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A paper that discusses the essential duality relationships in the calculus notation, and demonstrates them by use of examples. The development of duality theory has opened up powerful new approaches to applied econometric research in production economics, but these have not yet been widely used by agricultural economic researchers. One reason for this is that most of the theoretical literature employs mathematical notations not normally taught in an agricultural economics curriculum. Discussed here are the profit function, examples using the Cobb Douglas production function, the cost function as a special case of the profit function, one-to-one correspondence and theoretical restrictions, econometric application of duality concepts, extension to multiple inputs and outputs, the two main problems with the profit function approach, and the great flexibility that approach allows the investigator: Fixity of inputs or outputs is easily accommodated; deviations from pure profit-maximizing behavior can be systematically built into the theoretical framework; and factors affecting productivity, such as soil quality or climatic variables, can be built in as fixed variables and their contribution to production estimated. Finally, far more complex production processes can be modeled than is possible with the standard production functions.

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THE USE OF DUALITY BETWEEN PRODUCTION, PROFIT AND COST FUNCTIONS IN APPLIED ECONOMETRIC RESEARCH : A DIDACTIC NOTE

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THE USE OF DUALITY BETWEEN PRODUCTION, PROFIT AND COST FUNCTIONS IN APPLIED ECONOMETRIC RESEARCH : A DIDACTIC NOTE

1 INTRODUCTION

In recent years the development of duality theory between production, profit and cost functions [17, 14, 7, 6, 13] has opened up powerful new avenues to applied econometric research in production economics. However, these theoretical advances have not yet been widely used by agricultural economic researchers despite the fact that they allow a much richer specification of production relationships than, say, the traditional Cobb Douglas or CES production functions.

One reason for the slow acceptance of these tools may lie in the fact that most of the theoretical literature is in mathematical language not normally taught in an agricultural economics curriculum. This note will therefore present the essential duality relationships in the usual calculus notation relying as much as possible on well known theoretical tools. No proofs of these relationships will be given, but they will be demonstrated with the use of examples. No attempt is made to give an exhausting listing of the literature or the historical development of the theory since this can be found in [6]

1.1 What is a Profit or Cost Function ?

A profit (or cost function) relates maximised profits (or minimised costs) to factor and goods prices and to other exogenous variables such as fixed inputs, or climatic and soils variables. Once it is estimated, its parameter contains all the information about the underlying production function or production process. A given profit or cost function under certain conditions given in Section 3 corresponds uniquely to a given production function. Hence instead of starting economic or econometric model building from the production function one can start it directly from the profit function without worrying about the specific functional form of the production function. This is analogous to linear programming where the primal formulation and the dual formulation of the problem correspond to each other in a one to one relationship and one can start any solution either from the primal side or from the dual side.

In linear programming, however, not much may be gained by starting an emperical analysis from the dual side, whereas substantial simplification of econometric work is gained if one starts from the profit or cost function rather than from the production function. These simplifications will be discussed in detail in section (4).

Cost functions can be regarded as a special case of profit functions in which output is given rather than being variable. We will therefore concentrate primarily on the profit function.

Profit functions are derived from production functions in the following way : Consider the case where there are two variable inputs X_1 and X_2 and one fixed input Z which are related to output Y by production function $Y = f(X_1, X_2, Z)$. Since Z is fixed its opportunity cost in a given period is zero and the entrepreneur tries to maximise the sales value of output less the cost of variable inputs, i.e.

Variable profits (or returns to the fixed factors) π'

$$\pi' = Pf(X_1, X_2, Z) - Q_1X_1 - Q_2X_2 \qquad 1.1$$

where P is price of output and Q_1 and Q_2 are input prices. (The notations of the paper are summerized in table 1.)

The first order conditions of this problem are :

$$Pf_i = Q_i \quad \text{for } i = 1, 2 \qquad 1.2$$

where $f_i = \frac{\partial Y}{\partial X_i}$ is the marginal product of X_i.

