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PROGRESS REPORT
ON
NUMERICAL ANALYSIS
OF
ECONOMIC PLANNING MODELS

by

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and
Lance Taylor

August 1967

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1. Research Design*

1.1 Introduction

Economic planners in less developed countries are often in a position to affect the flow of investment to increase or decrease the rates of growth of different sectors in the economy. In so doing they would like to be able to select patterns of production and investment which would produce that set of paths of sectoral growth which would maximize a measure of welfare for the society. For example, they might want to encourage relatively slow growth of the housing industry and rapid growth of the machine tools industry at the beginning of the period and then reverse these relative rates of growth as time progresses.

1.2 Current State of Research in the Field

1.2.1 Empirical Growth Models

A number of linear programming models have been constructed to study feasible or optimal paths for the growth of multiple sector economies. Chenery and Clark (1959) developed one of the first single period linear programming models. Bruno (1966a) has constructed one of the most successful single period models and is presently working on a multiperiod model for Israel. The multiperiod model for India developed by Chakravarty, Eckaus, Lefeber and Parikh and completed by Eckaus and Parikh (1966) is a linear programming study disaggregate to eleven sectors. This study provides the most comprehensive empirical model of long-run economic growth yet developed.

These linear programming models force the analyst to use linear or piecewise linear functions to approximate inherently nonlinear relationships such as production functions, export revenue functions and welfare (criteria) functions, cf. Barr and Manne (1966). In principle, linear approximations to an underlying

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*This research has been supported in part by a grant from the Agency for International Development. We are indebted to Robert Kierr for very able programming assistance.

nonlinear relationship can be made arbitrarily close, but in practice close approximation becomes computationally infeasible.* In addition, linearized versions of nonlinear relatiohsips may be unstable with respect to small parameter changes which can cause large shifts in activity levels due to basis changes. The stability can be improved with better linear approximation, but at the cost of computation time.

To overcome these problems we propose to specify smooth nonlinear welfare and constraint functions that allow us to avoid the rigidities and instability problems of linear models. This approach also allows us to incorporate econometric work on estimation of substitution production functions and linear logarithmic formulations of export demand functions and consumption functions.

1.2.2 Theoretical Growth Models

Alongside the work on empirical models a number of theoretical studies of nonlinear multisectoral growth models have been completed. Samuelson and Solow (1956) worked out a capital model involving heterogeneous capital goods. Chakravarty (1962) analyzed a one-technique Leontief model without a labor input and Bruno (1966) has studied optimal growth in a multisectoral model of the discrete activity analysis type. The last two of these studies, those of Chakravarty and of Bruno, have employed techniques from the calculus of variations and from control theory to attempt to determine the optimal growth path over time in theoretical and multisectoral growth models.

1.3 Proposed Investigation

1.3.1 The Model

Kendrick and Taylor (1967) contains an outline of the model toward which we hope to work. We have begun our computational

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* A linearized version of a nonlinear program with easily approximated separable functions has a solution time about three times that of a strictly linear program of similar size. (Hadley, Nonlinear and Dynamic Programming, p. 119). Non-separable realtionships, such as production functions, require much more elaborate approximation and correspondingly more computation.

experimentation with greatly reduced forms of that model. We expect to add slowly to the complexity of the model and to the number of sectors. The complete model will contain the following elements:

1. A performance index which is an integral over time of utilities associated with the consumption path of $j = 1, \ldots\ldots, n$ different goods.
2. Distribution relations which relate total production and imports of each good to their uses. Production is determined by capital and labor inputs through a production function for each sector. We anticipate using production functions of either the Cobb-Douglas or constant elasticity of substitution form. The uses of the goods are divided by destination into intermediate products, exports, investments, and consumption. Intermediate products are required to obtain output, exports produce foreign exchange, investments result in increases in capital stocks, and consumption increases the performance index.
3. Investment relations, which relate the use of goods for investment purposes to increases in the sectoral capital stocks.
4. A foreign exchange equation which equates total foreign exchange earnings by exports and net capital inflow to total foreign exchange uses for imports.
5. An integral constraint on foreign aid which requires that net foreign capital inflow over the period of the model not exceed some specified value.
6. Labor availability constraints which define the labor force which is available to produce goods and services.

