AGENCY FOR INTERNATIONAL DEVELOPMENT
WASHINGTON, D. C. 20523

BIBLIOGRAPHIC INPUT SHEET

FOR AID USE ONLY
Batch 67

1. SUBJECT CLASSIFICATION

A. PRIMARY: TEMPORARY

B. SECONDARY:

2. TITLE AND SUBTITLE
Numerical solution of nonlinear planning models

3. AUTHOR(S)
Kendrick,D.A.; Taylor,Lance

4. DOCUMENT DATE
1968

5. NUMBER OF PAGES
38p.

6. ARC NUMBER
ARC

7. REFERENCE ORGANIZATION NAME AND ADDRESS
Harvard

8. SUPPLEMENTARY NOTES (*Sponsoring Organization, Publishers, Availability*)
(In Economic development rpt.no.98)

9. ABSTRACT

(ECONOMICS R&D)

10. CONTROL NUMBER
PN-AAE-191

11. PRICE OF DOCUMENT

12. DESCRIPTORS

13. PROJECT NUMBER

14. CONTRACT NUMBER
CSD-1543 Res.

15. TYPE OF DOCUMENT

AID 590-1 (4-74)

CSD-1543 Res. Harvard
PN-AAE-191

NUMERICAL SOLUTION
OF
NONLINEAR PLANNING MODELS

by

David A. Kendrick
Lance J. Taylor

Economic Development Report No. 98
May 1968.

PROJECT FOR QUANTITATIVE RESEARCH
IN ECONOMIC DEVELOPMENT
Center for International Affairs,
Harvard University,
Cambridge, Massachusetts.

NUMERICAL SOLUTION OF NONLINEAR PLANNING MODELS*

This paper discusses methods of numerical solution for nonlinear multisectoral planning models and their application to a particular model which could be used in the formulation of development programs for less developed countries. The numerical methods were originally developed by control theorists, and we devote some space to their derivation and evaluation since they are not well known to economists even though one can visualize a wide range of economic applications for them.

The model itself is based on ideas drawn from two recent lines of thought about economic growth over time. The first is that of neoclassical theoretical models designed to analyze the characteristics of an economy in asymptotic optimal growth, viz. Samuelson and Solow [25] and Koopmans [19]. The other line is that of finite horizon linear programming planning models, viz. Bruno [4], Eckaus and Parikh [10], and Chakravarty and Lefeber [7].

---

*This research has been financed in part by the Agency for International Development and in part by the Harvard Institute for Economic Research under a grant from the National Science Foundation. We are indebted to Rod Dobell, Hollis Chenery, J. A. Mirrlees, Louis Lefeber, Thomas Vietorisz, and Arthur Bryson for comments and suggestions and to Andy Szasz for programming assistance.

By drawing on the aforementioned numerical methods for solving dynamic nonlinear problems we have been able to blend the nonlinear production and welfare functions of the neoclassical models with the disaggregation and emphasis on foreign trade of the linear programs. In so doing we have employed an approach which offers growth theorists the possibility of using greater disaggregation than their present closed-form-solution methods permit and which offers economic planners the opportunity of specifying their models with nonlinear functions in both the performance index and the constraints.[1/]

In Section 1 we give a general statement of the discrete optimization problem we are interested in solving, along with the first order conditions for a constrained min-

---

[1/] While a number of authors have discussed models which share characteristics with ours, previous studies have not fully exploited the control theory approach to computer solution of dynamic problems. The pioneer in formulating such planning problems was Frisch, who in the 1950's proposed a number of models nonlinear in production and welfare functions (see, for example, [13]). At about the same time, Chenery and Uzawa [8] solved a model nonlinear in the foreign exchange constraint. Models with nonlinear welfare functions have been solved by Chakravarty [6], Barr and Manne [2], Brioschi and Rossi [3], and Johansen and Lindholt [16], while Mirrlees [21] has solved a one-sector model nonlinear in both welfare and production functions. Radner and Friedman [23] solved a modified multi-sectoral "log-linear" model with Cobb-Douglas welfare and production functions, but in published work have not discussed the applicability of their methods to more general functions. Finally, Dobell and Ho [9] and Stoleru [26] have obtained numerical solutions to models with "state variable" inequality constraints and with minimum time objective functions respectively by modified dynamic programming methods not akin to those we used.

imum. The statement and conditions serve as a basis for a discussion in Section 2 of the conjugate gradient and neighborhood extremal methods we employed to obtain numerical solutions. In Section 3 we sketch the four-sector economy-wide model to which we applied these techniques, and in Section 4 comment on their computational characteristics, especially speed of convergence. Finally, in Section 5 we demonstrate some of the effects of parameter variations on the solutions, concentrating on (i) effects of changes in the elasticity of substitution parameters and (ii) investigation of the turnpike properties of the model.

1. Formulation of the Problem in General Terms

There is a large literature, mainly in the engineering journals, on numerical methods for solving dynamic nonlinear optimizing models. In this and the next section we summarize some of the solution methods which control theorists have proposed, with some emphasis on their applicability to the kinds of problems which are likely to arise in economics.

An initial question in any study of dynamic problems is whether to work with discrete or continuous time. We formulated our models in discrete terms, largely for reasons of numerical convenience. One inevitably has to discretize problems for digital computer solution in any case, and an impressive body of oral tradition among numerical analysts led us to use difference equations <u>before</u> going to the computer, rather than letting the machine chop up time in differential equations according to the whims of a Runge-Kutta integration routine. Since the mechanics of setting up the discrete-time optimal control problem are not as well-known as the calculus of variations version, we review the problem here in general terms before discussing solution methods.[1/]

Consider the problem of finding the minimum of a scalar function

$$(1.1) \qquad J = \phi(x_{N+1}) + \sum_{i=1}^{N} L^i(x_i, u_i)$$

of n-vectors[2/] of state variables x_i and m-vectors of control variables u_i defined in each period $i = 1, 2, \ldots, N$ (and in period N+1 for the state variables). The evolution of the state variables is described by the vector-valued <u>system equations</u>

[1/] Much of the remainder of this section is based on the uncommonly clear control theory text of Bryson and Ho [5].

[2/] As usual, we treat all vectors as columns, and write x^T for the transpose of x.

(1.2) $$x_{i+1} = f^i(x_i, u_i).$$

We assume <u>initial conditions</u>

(1.3) $$x_1 \text{ specified}$$

and may require <u>terminal conditions</u>

(1.4) $$(x_{N+1})_j = \bar{x}_j$$

to be met by some components (indexed by j) of the vector x_{N+1}. (The constrained terminal components of x_{N+1} will not enter into the function $\phi(x_{N+1})$ in (1.1).)