For specific functional forms this system of two equations can be solved for the optimal input quantities X_i^* (or the demand functions for the inputs).

$$X_i^* = X_i^* (P, Q_1, Q_2, Z) \quad i = 1, 2 \qquad 1.3$$

If we substitute these functions into the expression for profits 1.1 we get the following :

$$\pi'^* = Pf(X_1^*, X_2^*, Z) - Q_1 X_1^* - Q_2 X_2^* \qquad 1.4$$

Clearly, profits can never be higher than when the profit maximizing quntities are used. Therefore, if we substitute the profit maximizing factor quantities into the expression for profits, π'^* of 1.18 becomes the maximum of variable profits. Furthermore, Since all X_i^* are functions of the prices and the fixed input quantity we can substitute prices and fixed quantities in the expressions of X_i^* and write, after collecting terms,

$$\pi'^* = \pi'^* (P, Q_1, Q_2, Z) \qquad 1.5$$

This profit function relates maximized profits to the prices of outputs and inputs and to the quantities, if any, of the fixed factors. In the next section we shall demonstrate this procedure to find a profit function by examples.

For econometric and theoretical work it is easier to work with a normalised profit function as developed by Lau [7, 11] because the number of variables is reduced by one and because the choice of functional forms is wider*. Note first that the profit maximizing problem is not altered if we divide both sides of 1.1 by a scalar or a fixed number. Choose the output price as the fixed number (but any price can be chosen) and write.

$$\frac{\pi'}{P} = \pi = f(X_1, X_2, Z) - q_1 X_1 - q_2 X_2 \qquad 1.6$$

* When working with profit functions one has to choose functional forms which are homogeneous of degree one in all prices, whereas this is not necessary for the normalised profit functions. For instance, the quadratic function is not homogeneous of degree one but has otherwise many desirable properties.

where $q_i = \frac{Q_i}{P}$ is the input price relative to the output price. The normalised profit π is profits divided by the output price and is now related to relative prices.

One can again derive first order conditions and factor demand equations as before but in terms of relative rather than absolute prices, substitute these factor demand equations into 1.6 and obtain the normalised profit function in terms of relative prices.

$$\pi^* = \pi^*(q_1, q_2, Z) \qquad 1.7$$

1.3 The Usefulness of the Profit Function

The usefulness of the normalised profit function arises out of Shephard's (or Hotelling's) lemma which states that

$$-\frac{\partial \pi^*}{\partial q_i} = X_i^* = X_i^*(q_1, q_2, Z) \qquad i = 1, 2 \qquad 1.8$$

The negative of the first derivative of the normalised profit function with respect to the normalised or relative prices is the optimal input quantity or the factor demand curve. Instead of having to solve a system of simultaneous equations as in the case of production function approach one can get the factor demand curves simply as the first derivatives of the profit function. This holds for any number of inputs or outputs (Section 5).

If one specifies the functional form of the normalised profit function one can estimate it directly in 1.7 or in the derived demand function 1.8.

Note that these functions are econometrically correctly specified so that only exogenous variables appear as independent variables of the regression equations. In production function estimation the right hand side always includes both dependent and independent variables since variable inputs such as fertilisers are jointly determined with output by the farmer. This misspecification never arises with profit functions where one can classify inputs and outputs as fixed or variable depending on the problem and situation at hand.

Note further that for functions which are more complicated (and more realistic) than the Cobb Douglas or the CES functions one cannot derive closed form expressions for the implied factor demand functions using the production function approach. But in the case of complicated functions of the profit function which allow all sorts of substitutability or complementarity relationships between inputs and outputs, this is easily possible. This aspect will be further discussed in section (4).

Once a normalised profit function has been estimated one can also estimate the optimal output quantity Y^*. Note that in (I.6) the first right hand side term is simply output so that we can estimate it as

$$Y^* = \pi^* - q_1 X_1^* - q_2 X_2^* \qquad 1.9$$

Another duality relationship which sometimes is important relates the derivatives of the normalised profit function with respect to the fixed factors to the marginal product of these factors, i.e.,

$$\frac{\partial \pi^*}{\partial Z_i} = \frac{\partial Y}{\partial Z_i} \qquad 1.10$$

With this relationship one can estimate the returns to or shadow price of the fixed factors from the normalised profit function. The three duality relationships 1.8, 1.9 and 1.10 show that all important economic magnitudes can be estimated from the profit function. Shephard's lemma provides factor demand elasticities. Output supply elasticities can be estimated from relationship 1.9 and marginal products of fixed factors from relationship 1.10.

