.

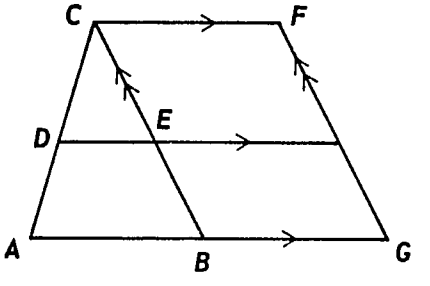
(c) $PR = 21$, $QR = 30$, $QT = 10$, find PS.



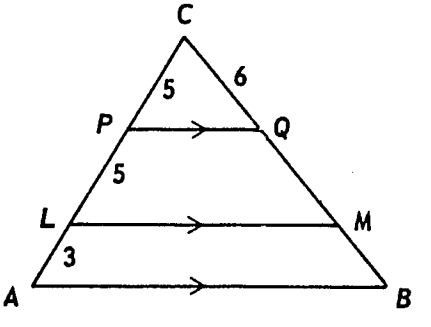
5. In the figure shown,

(a) If $AD = 3$, $AC = 9$, $FG = 12$ find CE.

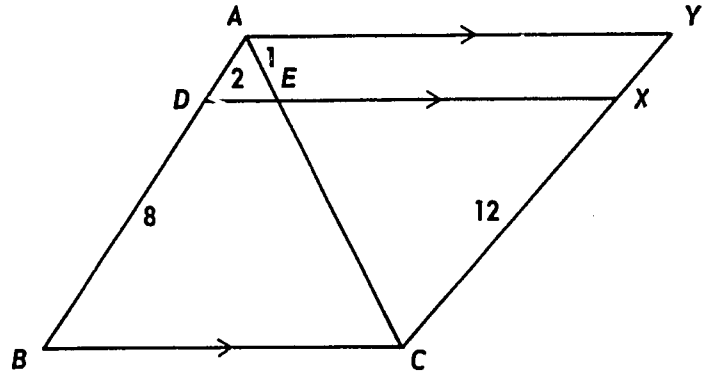
(b) If $AD = 1$, $CD = 2$, $CE = 4$ find FG.



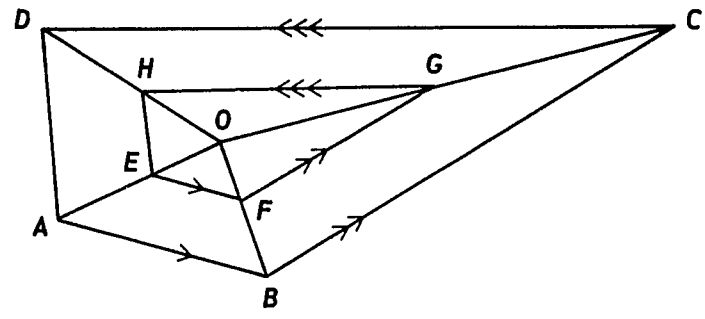
6. In the figure if the lengths are as shown how long are the segments $\overline{MQ}$ and $\overline{BM}$?



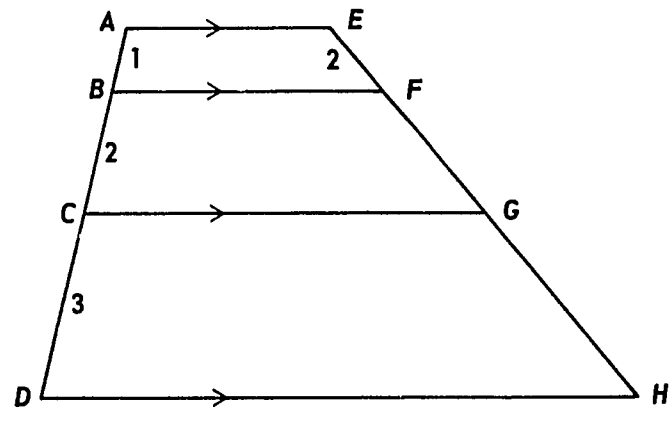
7. In the figure if the lengths of the segments are as indicated how long is $\overline{XY}$?



8. In the figure as shown prove that $\overleftrightarrow{HE} \| \overleftrightarrow{DA}$.

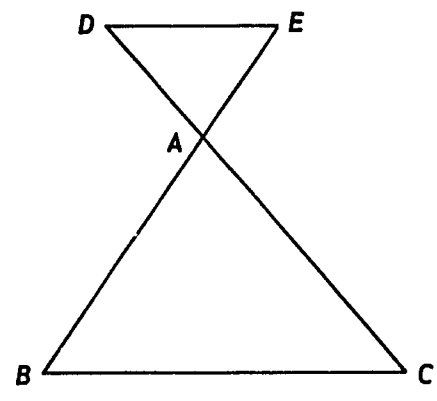


9. In the figure find the lengths of $\overline{FG}$ and $\overline{GH}$.



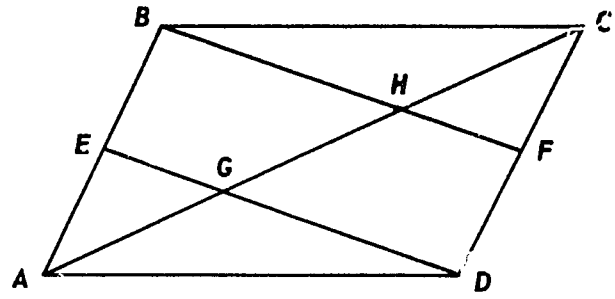
10. If, in the figure, $\overleftrightarrow{DE} \| \overleftrightarrow{BC}$, prove that $\frac{AE}{BE} = \frac{AD}{CD}$.

HINT: Draw $\overline{BD}$ and $\overline{EC}$. Consider the ratio of areas of ΔBDE, ΔDEC, ΔAED.



Challenge Problem

11. ▱$ABCD$ in a parallelogram and E and F are midpoints of $\overline{AB}$ and $\overline{CD}$ respectively. $\overline{AC}$ intersects $\overline{ED}$ and $\overline{BF}$ in G and H respectively. Prove that G and H trisect $\overline{AC}$. Prove also that $EG = \frac{1}{3} ED$.



HINT: Show that $\overleftrightarrow{BF} \parallel \overleftrightarrow{ED}$. Then consider ΔCHF and ΔCGD, and ΔAGE and ΔAHB. For the last part apply the result of Problem 10 to ΔCGD and ΔAGE.

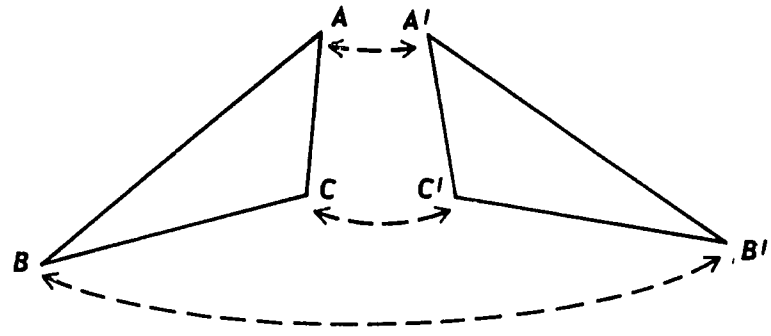
3-5 SIMILAR TRIANGLES.

We have been talking about geometric figures and, in particular, triangles that have the same shape but not necessarily the same size. As yet we have not made a formal definition of what we mean by "same shape". Later, in Section 3-11, we will define similar polygons. Triangles are a particular class of polygons and, since some of the properties of similar triangles are useful in discussing similar polygons, we define them first. As in congruence, similarity involves a correspondence between the two triangles such that corresponding sides and corresponding angles have a special relationship to each other.

> DEFINITION 3-3. Two triangles are similar if and only if a correspondence can be set up between the two triangles such that the corresponding angles are congruent and the corresponding sides are proportional. The correspondence between two similar triangles is called a *similarity*.

To be more explicit, two triangles ΔABC and $\Delta A'B'C'$ such that $\Delta ABC \leftrightarrow \Delta A'B'C'$ are similar if and only if

(1) $\widehat{A} \equiv \widehat{A'}, \widehat{B} \equiv \widehat{B'}, \widehat{C} \equiv \widehat{C'}$ and (2) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$.



Thus the definition of similarity requires two things: a correspondence set up such that (1) corresponding angles are congruent and (2) corresponding sides are proportional. For polygons, in general, both of these conditions must hold. However in the case of triangles it turns out that if one of the conditions holds then the other follows. In other words if corresponding sides are proportional then corresponding angles are congruent, and conversely. Hence only one of the conditions is required to determine similarity between triangles. In the theorems that follow we will establish the equivalence of the two relations (1) and (2). That is, we assume congruency of corresponding angles and deduce proportionality of corresponding sides and conversely.

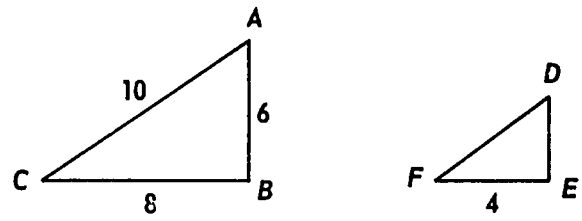
We note that if there is a similarity correspondence between two triangles then the sides opposite congruent angles correspond. Thus if ΔABC and $\Delta A'B'C'$ are similar such that $\widehat{A} \equiv \widehat{A'}$, $\widehat{B} \equiv \widehat{B'}$, $\widehat{C} \equiv \widehat{C'}$ the pairs of corresponding sides are $\overline{BC}$ and $\overline{B'C'}$, $\overline{AC}$ and $\overline{A'C'}$ and $\overline{AB}$ and $\overline{A'B'}$. $\overline{BC}$ and $\overline{B'C'}$ are opposite the congruent angles $\widehat{A}$ and $\widehat{A'}$ respectively. A similar statement holds for the other corresponding sides.

> *NOTATION*: If ΔABC and ΔXYZ are similar such that $A \leftrightarrow X$, $B \leftrightarrow Y$, $C \leftrightarrow Z$ we write $\Delta ABC \sim \Delta XYZ$.

PROBLEMS 3-5

1. If $\Delta ABC \sim \Delta B'A'C'$ write down the pairs of corresponding angles and the pairs of corresponding sides.

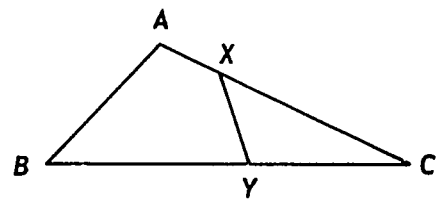
2. In the figure $\Delta ABC \sim \Delta DEF$. Find DE and DF.



3. If $\Delta ABC \sim \Delta A'B'C'$ and $AC = 20$, $A'C' = 15$, $B'C' = 12$, and $A'B' = 9$, find the lengths of the other sides of ΔABC.

4. The sides of a triangle are 4, 6 and a inches. The corresponding sides of a triangle similar to the first triangle are b, 12 and 8 inches respectively. What are the lengths of the sides of each triangle?

5. In the figure $\Delta ABC \sim \Delta YXC$. If $CX = 6$, $CY = 5$, $AX = 3$, and $AB = 8$ find BC and XY.



6. Prove that if, in ΔABC, D and E are the midpoints of $\overline{AC}$ and $\overline{BC}$ respectively, then $\Delta ABC \sim \Delta DEC$.

7. Prove that any two equilateral triangles are similar.

8. Are two congruent triangles similar? Why? Are two similar triangles congruent? Why?

9. Prove that if $\Delta ABC \sim \Delta RST$ and $\Delta STR \sim \Delta YZX$ then $\Delta ABC \sim \Delta XYZ$.

3-6 THE BASIC SIMILARITY THEOREMS.

In this section we prove fundamental theorems on similar triangles.

CLASS ACTIVITY

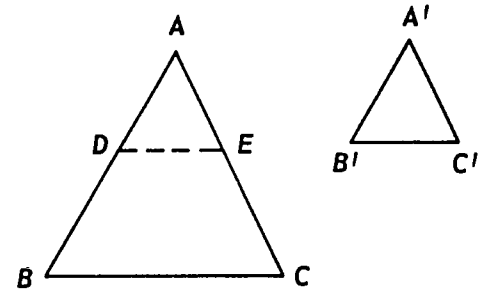
Draw several triangles ΔABC and $\Delta A'B'C'$ of different sizes such that $m(\widehat{A}) = m(\widehat{A'}) = 60°$, $m(\widehat{B}) = m(\widehat{B'}) = 40°$, $m(\widehat{C}) = m(\widehat{C'}) = 80°$. In each

case measure the lengths of the sides of the triangles and calculate the ratios $\frac{AB}{A'B'}$, $\frac{BC}{B'C'}$, $\frac{AC}{A'C'}$. What can you say about these ratios for each pair of triangles? Repeat this exercise for triangles with angles whose measures are 100°, 50°, 30°.

> THEOREM 3-6. (The AAA Similarity Theorem)
> If a correspondence between two triangles is such that the corresponding angles are congruent, then the correspondence is a similarity.

Given: $\triangle ABC$ and $\triangle A'B'C'$ such that $\hat{A} \equiv \hat{A}'$, $\hat{B} \equiv \hat{B}'$, $\hat{C} \equiv \hat{C}'$.

Prove: $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$ and therefore $\triangle ABC \sim \triangle A'B'C'$.



Proof:

If we can prove the first of these equations the second equation will follow merely by a cyclic interchange of letters. Hence we set out to prove that $\frac{AB}{A'B'} = \frac{AC}{A'C'}$.

1. Let D and E be points on $\overline{AB}$ and $\overline{AC}$ respectively such that $AD = A'B'$ and $AE = A'C'$.

2.	$\triangle ADE \equiv \triangle A'B'C'$	SAS congruence theorem
3.	$A\hat{D}E \equiv \hat{B}'$	Why?
4.	$A\hat{D}E \equiv \hat{B}$	Statement 3 and hypothesis
5.	$\therefore \overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$	Why?

6. If D and B coincide then $\triangle ADE \equiv \triangle ABC$ and therefore $\triangle ABC \equiv \triangle A'B'C'$. Thus $AB = A'B'$ and $AC = A'C'$ and hence $\frac{AB}{A'B'} = \frac{AC}{A'C'}$. Why?

7. If D is different from B then by the Basic Proportionality Theorem we have $\frac{AB}{AD} = \frac{AC}{AE}$ and therefore $\frac{AB}{A'B'} = \frac{AC}{A'C'}$. Why?

8. Similarly we have $\frac{AB}{A'B'} = \frac{BC}{B'C'}$.

9. Therefore $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$ and hence $\Delta ABC \sim \Delta A'B'C'$.

What have we assumed about the relative sizes of the ratios $\frac{AB}{A'B'}$ and $\frac{AC}{A'C'}$? Can you prove the theorem if $AB < A'B'$ and $AC < A'C'$?

> COROLLARY 3-6-1. (The AA Similarity Theorem)
>
> If two angles of one triangle are congruent to two angles of another triangle, then the two triangles are similar.

The proof of the corollary depends on the fact that if two pairs of corresponding angles of two triangles are congruent then (by the Angle-Sum Theorem for triangles) the third pair must also be congruent.

> COROLLARY 3-6-2.
>
> If a line parallel to one side of a triangle intersects the other two sides in distinct points, then it cuts off a triangle similar to the given triangle.

There are other corollaries which can be proved by the AAA Similarity Theorem:

COROLLARY 3-6-3.

If two right triangles have an acute angle of one congruent to an acute angle of the other, the triangles are similar.

COROLLARY 3-6-4.

If two triangles are similar to a third triangle, they are similar to each other.

COROLLARY 3-6-5.

Corresponding altitudes of two similar triangles are in the same ratio as any two corresponding sides.

COROLLARY 3-6-6.

Corresponding angle bisectors of two similar triangles are in the same ratio as any two corresponding sides.

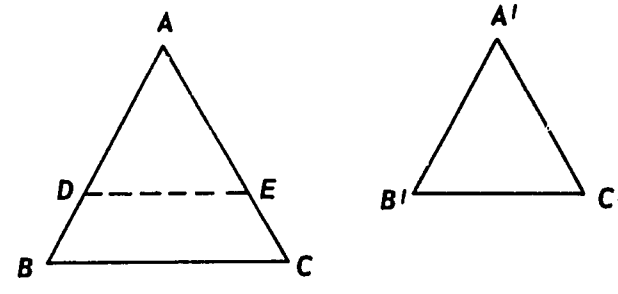
THEOREM 3-7. (The SAS Similarity Theorem)

Given a correspondence between two triangles. If two pairs of corresponding sides are proportional and the included angles are congruent, then the correspondence is a similarity.

Given: $\triangle ABC$ and $\triangle A'B'C'$

$\widehat{A} \equiv \widehat{A'}$ and $\dfrac{AB}{A'B'} = \dfrac{AC}{A'C'}$.

Prove: $\triangle ABC \sim \triangle A'B'C'$.



Proof:

1. Let D and E be points on $\overline{AB}$ and $\overline{AC}$ respectively such that $AD = A'B'$ and $AE = A'C'$.

2.	$\triangle ADE \equiv \triangle A'B'C'$	Why?
3.	$\frac{AB}{AD} = \frac{AC}{AE}$	Why?
4.	$\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$	Theorem 3-5.
5.	$\hat{B} \equiv \hat{ADE}$	Why?
6.	$\hat{A} \equiv \hat{A'}$	Hypothesis
7.	$\triangle ABC \sim \triangle ADE$	AA Similarity Theorem.
8.	$\triangle ABC \sim \triangle A'B'C'$	Statement 2 and Corollary 3-6-4.

In the proof of this theorem we have assumed that $AB > A'B'$ and $AC > A'C'$. Prove the theorem when

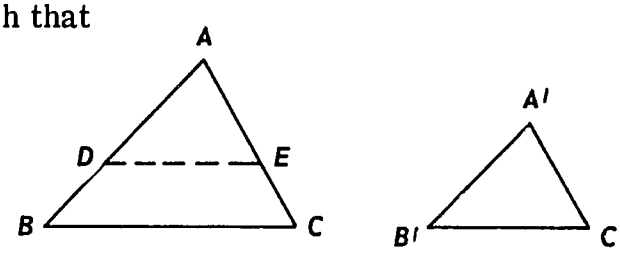
(1) $AB < A'B'$ and $AC < A'C'$ (2) $AB = A'B'$ and $AC = A'C'$.

> THEOREM 3-8. (The SSS Similarity Theorem)
>
> Given a correspondence between two triangles. If corresponding sides are proportional, then the correspondence is a similarity.

Given: $\triangle ABC$ and $\triangle A'B'C'$ such that

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$$

Prove: $\triangle ABC \sim \triangle A'B'C'$



Proof:

1. On $\overline{AB}$ construct $AD = A'B'$ and on $\overline{AC}$ construct $AE = A'C'$ and draw $\overline{DE}$.

2.	$\frac{AB}{AD} = \frac{AC}{AE}$	Why?
3.	$\hat{A} \equiv \hat{A}$	
4.	$\triangle ABC \sim \triangle ADE$	SAS Similarity Theorem.
5.	$\frac{DE}{BC} = \frac{AD}{AB}$	Definition of similarity.

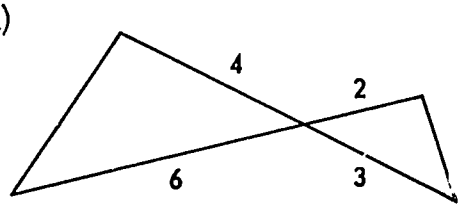
6.	$DE = BC \times \frac{AD}{AB} = BC \times \frac{A'B'}{AB}$	Statements 1 and 5.
7.	$B'C' = BC \times \frac{A'B'}{AB}$	Hypothesis.
8.	$DE = B'C'$	Statements 6 and 7.
9.	$\triangle ADE \equiv \triangle A'B'C'$	Statements 1, 8 and SSS Congruence Theorem.
10.	$\triangle ABC \sim \triangle A'B'C'$	Statements 4, 9 and Corollary 3-6-4.

What assumptions have we made in the proof about the relative sizes of the sides of $\triangle ABC$ and $\triangle A'B'C'$?

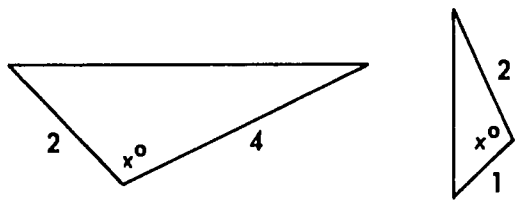
PROBLEMS 3-6

1. For each pair of triangles indicate whether the two triangles are similar and if they are similar state why.

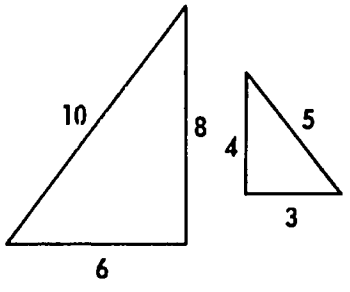
(a)



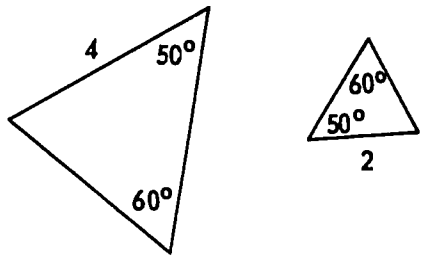
(c)



(b)



(d)

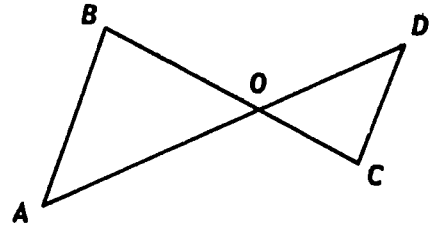


2. In $\triangle DEF$, $DF = 20$, $DE = 16$, $EF = 24$. Through H, a point in $\overline{DF}$, and K, a point in $\overline{EF}$, $\overline{HK}$ is drawn so that FH = 15 and $FK = 18$. Show that $\triangle HKF \sim \triangle DEF$. How long is $\overline{HK}$?

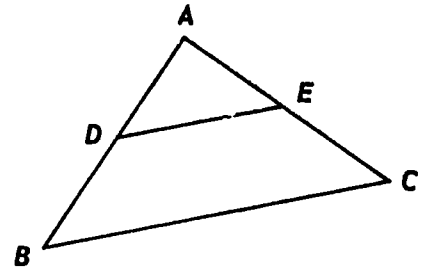
3. The legs of one right triangle are 3 and 4 and the legs of another are 6 and 8. Are the triangles similar?

4. In ΔABC and ΔDEF, $m(\hat{A}) = m(\hat{D}) = 60°$, $AB = 4$, $AC = 10$, $DE = 2$ and $DF = 5$. Are the triangles similar? If so, prove it.

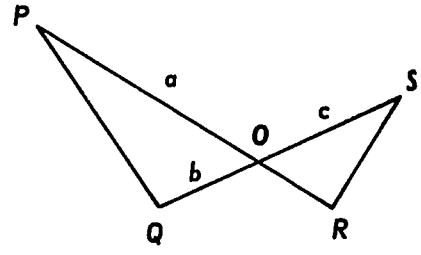
5. In the figure, $AO = 8$, $BO = 6$, $OD = 4$ and $OC = 3$. Prove that $\Delta AOB \sim \Delta DOC$.



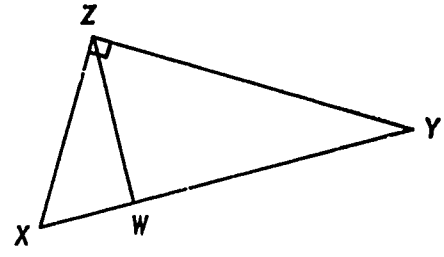
6. If in the figure $\frac{AE}{AB} = \frac{AD}{AC}$ prove that $\Delta ABC \sim \Delta AED$.



7. In the figure if $OP = a$, $OQ = b$, $OS = c$ and if $\Delta OPQ \sim \Delta OSR$ express OR in terms of a, b and c.



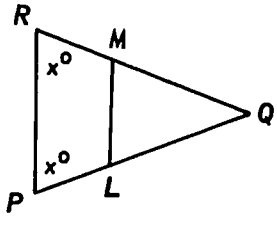
8. If ΔXYZ is a right triangle with right angle at Z and if $\frac{XW}{XZ} = \frac{XZ}{XY}$ prove that $\overline{ZW} \perp \overline{XY}$.



9. Prove that if two isosceles triangles have congruent vertex angles the triangles are similar.

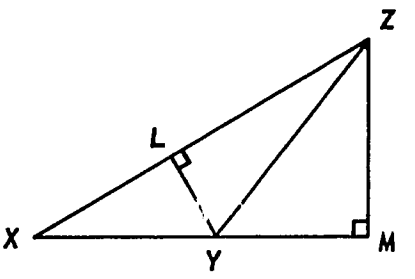
10. Prove that if two isosceles triangles have congruent base angles the triangles are similar.

11. If in the figure $RM = PL$ prove that $\overleftrightarrow{LM} \parallel \overleftrightarrow{PR}$.



12. Prove that if two triangles are similar then the ratio of their perimeters is the same as the ratio of two corresponding sides.

13. In the figure $\overline{MZ} \perp \overline{XY}$ and $\overline{LY} \perp \overline{XZ}$. Find two similar triangles. How many proportions can you write?



14. If $\triangle ABC \sim \triangle DEF$ and $\overline{AG}$ and $\overline{DH}$ are medians prove that the ratio of the medians is equal to the ratio of any two corresponding sides.

Challenge Problem

15. In $\triangle ABC$, D is the midpoint of $\overline{AB}$ and E is a point on $\overline{AC}$ such that $AE > EC$. $\overleftrightarrow{DE}$ and $\overleftrightarrow{BC}$ intersect at F. Prove that

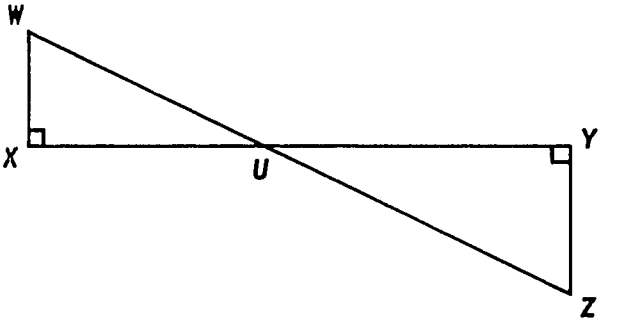
$$\frac{FB}{FC} = \frac{AE}{CE}.$$

HINT: Draw a line $\overleftrightarrow{CP}$ through C parallel to $\overleftrightarrow{AB}$ and intersecting $\overline{EF}$ at P.

3-7 METHODS OF PROVING LINE SEGMENTS PROPORTIONAL.

Many problems in similar triangles involve the setting up of ratios of line segments and proving line segments proportional.

Suppose that in this figure we want to prove $\frac{XW}{UW} = \frac{YZ}{UZ}$.



This proportion naturally makes us think of similar triangles because such proportions are a basic characteristic of similar triangles. If we could prove two triangles similar here the proportion might come as a consequence. What triangles might we try to prove similar? Do these triangles have as sides the segments involved in the proportion?

What correspondence between the two triangles would produce the required proportion? What information do we have which might help prove the triangles similar? We now give a formal proof of the above result.

Proof:

1. ΔXUW contains $\overline{XW}$ and $\overline{UW}$ and ΔUYZ contains segments $\overline{YZ}$ and $\overline{UZ}$.
2. $X\hat{U}W \equiv Y\hat{U}Z$ Why?
3. $W\hat{X}U \equiv U\hat{Y}Z$ Why?
4. $\Delta XUW \sim \Delta YUZ$ Why?
5. $\dfrac{XW}{UW} = \dfrac{YZ}{UZ}$ Why?

We can formulate the general method of attack on such problems. To prove that that four line segments are proportional we show that the line segments are corresponding sides of similar triangles. The steps in applying this procedure are:

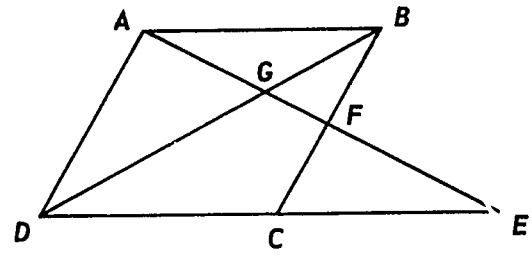
1. Find two triangles each of which has two of the four segments as sides.
2. Prove that these triangles are similar.
3. Form a proportion using the fact that these line segments are corresponding sides of the similar triangles.
4. If necessary, transform the proportion by an application of theorems on ratio and proportion.

Instead of proving the equality of two ratios of line segments we may be required to prove that the product of the lengths of two line segments is equal to the product of the lengths of two other line segments. In such a case we reduce the expression to the equality of two ratios and then apply the above procedure. For example in order to prove that $AD \times EF = BC \times DE$ we show that

$$\frac{AD}{DE} = \frac{BC}{EF} \qquad \text{or} \qquad \frac{AD}{BC} = \frac{DE}{EF}.$$

Sometimes the four line segments to be proved proportional may not happen to be sides of similar triangles. In that case we show that each of the given ratios is equal to a third ratio. This may involve the consideration of more than two triangles. The following example indicates the method of attack.

Example. In the figure ▱$ABCD$ is a parallelogram. Prove that $AG^2 = FG \times EG$.



Analysis: This means that we should prove that $\frac{AG}{FG} = \frac{EG}{AG}$.

Proof:

The line segments $\overline{AG}$, $\overline{FG}$ and $\overline{EG}$ do not all belong to any two triangles that we might be able to prove similar. So we consider each ratio separately.

First take $\frac{AG}{FG}$. These segments are sides of triangles ΔADG, ΔGFB and ΔAGB. Out of this set of three triangles the two that appear to be similar are ΔADG and ΔGFB. But for a proper correspondence we should consider ΔAGD and ΔFGB (why?). These two triangles are similar. Why? Hence we have $\frac{AG}{FG} = \frac{AD}{FB} = \frac{GD}{GB}$. (1)

Next consider $\frac{EG}{AG}$. These segments are sides of triangles ΔAGB, ΔDGE and ΔADG. Again for easy matching we see that we must drop ΔADG from our consideration. For a proper correspondence we consider ΔAGB and ΔEGD. These two triangles are similar. Why? Hence we have

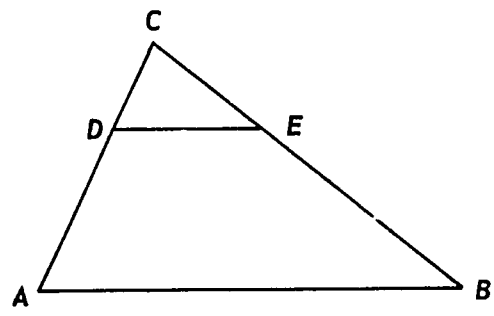
$$\frac{EG}{AG} = \frac{ED}{AB} = \frac{GD}{GB} \qquad (2)$$

From (1) and (2) we see that each of the ratios $\frac{AG}{FG}$ and $\frac{EG}{AG}$ is equal to $\frac{GD}{GB}$. Hence $\frac{AG}{FG} = \frac{EG}{AG}$ and therefore $AG^2 = EG \times FG$.

PROBLEMS 3-7

1. Prove that any two equiangular triangles are similar.

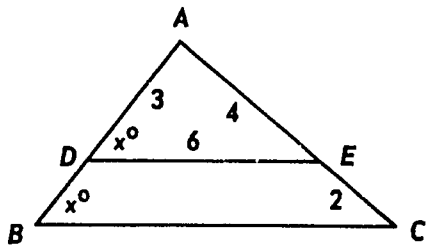
2. In the figure $AC = 18$, $BC = 24$, $DC = 6$ and $EC = 8$. Is $\overleftrightarrow{DE} \| \overleftrightarrow{AB}$? Is $\Delta CDE \sim \Delta CAB$?



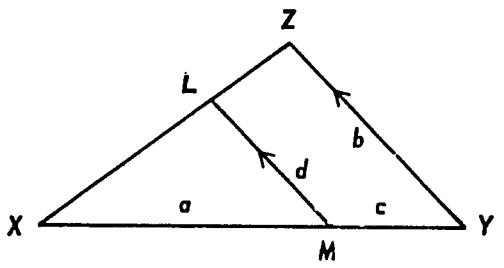
3. A tree casts a shadow 30 feet long when a 6-foot pole casts a shadow 9 feet long. How tall is the tree?

4. A line from the top of a cliff to the gound just passes over the top of a telephone pole 20 feet high and meets the ground at a point 15 feet from the base of the pole. If it is 120 feet from this point to the base of the cliff how high is the cliff?

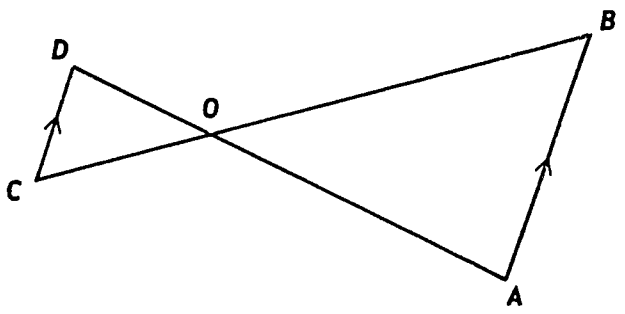
5. Given the figure as shown. Find AB and BC.



6. Given the figure as shown. Express d in terms of a, b and c.

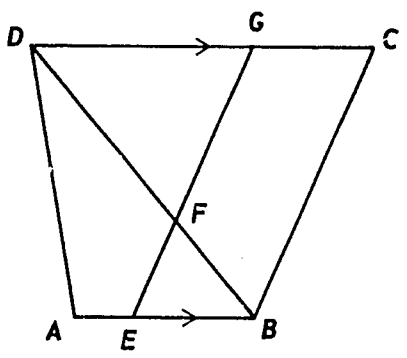


7. In the figure as shown prove that:
   (a) $\Delta ABO \sim \Delta DCO$
   (b) $AO \times OC = BO \times OD$.

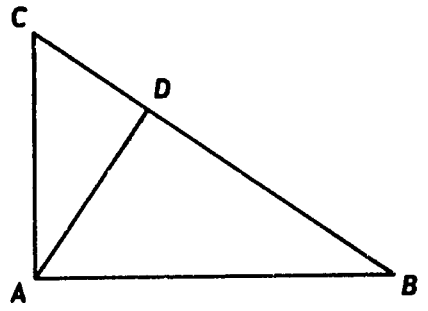


8. Prove that the diagonals of a trapezium divide each other into four segments that are proportional.

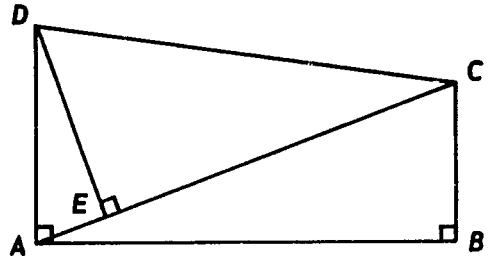
9. Given the figure as shown with E and G as any points on $\overline{AB}$ and $\overline{CD}$ respectively.
Prove that $DG \times FB = EB \times DF$.



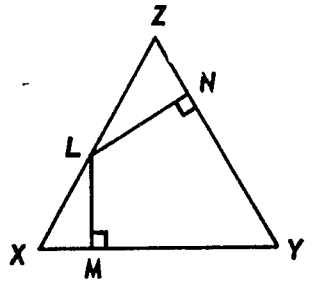
10. In the figure if $m(C\hat{A}B) = 90°$, $m(A\hat{D}B) = 90°$, prove that $\frac{AD}{AB} = \frac{CD}{AC}$ and $\frac{BD}{AD} = \frac{AB}{AC}$.



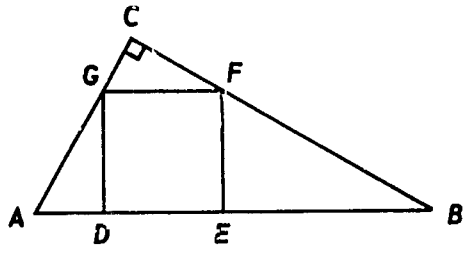
11. In the figure as shown prove that $\frac{DE}{AE} = \frac{AB}{BC}$.



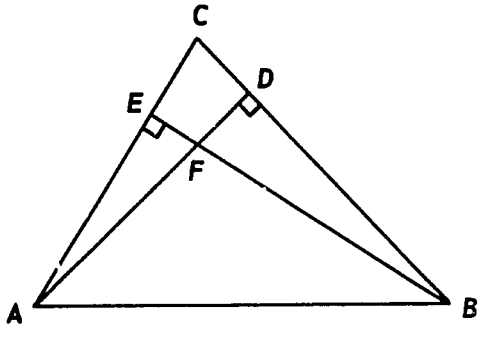
12. In the figure $XY = YZ$. Prove that $LM \times NZ = LN \times MX$.



13. In the figure $\square DEFG$ is a square.
Prove that:
(a) $AD \times EB = DG \times FE$.
(b) $AD \times CF = CG \times DG$.

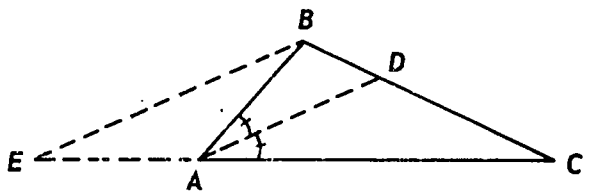


14. Using the given figure prove that
(a) $\Delta ADC \sim \Delta BEC$.
(b) $\Delta AFE \sim \Delta BFD$.



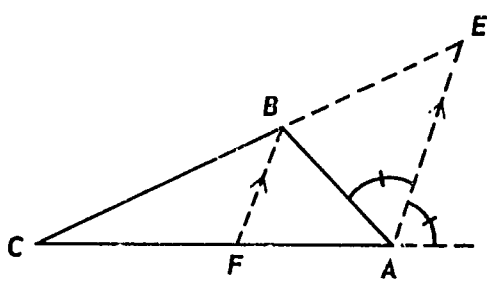
15. In ΔABC let E, F and G be the midpoints of the sides $\overline{AB}$, $\overline{BC}$ and $\overline{CA}$ respectively. Prove that $\Delta ABC \sim \Delta FGE$.

16. Prove that the bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides.



HINT: Given ΔABC, $\overline{AD}$ the bisector of $\hat{A}$ meeting $\overline{BC}$ in D. The problem is to prove that $\frac{CD}{BD} = \frac{AC}{AB}$. Draw $\overleftrightarrow{BE} \| \overleftrightarrow{AD}$.

17. Given ΔABC with $\overline{AE}$ the bisector of the external angle at $\hat{A}$ meeting $\overleftrightarrow{BC}$ in E. Prove that $\frac{BE}{CE} = \frac{BA}{CA}$.

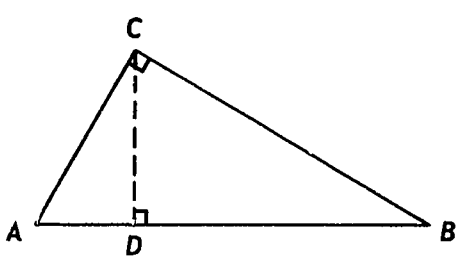


18. Given ΔABC. Let the bisectors of the internal and external angles at $\hat{A}$ meet $\overleftrightarrow{BC}$ in points D and E respectively. Prove that $\frac{BD}{CD} = \frac{BE}{CE}$.

HINT: Use the results of Problems 16 and 17.

3-8 SIMILARITY IN RIGHT TRIANGLES.

Consider the figure below where ΔABC has a right angle at C and the altitude is drawn from the right angle to the hypotenuse.



Are there any similar triangles in the figure? If you find two similar triangles, name them, being careful to put corresponding vertices in the same order. Can you find more than one pair of similar triangles? How many pairs of similar triangles are there in the figure?

Is $\triangle ADC \sim \triangle ACB$? Why?

Is $\triangle ACB \sim \triangle CDB$? Why?

Is $\triangle ADC \sim \triangle CDB$? Why?

Can you summarize the results above into one statement? We will call this statement Theorem 3-9.

> THEOREM 3-9. The altitude to the hypotenuse of a right triangle subdivides it into two right triangles that are similar to the given triangle and to each other.

Using the similarities developed in proving Theorem 3-9 we can write many different proportions involving corresponding sides of similar triangles. Let us list some of those proportions.

From $\triangle ADC \sim \triangle ACB$ we get (fill in the blanks):

(a) $\frac{AD}{AC} = \frac{?}{CB}$ (b) $\frac{DC}{CB} = \frac{AC}{?}$ (c) $\frac{?}{AC} = \frac{AC}{AB}$.

From $\triangle ACB \sim \triangle CDB$ we get:

(d) $\frac{AC}{?} = \frac{CB}{DB}$ (e) $\frac{?}{CB} = \frac{CB}{DB}$ (f) $\frac{?}{CD} = \frac{AB}{?}$

From $\triangle ADC \sim \triangle CDB$ we get:

(g) $\frac{AD}{?} = \frac{?}{DB}$ (h) $\frac{DC}{?} = \frac{AC}{?}$ (i) $\frac{?}{CD} = \frac{?}{CB}$.

If you look closely at the nine proportions you have written you will see that the pair (a) and (i) involve the same four numbers but in a different order; that is, one could be obtained from the other by interchanging the second and third terms. Similarly for the pair (b) and (f) and the pair (d) and (h). What do you notice about each of the three proportions (c), (e), and (g)? Do you see that each of them has the same number for its second and third terms? What do we call this number with respect to the other two terms? We find these latter three proportions are more useful than the others, so we state them as a theorem. Since this theorem is a natural consequence of Theorem 3-9, we list it as a corollary.

COROLLARY 3-9-1.

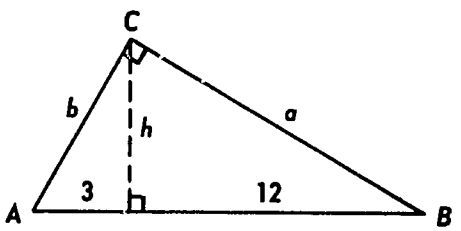
Given a right triangle and the altitude from the right angle to the hypotenuse:

(1) The altitude is the mean proportional between the two segments formed on the hypotenuse.

(2) Either leg is the mean proportional between the hypotenuse and the segment on the hypotenuse adjacent to the leg.

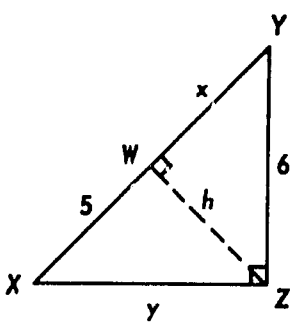
PROBLEMS 3-8

1. Find the length of a, b, and h in the figure on the right.



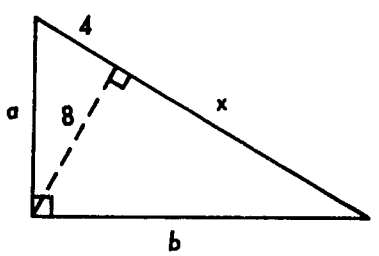
2. In Problem 1 above what is the ratio of b to a? What other pairs of segments in the figure have this same ratio?

3. In the figure on the right, find x, y, and h.



4. In Problem 3 above what is the ratio of h to y? What other pairs of segments in the figure have this same ratio?

5. In the figure, find x, a, and b.



Challenge Problem

6. In arriving at the statement of Theorem 3-9 we assumed that the altitude from the right angle intersects $\overleftrightarrow{AB}$ in a point D that is between A and B. Prove that this is correct.

