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SECONDARY FOUR GEOMETRY

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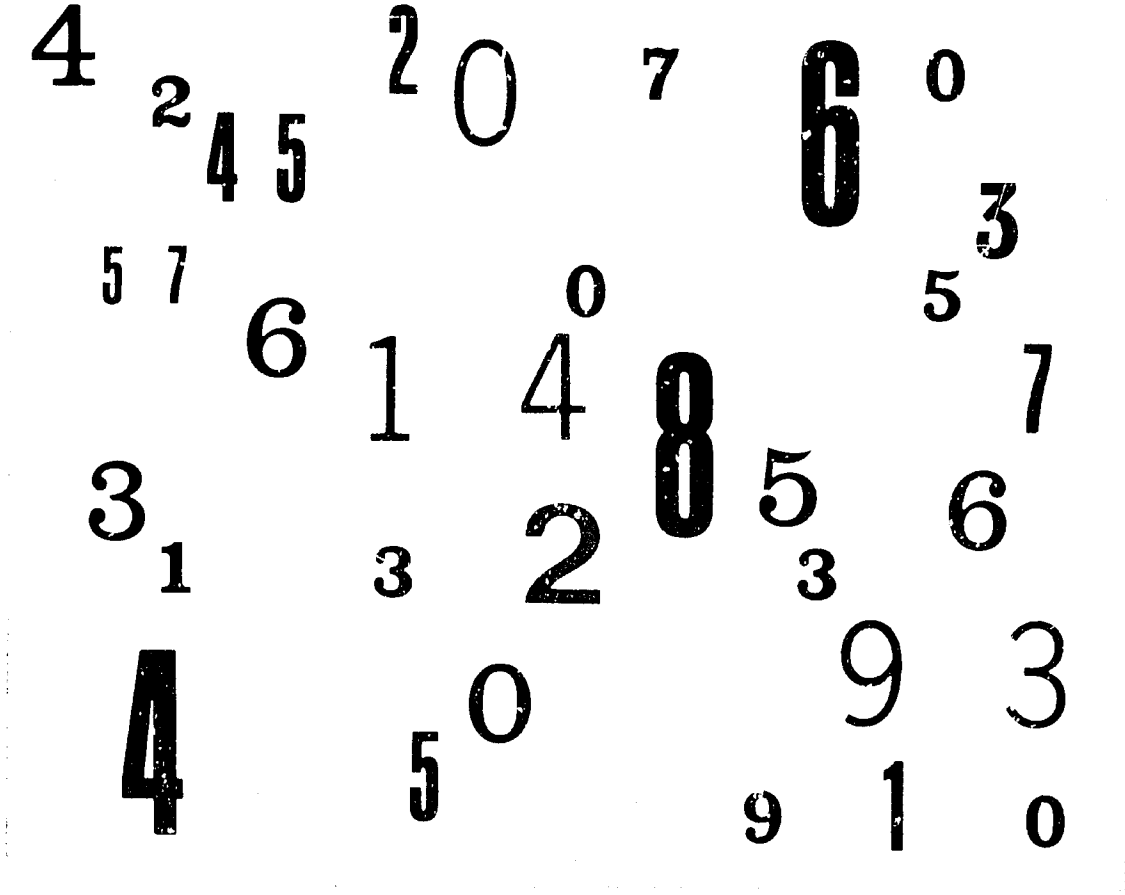

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Teachers'
Guide

ENTEBBE MATHEMATICS SERIES

Secondary Four Geometry
Preliminary Edition

Prepared at the 1965
Mombasa Mathematics Workshop

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PREFACE

This Teachers' Guide accompanies the text which was prepared at a study and writing workshop held during the summer of 1965 at Mombasa, Kenya, in which mathematicians and teachers from most English-speaking countries of Tropical Africa, the United States and the United Kingdom participated.

This material is intended to follow *Secondary Three*, *Secondary Two*, and *Secondary One* of the Entebbe Mathematics Series which were prepared at similar workshops during the summers of 1964, 1963, and 1962 at Entebbe, Uganda.

I. Preliminary Nature of This Material

The material must be considered to be preliminary until it has been tried out in class-rooms and modified in accordance with the experience gained from this experimentation. The Student Text and Teachers' Guide for SECONDARY FOUR ALGEBRA and GEOMETRY of the Entebbe Mathematics Series will be subject to correction and improvement in the light of suggestions from the teachers who use them.

This means that the teacher who uses this material has the responsibility, in addition to helping interpret the new material to students, of helping identify the areas where improvement is needed. To guide us in future revisions, we hope that teachers will fill out Secondary Chapter Reports. (See sample Report on pages ix, x, xi, xii.) In partial reward for this burden, the teacher will have the satisfaction of knowing that he or she is taking part in an experiment which is of great potential value.

II. Emphasis on Mathematical Ideas

In recent years there has been accumulated much evidence that young students are far more interested in mathematical *ideas* than they have usually been given credit for, and that they are far more competent to deal with such ideas than the current curricula would suggest. A presentation of mathematics which puts its emphasis on concepts rather than the rules of manipulation is likely to lead to far greater satisfaction on the part of the student, and will also lead to greater mathematical competence.

III. Content of the Text

The subjects of algebra and geometry are treated in two volumes; one is devoted to algebra and the other to geometry, as in *Secondary Three*. The material is arranged so that there is considerable flexibility in the order in which the topics can be studied, although Chapter 1 of ALGEBRA should be studied before beginning Chapter 6 of GEOMETRY.

The treatment of algebra continues the start made in *Secondary One*, *Two*, and *Three* to develop algebra as a unified structure. All five chapters are about functions. Chapter 1 treats the general concept of function. In Chapters 2 and 3 the pupil studies some of the most important types of functions; the rational functions and the exponential and logarithmic functions. As a result of having studied the polynomial functions in *Secondary Three* and the functions in Chapters 2 and 3 of this volume, the pupil will be familiar with all the basic types of elementary functions except the trigonometric functions which will be studied in *Secondary Five*. In Chapter 4 the domain is narrowed to the set of positive integers, with emphasis on two of the most important types, arithmetic progressions which are really just linear functions with the domain restricted to the positive integers, and geometric progressions, which are really just exponential functions with the domain restricted to the positive integers. Finally the central concept in Chapter 5 is that of a function whose domain is a finite set. These functions are studied in the context of descriptive statistics. The finite set is called a population and the functions are called population functions. Chapter 1 should be done first, so that the student will see the other four chapters as the study of special types of functions. The remaining four chapters are independent of each other and may be studied in any order.

As the pupils will have studied formal, deductive geometry in *Secondary Three*, there is now no need to be so formal in SECONDARY FOUR GEOMETRY. He will have already been exposed to formal proofs and will by this time have an idea about rigour in these proofs. Therefore, Chapter 1 of SECONDARY FOUR GEOMETRY indicates how much loosening up from *Secondary Three* occurs in this volume. The pupils however, are now equipped to be able to take any of the theorems or examples in the text and write out as formal a proof as they wish. Chapters 2 to 5 follow the pattern of *Secondary Three* except that not so much time is spent on the formal aspect of deductive reasoning. Chapter 6 is quite different both in content and spirit from the other chapters. The treatment of rigid motion is informal and intuitive. This approach avoids the complications of establishing a set of axioms from which the basic facts about rigid motion can be deduced. It should be mentioned that it is quite possible to develop all our geometric ideas by starting off with rigid motions. Except for Chapter 6 which could be studied any time after Chapter 1 of ALGEBRA, the chapters are best studied in the order in which they appear. Teachers who prefer the informal, intuitive treatment of geometry may wish to spread out the study of Chapter 6 throughout the school year.

IV. The General Approach

As in previous texts we try to be mathematically sound but at the same time the development is gradual with an effort to involve the pupils actively. Instead of just telling the ideas to the pupils outright, the teacher should use problems and careful class discussion to help the pupils recognize the ideas for themselves.

The best education is the education that the pupil creates from his own direct efforts. The teacher should resist the temptation to tell the class exactly what to do and how to do it. It is indeed a great temptation, for by such means a class will appear to be proceeding at a rapid pace. If the teacher takes the time and effort to lead the pupil to think through the ideas himself without telling him outright, there is considerably more assurance that the idea will be mastered and retained by the pupil and will become truly his own.

Here again the demands on the teacher are greater. The teacher, like the student, must be thinking at every moment, for it takes far more insight to lead than to tell. This Guide is intended primarily to assist the teacher in the actual conduct of the class. Methods are suggested which will encourage student imagination and generate student interest. In addition the Guide contains mathematical background and explanatory materials beyond that given in the Text. Answers to the problems are provided.

V. Relationship to Other Materials

This Text is based on the assumption that the pupil has studied from the previous Texts in the Entebbe Mathematics Series. If a pupil has not studied all of the earlier texts it may be necessary to go back and study portions of *Secondary Two* and *Secondary Three*. However, the pupil can make good progress in the present Text without having a complete mastery of these previous Texts.

This Text is designed to prepare the pupil for further mathematics in either the conventional school curriculum or in curricula evolving from this and similar experimental programmes. It is also designed to conform to the requirements of the present examinations.

It should be emphasized that this modified curriculum represents a great opportunity for teacher and pupil alike. In making use of it the school participates in a great experiment to help develop a strong African educational system of which we can be proud. It is a massive joint undertaking, in which pupils and teachers work side by side with mathematicians of international eminence, from their own country and from a dozen others, to create within Africa something that will be of major significance for Africa itself, and in a large measure for all the world.

ENTEBBE MATHEMATICS SERIES

SECONDARY _____ REPORT, CHAPTER _____

Would you please fill out one of these forms for each chapter of Secondary material as you complete the chapter. Send it by air mail to educational Services Incorporated, African Education Program, 55 Chapel Street, Newton, Massachusetts 02158, U. S. A. Your comments will help us to improve and to rewrite the material for further use.

Please respond to *all* parts of this report.

Name of Teacher _____

Name of School _____

School Address _____

Date _____

Chapter Number _____ in Secondary Book _____ Volume _____

Chapter Title _____

Number of classes in which you teach the Entebbe Mathematics Workshop Secondary Materials _____

_____

Number of pupils using material _____ . Number of periods each class meets per week _____ .

Length of class period _____ minutes. Class periods used teaching the chapter _____.

The pupils of my class are of the following ability level (tick one of the following):

fast _____ , slow _____ , medium _____ , mixed ability _____ .

My pupils have had, on the average, _____ years of schooling before this present year.

To the best of my judgment this chapter is (tick one of the following): excellent _____ ,

good _____ , fair _____ , poor _____ , or extremely poor _____.

Why did you rate the chapter in this way? (If you need more space than is provided here, please use an additional sheet.) _____

_____

_____

If you used any supplementary materials in teaching this unit, please describe them. _____

_____

_____

Please indicate the specific sections which need improvement and how *you* would recommend improving them.

Questions	Comments
1. Should this whole chapter be omitted? Why?	
2. Name specifically any ideas which you think ought to be in this chapter but which are not there. Why should they be included?	
3. Name specifically any topics in this chapter which you think should be omitted. Why should they be omitted?	
4. Give the section and page number of any part of this chapter where you think the explanation is not clear enough. What suggestions do you have on how to make it more clear?	
5. Indicate which sets of problems need more problems to give adequate practice. What kind of problems are needed?	
6. Indicate by page and number any problems which you think are too difficult and problems which you think are too easy.	

STUDENT TEXT

(continued)

Questions	Comments
7. Indicate any problems or illustrations which are ineffective because they use words or ideas which are unfamiliar to your students. For instance, a child who had never been to sea in a schooner would not much appreciate a problem about the area of the jib and the foresail and the mainsail. For the examples which are inappropriate for your children, can you suggest local situations which would be more appropriate in the example.	

TEACHERS' GUIDE

Questions	Comments
1. In what parts of the chapter were the suggestions on how to teach the material not sufficient to help you teach it effectively? What suggestions have you for improving those parts?	
2. In what parts of the chapter were the suggestions on teaching the material trivial and unnecessary?	

Questions	Comments
3. In some places in the Teachers' Guide there are discussions for the teacher of the mathematical background, going more deeply into the ideas than one would with the students. Indicate by section and page number: (a) which of these discussions were not clear enough to be helpful, (b) which were too brief and need to be expanded, (c) which were unnecessary for you and might be omitted, (d) where no such discussion was given where it was needed.	
4. Indicate any answers to problems which were not sufficiently clear or complete.	

Would you please write any other comments which you believe might be helpful in rewriting this chapter.

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Chapter 1

ANOTHER LOOK AT THE DEDUCTIVE METHOD

The purpose of this chapter is to refresh the pupils' understanding of inductive and deductive reasoning and to provide another critical look at the deductive method now that they have had substantial experience in *Secondary Three*. Much of the material is written for self-reading, and it is not expected, therefore, that more than four or five lessons be spent on this chapter.

The effect of this chapter on the pupils should be a de-emphasis of the postulational approach to geometry. Although it might appear that the geometry of *Secondary Three* was largely interested in such an approach, it actually had *three* main goals in mind:

(1) the discovery of new geometrical facts (through inductive reasoning),

(2) the pupils were to learn to use deductive proofs as a means of convincing themselves and others of the truth of these newly discovered facts, and

(3) geometry was to be revealed as a deductive science, all of its theorems being obtained by deductive arguments from a small number of initial statements called postulates.

The first two of these goals are by far the most important at this stage of the pupils' development. The third is largely of historical and aesthetic interest. Unfortunately, due to the large amount of detail and care required to achieve the third goal, it is frequently the case that the first two are obscured. There is also danger that the pupils may lose confidence in their own geometric intuition because of the fact that they spend a long period of

time proving theorems that they were already convinced of anyway. And they may come to feel that deductive proofs are something peculiar to geometry and that they are somehow inseparable from the postulates and long chains of theorems.

In studying this chapter, the pupils should be reassured that their common sense, imagination, and geometric intuition are supreme when it comes to discovering new facts. This is *inductive* reasoning, and they should not become bogged down in minute details during this activity. Only after they have discovered a new idea which they think likely to be true should their thoughts turn to constructing a proof to convince themselves and others that it is really true. And then they should realize that a deductive proof is nothing more than a reasonable, common sense argument that starts with certain hypotheses and shows *why* the conclusion should be believed if the hypotheses are believed.

In each of the examples of this chapter, the particular geometrical problem chosen for the example is inessential. What *is* important is the way in which inductive and deductive reasoning work together and the reduced emphasis placed on the postulational structure of geometry. The *methods* and *attitudes* and not the results should be emphasized. Only a few of the specific theorems are used in later work, although several of them are interesting in their own right. This is especially true of the discussion of Euler's formula and its application to regular polyhedra in Sections 1-5 and 1-6. These two sections provide an excellent opportunity to see inductive reasoning and deductive proofs used in a more relaxed context that cannot be confused with the narrower purpose of the postulational development of geometry. In this respect, these two sections are much more typical of more advanced mathematics as it is done today than is most of the geometry studied so far.

1-1 WHY DO WE PROVE THEOREMS?

This section is written for the pupil to read himself. The amount of discussion in class can be varied according to the amount of time available.

One of the main points of the section is that the pupils already have had enough experience with circles, triangles, and other geometrical objects to

discover and prove new theorems without being overburdened with new definitions and postulates. In studying geometry it is *not* necessary to be so highly organized and to prove theorems in exactly the right order as must be done when the postulational approach is the main item of interest. To make this point strongly, it is helpful to point out that the argument about the angle-in-circle theorem which accompanies Figure 4 can be made perfectly convincing to pupils who have studied only the geometry of Secondary One and Two but have *not* studied the geometry of Secondary Three at all.

It would be interesting to have the pupils prepare a list of additional geometrical facts which they feel are so natural and fundamental that they could be used without proof in starting the study of geometry. The postulates of last year will likely be placed in such a list, but in general so will many other facts which were proved from the postulates as theorems. This is the time to point out that one may choose as many statements to use as postulates as he wishes. But of course only those persons who accept these statements as true geometrical facts will necessarily be convinced that his theorems are also true.

In discussing the main example of this section, it is interesting to have several pupils measure the angle $B\hat{C}A$ in Figure 2 and to have several others measure the angle $B\hat{O}A$. It is unlikely that they agree exactly on their measurements or that the first group finds $B\hat{C}A$ to be exactly half as large as $B\hat{O}A$. The need for a deductive argument becomes very apparent.

Some pupils may recall the exterior angle theorem from *Secondary Three* and may want to use it in the proof that $m(B\hat{C}A) = \frac{1}{2}\, m(B\hat{O}A)$. This is very good, and the opportunity should be used to emphasize that different proofs may be given depending on the experience of the audience.

The essential ingredient of a correct proof will be discussed in Section 1-2.

Answers to
PROBLEMS 1-1

Student Text Pages 3-5

1. This is a numerical exercise using the result of the text. It should not

be emphasized.

(a) $m(B\hat{A}O) = \frac{1}{2}(105) = 52{\cdot}5°$

(b) $m(B\hat{O}A) = 180-105 = 75°$

(c) $m(B\hat{C}O) = \frac{1}{2}(75) = 37{\cdot}5°$

(d) $m(C\hat{B}O) = m(B\hat{C}O) = 37{\cdot}5°$

(e) $m(O\hat{B}A) = m(B\hat{A}O) = 52{\cdot}5°$

(f) $m(C\hat{B}A) = m(C\hat{B}O) + m(O\hat{B}A) = 37{\cdot}5 + 52{\cdot}5 = 90°$, or

$m(C\hat{B}A) = 180 - m(B\hat{C}O) - m(B\hat{A}O) = 180 - 37{\cdot}5 - 52{\cdot}5 = 90°$

2. The angle $C\hat{B}A$ is, of course, a right angle. This follows from the following computation: (this is a proof)

(a) $m(B\hat{O}C) = x$

(b) $m(B\hat{A}O) = \frac{1}{2}x$

(c) $m(B\hat{O}A) = 180 - x$

(d) $m(B\hat{C}A) = \frac{1}{2}(180 - x) = 90 - \frac{1}{2}x$

(e) $m(C\hat{B}A) = 180 - m(B\hat{C}A) - m(B\hat{A}O) = 180 - (90 - \frac{1}{2}x) - (\frac{1}{2}x)$

$= 90 - \frac{1}{2}x + \frac{1}{2}x = 90°.$

3. This result follows by adding the measures of the angles $B\hat{A}D$ and $D\hat{A}C$.

4. Again, this is a numerical exercise and should not be emphasized.

(a) $m(C\hat{A}D) = \frac{1}{2}m(C\hat{O}D) = m(C\hat{B}D) = 37°$

(b) $m(C\hat{O}D) = 2\,m(C\hat{B}D) = 2(37) = 74°$

(c) $m(O\hat{C}D) = m(O\hat{D}C)$, therefore $m(O\hat{C}D) = \frac{1}{2}(180 - m(C\hat{O}D)) =$

$= \frac{1}{2}(180 - 74) = 53°$

Angles $A\hat{C}B$ and $B\hat{D}A$ are congruent, because the measure of each is half of the measure of $A\hat{O}B$

(draw the segments $\overline{AO}$ and $\overline{BO}$).

5. The reasons to be given are the following: (the wording may vary, of course)

1. The base angles of an isosceles triangle are congruent.
2. Definition of congruence of segments.
3. Two right triangles are congruent if a leg and an acute angle of one are congruent respectively to a leg and an acute angle of the other.
4. Corresponding parts of congruent triangles.

5. By definition, the congruent angles $B\hat{A}D$ and $C\hat{A}D$ have the same measure.

The hypotheses used in the argument are 1 and 3.

6. The triangle which is suspended by the trisectors is equilateral. A routine proof using trigonometry or coordinate geometry can be given—but it is quite lengthy. Emphasize the use of inductive reasoning to discover this fact. It is hardly an obvious fact and should motivate the need for the stronger tools of coordinate geometry and trigonometry.

7. The triangle is equilateral. Again—a routine proof (although lengthy) can be given using coordinate geometry. It is important that students realize that there are many facts that they can discover easily but which are not readily proved with the tools they presently have available. This is one of the reasons why mathematics continues to grow and develop.

1-2 WHAT IS A CORRECT PROOF?

The pupils have written many proofs in *Secondary Three* and are expected to have some feeling for the nature of correct proofs. But this is the time to re-emphasize that the main purpose of a deductive proof is to *convince*. There is *no* set format, style, or rules that must be followed, and giving deductive proofs does *not* require the context of postulates. Nevertheless, all correct deductive arguments have two main features:

1) There are certain statements, called *hypotheses* or *assumptions*, which are accepted to start with, and

2) There is an argument which proceeds to show in a step-by-step manner *why* the *conclusion* must be true if the hypotheses are true.

In order that the proof be convincing for a given individual, he must accept the truth of the hypotheses and must understand and believe each step of the argument. Certainly both of these things depend on his experience and maturity. A proof that is convincing to a mathematician may not be at all convincing to the pupils. And similarly, a proof that is convincing to the pupils this year may have been incomprehensible to them last year.

The two-column form of proof is frequently used to teach pupils how to write proofs. It has the advantage of *reminding* the pupils that they must be able to give a reason for every statement that they make and that they must know clearly what hypotheses were used. But they should become aware that writing proofs in this way is largely a teaching device and that they should be able to carry over the same clarity and orderliness into a good English paragraph. They should learn to use discrimination as to which of the reasons to emphasize and which ones to leave for the reader to fill in for himself. And they should be reassured that there is no easy path to skill in writing proofs. It is very similar to learning to write good English—only practice and criticism and more practice lead to a good sense of proof.

Answers to
PROBLEMS 1-2

Student Text Pages 8-9

1. The reasons to be filled in are the following (the particular wording is not important).

   1. Construction.
   2. Both are right angles by construction.
   3. Definition of length of a segment.
   4. Two sides and the included angle of ΔPMA are congruent respectively to two sides and the included angle of ΔPMB (*SAS* Postulate).
   5. Corresponding parts of congruent triangles.

   The only hypothesis is reason number 4.

2.

STATEMENTS	*REASONS*
1. $\overline{OQ}$ is perpendicular to line ℓ	1. There is a unique perpendicular that can be drawn from a given point to a given line.
2. P is different from Q	2. Construction: $O\hat{P}Q$ is not a right angle.

3. $m(Q\hat{P}O) + m(P\hat{O}Q) + (P\hat{Q}O) = 180$	3. The sum of the measures of the angles in any triangle is 180°.
4. $m(Q\hat{P}O) + m(P\hat{O}Q) = 90$	4. Follows from 3 and the fact that $m(P\hat{Q}O) = 90$.
5. $m(Q\hat{P}O) < 90$	5. Follows from 4 and the fact that $m(P\hat{O}Q) > 0$.
6. $OQ < OP$	6. If two angles of a triangle are unequal, then the lengths of the sides opposite these angles are unequal, and the larger side is opposite the larger angle.
7. Q is inside the circle.	7. The inside of a circle is the set of points whose distance from the centre is less than the radius.
8. ℓ intersects the circle twice	8. A line which passes through a point inside a circle must intersect the circle twice.

The hypotheses in this argument are 1, 3, 6, and 8.

1-3 AN EXAMPLE OF AN INCORRECT PROOF.

As is often the case when we are learning about new methods and ideas, it is instructive to see what can happen when the methods are misused. The example in this section should help the pupil to understand the importance of knowing what all of his hypotheses are.

The configuration in Figure 5 is impossible, but it is not immediately obvious that this is the case. Drawing such a figure involves making certain assumptions, in this case the *false* assumption that the angle bisector and the perpendicular bisector intersect *inside* the triangle.

It is excellent exercise for the pupil to examine the proof carefully, list the hypotheses upon which the proof was built, and decide which hypothesis is at fault. The given proof contains, in fact, a perfectly correct deductive

argument But one need not be convinced that the conclusion is true unless he is convinced that all of the hypotheses are true.

In the argument accompanying Figures 5 and 6, the triangles ΔDEA and ΔDEC are congruent because two sides and the included angle of one are congruent respectively to two sides and the included angle of the other. The triangles ΔBDG and ΔBDF are congruent because they are right triangles with the hypotenuse and an acute angle of one congruent respectively to the hypotenuse and an acute angle of the other. And, finally, the triangles ΔADG and ΔCDF are congruent because they are right triangles with the hypotenuse and a leg of one congruent respectively to the hypotenuse and a leg of the other.

Answers to
PROBLEMS 1-3

Student Text Pages 10-11

1. The point of intersection will lie outside the triangle. It is worth mentioning that the two bisectors coincide if the triangle is isosceles but do indeed have a unique point of intersection otherwise.

2. The same proof holds verbatim. The false assumption made by the figure this time is that G is between A and B and F is between C and B. One of these points must fall on the *extension* of the side rather than on the side itself.

3. This time the figure hides no false assumptions. But one line of the proof is now false. Namely, after showing that $AG = CF$ and $GB = FB$ as before, we cannot *add* these equations to give AB and CB. For $AB = AG + GB$ whereas $CB = FB - CF$. The fact that one of the points G and F is on a side of the triangle ΔABC and the other is on a side *extended* is the key to explaining the "false proof".

1-4 ANOTHER EXAMPLE.

There are two main points to be made in reading this section. The first is to emphasize that inductive and deductive reasoning work together to

discover and then to establish a new theorem. And the second is that the deductive method depends in no way on the development of geometry from postulates.

Each individual, through his study and experience, has acquired a reservoir of geometrical intuition and knowledge in which he has confidence. The usual strategy for constructing a deductive proof of a new theorem is to work *backwards* from the theorem to be proved to statements in this reservoir, the latter statements being the hypotheses of the proof. It is in general a poor strategy to start with hypotheses from this reservoir and to try to "work up" to the theorem.

After having studied *Secondary Three,* the pupils have gained quite a bit of knowledge, and so it will be easier to construct proofs than it was a year ago. The geometry of *Secondary Three* began with a very small reservoir of only 26 statements, the so-called postulates. But it is essential to realize that all that was really required of this initial reservoir of facts was that they be acceptable as true and fundamental geometrical facts to as large a group of people as possible.

In the example, the pupil discovers, of course, that the medians are *concurrent*, that is, they meet in a single point. As is often the case in mathematics, it becomes even easier to prove this fact by discovering still more facts which make the theorem to be proved even stronger. In this case the new discovery is the fact that the intersection point O divides the medians in the ratio 1:3. After this discovery it is quite natural to try to construct parallel lines which cut $\overline{AA'}$ into three pieces, one of them intersecting $\overline{AA'}$ at the point O, and to show that these parallel lines are equally spaced.

The constructed segment $\overline{B'A'}$ is parallel to the base of $\triangle ABC$ and is half as long (this is used as a hypothesis—it was proved last year, of course, as an easy exercise in congruent triangles). Thus the quadrilaterals $B'A'EA$ and $B'A'DB$ are parallelograms because opposite sides are parallel. Then $AE = BD$ because they are both equal to $B'A'$, and $EB = B'A'$ because each of them is $\frac{1}{2}$ of AB. Thus $AE = EB = BD$. Since the four parallel lines $\overleftrightarrow{FE}$, $\overleftrightarrow{B'B}$, and $\overleftrightarrow{A'D}$ cut the transversal $\overleftrightarrow{AD}$ in three congruent segments, they will cut any other transversal in three congruent segments (again, this statement is used as a hypothesis). Thus $\overline{AA'}$ is cut into three congruent pieces as desired.

The proof will be convincing for anyone who is familiar with the geometric figures involved (triangles, parallelograms, parallel lines) and who accepts the two hypotheses.

Although the median concurrence theorem is interesting in its own right, the methods used to discover and prove it are the real points of emphasis.

Answers to
PROBLEMS 1-4

Student Text Pages 14-17

1. The hypotheses used in the proof are:
   1. The various parallel lines of the proof can be constructed, and they intersect the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ as stated.
   2. The line segment connecting the midpoints of two sides of a triangle is parallel to the third side and is half as long.
   3. If a set of parallel lines cuts off congruent segments on one transversal, then it cuts off congruent segments on every transversal.

   The first hypothesis is easily accepted by beginning students of geometry. The third is readily accepted after experience with the construction of parallel lines. The second hypothesis is accepted by pupils who have had more experience, for example, by those who have studied *Secondary Three* geometry.

2. The perpendicular bisectors are *concurrent*. A more observant pupil may also discover that their common point of intersection is equidistant from the three vertices of the triangle. The point of intersection may fall inside, on, or outside the triangle, and in the case of a right triangle it lies on the hypotenuse.

3. The altitudes are *concurrent*. For a right triangle the common point of intersection is the vertex of the right angle. The point of intersection may fall either inside, on, or outside the triangle; this is the reason for defining the altitudes here to be lines rather than line segments.

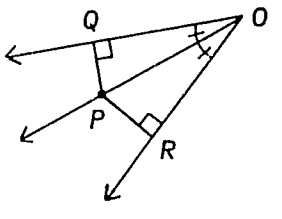
4. The angle bisectors are *concurrent*. Again, an observant pupil may also discover that their common point of intersection is equidistant from the three *sides* of the triangles.

5. The triangle $\Delta A'B'C'$ is equilateral. A proof may be given using the methods of Chapter 6.

6. (a) The two perpendicular bisectors intersect because they are respectively perpendicular to non-parallel lines and hence are not themselves parallel.
   (b) Any point on the perpendicular bisector of a segment is equidistant from the end points of the segment.
   (c) Same reason as (b).
   (d) Follows from (b) and (c).
   (e) Any point which is equidistant from the endpoints of a segment lies on the perpendicular bisector of the segment.

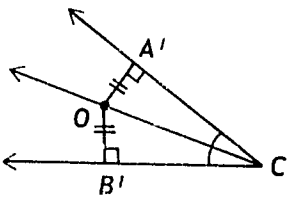
7. Three parallelograms in the figure are $ABA'C$, $AC'BC$ and $ABCB'$.
   (a) $\overline{AB}$ and $\overline{A'C}$ are opposite sides of a parallelogram.
   (b) $\overline{AB}$ and $\overline{B'C}$ are opposite sides of a parallelogram.
   (c) The point C is the midpoint of $\overline{A'B'}$, and the altitude of ΔABC through C is perpendicular to $\overline{AB}$ and hence also to $\overline{A'B'}$.
   (d) The remaining two altitudes of ΔABC are also perpendicular bisectors for the remaining two sides of $\Delta A'B'C'$.
   (e) The altitudes of ΔABC are the perpendicular bisectors of the sides of $\Delta A'B'C'$. They are therefore concurrent by Problem 6.

8. (a) If the angle bisectors of $\hat{A}$ and $\hat{B}$ were parallel, then the sum of the measures of $\hat{A}$ and $\hat{B}$ would be 360°.
   (b) Any point on the bisector of an angle is equidistant from the two sides. This is easily proved by showing that the right triangles ΔOPQ and ΔOPR shown in the figure are congruent.



   (c) The point O is equidistant from sides $\overline{BA}$ and $\overline{BC}$ since it is on the bisector of angle $\hat{B}$. It is also equidistant from sides $\overline{AC}$ and $\overline{AB}$ since it is on the bisector of angle $\hat{A}$. It is therefore equidistant

from the sides of angle $\hat{C}$. But it must therefore be on the bisector of $\hat{C}$. This may be proved by showing that the triangles $\Delta COA'$ and $\Delta COB'$ are congruent.

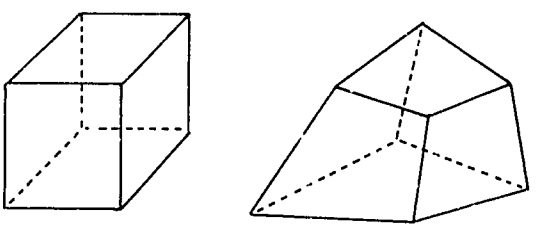


1-5* EULER'S FORMULA.

*This section can be considered as optional.

Euclidean geometry is primarily interested in the size and shape of geometrical objects, the basic relation between objects having the same size and shape being that of *congruence*. In Section 1-5, however, we are concerned with a quite different property of geometrical figures, namely the way in which they are put together from simpler figures.

In the case of polyhedra, for example, the two objects shown in the figure are very different from the viewpoint of Euclidean geometry. But they are both constructed from six quadrilaterals, three of which come together at each of the eight vertices. They have much in common—the same number of faces, the same number of edges, the same number of vertices. And the quadrilaterals are fastened together in the same way in each figure. Only the shapes and dimensions of the quadrilaterals differ.



It might seem that there is very little left to say once size and shape are overlooked. This section and the next, however, contain examples to show that this is not the case. And indeed a whole branch of modern mathematics—topology—has had its beginnings in just such considerations.

The table which the pupils are asked to complete is given below. It is immediately noticed that $V - E + F = 2$ for each of these polyhedra, and the main result of Section 1-5 is to prove that this formula, called Euler's formula, holds for *any* simple polyhedron. The additional example given in Figure 11 of the student's text also satisfies this formula, for the number of vertices, edges, and faces are given by $V = 12$, $E = 28$, and $F = 18$.

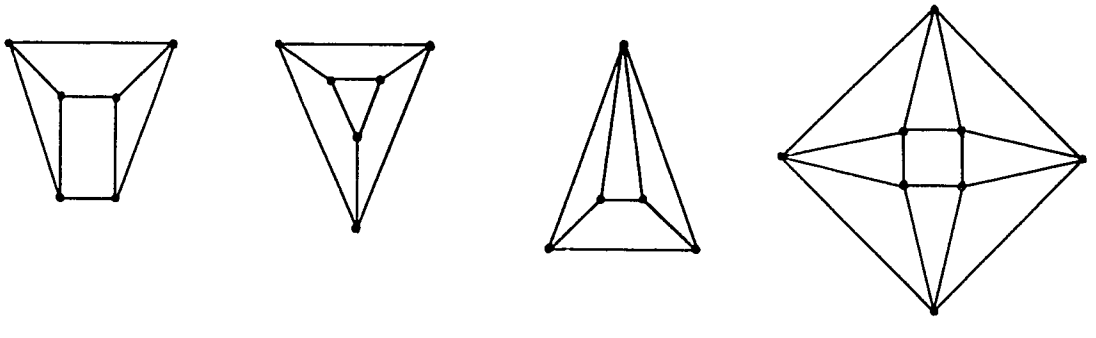
	Name	V	E	F	$(V-E+F)$
(a)	tetrahedron	4	6	4	2
(b)	cube	8	12	6	2
(c)	hexagonal pyramid	7	12	7	2
(d)	square pyramid	5	8	5	2
(e)	pentagonal prism	10	15	7	2
(f)	no name	10	17	9	2
(g)	octahedron	6	12	8	2

The *simple* polyhedra are exactly those for which the trick of removing one face and flattening out the remainder into a network can be accomplished. It takes some practice to visualize the appearance of the polyhedron after it has been flattened into a plane network. The examples in Figure 13 will help. It is rather easy to see the correspondence between the vertices of the polyhedron and the vertices of the network, and similarly for the edges and faces. The correspondence between the faces is indicated on Figure 13(g) of the text by means of numbers. Notice that in each example, the outside edges of the network correspond to the edges of the face that was removed, and this face in turn corresponds to the infinite region of the plane outside of the network.

Answers to
PROBLEMS 1-5 A

Student Text Pages 21-22

1.



2.

	V	E	F	$V-E+F$
(a)	11	15	6	2
(b)	10	15	7	2
(c)	19	33	16	2

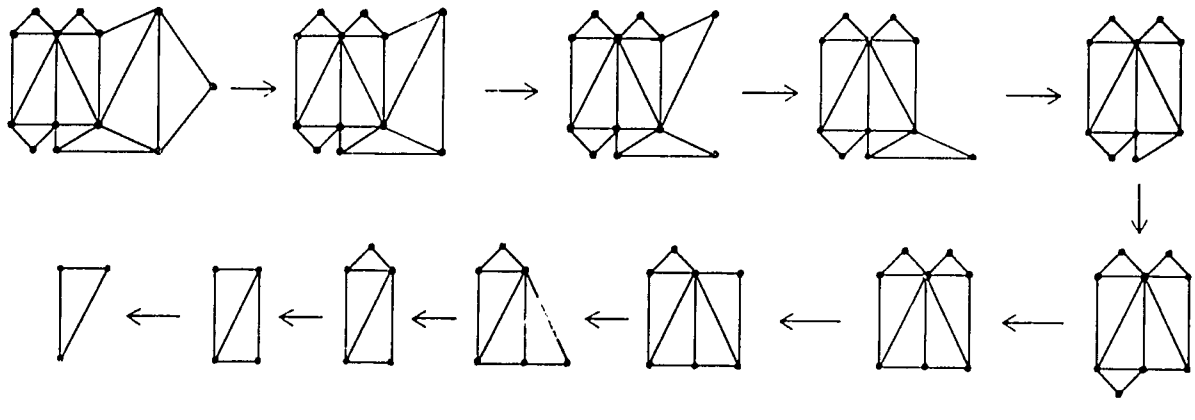
3. Making several of these polyhedra is an entertaining activity if materials are available and time permits.

4. Note the picture of the icosahedron in Figure 25 of the student's text.

The content of the two experiments which show that edges can be added to a network without changing the value of $V - E + F$ and that outer triangles can be "peeled away" without changing this value is most easily demonstrated at the blackboard with a single example. For example one could begin with the network shown in Figure 14 of the student's text and add edges one by one (as in Figures 15 and 16), pausing after each addition to recount the number of vertices, edges, and regions. The pupils can easily see that each time an edge is added, the number V remains unchanged and the numbers E and F *both* increase by 1. The value of $V - E + F$ does not change, therefore, since

$$V - (E + 1) + (F + 1) = V - E + F.$$

Now, using the eraser, begin removing triangles as described in the text.



By examining the number of vertices, edges, and regions removed at each step, the pupils can see that $V - E + F$ does not change, because the sum of the number of vertices and regions removed is always the same as the number of edges removed (see the table on Page 21 of the student's text).

The procedure described above for beginning with a polyhedron, removing a face and flattening it out, "triangulating" the regions, and peeling away the triangles, shows *why* the value of $V - E + F$ is the same for the polyhedron as for a single triangular region. Thus it becomes clear that $V - E + F$ must be 2 for the polyhedron.

Answers to
PROBLEMS 1-5 B

Student Text Page 25

1. $V = 28$ and $F = 27$
   Therefore $28 - E + 27 = 2$, and so $E = 28 + 27 - 2 = 53$.

2. Since the polyhedron is constructed from 20 equilateral triangles, we know that $F = 20$. If $V = 12$ and $E = 28$ as the pupil's friend suggests, then $V - E + F = 12 - 28 + 20 = 4$ and this violates Euler's formula. The correct value of E is 30.

3. $V = 21$, $E = 30$, and $F = 11$. Thus $V - E + F = 2$ as expected. Note that removing (or adding) a vertex in the middle of an edge does not change the value of $V - E + F$, for this would decrease the values of both E and V by 1 and does not change F at all.

4. $V = 16$, $E = 32$, and $F = 16$. Thus the value of $V - E + F$ is 0. Note that the "interior" of this polyhedron is shaped like a flat slab of stone with a square hole in it. It can be proved in general that if the interior of the polyhedron has p holes through it, then $V - E + F = 2 - 2p$. When $p = 0$, the polyhedron is called *simple* and this equation reduces to Euler's formula.

1-6* REGULAR POLYHEDRA.

*This section can be considered as optional.

The fact that there are only five kinds of regular polyhedra was already known to the ancient Greeks. The derivation of this fact from Euler's formula, however, came much later. It is an interesting example of proving a theorem in geometry using methods quite different from the traditional ones of postulational geometry.

It is, of course, not possible to construct a regular polyhedron from congruent squares by joining four squares at each vertex. For the sum of the measures of four right angles is 360, and hence the four squares would lie in a single plane. Similarly one cannot construct a regular polyhedron from congruent regular hexagons, because the measure of each vertex angle of a regular hexagon is 120, and thus three of them when joined together would lie in a plane.

Two equations relating the number of vertices, edges, and faces of a regular polyhedron are derived in the text. They are $E = \frac{1}{2} eF$ and $E = \frac{1}{2} nV$. The first is explained in the text, and the second follows from the observation that nV is the number of "ends" of edges, and since each edge has *two* ends, the number of edges is $\frac{1}{2} nV$.

Using the inequality $n < \frac{2e}{e-2}$, which is derived from these two equations together with Euler's formula, we find that:

When $e = 3$, n is less than 6.
When $e = 4$, n is less than 4.
When $e = 5$, n is less than $3\frac{1}{3}$.
When $e = 6$, n is less than 3.
When $e = 7$, n is less than $2\frac{4}{5}$.

But it is impossible for n to be less than 3 since it is the number of faces of a regular polyhedron which are joined at a vertex. Thus it is impossible for e to be greater than 5. This leads to the five cases listed in the student's text and consequently to the conclusion that there can be at most five kinds of regular polyhedra.

The table listing these five cases is easily filled in.

	e	n	E	V	F	name
Case 1	3	3	6	4	4	tetrahedron
Case 2	3	4	12	6	8	octahedron
Case 3	3	5	30	12	20	icosahedron
Case 4	4	3	12	8	6	cube
Case 5	5	3	30	20	12	dodecahedron

In each case (except 4) the name of the polyhedron is derived from the number of its faces. The cube could, of course, be called a *regular hexahedron.*

Answers to
PROBLEMS 1-6

Student Text Pages 30-31

1. There is an interesting duality between the cube and the octahedron. If one connects the midpoints of adjacent faces (that is, faces that have a common edge) of the cube as shown in the figure in the text, a regular octahedron results. Conversely, if one connects the midpoints of adjacent faces of the octahedron, a cube results. (Recall that the midpoint of a triangle is the point where its medians intersect.)

2. The tetrahedron is "self dual".

3. Connecting midpoints of adjacent faces of a dodecahedron results in an icosahedron, and conversely.

Chapter 2
AREAS OF POLYGONAL REGIONS

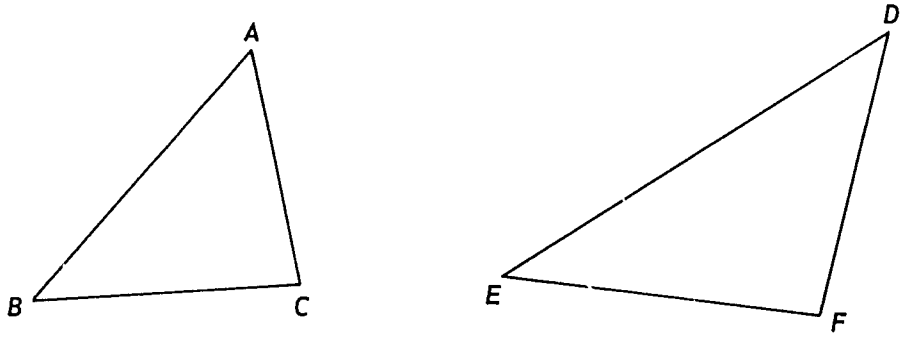
The subject matter of this chapter is the traditional one of areas of triangles, parallelograms, trapeziums and other plane regions bounded by segments. We also treat the very important theorem—Pythagoras' Theorem. We begin with an informal discussion of the idea of area (Sections 2-1 through 2-4). The discussion is a continuation of work done in earlier years, and is meant to develop further the pupils' intuitive understanding of the subject.

The viewpoint adopted in the formal treatment differs in certain essentials from the traditional one. First, we state explicitly as postulates the properties of area which we assume. This is much in keeping with the spirit of the geometry of Secondary Three.

The second point of difference is the introduction and the definition of the term "polygonal region". This certainly requires some explanation. One would expect such a term to refer to a region bounded by a polygon. One, however, soon encounters difficulties if one adopts this approach. It is fairly straightforward to define the concept of a polygon. The definition of the interior of a convex polygon does not present much difficulty either. But when one tries to define the interior of a general polygon, one faces an entirely different situation. It is not at all easy to define this more general concept. Moreover, the boundaries of the regions we study will not always consist of single polygons. So, if our definition of "polygonal regions" were to apply only to such regions, we would have to find a more general term to apply to the wider class of regions we study here. The definition we give in Section 2-5 avoids these difficulties, among others, at the cost of a feeling of strangeness, admittedly. But this feeling should wear off with usage.

Our definition allows such regions as depicted in Problem 2(d)-(f) of Problems 2-5 to be called polygonal regions. So, you will observe, does

it mit certain regions which are totally disconnected. The union of the two totally disjoint triangular regions shown below is a perfectly good example of a polygonal region.



Thirdly, in the development of the theorems (and these are basically those covered in a traditional course), the sequence adopted here is different. Traditionally, one first derived the area of a parallelogram, and from this deduced the area of a triangle. A main criticism of this approach is that the proof usually given for the area formula of a parallelogram is incomplete—only one out of four possibilities is treated. We reverse this particular sequence of derivations. In fact, we deduce the area formula for a parallelogram as a corollary of the theorem giving the area formula of a trapezium. We also give in the Appendix to this chapter two alternative sequences for the derivations of the various area formulae.

Some further remarks on the area postulates: As stated in the Student Text, we already have enough postulates to develop our theory of area. We assume as postulates the statements contained in the Area Postulates 1 to 4 only because it would be too difficult to prove them as theorems. Observe that the area postulates are similar to the postulates we assumed for distance and for angle measurement.

The question might be asked why we define our basic unit in terms of a square region, and not, for instance, relative to a triangular region—especially, since we define a polygonal region as the union of certain triangular regions. This is partly a matter of convenience. The area formula for a triangle (given in Theorem 2-3) is, in fact, equivalent to our Area Postulate 2. One could start by assuming that the area formula for a triangle

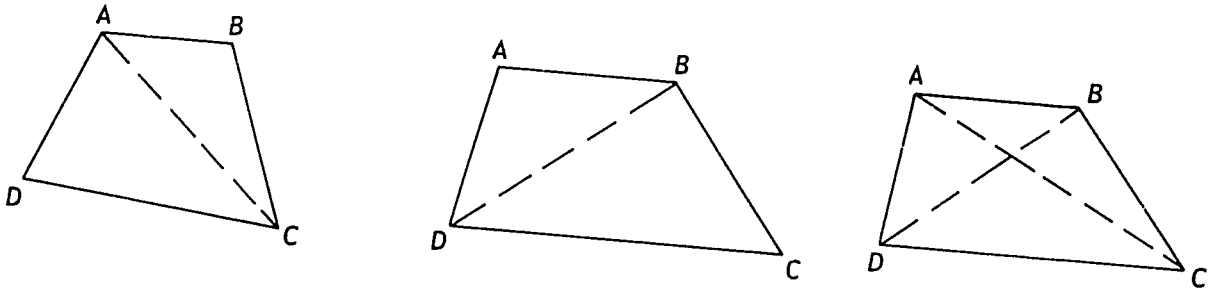
of base b and corresponding altitude h is $\frac{1}{2}bh$. One then has to show that the product "bh" is independent of the particular base chosen. The proof of this is simple, but involves a knowledge of similar triangles (Chapter 3). We give the proof in the Appendix.

The last result also provides us, incidentally, with a convenient starting point if we wish to prove the statements contained in the area postulates as theorems. The area formula for a triangle may then be taken as the definition of the area of a triangle.

It is, nevertheless, an interesting exercise (for the brighter pupils) to derive area formulae for various polygons, using as the unit of area an equilateral triangle, an isosceles right triangle, a rhombus, a regular hexagon, etc.

On the question of the area of a polygonal region being independent of the method of triangulation, it may help to convince the pupils of the truth of this if they consider specific examples. For instance, a quadrilateral $ABCD$ may be triangulated in each of the following three ways:

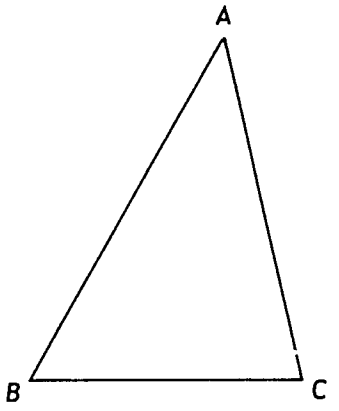
(1) by drawing the diagonal $\overline{AC}$;
(2) by drawing the diagonal $\overline{BD}$;
(3) by drawing both diagonals $\overline{AC}$ and $\overline{BD}$.

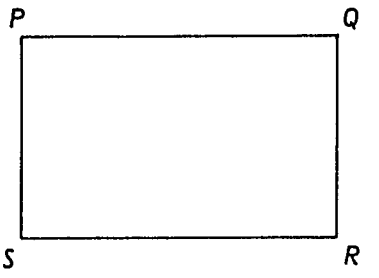


(1) (2) (3)

It is then pointed out that each of the triangulations (1) and (2) may be obtained by suitably combining in pairs the four triangular regions given by (3). So that, in a sense, triangulation (3) is the "sum" of triangulations (1) and (2).

Finally, the rather minor point may be made that whereas we may, in theory, compare the lengths of any two segments without "breaking up" either of them, we cannot, in general, compare the areas of two regions without in some way cutting up at least one of them. (Consider, for instance, the triangle and the rectangle in the figure.)





2-1 INTRODUCTION.

Student Text Page 32

PROBLEM:

The area of the first piece is 9×6 sq. ft. = 54 sq. ft.

The area of the second piece is 8×7 sq. ft. = 56 sq. ft. One would choose the second. It is a better value for one's money!

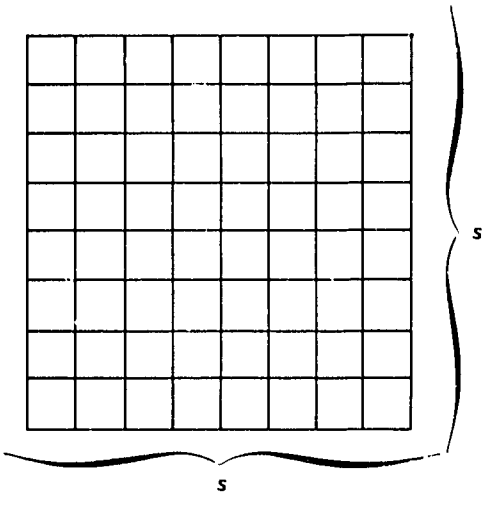
2-3 AREAS BY SUMMING UP SQUARES.

Student Text Page 33

Answers to PROBLEMS 2-3A

Each square should be divided into unit squares. (d) is illustrated in the figure. In (a) - (c), make sure that the units given in the answers are correct.

(a) 9 sq. cm.
(b) 64 sq. in.
(c) 196 sq. ft.
(d) s^2



The method for dividing the rectangles into unit squares is essentially the same as that for squares.

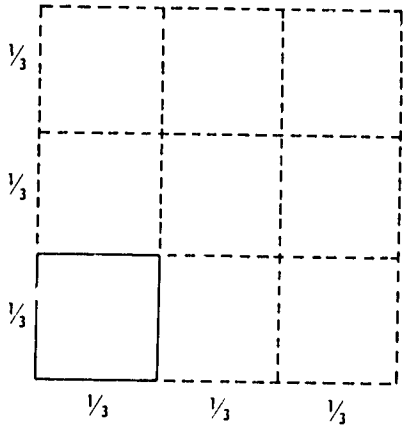
Answers: 15; ab

Answers to
PROBLEMS 2-3B

Student Text Page 34

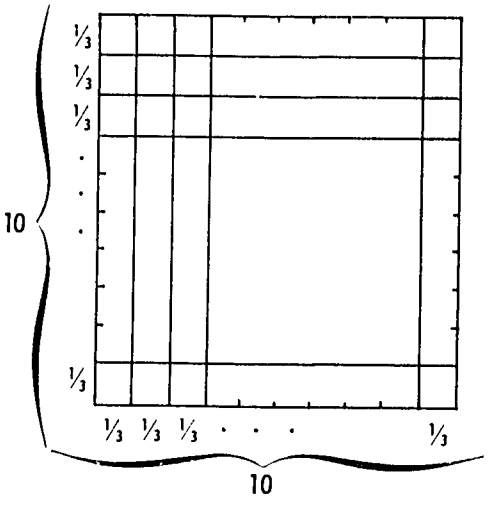
The point to note in these problems is that the unit square has to be divided into suitable congruent squares. These congruent squares we use in (ii) to (vi) to cover exactly the given region.

(i)



Area = $\frac{1}{9}$ sq. ft.

(ii)



$3\frac{1}{3} = 10 \times \frac{1}{3}$

Area = $100 \times \frac{1}{9}$ sq. cm. = $11\frac{1}{9}$ sq. cm.

(iii) Sides of rectangle are $\frac{1}{3}$ ft. by $\frac{1}{2}$ yd. $\left(= \frac{3}{2} \text{ ft.}\right)$.
We divide the rectangle into squares each of side $\frac{1}{6}$ ft. (L.C.M. of denominators is 6). Area = $\frac{1}{2}$ sq. ft. Alternately, we may work in yards. Area = $\frac{1}{18}$ sq. yd.

(iv) Divide the rectangle into squares each of side $\frac{1}{35}$ mile. Area = $\frac{6}{35}$ sq. mile.

(v) Divide the rectangle into squares each of side $\frac{1}{10}$ km. Area = $\frac{54}{5}$ sq. km. = $10\frac{4}{5}$ sq. km.

(vi) Divide the rectangle into squares each of side $\frac{1}{qs}$.

Area = $\frac{pr}{qs}$.