In the following section we will demonstrate Shephard's lemma with examples and simple functional forms. Note that for complicated functional forms where factor demand equations cannot be derived from the production function such demonstrations are impossible. But it is precisely the more complicated case in which Shephard's lemma is especially useful. The reason is that one can chose functional forms for the profit function which are (a) <u>linear in parameters</u> and (b) have the further characteristic that they provide local second order <u>approximations to arbitrary</u> production processes. One is thus provided with functional forms which can be estimated by least square method and which can accommodate much more complicated

production processes than was ever possible with simple production functions. This will be the topic of section (4). Hence the examples given are not intended for actual application but only for illustration.

TABLE 1 : NOTATIONS

Y_i	=	outputs for $Y_i > 0$
Y_i	=	inputs for $Y_i < 0$ in the multiple input-output case
X_i	=	variable inputs in the one output case
Z_i	=	fixed inputs or outputs
Y_i^*, X_i^*	=	profit maximizing output and input quantities
P_i^*	=	output prices (or input prices for $Y_i < 0$)
Q_i^*	=	input prices
p_i	=	normalized output or input prices, i.e., $p_i = \frac{P_i^*}{P_{n+1}^*}$
q_i	=	normalised input prices in the one output case, i.e. $q_i = \frac{Q_i}{P^*}$
π'	=	profits
π	=	normalised profits, i.e., $\pi = \frac{\pi}{P_{n+1}}$
π^*	=	maximised normalised profits, normalised profit function.

2 EXAMPLES WITH THE COBB DOUGLAS PRODUCTION FUNCTION

2.1 One Variable input

Assume that we have the following Cobb Douglas Production process

$$Y = AX^{\alpha} \qquad 2.1$$

where $\alpha < 1$. This production function is homogeneous of degree $k = \alpha$.

Profits to be maximized are :

$$\pi' = PY - QX \qquad 2.2$$

Note that maximising profits is equivalent to maximising normalised profits. Since the maximand is only divided by a scalar, i.e.,

$$\frac{\pi'}{P} = \pi = Y - qX = AX^{\alpha} - qX \qquad 2.3$$

where $q = Q/P$. This normalisation can be done regardless of the number of inputs. Taking the derivative with respect to X leads to the first-order condition

$$\frac{\partial \pi}{\partial X} = A\alpha X^{\alpha-1} - q = 0 \qquad 2.4$$

The profit maximising input quantity (the demand function) is the solution to this equation, i.e.,

$$X^{\alpha-1} = A^{-1}\left(\frac{q}{\alpha}\right)$$

$$X^* = A^{-\frac{1}{\alpha-1}} \left(\frac{q}{\alpha}\right)^{\frac{1}{\alpha-1}} = (\alpha A)^{-\frac{1}{\alpha-1}} q^{\frac{1}{\alpha-1}} \qquad 2.5$$

(Carrying α in the denominator of q rather than assembling it with the intercept a has advantages notationally, especially in more complicated cases).

Substituting (2.5) into the production function gives the supply function

$$Y^* = A^{-\frac{1}{\alpha-1}} \left(\frac{q}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \qquad 2.6$$

Substituting (2.5) and (2.6) into (2.3) leads to maximised profits as a function of the normalized price, i.e., the profit function :

$$\pi^* = A^{-\frac{1}{\alpha-1}} \left(\frac{q}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - \alpha \left(\frac{q}{\alpha}\right) A^{-\frac{1}{\alpha-1}} \left(\frac{q}{\alpha}\right)^{\frac{1}{\alpha-1}} \qquad 2.7\ a$$

and aggregating the terms in $\left(\frac{q}{\alpha}\right)$ we get the profit function :

$$\pi^* = A^{-\frac{1}{\alpha-1}} (1-\alpha) \left(\frac{q}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \qquad 2.7\ b$$