7. A set of boundary conditions which specify the capital stock available in each sector, the labor force, and the foreign debt at the original time and the target capital stocks for each sector at the terminal time.*

Mathematically we intend to formulate and solve the model as a control theory model. This will permit us to introduce a non-linear performance index (the consumption measure) and nonlinear constraints (viz., the distribution relations, and in particular the production functions). If this can be accomplished while retaining a sufficient degree of computational efficiency, a break could be made from some of the rigidities of the linear programming models which have been used for emperical studies of this sort in the past.

1.3.2 The Algorithm

We are presently using a conjugate gradient algorithm to solve the non-linear planning models.≠ This algorithm employs successive one dimensional searches in the "conjugate gradient" direction in seeking the optimum. The direction of search at the n + 1st step is a weighted average of the direction of search at the nth step and the gradient direction at the minimum point reached on the nth step. The weighting depends upon the relative stepness of the previous direction and the gradient direction. Figure 1 gives a schematic discussion of the algorithm, and Appendix A gives a more detailed description of the algorithm.

2. Results Obtained

We decided to check the algorithm and our programming on one

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*The terminal conditions may alternatively be set as some function of output levels or of intra-period capital stocks.

≠The algorithm employed is described in Lasdon, Mitter, and Warren (1966). It was programmed by Raman Mehra and has been modified by Robert Kierr.

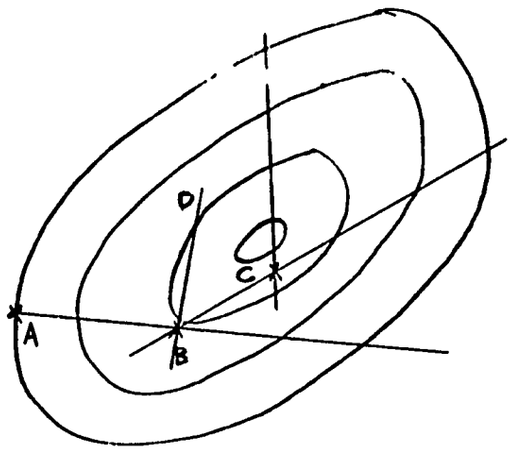

Figure 1

The Conjugate Gradient Algorithm

The algorithm is as follows:

1. Select an arbitrary starting point A.
2. Make a one dimensional search in the gradient direction from A to find the minimum point B.
3. Find the gradient direction BD at B.
4. Compute the conjugate gradient direction BC.
5. Make a one dimensional search along the conjugate gradient to the minimum point C.
6. Find the gradient direction at C.
7. Compute the conjugate gradient direction at C.
8. Make a one dimensional search along the conjugate gradient direction at C.
9. Etc.

sector growth models before proceeding to multisectoral problems. We did this because for these problems it is possible to obtain exact solutions in closed form. These solutions then served as a standard to which we could compare the solution obtained with the algorithm.

The two problems we have solved in this fashion are models described and analyzed by Chakravarty (1962) and which were later the subject of an exchange between Maneschi (1966) and Chakravarty (1966) primariliy in reference to the terminal conditions on the model. For both problems the algorithm solution and the exact solution were close together, although as might have been expected the algorithm performed relatively better on a problem with a non-linear production function than on a problem with a linear production function.

The basic model used by Chakravarty is

(1) $$\max J = \int_0^t \frac{1}{1-\eta} (C_t)^{1-\eta} \, dt$$

subject to

(2) $$C_t = bK_t^{\beta} - \dot{K}_t$$

and

(3) $$K_o = \bar{K}_o$$
$$K_t = \bar{K}_o (1+g)^t$$

where

C_t = consumption of time t

K_t = capital stock at time t

b = output capital ratio

β = coefficient for non-linear production function; when $\beta \neq 1.0$, b = 1.o

η = the elasticity of marginal utility

g = rate of growth used to compute the terminal capital stock

The two cases we have analyzed are for (1) $b = .33$, $\beta = 1.0$, and (2) $b = 1.0$, $\beta = .5$. Thus case (1) is with a linear production function

(4) $Y_t = bK_t$

where Y_t is output, and case (2) is with a Cobb-Douglas type production function

(5) $Y_t = K_t^{.5}$

exhibiting diminishing returns.