HINT: There are essentially five possible positions for D. They are

(1) A is between D and B

(2) B is between A and D

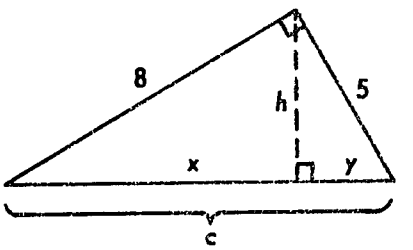
(3) $D = A$

(4) $D = B$

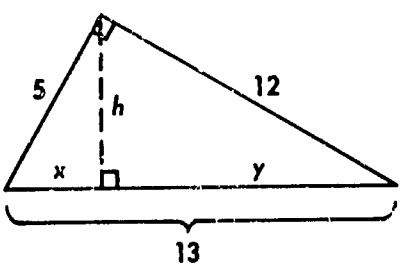
(5) D is between A and B.

Show that the first four of these contradict the fact that $\hat{A}$ is an acute angle, hence (5) is the only remaining possibility.

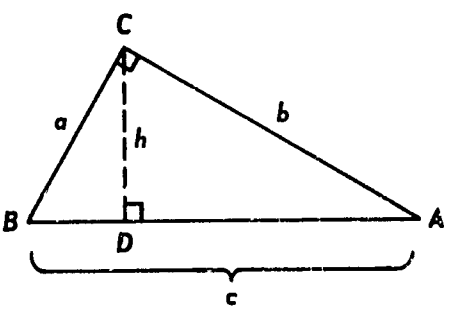
7. In the figure on the right find x, y, and h in terms of the side c.



8. In the figure on the right find x, y, and h.



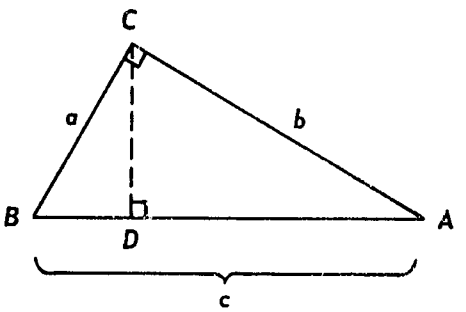
9. Use the figure on the right to show that $h = \frac{ab}{c}$, thus proving the theorem that the altitude to the hypotenuse of a right triangle is the product of the legs of the given triangle divided by the hypotenuse.



10. In the figure of Problem 9 find BD and DA in terms of the lengths a, b, and c of the sides.

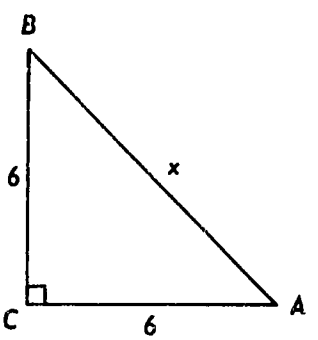
3-9 PYTHAGORAS' THEOREM.

You were shown a proof to Pythagoras' Theorem in Chapter 2 using area. Other area proofs were suggested in the exercises. There are over 300 known proofs to this famous theorem, not all of which employ the use of areas in their development. If you worked Problem 10 of Problems 3-8 above, you will have taken a major step toward a remarkable proof of Pythagoras' Theorem based on similar triangles. It is a very simple proof and you should feel the thrill of discovering it for yourself. Let us look at the figure of Problem 10 again.



In the problem you were asked to find BD and DA in terms of the lengths a, b, and c of the sides of right triangle ΔABC. Do this and see if you can prove, in the figure, that $c^2 = a^2 + b^2$. Be sure you try the proof yourself before reading the following hint: Use (2) of Corollary 3-9-1 and write expressions for BD and DA in terms of a, b, and c. Now, $BD + DA = c$. Why? Complete the proof.

PROBLEMS 3-9



1. In the figure on the right, ΔABC is an isosceles right triangle with sides as marked. Find the hypotenuse x.

2. Find the ratio of the leg to the hypotenuse in Problem 1. Is this ratio the same for all isosceles right triangles? Prove that it is.

3. Find the legs of an isosceles right triangle if the hypotenuse is $\sqrt{50}$.

4. One leg of a right triangle is 6 and the hypotenuse is 12. Find the other leg.

5. In a $30° - 60° - 90°$ right triangle the length of the hypotenuse is 10. Find the length of the longer leg.

6. What is the ratio of the shorter leg to the longer leg in Problem 5? Do you think this ratio is the same for all 30° - 60° - 90° right triangles? Prove that it is.

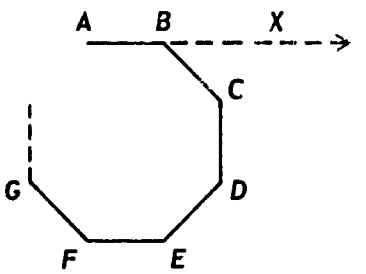
7. In a 30° - 60° - 90° right triangle the length of the hypotenuse is 2. Find the lengths of the legs.

3-10 POLYGONS.

Before defining similar polygons in the next section, we need to recall some of the definitions concerning polygons from your earlier work in geometry. We will then define a *regular* polygon and develop some formulas that will enable us to find the sum of the degree measures of the interior and exterior angles of a polygon as well as a formula for *each* angle of a regular polygon. In this and the following section when we use the word polygon, we will mean a simple, convex polygon.

You may recall that special names are given to polygons of 3, 4, 5, 6, 7, and 8 sides. In case you've forgotten they are called respectively a triangle, quadrilateral, pentagon, hexagon, heptagon, and octagon. In general a polygon having n sides is called an n-gon.

In the figure on the right, $ABCD\ldots$ indicates an n-gon with n sides and n vertices. Two vertices which are endpoints of the same side such as A and B, B and C, C and D, etc. are called *adjacent vertices*. Those that do not lie on the same side such as A and C, B and D, etc. are called non-adjacent or *opposite vertices*. Any line segment joining two opposite vertices is called a *diagonal*.



An angle whose rays contain adjacent sides of the polygon such as $A\hat{B}C$, $B\hat{C}D$, etc. is called an *interior* angle of the polygon. An angle which is adjacent to and supplementary to an interior angle, such as $C\hat{B}X$, is called an *exterior* angle of the polygon.

> DEFINITION 3-4. A polygon which has all of its sides congruent and all of its angles congruent is called a *regular polygon*.

CLASS ACTIVITY

1. Draw polygons of 4, 5, 6, 7, and 8 sides respectively. How many diagonals can be drawn from one vertex in each of the polygons you drew? How many diagonals can you draw from one vertex in an n-gon?

2. Into how many triangles do the diagonals from one vertex subdivide a polygon of 4 sides? 5 sides? 8 sides? n sides?

3. What is the sum in degrees of the interior angles of a polygon of 4 sides? 5 sides? 8 sides? n sides?

4. How many exterior angles are there at *each* vertex of a polygon? What is their relationship to each other? If we count only one exterior angle at each vertex, how many degrees are there in the sum of the exterior angles of a polygon of 4 sides? 5 sides? 8 sides? n sides?

Your work in the problems above should have convinced you that the degree measure of the sum of the *interior angles* of an n-gon is given by $(n-2) \times 180$. Since the sum of the degree measures of the interior and exterior angles at each vertex is 180 (Why?) and since there are n vertices in an n-gon, the sum of the degree measures of the interior and exterior angles is $180 \times n$. Thus if we subtract the interior angles from all the angles, interior and exterior, we have 2×180 or 360 as the sum of the degree measures of the *exterior angles* of any n-gon.

Since a *regular* n-gon has all of its angles congruent, their degree measures are the same. Thus the degree measure of *each interior angle* of a regular n-gon is given by $\frac{(n-2) \times 180}{n}$.

PROBLEMS 3-10

1. What is the degree measure of each interior angle of a regular polygon of 5 sides? 6 sides? 7 sides?

2. What is the degree measure of each exterior angle of a regular polygon of 5 sides? 6 sides? 8 sides?

3. The degree measure of each exterior angle of a regular n-gon is 36. How many sides does it have?

4. How many sides does a regular n-gon have if the degree measure of each interior angle is 150? 179?

5. Can each angle of a regular polygon have a degree measure of 130? Explain.

6. Name a polygon which has all of its sides congruent but does not have all of its angles congruent.

7. Name a polygon which has all of its angles congruent but does not have all of its sides congruent.

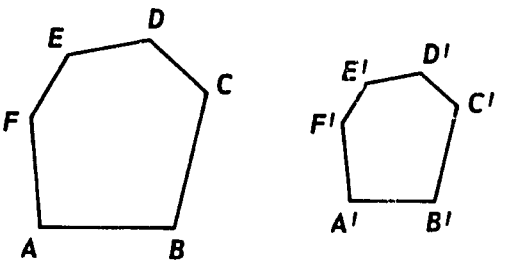
3-11 SIMILAR POLYGONS.

In Section 3-1 we asked you to recall the definition of similar polygons given in *Secondary Two.* In this section we will restrict our attention to simple, convex polygons and give a precise definition of what we mean when we say two such polygons are similar.

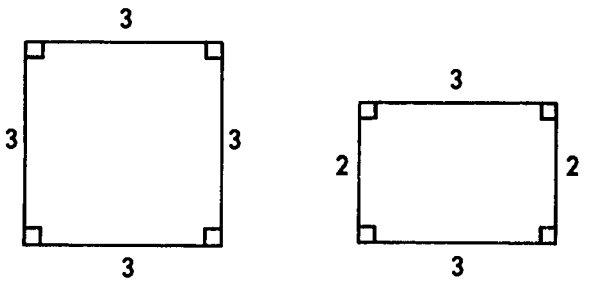
> DEFINITION 3-5. Two polygons are similar if their vertices can be put into one-to-one correspondence such that:
> (1) Corresponding angles are congruent and
> (2) Corresponding sides are proportional.

PROBLEM

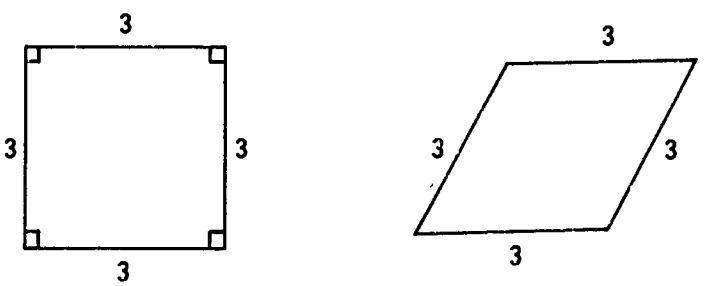
In the figure on the right, if $A \leftrightarrow A'$, $B \leftrightarrow B'$, etc. what else must be true before we can say that polygon $ABCDEF \sim$ polygon $A'B'C'D'E'F'$?



Example 1. For the two quadrilaterals shown in the figure below, although the corresponding angles are congruent (why?) the corresponding sides are not proportional. Are the polygons similar?



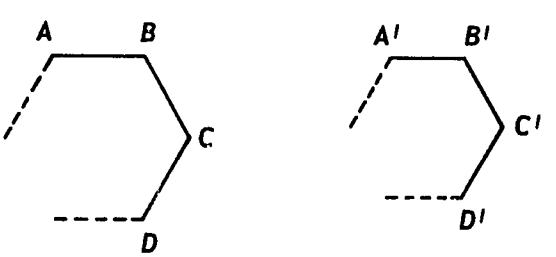
Example 2. For the two quadrilaterals shown below, although their corresponding sides are proportional, their corresponding angles are not congruent. Are these polygons similar?



The two examples above show us that the AAA, SAS, and SSS theorems for similar triangles do not hold for polygons of more than three sides and thus we see just how remarkable these theorems really are.

CLASS ACTIVITY

Consider two regular polygons, each having n sides, as in the figure below.



Each side of polygon $ABCD \ldots$ has the same length (Why?) which we will call r and each side of polygon $A'B'C'D' \ldots$ has the same length s.

1. What is the ratio of AB to $A'B'$? Is this ratio the same for *every* pair of corresponding sides in the two polygons? Why?

2. What is the measure of *each* angle of polygon $ABCD \ldots$? Is this measure the same for each angle of polygon $A'B'C'D' \ldots$?

3. Is polygon $ABCD \ldots \sim$ polygon $A'B'C'D' \ldots$? Why?

4. What theorem about regular polygons have we proven?

5. Were you careful in your statement of Problem 4? A square and an equilateral triangle are regular polygons. Are they similar?

We will write a formal statement of the above results and call it Theorem 3-10.

> THEOREM 3-10. Regular polygons having the same number of sides are similar.

PROBLEMS 3-11

1. Prove that if two polygons are similar, the ratio of their perimeters is the same as the ratio of any two corresponding sides. Suggestion: Use Theorem 3-2 (f).

2. Given two similar polygons. If for one pair of corresponding sides the measures are 15 in. and 18 in. and the perimeter of the smaller polygon is 40 in. find the perimeter of the larger polygon.

3. The sides of a polygon have lengths of 5, 7, 8, 11, and 19. The perimeter of a similar polygon is 75. Find the sides of the larger polygon.

4. A rectangular plot of ground has dimensions of 22 ft. and 35 ft. If we are to represent this on a map with a scale of $\frac{1}{5}$ in. to the foot, what would be the dimensions of the plot on the map?

5. The sides of a pentagon are 9, 11, 5, 14, and 17. If the shortest side of a similar pentagon is 15, find the rest of its sides.

6. If two polygons are similar, are they necessarily regular polygons?

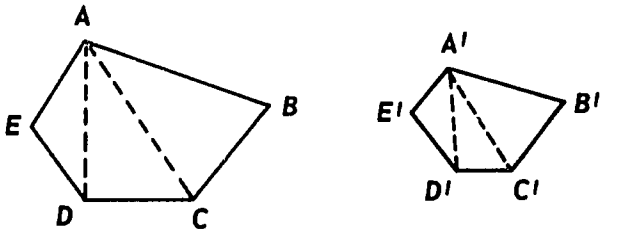
7. If two polygons are regular, are they necessarily similar?

8. Prove that if two pentagons are similar they can be subdivided into triangles so that corresponding triangles are similar.
HINT: In the figure we have $ABCDE \sim A'B'C'D'E'$. We want to prove

(1) $\Delta ABC \sim \Delta A'B'C'$

(2) $\Delta ACD \sim \Delta A'C'D'$

(3) $\Delta ADE \sim \Delta A'D'E'$



Proof:

(a) $\Delta ABC \sim \Delta A'B'C'$ Why?

(b) $\dfrac{AC}{A'C'} = \dfrac{CD}{C'D'}$ Why?

(c) $A\hat{C}D \equiv A'\hat{C}'D'$ Why?

Complete the proof.

9. Generalize the theorem of Problem 8 to any two similar polygons. Do you think it is still true?

10. Prove the converse of the theorem in Problem 8; that is, if two pentagons can be subdivided into triangles such that corresponding triangles are similar and similarly placed, then the pentagons are similar.

Chapter 4
CIRCLES

4-1 CIRCLES AND SPHERES.

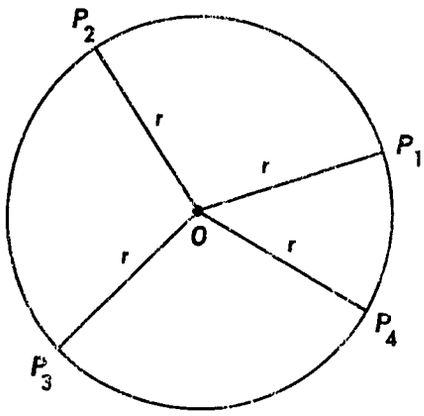
When we discussed simple plane and solid figures in *Secondary One*, we called the set of points in a plane, at a distance of 1 foot from a point A, a circle with centre A and radius 1 foot. We also used the surface of a globe as an example of a set of points which we called a sphere. We now give formal definitions of the circle and the sphere.

> DEFINITION 4-1: Let O be a point in space, and r a positive number. The *sphere* with *centre* O and *radius* r is the set of all points P in space such that $OP = r$. Let O be a point in a given plane, and r a positive number. Then the *circle* with *centre* O and *radius* r is the set of all points P of the given plane such that $OP = r$.

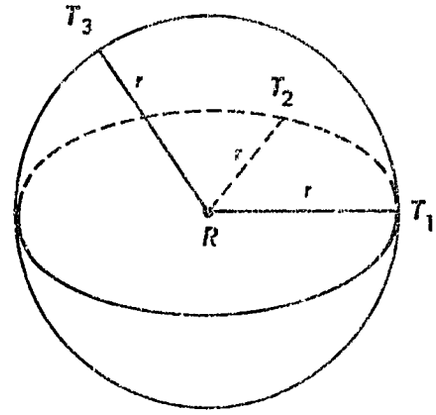
Two or more spheres (circles) with the same centre are called *concentric*.

We now prove a theorem which deals with one aspect of the more general problem of the intersection of a plane and a sphere.

> THEOREM 4-1: The intersection of a sphere and a plane containing the centre of the sphere is a circle with the same centre and the same radius as the sphere.

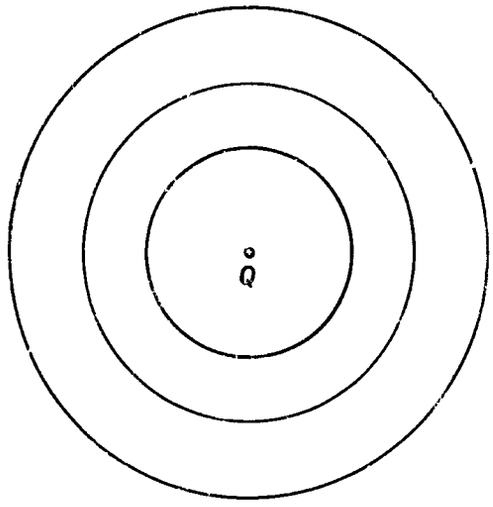


Circle with centre O.
$OP_1 = OP_2 = OP_3 = OP_4 = r$ = radius



Sphere with centre R.
$RT_1 = RT_2 = RT_3 = r$ = radius

Fig. 1.



Three concentric circles with common centre Q.

Fig. 2.

Proof: Let S be a given sphere with centre O and radius r, and P a given plane containing O. The intersection of S and P is the set of all points belonging to both S and P. That is, every member of the intersection is

(i) at a distance r from the point O.
(ii) in the plane P.

Hence the intersection is the set of all points of P at a distance r from the point O (which is also in P). This, by definition, is the circle in P with centre O and radius r.

Although we cannot prove it yet, it is also true that, if a plane intersects a sphere, then it does so in either a circle or a single point.

> DEFINITION 4-2: The intersection of a sphere and a plane which contains the centre of the sphere is called a *great circle*.

PROBLEMS 4-1A

1. What is (a) the union and (b) the intersection of a sphere and a circle with the same centre and the same radius?
2. Describe (i) the intersection and (ii) the union of (a) two concentric circles and (b) two concentric spheres.
3. How many great circles are there to a given sphere? On the Earth sphere, special names are given to certain great circles. What are these names?

Here are some further basic definitions.

> DEFINITION 4-3: A *chord* of a circle (sphere) is a segment whose end points are on the circle (sphere).
>
> A *diameter* of a circle (sphere) is a chord containing the centre of the circle (sphere).
>
> A *secant* of a circle (sphere) is a line which intersects the circle (sphere) in two distinct points.

Fig. 3.

In Figure 3, $\overline{AB}$ and $\overline{CD}$ are chords. $\overline{CD}$ is a diameter, and the lines $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ are secants.

We have defined the radius of a circle (sphere) as a certain positive number. The word "radius" is also used in a second sense. It is defined to be a segment from the centre of a circle (sphere) to a point on the circle (sphere). Thus, in Figure 1, the segments $\overline{OP_1}$, $\overline{OP_2}$, $\overline{OP_3}$ are all radii of the given circle, and the segments $\overline{RT_1}$, $\overline{RT_2}$, $\overline{RT_3}$ are radii of the sphere.

Similarly, the word "diameter" is used in two senses. We have already defined it as a certain segment. We also use it to mean the positive number $2r$, where r is the radius (in the first sense) of a circle (sphere). It should be clear which of the two senses of these words is meant by the context in which they are used.

PROBLEMS 4-1B

1. Two diameters of a circle, $\overline{AC}$ and $\overline{BD}$, intersect at right angles. Prove that $ABCD$ is a square.

2. A secant $\overleftrightarrow{BC}$ of a circle with centre O and radius r contains a chord $\overline{AB}$ of the circle. If $AB = r$, and $\overrightarrow{BC}$ is opposite to $\overrightarrow{BA}$, prove that $m(O\hat{B}C) = 120$.

3. $\overline{AB}$ is a chord (which is not a diameter) of a circle with centre O (Fig. 4). D is the foot of the perpendicular from O to $\overleftrightarrow{AB}$. Prove that $\overline{AD} \equiv \overline{DB}$.

4. $\overline{AB}$ is a chord of a circle with centre O and radius r. If $\overline{AB}$ is not a diameter, prove that $AB < 2r$.

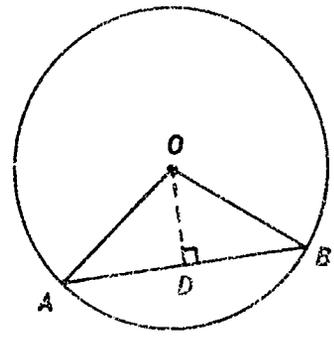


Fig. 4.

5. Prove that the diameters of a circle are the longest chords of the circle.

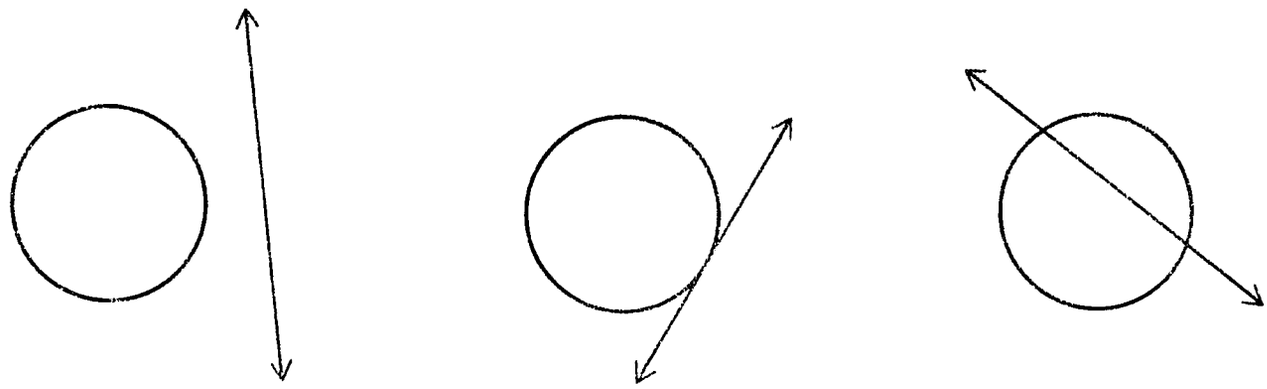
4-2 INTERSECTION OF A LINE WITH A CIRCLE.

We have studied the circle and the sphere together in Section 4-1, and you may have noticed the close relationship between the two. In the rest of this chapter, we will confine our study to the circle.

> DEFINITION 4-4: The *interior of a circle* is the set consisting of the centre of the circle and all the points of the plane of the circle whose distance from the centre of the circle is less than the radius.
>
> The *exterior of a circle* is the set of all points of the plane of the circle whose distance from the centre of the circle is greater than the radius.

From the above definitions we see that every point in the plane of a circle is either in the interior of the circle, on the circle, or in the exterior of the circle. Thus the circle separates its plane into an interior and an exterior part. We often use the expressions "inside the circle" and "outside the circle" for the "interior of the circle" and the "exterior of the circle" respectively.

Let us now consider the intersection of a line and a circle lying in the same plane. What do you think will be the possible intersections? It would seem that the only possible cases are those shown below in Figure 5.



Case I - The intersection is empty. (The line does not intersect the circle)

Case II - The intersection is a single point. (The line touches the circle)

Case III - The intersection consists of two distinct points.

Fig. 5.

Are these the only possible cases? Can you think of other possibilities? Try to give a reason for your answer. Consider a circle with centre O and let ℓ be a line in the same plane as the circle. Choose the point K on ℓ such that OK is the shortest distance from O to ℓ. How do we know that the point K exists? What is the relationship between the segment $\overline{OK}$ and the line ℓ? You may have found that since OK is the shortest distance from O to ℓ, $\overrightarrow{OK}$ is the unique perpendicular from O to ℓ. We now choose the point N on the ray $\overrightarrow{OK}$ such that N is also a point on the circle. How do we know that such a point exists and is unique?

Let us now consider the possible relationships between the segments $\overline{OK}$ and $\overline{ON}$. There are only three possibilities.

$$\text{(i)} \quad \overline{OK} > \overline{ON}, \qquad \text{(ii)} \quad \overline{OK} \equiv \overline{ON}, \qquad \text{(iii)} \quad \overline{OK} < \overline{ON}.$$

(Why?) Draw figures to show each of these possibilities.

Case(i): If $\overline{OK} > \overline{ON}$, then every point of ℓ (including K) is outside the circle (Why?), and so the line ℓ does not intersect the circle. This is Case I in Figure 5.

Case(ii): If $\overline{OK} \equiv \overline{ON}$, then the point K is the same as the point N.

Hence ℓ and the circle have at least the point K in common. Since OK is the unique shortest distance from O to ℓ, it follows that any other point P on ℓ is such that $OP > OK$. Hence P will lie outside the circle. (Why?) Thus ℓ intersects the circle in one and only one point. This is Case II in Figure 5.

> DEFINITION 4-5: A *tangent* to a circle is a line which intersects the circle in one and only one point. The point of intersection is called the *point of contact* (or *the point of tangency*).

Case(iii): If $\overline{OK} < \overline{ON}$, then at least one point K of ℓ is in the interior of the circle. When a line and a circle (which are in the same plane) are so related, it can be shown that the line intersects the circle in exactly two points and is therefore a secant. This is Case III in Figure 5. We shall prove this result later (as Theorem 4-3) and also collect the ideas above into the statement of Theorem 4-5.

We should remark, however, that in the above analysis, we have not proved that any of the three cases actually exist. Can you show that Cases I and III do exist? Our next theorem shows that a tangent can be drawn to a circle at any point of the circle.

> THEOREM 4-2: The line perpendicular to a radius of a circle at its intersection with the circle is a tangent to the circle.

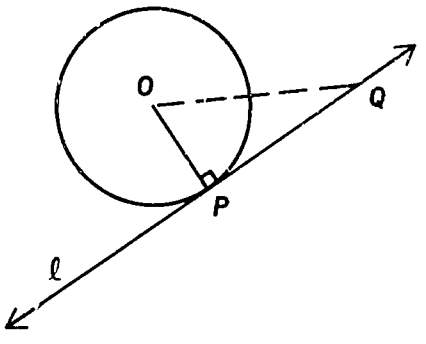


Fig. 6.

Proof: We have to show that if $\ell \perp \overline{OP}$ at the point P, then ℓ intersects the circle in exactly one point.

1.	By hypothesis P is a point of intersection of the line ℓ and the circle.	
2.	Let Q be any other point on ℓ. Then $\Delta\, OPQ$ is a right triangle.	Why?
3.	$\overline{OQ} > \overline{OP}$	Why?
4.	Q is in the exterior of the circle.	Why?
5.	P is the one and only one point of ℓ on the circle.	Statements 1 and 4.

Our next theorem will justify the conclusion we arrived at when we considered Case III of Figure 5.

> THEOREM 4-3: If a line intersects the interior of a circle, then it intersects the circle in exactly two points.

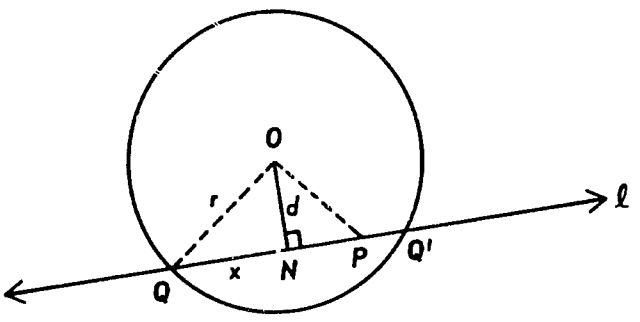


Fig. 7.

Proof: Suppose the line ℓ contains a point P, which is a point in the interior of the circle with centre O and radius r. Then $OP < r$, by definition. We drop a perpendicular $\overline{ON}$ from O to ℓ. Then $d = ON \leq OP < r$.

Now, if the line ℓ intersects the circle in a point Q, then applying Pythagoras' Theorem to the right triangle ΔONQ, we obtain $r^2 = x^2 + d^2$, where $NQ = x$. Conversely, if there is a positive number x such that $r^2 = x^2 + d^2$, then the point Q on ℓ, at a distance x from N, will lie on the circle, since its distance from O will be r. Hence ℓ will intersect the circle in a point Q if and only if there exists a positive solution of the equation $x^2 = r^2 - d^2$. Since $r > d \geq 0$, $r^2 - d^2 > 0$ and hence such a positive number x exists (it is $\sqrt{r^2 - d^2}$). Moreover, given the positive number x, there are exactly two points Q and Q' on ℓ such that $NQ' = NQ = x$. Hence ℓ intersects the circle in exactly these two points.

Our next theorem tells us that the converse of Theorem 4-2 is also true.

> THEOREM 4-4: A tangent to a circle is perpendicular to the radius at the point of contact.

Let P be the point of contact of the tangent $\overleftrightarrow{PQ}$ with the circle (with centre O). We have to prove that $O\hat{P}Q$ is a right angle. Our proof is by contradiction.

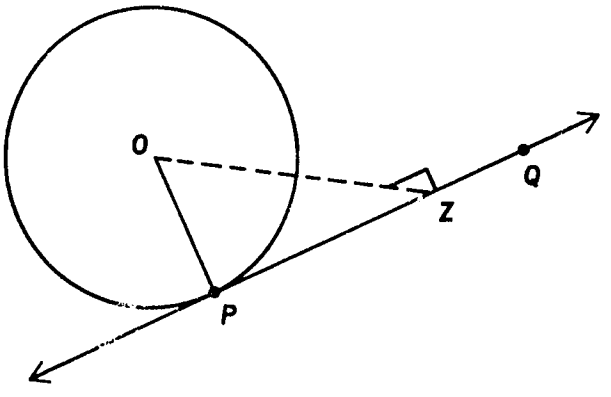


Fig. 8.

Proof:

1.	The point P of $\overleftrightarrow{PQ}$ lies on the circle, and every other point of $\overleftrightarrow{PQ}$ lies in the exterior of the circle.	Definition 4-5 and Theorem 4-3.

2.	If $\overline{OP}$ is not perpendicular to $\overleftrightarrow{PQ}$, we choose the point Z on $\overleftrightarrow{PQ}$ such that $\overline{OZ} \perp \overleftrightarrow{PQ}$.	Why is this possible?
3.	$\Delta\, OZP$ is a right triangle, with the right angle at Z.	Statement 2.
4.	$OZ < OP$.	Why?
5.	Z is in the interior of the circle.	OP equals the radius of the circle and Statement 4.

6. Statements 1 and 5 are contradictory. Hence we conclude that $\overline{OP}$ must be the perpendicular to the line $\overleftrightarrow{PQ}$, which proves the theorem.

We now state Theorem 4-5 to which we referred earlier. A number of interesting results can be proved using this theorem. We include some of these in the problems which follow.

> THEOREM 4-5: Let ℓ be a line in the plane of a circle with centre O, and let K be the foot of the perpendicular from O to ℓ. Then exactly one of the following three situations occurs:
>
> (1) K is outside the circle, and the line ℓ does not intersect the circle;
>
> (2) K is on the circle, and ℓ is a tangent to the circle;
>
> (3) K is inside the circle, and the line ℓ intersects the circle in exactly two points.

Can you write out a proof to this theorem? You should be able to do so, following closely the arguments we used earlier in this section and applying Theorems 4-2 and 4-3.

PROBLEMS 4-2A

1. Write out proofs for the following theorems.

   (a) THEOREM 4-6: The perpendicular from the centre of a circle to a chord bisects the chord.

   (b) THEOREM 4-7: The segment joining the centre of a circle to the mid-point of a chord is perpendicular to the chord.

   (c) THEOREM 4-8: If the perpendicular bisector of a chord lies in the plane of the circle, then it passes through the centre of the circle.

   (d) THEOREM 4-9: No circle contains three collinear points.

2. Use Theorem 4-5 to prove Theorem 4-4.

We have used the term "congruent" in connection with segments, angles and triangles. On each occasion, one underlying notion has been that of "the same size" or "the same size and shape." From a different point of view, we also found that two figures are congruent if they can be made to fit each other exactly.

Can you suggest a condition (or conditions) under which two circles will fit together exactly? Can you make a definition of "congruent circles" out of your answer? You will probably have a statement such as this:

> DEFINITION 4-6: Two circles with *equal radii* are said to be *congruent.*

PROBLEMS 4-2B

1. Prove the following theorems.

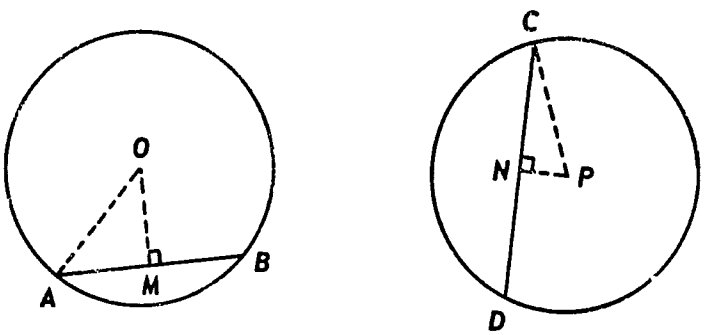
   (a) THEOREM 4-10: In the same circle, or in congruent circles, chords which are equidistant from the centre(s) are congruent.

   (b) THEOREM 4-11: In the same circle, or in congruent

circles, any two congruent chords are equidistant from the centre(s).

(c) THEOREM 4-12: In the same circle, or in congruent circles, if two chords are unequal, the shorter chord is at a greater distance from the centre.

Hint:

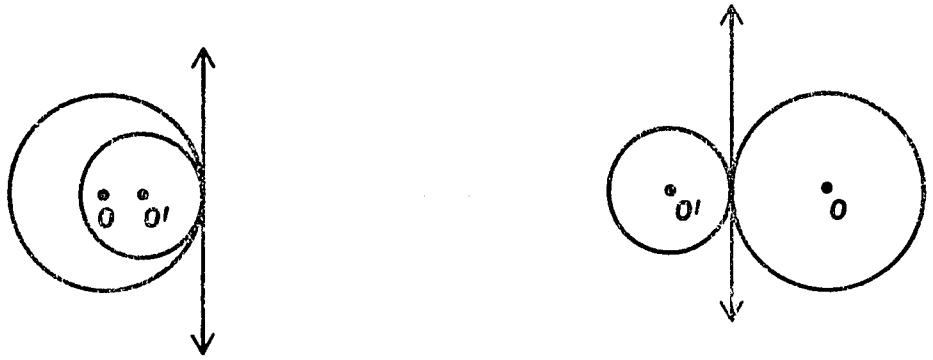


You are required to prove that, if $AB < CD$, then $OM > PN$. What is the relationship between AM and AB, and between CN and CD? How are the lengths AM and CN related? Now, apply Pythagoras' Theorem to the right triangles ΔOAM and ΔPCN, and use the relationship you have found between AM and CN.

2. The converse of Theorem 4-12 is also true. State and prove it.

DEFINITION 4-7: Two distinct coplanar circles are said to be *tangent* (or to *touch*) if they are both tangent to the same line with the same point of contact on that line.

Two circles which touch are *internally tangent* if their centres are on the same side of the common tangent; they are *externally tangent* if their centres are on opposite sides of the line. We also say that the circles *touch internally* (or *externally*).



Internally tangent Externally tangent

Fig. 9.

PROBLEMS 4-2C

1. Given a circle with centre O and radius 3 cm. Determine the intersection of the circle and a line ℓ, in the plane of the circle, if the shortest distance from O to ℓ is:
   (a) 5 cm., (b) 4 cm., (c) 3 cm., (d) 1 cm.

2. A circle of radius 5 cm. has its centre at a distance of 4 cm. from a chord. Find the length of the chord.

3. A chord of a circle of radius 13 cm. has length 10 cm. Find the distance of the chord from the centre.

4. A chord, 12 in. long, is 6 in. from the centre of the circle. What is the radius of the circle?

5. Prove that if a secant bisects a chord of a circle and is perpendicular to the chord, it contains a diameter of the circle.

6. If $\overline{AB}$ and $\overline{AC}$ are two distinct chords of a circle, show how you can locate the centre of the circle.

7. Given a circle, describe how you would locate its centre.

8. In Figure 10, A, B, C, D, E, F are points on the circle with

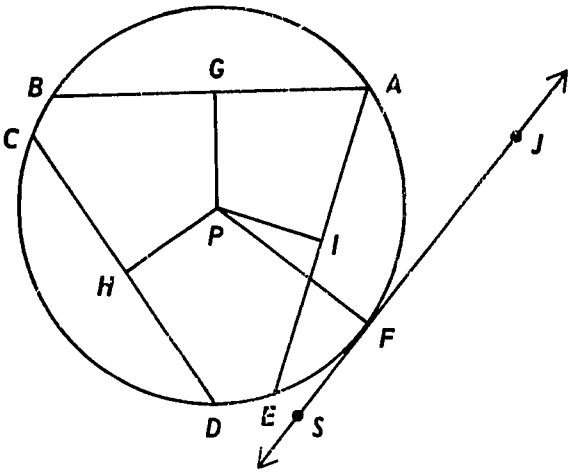


Fig. 10.

centre P. G, H, I are points on the chords $\overline{AB}$, $\overline{CD}$, $\overline{EA}$ respectively. The line $\overleftrightarrow{JS}$ contains the point F of the circle.

State the theorem or theorems which justify each of the following conclusions:

(a) If $\overline{PG} \perp \overline{AB}$, $\overline{PH} \perp \overline{CD}$, and $PG = PH$, then $\overline{AB} \equiv \overline{CD}$.

(b) If $\overline{PG} \perp \overline{AB}$, $\overline{PH} \perp \overline{CD}$, and $\overline{AB} \equiv \overline{CD}$, then $PG = PH$.

(c) If A, B, C are distinct points on the circle, then the line segment $\overline{AC}$ cannot contain B.

(d) If $\overline{AI} \equiv \overline{IE}$, then $\overline{PI} \perp \overline{AE}$.

(e) If $P\hat{H}D$ is a right angle, then $\overline{CH} \equiv \overline{HD}$.

(f) If $\overline{PH} \perp \overline{CD}$, $\overline{PI} \perp \overline{AE}$ and $PI < PH$, then $\overline{AE} > \overline{CD}$.

(g) If $\overline{PF} \perp \overleftrightarrow{JS}$ at F, then $\overleftrightarrow{JS}$ is a tangent to the circle.

(h) If $\overleftrightarrow{JS}$ intersects the circle in exactly the point F, then the degree measure of $J\hat{F}P$ is 90.

(i) If M is inside the circle, then $\overleftrightarrow{MA}$ intersects the circle in exactly one point other than A.

9. Prove that the tangents to a circle at the ends of a diameter are parallel.

10. Prove that the line segment joining the points of contact of two parallel tangents to a circle is a diameter of the circle.

11. In Figure 11, $\overleftrightarrow{AP}$ and $\overleftrightarrow{BP}$ are tangents to the circle at the points A and B respectively. Prove that $\overline{AP} \equiv \overline{BP}$.

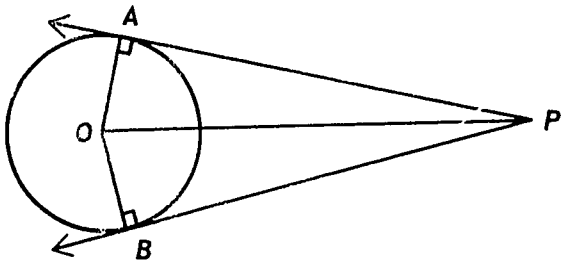


Fig. 11.

12. Prove that if, in a circle, a secant is parallel to a tangent, then the diameter through the point of contact of the tangent bisects the chord contained in the secant.

13. In Figure 12, we have two concentric circles with common centre O. $\overline{AB}$ is a chord of the larger circle and is also a tangent to the smaller circle at R. Prove that $AR = RB$.

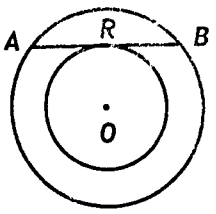


Fig. 12.

14. Prove the following theorem:

THEOREM 4-13: The line containing the centres of two tangent circles also contains the common point of tangency.

15. Prove the following theorem and its corollary.

THEOREM 4-14: Any three non-collinear points lie on exactly one circle. (You must prove there is one circle containing the three points and only one.)

COROLLARY 4-14-1: One and only one circle can be drawn through the vertices of a given triangle.

16. Prove that if two congruent circles touch externally, then any point equidistant from their centres lies on their common tangent.

17. In Figure 13, $\overline{AB}$ is a diameter of the circle with centre P. Line ℓ is a tangent to the circle at the point T. $\overline{AC}$, $\overline{BD}$ are drawn perpendicular to ℓ. Prove that $TC = TD$.

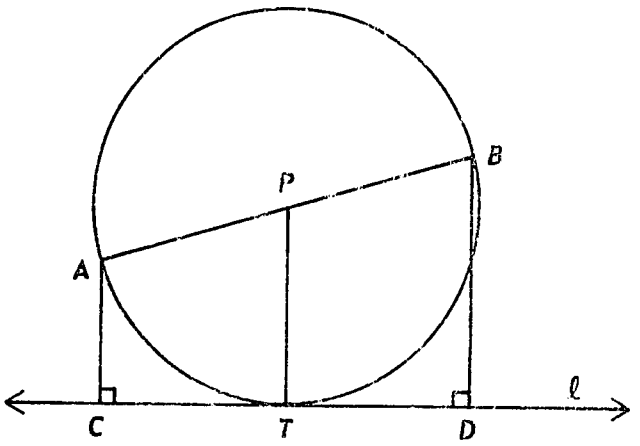


Fig. 13.

18. Prove the following theorem:

THEOREM 4-15: If two tangents to a circle intersect, the angles which they make with the line joining the point of intersection to the centre of the circle are congruent, and the segments from the point of intersection to the points of contact are also congruent. (See Figure 11 of Problem 11).

19. $\overline{AB}$ is a diameter and $\overline{AC}$ is a chord of a circle with centre P. A secant through P parallel to $\overline{AC}$ intersects the tangent at C in a point D. Prove that $\overleftrightarrow{DB}$ is tangent to the circle at B.

4-3 ARCS OF CIRCLES.

In *Secondary Three,* we defined the interior and the exterior of an angle. We now use these ideas to define an arc of a circle.

