AGENCY FOR INTERNATIONAL DEVELOPMENT
WASHINGTON, D. C. 20523

BIBLIOGRAPHIC INPUT SHEET

FOR AID USE ONLY
Batch 69

1. SUBJECT CLASSIFICATION

A. PRIMARY: Development and economics — DA00-0000-0000

B. SECONDARY: General

2. TITLE AND SUBTITLE
Income distribution and optimal growth; the case of open unemployment

3. AUTHOR(S)
McCabe,James; Lele,M.M.

4. DOCUMENT DATE	5. NUMBER OF PAGES	6. ARC NUMBER
1974	40p.	ARC

7. REFERENCE ORGANIZATION NAME AND ADDRESS
Yale

8. SUPPLEMENTARY NOTES (Sponsoring Organization, Publishers, Availability)
(In Center discussion paper no.197)

9. ABSTRACT
~~(EMPLOYMENT R&D)~~
~~(ECONOMICS R&D)~~

10. CONTROL NUMBER
PN-AAE-586

11. PRICE OF DOCUMENT

12. DESCRIPTORS
Growth
Income distribution
Unemployment

13. PROJECT NUMBER

14. CONTRACT NUMBER
CSD-2492 Res.

15. TYPE OF DOCUMENT

AID 590-1 (4-74)

CSD-2492 Res.
Yale PN-AAE-586

Econ 1
#32/52

ECONOMIC GROWTH CENTER

YALE UNIVERSITY

Box 1987, Yale Station
New Haven, Connecticut

CENTER DISCUSSION PAPER NO. 197

March, 1974

INCOME DISTRIBUTION AND OPTIMAL GROWTH: THE CASE OF OPEN UNEMPLOYMENT

James L. McCabe

Milind M. Lele

Note: Center Discussion Papers are preliminary materials circulated to stimulate discussion and critical comment. References in publications to Discussion Papers should be cleared with the author to protect the tentative character of these papers.

Portions of this research were financed by funds provided by the Agency for International Development under contract CSD/2492. However, the views expressed in this paper do not necessarily reflect those of AID.

INCOME DISTRIBUTION AND OPTIMAL GROWTH:

THE CASE OF OPEN UNEMPLOYMENT

Milind M. Lele
Harbridge House, Inc.
Boston, Massachusetts

James L. McCabe
Yale University
New Haven, Connecticut*

ABSTRACT

The investigation is based on a dynamic, development-planning model which assumes open unemployment of unskilled labor. The standard deviation of capital holdings across households enters as a state variable and feeds back on the system through its effect on demand composition and the labor-intensity of the output mix. The system is controlled by varying the average level and progressivity of taxes and subsidies which depend on household income. From this model, the behavior of summary measures of capital and income inequality along a consumption-optimal trajectory is determined. For example, conclusions are drawn concerning possibly divergent changes in capital and income inequality and the relationship between these dispersion measures and output growth.

---

*Economic Growth Center, Yale University, 52 Hillhouse Avenue, New Haven, Connecticut 06520.

This paper analyzes the dispersion as well as the level of income and capital per household along a consumption-optimal growth trajectory. In particular, we examine cases where there is a conflict between growth in output per head and equity growth. The underlying economic model assumes open unemployment due to factor-price rigidity; this formulation describes the situation in many developing economies. In these countries, the rental rate on capital is often determined by prices in the international market and is insensitive to internal movements in the capital-labor ratio [4, pp. 81-88].

Two variations of the same basic model are considered: one where the savings of unskilled laborers is subsidized and one where unskilled laborers do not accumulate capital, though skilled do. The steady-state equilibrium solutions of these models exhibit various interesting characteristics. Take, for example, the case where unskilled labor savings is subsidized. In this case, we are able to demonstrate the existence of situations wherein income equity and a more even distribution of assets are not consistent objective along a consumption-optimal growth path. We show, for one thing, that decreases in the level and dispersion of capital are accompanied by increases in income dispersion. Alternatively, should the employment rate fall as capital intensity increases, paradoxical situations may well arise in which increasing capital per household causes further inequalities in income distribution despite a more even distribution of capital holdings. This conflict between output and equity growth occurs only with a decreasing unskilled employment rate in the case where taxes and subsidies affect all households and all households save. We show that

this result breaks down and conflicts between output and equity become more commonplace if it is assumed that unskilled laborers are insulated from "negative" income taxes and therefore cannot be induced to save by fiscal re-distribution.

Dynamic equations describing the change in capital intensity and capital dispersion are derived in Section I. Section II outlines the main arguments in the derivation of a production-demand relationship, which related output to capital intensity and dispersion. This is an important prerequisite to any considerations of optimal growth, inasmuch as capital and income dispersion may have a significant impact on output through demand composition. Equilibrium conditions are obtained in Section III for the case where savings behavior is the same across all households. In Section IV, we examine the implications for the optimal growth trajectory of assuming that unskilled laborers do not accumulate capital. Main conclusions and possible extensions of the model are discussed in the final section.

I. THE DYNAMIC SYSTEM

It will be convenient to assume that the economy is divided into m groups of varying size but growing at the same rate. The capital of a household is assumed to be spread evenly among its heirs; the number of heirs is assumed to be the same for each household, so that the capital holdings of each of the households within a group remain equalized.

Assuming a total population of m groups whose household are growing at rate n, let us define the following quantities <u>for a representative household in the jth group</u>:

(1.1) $c_j \triangleq$ consumption

(1.2) $s_j \triangleq$ savings

(1.3) $k_j \triangleq$ capital stock (including human capital)

(1.4) $\hat{w}_j \triangleq$ wage income of unskilled labor assumed allocated completely to consumption

(1.5) $\Pi_j \triangleq$ capital income before taxes and subsidies

(1.6) z_j = disposable capital income.

Then we have savings, s_j equal to the rate of change of capital, i.e.

(1.7) s_j = gross changes in capital stock.

But

(1.8) $$s_j = \text{total disposable income-consumption} = z_j + \hat{w}_j - c_j \ .$$

We assume that the function determining the consumption per household of the jth group is of the form

(1.9) $$c_j = \alpha k_j + \hat{w}_j$$

where

$$0 \leq \alpha \leq 1$$

i.e. all unskilled wage income--which is untaxed--and a constant fraction of capital income are allocated completely to consumption. The effects on optimal growth results of relaxing the assumption that the marginal propensity to consume out of unskilled wage income is unity and that out of disposable capital income zero are considered later in the paper and in the Appendix.

The expression for savings s_j now becomes, from (1.7) and (1.8)

(1.10) $$s_j = z_j - \alpha k_j \ .$$

Denote aggregate domestic savings per household by s, disposable capital income by z, and capital stock per household by k. Then, from (1.9) we obtain the following aggregate savings function

(1.11) $$s = z - \alpha k \ .$$

Then from (1.8) assuming a constant (exponential) depreciation rate δ and a rate of household growth n, we have

(1.12) $$\frac{dk_j}{dt} = \text{net change in capital stock per household}$$

$$= z_j - (\alpha + \delta + n)k_j \ .$$

Denote taxes (net of subsidies) levied on the jth household by NT_j . Taxes are assessed only on capital income, which includes the income of skilled labor. We now have

(1.13) $$z_j = \Pi_j - NT_j$$

$$= rk_j - NT_j$$

where r is the return to capital (assumed constant).