Answers to
PROBLEMS 2-3C

Student Text Page 36

1. (a) $2\sqrt{3}$ (b) $\sqrt{6}$

2. Yes; 3.

2-4 APPROXIMATIONS TO AREAS OF SOME PLANE REGIONS.

Student Text Page 36

CLASS ACTIVITY

1. By calculation, the area is $8\sqrt{3}$ sq. cm. $\approx$ 13·85 sq. cm.

2. The length of the other diagonal is 4 in. Hence, by calculation, the area is 6 sq. in.

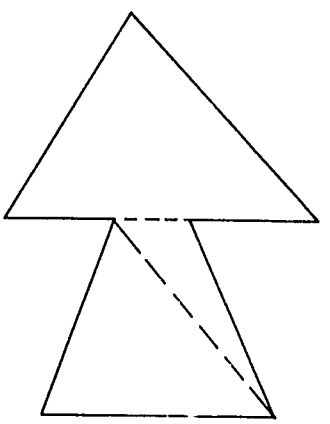
Accuracy is important, but the real point of the exercises is to teach the method whereby we obtain successive approximations to the area we want. Note that the actual position of figures on the squared paper will affect the results obtained. It may be pointed out that this becomes less and less important with successive approximations.

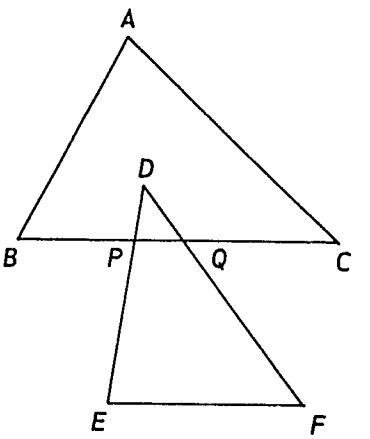
2-5 POLYGONAL REGIONS.

Answers to
PROBLEMS 2-5

Student Text Page 40

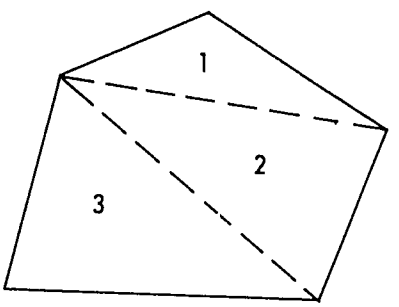
1. It does not, because the two triangular regions ABC and DEF have the triangular region DPQ in common.



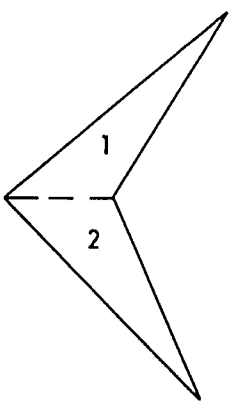


2. *General remark*: Each region may be subdivided into triangular regions in an infinite number of ways. (This is the answer to Problem 4i). Ask the students to subdivide each region in different ways. They may also try to discover the minimum number of triangular regions into which each region may be subdivided.

(a)



(b)



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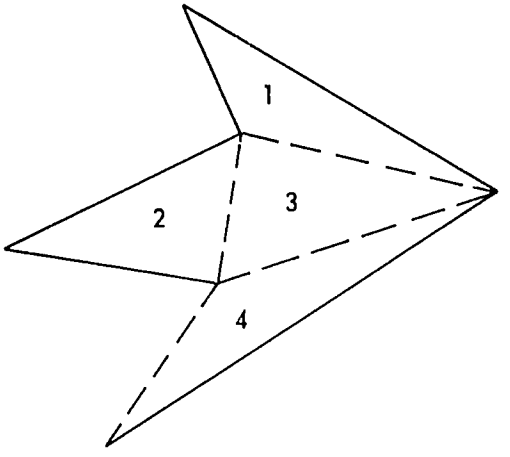

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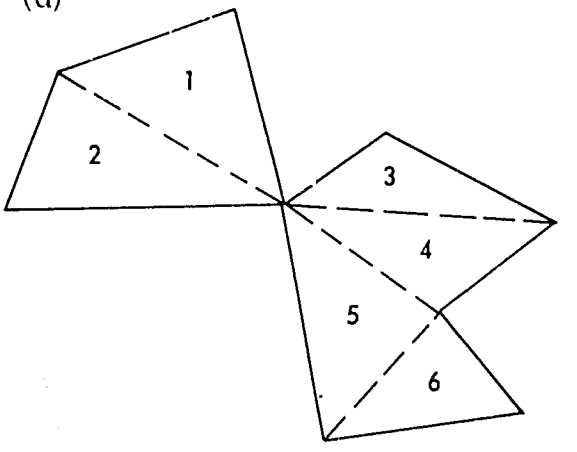

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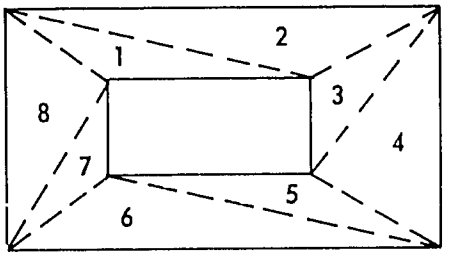

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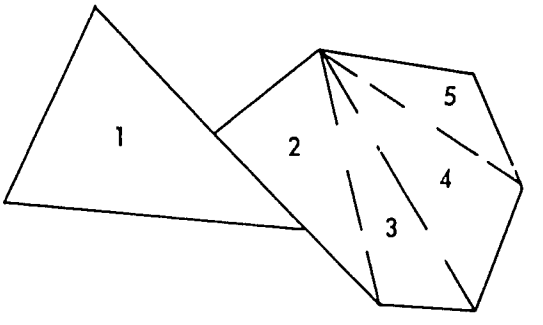

2-6 AREAS OF POLYGONAL REGIONS.

Answers to PROBLEMS 2-6

Student Text Page 43

1. Yes
2. No
3. Yes; (according to the convention stated in the text.)
4. No.
5. False

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2-7 SOME AREA FORMULAE.

Reasons for the statements in the proof of Theorem 2-2

1. $EFDF'$ is a parallelogram ($\overline{EF} \parallel \overline{F'D}$ and $\overline{EF'} \parallel \overline{FD}$, by construction) and the angle at F is a right angle.

2. SSS

3. Area Postulate 3.

4. Area Postulate 4 and statement 3.

5. Statement 1 and Theorem 2-1

6. Statements 5 and 6.

Answers to
PROBLEMS 2-7

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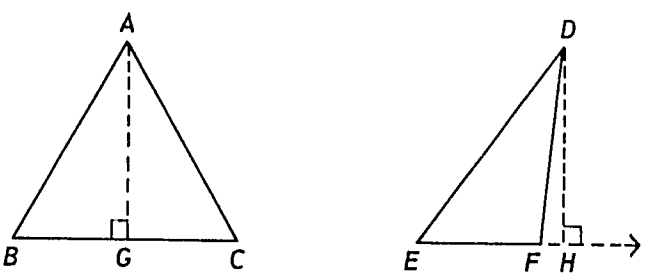
1. Triangles $\triangle ABC$ and $\triangle DEF$ with altitudes $\overline{AG}$ and $\overline{DH}$ respectively such that $AG = DH$.

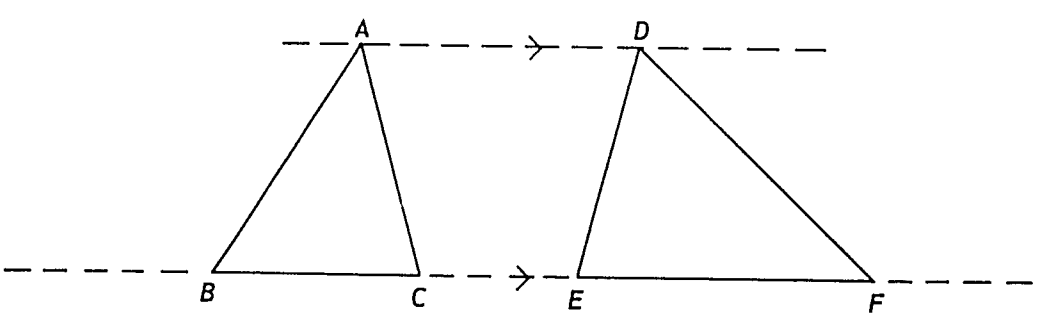


$$\frac{ABC}{DEF} = \frac{\frac{1}{2} AG \times BC}{\frac{1}{2} DH \times EF}$$

$$= \frac{BC}{EF}$$

2. *Statement*: If two triangles have equal bases, the ratio of their areas is equal to the ratio of the altitudes to these bases. The proof is similar to that given in Problem 1.

3. (a) The ratio of the areas is equal to the ratio of the bases, BC and EF. (The triangles have equal altitudes.)



(b) $ABC = DEF$

4. Let a be the constant area.

(i) P is a point on either of the two lines parallel to $\overleftrightarrow{BC}$ and distant $\frac{2a}{BC}$ from it.

(ii) P is a point on the cylinder with $\overleftrightarrow{BC}$ as axis and radius $\frac{2a}{BC}$, stretching infinitely in both directions.

5. (i) 30

(ii) $\frac{60}{13} = 4\frac{8}{13}$

6. (a) $\frac{168}{25} = 6\frac{18}{25}$

(b) 7

(c) 24 (Note that this is just the given leg!)

7. 10

8. $\frac{96}{10} = 9\frac{3}{5}$

9. 156

10. 58 yards

11. $\frac{5}{18}$ ft.

12. This may be considered as a special case of Problem 3(b). A direct proof may be given, of course:

$$\frac{ABD}{ADC} = \frac{BD}{DC}$$
$$= 1$$
$$\therefore ABD = ADC.$$

No;

The triangles will be congruent only if ΔABC is isosceles, with base $\overline{BC}$.

13. By the mid-point theorem,

$$B'C' = BA' = A'C$$
$$C'A' = CB' = B'A$$
$$A'B' = AC' = C'B$$

$\Delta A'B'C' \equiv \Delta AB'C'$ SSS

$\equiv \Delta BC'A'$ SSS

$\equiv \Delta CA'B'$ SSS

$$A'B'C' = AB'C'$$
$$= BC'A'$$
$$= CA'B'$$
$$A'B'C' = \frac{1}{4}(A'B'C' + AB'C' + BC'A' + CA'B')$$
$$= \frac{1}{4}(ABC)$$

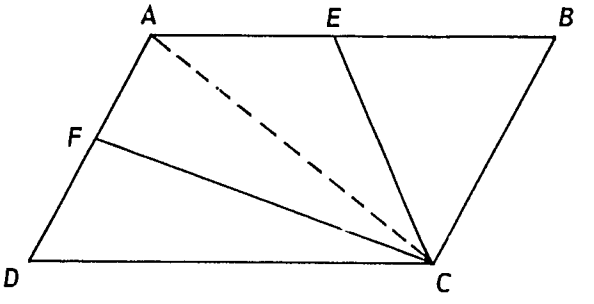
14. $AEC = \frac{1}{2}\ ABC$

$AFC = \frac{1}{2}\ ADC$

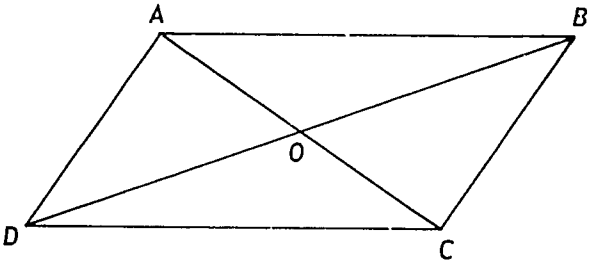
$AECF = AEC + AFC$

$= \frac{1}{2}\ (ABC + ADC)$

$= \frac{1}{2}\ ABCD$



15. $\overline{AO}$ is a median of the ΔADB. Hence $ADO = ABO$.
Similarly $ABO = BCO$
and $BCO = DCO$.



16. $ABD = \frac{1}{2} AO \times BD$

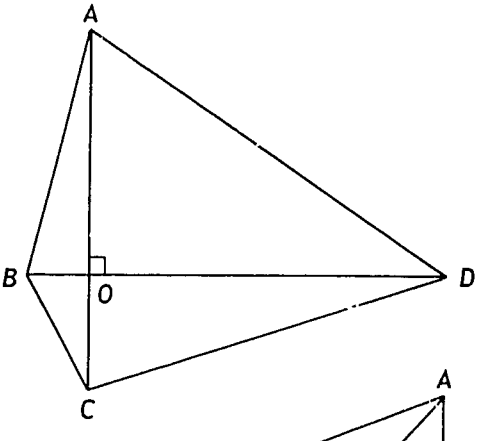
$BCD = \frac{1}{2} CO \times BD$

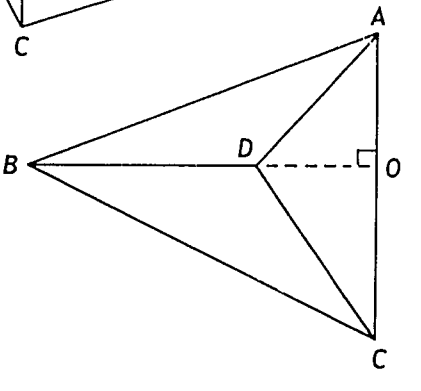
$ABCD = ABD + BCD$

$= \frac{1}{2} AO \times BD + \frac{1}{2} CO \times BD$

$= \frac{1}{2}(AO + CO)\ BD$

$= \frac{1}{2} AC \times BD$





16. Cont.

Yes. (A rhombus is a quadrilateral satisfying the condition of the theorem.) If d_1, d_2 are the lengths of the diagonals of a rhombus, the area of the rhombus is $\frac{1}{2} d_1 d_2$.

17. (i) S^2

(ii) $\frac{1}{2} d^2$ (using the result of Problem 16).

$d^2 = 2s^2$ or $d = s\sqrt{2}$.

18. (i) $a\sqrt{2}$

(ii) $\frac{1}{2} a^2$

(iii) $\frac{a}{\sqrt{2}}$

2-8 PYTHAGORAS' THEOREM.

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Reasons for the statements in the proof of Theorem 2-6.

(1) The four triangles ΔHEP, ΔEFQ, ΔFGR and ΔGHS are each congruent to the original triangle ΔABC. Hence $HE = EF = FG = GH = AB = c$. Also, from the congruent triangles ΔHEP and ΔEFQ, we have

$P\hat{H}E \equiv Q\hat{E}F$

Again $m(P\hat{H}E) + m(P\hat{E}H) = 90$

and $m(P\hat{E}H) + m(Q\hat{E}F) + m(H\hat{E}F) = 180.$

$m(H\hat{E}F) = 90.$

(2) Area Postulate 2.

(3) Area Postulate 4.

(4) Statement 1 and Area Postulate 3.

(5) Theorem 2-2.

Reasons for statements (4) and (6) in the proof of Theorem 2-7.

(4) From statement 3 and the fact that both c and c' are positive numbers.

(6) Corresponding angles of the congruent triangles ΔABC and $\Delta A'B'C'$.

		a	b	c
1.	(i)	12	5	13
	(ii)	7	24	25
	(iii)	16	20	$4\sqrt{41}$
	(iv)	9	9	$9\sqrt{2}$
	(v)	$\frac{15}{4}$	5	$\frac{25}{4}$
	(vi)	8	15	17
	(vii)	6	$2\sqrt{7}$	8
	(viii)	0·9	4	4·1

2. $QR = QN + NR$
$= 10 + 18$
$= 28$

3. 15 feet

4. (b), (d), (g).

We use *both* Pythagoras' Theorem and its converse. More precisely, we conclude from Pythagoras' Theorem that if $c^2 \neq a^2 + b^2$, the triangle is not right-angled (at C); the converse tells us that if $c^2 = a^2 + b^2$, the triangle is right-angled (at C).

5. $(p^2 + q^2)^2 = p^4 + 2p^2q^2 + q^4$
$= (p^4 - 2p^2q^2 + q^4) + 4p^2q^2$
$= (p^2 - q^2)^2 + (2pq)^2$.

(The numbers involved are all positive.) $p^2 + q^2$. (This is the largest of the three numbers.)

6.

p	q	$p^2 - q^2$	$2pq$	$p^2 + q^2$
2	1	3	4	5
3	1	8	6	10
3	2	5	12	13
4	1	15	8	17
4	2	12	16	20
4	3	7	24	25
5	1	24	10	26
5	2	21	20	29
5	3	16	30	34
5	4	9	40	41

There are 10 such triangles.

If n is a positive integer, there are $\frac{n(n-1)}{2}$ such triangles where the hypotenuse has length less than or equal to $n^2 + (n-1)^2 = 2n^2 - 2n + 1$.

7. $\{3, 4, 5\}$; $\{5, 12, 13\}$; $\{8, 15, 17\}$;
$\{7, 24, 25\}$; $\{20, 21, 29\}$; $\{9, 40, 41\}$.

8. $BH^2 = HF^2 + BF^2$ ($\triangle HFB$ is right-angled at F)
$= 20^2 + 21^2$ ($HF^2 = HG^2 + GF^2$
$= 29^2$ $= 12^2 + 16^2$
$BH = 29$ $= 20^2$)

9. $\overline{DE} \parallel \overline{CA}$, since each segment $\perp \overline{DC}$.
$\triangle ABC \equiv \triangle BED$
$A\hat{B}C \equiv B\hat{E}D$
$m(D\hat{B}E) + m(B\hat{E}D) = 90$
$m(A\hat{B}C) + m(D\hat{B}E) + m(A\hat{B}E) = 180$
$m(A\hat{B}E) = 90$
$ACDE = ABC + ABE + BED$
$\frac{1}{2}(a+b)(a+b) = \frac{1}{2}ab + \frac{1}{2}c^2 + \frac{1}{2}ab$
$a^2 + 2ab + b^2 = 2ab + c^2$
$a^2 + b^2 = c^2$

10. Each of the triangles $\triangle EAH$, $\triangle DEG$, $\triangle BDF$ is congruent to the triangle $\triangle ABC$ (the hypotenuse and an acute angle of each of them are congruent to the hypotenuse and an acute angle of $\triangle ABC$).

$$CH = AC - AH$$
$$= b - a$$

Similarly, $HG = GF = FC = b - a$.
Moreover, the angle at C is a right angle.

$$ABDE = ABC + BDF + DEG + EAH + CFGH$$
$$c^2 = 4\left(\frac{1}{2}ab\right) + (b-a)^2$$
$$= a^2 + b^2$$

11. (1) *SAS* ($C\hat{A}E \equiv H\hat{A}B$, the measure of each is $90 + m(C\hat{A}B)$).
(2) Statement (1) and Area Postulate 3.

(3) The square $CAHJ$ and the triangle ΔABH have equal bases (AH) and equal altitudes (CA).

(4) Similar to reason for statement (3)

(5) From statements (2), (3) and (4).

(6) The steps involved are similar to those used in proving statement (5).

(7) Statements (5) and (6) and Area Postulate 4.

12. SAS

Mid-point theorem applied to ΔABD.

$CE = \frac{1}{2} AD$

$= \frac{1}{2} AB$ ($\overline{AB} \equiv \overline{AD}$, corresponding segments of congruent triangles ΔACB and ΔACD).

13. In ΔACD, $AD = CD$ and $m(\hat{A}) = 60$. Hence ΔACD is equilateral and $AC = AD = \frac{1}{2} AB$.

14. $\frac{s\sqrt{3}}{2}$

$\frac{s^2\sqrt{3}}{4}$

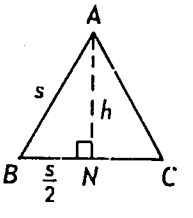
15. $\frac{16\sqrt{3}}{3}$ sq. cm.

16. $s^2 - \frac{s^2}{4} = h^2$

$\frac{3s^2}{4} = h^2$

$s = \frac{2\sqrt{3}}{3} h$

Area $= \frac{h^2\sqrt{3}}{3}$



17. 6, $\frac{60}{13} = 4\frac{8}{13}$

18. $ABX = \frac{1}{2} ABCD$

$= 72$

$BX = 12{\cdot}5$

$AN = \frac{288}{25}$

$= 11\frac{13}{25}$ (11·52)

19. $PR^2 = PQ^2 + QR^2$

$= 5^2 + 12^2 \qquad (PQ = QS = 5)$

$PR = 13$

$PSR = \frac{1}{2} \times 5 \times 7$

$= \frac{35}{2}$

$SN = \frac{35}{13}$

$= 2\frac{9}{13}$

20. $\Delta ABP \equiv \Delta ADQ$

$B\hat{A}P \equiv Q\hat{A}D$

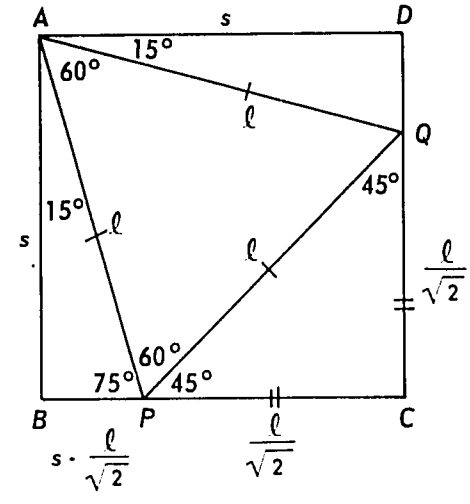
$m(B\hat{A}P) = m(Q\hat{A}D)$

$= \frac{1}{2}(90 - m(P\hat{A}Q))$

$= 15$

$m(B\hat{P}A) = 75$

$m(C\hat{P}Q) = 45$



ΔPQC is an isosceles right triangle.

Let $AB = s$ and $AP = \ell$. Then

$CP = \frac{\ell}{\sqrt{2}}$ and $BP = s - \frac{\ell}{\sqrt{2}}$

Apply Pythagoras' Theorem to right triangle ΔABP to give

$\ell^2 = s^2 + \left(s - \frac{\ell}{\sqrt{2}}\right)^2$

$\ell^2 = 2\sqrt{2}\, s\ell - 4s^2 = 0$ and the quadratic formula gives

$\ell = (-\sqrt{2} \pm \sqrt{6})s$

$\therefore \frac{\ell}{s} = \sqrt{6} - \sqrt{2} \qquad \left(\frac{\ell}{s} > 0\right)$

[Alternately, we may use

$ABCD = ABP + PCQ + QDA + APQ.$

$s^2 = \frac{\ell}{2}s\left(s - \frac{\ell}{\sqrt{2}}\right) + \frac{\ell^2}{4} + \frac{\ell}{2}s\left(s - \frac{\ell}{\sqrt{2}}\right) + \frac{\ell^2\sqrt{3}}{4}$

$\frac{s}{\sqrt{2}} = \frac{\ell(\sqrt{3}+1)}{4}$

$\frac{\ell}{s} = \frac{2\sqrt{2}}{\sqrt{3}+1}$

$= \sqrt{2}\,(\sqrt{3} - 1).$]

21. $XL = a\sqrt{3}$

$XN = a(2 + \sqrt{3})$

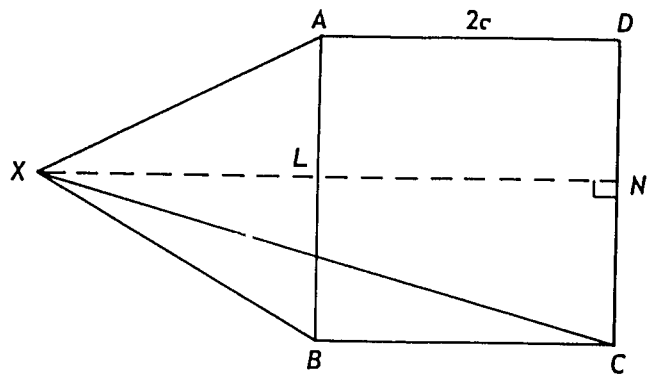
$(XC)^2 = (XN)^2 + (NC)^2$

$= a^2(2 + \sqrt{3})^2 + a^2$

$= a^2(8 + 4\sqrt{3})$

$= a^2(8 + 2\sqrt{12})$

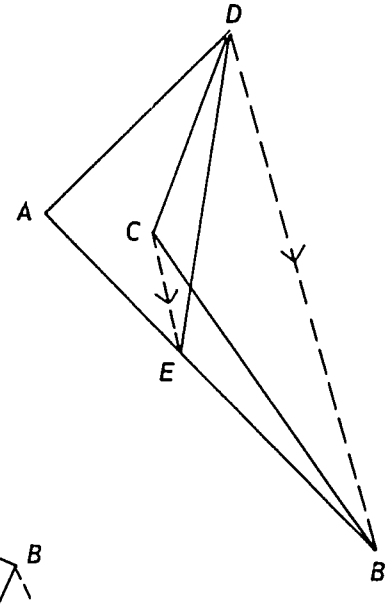
$XC = a(\sqrt{6} + \sqrt{2})$



22. $ADE = ADB + BDE$

$= ADB + DCB$

$= ABCD$



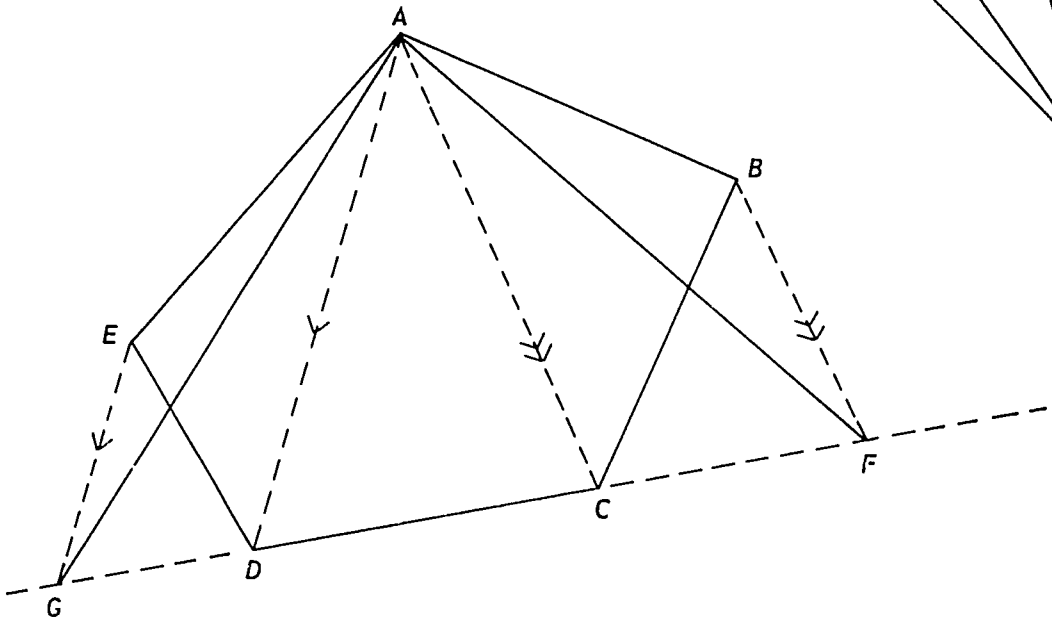
23. Yes.

$ADE = ADB - BDE$

$= ADB - DCB$

$= ABCD$

24.



Through E draw $\overleftrightarrow{EG} \parallel \overleftrightarrow{AD}$, and let $\overleftrightarrow{EG}$ intersect $\overleftrightarrow{DC}$ in G.
Through B draw $\overleftrightarrow{BF} \parallel \overleftrightarrow{AC}$, and let $\overleftrightarrow{BF}$ intersect $\overleftrightarrow{DC}$ in F.
ΔAFG is the required triangle.

25. We use the "angle-sum" theorem to deduce that HEF is a right angle.

A. Alternative sequences for the derivation of the area formulae given in Theorems 2-1 through 2-5.

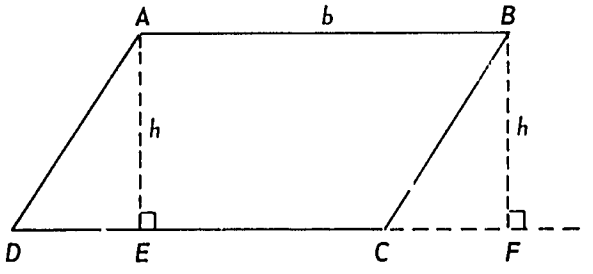
First Sequence—Theorem 2-1': Same as Theorem 2-1.

Theorem 2-2': The area of a parallelogram is the product of any base and the corresponding altitude.

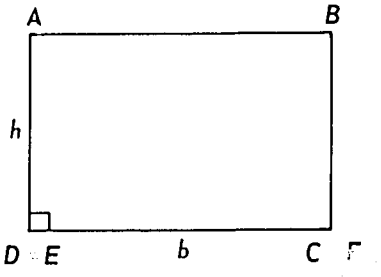
Given: A parallelogram $ABCD$ with base $DC = b$ and corresponding altitude $AE = h$.

Prove: $ABCD = bh$

Proof: Draw the altitudes $\overline{AE}$ and $\overline{BF}$



(1)



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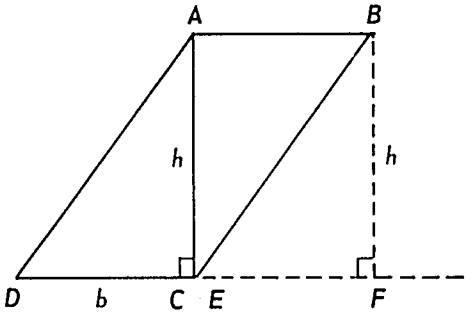

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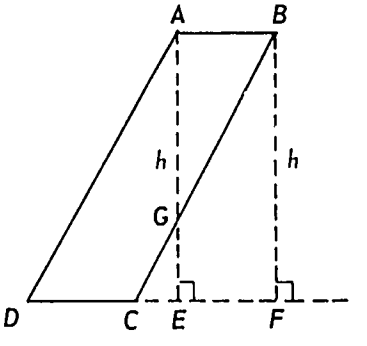


(4)

As shown in the figure, there are four cases to consider:

Case 1: (E lies between D and C.)

1. $\Delta ADE \equiv \Delta BCF$ (hypotenuse and one leg of right ΔADE are congruent to hypotenuse and one leg of right ΔBCF)

2. $ADE = BCF$ (Statement 1 and Area Postulate 3)

3. $ABCD = ADE + AECB$ (Area Postulate 4)
$= BCF + AECB$ (Statement 2)
$= AEFB$ (Area Postulate 4)
$= bh$ (Theorem 2-1': $AEFB$ is a rectangle)

Case 2:

E coincides with D. Parallelogram $ABCD$ is a rectangle and the result follows immediately by Theorem 2-1'.

Case 3:

E coincides with C.

The proof is the same as for case 1.

Case 4:

Both E and F fall outside the segment $\overline{DC}$.

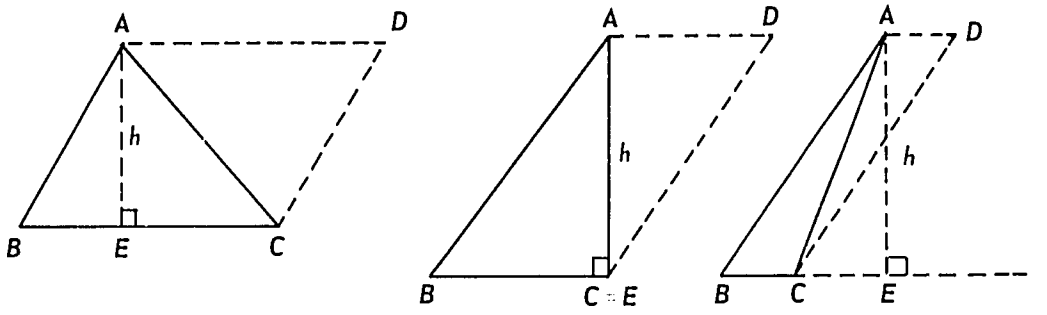
1. $\Delta ADE \equiv \Delta BCF$ (hypotenuse and one leg of right triangle ADF are congruent to hypotenuse and one leg of right triangle ΔBCF.)
2. $ADE = BCF$ (Statement 1 and Area Postulate 3)
3. $ADE = ADCG + GCE$ (Area Postulate 4)
4. $BCF = BGEF + GCE$ (Area Postulate 4)
5. $ADCG + GCE = BGEF + GCE$ (Statements 2 to 4)
6. $ADCG = BGEF$ (Statement 5)
7. $ADCG + ABG = BGEF + ABG$ (Statement 6)
8. $ABCD = ABEF$ (Statement 7 and Area Postulate 4)
9. $ABEF = bh$ (Theorem 2-1'; $ABEF$ is a rectangle)
10. $ABCD = bh$ (Statements 8 and 9)

Theorem 2-3': The area of a triangle is half the product of any base and the corresponding altitude.

Given: A triangle ΔABC with base $BC = b$ and corresponding altitude $AE = h$.

Prove: $ABC = \frac{1}{2} bh$

Proof:



Through A we draw $\overline{AD} \parallel \overline{BC}$ and through C we draw $\overline{CD} \parallel \overline{BA}$.

1.	$ABCD$ is a parallelogram	(Construction)
2.	$\triangle ABC \equiv \triangle ADC$	(SSS)
3.	$ABC = ADC$	(Statement 2 and Area Postulate 3)
4.	$ABCD = ABC + ADC$	(Area Postulate 4)
5.	$ABCD = bh$	(Theorem 2-2')
6.	$ABC = \frac{1}{2} bh$	(Statements 3, 4 and 5)

Theorem 2-4': Same as Theorem 2-4.

Second sequence:

Theorem 2-1'': Same as Theorem 2-1

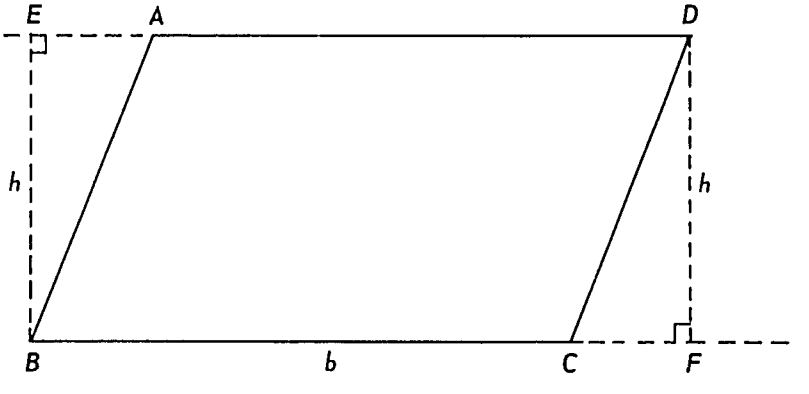
Theorem 2-2'': Same as Theorem 2-2

Theorem 2-3'': The area of a parallelogram is the product of any base and the corresponding altitude.

Given: A parallelogram $ABCD$ with base $BC = b$ and corresponding altitude h.

Prove: $ABCD = bh$

Proof:



If $ABCD$ is a rectangle, the result follows immediately from Theorem 2-2. Otherwise we choose the opposite vertices B and D which are such that the two altitudes $\overline{BE}$ and $\overline{DF}$ lie entirely in the exterior of the parallelogram, except for the end-points B and D.

1.	$\Delta ABE \equiv \Delta CDF$	(hypotenuse and one leg of right ΔABE are congruent to hypotenuse and one leg of right ΔCDF.)
2.	$ABE = CDF$	(Statement 1 and Area Postulate 3)
3.	$BEDF = BEA + ABCD + DCF$	(Area Postulate 4)
4.	$BEDF = (b + AE)h$	(Theorem 2-1'': $BEDF$ is a rectangle)
5.	$BEA = \frac{1}{2} AE \times h$	(Theorem 2-2'')
6.	$DCF = \frac{1}{2} AE \times h$	(Statements 2 and 5)
7.	$(b + AE)\ h = \frac{1}{2} AE \times h + ABCD + \frac{1}{2} AE \times h$	(Statements 2 to 6)
8.	$ABCD = bh$	(Statement 7)

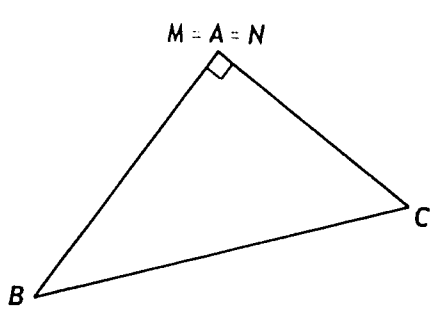
Theorem 2-4'': Same as Theorem 2-3'

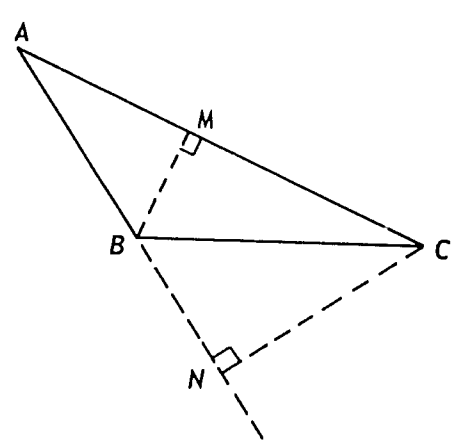
Theorem 2-5'': Same as Theorem 2-4.

B. Theorem: In any triangle, the product of a base and the corresponding altitude is independent of the choice of the base.

Given: A triangle ΔABC with altitudes $\overline{BM}$ and $\overline{CN}$.

Prove: $AB \times CN = AC \times BM$





If $\widehat{A}$ is a right angle, $M = A = N$, and the result is trivial. Suppose now that $\widehat{A}$ is not a right angle. Then $M \neq A$ and $N \neq A$. In triangles ΔAMB, ΔANC,

$\widehat{A} \equiv \widehat{A}$

$A\widehat{M}B \equiv A\widehat{N}C$ (each is a right angle)

$\Delta AMB \sim \Delta ANC$ (AA Similarity Theorem)

$\frac{AB}{AC} = \frac{BM}{CN}$

$AB \times CN = AC \times BM.$

Chapter 3

SIMILAR TRIANGLES AND POLYGONS

3-1 INTRODUCTION.

Your pupils will have done some work with similar polygons in *Secondary Two* and therefore will have some understanding of what is meant by two figures having the same shape but not necessarily the same size. They will already have had considerable experience in drawing scale models of larger figures. They may not have been aware at the time that the figures they drew on their papers (if they were drawn to scale) were exact models of larger figures. You should remind them of this fact here and perhaps pursue the idea a little further by asking them how they would draw on their paper a picture of a triangle whose sides are 10 ft., 12 ft., and 15 ft.

In similar triangles, as in congruent triangles, the notion of a one-to-one correspondence between the vertices is important. We call the pupil's attention to this correspondence here by repeating the definition of congruent triangles so that later, when similar triangles are formally defined, he will have a basis for comparison. Actually, we could define congruent triangles in terms of similar triangles by observing that when two similar triangles have the ratio of their corresponding sides equal to 1, the triangles are congruent.

Comments on
CLASS ACTIVITY

Student Text Page 62

You may want your pupils to do more of the kind of activity suggested here. It is important for them to realise that one way of classifying geometric figures is by similarity and in order to make this classification easy and exact we need a precise definition of similarity.

In Problems 1 and 2, of course, all circles are similar to each other as are squares. In Problem 3 some of your pupils may have been fooled by their pictures into thinking that all rectangles are similar. Once they have seen an example of two rectangles like the figure in the text they should be convinced of the error in their thinking.

3-2 RATIO AND PROPORTION.

If your pupils have studied Rational Functions in *Secondary Four - Algebra*, they will have already done considerable work with ratios and you may want to omit part or all of the exercises in Problems 3-2A. The only thing new here is the word ratio and its representation by the symbol 2:3 or the words 2 to 3. Be sure that your pupils understand that a ratio is a number and that it is usually described as the quotient of two numbers, even when the denominator is 1. Thus if k is a number, k can be expressed as a ratio since $k = \frac{k}{1}$.

We have defined ratio in terms of positive numbers for two reasons. The first is that the numbers used in geometry represent measurements and are therefore positive. The second is that it eliminates the necessity of always having to restrict our denominators to non-zero numbers in our discussions and problems. We adopt this same convention later, when we define proportion.

Answers to
PROBLEMS 3-2A

Student Text Pages 64-65

1. (a), (c), (d)

2. (a) $\frac{5}{1}$ (c) $\frac{17\,bc}{23}$

   (b) $\frac{14x}{9z}$ (d) $\frac{x-1}{x+1}$

3. (a) $x = 9$ (c) $x = \frac{1}{2}$

   (b) $x = 6\frac{2}{3}$ (d) $x = 7$

4. (a) $\frac{x}{y} = \frac{4}{3}$ (c) $\frac{x}{y} = \frac{3}{2}$

(b) $\frac{x}{y} = \frac{a}{b}$ (d) $\frac{x}{y} = \frac{1}{5}$

5. (a) $\frac{7}{5}$ (c) $\frac{7}{1}$

(b) $\frac{2}{5}$ (d) $\frac{2x + 3}{1}$

6. Two such sequences might be 9, 15, 21 or 15, 25, 35. (Any sequence which is a multiple of 3, 5, 7.)

7. $\frac{4s}{s} = \frac{4}{1}$

8. Since the length of the diameter of the circle equals the length of the side of the square, the area of the square is given by $(2r)^2 = 4r^2$. The ratio of the area of the square to the area of the circle is then $\frac{4r^2}{\pi r^2} = \frac{4}{\pi}$.

Answers to the
PROBLEM (Page 66)

$\frac{1}{3}$; 3

We have treated proportions rather extensively in this section. Your pupils may already have a fine understanding of the algebra of proportions. If this is the case, you may want to skip over this section rather hurriedly, concentrating on the definitions, but omitting many of the exercises—especially if you are faced with a shortage of time. On the other hand, here is a good example of how algebra and geometry are interrelated. If your pupils are weak in the algebraic manipulation of fractions, the exercises here should provide ample revision for them.

By a *mean proportional* between a and c or a *geometric mean* between a and c we mean *any* number b for which $a : b = b : c$ is a true statement. Since $\frac{a}{b} = \frac{b}{c} \Rightarrow b^2 = ac$, we get two numbers, $b = \sqrt{ac}$ and $b = -\sqrt{ac}$, as

possible geometric means between a and c. We have defined *the* geometric mean or *the* mean proportional between a and c to be the *positive number* $\sqrt{ac}$. Perhaps the phrase "between a and c" for the geometric mean is not a wise choice of words since the word "between" implies that $\sqrt{ac}$ is between a and c: that is, $a < \sqrt{ac} < c$ or $a > \sqrt{ac} > c$. If a and c are positive and $a \neq c$, this is actually the case. However, if $a = c$, then $\sqrt{ac} = a = c$ and the point representing the geometric mean is *not* between the points representing a and c on the number line. If your pupils are not confused by this use of the word between, I would not bother them with the above discussion. The word "between" arises from the fact that, when writing the proportion $a : b = b : c$, b appears in order between a and c.

The basic theorem on proportions is Theorem 3-1. There are many other theorems concerning proportions which are consequences of Theorem 3-1 and the laws of algebra. We list the more important of these in Theorem 3-2 following Problems 3-2B.

Answers to
PROBLEMS 3-2B

Student Text Page 67

1. (a) If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$.

   Proof:

   By hypothesis, $\frac{a}{b} = \frac{c}{d}$. Multiplying both sides of this equation by bd ($bd \neq 0$ because $b \neq 0$ and $d \neq 0$) we get $ad = bc$.

   (b) If $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$.

   Proof:

   By hypothesis, $ad = bc$. If $b \neq 0$ and $d \neq 0$, we can divide by bd obtaining $\frac{a}{b} = \frac{c}{d}$.

2. No; $\frac{2}{5} \neq \frac{6}{20}$ because $2 \times 20 \neq 5 \times 6$.

3. (a), (c), (d).

4. (a) $x = 6{\cdot}3$ or $6\frac{3}{10}$.

4. Cont.

(b) $x = 6$. (As agreed in Problems 3-2A, the letters in this Problem set represent positive numbers, so $x = -6$ is not a permissible solution.)

(c) $x = \frac{83}{6}$ or $13\frac{5}{6}$.

(d) $x = 4$.

5. $4\sqrt{3}$.

6. $\frac{x}{y} = \frac{10}{3}$.

7. If $\frac{a}{x} = \frac{a}{y}$ and $a \neq 0$, we have by Theorem 3-1 $ax = ay$ and, dividing by a, $x = y$.

8. $x = \frac{9}{5}$; $y = \frac{10}{3}$.

The equivalent forms of the given proportion $\frac{a}{b} = \frac{c}{d}$ listed in Theorem 3-2 are often very useful and if the pupil recognizes how they are obtained from the given proportion he can save himself many steps in the proofs of certain theorems involving proportions.

Prior to discussing Theorem 3-2 you might want to give your pupils the proportion $\frac{a}{b} = \frac{c}{d}$ and ask them how they would obtain other proportions from it by rearranging the order of the terms a, b, c, d so that the cross-product $ad = bc$ is retained. This should produce parts (a), (b), and (c) of Theorem 3-2. You could then ask them to try to discover other proportions whose cross-products are reducible to $ad = bc$ by using only the four terms a, b, c, d and the operations of addition and subtraction on them in setting up the proportions. Once they have discovered parts (d), (e), and (f) of Theorem 3-2 you could ask them to use these results to generate even more proportions equivalent to $\frac{a}{b} = \frac{c}{d}$. This kind of pupil activity should make them thoroughly familiar with the various equivalent forms they will find useful in later work with proportions.

1. (a) is obtained by interchanging the first and fourth terms of the proportion $\frac{a}{b} = \frac{c}{d}$.
   (b) is obtained by interchanging the second and third terms.
   (c) is obtained by inverting the two ratios.
   (d) is obtained by denominator-addition.
   (e) is obtained by denominator-subtraction.
   (f) is obtained by numerator-denominator-addition.

2. (a) $\frac{13}{5}$; $\frac{3}{5}$
   (b) $\frac{9}{4}$; $\frac{5}{4}$

3. If $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$, then by an extension of Theorem 3-2(f) we get $\frac{a+b+c}{a'+b'+c'} = \frac{a}{a'}$, and hence $\frac{p}{p'} = \frac{a}{a'}$.

4. If $\frac{DB}{AD} = \frac{EC}{AE}$, then by Theorem 3-2(d) we get $\frac{AD+DB}{AD} = \frac{AE+EC}{AE}$. By one of our segment measurement postulates $AD + DB = AB$ and $AE + EC = AC$. Therefore, $\frac{AB}{AD} = \frac{AC}{AE}$.

3-3 RATIO AND PROPORTION IN GEOMETRY.

Be sure that your pupils understand Definition 3-2 and the discussion that follows it. There is nothing new about the algebraic properties of proportions in this section, but the agreements made in Definition 3-2 and the paragraph that follows it are important to the work that comes in Sections 3-4, 3-5, and 3-6.

Stress the importance of two segments being measured in terms of the same unit before a number can be obtained for their ratio. It would be absurd, for example, to say that the ratio of $\overline{AB}$ to $\overline{CD}$ is $\frac{2}{3}$ if the measure of $\overline{AB}$ is 2 feet and the measure of $\overline{CD}$ is 3 inches. When we say that the ratio of two segments is $\frac{2}{3}$ we mean that the first segment is $\frac{2}{3}$ as long as the

second, and, of course, this would not be true at all about the two segments above.

The answer to the question posed on page 69 is important and it may seem obvious to some of your pupils while others may not be so certain. Some may feel that since measuring two segments with different units gives different numbers for their lengths, it may also give a different number for their ratio. A little thought and some discussion of examples like those in the text should convince everyone of the truth of Theorem 3-3. Once the pupils have arrived at the statement of Theorem 3-3 through looking at examples, there is not much to be gained by discussing the formal proof as given in the text. It is sufficient for the poor pupil to know that if one segment is $\frac{2}{5}$ as long as another, it remains so regardless of what unit is used to measure them. It is more important for the good pupil to recognise the need for Theorem 3-3 than it is for him to be able to prove it.

Answers to
PROBLEMS 3-3

Student Text Page 71

1. $\frac{AB}{FD} = \frac{BC}{DE} = \frac{AC}{FE}$.

2. $B'C' = 27$.
   $A'C' = 39$.

3. Their ratio is $\frac{24}{36} = \frac{2}{3}$ regardless of whether they are measured in inches, feet, yards, or miles.

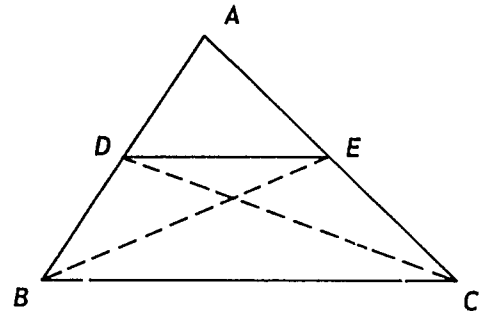
4. $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{7}{3}$.

3-4 THE BASIC PROPORTIONALITY THEOREM.

THEOREM 3-4. If a line parallel to one side of a triangle intersects the other two sides in distinct points, then it cuts off segments which are proportional to these sides.

Given: ΔABC with points D and E on $\overline{AB}$ and $\overline{AC}$ respectively such that $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$.

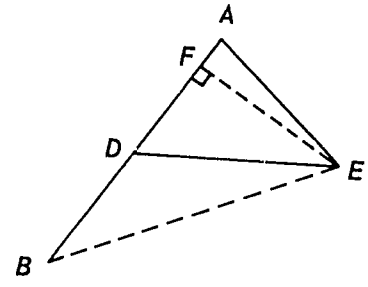
Prove: $\frac{AB}{AD} = \frac{AC}{AE}$.



Proof:

(1) Draw $\overline{DC}$ and $\overline{BE}$.

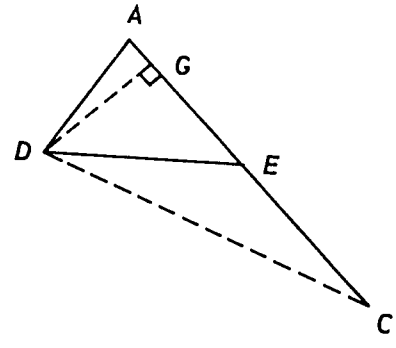
(2) In ΔADE and ΔBED let us regard $\overline{AD}$ and $\overline{BD}$ as the bases. The triangles have the same altitude, namely $\overline{EF}$. Therefore by Corollary 2-3-1 the ratio of their areas is the ratio of their bases, and we have



$$\frac{\text{Area } \Delta BED}{\text{Area } \Delta ADE} = \frac{BD}{AD}.$$

(3) In ΔADE and ΔCED let us regard $\overline{AE}$ and $\overline{CE}$ as the bases.

The triangles have the same altitude DG. Therefore by Corollary 2-3-1 we have as before



$$\frac{\text{Area } \Delta CED}{\text{Area } \Delta ADE} = \frac{CE}{AE}.$$

(4) Now ΔBED and ΔCED have the same base $\overline{DE}$ and equal altitudes.

$\therefore$ Area ΔBED = Area ΔCED.

(5) Combining the statements in (2), (3) and (4) we have

$$\frac{BD}{AD} = \frac{CE}{AE}.$$

(6) $\therefore \frac{BD}{AD} + 1 = \frac{CE}{AE} + 1$ or $\frac{BD + AD}{AD} = \frac{CE + AE}{AE}$.

(7) $\therefore \frac{AB}{AD} = \frac{AC}{AE}$.

The proof of this theorem is mainly based on areas.

COROLLARY 3-4-1.

The first part of the corollary follows by inverting the result obtained in step 5 of the proof of Theorem 3-4 which gives

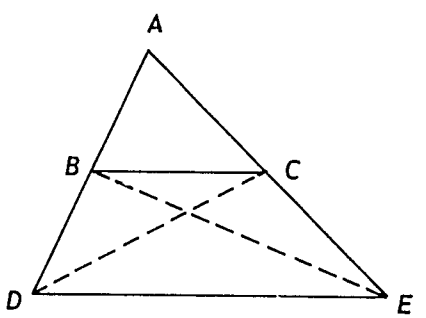
$\frac{AD}{BD} = \frac{AE}{CE}$. From this we get $\frac{AD}{AE} = \frac{BD}{CE}$ and from Theorem 3-4 we get $\frac{AB}{AC} = \frac{AD}{AE}$. These last two results give $\frac{AB}{AC} = \frac{BD}{CE}$.

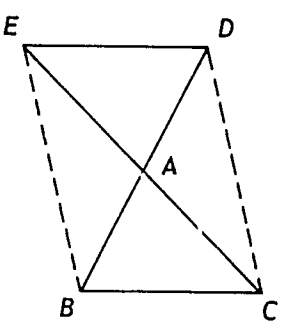
Therefore, $\frac{AB}{BD} = \frac{AC}{CE}$,

which is the second part of the corollary. (We have used Theorem 3-2 in transforming these proportions.)

COROLLARY 3-4-2.

There are two cases as indicated in the two figures.





In both cases draw $\overline{BE}$ and $\overline{DC}$.