The profit function is homogeneous of degree $\alpha/\alpha-1$. It is again a Cobb-Douglas function of the form

$$\pi^* = B\, q^{\beta} \qquad 2.8$$

where $B = A^{-\frac{1}{\alpha-1}} (1-\alpha)\alpha^{\frac{-\alpha}{\alpha-1}}$ and $\beta = \frac{\alpha}{\alpha-1} < 0$

Shephard's lemma can be demonstrated easily by taking the first derivative with respect to q

$$-\frac{\partial \pi^*}{\partial q} = A^{-\frac{1}{\alpha-1}} \left(\frac{q}{\alpha}\right)^{\frac{1}{\alpha-1}} = X^*$$

Which is identical to (2.5). Hence Shephard's lemma has been demonstrated. Optimal output level is formed by (1.9) i.e. $Y^* = P^* - qX^*$ which the reader can compute himself as an exercise.

2.2 The Fixed Input Case

In the short run, when capital or land inputs cannot be altered, profit maximising behaviour implies the maximisation of the sum of sales receipts less the cost of the variable inputs. The cost of the fixed inputs is immaterial.

Let the production function again be Cobb-Douglas

$$Y = X^{\alpha} Z^{\beta} \qquad 2.9$$

where Z is the fixed input. Variable profits are :

$$\pi' = P' Y - QX$$

Normalising on the output price and substituting the production function for Y leads to :

$$\pi = X^{\alpha} Z^{\beta} - qX \qquad 2.10\ a$$

We can rewrite this as

$$\pi = AX^{\alpha} - qX \qquad 2.10\ b$$

Where $A = Z^{\beta}$, Note that (2.10 b) is identical to (2.3) ie. the problem reduced to the one input case.

We therefore know from (2.7) that profit function is

$$\pi^* = A^{\frac{1}{\alpha-1}} (1-\alpha) \left(\frac{q}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} = Z^{\frac{-\beta}{\alpha-1}} (1-\alpha) \left(\frac{q}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \qquad 2.11$$

Of course, $-\frac{\partial \pi^*}{\partial q} = X^*$ just as in the one input case, as the reader can verify for himself.

But in this problem we can demonstrate the relationship (1.10) of the derivatives of the marginal product (or opportunity cost of a fixed factor) with the derivatives of π^* with respect to Z. From the production function we have

$$\frac{\partial Y}{\partial Z} = \beta X^{\alpha} Z^{\beta-1} \qquad 2.12$$

Substituting into this the optimal input quantity $X^* = Z^{\frac{-\beta}{\alpha-1}} \left(\frac{q}{\alpha}\right)^{\frac{1}{\alpha-1}}$ leads to marginal product as estimated from the production function

$$\frac{\partial Y}{\partial Z} = \beta\, Z^{\frac{1-\alpha-\beta}{\alpha-1}} \left(\frac{q}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} \qquad 2.13$$

The same can be found by taking the derivation of the profit function (2.11) with respect to Z :

$$\frac{\partial \pi^*}{\partial Z} = \beta Z^{\frac{1-\alpha-\beta}{\alpha-1}} \left(\frac{q}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} = \frac{\partial Y}{\partial Z} \qquad 2.14$$

The expressions are identical, which completes the demonstration of relationship (1.10).

2.3 The Cost function as a special case of the profit function

It has been mentioned previously that the minimum cost function is a special case of the normalized profit function with the output level fixed.

Shephard's lemma holds as well for the cost function.

The cost minimising problem

$$\min.\ C = Q_1X_1 + Q_2X_2 - \lambda(X_1^{\alpha}\, X_2^{1-\alpha} - Y)$$

can be transformed into a "profit" maximising problem

$$\max.\ \pi' = -Q_1X_1 - Q_2X_2 + \lambda(X_1^{\alpha}X_2^{1-\alpha} - Y)$$

Note that here the production function may be homogeneous of degree one or greater than one.