The problem as stated can be solved in principle by standard methods of elementary calculus. We adjoin the system equations (1.2) to (1.1) with a sequence of multiplier vectors λ_{i+1},

(1.5) $$J^* = \phi(x_{N+1}) + \sum_{i=1}^{N} \{L^i(x_i, u_i) + \lambda_{i+1}^T [f^i(x_i, u_i) - x_{i+1}]\}.$$

For convenience, we define a scalar sequence H^i (analogous to the Hamiltonian in the continuous formulation) as

(1.6) $$H^i = L^i(x_i, u_i) + \lambda_{i+1}^T f^i(x_i, u_i)$$

and substitute into (1.5), also changing the indices of summation on the last term

(1.7) $$J^* = \phi(x_{N+1}) - \lambda_{N+1}^T x_{N+1} + \sum_{i=2}^{N} [H^i - \lambda_i^T x_i] + H^1.$$

Taking differential changes in J* due to differential changes in u_i, we get[1/]

$$(1.8) \qquad dJ^* = \left[\frac{\partial\phi}{\partial x_{N+1}} - \lambda_{N+1}^T\right] dx_{N+1} + \sum_{i=2}^{N} \left\{\left[\frac{\partial H_i}{\partial x_i} - \lambda_i^T\right] dx_i + \frac{\partial H_i}{\partial u_i} du_i\right\} + \frac{\partial H^1}{\partial x_1} dx_1 + \frac{\partial H^1}{\partial u_1} du_1$$

Our aim is to find conditions on the x's, u's and λ's which will guarantee that the standard first order optimality condition dJ*= 0 is satisfied. This task is considerably simplified if we <u>choose</u> the multiplier sequence λ_i so that

$$\lambda_i^T - \frac{\partial H^i}{\partial x_i} = 0 \qquad\qquad i = 2, \ldots, N$$

which implies that

$$(1.9) \qquad \lambda_i^T = \frac{\partial L^i(x_i, u_i)}{\partial x_i} + \lambda_{i+1}^T \frac{\partial f^i(x_i, u_i)}{\partial x_i} .$$

To simplify (1.8) further, we impose boundary conditions on the <u>costate equations</u> (1.9) as follows:

$$(1.10) \qquad \lambda_{N+1}^T = \frac{\partial\phi}{\partial x_{N+1}} .$$

[1/] For a scalar function $f(x_1, \ldots, x_n)$, define $\frac{\partial f}{\partial x}$ as the row vector $[\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}]$.

We have now reduced equation (1.8) to the form

$$dJ^* = \sum_{i=1}^{N} \frac{\partial H^i}{\partial u_i} du_i + \frac{\partial H^1}{\partial x_1} dx_1.$$

The vector x_1 is assumed given, so the second term vanishes. To set $dJ^* = 0$ and satisfy the first order necessary conditions for a minimum we finally specify <u>optimality conditions</u>

$$\frac{\partial H^i}{\partial u_i} = 0.^{1/} \tag{1.11}$$

Summarizing, to find the control sequence that gives a stationary value of the performance index J, we must choose at each step i

(a) The n state variables, x_i

(b) The n adjoint variables, λ_i

(c) The m control vectors, u_i

to satisfy, simultaneously,

(i) The n system equations, (1.2)

(ii) The n costate equations, (1.9)

1/ The discrete programming problem discussed here can be generalized by the addition of inequality constraints, in which case the Kuhn-Tucker conditions apply. For example, a minimum principle holds for the controls in a discrete system, i.e., if the vector u_i is constrained to lie within some set Ω, and if the set $\{f^i(x_i, u_i) : u_i \varepsilon \Omega\}$ is convex for all x_i, then simply minimizing H^i, $i = 1, \ldots, N$ for $u_i \varepsilon \Omega$ is a necessary condition for a minimum. (See Halkin [14] for details.)

(iii) The m optimality conditions, (1.11)

(iv) The initial and final boundary conditions, and (1.3) and (1.10).[1/]

2. Solution Algorithms

Finding a set of variables which satisfies all of these equations is not an easy task in any non-trivial problem. Most solution methods in fact are based on not satisfying some of the first order conditions. Rather, a nominal control history is chosen which satisfies some set of the conditions (i)-(iv). A system of difference equations linearized about this nominal is then formulated and solved in such a way as to indicate changes in the nominal solution which will bring it closer to satisfying the violated conditions. There are a number of possible sets of the conditions (i)-(iv) which could be relaxed in these "successive linearization" algorithms. Three that have been used are shown in Table 1.

[1/] If some of the terminal states are specified as in (1.4), the corresponding terminal costates satisfy the conditions

(1.10a) $(\lambda_{N+1})_j = \nu_j$

where the ν_j can be interpreted as Lagrange multipliers on the constraints $(x_N)_j = \bar{x}_j$.

Table 1

NOMINAL SOLUTION SATISFIES

	(i) System Equations	(ii) Costate Equations	(iii) Optimality Conditions	(iv) Boundary Conditions
Neighboring Extremal Methods	Yes	Yes	Yes	No
Gradient Methods	Yes	Yes	No	No
Quasilinearization Methods	No	No	Yes	Yes

We have experimented with the first two of these methods, and will sketch out both how they work and what we have learned about their applicability to economic optimization problems.

2.1 Neighboring Extremal Algorithms

This class of algorithms is based on the most straightforward (perhaps naive is a better word) method of solving optimal control problems. A typical recipe goes as follows:

(i) Choose an initial λ_2. Use (1.3) to obtain x_1. Use x_1 and λ_2 in (1.11) to solve for u_i. Use x_1 and u_1 in (1.2) to obtain x_2. Set $i = 2$.

(ii) Use x_i and λ_i in (1.9) and (1.11) to solve for u_i and λ_{i+1}.[1/]

[1/] In practice we found it useful to solve (1.9) and (1.11) separately, iterating between them until the λ_{i+1}'s converged. We did this by choosing λ_{i+1}'s, using these in (1.11) with an unconstrained minimization technique (see Fletcher-Powell [11]) to obtain u_i's, computing new λ_{i+1}'s with (1.9), and repeating this process until the λ_{i+1}'s converged.

(iii) Use u_i and x_i in (1.2) to obtain x_{i+1}. Set $i = i+1$.

(iv) Repeat steps (ii) and (iii) until the final state variables and adjoint variables are obtained. Observe how far these values miss their respective boundary conditions (1.10)(or (1.4) and (1.10a)) and use this information to modify the initial λ_2.

(v) Repeat steps (i)-(iv) until the final state and adjoint variables obtained in (iv) are "close enough" to the boundary conditions.