The proof to the first case follows directly from the proof of Theorem 3-4 by interchanging B and D, and C and E respectively. To prove the second case we use Corollary 2-3-1 on the proportionality of the ratio of areas of triangles to the ratio of their bases. We get

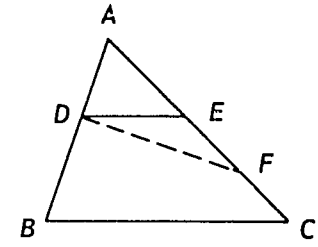
(1) $\frac{\text{Area } \Delta BED}{\text{Area } \Delta CED} = 1.$ They have the same base and equal altitudes.

(2) $\frac{\text{Area } \Delta BED}{\text{Area } \Delta ADE} = \frac{BD}{AD}.$ Corollary 2-3-1.

(3) $\frac{\text{Area } \Delta CED}{\text{Area } \Delta ADE} = \frac{CE}{EA}.$ Corollary 2-3-1.

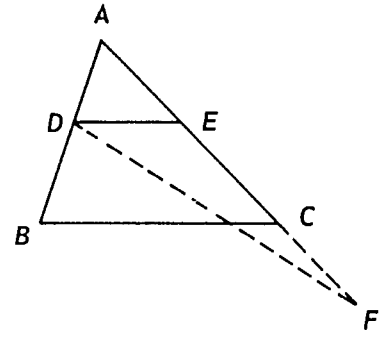
(4) Combining (1), (2) and (3) gives the required result.

THEOREM 3-5



In the construction for the proof of the theorem we assumed that F lies between A and C.

Obviously it is impossible for C to be between A and F. It can be seen that if C lies between A and F then $\overleftrightarrow{DF}$ cannot be parallel to $\overleftrightarrow{BC}$ since it intersects it. Now suppose F is such that A is between F and C. Then since $\overleftrightarrow{DF} \parallel \overleftrightarrow{BC}$ it follows that

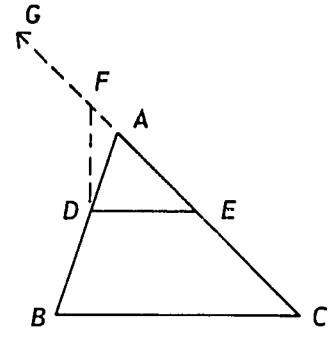


(1) $\widehat{DFG} \equiv \widehat{BCA}$ by corresponding angles.

(2) $m(\widehat{DFG}) > m(\widehat{DAG})$ by the Exterior Angle Theorem.

(3) $m(\widehat{DAG}) > m(\widehat{BCA})$ by the Exterior Angle Theorem.

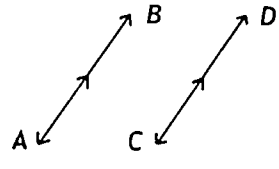
(4) $m(\widehat{DFG}) > m(\widehat{BCA})$ by Statements (2) and (3).



This leads to a contradiction of the fact that $\overleftrightarrow{DF} \parallel \overleftrightarrow{BC}$.

Hence if $\overleftrightarrow{DF} \parallel \overleftrightarrow{BC}$ then F must lie between A and C.

There should be a short discussion on "the method of indirect proof" to refresh the notion in the minds of the pupils.

NOTATION: The notation 

for $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ must not be confused with the notation for rays. The arrows on rays are at the end of the segment: .

Answers to PROBLEMS 3-4

Student Text Pages 75-78

1. $\frac{XM}{XY}$, $\frac{LZ}{LX}$, $\frac{XY}{XZ}$, $\frac{XM}{XL}$, $\frac{XY}{MY}$, $\frac{XY}{XZ}$.

2. $AC = 5$.

3. $\frac{AE}{EC}$, $\frac{AB}{AD}$, $\frac{AB}{AC}$, $\frac{AE}{AD}$.

4. (a) $QT = 4$, (b) $PR = 18$, (c) $PS = 7$.

5. (a) $CE = 8$, (b) $FG = 6$.

6. $MQ = 6$, $BM = \frac{18}{5}$.

7. From ΔADE and ΔABC: $\frac{AE}{EC} = \frac{AD}{BD}$

From ΔCEX and ΔCAY: $\frac{AE}{EC} = \frac{XY}{CX}$

$\therefore \frac{XY}{CX} = \frac{AD}{BD}$ $\therefore XY = 3$

8. (1) In ΔOAB, $\frac{OE}{OA} = \frac{OF}{OB}$ Theorem 3-4

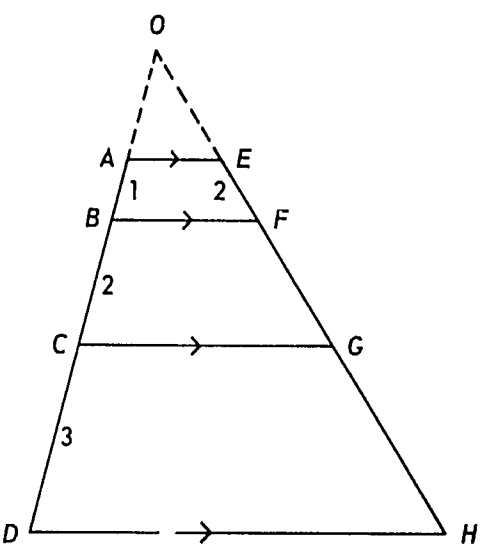
(2) In ΔOBC, $\frac{OF}{OB} = \frac{OG}{OC}$ Theorem 3-4

(3) In ΔOCD, $\frac{OG}{OC} = \frac{OH}{OD}$ Theorem 3-4

(4) $\therefore \frac{OE}{OA} = \frac{OH}{OD}$ Statements 1, 2 and 3

(5) In ΔODA, $\overleftrightarrow{HE} \parallel \overleftrightarrow{DA}$ Theorem 3-5

9. Let $\overrightarrow{DA}$ and $\overrightarrow{HE}$ intersect in the point O.



(1) From ΔOAE and ΔOBF

$$\frac{OA}{OE} = \frac{AB}{EF}$$

(2) From ΔOAE and ΔOCG

$$\frac{OA}{OE} = \frac{AC}{EG}$$

(3) From ΔOAE and ΔODH

$$\frac{OA}{OE} = \frac{AD}{EH}$$

(4) From statements (1) and (2)

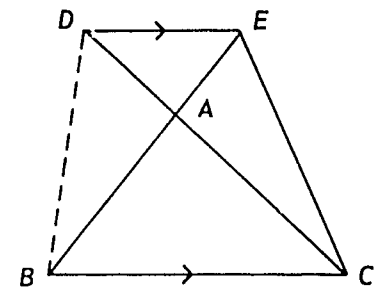
$\frac{AC}{EG} = \frac{AB}{EF}$ $\therefore EG = 6$ and $FG = 4$.

(5) From statements (1) and (3)

$\frac{AD}{EH} = \frac{AB}{EF}$ $\therefore EH = 12$ and $HG = 6$.

10. (1) Since ΔBDE and ΔCDE have the same base and equal altitudes their areas are equal.

$\therefore$ Area ΔBDE = Area ΔCDE



(2) $\dfrac{\text{Area } \Delta ADE}{\text{Area } \Delta BDE} = \dfrac{AE}{BE}$ Corollary 2-3-1.

(3) $\dfrac{\text{Area } \Delta ADE}{\text{Area } \Delta CDE} = \dfrac{AD}{CD}$ Corollary 2-3-1.

(4) Combining statements (1), (2) and (3) gives $\dfrac{AE}{BE} = \dfrac{AD}{CD}$.

11. Since, by hypothesis, $EB = FD$ and $\overleftrightarrow{EB} \parallel \overleftrightarrow{FD}$ it follows that $\overleftrightarrow{BF} \parallel \overleftrightarrow{ED}$. Therefore in ΔCGD and ΔCHF by Theorem 3-4,

$$\frac{CH}{GH} = \frac{CF}{FD} = 1.$$

$\therefore CH = GH.$

Similarly in ΔABH and ΔAEG,

$$\frac{AG}{GH} = \frac{AE}{EB} = 1.$$

$\therefore AG = GH.$

Hence $AG = GH = HC$.

Hence G and H trisect $\overline{AC}$.

For the last part consider ΔAEG and ΔCDG. By hypothesis, $\overleftrightarrow{AE} \parallel \overleftrightarrow{CD}$. Therefore by application of the result of Problem 10 we have

$$\frac{EG}{GD} = \frac{AG}{GC} = \frac{1}{2}$$

$$\therefore EG = \frac{1}{2} GD$$

$$\therefore EG = \frac{1}{2}(ED - EG)$$

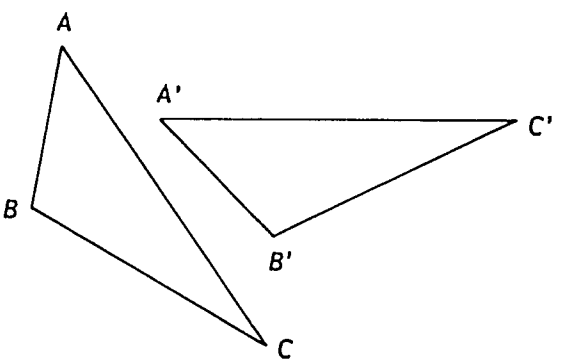
$$\therefore EG = \frac{1}{3} ED$$

3-5 SIMILAR TRIANGLES.

The definition of a similarity, like the definition of a congruence, requires two things: a congruence between corresponding angles and a proportionality between corresponding sides. For similar triangles we could have based our definition on either one of the two conditions, and proved the other. However it is better to define similar triangles in a manner that can be generalized to other polygonal figures.

Two triangles are similar if they have the same shape but not necessarily the same size. If two triangles are similar it ought to be possible to "magnify" or "stretch" one of them, without changing its shape, so as to get the second triangle. Also the second triangle could be "shrunk" or "diminished", without altering its shape, so as to get the first triangle.

The setting up of the correspondence between two similar triangles must be carefully dealt with. A lot of diagrams should be drawn to explain how the corresponding parts of two similar triangles can be picked out. For example, in ΔABC, by inspection $m(\hat{B}) > m(\hat{A}) > m(\hat{C})$. In $\Delta A'B'C'$, by inspection $m(\hat{B'}) > m(\hat{A'}) > m(\hat{C'})$. Hence if $\Delta ABC \sim \Delta A'B'C'$ we should expect that $\hat{A} \equiv \hat{A'}$, $\hat{B} \equiv \hat{B'}$ and $\hat{C} \equiv \hat{C'}$.



Some pupils may tend to think that corresponding angles in two similar triangles are the ones which occupy the same relative position on the blackboard. It is necessary to give several examples to dispel this idea.

It must be stressed to the pupils that the notation for a triangle similarity must be such that letters representing corresponding vertices in the similarity are relatively situated within the expressions denoting the triangles. Thus if $A \leftrightarrow X$, $B \leftrightarrow Y$, and $C \leftrightarrow Z$, then $\Delta ABC \sim \Delta XYZ$.

Answers to
PROBLEMS 3-5

Student Text Page 80

1. $\hat{A} \leftrightarrow \hat{B'}$, $\hat{B} \leftrightarrow \hat{A'}$, $\hat{C} \leftrightarrow \hat{C'}$, $\overline{AB} \leftrightarrow \overline{B'A'}$, $\overline{BC} \leftrightarrow \overline{A'C'}$, $\overline{CA} \leftrightarrow \overline{C'B'}$.

2. $DE = 3$; $DF = 5$.

3. Since $\Delta ABC \sim \Delta A'B'C'$

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$$

$\therefore AB = 12$ and $BC = 16$

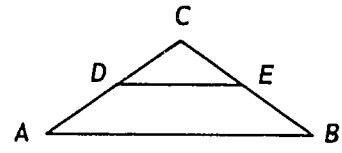
4. $\frac{4}{b} = \frac{6}{12} = \frac{a}{8}$. Therefore, $b = 8$ in. and $a = 4$ in.

5. Since $\Delta ABC \sim \Delta YXZ$

$$\frac{AB}{XY} = \frac{BC}{XC} = \frac{AC}{YC}$$

$\therefore \frac{8}{XY} = \frac{BC}{6} = \frac{9}{5}$. $\therefore XY = \frac{40}{9}$ and $BC = \frac{54}{5}$.

6. (1) $\frac{CD}{CA} = \frac{CE}{CB} = \frac{DE}{AB} = \frac{1}{2}$ (by hypothesis D and E are the midpoints of AC and BC.)



(2) $\overleftrightarrow{DE} \parallel \overleftrightarrow{AB}$ by Theorem 3-5

(3) $\angle CDE \cong \angle CAB$ and $\angle DEC \cong \angle ABC$ by the Alternate Angle Theorem

(4) $\angle C \cong \angle C$

(5) $\Delta ABC \sim \Delta DEC$ by Definition 3-3.

7. Let ΔABC and $\Delta A'B'C'$ be two equilateral triangles.

$\therefore$ (1) $\angle A \cong \angle B \cong \angle C$ and $\angle A' \cong \angle B' \cong \angle C'$.

(2) $AB = BC = CA$; $A'B' = B'C' = C'A'$.

then from (1)

$\angle A \cong \angle A'$, $\angle B \cong \angle B'$, and $\angle C \cong \angle C'$.

and from (2)

$$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$$

$\therefore \Delta ABC \sim \Delta A'B'C'$ by Definition 3-3.

From (1) and (2) we have also $\Delta ABC \sim \Delta B'C'A'$,

$\Delta ABC \sim \Delta C'A'B'$.

8. Yes. Let $\Delta ABC \cong \Delta DEF$. Therefore

(1) $\angle A \cong \angle D$, $\angle B \cong \angle E$, $\angle C \cong \angle F$

(2) $AB = DE$, $BC = EF$ and $AC = DF$.

Hence from (2) we get

(3) $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$.

From (1) and (3) we conclude that $\Delta ABC \sim \Delta DEF$.

Not every pair of similar triangles are congruent. For example, if $\Delta ABC \sim \Delta DEF$ then (1) and (3) above hold. But (3) does not necessarily imply (2) which is necessary for congruence of the two triangles.

9. Since $\Delta ABC \sim \Delta RST$, by definition

8. Cont.

(1) $\hat{A} \equiv \hat{R}$, $\hat{B} \equiv \hat{S}$, $\hat{C} \equiv \hat{T}$, $\frac{AB}{RS} = \frac{BC}{ST} = \frac{AC}{RT}$

Since $\Delta STR \sim \Delta YZX$, by definition

(2) $\hat{S} \equiv \hat{Y}$, $\hat{T} \equiv \hat{Z}$, $\hat{R} \equiv \hat{X}$, $\frac{ST}{YZ} = \frac{TR}{ZX} = \frac{SR}{XY}$

Combining (1) and (2) we have

(3) $\hat{A} \equiv \hat{X}$, $\hat{B} \equiv \hat{Y}$, $\hat{C} \equiv \hat{Z}$, $\frac{AB}{XY} = \frac{BC}{YZ} = \frac{AC}{ZX}$

Hence it follows from (3) that $\Delta ABC \sim \Delta XYZ$.

3-6 THE BASIC SIMILARITY THEOREMS.

THEOREM 3-6 (The AAA Similarity Theorem).

This theorem may be stated briefly as:

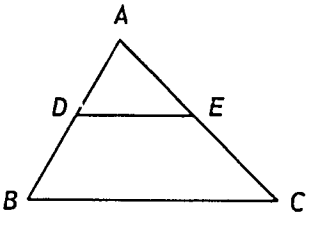
If the angles of one triangle are congruent respectively to the angles of another triangle, then the triangles are similar.

In the proof of the theorem we assume that $\frac{AB}{A'B'} > 1$ and $\frac{AC}{A'C'} > 1$. If $AB < A'B'$ and $AC < A'C'$ the proof follows in the same manner with the roles of ΔABC and $\Delta A'B'C'$ interchanged.

COROLLARY 3-6-1.

If two angles of one triangle are congruent to two angles of another triangle then by the Angle Sum theorem for triangles the third pair of angles are also congruent. Hence the three angles of one triangle are congruent to the three angles of the other triangle. It follows from Theorem 3-6 that the two triangles are similar.

COROLLARY 3-6-2.



Given: ΔABC and $\overleftrightarrow{DE} \parallel \overleftrightarrow{BC}$

Prove: $\Delta ABC \sim \Delta ADE$

Proof:

(1)	$\hat{ADE} \equiv \hat{B}$	Corresponding Angle Theorem
(2)	$\hat{DEA} \equiv \hat{C}$	Corresponding Angle Theorem
(3)	$\hat{A} \equiv \hat{A}$	
(4)	$\Delta ABC \sim \Delta ADE$	AAA Similarity Theorem.

COROLLARY 3-6-3.

Let ΔABC and ΔDEF be the two right triangles with right angles at A and D and $\hat{B} \equiv \hat{E}$. It follows from the AA Similarity Theorem that

$$\Delta ABC \sim \Delta DEF.$$

COROLLARY 3-6-4.

Given: $\Delta ABC \sim \Delta DEF$ and $\Delta A'B'C' \sim \Delta DEF$

Prove: $\Delta ABC \sim \Delta A'B'C'$

Proof:

(1) $\hat{A} \equiv \hat{D}$, $\hat{B} \equiv \hat{E}$, $\hat{C} \equiv \hat{F}$ since $\Delta ABC \sim \Delta DEF$.

(2) $\hat{D} \equiv \hat{A'}$, $\hat{E} \equiv \hat{B'}$, $\hat{F} \equiv \hat{C'}$ since $\Delta DEF \sim \Delta A'B'C'$.

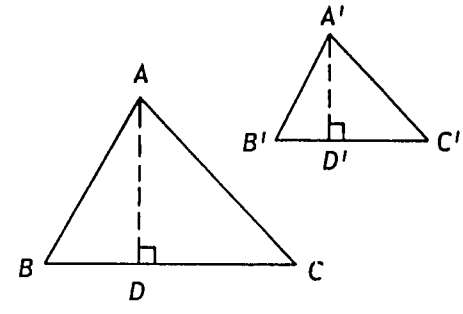
(3) $\hat{A} \equiv \hat{A'}$, $\hat{B} \equiv \hat{B'}$, $\hat{C} \equiv \hat{C'}$ Statements (1) and (2)

(4) $\therefore \Delta ABC \sim \Delta A'B'C'$, by the AAA Similarity Theorem.

COROLLARY 3-6-5

Given: $\Delta ABC \sim \Delta A'B'C'$ with altitudes $\overline{AD}$ and $\overline{A'D'}$ respectively.

Prove: $\dfrac{AD}{A'D'} = \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$



Proof:

(1) $\hat{B} \equiv \hat{B'}$ since $\Delta ABC \sim \Delta A'B'C'$.

(2) $\Delta ABD \sim \Delta A'B'D'$ by Corollary 3-6-3.

(3) $\dfrac{AD}{A'D'} = \dfrac{AB}{A'B'}$.

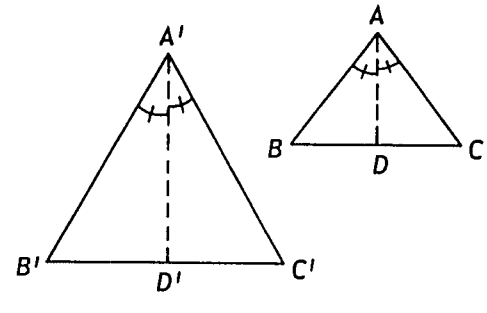
(4) $\dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$ since $\Delta ABC \sim \Delta A'B'C'$, and

(5) $\dfrac{AD}{A'D'} = \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$.

COROLLARY 3-6-6

Given: $\Delta ABC \sim \Delta A'B'C'$ with angle bisectors $\overline{AD}$ and $\overline{A'D'}$.

Prove: $\dfrac{AD}{A'D'} = \dfrac{AB}{A'B'} = \dfrac{BC}{B'C'} = \dfrac{AC}{A'C'}$.



Proof:

(1) $\angle B \equiv \angle B'$, $\angle A \equiv \angle A'$ and $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$ since $\Delta ABC \sim \Delta A'B'C'$.

(2) $\angle BAD \equiv \angle B'A'D'$ by Statement (1) and hypothesis.

(3) $\Delta ABD \sim \Delta A'B'D'$ by the AA Similarity Theorem.

(4) $\frac{AD}{A'D'} = \frac{AB}{A'B'}$ by Definition of similarity of triangles.

(5) $\frac{AD}{A'D'} = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$ by Statements (1) and (4).

Answers to
PROBLEMS 3-6

Student Text Pages 85-87

1. (a) Similar; SAS (b) Similar; SSS
   (c) Similar; SAS (d) Similar; AAA

2. $\frac{FH}{DF} = \frac{FK}{EF}$ $\left(= \frac{3}{4}\right)$ and F is common to the two triangles.
   $\therefore$ $\Delta HKF \sim \Delta DEF$ by the SAS Similarity Theorem.
   $\therefore$ $\frac{HK}{DE} = \frac{FH}{DF}$.
   $\therefore$ $HK = 12$.

3. They are similar by the SAS Similarity Theorem.

4. They are similar by the SAS Similarity Theorem.
   $\angle A \equiv \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$.

5. $\frac{AO}{OD} = \frac{OB}{OC}$ $\left(= \frac{2}{1}\right)$ and $\angle AOB \equiv \angle COD$.
   $\therefore$ $\Delta AOB \sim \Delta DOC$ by the SAS Similarity Theorem.

6. $\Delta ABC \sim \Delta AED$ by the SAS Similarity Theorem.

7. $\frac{OR}{OQ} = \frac{OS}{OP}$. $\therefore$ $OR = \frac{bc}{a}$

8. Consider ΔXYZ and ΔXZW:
   $\frac{XW}{XZ} = \frac{XZ}{XY}$ and $\angle ZXY \equiv \angle WXZ$ by hypothesis.
   $\therefore$ $\Delta XYZ \sim \Delta XZW$ by the SAS Similarity Theorem.
   $\therefore$ $\angle XZY \equiv \angle XWZ$ Corresponding angles in two similar triangles.
   $\therefore$ $\angle XWZ$ is a right angle since $\angle XZY$ is a right angle.
   $\therefore$ $\overline{ZW} \perp \overline{XY}$

9. In the isosceles triangles ΔABC and ΔDEF let $AB = AC$ and $DE = DF$.

$\therefore \dfrac{AB}{DE} = \dfrac{AC}{DF}$

$\widehat{A} \equiv \widehat{D}$ by hypothesis

$\therefore \Delta ABC \sim \Delta DEF$ by the SAS Similarity Theorem.

10. In the isosceles triangles ΔABC and ΔDEF if the base angles $\widehat{B}$, $\widehat{C}$, $\widehat{E}$ and $\widehat{F}$ are congruent it follows that $\widehat{A} \equiv \widehat{D}$ and by the result of Problem 9 we have $\Delta ABC \sim \Delta DEF$.

11. Since $\widehat{P} \equiv \widehat{R}$ it follows that $PQ = RQ$.

By hypothesis $RM = PL$. Hence $MQ = LQ$.

$\therefore \dfrac{MQ}{RQ} = \dfrac{LQ}{PQ}$

$\therefore \Delta PQR \sim \Delta LQM$ by the SAS Similarity Theorem

$\therefore \widehat{P} \equiv Q\widehat{L}M$ and $\widehat{R} \equiv L\widehat{M}Q$

$\therefore \overleftrightarrow{LM} \parallel \overleftrightarrow{PR}$

12. Let $\Delta ABC \sim \Delta XYZ$. Then by definition

$\dfrac{AB}{XY} = \dfrac{BC}{YZ} = \dfrac{AC}{XZ} = k$, say.

$\therefore \dfrac{AB + BC + AC}{XY + YZ + XZ} = \dfrac{kXY + kYZ + kXZ}{XY + YZ + XZ} = k$

Hence the result follows.

13. $\Delta MXZ \sim \Delta LXY$ by Corollary 3-6-3.

$\dfrac{MX}{LX} = \dfrac{XZ}{XY} = \dfrac{MZ}{LY}$.

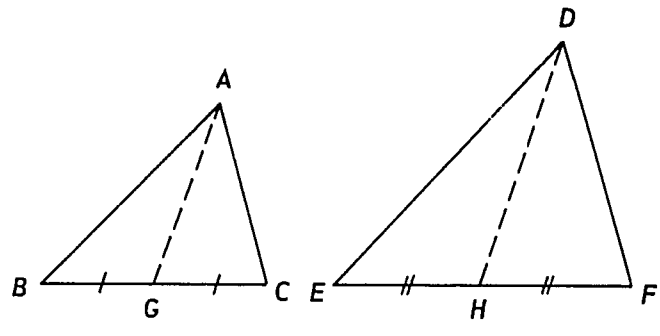
14. Since $\Delta ABC \sim \Delta DEF$ we have

$\widehat{B} \equiv \widehat{E}$ and $\dfrac{AB}{DE} = \dfrac{BC}{EF} = \dfrac{AC}{DF}$.

$\therefore \dfrac{AB}{DE} = \dfrac{2BG}{2EH} = \dfrac{BG}{EH}$.

$\therefore \Delta ABG \sim \Delta DEH$ by the SAS Similarity Theorem.

$\therefore \dfrac{AG}{DH} = \dfrac{AB}{DE}$.

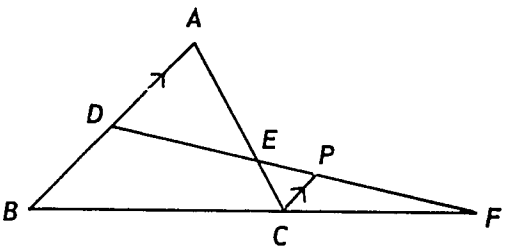


Challenge Problem

15. $D\hat{A}E \equiv E\hat{C}P$ and $A\hat{D}E \equiv C\hat{P}E$ (alternate angles).

$A\hat{E}D \equiv C\hat{E}P$ (vertically opposite angles).

$\therefore$ $\Delta ADE \sim \Delta CPE$ by the AAA Similarity Theorem.



(1) $\therefore$ $\dfrac{AE}{CE} = \dfrac{AD}{CP}$

$D\hat{B}F \equiv P\hat{C}F$ and $B\hat{D}F \equiv C\hat{P}F$ (alternate angles).

$\therefore$ $\Delta BFD \sim \Delta CFP$ by the AAA Similarity Theorem

(2) $\therefore$ $\dfrac{FB}{CF} = \dfrac{BD}{CP}$

Since $BD = AD$ by hypothesis, we have from (1) and (2)

$$\frac{FB}{CF} = \frac{AE}{CE}$$

3-7 METHODS OF PROVING LINE SEGMENTS PROPORTIONAL.

This section is very important as it shows the student how to set out attacking problems involving similar triangles. Great pains should be taken to show the student why, given a problem involving several triangles, we pick out two of the triangles and try and prove them similar. These triangles are picked out, not arbitrarily, but so as to involve the ratios or proportions we are interested in. The second example illustrates the method to follow when no pair of triangles involve all the sides in the proportion. We then pick two sides in one of the ratios and look for triangles involving them. Do the same for the other two sides and try to find a connecting link. Some students find this fascinating and relatively easy. Others find it extremely difficult. Be patient but do not spend too much time trying to get every student to master this technique.

Answers to PROBLEMS 3-7

Student Text Pages 89-92

1. Let ΔABC and ΔXYZ be two equiangular triangles. Then $\hat{A} \equiv \hat{B} \equiv \hat{C}$ and $\hat{X} \equiv \hat{Y} \equiv \hat{Z}$

1. Cont.

and by the Angle-Sum Theorem for triangles the measure of each angle must be 60°. Hence we have

$\hat{A} \equiv \hat{X}$, $\hat{B} \equiv \hat{Y}$, $\hat{C} \equiv \hat{Z}$

Hence $\Delta ABC \sim \Delta XYZ$ by the AAA Similarity Theorem.

Also $\Delta ABC \sim \Delta YZX$ and $\Delta ABC \sim \Delta ZXY$.

2. $\frac{CD}{AC} = \frac{EC}{BC} \left(= \frac{1}{3}\right)$

$\overleftrightarrow{DE} \parallel \overleftrightarrow{AB}$ by Theorem 3-5.

$\therefore$ $C\hat{D}E \equiv \hat{A}$ and $D\hat{E}C \equiv \hat{B}$ by the Corresponding Angles Theorem

$\therefore$ $\Delta CDE \sim \Delta CAB$ by the AAA Similarity Theorem.

3. Let $\overline{AB}$, $\overline{CD}$, $\overline{OA}$, and $\overline{OC}$ represent the tree, the pole, the shadow cast by the tree, and the shadow cast by the pole respectively. Then

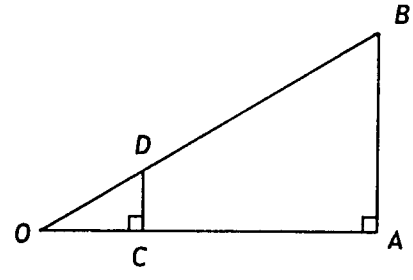
$CD = 6$, $OA = 30$, $OC = 9$.

$\Delta OAB \sim \Delta OCD$ by the AAA Similarity Theorem

$\therefore$ $\frac{AB}{CD} = \frac{OA}{OC}$

$\therefore$ $AB = 20$

Hence the tree is 20 feet tall.



4. If $\overline{AB}$ and $\overline{CD}$ represent the cliff and the telephone pole respectively, then

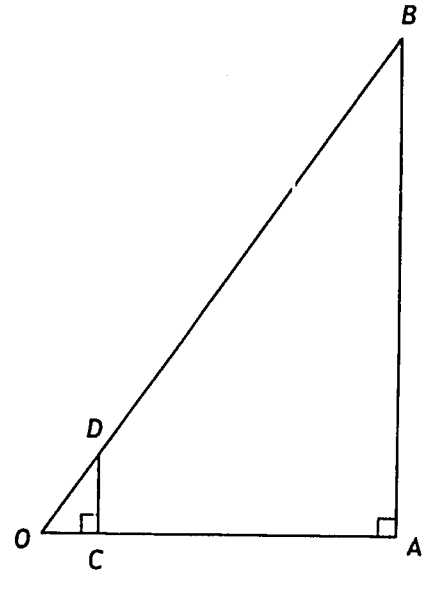
$CD = 20$, $OC = 15$ and $OA = 120$.

$\Delta OAB \sim \Delta OCD$ by the AAA Similarity Theorem

$\therefore$ $\frac{AB}{CD} = \frac{OA}{OC}$

$\therefore$ $AB = 160$,

Hence the cliff is 160 feet high.



5. $\Delta ABC \sim \Delta ADE$ by the AA Similarity Theorem

$\therefore \dfrac{AB}{AD} = \dfrac{BC}{DE} = \dfrac{AC}{AE}$

$\therefore AB = 4\frac{1}{2}, BC = 9.$

6. $X\hat{M}L \equiv \hat{Y}$ and $M\hat{L}X \equiv \hat{Z}$ since $\overleftrightarrow{LM} \parallel \overleftrightarrow{YZ}$.

$\therefore \Delta XYZ \sim \Delta XML$ by the AAA Similarity Theorem

$\therefore \dfrac{ML}{YZ} = \dfrac{XM}{XY}$.

$\therefore \dfrac{d}{b} = \dfrac{a}{a+c}$.

$\therefore d = \dfrac{ab}{a+c}$.

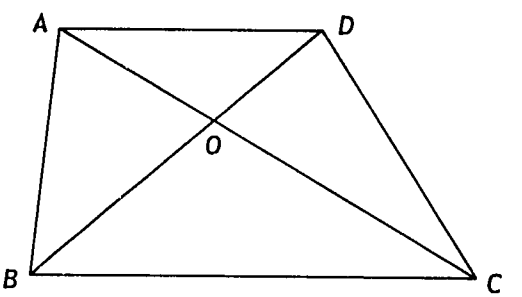
7. $\hat{C} \equiv \hat{B}$ and $\hat{D} \equiv \hat{A}$ by the Alternate Interior Angle Theorem.

$\therefore \Delta ABO \sim \Delta DCO$ by the AAA Similarity Theorem

$\therefore \dfrac{AO}{DO} = \dfrac{BO}{CO}$

$\therefore AO \times CO = BO \times DO$

8. Let the trapezium be as shown with $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$.



$\therefore D\hat{A}O \equiv B\hat{C}O$ and $A\hat{D}O \equiv O\hat{B}C$.

$\therefore \Delta AOD \sim \Delta COB$ by the AAA Similarity Theorem.

$\therefore \dfrac{AO}{CO} = \dfrac{OD}{OB}$.

9. $E\hat{B}F \equiv F\hat{D}G$ and $B\hat{E}F \equiv F\hat{G}D$ by the Alternate Interior Angle Theorem.

$\therefore \Delta DFG \sim \Delta BFE$ by the AAA Similarity Theorem

$\therefore \dfrac{DG}{BE} = \dfrac{DF}{BF}$

$\therefore DG \times BF = DF \times BE$

10. Consider ΔABC and ΔDAC

$C\hat{A}B \equiv C\hat{D}A$ both are right angles

$A\hat{B}C = D\hat{A}C$ both are complementary to $B\hat{A}D$

$\therefore \Delta ABC \sim \Delta DAC$ by the AAA Similarity Theorem

$\therefore \dfrac{AD}{AB} = \dfrac{CD}{AC}$.

10. Cont.

Similarly we can show that $\Delta ABC \sim \Delta DBA$. From this we obtain

$\frac{BD}{AB} = \frac{AD}{AC}$. Hence $\frac{BD}{AD} = \frac{AB}{AC}$.

11. Consider ΔABC and ΔAED.

$\hat{B} \equiv \hat{E}$ both are right angles.

$\hat{C} = D\hat{A}E$ both are complementary to $B\hat{A}C$.

$\therefore$ $\Delta ABC \sim \Delta DEA$ by the AA Similarity Theorem.

$\therefore$ $\frac{DE}{AB} = \frac{AE}{BC}$ and hence $\frac{DE}{AE} = \frac{AB}{BC}$.

12. $XY = YZ$ by hypothesis.

$\hat{X} \equiv \hat{Z}$ base angles of isosceles triangle.

$X\hat{M}L \equiv L\hat{N}Z$ both are right angles.

$\therefore$ $\Delta XML \sim \Delta ZNL$ by the AA Similarity Theorem.

$\therefore$ $\frac{LM}{NL} = \frac{XM}{NZ}$.

$\therefore$ $LM \times NZ = MX \times NL$.

13. Consider ΔADG and ΔBEF.

$A\hat{D}G \equiv B\hat{E}F$ both are right angles.

$D\hat{A}G \equiv B\hat{F}E$ both are complementary to $E\hat{B}F$.

$\therefore$ $\Delta ADG \sim \Delta FEB$ by the AA Similarity Theorem.

$\therefore$ $\frac{AD}{FE} = \frac{DG}{BE}$

$\therefore$ $AD \times BE = DG \times EF$.

Similarly it can be easily shown that

$\Delta ADG \sim \Delta GCF$ by the AA Similarity Theorem.

$\therefore$ $\frac{AD}{CG} = \frac{DG}{CF}$

$\therefore$ $AD \times CF = DG \times CG$.

14. $A\hat{D}C \equiv B\hat{E}C$ both are right angles.

$\hat{C} \equiv \hat{C}$

$\therefore$ $\Delta ADC \sim \Delta BEC$ by the AA Similarity Theorem.

In ΔAFE and ΔBFD,

$A\hat{E}F \equiv B\hat{D}F$ both are right angles.

14. Cont.

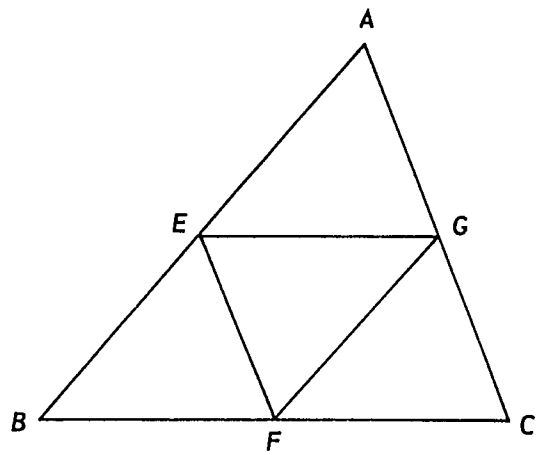
$A\hat{F}E \equiv B\hat{F}D$ vertically opposite angles.

$\therefore \ \Delta AFE \sim \Delta BFD$ by the AA Similarity Theorem.

15. From ΔABC and ΔBFE

$\frac{BE}{AB} = \frac{BF}{BC} \left(= \frac{1}{2}\right)$ by hypothesis

$\therefore \ \overleftrightarrow{EF} \parallel \overleftrightarrow{AC}$ by Theorem 3-5



$\therefore \ B\hat{F}E \equiv \hat{C}$ and $B\hat{E}F \equiv \hat{A}$ by the Corresponding Angles Theorem

Similarly we obtain

$C\hat{F}G \equiv \hat{B}$ and $C\hat{G}F \equiv \hat{A}$

and

$A\hat{E}G \equiv \hat{B}$ and $A\hat{G}E \equiv \hat{C}$

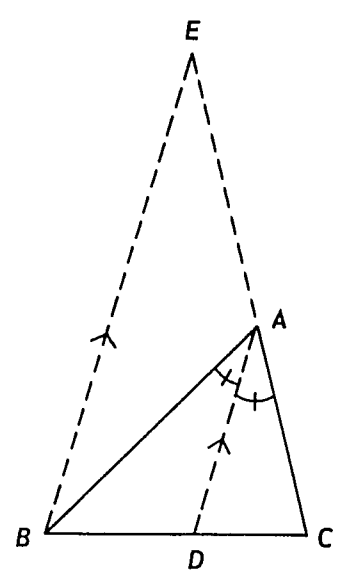
$\therefore$ by using the Angle Sum Theorem for triangles we get

$E\hat{F}G \equiv \hat{A}$, $F\hat{G}E \equiv \hat{B}$, $F\hat{E}G \equiv \hat{C}$

$\therefore \ \Delta ABC \sim \Delta FGE$ by the AAA Similarity Theorem.

16.

1.	$B\hat{A}D \equiv A\hat{B}E$	by the Alternate Interior Angles Theorem
2.	$C\hat{A}D \equiv A\hat{E}B$	corresponding angles.
3.	$B\hat{A}D \equiv C\hat{A}D$	by hypothesis.
4.	$\therefore A\hat{B}E \equiv A\hat{E}B$	statements 1, 2, and 3.
5.	$\therefore \ \overline{AB} \equiv \overline{AE}$	
6.	$\frac{CD}{BD} = \frac{AC}{AE}$	by Corollary 3-4-1.
7.	$\therefore \ \frac{CD}{BD} = \frac{AC}{AB}$	by Statements 5 and 6.



17. Since $\overleftrightarrow{AE} \parallel \overleftrightarrow{BF}$

$A\hat{B}F \equiv B\hat{A}E$ alternate angles.

$A\hat{F}B \equiv G\hat{A}E$ corresponding angles

$B\hat{A}E \equiv G\hat{A}E$ by hypothesis

17. Cont.

$\therefore\ A\hat{B}F \equiv A\hat{F}B$

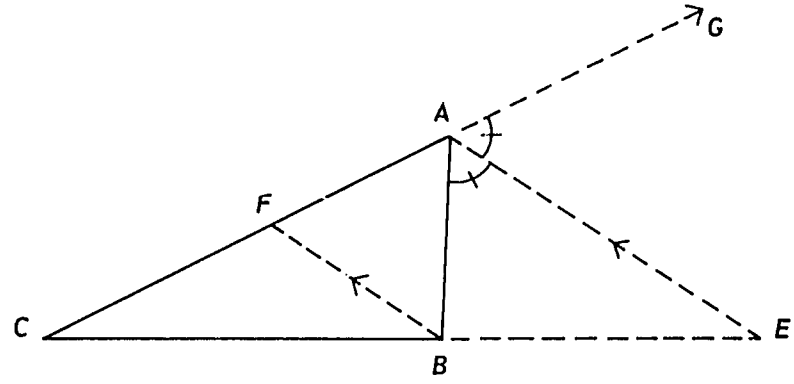
$\therefore\ AF = AB$

Since $\overleftrightarrow{AE} \parallel \overleftrightarrow{BF}$

$$\frac{BE}{CE} = \frac{AF}{AC}$$

by Corollary 3-4-1

$$\therefore\ \frac{BE}{CE} = \frac{AB}{AC}.$$



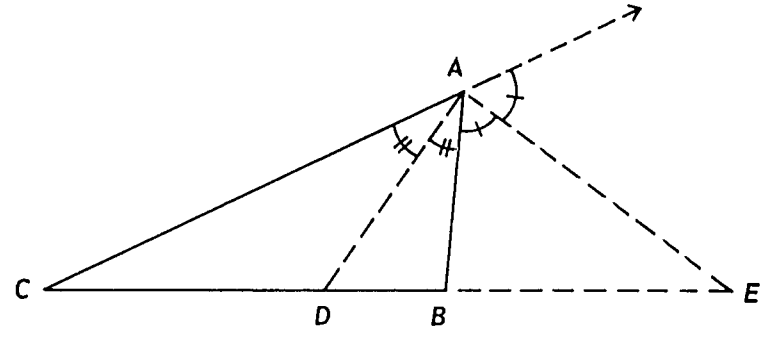
18. Using the results of Problems 16 and 17 we had

$$\frac{BD}{CD} = \frac{AB}{AC}$$

and

$$\frac{BE}{CE} = \frac{AB}{AC}.$$

$$\therefore\ \frac{BD}{CD} = \frac{BE}{CE}.$$



You may draw the attention of the better pupils to the following important point which arises from Problem 18:

> DEFINITION: D and E are said to divide $\overline{BC}$ *harmonically*; conversely B and C are said to divide $\overline{DE}$ *harmonically*. The points C, D, B and E are said to form a *harmonic range* if $\frac{BD}{CD} = \frac{BE}{CE}$.

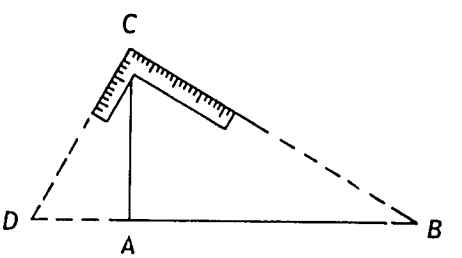
The notion of harmonic ranges is important in the study of projective geometry.

3-8 SIMILARITY IN RIGHT TRIANGLES.

The main purpose of this section is to derive the mean proportional theorems of Corollary 3-9-1 from the similarities of the triangles formed by dropping the altitude to the hypotenuse of a right triangle. Your pupils should have no difficulty in seeing that this altitude subdivides the given right triangle into two right triangles that are similar to it and to each other.

The proportions that result from the similarities of Theorem 3-9 are easy to discover by just looking at the correspondences set up between the sides of the similar triangles. Some of these proportions are useful in physics and carpentry.

For example, a carpenter's square can be used to find the distance between two points. By placing a pole of convenient height at one point (A) and sighting the second point (B) along the edge of the square as in the figure below, we can measure the distance DA and by a mean proportional theorem find AB. Thus if $AC = 6$ ft. and $DA = 2\frac{1}{4}$ ft., we can find $AB = 16$ ft.



We have an application of similar right triangles in physics when we study the principle of the inclined plane. Perhaps one of your pupils could show the rest of the class how this principle works and how similar right triangles are used.

The exercises of Problems 3-8 may seem repetitious and certainly not all of them need be assigned. They simply give the pupil practice in applying the correspondences of the triangle similarities developed in the theorems and are not important in themselves. Do not worry your poor pupils with Problem 6. It is obvious from the figure that point D falls between points A and B and they should not even feel the need for proving it. Your good pupils should know that we do have the tools in our geometry for proving this obvious fact and the problem provides another good example of proof by contradiction. Be sure to assign Problem 10 to all the pupils as we will use this problem as a beginning to a proof of Pythagoras' Theorem using similar triangles in the next section.

Answers to
PROBLEMS 3-8

Student Text Pages 94-95

1. $a = 6\sqrt{5}$; $b = 3\sqrt{5}$; $h = 6$

2. $\frac{1}{2}$; $\overline{CD}$ and $\overline{DB}$; $\overline{AD}$ and $\overline{CD}$.

3. $x = 4$; $y = 3\sqrt{5}$; $h = 2\sqrt{5}$.

4. $\frac{2}{3}$; $\overline{WY}$ and $\overline{ZY}$; $\overline{ZW}$ and $\overline{XY}$.

5. $x = 15$; $a = 4\sqrt{5}$; $b = 8\sqrt{5}$

6. There are actually 5 cases to consider. They are:

I. (1) A is between D and B (See Figure I below.).

(2) B is between A and D.

The proofs to these two cases are similar, so only one of them will be treated here.

II. (1) $A = D$ (See Figure II below.)

(2) $B = D$

Again, the proofs to these two cases are similar so we treat only one of them.

III. D is between A and B.

We will show that cases I and II are impossible, so III is true.

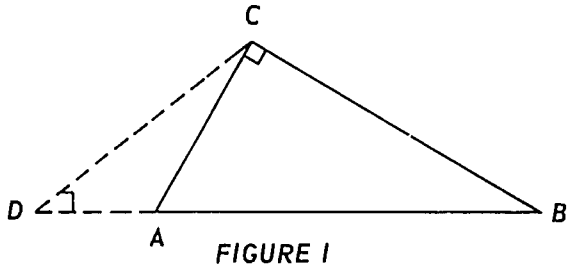


FIGURE I

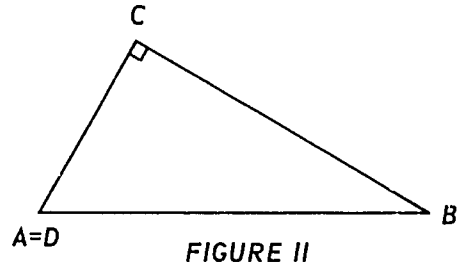


FIGURE II

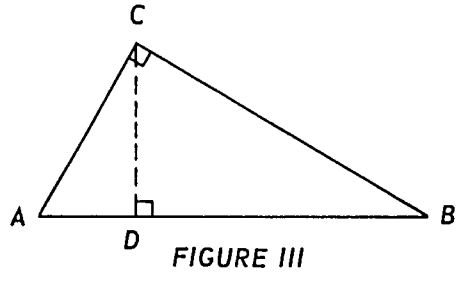


FIGURE III

Proof:

I. In Figure I, $A\hat{D}C$ is a right angle since we are given $\overline{CD} \perp \overleftrightarrow{AB}$ and $B\hat{A}C$ is acute since ΔABC has a right angle at C. This contradicts the exterior angle theorem (we cannot have $B\hat{A}C < A\hat{D}C$), so A is not between D and B. Similarly, B is not between A and D.

II. In Figure II, if $A = D$, then we have $m(C\hat{D}B) = m(C\hat{A}B) = 90°$, and this contradicts the fact that $C\hat{A}B$ is an acute angle. So $A \neq D$. Similarly, $B \neq D$.

Since cases I and II are false, case III is the only remaining possibility.

7. $x = \frac{64}{c}$; $y = \frac{25}{c}$; $h = \frac{40}{c}$.

8. $x = \frac{25}{13}$; $y = \frac{144}{13}$; $h = \frac{60}{13}$.

9. Since, in the figure, ΔCDB ΔACB, we have $\frac{CD}{AC} = \frac{BC}{AB}$. Substituting, $\frac{h}{b} = \frac{a}{c}$ and $h = \frac{ab}{c}$.

10. By the mean proportional theorems, $\frac{BD}{a} = \frac{a}{c}$ and $\frac{DA}{b} = \frac{b}{c}$. Therefore, $BD = \frac{a^2}{c}$ and $DA = \frac{b^2}{c}$.

3-9 PYTHAGORAS' THEOREM.

The mean proportional theorems of the last section provide us with a remarkably simple proof of Pythagoras' Theorem. If you assigned Problem 10 of Problems 3-8, ask your pupils to take their results for BD and DA and, using the figure in the text, prove $c^2 = a^2 + b^2$. Do not spoil your pupils' fun of discovering the proof for themselves by giving them any more information than is suggested in Problem 10. Those who are unable to complete the proof should be able to do so when given the hint that $BD + DA = BA$. We have $BD = \frac{a^2}{c}$ and $DA = \frac{b^2}{c}$ by Corollary 3-9-1 and algebra. By a segment measurement postulate $BD + DA = BA$, and by substitution, $\frac{a^2}{c} + \frac{b^2}{c} = c$. Multiplying by c gives $a^2 + b^2 = c^2$ and the theorem is proved.

Your pupils might reasonably ask why we bother to prove the same theorem so many times. You could answer them by saying that this is one of the theorems in geometry that lends itself to many different kinds of proof and is a good example of the flexibility of our geometry. We are interested, not in proving the theorem for its own sake, but in finding various proofs that are logical consequences of the foundations that have already been laid. Finding new proofs to this theorem has occupied the minds of mathematicians and laymen alike for many centuries, which is one reason why this theorem is said to have more different proofs than any other in mathematics. Each new proof has its own beauty which only the mathematician and those interested in logical reasoning and the thrill of discovery can appreciate.

The exercises of Problems 3-9 are intended to give the pupil more practice in using the theorem and to call his attention again to the properties of the two special right triangles—the isosceles right triangle and the 30° - 60° - 90° right triangle.

Answers to
PROBLEMS 3-9

Student Text Pages 96-97

1. $x = 6\sqrt{2}$.

2. $\frac{1}{\sqrt{2}}$; We can prove that this ratio is the same for all isosceles right triangles by letting a be the length of each leg and x the length of the hypotenuse. We have, by Pythagoras' Theorem,

   $x^2 = a^2 + a^2$ and

   $x = a\sqrt{2}$.

   Thus the ratio of a leg to the hypotenuse is $\frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$.

3. Each leg is 5.

4. The remaining leg is $6\sqrt{3}$

5. Here, the pupil must remember that the shorter leg is $\frac{1}{2}$ as long as the hypotenuse. The longer leg, then, is $5\sqrt{3}$.

6. $\frac{5}{5\sqrt{3}} = \frac{1}{\sqrt{3}}$: This ratio is the same for all 30° - 60° - 90° right triangles.

   Proof:

   Let x be the length of the hypotenuse. Then the shorter leg is $\frac{1}{2}x$,

and, if y represents the longer leg we have, by Pythagoras' Theorem, $y^2 + \left(\frac{1}{2}x\right)^2 = x^2$ from which we get $y = \frac{\sqrt{3}}{2}x$. The ratio of the shorter leg to the longer leg is $\frac{\frac{1}{2}x}{\frac{\sqrt{3}}{2}x} = \frac{1}{\sqrt{3}}$.

7. The shorter leg is 1 and the longer leg is $\sqrt{3}$.

3-10 POLYGONS.

We include this section on polygons in order to develop some of the definitions and properties pertaining to polygons that will be needed in the next section. The main purpose of this section is to define *regular polygons* and to derive the formulas for the sum of the interior angles of a simple, convex polygon and for each angle of a regular polygon. We try to get the pupil to discover these formulas for himself through the Class Activity exercises.

The first part of this section is a revision of many of the definitions given in Chapter 11 of *Secondary Three*. It has been some time since the pupil has used these definitions, and he may have forgotten many of them.

Comments on
CLASS ACTIVITY

Student Text Page 98

1. We can draw 1, 2, 3, 4, and 5 diagonals respectively from one vertex for polygons of 4, 5, 6, 7, and 8 sides. In the general n-gon we can draw n-3 diagonals from one vertex. This can be seen from the fact that there can be no diagonal drawn to 3 of the vertices, namely the vertex itself and the two vertices on either side of it. Thus there are n-3 vertices to which diagonals can be drawn.

2. If we draw the diagonals from one vertex for polygons of 4, 5, 8 and n sides, the polygons are subdivided into 2, 3, 6, and n-2 triangles respectively. We observe that each side of the polygon, along with two diagonals, determines a triangle except for the *two sides* adjacent to the vertex from which the diagonals are drawn. Thus there are n-2 triangles formed in each n-gon.

3. 360; 540; 1080; $(n-2) \times 180$. Since each n-gon is divided into n-2 triangles by the diagonals from one vertex, and the sum of the degree measures of the interior angles of the polygon is the same as the sum of the degree measures of all the angles of the triangles, we have the number of triangles (n-2) times the sum of the degree measures of the angles of each triangle (180) for our formula.

4. By definition, there are two exterior angles at each vertex. They are vertically opposite each other, hence are congruent. The sum of the degree measures of the exterior angles of any n-gon is 360. This is seen from the fact that the sum of the interior and exterior angles at *each* vertex is a straight angle. Since there are n vertices in an n-gon, we have n straight angles for the sum of all the interior and exterior angles. But the interior angles have been shown to have a sum of $(n-2)$ straight angles. This leaves 2 straight angles, or a degree measure of 360, for the sum of the exterior angles and this result is independent of the number of sides the polygon may have.