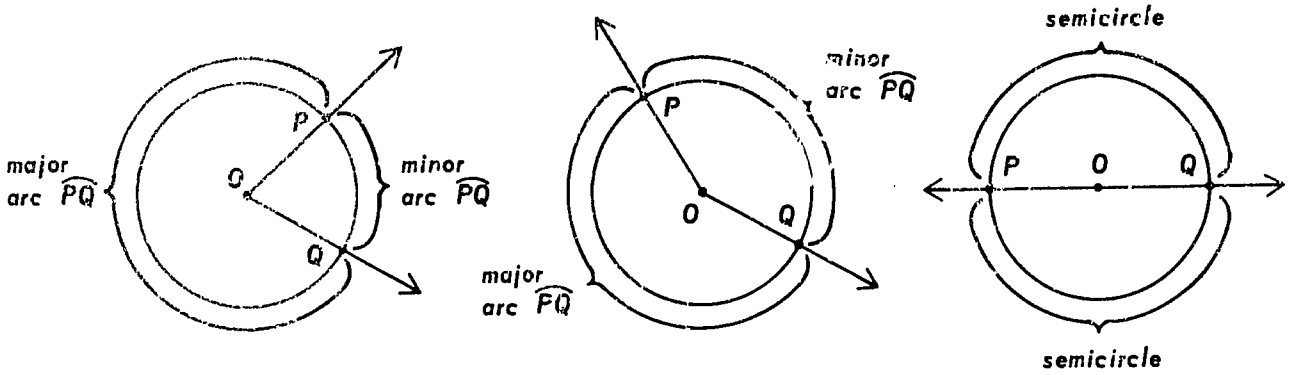


Fig. 14.

> DEFINITION 4-8: In a circle with centre O, let P and Q be two points of the circle which are not the endpoints of a diameter. The union of P, Q, and all the points of the circle in the interior of the angle $P\hat{O}Q$ is called the *minor arc* $\widehat{PQ}$. The union of P, Q, and all the points of the circle in the exterior of angle $P\hat{O}Q$ is called the *major arc* $\widehat{PQ}$.

NOTATION: The symbol $\widehat{PQ}$ (which is read "arc PQ") will be used to denote an arc (of a circle) whose endpoints are P and Q.

If P, Q are endpoints of a diameter of a circle, the union of P, Q, and all points of the circle in one of the half-planes (with edge $\overleftrightarrow{PQ}$) of the plane of the circle is called a *semicircle*.

By an *arc of a circle*, we shall mean a minor arc, a major arc or a semicircle. The points P and Q in Definition 4-8 are called the endpoints of the arc.

You may have noticed the notation ($\widehat{PQ}$) for an arc with endpoints P, Q is ambiguous. It does not distinguish between the minor arc $\widehat{PQ}$ and the major arc $\widehat{PQ}$, nor if P and Q are the ends of a diameter, does it tell us which of the two semicircles to consider. One way of avoiding this ambiguity is to label a third point on the arc under consideration, the two other points labelled being the endpoints. Thus in Figure 15 below, we can refer unambiguously to the semicircle $\widehat{PXQ}$, or the arc $\widehat{YXQ}$ (which is the minor arc $\widehat{YQ}$). In what other way can you designate the major arc $\widehat{YQ}$?

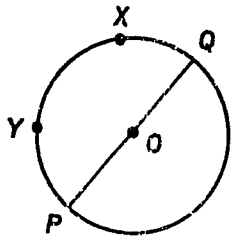


Fig. 15.

We now recall certain definitions, which you have already met in *Secondary Two*.

> DEFINITION 4-9: A *central angle* of a circle is an angle whose vertex is the centre of the circle.

Can you name any central angles in the circle of Figure 15?

A central angle is said to be *subtended* by a chord (minor arc) if the sides of the angle contain the endpoints of the chord (minor arc). In Figure 16, $\overline{PQ}$ is a diameter. The minor arc $\widehat{PR}$ subtends the central angle $P\hat{O}R$. The chord $\overline{PR}$ also subtends the central angle $P\hat{O}R$. The minor arc $\widehat{RQ}$ subtends the central angle $R\hat{O}Q$. Name the chord that subtends the central angle $R\hat{O}Q$.

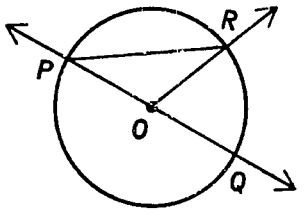


Fig. 16.

In Figure 17, the sides of the angle $C\hat{A}B$ contain the endpoints C, B of the arc $\widehat{CAB}$ and the vertex A of the angle is a point of the arc. When this is the case, we say that the angle $C\hat{A}B$ is *inscribed* in the arc $\widehat{CAB}$. If $\widehat{CEB}$ is the other arc determined by the points C and B, we say that the arc

$\widehat{CEB}$ *subtends* the angle $C\hat{A}B$. In Figure 17, $C\hat{D}A$ is inscribed in the arc $\widehat{CDA}$, and is subtended by the arc $\widehat{CBA}$. In what arcs are the angles $D\hat{A}C$ and $B\hat{A}D$ inscribed, and what arcs subtend them?

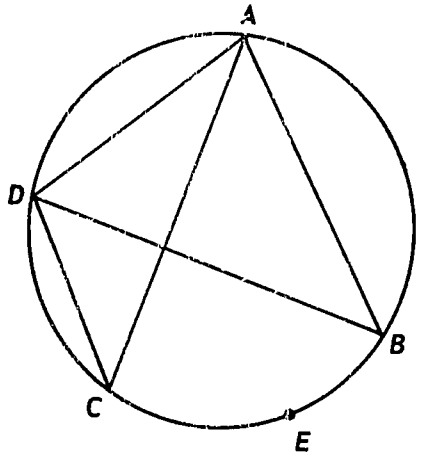


Fig. 17.

> DEFINITION 4-10: An angle is said to be *inscribed* in an arc of a circle if the sides of the angle contain the endpoints of the arc and if the vertex of the angle is a point, but not an endpoint, of the same arc.

When an angle is inscribed in a given major (minor) arc, we also say that it is *subtended* by the minor (major) arc having the same endpoints as the given arc.

We now want to define the *degree measures* of arcs.

> DEFINITION 4-11: The *degree measure of a minor arc* is the degree measure of the central angle which it subtends. The *degree measure of a major arc* is 360 minus the degree measure of the minor arc having the same endpoints as the major arc.
>
> The *degree measure of a semicircle* is 180.

We will, in this chapter, refer to the degree measure of an arc simply as its measure. If the degree measure of arc $\widehat{AB}$ is 30, we will denote this by writing $m(\widehat{AB}) = 30$. You may have noticed that the measure of a semicircle is independent of the size of the circle of which it is a part. This is also true of the measures of minor and major arcs. In Figure 18, the minor arcs $\widehat{PQ}$, $\widehat{P'Q'}$, $\widehat{P''Q''}$ all have measure 60. What is the measure of the corresponding major arcs?

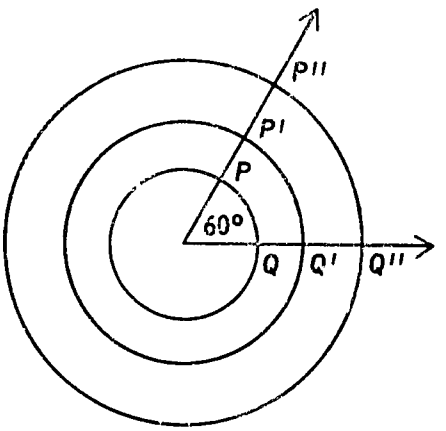


Fig. 18.

We now state an important result that seems reasonable, but whose proof will be omitted as it is a little tedious.

> THEOREM 4-16: If the arc $\widehat{PQR}$ is the union of the arcs $\widehat{PSQ}$ and $\widehat{QTR}$ (with only the endpoint Q in common), then $m(\widehat{PQR}) = m(\widehat{PSQ}) + m(\widehat{QTR})$.

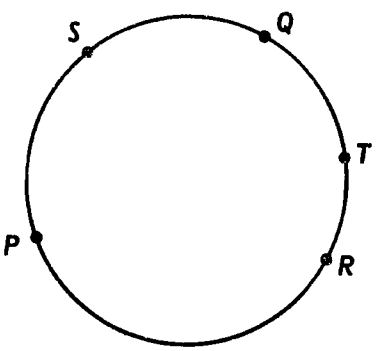


Fig. 19.

We note that if $\widehat{PQR}$ is a minor arc or a semicircle, the proof of Theorem 4-16 follows immediately from the Angle Addition Postulate. However, if $\widehat{PQR}$ is a major arc there are several cases to consider, but each of the cases is easy. If you are interested, perhaps your teacher will show you what they are.

PROBLEMS 4-3A

1. In Figure 20, O is the centre of the circle, and A, B, C are points of the circle. Name the central angles, and the arcs which subtend them. Name the inscribed angles, the arcs in which they are inscribed, and the arcs which subtend them.

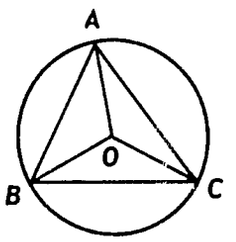


Fig. 20.

2. In Figure 21, D and E are the endpoints of a diameter.
   (a) Name the minor arcs.
   (b) Name the major arcs.
   (c) Name the semicircles.

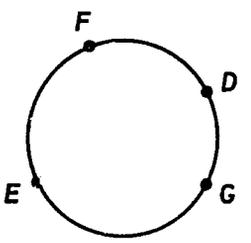


Fig. 21.

3. Prove that in the same circle, or in congruent circles, congruent chords cut minor arcs which are equal in measure.

4. Prove that the line from the centre of a circle which bisects a chord bisects the minor arc determined by the chord.

5. In Figure 22, P is the centre of the circle, $\overline{CB}$ is a diameter, and D, A, E are also points of the circle. If $\overline{AB} \equiv \overline{PC}$, find $m(\widehat{AEB})$, $m(\widehat{ADC})$, $m(\widehat{CDB})$.

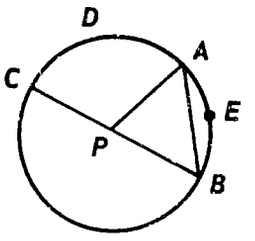


Fig. 22.

6. In Figure 23, find $m(\widehat{BCA})$, where O is the centre of the circle.

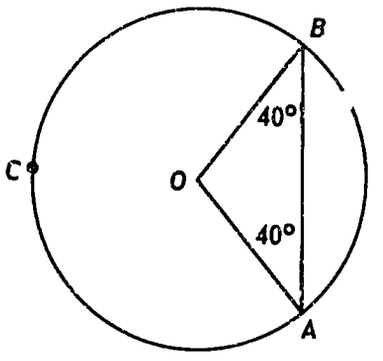


Fig. 23.

7. If $m(\widehat{PQR})$ of a circle with centre O is 300, find $m(\widehat{OPR})$.

8. If the measure of a major arc $\widehat{AB}$ is x, where x is a positive real number ($180 < x < 360$), find the measure of the minor arc $\widehat{AB}$.

9. In Figure 24, $\overline{AC}$ is a diameter of a circle with centre O. $\overleftrightarrow{CE}$ is a tangent at C, and B is a point on the circle. If $m(B\hat{C}E) = 60$, find the measures of

(a) the minor arc $\widehat{CB}$
(b) the minor arc $\widehat{AB}$
(c) angle $A\hat{B}C$.

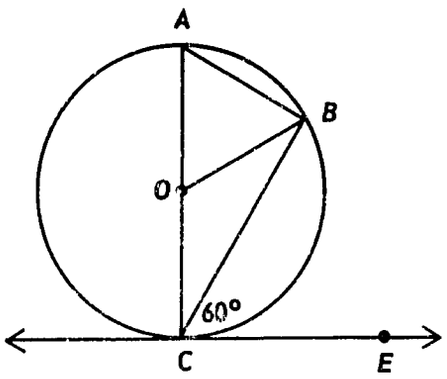


Fig. 24.

10. In Figure 25, $\widehat{AB}$ and $\widehat{PQ}$ are semicircles of two concentric circles with common centre C. $\overline{EC} \perp \overline{AB}$ and $\overline{DC} \perp \overline{CF}$. Prove that $m(\widehat{AD}) + m(\widehat{QT}) = m(\widehat{EF}) + m(\widehat{RS})$. (All the arcs involved are minor arcs).

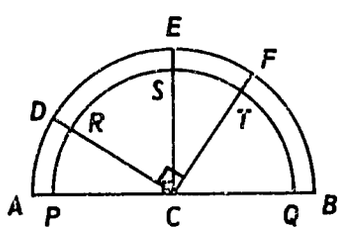


Fig. 25.

We now prove a theorem which will give us a relationship between the measure of an angle inscribed in an arc and the measure of the arc subtending the angle.

> THEOREM 4-17: The measure of an angle inscribed in an arc is half the measure of the arc which subtends the angle.

We first restate the theorem. In Figure 26, $B\hat{A}C$ is inscribed in the arc $\widehat{BAC}$. We have to prove that $m(B\hat{A}C) = \frac{1}{2}\, m(\widehat{BDC})$.

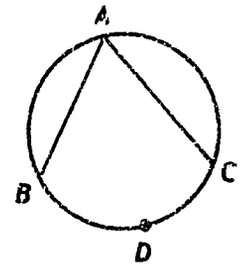


Fig. 26.

Proof: There are three cases, and we consider them separately.

Case 1: A side of $B\hat{A}C$ contains a diameter $(\overline{AC})$ of the circle.

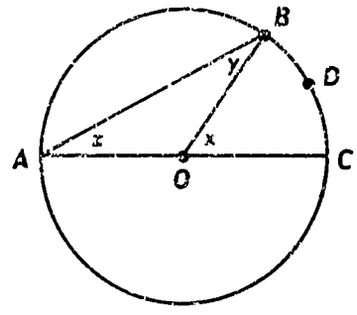


Fig. 26(a).

Let $x = m(B\hat{O}C)$, $y = m(A\hat{B}O)$, and $z = m(B\hat{A}C)$. Then

1. $x = y + z$	1. (Exterior Angle Theorem, applied to ΔACB).

2. $y = z$	2. Why?
3. $x = 2z$	3. Statements 1 and 2.
4. $m(B\hat{A}C) = \frac{1}{2} m(B\hat{O}C)$	4. Statement 3 and algebra.
5. $m(B\hat{A}C) = \frac{1}{2} m(\widehat{BDC})$	5. Statement 4 and Definition 4-11.

Case 2: Points B and C are on opposite sides of the line $\overleftrightarrow{AD}$ containing the diameter through A.

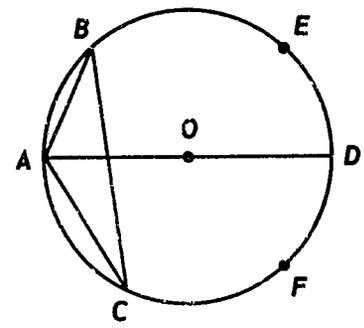


Fig. 26(b).

From Case 1,

$$m(B\hat{A}D) = \frac{1}{2} m(\widehat{BED}) \quad \text{and}$$

$$m(D\hat{A}C) = \frac{1}{2} m(\widehat{DFC})$$

$$\begin{aligned} m(B\hat{A}C) &= m(B\hat{A}D) + m(D\hat{A}C) && \text{Why?} \\ &= \frac{1}{2}\left[m(\widehat{BED}) + m(\widehat{DFC}) \right] \\ &= \frac{1}{2} m(\widehat{BDC}) && \text{Theorem 4-16.} \end{aligned}$$

Case 3: B and C are on the same side of the line $\overleftrightarrow{AE}$ containing the diameter through A.

From Case 1,

$$m(B\hat{A}C) + m(C\hat{A}E) = \frac{1}{2}\left[m(\widehat{BDC}) + m(\widehat{CFE}) \right]$$

and $$m(C\hat{A}E) = \frac{1}{2} m(\widehat{CFE}).$$

Hence $$m(B\hat{A}C) = \frac{1}{2} m(\widehat{BDC}).$$

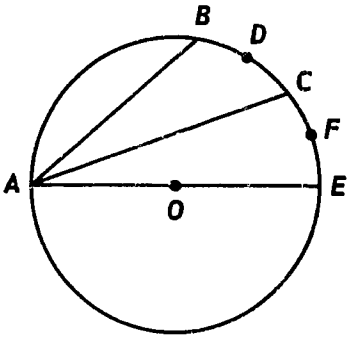
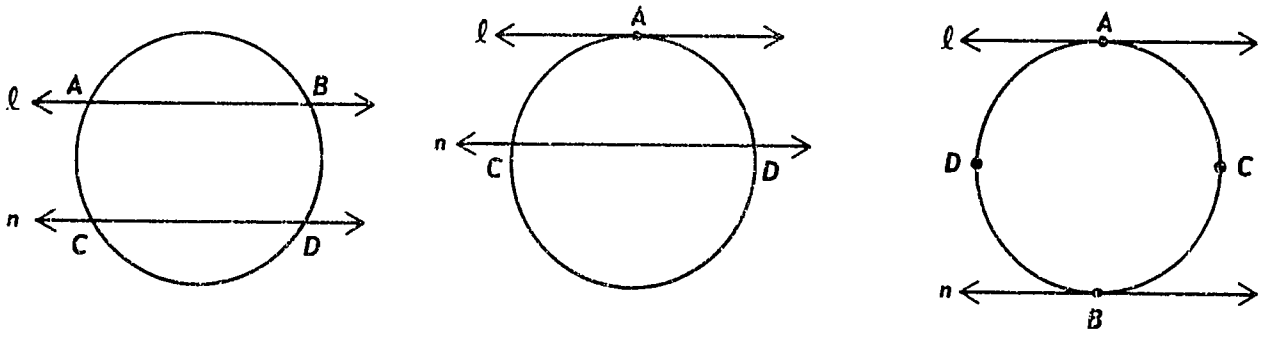


Fig. 26(c).

The following two corollaries follow immediately from Theorem 4-17.

> COROLLARY 4-17-1: An angle inscribed in a semicircle is a right angle.
>
> COROLLARY 4-17-2: Two angles inscribed in the same arc are congruent.

When each of two lines (secants, tangents, or a secant and a tangent) intersects a given circle, they determine certain arcs which we call *intercepted arcs*. If the lines are parallel, as in Figure 27 below, the intercepted arcs are those arcs all of whose points (except the endpoints) are between the parallel lines.



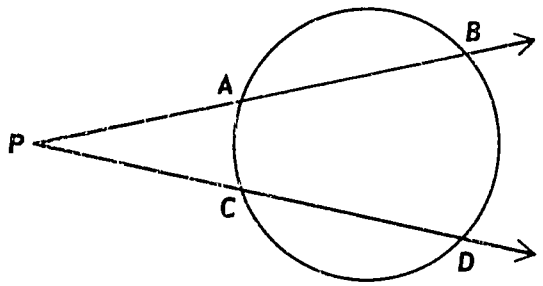
secants ℓ and n intercept minor arcs $\widehat{AC}$ and $\widehat{BD}$

secant line n and tangent line ℓ intercept minor arcs $\widehat{AC}$ and $\widehat{AD}$

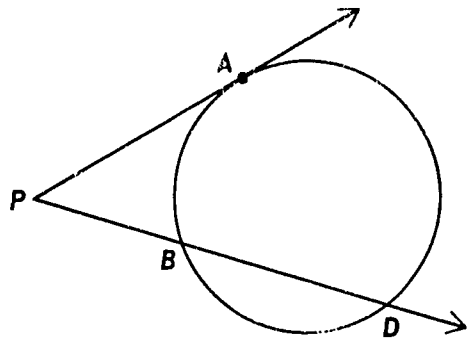
tangent lines ℓ and n intercept semicircles $\widehat{ACB}$ and $\widehat{ADB}$

Fig. 27.

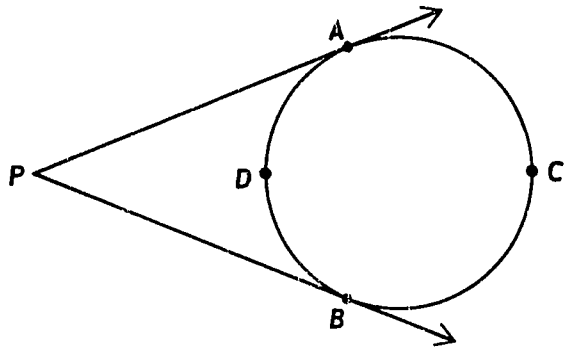
If the two lines are not parallel, as in Figure 28 below, the intercepted arcs are those arcs all of whose points (except the endpoints) are in the interior of the angle formed by the two intersecting lines.



$B\hat{P}D$ intercepts minor arcs $\overset{\frown}{AC}$ and $\overset{\frown}{BD}$



$A\hat{P}D$ intercepts minor arcs $\overset{\frown}{AB}$ and $\overset{\frown}{AD}$



$A\hat{P}B$ intercepts arcs $\overset{\frown}{ACB}$ and $\overset{\frown}{ADB}$

Fig. 28.

We may have two secants intersecting inside the circle as in Figure 29 below.

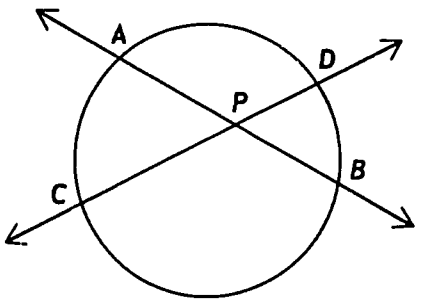


Fig. 29.

In this case, we say that $A\hat{P}C$ intercepts the minor arc $\widehat{AC}$. Name the other angles at P and the arcs they intercept.

Figure 30 below shows two secants and a secant and a tangent intersecting *on* the circle. Name the arcs intercepted by each of the angles $A\hat{P}B$, $A'\hat{P}'B'$, and $A'\hat{P}'C'$.

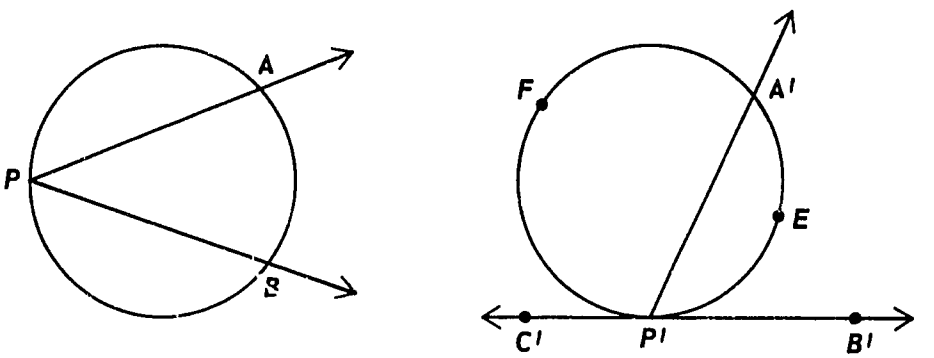


Fig. 30.

We will now collect the above results into a definition.

> DEFINITION 4-12: An angle *intercepts* an arc of a circle if:
>
> (1) the endpoints of the arc lie on the sides of the angle,
>
> (2) all other points of the arc are in the interior of the angle, and
>
> (3) each side of the angle contains an endpoint of the arc.

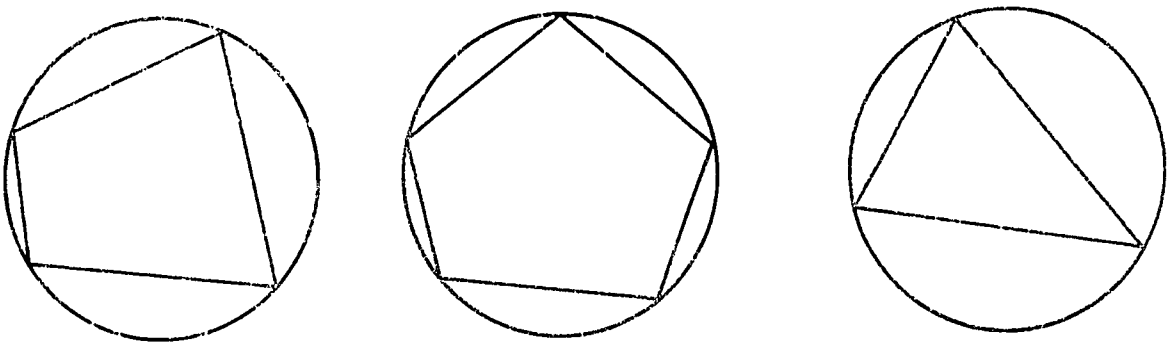
We could now restate Theorem 17 as follows:

The measure of an angle inscribed in an arc of a circle is half the measure of the arc which it intercepts.

Definition 4-10 defines an angle inscribed in the arc of a circle, or more simply, an inscribed angle. We now define what is meant by a polygon inscribed in a circle.

DEFINITION 4-13: A polygon having all of its vertices *on* a circle is said to be *inscribed* in the circle.

We also say that the circle *circumscribes* the polygon.



Inscribed quadrilateral | Inscribed pentagon | Inscribed triangle

Fig. 31.

PROBLEMS 4-3B

1. Prove that an inscribed triangle is a right triangle if one of its sides contains the centre of the circle.

2. Given an arc of a circle, explain carefully how you would determine its degree measure.

3. Prove the following theorem:

   THEOREM 4-18: The opposite angles of an inscribed quadrilateral are supplementary.

4. Prove that a parallelogram inscribed in a circle is a rectangle.

5. In Figure 32 below, find $m(\widehat{BCD})$ and $m(\widehat{CDA})$.

6. Prove the following converse to Corollary 4-17-1:
   If an angle inscribed in an arc of a circle is a right angle, then the arc is a semicircle.

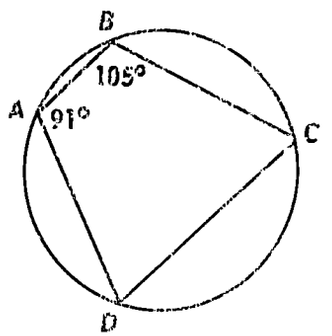


Fig. 32.

7. In Figure 33, $\overline{AC}$ is a diameter of the circle, and $\overleftrightarrow{CE}$ is the tangent at C. If D and B are points of the circle, and $m(B\hat{C}E) = 40$, find $m(C\hat{A}B)$ and $m(C\hat{D}B)$.

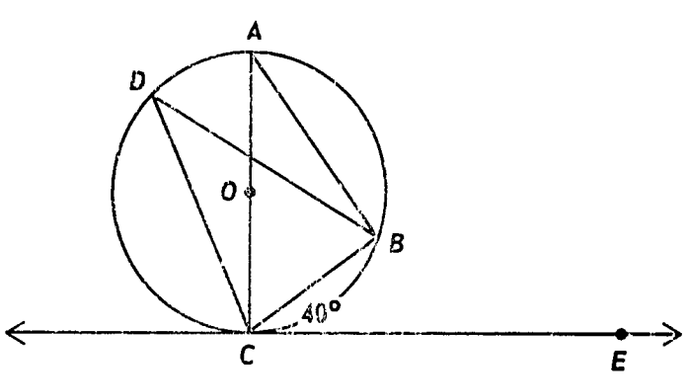


Fig. 33.

8. Secants $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at a point O inside the circle. If $A\hat{O}D$ intercepts an arc $\widehat{AD}$ whose measure is $2x$, and $C\hat{O}B$ intercepts an arc $\widehat{CB}$ whose measure is $2y$, prove that the measure of $C\hat{O}B$ is $x + y$. Hint: Draw $\overline{AC}$ and use the Exterior Angle Theorem on $\triangle AOC$.

We will state this result in the following theorem:

THEOREM 4-19: The measure of an angle formed by two secants of a circle intersecting at a point in the interior of the circle is one-half the sum of the measures of the arcs intercepted by the angle and its vertically opposite angle.

9. In Problem 8, find $m(C\hat{O}B)$, if $m(\widehat{BC}) = 203$ and $m(\widehat{AB}) = 59$.

10. Prove that the measure of the angle formed by two intersecting

tangents is equal to 180 minus the measure of the minor arc intercepted. Hint: Draw the two radii to the points of tangency and consider the quadrilateral thus formed.

11. $\overline{AB}$ and $\overline{CD}$ are two perpendicular chords of a circle which intersect inside the circle at X. E is a point on $\overleftrightarrow{XC}$ such that $\overline{EX} \equiv \overline{XD}$. Prove that

(i) $\triangle EAX \equiv \triangle DAX$

(ii) $E\hat{A}X \equiv X\hat{C}B$

(iii) $\overleftrightarrow{CB} \perp \overleftrightarrow{AE}$

An angle formed by a secant and a tangent with the vertex at the point of tangency is not an inscribed angle (why?), but our next theorem will show that such an angle is also measured by one-half the measure of its intercepted arc.

> THEOREM 4-20: The measure of the angle formed by a tangent and a secant with the vertex at the point of tangency is one-half the measure of its intercepted arc.

There are two cases to consider, depending on whether the angle is acute or obtuse. We shall prove the first case, and leave the second to you as exercise.

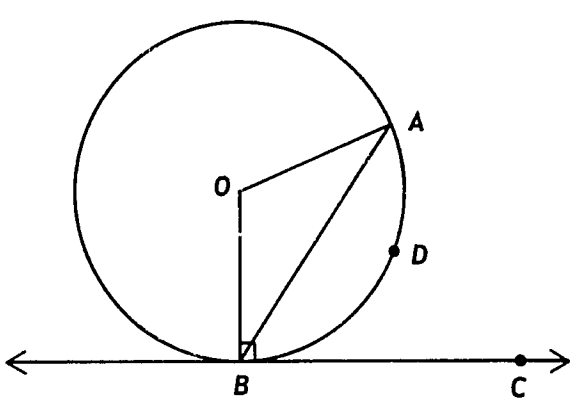


Fig. 34.

Proof: Using Figure 34, we have to prove that $m(A\hat{B}C) = \frac{1}{2} \times m(\widehat{ADB})$.

1.	$m(A\hat{B}C) + m(A\hat{B}O) = 90$	Why?
2.	$m(A\hat{B}O) = m(B\hat{A}O)$	Why?
3.	$m(A\hat{O}B) + 2 \times m(A\hat{B}O) = 180$	Angle sum of ΔAOB, and Statement 2.
4.	$m(A\hat{O}B) + 2 \times (90 - m(A\hat{B}C)) = 180$	Statements 1 and 3.
5.	$m(A\hat{O}B) = 2 \times m(A\hat{B}C)$	Statement 4.
6.	$m(A\hat{B}C) = \frac{1}{2} \times m(A\hat{O}B) = \frac{1}{2} \times m(\widehat{ADB})$	Statement 5 and Definition 4-11.

Theorems 4-17, 4-19, and 4-20, along with Problem 10 of Problems 4-3B, tell us how to find the measures of certain angles formed by two lines (secants, tangents, or a secant and a tangent) in terms of the arcs of a circle these angles intercept. Our next theorem will complete this list when we consider an angle formed by two secants or a secant and a tangent intersecting outside a circle.

> THEOREM 4-21: The angle formed by two secants or by a secant and a tangent intersecting outside a circle has a measure that is one-half the difference of the measures of the intercepted arcs.

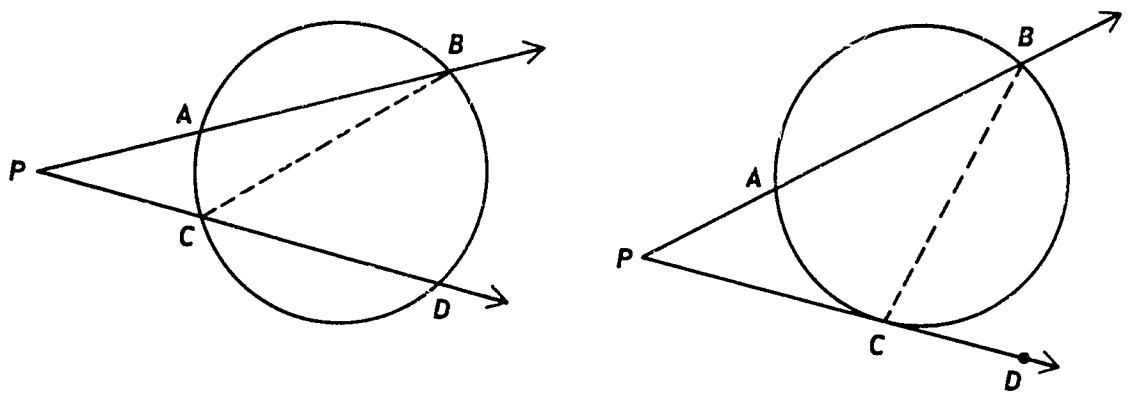


Fig. 35.

We will treat both cases together, since the statements in the following proof could apply to either figure above. In Figure 35(a) we want to prove that $m(\widehat{P}) = \frac{1}{2}[\, m(\widehat{BD}) - m(\widehat{AC})\,]$ and in Figure 35(b) we want to prove that $m(\widehat{P}) = \frac{1}{2}[\, m(\widehat{BC}) - m(\widehat{AC})\,]$. In either case, we draw $\overline{BC}$.

Proof:

1.	In both figures, $m(B\widehat{C}D) = m(\widehat{P}) + m(P\widehat{B}C)$	Why?
2.	$m(\widehat{P}) = m(B\widehat{C}D) - m(P\widehat{B}C)$	Statement 1 and algebra.
3.	$m(P\widehat{B}C) = \frac{1}{2}\, m(\widehat{AC})$	Why?
4.	In Figure 35(a), $m(B\widehat{C}D) = \frac{1}{2}\, m(\widehat{BD})$	Why?
	and in Figure 35(b), $m(B\widehat{C}D) = \frac{1}{2}\, m(\widehat{BC})$	Why?
5.	In Figure 35(a), $m(\widehat{P}) = \frac{1}{2}[\, m(\widehat{BD}) - m(\widehat{AC})\,]$	Statements 2, 3, and 4.
	and in Figure 35(b), $m(\widehat{P}) = \frac{1}{2}[\, m(\widehat{BC}) - m(\widehat{AC})\,]$	Statements 2, 3, and 4.

Consider the arcs $\widehat{AB}$ and $\widehat{CD}$ of the two concentric circles in Figure 36 below. Each arc is measured by the central angle it subtends, hence each arc has a measure of 60. Would you say the two arcs are congruent?

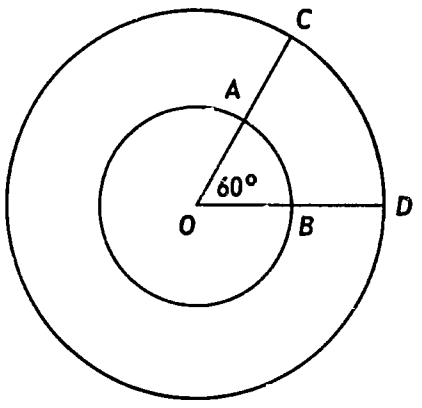


Fig. 36.

Besides having equal degree measures, what else must be true about two arcs in order for them to be congruent? We now define congruent arcs.

> DEFINITION 4-14: Two *arcs* are said to be *congruent* if they are arcs of the same circle or congruent circles and if they have the same measure.

PROBLEMS 4-3C

1. In Figure 37, find $m(\widehat{B})$, $m(\widehat{A})$ and $m(\widehat{BCE})$.

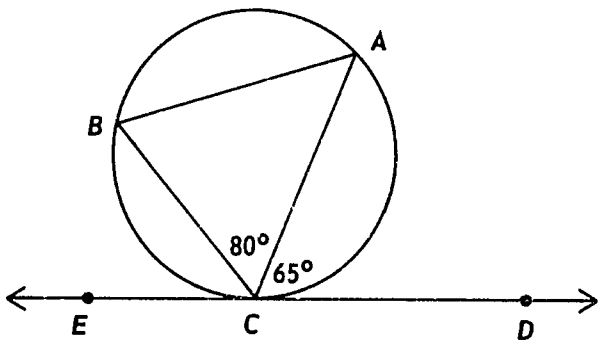


Fig. 37.

2. An isosceles triangle ΔABC (with $\overline{AC} \equiv \overline{BC}$) is inscribed in a circle. Prove that the base $\overline{AB}$ is parallel to the tangent at C.

3. Prove that if two parallel lines intercept arcs of a circle, the intercepted arcs have equal measure.

4. Prove the following theorem:

   THEOREM 4-22: In the same circle, or in congruent circles, if two chords are congruent, then so are the minor arcs they determine.

5. State the converse of Theorem 4-22. Prove that the statement is true.

6. $ABCD$ is a quadrilateral inscribed in a circle such that the tangent to the circle at A is parallel to $\overleftrightarrow{BD}$. Prove that the diagonal $\overline{AC}$ bisects $\widehat{BCD}$.

7. Two circles are tangent externally at P. A common tangent $\overleftrightarrow{AB}$ is tangent to one circle at A and to the other at B. Prove that $A\hat{P}B$ is a right angle.

8. Prove that if a pair of opposite angles of a quadrilateral are supplementary, then the quadrilateral can be inscribed in a circle. What relationship does this result have to Theorem 4-18?

9. $ABCD$ is a quadrilateral inscribed in a circle. $\overline{AB} \equiv \overline{AD}$ and the tangent at A meets the line $\overleftrightarrow{CD}$ at T. Prove that triangles ΔABC and ΔTDA are similar.

10. In a circle with centre O, $\overline{OC}$ is a radius perpendicular to a chord $\overline{AB}$. Prove that the point C is equidistant from $\overline{AB}$ and the tangent at A.

11. Two circles with centres O and O' are externally tangent at A. $\overline{OE}$ and $\overline{O'F}$ are parallel radii, and $\overline{AB}$ is a diameter of the circle with centre O'. Prove that the chords $\overline{AE}$ and $\overline{BF}$ are parallel.

12. Prove that if two circles are tangent internally, and if the radius of one is equal to the diameter of the other, all chords of the larger circle drawn from the point of contact are bisected by the smaller circle.

*4-4 SEGMENTS OF CHORDS, SECANTS AND TANGENTS.

We have, up to now, been studying the properties of circles and their relationships with certain lines, arcs and angles. We have learned how to determine the measure of angles formed by chords, tangents and secants whether the point of intersection lies on the circle, in its interior or in its exterior. We now want to consider how the lengths of certain segments of chords, secants and tangents are related. For this the main tools will be the similarity theorems studied in Chapter 3.

If, in Figure 38, $\overline{AB}$ and $\overline{CD}$ intersect at P, is there any relationship between AB and CD? About all we could say is that each is less than or equal to the length of the diameter of the circle and greater than or equal to the length of the chord through P which is perpendicular to the diameter

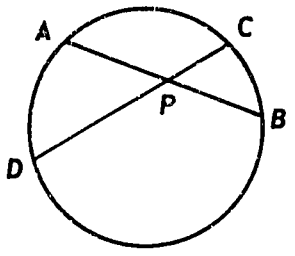


Fig. 38.

through P. This is an immediate consequence of Theorem 4-12 as you will be asked to prove in the problems below.

If we cannot say much about AB and CD, perhaps we can say something about AP, BP, CP and DP.

PROBLEMS 4-4A

1. In a circle with centre O, $\overline{AB}$ is a diameter, X is a point on $\overline{AB}$ different from O, $\overline{CD} \perp \overline{AB}$ at X, $\overline{PQ}$ is any other chord through X. Prove that $PQ < AB$, and $PQ > CD$.

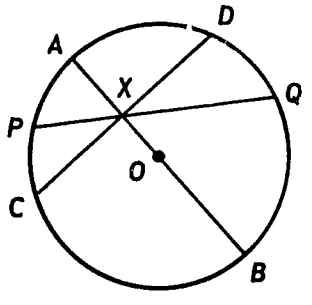


Fig. 39.

2. If the radius of a circle is 13 inches long and a point P is 5 inches from the centre, determine the length of the longest and shortest chords which can be drawn through P.

3. $\overline{AB}$ is a chord of a circle and $AB = 16$ in. If $\overline{AB}$ is 15 inches from the centre of the circle what is the length of the radius of the circle?

4. In each case determine the lengths of the segments as indicated.

	AB	OP	CO	CD
a)	12	3	?	?
b)	?	5	?	20
c)	18	?	12	?
d)	48	7	?	?
e)	?	9	18	?

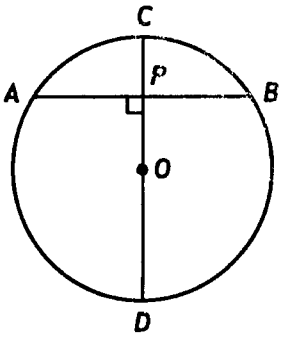


Fig. 40.

Consider now a circle, any point P, and two lines passing through P which intersect or are tangent to the circle (Figure 41). If P is in the interior of the circle we have two chords $\overline{AB}$ and $\overline{CD}$. If we draw $\overline{AD}$ and $\overline{BC}$ we see that $\hat{D} \equiv \hat{B}$ and $\hat{A} \equiv \hat{C}$ (Why?) making $\Delta ADP \sim \Delta CBP$. (Why?)

It follows that $\dfrac{CP}{AP} = \dfrac{BP}{DP}$ and therefore

$$AP \times BP = CP \times DP$$

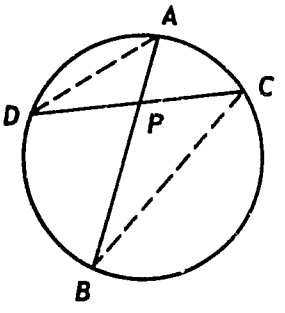


Fig. 41.

Would the result still be true if $\overline{AB}$ were a diameter and $\overline{CD} \perp \overline{AB}$?

Suppose now the point P is on the circle. We could have either of the two situations illustrated in Figure 42.

In either case P, B and C are all the same point. Although we do not usually consider $\overline{PB}$ as a segment when P and B are the same point, it is sometimes convenient to do so. If we adopt this convention here and define $PB = 0$ when $P = B$, we have again

$$PA \times PB = PC \times PD$$

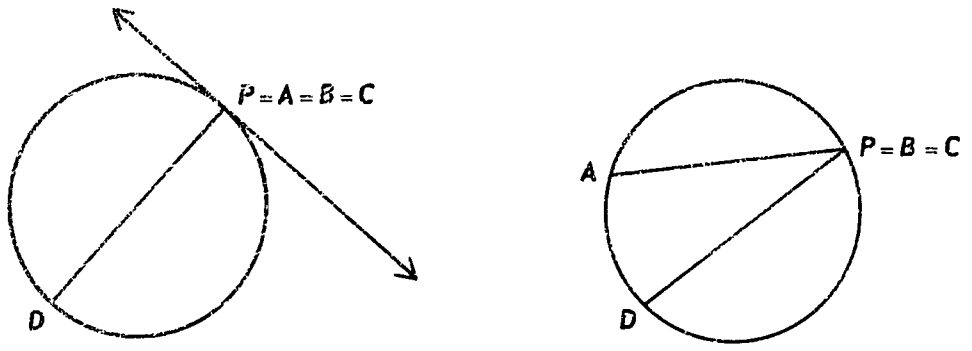


Fig. 42.

If P is outside the circle the two lines through P may both be secants, one may be a secant and the other a tangent or they may both be tangents. Let us look at each of these cases in turn.

Suppose both lines are secants.

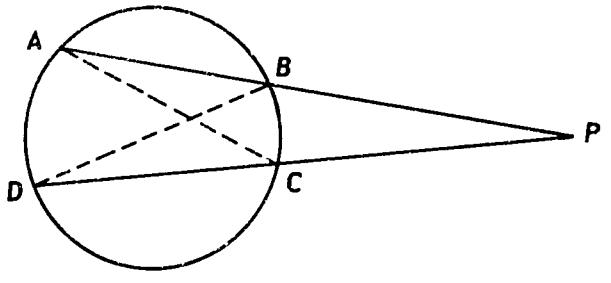


Fig. 43.