We now normalise on the price of the last input, i.e.,

$$\frac{\pi'}{Q_2} = \pi = -q_1X_1 - X_2 + \lambda(X_1^{\alpha}X_2^{1-\alpha} - Y) \qquad 2.15$$

where $q_1 = Q_1/Q_2$ is the relative input price.

The first-order conditions are :

From $\frac{\partial\pi}{\partial X_1} \rightarrow \lambda\ \alpha\ X_1^{\alpha-1}\ X_2^{1-\alpha} = q_1$ 2.16

From $\frac{\partial\pi}{\partial X_2} \rightarrow \lambda\ (1-\alpha)\ X_1^{\alpha}X_2^{-\alpha} = 1$ 2.17

From $\frac{\partial\pi}{\partial\lambda} \rightarrow X_1^{\alpha}X_2^{1-\alpha} = Y$ 2.18

Solving the production function of 2.18 for X_1 we find the solution for the isoquant

$$X_1 = Y^{\frac{1}{\alpha}}\ X_2^{-\frac{1-\alpha}{\alpha}} \qquad 2.19$$

Eliminate λ by dividing (2.16) by (2.17) and solve for X_2

$$X_2 = \left(\frac{1-\alpha}{\alpha}\right) q_1 X_1 \qquad 2.20$$

Substituting (2.19) into (2.20) leads to

$$X_2 = \left(\frac{1-\alpha}{\alpha}\right) q_1 Y^{\frac{1}{\alpha}} X_2^{-\frac{1-\alpha}{\alpha}}$$

Solving this equation for X_2 leads to the factor demand curve

$$X_2^* = \left(\frac{\alpha}{1-\alpha}\right)^{-\alpha} Y q_1^{\alpha} \qquad 2.21$$

And similarly for X_1

$$X_1^* = \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} Y q_1^{-(1-\alpha)} \qquad 2.22$$

Substituting back into (2.15)

$$\pi^* = -\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} Y q_1^{\alpha} - \left(\frac{\alpha}{1-\alpha}\right)^{-\alpha} Y q_1^{\alpha}$$

Aggregating terms in q_1 leads to

$$\pi^* = -Y q_1^{\alpha} \left(\frac{\alpha}{1-\alpha}\right)^{-\alpha} \left(\frac{\alpha}{1-\alpha} + 1\right)$$

and simplifying further

$$\pi^* = -Y \left(\frac{1}{1-\alpha}\right)^{1-\alpha} \left(\frac{q_1}{\alpha}\right)^{\alpha} \qquad 2.23$$

The cost function is proportional to output because the production function is homogeneous of degree 1.

Note again that

$$-\frac{\partial \pi^*}{\partial q_1} = Y\left(\frac{1}{1-\alpha}\right)^{1-\alpha}\left(\frac{q_1}{\alpha}\right)^{-(1-\alpha)} = X_1^*$$

i.e. the first derivative of the "profit" function with respect to the relative input price again gives us the derived demand function. This profit function gives us the negative of total costs per unit price of the second input. If we want to go back to a more straightforward concept of a minimum cost function 1/ we can multiply negative normalized profits by $(-P_2)$.

$$C^* = -P_2\pi^* = Y\left(\frac{P_1}{\alpha}\right)^{\alpha}\left(\frac{P_2}{1-\alpha}\right)^{1-\alpha} \qquad 2.24$$

In this function minimum cost is related to output and absolute input prices. In general for a Cobb Douglas production function $Y = A\prod_{i=1}^{n} X_i^{\alpha_i}$ homogeneous of degree $\Sigma\,\alpha_i = 1$ the cost function becomes

$$C^* = \left(\frac{Y}{A}\right)\prod_{i=1}^{n}\left(\frac{P_i}{\alpha_i}\right)^{\alpha_i} \qquad 2.25$$

For this form of the cost function Shephard's lemma is written

$$\frac{\partial C^*}{\partial P_i} = X_i^* \quad .$$

1/ Such as used in [1, 2].