Answers to PROBLEMS 3-10

Student Text Pages 98-99

1. 108; 120; $128\frac{4}{7}$.
2. 72; 60; 45.
3. 10.
4. 12; 360:
5. No; using the formula, if it were possible for a regular n-gon to have each of its interior angles 130°, we would have $\frac{(n-2)180}{n} = 130$ which gives $n = 7\frac{1}{5}$. This is impossible since n represents the number of sides and must be a positive integer greater than 2.
6. A rhombus is such a polygon.
7. A rectangle is such a polygon.

3-11 SIMILAR POLYGONS.

Definition 3-5 of this section generalizes the definition of similar triangles given in Section 3-5 to polygons of any number of sides. Stress again the importance of the one-to-one correspondence between the vertices of the two polygons and the naming of the two polygons so that their vertices are in the proper order; that is, corresponding vertices are named in

corresponding positions. Of course, because of this correspondence, two polygons cannot be similar unless they have the same number of sides.

Emphasize to your pupils that the AAA, SAS, and SSS theorems for similar triangles do not hold for polygons of more than 3 sides by showing them examples like the ones in the text.

Answers to
PROBLEM (Page 99)

We must also have $\hat{A} \equiv \hat{A}'$, $\hat{B} \equiv \hat{B}'$, $\hat{C} \equiv \hat{C}'$, $\hat{D} \equiv \hat{D}'$, $\hat{E} \equiv \hat{E}'$, $\hat{F} \equiv \hat{F}'$, and $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EF}{E'F'} = \frac{FA}{F'A'}$.

Comments on
CLASS ACTIVITY

Student Text Page 100

1. Since each side of a regular polygon has the same length, if we denote the lengths of each side of polygons $ABCD\ldots$ and $A'B'C'D'\ldots$ by r and s respectively, then the ratio of any two corresponding sides is $\frac{r}{s}$. Therefore, the corresponding sides of the two polygons are proportional.

2. The degree measure of each angle of a regular polygon is given by the formula $\frac{(n-2)\times 180}{n}$. Since n is the same number for both polygons, each of their angles has the same measure and so corresponding angles are congruent.

3. By Definition 3-5 polygon $ABCD\ldots \sim$ polygon $A'B'C'D'\ldots$.

4. If your pupils are not careful in forming the statement of this theorem, they may make a statement to the effect that "all regular polygons are similar." If this is the case, the questions in (5) should show them that they have left out a very important part of the theorem.

Some of the more important theorems regarding similar polygons are stated as exercises in Problems 3-11. Problem 1 should be assigned to everyone. Problems 7, 8, and 9 are theorems that tell us we can subdivide similar polygons into correspondingly placed similar triangles and if two polygons can be so subdivided, they are similar. Thus the problem of proving two polygons similar is reduced to that of proving corresponding triangles similar, and we have all the theorems on similar triangles at our disposal to do this.

Answers to
PROBLEMS 3-11

Student Text Page 101-102

1. Let $ABCD \ldots$ and $A'B'C'D' \ldots$ be the two polygons such that $ABCD \ldots \sim A'B'C'D' \ldots$. By definition of similar polygons we have $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \ldots$ and by Theorem 3-2(f) $\frac{AB + BC + CD + \ldots}{A'B' + B'C' + C'D' + \ldots} = \frac{AB}{A'B'}$. But, $AB + BC + CD + \ldots$ equals the perimeter (P) of the first polygon and $A'B' + B'C' + C'D' + \ldots$ equals the perimeter (P') of the second polygon.
   Thus $\frac{P}{P'} = \frac{AB}{A'B'}$ and the theorem is proved.

2. The perimeter of the larger polygon is 48 in.

3. $7\frac{1}{2}$; $10\frac{1}{2}$; 12; $16\frac{1}{2}$; and $28\frac{1}{2}$ respectively.

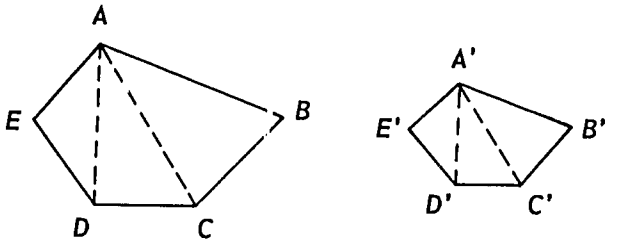
4. $4\frac{2}{5}$ in. $\times$ 7 in.

5. 27; 33; 42; 51.

6. No.

7. No.

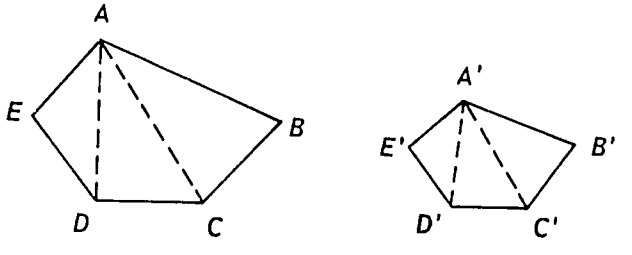
8. That the polygons can be subdivided into triangles by drawing the diagonals from one vertex has been shown in the Class Activity of Section 3-10. To prove the corresponding triangles are similar, we proceed as follows.



We are given (in the figure above) $ABCDE \sim A'B'C'D'E'$ and so $\widehat{A} \equiv \widehat{A'}$, $\widehat{B} \equiv \widehat{B'}$, $\widehat{C} \equiv \widehat{C'}$, $\widehat{D} \equiv \widehat{D'}$, $\widehat{E} \equiv \widehat{E'}$ and $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CD}{C'D'} = \frac{DE}{D'E'} = \frac{EA}{E'A'}$ by definition of similar polygons. We have $\Delta ABC \sim \Delta A'B'C'$ by S.A.S. and so $\frac{AC}{A'C'} = \frac{BC}{B'C'} = \frac{CD}{C'D'}$. Also, $\widehat{BCA} \equiv \widehat{B'C'A'}$ and by subtraction of angles, $\widehat{ACD} \equiv \widehat{A'C'D'}$. Therefore, $\Delta ACD \sim \Delta A'C'D'$ by S.A.S. and $\frac{AD}{A'D'} = \frac{CD}{C'D'} = \frac{DE}{D'E'}$. Since, by the second pair of similar triangles, $\widehat{CDA} \equiv \widehat{C'D'A'}$, again by subtraction of angles we have $\widehat{ADE} \equiv \widehat{A'D'E'}$ and $\Delta ADE \sim \Delta A'D'E'$ by S.A.S. Thus each pair of corresponding triangles are similar and the theorem is proved.

9. We could state the general theorem as follows: "If two polygons are similar they can be subdivided into triangles so that corresponding triangles are similar." This theorem is true and can be proved in a manner similar to the proof of the theorem in Problem 8.

10.



In the figure above we are given

$\Delta ABC \sim \Delta A'B'C'$,

$\Delta ACD \sim \Delta A'C'D'$,

$\Delta ADE \sim \Delta A'D'E'$.

We want to prove $ABCDE \sim A'B'C'D'E'$.

Proof:

From the similar triangles we have

$\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'} = \frac{CD}{C'D'} = \frac{AD}{A'D'} = \frac{DE}{D'E'} = \frac{AE}{A'E'}$, so the corresponding sides of the two pentagons are proportional.

The corresponding angles of the three pairs of similar triangles are congruent, and by addition of angles we get $\widehat{B} \equiv \widehat{B'}$, $\widehat{BCD} \equiv \widehat{B'C'D'}$, $\widehat{CDE} \equiv \widehat{C'D'E'}$, $\widehat{DEA} \equiv \widehat{D'E'A'}$, $\widehat{EAB} \equiv \widehat{E'A'B'}$. Thus the corresponding angles of the two pentagons are congruent, and $ABCDE \sim A'B'C'D'E'$ by Definition 3-5.

Chapter 4
CIRCLES

This chapter may be divided into three parts. The first deals with the intersection properties of a circle and a line in the plane of the circle (here we touch briefly on similar properties of a sphere and a plane in space). The second studies the degree measure of arcs of a circle and properties of arcs, chords, secants and tangents in relation to certain angles. The third part is concerned with distance relations between segments of chords, secants, and tangents.

Most of the theorems of this chapter are contained in the exercises, and certainly not all of them need to be proved. It is important that your pupils know the meaning of terms like circle, radius, diameter, chord, secant, tangent, interior and exterior of a circle, internally and externally tangent circles, minor arc, major arc, semicircle, central angle, angle inscribed in an arc of a circle, degree measures of arcs, and congruent arcs and circles.

The important theorems are Theorems 4-2, 4-5, 4-6, 4-14, 4-16, 4-17 and its corollaries, 4-19, 4-20, and Problem 10 or Problems 4-3B. Be sure that you discuss these theorems with your pupils and that they work problems involving them.

4-1 CIRCLES AND SPHERES.

Observe that, as with a triangle, we define a circle to be the boundary of a certain region, and not the region itself. Similarly a sphere is the surface of a certain solid figure.

Answers to
PROBLEMS 4-1A

Student Text Page 105

1. (a) The sphere
   (b) The circle

2. (a) (i) The empty set
      (ii) The two circles
   (b) (i) The empty set
      (ii) The two spheres

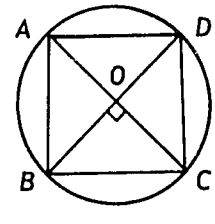
3. Infinitely many;
   Longitudes (or meridians) and the equator.

Some examples of words used to denote both a segment and a length are diagonal, base, altitude, hypotenuse, side. Note that "altitude" was also used in Chapter 1 to denote a line.

Answers to
PROBLEMS 4-1B

Student Text Page 106

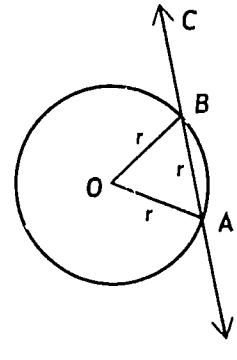
1. The diagonals $\overline{AC}$, $\overline{BD}$ of the quadrilateral $ABCD$ bisect each other at right angles. ($AO = OC = BO = OD$ = the radius of the circle; and, by hypothesis, $\overline{AC} \perp \overline{BD}$). Hence $ABCD$ is a square.



2. $\Delta\, OAB$ is equilateral, and hence

   $m(O\hat{B}A) = 60$

   $m(O\hat{B}C) = 180 - m(O\hat{B}A)$
   $= 120.$



3. In the right triangles ΔADO, ΔBDO

$AO = BO$ (radii of the same circle)

$OD = OD$

$\therefore \Delta ADO \equiv \Delta BDO$

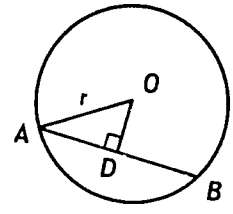
$\therefore \overline{AD} \equiv \overline{BD}$

4. Draw $\overline{OD} \perp \overline{AB}$.

In right triangle ΔAOD,

$AO = r > AD$.

$2r > 2AD = AB$ (using the result of Problem 3).



5. This follows immediately from Problem 4 and the definition of a diameter of a circle.

4-2 INTERSECTION OF A LINE WITH A CIRCLE.

The important result here is the one contained in Theorem 4-5. The students will readily accept that there are only the three possibilities for the intersection of a circle with a line lying in the plane of the circle. They may, however, find some difficulty with the proof of Theorem 4-3, which enables us to deduce the third case. It is important that they realise the necessity for proving the statement:

"If the line ℓ intersects the circle in a point Q, then $r^2 = x^2 + d^2$, where $NQ = x$," as well as its converse.

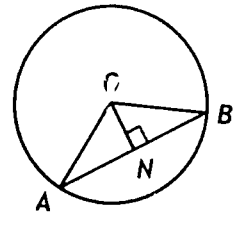
Answers to
PROBLEMS 4-2A

Student Text Page 113

1. (a) Theorem 4-6.

ΔOAB is isosceles,

and $\overline{ON}$ is the perpendicular from the vertex O to the base $\overline{AB}$. Hence $\overline{AN} \equiv \overline{NB}$.



(b) Theorem 4-7.

Using the figure for (a), $\overline{ON}$ is the median to the base $\overline{AB}$ of the isosceles ΔOAB. Hence $\overline{ON} \perp \overline{AB}$.

(c) Theorem 4-8.

Again, using the figure for (a), the perpendicular bisector of the chord $\overline{AB}$ in the plane of the circle is the set of all points in that plane which are equidistant from A and B. The centre O of the circle is one such point, and hence the perpendicular bisector contains O.

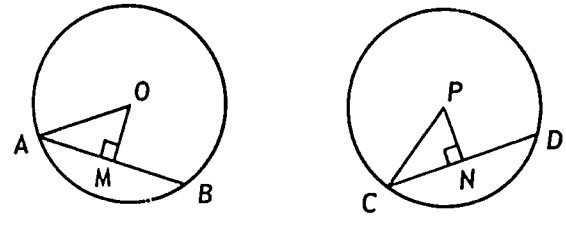
(d) From Theorem 4-5, it follows immediately that no line has more than two points of intersection with a circle. [Note that in Theorems 4-6 and 4-7, we should exclude chords which are diameters]

2. From Theorem 4-5, if a line ℓ is tangent to a circle, then case (2) of Theorem 4-5 must hold. Theorem 4-4 follows immediately, since the point K is the point of contact.

Answers to
PROBLEMS 4-2B

Student Text Page 113

1. (a) Theorem 4-10.



In right triangles
$\Delta\, OAM$, $\Delta\, PCN$
$\overline{OA} \equiv \overline{PC}$ (radii of congruent circles)
$\overline{OM} \equiv \overline{PN}$ (hypothesis)
$\therefore\ \Delta\, OAM \equiv \Delta\, PCN$
$\therefore\ \overline{AM} \equiv \overline{CN}$
But $\overline{AB} \equiv 2\ \overline{AM}$ and $\overline{CD} \equiv 2\ \overline{CN}$ (Theorem 4-6)
$\therefore\ \overline{AB} \equiv \overline{CD}$.

(b) Theorem 4-11.

(Same figure as for (a)). We take M and N as the mid-points of the congruent chords $\overline{AB}$ and $\overline{CD}$.

In triangles $\Delta\, OAM$, $\Delta\, PCN$,

$\overline{OA} \equiv \overline{PC}$ (radii of congruent circles)

$\overline{AM} \equiv \overline{CN}$ ($\overline{AM} \equiv \frac{1}{2}\,\overline{AB}$, $\overline{CN} \equiv \frac{1}{2}\,\overline{CD}$; $\overline{AB} \equiv \overline{CD}$, by hypothesis)

$O\hat{M}A \equiv P\hat{N}C$ (each is a right angle by Theorem 4-7)

$\therefore\ \Delta\, OAM \equiv \Delta\, PCN$

$\therefore\ \overline{OM} \equiv \overline{PN}$.

(c) Theorem 4-12. (Same Figure as for (a)).

$AM = \frac{1}{2} AB$ and $CN = \frac{1}{2} CD$ (Theorem 4-6)

$\therefore AM < CN$ ($AB < CD$ by hypothesis)

In right triangle ΔOAM

$OA^2 = OM^2 + AM^2$ (Pythagoras' Theorem)

Similarly, $PC^2 = PN^2 + CN^2$

But $OA = PC$ (radii of congruent circles)

$\therefore OM^2 + AM^2 = PN^2 + CN^2$

$OM^2 - PN^2 = CN^2 - AM^2 > O$ ($CN > AM$)

$\therefore OM^2 > PN^2$

$\therefore OM > PN$. (Both OM and PN are positive numbers)

2. If in the same circle, or in congruent circles, two chords are at unequal distances from the centre(s), the chord nearer the centre is the longer of the two chords.

The proof is similar to that for Theorem 4-12.

Answers to
PROBLEMS 4-2C

Student Text Pages 115-118

1. (a) The empty set (b) The empty set
   (c) One point (d) Two points

2. $OA = 5$ cm., $ON = 4$ cm.

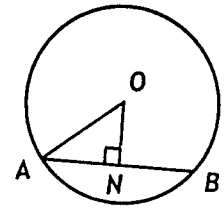
   By Pythagoras' Theorem,

   $AN^2 = OA^2 - ON^2$

   $= 5^2 - 4^2$

   $= 3^2$

   $AN = 3$ cm., and $AB = 2 \times AN = 6$ cm.



3. 12 cm.

4. $6\sqrt{2}$ in.

5. The secant is the perpendicular bisector of the given chord and lies in the same plane as the circle. By Theorem 4-8, the secant contains the centre of the circle, and hence cuts the circle in a chord which contains the centre; that is, in a diameter.

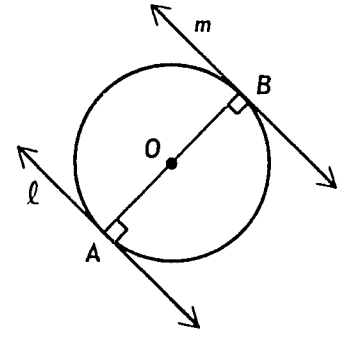
6. Find the intersection of the perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$. This intersection exists, since the two chords are not parallel (they have the common point A).

7. Draw any two chords of the circle which are not parallel, and determine the intersection of their perpendicular bisectors. This intersection is the centre of the circle.

8. (a) Theorem 4-10
(b) Theorem 4-11
(c) Theorem 4-9
(d) Theorem 4-7
(e) Theorem 4-6
(f) Converse of Theorem 4-12
(g) Theorem 4-2
(h) Theorem 4-4
(i) Theorem 4-3

9. Each tangent is perpendicular to the given diameter.

10. Let O be the centre of the circle and A, B the points of contact of the parallel tangents ℓ, m respectively. Draw the segments $\overline{OA}$, $\overline{OB}$. Then $\overline{OA} \perp \ell$ and $\overline{OB} \perp m$. Since $\ell \parallel m$, $\overline{OA}$ and $\overline{OB}$ are either parallel or segments of the same line. But $\overline{OA}$ and $\overline{OB}$ cannot be parallel since they have a common point O. Hence O, A, B are collinear points, and $\overline{AB}$ is a diameter of the circle.



11. In triangles ΔOAP, ΔOBP

$\overline{OA} \equiv \overline{OB}$ (radii of the same circle)

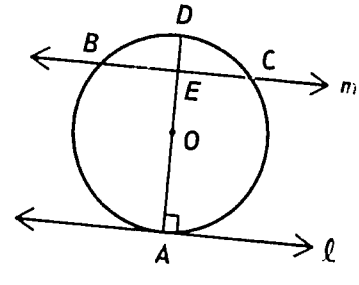
$\overline{OP} \equiv \overline{OP}$

$O\hat{A}P \equiv O\hat{B}P$ (each is a right angle, by Theorem 4-4)

$\therefore \Delta OAP \equiv \Delta OBP$

$\therefore \overline{AP} \equiv \overline{BP}$

12. Given: A secant m containing chord $\overline{BC}$, and a tangent ℓ with point of contact A. O is the centre of the circle and $\ell \parallel m$. We want to prove that the diameter $\overline{AD}$ through A bisects the chord $\overline{BC}$.

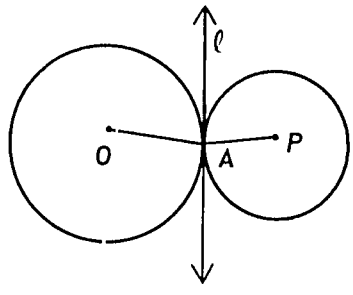


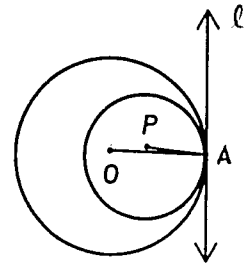
Proof: The line $\overleftrightarrow{OA} \perp \ell$ and hence $\overleftrightarrow{OA} \perp m$. The diameter $\overline{AD}$ is contained in the line

$\overleftrightarrow{OA}$. Hence $\overline{AD} \perp m$. If $\overline{AD}$ intersects $\overline{BC}$ in E, $\overline{OE}$ is a segment from the centre O to the chord $\overline{BC}$ and $\overline{OE} \perp \overline{BC}$. Hence $\overline{OE}$ bisects $\overline{BC}$, or $\overline{AD}$ bisects $\overline{BC}$.

13. $\overline{OR}$ is a segment from the centre to the chord $\overline{AB}$ of the larger circle. Also $OR \perp AB$, since R is the point of contact of the tangent $\overleftrightarrow{AB}$ to the smaller circle. Hence $\overline{AR} \equiv \overline{RB}$ and $AR = RB$.

14. Theorem 4-13.





There are two cases to consider and we will treat both cases at once.

Given: Two circles with centres O and P, which touch at a point A. Line ℓ is the common tangent. We want to prove that the line $\overleftrightarrow{OP}$ contains A.

Proof: $\overleftrightarrow{OA} \perp \ell$ at the point A and $\overleftrightarrow{PA} \perp \ell$ at the point A.

Hence $\overleftrightarrow{OA}$ and $\overleftrightarrow{PA}$ are one and the same line; that is, $\overleftrightarrow{OP}$ contains A.

15. Given: Three non-collinear points A, B, C.

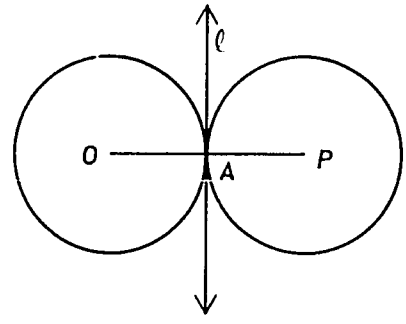
We want to prove that there is one and only one circle containing the three points.

Proof: *Existence* - Let the perpendicular bisectors of the segments $\overline{AB}$, $\overline{AC}$ meet in a point O. Then $OA = OB = OC$, and the circle with centre O and radius OA contains the three points A, B, C.

Uniqueness—Suppose there is a second circle with centre O' containing the points A, B, C. Since $\overline{AB}$ is a chord of this circle, the centre O' lies on the perpendicular bisector of $\overline{AB}$. Similarly, O' lies on the perpendicular bisector of $\overline{AC}$. Hence O' is the (unique) intersection of the two perpendicular bisectors. But from the first part of the proof, this intersection is also the point O. Hence $O' = O$ and $O'A = OA$. Therefore the two circles are the same.

Corollary 4-14-1: The corollary follows immediately from the theorem, since the vertices of a triangle are three non-collinear points.

16. Given: Two congruent circles with centres O and P, which touch externally at A. We must prove that any point equidistant from O and P lies on the common tangent ℓ.



Proof: By Theorem 4-13, the line $\overleftrightarrow{OP}$ contains the point A. Also $\overline{OA} \equiv \overline{PA}$, and $\overline{OP} \perp \ell$. Hence ℓ is the perpendicular bisector of the segment $\overline{OP}$. Any point equidistant from O and P must, therefore, lie on ℓ.

17. $\overline{AC} \perp \ell$, $\overline{BD} \perp \ell$ (hypothesis)

$\overline{PT} \perp \ell$ ($\overline{PT}$ is the radius through the point of contact T of the tangent ℓ)

$\therefore \overleftrightarrow{AC} \parallel \overleftrightarrow{BD} \parallel \overleftrightarrow{PT}$

Now, the parallel lines $\overleftrightarrow{AC}$, $\overleftrightarrow{PT}$, $\overleftrightarrow{BD}$ cut off congruent segments on the transversal $\overline{AB}$, hence they cut off congruent segments on $\overline{CB}$; that is, $\overline{CT} \equiv \overline{TD}$ and $CT = TD$.

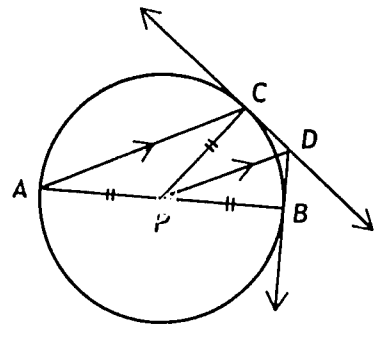
18. Theorem 4-15: The second part is just the result of Problem 11, and the first part follows from the congruence of the triangles ΔOAP and ΔOBP (proved in the solution of that problem).

19. We draw the segment $\overline{PC}$.

$C\hat{A}P \equiv D\hat{P}B$ (corresponding angles: $\overleftrightarrow{AC} \parallel \overleftrightarrow{PC}$)

$P\hat{C}A \equiv C\hat{P}D$ (alternate angles $\overleftrightarrow{AC} \parallel \overleftrightarrow{PD}$)

$C\hat{A}P \equiv D\hat{C}A$ (base angles of the isosceles ΔPAC)



$\therefore D\hat{P}B \equiv C\hat{P}D$

In triangles ΔPDC, ΔPDB,

$\overline{PD} \equiv \overline{PD}$

$\overline{PC} \equiv \overline{PB}$ (radii of the same circle)

$C\hat{P}D \equiv B\hat{P}D$ (proved)

$\therefore\ \Delta PDC \equiv \Delta PDB$

$\therefore\ \ P\hat{C}D \equiv P\hat{B}D$

But $P\hat{C}D$ is a right angle, since $\overline{PC}$ is the radius through the point of contact C of the tangent $\overleftrightarrow{CD}$. Hence $P\hat{B}D$ is a right angle. Since $\overline{PB}$ is a radius this implies that $\overleftrightarrow{DB}$ is a tangent to the circle at B.

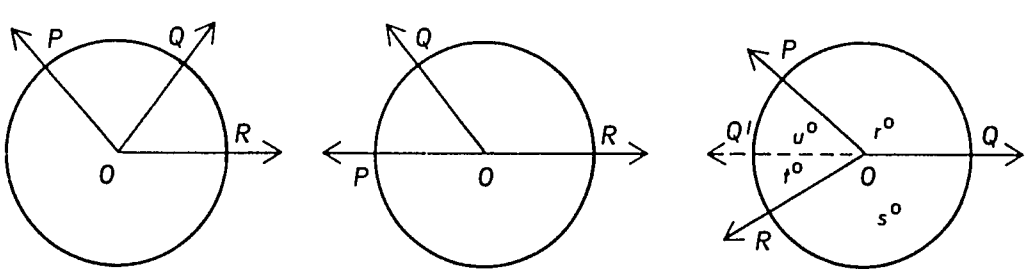
4-3 ARCS OF CIRCLES.

In this section we develop the theorems pertaining to the degree measures of certain angles formed by two lines (secants, tangents, or a secant and a tangent) intersecting a circle. These angles are measured by the arcs they intercept on the circle. Once we have proven Theorem 4-17, the proofs of Theorems 4-19, 4-20, and Problem 10 of Problems 4-3B are all based on this result.

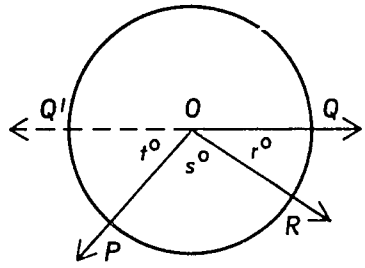
The important thing here, along with the theorems mentioned above, is that the pupils understand Definition 4-11.

In *Secondary Three* we postulated that a point separates a line. Such is not the case with a circle. However, two distinct points on a circle separate the circle into two sets of points which we define to be the minor and major arcs having the two points as endpoints. The ambiguity in naming an arc mentioned in the Student Text should not cause any real problem if pupils are careful to use a third point of an arc along with its endpoints when naming the arc.

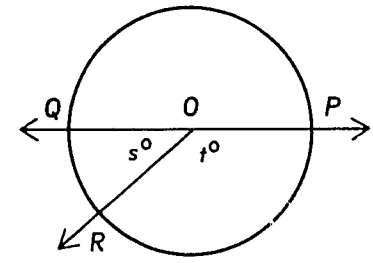
Theorem 4-16 enables us to add the measures of two arcs which have only their endpoints in common. We sometimes refer to this as *The Arc Addition Theorem*. The proof is rather tedious because we need to examine five cases, but each of the cases is easy. We want to prove in each of the five cases that $m(\widehat{PQR}) = m(\widehat{PQ}) + m(\widehat{QR})$.



Case 1 Case 2 Case 3



Case 4



Case 5

Case 1. $\widehat{PQR}$ is a minor arc. The conclusion follows from the Angle Addition Postulate.

Case 2. $\widehat{PQR}$ is a semicircle. The conclusion follows from the Angle Addition Postulate.

Case 3. $\widehat{PQR}$ is a major arc, and P and R are on opposite sides of the diameter that contains Q. We have $360 = u + r + s + t$

and $360 - (u + t) = r + s$

or $m(\widehat{PQR}) = m(\widehat{PQ}) + m(\widehat{QR})$.

Case 4. $\widehat{PQR}$ is a major arc and P and R are on the same side of the diameter that contains Q.

We have $m(\widehat{PQR}) = 360 - s = [360 - (r + s)] + r = m(\widehat{PQ}) + m(\widehat{QR})$.

Case 5. $\widehat{PQR}$ is a major arc, and one of the arcs $\widehat{PQ}$, $\widehat{QR}$, is a semicircle.

We have $m(\widehat{PQR}) = 360 - t = 180 + 180 - t = 180 + s = m(\widehat{PQ}) + m(\widehat{QR})$.

Answers to
PROBLEMS 4-3A

Student Text Pages 123-125

1. $A\hat{O}C$ is subtended by the arc $\widehat{CYA}$

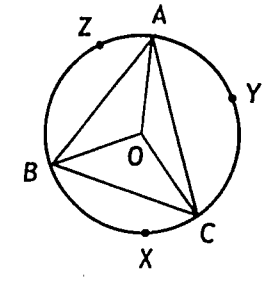
   $C\hat{O}B$ is subtended by the arc $\widehat{BXC}$

   $B\hat{O}A$ is subtended by the arc $\widehat{AZB}$

   $A\hat{B}C$ is inscribed in arc $\widehat{ABC}$ and is subtended by arc $\widehat{AYC}$

   $A\hat{C}B$ is inscribed in arc $\widehat{ACB}$ and is subtended by arc $\widehat{AZB}$

   $B\hat{A}C$ is inscribed in arc $\widehat{BAC}$ and is subtended by arc $\widehat{BXC}$



2. (a) The minor arcs are:

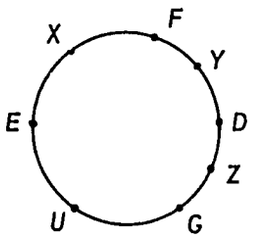
$\overset{\frown}{EXF}$, $\overset{\frown}{FYD}$, $\overset{\frown}{DZG}$, $\overset{\frown}{GUE}$.

(b) The major arcs are:

$\overset{\frown}{EDF}$, $\overset{\frown}{FED}$, $\overset{\frown}{DEG}$, $\overset{\frown}{GDE}$.

(c) The semicircles are:

$\overset{\frown}{EFD}$, $\overset{\frown}{EGD}$.



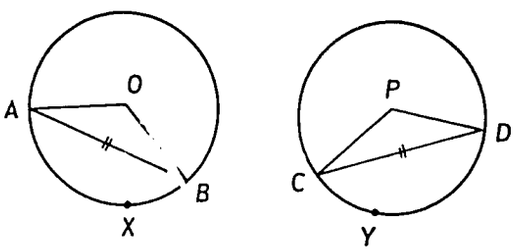
3. Given: Two congruent circles with centres O and P. A chord $\overline{AB}$ of one circle and a chord $\overline{CD}$ of the other are congruent.

To prove that $m(\overset{\frown}{AXB}) = m(\overset{\frown}{CYD})$.

Proof: $\Delta OAB \equiv \Delta PCD$ (SSS)

$\therefore A\hat{O}B \equiv C\hat{P}D$

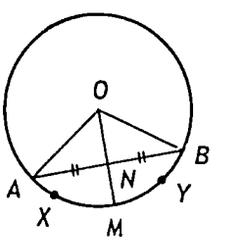
$\therefore m(\overset{\frown}{AXB}) = m(\overset{\frown}{CYD})$ (by definition)



4. Given: A circle with centre O. $\overline{AB}$ is a chord and $\overline{AN} \equiv \overline{NB}$. $\overline{ON}$ meets the minor arc $\overset{\frown}{AB}$ in M. To prove that $m(\overset{\frown}{AXM}) = m(\overset{\frown}{MYB})$.

Proof: ΔOAB is isosceles and $\overline{ON}$ is the median to the base $\overline{AB}$. Hence $A\hat{O}N \equiv B\hat{O}N$.

By definition, this implies $m(\overset{\frown}{AXM}) = m(\overset{\frown}{MYB})$.



5. $m(\overset{\frown}{AEB}) = 60$, $m(\overset{\frown}{ADC}) = 120$, $m(\overset{\frown}{CDB}) = 180$

6. $m(\overset{\frown}{BCA}) = 260$

7. $m(O\hat{P}R) = 60$

8. $360 - x$

9. (a) 120
   (b) 30
   (c) 90

10. 1. $m(\overset{\frown}{AB}) + m(\overset{\frown}{DE}) = 90$
    2. $m(\overset{\frown}{DE}) + m(\overset{\frown}{EF}) = 90$
    3. $m(\overset{\frown}{AB}) = m(\overset{\frown}{EF})$ (from statements 1 and 2)
    4. $m(\overset{\frown}{DE}) = m(\overset{\frown}{FB})$ similar to the deduction of 3.

5. $m(\widehat{FB}) = m(\widehat{QT}) = m(F\hat{C}B)$ (definition)

6. $m(\widehat{AD}) + m(\widehat{QT}) = 90$ (from statements 1, 4, and 5)

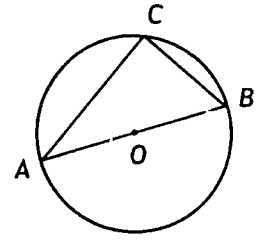
7. $m(\widehat{EF}) + m(\widehat{RS}) = 90$ (similar to deduction of 6)

8. $m(\widehat{AD}) + m(\widehat{QT}) = m(\widehat{EF}) + m(\widehat{RS})$ (from statements 6 and 7)

Answers to
PROBLEMS 4-3B

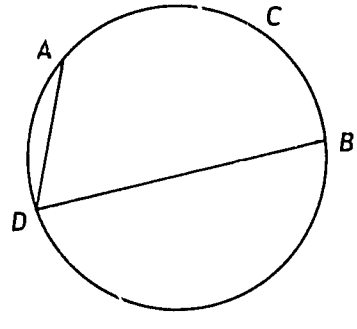
Student Text Pages 131-133

1. Given: A triangle ΔABC inscribed in a circle with centre O. A side $\overline{AB}$ of the triangle contains O. To prove that ΔABC is a right triangle.



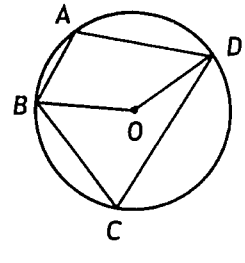
Proof: $A\hat{C}B$ is inscribed in the semicircle $\widehat{ACB}$. Hence by Corollary 4-17-1, $A\hat{C}B$ is a right angle.

2. Given an arc $\widehat{ACB}$ of a circle. To determine $m(\widehat{ACB})$: we find the measure of any inscribed angle ($A\hat{D}B$ for example) subtended by the arc $\widehat{ACB}$. The measure of the angle subtended by $\widehat{ACB}$ is $\frac{1}{2}\, m(\widehat{ACB})$.



Hence $\frac{1}{2}\, m(\widehat{ACB}) = m(A\hat{D}B)$ and $m(\widehat{ACB}) = 2\, m(A\hat{D}B)$.

3. Theorem 4-18.
Given: A quadrilateral $ABCD$ inscribed in a circle with centre O.



To prove that $m(\hat{A}) + m(\hat{C}) = 180$

$m(\hat{B}) + m(\hat{D}) = 180$

Proof: $m(\hat{A}) = \frac{1}{2}\, m(\widehat{BCD})$

$m(\hat{C}) = \frac{1}{2}\, m(\widehat{BAD})$

$$m(\widehat{A}) + m(\widehat{C}) = \frac{1}{2}\left\{m(\widehat{BCD}) + m(\widehat{BAD})\right\}$$
$$= \frac{1}{2} \times 360$$
$$= 180.$$

Similarly, $m(\widehat{B}) + m(\widehat{C}) = 180$.

4. The opposite angles of a parallelogram are congruent. If the parallelogram is inscribed in a circle, by Theorem 4-18, the sum of the measures of the opposite angles is 180. Hence each of the angles of the parallelogram is a right angle, and the parallelogram is a rectangle.

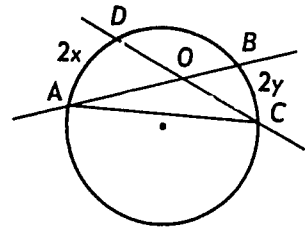
5. $m(B\widehat{C}D) = 89$, $m(C\widehat{D}A) = 75$

6. By Theorem 4-17, the measure of the arc intercepted by the inscribed angle is 180. The intercepted arc is, therefore, a semicircle. From this it follows that the given arc is itself a semicircle.

7. $m(C\widehat{A}B) = 40$

   $m(C\widehat{D}B) = 40$

8. Draw the segment $\overline{AC}$.



$$m(A\widehat{C}D) = \frac{1}{2}\, m(\widehat{AD})$$
$$= x$$

Similarly, $m(B\widehat{A}C) = y$

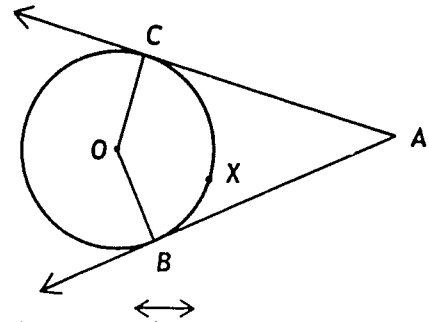
$m(C\widehat{O}B) = m(A\widehat{C}D) + m(B\widehat{A}C)$

Exterior angle theorem applied to ΔAOC.

$\therefore m(C\widehat{O}B) = x + y = \frac{1}{2}\left[m(\widehat{AD}) + m(\widehat{BC})\right]$

9. $\frac{1}{2}(203 + 59) = 131$

10. Given: A circle with centre O and two tangents $\overleftrightarrow{BA}$, $\overleftrightarrow{CA}$ with points of tangency B, C and intersecting at A. To prove that $m(B\widehat{A}C) = 180 - m(\widehat{BXC})$.



Proof: In the quadrilateral $ABOC$,

$m(\widehat{B}) = 90$ ($\overline{OB}$ is the radius to the tangent $\overleftrightarrow{AB}$ at its point of tangency B)

Similarly $m(\widehat{C}) = 90$

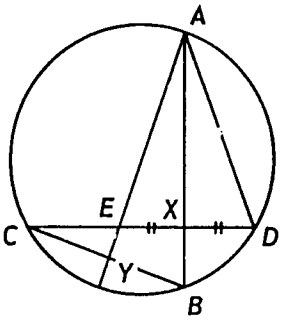
Hence $m(\widehat{A}) + m(\widehat{O}) = 180$

But $m(\widehat{O}) = m(\overset{\frown}{BXC})$ (by definition)

$\therefore\ m(B\widehat{A}C) = 180 - m(\widehat{O})$

$= 180 - m(\overset{\frown}{BXC})$.

11. (i) In triangles ΔEAX, ΔDAX,



$\overline{EX} \equiv \overline{DX}$ (hypothesis)

$\overline{AX} \equiv \overline{AX}$

$E\widehat{X}A \equiv D\widehat{X}A$ (each is a right angle, hypothesis)

$\therefore\ \Delta EAX \equiv \Delta DAX$ (SAS)

$\therefore\ E\widehat{A}X \equiv D\widehat{A}X$ (corresponding angles of congruent triangles ΔEAX, ΔDAX.

(ii) But $X\widehat{C}B \equiv D\widehat{A}X$ (angles inscribed in the same arc $\overset{\frown}{DCB}$)

$\therefore\ E\widehat{A}X \equiv X\widehat{C}B$

(iii) Let $\overleftrightarrow{AE}$ intersect $\overleftrightarrow{CB}$ in Y.

$m(E\widehat{A}X) + m(A\widehat{E}X) + m(E\widehat{X}A) = 180$ (angle-sum of ΔEAX)

Similarly, $m(X\widehat{C}B) + m(Y\widehat{E}C) + m(E\widehat{Y}C) = 180$

But $m(E\widehat{A}X) = m(X\widehat{C}B)$ (proved)

$m(A\widehat{E}X) = m(Y\widehat{E}C)$ (they are vertically opposite angles)

$\therefore\ m(E\widehat{X}A) = m(E\widehat{Y}C)$

But $E\widehat{X}A$ is a right angle, by hypothesis.

$\therefore\ E\widehat{X}C$ is a right angle and $\overleftrightarrow{CB} \perp \overleftrightarrow{AE}$.

Answers to
PROBLEMS 4-3C

Student Text Pages 136-137

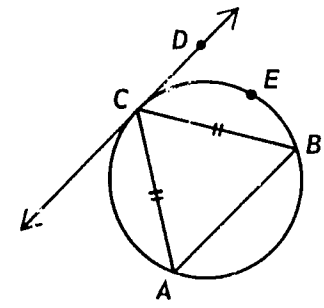
1. $m(\widehat{A}) = 35$, $m(\widehat{B}) = 65$, $m(B\widehat{C}E) = 35$

2. $C\hat{B}A \equiv C\hat{A}B$ (base angles of Isosceles ΔABC)

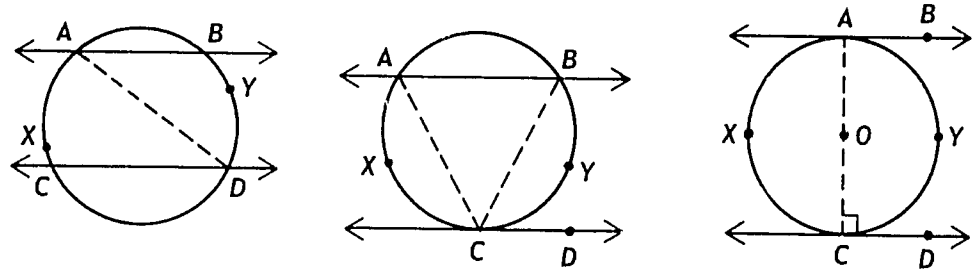
$C\hat{A}B \equiv B\hat{C}D$ (each is $\frac{1}{2}\, m(\widehat{BEC})$)

$\therefore C\hat{B}A \equiv B\hat{C}D$

$\therefore \overleftrightarrow{CD} \| \overleftrightarrow{AB}$ by the converse of the Alternate Interior Angle Theorem.



3.



Case 1 Case 2 Case 3

Given: Two parallel lines $\overleftrightarrow{AB}, \overleftrightarrow{CD}$ which intercept arcs $\widehat{AXC}$, $\widehat{BYD}$ of a circle.

To prove that $m(\widehat{AXC}) = m(\widehat{BYD})$

Proof: As shown in the figure, there are three cases to consider.

Case (1): $\overleftrightarrow{AB}$, $\overleftrightarrow{CD}$ are both secants.

We draw the segment $\overline{AD}$.

$B\hat{A}D \equiv A\hat{D}C$ (alternate angles)

$m(B\hat{A}D) = \frac{1}{2}\, m(\widehat{BYD})$ (the arc $\widehat{BYD}$ is intercepted by the angle $B\hat{A}D$)

Similarly, $m(A\hat{D}C) = \frac{1}{2}\, m(\widehat{AXC})$

$\therefore \quad m(\widehat{AXC}) = m(\widehat{BYD})$

Case (2): $\overleftrightarrow{AB}$ is a secant and $\overleftrightarrow{CD}$ is the tangent at C.

We draw the segments $\overline{AC}$, $\overline{BC}$.

$A\hat{B}C \equiv B\hat{C}D$ (alternate angles)

$B\hat{C}D \equiv B\hat{A}C$ (both are measured by $\frac{1}{2}\, m(\widehat{BYC})$)

$\therefore A\hat{B}C \equiv B\hat{A}C$

But $m(A\hat{B}C) = \frac{1}{2} m(\widehat{AXC})$ (the arc $\widehat{AXC}$ is intercepted by the angle $A\hat{B}C$)

and $m(B\hat{A}C) = \frac{1}{2} m(\widehat{BYC})$.

Hence $m(\widehat{AXC}) = m(\widehat{BYC})$.

Case (3): $\overleftrightarrow{AB}$ is the tangent at A and $\overleftrightarrow{CD}$ is the tangent at C.

In this case, the chord $\overline{AC}$ contains the centre of the circle and hence is a diameter of the circle. (We have $\overleftrightarrow{OA} \perp \overleftrightarrow{AB}$ at A, hence $\overleftrightarrow{OA} \perp \overleftrightarrow{CD}$. But $\overleftrightarrow{OC} \perp \overleftrightarrow{CD}$ at C, and there is just the one line through O perpendicular to $\overleftrightarrow{CD}$, so $\overleftrightarrow{OA} = \overleftrightarrow{OC}$.)

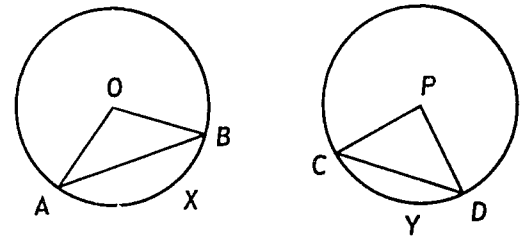
The arcs $\widehat{AXC}$, $\widehat{BYC}$ are semicircles and hence have equal measures.

4. This is Problem 3 of Problem Set 4-3A.

5. In the same circle, or in congruent circles, if two minor arcs are congruent the chords joining the endpoints of the respective arcs are congruent.

Given: Two congruent circles with centres O and P. The minor arcs $\widehat{AXB}$ and $\widehat{CYD}$ are congruent.

To prove that $\overline{AB} \equiv \overline{CD}$.



Proof: $m(\widehat{AXB}) = m(\widehat{CYD})$ (hypothesis)

$m(A\hat{O}B) = m(\widehat{AXB})$ (definition)

$m(C\hat{P}D) = m(\widehat{CYD})$ (definition)

$\therefore$ $m(A\hat{O}B) = m(C\hat{P}D)$

$\therefore$ $A\hat{O}B \equiv C\hat{P}D$.

In triangles ΔAOB, ΔCPD,

$\overline{AO} \equiv \overline{CP}$ (radii of congruent circles)

$\overline{OB} \equiv \overline{PD}$

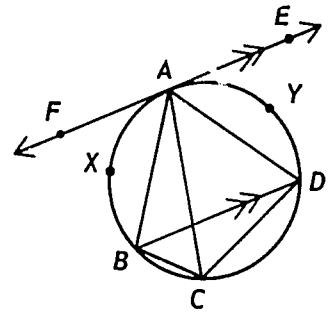
$A\hat{O}B \equiv C\hat{P}D$ (proved)

$\therefore \Delta AOB \equiv \Delta CPD$ (SAS)

$\therefore \overline{AB} \equiv \overline{CD}$.

6. $D\hat{A}E \equiv A\hat{D}B$ (alternate angles)

$A\hat{D}B \equiv F\hat{A}B$ (both are measured by $\frac{1}{2}\, m(\widehat{BXA})$)



$\therefore D\hat{A}E \equiv A\hat{D}B \equiv F\hat{A}B$

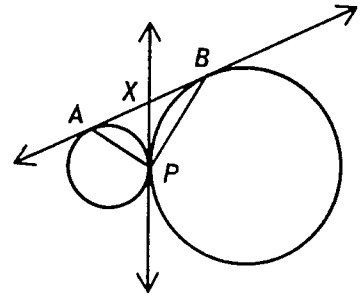
$D\hat{A}E \equiv A\hat{C}D$ (both are measured by $\frac{1}{2}\, m(\widehat{DYA})$)

Similarly $F\hat{A}B \equiv A\hat{C}B$

$\therefore A\hat{C}D \equiv A\hat{C}B$

and hence $\overline{AC}$ bisects $B\hat{C}D$.

7. Let the common tangents $\overleftrightarrow{AB}$ and $\overleftrightarrow{PX}$ intersect at X.



$\overline{AX} \equiv \overline{PX}$ (Theorem 4-15)

$\therefore X\hat{A}P \equiv X\hat{P}A$

Similarly $X\hat{B}P \equiv X\hat{P}B$.

In ΔAPB,

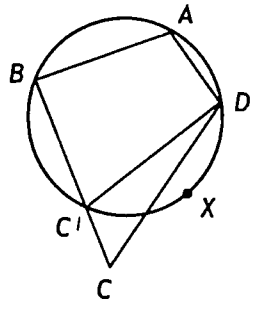
$$m(X\hat{A}P) + m(X\hat{P}A) + m(X\hat{B}P) + m(X\hat{P}B) = 180.$$

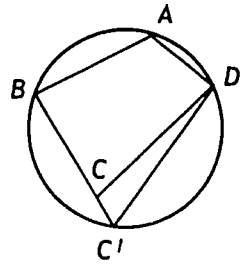
Hence $$2m(X\hat{P}A) + 2m(X\hat{P}B) = 180$$

$$m(X\hat{P}A) + m(X\hat{P}B) = 90$$

or, $$m(A\hat{P}B) = 90.$$

8.





Given: A quadrilateral $ABCD$ in which $m(\hat{A}) + m(\hat{C}) = 180$.

To prove that a circle can be drawn through the four points A, B, C, D.

Proof: We draw the circle through the points A, B, D.

This is possible by Theorem 4-14. $\overleftrightarrow{BC}$ is not a tangent to this circle, for, if it were, we would have:

$B\hat{A}D \equiv D\hat{B}C$ (both would be measured by $\frac{1}{2}\, m(\widehat{DXB})$)

Since $m(B\hat{A}D) + m(B\hat{C}D) = 180$, this would imply that

$m(D\hat{B}C) + m(B\hat{C}D) = 180.$

This would in turn imply that $\overleftrightarrow{BD}$, $\overleftrightarrow{CD}$ were parallel lines. This is a contradiction, since $\overleftrightarrow{BD}$ and $\overleftrightarrow{CD}$ intersect at D. Since $\overleftrightarrow{BC}$ intersects the circle in B, we deduce from Theorem 4-5 that $\overleftrightarrow{BC}$ intersects it on a second point C'. If $C' = C$, our result is proved.

So suppose $C' \neq C$. Then

$m(\hat{A}) + m(\hat{C}) = 180$ (hypothesis)

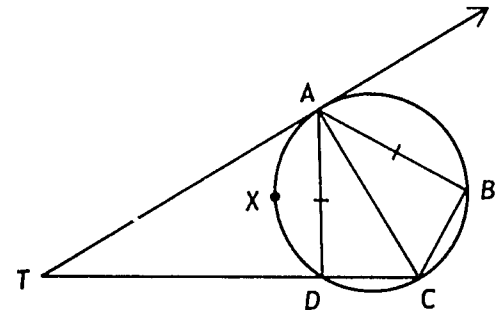
$m(\hat{A}) + m(\hat{C'}) = 180$ (Theorem 4-18)

$\therefore\ m(\hat{C}) = m(\hat{C'}).$

But this is impossible, since one of the angles $\hat{C}$, $\hat{C'}$ is an exterior angle and the other is one of the opposite interior angles of the $\Delta DCC'$. Hence $C' = C$ and the problem is proved. The statement of this problem is the converse of Theorem 4-18.

9. $A\hat{C}B \equiv A\hat{C}D$ (by Theorem 4-22, the minor arcs $\widehat{AB}$, $\widehat{AD}$ which they intercept are congruent)

$A\hat{C}D \equiv T\hat{A}D$ (both are measured by $\frac{1}{2}\, m(\widehat{AXD})$)



$\therefore A\hat{C}B \equiv T\hat{A}D$

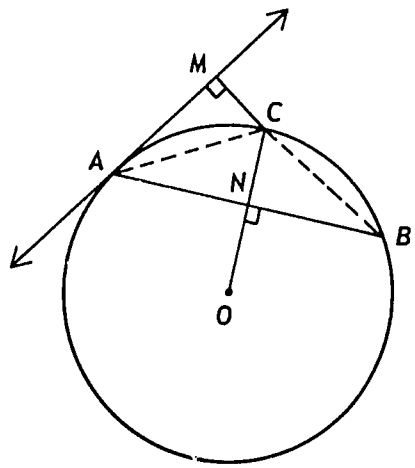
In triangles ΔABC, ΔTDA

$A\hat{C}B \equiv T\hat{A}D$ (proved)

$A\hat{B}C \equiv T\hat{D}A$ ($T\hat{D}A$ is the exterior angle of an inscribed quadrilateral, and both $A\hat{B}C$ and $T\hat{D}A$ are supplementary to $A\hat{D}C$.)

$\therefore\ \Delta ABC \sim \Delta TDA.$ (AA Similarity Theorem).

10. $\overline{AB}$ is a chord of a circle with centre O. $\overline{OC}$ is a radius $\perp \overline{AB}$ and intersects $\overline{AB}$ at N. AM is the tangent at A and $\overline{CM} \perp \overleftrightarrow{AM}$.



To prove that $\overline{CN} \equiv \overline{CM}$.

Proof: Since $\overline{ON} \perp \overline{AB}$, $\overline{AN} \equiv \overline{NB}$

In triangles ΔANC, ΔBNC.