As one might expect, this approach is a bit too simple to be of general applicability.[1/] The problem is that state variables at the terminal time may be highly sensitive to changes in the guesses of the initial costates. This sensitivity

---

[1/] However, we have observed that multisectoral programming models in which (i) all the distributional relations hold as equalities, (ii) the production functions are linear, and (iii) the welfare functions are nonlinear in the controls, can be solved very quickly by naive methods. With a terminal scrap value in each sector we obtain λ^T_{N+1} from (1.10) and sweep the equation (1.9) backward to provide the λ path for each sector. The linearity of the production functions and the fact that the capital stocks (i.e., state variables) do not enter the performance index makes it possible to sweep (1.10) backward without computing the control or state variables simultaneously. The λ's can then be used to compute the control variables using (1.11) and the control variables can be used to integrate (1.2) forward to obtain the state variables.

Even in the case where terminal capital stocks are specified instead of terminal unit scrap values, the procedure outlined above leads to a very effective iterative scheme for solving this class of optimizing planning models.

problem arises because equations (1.9) and (1.2) when linearized about the optimal path are adjoint, with the property that solutions to the homogeneous parts of the two sets of equations may diverge from each other over time by many orders of magnitude. This divergence of fundamental solutions makes transmission of information about appropriate changes in λ_2 from the terminal time to the initial time very much subject to problems of numerical round-off error. In simple one-sector capital accumulation models with fifty-year planning horizons we found that under certain conditions changes of one unit in the eight significant digit of λ_2 would change the level of terminal capital stock by a factor of two or more!

These conditions may but do not necessarily occur and we found the neighborhood extremal method more efficient than the conjugate gradient method for solving one sector models, but less efficient than gradient methods on our multisectoral models. Had we undertaken the extensive programming required to translate the effects of differential errors in missing boundary conditions into differential changes in λ_2, via the backward integration of quadratic Riccati matrix differential equations, this method might have surpassed the gradient approach.[1/]

[1/] See Bryson and Ho [5], Section 7.3, for a discussion of such an algorithm.

2.2 Gradient Methods[1/]

As indicated in Table 1, most gradient algorithms work with a nominal solution which satisfies neither the optimality condition (1.11) nor terminal boundary conditions of the form (1.4) and (1.10a). The algorithms operate by iteratively improving the nominal control histories to meet these conditions. The boundary conditions may be approached by successive modification of estimates of the multipliers ν_j in (1.10a) or else by adding to the performance index (1.1) quadratic penalty functions of the form $\frac{1}{2} a_j[(x_{N+1})_j - \bar{x}_j]^2$ on deviations of the final states $(x_{N+1})_j$ from their specified terminal values $\bar{x}_j$. Modifying the multipliers is a somewhat neater approach since it does not distort the "true" shadow prices λ as do the penalty functions which substitute a condition of the form

$$(2.1) \qquad [\lambda^T_{N+1}]_j = a_j[(x_{N+1})_j - \bar{x}_j]$$

for the condition (1.10a). However, penalty functions are useful since they are easy to apply, proving quite feasible in our planning model with five state variables specified at the terminal time.

Given a nominal control history u_i, the steps of the gradient algorithm we used go as follows:

(i) Integrate the system equations (1.2) forward in time using the nominal u_i history.

[1/] We are grateful to Raman Mehra for giving us a copy of his conjugate gradient code, and to Robert Kierr for excellent programming work in helping us to modify this code for our purposes.

(ii) At terminal time, evaluate λ_{N+1} using (2.1) for the constrained state variables, and (1.10) for the unconstrained states.

(iii) Integrate the costate equations backward in time, and using the calculated λ_i and x_i histories, calculate the Hamiltonian (1.6) and its gradient with respect to the u_i.

(iv) Make a one-dimensional search in the gradient direction[1/] until the Hamiltonian is minimized; more precisely, this means first choosing a parameter α to minimize

$$g(\alpha) = H[u - \alpha \nabla H^T(u)]$$

where u is the current control history vector, H is the Hamiltonian, and ∇H is its gradient, and then calculating the new control history from the relationship

$$u_{new} = u - \alpha \nabla H(u).$$

We actually only approximated α, which is given by the condition

$$0 = \frac{dg(\alpha)}{d\alpha} = - \nabla H(u) \cdot [\nabla H(u - \alpha \nabla H(u))]^T, \tag{2.2}$$

by varying u in the gradient direction until

[1/] In actual practice, it is often preferable to search in a modified gradient direction -- the modifications attempting to take account of curvature properties of the Hamiltonian as a function of the u_i. We used a "conjugate gradient" procedure which at each step changes the u_i according to a weighted average of this step's gradient and last step's direction of movement. For details, see Lason, Mitter and Warren [20], and Fletcher and Reeves [12].

the quantity on the right of (2.2) changed signs, and then fitting an interpolating polynomial in α to the control vectors calculated just before and after the sign change.[1/]

(v) Return to step (i) with the new u-history, after possible modifications to the penalty function constants a_j in (2.1).

First order gradient methods of this type typically show rapid convergence in the first few iterations, but perform relatively poorly near the optimum. In addition, they have the drawback of not picking out with complete accuracy the optimal control histories. Both of these problems can be rectified near the optimum by use of either neighboring extremal methods (if they converge), or of second order gradient methods which take account of curvature properties of the Hamiltonian. However, both second order methods and neighboring extremal algorithms with modifications to counteract numerical instability require large inputs of programming logic and computation time. It is not clear in view of the high marginal costs of these inputs that the improvements which can be made on first order solutions are economically justifiable. We have found that first order methods will typically determine control histories to three significant

---

[1/] We imposed non-negativity constraints on some control variables, and when these applied we simply set the negative controls equal to zero after fitting the polynomial. A more complete procedure would involve (a) minimizing the unconstrained discrete Hamiltonian, (b) checking for negative controls and setting them to zero, (c) re-minimizing the Hamiltonian thus constrained. We did not go through these additional calculations largely because our problems seemed to converge well with the simpler procedure.

digits in a score or so of iterations.[1/] Since the parameters in our problems are known at best to this degree of precision, better solutions seem somewhat extravagant.

3. Application to a Four-Sector Model[2/]

The numerical methods can be demonstrated on a four-sector model which is large enough to provide a good test of their computational feasibility.

The four sectors are (1) agriculture and mining, (2) heavy industry, (3) light industry, and (4) services. The basic structure of the model is to maximize a welfare function over a thirty year period subject to constraints in the form of distribution relations, production functions, absorptive capacity functions, foreign exchange constraints, and initial and terminal capital stock and foreign debt constraints.

3.1 The Welfare Function

In a number of linear programming models (e.g., Bruno [4]) the welfare function has been specified in something like the following form:

$$(3.1) \qquad \xi = \sum_{i=1}^{N} (1+z)^{-i} \sum_{j=1}^{4} c_{ji}$$

[1/] See Section 4 for a discussion of convergence of our solution methods for the model considered in this paper.

[2/] We specified parameters for the model on the basis of recent South Korean data. For more detail, see [18], and footnote 2, p. 26.

where z = consumption discount rate

i = time period index

j = sector index

c = consumption

That is, the discounted sum over time of each year's total consumption is the maximand. To make the problem well-posed, each c_{ji} has been constrained by a linearized income elasticity formula to bear a certain relationship to $\sum_j c_{ji}$, i.e., total consumption in period i.