Chakravarty (1962) obtained the closed form solution for the capital stock and consumption paths for both the linear and Cobb-Douglas cases but did not work out numerically and display these paths. Maneschi (1966) pointed out that the capital stock path for the linear case followed the very strange path which is shown in Figure 2. Chakravarty (1966) countered that a different terminal condition would produce a more realistic capital stock path.

We have obtained a numerical solution for Chakravarty's closed form solution for the Cobb-Douglas case. In solving this problem Chakravarty comments that he uses a Cobb-Douglas production function with a coefficient of .5 because for this special case he is able to integrate the Euler equation in closed form. His solution path for the capital stock for this problem is

(6) $$\left[K^{\frac{1}{2}} - \frac{t-a_1}{2\eta}\right] = a_2\left[K^{\frac{1}{2}} - \frac{t-a_1}{2}\right]^{\eta}$$

where K is capital stock

η is the elasticity of marginal utility

t time

a_1 and a_2 arbitrary constants of integration

We used the initial and terminal conditions on K(t) to obtain two equations from (6) through which we could evaluate the constants a_1 and a_2. Then (6) was solved for each time period to obtain

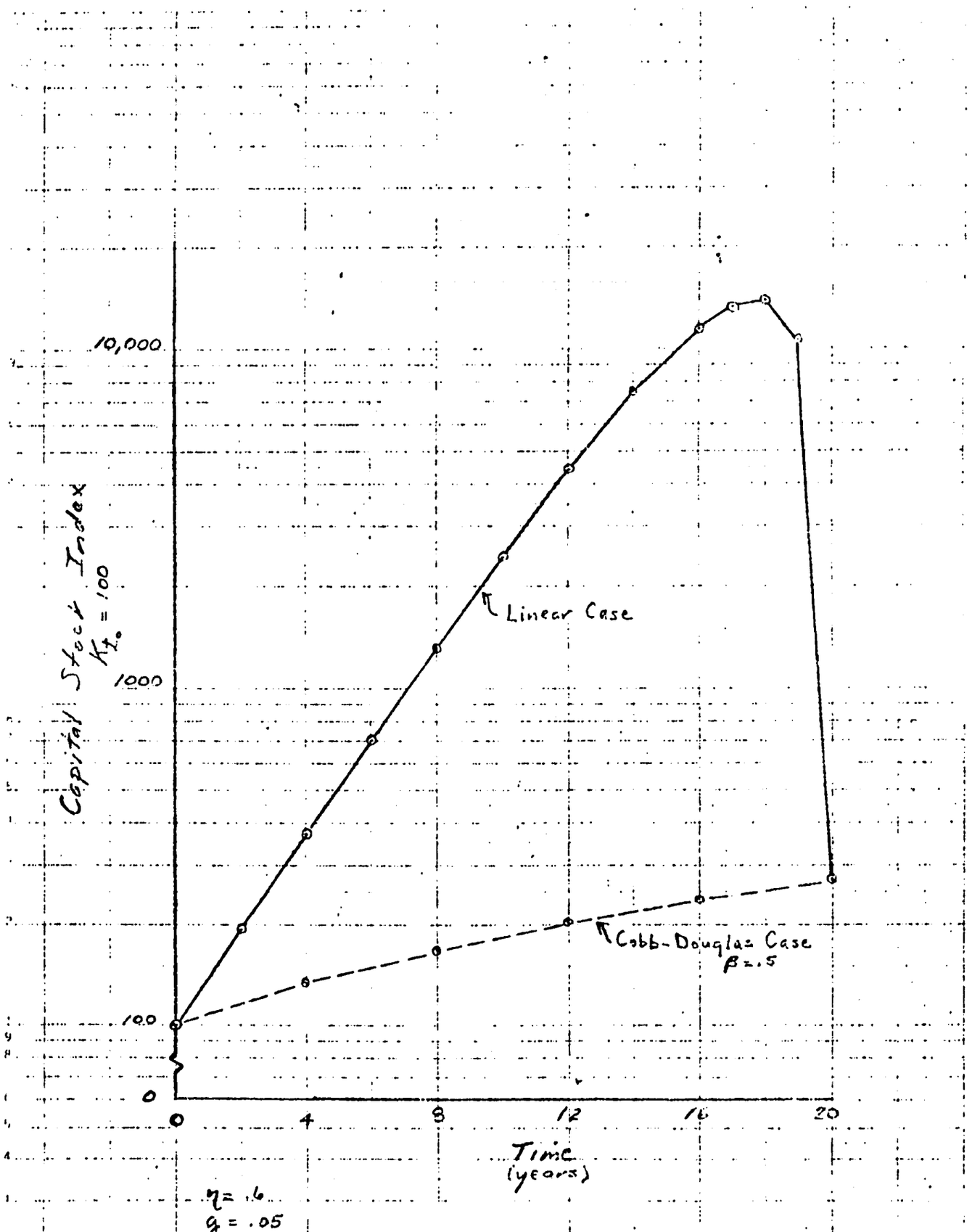

Figure 2
Capital Stock Path

the capital stock path K(t) which is shown in Figure 2*.

Figure 2 shows that the substitution of a non-linear production function for a linear production function in a one sector growth model with fixed terminal capital stock results in changing the capital stock path from one that violates our expectations to one that is more in keeping with our expectations.

At present we are attempting to solve this planning model with a Cobb-Douglas type production function and a coefficient of $\beta = .75$. Perhaps this will provide us with an intermediate case between the linear case and the extreme Cobb-Douglas case with $\beta = .5$.

Figure 3 gives the consumption paths obtained in each of the two models.

3. Our Agenda

3.1 A Model for Primary Exporters

We have started to work with a problem involving two control variables≠ which was suggested to us by Hossein Mahdavi, an Iranian graduate student. The problem admits a fairly straightforward analytical solution, so we propose to undertake with it the same program as we did with the Chakravarty model, i.e. run out numerical values for the analytical solution, solve the problem with the conjugate gradient method, and compare results.

The model itself is motivated by the planning problem facing the oil-exporting countries, and some other natural-resource exploiters. These countries have exceptionally high current foreign exchange revenues which may decline in the future. Their current foreign exchange earnings, to a first approximation, are generated at no domestic resource cost and can be used to supplement the relatively low domestic production and thereby allow fairly