$\overline{AN} \equiv \overline{BN}$ (proved)

$\overline{CN} \equiv \overline{CN}$

$A\hat{N}C \equiv B\hat{N}C$ (each is a right angle)

$\therefore \Delta ANC \equiv \Delta BNC$

$\therefore C\hat{A}B \equiv C\hat{B}A$.

(Alternately, the perpendicular $\overline{CN}$ to the base $\overline{BC}$ of ΔABC bisects the base. Hence ΔABC is isosceles with $\overline{AC} \equiv \overline{BC}$.)

But $C\hat{B}A \equiv C\hat{A}M$ (both are measured by $\frac{1}{2}\, m(\widehat{AC})$)

$\therefore C\hat{A}B \equiv C\hat{A}M$.

In triangles ΔCAN, ΔCAM

$C\hat{N}A \equiv C\hat{M}A$ (each is a right angle)

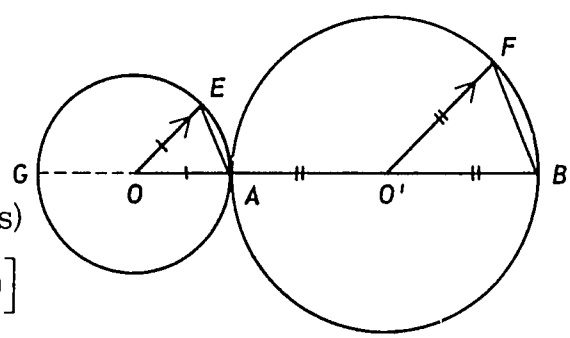
$C\hat{A}N \equiv C\hat{A}M$ (proved)

$\overline{AC} \equiv \overline{AC}$

$\therefore \Delta CAN \equiv \Delta CAM$ (ASA)

$\therefore \overline{CN} \equiv \overline{CM}$

11. A is collinear with O, O', and hence the four points O, A, O', B are all collinear.



$E\hat{O}A \equiv F\hat{O'}B$ (corresponding angles)

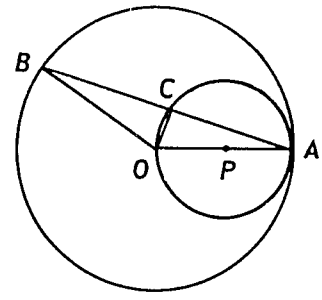
$m(E\hat{A}O) = \frac{1}{2}\, m(\widehat{GE}) = \frac{1}{2}\left[180 - m(\widehat{EA})\right]$

$= \frac{1}{2}\left[180 - m(E\hat{O}A)\right]$.

Similarly, $m(F\hat{B}O') = \frac{1}{2}\left\{180 - m(F\hat{O'}B)\right\}$

Hence $E\hat{A}O \equiv F\hat{B}O'$, from which it follows that $\overline{AE} \parallel \overline{BF}$, since the two angles are corresponding angles.

12. Two circles with centres O and P touch internally at A. The radius of the larger circle is congruent to the diameter of the smaller one. $\overline{AB}$ is any chord of the larger circle intersecting the smaller one again at C. To prove that $\overline{AC} \equiv \overline{BC}$.



Proof: O, P, A are collinear, and ΔOBA is isosceles ($\overline{OB} \equiv \overline{OA}$ by hypothesis).

$O\hat{C}A$ is a right angle (angle inscribed in a semicircle).

$\overline{OC}$ is therefore the altitude from the vertex O of the isosceles ΔOBA to the base $\overline{AB}$. Hence $\overline{OC}$ bisects $\overline{AB}$, and $\overline{AC} \equiv \overline{BC}$.

*4-4 SEGMENTS OF CHORDS, SECANTS AND TANGENTS.

The last two sections of this chapter, while interesting in themselves, may well be considered as supplementary material outside the main stream of the treatment of circles. We have starred these sections to indicate that you may omit them if you are pressed for time. On the other hand your good students will find many interesting properties of circles developed in the examples and exercises of these sections. The properties are those of the relationships of the measures of segments of lines which intersect one or two circles in various ways. We also use the triangle similarity theorems and the angle measurement theorems to arrive at solutions of many problems about circles, intersecting circles, circles and triangles, etc.

First we are interested in computing the lengths of various segments of chords, secants and tangents passing through a fixed point P and determined by the intersections with the circle. The point P may be in the interior or exterior of the circle.

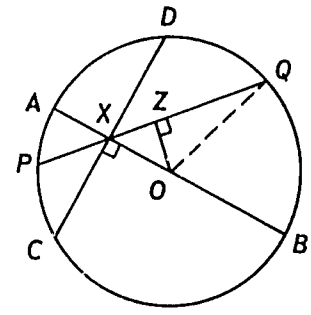
We start with P in the interior and try to determine the largest and shortest chords containing P.

Answers to
PROBLEMS 4-4A

Student Text Pages 138-139

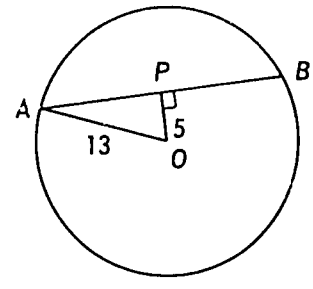
1. Construct $\overline{OZ} \perp \overline{PQ}$. Since $\overline{OX} \perp \overline{CD}$, if it were also $\perp$ to $\overline{PQ}$, $\overline{PQ}$ would be $\parallel$ to $\overline{CD}$ but this is impossible since they both contain X. Then

$OZ < OX$ because a perpendicular is the shortest distance from a point to a line. $\therefore PQ > CD$ by the theorem that if two chords are unequally distant from the centre, the closer chord is the greater. Since $\overline{AB}$ is a diameter its distance to the centre is O. $\therefore$ It is greater than $\overline{PQ}$ whose distance $OZ > O$. Another proof of the fact $AB > PQ$ would be:

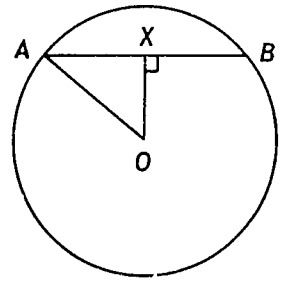


$$AB = 2OB = 2OQ \text{ but } OQ > ZQ \quad \therefore AB > 2ZQ = PQ.$$

2. $\overline{OP} \perp \overline{AB}$ since the distance from a point to a line is measured along the perpendicular.
$AO = 13$, $OP = 5$.
$AP^2 = AO^2 - PO^2$ by Pythagoras' Theorem
$AP = \sqrt{13^2 - 5^2} = \sqrt{169 - 25} = \sqrt{144} = 12$
$\therefore AB = 24$. By Problem 1, $\overline{AB}$ is the shortest chord through P. Also the longest chord through P is the diameter through P and its length is twice the radius. It is therefore 26 inches.

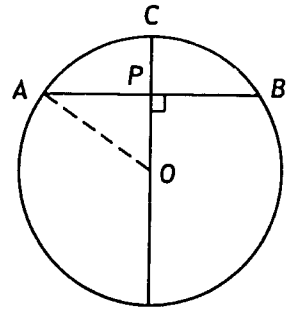


3. Draw $\overline{OX} \perp \overline{AB}$, $OX = 15$.
$AX = \frac{1}{2} AB = \frac{1}{2}(16) = 8$
$AO^2 = OX^2 + AX^2$
$= 15^2 + 8^2 = 225 + 64 = 289$
$\therefore AO = 17$



4.

	AB	AP	OP	$AO = OC$	$CD = 2AC$
(a)	12	6	3	$\sqrt{36+9} = \boxed{3\sqrt{5}}$	$\boxed{6\sqrt{5}}$
(b)	$\boxed{10\sqrt{3}}$	$\sqrt{75}$	5	$\boxed{10}$	20
(c)	18	9	$\sqrt{144-18}$ $\boxed{3\sqrt{7}}$	12	$\boxed{24}$
(d)	48	24	7	$\sqrt{24^2 + 7^2}$ $\boxed{25}$	$\boxed{50}$
(e)	$\boxed{18\sqrt{3}}$	$9\sqrt{3}$	9	18	$\boxed{36}$



In each part we use the formula $AO^2 = AP^2 + PO^2$

Now we begin to use the angle measurement theorems and the basic similarity theorems. In developing Theorem 4-23 we first assume P is inside the circle. In this case $\hat{B} \equiv \hat{D}$ because they intercept the same arc $\widehat{AC}$ while $\hat{A} \equiv \hat{C}$ because they intercept arc $\widehat{BD}$. The triangles are similar by the AA Similarity Theorem. The proportion follows since the segments involved are corresponding sides of the similar triangles. The result is true of course if $\overline{AB}$ were a diameter and $\overline{CD} \perp \overline{AB}$ since this would be only a special case of our theorem. Of course if P is on the circle one of the segments degenerates to a single point which is not usually said to have a length so the theorem no longer has meaning. But if we do for the moment say the length of such a "segment" is 0, the same formula still holds.

Proof in the case of two secants.

1. $\hat{A} \equiv \hat{D}$	1. Each angle is inscribed in $\widehat{BC}$.
2. $A\hat{P}C \equiv D\hat{P}B$	2. They are the same angle
3. $\Delta APC \sim \Delta DPB$	3. AA Similarity Theorem.
4. $\frac{AP}{PC} = \frac{DP}{PB}$	4. Definition of similar triangles
5. $PA \times PB = PC \times PD$	5. Property of a proportion.

Proof for the case of a tangent and a secant.

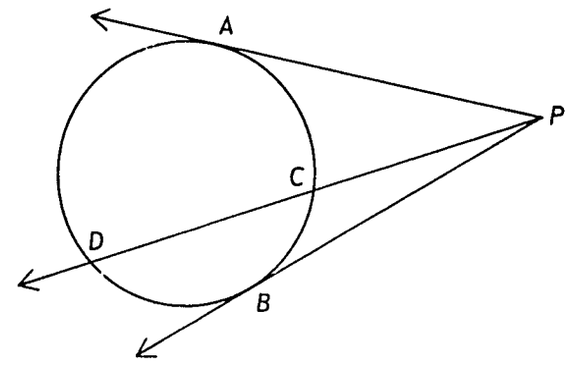
1. $m(A\hat{D}P) = \frac{1}{2} m(\widehat{AC})$	1. Inscribed angle theorem

2. $m(C\hat{A}P) = \frac{1}{2} m(\widehat{AC})$	2. The angle formed by a tangent and a chord at the point of contact is measured by one half the intercepted arc.
3. $A\hat{D}P \equiv C\hat{A}P$	3. They have the same measure.
4. $A\hat{P}D \equiv C\hat{P}A$	4. They are the same angle.
5. $\Delta APD \sim \Delta CPA$	5. AA Similarity Theorem.
6. $\frac{AP}{PD} = \frac{CP}{PA}$	6. Definition of similar triangles.
7. $AP^2 = PC \times PD$	7. Property of a proportion

Answers to
PROBLEMS 4-4B

Student Text Pages 142-146

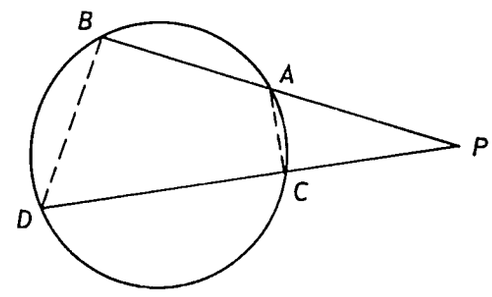
1. See above
2. See above
3. Draw any secant through P cutting the circle in points C and D. By Theorem 4-23 $PA^2 = PC \times PD$ and $PB^2 = PC \times PD$ $PA^2 = PB^2$ and since PA and PB are positive real numbers, being the lengths of $\overline{PA}$ and $\overline{PB}$, we know $PA = PB$.



4. 1. $A\hat{P}C \equiv D\hat{P}B$

   2. $P\hat{A}C$ is a supplement of $C\hat{A}B$

   3. $P\hat{D}B$ is a supplement of $C\hat{A}B$



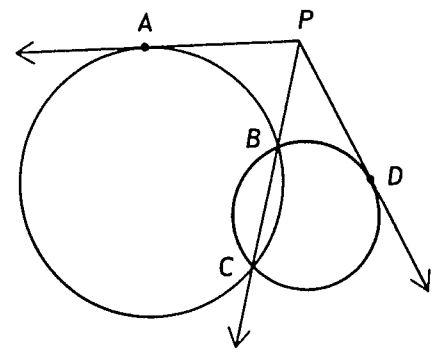
3. They are opposite angles of the inscribed quadrilateral $ACDB$.

4. $P\hat{D}B \equiv P\hat{A}C$	4. Steps 2 and 3.
5. $\Delta PDB \sim \Delta PAC$	5. AA Similarity Theorem.
6. $\frac{PD}{PB} = \frac{PA}{PC}$	6. Definition of similar triangles
7. $PA \times PB = PC \times PD$	7. Property of a proportion

5. $PA^2 = PB \times PC = PD^2$ by segment product theorem

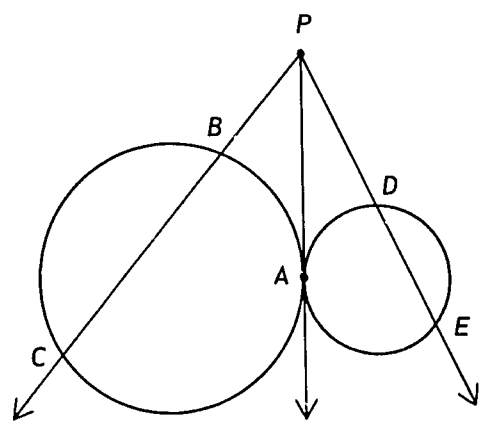
$PA^2 = PD^2$

$PA = PD$



6. $PB \times PC = PA^2 = PD \times PE$

$\therefore PB \times PC = PD \times PE$



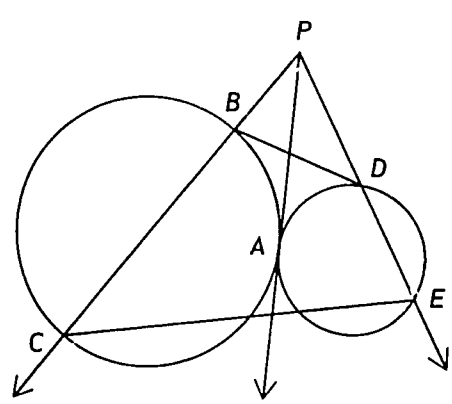
7. $PB \times PC = PA^2 = PD \times PE$

$PB \times PC = PD \times PE$

$\frac{PB}{PE} = \frac{PD}{PC}$ property of a proportion

$B\hat{P}D \equiv E\hat{P}C$

$\Delta PBD \sim \Delta PEC$ by the SAS Similarity Theorem.

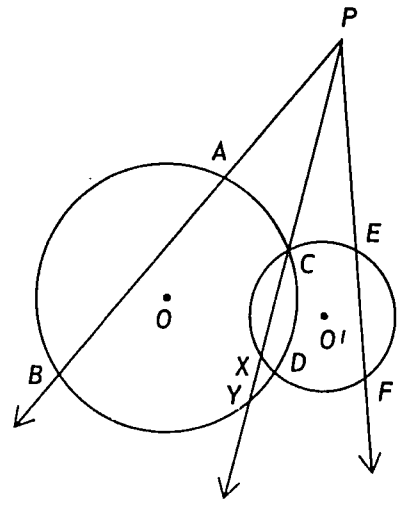


8. 1. $BC^2 = BD \times BE$	1. Segment product theorem
2. $= (BA - DA) \times (BA + AE)$	2. Measurement addition postulate

3.	$= (BA - AC) \times (BA + AC)$	3. All radii of the same circle have the same measure.
4.	$= BA^2 - AC^2$	4. Algebraic product of sum and difference of two numbers.
5.	$BA^2 = AC^2 + BC^2$	5. Addition

Challenge Problems

9. (a) Given $PA \times PB = PE \times PF$. To prove P lies on the line of the common chord $\overleftrightarrow{CD}$.



Proof: Draw the secant $\overleftrightarrow{PC}$.

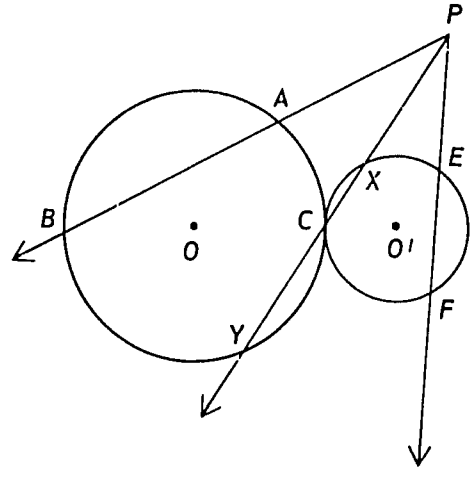
Since $\overleftrightarrow{PC}$ intersects the circle with centre O at C it must intersect it again, say at Y. In the same manner it intersects the circle with centre O' at some point, say at X. Then by the segment product theorem:

$$PC \times PY = PA \times PB \text{ and } PC \times PX = PE \times PF.$$

Since $PA \times PB = PE \times PF$, then $PC \times PY = PC \times PX$ and $PY = PX$. This means that X and Y are the same point since they are on the same side of P as C on line $\overleftrightarrow{PC}$ and there is only one point on $\overleftrightarrow{PC}$ at a certain distance from P on a given side of $\overleftrightarrow{PC}$. But this point must then be D since the two circles intersect only in the two points C and D. This means $\overleftrightarrow{PC}$ passes through D or P lies on $\overleftrightarrow{CD}$.

(b) The proof is identical with that in (a) except that both X and Y are on the opposite side of P as C.

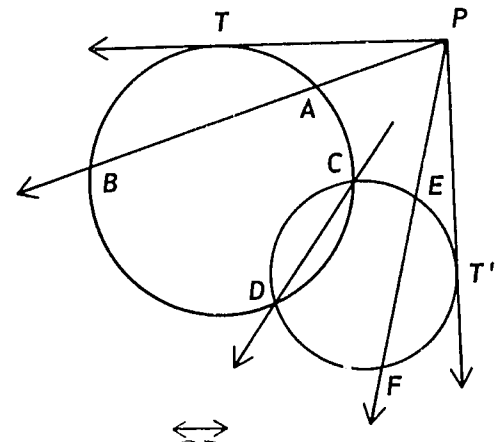
(c) The two circles are tangent at C. Draw $\overleftrightarrow{PC}$. If $\overleftrightarrow{PC}$ is tangent to the circles we are finished.



If $\overleftrightarrow{PC}$ is not tangent to the two circles it intersects each one at C and therefore at another point X on O' and Y on O. X and Y are different points since C is the only point on both circles.

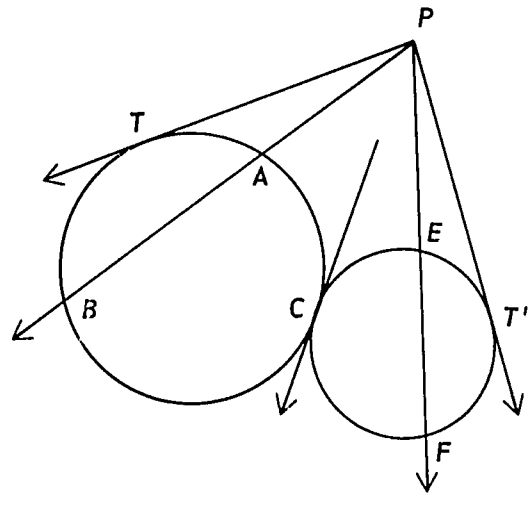
But now $PE \times PF = PC \times PX$ and $PA \times PB = PC \times PY$. Since $PE \times PF = PA \times PB$, $PC \times PX = PC \times PY$ making $PX = PY$. This means X and Y are the same point contradicting the assumption that they are distinct. Therefore the hypothesis that $\overleftrightarrow{PS}$ is not tangent is false and $\overleftrightarrow{PC}$ is tangent to both circles.

(d) This result is an immediate consequence of Problems 5 and 9(a). In Problem 5 we proved that the tangents from any point in $\overleftrightarrow{CD}$ are equal. Using Problem 9(a) we can show the converse which is required to establish the locus.

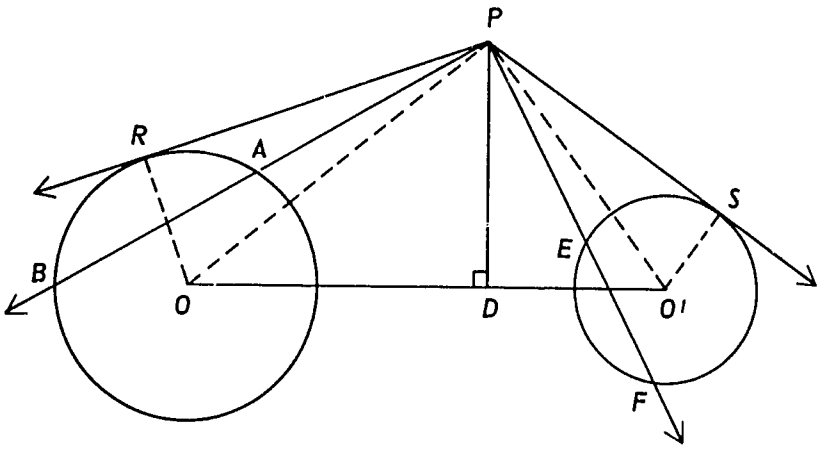


For if $PT = PT'$ then since $PT^2 = PA \times PB$ and $PT'^2 = PE \times PF$ it follows that $PA \times PB = PE \times PF$ and by Problem 9(a) P lies on $\overleftrightarrow{CD}$.

(e) If the tangents $\overleftrightarrow{PT}$ and $\overleftrightarrow{PT'}$ are drawn from any point in the common tangent $\overleftrightarrow{PC}$ then $PT = PC = PT'$. Conversely if the tangents $\overline{PT}$ and $\overline{PT'}$ are congruent, again $PA \times PB = PE \times PF$ and by Problem 9(c) P lies on the common tangent.



(f)



Given $PA \times PB = PE \times PF$. Draw $\overleftrightarrow{PR}$ and $\overleftrightarrow{PS}$ tangent to the circles. Also $\overline{PO}$, $\overline{PO'}$, $\overline{OR}$ and $\overline{O'S}$. We are going to prove that if the hypothesis is true the perpendicular from P to $\overline{OO'}$ always passes through a certain fixed point D and therefore P always lies on the fixed line which is perpendicular to $\overline{OO'}$ at D. To do this we will show that OD = a fixed constant. We know $\overline{OR}$ is the radius of one circle and $\overline{O'S}$ of the other. Therefore OR = a constant, call it a, and $O'S$ a constant, call it b. We use the theorem of Pythagoras.

$$PO^2 = PR^2 + RO^2$$

$$PO'^2 = PS^2 + O'S^2. \text{ But } PR^2 = PA \times PB = PE \times PF = PS^2$$

$$\therefore \quad PO^2 = PO'^2 = RO^2 - O'S^2 = a^2 - b^2 = \text{a constant, call it } c.$$

Again, $PO^2 = OD^2 + PD^2$

$$PO'^2 = O'D^2 + PD^2$$

$$\therefore \quad PO^2 - PO'^2 = OD^2 - O'D^2 = c$$

but $\quad OD^2 - O'D^2 = (OD + O'D)(OD - O'D) = OO' \times (OD - O'D)$

and OO' is a constant we can call d.

$$\therefore \quad d \times (OD - O'D) = c \text{ or } OD - O'D = \frac{c}{d} \text{ since } d > 0$$

$$\therefore OD + O'D = d$$

$$OD - O'D = \frac{c}{d}$$

$2 \times OD = d + \frac{c}{d}$ or $OD = (d + \frac{c}{d}) \times \frac{1}{2}$. Everything on the right is a constant. $\therefore$ OD = a constant and D is a fixed point.

In doing the next set of numerical problems be careful to pick the segments correctly for the segment product theorem.

Answers to
PROBLEMS 4-4C

Student Text Pages 147-148

1. In each problem $MR \times RN = PR \times RQ$.

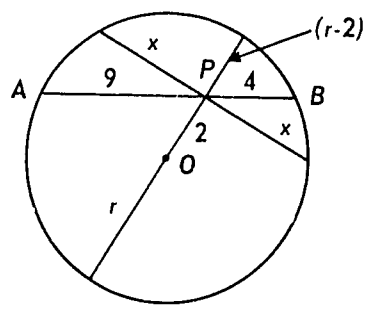
	MN	MR	RN	PQ	PR	RQ	
a)	6	4	2		$x = \boxed{2}$	4	$4x = 2 \times 4$
b)		9	6		4	$x = \boxed{\frac{27}{2}}$	$4x = 9 \times 6$
c)		$x = \boxed{6}$	x		4	9	$x^2 = 4 \times 9$
d)	15	12	3	13	$x = \boxed{9, 4}$	$13-x$	$x(13-x) = 3 \times 12$ $x^2 - 13x + 36 = 0$ $x = 9, 4$
e)	$x = \boxed{\frac{27}{2}}$	10	$x - 10$	12	5	7	$10(x - 10) = (5 \times 7)$ $x - 10 = \frac{7}{2}$ $x = \frac{27}{2}$
f)	6	4	2	7	$7-x$	x	$x(7-x) = 4 \times 2$ $x^2 - 7x + 8 = 0$

This equation does not factor so we solve it by the quadratic formula $x = \frac{7 \pm \sqrt{49 - 32}}{2} = \frac{7 \pm \sqrt{17}}{2}$. Then $\frac{7 + \sqrt{17}}{2}$ is the length of the longer segment and $\frac{7 - \sqrt{17}}{2}$ that of the shorter segment. The approximate values are 5·6 and 1·4.

2.

	RS	RT	ST	RV	RW	WV	RP	
a)	2	16		4	$x = \boxed{8}$			$4x = 32$
b)	2	16		4	$x + 4$	$x = \boxed{4}$		$4(x + 4) = 32$
c)	6	12	6	$x = \boxed{5{\cdot}1}$	$9 + x$	9		$x(x + 9) = 72$ $x = \dfrac{-9 \pm \sqrt{369}}{2}$
d)	2	16					$x = \boxed{4\sqrt{2}}$	$x^2 = 32$
e)				8	$x + 8$	$x = \boxed{10}$	12	$8(x + 8) = 144$
f)	5	11	6	$\boxed{5}$		6	$x = \boxed{\sqrt{55}}$	$x^2 = 5 \times 11$

3. The shortest chord is the one $\perp \overline{PO}$. Its segments are equal in length, therefore $x^2 = 9 \times 4$, $x = 6$, and the length is 12. If $PO = 2$ then $r^2 - 4 = 36$, $r = 2\sqrt{10}$. The longest chord through P is of course the diameter whose length is $4\sqrt{10}$.



Challenge Problem

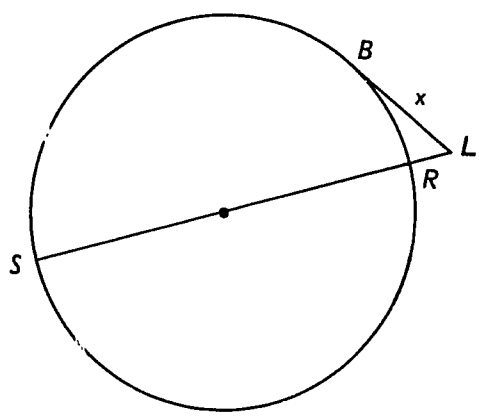
4. $x^2 = LS \times LR$

$LR = 200 \text{ feet} = \dfrac{200}{5280} \text{ miles}$

$LS = 8000 + \dfrac{200}{5280} \text{ miles}$

$x^2 = \dfrac{200}{5280} \times \left(8000 + \dfrac{200}{5280}\right)$

$x^2 \approx \dfrac{200}{5280} \times 8000$



Note that the other term $\left(\dfrac{200}{5280}\right)^2$ is too small to effect the result to the appropriate degree of accuracy.

$x^2 \approx \dfrac{160000}{528}$, $x \approx \dfrac{400}{23} \approx 16{\cdot}5$ miles.

*4-5 SIMILAR TRIANGLES AND CIRCLES.

These exercises and problems give the student a chance to use together many different ideas and methods of proof, so that he gets some idea of how the various aspects of geometry fit together to give more powerful tools to solve problems. We have only outlined solutions believing that you can in most cases easily supply the reasons and details.

Answers to
PROBLEMS 4-5

Student Text Pages 150-153

1. Draw $\overline{AD}$.

$A\hat{D}E \equiv A\hat{B}E$ both are measured by $\frac{1}{2}\, m(\widehat{AE})$.

$A\hat{D}F \equiv A\hat{C}F$ both are measured by $\frac{1}{2}\, m(\widehat{AF})$.

$\therefore\ A\hat{B}E \equiv A\hat{C}F$ since $A\hat{D}E = A\hat{D}F$.

$\therefore\ \overline{BE} \parallel \overline{CF}$ since corresponding angles are congruent.

2. Draw $\overline{AD}$ and choose point G on ray $\overrightarrow{BE}$ such that E is between G and B.

$A\hat{E}G$ is supplementary to $A\hat{E}B$.

$A\hat{D}B$ is supplementary to $A\hat{E}B$ since they are opposite angles of an inscribed quadrilateral.

(1) $\therefore\ A\hat{E}G \equiv A\hat{D}B$.

$A\hat{D}B$ is supplementary to $A\hat{D}F$.

$A\hat{C}F$ is supplementary to $A\hat{D}F$ since they are opposite angles of an inscribed quadrilateral.

(2) $\therefore\ A\hat{D}B \equiv A\hat{C}F$.

from (1) and (2) we have $A\hat{E}G \equiv A\hat{C}F$, hence $\overline{BE} \parallel \overline{CF}$ by the Alternate Interior Angle Theorem.

3. See pages 96-97, first five steps of proof for Problem 4.

4. Draw $\overline{AB}$

$\hat{1} \equiv \hat{2}$ both are measured by $\frac{1}{2}\, m(\widehat{AP})$.

$\hat{2}$ is supplementary to $\hat{3}$.

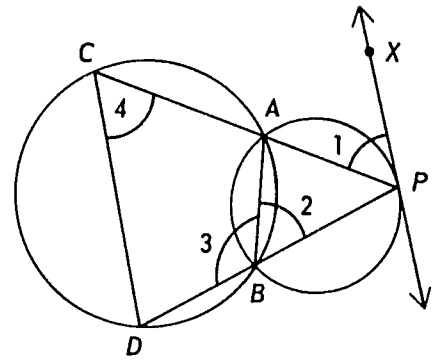
4. Cont.

$\hat{4}$ is supplementary to $\hat{3}$.

$\hat{2} \equiv \hat{4}$

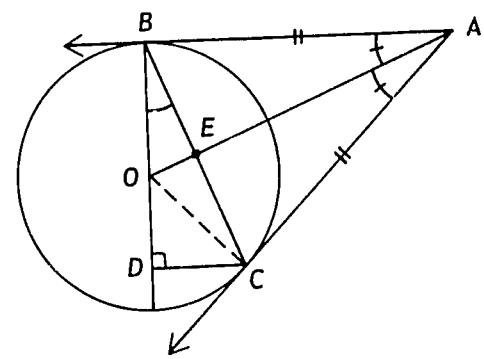
$\therefore \hat{1} \equiv \hat{4}$

$\therefore \overleftrightarrow{PX} \parallel \overleftrightarrow{CD}$



5. Drawing $\overline{AP}$ we see that $A\hat{P}B$ is a right angle since it is inscribed in the semicircle $\widehat{APB}$. Similarly $A\hat{P}C$ is a right angle. But this means $B\hat{P}C$ is a straight angle and so P lies on $\overleftrightarrow{BC}$.

6. $\overline{AB} \perp \overline{BD}$ because they are a tangent and a radius drawn to the same point on a circle. $O\hat{B}C$ is a complement of $C\hat{B}A$. $\Delta ABE \equiv \Delta ACE$ by SAS, hence $\overline{BC} \perp \overline{OA}$ and $B\hat{A}O$ is a complement of $C\hat{B}A$. $\therefore D\hat{B}C \equiv B\hat{A}O$ but $m(B\hat{D}C) = 90 = m(A\hat{B}O)$



$\therefore \Delta BCD \sim \Delta AOB$ by the AA Similarity Theorem.

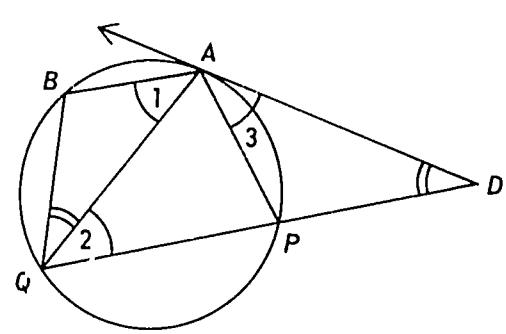
7. Since $\overline{AB} \parallel \overline{DQ}$, $\hat{1} \equiv \hat{2}$ and $\widehat{BQ} \equiv \widehat{AP}$.

$\hat{3} \equiv \hat{2}$ since they are both measured by $\frac{1}{2} m(\widehat{AP})$. $\therefore \hat{1} \equiv \hat{3}$.

$$m(\hat{D}) = \frac{1}{2}\left[m(\widehat{ABQ}) - m(\widehat{AP})\right]$$

$$= \frac{1}{2}\left[m(\widehat{AB}) + m(\widehat{BQ}) - m(\widehat{AP})\right]$$

$$= \frac{1}{2} m(\widehat{AB}) = m(B\hat{Q}A).$$



$\therefore B\hat{Q}A \equiv P\hat{D}A$ making $\Delta BQA \sim \Delta PDA$.

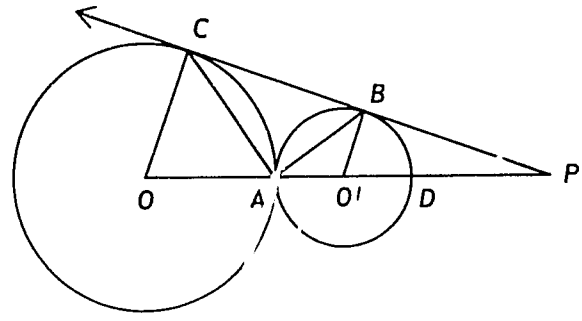
It is interesting to note that ΔBQA is also similar to ΔADQ.

8. To prove:

$PA^2 = PB \times PC$.

Draw $\overline{CA}$ and $\overline{BA}$.

8. Cont.



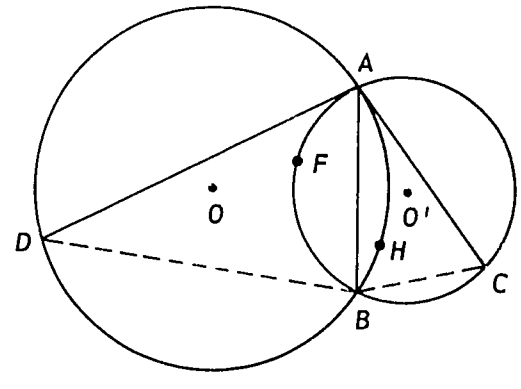
Consider ΔCAP and ΔBAP. Draw $\overline{OC}$ and $\overline{O'B}$. They are both $\perp \overline{PC}$ and are parallel.

$\therefore m(\widehat{AC}) = m(\widehat{BD})$ since they have the same measure as that of the congruent angles $C\hat{O}P$ and $B\hat{O'}P$. But now $m(P\hat{C}A) = \frac{1}{2} m(\widehat{CA})$ and $m(P\hat{A}B) = \frac{1}{2} m(\widehat{BD})$. This means $P\hat{C}A \equiv P\hat{A}B$. $A\hat{P}C \equiv B\hat{P}A$. $\therefore \Delta APC \sim BPA$ and $\frac{PA}{PC} = \frac{PB}{PA}$.

9. Frequently it is very important to pick out the correct triangles to try to prove similar. Here we pick ΔEFD and ΔEBC. In either of the two figures, $E\hat{F}D \equiv E\hat{B}C$ since they each intercept the same arc $\widehat{EC}$. In figure (a) $E\hat{C}B \equiv E\hat{A}B \equiv E\hat{D}F$; the second congruence resulting from the parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{DC}$.

In figure (b) $E\hat{C}B$ is supplementary to $\hat{A}$ because they are opposite angles in an inscribed quadrilateral and $\hat{D}$ is supplementary to $\hat{A}$ because they are adjacent angles in a parallelogram. In either case we now have $\Delta EBC \sim \Delta EFD$ and $\frac{EC}{EB} = \frac{ED}{EF}$.

10. Draw $\overline{BD}$ and $\overline{BC}$ and consider ΔABD and ΔABC. $\overline{AD}$ is tangent to the circle with centre O'. $\therefore m(D\hat{A}E) = \frac{1}{2} m(\widehat{AFB})$



$= m(A\hat{C}B)$

$\overline{AC}$ is tangent to the circle with centre O.

$\therefore m(B\hat{A}C) = \frac{1}{2} m(\widehat{AHB}) = m(A\hat{D}B)$

$\therefore D\hat{A}B \equiv A\hat{C}B$ and $B\hat{D}A \equiv B\hat{A}C$

$\Delta DAB \sim \Delta ACB$

$\frac{DB}{AB} = \frac{AB}{CB}$ or $AB^2 = DB \times CB$

In this problem it is very easy to get confused as to which are the corresponding sides in the two triangles.

Chapter 5
CIRCLE MEASURES

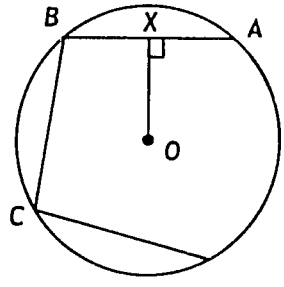
5-1 REGULAR POLYGONS AND CIRCLES.

In the first section of this chapter we study the relationship between regular polygons and circles. On the basis of this relationship we then explore the problem of determining the length of the circumference of a circle and the area of a circular region. This is done on quite an intuitive basis without any reference to limits which would be necessary for a more rigorous treatment. Next we examine parts of circles and circular regions and decide how to measure them. In the final section we introduce the idea of radian measure of angles in anticipation of its use later in trigonometry and calculus if pupils continue their study of mathematics.

The discussion of the relationship of regular polygons and their circumscribed and inscribed circles while interesting in its own right is primarily aimed at providing an introduction to computing the circumference and area of circles. We first find certain properties common to all regular polygons of n sides. Then we shall observe that by increasing n we can make the n-gons get closer and closer to either the circumscribed or inscribed circle. Thus it should seem natural to the pupils that many properties of regular n-gons will also be properties of circles.

Theorems 5-1 and 5-2 tell us that given a regular n-gon we can always circumscribe and inscribe circles in it and that these two circles have the same center. In the proof of Theorem 5-1, $A\hat{B}C \equiv D\hat{C}B$ by definition of a regular n-gon. $\overline{OB} \equiv \overline{OC}$ because they are radii of the circle drawn through A, B, and C; $O\hat{C}B \equiv O\hat{B}C$ because of the isosceles ΔOBC; and $O\hat{C}D \equiv O\hat{B}A$ by subtraction. In Theorem 5-2 the chords $\overline{AB}$, $\overline{BC}$ are each at the distance r from O. But distance from a point to a line is measured along a perpendicular. In the following figure $\overline{OX}$ is this perpendicular. So if a circle is drawn with centre O and radius $\overline{OX}$, $\overline{AB}$, being perpendicular to the radius at

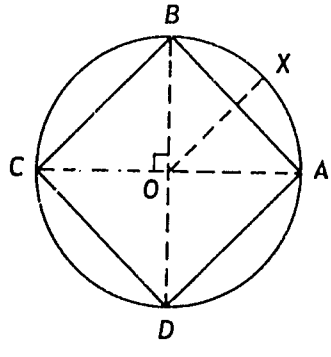
its endpoint X, is tangent to the circle. Similarly each other chord is tangent to this circle.



Theorem 5-3 reverses the situation and starts with a given circle. Note that this theorem does *not* tell us how to carry out the construction. It simply says that if n points $A, B, \ldots, N$ equally spaced on the circle are given, then the chords and tangents will form regular n-gons. It does not tell us how to find these points for a certain n. As a matter of fact this is fairly easy to do for $n = 3, 4, 5, 6, 8, 10$ which are therefore the polygons usually used in designs. Of course we can measure a central angle approximately equal to $\frac{360}{n}$ for any integer n by using a protractor and this will enable us to approximate any regular n-gon.

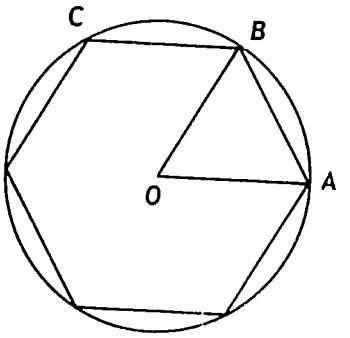
Some students like to make designs using regular n-gons so we show quickly how to draw them in the cases mentioned above.

(a) $n = 4$. Construct two diameters perpendicular to each other and connect the end points. $\widehat{AB}$ is the required arc since $m(\widehat{AB}) = m(\widehat{AOB}) = 90 = \frac{360}{4}$.



(b) $n = 8$. Bisect the $\widehat{AOB}$ in the previous construction. $\widehat{BX}$ is the required arc since $m(\widehat{BX}) = 45 = \frac{360}{8}$.

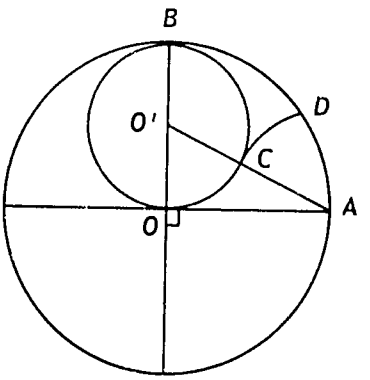
(c) $n = 6$.



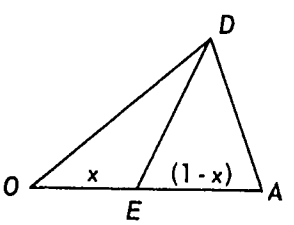
With A as center and $\overline{OA}$ as radius determine $\overline{AB} \equiv \overline{OA} \equiv \overline{OB}$. $\widehat{AB}$ is the required arc since $m(\widehat{AB}) = m(\widehat{AOB}) = 60 = \frac{360}{6}$.

(d) $n = 3$. $\widehat{AC}$ in the previous figure is the required arc since $m(\widehat{AC}) = 2m(\widehat{AB}) = 120 = \frac{360}{3}$.

(e) $n = 10$.



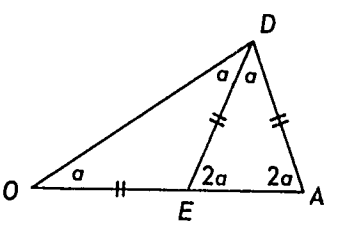
Construct two perpendicular diameters. Construct the circle with diameter $\overline{OB}$ and center O'. Draw $\overline{AO'}$ cutting the small circle at C. With A as centre and radius $\overline{AC}$ determine $\overline{AD} \equiv \overline{AC}$. $\widehat{AD}$ is the required arc. The proof is a fairly complicated algebraic computation to show $m(\widehat{AD}) = 36$. First consider $\Delta AOO'$ and let $OA = 1$. Then $OO' = \frac{1}{2}$ and $AO' = \sqrt{1 + \frac{1}{4}} = \frac{1}{2}\sqrt{5}$. $AC = \frac{1}{2}\sqrt{5} - \frac{1}{2} = AD$. Now consider ΔAOD.



With D as centre and radius $\overline{DA}$ determine E between O and A. Then ΔODA and ΔDEA are each isosceles triangles with a common base angle at A. They are therefore similar.

$\Delta ODA \sim \Delta DEA \therefore \frac{OA}{DA} = \frac{DA}{EA}$. Let $OE = x$ and remember $OA = 1$ and $DA = \frac{1}{2}(\sqrt{5} - 1)$. Then $EA = 1 - x$.

$$\frac{1}{\frac{1}{2}(\sqrt{5} - 1)} = \frac{\frac{1}{2}(\sqrt{5} - 1)}{1 - x}$$



$$\therefore 1 - x = \frac{1}{4}(\sqrt{5} - 1)^2$$

$$x = 1 - \frac{1}{4}(5 - 2\sqrt{5} + 1)$$

$$x = \frac{4 - 6 + 2\sqrt{5}}{4} = \frac{2\sqrt{5} - 2}{4}$$

$$x = \frac{1}{2}(\sqrt{5} - 1)$$

This means $\overline{OE} \equiv \overline{AD} \equiv \overline{DE}$. $\therefore$ Letting $a = m(A\hat{O}D) = m(E\hat{O}D)$ we get $m(O\hat{D}E) = a$. $m(D\hat{E}A) = 2a$ by the exterior angle theorem. $m(\hat{A}) = 2a = m(O\hat{D}A)$, since $OA = OD$. Then in ΔAOD we have $a + 2a + 2a = 180$ or $a = 36$. This finally establishes that $m(\widehat{AD}) = 36 = \frac{360}{10}$ and $\widehat{AD}$ is the required arc.

(f) $n = 5$. If A, D and F are successive vertices of the regular 10-gon $\widehat{AF}$ is the required arc since $m(\widehat{AF}) = 2m(\widehat{AD}) = 72 = \frac{360}{5}$.

Theorem 5-4 is proved easily and leads directly to the important conclusion that for any two regular n-gons $\frac{p}{r} = \frac{p'}{r'}$. Since $\frac{p}{r}$ is the same number for any two regular n-gons it is always a constant for a given n. It is important however to see that this ratio does change as n changes and in the problems we ask the students to compute it for the cases $n = 3$, 4 and 6. What we are after is to see the pattern of the change so that as n gets larger and larger and p gets closer and closer to C, the circumference of the circle, we might hopefully be able to see what $\frac{C}{r}$ is, since if p gets close to C we would expect $\frac{p}{r}$ to get close to $\frac{C}{r}$. Although we do not ask the students to do this we will show a little later on in this chapter how we can start with p_3 and from it compute successively p_6, p_{12}, p_{24}, ... or from p_4 compute p_8, p_{16}, p_{32}, ... thus getting $\frac{p_n}{r}$ for larger and larger n and so closing in on $\frac{C}{r}$.

Answers to
PROBLEMS 5-1

Student Text Pages 159-160

1. $r = 8$

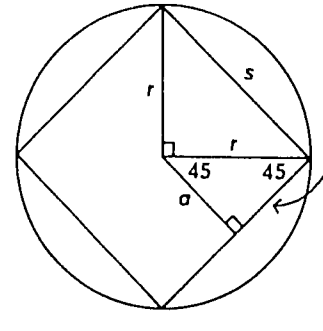
$s = \sqrt{64 + 64} = 8\sqrt{2}$

$a = 4\sqrt{2}$

$P = 4 \times 8\sqrt{2} = 32\sqrt{2}$

$\frac{P_4}{r} = \frac{32\sqrt{2}}{8} = 4\sqrt{2} \approx 4 \times 1{\cdot}414 \approx 5{\cdot}7$

$\frac{P_4}{a} = \frac{32\sqrt{2}}{4\sqrt{2}} = 8.$



$\frac{s}{2} = a$

$\frac{P_4}{r} \approx 5{\cdot}7$

$\frac{P_4}{a} = 8$

2. $\frac{p}{p'} = \frac{r}{r'}$ $\therefore$ $\frac{63}{p'} = \frac{7}{5}$ $\therefore$ $7p' = 5 \times 63, p' = 45$ but in any regular 6-gon $s = r$. $p' = 6s'$ $s' = \frac{45}{6}$ $r' = \frac{15}{2}$ $\frac{p'}{r} = \frac{45}{\frac{15}{2}} = 6$. Of course $p_6 = 6s = 6r$ so $\frac{p}{r}$ is always 6.

3. In the equilateral ΔAOB

$AB = OB = r$, $BC = \frac{r}{2}$.

$\therefore a^2 + \left(\frac{r}{2}\right)^2 = r^2$

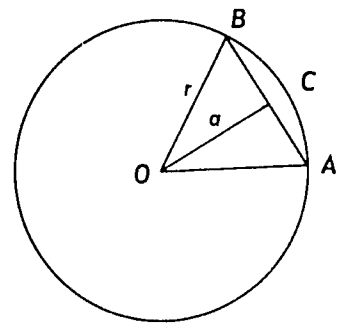
since $a = 6$, $36 + \frac{r^2}{4} = r^2$

$$\frac{3r^2}{4} = 36$$

$$r^2 = 48$$

$$r = 4\sqrt{3}$$

$p = 6s = 6r = 24\sqrt{3}$



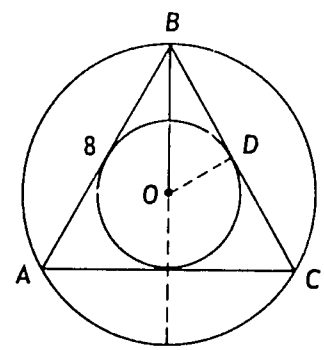
4. (a) $OB = r$ $BD = 4$

$OD = a$

$a = \frac{r}{2}$ $\therefore$ $\left(\frac{r}{2}\right)^2 + 16 = r^2$

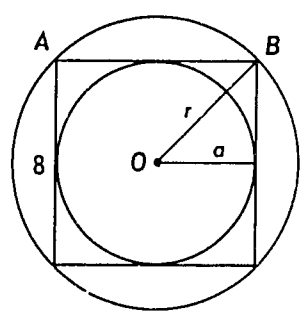
$\frac{3r^2}{4} = 16$ $r^2 = \frac{64}{3}$

$r = \frac{8}{3}\sqrt{3}$ $a = \frac{4}{3}\sqrt{3}$



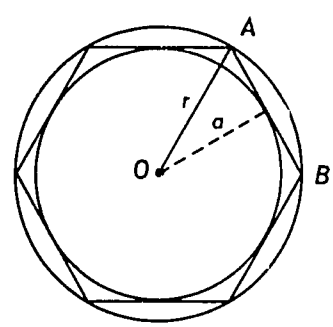
(b) $a = \frac{1}{2}AB = 4$

$r = OB = 4\sqrt{2}$



(c) $r = OA = AB = 8$

$a = 4\sqrt{3}$

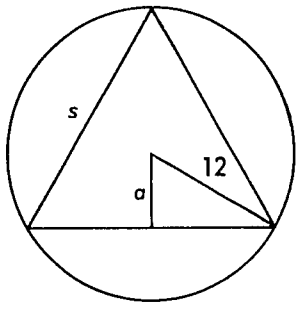


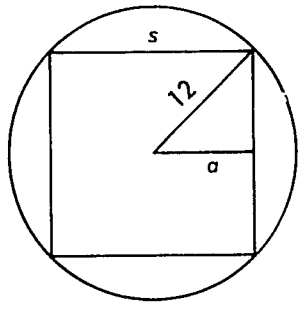
5. (a) $r = \frac{s}{3}\sqrt{3}$; $a = \frac{s}{6}\sqrt{3}$

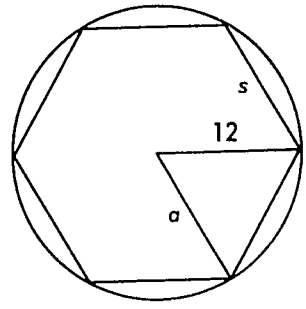
(b) $r = \frac{s}{2}\sqrt{2}$; $a = \frac{s}{2}$

(c) $r = s$; $a = \frac{s}{2}\sqrt{3}$

6.



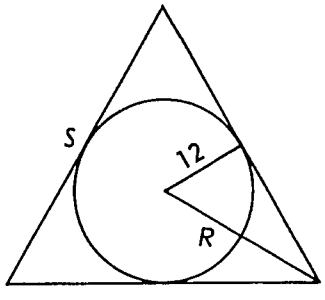


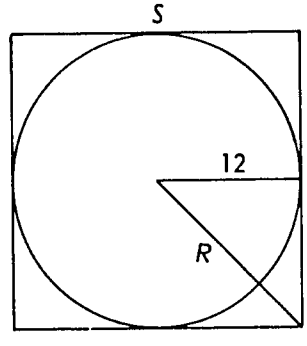


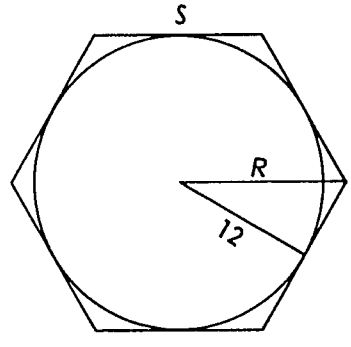
In the following answers, a represents the radius of the inscribed circle.

$a = 6$	$a = 6\sqrt{2}$	$a = 6\sqrt{3}$
$s = 12\sqrt{3}$	$s = 12\sqrt{2}$	$s = 12$
$p = 36\sqrt{3}$	$p = 48\sqrt{2}$	$p = 72$

7.