We have adopted a superficially different but actually rather similar welfare function,

$$(3.2) \qquad \mathcal{E} = \sum_{i=1}^{N} (1+z)^{-i} \sum_{j=1}^{4} a_j c_{ji}^{b_j} \qquad 0 \leq b_j < 1 \quad a_j > 0.$$

Connoisseurs of consumption theory will recognize that the inner sum is just Houthakker's "direct addilog" utility function [15], which gives rise to fairly simple demand functions for the c_{ji} in terms of total consumption and the parameters a_j and b_j. We specified these parameters by setting nominal Korean 1965-66 prices to unity, and then choosing the a_j and b_j (non-uniquely) so that income elasticities and consumption shares from the direct addilog function approximately equaled those in Korea in that year.

3.2 Capital Stock Accumulation Equations

In most planning models, capital stock accumulation equations are written in the form

$$k_{j,i+1} = k_{ji} + \delta_{ji}$$

where j indexes sectors and i indexes time.[1/] The linear programmers usually put upper bounds (for "absorptive capacity") on the δ_{ji}, while constraints of the form $\delta_{ji} \geq 0$ (which guarantee non-shiftability of capital) appear routinely in linear programs and were used explicitly by Johansen and Lindholt [16] in their nonlinear model.

We have adopted both of these assumptions, by specifying that the δ_{ij} should be non-negative,[2/] and by rewriting our accumulation equations in the form

$$k_{j,i+1} = k_{ji} + g_j(\delta_{ji}, k_{ji}) \tag{3.3}$$

where $g(\delta_{ji}, k_{ji})$ is a function suggested by Robert Dorfman[3/]

[1/] The addition of a depreciation term for the capital stock in each sector adds no essential difficulty to the problem.

[2/] See footnote 1, p. 14, for details on how we imposed non-negativity constraints in computation.

[3/] The function was suggested for a different model and we have adopted it for use here. Both Sam Bowles and Louis Lefeber have suggested to us that an absorptive capacity relationship should include educated or highly skilled labor as one of the inputs. While we are in agreement, we have not implemented that suggestion in the present model.

which specifies decreasing returns to investment δ_{ji} in the creation of new capital stock. More explicitly, we write

$$(3.4) \qquad g_j(\delta_{ji}, k_{ji}) = \Delta k_{ji} = \mu_j k_{ji} \left[1 - \left(1 + \frac{\varepsilon_j}{\mu_j} \frac{\delta_{ji}}{k_{ji}} \right)^{-\frac{1}{\varepsilon_j}} \right]$$

$$\varepsilon_j \geq -1$$

$$\mu_j \geq 0.$$

The assumption behind this function is simply that as the increase in capacity (Δk) approaches some fraction μ of existing capacity k, then investment (δ) becomes less and less effective in increasing Δk. For values of $\varepsilon \geq 0$, the function $g(\delta,k)$ is concave, and increases asymptotically to μ, the absolute upper bound on percentage capacity expansion. For $\varepsilon < 0$, g is still concave, but becomes conplex-valued at the ordinate μ. On grounds of numerical convenience, we set $\varepsilon = 0.5$, and chose μ in the vicinity of 0.3.[1/]

3.3 Distribution and Production Functions

The distribution relations are of the standard input-output type:

$$(3.5) \qquad q + Dq + m = Aq + B\delta + e + c$$

where

q = vector of sectoral output levels

δ = vector of investment levels

m = vector of untied imports into sectors

e = vector of exports

c = vector of consumption levels

D = diagonal matrix of marginal propensities to import for production

A = Leontief matrix

B = capital coefficient matrix

We have specified the top and bottom rows of B to consist of zero elements, i.e., the agriculture and mining sector and the services sector provide negligible amounts of inputs to capital formation.[1/]

In line with recent empirical work, we assume that production functions are CES (constant elasticity of substitution):[2/]

[1/] Actually this assumption is also an empirical result. We aggregated an 18-sector Korean input-output "B" matrix to get our matrix. Only 5 out of the 18 sectors actually produce capital goods, and these were all aggregated into our "Heavy" and "Light" industry sectors.

[2/] We use the constant returns to scale form. Diminishing returns specification would add no essential complication but the increasing returns specification would make the problem nonconvex.

$$(3.6) \qquad q_{ji} = \tau_j (1+\nu_j)^i \left[\beta_j k_{ji}^{-\rho_j} + (1-\beta_j)\ell_{ji}^{-\rho_j}\right]^{-\frac{1}{\rho_j}}$$

where

q = output

τ = efficiency parameter

ν = rate of technical progress

β = distribution parameter

k = capital input

ℓ = labor input

$\rho_j = \frac{1}{\sigma_j} - 1$ where σ_j is the elasticity of substitution for the jth sector.

We assume full employment in the model, so the sector labor forces are constrained to add up to the total labor available in any given period:

$$(3.7) \qquad \sum_{j=1}^{4} \ell_{ji} = \ell_i .$$

In the interest of minimizing the number of control variables in the model, we specified the sectoral exports exogenously,

$$(3.8) \qquad e_{ji} \text{ given, } \quad \text{all } j,i$$

using constant exponential growth paths in 10-year periods to make projections on the basis of current export levels.[1/]

[1/] If we were to make exports endogenous, we could use price elasticities to generate foreign exchange revenues from real exports for use in the trade constraint (3.15). Lacking data to do this, and wishing to keep our model to an easily manageable size, we adopted the alternative of "parametric programming" about a given export path.

Also in the interests of simplicity, we allowed no untied imports into sectors one (primary production) and four (services):

$$(3.9) \qquad m_{1i} = m_{4i} = 0, \quad \text{all } i$$

Using the given export paths and all the different kinds of imports, we may write a foreign debt "accumulation equation" of the form

$$(3.10) \qquad \gamma_{i+1} = (1 + \theta)\gamma_i + \sum_{j=1}^{4} (d_{jj}q_{ji} - e_{ji} + \pi_j\delta_{ji} + m_{ji})$$

where

γ_i = foreign debt

θ = interest rate on foreign debt

d_{jj} = elements of D, i.e., marginal propensities to import for production

π_j = marginal differential propensity to import for capital formation.

We know initial foreign debt and can constrain terminal debt to be at a given level,

$$(3.11) \qquad \gamma_1 \text{ known}; \quad \gamma_{N+1} \text{ chosen},$$

but we have no explicit constraints on the level of debt at any intermediate time period.[1/]

[1/] We are thus solving an isoperimetric problem with respect to foreign debt, specifying a given change in debt (from γ_1 to γ_{N+1}) and letting the model optimally allocate this change over time. (An analogous problem in the classical calculus of variations is finding the maximum area which can

The system has 5 state variables (4 k_{ji}'s and γ_i) and 14 control variables (4 δ's, 4 c's, 2 m's, and 4 ℓ's). However, there are effectively only nine controls because (3.5) and (3.7) can be used to express the four c_j's and ℓ_1 in terms of the other controls.