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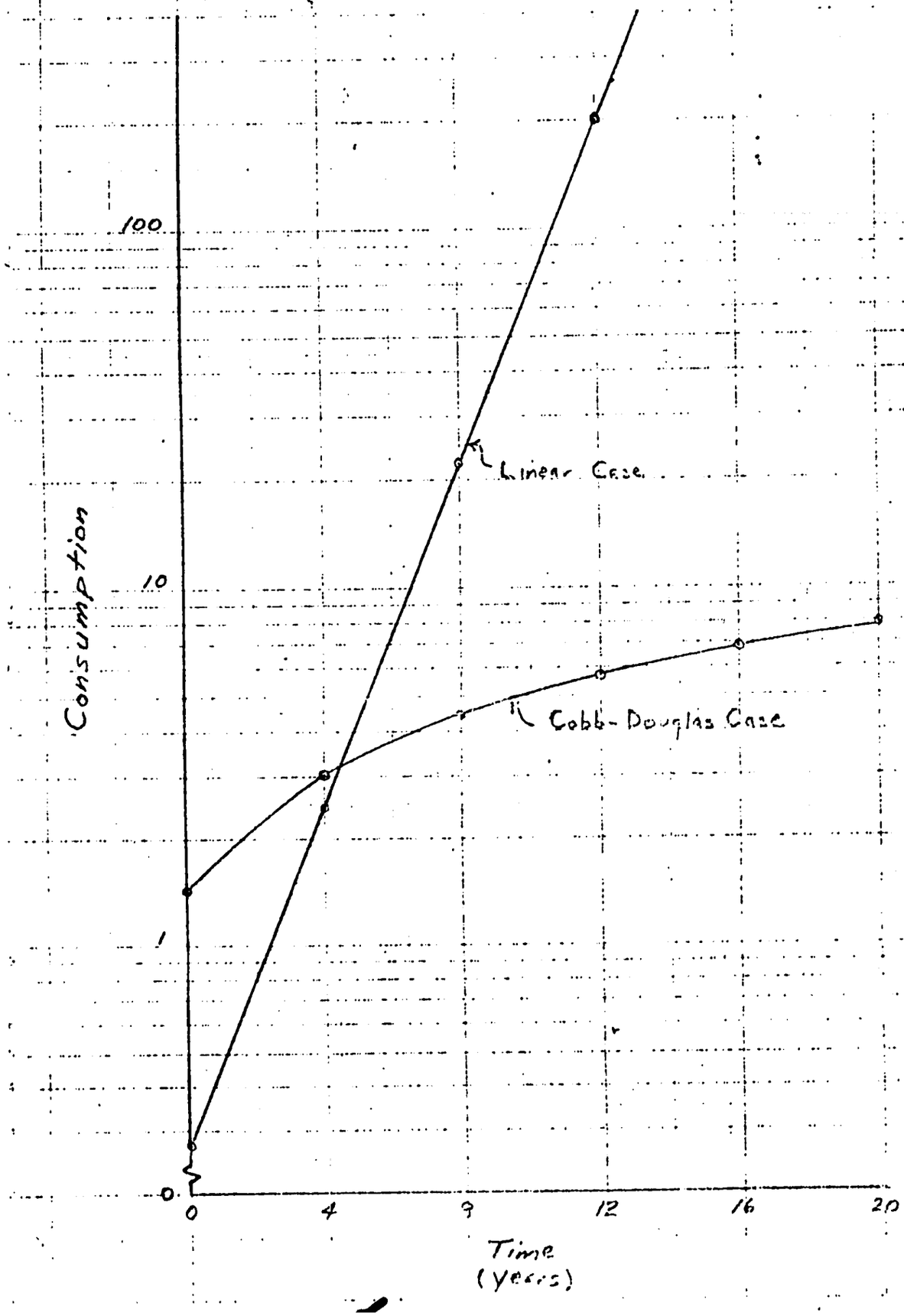
*For the path shown $\eta = .6$ and $g = 5\%$.

≠The computational difficulty of solving control theory problems seems to increase rapidly with the number of control variables.

Figure 3

Consumption Path



high consumption levels. This situation poses a difficult allocation-over-time problem for these countries. How much consumption should be allowed now? How much in the future? How much investment should be undertaken at what times, and how should it be coordinated with the path of expected foreign exchange earnings?

To begin setting up a model for investigating these questions, we assume an unusual form of the national income identity,

(7) $Y_t + M_t = C_t + I_t$

which states that aggregate supply (domestic production plus imports) equals aggregate demand (consumption plus investment). Exports do not enter the accounting identity since they are assumed not to use domestic resources.

Let b be the output-capital ratio, set $I_t = \dot{K}_t$ (ignoring depreciation), and write (7) as

(8) $\dot{K}_t = b\,K_t + M_t - C_t$.

This is the accumulation equation for the model. (We assume knowledge of initial and terminal capital stocks.)

Following established custom (as we did with the Chakravarty problem) we seek the maximum of the integral of some utility function However, we modify this approach somewhat to take account of two additional constraints imposed by opening the model to foreign trade:

(i) Over the planning period, we require total imports to equal total exports;

(ii) At any time, we want imports to be "reasonably" close to exports, although equality need not be required. This restriction is imposed as an institutional constraint on the ground that (a) it is difficult for most governments to build up and hold large foreign exchange reserves, and (b) most international lending agencies tie their loans to current and expected export performance.