One possible tax function is

(1.14) $$NT_j = a_0'' + a_1''(\Pi_j - \Pi)$$

where Π is the mean capital income per household for the entire society. This can be written as

(1.15) $$NT_j = a_0'' + a_1''r(k_j - k) \ .$$

The coefficient a_0'' determines the revenue impact of the tax, while a_1'' determines the redistributive effects.

From (1.13), (1.15), equation (1.12) can be written as

(1.16) $$\frac{dk_j}{dt} = r\left[k_j - \frac{a_0''}{r} - a_1''(k_j - k)\right] - (\alpha + \delta + n)k_j \ .$$

Define aggregate quantities as below

$$k \triangleq \frac{1}{N} \sum_{j=1}^{m} k_j N_j \ , \tag{1.17}$$

N = total number of households, N_j = the number of households in the jth group, and m is the number of groups

$$\sigma^2 = \frac{1}{N} \sum_{j=1}^{m} N_j (k_j - k)^2 \tag{1.18}$$

where k is the weighted average of capital per household in each group and σ^2 is the weighted variance. We have, assuming an exponential growth rate n for the number of households

$$\frac{dk}{dt} = r\left(k - \frac{a_0''}{r}\right) - (n + \delta + \alpha)k \tag{1.19}$$

and

$$\frac{d\sigma}{dt} = \{r(1 - a_1'') - (n + \alpha + \delta)\}\sigma \ . \tag{1.20}$$

These are the basic dynamic equations describing the manner in which capital intensity and capital dispersion vary over time. Note that, in the derivation of these equations, <u>no assumptions have been made regarding the form of the distribution of</u> k_j . It is only necessary that the quantities in (1.17) and (1.18) exist.

II. THE PRODUCTION-DEMAND RELATIONSHIP

The effects of income or asset distribution on the level of output are generally excluded from consideration in the literature on consumption-optimal growth. Such effects, it is felt, are negligible and of little consequence in the determination of the steady-state equilibrium solutions for the economic models usually considered.

Inasmuch as we are interested in examining the impact of changes in capital intensity on income dispersion through its effect upon the employment rate, we must therefore consider possibility that changes in dispersion feed back on employment, however small the magnitude of this effect. This closes the causal chain and permits us to formulate a well-posed problem.

We shall make the following assumptions; these can be eased considerably as shown in [5].

A1: There are two commodities in the system. Commodity 1 is used for both investment and consumption. Commodity (or good) 2 is used for consumption only.

A2: The production functions for the two commodities are differentiable and homogeneous of degree one, with unskilled labor and capital being the factors of production.

A3: Commodity 2 is more labor intensive compared to Commodity 1.

A4: Open unemployment exists in the system even in equilibrium; either the minimum real wage is (exogenously) specified or the rental rate on capital is determined by prices in the international market. Each household has one laborer (either skilled or unskilled).

A5: All government tax revenues are expended on Commodity 1.

and, provisionally[1]

A6: A closed economy.

From A5, we see that the total expenditure on Commodity 1, per household, other than private consumption, is determined by

$$G + i = (r_1 - \alpha)k \tag{2.1}$$

where $G \triangleq$ government expenditure per household

$i \triangleq$ gross investment per household

and $r_1 \triangleq$ rental rate on capital (expressed in terms of good 1).

This expression indicates that the sum of investment and government expenditure depends only on the capital intensity k, since r_1 and α are specified. Consequently, the only relevant changes in output mix are those effected by changes in the composition of private consumer demand. Since good 2 is labor intensive relative to good 1 (A3), these changes in the bill of goods demanded will influence the level of aggregate output and

---

[1]The closed economy assumption (A6) has been introduced only to simplify the exposition of the general equilibrium analysis. It is abandoned later The detailed analysis, with trade included, is presented in [5].

employment. The importance of capital dispersion as a variable affecting growth therefore depends on its relationship to the composition of consumer demand.

We will therefore first derive expressions for the aggregate commodity demand functions. We will then solve a general equilibrium model to determine the employment rate and capital intensity, as well as the relative shares of the two goods, when demand equals supply. Solution of the general equilibrium system will yield expressions for the GDP per laborer (y) and the employment rate $(\tilde{e})$ in terms of the capital intensity (k), and the dispersion of capital (σ).

The Commodity Demand Functions

Aggregate private consumption per household for Commodity i may be written in the form

$$c_i = c_i(c, \sigma_c, P) \tag{2.2}$$

where σ_c is the standard deviation of the distribution and P is the commodity-price ratio (good 2 to good 1). Exact derivations of the functional form of (2.2) exist in the case of the quadratic and the exponential demand functions for the individual household. See Klein [3]. Substituting the aggregate version of (1.9) in (2.2) we have

$$c_i = c_i'(\hat{w}, k, \sigma_c, P) . \tag{2.3}$$

Well-behaved household demand functions relating consumption of good i to expenditure are monotone over the range being considered, and pass

through the origin. It can be shown that the expenditure elasticities are greater than (less than) unity according as the functions are convex (concave) to the origin. Denoting the expenditure elasticity of demand of the ℓth household for the ith commodity by η_i^ℓ, we have

$$\frac{\partial c_i}{\partial \sigma_c} \gtreqless 0 \quad \text{according as} \quad \eta_i^\ell \gtreqless 1 \tag{2.4}$$

i.e., according as the household's commodity demand functions are convex or concave.

The standard deviation of consumption σ_c is related to measures of income dispersion through the variance-covariance matrix of the joint distribution of household capital holdings and employment. (Every household is assumed to face the same wage rate for unskilled labor.) The unskilled labor employment of a particular household may well be an increasing function of its capital stock, particularly if family enterprises exist. However, for simplicity, we make the assumption

A7: Employment of unskilled labor is distributed independently of capital holdings and the employment rate of skilled labor remains fixed at unity.