Drawing $\overline{AC}$ and $\overline{BD}$ we can prove $\Delta ACP \sim \Delta DBP$ and thus

$$PA \times PB = PC \times PD$$

If you think of the secant $\overleftrightarrow{PBA}$ turning slowly on P in a clockwise direction you will see that as it turns B and A get closer and closer together so that $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$ get more and more nearly equal. (Fig. 44)

You might guess that if $\overleftrightarrow{PA}$ were a tangent we would still have $PA \times PB = PC \times PD$ or

$$PA^2 = PC \times PD.$$

You should be able to prove this by looking at Figure 45 below and proving $\Delta ACP \sim \Delta DAP$.

If $\overleftrightarrow{PA}$ and $\overleftrightarrow{PC}$ are both tangents to the circle we would have $PA^2 = PC^2$ or $PA = PC$ and this is a new proof that two tangents drawn to a circle from an exterior point are equal in length.

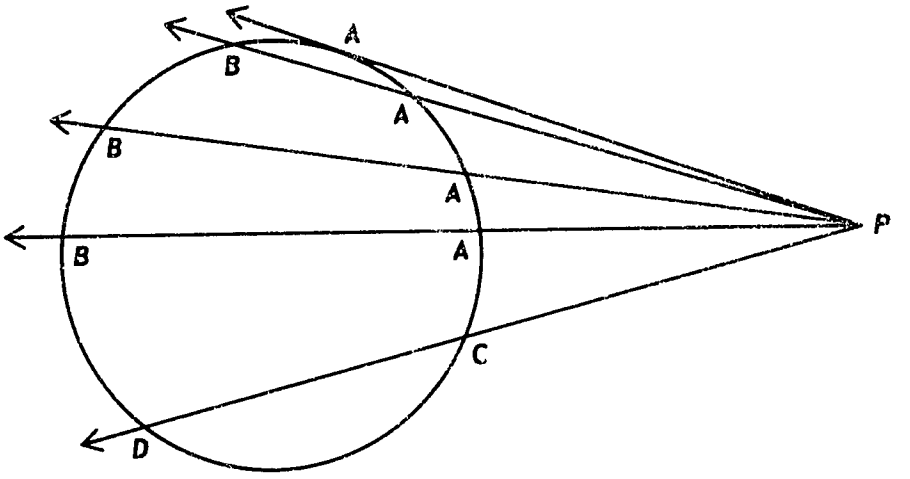


Fig. 44.

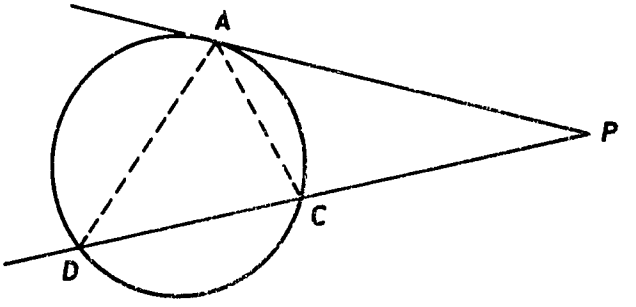


Fig. 45.

We have considered what happens when P is inside, on and outside the circle and in each case we have found that $PA \times PB = PC \times PD$ although, when we are considering tangents, PA may be equal to PB or PC equal to PD. We can combine all these situations into one theorem which we will call the "segment product theorem."

> THEOREM 4-23: If through any point P two lines are drawn intersecting a circle, the product of the lengths of the segments of one is equal to the product of the lengths of the segments of the other.

PROBLEMS 4-4B

1. Write out the details of the proof of Theorem 4-23 if both lines are secants.

2. Write out the details of the proof of Theorem 4-23 if one line is a secant and one a tangent.

3. Prove, using Theorem 4-23, that if two tangents are drawn to a circle from a point in the exterior of the circle, the tangents are equal in length.

4. Prove Theorem 4-23 for two secants using Figure 46 below.

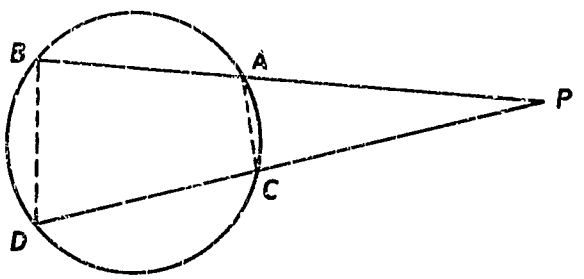


Fig. 46.

5. Prove: $PA = PD$

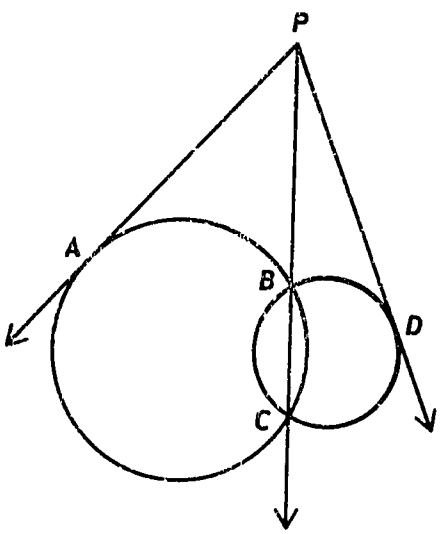


Fig. 47.

6. If 2 circles are each tangent to $\overleftrightarrow{PA}$ at A:

   Prove: $PB \times PC = PD \times PE$

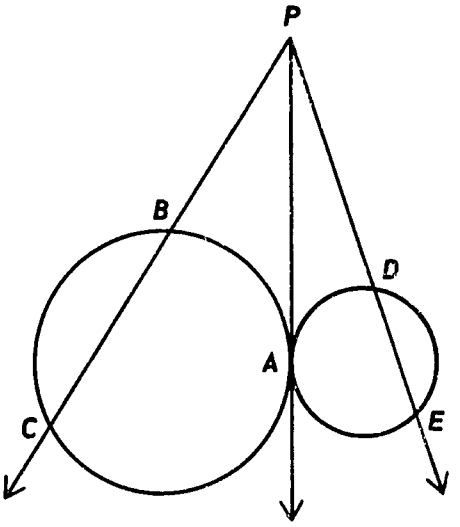


Fig. 48.

7. If 2 circles are each tangent to $\overleftrightarrow{PA}$ at A:

Prove: $\Delta PBD \sim \Delta PEC$

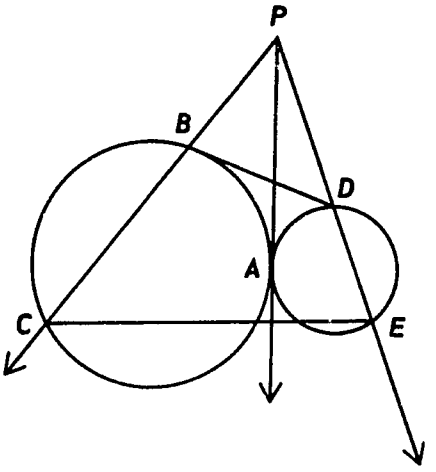


Fig. 49.

8. An alternate proof of Pythagoras' Theorem may be given as follows: If $\widehat{C}$ is a right angle in ΔABC, draw a circle with centre A and radius AC (Fig. 50). The circle will intersect $\overleftrightarrow{BA}$ at two points. (Why?) Call them D and E. Then:

$$\begin{aligned} BC^2 &= BD \times BE \\ &= (BA - DA) \times (BA + AE) \\ &= (BA - AC) \times (BA + AC) \\ &= BA^2 - AC^2 \\ \therefore BA^2 &= AC^2 + BC^2 \end{aligned}$$

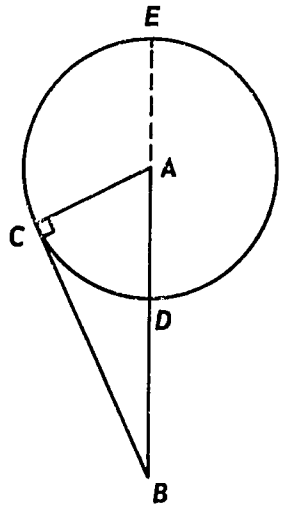


Fig. 50.

Write out the reasons for the steps of this proof.

Challenge Problems

9. a. If two circles intersect at C and D, if P is any point on $\overleftrightarrow{CD}$ outside the circles, and if secants $\overleftrightarrow{PAB}$ and $\overleftrightarrow{PEF}$ are drawn, we can easily prove $PA \times PB = PC \times PD = PE \times PF$.

   Prove the converse: If P is a point outside each of two intersecting circles, if secants $\overleftarrow{PAB}$ and $\overleftarrow{PEF}$ are drawn, one to each circle, and if $PA \times PB = PE \times PF$, then P lies on the line determined by the common chord.

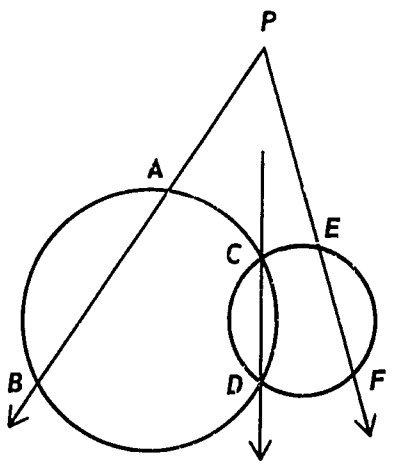


Fig. 51.

b. Prove the corresponding statement if P is in the interior of both circles.

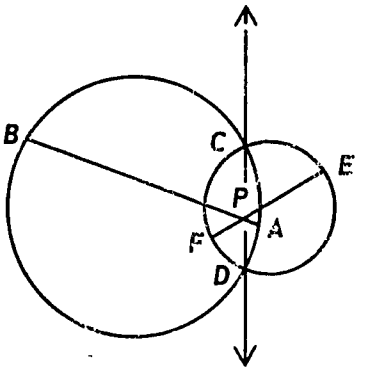


Fig. 52.

c. Prove: If two circles are tangent at C and from a point P secants are drawn to each circle such that $PA \times PB = PE \times PF$ then P lies on the common tangent at C.

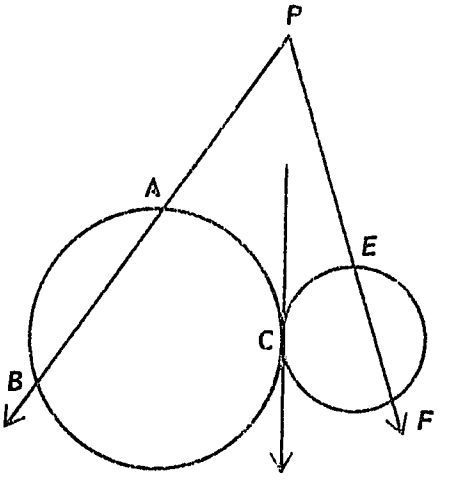


Fig. 53.

d. Prove: If two circles intersect at C and D, the locus of points from which the tangents to the two circles are congruent is all of the line determined by the common chord $\overline{CD}$ except for $\overline{CD}$ itself.

e. Prove: If two circles are tangent at C the locus of points from which tangents to the two circles are congruent is the common tangent at C.

f. Two circles do not intersect and each is in the exterior of the other. From P, a point outside both circles, secants $\overleftrightarrow{PAB}$ and $\overleftrightarrow{PEF}$ are drawn. If $PA \times PB = PE \times PF$ prove that for all positions of P, P lies in a certain fixed line which is perpendicular to the line of centres of the two circles. Hints: Draw tangents $\overleftrightarrow{PR}$ and $\overline{PS}$, also $\overline{PD} \perp \overline{OO'}$.

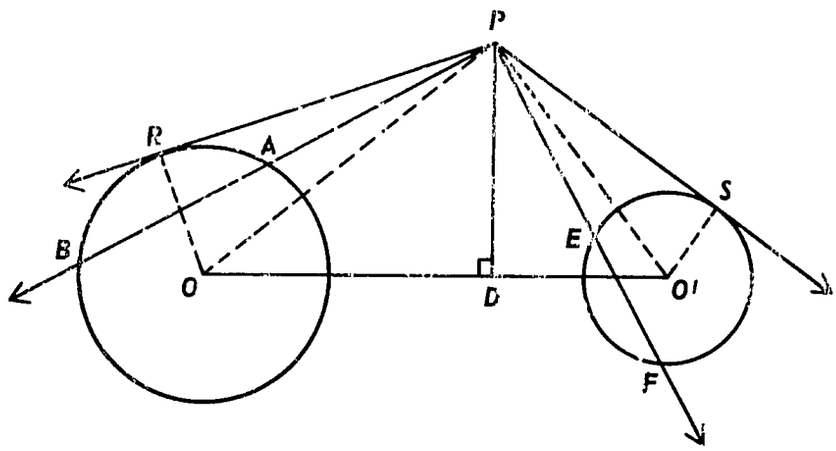


Fig. 54.

Try to show OD is a constant and therefore D is a fixed point. Then P lies on the line perpendicular to $\overline{OO'}$ at D.

The theorems we have just proved enable us to do many numerical problems.

Example 1. In circle O, $\overline{AB}$ and $\overline{CD}$ intersect at P, $AP = 3$, $CP = 4$, $DP = 6$, we want to find AB. Let $BP = x$. Then by the segment product theorem $3 \times x = 6 \times 4$, $3x = 24$, $x = 8$. Therefore $AB = 3 + 8 = 11$.

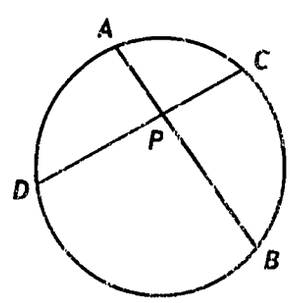


Fig. 55.

Example 2. Segments are given as marked (in Figure 56). Find AP. Again by the segment product theorem $AP^2 = CP \times BP$, $x^2 = (4 + 5) \times 4$, $x^2 = 36$, $x = 6$, $AP = 6$. Note that since x represents the length of $\overline{AP}$ we need consider only the positive root of $x^2 = 36$.

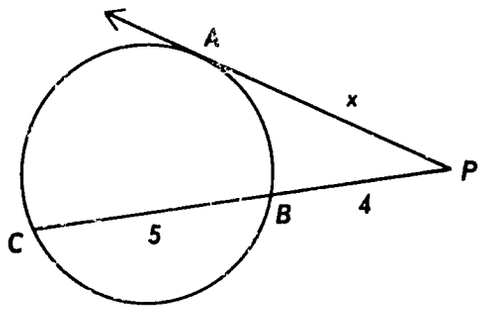


Fig. 56.

PROBLEMS 4-4C

1. a. $MN = 6$, $RM = 2$, $QR = 4$, $RP = ?$
   b. $MR = 9$, $RN = 6$, $RP = 4$, $QR = ?$
   c. $MR = RN$, $QR = 9$, $PR = 4$, $MR = ?$
   d. $MN = 15$, $RN = 3$, $PQ = 13$, $PR = ?$
   e. $QR = 7$, $QP = 12$, $MR = 10$, $MN = ?$
   f. $MN = 6$, $MR = 4$, $QP = 7$, $QR = ?$

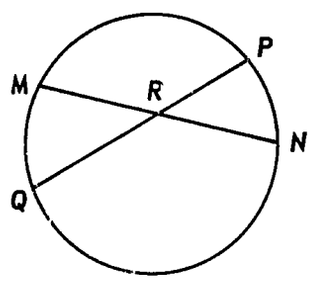


Fig. 57.

2. a. $RT = 16$, $RS = 2$, $RV = 4$, $RW = ?$
   b. $RT = 16$, $SR = 2$, $RV = 4$, $WV = ?$
   c. $TS = 6$, $SR = 6$, $WV = 9$, $RV = ?$
   d. $RT = 16$, $SR = 2$, $RP = ?$

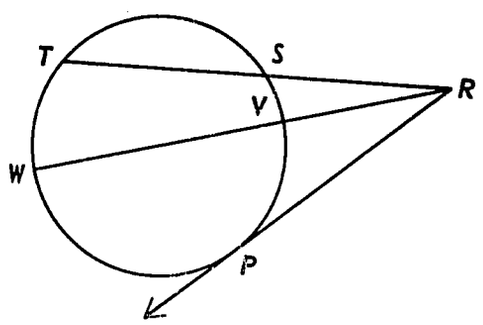


Fig. 58.

e. $RP = 12$, $RV = 8$, $WV = ?$

f. $TS = 6$, $WV = 6$, $SR = 5$, $VR = ?$ $RP = ?$

3. In Figure 59, $AP = 9$, $PB = 4$. What is the length of the shortest chord which can be drawn through P? If the distance of P from the centre of the circle is 2, what is the radius of the circle? What is the length of the longest chord which can be drawn through P?

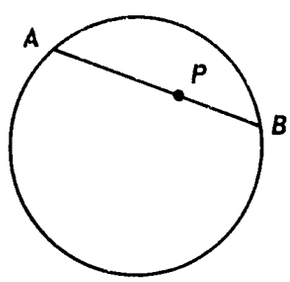


Fig. 59.

Challenge Problem

4. A man on the bridge of a ship is 20 feet above the sea. He has just caught sight of the top of a lighthouse which his charts tell him is 220 feet above sea level. How far is the ship from the lighthouse? (Assume that the diameter of the earth is about 8000 miles long). Note that the distance along the surface of the earth is almost equal to x the length of the tangent from B to L.

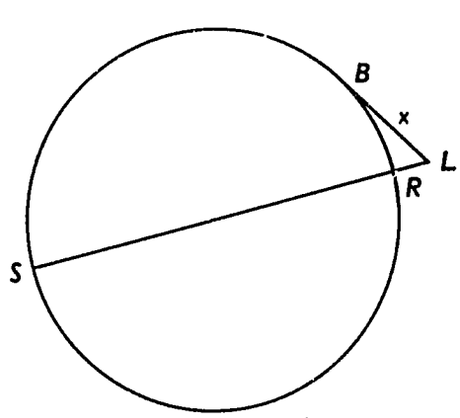


Fig. 60.

*4-5 SIMILAR TRIANGLES AND CIRCLES.

Many problems involving circles and triangles can be solved by using together the theorems about angle measurement and the theorems about similar triangles.

This illustrates a fact which is of great importance in mathematics and also in other fields. One bit of information you know or one theorem you have proved may enable you to do certain problems. Another theorem or another bit of information helps you to solve another set of problems. But the two ideas used together may open up a whole new method of attack leading to the solution of many problems you could not begin to handle before.

Example 1. Two circles are tangent at P. Two secants are drawn through P. Prove: $\overline{AC} \parallel \overline{BD}$ and $\triangle APC \sim \triangle BPD$. We draw the common tangent $\overleftrightarrow{XY}$. Then $A\hat{C}P \equiv A\hat{P}X \equiv B\hat{P}Y \equiv B\hat{D}P$ and $\overline{AC} \parallel \overline{BD}$. Also $A\hat{P}C \equiv B\hat{P}D$ so $\triangle APC \sim \triangle BPD$.

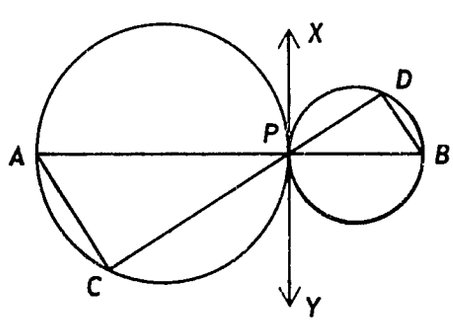


Fig. 61.

Note that the result holds whether the circles are tangent externally or internally and the proof varies only slightly in that this time $A\hat{P}X \equiv B\hat{P}X$.

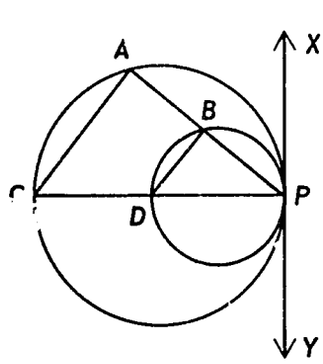


Fig. 62.

Example 2. A is the midpoint of arc $\widehat{BC}$. $\overline{BC}$ and $\overline{AD}$ meet at E. Prove: $AC^2 = AE \times AD$. Draw $\overline{CD}$. Since $A\hat{C}E \equiv A\hat{D}C$ (Why?) $\Delta AEC \sim \Delta ACD$ ($E\hat{A}C = C\hat{A}D$). Therefore $\frac{AE}{AC} = \frac{AC}{AD}$ and the result follows.

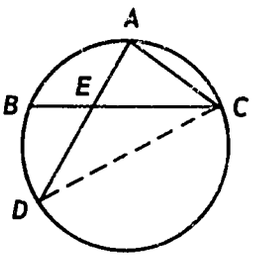
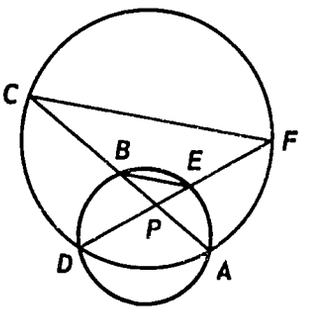


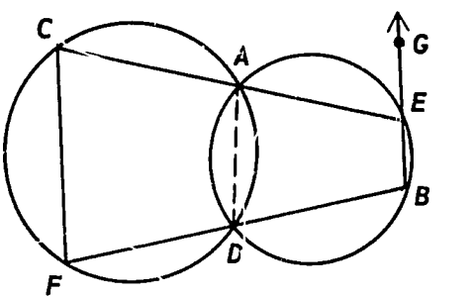
Fig. 63.

PROBLEMS 4-5

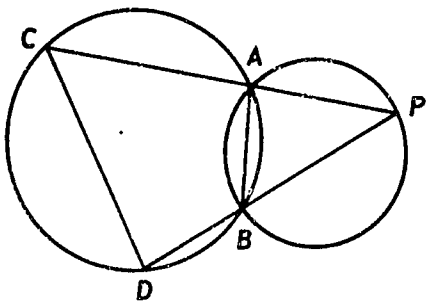
1. Prove $\overline{BE} \parallel \overline{CF}$



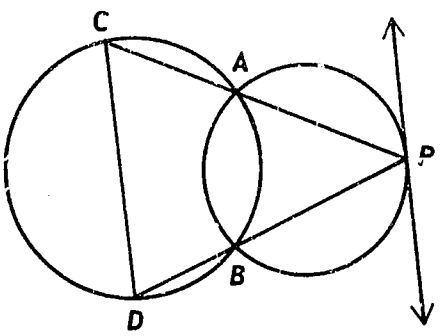
2. Prove $\overline{BE} \parallel \overline{CF}$



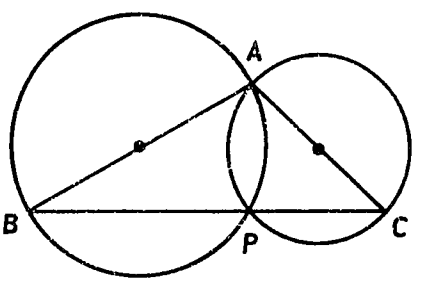
3. Prove: $\Delta PAB \sim \Delta PDC$



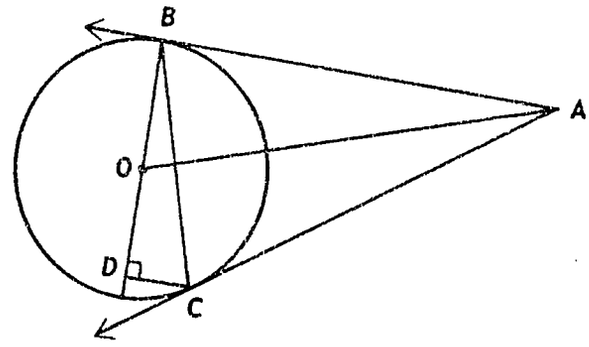
4. Prove: The tangent at P is parallel to $\overline{CD}$.



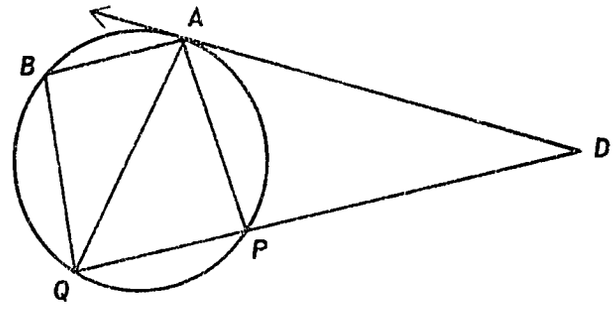
5. If two circles intersect at A and P and if $\overline{AB}$ and $\overline{AC}$ are diameters prove $\overline{BP}$ and $\overline{PC}$ lie on $\overleftrightarrow{BC}$. (Hint: Draw $\overline{AP}$).



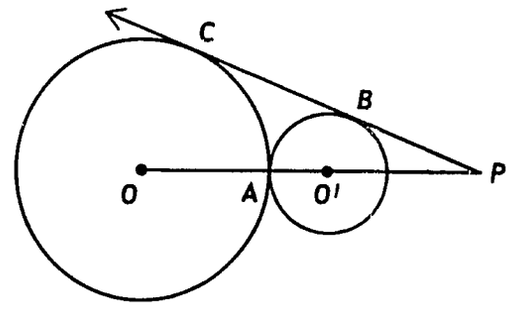
6. $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ are tangents to the circle with centre O. $\overline{CD} \perp \overline{BO}$. Prove $\Delta BCD \sim \Delta AOB$.



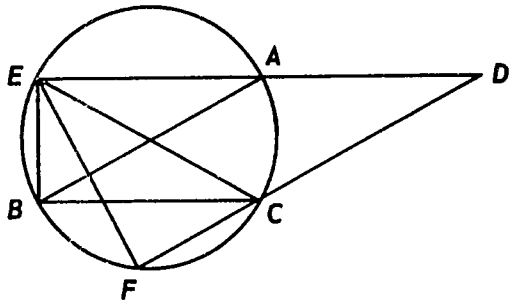
7. $\overleftrightarrow{AD}$ is tangent to the circle. $\overleftrightarrow{DQ} \parallel \overleftrightarrow{AB}$. Prove: $\Delta ADP \sim \Delta AQB$.
Hint: What do you know about $\overset{\frown}{AP}$ and $\overset{\frown}{BQ}$?



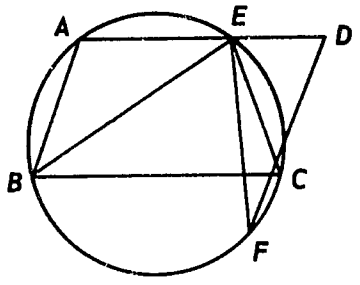
8. Two circles are tangent at A. The common external tangent $\overleftrightarrow{CB}$ meets the line of centres $\overline{OO'}$ at P. Prove: $PA^2 = PB \times PC$.



9. In each of figures (a) and (b) below, $ABCD$ is a parallelogram.
Prove: $\dfrac{ED}{EF} = \dfrac{EC}{EB}$
Hint: Prove $\Delta EFD \sim \Delta EBC$



(a)



(b)

10. Two circles intersect at A and B. At A tangents are drawn, one to each circle, meeting the circles again at C and D. Prove $AB^2 = CB \times DB$.

Chapter 5
CIRCLE MEASURES

5-1 REGULAR POLYGONS AND CIRCLES.

In Chapter 4 you learned how to circumscribe a circle about any triangle. Do you think you could circumscribe a circle about any quadrilateral? Certainly you could about some quadrilaterals. For example you could always do it for a rectangle. In fact we have proved that a circle can be circumscribed about a quadrilateral if and only if its opposite angles are supplementary. (See Theorem 4-18 and its converse, Problem 8 of Problems 4-3C).

But for pentagons, hexagons and other polygons we have as yet no theorem. Are there any special polygons for which we would have some hope of being able to circumscribe a circle? We shall see there are. In fact there are certain ones for which we can always both circumscribe and inscribe circles. These are the *regular polygons*. You will remember that they are like equilateral triangles or squares in that all their angles are congruent, and all their sides are congruent to each other. A restatement might be: a regular polygon is a polygon which is both equilateral and equiangular. Thus a pentagon $ABCDE$ is regular if and only if $\overline{AB} \equiv \overline{BC} \equiv \overline{CD} \equiv \overline{DE} \equiv \overline{EA}$ and $\widehat{A} \equiv \widehat{B} \equiv \widehat{C} \equiv \widehat{D} \equiv \widehat{E}$.

Suppose now $ABCD \ldots X$ is a regular polygon of n sides. We know from Theorem 4-14 that there is one and only one circle through A, B and C. The question is, does this circle also contain all the vertices D, E, . . . to X? Suppose O is the centre of the circle passing through A, B and C.

It follows that $OA = OB = OC$ and if $OD = OA$, D will also lie on the circle. But this will be true if $\Delta AOB \equiv \Delta DOC$. Since $\overline{AB} \equiv \overline{DC}$ and $\overline{BO} \equiv \overline{CO}$, the triangles will be congruent by SAS if we can show that $A\widehat{B}O \equiv D\widehat{C}O$. But $A\widehat{B}C \equiv D\widehat{C}B$ (Why?) and $O\widehat{B}C \equiv O\widehat{C}B$ (Why?). Our result follows; i.e., D does lie on the circle. We can continue this argument

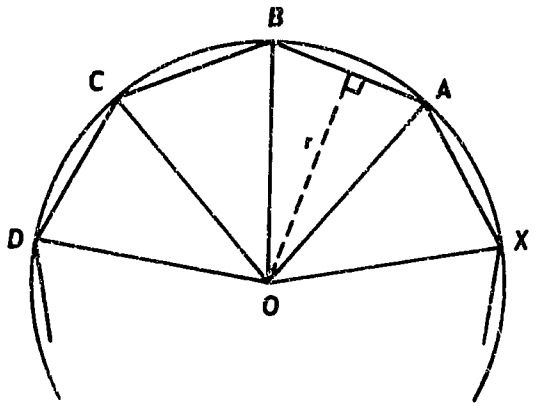


Fig. 1.

to show that the next vertex E is on the circle and so on for all the vertices. We have thus established the following theorem.

> THEOREM 5-1: One and only one circle can be circumscribed about any regular polygon.

Now suppose O is the centre of the circle circumscribed about the regular polygon $ABC \ldots X$ of Figure 1 above. Since $\overline{AB} \equiv \overline{BC} \equiv \ldots \equiv \overline{XA}$, each of these segments is the same distance r from O. (Why?) A circle drawn with centre O and radius r will have each segment as a tangent. (Why?) Thus this circle is inscribed in the regular polygon and we have:

> THEOREM 5-2: A circle can be inscribed in any regular polygon.

In the exercises at the end of this section you will be asked to show that there is only one such inscribed circle in any regular polygon.

To complete the relationship between circles and regular polygons we need the following theorem.

THEOREM 5-3: If n points $A, B, C, \ldots N$ are taken in order on a circle so that $\overset{\frown}{AB} \equiv \overset{\frown}{BC} \equiv \overset{\frown}{CD} \equiv \ldots \equiv \overset{\frown}{NA}$, the chords $\overline{AB}, \overline{BC}, \ldots, \overline{NA}$ are the sides of a regular inscribed polygon. The tangents drawn at $A, B, C, \ldots, N$ form the sides of a regular circumscribed polygon.

Given: $\overset{\frown}{AB} \equiv \overset{\frown}{BC} \equiv \overset{\frown}{CD} \equiv \ldots \equiv \overset{\frown}{NA}$, $\overline{XY}$ tangent at B, $\overline{YZ}$ tangent at C etc. in Figure 2 below.

Prove: $ABC \ldots N$ and $XYZ \ldots W$ are regular polygons.

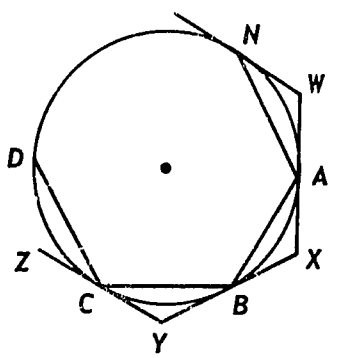


Fig. 2.

Proof:

1. $\overline{AB} \equiv \overline{BC} \equiv \ldots \equiv \overline{NA}$	1. Congruent arcs determine congruent chords.
2. $m(A\hat{B}C) = \frac{1}{2}\{360 - m(\overset{\frown}{ABC})\}$ $= \frac{1}{2}\{360 - 2m(\overset{\frown}{BC})\}$ $m(B\hat{C}D) = \frac{1}{2}\{360 - m(\overset{\frown}{BCD})\}$ $= \frac{1}{2}\{360 - 2m(\overset{\frown}{BC})\}$	2. Why?
3. $A\hat{B}C \equiv B\hat{C}D$, etc, for all the angles of the polygon.	3. Why?
4. $ABC \ldots N$ is a regular polygon.	4. By definition.

5.	$X\hat{A}B \equiv X\hat{B}A \equiv Y\hat{B}C \equiv Y\hat{C}B \equiv Z\hat{C}D \equiv Z\hat{D}C$ etc.	5.	They are all measured by congruent arcs.
6.	$\Delta XAB \equiv \Delta XBA \equiv \Delta YBC \equiv \Delta YCB$ etc.	6.	By ASA congruence theorem.
7.	$\hat{X} \equiv \hat{Y} \equiv \hat{Z} \equiv \ldots$ etc. $\overline{XA} \equiv \overline{XB} \equiv \overline{YB} \equiv \overline{YC} \equiv \ldots$ etc.	7.	Def. of congruent triangles.
8.	$\overline{XY} \equiv \overline{YZ} \equiv$ etc.	8.	Why?
9.	$XYZ \ldots W$ is a regular polygon.	9.	Steps 7 and 8.

In Theorem 5-2 we proved that the centre of the circle circumscribed about a regular polygon is the same point as the centre of the inscribed circle.

> DEFINITION 5-1: The *centre of a regular polygon* is the common centre of its circumscribed and inscribed circles.

Thus, in the figure below, $ABCDE$ is a regular pentagon, O is its centre, OA the radius of the circumscribed circle, and OP the radius of the inscribed circle.

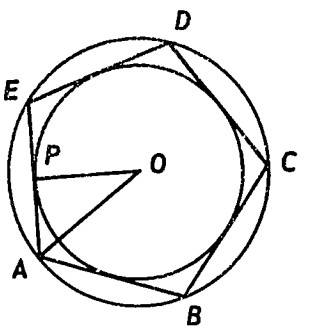


Fig. 3.

We know by Theorem 3-10 that two regular n-gons are similar. From this we can prove:

THEOREM 5-4: The ratio of the perimeters of two regular n-gons is equal to the ratio of the radii of their circumscribed or inscribed circles as well as the ratio of their sides.

In Figure 4 below, if $ABC \ldots$ and $A'B'C' \ldots$ are each regular n-gons whose centres are O and O' we know that $OA = OB = OC = \ldots = OF$ and $AB = BC = \ldots = FA$. Also $O'A' = O'B' = O'C' = \ldots = O'F'$ and $A'B' = B'C' = \ldots = F'A'$. This means that by SSS, $\Delta AOB \equiv \Delta BOC \equiv \ldots \equiv \Delta FOA$ and $\Delta A'O'B' \equiv \Delta B'O'C' \equiv \ldots \equiv \Delta F'O'A'$. Since there are n of these triangles and the sum of the n congruent angles $A\hat{O}B$, $B\hat{O}C$ etc. makes two straight angles we know that $m(A\hat{O}B) = \frac{360}{n}$ and similarly $m(A'\hat{O'}B') = \frac{360}{n}$.

Therefore $A\hat{O}B \equiv A'\hat{O'}B'$.

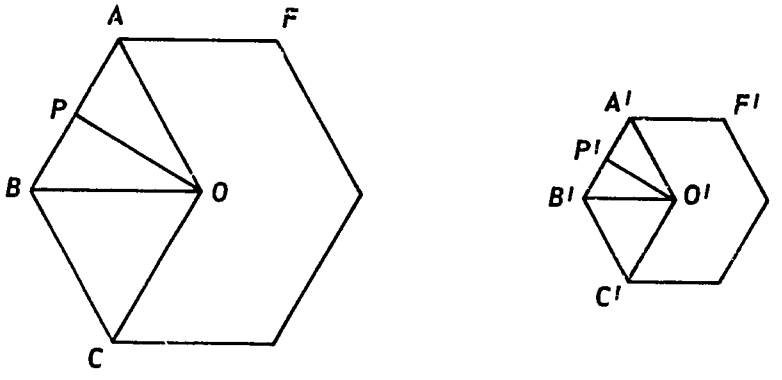


Fig. 4.

Also $OA = OB$ and $O'A' = O'B'$ so $\frac{OA}{O'A'} = \frac{OB}{O'B'}$. Then by the SAS Similarity Theorem $\Delta AOB \sim \Delta A'O'B'$. It follows that $\frac{AB}{A'B'} = \frac{OA}{O'A'} = \frac{OP}{O'P'}$. The last proportion is true since $\overline{OP}$ and $\overline{O'P'}$ are corresponding altitudes of similar triangles and we know their ratio is equal to the ratio of corresponding sides. The perimeter of $ABC \ldots$ is equal to $n(AB)$ and the perimeter of $A'B'C' \ldots$ is equal to $n(A'B')$. Letting p and p' stand for these perimeters we finally have, as required:

$$\frac{p}{p'} = \frac{n(AB)}{n(A'B')} = \frac{AB}{A'B'} = \frac{OA}{O'A'} = \frac{OP}{O'P'}.$$

If we let r and r' stand for the radii of the circumscribed circles and a and a' for the radii of their inscribed circles we can write

$$\frac{p}{p'} = \frac{r}{r'} = \frac{a}{a'} .$$

It follows that

$$\frac{p}{r} = \frac{p'}{r'} \quad \text{or} \quad \frac{p}{a} = \frac{p'}{a'} .$$

Stated in words and calling r the radius of the polygon, the first of these two proportions says that:

> THEOREM 5-5: In any regular polygons of the same number of sides the ratio of the perimeter to the radius is a constant.

Notice that this does *not* say that the ratio $\frac{p}{r}$ for a square is the same as for a regular hexagon. It does say that if you have two regular hexagons of different sizes the ratio $\frac{p}{r}$ is the same for each one, or if you have several different regular pentagons and compute $\frac{p}{r}$ for each one you will get the same number although it will be different from the one you got for the hexagons.

PROBLEMS 5-1

Note: In the following problems, r represents the radius of the polygon (circumscribed circle), and a represents the radius of the inscribed circle.

1. The radius of a regular polygon of four sides is 8. Find its perimeter and the radius of the inscribed circle. Find $\frac{p}{r}$ and $\frac{p}{a}$.

2. Two regular 6-gons have radii whose ratio is $\frac{7}{5}$. If the perimeter of the larger polygon is 63 find the perimeter, the side, and the radius of the smaller one. Find $\frac{p}{r}$.

   Hint: What kind of a triangle is the triangle formed by two radii and a side?

3. The radius of the inscribed circle of a regular hexagon is 6. Find r and p.

4. (a) If the side of an equilateral triangle is 8 find the radii of its inscribed and circumscribed circles.

   (b) If the side of a square is 8 find the radii of its inscribed and circumscribed circles.

   (c) If the side of a regular hexagon is 8 find the radii of its inscribed and circumscribed circles.

5. Repeat each part of Problem 4 if the side is s.

6. If the radius of a circle is 12, compute the side, radius of the inscribed circle, and perimeter of the regular inscribed

   (a) triangle, (b) quadrilateral, (c) hexagon.

7. If the radius of a circle is 12, compute the side, radius and perimeter of the regular circumscribed

   (a) triangle, (b) quadrilateral, (c) hexagon.

8. Repeat Problems 6 and 7 if the radius is r.

9. Compute, to the nearest tenth, the ratio $\frac{p}{r}$ for the regular triangle, quadrilateral and hexagon, using the results of Problem 8.

10. If $ABC \ldots$ is a regular polygon and O is the centre of an inscribed circle, prove $\overline{OQ}$, the radius to the point of tangency, is the perpendicular bisector of $\overline{AB}$. Similarly $\overline{OR}$ is the perpendicular bisector of $\overline{BC}$. Since these perpendicular bisectors are unique, the point O is unique and there exists only one circle inscribed in $ABC \ldots$.

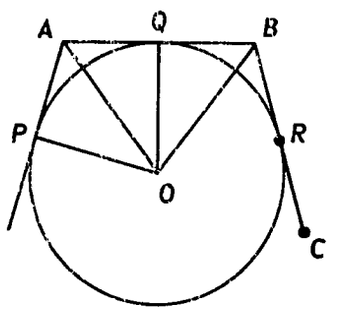


Fig. 5.

5-2 CIRCUMFERENCE OF A CIRCLE.

We now investigate how to find the length of the circumference of a circle when we know its radius. By wrapping a piece of string around a tin can and measuring the length needed we see that it is about three times as long as the diameter of the tin can or about six times the radius.

But we can easily see that this is not exactly right. To do this draw a circle of the same size as the circular end of the can and inscribe a regular hexagon $ABCDEF$. We know that for the hexagon

$$\frac{p}{r} = 6 \qquad \text{(See Problem 9 in Section 5-1)}$$

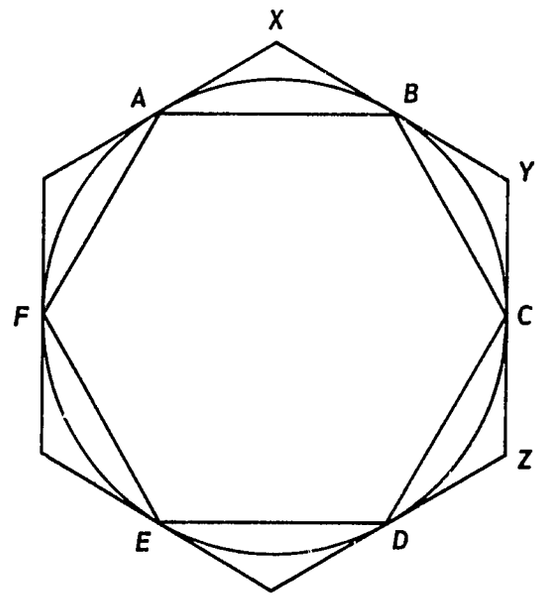


Fig. 6.

In Figure 6 above we have drawn a circle with regular inscribed and circumscribed hexagons. How do you think the length of arc $\overset{\frown}{AB}$ compares with the length of segment $\overline{AB}$? How do you think the length of arc $\overset{\frown}{AB}$ compares with the length $AX + XB$? Take a piece of string or thread and fit it on arc $\overset{\frown}{AB}$ so that it just covers the arc. You will need to cut the string so that it represents just the length of arc $\overset{\frown}{AB}$. Do the same for segment $\overline{AB}$ and segments $\overline{AX} + \overline{XB}$. Now compare the lengths of the strings you have. Do you see that the length of the string representing arc $\overset{\frown}{AB}$ is greater than that representing the length of $\overline{AB}$, but is less than the length of string representing the sum of the lengths $AX + XB$? Do you think the above result would be the same for any circle and any inscribed and circumscribed polygons of n sides? Try it for a circle of radius 2 inches and inscribed and circumscribed equilateral triangles. You will find that for each such trial you make, the length of the arc is greater than the length of the side of the regular inscribed polygon but less than the sum of the half-lengths of two adjacent sides of the regular circumscribed polygon.

The fact that this is true for *all* circles and their regular inscribed and circumscribed polygons of n sides is very difficult to prove. Since, however, it can be proved by methods involving the calculus, we will assume this fact here without proof. That is, we assume that for any given circle, and any given inscribed and circumscribed regular polygons of n sides, if $\overline{AB}$ is the side of the inscribed polygon and $\overline{XY}$ is a side of the circumscribed polygon as in Figure 6 above, then $AB <$ length of arc $\widehat{AB} < AX + XB$.