The first constraint is easily stated as

$$(9) \quad \int_{t_o}^{t_f} (M_t - X_t)\, dt = 0$$

where X_t is the (known) time path of exports and t_o and t_f are the initial and final years of the planning period.

The second constraint leads to the use of a welfare functional of the form

$$(10) \quad J = \int_{t_o}^{t_f} [e^{-qt} U(C_t) - P(M_t - X_t)]\, dt.$$

The $U(C_t)$ term is the consumption utility function, discounted at rate q, while $P(M_t - X_t)$ is a "penalty function" on deviations of imports from exports. (Note that we could handle the problem of keeping imports "close" to exports by use of inequality constraints on the variables. However, this is computationally tricky, as well as being somewhat inelegant In practice, of course, the two approaches would tend to converge.)

To enable us to get a closed form solution to the model, we chose simple functions to use in the welfare functional. Specifically, we write J as

$$(11) \quad J = \int_{t_o}^{t_f} \left[e^{-qt} \frac{1}{1-\eta} C_t^{1-\eta} - P(M_t - X_t)^2 \right] dt.$$

The constant elasticity utility function is used routinely in the literature (in the Chakravarty problem, for instance). The symmetric quadratic penalty function $p(M_t - X_t)^2$ is used largely on grounds of expediency. Presumably an asymmetric function penalizing exchange deficits and surpluses unequally would be preferable, but in this exercise the mathematical tractability of the simple quadratic offsets its drawbacks.

The problem, then, is to maximize (11) subject to the integral constraint (9), and the differential equation (8). After going through the usual hocus-pocus with the Hamiltonian, the solution for K can be written out explicitly. It turns out to be

$$(12)\quad K(t) = -\frac{\mu}{2pb} - \frac{E}{4pb}e^{-bt} - \frac{E^{\frac{-1}{\eta}}\ e^{\gamma t}}{d - b}$$

$$+ e^{bt}\int_{t_o}^{t} X(\tau)e^{-b\tau}\,d\tau + Ge^{bt}$$

where μ, E, and G are constants to be evaluated from boundary conditions, and $\gamma = (b - q)/\eta$. The solution is a sum of exponentials modified by a term derived from the export time path. All parameters influence the growth rate, so analysis of solutions is difficult.

3.2 Some Extensions

The model sketched in the previous section has a number of defects and one cardinal virtue. The virtue, of course, is that the thing is soluble in closed form. The defects include (a) no explicit differentiation between imports of consumption goods and imports of producers' goods; (b) a makeshift assumption of a constant capital-output ratio; (c) no comparison in the utility function of the relative benefits of imported and home-produced consumption goods.

In an attempt to remedy some of these defects, we have devised another model using (conveniently) three control variables. In skeleton form, it looks like this:

Let d_1 be the fraction of domestic output assigned to consumption, with

$$(13)\quad \underline{d_1(t)} \leq d_1 \leq \overline{d_1(t)}$$

where the bounds are imposed in line with the effective limits on policy measures. Similarly, let d_2 be the fraction of imports which can be consumed, with

$$(14)\quad \underline{d_2(t)} \leq d_2 \leq \overline{d_2(t)}.$$

The problem is to maximize the integral

$$(15) \quad J = \int_{t_o}^{t_f} \left\{ e^{-qt} \left[U_1(d_1 f(K)) + U_2(d_2 M) \right] - p(M - X)^2 \right\} dt$$

subject to (13), (14), and

$$(16) \quad \dot{K} = (1 - d_1) f(K) + (1 - d_2) M$$

and

$$(17) \quad \int_{t_o}^{t_f} (M - X)\, dt = o,$$

where $K(t_o)$ and $K(t_f)$ are known, as is the time path of X.

Note several points.

(i) We have purposely left the production function f(K) unspecified, to allow a choice between a constant capital-output ratio and decreasing returns (perhaps modified by some sort of technical progress).