This implies that the household unskilled employment rate is independent of capital holdings and is binomially distributed. (From A4, workers are either employed or unemployed.) Under these conditions, σ_w^2, the variance of unskilled wage income per household is seen to be

$$(2.5)\quad \sigma_w^2 = w_1^2(\tilde{e} - \tilde{e}^2) = \underbrace{\lambda w_1^2(e_u)(1 - e_u)}_{\substack{\text{(sum of the weighted}\\ \text{variances of}\\ \text{unskilled income}\\ \text{within groups)}}} + \underbrace{\left[\frac{(w_1\tilde{e})^2}{\lambda} - (w_1\tilde{e})^2\right]}_{\substack{\text{(weighted variance}\\ \text{of unskilled wage}\\ \text{income between}\\ \text{groups)}}},$$

where w_1 is the wage rate in terms of good 1, $\tilde{e}$ is the ratio of unskilled laborers employed to total households, λ is the ratio of unskilled to total laborers, and e_u $(= \tilde{e}/\lambda)$ is the unskilled employment rate. This expression is derived from the total variance formula for two mutually-exclusive groups--households with skilled and households with unskilled labor. Denote average capital per household for the unskilled-labor group by k_u and the variance of household consumption by σ_c^2. Then, by means of the total-variance formula for grouped data, we obtain from the household consumption function and (2.5)

$$(2.6)\qquad \sigma_c^2 = \sigma_w^2 + \alpha^2\sigma^2 + 2w_1\tilde{e}\alpha(k_u - k) .$$

Given the function for capital accumulation of a representative household the percentage change in $(k_u - k)^2$ can be shown to be the same as the percentage change in σ^2. This implies that $(k_u - k)$ is a fixed proportion of σ. Thus (2.6) may be rewritten as

$$(2.7)\qquad \sigma_c^2 = \sigma_w^2 + \alpha^2\sigma^2 + 2w_1\tilde{e}\alpha\beta\sigma$$

where $-1 < \beta < 0$. We assume

(2.8a) $$\sigma \geq -\frac{w_1\tilde{e}\beta}{\alpha}, \quad \text{implying} \quad \frac{\partial\sigma^2 c}{\partial\tilde{e}} \geq 0$$

and

(2.8b) $$\frac{.5}{(1-\beta^2)\lambda} < e_u = \tilde{e}/\lambda, \quad \text{implying} \quad \frac{\partial\sigma^2 c}{\partial\tilde{e}} < 0 .$$

We can therefore write (2.3) in the form

(2.9) $$c_i = c_i''(k, P, \sigma, \tilde{e})$$

with

(2.10) $$\frac{\partial c_i}{\partial\sigma} \geq 0 \quad \text{if} \quad \eta_i^{\ell} \geq 1$$

(2.11) $$\frac{\partial c_i}{\partial\tilde{e}} > 0 \quad \text{if} \quad c_i > 0$$

(in the case of a quadratic demand function).

Demand and Supply Equilibrium

How, then, do changes in the standard deviation of the distribution of capital, σ, affect the demand curve and the equilibrium point? Assume that the expenditure elasticity of good 1, η_1^ℓ is greater than unity and that $\eta_2^\ell < 1$ and that $\partial\sigma c^2/\partial\sigma > 0$. Then a reduction in the household asset dispersion from σ' to σ'' $(\sigma'' < \sigma')$ will cause the demand curve D_2D_1 to shift towards good 2 as shown in Figure 2.[2] This is seen from (2.10). The equilibrium point will shift upwards along the transformation surface (from 1 to 2); this transformation surface is linear because of the specified factor price. The new equilibrium point, 2, will represent a higher level of GNP per household and a higher unskilled employment per household, $\tilde{e}$, than those attained at point 1.[3]

By solving such a system we may derive expressions for the GNP per household (y) and average unskilled employment per household

$$y = f(k, \sigma; \theta) \tag{2.12}$$

$$\tilde{e} = g(k, \sigma; \theta) \tag{2.13}$$

where θ is the exogeneously specified factor price expressed in terms of one of the goods. Noting that $y = r_1k + w_1\tilde{e}$,

---

[2]Note that each point on D_2D_1 represents a fixed asset dispersion σ, but a different wage income distribution due to changes in unskilled employment per household $\tilde{e}$.

[3]Introducing trade into model does not affect the qualitative results, provided there are no intermediate good imports. See [5].

we have

$$\frac{\partial f}{\partial \sigma} \gtreqless 0 \quad \text{as} \quad \frac{\partial g}{\partial \sigma} \gtreqless 0 \tag{2.14}$$

and

$$\frac{\partial f}{\partial k} \gtreqless r_1 \quad \text{as} \quad \frac{\partial g}{\partial k} \gtreqless 0 \tag{2.15}$$

where r_1 is the rental rate on capital and w_1 the wage rate in terms of good 1.

From the equilibrium analysis carried out above, we see that, as long as σ lies above the point where $\partial\sigma_c/\partial\sigma = 0$.

$$\frac{\partial \tilde{e}}{\partial \sigma} \equiv \frac{\partial g}{\partial \sigma} \gtreqless 0 \quad \text{according as} \quad \frac{\partial c_1}{\partial \sigma_c} \lesseqgtr 0$$

i.e.,

$$\frac{\partial \tilde{e}}{\partial \sigma} = \frac{\partial g}{\partial \sigma} \gtreqless 0 \quad \text{according as} \quad \eta_1^{\ell} \lesseqgtr 1 \ . \tag{2.16}$$

If $\partial\sigma_c/\partial\sigma = 0$, then $\partial\tilde{e}/\partial\sigma = 0$.

The compatibility of output and employment objectives implied by (2.14) and (2.15) may well break down when the good intensive in <u>unskilled</u> labor is inferior. In this situation, under certain conditions, $\partial g/\partial k$ is negative and increases in capital intensity may cause employment to decline even as GDP per household increases. The details are presented in McCabe [5].

III. EQUILIBRIUM GROWTH PATHS

Denoting total consumption (both private and public) by $\hat{c}$, instantaneous welfare is given by

$$U(\hat{c}) = Nu(\hat{c}) \tag{3.1}$$

$$\hat{c} = f(k,\sigma) - r_1 \quad k - \frac{a_0''}{r_1} \quad + \alpha k$$

where $u(\hat{c})$ determines aggregate utility per household. The possible effects of changes in expenditure distribution on social welfare are not considered in this function.[4] We are concentrating on the analysis of possible growth-equity conflicts along a consumption-optimal growth path.

Then the optimal growth paths are those that maximize discounted future consumption per household, i.e., the quantity.

$$\int_0^T e^{-\rho t} N_0 e^{nt} u(\hat{c})dt = \int_0^T e^{-\gamma t} U(\hat{c})dt \tag{3.2}$$

where ρ is the rate of social discount, n is the rate of household growth (size constant), γ ($= \rho - n$) the net rate of social discount,

---

[4]Since the commodity price ratio is fixed, private consumption (in terms of good 1) is treated as a single good; public and private consumption expenditures are perfectly substitutable in this function.

N_0 is the initial number of households and T is the planning horizon --assumed finite for simplicity.