$R = 24$	$R = 12\sqrt{2}$	$R = 8\sqrt{3}$
$S = 24\sqrt{3}$	$S = 24$	$S = 8\sqrt{3}$
$P = 72\sqrt{3}$	$P = 96$	$P = 48\sqrt{3}$

8. If the radius is r,

(6.) $a = \frac{r}{2}$	$a = \frac{r}{2}\sqrt{2}$	$a = \frac{r}{2}\sqrt{3}$
$s = r\sqrt{3}$	$s = r\sqrt{2}$	$s = r$
$p = 3r\sqrt{3}$	$p = 4r\sqrt{2}$	$p = 6r$
(7.) $R = 2r$	$R = r\sqrt{2}$	$R = \frac{2r}{3}\sqrt{3}$
$S = 2r\sqrt{3}$	$S = 2r$	$S = \frac{2r}{3}\sqrt{3}$
$P = 6r\sqrt{3}$	$P = 8r$	$P = 4r\sqrt{3}$

9. Note that when talking about a regular n-gon, r stands for its radius which by definition is the radius of its circumscribed circle. Referring back to the results of Problem 8, we find

$$\frac{p_3}{r} = \frac{3r\sqrt{3}}{r} = 3\sqrt{3} \approx 3 \times 1{\cdot}73 \approx 5{\cdot}2$$

$$\frac{p_4}{r} = \frac{4r\sqrt{2}}{r} = 4\sqrt{2} \approx 4 \times 1{\cdot}41 \approx 5{\cdot}6$$

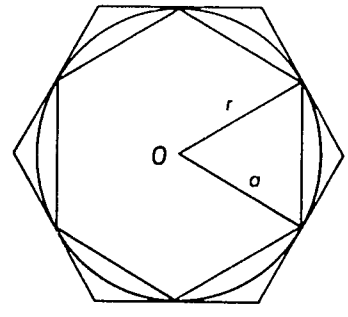
$$\frac{p_6}{r} = \frac{6r}{r} = 6 \qquad = 6{\cdot}0$$

We note that this ratio is increasing as n increases. In the next section we will study, for a given circle, the ratios $\frac{p}{r}$ for the inscribed regular n-gons and $\frac{P}{a}$ for the circumscribed regular n-gons, where a is the radius of the inscribed circle and is equal to the radius, r, of the inscribed polygon. (See the figure below.) To anticipate this, look at $\frac{P}{a}$ for the three cases above.

$$\frac{P_3}{a} = \frac{6r\sqrt{3}}{r} = 6\sqrt{3} \approx 10{\cdot}4$$

$$\frac{P_4}{a} = \frac{8r}{r} \qquad = 8$$

$$\frac{P_6}{a} = \frac{4r\sqrt{3}}{r} = 4\sqrt{3} \approx 6{\cdot}9$$



This ratio decreases as n increases. The perimeters of the circumscribed n-gons also get close to the circumference of the circle so again we get an estimate of $\frac{C}{r}$. Hopefully we can squeeze $\frac{C}{r}$ between two estimates and thus get a better idea of its value.

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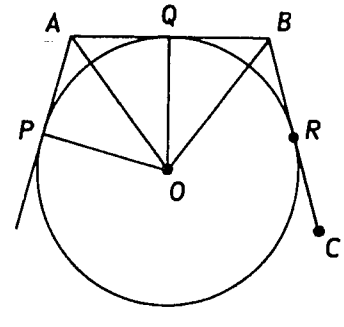
10. $OQ = OP$ and $A\hat{P}O \equiv A\hat{Q}O$ since a tangent is perpendicular to the radius at the point of contact. This makes $\Delta APO \equiv \Delta AQO$ and therefore $P\hat{A}O \equiv Q\hat{A}O$ and $A\hat{O}P \equiv A\hat{O}Q$. Since $m(P\hat{A}Q) = \frac{n-2}{n} \times 180$ we know $m(Q\hat{A}O) = \frac{n-2}{n} \times 90$. This makes $m(A\hat{O}Q) =$



$90 - \left(\frac{n-2}{n} \times 90\right) = \frac{180}{n}$. Similarly $m(B\hat{O}Q) = \frac{180}{n}$ and therefore $A\hat{O}Q \equiv B\hat{O}Q$. By ASA this makes $\Delta AOQ \equiv \Delta BOQ$ and so $\overline{AQ} \equiv \overline{BQ}$. $\overline{OQ}$ is therefore the unique perpendicular bisector of $\overline{AB}$. Similarly, $\overline{OR}$ is the unique perpendicular bisector of $\overline{BC}$. Therefore, the point O is unique and there exists only one circle inscribed in ABC . . . .

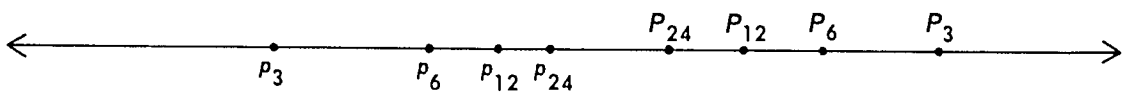
5-2 CIRCUMFERENCE OF A CIRCLE.

In this section we assume the existence of "length" as applied to a circle, even though length so far has only been used for line segments. Obviously a circle is not made up of line segments so this is a new use of the word. What is done in advanced mathematics is to define the length of a circle as the limit of the lengths of the perimeters of inscribed n-gons as n increases in such a fashion that the length of every side of the n-gon decreases towards zero. This limit is proved to exist and to have a certain value. We do not make any attempt to prove this but make use of the fact that most students feel intuitively that such a length does exist and can be approximated in some such fashion as we indicate here.

Using the fact that the circle has an inscribed regular n-gon and a circumscribed regular n-gon and assuming that the circumference of the circle lies between the perimeters of these two n-gons we find for $n = 6$ that $6 < \frac{C}{r} < 6{\cdot}9$.

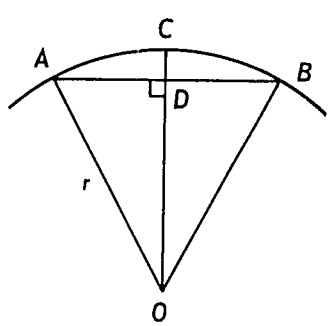
In order to determine $\frac{C}{r}$ more accurately we present here a method for computing the perimeters of certain inscribed and circumscribed regular n-gons in terms of the radius of the given circle. Let us indicate the perimeters of these n-gons by p_n and P_n respectively, and their sides by s_n and S_n. Since the ratios $\frac{p_n}{r}$ and $\frac{P_n}{r}$ are each constant for different values of r we will choose $r = 1$. This will make our computations easier. Before going to the trouble of doing all this computing we want to be reasonably sure it will be worthwhile. If p_n increases with n will it get too big or will there be an upper bound, beyond which it cannot go? We show that p_n is always less than P_n, that $p_n < p_{2n}$ and $P_n > P_{2n}$. Starting with $n = 3$ this will give us a series of inequalities for p_n and P_n which will be a help to answer the question.

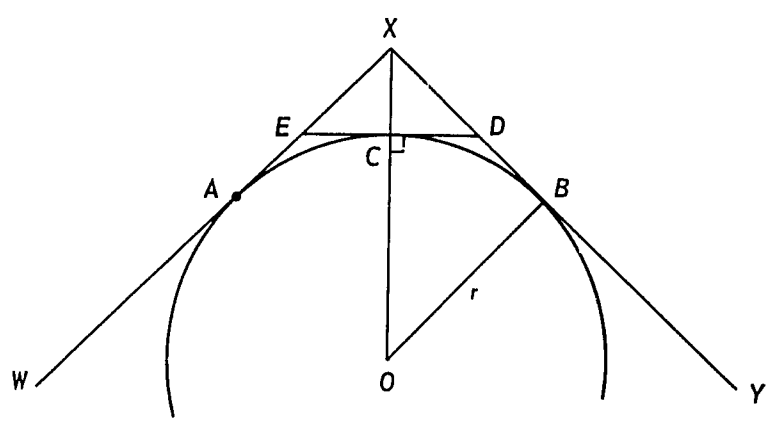
We can write all these inequalities together in the following way: If $i < j$ and i and j are both multiples of 3, $p_i < p_j$; $P_i > P_j$; and $p_i < P_i$. From this we can see that any p is less than any P. Thus $p_3 < p_6 < p_{12} < P_{12}$ while $p_{24} < P_{24} < P_{12} < P_6 < P_3$. If we plot these values on the number line we get a figure something like this



But we have not found the actual values. How can we do this so that we can draw the above graph reasonably accurately? We do this by finding a formula for the side s_{2n} for the *inscribed* $2n$-gon in terms of s_n and r, and for S_{2n} for the *circumscribed* $2n$-gon in terms of S_n and r. Then knowing $s_3 = r\sqrt{3}$ and $S_3 = 2r\sqrt{3}$ and with $r = 1$ we make our computations.

Careful diagrams will help us as always. If $\overline{AB}$ is the side of the regular inscribed n-gon and C is the midpoint of $\widehat{AB}$, $\overline{AC}$ will be the side of the $2n$-gon. We know $AD = \frac{1}{2}s_n$ and we can compute in succession DO, DC and $AC = s_{2n}$. We get $s_{2n} = \sqrt{2r^2 - r\sqrt{4r^2 - s_n^2}}$.





On the other hand if $\overline{WX}$ and $\overline{XY}$ are sides of a regular circumscribed n-gon and C is the midpoint of $\widehat{AB}$, then $\overline{DE}$ is one side of the $2n$-gon. $XB = \frac{1}{2}S_n$, $CD = \frac{1}{2}S_{2n} = DB$. We compute in succession XO, XC, XD, CD and $ED = S_{2n}$. The answer is $S_{2n} = \frac{2r}{S_n}\left[\sqrt{4r^2 + S_n^2} - 2r\right]$.

If $r = 1$ we know $s_3 = \sqrt{3}$ and $S_3 = 2\sqrt{3}$. We compute

$$s_6 = \sqrt{2 \times 1^2 - 1\sqrt{4 \times 1^2 - 3}} \quad \text{and} \quad S_6 = \frac{2}{2\sqrt{3}}\left[\sqrt{4 + (2\sqrt{3})^2} - 2\right]$$

$$= 1 \qquad\qquad = \frac{2\sqrt{3}}{3}$$

Both of these agree with the previously known values for the sides of these polygons and this provides a check on our formulas. Applying the formulas again we find

$$s_{12} = \sqrt{2 - \sqrt{4 - 1}} \qquad \text{and} \qquad S_{12} = \frac{2}{\frac{2\sqrt{3}}{3}}\left[\sqrt{4 + \frac{4}{3}} - 2\right]$$

$$\approx \cdot 518 \qquad\qquad \approx \cdot 536$$

Also $s_{24} \approx \sqrt{2 - \sqrt{4 - \cdot 2680}}$ and $S_{24} \approx \frac{2}{\cdot 536}\left[\sqrt{4 + (\cdot 536)^2} - 2\right]$

$$\approx \cdot 260 \qquad\qquad \approx \cdot 264$$

From these values we find

$$p_3 = 3 \times \sqrt{3} \approx 5\cdot 196 \qquad P_3 = 3 \times 2\sqrt{3} \approx 10\cdot 392$$

$$p_6 = 6 \times 1 = 6\cdot 00 \qquad P_6 = 6 \times \frac{2\sqrt{3}}{3} \approx 6\cdot 928$$

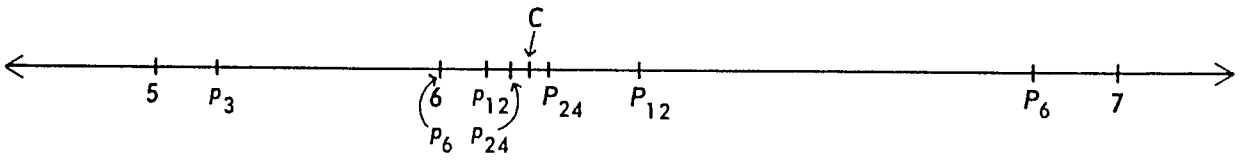
$$p_{12} \approx 12 \times (\cdot 518) \approx 6\cdot 216 \qquad P_{12} \approx 12 \times \cdot 536 = 6.432$$

$$p_{24} \approx 24 \times (\cdot 260) \approx 6\cdot 240 \qquad P_{24} \approx 24 \times \cdot 264 = 6\cdot 336$$

We see that $P_3 - p_3 = 5\cdot 18$; $P_6 - p_6 = 0\cdot 92$
$P_{12} - p_{12} = 0\cdot 216$; $P_{24} - p_{24} = 0\cdot 096$. These differences are getting smaller and smaller as n increases. Since $p_n < C < P_n$ the average of p_n and P_n might be a good estimate of C. We note $\frac{p_3 + P_3}{2} = 7\cdot 79$, $\frac{p_6 + P_6}{2} = 6\cdot 46$, $\frac{p_{12} + P_{12}}{2} = 6\cdot 32$ and $\frac{p_{24} + P_{24}}{2} = 6\cdot 28$.

If we continued the computations for larger and larger n and assumed that r was exactly 1 so the results could be given to as many figures as we wanted, the differences $P_n - p_n$ would get closer and closer to 0. We already know that $6\cdot 240 < C < 6\cdot 336$ and we suspect that the average of these, i.e., $6\cdot 28$ would be a pretty good value for C. As a matter of fact since $\pi = \frac{C}{2r}$ we would have $\pi \approx \frac{6\cdot 28}{2 \times 1} = 3\cdot 14$ and this is correct to three digits.

We now plot the values of p_i and P_i as we have found them so that we get a better picture of their relationship.



We could of course have started with $r = 1$, $s_4 = r\sqrt{2}$ and $S_4 = 2r$. In this way we would have obtained somewhat different approximations for C but by the time we had done the computations for $n = 8$, 16 and 32 we would again have obtained the value of C to be 6·28 to three digits or the value of π to be 3·14. There are much easier ways to compute π to many more places. These involve more advanced mathematics than your students have yet had but at least this relatively straightforward method gives us a reasonable approximation.

5-3 COMPUTING WITH π.

Many pupils will be tempted to give the circumference of a circle to many more digits than are justified. Do not ask for and do not allow pupils to give answers with greater accuracy than the initial data allow. The discussion in the text should make this clear. Usually answers in terms of π should be allowed unless otherwise specified.

Answers to
PROBLEMS 5-3

Student Text Pages 165-166

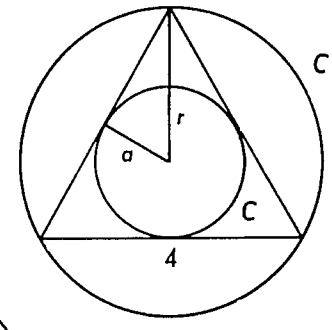
1.	a.	6π	d.	$2\sqrt{2}\,\pi$	g.	6π	j.	$\frac{6\pi}{7}$
	b.	10π	e.	$6\sqrt{5}\,\pi$	h.	π	k.	$\frac{4\pi}{5}$
	c.	34π	f.	$\frac{6\pi}{5}$	i.	$2\sqrt{2}\,\pi$	l.	14π

2.	a.	6	c.	$\frac{25}{2}$	e.	9	g.	$\frac{5}{\pi}$	i.	$\frac{11}{\pi}$
	b.	$\frac{17}{2}$	d.	$\frac{45}{4}$	f.	$\frac{\sqrt{2}}{2}$	h.	$\frac{18}{\pi}$	j.	$\frac{45}{2\pi}$

3. a. $\underline{20}$ c. 52·5 e. 45 g. 21 i. ·204
b. 38 d. 45·3 f. 19·7 h. 27·1 j. 204

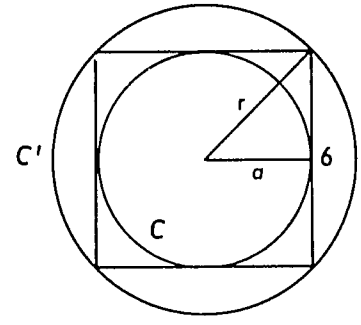
4. $s = 4$ $a = \frac{2}{3}\sqrt{3}$ $C = \frac{4\sqrt{3}\,\pi}{3}$

$r = \frac{4}{3}\sqrt{3}$ $C' = \frac{8\sqrt{3}\,\pi}{3}$



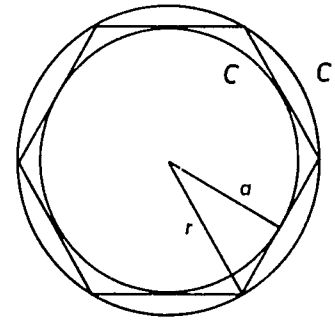
5. $s = 6$ $a = 3$ $C = 6\pi$

$r = 3\sqrt{2}$ $C' = 6\sqrt{2}\,\pi$



6. $s = 10$ $a = 5\sqrt{3}$ $C = 10\sqrt{3}\,\pi$

$r = 10$ $C' = 20\pi$



7. The polygon need not be regular.

Area $\Delta AOB = \frac{1}{2} r \times AB$

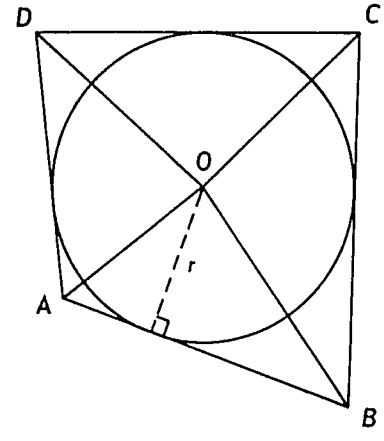
Area $\Delta BOC = \frac{1}{2} r \times BC$ etc.

Area of polygon = area ΔAOB + area $\Delta BOC + \ldots$

$= \frac{1}{2} r \times AB + \frac{1}{2} r \times BC + \ldots$

$= \frac{1}{2} r(AB + BC + \ldots + DA)$

$= \frac{1}{2} rp$



8. We know a circle may always be inscribed in a regular polygon. As in Problem 7,

Area $= \frac{1}{2}p \times$ (radius of inscribed circle)

$= \frac{1}{2} rp$

5-4 AREAS OF CIRCLES.

The area of a circular region, just as the circumference of a circle, is assumed to have a measure which may be approximated by the areas of the associated regular n-gons. Although we might carry out these successive approximations again it seems better simply to note that for the circumscribed polygons $A_n = \frac{1}{2} r P_n$ and that as n increases A_n gets close to A and P_n to C. The logical result is that $A = \frac{1}{2} r C$ which leads immediately to the familiar formula $A = \pi r^2$. The expression "area of a circle" is inaccurate since the circle has no area but nevertheless the meaning is clear and the phrase is much shorter than any more accurate one so we will use it freely.

Answers to
PROBLEMS 5-4

Student Text Pages 167-170

1.	a.	25π	d.	2π	g.	$\frac{25\pi}{4}$	j.	64π
	b.	49π	e.	36π	h.	$\frac{49\pi}{4}$	k.	9π
	c.	$\frac{4\pi}{9}$	f.	144π	i.	$\frac{\pi}{9}$	l.	36π
2.	a.	8	d.	$\frac{3}{2}$	g.	$6\sqrt{2}$	j.	$\frac{5}{\pi}\sqrt{3\pi}$
	b.	11	e.	18	h.	$3\sqrt{5}$	k.	$\frac{10}{\pi}\sqrt{\pi}$
	c.	13	f.	$5\sqrt{2}$	i.	$2\sqrt{22}$	l.	1

3.	a.	$A = 75 = \pi r^2$	$r = \frac{5}{\pi}\sqrt{3\pi}$	$C = 2\pi r = 10\sqrt{3\pi}$
	b.	$\pi r^2 = 64\pi$	$r = 8$	$C = 16\pi$
	c.	$\pi r^2 = 8\pi$	$r = 2\sqrt{2}$	$C = 4\sqrt{2}\pi$
	d.	$\pi r^2 = 9\pi$	$r = 3$	$C = 6\pi$
	e.	$\pi r^2 = 1000\pi$	$r = 10\sqrt{10}$	$C = 20\sqrt{10}\,\pi$
	f.	$\pi r^2 = 125$	$r = \frac{5}{\pi}\sqrt{5\pi}$	$C = 10\sqrt{5\pi}$
	g.	$\pi r^2 = 324$	$r = \frac{18}{\pi}\sqrt{\pi}$	$C = 36\sqrt{\pi}$
	h.	$\pi r^2 = 425$	$r = \frac{5}{\pi}\sqrt{17\pi}$	$C = 10\sqrt{17\pi}$

4. Area of ring $= \pi \times 36 - \pi \times 16 = 20\pi$

Area of smaller circle is 16π $\therefore \dfrac{\text{area of ring}}{\text{area of smaller circle}} = \dfrac{20\pi}{16\pi} = \dfrac{5}{4}$.

5. Area of first circle $= \pi \cdot 2^2 = 4\pi$

Area of second circle $= 8\pi$

$\therefore \pi r^2 = 8\pi$ and the radius of the second circle $= 2\sqrt{2}$ inches

6. $\dfrac{r}{r'} = \dfrac{2}{3}$. If $r' = 3a$ then $r = 2a$

$A' = 9a^2\pi$, $A = 4a^2\pi$. So $\dfrac{A}{A'} = \dfrac{4}{9}$

7. $\dfrac{A}{A'} = \dfrac{50}{72} = \dfrac{25}{36}$. If $r = 15$ $A = \pi r^2 = \pi \times 225$

$A' = \pi r'^2$

$\therefore \dfrac{25}{36} = \dfrac{225\pi}{(r')^2 \times \pi} = \dfrac{225}{(r')^2}$ $\therefore \dfrac{5}{6} = \dfrac{15}{r'}$ $r' = 18$

The radius of the larger circle is 18 inches.

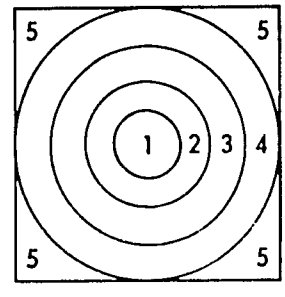
8. $A_1 = \dfrac{\pi}{4}$ $\approx \cdot 79$

$A_2 = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$ $\approx 2\cdot 26$

$A_3 = \dfrac{9\pi}{4} - \pi = \dfrac{5\pi}{4}$ $\approx 3\cdot 93$

$A_4 = 4\pi - \dfrac{9\pi}{4} = \dfrac{7\pi}{4}$ $\approx 5\cdot 50$

$A_5 = 16 - 4\pi$ $\approx 3\cdot 44$

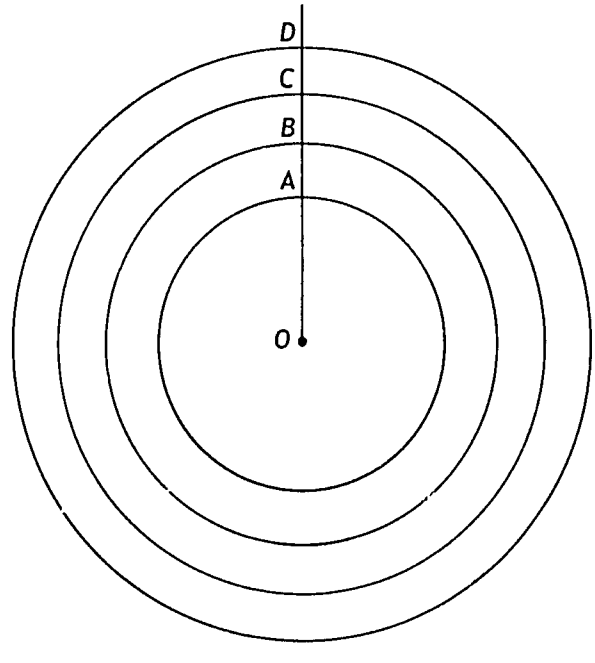


9. $\pi r_1^2 = \dfrac{1}{4}\, 4\pi = \pi$. $\therefore r_1 = 1 = OA$

$\pi r_2^2 = \dfrac{1}{2}\, 4\pi = 2\pi$. $\therefore r_2 = \sqrt{2} = OB$

$\pi r_3^2 = \dfrac{3}{4} \cdot 4\pi = 3\pi$ $\therefore r_3 = \sqrt{3} = OC$

Quite a surprising looking target isn't it?



10. $A = \pi r^2$
$A' = \pi r'^2$ $\quad \therefore \frac{A}{A'} = \frac{\pi r^2}{\pi r'^2} = \frac{r^2}{r'^2} = \left(\frac{r}{r'}\right)^2$

11. a. $\frac{1}{2}\pi r^2 = \frac{1}{2}100\pi = 50\pi$ since a diameter divides a circle in half.

b. $\frac{1}{4}\pi r^2 = \frac{1}{4}100\pi = 25\pi$. This is obviously half of the semicircular region.

c. $\frac{1}{2}2\pi r = 10\pi$

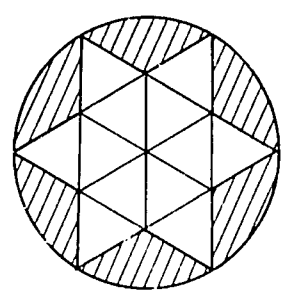
d. $\frac{1}{4} \cdot 2\pi r = 5\pi$

12. a. The two parts (shaded and unshaded) are the same shape so each must be half the circle; i.e., area = 50π.

b. Perimeter = half of the large circle plus 2 halves of the small circle

$$= \frac{1}{2} \times 2\pi \times 10 + 2 \times \frac{1}{2} \times 2\pi \times 5$$
$$= 10\pi + 10\pi = 20\pi.$$

13. If $r = 6$, the side of the large equilateral triangles is $6\sqrt{3}$. The small triangles are also equilateral so each side = $2\sqrt{3}$. The unshaded part is made up of 12 such triangles. Its area is therefore $12 \times 3\sqrt{3} = 36\sqrt{3}$.

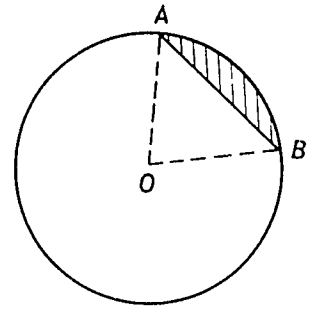


Area required is: $36\pi - 36\sqrt{3} = 36(\pi - \sqrt{3})$

5-5 LENGTHS OF ARCS, AREAS OF SECTORS AND SEGMENTS.

This section takes up lengths and areas of certain subsets of circles and circular regions. The fundamental technique is the use of the additive property of measure. We express the required point set as the union or intersection of point sets whose measures we already know and then add and subtract the corresponding measures as required. Thus the union of two semicircles of the same radius which have only their end points in common is the whole circle. The length of each semicircle is therefore half the

length of the circle. Similarly if we have an arc of measure m in degrees, where m is an integer, the arc is the union of m arcs each of measure 1. The length of the arc of 1 degree measure is $\frac{1}{360}$ of the circle since the union of 360 such congruent arcs with only end points in common will be the circle. If m is not an integer we simply define the length of the arc as $\frac{m}{360} \times 2\pi r$ so that the same formula will always hold true. The same technique yields the formula for area of a sector.



For a segment bounded by $\overline{AB}$ and $\overset{\frown}{AB}$ we note that {Sector AOB} = the union of {ΔAOB} and {Segment AB}. Note the new use of the word segment. The "segment" $\overline{AB}$ is part of a line; "segment of a circle" is that part of a circular region cut off by a chord. Area of segment = area of sector - area of triangle. The difficulty here is that the area of ΔAOB is difficult to compute if only the radius of the circle and the measure of $\overset{\frown}{AB}$ are known. It can be done of course by use of the trigonometric formula area $\Delta AOB = \frac{1}{2} AO \times BO \times \sin A\hat{O}B$.

Answers to
PROBLEMS 5-5

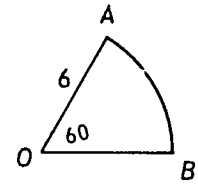
Student Text Pages 173-175

1. Perimeter $= OA +$ length of $\overset{\frown}{AB} + BO$

$$= 6 + \frac{1}{6} 2\pi \times 6 + 6$$

$$= 12 + 2\pi$$

Area $= \frac{1}{6}\pi \times 36 = 6\pi$

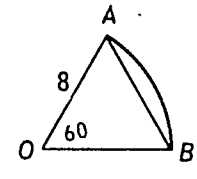


2. Perimeter $= m(\overline{AB}) + m(\overset{\frown}{AB}) = 8 + \frac{1}{6} 16\pi = 8 + \frac{8}{3}\pi$

Area $= m(\text{Sector } AOB) - m(\Delta AOB)$

$$= \frac{64\pi}{6} - 16\sqrt{3}$$

$$= \frac{32\pi}{3} - 16\sqrt{3}.$$



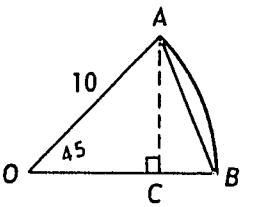
3. Area of Sector $AOB = \frac{45}{360} \times \pi \times 100 = \frac{25\pi}{2}$.

$m(\Delta AOB) = \frac{1}{2}\, OB \times AC$, but $AC = OC$

since $m(O\hat{A}C) = 45$. $\therefore AC = 5\sqrt{2}$

$m(\Delta AOB) = \frac{1}{2}\, 10 \times 5\sqrt{2} = 25\sqrt{2}$.

Area of segment $= \frac{25\pi}{2} - 25\sqrt{2}$



4. $\frac{m}{360} \times \pi \times 100 = 40\pi$

$\frac{10m}{36} = 40$

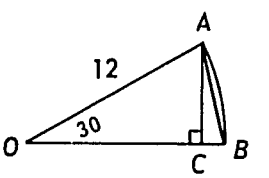
$m = 144$. (Degree measure of arc.)

length of arc $= \frac{144}{360} \times 2\pi \times 10 = \frac{2}{5} \times 20\pi = 8\pi$

5. Area $AOB = \frac{30}{360}\pi \times 12^2 = 12\pi$.

Area $\Delta AOB = \frac{1}{2}\, OB \times AC = \frac{1}{2}\, 12 \times 6 = 36$.

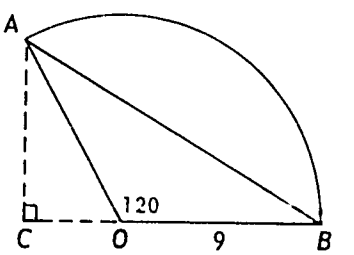
Area Segment $= 12\pi - 36$.



6. Area $AOB = \frac{120}{360}\pi \times 81 = 27\pi$.

Area $\Delta AOB = \frac{1}{2}\, OB \times AC = \frac{1}{2}\, 9 \times \frac{9}{2}\sqrt{3} = \frac{81\sqrt{3}}{4}$.

Area segment $= 27\pi - \frac{81\sqrt{3}}{4}$.



7. (3.) Major sector $= 100\pi - \frac{25\pi}{2} = \frac{175\pi}{2}$.

Major segment $= 100\pi - \left(\frac{25\pi}{2} - 25\sqrt{2}\right) = \frac{175\pi}{2} + 25\sqrt{2}$.

(5.) Major sector $= 144\pi - 12\pi = 132\pi$.

Major segment $= 144\pi - (12\pi - 36) = 132\pi + 36$.

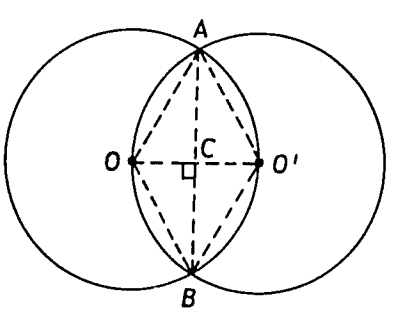
(6.) Major sector $= 81\pi - 27\pi = 54\pi$.

Major segment $= 81\pi - \left(27\pi - \frac{81\sqrt{3}}{4}\right) = 54\pi + \frac{81\sqrt{3}}{4}$.

8. Required area of intersection is sum of 2 segments.

$OA = OO' = AO' = 12$

$m(A\hat{O}B) = 120$



Area segment $AO'B = \frac{120}{360} \times \pi 144 - \frac{1}{2} AB \times OC = 48\pi - \frac{1}{2} 12\sqrt{3} \times 6$

$= 48\pi - 36\sqrt{3}$

$\therefore$ Required area $= 96\pi - 72\sqrt{3}$.

Area of union = area of circle O + area of circle O' - area already found

$$= 144\pi + 144\pi - (96\pi - 72\sqrt{3}) = 192\pi + 72\sqrt{3}$$

9. $\frac{10}{360} \times \pi r^2 = \pi 15^2 - \pi 12^2$

$\frac{\pi r^2}{36} = 225\pi - 144\pi$

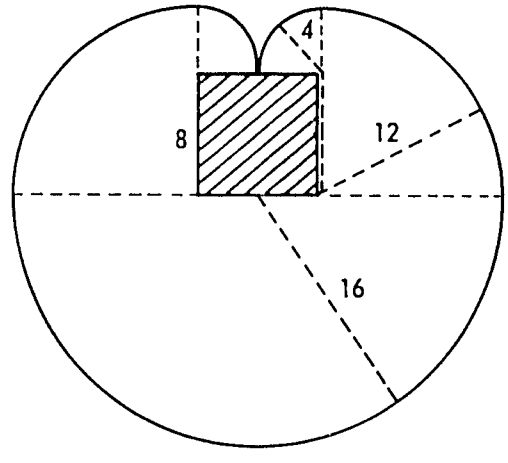
$r^2 = 36(81)$

$r = 6 \times 9 = 54.$

10. The area the goat can graze over is made up of one semicircular region of radius 16 feet, two quarter circle regions of radius 12 and two of radius 4. The total area is

$\frac{1}{2}\pi\ 16^2 + 2 \times \frac{1}{4}\pi\ 12^2 + 2 \times \frac{1}{4}\pi\ 4^2 =$

$128\pi + 72\pi + 8\pi = 208\pi$ sq. ft.

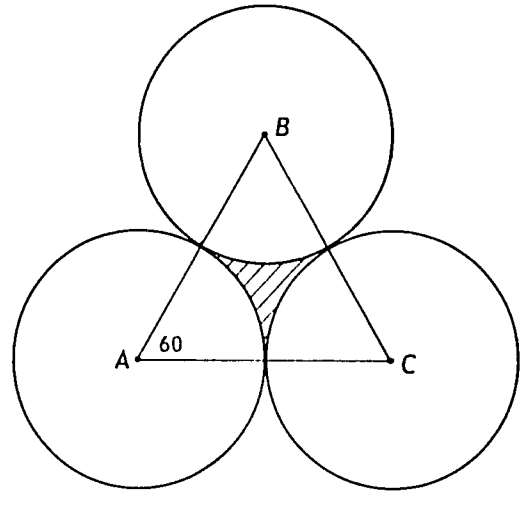


11. The required area is the difference between the areas of ΔABC and the three 60° sectors.

Area $\Delta ABC = \frac{1}{2} \times 4 \times 2\sqrt{3} = 4\sqrt{3}$

Area of each sector $= \frac{1}{6}\pi \times 4 = \frac{2\pi}{3}$

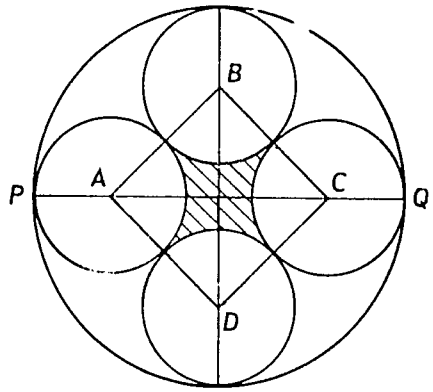
Answer $4\sqrt{3} - 2\pi$



12. The required area is the difference between the areas of the square $ABCD$ and the four 90° sectors. If the radius of each small circle is equal to x, $AB = 2x$.

Area $= 4x^2 - \pi x^2$.

We still have to find x. $\overline{PQ}$ is a diameter. $PQ = 16$



$$AC^2 = AB^2 + BC^2 = (2x)^2 + (2x)^2 = 8x^2$$

$$AC = 2\sqrt{2}x,\ PA = x,\ CQ = x,$$

$$PQ = x + 2\sqrt{2}\,x + x = x(2\sqrt{2} + 2)$$

$$\therefore x(2\sqrt{2} + 2) = 16$$

$$x = \frac{8}{\sqrt{2} + 1} = 8(\sqrt{2} - 1)$$

$$x^2 = 64(2 - 2\sqrt{2} + 1)$$

$$= 64(3 - 2\sqrt{2})$$

$$\text{Area} = (4 - \pi)64(3 - 2\sqrt{2}) \approx 9{\cdot}3.$$

Since the area of the large circle ≈ 201, the shaded part is about 4·5% of the total.

5-6 RADIAN MEASURE.

This last section takes up the radian measure of an angle in anticipation of its use in trigonometry and, by those who take advanced mathematics, in calculus. The concept of using different units for angle measurement should not be too difficult for pupils who are used to using such different units as inches or centimetres for measuring segments. However they will need a lot of practice in converting from one type of unit to the other.

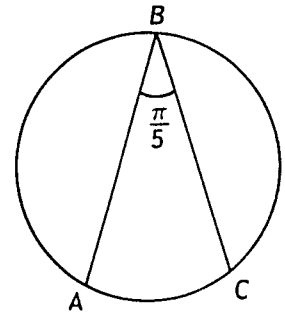
Answers to PROBLEMS 5-6

Student Text Pages 177-178

1.	a. $\frac{\pi}{6}$	d. $\frac{7\pi}{180}$	g. $\frac{3\pi}{4}$	j. $\frac{\pi}{5}$
	b. $\frac{2\pi}{9}$	e. $\frac{\pi}{8}$	h. $\frac{7\pi}{8}$	k. $\frac{2\pi}{5}$
	c. $\frac{\pi}{12}$	f. $\frac{5\pi}{12}$	i. $\frac{179\pi}{180}$	l. $\frac{4\pi}{5}$

2.	a.	90	d.	$\frac{432}{\pi}$	g.	$\frac{108}{\pi}$	j.	100
	b.	36	e.	50	h.	$\frac{234}{\pi}$	k.	157·5
	c.	$\frac{270}{\pi}$	f.	120	i.	$\frac{180}{7}$	l.	$\frac{180}{\pi}$
3.	a.	π	d.	$\frac{2\pi}{3}$	g.	·1225	j.	2
	b.	$\frac{8\pi}{3}$	e.	6	h.	4		
	c.	π	f.	9·1	i.	9		

4. $m(A\hat{B}C) = \frac{\pi}{5}$

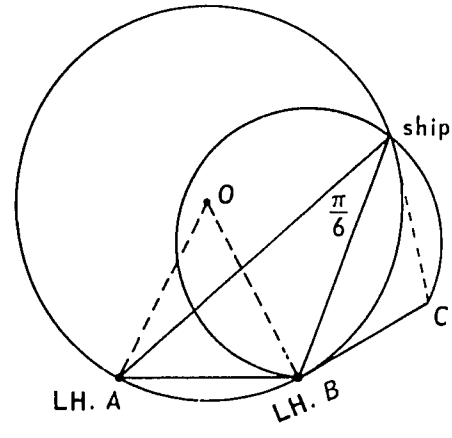


$\therefore m(\overset{\frown}{AC}) = \frac{2\pi}{5}$ or

$\overset{\frown}{AC}$ is one fifth of the circle. The length of $\overset{\frown}{ABC}$ is $\frac{4}{5}$ of the circumference

$\therefore$ Length of $\overset{\frown}{ABC} = \frac{4}{5}\ 24\pi = \frac{96\pi}{5}$.

5. The angle at the ship is $\frac{\pi}{6}$ radians.



Suppose we could draw the circle determined by the ship and the two lighthouses.

Then $m(A\hat{O}B) = 2m(A\hat{S}B)$

$m(A\hat{O}B) = \frac{\pi}{3}$

This means that $\overset{\frown}{AB}$ is an arc whose degree measure is 60 and ΔAOB is an equilateral triangle. We can then find the position of O by drawing two circles with centres at A and B and each with radius 10 miles. O will be the intersection of these circles on the seaward side of $\overline{AB}$. Draw the circle with centre O and radius 10. The ship is somewhere on the major arc of this circle since at any point on this arc the measure of the angle subtended by $\overset{\frown}{AB}$ will be equal to $\frac{\pi}{6}$.

If a third lighthouse were visible at some point C a similar construction would put the ship on a circle of which $\overline{BC}$ was a chord. The intersection of these two circles would be the position of the ship.

Chapter 6
MOTIONS AND TRANSFORMATIONS

INTRODUCTION.

Introductory Comments. The ideas presented in this chapter have been well known in advanced mathematics for a long time. It is only in the past few years, however, that they have begun to appear in secondary school texts. As you will see, these ideas are simple, natural, and directly related to other major areas of mathematics, such as algebra. Many educators and mathematicians believe that, in years to come, these ideas will become more and more important in secondary mathematics.

Because these ideas are probably new to you as well as to your pupils, they are presented slowly and with care in the Student Text. *It is vitally important that you give the entire Student Text and the entire Teacher's Guide (for this chapter) a careful reading before you begin to teach this material.* In this way you will get an over-all view of the direction and main ideas of the material, and you will see how very simple and elementary these ideas are, once they are properly understood. In your careful preliminary reading, pay special attention to Sections 6-1 to 6-6. Also, be sure to look at all the problems and at the discussion of answers given in this Guide. The problems are meant to be especially helpful for obtaining an easy and correct understanding.

We now give a brief general summary of the over-all content of Chapter 6. We follow this with a brief summary by sections. We then give some more general comments. We then discuss various ways of omitting material to shorten the program. We then give several suggested programs. Finally, we end the Introduction with a discussion of how to plan your time in presenting Chapter 6 to your pupils. After the Introduction, we take up the Student Text, section by section, giving answers to problems and giving other general comments on the material.

You will probably find that you will understand the summary below better after you have worked through the entire Text and Guide.

Summary. This chapter presents the concept of *rigid motion* in the plane. Various kinds of rigid motion are considered, certain mathematical facts about rigid motions are obtained, and a number of applications are described. A rigid motion is a *mapping* (that is to say, *function*) from the plane onto the plane such that the distance between the images (under the mapping) of any two points is the same as the distance between the two points. Special kinds of rigid motion include direct motions, reversing motions, translations, rotations, and reflections. One of the chief mathematical facts presented is that every rigid motion can be viewed either as a translation, a rotation, a reflection, or a combination of reflection and translation. (Which of these a motion is will depend upon the particular motion.) This fact and others lead to a variety of useful applications in geometry. Also, certain algebraic notations and ideas can be used in working with rigid motions.

Summary by Sections. Sections 6-1 and 6-2 use exercises with tracing paper to help prepare and build the pupil's thinking. Questions of congruence and of "moving" geometrical figures in the plane are the immediate subject matter. In Section 6-3, the concept of rigid motion is introduced and several basic mathematical facts about rigid motions are given. Section 6-4 gives two illustrations of the way in which the concept of rigid motion can help our thinking in solving geometrical problems. In Section 6-5, the basic kinds of rigid motion are described and illustrated. In Section 6-6, we use the idea of rigid motion to give a general definition of congruence for geometrical figures in the plane. Sections 6-7, 6-8, and 6-9 give further information and examples about translations, rotations, and reflections. In Section 6-10, we consider ways that different rigid motions can be combined, one after the other, to give new rigid motions, and we see the usefulness of algebraic ideas in working with such combined motions. In Section 6-10, we also present some of the main theoretical facts, such as, for example, the fact that every direct motion is either a translation or a rotation. Finally, in Section 6-11, we give several further applications of rigid motions to solving geometrical problems.

General Comments. (1) As you read through the Text and Guide, you see that our work on rigid motions in Chapter 6 is closely related to the work on algebra in *Secondary Four*. Both in the algebra and in the geometry of rigid

motions, the concept of *function* is fundamental. In algebra, we have functions from real numbers to real numbers. In geometry, a rigid motion is a special kind of function from points in the plane to points in the plane. In Section 6-3 of the Student Text, this connection between the pupil's work in geometry and algebra is emphasized, and various notations used in the pupil s work are also used for rigid motions. In studying Section 6-3, you should read again the basic material on *functions* in the algebra Student Text and Teacher's Guide.

(2) A main reason for studying rigid motions is that they have a close connection with more advanced work in algebra as well. This is shown a little bit in the work on coordinate axes in Sections 6-7, 6-8, and 6-9, and it is shown in the work in Section 6-10 on combinations of motions. It will become much clearer in *Secondary Five* algebra where the pupil will be introduced to matrices and to the concepts of linear algebra which are fundamental for so much of modern mathematics and its applications. The pupil will then find that his study of rigid motions will have provided him with a valuable preparation for this later work in algebra.

(3) The value of this study of rigid motions comes, in large part, from the simplicity and naturalness of the geometric ideas. The work in Chapter 6 is intended to show this. If your pupils find Chapter 6 easy, do not be dismayed. This only means that you are doing a successful job in teaching the material to them.

Material that can be omitted. The Student Text is arranged so that, depending on time available, on the background and enthusiasm of your pupils, and on your personal preferences, certain material can be omitted. This material, which we list below, is in part theoretical. While of value, it is not essential. Even if you omit all of the material below, many important points and ideas will remain, and the student will still have a sound and worthwhile program upon which to build later mathematical work. Do not hesitate to drop any of this material that does not suit your class, your time, or your preferences.

Material can be omitted, as you choose, according to the following outline, which proceeds section by section.

6-1. All of this section should be used.

6-2. The last part, after Problems 6-2B, can be omitted or treated lightly. This part, on "using ruler and compasses," will be of more interest

to pupils who have already had ruler and compasses constructions up to the construction of a line through a given point parallel to a given line. Note the important final sentence of Section 6-2 (just before Problems 6-2C). Rigid motion ideas are very useful for solving problems about ruler and compasses constructions, but knowledge of ruler and compasses constructions is not necessary for understanding the theory of rigid motions.

6-3. All of this material should be used. Note, however, that certain proofs (of properties (2)-(6) from property (1)) have been placed in an Appendix. These proofs are intended as "footnotes" to the text. Later work does not depend on them. Except for pupils with a special interest in them, the proofs in this Appendix should be omitted.

6-4. The first example should be used, but the second can be omitted or treated lightly, especially if pupils have not studied ruler and compasses constructions. (See comment above on possible omission of material from Section 6-2.)

6-5. All of this section should be used.

If you are pressed for time, you can stop with 6-5 and omit the rest of Chapter 6. Your class will still have obtained much of value. If you go further, you can take one of the following alternatives.

(a) Do Sections 6-6 through 6-9.
(b) Do Sections 6-6 through 6-10.
(c) Do Sections 6-6 through 6-11

In each of these alternatives, there are partial omissions possible in the sections listed. We now describe these.

6-6. The "Note on isometric mappings" at the end of the section can be omitted. It is not difficult, however, and should be kept if possible.

6-7. The proof following Definition 6-2 can be either omitted or treated briefly. The material on coordinate axes can be omitted.

6-8. The proof following Definition 6-3 can be either omitted or treated briefly. The material on symmetry and coordinate axes can be omitted.

6-9. The proof following Definition 6-4 can be either omitted or treated briefly. The material on symmetry and coordinate axes can be omitted.

6-10. This section can be done even if all the above omissions in Sections 6-7, 6-8, and 6-9 have been made. Section 6-10 is necessary for Section 6-11. The material under "A basic theorem" (after Problems 6-10D) should be included, except that the construction after the statement of Theorem 6-1 can be omitted.

6-11. Either or both of Examples 3 and 4 can be omitted.

Appendix. These proofs can be omitted as noted above in the comment on Section 6-3.

Suggested program. What program you choose, and how much you omit, will depend on four things: (i) your own preferences as to emphasis; (ii) how much time you have available; (iii) the interest and previous preparation of your pupils; and (iv) the examination syllabus for which your pupils are preparing. The minimum program noted above (Sections 6-1 through 6-5 with some omissions) provides the pupil with much of value for later work in mathematics. At the present writing (1965) an examination syllabus has not been prepared for Secondary 4. When such a syllabus is drawn up, we would expect it to cover material in Sections 6-7, 6-8, and 6-9 as well, and possibly some material from Section 6-10. The following comments on time may help you in planning a program for your class.

Planning your time. We assume five class meetings per week. We estimate that the sections will take the following amounts of time. (You may find that the material goes somewhat faster than this.)

Sections 6-1 and 6-2 together will take about a week. There is not much mathematical content, but the exercises need to be covered with some care. These sections will take less than a week if the ruler and compasses part of Section 6-2 is omitted.

Sections 6-3 and 6-4 together will take about a week. Again, the total amount of material is not great, but there are new ideas and new terminology that need to be treated with care. These sections will take less that a week if the second illustration in Section 6-4 is omitted.

Sections 6-5 and 6-6 together will take about a week, and less if the last part of Section 6-6 is omitted.

(Thus the minimum program mentioned above, of Sections 6-1 through 6-5 with omissions, can be covered in two to two and a half weeks.)

Sections 6-7, 6-8, and 6-9 together can be covered in two weeks. With omissions, they can be covered in a week and a half.

Section 6-10 will take about a week.

(Thus a minimum program leading through Section 6-10 would take from four to five weeks.)

Section 6-11 will take a little less than a week.

(Thus a maximum program through Section 6-11 and including all material could take as much as seven weeks.)

Special materials needed. In addition to the usual materials for geometry, each pupil should be supplied with about twenty sheets of tracing paper. Any paper thin enough to make tracings from a pencilled drawing will serve.

6-1 FIGURES WITH THE SAME SIZE AND SHAPE.

General discussion. Read this entire section with care. You will need two class meetings to cover this section. In the first meeting you can discuss the section with the class, go over the examples in the text, and carry out the class activity. In the second meeting you can go over the problems with the class. These problems, while easy, are of central importance for giving the pupil a full understanding. You can also invent additional problems as we point out below.

The main idea of this section is simple. In previous work two triangles were defined to be "congruent" if they had corresponding sides congruent, and corresponding angles congruent. The pupil proved as a theorem (SSS) that if corresponding sides are congruent, then corresponding angles are congruent also. The notion of congruence was important because it made precise the idea of two triangles having "the same size and shape". Can we state the idea of "same size and shape" in a more general way that will apply to other figures besides triangles? Section 6-1 answers this question in a simple and obvious way. Two figures have the same size and shape if a tracing of one figure can be made to coincide with the other. This answer is not a purely mathematical one, but it leads us, in later sections, to a variety of interesting mathematical ideas. Sections 6-3 and 6-6 will show how the answer can be made purely mathematical. If pupils ask you for a more precise answer at this stage, probably the best reply is to tell them to imagine a "perfect" tracing, and then to say to them that two figures have the "same size and shape" if a perfect tracing of one figure can be made perfectly to coincide with the other.

As the text makes clear, there is one aspect of our answer that the pupil may find unnatural at first. This is the fact that we allow our tracing to be turned over. (And thus we say that in Figure 7, for example, A and B have the same size and shape.) The pupil may not feel that this agrees with his own private idea of "same shape". In any case, he must understand that this is the way the words "same shape" are to be used in the present chapter.

Warning. There are two places where your pupils may be confused by the words we use. (1) We use the word "figure" sometimes to mean a single

geometrical figure such as a triangle, a square, or a circle, and sometimes we use the word "figure" to mean an entire drawing in the text such as "Figure 7" or "Figure 11." Once the pupil is aware that "figure" is used in these two distinct ways, he will have no trouble. Which of the two meanings is intended will always be clear from surrounding statements. Thus, in the title of Section 6-2 ("Moving a figure in the plane"), it is the first of the above meanings that is intended. (2) In the paragraph following Figure 7, we speak of "turning over" the tracing paper. By this we mean turning the paper entirely over so that the side of the paper which was previously underneath and facing the figure being traced is now on top and facing you. (Some pupils may make the mistake at first of thinking that "turning over" means sliding the paper through a rotation of 180° while always keeping the same side of the paper in contact with the figure being traced.)

Answers to questions raised in text.

The triangles in Figure 1 do have the same size and shape. So do the triangles in Figure 2, the circles in Figure 3, and the circles in Figure 4. The ruler and compasses construction to find the centre of a circle is to draw two non-parallel chords and find their perpendicular bisectors. The intersection of these bisectors is the centre of the circle. You may wish to omit the exercise with Figure 4 (since it will require drawing on the pages of the text) and to put a blackboard exercise in its place.

The two figures in Figure 5 have the same size and shape, as we can see by sliding a tracing of one over the other. In the Class Activity, the figures in (a) and (c) have the same size and shape, but the figures in (b) do not.

Answers to
PROBLEMS 6-1

Student Text Pages 183-185

1. (a) yes; (b) yes; (c) yes (in this case, the tracing paper must be turned over.)

2. (a) 1; (b) 2; (c) 6; (d) 4; (e) 4.

Comment. The purpose of these problems is not to teach a special technique of using tracing paper, but rather to prepare the student for the mathematical idea of a *rigid motion*. Once the student sees how the tracing paper is used, he may find that he is able to answer certain questions (like Problem 2 above) by imagining the correct motion of the tracing paper without actually carrying it out physically. This should be encouraged. Problems similar to Problem 2 can be easily made up by you for use in class. For example, the following figures can be used.