(ii) Equation (16), which says that capital goods, imports and domestic capital formation are two perfectly substitutable "factors" adding up to total capital formation is unrealistic. We know that some 40-60% of capital formation is construction, which <u>has</u> to be produced domestically, while some other given portion of capital formation <u>has</u> to be made up of imports. This prior knowledge suggests that (16) should be replaced by

$$(18) \quad \dot{K} = g\, [(1 - d_1) f(K),\ (1 - d_2) M],$$

where g is an index function (or production function, if you like) which catalogs permissible trade-offs between imports and domestic production. A CES-type function with shrewdly chosen parameters might be an appropriate choice for g -- the distribution parameters could be chosen close to current shares of imports and domestic capital formation, and the elasticity of substitution between the two types of capital is presumably low. (As far as our limited knowledge of the literature goes, no one has estimated functions such as we have in mind, although Sato (1967) does estimate an elasticity of substitution between United States plant and equipment, coming out with $\sigma > 1$ by quite a margin.)

(iii) The upper and lower bounds on d_1 and d_2 are arbitrary. One can, of course, set them equal to one and zero respectively, which allows maximum freedom. The chief justification for the bounds is that they provide some way of evaluating the benefits of policy options like forcing consumption imports down to a certain percentage of total imports in a certain time. These benefit calculations come from shadow prices on the inequality constraints, which enter into the control theory formulation in a way quite analagous to linear programming shadow prices.

3.3 More Extensions

Beyond our souped-up oil-exporter problem, we have some more extensions in mind. Briefly, these are

(i) Making exports into a control variable, with export revenue a declining function of total volume.

(ii) Adding sectoral labor forces into the production functions, originally in a full general equilibrium setting with perfectly flexible everything, and the perhaps with institutional constraints along "surplus labor" lines.

(iii) Using an input-output table and a B-matrix to build up a non-linear dynamic Leonteif model. The 2-sector version of this involves two capital stock accumulation differential equation and six controls (two sectoral consumptions, two labor forces, one "primary" export and one "manufactured" import). This is probably about as many controls as we can handle with present computer technology and programs. But when third generation computer finally come along, who knows what we can do?

APPENDIX A

The Conjugate Gradient Algorithm

Robert Kierr

Below is a brief description of the conjugate gradient algorithm. The reader who wishes a more detailed explaination is referred to Fletcher and Reeves (1964) and Lasdon, Mitter, and Warren (1966). The first derives the technique for finite dimensional vector spaces and proves that it is quadratically convergent for quadratic f(x) (they also define quadratic convergence). The latter extends the technique to Hilbert spaces.

The conjugate gradient procedure is designed to find the minimum of a scalar field, f(x), where x is an n dimensional vector. Let h be the (unknown) location of the minimum. The h is found by progressing from some initial guess, x_o, to a better guess, x_1, to x_2,..., to h. If f(x) is a quadratic function (i.e. f(x) can be expressed in a form that looks like a parabola except that the terms are vectors and matrices) then it can be shown that the conjugate gradient method will find the location of the minimum in n steps.

The procedure is similar to the familiar first order gradient method; except that at every step the one dimensional search is done in the "conjugate gradient" direction instead of in the "gradient direction."

The conjugate gradient direction will be called p below. Then the algorithm is:

x_o = arbitrary starting point

g_o = grad f(x)

$p_o = - g_o$

Then

$$x_1 = x_o + \alpha_o p_o ,$$

where α_o is found in the one-dimensional search alluded to above. The criteria for α_i (at any step) is that $f(x_i + \alpha\ p_i)$ = a minimum for all possible values of α.

For successive values of x_i, the formulas are:

$$x_{i+1} = x_i + \alpha_i\ p_i$$

when

$$p_i = -g_i + \left(\frac{\|g_i\|}{\|g_{i-1}\|}\right)^2 p_{i-1}$$

and

$$g_i = \text{grad } f(x_i)$$

As stated above, and proven in Fletcher and Reeves, the algorithm is guaranteed to converge on the minimum in n steps when f(x) is a quadratic function. Since the function with which we deal in general, quadratic, we do not have this guarantee. However, in our experience convergence has been rapid.

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