Thus the paths of optimal capital accumulation and dispersion are given by the solution of the following optimal control problem:

$$\text{(3.3)} \qquad \underset{u_1,\, u_2}{\text{Maximize}} \int_0^T e^{-\gamma t} U[f(k,\sigma) - r_1(k - u_1) + \alpha k]dt$$

subject to

$$\text{(3.4)} \qquad \frac{dk}{dt} = r_1(k - u_1) - (n + \alpha + \delta)k$$

$$\text{(3.5)} \qquad \frac{d\sigma}{dt} = (r_1 u_2 - \alpha - \delta - n)\sigma$$

and the constraints

$$\text{(3.6)} \qquad k \geq 0 \ , \quad \sigma \geq 0$$

$$\text{(3.7)} \qquad 0 \leq u_1 \leq (r_1 - \alpha/r_1)k$$

$$\text{(3.8)} \qquad 0 < A \leq u_2$$

$$\text{(3.9)} \qquad u_1 \leq (1 - u_2)k \ .$$

Here u_1 and u_2, the control variables have been defined as (equation (1.20)).

$$\text{(3.10)} \qquad u_1 = \frac{a_0''}{r_1} \ ; \quad u_2 = 1 - a_1'' \ .$$

The upper bound on u_1 given in (3.7) is obtained from (2.1), as gross investment per household has to exceed zero $[G = a_0'']$. The lower bound on u_1 given in (3.8) takes note of the fact that government expenditures are non-negative. The lower bound, A , insures that the marginal net tax rate a_1'' will be less than unity, thereby retaining the incentive to save. Restrictions such as (3.9) must also be placed on the control variables to insure that no household is taxed more than its gross capital income.[5] Although such restrictions may be binding during the transition to an equilibrium point, they will not affect the values of k and σ once an optimal steady-state has been attained.

The above optimal control problem is linear in the controls u_1 and u_2 ; thus the optimal policies will be of the "bang-singular-bang" type [1, pp. 261-265], i.e., the control variables will move between their boundary values and an interior value(s) corresponding to the singular arc(s).

---

[5] From expressions (3.17) and (3.18) derived later in the text, it is clear that this condition will always hold as an equality in steady-state equilibrium. In situations where k is rising and σ is falling, this constraint will not be binding. However, if k is falling, the lower bound on disposable income may dominate the upper bound on u_1 , derived from the non-negativity constraint on gross investment. In any case, our main qualitative results will not be altered, provided, as is the case in the problem being considered, the transition to the optimal steady state always implies a decline in u_2 .

We will only consider the case where, for all feasible values of k and σ

$$\frac{\partial f(k,\sigma)}{\partial \sigma} \leq 0 \ . \tag{3.11}$$

The function $f(k,\sigma)$ is only concave if the output share of the labor-intensive good declines as k increases [5]. Denote the undiscounted costate variables by λ_k and λ_σ respectively; then the Hamiltonian has the form [1]

$$H \triangleq e^{-\gamma t}[U(\hat{c}) + \lambda_k \dot{k} + \lambda_\sigma \dot{\sigma}] \ . \tag{3.12}$$

Then the optimality conditions yield ([1], pp. 261-265)

$$\frac{\partial H}{\partial u_1} = 0 = U_c \cdot r_1 - r_1 \cdot \lambda_k \tag{3.13}$$

$$\frac{\partial H}{\partial u_2} = 0 = r_1 \cdot \sigma \cdot \lambda_\sigma \tag{3.14}$$

and

$$-\dot{\lambda}_k = U_c(f_k - r_1 + \alpha) + \lambda_k(r_1 - \alpha - n - \delta - \gamma) \tag{3.15}$$

$$-\dot{\lambda}_\sigma = U_c(f_\sigma) - \lambda_\sigma(n + \delta + \gamma - r_1 u_2) \ . \tag{3.16}$$

In addition, we have the dynamics which are given by (3.4) and (3.5). In equilibrium, $\dot{k} = \dot{\sigma} = \dot{\lambda}_k = \dot{\lambda}_\sigma = 0$. This gives

$$u_1 = \left\{ 1 - \frac{n + \alpha + \delta}{r_1} \right\} k \tag{3.17}$$

$$u_2 = \frac{n + \delta + \alpha}{r_1} . \tag{3.18}$$

Note that $n + \delta + \alpha/r_1 > A$ for a steady-state to exist. Since $r_1 > 0$ and $\sigma > 0$

$$\lambda_\sigma = 0 \tag{3.19}$$

$$\lambda_k = U_c . \tag{3.20}$$

The two remaining variables k and σ can be obtained from (3.15) and (3.16)

$$\frac{\partial f(k,\sigma)}{\partial k} = n + \delta + \gamma = \rho + \delta \tag{3.21}$$

$$\frac{\partial f(k,\sigma)}{\partial \sigma} = 0 . \tag{3.22}$$

Given that $g_k > 0$, condition (3.21) will hold only if the net social rate of discount (γ) exceeds the private rate of return (r_1) , from (2.17). Otherwise, g_k must be negative. This may well be the case, as indicated in Section II, if Commodity 2 is inferior at the equilibrium point.

As yet we have not determined the standard deviation of the distribution of income. Total income for the jth household, $\hat{y}_j$, is given by

(3.24) $$\hat{y}_j = \text{net capital income and wage income}$$
$$= r_1 k_j - NT_j + \hat{w}_j \ .$$

From the formula for the variance of grouped data, we have from (2.5) and (3.18)

(3.25) $$\sigma_{\hat{y}} = \sqrt{(n+\delta+\alpha)^2\sigma^2 + w_1^2\tilde{e}(1-\tilde{e}) + 2(n+\delta+\alpha)\beta\sigma w_1\tilde{e}} \ .$$

With σ greater than or equal to its optimum value, it can be shown that $\partial\sigma_{\hat{y}}/\partial\sigma > 0$; further, if $\partial\sigma_c/\partial\tilde{e} < 0$, then $\partial\sigma_{\hat{y}}/\partial\tilde{e} < 0$.

We can now plot the movement of key target variables along the optimal trajectory. For the case $g_k > 0$ (neither good inferior), the curves of constant GDP per household, employment rate and $\sigma_{\hat{y}}$, the standard deviation of the distribution of after-tax income are as shown[6] (for a specified equilibrium point (k^*,σ^*)) in Figure 3. The behavior of the various quantities k , σ , $\sigma_{\hat{y}}$, e and GDP per household along the optimal trajectory clearly depends upon the location of the initial

---

6

$$\text{GDP} = \text{constant} = \left(\frac{d\sigma}{dk}\right)_{GDP} = \left|\frac{g_k}{g_\sigma}\right|\left(1 + g_k\frac{r_1}{w_1}\right) \ ; \quad g_k > 0$$

$$e = \text{constant} = \left(\frac{d}{dk}\right)_e = |g_k/g_\sigma|$$

$$\sigma_y = \text{constant} = \left(\frac{d\sigma}{dk}\right)_y = \left|\frac{g_k}{g_\sigma}\right| \frac{1}{1+\theta} \ ; \ \theta > 0$$

values of k and σ (say) k_0 , σ_0 . This is summarized in Table 1 for the cases where $\sigma_0 > \sigma^*$.