In summary, our problem is to maximize the discounted utility sum (3.2), subject to the capital stock and foreign debt accumulation equations (3.3) and (3.10). The control variables in these equations are the four investment "activities" (δ_{ji}), the labor forces in sectors two through four, and untied imports into sectors two and three. Consumptions and the labor force in sector one are given in terms of these controls by equations (3.5) and (3.7) while (3.4) and (3.6) describe the absorptive capacity and production functions. We know initial foreign debt and initial capital stocks, and specify terminal levels for these state variables in (3.11) and in

(3.12) $\quad k_{j1}$ known; $k_{j,N+1}$ chosen $\qquad j = 1, \ldots, 4.$

The problem is thus completely stated (with second order

---

be enclosed by a given length of rope.) Possible alternative treatments of foreign debt are (a) using penalty functions to hold debt at any time "close" to some predetermined level; (b) putting inequality constraints on the level of debt in each period. The former alternative is computationally feasible, although our debt paths seemed well enough behaved for us not to bother with it. The latter approach, involving state variable inequality constraints, is difficult to handle computationally.

conditions satisfied because of the linearly homogeneous production functions, concave absorptive capacity functions, and convex utility function), and can be solved by an application of the methods of last section.

4. Computational Properties

All of our computational work was done on the time-sharing system of the Massachusetts Institute of Technology. Time-sharing proved a useful adjunct to our gradient algorithms, since it facilitated operator intervention to change both the step size in the one-dimensional search procedure, and the constants in the quadratic penalty functions on deviations of the terminal state variables from their targets.

Using the M.I.T. system, each full iteration of the algorithm for the four-sector, thirty-period model took between thirty and fifty seconds of computation time (with an IBM 7094), depending on the number of steps made in the one-dimensional search.[1/] By comparison of the four-sector model to simpler ones, it appears that computation time per iteration increases roughly in proportion to the length of the planning horizon and the number of state variables. The number of iterations taken until an "optimal" solution is reached naturally depends

---

[1/] MAD, the coding language used, tends to produce inefficient machine-language computer programs in comparison to, say, FORTRAN IV. Hence, there is every reason to believe that our computation times per iteration could well have been reduced by a factor ranging from two to five if we had worked with another programming language.

on the initial nominal path chosen, and on the stopping criteria. Our stopping criteria were (a) improvement in the objective function had to be less than two tenths of a percent for at least five iterations; (b) the difference of the terminal state variables from their targets had to be less than one percent of the target for the same number of iterations. Using these criteria, and choosing nominal paths from closely related versions of the model (e.g., the nominal path would be the optimal solution for a model differing by only one or two parameters from the one actually being solved), we usually obtained convergent solutions in 20 or 30 iterations.

In general, we found convergence to be slightly slower for the state variables than for the objective function, and somewhat slower still for some of the control variables, especially the labor forces in the two industrial sectors and investment in primary production and heavy industry. However, the relative variation for these controls had usually been reduced to the order of 10^{-2} well before our stopping criteria took effect.[1/] Figure 1 illustrates the control paths (plotted at five-year intervals) calculated for two of the more volatile sectors at various points during a run on which we

---

[1/] Rapid approximate convergence of the controls is a characteristic to be expected of solutions from first order gradient algorithms. Exact convergence can usually not be obtained with such algorithms, although it is possible with second order methods which in effect use first derivative information on the controls.

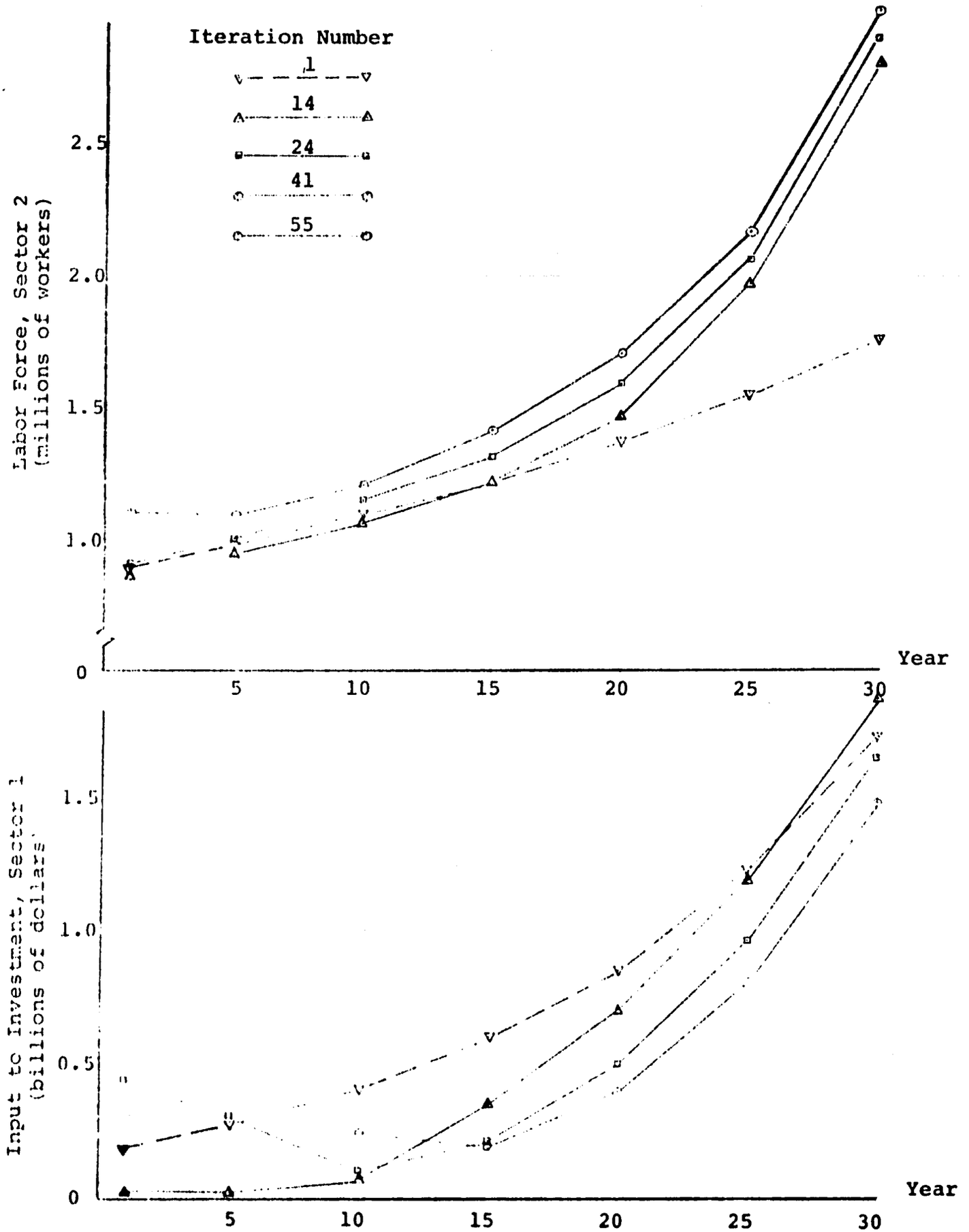


Figure 1

CONVERGENCE

took fifty-five iterations. The graphs indicate that by about twenty-four iterations, the algorithm had modified the pre-guessed control paths until their "shape" was fairly close to optimal. By forty-one iterations, the solution was essentially complete to the scale of the graph, since the paths for forty-one and fifty-five iterations are indistinguishable.