The main use for the cost function is precisely the case in which the production function is homogeneous of degree one or greater than one and when no inputs are fixed. Whenever there are fixed inputs and the production function is homogeneous of degree less than one in the variable inputs one might as well work directly with the profit function with fixed inputs as discussed previously.

2.4 Two Variable Inputs

The interested reader can check his understanding of the concepts used before by deriving the profit function for the following two variable production functions : $Y = X_1^{\alpha} X_2^{\beta}$ where $\alpha+\beta < 1$ 2.26

The solution of this problem is a profit function of the form

$$\pi^* = (1-\alpha-\beta) \left(\frac{q_1}{\alpha}\right)^{\frac{-\alpha}{1-\alpha-\beta}} \left(\frac{q_2}{\beta}\right)^{\frac{-\beta}{1-\alpha-\beta}} \qquad 2.27$$

3. ONE TO ONE CORRESPONDENCE AND THEORETICAL RESTRICTIONS

Profit functions are not very useful when the underlying production function is simple and known. The most powerful use can be made of them when one chooses directly an econometrically convenient form of the profit function in such a way that one makes sure that the corresponding production function obeys the restrictions implied by production theory. One would like to attain as much computational ease and as much flexibility for the under

lying production process as possible, but is in no way interested to know the precise functional form of the production functions. The theory of one to one correspondence between production and profit function as discussed in [7, 6, 11] is the foundation on which this neglect of the functional form is based. The following conditions are imposed on the production process $Y = F(X, Z)$, where X and Z are vectors of variable and fixed inputs (Lau[7]p. 2).

a. The production function is continuous in X and Z (twice differentiable in X and once differentiable in Z).

b. The production function is strictly increasing in X and Z, i.e.

$$\frac{\partial Y}{\partial X} > 0 \quad , \quad \frac{\partial Y}{\partial Z} > 0 \qquad 4.1$$

In addition

$$\lim_{X_i \to 0} \frac{\partial F}{\partial X_i} \to \infty \qquad i = 1, \ldots\ldots, m$$

c. The production function is strictly concave in X in the non-negative orthant.

d. $Y = F(X, Z)$ is finite for all finite X and Z. F is unbounded as X and Z approach infinity.

Assumptions (a) to (d) are sufficient to ensure that there exists a unique, optimal solution to the profit maximising problem in the interior of the negative orthant. Consequently they also guarantee the existence of single - valued supply and derived demand functions as continuously differentiable functions of normalised prices and fixed inputs.

One to One Correspondence

Corresponding to any production function $Y = F(X, Z)$ satisfying the assumptions (a) to (d) there exists a unique normalised profit function $\pi^* = G(q, Z)$ satisfying the constraints (a') to (d') below and vice versa.

(a') The normalised profit function is continuous in the normalised prices q_i and the fixed input quantities Z_j. It is twice differentiable in the q_i's and once differentiable in the Z_j's. These properties follow from the existence of continuous demand functions conditional on the levels of fixed inputs.

(b') The normalised profit function is strictly decreasing in the q_i's and strictly increasing in the Z_j's. This also implies that the normalised profit function is increasing in the money price of output and decreasing in the money prices of the variable inputs.

(c') The normalised profit function is strictly convex in the q_i's. No convexity conditions are imposed on the Z_j's.

(d') The normalised profit function is finite for all finite normalised prices and non-positive as all normalised prices approach infinity.

In econometric applications one chooses a normalised profit function which follows assumptions (a') to (d'). Then one is sure that all assumptions (a) to (d) hold for the production process.

In addition to the assumption (a) to (d) on the production process it is well known that there are other theoretical constraints which have to hold for production functions. The most important in this context is the symmetry constraint which states that

$$\frac{\partial F}{\partial X_i \partial X_j} = \frac{\partial F}{\partial X_j \partial X_i} \quad \text{for all } i, j \qquad 4.2$$

A similar constraint applies to profit functions i.e.

$$\frac{\partial \pi^*}{\partial q_i \partial q_j} = \frac{\partial \pi^*}{\partial q_j \partial q_i} \qquad 4.3$$

Of course these constraints simply come from the laws of calculus but they are extremely important in econometric applications to increase estimation efficiency and to test the assumption of production theory.