From Table 1 it is clear that, in the case when $g_k > 0$, equity and growth are always consistent objectives, in the sense that increases in k do not bring about increases in the dispersion of after-tax income, as measured by $\sigma_{\hat{y}}$. However, it is worth noting that in Region III, where k and $\sigma_{\hat{y}}$ both decline, $\sigma_{\hat{y}}$ may well decline more slowly, especially so, since the employment rate $\tilde{e}$, is tending to reverse this decline. <u>In this situation, the income dispersion as measured by the coefficient of variation $\sigma_{\hat{y}}/\hat{y}$ may increase.</u>[7]

TABLE 1

Region	k	σ	GDP	$\tilde{e}$	$\sigma_{\hat{y}}$
I	⊕	↑	⊕	⊕	⊖
II	⊖	⊖	⊖	⊕	⊖
III	⊖		⊖	⊖	⊖
IV	⊖	↓	⊖	⊖	⊕

⊕ increasing

⊖ decreasing

[7]The expression for aggregate disposable income per household in steady-state equilibrium, derived from (3.13), is

$$\hat{y} = y - r_1 u_1 = f(k,\sigma) - r_1 - (n + \alpha + \delta)k$$

$$\partial\hat{y}/\partial k = w_1 g_k + (n + \alpha + \delta) .$$

In the case when $g_k < 0$, owing (say) to the good intensive in unskilled labor being inferior, the location of the level curves is for the most part reversed, as shown in Figure 4. The behavior of the key variables is summarized in Table 2. Here, growth and equity are not consistent objectives over a significant portion of the (k,σ) plane. In Region IV, $\sigma_{\hat{y}}$ is unequivocally increasing as k increases. Thus, while capital intensity is increasing, GDP per laborer, the employment rate and income equity are decreasing. In Regions II and III, there may exist situations where $\hat{y}$ is declining faster than $\sigma_{\hat{y}}$; this dispersion, as measured by the coefficient of variation, $\sigma_{\hat{y}}/\hat{y}$, is increasing along with k and possibly GDP per household.

TABLE 2

$g_k < 0$

Region	k	σ	GDP	$\tilde{e}$	$\sigma_{\hat{y}}$
I	⊖	↑	⊕	⊕	⊖
II	⊕	⊖	⊕	⊖	⊖
III	⊕		⊖	⊖	⊖
IV	⊕	↓	⊖	⊖	⊕

⊕ $\triangleq$ increasing

⊖ $\triangleq$ decreasing

IV. A TWO-CLASS MODEL

In the model just presented, it is possible for households headed by unskilled laborers to own capital even though the marginal propensity to save out of unskilled wage income is assumed to be zero. This is true simply because savings in these households can be financed out of subsidies. Suppose, by contrast, that unskilled households are unaffected by taxes and subsidies and thus never own capital; then a major change will occur in the optimal growth trajectory. Specifically, a conflict may well arise between output and equity even in the case where $\tilde{e}$, unskilled employment per household, is an increasing function of k.

Given a model in which unskilled laborers are assumed to be propertyless, let us define σ' as the standard deviation of capital among skilled-labor households (which constitute the capitalist class). We define σ'_w as the standard deviation of wage income within the unskilled labor group. In such a model, the standard deviation of disposable income, $\sigma_{\hat{y}}$, is given by

$$(4.1) \qquad \sigma_{\hat{y}} = \sqrt{(1-u_2)^2 r_1^2 \sigma'^2 (1-\lambda) + \lambda \sigma_w'^2 + V\bar{z}\bar{w}}$$

$$= \sqrt{(n+\delta+\alpha)^2 \sigma'^2 (1-\lambda) + w_1^2 (\lambda\tilde{e} - \tilde{e}^2) + V\bar{z}\bar{w}}$$

where

$$V\bar{z}\bar{w} = \frac{(w_1\tilde{e})^2}{\lambda} + \frac{[(n+\delta+\alpha)k]^2}{(1-\lambda)} - \hat{y}^2$$

$$= \lambda\left[\frac{w_1\tilde{e}}{\lambda} - \hat{y}\right]^2 + (1-\lambda)\left[\frac{(n+\delta+\alpha)k}{(1-\lambda)} - \hat{y}\right]^2 .$$

The parameter $V\bar{z}\bar{w}$ represents the weighted variance of disposable income per household in the skilled and unskilled-labor groups around the grand mean, $\hat{y}$. Under not very stringent conditions, it can be shown that the slope of the level curve for a fixed $\sigma_{\hat{y}}$ is negative in the (σ', k) plane where $\partial\tilde{e}/\partial k > 0$. In other words, while $\partial\sigma_{\hat{y}}/\partial\sigma$ is unequivocally positive, $\partial\sigma_{\hat{y}}/\partial k$ may also be positive.[8]

Since the variable σ' is a measure of capital dispersion only within the skilled labor group, expenditure inequality always decreases with σ' . Therefore, the long-run optimal value of σ' is its lower bound, zero. Given a level curve for $\sigma_{\hat{y}}$ in the (k, σ') plane which is negatively-sloped, the long-run optimum value of $\sigma_{\hat{y}}$ may be greater than its initial value when $k_0 < k^*$ and $\sigma'_0 > 0$. Since in this case it is possible for the percentage increase in $\sigma_{\hat{y}}$ to be greater than the percentage increase in $\hat{y}$, the coefficient of variation of disposable

---

[8] It can be shown that sufficient conditions for $\partial\sigma_{\hat{y}}/\partial k > 0$, when $\partial\tilde{e}/\partial k > 0$, are

(a) $\frac{w_1\tilde{e}}{\hat{y}} \leq \lambda B$, where $B = \frac{1}{\left[1 - \frac{(1-\lambda)\lambda}{(1-2\lambda)}\right]}$; and

(b) $\frac{(n+\delta+\alpha)}{w_1}\frac{k}{\tilde{e}} \geq 1$.

Condition (b) implies that average disposable income per household for the skilled-labor group, $(n+\delta+\alpha)k$, must exceed or equal that for the unskilled labor group, $\tilde{e}w_1$, an eminently plausible assumption. Assuming the ratio of unskilled to total labors, λ , is greater than .5, condition (a) indicates that the unskilled wage share of disposable income must be less than or equal to some fraction smaller than λ ; e.g., if

$$\lambda = .6 \text{ , then } \frac{w_1\tilde{e}}{\hat{y}} \leq .454 \text{ .}$$

income may increase as k rises. Such a conflict between disposable income and equity growth can occur, even though the unskilled employment rate is rising. The difference in average disposable capital income per household between the two classes rises as k increases, and this may more than counteract the effects of a reduced variance and a higher mean level of unskilled wage income associated with growing capital per household.

V. SUMMARY AND CONCLUSIONS

In this paper, we have developed a normative model for the size distribution of capital and income essentially free of assumptions regarding the underlying distributions. We proceeded to examine the impact of taxes and subsidies upon capital intensity, capital and income dispersion, employment rate and GDP per household, along consumption-optimal growth paths. We found cases in which output growth and income equity were not consistent objectives even though it was assumed that output was a decreasing function of the dispersion of capital holdings. In the case where unskilled laborers own capital and save out of capital income, such a conflict can occur when the employment rate for unskilled labor is a decreasing function of average capital per household. Alternatively, in situations where unskilled laborers do not accumulate capital, increases in capital equity and the unskilled employment rate may well be accompanied by increases in the coefficient of variation of disposable income.