It follows, then, that the perimeter, p, of the inscribed hexagon is less than the circumference, C, of a circle; that is $p < C$ and $\frac{p}{r} = 6 < \frac{C}{r}$. Similarly, if P is the perimeter of the circumscribed hexagon, we have $C < P$. In Problem 8 of Section 5-1 we found $P = \frac{12r}{\sqrt{3}}$. Therefore $\frac{C}{r} < \frac{P}{r} = \frac{12}{\sqrt{3}} = 4\sqrt{3} \approx 6 \cdot 9$. Since $\frac{p}{r} = 6$ and $\frac{P}{r} = 4\sqrt{3}$ for the corresponding inscribed and circumscribed regular hexagons for a circle of any radius, we find that for any circle, $6 < \frac{C}{r} < 6 \cdot 9$.

In as much as we are going to be working with the perimeters of the inscribed and circumscribed regular n-gons, a convenient notation is useful. We shall use p_n and P_n respectively for these two numbers. Thus p_3 is the perimeter of the *inscribed* equilateral triangle, P_6 the perimeter of the *circumscribed* regular hexagon, etc.

In Problem 8 of Section 5-1 we found that

$$p_3 = 3\sqrt{3}\,r \approx 5 \cdot 196r$$

$$p_4 = 4\sqrt{2}r \approx 5 \cdot 656r$$

$$p_6 = 6r \quad = 6 \cdot 000r$$

It is true that $p_3 < p_4 < p_6$ and it seems as though p_n increases as n increases. Do you think this would always be true? Would $p_6 < p_7$, $p_{22} < p_{23}$, etc? As a matter of fact this is true but it is not at all easy to prove. What is easy to show is that $p_n < p_{2n}$ for any n. This is true since in going from the inscribed regular n-gon to the inscribed regular $2n$-gon we replace each line segment $\overline{AB}$ in p_n by two line segments $\overline{AP}$ and $\overline{PB}$ and, by the triangle inequality, $AB < AP + PB$.

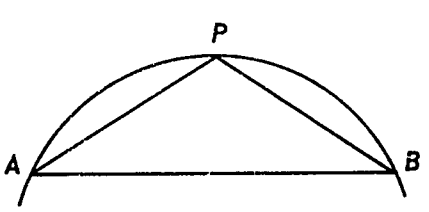


Fig. 7.

Also $p_n < P_n$ since in this comparison the segment $\overline{AB}$ is replaced by $\overline{AX}$ and $\overline{XB}$ and $AB < AX + XB$.

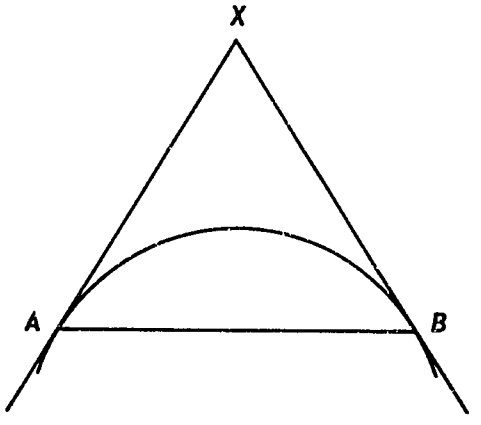


Fig. 8.

On the other hand $P_{2n} < P_n$ since this time we replace the two segments $\overline{PX}$ and $\overline{XQ}$ by the single segment $\overline{PQ}$ and $PQ < PX + XQ$.

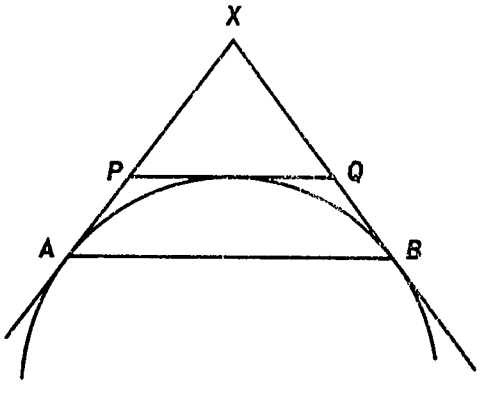


Fig. 9.

Take a circle of radius r and circumference C. Consider the series $p_3, p_6, p_{12}, \ldots,$ of perimeters of inscribed regular polygons. We know this series increases. But every member of the series is less than every member of the the decreasing series $P_3, P_6, P_{12}, \ldots$. For example $p_{48} < P_{48} < P_{24} < P_{12} < P_6$ or $p_6 < p_{12} < \ldots < p_{96} < P_{96}$.

As n gets larger and larger p_n increases and gets closer and closer to C. So also $\frac{p_n}{r}$ gets closer and closer to $\frac{C}{r}$.

Since for regular n-gons of different sizes but with n fixed we have proved $\frac{P_n}{r}$ is a constant it would seem that likewise for circles of different sizes $\frac{C}{r}$ should be a constant. It will also always be true that for any n, $\frac{p_n}{r} < \frac{C}{r} < \frac{P_n}{r}$ and p_n and P_n get closer and closer together as n increases, always squeezing C between them. Thus by computing p_n and P_n for increasing values of n we can determine the value of $\frac{C}{r}$ to as many significant digits as we wish.

If $\frac{C}{r}$ is a constant then certainly $\frac{C}{2r} = \frac{C}{d}$ is a constant. We give this last constant a name and call it π. (π is a Greek letter pronounced pi).

5-3 COMPUTING WITH π.

What is the value of π? We know that $6 < \frac{C}{r} < 6 \cdot 9$. Therefore since $\pi = \frac{C}{2r}$ we have $3 < \pi < 3{\cdot}45$.

There are methods in advanced mathematics for computing π quickly to many digits but the important thing for us to know and remember is that π is a real number which represents the ratio of the circumference to the diameter of any circle. Its value to five significant digits is 3·1416, but this is only an approximation to π since π is an irrational number and you know that it cannot be represented exactly by any numeral with a finite number of digits. Another approximation to π which is sometimes used is $\frac{22}{7}$.

When you are asked to compute the circumference of a circle whose radius is given use the formula

$$\boxed{C = 2\pi r,}$$

and leave your answer in terms of π unless you are specifically asked to give a numerical approximation. In this case you must be careful not to give an answer to a greater degree of accuracy than you had in the original information.

Thus if you measure a piece of string to be five inches long and use it as a radius to draw a circle, how long a piece of string will it take to measure the circumference? Our formula tells us $C = 2\pi r = 10\pi$.

To approximate π we might use 3 or 3·1 or 3·14 or 3·141 or 3·1416 etc. and in each case we would get a different value for C; i.e., 30 in., 31 in., 31·4 in., 31·41 in., 31·416 in., etc. Which one should we give as our answer?

If you measured the radius very roughly, that is if 5 inches meant somewhere between 4·5 and 5·5, then you have only one significant digit in this factor and the product should have only one: i.e., the answer should be 30 inches. But if you measured the radius carefully with a ruler and really meant that it was between 4·95 and 5·05 inches long you could have said it was 5·0 inches long. Since the number representing this measurement has two significant digits your answer also should have two significant digits and should be given as 31 inches. The general rule is to use the approximation to π with one more significant digit than you want in the answer and then round off the result to the correct number of digits.

PROBLEMS 5-3

1. Find the circumference of each circle if the radius r or diameter d is as given.

a.	$r = 3$	d.	$r = \sqrt{2}$	g.	$d = 6$	j.	$r = \frac{3}{7}$
b.	$r = 5$	e.	$r = 3\sqrt{5}$	h.	$d = 1$	k.	$d = \frac{4}{5}$
c.	$r = 17$	f.	$r = \frac{3}{5}$	i.	$d = 2\sqrt{2}$	l.	$r = 7$

2. Find the radius of each circle if the circumference is

a.	12π	c.	25π	e.	18π	g.	10	i.	22
b.	17π	d.	$\frac{45\pi}{2}$	f.	$\sqrt{2}\pi$	h.	36	j.	45

3. A draftsman draws circles with the following radii. Find the circumference correct to the appropriate number of significant digits in each case.

a.	3·2	c.	8·35	e.	7·2	g.	3·4	i.	·0325
b.	6·0	d.	7·21	f.	3·14	h.	4·32	j.	32·5

4. An equilateral triangle has each side 4 in. long. Find the circumferences of the inscribed and circumscribed circles.

5. A square has each side 6 in. long. Find the circumferences of the inscribed and circumscribed circles.

6. A regular hexagon has each side 10 in. long. Find the circumferences of the inscribed and circumscribed circles.

7. A polygon is circumscribed about a circle of radius r. Prove the area of the polygon is $\frac{rp}{2}$ where p stands for the perimeter.

8. Prove that the area of any regular polygon is $\frac{1}{2}r \times p$, where r is the radius of the inscribed circle and p is the perimeter of the polygon.

5-4 AREAS OF CIRCLES.

We know (see Problem 8 in Section 5-3) that the area of any regular polygon is $\frac{1}{2}r \times p$, where r is the radius of the inscribed circle and p is the perimeter. In Figure 10 below we have drawn a circle and its regular circumscribed pentagon as an illustration.

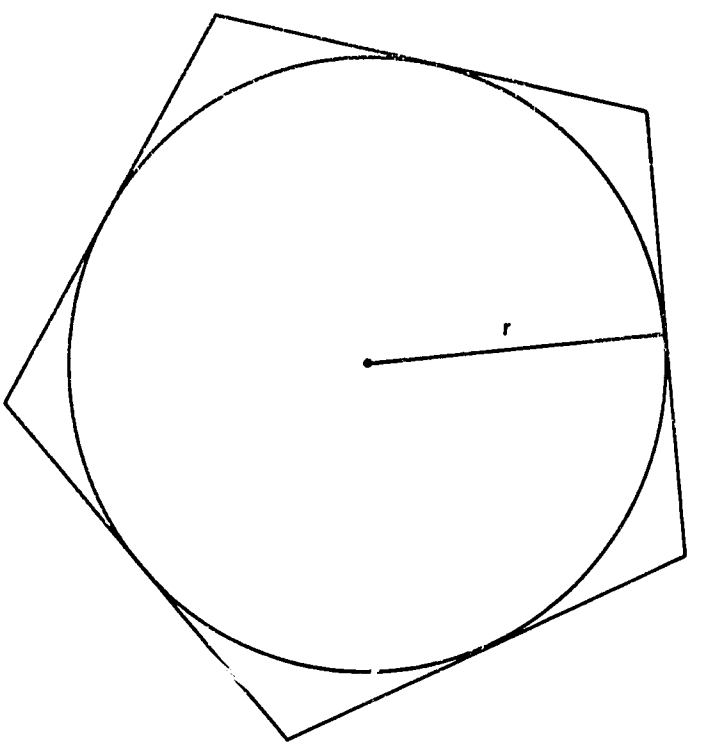


Fig. 10.

The area of the circumscribed pentagon is $\frac{1}{2} r \times P_5$. If we draw the tangents at the midpoints of the arcs we get the regular circumscribed 10-gon. Its area is $\frac{1}{2} r \times P_{10}$.

Let us consider the regular circumscribed n-gons. We know from Section 5-3 that as n increases P_n gets closer and closer to C, the circumference of the circle. It seems reasonable to suppose that the *areas* of the polygons get closer and closer to the area of the region bounded by the inscribed circle. That this is actually the case can be proved by methods which you have not yet studied. Since we do not wish to go into the details of such a proof here, we assume its truth without proof. Thus we have A_n (the area of the regular circumscribed n-gon) approaching A (the area of the inscribed circle) and P_n (the perimeter of the regular circumscribed n-gon) approaching C (the circumference of the circle) for large values of n. Therefore our formula for the area of a regular circumscribed n-gon, $A_n = \frac{1}{2} r \times P_n$, gives a number close to $A = \frac{1}{2} r \times C$ for large values of n. We take this latter formula for the area of a circle.

Since $C = 2\pi r^2$ we find that $A = \frac{1}{2}\, 2\pi r \times r$ or

$$\boxed{A = \pi r^2.}$$

Usually we call A the *area of the circle* even though we really mean the area of the circular region or the area of the region bounded by the circle.

PROBLEMS 5-4

1. Find the area of each circle if

a.	$r = 5$	d.	$r = \sqrt{2}$	g.	$d = 5$	j.	$d = 16$
b.	$r = 7$	e.	$r = 6$	h.	$d = 7$	k.	$d = 6$
c.	$r = \frac{2}{3}$	f.	$r = 12$	i.	$d = \frac{2}{3}$	l.	$d = 12$

2. Find the radius of each circle whose area is

a.	64π	d.	$\frac{9\pi}{4}$	g.	72π	j.	75

b. 121π e. 324π h. 45π k. 100

c. 169π f. 50π i. 88π l. π

3. Find the circumference of the circle whose area is

a. 75 c. 8π e. 1000π g. 324

b. 64π d. 9π f. 125 h. 425

4. Two concentric circles are drawn (Figure 11). The radius of the larger is 6 inches while the radius of the smaller is 4 inches. What is the area of the ring-shaped region between the two circles? How does this area compare to the area of the smaller circle?

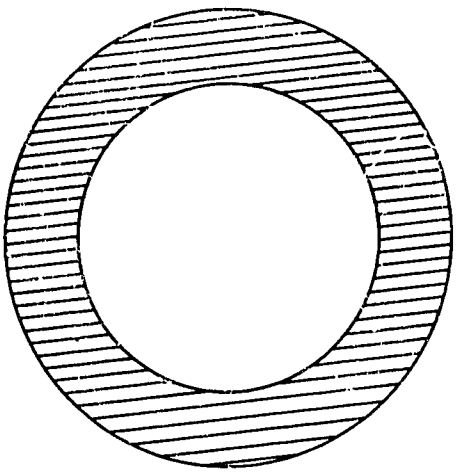


Fig. 11.

5. A circle has radius 2 inches. A second circle is to be drawn whose area is to be just twice the area of the first. What is the radius of the second circle?

6. Two circles have radii whose ratio is $\frac{2}{3}$. What is the ratio of the areas of the circles?

7. The areas of two circles are in the ratio of $\frac{50}{72}$. If the radius of the smaller circle is 15 inches what is the radius of the larger circle?

8. A target for an archer has concentric circles painted on it (Figure 12). If the target is a square four feet on a side and the

circles have diameter 1, 2, 3 and 4 feet, find the areas of each of the regions of the target numbered from 1 to 5.

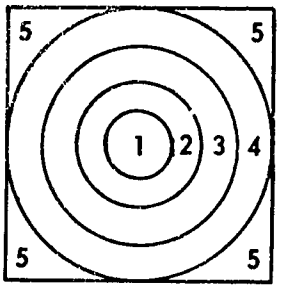


Fig. 12.

9. A target for an archer is a circle of diameter 4 feet. Three smaller, concentric circles are painted on it so that the four resulting areas have equal measure. Determine the radius of each of the three circles.

10. Prove that the ratio of the areas of any two circles is equal to the square of the ratio of their radii.

11. Given a circle whose radius is 10.

    a. What is the area of the region bounded by a semicircle and its diameter?

    b. What is the area of the region bounded by a 90° arc and the two radii drawn to the end points of the arc?

    c. What is the length of the semicircle?

    d. What is the length of the 90° arc?

12. The larger circle in Figure 13 has radius 10 and the two smaller semicircles each have radius 5. The circles are tangent at A, O, and B and $\overline{AB}$ is a diameter. Find the area of the shaded portion of the figure. Find the perimeter of the shaded portion.

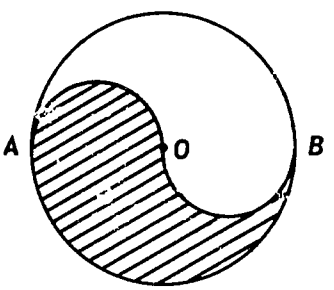


Fig. 13.

Challenge Problem

13. A, B, C, D, E and F are equally spaced on the circumference of a circle of radius 6 (Figure 14). $\overline{AC}$, $\overline{BF}$, $\overline{BD}$ etc. intersect to form a star. Find the area of that portion of the circle outside the star (the area of the shaded part of the figure).

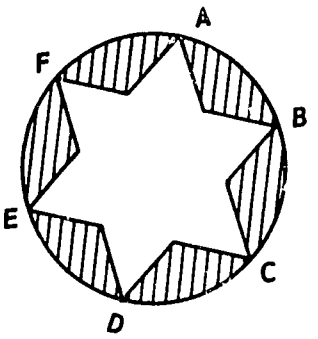


Fig. 14.

5-5 LENGTHS OF ARCS, AREAS OF SECTORS AND SEGMENTS.

DEFINITION 5-2: A *sector of a circle* is a subset of the circular region bounded by an arc of the circle and the radii drawn to the ends of the arc. If the arc is a minor (major) arc the sector is a minor (major) sector.

DEFINITION 5-3: A *segment of a circle* is a subset of the circular region bounded by an arc of the circle and the chord connecting the two end points of the arc. If the arc is a minor (major) arc the segment is a minor (major) segment.

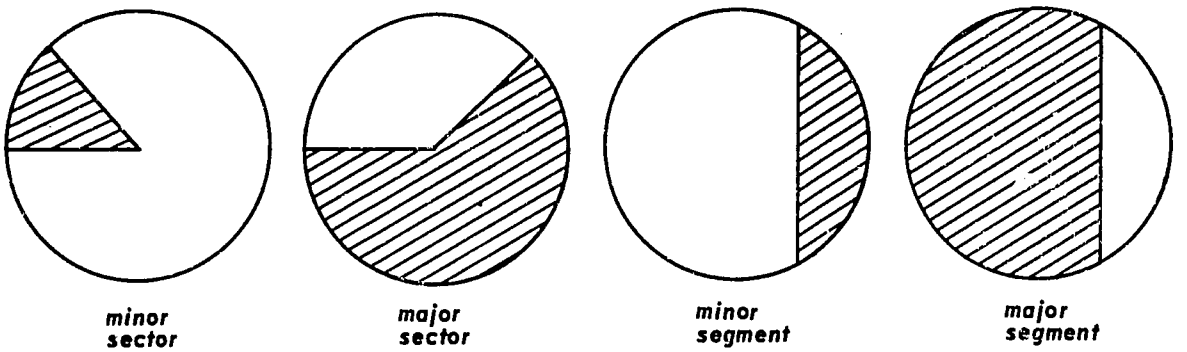


Fig. 15.

Note that the word "segment" here is used in an entirely different sense from that intended when you speak of a "line segment."

How do we find the length of an arc and the area of a sector or a segment of a circle if the radius of the circle and the measure of the arc are given?

The answer depends on two fundamental properties of any kind of measure. The first is that the measures of two congruent figures are the same. The second is the addition property of measure. This means that if, for instance, $\overline{AC}$ is the union of $\overline{AB}$ and $\overline{BC}$, then $AC = AB + BC$, or if $\overrightarrow{BD}$ is in the interior of $A\hat{B}C$ then $m(A\hat{B}C) = m(A\hat{B}D) + m(D\hat{B}C)$. For areas the same property holds: the area of a region composed of two non-overlapping regions is the sum of the areas of the regions.

A diameter subdivides a circle into two congruent arcs and the circular region into two congruent regions (Figure 16). Since $\overset{\frown}{ACB} \equiv \overset{\frown}{ADB}$ their

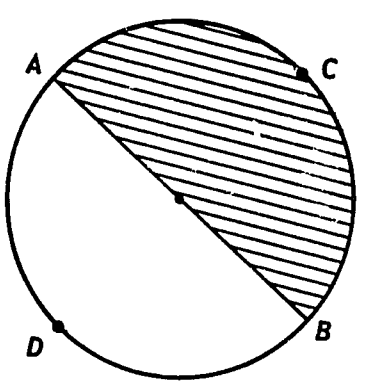


Fig. 16.

lengths must be the same and since together they form the circle the length of each is one half the circumference. That is, the length of $\widehat{ACB} = \frac{1}{2} \times 2\pi r$. Similarly the area of the region bounded by the semicircular arc $\widehat{ACB}$ and the diameter $\overline{AB}$ is half the area of the circle.

Area of the semicircular region $= \frac{1}{2} \times \pi r^2$.

An arc of 60° is one of six possible congruent arcs which together make up the circle. Its length must be one sixth the circumference

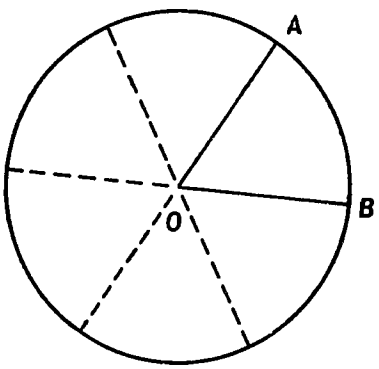


Fig. 17.

of the circle. $\ell(\widehat{AB}) = \frac{1}{6} \times 2\pi r$. The sector AOB is one of six congruent sectors that make up the circle so its area is $\frac{1}{6} \times \pi r^2$.

Similarly if the arc has degree measure 1, then it is one of 360 possible congruent arcs which make up the circle. Its length is $\frac{1}{360} \times 2\pi r$ and the area of its sector $\frac{1}{360} \times \pi r^2$.

If the degree measure of $\widehat{AB}$ is m, where m is an integer, $\widehat{AB}$ can be divided into m congruent arcs of measure 1. By the additive property of measure:

$$\ell(\widehat{AB}) = \frac{m}{360} \times 2\pi r \text{ and area of sector } AOB = \frac{m}{360} \times \pi r^2.$$

If m is an arbitrary real number (not necessarily an integer) between 0 and 360 it is the degree measure of some arc and we define the length of this arc and the area of its sector by these same formulas.

Example 1. In a circle of radius 6 the length of a 45° arc is $\frac{45}{360} \times (2\pi \times 6) = \frac{1}{8} \times 12\pi = \frac{3\pi}{2}$.

Example 2. Find the area of the minor sector and the minor segment determined by a 60° arc in a circle whose diameter is 24. We have $d = 24$ and $r = 12$. In Figure 18 below,

$$\text{Area of sector} = \frac{60}{360} \times \pi \times 144 = 24\pi.$$

$$\begin{aligned}\text{Area of segment} &= \text{Area of sector} - \text{area of } \Delta AOB \\ &= 24\pi - \frac{1}{2} \times 12 \times 6\sqrt{3} = 24\pi - 36\sqrt{3}\end{aligned}$$

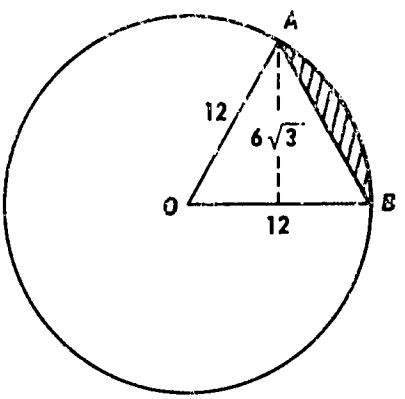


Fig. 18.

PROBLEMS 5-5

1. Find the perimeter and area of a 60° sector in a circle of radius 6. (Perimeter is made up of 2 radii and the arc.)

2. Find the perimeter and area of a 60° segment in a circle of radius 8. (Perimeter is made up of a chord and the arc).

3. Find the area of a 45° sector and a 45° segment in a circle of radius 10. (Hint: To find the area of ΔAOB in Figure 19 drop the altitude from A to $\overline{OB}$).

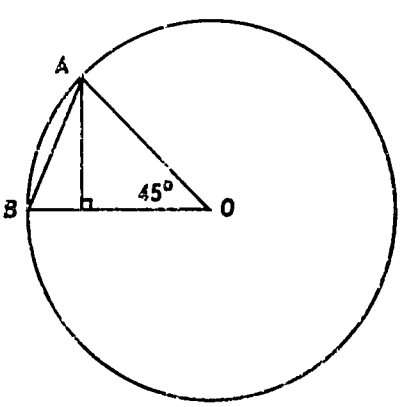


Fig. 19.

4. In a circle of radius 10 a sector has area 40π. Find the degree measure and length of the arc of the sector.

5. Find the areas of a 30° sector and a 30° segment each in a circle of radius 12.

6. Find the areas of a 120° sector and a 120° segment in a circle of radius 9.

7. To find the areas of a major sector or segment determine the areas of the corresponding minor sector or segment and subtract from the area of the circle.

   Find the major sectors and segments determined by the arcs given in Problems 3, 5 and 6.

8. Two circles whose radii are each 12 inches have their centres 12 inches apart. Find the area of the intersection of the two regions. Find the area of the union of the two regions.

9. The area of a 10° sector of a certain circle is equal to the area of the ring between two concentric circles whose radii are 15 and 12. What is the radius of the first circle?

10. A goat is tied by a rope 16 feet long. The other end of the rope is tied to a post at the middle of one side of a shed which is 8 feet square (Figure 20). How many square feet of ground are available for the goat to wander over if he can never get inside the shed?

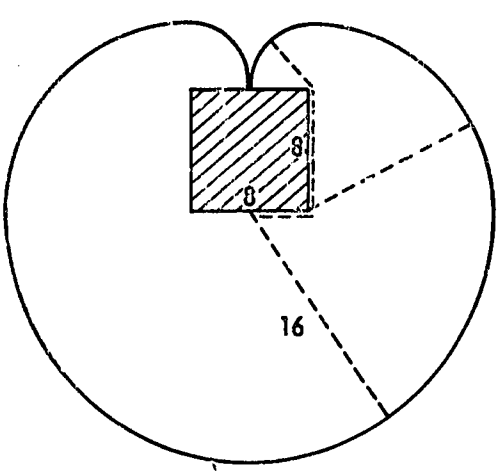


Fig. 20.

11. Three coins are laid on a table next to each other so as to be tangent to each other. If the radius of each coin is 2 cm. what is the area of the region shaded in Figure 21? Hint: Draw the equilateral triangle formed by the centres of the three circles.

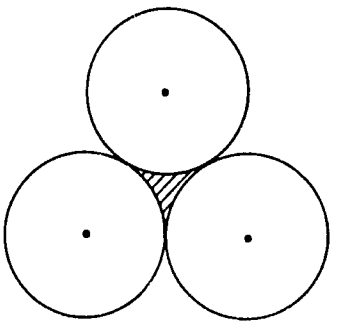


Fig. 21.

12. The larger circle in Figure 22 has radius 8. The radii of the four small circles are equal. Each of them is tangent to the large circle and to two small circles. Find the area of the shaded region. Hint: Draw the square formed by the centres of the four smaller circles. Find the area of the square and subtract from it the areas of the four 90° sectors.

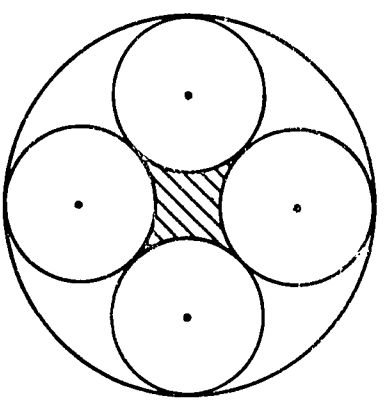


Fig. 22.

5-6 RADIAN MEASURE.

You will remember that several different units are customarily used in measuring a line segment. Sometimes the unit is a foot, sometimes a meter,

sometimes an inch or again a centimeter. However in measuring an angle we have up to now used only a degree as the unit, although at one time we said a more natural unit might be a straight angle. Now that you know what is meant by the length of an arc we can introduce another unit for the measure of an angle. This unit angle is that angle which intercepts on any circle for which it is a central angle an arc whose length is equal to the radius of the circle. If an angle intercepts an arc of length $2r$ its measure in terms of this new unit is 2. If the arc length is some constant k times r the central angle has measure k. Since a right central angle intercepts an arc which is one fourth of the circumference; i.e., $\frac{1}{4} \times 2\pi r = \frac{\pi}{2} r$ the measure of a right angle in terms of this new unit is $\frac{\pi}{2}$.

The unit of angle measure we have defined in this way is called a *radian*. Since the length of a 60° arc is $\frac{1}{6}\, 2\pi r = \frac{\pi}{3} r$ the radian measure of a 60° angle is $\frac{\pi}{3}$. Similarly the length of an arc of m degrees is $\frac{m}{360} \times 2\pi r = \frac{m\pi}{180} r$, so the radian measure of an angle of m degrees is $\frac{m\pi}{180}$.

We can set up a table to help convert degree measure of angles to radian measure.

degree measure		radian measure
90°	$\longleftrightarrow$	$\frac{\pi}{2}$
45	$\longleftrightarrow$	$\frac{\pi}{4}$
60	$\longleftrightarrow$	$\frac{\pi}{3}$
30	$\longleftrightarrow$	$\frac{\pi}{6}$
1	$\longleftrightarrow$	$\frac{\pi}{180}$
m	$\longleftrightarrow$	$\frac{m\pi}{180}$
$\frac{180}{\pi}$	$\longleftrightarrow$	$\frac{180}{\pi} \times \frac{\pi}{180} = 1$
$\frac{\theta \times 180}{\pi}$	$\longleftrightarrow$	θ

To convert from degree measure to radian measure multiply the degree measure by $\frac{\pi}{180}$.

To convert from radian measure to degree measure multiply the radian measure by $\frac{180}{\pi}$.

An angle of one radian is an angle of $\frac{180}{\pi}$ degrees. This is an angle of about 57° 17′ 45″ or 57·3°.

In most work in geometry we will use degree measure for angles but when you study trigonometry again in *Secondary Five* you will find that radian measure of angles becomes very useful and important. Also if you study advanced mathematics in Sixth Form and at the universities you will use radian measure almost exclusively so it is well to begin to learn about this new unit.

PROBLEMS 5-6

1. Find the radian measure of the angles whose degree measures are

a.	30	d.	7	g.	135	j.	36
b.	40	e.	22·5	h.	157·5	k.	72
c.	15	f.	75	i.	179	l.	144

2. Find the degree measure of the angles whose radian measures are

a.	$\frac{\pi}{2}$	d.	2·4	g.	·6	j.	$\frac{5\pi}{9}$
b.	$\frac{\pi}{5}$	e.	$\frac{5\pi}{18}$	h.	1·3	k.	$\frac{7\pi}{8}$
c.	1·5	f.	$\frac{2\pi}{3}$	i.	$\frac{\pi}{7}$	l.	1

3. The radian measure of $\widehat{AB}$ = radian measure of $A\hat{O}B$, where O is the centre of the circle. Find the length of the arc $\widehat{AB}$ if the radian measure of $\widehat{AB}$ and the radius of the circle are given.

	radian $m(\widehat{AB})$	radius
a)	$\frac{\pi}{6}$	6
b)	$\frac{\pi}{3}$	8
c)	$\frac{\pi}{10}$	10
d)	$\frac{\pi}{12}$	8
e)	2	3
f)	·35	2·6
g)	·35	·35

	radian $m(AB)$	radius
h)	1	4
i)	1·5	6
j)	$\frac{1}{2}$	4

4. An inscribed angle $A\hat{B}C$ has radian measure $\frac{\pi}{5}$ in a circle whose radius is 12. Determine the length of $\widehat{ABC}$.

5. Two lighthouses are situated 10 miles apart. A ship at sea measures the angle between the lines of sight to the two lighthouses as being equal to $\frac{\pi}{6}$ radians. Determine the possible position of the ship in relation to the lighthouses.

Chapter 6
MOTIONS AND TRANSFORMATIONS

6-1 FIGURES WITH THE SAME SIZE AND SHAPE.

Look at the following triangles. Do they have the same size and shape?

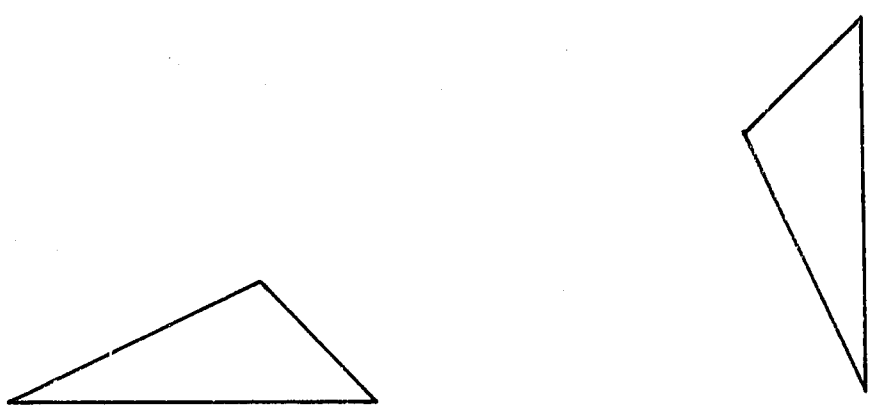


Fig. 1.

One way to test whether or not they have the same size and shape is to measure the sides of each triangle with a ruler and the angles of each triangle with a protractor. If you obtain the same measurements for each triangle, then, as far as you can tell by your measurements, the triangles do have the same size and shape.

In work on geometry that you have already done, you learned that if you measure the sides (but not the angles) of two triangles, this is enough.

If the measurements of the sides for one triangle are the same as the measurements of the sides for the other triangle, then the measurements of the angles for one triangle must also be the same as for the other triangle. Can you state the theorem which showed this?

Using dividers or a pair of compasses to compare sides, test whether the following two triangles have the same size and shape.

Fig. 2.

Look at the following circles. Do they have the same size and shape?

Fig. 3.

One way to test whether or not they have the same size and shape is first to find the centre of each circle and then to measure the radius of each circle. If the measurements are the same, then the two circles have the same size and shape.

Using compasses and a ruler, test whether or not the following two circles have the same size and shape.

Fig. 4.

(Remember to use the construction for finding the centre of a circle which you learned in Chapter 4.)

In our earlier work, if two triangles had the same size and shape we said that the two triangles were *congruent*. If two circles had the same size and shape, we said that the two circles were *congruent*.

Now look at the following two figures. Do these figures have the same size and shape?

Fig. 5.

Clearly there are no simple measurements with ruler and compasses that you can use to answer this question. Think hard and decide what you would do if you had to get an answer.

One good way to get an answer is the following. Take a piece of tracing paper (any thin paper that you can see through a little bit will do),

and place this paper over figure A. Then, holding this paper so that it does not move while you are writing on it, carefully trace, with your pencil, an outline of figure A. Next, move the paper over to figure B and try to find a position where the traced figure falls exactly on top of figure B. If you succeed, then you will know that, as far as you can tell by your drawings, the two figures do indeed have the same size and shape.

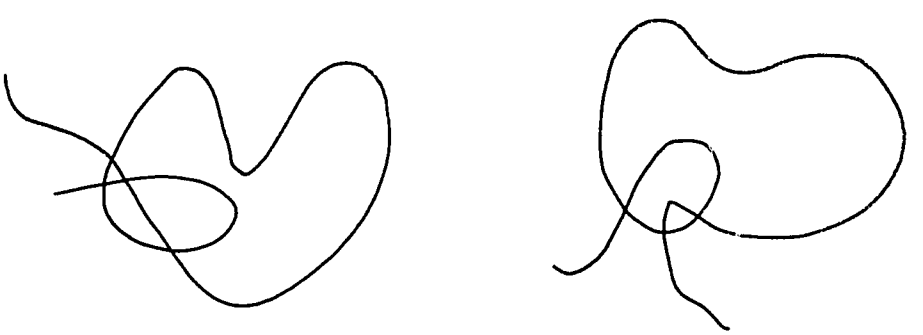
Do the above figures have the same size and shape?

CLASS ACTIVITY

For each of the following pairs of figures, test whether or not the two figures have the same size and shape.

a.

b.



c.

Fig. 6.

Consider the following pair of figures.

Fig. 7.

Make a tracing of figure A above. Now compare this tracing with figure B. You will find that you cannot make the tracing of A fall exactly on B. Now turn the tracing paper over so that the tracing of A is on the bottom. Place this over B and, looking through the paper, see if you can make the tracing of A fall on B. You will find that the tracing can now be made to fall exactly on B.

Shall we say that A and B have the same size and shape? In geometry, mathematicians long ago agreed to use the words "same size and shape" in such a case. We therefore do say that A and B have the same size and shape.

In general, we say that two figures in the plane have the *same size and shape* if a tracing of one can be made to fall exactly on the other, where we may turn the tracing over if needed.

PROBLEMS 6-1

1. Do the following figures have the same size and shape?

a.

b.

c.

Fig. 8.

2. For each of the figures in Figure 9, make a tracing of the first figure. For how many distinct positions of the tracing paper can you make the tracing fall exactly on the second figure? (Include positions where the tracing paper is turned over.)

a.

b.

c.

d.

e.

Fig. 9.

6-2 MOVING A FIGURE IN THE PLANE.

By using tracing paper, we can take a figure in the plane and make a copy of it at a new position in the plane. Here is an example.

Consider the following picture.

Fig. 10.

Make a copy of this picture on your own paper, and call the plane of the picture **P**. We are now going to make a copy of the triangle ABC at a new position in **P**. We shall place this copy of the triangle ABC so that point C falls on point F and line segment $\overline{AC}$ falls on the line of segment $\overline{EF}$. We do this as follows. Take a piece of tracing paper. Call this paper **T**. Place paper **T** on top of **P** and make a copy of triangle ABC on paper **T**. Next, move paper **T** so that point C of paper **T** falls above point F of plane **P**, and line segment $\overline{AC}$ of paper **T** falls along line segment $\overline{EF}$ of **P**. Keeping the paper in this position, copy triangle ABC from paper **T** onto **P**. This may be done by using a point of your compasses or of your dividers to prick through from paper **T** to **P** at points A, B, and C, of paper **T**, and then using a ruler to draw the line segments between points on **P**. Plane **P** will now look like this.

Fig. 11.

Here we have used A', B', and C' to name the vertices of the triangle in its new position.

When we have done all this, we say that we have *moved* the triangle to a new position *in the plane of* **P**. There are now two triangles in **P**: triangle ABC and triangle $A'B'C'$. In such a case, we shall often think of these as the *same* triangle in two different *positions*.

PROBLEMS 6-2A

1. Copy the following picture. In this picture, line L is parallel to $\overline{AD}$. Use tracing paper to move the rectangle $ABCD$ so that point D falls on point E of the circle and each side of the rectangle in its new position is parallel to its first position.

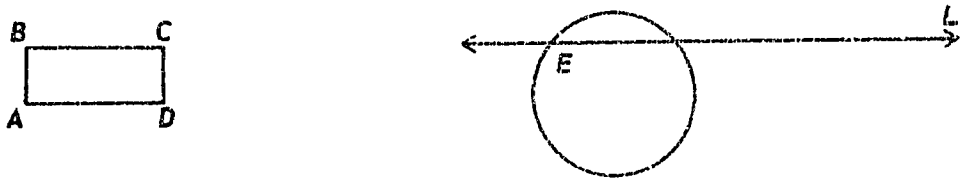


Fig. 12.

2. In the same way, in the following picture, move the triangle ABC so that the point C falls on its original position and the segment $\overline{CB}$ falls along the line of segment $\overline{CD}$.

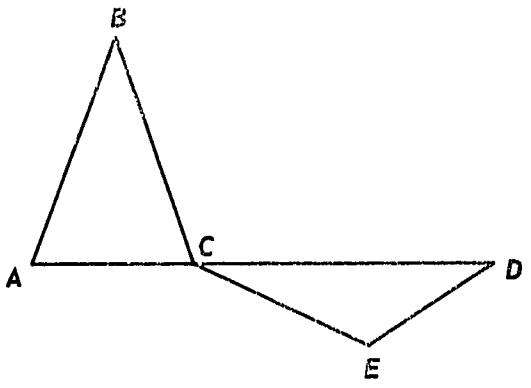


Fig. 13.

3. In the same way, in the following picture, move triangle ABC so that segment $\overline{AB}$ falls along segment $\overline{DE}$, point A falls on point F, and triangle ABC falls entirely inside the circle.

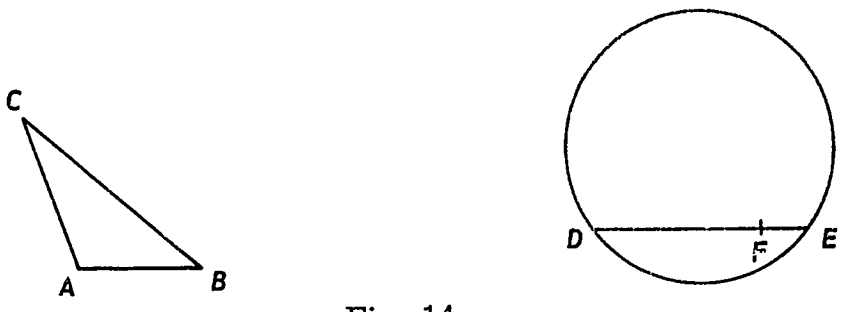


Fig. 14.

(Note that you will have to turn the tracing paper over before you copy the triangle back onto the picture.)

4. Show that Problem 1 has four different answers and that Problem 2 has four different answers.

Consider the following figure,

Fig. 15.

and the movement of the tracing paper which carries A to C and $\overline{AB}$ onto the line of $\overline{CD}$. This gives Figure 16 below.

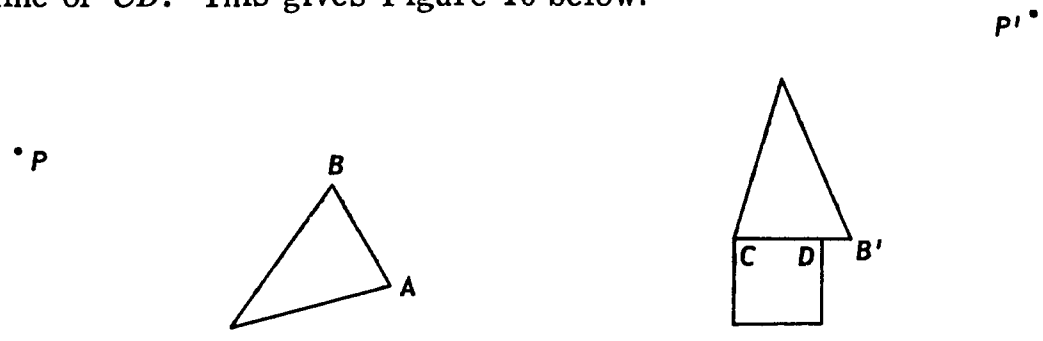


Fig. 16.

Now take any point in the original figure. Call it P. The same movement of the tracing paper which carries the triangle to the top of the square also carries the point P to a new position P'. We can get P' by marking P on the tracing paper at the same time that we copy the triangle and then pricking through, after we have moved the tracing paper.

The following picture shows the result for several different points P_1, P_2, P_3. P_1' is the new position of P_1; P_2' is the new position of P_2; and P_3' is the new position of P_3.

Note that in this movement of the tracing paper, some points may move farther than others.

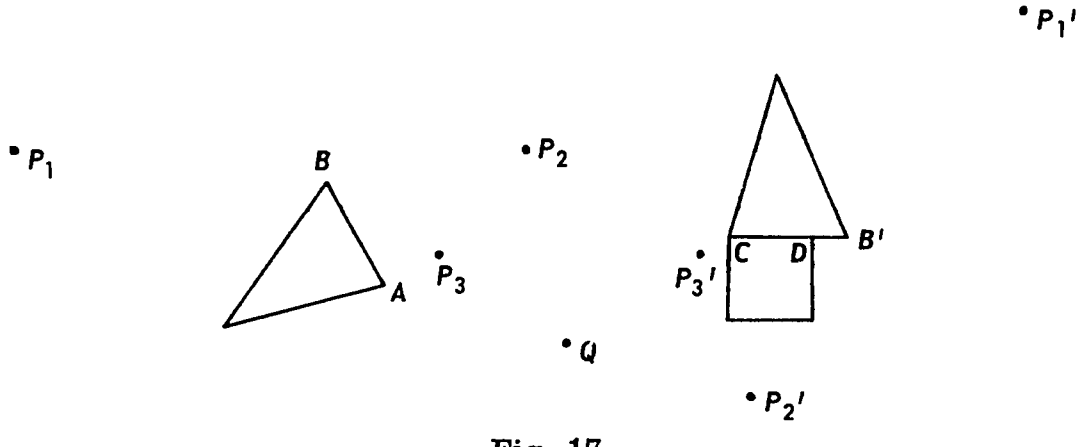


Fig. 17.