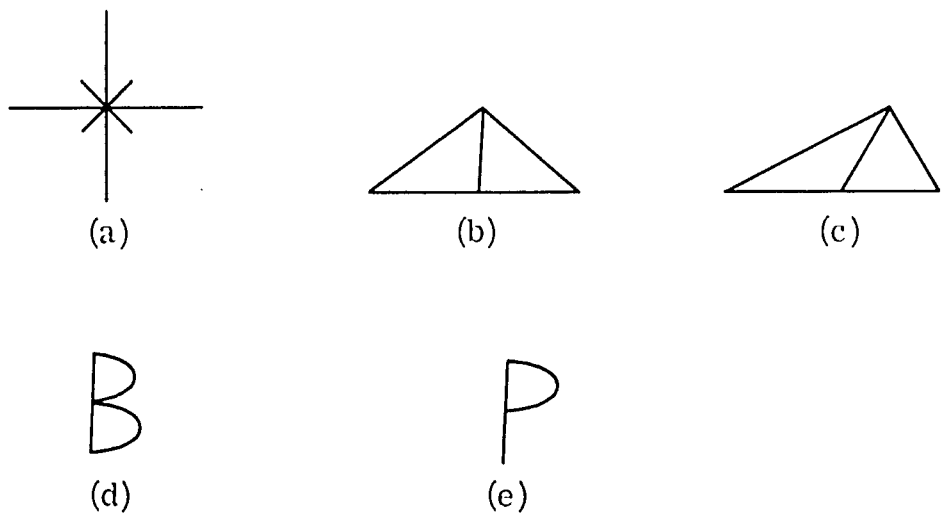


Fig. G-1.

Here the answers are: (a) 8; (b) 2; (c) 1; (d) 2; (e) 1. (Letters of the alphabet make a good source of problems.)

6-2 MOVING A FIGURE IN THE PLANE.

General discussion. The material in Section 6-2 falls into three parts. The first part (up to Problems 6-2A) shows how we can use tracing paper to "move" a figure from one place to another in the plane. The second part (up to Problems 6-2B) shows that, along with the figure being moved, any such "movement" by tracing paper carries every other point in the plane to a new position. The third part (up to Problems 6-2C) shows that ruler and compasses can often be used without tracing paper to get the same result, in moving a figure or a point, that we would get with tracing paper.

The first two parts will take one class meeting each. The third part, if you decide to use it, will take one or possibly two class meetings. As with Section 6-2, the problems are especially important for the pupil to get a full understanding and you should spend time in class discussing them. A good class activity in the second part is to have your pupils use tracing paper to check the statements made about Figures 16, 17, and 18 in the text.

As we remarked in the Introduction, the third part can be omitted. If you omit the third part, Problems 4 and 5 (and possibly 6) in Problems 6-2C should still be given to your pupils.

Warning. Two difficulties can arise in connection with the basic steps described at the beginning of Section 6-2. (1) How is the student to make a copy of Figure 10? One answer is to have the student make a tracing of Figure 10 on tracing paper. If he does this, however, he may become confused between his two pieces of tracing paper in carrying out the steps described in the text. A better way is to have him make the copy on a sheet of paper which is somewhat thicker than tracing paper, but which is still thin enough to trace onto from the Figure in the book. A third and still better way is to have a copy of Figure 10 already prepared for his use. (These remarks also apply to later Figures and Problems.) (2) How can we be sure that the student is following the basic steps correctly? The main thing here is to have him see

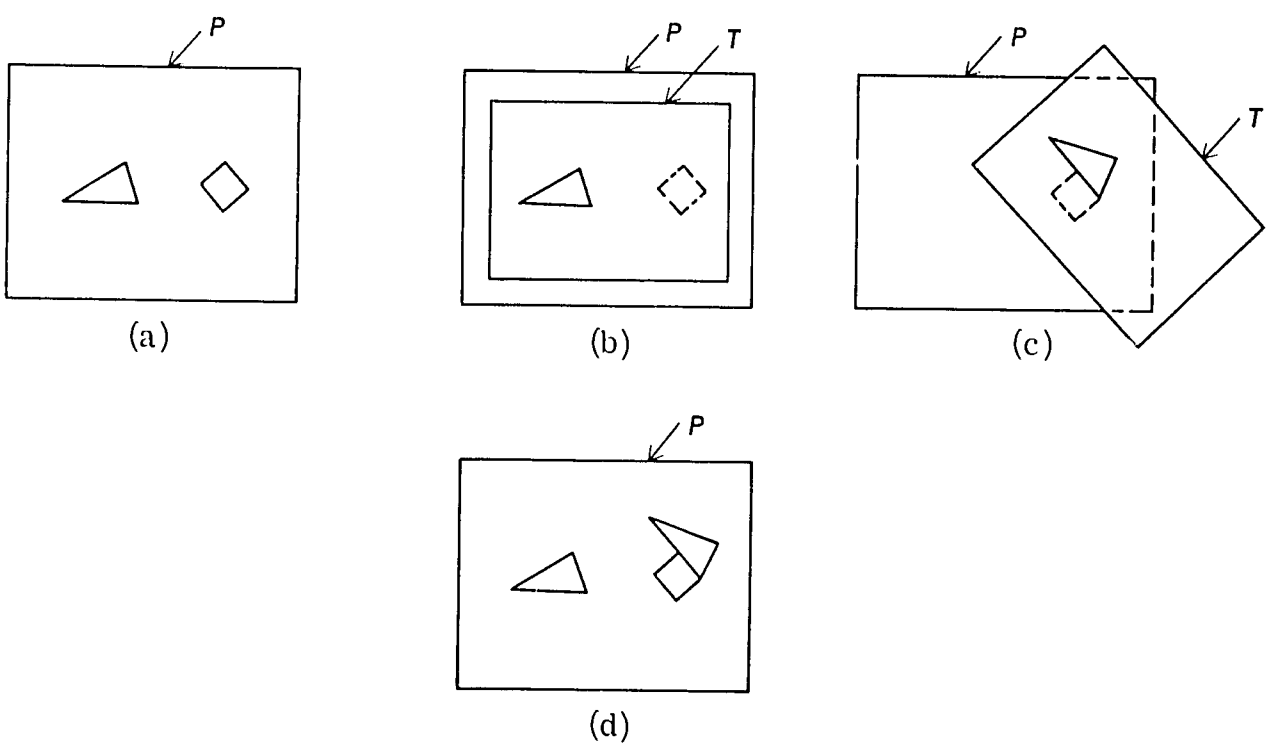


Fig. G-2.

that P is the piece of paper with the original copy of Figure 10 and T is the piece of tracing paper which he is using to move the triangle to a new position on P. The successive steps can be pictured as in Figure G-2. In (a) we have the copy P of Figure 10. In (b) we have placed the tracing paper on top and made a tracing of the triangle. In (c) we have moved the tracing paper. (d) gives the result on P after we have pricked through the vertices of the triangle from T onto P and then drawn its sides on P.

Be sure to emphasize to your pupils that for certain motions (Problem 3 below, for example) they will have to turn paper T over before pricking through onto P.

Answers to
PROBLEMS 6-2A

Student Text Pages 186-187

1. Figure 44 (in the Student Text) gives one solution.
2. Figure 45 (in the Student Text) gives one solution.
3. Figure 46 (in the Student Text) gives the solution.
4. The four answers to Problem 1 are indicated in Figure G-3. Note that for two of these answers the tracing paper must be turned over. The four answers to Problem 2 are indicated in Figure G-4.

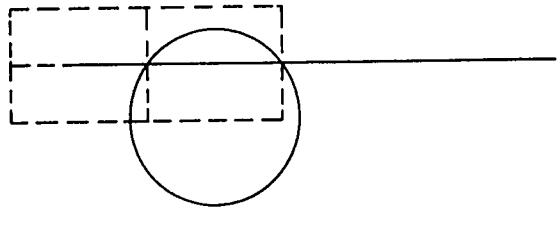


Fig. G-3.

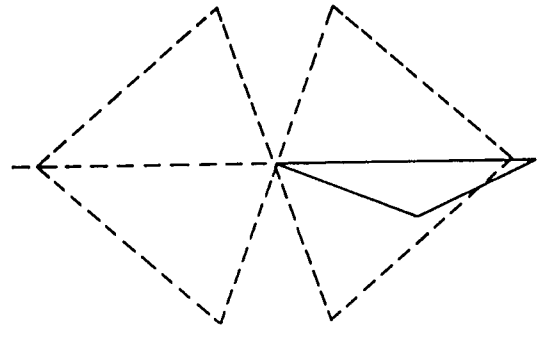


Fig. G-4.

1. In this movement every point moves the same distance (such a movement will be called a *translation* in Section 6-5), so that there is no fixed point. Figure G-5 shows the new positions of P_1, P_2, and P_3 as P_1', P_2' and P_3'.

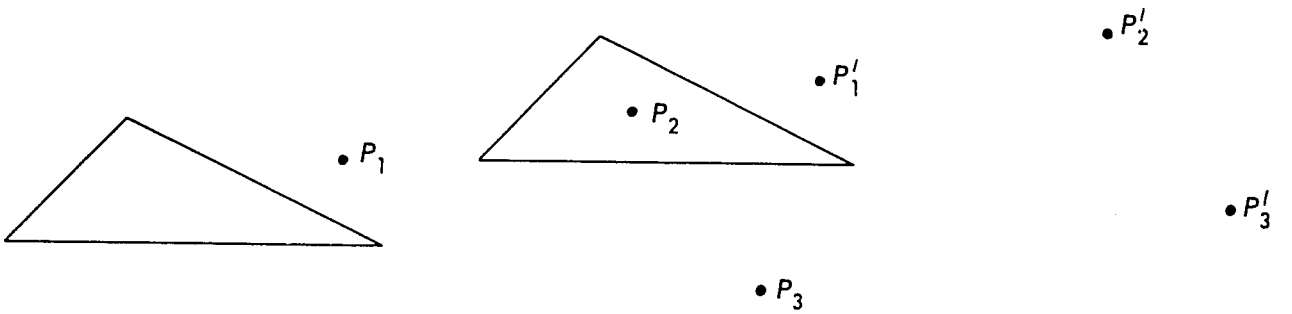


Fig. G-5.

2. Points P_1, P_2 and P_3 are moved as shown in Figure G-6. The circle D is invariant under this movement as can be checked with tracing paper. The centre of the circle is a fixed point of the movement. (This movement is an example of what will be called, in Section 6-5, a *rotation*.)

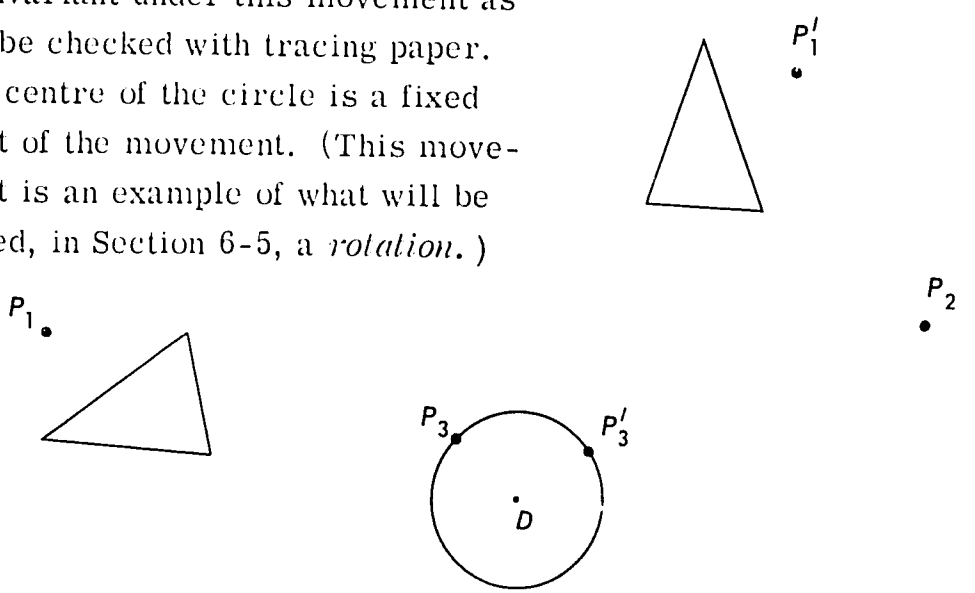


Fig. G-6.

3. This carries L_1 and L_2 onto themselves, and L_3 to the position L_3' in Figure G-7. (This movement is an example of what will be called, in Section 6-5, a *reflection.*)

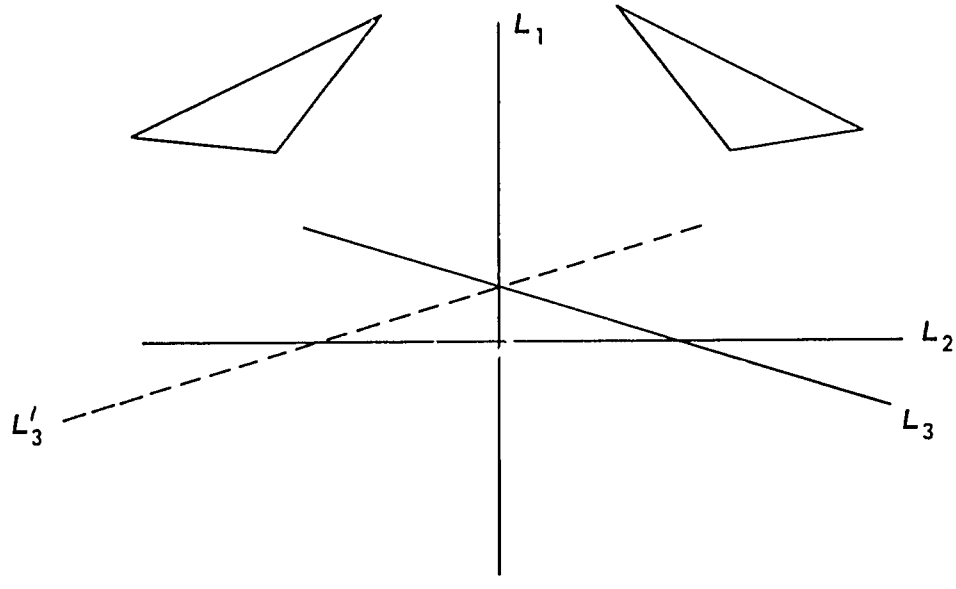


Fig. G-7.

4. (a) The fixed points are the points of line L_1.
   (b) L_1 is invariant and any line perpendicular to L_1 (such as L_2) is invariant. (Note, however, that the only fixed point on L_2 is its point of intersection with L_1.)
5. There are six motions, and the new positions of P are as in Figure G-8.

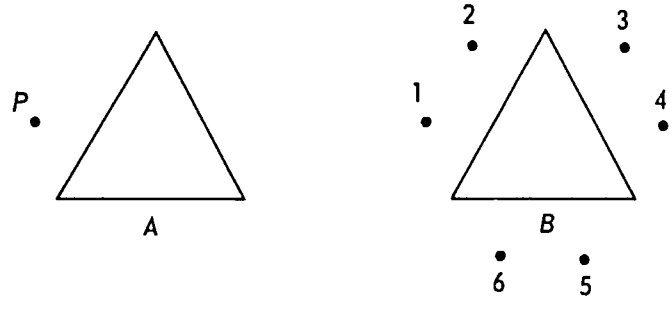


Fig. G-8.

1. Find B' on CD so that $CB' = CB$. Then draw an arc of radius AB about B' and an arc of radius AC about C. Take an intersection of these arcs as A'.

2. Find B' between D and F so that $B'F = BA$. Then take an arc of radius AC about F and an arc of radius BC about B'. Their intersection inside the given circle yields the third vertex C'.

3. Construct the desired rectangle on L by constructing a perpendicular to L at E, laying off $A'E = AD$ on L and $C'E = CD$ on the perpendicular, and then getting B' as the intersection of two appropriate arcs with centers at A' and C'.

4. Let D' be the new position of D. To locate D', we take an arc of radius AD about A' and an arc of radius CD about C'. These arcs will intersect at two points. D' will be the intersection which lies on the opposite side of $\overline{A'C'}$ from B' (since D lies on the opposite side of $\overline{AC}$ from B.) The new positions of E and F can be found similarly. Note that we could have equally well used arcs around A' and B' or around C' and B' to locate D'.

5. (a) The points on a circle of radius AP about A'.
   (b) The two points given by the two intersections of an arc of radius AP about A' with an arc of radius BP about B'.
   (c) The single point given when we take the two intersections as in (b) and then choose that intersection which lies on the opposite side of $\overline{A'B'}$ from C'.

6. There are a variety of acceptable answers. We give two examples here.

   Example 1. "If we are given three points not all on the same straight line, and if we know where a movement carries each of those three points, then we can find where the movement carries any fourth point."

   Example 2. "Let A and A' be any two points, then there are many different movements which carry A to A'. Let A, B, A', and B' be four points such that A is different from B and $AB = A'B'$.

Then there are exactly two movements which carry A to A' and B to B'. Let A, B, and C be any three points not on a straight line, and let A', B', and C' be any three points such that $AB = A'B'$, $BC = B'C'$, and $AC = A'C'$. Then there is exactly one movement which carries A to A', B to B', and C to C'."

6-3 RIGID MOTIONS.

General discussion. This section has two main parts. In the first part, we see that any "movement" of tracing paper gives us a *function* or *mapping* from the plane onto itself. (For the purpose of seeing this, we imagine that our tracing paper is infinitely large and covers the whole plane.) Any such function, given by a tracing paper movement, is called a *rigid motion.* We here use the word "function" in exactly the same way that the word "function" is used and discussed in *Secondary Four* algebra. Ask your pupils to go back and revise their work on functions in algebra. The *domain* of a rigid motion is the whole plane, and the *range* of a rigid motion is the whole plane. In the first part of Section 6-3, we give notations and terms that will be used in the study of rigid motions. The pupil will have met some of these already in his work on algebra. If T is a rigid motion and P is a point, and if T carries P to Q, then we use the functional notation of algebra and write "$T(P) = Q$." We call Q the *image* of P under the rigid motion T. P is sometimes called a "pre-image" or "inverse image" of Q under the rigid motion T. The result of not moving the tracing paper at all is also taken to be a rigid motion and is called the "identity motion." It is like the identity function in algebra.

The second part of the section begins with the list of six properties that all rigid motions have. It presents these properties to the pupil and introduces several additional new words. The most important property is the first property, which states that given any rigid motion T and given any two points P and Q, the distance between $T(P)$ and $T(Q)$ is the same as the distance between P and Q. This property of rigid motions is called the *isometric* property. (The word "isometric" comes from the Greek: *iso* + *metron* where *iso* = same and *metron* = measure.) Later, in the part of

Section 6-6 called *Note on Isometric Mappings,* we shall show a fundamental fact about this property: not only does every tracing paper motion have this property, but, also, every function from the plane into the plane with this property can be obtained from an appropriate tracing paper motion. We state this fundamental fact for the pupil in Definition 6-1.

The Student Text has been written so that your pupils can approach the subject of rigid motions in either of two quite different ways. (1) The pupil can continue to think of a rigid motion as something got by moving tracing paper. In the second part of Section 6-3, he then simply observes, as a fact about tracing paper, that every rigid motion must have each of the six properties listed. For him, the definition of *rigid motion* is like the definition of *polynomial function* in algebra--namely, it is a function which can be actually carried out in a certain way (by a certain movement of tracing paper, which is like, in algebra, calculating values of a certain polynomial.) This approach to rigid motions has the advantage that it is easier for the pupil to follow and understand. You will probably wish to have most of your pupils take this approach. The Text is written so that pupils can follow this approach to the end of the Chapter. This approach has the disadvantage, however, that it depends on an essentially non-mathematical idea: the idea of tracing paper. (2) The pupil can *define* rigid motions to be those functions from the plane into the plane which happen to be isometric. This makes the concept of rigid motion purely mathematical, but at the expense of requiring that the pupil accept and use a more abstract idea. Probably, most of your pupils will not be ready for this approach. The Text is written, however, so that pupils who wish to, can also follow this approach to the end of the chapter. If this approach is taken, then the remaining five properties of rigid motions become mathematical facts to be proved. Geometric proofs of these facts are given in the Appendix at the end of the chapter. If your pupils follow the first approach, you may still wish to point out to them that the second approach exists, even though you are not using it. (Tell them that Definition 6-1, using the isometric property, is just a way of making the idea of tracing paper mathematically precise.) As the Text remarks, the more abstract approach is necessary when we study rigid motions in three-dimensional (solid) geometry.

Although the ideas are simple (especially if you use the first approach mentioned above), you will want to be sure that your students can use these ideas carefully and correctly. This means that you should spend three or

four days on Section 6-3: one day for each of the two parts of the section, and one or two days to go over problems and examples.

Warning. The word "movement" may be confusing to some of your pupils. They may think that "rigid motion" means a particular *path* of movement. This is not true. A rigid motion is determined by the initial and final positions of the tracing paper. If the tracing paper moves through two different paths to get to the same final position, then it is the same rigid motion, whichever path is used.

Comments on motion geometry. In the study of plane geometry, other functions besides the rigid motions are sometimes used, and additional terminology is used. A function with properties (3) and (4) (and hence also (2) and (6), see Problem 12) is called a *transformation*, or, sometimes, an *affine transformation*. A function with these properties and property (5) as well is sometimes called a *homothetic transformation*, or a *similarity transformation*, or, simply, a *motion*. Functions with property (1) as well, that is to say, rigid motions, are sometimes also called *Euclidean transformations*. In recent years, these various functions have been used more and more in presenting geometry to secondary pupils. Problem 7 gives an example of a transformation which is not homothetic, and Problem 9 gives an example of a homothetic transformation which is not a rigid motion.

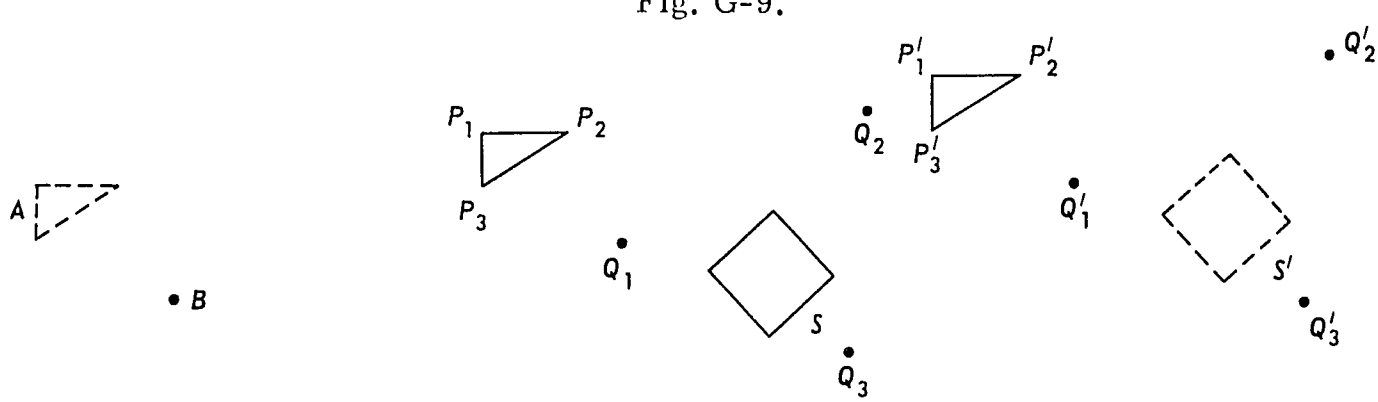
Answers to
PROBLEMS 6-3.

Student Text Pages 200-204

1. In Figure G-9, Q_1', Q_2' and Q_3' are the images of Q_1, Q_2 and Q_3 and S' is the image of S.

2. In Figure G-9, B has Q_1 as its image, and triangle A has $\Delta P_1P_2P_3$ as its image. (This rigid motion is an example of what will later be called a *translation*.)

Fig. G-9.



3. The image of $\Delta P_1'P_2'P_3'$ is $\Delta P_1P_2P_3$, hence $\Delta P_1'P_2'P_3'$ is, itself, the triangle which has $\Delta P_1P_2P_3$ as its image. (This rigid motion is an example of what will later be called a *reflection.*)

4. In Figure G-10, $\Delta P_1''P_2''P_3''$ is the image of $\Delta P_1'P_2'P_3'$, and triangle A has $\Delta P_1P_2P_3$ as its image. (This rigid motion is an example of what will, in Section 6-10, be called a *glide reflection.*)

5. (a) No. A shift of the tracing paper, such as in Problem 1 above, has no fixed point. Every point gets moved the same distance.
(b) Yes. In the identity motion, for example, every line is identical with its image. Are there examples other than the identity motion? Yes, the shift of Problem 1 is such a motion.
(c) No. A rotation of the tracing paper through 90° will, for example, make every line perpendicular to its own image. (See Problem 2 of Problems 6-2B.)
(d) No. See (c) above.
(e) Yes. In Problem 1, the line $\overleftrightarrow{P_1P_1'}$, for example, is its own image. (If students ask for a "proof" of this, tell them to wait until Section 6-7.)
(f) No. Assume that a motion maps some triangle onto itself. Either it maps all three vertices onto themselves (and we have the identity motion) or it maps some one vertex onto another and we have a triangle that must be isosceles or equilateral. In the former case, every point is a fixed point. In the latter case, the intersection of the angle bisectors must be a fixed point. (Since every angle bisector gets mapped onto an angle bisector, a point that lies on all three angle bisectors must lie on all three images of angle bisectors.)

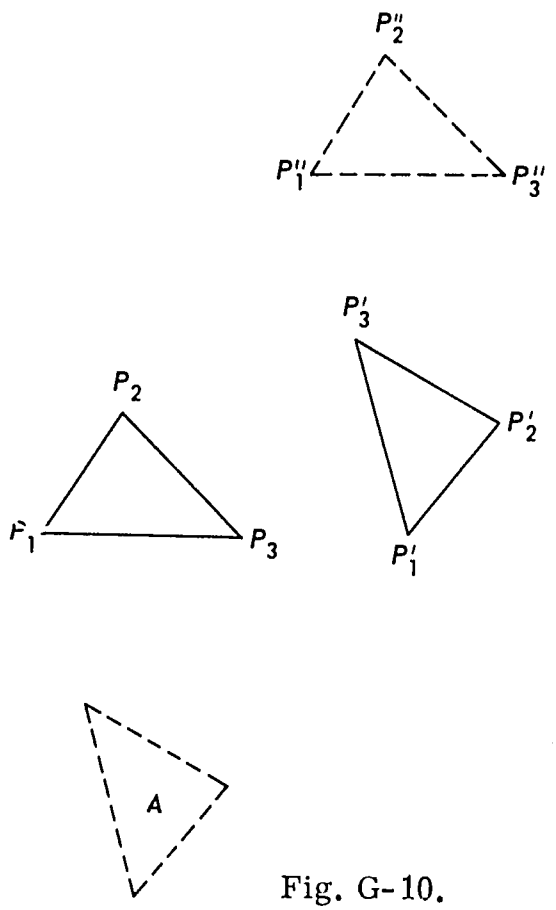


Fig. G-10.

Hence any motion which maps a triangle onto itself must have a fixed point.

6. We use the notation $\begin{pmatrix} ABC \\ DEF \end{pmatrix}$ to stand for that rigid motion which takes A to D, B to E, and C to F. Similarly we use $\begin{pmatrix} ABC \\ EFD \end{pmatrix}$ to stand for that rigid motion which takes A to E, B to F, and C to D. Similarly for the other four possibilities. We consider each of the six possibilities in turn.

(i) $\begin{pmatrix} ABC \\ DEF \end{pmatrix}$. This motion shifts every point the same distance to the right, and there are no fixed points.

(ii) $\begin{pmatrix} ABC \\ EFD \end{pmatrix}$. This motion is given by a rotation of the tracing paper about a fixed point P_1 in Figure G-11. It has P_1 as its one and only fixed point.

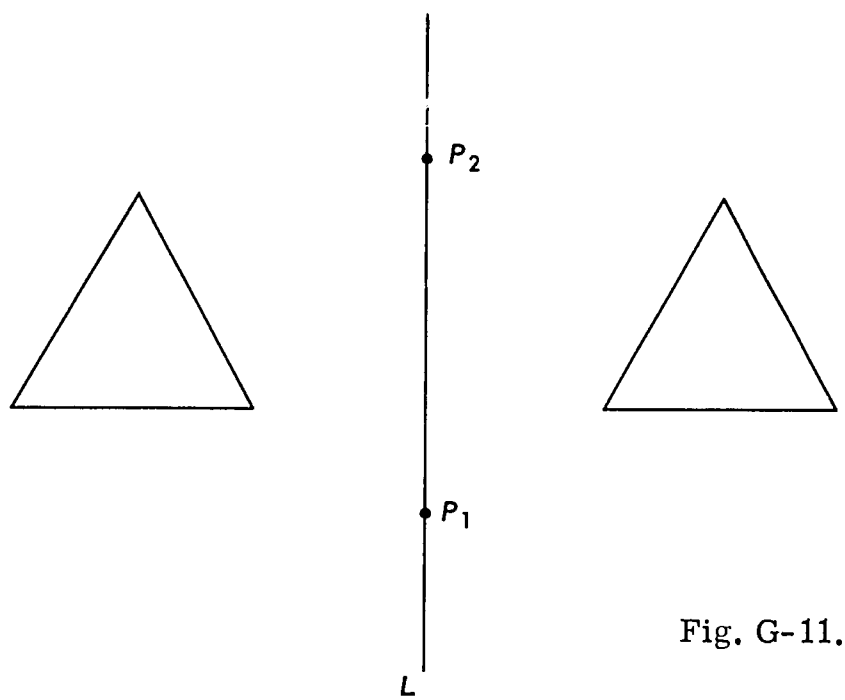


Fig. G-11.

(iii) $\begin{pmatrix} ABC \\ FDE \end{pmatrix}$. This motion is given by a rotation about point P_2. It has P_2 as its one and only fixed point.

(iv) $\begin{pmatrix} ABC \\ FED \end{pmatrix}$. This motion requires that the tracing paper be turned over and put back down in such a way that it has the line L as its set of fixed points.

(v) $\begin{pmatrix} ABC \\ EDF \end{pmatrix}$. This motion requires that every line parallel to $\overleftrightarrow{AB}$ be moved perpendicular to itself. Hence there are no fixed points.

(vi) $\begin{pmatrix} ABC \\ DFE \end{pmatrix}$. This motion requires that every line parallel to $\overleftrightarrow{BC}$ be moved perpendicular to itself. Hence there are no fixed points.

7. This problem is easier than the pupil may expect. Let P be any point not on L and let Q be the foot of the perpendicular from P to L. Then $P'Q' = 2(PQ)$. Since $PQ \neq O$, $PQ \neq P'Q'$. Hence the mapping T is not isometric. Hence it is not a rigid motion.

8. This construction amounts to turning the tracing paper over and placing it back down so that every point of L falls on its original position. The "Hint" suggests a more formal, geometrical proof. This proof is given with Figure 80 in Section 6-9 of the Student Text.

9. The proof that T is not a rigid motion is similar to that in Problem 7. Take P different from O. Then $OP \neq O$ and $O'P' = OP' = 2(OP)$.

To show that T is a transformation, we must show that it has properties (3) and (4).

Any point P is the image of the point halfway between P and O. This shows that (3) holds.

To show (4), we must show that the image of every line is a line. To do this, let L_1 be any given line. If L_1 contains O, then L_1 is immediately seen to be its own image. If L_1 does not contain O, let P and Q be any two distinct points on L_1. Let L_2 be the line through the points $T(P)$ and $T(Q)$. Let M be a line through O intersecting L_1 and L_2 at R and R'. Then, by a similar triangles argument, $OR = RR'$. Hence $R' = T(R)$. This shows that every point on L_1 is carried to a point on L_2. and that every point on L_2 comes from some point on L_1. Hence L_2 is the image of L_1. See Figure G-12. To show that T has property (5), it is enough to show that the image of any triangle A is a triangle similar to A. To show this, it is enough (because of a basic SSS theorem on similar triangles) to show that the image of any segment is a segment that is twice as long. To show this latter fact, consider the segment $\overline{PQ}$ and its image in Figure G-12.

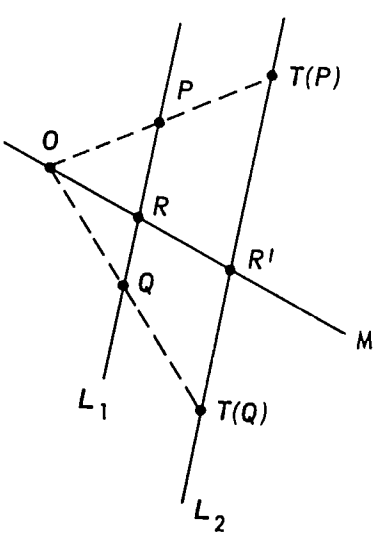


Fig. G-12.

10. The figure on the right shows that property (5) fails, since $P'\widehat{Q'}R'$ is contained in, and hence smaller than, $P\widehat{Q}R$. (Here Q' is the same point as Q.)

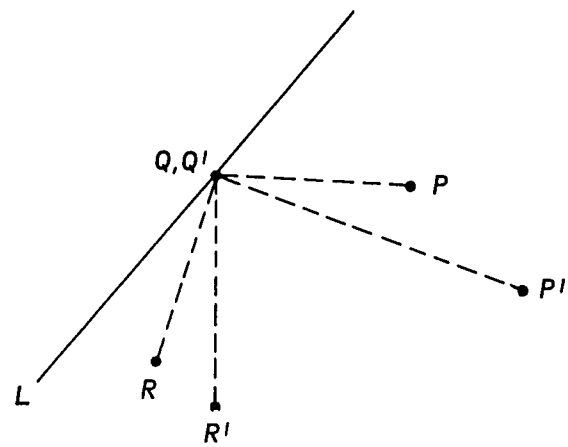


Fig. G-13.

11. Any point P is the image of the point halfway between P and L. This shows that property (3) holds. To show (4), we must show that the image of every line is a line. The proof is similar to that for Problem 9. Let L_1 be any given line. If L_1 is identical with L, then clearly the image of L_1 is L_1 itself. If L_1 is parallel to L, then clearly the image of L_1 is a line parallel to L and twice as far from L. If L_1 intersects L at some point Q, let P be any point on L_1 different from Q. Find $T(P)$ and let L_2 be the line determined by Q and $T(P)$. Let M be any line perpendicular to L, and let R and R' be its intersections with L_1 and L_2. Let P_1 be the foot of the perpendicular from P to L and let R_1 be the intersection of M with L. By the construction, we have P_1P = distance from P to $T(P)$. Hence by similar triangles, $R_1R = RR'$. Hence $R' = T(R)$. This shows that every point on L_1 is carried to a point on L_2, and that every point on L_2 comes from some point on L_1. See Figure G-14.

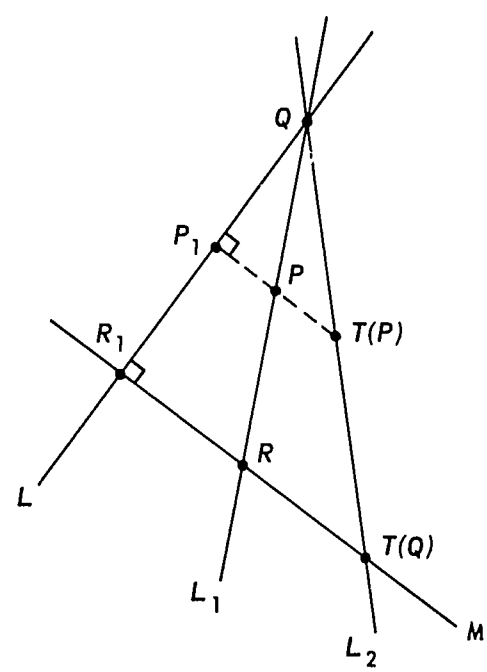


Fig. G-14.

12. We first show (2) and then deduce (6) from (2). Let T be any transformation. (That is to say, T has properties (3) and (4).) We show (2) by assuming that (2) is false and getting, from this, a contradiction.

If (2) is false, there are distinct points P_1 and P_2, and a point Q, such that $T(P_1) = T(P_2) = Q$. Let L be $\overleftrightarrow{P_1P_2}$. Then by (4), the image of L is a straight line M. Let Q' be a point not on M. Then, by (3), Q' has a pre-image under T. Let P' be a pre-image of Q'. Then P' cannot lie on L. Let L_1 be $\overleftrightarrow{P_1P'}$ and let L_2 be $\overleftrightarrow{P_2P'}$. Let M' be $\overleftrightarrow{QQ'}$. Then M' must be the image of both L_1 and L_2. Let M'' be any line parallel to M'. Take distinct points Q_1 and Q_2 on M''. Let P_3 be a pre-image of Q_1 and P_4 be a pre-image of Q_2. P_3 and P_4 must be distinct, since Q_1 and Q_2 are distinct. Let L'' be $\overleftrightarrow{P_3P_4}$. Then L'' must intersect L_1 or L_2. Also L'' has M'' as its image. As we saw on the preceeding page, L_1 and L_2 have M' as their common image. Hence the point of intersection between L'' and L_1 or L_2 must map to a point of intersection between M' and M''. But M' and M'' are parallel. This is a contradiction. We have hence shown (2). See Figure G-15.

It is easy now to go and show (6). See the last proof in the Appendix.

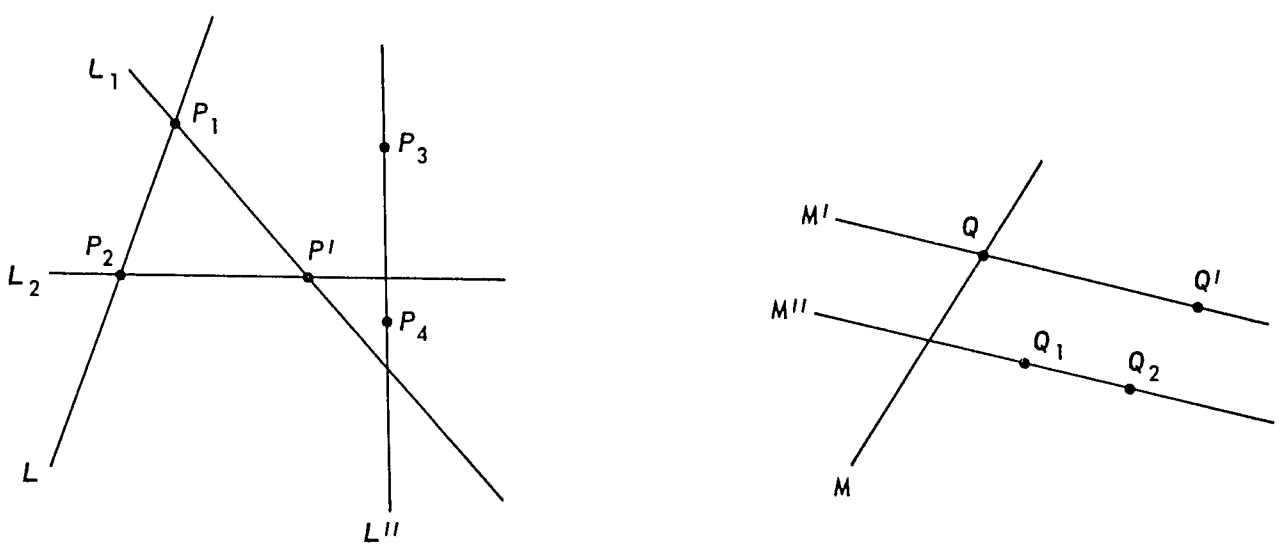


Fig. G-15.

Note. Even though the proof uses only elementary ideas, Problem 12 above is perhaps the most difficult of all the problems in Chapter 6. If any of your pupils get a correct solution to this problem, it is likely that they have mathematical abilities of a high order, and that they will be able to pursue a university course in advanced mathematics with distinction.

A somewhat easier problem that you can also give to your students is the following: show that if a function from the plane into the plane has properties (2) and (4), then it must have property (3). An outline of the solution is as follows. Let Q be any point. We wish to show that Q is the image of some point. Take any two lines L_1 and L_2. Let M_1 and M_2 be the images of L_1 and L_2. Take any line through Q that intersects M_1 and M_2, and let Q_1 and Q_2 be the points of intersection. Let P_1 and P_2 be the pre-images of Q_1 and Q_2. Then $\overleftrightarrow{P_1P_2}$ must have $\overleftrightarrow{Q_1Q_2}$ as its image, and hence the point Q since it lies on $\overleftrightarrow{Q_1Q_2}$ must have a pre-image.

6-4 USING MOTIONS TO SOLVE PROBLEMS

General discussion. This section can be treated rather briefly. It is intended to show the pupil some of the ways in which the idea of rigid motion can be useful. It is separate from the work in other sections, and later sections do not depend on it. You should allow one or two class meetings for it (only one meeting if the second example is treated lightly or omitted.) Example 1 is a practical problem where the use of a rigid motion gives a quick and neat answer. Example 2 is a geometrical construction which appears difficult until we find that, by thinking about the correct motions, we can organize our attack on it in a rather simple way. The final ruler and compasses solution which we get does not speak of motions, even though we use motions to discover it.

The main point of this section is not the particular examples given, and it is not to give special ways of solving problems. The main point is that the general idea of motion can often be surprisingly helpful in both practical and mathematical thinking.

Answer to Challenge Problem. Let A be the centre of C_1. Construct a line parallel to L through A. (See the construction for Figure 28 in Section 6-2.) Find a point B on this line such that $AB = PQ$. Draw a circle with centre B and radius same as C_1. Let P_1 and P_2 be the points of intersection of this circle with C_2. Draw lines through P_1 and P_2 parallel to L. Let Q_1 and Q_2 be the intersections of these lines with C_1 as shown in the figure below. Then $\overline{P_1Q_1}$ and $\overline{P_2Q_2}$ are the two possible segments.

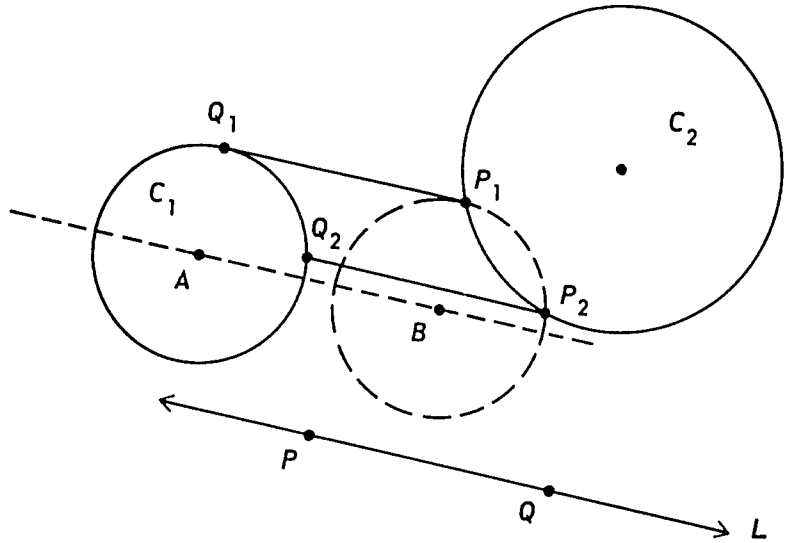


Fig. G-16.

6-5 SPECIAL KINDS OF RIGID MOTION

General discussion. In this section we give an informal description of various kinds of rigid motion that will be considered further in later sections. All rigid motions can be divided into two special and separate kinds: *direct* and *reversing*. The direct rigid motions include two special kinds: *translations* and *rotations*. We shall see in Section 6 10 (Theorem 6-1) that every direct rigid motion must either be a translation or a rotation. (The identity motion is usually counted as both a translation and a rotation.) The reversing rigid motions include two special kinds: *reflections* and *glide reflections*. In Section 6-10 (Theorem 6-2) we see that every reversing rigid motion must either be a reflection or a glide reflection. The present section gives the pupil a first introduction to the ideas: direct; reversing; translation; rotation; and reflection. Translations are considered in more detail in Section 6-7, rotations in Section 6-8, and reflections in Section 6-9. Section 6-10 gives further facts about these ideas, including the two theorems mentioned above. Glide reflections are not discussed in this section or in later sections except for a brief treatment in Section 6-10. The motion in Figure 47 is a glide reflection.

Although this section is short, and the ideas are easy, it is important that the student have a clear understanding of the new words given to him. You should allow two or possibly even three class meetings to discuss this material with your class and to go over the problems.

Warning. The new ideas introduced in this section are defined in terms of tracing paper movement. More purely mathematical definitions for translation,

rotation, and reflection will be given in Sections 6-7, 6-8, and 6-9. The use of "clockwise order" and "counterclockwise order" helps to put the definitions of *direct* and *reversing* in somewhat more mathematical form, but the ideas "clockwise" and "counterclockwise" are still not perfectly mathematical. Perfectly mathematical definitions for *direct* and *reversing* can be given, but they require a long treatment and rather complex proofs to show that they are satisfactory. Since the underlying idea is very simple when thought of in terms of tracing paper, we shall, in Chapter 6, omit the purely mathematical definitions. If a pupil asks for a more mathematical definition, tell him that he can get definitions using "clockwise" and "counterclockwise" by taking the two statements before Problems 6-5 which are called *facts,* and thinking of them as *definitions.* If he does this, he is then left with the problem of proving, as a theorem, that every rigid motion must be either direct or reversing. The proof is not hard, but it is rather long and we do not give it here. It amounts to showing that if the vertices of some one triangle are carried from clockwise order to clockwise order by a rigid motion, then the vertices of any other triangle are carried from clockwise order to clockwise order by the same rigid motion.

Note. Direct motions are sometimes called "orientation-preserving," and reversing motions are sometimes called "orientation-reversing." Rigid motions in three dimensions can also be divided into direct and reversing, although the informal definitions are not quite as simple as for rigid motions in the plane. The book, *Through the Looking Glass* , by Lewis Carroll tells what it would be like to live in a world after it had been subjected to a reversing rigid motion.

Answers to
PROBLEMS 6-5

Student Text Pages 210-211

1. In 1 of Problems 6-2B, the motion is direct and is a translation.
   In 2 of Problems 6-2B, the motion is direct and is a rotation about the centre of the circle D.
   In 3 of Problems 6-2B, the motion is reversing and is a reflection about the line L_1.
   For 5 of Problems 6-2B, we use Figure 38 and the notation given in the Guide for the answer to Problem 6 of Problems 6-3.

$\begin{pmatrix} ABC \\ DEF \end{pmatrix}$ is direct and a translation. $\begin{pmatrix} ABC \\ EFD \end{pmatrix}$ is direct and a rotation about point P_1 in Figure G-11. $\begin{pmatrix} ABC \\ FDE \end{pmatrix}$ is direct and a rotation about P_2 in Figure G-11. $\begin{pmatrix} ABC \\ FED \end{pmatrix}$ is reversing and a reflection about line L in Figure G-11. $\begin{pmatrix} ABC \\ EDF \end{pmatrix}$ is reversing but not a reflection. $\begin{pmatrix} ABC \\ DFE \end{pmatrix}$ is reversing but not a reflection. (Both $\begin{pmatrix} ABC \\ EDF \end{pmatrix}$ and $\begin{pmatrix} ABC \\ DFE \end{pmatrix}$ are glide reflections.)

2. In Figure G-17 below, $Q_1 = T(Q)$ if T is direct, and $Q_2 = T(Q)$ if T is reversing. Note, as a general principle, that if we are given the images of two points, then there are exactly two motions possible, one of them direct and one of them reversing. See Problems 5 and 6 in Problems 6-2C.

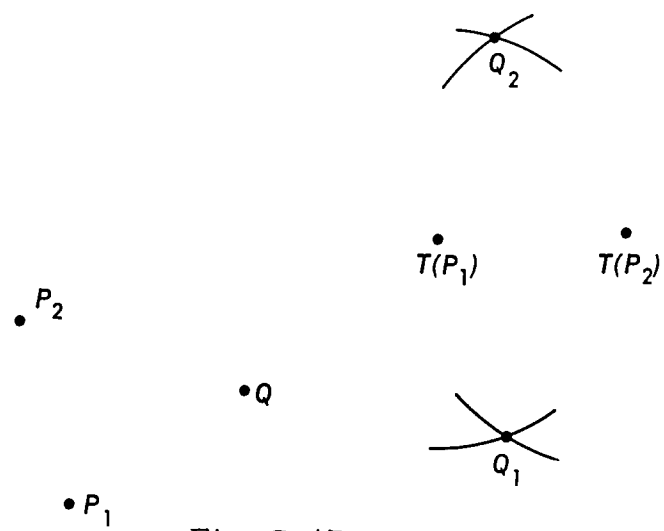


Fig. G-17.

In the figure above, the points Q_1 and Q_2 are found by drawing a circle about $T(P_1)$ with radius P_1Q, and a circle about $T(P_2)$ with radius P_2Q. Q_1 and Q_2 are the two intersections obtained. Q_1 is selected so that $T(P_1)$, $T(P_2)$, Q_1 occur in clockwise order (since P_1, P_2, Q occur in clockwise order.) Q_2 is selected so that $T(P_1)$, $T(P_2)$, Q_2 occur in counterclockwise order.

3. (a) Yes. (b) Yes. See the solution to Problem 2 above.

6-6 CONGRUENT FIGURES

General discussion. You should allow one or possibly two class meetings for the material in Section 6-6. There are no problems to be solved, and the

text presents ideas that will already be fairly obvious to some of the pupils. The section is in two parts. The first part (up to the *Note on Isometric Mappings*) takes the ideas of Section 6-1 and restates them using the mathematical words of Section 6-3. The first part begins by noting that a triangle must be congruent to its image under any rigid motion (by SSS), and then showing that given any two congruent triangles, there is a rigid motion which makes one triangle the image of the other. Thus we see that two triangles are congruent if and only if there is a rigid motion which makes one triangle the image of the other. Finally, we use the idea of rigid motion to make a *general definition* of congruence for any pair of figures in the plane: we say that two figures are *congruent* if there is a rigid motion which makes one figure the image of the other. Because of our result above about triangles, we know that our now general definition of congruence agrees with our old definition for the special case of triangles.

In previous sections, we have taken "rigid motion" to mean "mapping given by a movement of tracing paper." In Section 6-3, we remarked that the idea of *isometric mapping* puts the idea of rigid motion in purely mathematical form. It was clear in Section 6-3 that every tracing paper movement gives an isometric mapping. Now, in the second part of Section 6-6, we complete the justification of our remarks in Section 6-3 by showing that for every isometric mapping there is a movement of tracing paper that gives that mapping. This is not a purely mathematical result, since it talks about tracing paper. Even though the result is not purely mathematical, a convincing argument (or "justification") for it can be given: and this is the argument given here in Section 6-6.

The argument refers to a construction in Section 6-2. This is the construction for Figures 29 and 30 in the Text. It begins with the words "Let ABC be a triangle." If you omitted this construction when you covered Section 6-2, and if you are not omitting the second part of Section 6-6, then you should go back and do this construction with your pupils now.

Note. The result about triangles in the first part of Section 6-6 uses the tracing paper idea of rigid motion. To get a purely mathematical proof of this result (using "rigid motion" to mean "isometric mapping"), proceed as follows. (1) Show that a triangle must be congruent to its image under any isometric mapping. (The proof is immediate by SSS exactly as before). (2) Given two congruent triangles, use the construction from Section 6-2 (mentioned above) to define a mapping which you can show to be isometric.

In comment (b), just before the *Note on Isometric Mappings*, the words

"original study of congruent triangles" refer not to earlier portions of this chapter, but rather to work in a previous year where the pupil was first introduced to the idea of congruent triangles.

6-7 TRANSLATIONS.

General discussion. Section 6-7 on translations, Section 6-8 on rotations, and Section 6-9 on reflections each follow the same general outline. First, a particular kind of construction is presented. (In Section 6-7, this is the construction for "translating" a point P by a given vector $\mathbf{U}$. In Section 6-8, this is the construction for "rotating" a point P through a given angle β about a given point O. In Section 6-9 this is the construction for "reflecting" a point P in a given line L.) A construction of this kind (with a fixed $\mathbf{U}$ in the case of 6-7, a fixed β and O in the case of 6-8, a fixed L in the case of 6-9) gives a mapping from points of the plane to points of the plane. Second, a proof is given that this mapping must be isometric and hence a rigid motion. (As we mentioned in the Introduction, the proof in this second part can be omitted or treated lightly.) Third, this kind of mapping is given a name (*translation* in 6-7, *rotation* in 6-8, *reflection* in 6-9). Fourth, various facts about mappings of this kind are given. These include some geometrical facts (like, for example, the fact that every line gets moved onto itself or else parallel to itself by a translation), some facts about coordinate axes, and, in the case of rotations and reflections some facts about symmetry.

In case pupils have already done some coordinate geometry, you will find that the parts on coordinate axes can be treated rather briefly. Otherwise (unless you omit the parts on coordinate axes) the meaning of locating a point P on the plane by $P(a,b)$ should be discussed in more detail. Once the meaning of (a,b) has been made clear, the solutions to the problems on coordinate axes will become quite simple.