5. Some Numerical Solutions[1/]

In a companion paper [18], we describe in detail the derivation of numerical parameters for the model,[2/] and solutions under a number of parameter variations. Here we discuss briefly two aspects of the solutions which are of

---

[1/] David Cole and Larry Westphal generously helped us with accumulating the data we needed and also made useful suggestions about the formulation of the model.

[2/] As much as possible, we used recent data for South Korea for our parameters, in particular the input-output "A" and "B" matrices, the initial capital stock and debt levels, and the export projections. We assumed values for the elasticities of substitution and the distribution parameters in the production functions, and chose the efficiency parameters to equate production function output to Korean gross sectoral output in 1965-66 (assuming the labor force levels of that year). The welfare function parameters were chosen to give consumption income elasticities and shares consistent with those of Korea at her current income level (this choice was non-unique), while terminal capital stocks were estimated by applying "reasonable" growth rates over the thirty-year planning period to the initial stocks. Terminal debt was set according to the rule of thumb that interest plus amortization payments (a total of about 10% of the debt) should equal 20% of exports in the terminal year. In the absorptive capacity constraints, we set the absolute upper bound on the percentage change in capital stock (μ) at a level of about 30% (varying slightly from sector to sector), and set the parameter ε equal to 0.5 in all sectors.

general interest--the turnpike properties (or lack of them) of the model, and the effects on sectoral labor allocations of changes in the production elasticities of substitution.

5.1 Turnpike Properties

It is well known from economic theory that certain types of closed economy neoclassical models exhibit turnpike behavior in the sense that for most of a sufficiently long planning period, such a model will be in a balanced growth state with resource allocations approximately equal to those prevailing asymptotically in an infinite horizon plan [25]. As a corollary to this theorem, one might expect that the initial stages of a sufficiently long plan would be quite insensitive to terminal conditions. In some previously reported experiments with one-sector closed economy models [17], we found this type of behavior--in a model with a fifty-year planning horizon, the first twenty years of the plan were essentially unaffected by a wide range of terminal conditions.

The more complex model of this paper also displays the hypothesized getting-to-the-turnpike properties, but to a more limited extent. Figures 2 and 3 illustrate.

Figure 2a shows capital stock accumulation paths for sector one. We see that for the two lower terminal stocks (based on capital growth rates of 5.0% and 6.5%), the first