The major assumption, explicitly stated in Section III, is that $\partial g/\partial\sigma < 0$, implying that the income elasticity of good 1, the capital intensive good, is greater than unity, with $\eta_1 > \eta_2$ and $\eta_2 < 1$. Only in this case is $g(k,\sigma)$ concave and the optimal control problem well-posed [5]. We do feel that this is not an overly restrictive assumption, inasmuch as all that is required is that g_σ be negative, however small it may be in magnitude. It is also supported by the empirical investigation of Cline [2, p. 176] and is consistent with Engel's law provided agricultural goods as a whole are more labor intensive than non-agricultural goods. In any case, the main purpose of this paper is to show that conflicts between

output and equity growth exist along a consumption optimal path <u>even when</u> decreases in capital dispersion have a positive effect on output in the static context.

Some apparently crucial assumptions may be altered considerably without affecting the qualitative results presented in Section III. In particular, we may adopt the alternative formulation in which the household savings function has a Keynesian form with identical marginal propensities to save out of unskilled wage and capital income. We show, in the Appendix, that under these conditions the expressions the long-run optimal values of k and σ are modified somewhat, but the main qualitative results of the comparative equilibrium analysis presented in Tables I and II in Section III do not necessarily break down.

ACKNOWLEDGMENT

We are grateful to David Sibley and to Professors Kenneth Arrow and John Fei for their helpful comments and criticisms.

REFERENCES

[1] Bryson, R. E., Jr. and Y. C. Ho. Applied Optimal Control (Blaisdell, 1970).

[2] Cline, W. R. Potential Effects of Income Redistribution on Economic Growth (Praeger, New York, 1972).

[3] Klein, L. R. An Introduction to Econometrics (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1962).

[4] Little, I.; R. Scitovsky; and M. Scott. Industry and Trade in Some Developing Countries: A Comparative Study (Oxford University Press, London, 1970).

[5] McCabe, J. L. "Effects of Changes in the Level and Distribution of Capital on Income Inequality," Growth Center, Yale University, Discussion Paper No. 169, February 1973.

APPENDIX

The Keynesian Savings-Function Case

Suppose that the savings function for a representative household in the jth group takes the form

$$\text{(i)} \qquad s_j = -\alpha_0 + \alpha_1(y_j - NT_j)$$

where y_j is the total gross income of the jth household and NT_j is the net tax. Then the change in the capital per household of the jth group will be determined by the relationship

$$\text{(ii)} \qquad \frac{dk_j}{dt} = -\alpha_0 + \alpha_1\{r_1[k_j - a_0''/r_1 - a_1''(k_j - k)] + w_1\tilde{e}_j\} - (\delta+n)k_j \ .$$

Let us assume that a_0'' is held constant and government savings per household, s_g, is controlled instead. Then the accumulation equation for aggregate capital per household, $\hat{k}$, may be written

$$\text{(iii)} \qquad \frac{d\hat{k}}{dt} = -\alpha_0 + \alpha_1 r_1 k_p + \alpha_1 w_1 \tilde{e} + s_g - (n+\delta)\hat{k}$$

$$= -\alpha_0 + \alpha_1 r_1 \hat{k} + \alpha_1 w_1 \tilde{e} + s_g - (n+\delta)\hat{k}$$

where k_p is private capital per household assumed equal to $\hat{k}$ [1] and $\alpha^* = \alpha_0 + \alpha_1 \bar{a}_0''$. The percentage change in the variance of total capital is given by the following expression (provided each household receives the same amount of capital from the government):

$$\text{(iv)} \qquad (d\sigma^2/dt)/\sigma^2 = 2\cdot\{[(1 - a_1'')\alpha_1 - (n+\delta)] + \alpha_1 w_1 (k_u - k_p)/\sigma^2\}$$

where k_u is average private capital per household for the unskilled-labor

---

[1] The simplifying assumption that $\hat{k} = k_p$ makes sense if government savings takes the form of public education and health expenditures; or if it takes the form of investments yielding services otherwise bought by consumers.

group. The expression for the percentage change in $(k_u - k_p)$ is given by

$$\text{(v)} \quad \frac{d(k_u - k_p)^2}{dt} \Big/ (k_u - k_p)^2 = 2\left\{[(1 - a_1'')\alpha_1 - (n+\delta)] + \frac{w_1(1/\lambda - 1)\tilde{e}\alpha_1}{(k_u - k_p)^2}\right\} .$$

It can be shown that a stable equilibrium, independent of a_1'', will be attained where the percentage change in σ^2 is equal to the percentage change in $(k_u - k_p)^2$. This will occur where

$$\text{(vi)} \qquad (1/\lambda - 1) = \frac{(k_u - k_p)^2}{\sigma^2} .$$

Therefore, with $(k_u - k_p) = -(1/\lambda - 1)^{1/2} \cdot \sigma$, we may write

$$\text{(vii)} \qquad \dot{\sigma} = [(1 - a_1'')\alpha_1 r_1 - (n+\delta)]\sigma - w_1\tilde{e}(1/\lambda - 1)^{1/2} .$$

The sum of government expenditure plus private savings is given by the relationship

$$\text{(viii)} \qquad c_g + s_g + s_p = (1 - \bar{a}_0'')\alpha_0 + \alpha_0 + \alpha_1(r_1 k + w_1\tilde{e})$$

where $\bar{a}_0''$ is the fixed value of a_0'', where s_p is private savings per household and c_g is government consumption per household. Define the control variables as

$$u_1 = s_g$$

$$u_2 = 1 - a_1'' .$$

If all the expenditures on the left-hand side of (viii) are included in

the final demand for the same good, then a function determining average unskilled employment per household from k and $u_2\sigma$ may be derived in a manner similar to the one outlined in Section II. With

$$\tilde{e} = g(k, u_2\sigma)$$

and the dynamics given by (iii) and (vii), the steady-state optimum solution involves the modified golden-rule condition and an equation

$$\text{(ix)} \qquad g_2(k, u_2\sigma) = 0$$

which only give the optimum value of the product $u_2 \cdot \sigma$.[1] From the steady

---

[1]The condition (ix) is obtained from the optimality condition

$$\frac{\partial H}{\partial u_2} = U_c(1-\alpha_1)g_2\sigma + \lambda_k\alpha_1 g_2\sigma + \lambda_\sigma\{\alpha_1 r_1\sigma - w_1 g_2\sigma(1/\lambda - 1)^{1/2}\} = 0$$

by proving that $\lambda_k = U_c$ and $\lambda_\sigma = 0$. The equality between λ_k and U_c is obvious from the condition

$$\frac{\partial H}{\partial u_1} = -U_c + \lambda_k = 0 .$$

It follows from the optimality condition $\partial H/\partial u_2 = 0$ that, if $\lambda_k = U_c$,

$$g_2 U_c = \lambda_\sigma\{\alpha_1 r_1 - w_1 g_2(1/\lambda - 1)^{1/2}\} .$$

The adjoint equation multiplied by $(1/u_2)$ may be written as

$$\partial H/\partial\sigma(1/u_2) = U_c g_2 + \lambda_\sigma\{\alpha_1 r_1 - w_1 g_2(1/\lambda - 1)^{1/2}\} - \lambda_\sigma \frac{(n+\delta+\gamma)}{u_2} = 0 .$$