(Indeed, we can see that there is one point which does not get moved at all. Its new position is the same as its old position. This point is marked Q in Figure 17.)

Sometimes a movement of the tracing paper will move the individual points in a figure, but leave the final form and position of the entire figure unchanged. For example, if we add the following equilateral triangle to Figure 17,

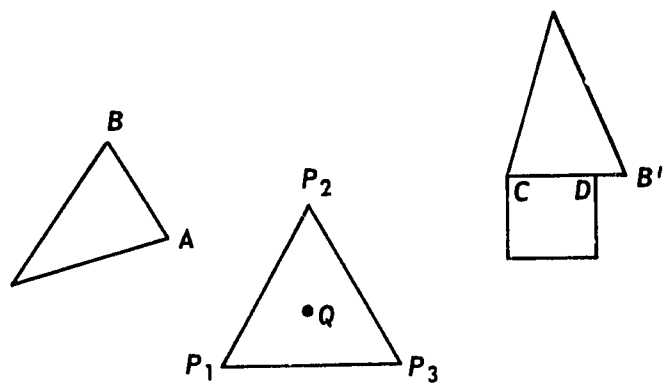


Fig. 18.

then the movement described before gives the following result.

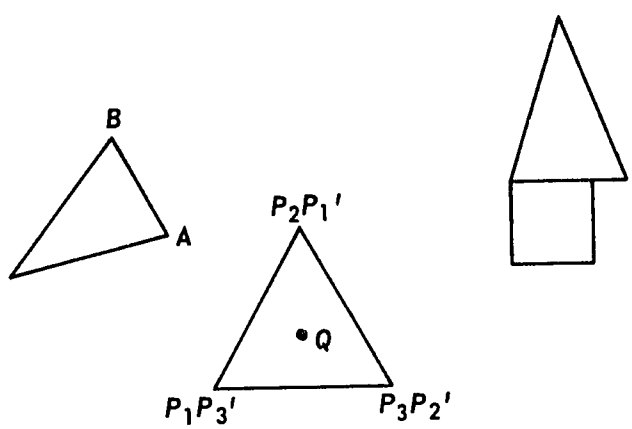


Fig. 19.

Here the new position of P_1 is the same as the old position of P_2, the new position of P_2 is the same as the old position of P_3, and the new position of P_3 is the same as the old position of P_1. Each point of the triangle has moved, but the whole triangle in its new position has come out exactly on top of its old position.

If, in a movement, a point (like the point Q in the example above) has its new position the same as its old position, we say that it is a *fixed-point* of that movement.

If, in a movement, a figure (like the triangle $P_1P_2P_3$ in the example above) has its new position fall exactly on its old position, we say that the figure is *invariant* under the movement.

PROBLEMS 6-2B

1. Consider the movement which carries ΔABC to $\Delta A'B'C'$ in the following picture.

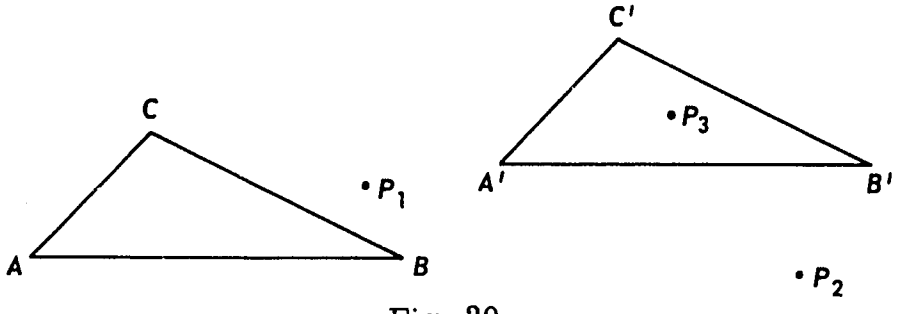


Fig. 20.

Where does it carry points P_1, P_2, and P_3? Does this movement have any fixed points?

2. Consider the movement which carries ΔABC *to* $\Delta A'B'C'$.

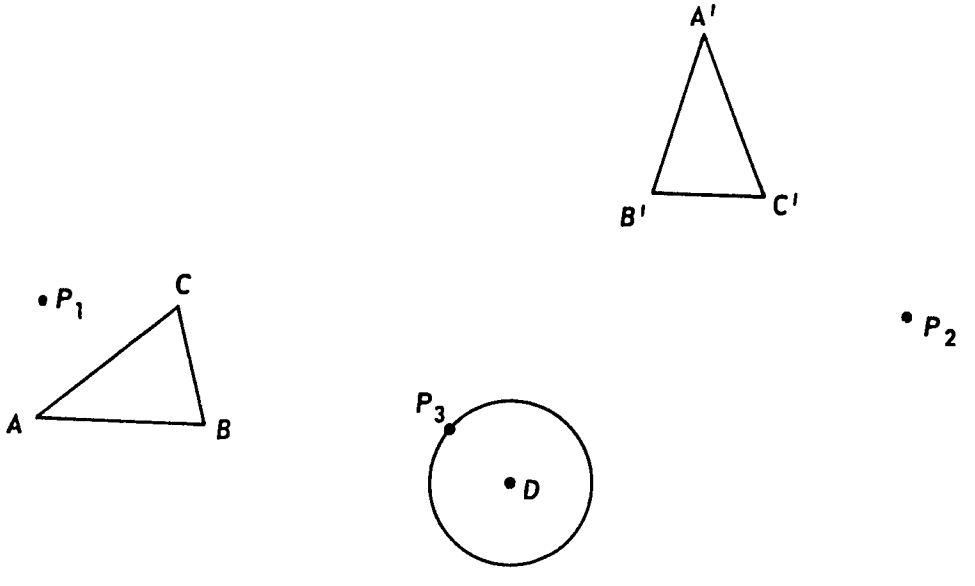


Fig. 21.

What does it do to the point P_1? To the point P_2? To the point P_3? To the circle D?

3. Consider the movement which carries ΔABC to $\Delta A'B'C'$ in the following picture.

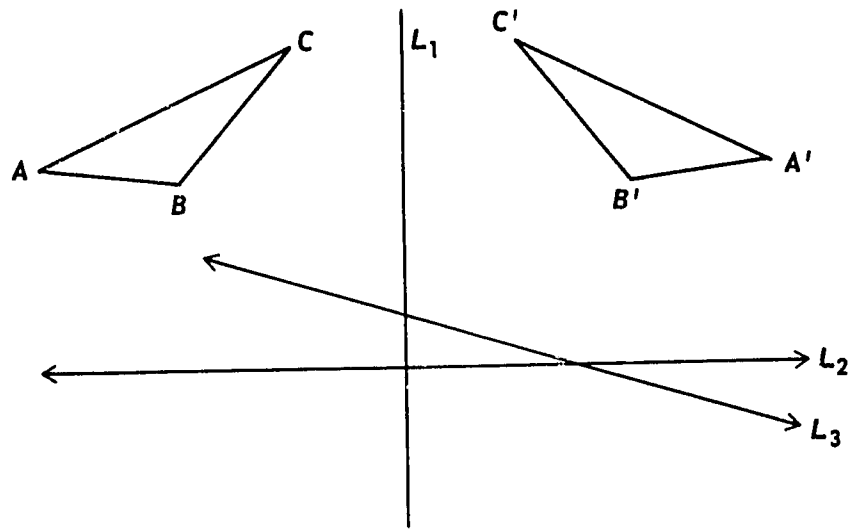


Fig. 22.

Where does it carry the line L_1? Where does it carry the line L_2? Where does it carry the line L_3? (Note: you will have to turn your tracing paper over to obtain this movement.)

Challenge Problem

4. a. In Problem 3, what are the fixed points of the movement?
   b. For the movement of Problem 3, describe all lines that are invariant under the movement.

5. How many distinct motions carry triangle A onto triangle B?

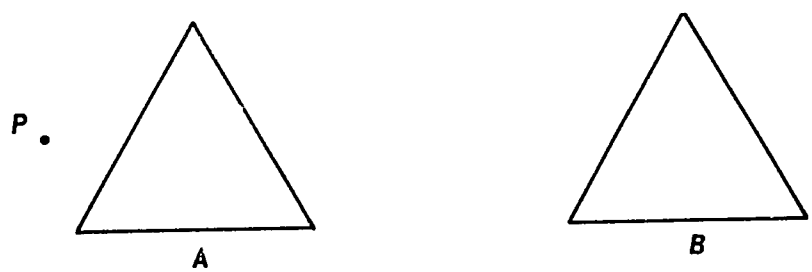


Fig. 23.

What does each of these movements do to the point P?

USING RULER AND COMPASSES TO MOVE FIGURES AND POINTS

If we wish to move a figure like either of the two below,

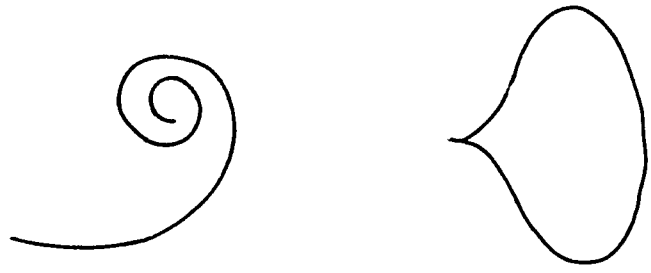


Fig. 24.

we must use tracing paper and prick through many points of the figure. But if we have a figure made up entirely of straight line segments and circular arcs, we can move it by ruler and compasses without using tracing paper at all. Our earlier work in geometry shows us how to do this. For example, in the following picture (which is the same as Figure 10 in this section)

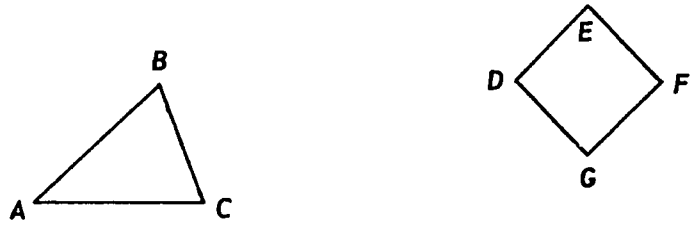


Fig. 25.

we can move triangle ABC so that C falls on F and $\overline{AC}$ falls on the line of $\overline{EF}$ as follows. Extend segment $\overline{EF}$. Measure distance AC and mark off an equal distance from F on $\overline{EF}$ extended. This gives a point A' on $\overline{EF}$ extended. Draw an arc with centre F and radius BC. Draw an arc with centre A' and radius AB. Take the intersection of these arcs as B'. Draw the segments $\overline{A'B'}$ and $\overline{FB'}$. When you finish, your figure should look like this.

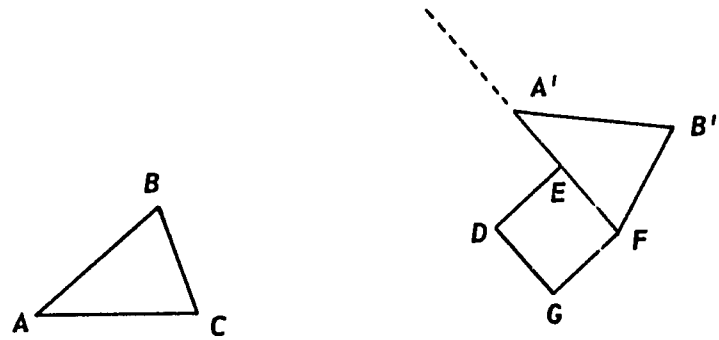


Fig. 26.

Here is another example. Move the line segment $\overline{AB}$ parallel to itself so that point A falls on point C in the following picture.

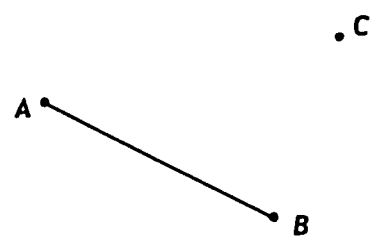


Fig. 27.

To solve this, draw segment $\overline{AC}$ and extend it beyond C to E. Then use compasses to obtain point D so that $D\hat{C}E$ equals $B\hat{A}C$ in measure. Draw $\overline{CD}$ and locate B' on its extension so that $CB' = AB$. When you are finished your figure will look like this.

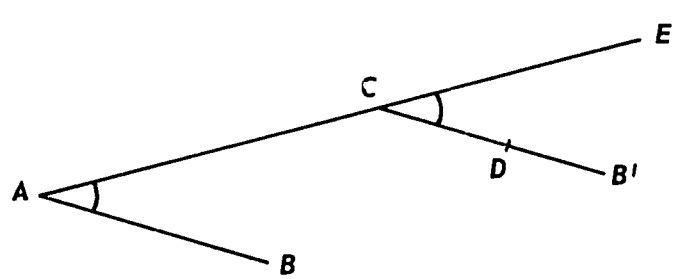


Fig. 28.

$\overline{CB'}$ is the new position of $\overline{AB}$ that we wanted.

Let ABC be a triangle. Assume that a certain movement of the tracing paper carries A to A', B to B', and C to C'. If we know the positions of ΔABC and $\Delta A'B'C'$, and if we are given any point P, we can then use

compasses to find where the same movement carries P. The construction is very simple. (See Figures 29 and 30 below).

With B' as centre, draw a circle of radius BP. With A' as centre, draw a circle of radius AP. If P lies on $\overline{AB}$, then these circles intersect at one point, and we take this point of intersection for the new position of P.

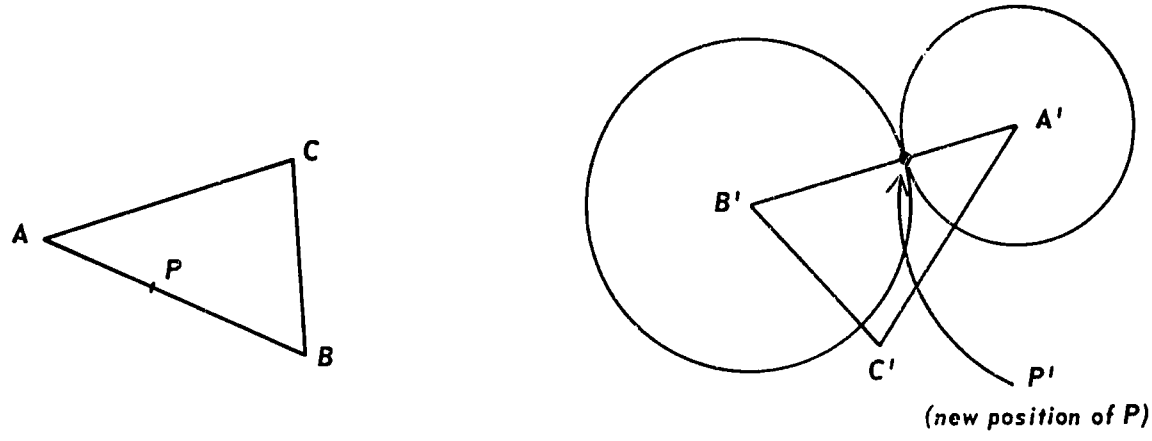


Fig. 29.

If P does not lie on $\overline{AB}$, then these circles intersect at two points which lie on opposite sides of $\overline{A'B'}$. In this case, see if P lies on the same side of $\overline{AB}$ as C. If so, choose, as the new position of P, that point of intersection which lies on the same side of $\overline{A'B'}$ as C'. If not, choose that point of intersection which lies on the opposite side of $\overline{A'B'}$ from C'.

The following picture shows this construction for the case where P is not on the same side of $\overline{AB}$ as C.

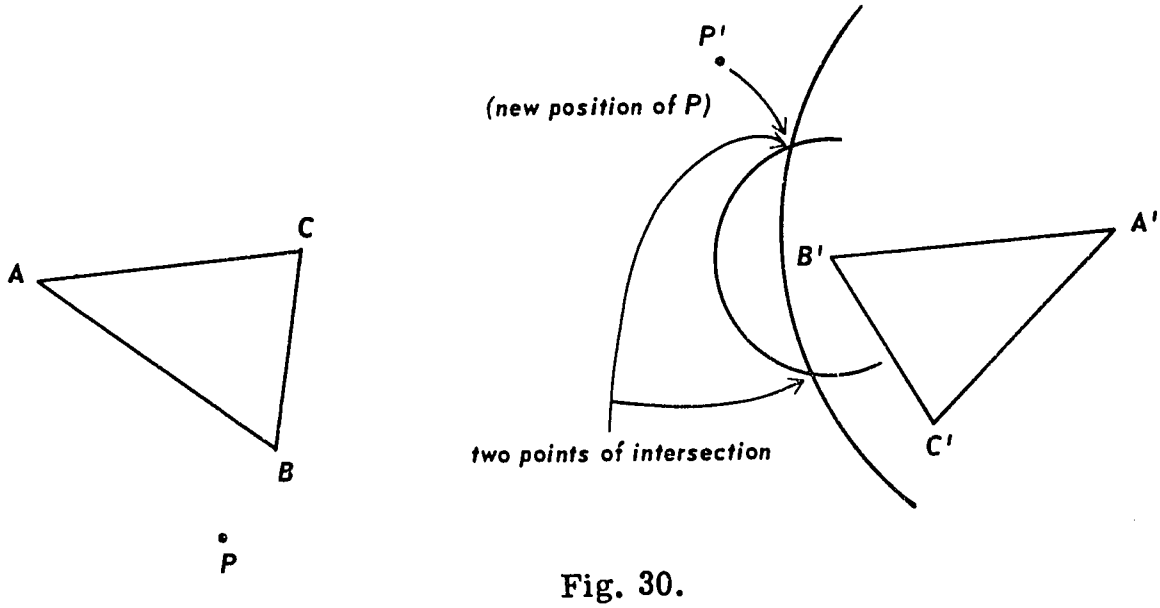


Fig. 30.

The use of ruler and compasses to move a figure or point often gives a more accurate result than the use of tracing paper. In the rest of this chapter we are going to study some of the things that happen when we move figures and points in the plane. We shall find that if we move a figure, or part of a figure, this can often help us to find new facts about the original figure. Often, in order to carry out a proof, it will be enough to imagine the movement and then make a rough sketch of the result, without using tracing paper or doing all the details of a ruler and compasses construction.

PROBLEMS 6-2C

1. Use ruler and compasses to do Problem 2 in Problems 6-2A.

2. Do the same for Problem 3 in Problems 6-2A.

3. Do the same for Problem 1 in Problems 6-2A.

4. In the following figure, a certain movement of the tracing paper carries $\triangle ABC$ to $\triangle A'B'C'$. Use compasses to find where this movement carries the points D, E, and F.

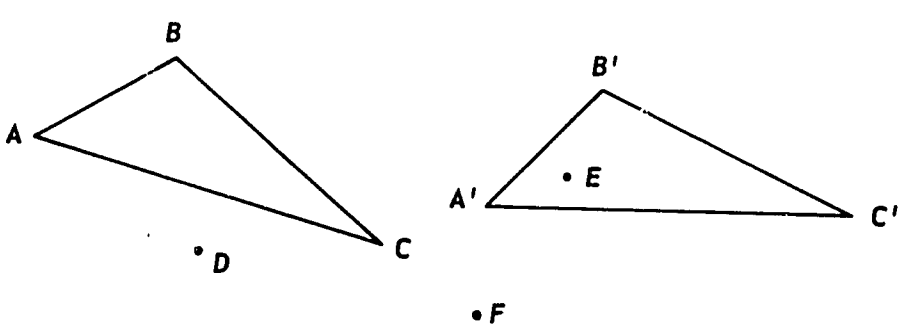


Fig. 31.

Where does this movement carry the triangle whose vertices are D, E, and F?

5. a. If a movement carries point A to point A' in the following picture, what are all the possible new positions of point P?

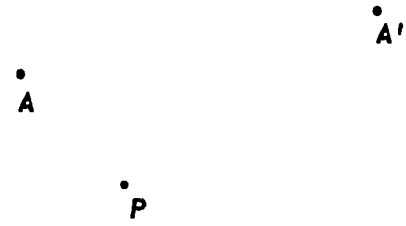


Fig. 32.

b. If a movement carries A to A' and B to B', what are all the possible new positions of the point P?

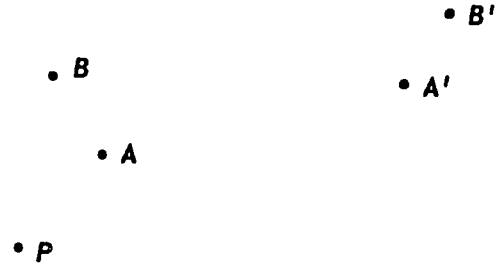


Fig. 33.

c. If a movement carries A to A', B to B', and C to C', what are all the possible new positions of the point P?

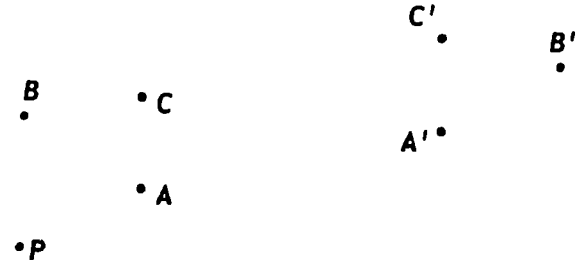


Fig. 34.

Challenge Problem

6. State a general rule which covers the cases described in Problem 5.

6-3 RIGID MOTIONS.

In the last section, we moved figures and points in the plane by sliding a piece of tracing paper to a new position. Let us now imagine a large (in fact infinite) piece of tracing paper on top of the plane. By copying from the plane onto the tracing paper and then sliding the tracing paper to a new position (or possibly turning it over and sliding it) we can move a figure in the plane. Indeed, given a change in position of the tracing paper, we can see where, in the plane, that change of position carries *any* given point, line, or figure that we are interested in. Such a change in position over the entire plane is called a *rigid motion*. Given a rigid motion and given a point P, the new position of P is called the *image* of the old position of P. Given a figure, the new position of the figure is called the *image* of the old position of that figure. If P' is the image of a point P under a given rigid motion, we sometimes say that the motion *maps* P to P'.

A rigid motion in geometry is much like a function in algebra. Just as a function on the real numbers is a way of assigning to every real number x a real number y called the *value* of the function at x, so a rigid motion T is a way of assigning to every point P in the plane a point P' called the *image* of P under the rigid motion. In algebra, if f is a function and x is a real number, we sometimes write the value of f at x as $f(x)$. Similarly, in geometry, if T is a rigid motion and P is a point, we sometimes write the image of P under T as $T(P)$. This notation gives us a short and useful way to say things about rigid motions. For example we can now restate the definition of *fixed-point* as follows.

P is a fixed-point of T if and only if $T(P) = P$.

We have spoken of a rigid motion as a change in position of an imaginary piece of tracing paper, and we have talked about *movement* of this paper. It would be better to speak of a rigid motion simply as a *new position* of the entire piece of tracing paper. In our study of geometry it does not matter what path we follow during the actual movement of the paper. All that matters is the final position of the paper. Thus we see that a rigid motion is, in fact, a function (of a certain kind) from the plane into itself.

What if we take our infinite tracing paper and do not move it at all? In this case, the new position of the infinite sheet of tracing paper is the same as the old position. Since there is no *change* in position, it is natural to ask whether or not we wish to call this function from the plane into the plane (where, for every point P, the image of P is P itself) a rigid motion. We shall find it useful to call this function a rigid motion, even though no change in position has occurred. This rigid motion is called the *identity motion*, and we often write it as I. Thus we have, in our notation:

$$\text{for any point } P, \quad I(P) = P.$$

(Hence every point is a fixed point of the identity motion.)

We now list several important facts about rigid motions. These facts hold for all rigid motions. Since they hold for all rigid motions, we say that they are *properties* of rigid motions. Let T be a rigid motion defined by the change in position of an infinite piece of tracing paper.

(1) If P and Q are points in the plane, if $P' = T(P)$, and if $Q' = T(Q)$, then the distance PQ is the same as the distance $P'Q'$.

(2) If P and Q are distinct points, if $P' = T(P)$, and if $Q' = T(Q)$, then P' is distinct from Q'.

(3) For every point Q there is a point P such that $Q = T(P)$. That is to say, every point is an image of some point.

(4) The image of any straight line is a straight line.

(5) If P, Q, and R are distinct points, and if $P' = T(P)$, $Q' = T(Q)$, and $R' = T(R)$, then $P'\widehat{Q'}R'$ is the image of $P\widehat{Q}R$ and $P'\widehat{Q'}R' = P\widehat{Q}R$ in measure.

(6) If two straight lines are parallel, then their images are parallel.

These six properties are obvious from the tracing paper idea of a rigid motion. Consider Property (6) for example. If two lines are parallel, then their copies on the tracing paper are parallel, and these copies remain parallel as we move the tracing paper. Hence the images of the lines must be parallel.

It is an interesting and surprising fact that, if we assume Property (1) for rigid motions (in other words, if we take Property (1) as a postulate), then we can prove Properties (2), (3), (4), (5), and (6) as theorems. We can do this *without using the idea of tracing paper at all.* These proofs will not be given here as part of your geometry course. If you are interested in looking at these proofs, you will find them in the appendix at the end of this chapter.

Since Property (1) is so basic, we give it a special name. It is called the *isometric property*. If a function from the plane into the plane has Property (1), we say that the function is *isometric*.

If a function is isometric, must there be a change in position of our infinite tracing paper which will give us the function? We shall see, in Section 6-6, that the answer to this question is *yes*. The idea of an *isometric function* and the idea of a *rigid motion* (as given by change in position of tracing paper) are therefore completely equivalent. Hence we could, if we wished, study rigid motions without using the idea of tracing paper at all. We would begin with the following formal definition.

> DEFINITION 6-1. A *rigid motion* is a function from the plane to the plane which is isometric.

We could then base our whole study on this formal definition. We will not do this here. In our work on rigid motions we will keep on using the tracing paper idea to help us to understand important facts and constructions.

A function from the plane into itself is often called a *mapping* on the plane. In geometry, this word "mapping" has exactly the same meaning as the word "function". As we have seen, if a mapping has the isometric property, it is called a *rigid motion*.

If a mapping has both Property (3) and Property (4) it is called a *transformation* on the plane. We have seen above that every rigid motion is a transformation. In Problem 6 below, we shall give an example of a transformation which is not a rigid motion.

In the rest of this chapter we shall study rigid motions in the plane. In later work in mathematics, we shall study other kinds of transformations as well. Transformations are important in both algebra and geometry.

In this chapter, we consider rigid motions of the plane onto itself. Rigid motions can also be used in space. In space, we can no longer use the tracing paper idea, but the formal definition of rigid motion can be given exactly as above. That is to say, rigid motion is a mapping from space into space with the property that for any points P_1 and P_2, if P_1' is the image of P_1 and P_2' is the image of P_2, then $P_1'P_2' = P_1P_2$. In what follows, our main study will be of rigid motions in the plane.

PROBLEMS 6-3

1. Let T be the rigid motion which takes $\Delta P_1P_2P_3$ to $\Delta P_1'P_2'P_3'$ in the following figure. Find the images of the points Q_1, Q_2, and Q_3 under T. What is the image of the square S under T?

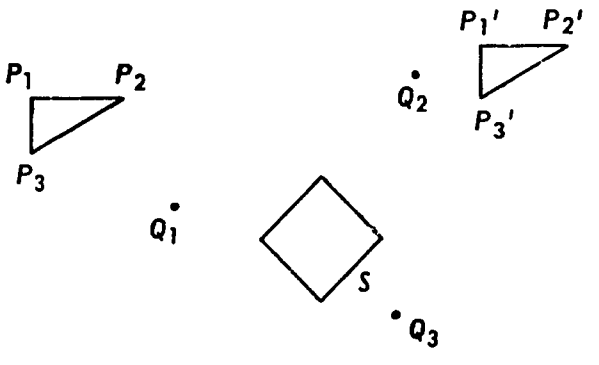


Fig. 35.

2. In the figure of Problem 1, what point has Q_1 as its image under T? Draw the triangle that has $\Delta P_1P_2P_3$ as its image under T.

3. Let T be the rigid motion which takes $\Delta P_1 P_2 P_3$ into $\Delta P_1' P_2' P_3'$ in the following figure.

Fig. 36.

What is the image of $\Delta P_1' P_2' P_3'$ under T? Can you draw the triangle which has $\Delta P_1 P_2 P_3$ as its image?

Challenge Problem

4. Let T be the rigid motion which takes $\Delta P_1 P_2 P_3$ into $\Delta P_1' P_2' P_3'$ in the following figure.

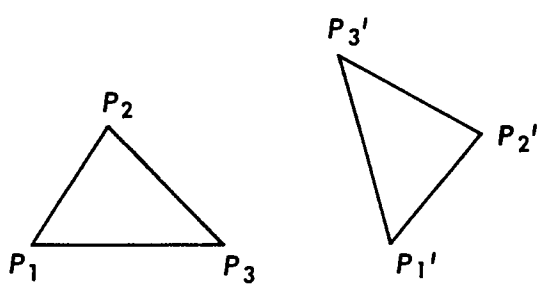


Fig. 37.

What is the image of $\Delta P_1' P_2' P_3'$ under T? Can you draw the triangle which has $\Delta P_1 P_2 P_3$ as its image?

5. Tell which of the following statements are true and which are false.
   (a) Every rigid motion has a fixed point.
   (b) There exists a rigid motion such that the image of each straight line is either identical with that line or parallel to it.
   (c) In every rigid motion there is some straight line which is parallel to its image.
   (d) In every rigid motion there is a line which is its own image.
   (e) There is a rigid motion which has no fixed point, but has a line which is its own image.
   (f) There is a rigid motion that has no fixed point, but which has a triangle that is its own image.

Challenge Problem

6. There are six different rigid motions which carry ΔABC to ΔDEF in the following figure.

Fig. 38.

Locate the fixed points, if any, for each of these rigid motions.

7. Let L be a given line. We define a certain mapping T by showing, for any point P, how to find $T(P)$.
Let P be given. If P is on L, take $T(P)$ to be P itself. If P is not on L, drop a perpendicular from P to L, and locate a point on the perpendicular which is on the same side of L and twice as far from L as the point P. Take this new point to be $T(P)$.

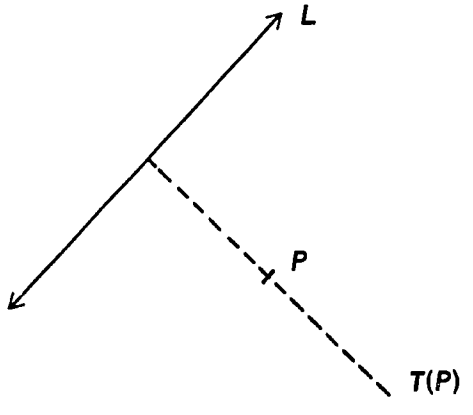


Fig. 39.

Show that this mapping is not a rigid motion. (Hint: show that this mapping is not isometric by finding two points P and Q such that

$PQ \neq P'Q'$, where $P' = T(P)$ and $Q' = T(Q)$. It then follows that T cannot be a rigid motion.)

NOTE: Tracing paper cannot be used to get this mapping. We can get it if, in place of tracing paper, we use a thin rubber sheet that can be stretched after we trace onto it.

Challenge Problem

8. Let L be a given line. We define a certain mapping T by showing, for any point P, how to find $T(P)$.

Let P be given. If P is on L, take $T(P)$ to be P itself. If P is not on L, drop a perpendicular from P to L, extend the perpendicular beyond L, and locate a point on the perpendicular which is on the opposite side of L from P and at the same distance from L as P. Take this new point to be $T(P)$.

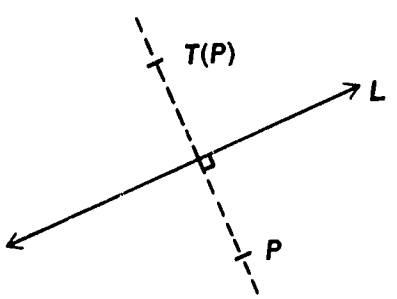


Fig. 40.

Show that this mapping is a rigid motion.

(Hint: show that the mapping is isometric by showing that for any two points P and Q, $PQ = P'Q'$, where $P' = T(P)$ and $Q' = T(Q)$. It then follows that T must be a rigid motion.)

How can we move our tracing paper to get the mapping T?

Challenge Problem

9. Let O be a given point in the plane. Then for every point P in the plane, we can locate a point P' (which will depend on P) as follows.

If P is O, we take P' to be O. If P is not O, we draw a line from O through P and extend it beyond P. We then find a point P' on this line such that $OP = PP'$.

This procedure defines a mapping T such that for any P, $T(P) = P'$. Show that T is a transformation but not a rigid motion. (Hint: use similar triangles for proving Property (4).)

NOTE: As in Problem 7, tracing paper cannot be used to get this transformation, but we can get it if, in place of tracing paper, we use a thin rubber sheet that can be stretched after we trace onto it. The transformation defined in Problem 9 also has Property (5). (Can you prove this?) If a transformation has Property (5), it is called a *motion*. Hence T, in Problem 9, is a motion which is not a rigid motion. Problem 7 gives an example of a transformation which is not a motion. See Problems 10 and 11.

10. Show that T in Problem 7 does not have Property (5).

Challenge Problem

11. Show that T in Problem 7 has Properties (3) and (4). (T is therefore a transformation.)

Challenge Problem

12. Prove that every transformation has Property (2) and Property (6).

6-4 USING MOTIONS TO SOLVE PROBLEMS.

There are many ways in which rigid motions can be used to help solve problems in geometry. We shall look at some of these in later sections. We give two examples here.

Example 1

The following figure shows a map of a river and a road. The figure also shows a scale of distance. It is desired to build a factory at a point on the river that is within two kilometers of the road. What points of the river can be used?

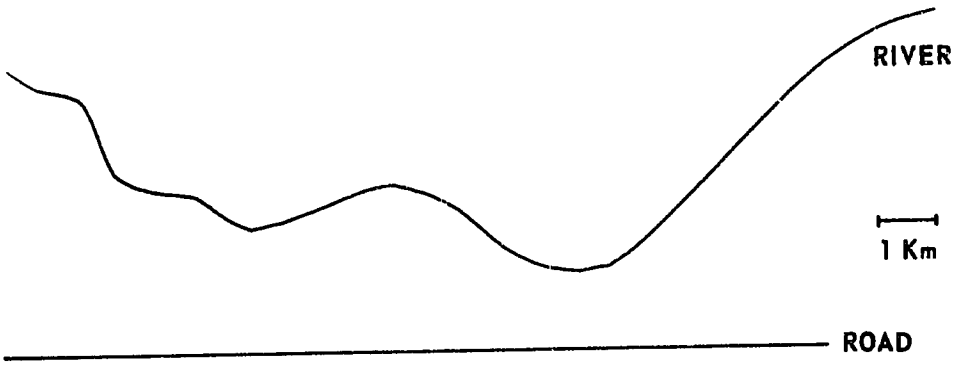


Fig. 41.

Solution to Example 1

The road is a straight line. Use the rigid motion which moves this straight line parallel to itself towards the river for a distance of two kilometres (on the scale). This gives the following figure.

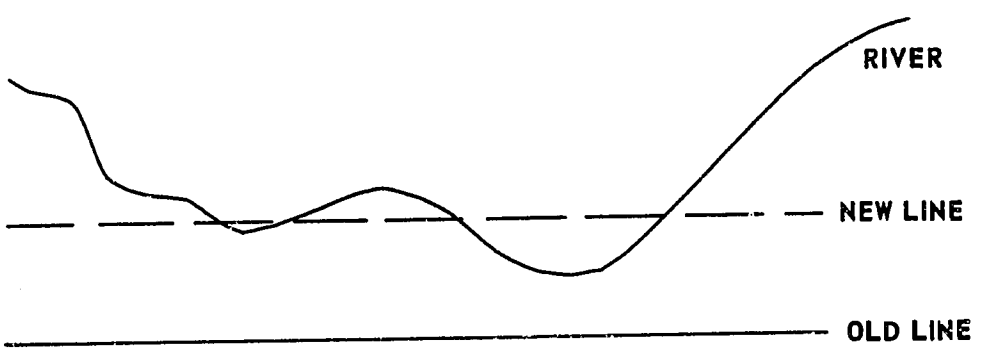


Fig. 42.

The portions of the river that lie below the image of the road now give the points that can be used for building the factory.

Example 2

In the following figure, construct a line segment which is parallel to line L, has length PQ, has one of its endpoints on circle C_1 and has one of its endpoints on circle C_2.

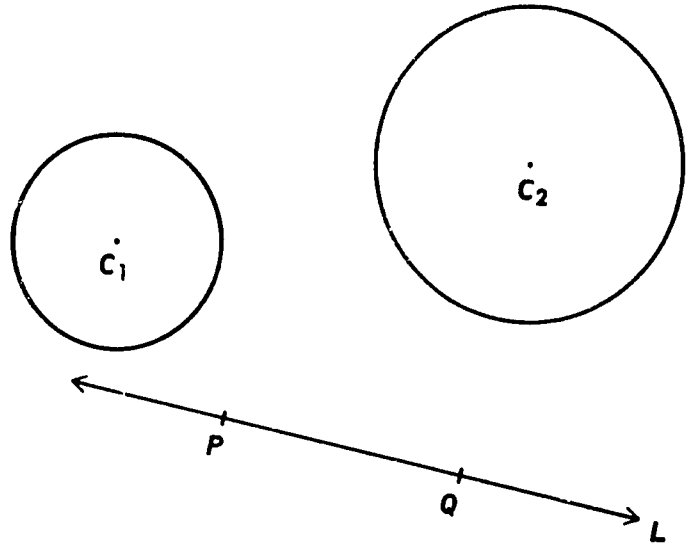


Fig. 43.

Solution to Example 2

Use a rigid motion which carries the circle C_1 a distance PQ towards C_2 in a direction parallel to L. The points of intersection of C_2 with the image of C_1 now give the possible points on C_2 for the segment to be constructed.

Challenge Problem

Use ruler and compasses to construct a line segment as asked for in the second example above. (There are two segments possible).

6-5 SPECIAL KINDS OF RIGID MOTION.

In the sections above, we have looked at different examples of rigid motions. Some of these rigid motions had special properties that others did not have. For example, some rigid motions had fixed points, others did not. It is helpful to have special names for some of the different kinds of rigid motions that are possible. In this section, we use the tracing paper idea to show the meaning of some of these names. In following sections we shall give more exact definitions.

(1) A rigid motion is called *direct* if we do not need to turn the tracing paper over in order to carry it out. If we do have to turn the tracing paper over, the rigid motion is called *reversing*.

(2) A rigid motion is called a *translation* if it is direct and can be obtained by sliding the tracing paper, without rotation, so that every point moves the same fixed distance.

(3) A rigid motion is called a *rotation* if it is direct and can be obtained by rotating the tracing paper about some chosen fixed point.

(4) A rigid motion is called a *reflection* if it is reversing and can be obtained by choosing a straight line and then turning the tracing paper over and putting it back down so that every point on the chosen line falls on its original position. The chosen line is called the *line of reflection* of the rigid motion.

Examples

(1) In Problem 1 of Problems 6-2A, the rigid motion which yields Figure 44 below

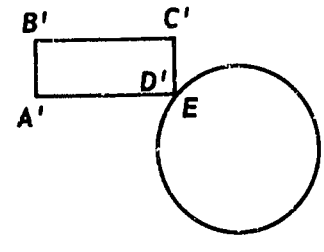


Fig. 44.

is a translation.

(2) In Problem 2, the rigid motion which yields Figure 45 below

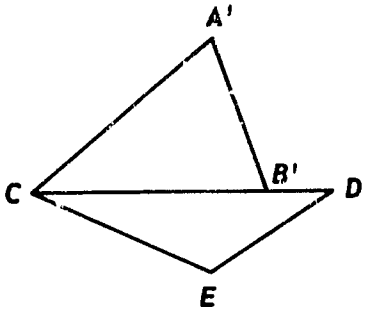


Fig. 45.

is a rotation. (Here C is the fixed point of the rotation.)

(3) In Problem 3, the transformation which yields Figure 46 below

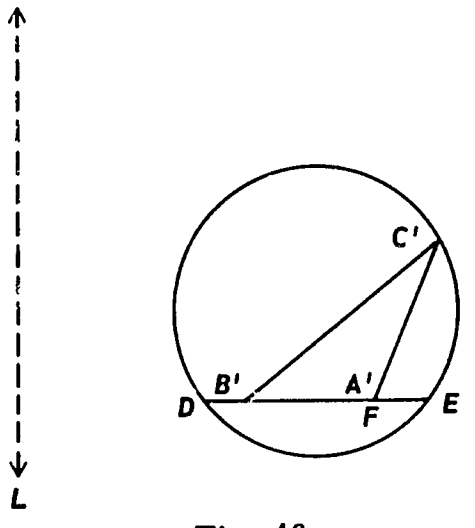


Fig. 46.

is a reflection. (Here L is the line of fixed points which we call the *line of reflection.*)

(4) In Problem 4 of Problems 6-3 we get an example of a rigid motion which is neither a translation nor a rotation nor a reflection. The rigid motion in this problem *can* be obtained by carrying out first a reflection and then a translation. In the following figure, the straight line indicates the line of reflection. Note that the translation leaves the line of reflection invariant.

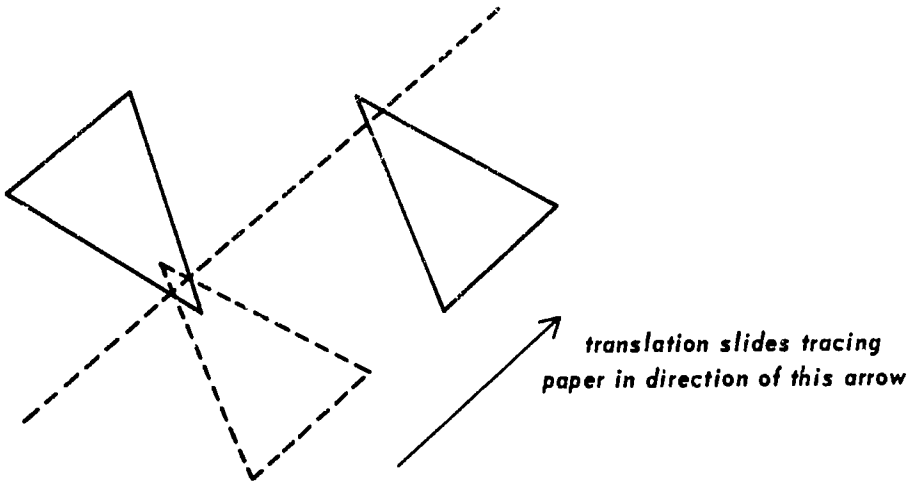


Fig. 47.

We shall see later that *every* rigid motion is either a translation, a rotation, a reflection, or can be obtained by carrying out first a reflection, and then a translation as in the above example.

Certain words and ideas are especially useful in talking about whether or not a motion is reversing. We give these words and ideas here.