Figure 2

CAPITAL STOCK PATHS WITH VARIOUS TARGET CAPITAL STOCKS

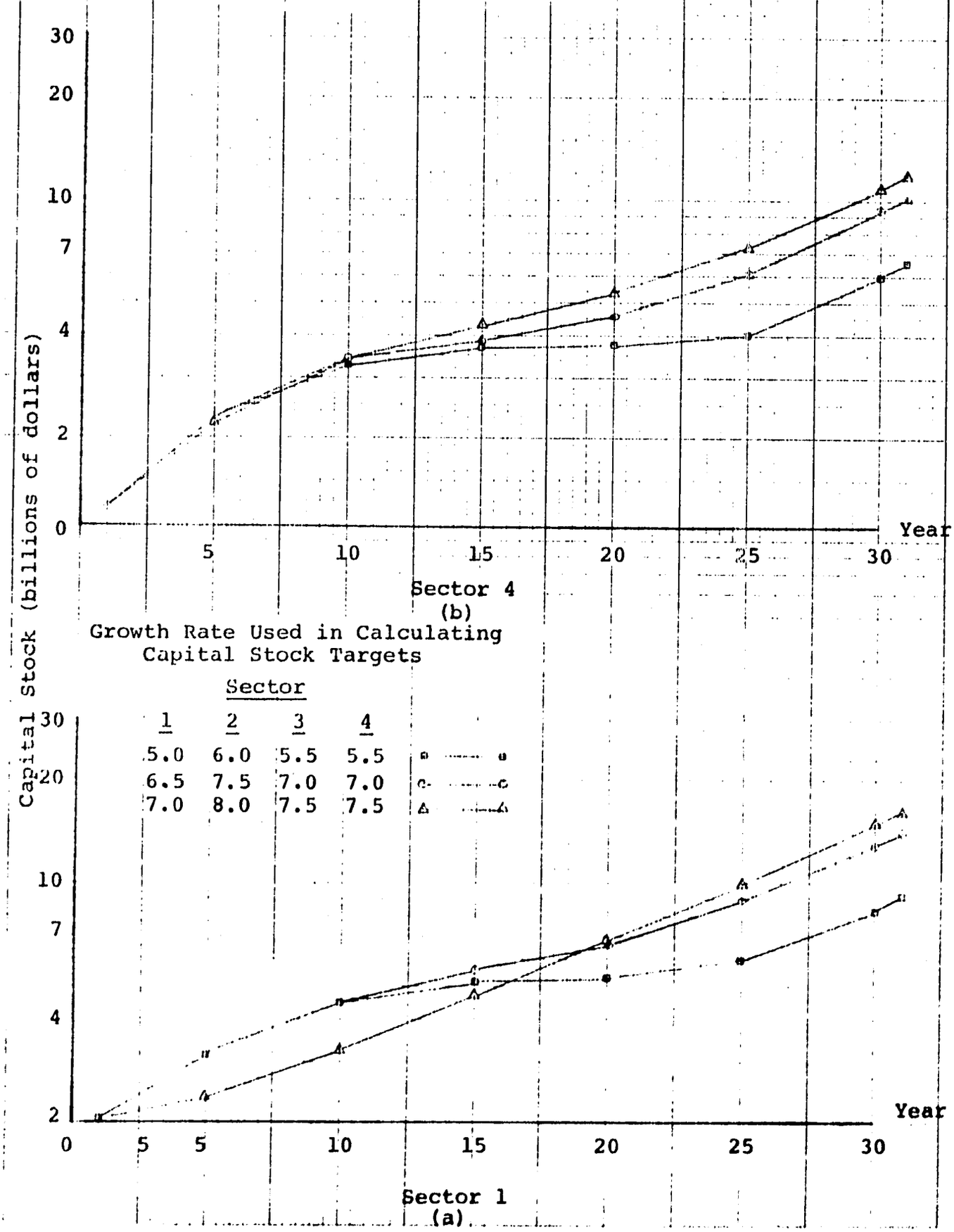
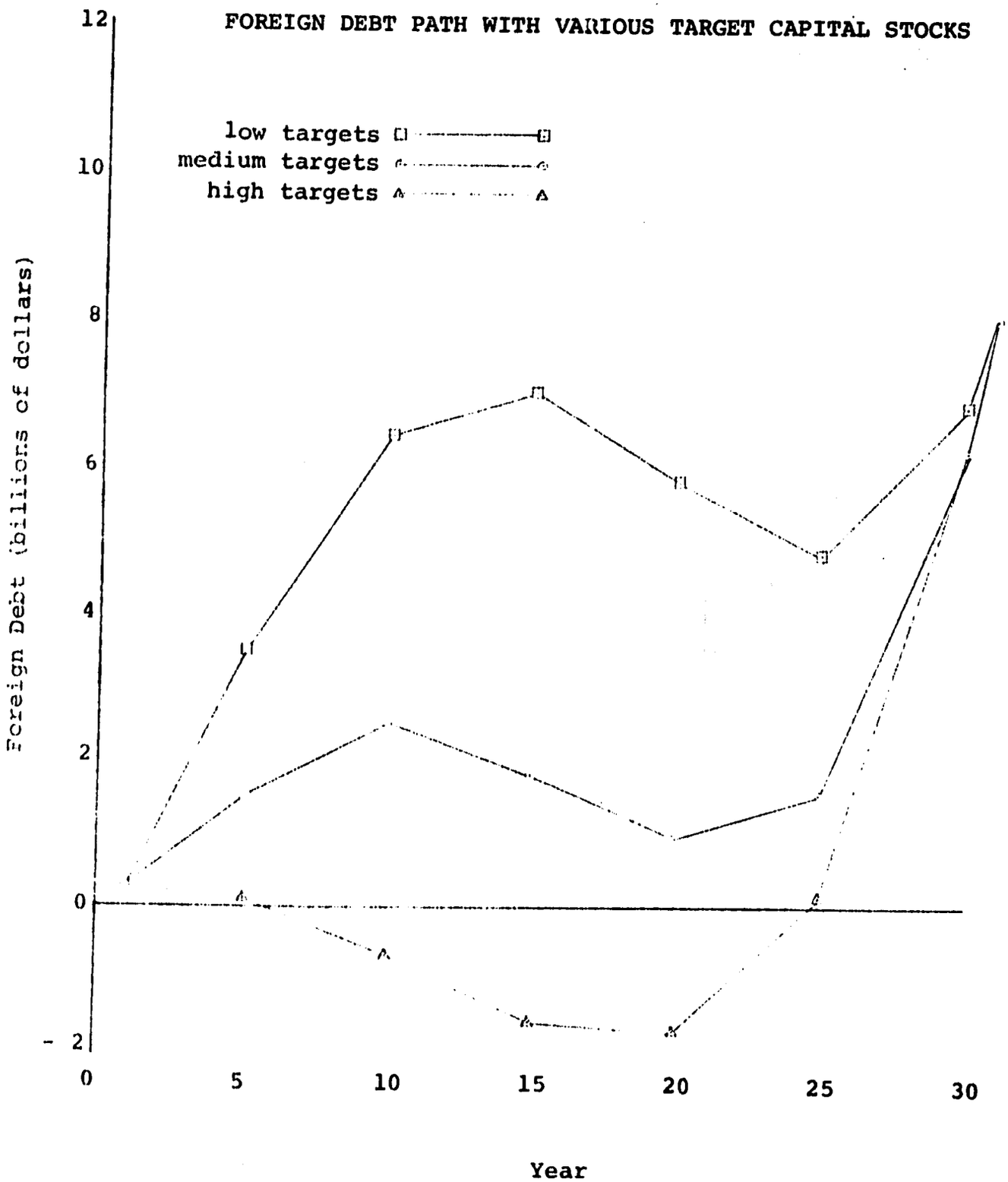


Figure 3

FOREIGN DEBT PATH WITH VARIOUS TARGET CAPITAL STOCKS



10 to 15 years of the plan are largely independent of the terminal conditions, while for the high terminal stock (based on a 7.0% growth rate), the accumulation path differs greatly from the other two. In Figure 2b, by contrast, the first ten years of the plan for sector four are unaffected by terminal conditions for all three terminal stock targets (based on growth rates of 5.5%, 7.0%, and 7.5%).

Figure 3, which shows foreign debt paths, helps explain this contrasting behavior. As it turns out, sector one has a high foreign exchange component in investment (i.e., π_1 is relatively large in equation (3.10)), while sector four's investment importation coefficient is relatively low. It can be seen from Figure 3 that as the terminal stock targets are increased, total foreign debt over the planning period is reduced, even becoming negative during the middle years of the high target plan. The mechanism by which the reduction of debt between the low target and medium target plans takes place involves the untied imports m_{2i} and m_{3i}. (See again equation (3.15).) These are drastically reduced between the two lower target solutions. In the high target solution, these "slack" import variables are forced to zero, and other things must be adjusted by the model in order for it to hold terminal debt down to the required level of 8.0 (billion dollars). As it turns out, investment in sectors with a high import component in capital formation is deferred, and the type of anomaly displayed in Figure 2a results. The

initial accumulation pattern in sectour four (Figure 2b) is not much affected affected by the target increase, again because the import coefficient π_4 is relatively small.

One might conjecture that if more slack were built into the debt constraint (e.g., by the inclusion of activities allowing import substitution and/or export promotion), the initial phases of an optimal plan would be independent of a wider range of terminal capital stocks. In any event, the examples given here demonstrate that generalization of the desirable getting-to-the-turnpike property to open economy models is likely not to be a completely straightforward process.

5.2 Varying Elasticities of Substitution

Conjectures vary as to the importance of differential elasticities of substitution in influencing the economic growth process. On the aggregate level, Nelson (as summarized by Nerlove [22]) has shown that when capital and labor are growing at roughly equal rates, changes in the aggregate elasticity of substitution will have little influence on the overall growth rate. In a disaggregated analysis, however, Arrow, et al. [1], point out that differences in elasticities of substitution among industries will have significant effects on sectoral allocations of capital and labor (and ultimately on the aggregate elasticity of substitution). In particular high elasticities in the primary sector and lower elasticities in the secondary and tertiary sectors are a means of explaining the well-known shift of labor from the former sector toward the latter.