Substituting the expression for $g_2 U_c$ into this equation yields

$$\lambda_\sigma = 0 \quad \text{if} \quad \frac{(n+\delta+\gamma)}{u_2} > 0 .$$

state equilibrium expression for u_2 derived from (vii),

$$\text{(x)} \qquad u_2 = \frac{(n+\delta)}{\alpha_1 r_1} + \frac{w_1 g(k,\ u_2 \cdot \sigma)(1/\lambda - 1)^{1/2}}{\alpha_1 r_1 \sigma}$$

it can be shown that[2]

$$\text{(xi)} \qquad u_2\sigma = \psi(\sigma, k) \quad , \quad \psi_\sigma > 0 \ , \quad \psi_k > 0 \ .$$

Thus, σ will tend to decline along the optimal trajectory until (xi) is met. From (x) it is clear that the effect of a change in the steady-state value of $\tilde{e}$ is given by

$$\text{(xii)} \qquad \frac{\partial \tilde{e}}{\partial k} = g_1 + g_2\sigma \frac{\partial u_2}{\partial k}$$

$$= g_1(1 + Q)$$

---

[2]Multiplying both sides of (x) by σ yields

$$u_2\sigma = \frac{(n+\delta)}{\alpha_1 r_1}\sigma + \frac{w_1 g(k,\ u_2\sigma)(1/\lambda - 1)^{1/2}}{\alpha_1 r_1}$$

By differentiating this equation totally with respect to σ and $u_2\sigma$ and rearranging, we obtain

$$\frac{\partial(u_2\sigma)}{\partial\sigma} = \frac{(n+\delta)}{\alpha_1 r_1 - w_1 g_2(k,\ u_2\sigma)(1/\lambda - 1)^{1/2}} > 0$$

if $g_2(k,\ u_2\sigma) < 0$. In a similar manner, we derive the expression

$$\frac{\partial(u_2\sigma)}{\partial k} = \frac{w_1 g_1 (1/\lambda - 1)^{1/2}}{\alpha_1 r_1 \sigma - w_1 g_2 (1/\lambda - 1)^{1/2}} > 0$$

if $g_2(k,\ u_2\sigma) < 0$.

where $$Q = \frac{g_2 w_1 (1/\lambda - 1)^{1/2}}{\alpha_1 r_1 - w_1 (1/\lambda - 1)^{1/2} g_2} .$$

If $g_2 < 0$, then $Q > -1$ and $\partial\tilde{e}/\partial k \gtreqless 0$ as $g_1 \gtreqless 0$. The change in the steady-state equilibrium value of $\tilde{e}$ with respect to σ is simply $u_2 g_2$, which is assumed to be negative. Since k and σ affect the steady-state value of $\tilde{e}$ in the same manner as they did in Section III when $g_k > 0$ or $g_k < 0$, the relative slopes of the level curves in the (k,σ) plane will be qualitatively similar to those shown in Figures 3 and 4, and the value of the absolute slopes will be positive if $g_k > 0$ and negative if $g_k < 0$.

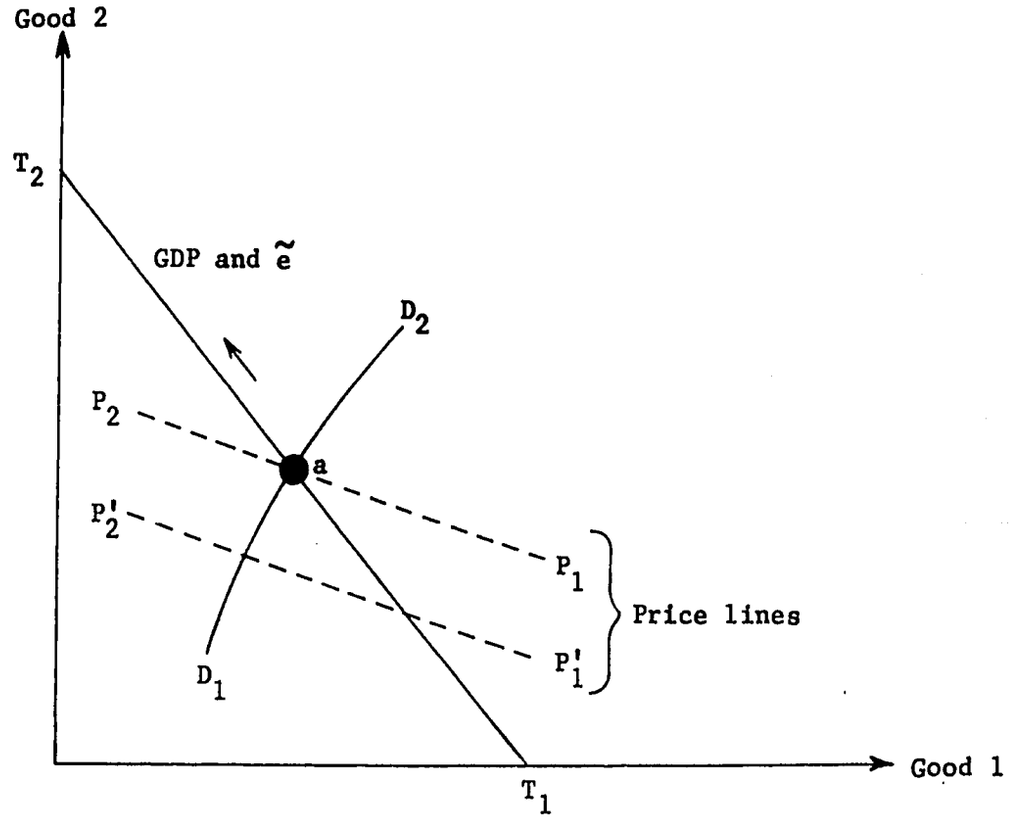
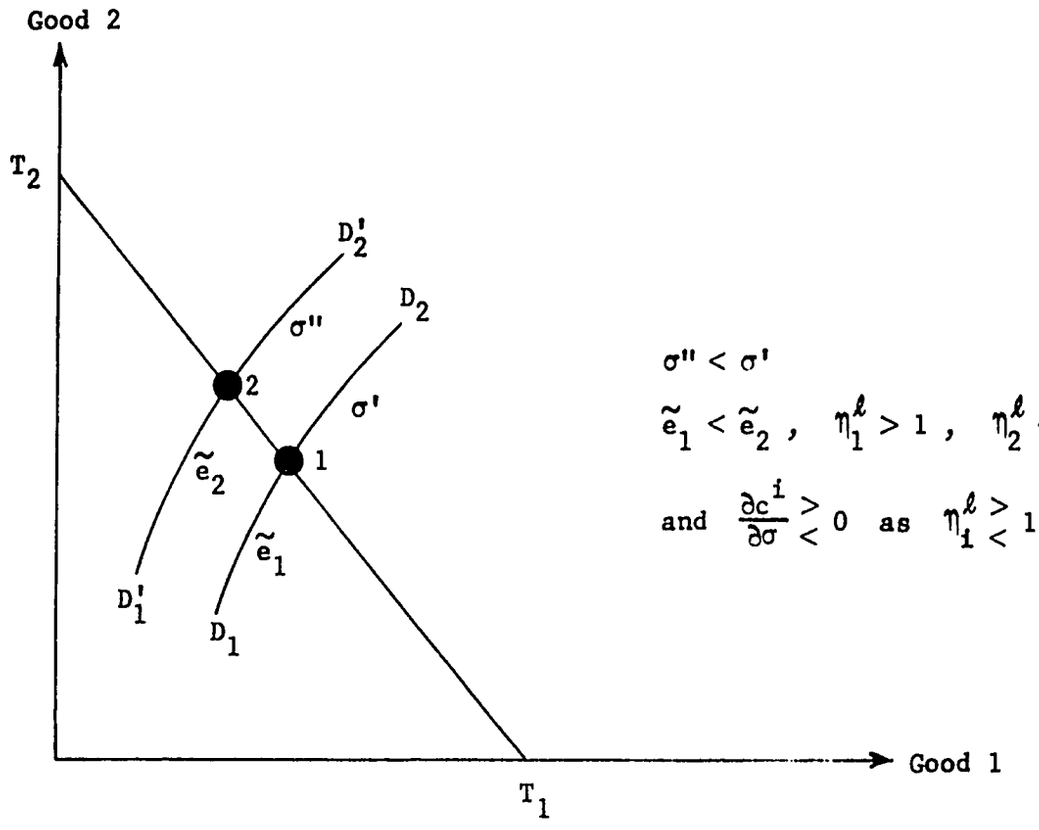