Assume that we are looking down at the plane from above. Let A, B, C be the vertices of a triangle in the plane. We say that we see the vertices A, B, C in *clockwise order* if, as we look from A to B to C and back to A, our eyes follow the same direction that they would follow if we were watching hands move on the face of a clock. In the following picture, A, B, C occur in clockwise order.

Fig. 48.

In the following picture, A, B, C do not occur in clockwise order.

Fig. 49.

Whether or not vertices occur in clockwise order depends upon the order in which we name them. In the last picture above, A, C, B occur in clockwise order, while A, B, C do not.

If vertices do not occur in clockwise order, we say that they occur in *counter-clockwise order*. Thus, in the last picture above, we say that A, B, C occur in counter-clockwise order.

The following facts are now clear.

A rigid motion is direct if, for every triangle ABC in which A, B, C occur in clockwise order the images A', B', C' occur in clockwise order.

A rigid motion is reversing if, for every triangle ABC in which A, B, C occur in clockwise order, the images A', B', C' occur in counter-clockwise order.

PROBLEMS 6-5

1. Look at the rigid motions described in Problems 1, 2, 3 and 5 of Problems 6-2B. Which of these are direct and which are reversing? Which are translations, which are rotations and which are reflections?

2. We are given a rigid motion T. In the following figure, find $T(Q)$ assuming that T is a direct motion.

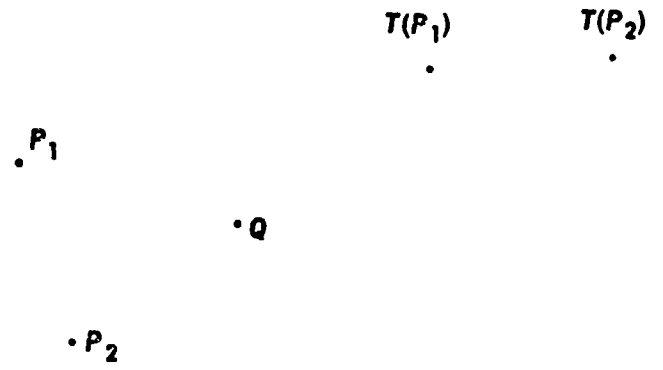


Fig. 50.

In the same figure, find $T(Q)$ assuming that T is a reversing motion.

3. T is a rigid motion; P_1, P_2 and Q are three distinct points.
(a) You are given $T(P_1)$ and $T(P_2)$, and you are told that T is direct. Can you find $T(Q)$?
(b) You are given $T(P_1)$ and $T(P_2)$, and you are told that T is reversing. Can you find $T(Q)$?

6-6 CONGRUENT FIGURES.

As we have seen, the image of a triangle under a rigid motion is congruent to the original triangle (by SSS). It is also true that for any two congruent triangles, there is a rigid motion which carries one triangle to the other. This fact is easy to see from the tracing paper idea of rigid motion.

Let ABC and $A'B'C'$ be two congruent triangles. We must show that there is a movement of tracing paper which carries ΔABC to $\Delta A'B'C'$.

Imagine that we have made a tracing of ΔABC. Since the triangles are congruent, $AB = A'B'$. Slide the tracing paper so that A falls on A' and B falls on B'.

(i) If C (on the tracing) now falls on the same side of $\overline{A'B'}$ as C', we see that C must fall exactly on C' (since $C\hat{A}B = C'\hat{A}'B'$ in measure, and $C\hat{B}A = C'\hat{B}'A'$ in measure).

(ii) If, after the tracing paper is moved, C happens to fall on the opposite side of $\overline{A'B'}$ from C', turn the tracing paper over and place A over A' and B over B'. Now the new position of C falls on the same side of $\overline{A'B'}$ as C', and, as before, C must fall exactly on C'.

Thus, in either case, we have a movement of the tracing paper which carries $\triangle ABC$ to $\triangle A'B'C'$. This movement defines the rigid motion that we wanted.

We thus have the following:

Two triangles are congruent if and only if there is a rigid motion which carries one triangle on to the other.

In the same way, we could show that two circles are congruent if and only if there is a rigid motion which carries one circle on to the other.

These facts suggest that we use the idea of rigid motion to *define* congruence for any two figures in the plane.

> *General definition of congruence.* Let S and S' be two sets of points in the plane. We say that S is *congruent* to S' if there is a rigid motion under which S' is the image of S.

This definition is different from the definitions of congruence for triangles and circles that you were given in your earlier work in geometry. We make the following two comments.

(a) For triangles and circles, the new definition agrees with the old definition. (That is to say, two triangles are congruent under the new definition if, and only if, they are congruent under the old definition.)

(b) The new definition is closer to the informal idea of "same size and shape" which led to our original study of congruent triangles. This informal idea was, simply, that two triangles are congruent if we can move one of them so that it falls exactly on the other.

NOTE ON ISOMETRIC MAPPINGS

In Section 6-3, we saw that every mapping obtained by using tracing paper must be isometric. We also stated that every isometric mapping

can be obtained by using tracing paper. We can now see why this last fact is true.

Let T be a given isometric mapping. Then T has the property that it carries every straight line to a straight line. (This was Property (4) in Section 6-3. The fact that every isometric mapping has this property is proved in the Appendix to this chapter.) Let ABC be any given triangle. Let $A' = T(A)$, $B' = T(B)$, and $C' = T(C)$. Then the image of ΔABC under T must be $\Delta A'B'C'$, (by Property (4) of T); hence $\Delta A'B'C'$ is congruent to ΔABC (by the isometric property of T and SSS). What we did at the beginning of this section shows that there is a movement of tracing paper which carries A to A', B to B', and C to C'. Call the mapping over the plane given by this movement: T'. Then $T'(A) = T(A)$, $T'(B) = T(B)$, and $T'(C) = T(C)$. Now look at the construction given at the end of Section 6-2. This construction shows that if we know where an isometric mapping carries the vertices of a triangle, then we can find out exactly where that mapping carries any other point. Given any point P, the constructions for T and for T' must be the same. Hence, for any point P, $T(P) = T'(P)$. Hence T and T' are the same mapping. Thus our given mapping T can be obtained by a movement of tracing paper, which is what we set out to show.

6-7 TRANSLATIONS.

Let two points A and B be given. Draw an arrow from A to B.

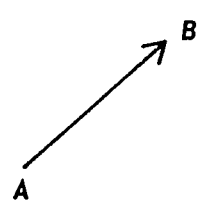


Fig. 51.

For any point P we carry out the following construction. Through P draw a line parallel to $\overleftrightarrow{AB}$. Call this line L'. (In case P lies on $\overleftrightarrow{AB}$, we take L' to be $\overleftrightarrow{AB}$ itself.) On L', we take a point P' so that $PP' = AB$ and P' lies in the same direction from P that B lies from A.

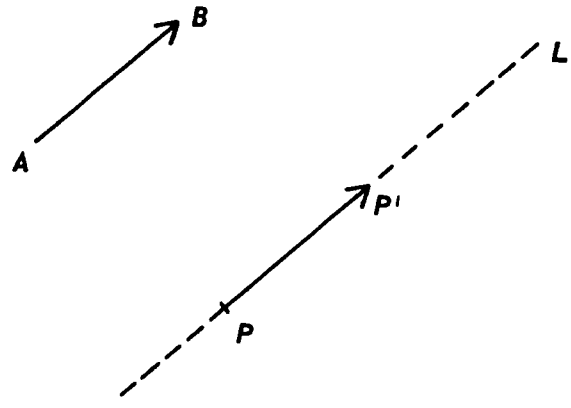


Fig. 52.

Note that in this construction, if we draw an arrow from P to P', we get an arrow with the same length and same parallel direction as the arrow from A to B.

An arrow from one point to another point in the plane is called a *vector*. We shall use the symbols **U** and **V** to stand for vectors. *NOTE*: Do not confuse a vector with a ray. A vector has *direction* and *length*, whereas a ray has direction only.

Let us give the name **U** to the vector from A to B in the above construction.

> DEFINITION 6-2. If P' is obtained from P as in the above construction, we say that P' is the *result of translating* P *by the vector* **U**.

Let a vector **U** be given in the plane. The construction above can be carried out for any point P. Hence we have a mapping T which we get by

taking $T(P)$, for any point P, to be the result of translating P by the vector $\mathbf{U}$.

Must the mapping T, which we get in this way, be a rigid motion? If we can show that T is isometric, then, by our work in Sections 6-3 and 6-6, we will know that T is a rigid motion. We can show that T is isometric as follows. Take any points P_1 and P_2. The construction of P_1' and P_2' gives the following figure.

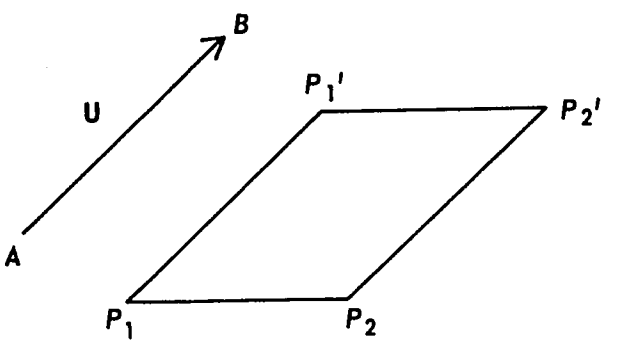


Fig. 53.

To show that T is isometric, we must show that $P_1P_2 = P_1'P_2'$.

Consider the quadrilateral $P_1P_1'P_2'P_2$. $\overline{P_1P_1'}$ is parallel to $\overline{P_2P_2'}$, since both are parallel to $\mathbf{U}$ by construction. $P_1P_1' = P_2P_2'$, since both are equal to the length of $\mathbf{U}$ by the construction. Hence $P_1P_1'P_2P_2'$ is a parallelogram. (Recall that any convex quadrilateral with a pair of opposite sides which are equal in length and parallel must be a parallelogram.) Hence $P_1P_2 = P_1'P_2'$, since opposite sides of a parallelogram must be of equal length. (In the figure, we have assumed that $\overleftrightarrow{P_1P_1'}$ and $\overleftrightarrow{P_2P_2'}$ are distinct lines. In case they are the same line, the proof is even simpler.)

We have shown T is isometric, and we see that T is a rigid motion. Is the rigid motion T direct or reversing? The following figure makes it clear that T must be direct.

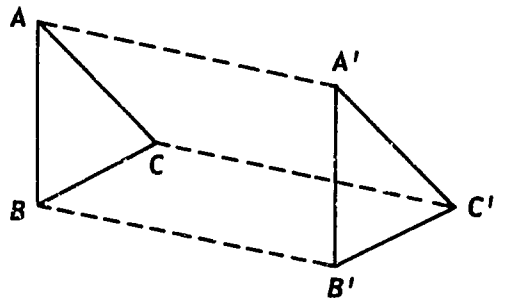


Fig. 54.

What kind of rigid motion is T? *Clearly, every such T is what we called, in Section 6-5, a translation, and every translation can be obtained by using a suitable vector* **U** *in the above construction for T.* The vector **U** simply gives the distance and direction through which we slide the tracing paper to get the rigid motion T.

We now list some more important facts about translations. In a translation, every point moves the same distance. This is clear from the construction.

Given any line L, the image of L in a translation is either parallel to L or else is L itself. To see that this is true, look at Figure 53 where we showed $P_1P_2 = P_1'P_2'$. If P_1 and P_2 are two points on L, then P_1' and P_2' are two points on the image of L, and, since $P_1P_1'P_2'P_2$ is a parallelogram, we have that L and its image are parallel. For what lines is it true that the image of L is L itself?

Is the identity motion a translation? We agree to call it a translation, although it is the translation in which every point moves zero distance. It is given by a vector whose length is zero.

TRANSLATIONS AND COORDINATE AXES

Suppose we take a point O which is the origin of a coordinate system. Then we can draw our coordinate axes as in Figure 55.

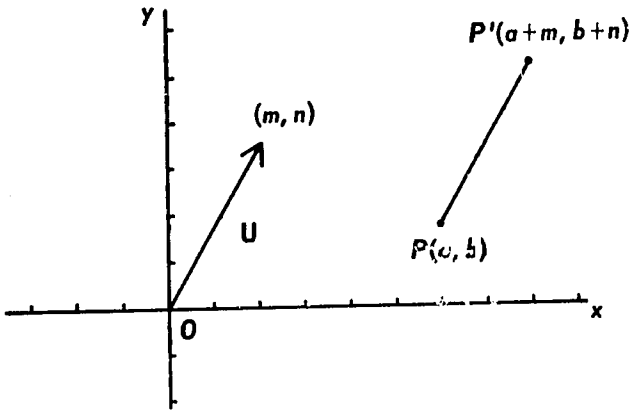


Fig. 55.

Suppose we are given a vector **U** from the point O to some point with coordinates (m, n). Let P have coordinates (a, b). Let P' be the result of translating P by **U**. What are the coordinates of P'? Clearly P' has coordinates $(a + m, b + n)$. Can you show this?

PROBLEMS 6-7

1. Let T be the translation given by the vector **U** in the following figure. Find the image of the triangle. Find the image of the circle. Find the figure which has the circle as its image.

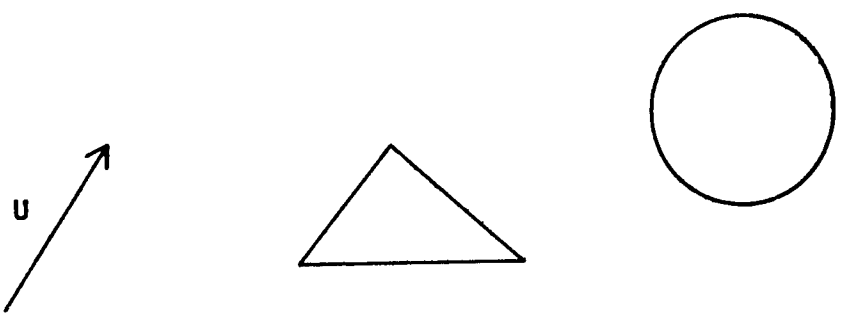


Fig. 56.

2. In Problem 1, what lines are carried into themselves by T (are invariant under T)?

3. What is the result of applying first translation **U** and then translation **V** in the following figure?

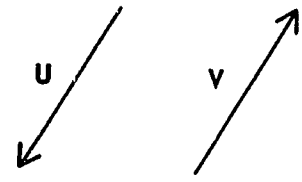


Fig. 57.

4. What is the result of applying first translation **U** and then translation **V** in the following figure?

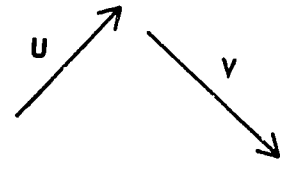


Fig. 58.

5. Let **U** be the vector between (0, 0) and (2, -4). For each of the following points P, find the coordinates of P' where P' is the result of translating P by **U**.

P.	(1, 1)
	(-2, 4)
	(7, -10)

6. In the last paragraph of this section (Figure 55), prove that P' has coordinates $(a + m,\ b + n)$.

Challenge Problem

7. You are told that a certain rigid motion T has the property that it carries every point the same distance. Show that T must be a translation.

8. You are told that a certain rigid motion T has the property that, for every line L, the image of L is either parallel to L, or is L itself. Show that T need not be a translation.

9. You are given two points P_1 and P_2, you are told that a certain rigid motion T carries P_1 to P_1' and P_2 to P_2', and you are told that $P_1P_1'P_2'P_2$ is a parallelogram. Show that T need not be a translation.

Challenge Problem

10. You are given the same information as in Problem 9 and you are also told that T is direct. Show that T must be a translation.

6-8 ROTATIONS.

Suppose we have a fixed point O and a point P. If we move P to the position P' (as in Figure 59) in such a way that $OP = OP'$ and $m(\widehat{POP'}) = \beta$, then we say that we have turned P through an angle β about O.

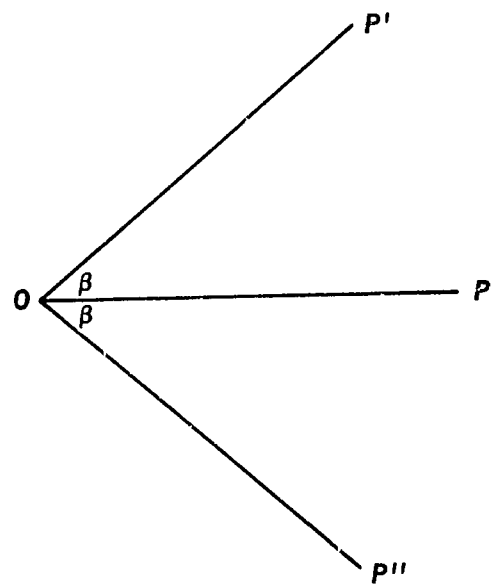


Fig. 59.

We can also turn P through an angle β to get P''. So it is possible to turn P through the same angle β, but getting two different positions as P' and P'' in Figure 59.

In order to get just one result when we talk about rotating a point through a given angle β about a fixed point, we need to give also the direction in which the angle β is turned.

For example in the above case when P moves to P', we actually turned in the direction that is the counter-clockwise direction. To get P'' we would turn in the direction . We see clearly that is in the opposite direction to . So we use the following sign convention: the direction (counter-clockwise) will be taken as *positive* and the opposite direction (clockwise) will be taken as *negative*.

Whenever we say that a point (or for that matter anything else) is turned through an angle $\beta > O$, we mean that the turning is in the counter-clockwise direction, and when we say that a point is turned through an angle $\beta < O$, we mean that the turning is in a clockwise direction.

> DEFINITION 6-3. The point P is *rotated about O through an angle* β, if P moves in such a way that it remains a fixed distance from O while $\overrightarrow{OP}$ turns through the angle β.

Let a point O be given in the plane and let a value of β be given. For any point P we can find a new point P' by rotating P about O through an angle β. Hence we have a mapping T which we get by taking $T(P)$, for any point P, to be the result of rotating P about O through an angle β.

Must the mapping T, which we get in this way, be a rigid motion? If we can show that T is isometric, then, by our work in Sections 6-3 and 6-6, we will know that T is a rigid motion. We can show that T is isometric as follows. Take any points P_1 and P_2. The construction of P_1' and P_2' gives the following figure.

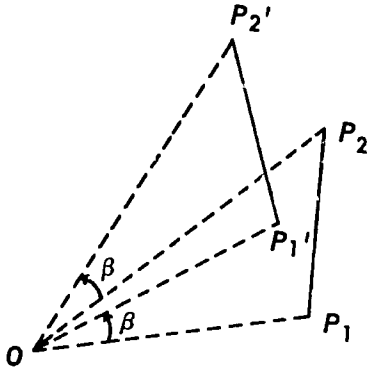


Fig. 60.

To show that T is isometric, we must show that $P_1P_2 = P_1'P_2'$.

In this figure, $m(P_1\widehat{O}P_2) = \beta + m(P_1'\widehat{O}P_2)$ and $m(P_1'\widehat{O}P_2') = \beta + m(P_1'\widehat{O}P_2)$. Hence $m(P_1\widehat{O}P_2) = m(P_1'\widehat{O}P_2')$. Also, $OP_2 = OP_2'$ and $OP_1 = OP_1'$, by the construction. Hence, by SAS, $\Delta P_1'OP_2'$ is congruent to

ΔP_1OP_2. It follows that $P_1P_2 = P_1'P_2'$. (In the figure, we have assumed that the interiors of $P_1\widehat{O}P_2$ and $P_1'\widehat{O}P_2'$ overlap. In case they do not overlap, the proof is the same except that $m(P_1\widehat{O}P_2) = m(P_1'\widehat{O}P_2') = \beta - m(P_1'\widehat{O}P_2)$.)

Therefore, T is isometric, and we see that T is a rigid motion. Is the rigid motion T direct or reversing? The following figure makes it clear that T must be direct.

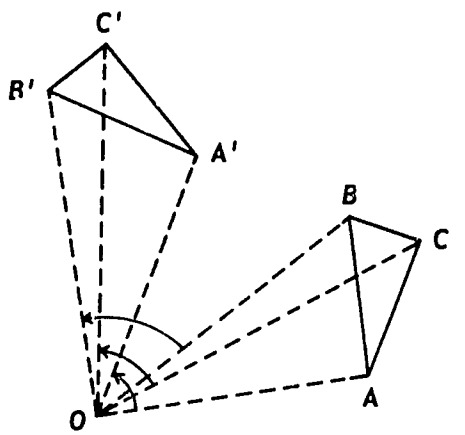


Fig. 61.

What kind of rigid motion is T? Clearly every such T is what we called, in Section 6-5, a rotation, and every rotation can be obtained by using a suitable point O and value β in the above construction of T. The point O is simply the point about which we rotate the tracing paper (imagine a pin stuck through the tracing paper at this point), and β is the measure of the angle (positive or negative) through which we turn the tracing paper.

If $\beta = 90^0$ in a rotation, we call that rotation a *quarter turn*. If $\beta = 180^0$ in a rotation, we call that rotation a *half turn*. Some of the problems below have to do with these special rotations.

The identity motion is the rotation that we get for $\beta = 0^0$. (The identity motion is thus called *both* a rotation and a translation.)

Note the following fundamental fact about rotations. Let T be a rotation through an angle β. Let L be any line. Let L' be the image of L under T. Then L and L' form an angle whose measure is β.

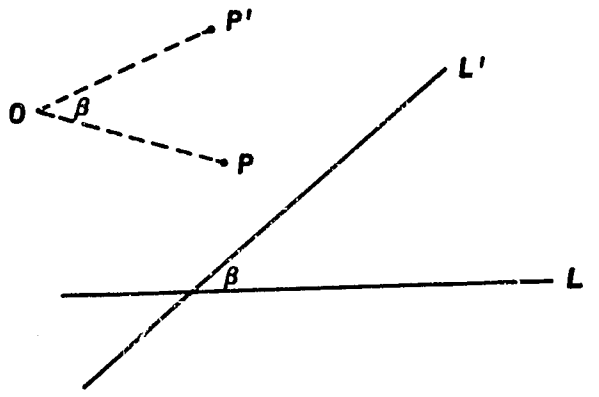


Fig. 62.

PROBLEMS 6-8A

1. If A is rotated 90^0 about O, where will its image A' be?

Fig. 63.

2. If B is rotated 180^0 about X, where will its image B' be?

Fig. 64.

3. If $\overline{AB}$ is rotated 90^0 about O where will $\overline{A'B'}$ be?

Fig. 65.

4. Where will a half-turn about O bring $\overline{AB}$ in Figure 66? If $\overline{A'B'}$ is the image of $\overline{AB}$ from this rotation and you moved towards the right to get from A to B, in what direction would you move to get from A' to B'?

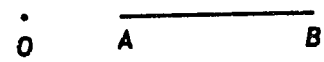


Fig. 66.

5. Where will a quarter turn about O bring $\overline{AB}$?

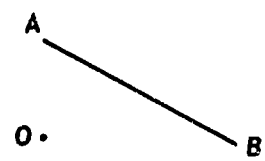


Fig. 67.

6. Draw $\overline{A'B'}$, the segment resulting from a half turn of $\overline{AB}$ about O. Can you show that $ABA'B'$ is a parallelogram?

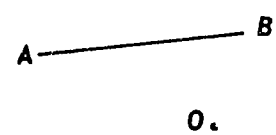


Fig. 68.

7. Let T be a rotation of 120^0 about O, the centre of the equilateral triangle. Find the images under T of the circle, and the two triangles in the following figure.

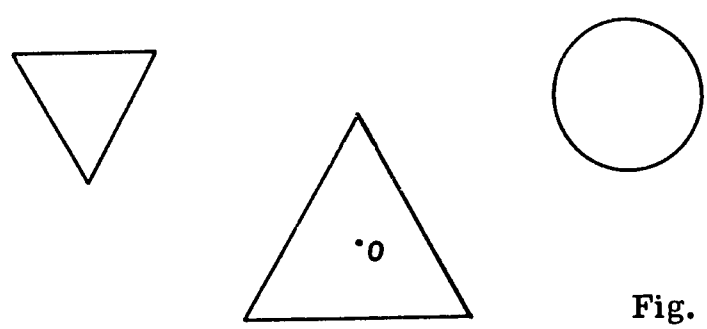


Fig. 69.

8. If, in the figure, A' is the image of A under a rotation about a point O which lies along $\overline{AA'}$, where is this point O?

Fig. 70.

9. Let A and A' be given. If A is moved to A' by a rotation about some point O, what is the locus of possible points O?

Challenge Problem

10. If $\overline{AB}$ is transformed into $\overline{A'B'}$ by a rotation about a point O, how can you find the position of O? Is there a rigid motion which carries $\overline{AB}$ to $\overline{A'B'}$, but is not a rotation about some point O?

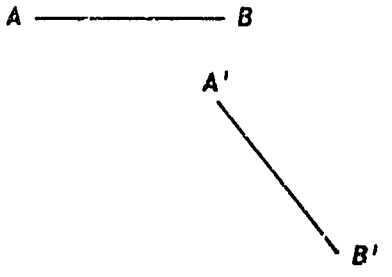


Fig. 71.

11. Triangle $A'B'C'$ is the image of $\triangle ABC$ under a rotation. How would you find the point about which it was rotated?

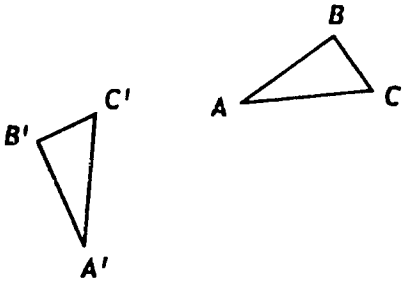


Fig. 72.

NOTE: The point about which a rotation is made is known as a centre of rotation.

12. Prove that in a rotation through angle β any line makes an angle β with its image.

ROTATIONAL SYMMETRY

Consider the equilateral triangle ABC, with centre O. Suppose we rotate the triangle through 120^0 about O. Where will the sides of the image, $\Delta A'B'C'$, lie?

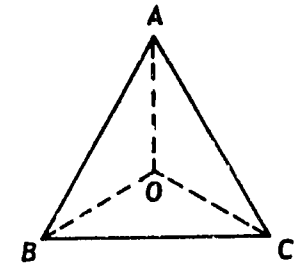


Fig. 73.

You will find they lie on the sides of ΔABC, although now C' will be where A was, A' will be where B was and B' will be where C was. That is, this image will fall exactly on the original triangle. Also if we rotate the triangle through 240^0, we find that the sides of the image again fall exactly on the original triangle.

Suppose we next take a square $ABCD$ with centre O. If we rotate the square 120^0 about O, will the sides of the image fall on the sides of the original square? What happens when the square makes a quarter-turn about O?

Consider the following figure. Could you find a rotation about O which will give an image lying on top of the original figure? (Find as many angles as possible lying between O and 360^0 for which this could be done.)

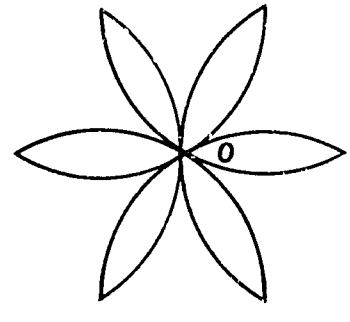


Fig. 74.

If we restrict ourselves to a rotation less than a complete turn (that is, less than 360^0) do you think you can find a rotation about O which will give an image coincident with the original figure in this illustration?

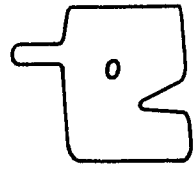


Fig. 75.

When we do a complete turn, every point returns to itself (and so we have the identity motion). So a complete turn will always leave a point, a line or any figure unchanged. If it is possible to rotate a figure less than 360^0 about a point O and bring it into coincidence with itself, then we say that the figure has *rotational symmetry* about O.

ROTATIONS AND COORDINATE AXES

Suppose we consider a point O which is the origin of a coordinate system, and we draw the coordinate axes as in the figure. A point R on the x-axis, when rotated 90^0 about O, goes to R'. What is the distance of R' from O? Suppose we considered another point P distant p from O along the y-axis. Where will its image P' lie under a quarter-turn about O? Where will its image P'' lie under a half-turn about O?

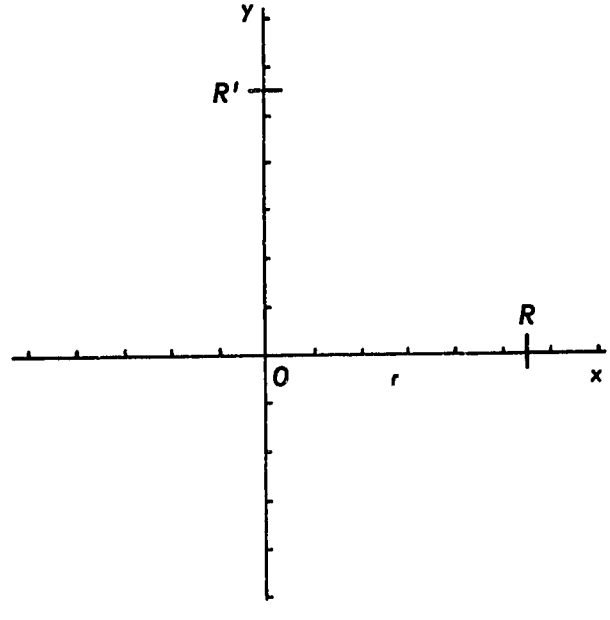


Fig. 76.

We can see that a half-turn will take R to R'', where R is a distance r from O along the positive x-axis and R'' is a distance r from O along the negative x-axis. A quarter turn will move R to R', where R' is

a distance r from O along the positive y-axis. Can you make similar statements for the point P?

Let a point P have coordinates (a, b). What do the coordinates of P become under a half turn about the origin? Under a quarter-turn about the origin?

PROBLEMS 6-8B

1. Which of the following figures have rotational symmetry?

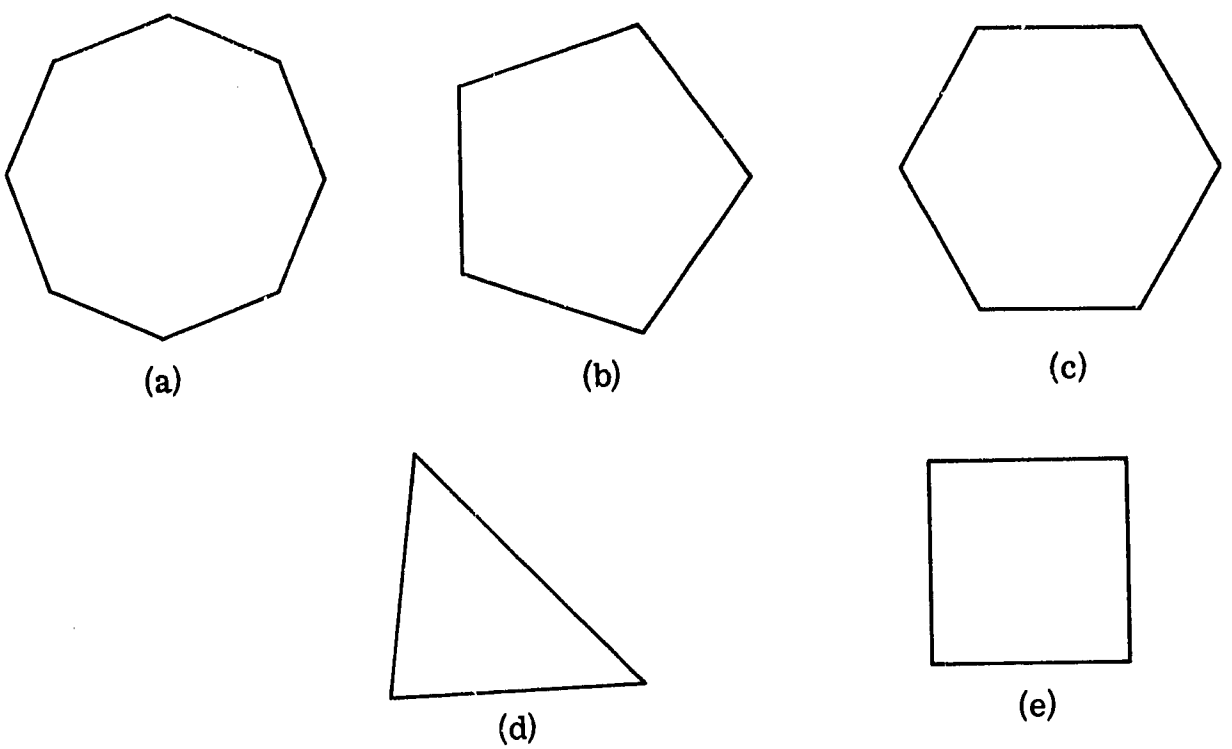


Fig. 77.

2. The number of distinct rotations that carry a figure onto itself for rotations less than 360° is called the *order of symmetry* of that figure. Find the order of symmetry of each of the figures in Problem 1. Note that a figure whose order of symmetry is 1 does not have rotational symmetry.

Challenge Problem

3. You are told that a rigid motion T has at least one fixed point. You are also told that T is direct. Show that T must be a rotation.

6-9 REFLECTIONS.

Let a line L be given.

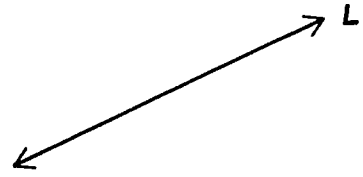


Fig. 78.

For any point P, we carry out the following construction. Through P we draw a line perpendicular to L. On this perpendicular we take a point P' so that P' is on the opposite side of L from P, and P' is at the same distance from L as P. (In case P is on L, we take P' to be P itself).

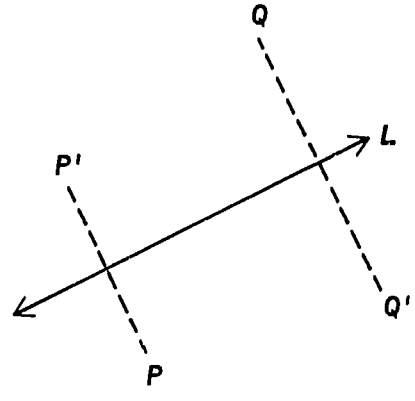


Fig. 79.

> DEFINITION 6-4. If P' is obtained from P as in the above construction, we say that P' is the result of *reflecting* P *in* L.

Let a line L be given in the plane. The construction above can be carried out for any point P. Hence we have a mapping T which we get by taking $T(P)$, for any point P, to be the result of reflecting P in L.

Must the mapping T, which we get in this way, be a rigid motion? If we can show that T is isometric, then, by our work in Section 6-3, we will know that T is a rigid motion. We can show that T is isometric as follows.

Take any points P_1 and P_2. The construction of P_1' and P_2' gives Figure 80 below.

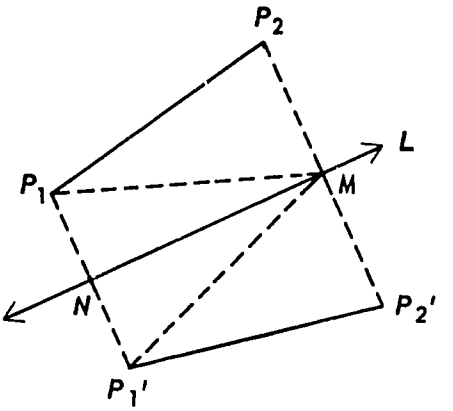


Fig. 80.

To show that T is isometric, we must show that $P_1P_2 = P_1'P_2'$.

In the above figure, $P_1N = P_1'N$ by the construction. Hence ΔP_1NM is congruent to $\Delta P_1'NM$ by SAS. Hence $P_1M = P_1'M$ and $m(P_1\widehat{M}N) = m(P_1'\widehat{M}N)$. Thus $m(P_1\widehat{M}P_2) = m(P_1'\widehat{M}P_2')$. Since $P_2M = P_2'M$ by the construction, we have ΔP_1MP_2 congruent to $\Delta P_1'MP_2'$ by SAS, and $P_1P_2 = P_1'P_2'$. (In the figure we have assumed that P_1 and P_2 lie on the same side of L. The proof is similar in case P_1 and P_2 lie on opposite sides of L.)

Hence T is isometric, and we see that T is a rigid motion. Is the rigid motion T direct or reversing? The following figure makes it clear that T must be reversing.

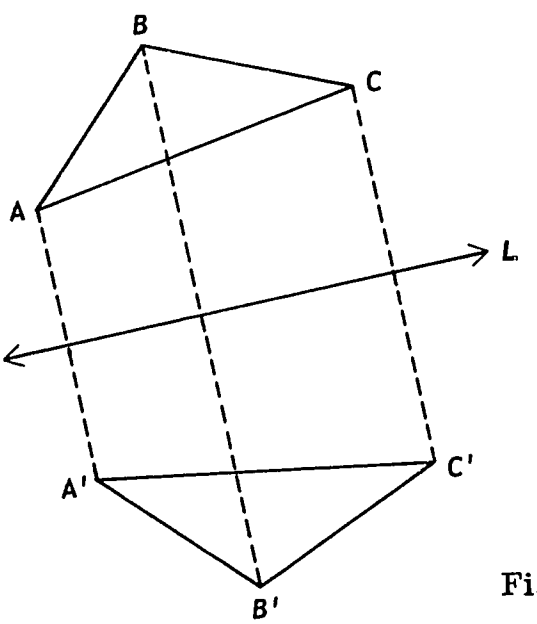


Fig. 81.

What kind of rigid motion is T? Clearly, T is what we called, in Section 6-5, a reflection, and every reflection can be obtained by using a suitable line L in the above construction for T. The line L is simply the line which remains fixed after we have turned the tracing paper over.

The rigid motions which we call reflections are closely related to the idea of reflection in a mirror. You have no doubt had the experience of standing in front of a mirror and observing your image. The image looks exactly like you and appears to be situated as far behind the mirror as you are standing in front of it. One big difference between you and your image, however, is that your left side appears as the right side of the image and vice versa. If you have been studying General Science or Physics in your course, you would have come across very many interesting experiments and examples dealing with reflection.

If T is one of the rigid motions in the plane which we are calling *reflections*, then the line L is much like the mirror-image that you would seem to see if you were to look at P in a mirror. The fact that T is reversing is like the fact that left and right are interchanged in a mirror-image.

PROBLEMS 6-9A

1. The line m is parallel to a reflecting line L. Is its image m' going to be a parallel to L also?

2. If $\overrightarrow{AB}$ is parallel to the reflecting line L, in what direction will the image $\overrightarrow{A'B'}$ point?

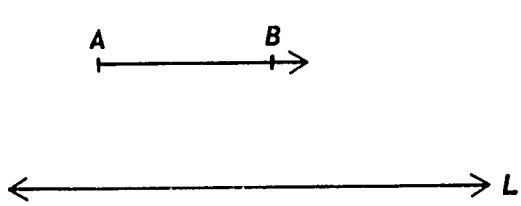


Fig. 82.

3. The line $\overleftrightarrow{AB}$ is perpendicular at O to the reflecting line L. Draw the image $\overline{A'B'}$ of $\overline{AB}$.

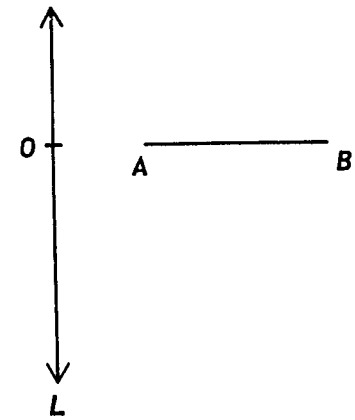


Fig. 83.

4. If, in Problem 3, one moves from left to right in going from A to B, in what direction would he move in going from A' to B'?

5. Is the result in Problem 4 similar to that obtained for a half-turn about O?

6. Draw the image of $\triangle ABC$ under a reflection along L. Is $\triangle ABC \equiv \triangle A'B'C'$, where $\triangle A'B'C'$ is the image of $\triangle ABC$? How do you know this?

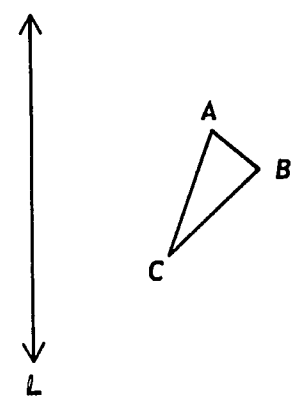


Fig. 84.

7. Draw the image of $ABCD$ under a reflection along L. Is this image congruent to $ABCD$?

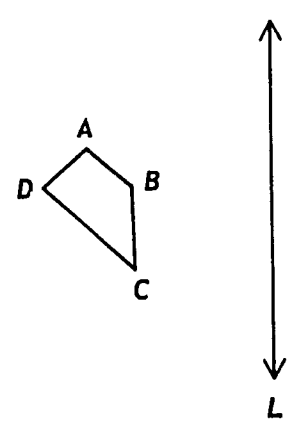


Fig. 85.

8. What will the image of a circle be under reflection on a line? The arrow in the circle is clockwise. In what direction will the arrow point in the image?

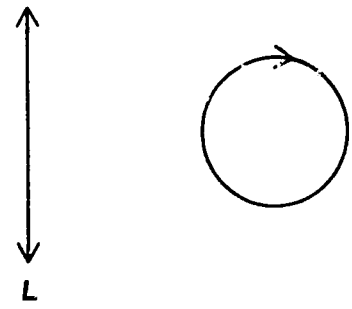


Fig. 86.

9. If $\overline{A'B'}$ is the image of $\overline{AB}$ under a line reflection, where will the line of reflection lie in the following diagrams?

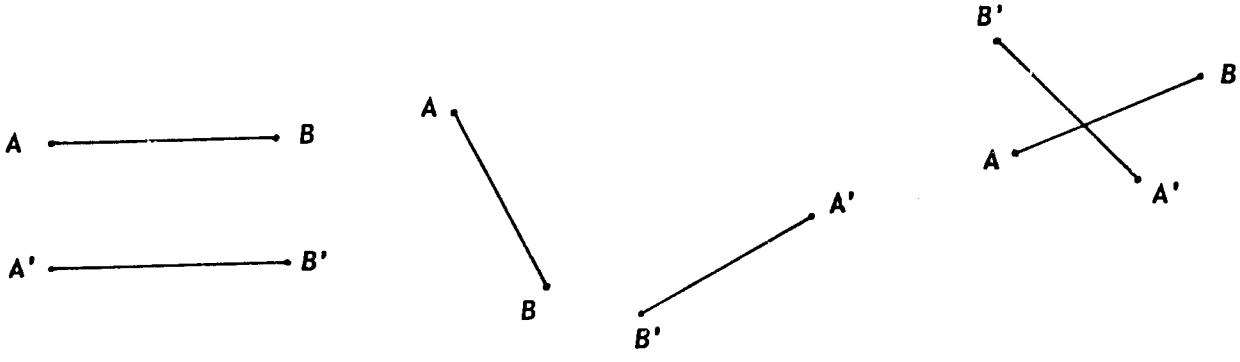


Fig. 87.

10. If $\Delta A'B'C'$ is the image of ΔABC under a line reflection, where will the line of reflection be?

Fig. 88.

11. Is it possible to get a line of reflection which will make $\Delta A'B'C'$ an image of ΔABC in Figure 89? If not, could you find a reason?

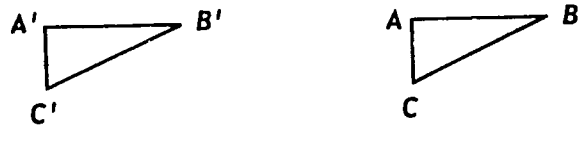


Fig. 89.

AXIS OF SYMMETRY

Consider the isosceles triangle ABC with $\overline{AB} \equiv \overline{AC}$ and $\overline{AD}$ an altitude. A reflection of this triangle along $\overleftrightarrow{AD}$ will give a triangle $AC'B'$ which is the same as triangle ABC with C' falling on B and B' falling on C, since the reflection of A in $\overleftrightarrow{AD}$ still gives A.