Using our four-sector laboratory, we made some partial tests of these hypotheses, especially the latter one, by varying sectoral elasticities of substitution while at the same time recalculating the efficiency parameters (τ_j in equation (3.6)) to bring initial outputs in line with those of Korea in 1965-66. Given this means of normalizing our three-parameter production functions to fit three pieces of data (initial capital stocks, labor forces, and gross production levels), we calculated optimal solutions to the model under the conditions shown in Table 2:

Table 2

VARYING ELASTICITIES OF SUBSTITUTION

Solution	Sector			
	1	2	3	4
a	1.2	0.9	0.9	0.6
b	1.2	1.3	0.5	0.6
c	0.1	0.9	0.9	3.0
d	3.0	0.9	0.9	0.1

In the first three of these solutions, sectoral labor forces were essentially unchanged, although there were some shifts in the time-phasing of investment. There were major labor force shifts only in solution d where the elasticity of substitution in primary production was raised by a factor of more than two. In this case labor in the primary sector decreased by 0.6 million workers in the terminal year, representing a shift of about three percent of the total labor

force of 18 million. In terms of the conventional GNP aggregate, the growth rate in solution a over the thirty-year plan was 6.7%, while it was 6.8% in solution d. Here again the effects of changing substitution elasticities were relatively minor.

How well these preliminary results would stand up under further experimentation is, needless to say, open to question. In particular, shifts of the elasticities of substitution in connection with different welfare functions and/or different normalizations for the efficiency parameter might have more significant effects. It does appear, however, that further experimentation along the lines suggested here would provide a partial answer to the troubling empirical questions regarding the relevance of the elasticity of substitution to actual planning exercises.

6. Summary and Conclusions

We have summarized some of the recent advances in the formulation of algorithms for the solution of optimal control problems, and demonstrated the feasibility of applying them to the solution of a moderately complex economic planning model. We also report on the results of some modest experimentation with the model as an applied theoretical tool for the analysis of getting-to-the-turnpike properties of optimal growth, and of changes in the elaticities of substitution. Further experimentation along these lines appears promising, as does the eventual application of nonlinear models as actual planning tools.

REFERENCES

1. Arrow, K. J., H. B. Chenery, B. S. Minhas, and R. M. Solow, "Capital-Labor Substitution and Economic Efficiency," Review of Economics and Statistics, Vol. 43, August 1961.

2. Barr, James L. and Alan S. Manne, "Numerical Experiments with a Finite Horizon Planning Model," Research Center in Economic Growth, Memorandum #51, Stanford University, Stanford, California, May 1966.

3. Brioschi, F. and S. Rossi, "The Complementariety of Consumption in the Optimal Savings Problem," Estratto da Calcolo, Vol. 4, No. 2, April-June 1967.

4. Bruno, Michael, "A Programming Model for Israel," in I. Adelman and E. Thorbecke (eds.) The Theory and Design of Economic Development, The Johns Hopkins Press, 1966.

5. Bryson, Arthur E. and Yu-Chi Ho, Optimization, Estimation, and Control, (memo), Division of Engineering and Applied Physics, Harvard University, November, 1966; to be published by the Blaisdell Publishing Company, Waltham, Massachusetts.

6. Chakravarty, S., "Optimum Saving with a Finite Planning Horizon," International Economic Review, September 1962.

7. Chakravarty, S. and Louis Lefeber, "An Optimizing Planning Model," The Economic Weekly, Annual Number, February 1965.

8. Chenery, Hollis and Hirofumi Uzawa, "Nonlinear Programming in Economic Development," in Studies in Linear and Nonlinear Programming, by Arrow, Hurwicz, and Uzawa (eds.), Stanford University Press, 1958.

9. Dobell, A. R. and Y. C. Ho, "Optimal Investment Policy: An Example of a Control Problem in Economic Theory," IEEE Transactions on Automatic Control, Vol. AC-12, No. 1, February 1967.

10. Eckaus, R. S. and Kirit Parikh, Planning for Growth, Multisectoral Intertemporal Models Applied to India, (memo), Center for International Studies, Massachusetts Institute of Technology, Cambridge, Massachusetts, 1967, forthcoming, M.I.T. Press, Summer 1968.

11. Fletcher, R. and M. J. D. Powell, "A Rapidly Convergent Descent Method for Minimization," The Computer Journal, Vol. 6, 1963, pp. 163-168.

12. Fletcher, R. and C. M. Reeves, "Function Minimization by Conjugate Gradients," The Computer Journal, July 1964.

13. Frisch, R., "A Survey of Types of Economic Forecasting and Programming and a Brief Description of the Oslo Channel Model," (memo), Institute of Economics, University of Oslo, Oslo, Norway, May 1961.

14. Halkin, H., "A Maximum Principle of the Pontryagin Type for Systems Described by Nonlinear Difference Equations," SIAM Journal on Control, 1966, pp. 90-111.

15. Houthakker, H., "A Note on Self-Dual Preferences," Econometrica, October 1965.

16. Johansen, L. assisted by T. Lindholt, "Savings and Growth in Long Term Programming Models. Numerical Examples with a Nonlinear Objective Function," (memo), Oslo Institute of Economics, Oslo, Norway, August 1964.

17. Kendrick, D. A. and Lance J. Taylor, "Numerical Methods and Nonlinear Optimizing Models for Economic Planning," Economic Development Report #65, Project for Quantitative Research in Economic Development, Center for International Affairs, Harvard University, November 1967.

18. Kendrick, D. A. and Lance J. Taylor, "A Dynamic Nonlinear Planning Model for Korea," Economic Development Report #99, Project for Quantitative Research in Economic Development, Center for International Affairs, Harvard University, May 1968.

19. Koopmans, T., "On the Concept of Optimal Economic Growth," in The Econometric Approach to Development Planning, North Holland and Rand McNally, 1966.

20. Lasdon, E. S., S. K. Mitter and A. D. Warren, "The Conjugate Gradient Method for Optimal Control Problems," IEEE Transactions on Automatic Control, April 1967.

21. Mirrlees, J. A., "Optimum Growth when Technology Is Changing," Review of Economic Studies, Vol. 34, No. 97.

22. Nerlove, Marc, "Recent Empirical Studies of the CES and Related Production Functions," in The Theory and Empirical Analysis of Production, National Bureau of Economic Research, 1967.

23. Radner, Roy and S. Friedman, "An Algorithm for Dynamic Programming of Economic Growth," Working Paper #99, Center for Research in Management Science, University of California, Berkeley, California.

24. Samuelson, P. A., "A Catenary Turnpike Theorem Involving Consumption and the Golden Rule," American Economic Review, June 1965.

25. Samuelson, P. A. and Robert M. Solow, "A Complete Capital Model Involving Heterogeneous Capital Goods," Quarterly Journal of Economics, November 1956.

26. Stoleru, L. G., "An Optimal Policy for Economic Growth," Econometrica, Vol. 33, No. 2, April 1965.