FIGURE 1



$\sigma'' < \sigma'$

$\tilde{e}_1 < \tilde{e}_2$, $\eta_1^{\ell} > 1$, $\eta_2^{\ell} < 1$

and $\frac{\partial c^i}{\partial \sigma} \gtreqless 0$ as $\eta_i^{\ell} \gtreqless 1$

FIGURE 2

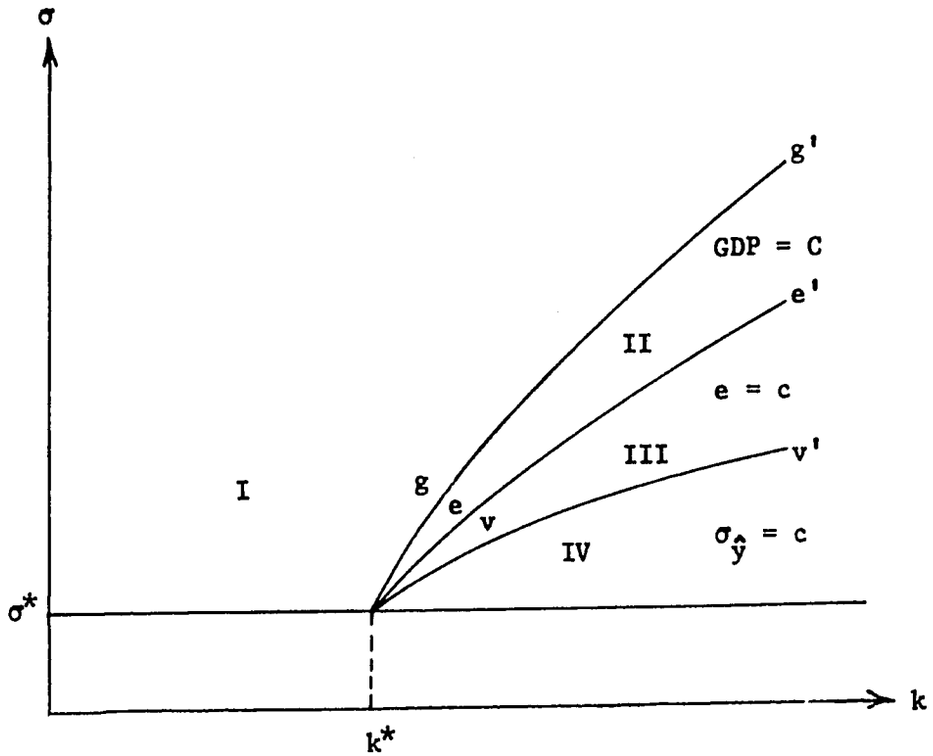


FIGURE 3. $g_k > 0$

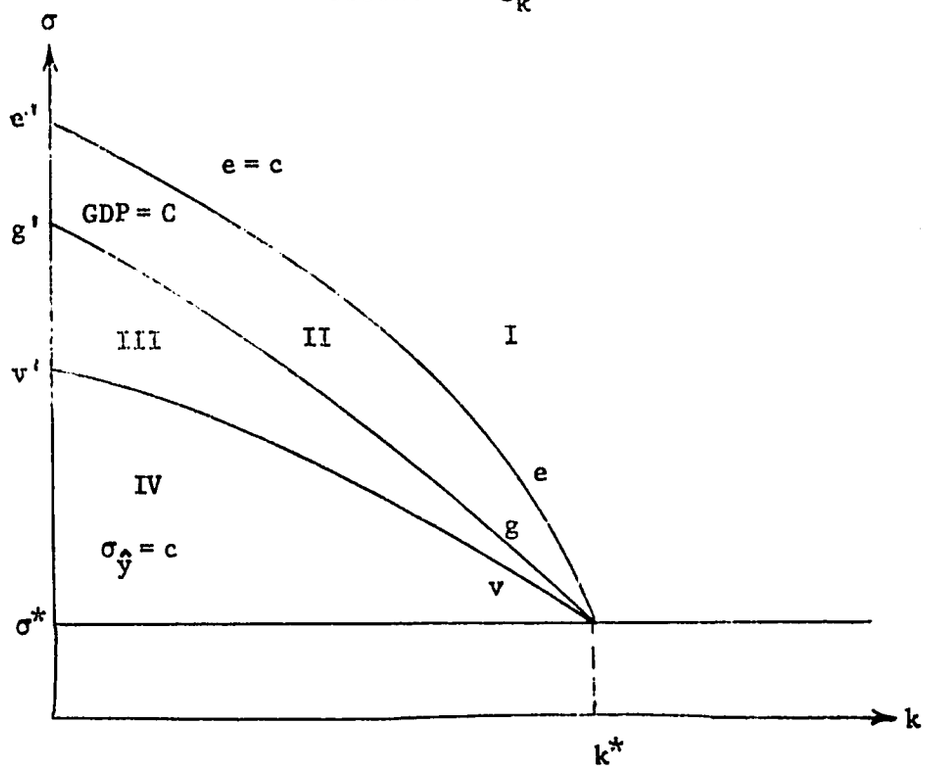


FIGURE 4. $g_k < 0$