We see that, in the above reflection, each point P of the triangle is reflected to give an image point P' which is on the triangle. That is, every point P and its image belong to the triangle.

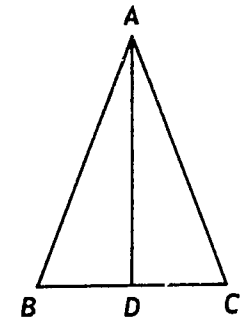


Fig. 90.

> DEFINITION 6-5. If for every point P of a figure its image P' under a reflection on a line L is also a point of the figure, the line L is an *axis of symmetry* of the figure.

REFLECTION ALONG COORDINATE AXES

A point R at a distance r along the positive x-axis becomes point R' at a distance r along the negative x-axis. Similarly a point Q at a distance q along the positive y-axis will become point Q' at a distance q along the negative y-axis.

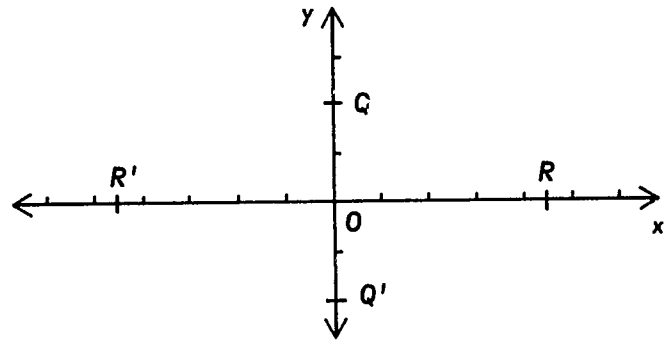


Fig. 91.

A point $P(a, b)$ when reflected in the x-axis becomes point P' $(a, -b)$. A point $P(a, b)$ when reflected in the y-axis becomes point P'' $(-a, b)$.

In general, the point (x, y) becomes $(x, -y)$ under a reflection in the x-axis and the point (x, y) becomes $(-x, y)$ under a reflection in the y-axis.

PROBLEMS 6-9B

1. Lines L and m are the bisectors of the angles formed by the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ in Figure 92 below. Show that L is an axis of symmetry of the figure.

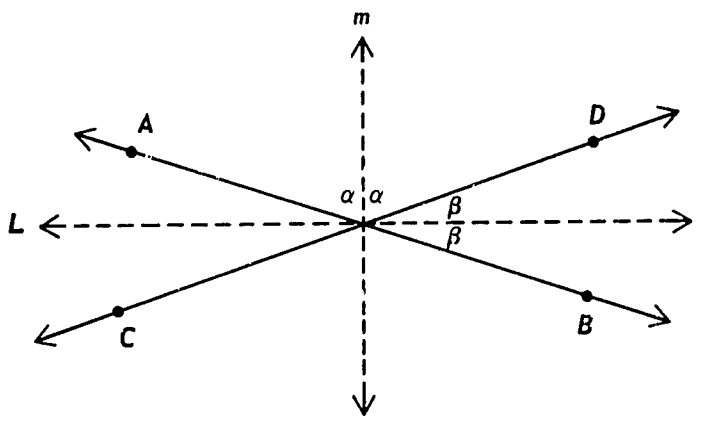


Fig. 92.

Show that L and m are the locus of points equidistant from the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

2. We say that a point P is *reflected in a point* O, if its image P' is such that $\overleftrightarrow{POP'}$ is a straight line and $PO = OP'$. Show that this is the same as doing a half turn about O.

Challenge Problem

3. What will the coordinates of the point (a, b) become under a reflection along the line $y = x$?

6-10 ONE MOTION FOLLOWED BY ANOTHER.

Let a rigid motion R and a rigid motion S be given. For example, R might be the rotation about O which carries ΔABC to $\Delta A'B'C'$ in the following figure, and S might be the translation given by the vector $\mathbf{U}$ in the figure.

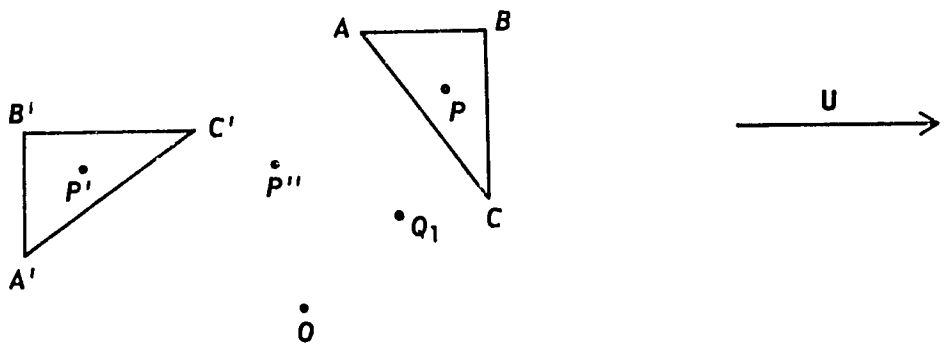


Fig. 93.

Now for any point P, we carry out the following construction. First we find the point $R(P)$, which we call P'. Then we take P' and find the point $S(P')$, which we call P''. We can carry out this construction for any point P. Hence we have a mapping T which we get by taking $T(P)$, for any point P, to be the point P'' obtained from P by this construction.

Must the mapping T, which we get in this way, be a rigid motion? If we can show that T is isometric, then, by our work in Sections 6-3 and 6-6, we will know that T is a rigid motion. Take any points P_1 and P_2. To show that T is isometric, we must show that $P_1P_2 = P_1''P_2''$. But $P_1P_2 = P_1'P_2'$, since R is isometric, and $P_1'P_2' = P_1''P_2''$, since S is isometric. Hence $P_1P_2 = P_1''P_2''$, and we have that T is isometric. Therefore T must be a rigid motion.

> DEFINITION 6-6. Given any two rigid motions R and S, the new rigid motion T which we get when R is followed by S is called the *combination of R with S*. We shall also refer to is as simply "R followed by S".

Think for a moment of the tracing paper idea of a rigid motion. R results from a certain movement of the tracing paper and S results from a certain movement of the tracing paper. If we make the first of these movements and then follow it with the second, we reach a final position (after the two movements) which could have been reached by a single movement of the tracing paper. T is the rigid motion which results from this single movement.

What kind of rigid motion is T? This will depend, of course, on the rigid motions R and S. In Figure 93 above, T is a rotation about the point marked Q_1. (You can check this with some tracing paper.)

What would happen if we carried out the two motions R and S in reverse order? Then the figure would be as follows.

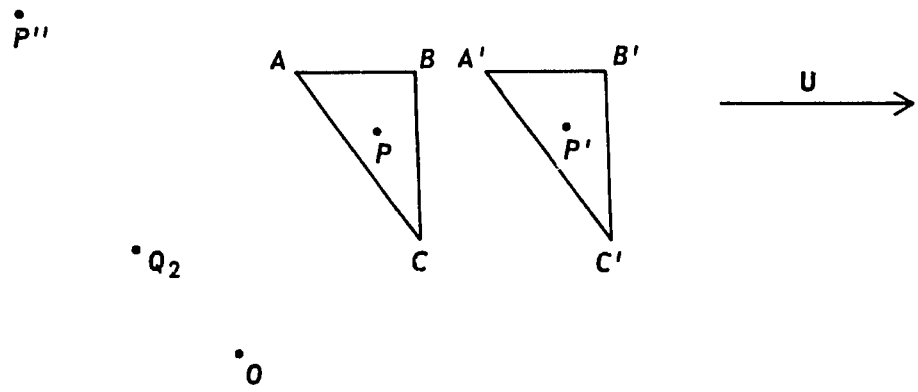


Fig. 94.

The resulting rigid motion, as you can check with tracing paper, is a rotation about the point Q_2. Clearly we get two different motions, depending upon the order in which we take R and S.

This example tells us an important fact about combinations of rigid motions: if R and S are *rigid motions, R followed by S need not be the same as S followed by R.*

In special cases, R followed by S and S followed by R may be the same. For example, if R and S are the translations given by **U** and **V** in the following figure, you can check that, in this case, R followed by S is the same as S followed by R. Each is the transformation T given by **W**.

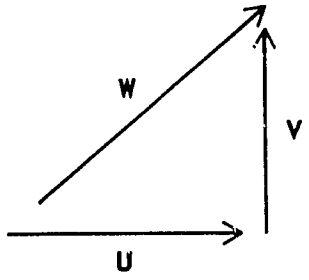


Fig. 95.

There are some general rules which we can use to tell what kind of rigid motion T is if we already know what kinds of motion R and S are. We now look at some of these rules.

TRANSLATION FOLLOWED BY TRANSLATION

We remind ourselves that in a translation every point in a line or figure moves the same distance and in the same direction, and that a translation can be described by means of a vector whose length gives the distance that every point moves and whose direction gives the direction that every point moves.

Fig. 96.

In Figure 96 the triangle A can be moved to A' by the translation given by vector **U** and A' moved to A'' by the translation given by vector **V**. Hence A is moved to A'' by translating first by **U** and then by **V**. But clearly A is also moved to A'' by the translation given by **W**. The three vectors are drawn to the right in Figure 96.

This example suggests that a translation followed by another translation must always be a translation. We can state this precisely as follows:

A translation R followed by a translation S equals a translation T. If R is given by a vector **R** and S is given by a vector **S** whose initial point is the end point of **R** (see Fig. 97) then T is given by vector **T**, with initial point at the initial point of **R** and end point at the end point of **S**.

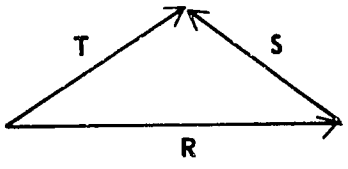


Fig. 97.

Try several pairs of translations like R and S on different figures to test the correctness of the above statement.

PROBLEMS 6-10A

1. In Figure 96 above, find two other translations besides **U** and **V** which have the same effect as **W** when used one after the other.

2. Draw a triangle A and two vectors **U** and **V**. Draw the image of A under **U**; i.e., draw A' where A is moved by **U**. Draw A'' where A' is moved by **V**. Draw a vector showing a single translation moving A to A''. Is this equal to **U** followed by **V**?

3. Draw three vectors **S**, **T**, **U** in any position. Draw a vector **V** showing a translation equal to translation **S** followed by translation **T**, followed by translation **U**.

4. Draw a vector **V** showing a translation. Draw two vectors **S** and **T** at right angles showing translations such that **V** equals **S** followed by **T**.

5. If A is moved to A' by translation **T**, where does **T** move A'? If A' is moved to A'' by **T**, where does **T** move A''? Draw a diagram showing A, A', A'', A''', A^{iv} where **T** moves each point to the next one to its right.

EXAMPLE

In Figure 98, the line segment $\overline{CD}$ can be moved to $\overline{C'D'}$ by a half-turn about H followed by a translation **W**. $\overline{CD}$ can also be moved to $\overline{C'D'}$ by a half-turn about K. Any other figure will be moved to the same position by the half-turn about K as by the half-turn about H followed by translation **W**. We say the half-turn about K equals the half-turn about H followed by the translation **W**.

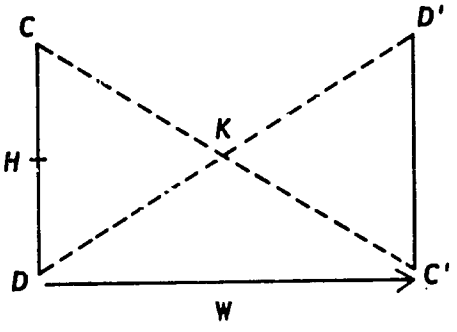


Fig. 98.

PROBLEMS 6-10B

1. In Figure 98, suppose $\overline{CD}$ is moved by translation **W** to a new position, say $\overline{C''D''}$. Can you find a half-turn which will move $\overline{C''D''}$ into $\overline{CD}$?

2. In Figure 98, can you find a half-turn about another point and a translation to follow it which is equal to a half-turn about K; i.e., moves $\overline{CD}$ to $\overline{C'D'}$?

3. Suppose $\overline{C''D''}$ in Problem 1 is moved by the half-turn at H; i.e., $\overline{CD}$ is moved by **W** followed by a half-turn at H. Where is the final position of $\overline{CD}$? Is it the same as $\overline{CD}$?

ROTATION FOLLOWED BY ROTATION

Consider a point P rotated about a point O through angle α to give P' followed by another rotation of P' through angle β to give P''. Clearly a rotation of P through angle $\alpha + \beta$ will also give P''.

However if the rotations are about two different centres the result is not so simple to obtain. Here we shall only consider combinations of some special rotations—such as half-turns—about different centres. We give two facts about such combinations.

(1) A half-turn about point H followed by another half-turn about H is the identity.

Can you show this?

(2) The result of a half-turn about a point H followed by a half-turn about a point K is a translation T, shown by a vector twice the vector from H to K.

To show this, let P be moved to P' by H and let P' be moved to P'' by K in Figure 99. Since H, K are the mid-points of the sides, $\overline{P'P}$ and $\overline{P'P''}$ of the triangle $PP'P''$, then $PP'' = 2HK$.

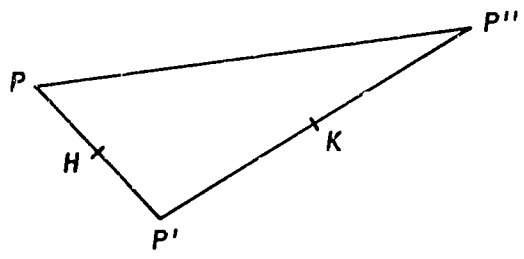


Fig. 99.

PROBLEMS 6-10C

1. Mark two points H, K about one inch apart as centres of half-turns. Take any three points P, Q, R and construct their images under the half-turn about H followed by the half-turn about K; i.e., find their images P', Q', R' respectively under the half-turn about H and then the images of these points P', Q', R' under the half-turn about K, which could be labelled P'', Q'', R'' respectively. What can you say about the length of the segments $\overline{PP''}$, $\overline{QQ''}$, $\overline{RR''}$? Are these segments parallel to each other?

2. Let S be a translation shown by a vector LL'. Let F and G be any two points whose distance apart is half the distance LL', and $\overline{FG}$ is parallel to $\overline{LL'}$. Show that the translation $S = GF$, where GF means performing the half-turn about F followed by the half-turn about G.

3. $ABCD$ are the vertices of a parallelogram. Show that the half-turn about A followed by the half-turn about B is the same motion as the half-turn about D followed by the half-turn about C.

4. Let KH be a half-turn about H followed by a half-turn about K. What would you mean by HK? Is $HK = KH$? If not, what is the difference?

5. Let H, J, K be three half-turns about points which lie on a line, equally spaced with J between H and K. Show that $KJH = J$. (How would you define KJH?)

6. P, Q, R, S are half-turns about points on a line shown in the diagram with $\overline{PQ} = \overline{RS}$.

   Show that $SQP = R$.

Fig. 100.

7. Let A, B, C, D be four half-turns about successive vertices of a parallelogram. Show that $CBA = D$ and that $ABCD$ is the identity.

REFLECTION FOLLOWED BY REFLECTION

We shall get information about a reflection followed by a reflection from the following problems.

PROBLEMS 6-10D

Consider two parallel reflecting lines L and m, with a point P first reflected along L to give P' and P' then reflected along m to give P''.

1. Suppose the distance of P from L is p, and the distance between L and m is d. What is the length of $\overline{PP''}$?

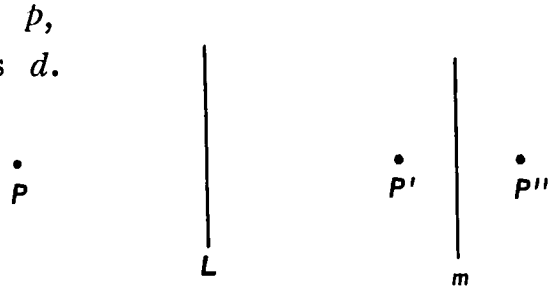


Fig. 101.

2. Use the same values as in Problem 1, but now P is reflected first along m to give P_1' and then P_1' is reflected along L to give P_1''. What is the length of $\overline{PP_1''}$?

3. Consider the line segment $\overline{PQ}$ in Figure 102. Can you determine the image $\overline{P''Q''}$ which results from reflecting first along L and then along m? Can you show that this double reflection is equivalent to a translation?

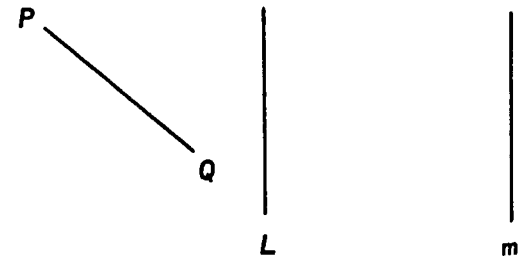


Fig. 102.

4. Replace the numbers 1 and 2 in Figure 103 by the image of B under reflection on f and the image of B under reflection on f, followed by reflection on g.

Fig. 103.

5. Let f and g be parallel lines three units apart. Find the second images of the following points when reflected first in f, then in g, where f is to the left of g.
   (a) P, one unit to the right of f.
   (b) Q, one unit to the left of f.
   (c) R, four units to the left of f.
   (d) S, one unit to the right of g.

6. Find the second images of the above points if they are reflected first in g, then in f.

We see from the solutions to the above problems that the result of two reflections along parallel lines is a translation shown by a vector perpendicular to the lines of reflections of length twice the distance between the parallel lines.

---

7. Let T be a translation shown by the vector **T**. Find two parallel lines of reflection f and g, so that f followed by g equals the translation T.

Fig. 104.

A BASIC THEOREM

Let R and S be rigid motions, and let T be the rigid motion R followed by S. We looked at some rules above for finding out what kind of motion T is, if we know what kinds of motion R and S are. We now look at several further facts that can be useful in getting information about T. These facts will not always tell us exactly what T is, but they can give some helpful information. The first of these facts is the following.

> If R and S are both direct, then T must be direct. If R and S are both reversing, then T must be direct. If R is direct and S is reversing, or if R is reversing and S is direct, then T must be reversing.

We see from the definition of direct and reversing why this fact is true. For example, if R requires that the tracing paper be turned over and S requires that the paper be turned over again, then clearly, in the resulting rigid motion T, the tracing paper has the same side up that it had to start with.

The following basic theorem gives an important and useful fact about direct rigid motions, as we shall see.

> THEOREM 6-1. Let T be a rigid motion. If T is direct, then either T is a translation or T is a rotation.

We shall not give the proof of this theorem here. Instead, we shall tell you how to find out exactly what kind of rigid motion T is, if you know that T is direct, and if you are given two triangles ABC and $A'B'C'$ such that $A' = T(A)$, $B' = T(B)$, and $C' = T(C)$.

Do the following:

Draw the segments $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$. At least two of these segments have positive length (otherwise the triangles are the same). Draw the perpendicular bisector line for each of the segments with positive length. At least two of these perpendicular bisector lines will be distinct. (Can you prove this?) We show some of the possibilities in the following figures. (In these figures, the perpendicular bisector lines are marked L_1, L_2, and L_3.)

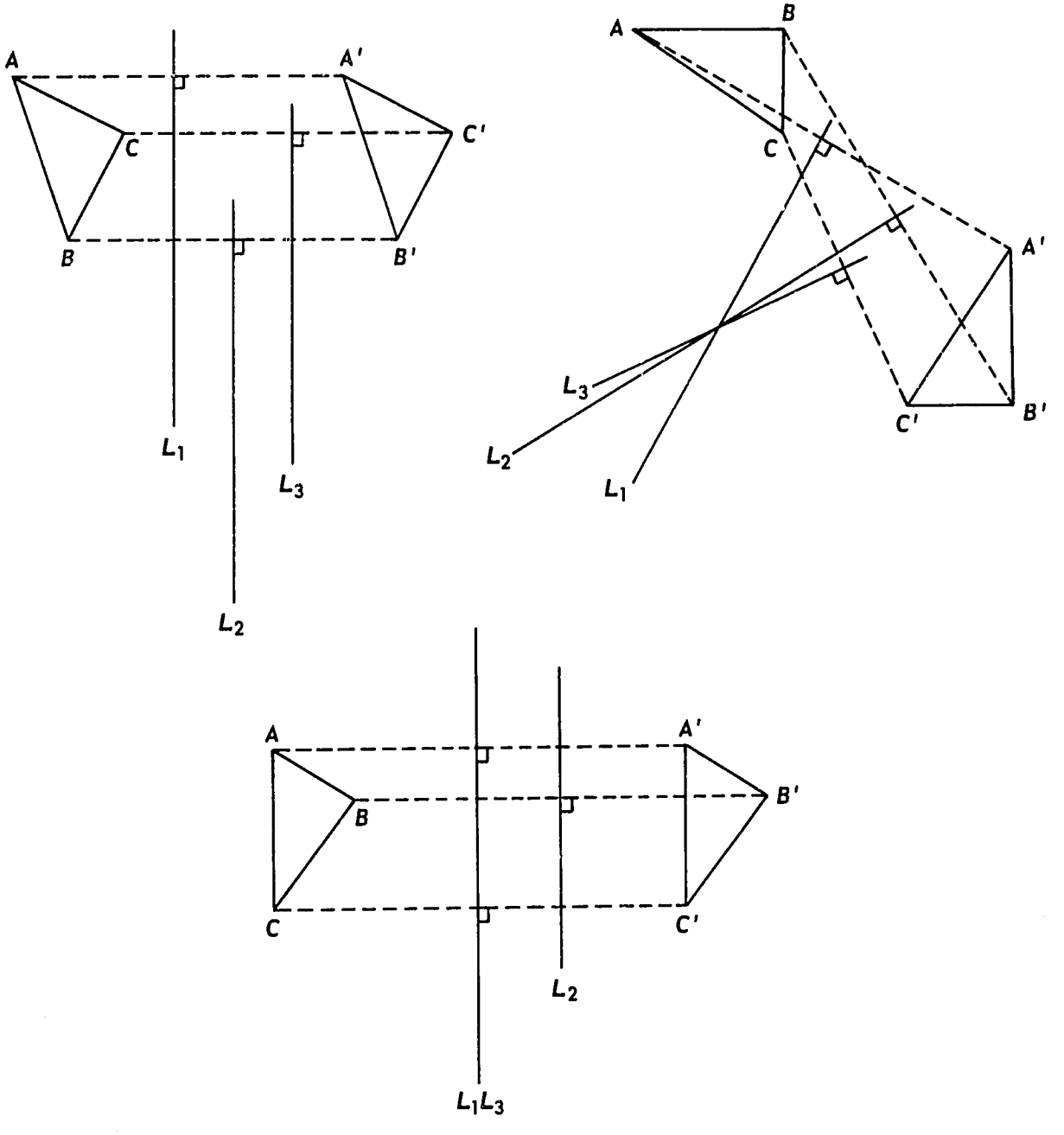


Fig. 105.

Choose two distinct perpendicular bisector lines. If these lines are parallel, then you have T as the translation which moves every point parallel to $\overline{AA'}$ through a distance AA'. If these lines intersect (call this point of intersection O), then you have T as the rotation about O which carries A to A'.

The method will always work. We do not prove this here. If you can prove that the method always works, you will be very close to having a proof for Theorem 6-1.

Theorem 6-1 is a strong tool for getting information about combinations of rigid motions. Look at the following example. In Figure 106, let R be a reflection about L_1 and let S be a reflection about L_2. Let T be R followed by S.

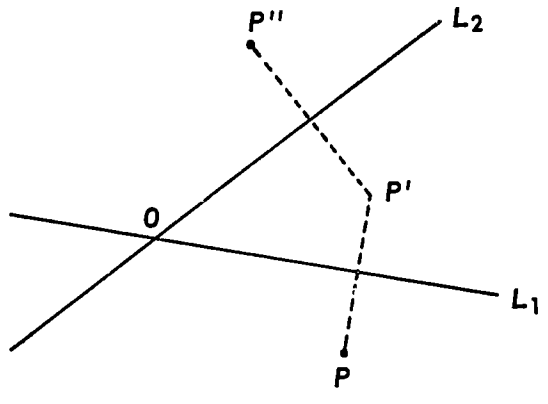


Fig. 106.

What kind of motion is T? We can find this very simply as follows. Since R and S are each reversing, we know that T must be direct. Hence, by Theorem 6-1, T must be either a translation or a rotation. But the point O is a fixed point of T. Hence T must be a rotation about O. (It cannot be a translation, since no translation, except the identity motion, can have a fixed point. T is not the identity motion, as we see from the points P, P', and P'' in the figure.) To find the angle of rotation, we need only measure the angle $P\hat{O}P''$ in the figure.

A second basic theorem can be given for rigid motions that are reversing. We state it below as Theorem 6-2, but do not consider it further.

DEFINITION 6-7. T is called a *glide-reflection*, if T is R followed by S, where R is a reflection and S is a translation given by a vector parallel to the line of reflection of R.

For example, if R is reflection in L, and S is the translation given by $\mathbf{U}$ in the following figure, then the motion R followed by S, which we are calling a glide-reflection, is the motion which takes ΔABC to $\Delta A''B''C''$.

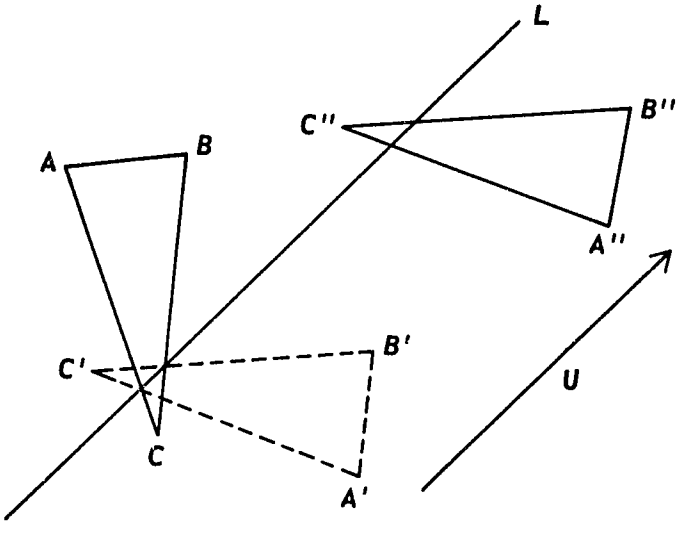


Fig. 107.

THEOREM 6-2. Let T be a rigid motion. If T is reversing, then either T is a reflection or T is a glide-reflection.

Hence we know from these two theorems that *every* rigid motion is either a translation, a rotation, a reflection, or a glide-reflection.

6-11 MORE EXAMPLES OF THE USE OF MOTIONS.

In Section 6-4 we looked at two examples of the use of rigid motions to solve problems. Both of these examples used translations. We now look at

several more examples. These new examples will use rotations and reflections as well as translations.

EXAMPLE 1. Take any triangle ABC. Construct equilateral triangles on $\overline{AB}$ and $\overline{BC}$ as in the figure.

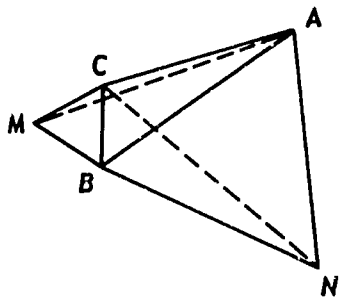


Fig. 108.

We wish to show that $AM = CN$.

A congruent triangle proof can be used to show this (by using SAS to get ΔABM congruent to ΔNBC). But the idea of a rotation can be used to get an even quicker proof as follows. *Let T be a rotation about B through an angle of 60^0.* Clearly, $T(N) = A$ (since $BN = BA$ and $m(N\hat{B}A) = 60^0$), and $T(C) = M$ (since $BC = BM$ and $m(C\hat{B}M) = 60^0$). Thus segment $\overline{AM}$ is the image of segment $\overline{CN}$ under T, and $AM = CN$ by the isometric property of rigid motions.

PROBLEMS 6-11A

1. In the figure for Example 1, let Q be the point of intersection of $\overline{AM}$ and $\overline{CN}$. What is the measure of $N\hat{Q}A$?

2. In the following figure, ABC is any triangle, and squares have been constructed on $\overline{AB}$ and $\overline{BC}$.

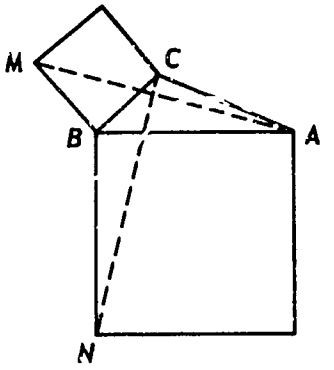


Fig. 109.

Use a rotation to show that $AM = CN$.

3. In Problem 2, show that $\overline{AM}$ is perpendicular to $\overline{CN}$.

EXAMPLE 2. Let ABC be a triangle (Figure 110) in which $m(B\hat{A}C) = m(B\hat{C}A)$.

It is a fact of elementary geometry that AB must be equal to BC. Let us assume, for the moment, that you have forgotten this fact, but that you do remember the basic properties of rigid motions. (These were listed as Properties (1) to (6) in Section 6-3). How could you use what you do know about rigid motions to prove that $AB = BC$?

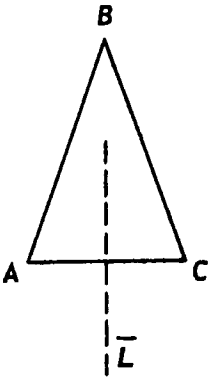


Fig. 110.

One very simple proof would be as follows. Take the perpendicular bisector of $\overline{AC}$, call it $\overline{L}$. Let T be the rigid motion: *reflection in* $\overline{L}$. Since T leaves the measure of an angle unchanged (Property (6) in Section 6-3), the image of $\overline{AC}$ under T must also be perpendicular to $\overline{L}$. Hence, since $\overline{L}$ is a bisector, the image of A is C and the image of C is A. Since angles $B\hat{A}C$ and $B\hat{C}A$ have equal measure, the image of angle $B\hat{A}C$ must be angle $B\hat{C}A$, and the image of angle $B\hat{C}A$ must be angle $B\hat{A}C$. Hence the image of line $\overleftrightarrow{AB}$ is line $\overleftrightarrow{CB}$ and the image of line $\overleftrightarrow{CB}$ is line $\overleftrightarrow{AB}$. Since point B lies on both $\overleftrightarrow{AB}$ and $\overleftrightarrow{CB}$, the image of B must lie on both $\overleftrightarrow{CB}$ and $\overleftrightarrow{AB}$. But this means that the image of B has to be B itself. Thus the image of segment $\overline{AB}$ is segment $\overline{CB}$, and the image of segment $\overline{CB}$ is segment $\overline{AB}$. By the isometric property, we have $AB = CB$, which is what we wanted.

PROBLEMS 6-11B

1. Let ABC be a triangle in which $AB = CB$. It is a fact of elementary geometry that $m(C\hat{A}B)$ must be equal to $m(A\hat{C}B)$. Assume, for the moment, that you have forgotten this fact, but that you do remember the basic properties of rigid motions. Use a rigid motion to prove that $m(C\hat{A}B) = m(A\hat{C}B)$. (Hint: let L be the bisector of $A\hat{B}C$ in Figure 111, and make a reflection in L.)

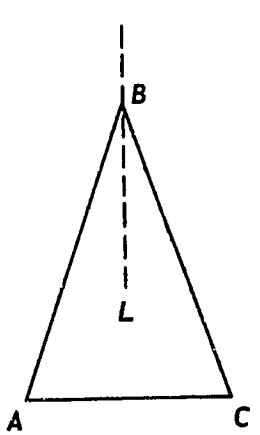


Fig. 111.

Note: As we saw in *Secondary Three*, it is possible to get the facts of geometry from a certain set of *axioms* about points, lines, angles, betweenness, and congruence. It is also possible to get the facts of geometry in an entirely different way by taking the idea of rigid motion as basic (along with the ideas of point, line, and angle), and by starting with a set of axioms which includes basic facts about rigid motions (such as those listed in Section 6-3). This set of axioms is quite different from the set of axioms used in *Secondary Three*. In this new deductive theory, we can get all the same facts of geometry that we got before, but the proofs are quite different, and the theorems appear in a different order from before. Example 2 above and the problem which follows it shows us what such proofs would be like. We do not consider this new theory any further, and we do not give the set of axioms for it here.

In the next two examples, we shall look at more difficult problems. In each of these problems, rigid motions can be used to get a solution which is shorter than any other solution known. In solving each problem, we will allow ourselves to use facts that we know about rigid motions together with any other facts of geometry that are already known to us.

EXAMPLE 3.

Let $ABCD$ be any convex quadrilateral. Construct equilateral triangles APB, BQC, CRD, and DSA as in Figure 112. Note that two triangles are constructed towards the inside of the quadrilateral and two are constructed towards the outside.

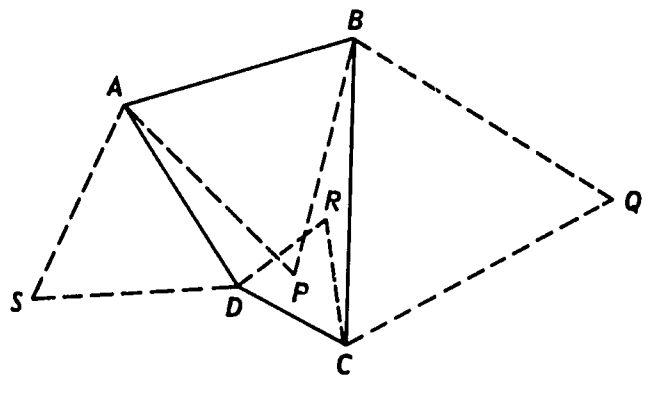


Fig. 112.

We wish to show that the points R, Q, P, and S must form the vertices of a parallelogram. We use rigid motions to prove this as follows.

Let T_1 be a rotation through 60^0 about A, and let T_2 be a rotation through -60^0 about C. Let U be the rigid motion: T_1 followed by T_2. Since T_1 and T_2 are direct, U must be direct. Hence by Theorem 6-1 of Section 6-10, U must be a translation or a rotation. If it is a rotation through an angle β, then, by our work at the end of Section 6-8, every line makes an angle β with its own image. Let L be any line. Let L_1 be the image of L under T_1 and let L_2 be the image of L_1 under T_2. Then L_1 forms an angle of 60^0 with L, and L_2 forms an angle of -60^0 with L_1. Hence L_2 forms an angle of 0^0 with L. *Hence, if* U *is a rotation, it can only be a rotation through* 0^0, *which is the identity motion.* Thus U must be either the identity motion or a translation.

Let us see where U carries the points S and P. T_1 carries S to D and T_2 carries D to R. Hence $U(S) = R$. Also, T_1 carries P to B, and T_2 carries B to Q. Hence $U(P) = Q$. Thus U is a translation which carries segment $\overline{SP}$ to segment $\overline{RQ}$.

By the isometric property of U, $SP = RQ$. Since every translation carries a line parallel to itself (or else onto itself), $\overleftrightarrow{SP}$ is parallel to $\overleftrightarrow{RQ}$. Since the quadrilateral $SQPR$ has a pair of opposite sides which are equal and parallel, it must be a parallelogram. This is what we set out to show.

EXAMPLE 4. In the following figure, let ABC be any triangle. ABM, BCN and CAO are equilateral triangles constructed on the sides of triangle ABC. Let P, Q, R be the centres of these three equilateral triangles. (By *centre* of an equilateral triangle we mean the point at which the three perpendicular bisectors of the sides meet. This is the point about which an equilateral triangle has rotational symmetry.)

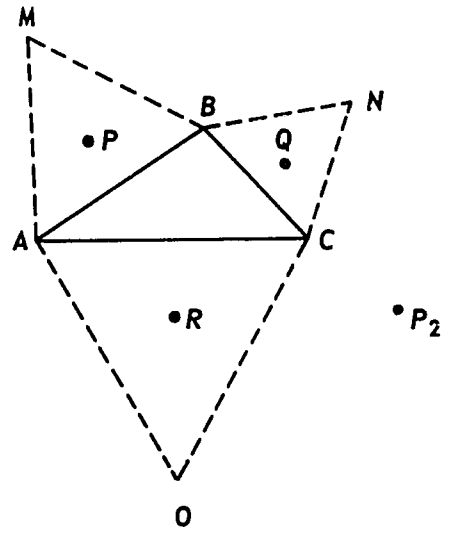


Fig. 113.

We wish to show that P, Q, and R must form the vertices of an equilateral triangle.

Let T_1 be a rotation through 120^0 about P. Let T_2 be a rotation through 120^0 about Q, and let T_3 be a rotation through 120^0 about R. Let U be the rigid motion: T_1 followed by T_2 followed by T_3. U is the result of three rotations, each of 120^0. Hence, in the same way as in Example 3, we see that U is either a translation or a rotation through 360^0. But this means that U must be a translation or the identity motion.

From the figure, we see that $T_1(A) = B$, $T_2(B) = C$, and $T_3(C) = A$. Thus $U(A) = A$, and A is a fixed point of U. But the only translation which has a fixed point is the identity motion. Hence U must be the identity motion.

Now take the point P. Let P_1 be the image of P under T_1, let P_2 be the image of P_1 under T_2, and let P_3 be the image of P_2 under T_3. Since T_1 is a rotation about P, we see that P_1 is the same as P. Since U is the identity motion, we see that P_3 is the same as P. Hence P, Q, P_2 and R form the following figure.

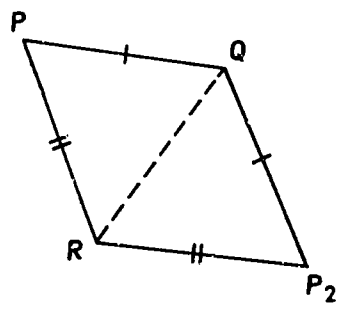


Fig. 114.

In this figure, $PQ = P_2Q$ by the isometric property of T_2, and $P_2R = PR$ by the isometric property of T_3. Hence, by SSS, triangles PQR and P_2QR are congruent. Hence $m(P\hat{Q}R) = m(P_2\hat{Q}R)$ and $m(P\hat{R}Q) = m(P_2\hat{R}Q)$. But $m(P\hat{Q}P_2) = 120^0$ since T_2 is a rotation through 120^0, and $m(P\hat{R}P_2) = 120^0$ since T_3 is a rotation through 120^0. Hence $m(P\hat{Q}R) = \frac{1}{2}\, m(P\hat{Q}P_2) = 60^0$, and $m(P\hat{R}Q) = \frac{1}{2}\, m(P\hat{R}P_2) = 60^0$. But any triangle with two angles of 60^0 must be equilateral. This is what we set out to prove.

*APPENDIX TO CHAPTER 6

In Section 6-3, we listed six properties of rigid motion, and we said that we could prove Properties (2), (3), (4), (5), and (6) from Property (1) (the isometric property) without using the idea of tracing paper at all. We give these proofs here.

Let T be a mapping from the plane to the plane which has Property (1).

Proof of Property (2). If P and Q are distinct, then $Q > O$. By Property (1), $P'Q' = PQ$. Hence $P'Q' > O$, and P' and Q' must be distinct.

Proof of Property (3). Let Q be any given point. We shall find a point P such that $T(P) = Q$. Take any triangle in the plane and let P_1, P_2, and P_3 be its vertices. Let P_1', P_2', and P_3' be the images of P_1, P_2, and P_3 under T. By Property (1), $P_1'P_2' = P_1P_2$, $P_2'P_3' = P_2P_3$, and $P_3'P_1' = P_3P_1$. Hence, by SSS, P_1', P_2' and P_3' form the vertices of a triangle (and $\Delta P_1P_2P_3 \equiv \Delta P_1'P_2'P_3'$). This means that the points P_1', P_2', P_3' do not all lie on the same straight line. Now at least one out of the following three cases must occur. Either (i) P_1 is the same point as Q, or (ii) Q does not lie on $\overleftrightarrow{P_1'P_2'}$, or (iii) Q does not lie on $\overleftrightarrow{P_1'P_3'}$.

(i) If P_1 is the same point as Q, then P_1 is the point P that we want.

(ii) If Q does not lie on $\overleftrightarrow{P_1'P_2'}$, we do the following. With P_1 as centre, we draw a circle of radius $P_1'Q$, and with P_2 as centre, we draw a circle of radius $P_2'Q$. Let M and N be the points of intersection of these two circles. We now have a figure that looks like the following.

Fig. 115.

Take the image of M, which we call M', and the image of N, which we call N'. By Property (1), $P_1'M' = P_1'N' = P_1Q$, and $P_2'M' = P_2'N' = P_2'Q$. Since M and N are distinct, M' and N' must be distinct (by Property (2) which we have already proved). Hence M' and N' fall on opposite sides of $\overleftrightarrow{P_1'P_2'}$. By SSS, $\Delta P_1'P_2'Q \equiv \Delta P_1'P_2'M' \equiv \Delta P_1'P_2'N'$. Hence, if Q is on the same side of $\overleftrightarrow{P_1'P_2'}$ as M', then Q must be the same point as M', and if Q is on the opposite side of $\overleftrightarrow{P_1'P_2'}$ from M', then Q must be the same point as N'. If Q is the same as M', then M is the point P that we want; if Q is the same as N', then N is the point P that we want.

(iii) If Q does not lie on $\overleftrightarrow{P_1'P_3'}$, we carry out the same proof as for (ii), but use P_3' in place of P_2'. We again find the point P that we want.

Thus in all three cases we can get the point P that we want. This ends the proof of Property (3).

Proof of Property (4). Let L be a given straight line. Take two distinct points P_1 and P_2 on L. Let P_1' be the image of P_1 and let P_2' be the be the image of P_2. Let L' be the line determined by P_1' and P_2'. To prove (4) we need to show that every point on L has its image on L', and that every point on L' is the image of some point on L.

Let Q be any point. By Property (3), which we have already proved, there is a point P such that $Q = T(P)$. If Q is not on L' then, since the triangle $P_1P_2P_3$ is congruent to the triangle $P_1'P_2'O$, P is not on L. Conversely, if P is not on L, then, by the same congruent triangles, Q is not on L'. This gives us the result we want, because it shows that every

point on L must have its image on L' and, by Property (3), it shows that every point on L' must be the image of some point on L.

Proof of Property (5). By Property (4), which we have proved, $P'\widehat{Q'}R'$ is the image of $P\widehat{Q}R$. Since, by Property (1), and SSS, $\Delta PQR \equiv \Delta P'Q'R'$, we have that $P'\widehat{Q'}R' = P\widehat{Q}R$ in measure.

Proof of Property (6). Let L_1 and L_2 be two parallel lines. Let L_1' be the image of L_1, and let L_2' be the image of L_2. By Property (4), which we have proved, L_1' and L_2' are straight lines. If L_1' and L_2' intersect at some point Q, then, by Property (3) and Property (4) there is a point P such that $T(P) = Q$, P lies on L_1, and P lies on L_2. But if P lies on L_1 and L_2, L_1 and L_2 cannot be parallel. This shows that L_1' and L_2' must be parallel.

Thus we have proved Properties (2), (3), (4), (5), and (6) from Property (1).

LIST OF THEOREMS AND OTHER IMPORTANT RESULTS

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THEOREM.	The medians of a triangle meet in a single point O. For each median this point is located $\frac{2}{3}$ of the way from the vertex to the opposite side.	13
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THEOREM 2-6.	If a right triangle has legs of lengths a, b, and hypotenuse of length c, then $c^2 = a^2 + b^2$.	53
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