Note. In 6-7, the Text asks the question: "For what lines is it true that the image of L is L itself?" The answer is: "those lines which are parallel to $\mathbf{U}$."

You should allow about three class meetings to cover the material and problems in Section 6-7.

1. In the following figure, A is the image of the triangle, B is the image of the circle, and C is the figure whose image is the circle.

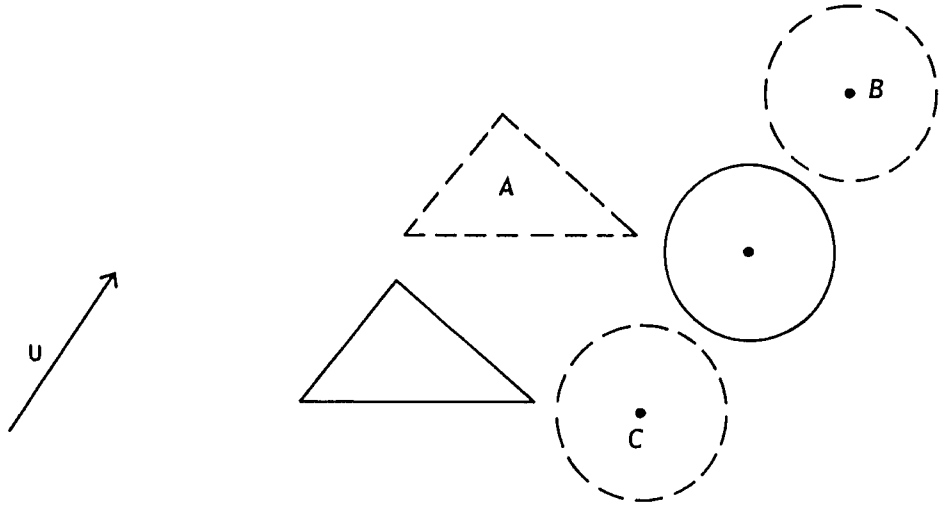


Fig. G-18.

2. All lines parallel to **U**.

3. The identity motion.

4. The result is a translation by the vector **W** given in the following Figure G-19.

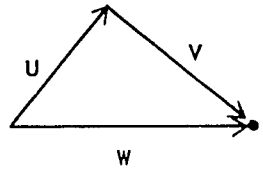


Fig. G-19.

5. (3, -3), (0, 0), (9, -14).

6. In the following figure, right triangle I is congruent to right triangle II

(by ASA), hence the two legs of II have lengths m (for the horizontal leg), and n (for the vertical leg), and hence P' has the coordinates $(a + m, b + n)$.

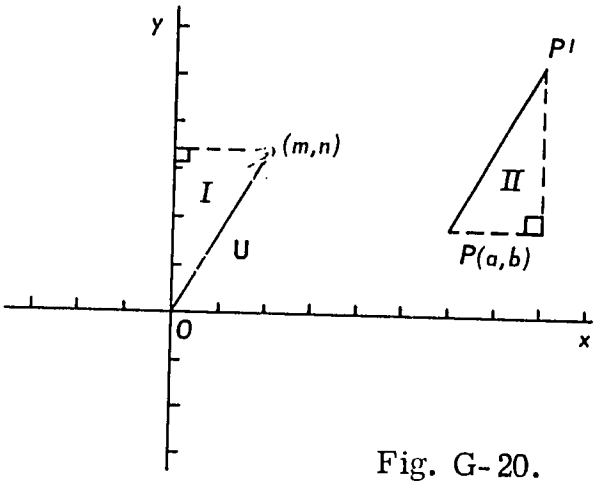


Fig. G-20.

7. Let P be any point and let P' be $T(P)$. Let U be the vector from P to P'. Let Q be any point, and let $Q' = T(Q)$. By assumption $PP' = QQ'$. Since T is a rigid motion, we also have $PQ = P'Q'$. It follows that one of the two possibilities given in Figure G-21 must occur.

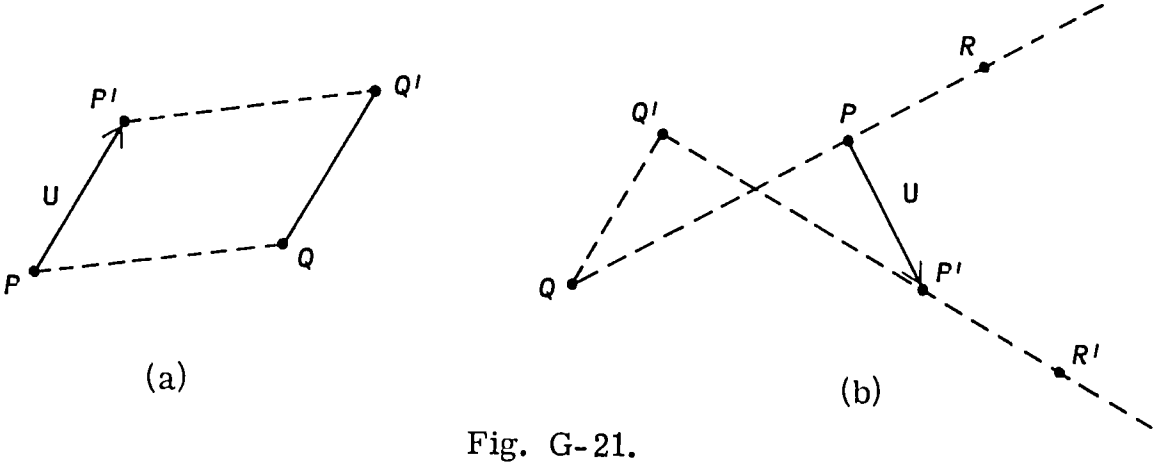


Fig. G-21.

We next show that possibility (b) cannot occur. Let R be any point far out along line $\overleftrightarrow{PQ}$ on the side of P opposite to Q. Then $R' = T(R)$ must be a point far out along $\overleftrightarrow{P'Q'}$ on the side of P' opposite to Q'. As we take R farther and farther out, the distance RR' continues to increase. This is contrary to our assumption that all points move the same distance. Thus the only possibility is (a). Since a simple quadrilateral with opposite sides equal in pairs must be a parallelogram, we have $\overline{QQ'}$

parallel to $\overline{PP'}$. But this means that Q' is the result of translating Q by the vector **U**. Since Q was any point, we see that T is a translation. (Note that Problem 7, together with the facts given in the text, tells us that the translations are exactly those rigid motions in which every point moves the same distance. We therefore say that the translations can be *characterized* as those rigid motions which move every point the same distance.)

8. Consider a rotation of the tracing paper through 180° about some fixed point Q. This motion is not a translation since not every point is moved the same distance. (In fact one point, namely Q, is not moved at all.)

9. Let the points be given as in the following figure. Let T be the result of a reflection in L followed by a translation by **U**. T will then carry P_1 to P_1' and P_2 to P_2'. The motion T is reversing (since the tracing paper must be turned over to carry it out). Hence T cannot be a translation.

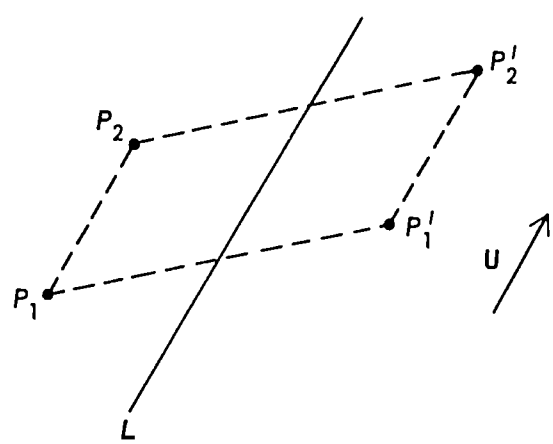


Fig. G-22.

10. Let Q be any third point. By Problem 5 of Problems 6-2C, T must carry Q to either Q' or Q'' in the following figure. Because T is direct, it must carry Q to Q'. But triangle P_1P_2Q is congruent to triangle $P_1'P_2'Q'$ (by SSS). Hence, using equal angles from these triangles, we get $\overline{P_1Q}$ parallel to $\overline{P_1'Q'}$. Since $P_1Q = P_1'Q'$, we have that $P_1QQ'P_1'$ is a parallelogram (because a simple quadrilateral with one pair of opposite sides equal and parallel must be a parallelogram). Hence $QQ' = P_1P_1'$. But Q was any point. Hence T moves every point

the same distance. Hence, by Problem 7, T must be a translation.

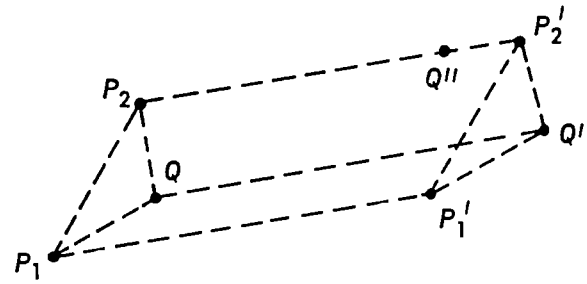


Fig. G-23.

Note. In the answers to Problems 7 and 10, "simple quadrilateral" means quadrilateral in which no pair of opposite sides intersects.

6-8 ROTATIONS.

General discussion. The text for Section 6-8 follows the same general outline as for Section 6-7. See the general discussion above for Section 6-7. Note that we pay special attention in the problems to *quarter-turns* and *half-turns*. Note also that Problem 12 of Problems 6-8A asks for a proof for the "fundamental fact" mentioned just before Problems 6-8A. (See below for a solution.)

After Problems 6-8A, rotational symmetry is discussed.

The idea of symmetry is quite important in bringing out geometrical properties of figures. For example in dealing with the equilateral triangle, we could infer that the centre is the same distance from each vertex, in case the pupils had not come across this fact previously, and so on.

We see that the square cannot be brought into coincidence with itself by a rotation of 120°, however a rotation of 90° does.

In Figure 74, rotations of 60°, 120°, 180°, 240° will bring the diagram into coincidence with itself. The idea of *order of symmetry* can be introduced at this stage. However the text does not choose to emphasize this idea. Figure 75 does not have rotational symmetry about O.

In connection with coordinate axes, note that, since a half-turn would take a point distant r along the positive x-axis to a point distant r along the negative x-axis and similarly for a point on the y-axis, *a half-turn would send*

a point $P(a,b)$ *to point* P' $(-a, -b)$. A quarter turn would send a point $P(a,b)$ to point P' $(-b,a)$.

You should allow three or four class meetings to cover the material and problems in Section 6-8.

Answers to
PROBLEMS 6-8A

Student Text Pages 222-225

1. $m(A\hat{O}A') = 90°$
   $\overline{OA} \equiv \overline{OA'}$

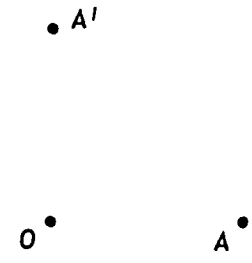


Fig. G-24.

2. BXB' is a straight line
   $\overline{BX} \equiv \overline{B'X}$

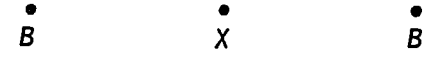


Fig. G-25.

3. $B'A'$ is at right angles to $\overline{AB}$
   $\overline{AB} \equiv \overline{A'B'}$

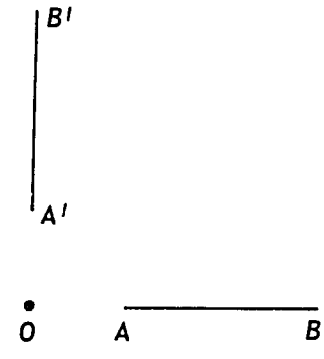


Fig. G-26.

4. B', A', O, A, B, all lie on a straight line.
   $\overline{AB} \equiv \overline{A'B'}$
   Move from right to left to get from A' to B'.

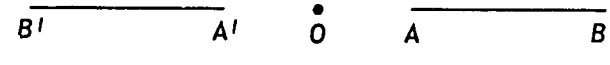


Fig. G-27.

5.

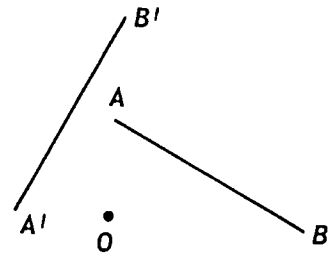


Fig. G-28.

6. Join $\overline{AB'}$, $\overline{BA'}$ to get the quadrilateral $AB'A'B$.

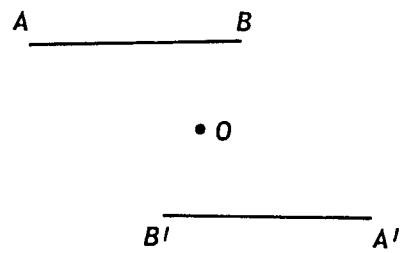
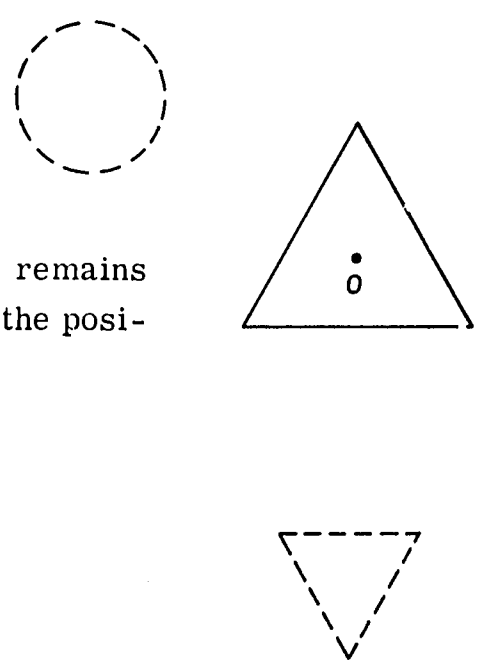


Fig. G-29.

BOB', AOA' are straight lines.
We prove $\Delta AOB \equiv \Delta A'OB'$.
[Since $\overline{OA} \equiv \overline{OA'}$; $\overline{OB} \equiv \overline{OB'}$; $\overline{AB} \equiv \overline{A'B'}$.]
Hence $A\hat{B}B'$ (same as $A\hat{B}O$) $\equiv A'\hat{B}'B$ (same as $A'\hat{B}'O$).
So $\overline{AB}$ is congruent and parallel to $\overline{B'A'}$.
Therefore $AB'A'B$ is a parallelogram.



7. The equilateral triangle with centre O remains invariant. The other figures move to the positions indicated by the broken lines.

Fig. G-30.

8. O lies at the mid point of $\overline{AA'}$.
9. O lies on the perpendicular bisector of $\overline{AA'}$.

Challenge Problem

10. If rotation is about O, $\overline{OA} \equiv \overline{OA'}$ and $\overline{OB} \equiv \overline{OB'}$.
For $\overline{OA} \equiv \overline{OA'}$, the locus of O is the perpendicular bisector of $\overline{AA'}$.
For $\overline{OB} \equiv \overline{OB'}$, the locus of O is the perpendicular bisector of $\overline{BB'}$.
So O will be given by the intersection of the perpendicular bisectors of $\overline{AA'}$ and $\overline{BB'}$. There is a rigid motion which carries $\overline{AB}$ to $\overline{A'B'}$ and is not a rotation, namely the reversing motion which takes A to A' and B to B'.

11. By similar reasoning as in Problem 10, we get O to be given by the point of intersection of the perpendicular bisectors of $\overline{AA'}$, $\overline{BB'}$ (and $\overline{CC'}$). All three are concurrent. (Note that, unlike Problem 10, there is no other motion which takes ΔABC to $\Delta A'B'C'$. See Problems 5 and 6 of Problems 6-2C.)

12. Let L be a given line and let L' be its image under a rotation about O. Make the construction indicated in the following figure by dropping perpendiculars from O to L and L'. Then $m(\widehat{COB}) + m(\widehat{OBC}) = 90°$ and $m(\widehat{DAB}) + m(\widehat{OBC}) = 90°$. Hence $m(\widehat{COB}) = m(\widehat{DAB})$, which is the fact to be proved.

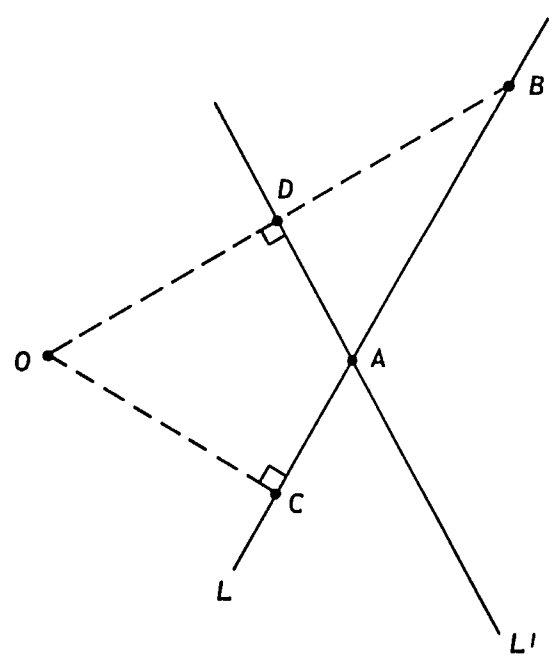


Fig. G-31.

1. All the figures are regular except the triangle, so each has rotational symmetry except the triangle.

2. The regular octagon (1st figure) has order of symmetry 8.
The regular pentagon (2nd figure) has order of symmetry 5.
The regular hexagon (3rd figure) has order of symmetry 6.
The triangle has no rotational symmetry and so its order of symmetry is 1. (The identity motion is the only rotation carrying this triangle onto itself.)
The square has order of symmetry 4.

3. Let O be a fixed point of the given rigid motion. If every other point is also a fixed point, then we have the identity motion which is a rotation. Otherwise let P be a point which is not a fixed point. Let $P' = T(P)$. Let $\beta = m(\widehat{POP'})$. Let Q be any other point. (See Figure G-32 below). By the isometric property, $T(Q)$ must be either Q' or Q''. Since T is direct, $T(Q)$ must be Q'. By SSS, $\Delta\, OPQ$ is congruent to $\Delta\, OP'Q'$. Therefore (by a simple equation) we get $m(\widehat{QOQ'}) = m(\widehat{POP'}) = \beta$. Thus Q' is obtained from Q by a rotation through the angle β. Since Q was any point, we have that T is a rotation.

Note. The solution to Problem 3 puts us very close to a proof of Theorem 6-1 (in Section 6-10) which states that every direct rigid motion is either a rotation of a translation. To prove Theorem 6-1, it only remains to show that any direct motion without fixed points must be a translation. For more on this, see the discussion below on Section 6-10.

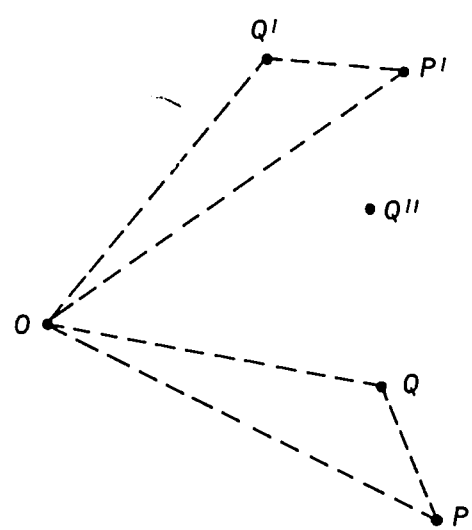


Fig. G-32.

6-9 REFLECTIONS.

General discussion. The text for Section 6-9 follows the same general outline as for Section 6-7. See the general discussion above for Section 6-7.

Although some of the pupils will have come across reflections in Physics, it would not be a waste of time to go through it thoroughly again. The idea of reflection along a line instead of a plane mirror may not be easy to get across initially. The construction given in the definition of reflection is the most practical for obtaining images of reflection. The pupils' attention must be drawn to the fact that this construction is equivalent to turning the tracing paper over along the line of reflection.

After problems 6-9A, the idea of *axis of symmetry* is discussed. Another way of stating Definition 6-4 is as follows. "If a figure is invariant under a reflection, then the line of reflection is called an *axis of symmetry* of the figure."

You should allow about three class meetings to cover the material and problems in Section 6-9.

Answers to
PROBLEMS 6-9A

Student Text Pages 230-232

1. Yes
2. Same direction as $\overrightarrow{AB}$, i.e., left to right.

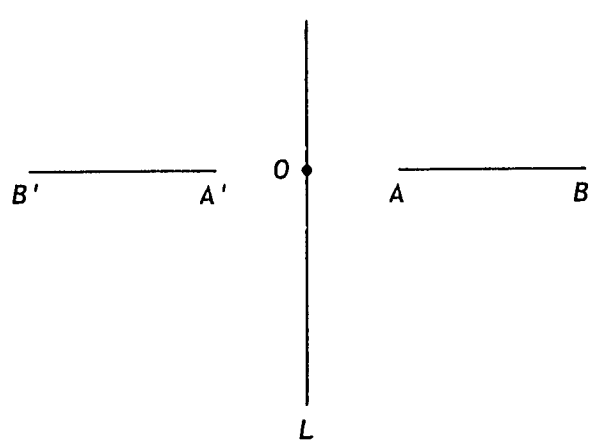


Fig. G-33.

4. Right to left.

5. Yes.

6. $\triangle ABC \equiv \triangle A'B'C'$

Because reflection is an isometric transformation.

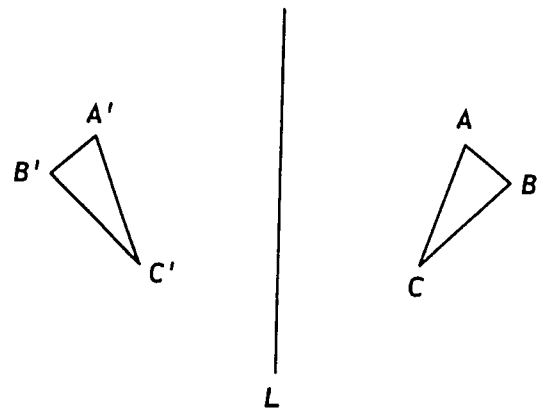


Fig. G-34.

7. Yes.

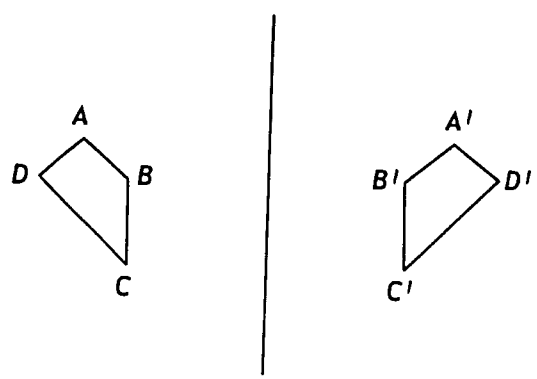


Fig. G-35.

8. Image is a circle.

Arrow in image will point counterclockwise.

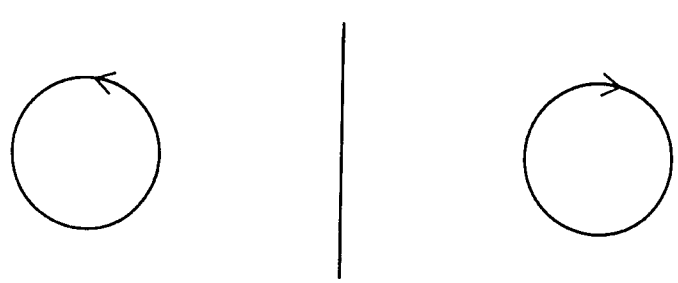


Fig. G-36.

9. The line of reflection is obtained by joining the mid-points of $\overline{AA'}$ and $\overline{BB'}$ in each case.

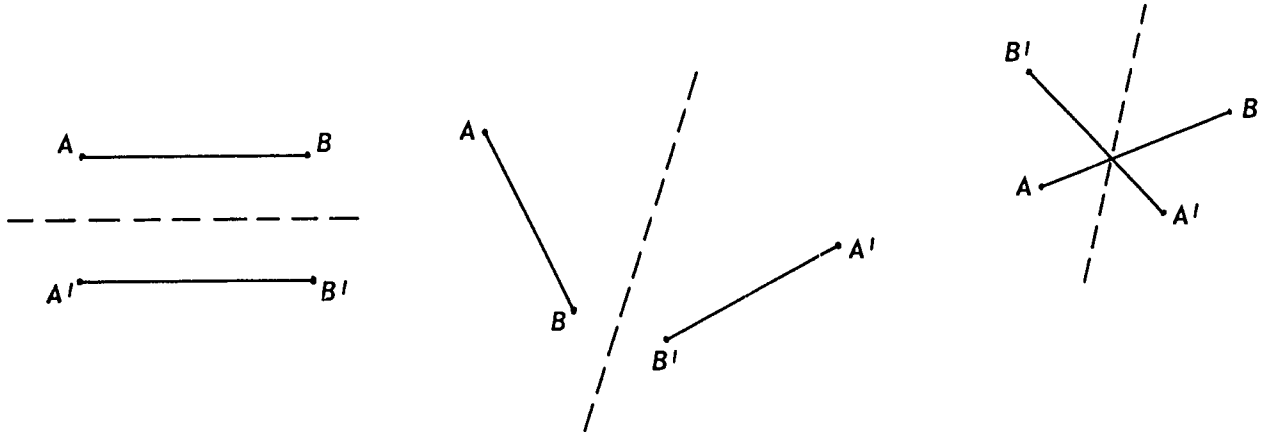


Fig. G-37.

10.

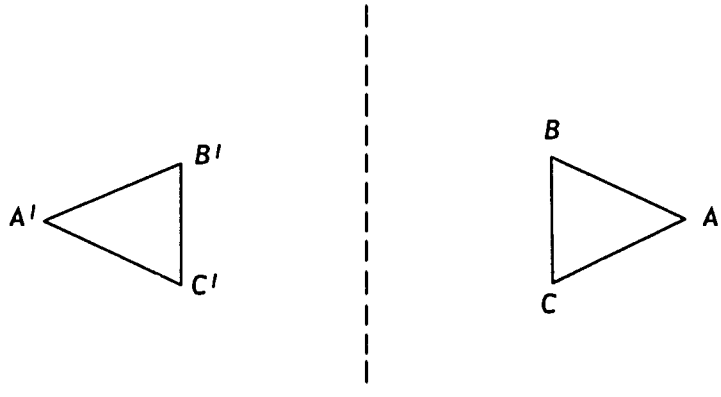


Fig. G-38.

11. Not possible.
It is not a reflection since the image shown in the problem results from a direct rigid motion while a reflection is reversing.

Answers to
PROBLEMS 6-9B

Student Text Page 234

1. Through every point X of $\overleftrightarrow{AB}$, we drop a perpendicular to L, which cuts L at P and produce the perpendicular to cut $\overleftrightarrow{CD}$ at X'. $\overline{XP} \equiv \overline{X'P}$. (This is easily shown by congruent triangles.) Alternately, from any point Y on $\overleftrightarrow{CD}$ we can get Y' on $\overleftrightarrow{AB}$ in the same way as above with $\overline{YP} \equiv \overline{Y'P}$. So we have shown that every point of $\overleftrightarrow{AB}$ has its image on $\overleftrightarrow{CD}$ and vice

versa. Hence L is an axis of symmetry.

Line m is also an axis of symmetry.
This is enough to show that L and m are the loci of points equidistant from $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$.

2. A half-turn of P about O gives P' where $P'OP$ is a straight line and $\overline{OP} \equiv \overline{OP'}$. This is the same as the result given in the problem.

Challenge Problem

3. $y = x$ is an axis of symmetry of this diagram. So P' is the point (b,a).

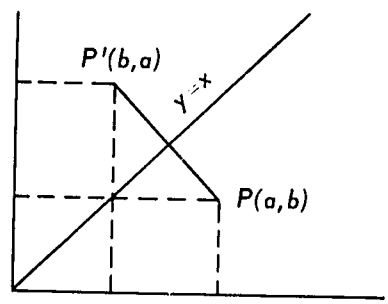


Fig. G-39.

6-10 ONE MOTION FOLLOWED BY ANOTHER.

General discussion. You should allow five or six class meetings to cover the material in this section. The first two class meetings would cover up through Problems 6-10B. The third and fourth class meetings would cover up through Problems 6-10D. The fifth and possibly sixth class meetings would cover the material given under the heading "A Basic Theorem."

The material given in Section 6-10 provides only the briefest and shallowest introduction to a rich mathematical subject. A full year could easily be spent in further and deeper study of some of the ideas that are touched on here. The basic idea is that one rigid motion can be *combined* with another by carrying out the two motions one after the other. This combination yields a resulting mapping which is, by itself, a rigid motion. We say that this third rigid motion is the "result of combining" the first two rigid motions. If the first two rigid motions are R and S and if T is the third rigid motion which results from carrying out first R and then S, we say that "T equals R *followed by* S." The act of combining two rigid motions is sometimes called an *operation*. It is a way of putting two motions together to get

a motion, just as, in arithmetic, the *operations* of addition and multiplication are ways of putting two numbers together to get a number.

In Section 6-10, we first define the operation of combining rigid motions, and then we look at some facts concerning some of the simplest combinations of different kinds of rigid motion. This material is covered in Problems 6-10A on translations combined with translations, Problems 6-10B on translations combined with half-turns, Problems 6-10C on half-turns combined with half-turns, and Problems 6-10D on reflections combined with reflections in the special case where the two lines of reflection are parallel. Finally, in the last part of the section, we give some general facts and rules about combinations of rigid motions. (Such as, for example, the rule that any combination of two reversing motions must be a direct motion.)

If we were to carry out further and deeper study of the operation of combining rigid motions, there are two directions in which we could proceed. (1) We could look at ways in which this operation (and facts about it) can be used to solve geometrical problems. Section 6-11 gives some examples of this (Examples 3 and 4) where we see that the use of this operation leads to quicker and easier solutions than we would otherwise get. (2) We could study the "algebra" of this operation in much the same way as, in previous years, the pupil has studied the algebra of the addition and multiplication operations from arithmetic. Except for a few brief comments in some of the problems, Chapter 6 does not take up this "algebra" of rigid motions. In *Secondary Five*, both in algebra and geometry, the pupil will study it further. The rigid motions form what is called, in higher algebra, a *group*.

Comments. In considering Definition 6-6, remind the student that a rigid motion is a *mapping* given by a tracing paper movement. It is not the actual path of the movement. This is important if the student is to understand, for example, how the combination of two reflections (in Figure 101) can be a translation.

Note that a number of facts are given without proof. (For example, the fact that a translation followed by a translation must be a translation.) In most of these cases, proofs are easy to find, and can be given to your pupils as added exercises. (For example, use of Problem 7 in Problems 6-7, together with the fact that each translation moves all points the same distance, gives an immediate proof that a translation followed by a translation must be a translation.)

Algebraic notation for the operation of combining rigid motions is introduced, to a small extent, in some of the problems. (Problem 2 of Problems 6-10C is the first of these.)

Warning. Note that, in the algebraic notation used in Chapter 6, the motion R followed by S is written as SR. Thus we abbreviate "R followed by S" by having R follow S on the page. This reversal is in some ways bad and can cause confusion to the pupil. It has, however, the following advantage, which explains why we use it: if T is R followed by S, then for any point $P, T(P) = S\big(R(P)\big)$ If we write $T = SR$, then we can abbreviate $S\big(R(P)\big)$ as $SR(P)$. For us, the convenience of this notation outweighs the dangers of confusion.

Note that RS need not be the same as SR. (In the special case where R and S are both translations, it is true that $RS = SR$.) Problems 6-10B, 6-10C, and 6-10D give various examples where RS and SR are not the same rigid motion. Thus, to use a word from algebra, we would say that the operation of combining rigid motions is not *commutative*. (It is, however, *associative*, that is to say, $(QR)S = Q(RS)$; because, for any point P, both $[(QR)S]\ (P)$ and $[Q(RS)]\ (P)$ are the same as $Q\Big(R\big(S(P)\big)\Big)$.

The basic facts and rules given at the end of the section (under the heading "A Basic Theorem") are not proved, although the first of these, the rule on direct and reversing motions, is immediately clear from the tracing paper idea of motion. These facts and rules are, however, an important and central part of Chapter 6. Please make every effort to persuade your pupils that they are important and true.

How could a proof for Theorem 6-1 be found? It would amount to showing that the method illustrated in Figure 105 always worked (that is to say, that if two of the perpendicular bisectors meet, then all three meet at a common point--the fact that we then have a rotation is immediate). This proof is based on congruent triangles. It is lengthy, but not hard.

What about Theorem 6-2? Here the construction (corresponding to the construction in Figure 105 for Theorem 6-1) is as follows. Take the midpoints of $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$. At least two of these midpoints are distinct (otherwise we have a half-turn, which is not reversing). All three midpoints are on the same line (a lengthy congruent triangles proof). Then T is either a reflection, or else a glide reflection, with this line as its line of reflection.

Answers to questions raised in Text. In the text on "Rotation Followed by Rotation," how do we show that the combination of two half-turns about the same point is the identity? Since a point gets rotated 180° by the first turn and then 180° by the second turn, it must be returned to its original position. The combination leaves every point fixed and we have the identity.

In the text for Figure 105, why must at least two of the perpendicular bisector lines be distinct? If all three coincide, it is easy to see (draw a figure) that the given motion has to be reversing, contrary to assumption.

Answers to
PROBLEMS 6-10A

Student Text Pages 238-239

To describe a translation completely we had to introduce vectors. Combining translations is, in effect, addition of vectors. We see in the diagram below that **U** + **V** = **W** = **V** + **U**. That is, translation **U** followed by translation **V** gives translation **W**, which is also obtained from translation **V** followed by translation **U**.

It is a good idea to get the pupils to try several pairs of translations to demonstrate the rule for combining translations.

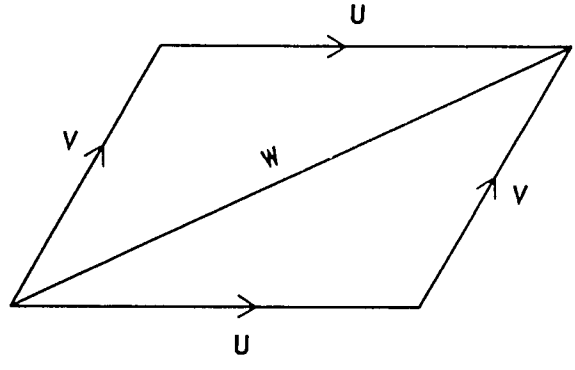


Fig. G-40.

1. The two sides of any triangle whose third side is **W** give a solution. (Of course, **U** and **V** are not included.) Hence there are an unlimited number of acceptable answers.

2. **W** is the single vector and **W** = **U** + **V**.

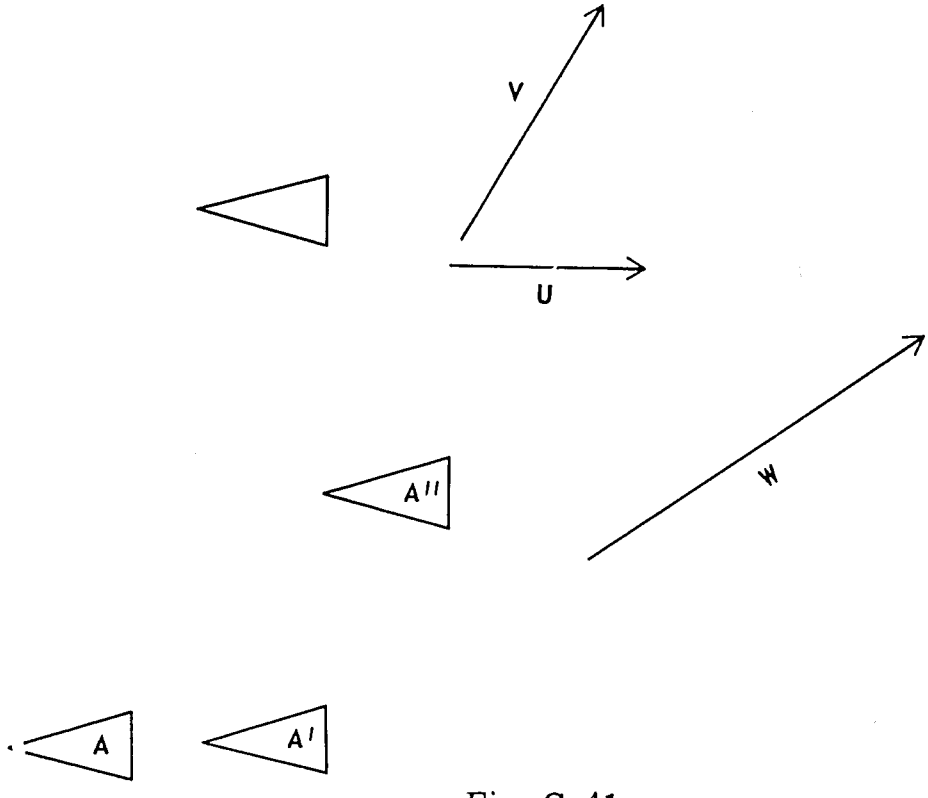


Fig. G-41.

3.

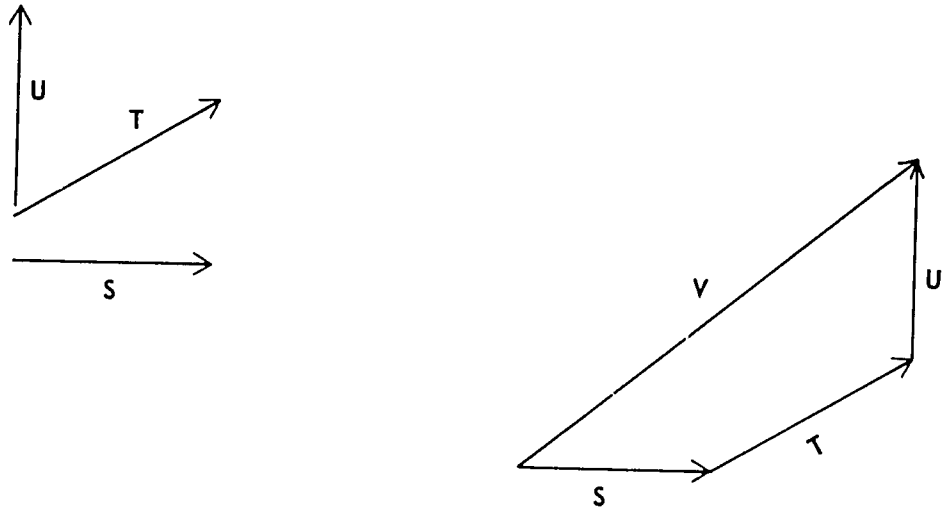


Fig. G-42.

4. Any right-angled triangle having **V** as hypotenuse will be an acceptable solution.

5. T will move A' to A'', 2T away from A.
T will move A'' to A'''. 3T away from A.

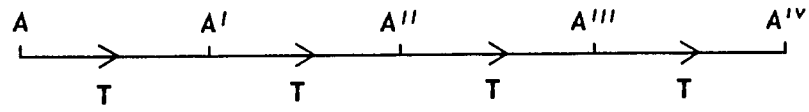


Fig. G-43.

In the preceeding problems try to see if the students can give a general solution.

Answers to
PROBLEMS 6-10B

Student Text Page 239

1. No. (Note that $C''=D'$ and that $D''=C'$. Note further that only a translation or a reflection will take C'' back to C and D'' back to D.

2. Yes.
For example: Take a point P along $\overline{HK}$ distant d from $\overline{CD}$. A half-turn about P followed by a translation T parallel to W, where the length of T is $W - 2d$, will move $\overline{CD}$ to $\overline{C'D'}$.

3. The final position will be at a distance W from $\overline{CD}$ but on the opposite side of $\overline{CD}$ from $\overline{C'D'}$. Not the same as $\overline{CD}$.

1.

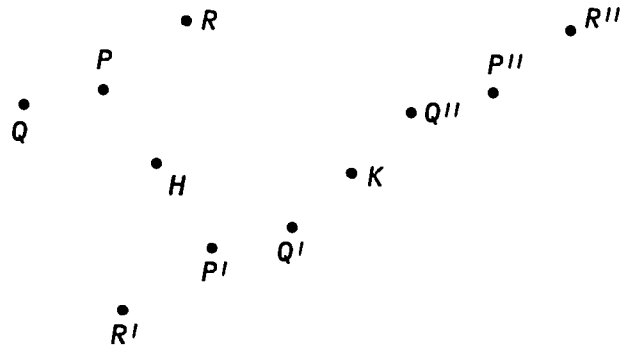


Fig. G-44.

$\overline{PP''} \equiv \overline{QQ''} \equiv \overline{RR''} = (2\overline{HK})$

They are parallel to each other, since each is parallel to $\overline{HK}$.

2. From Problem 1, GF is a translation given by $2\overrightarrow{GF}$, i.e., a translation parallel to $\overline{GF}$ and twice the length of $\overline{GF}$.
So $S = GF$,
since $\overrightarrow{LL'} = 2\overrightarrow{GF}$.

3. Calling the half-turn about A, A, half-turn about B, B, etc.
A followed by B = **T** where **T** = $2\overrightarrow{AB}$.
D followed by C = **S** where **S** = $2\overrightarrow{DC}$.
But $\overrightarrow{AB} = \overrightarrow{DC}$ ($ABCD$ is a parallelogram.)
Hence **T** = **S**,
i.e., $AB = DC$.

4. HK is a half-turn about K followed by a half-turn about H.
$HK \neq KH$.
The translation HK is equal in length to the translation KH but opposite in direction.

5. Define KJH as a half-turn about H, followed by a half-turn about J followed by a half-turn about K.

Suppose H carries P to P'
J carries P' to P''
K carries P'' to P'''

So KJH carries P to P'''.
KJ carries P' to P''' so $\overrightarrow{P'P'''} = 2\overrightarrow{JK}$.
JH carries P to P'' so $\overrightarrow{PP''} = 2\overrightarrow{HJ}$.
$\overline{P'P'''}$ is equal and parallel to $\overline{PP''}$
(because $\overline{JK} \equiv \overline{HJ}$).
Hence $PP''P'''P'$ is a parallelogram.
So $\overline{PJ} \equiv \overline{JP'''}$ (intersection of diagonals).
Hence J carries P to P''',
i.e., $KJH = J$.

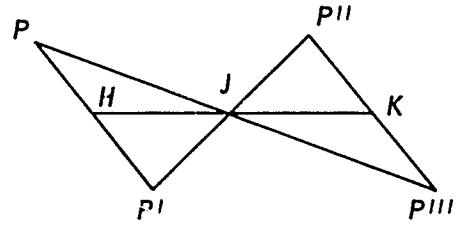


Fig. G-45.

6. $P(B) = B'$.
$Q(B') = B''$.
$S(B'') = B'''$.
$\overrightarrow{BB''} = 2\overrightarrow{PQ}$.
$\overline{PQ} \equiv \overline{RS}$ and $PQRS$ is a straight line.
Hence $BB'' = 2RS$ and $\overline{BB''}$ is parallel to $\overline{RS}$.
Since $\overline{B''S} \equiv \overline{SB'''}$, we have in $\Delta\, \overline{BB'B'''}$ that point R lies on $\overline{BB'''}$ and that $\overline{BR} \equiv \overline{RB'''}$.
Hence $R(B) = B'''$,
i.e. $SQP(B) = R(B)$.

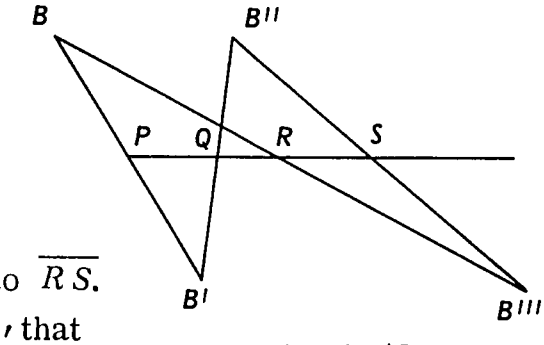
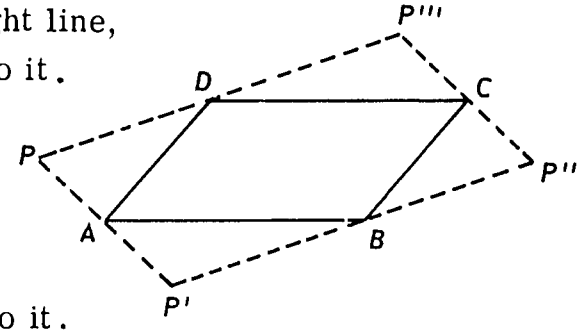


Fig. G-46.

7. $A(P) = P'$.
$BA(P) = P''$.
$C(P'') = P'''$.
In $\Delta\, PP'P''$,
$\overrightarrow{PP''} = 2\overrightarrow{AB}$ $(= 2\overrightarrow{DC})$,
since $ABCD$ is a parallelogram.
In $\Delta\, PP''P'''$, PDP''' is a straight line,
since $PP'' = 2\,DC$ and parallel to it.
So $PD = DP'''$.
Hence $D(P) = P'''$,
i.e., $CBA(P) = D(P)$.
Also, $CD(P) = P_1$,
where $PP_1 = 2\,DC$ and parallel to it.
$AB(P_1) = P_2$,
where $P_1P_2 = 2\,BA$ and parallel to it
But $CD = BA$; so $\overline{P_1P_2} \equiv \overline{PP_1}$.
Also $\overrightarrow{P_1P_2}$ points in the same direction as $\overrightarrow{P_1P}$.
Hence P_2 and P coincide,
i.e., $ABCD(P) = P$.



1. Distances are as indicated in the diagram.

$PP'' = p + p + (d - p) + (d - p)$
$= 2d.$

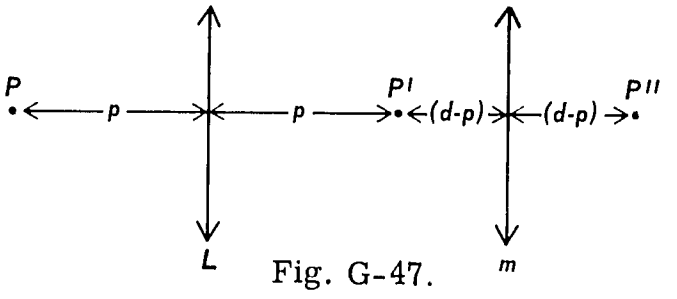


Fig. G-47.

2. Distances are as indicated in the diagram.

$PP_1'' = p + 2d - p$

$= 2d.$

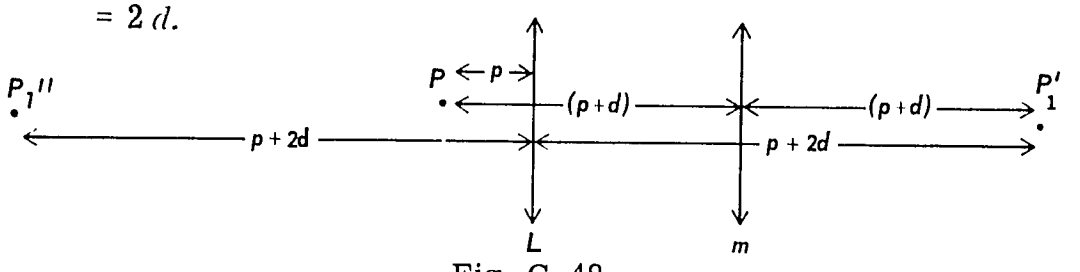


Fig. G-48.

3. Each point moves a distance $2d$ (if d is the separation of L and m). Hence a translation (see Problem 7 in Section 6-7).

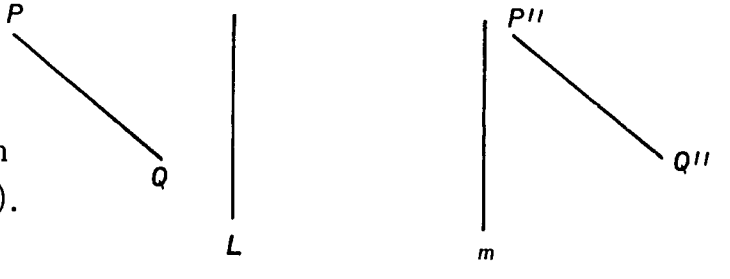


Fig. G-49.

4.

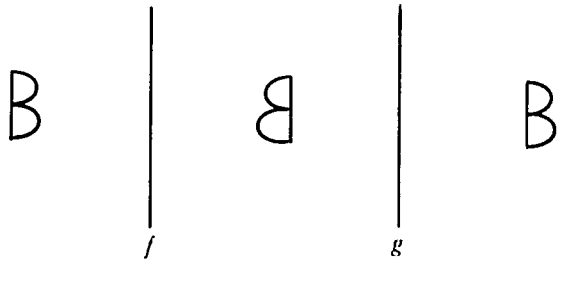


Fig. G-50.

5. S'', R'', Q'' and P'' give the required images. $SS'' = RR'' = QQ'' = PP'' = 6$ units.

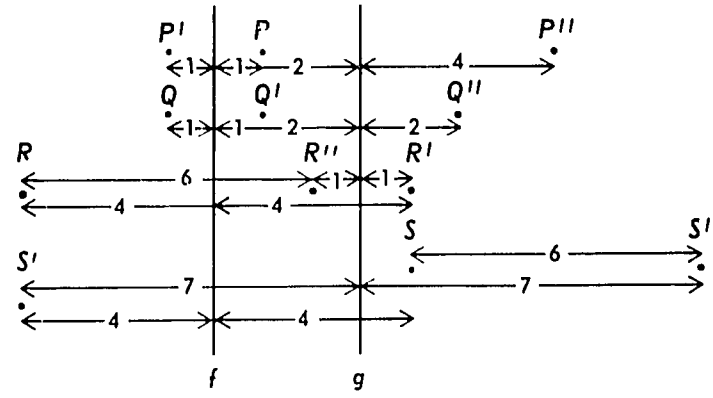


Fig. G-51.

6. S'', R'', Q'', P'' are the required images. $SS'' = RR'' = QQ'' = PP'' = 6$ units.

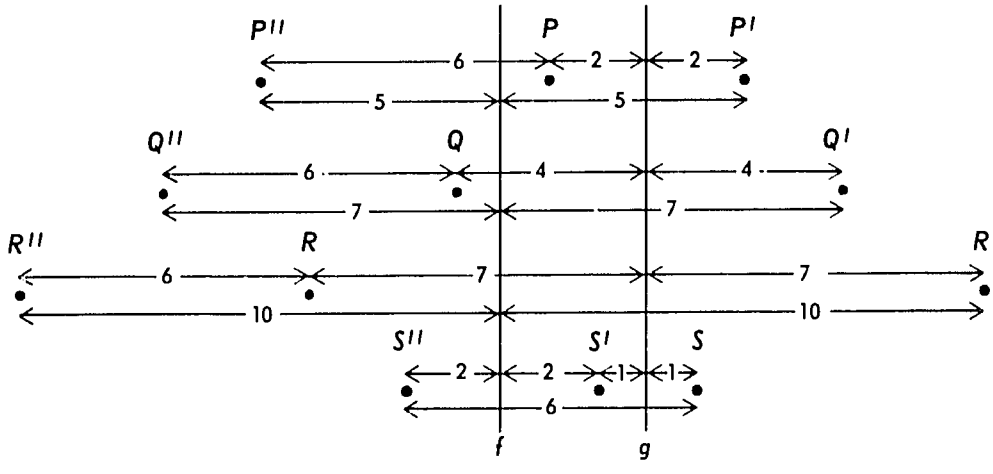


Fig. G-52.

7. Any two parallel lines perpendicular to T and whose distance of separation is half the length of T will be an acceptable solution.

6-11 MORE EXAMPLES OF THE USE OF MOTIONS.

General discussion. In this section we give four illustrations of the power of rigid motion ideas in solving geometrical problems. If you have had a traditional training in Euclidean geometry, you will find these examples especially impressive.

You should allow four class meetings if you wish to cover all four examples. You may select any subset of the four if you prefer, for the examples do not depend upon each other. You may wish to use only the first two examples. These are simpler than the last two.

Note that both of the first two examples have very simple congruent

triangle solutions. What is important in these examples is not so much that rigid motions give the pupil shorter proofs (though they do do that) but rather that they give him an entirely new way of thinking about the problems. This new way of thinking about geometrical problems makes the solutions of many problems easier to find.

Note in Examples 3 and 4 the very powerful use made of the basic facts and rules given at the end of Section 6-10.

Answers to
PROBLEMS 6-11A

Student Text Pages 247-248

1. 60° (See the fact stated just before Problems 6-8A and proved in Problem 12 of Problems 6-8A).
2. $\overline{NC}$ goes to $\overline{AM}$ by a rotation through 90° about B. The isometric property of this rotation gives the result immediately.
3. Immediate by fact stated just before Problems 6-8A.

Answers to
PROBLEMS 6-11B

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1. The reflection in L carries $\overline{AB}$ onto $\overline{CB}$ and hence A onto C. By property (5) of rigid motions, we have the desired result.

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