Let us now demonstrate the one to one relationship by finding the production function implied in a given simple normalised profit function, or more specifically find the production function implicit in the example given in section 2.4.

We want to derive the production function implied in a profit function by a minimizing procedure. We know that for any given input (not necessarily the profit maximizing ones) levels of actual normalised profits are always less than or equal to the maximum profits, which implies the following inequality :

$$f(X_1, X_2) - q_1X_1 - q_2X_2 \leq \pi^*(q_1, q_2) \qquad 4.4$$

Note that $Y = f(X_1, X_2)$ is a fixed number for fixed values of X_1 and X_2. We can rewrite the inequality as follows :

$$f(X_1, X_2) \leq \pi^* (q_1, q_2) + q_1X_1 - q_2X_2 \qquad 4.5$$

In particular this inequality has to hold when we minimize the right hand side with respect to the q_i, i.e., when we find the values of q_1 and q_2 which minimizes the right hand side for given values of the X's. Such problems, of course, do not usually occur in economics. Suppose the profit function is the Cobb-Douglas function of Section 2.4. Then the minimizing problem has the form :

$$\text{min. } f(X_1, X_2) = C\, q_1^{\ \gamma}\, q_2^{\ \delta} + q_1X_1 + q_2X_2 \qquad 4.6$$

Now choose C and γ and δ such that they correspond to the profit function of Section 2.4 and rewrite

$$\text{min. } Y = (1-\alpha-\beta) \left(\frac{q_1}{\alpha}\right)^{\frac{-\alpha}{1-\alpha-b}} \left(\frac{q_2}{\beta}\right)^{\frac{-\beta}{1-\alpha-\beta}} + q_1X_1 + q_2X_2 \qquad 4.7$$

By deriving the first order conditions with respect to q and solving them for the optimal q the reader can verify that the optimal normalized price levels for this minimizing problem are :

$$q_1^* = \alpha\, X_1^{\alpha-1}\, X_2^{\beta} \qquad 4.8$$

$$q_2^* = \beta\, X_1^{\alpha}\, X_2^{\beta-1} \qquad 4.9$$

Substituting these equations into (4.7)

$$Y = (1-\alpha-\beta)\, X_1^{-\frac{\alpha(\alpha-1)}{1-\alpha-\beta}}\, X_2^{-\frac{\alpha\beta}{1-\alpha-\beta}}\, X_1^{-\frac{\alpha\beta}{1-\alpha-\beta}}\, X_2^{-\frac{\beta(\beta-1)}{1-\alpha-\beta}}$$

$$+ \alpha\, X_1^{\alpha}\, X_2^{\beta} + \beta\, X_1^{\alpha}\, X_2^{\beta}$$

Or simplified

$$Y^* = X_1^{\alpha}\, X_2^{\beta}\, (1-\alpha-\beta+\alpha+\beta) = X_1^{\alpha}\, X_2^{\beta}$$

and this is the production function of example 2.4

This completes the demonstration of the one to one correspondence since we derived the production function of example 2.4 simply from the knowledge of the profit function.

4. ECONOMETRIC APPLICATION OF DUALITY CONCEPTS

In this section four empirical application to agriculture are briefly discussed first. Then a few new functional forms and econometric estimation procedures are discussed.

To my knowledge the first applications of profit functions to agriculture by Lau and Yotopoulos are in three articles. In "Profit, Supply and Factor Demand Functions" [13], they use a Cobb-Douglas normalised profit function (corresponding to a Cobb-Douglas production function) and jointly estimates a profit and a factor demand function with data from Indian Farm Management studies. For given land inputs they estimate a labour demand elasticity - 2.166 and an output supply elasticity of 1.66 and tests various restrictions of economic theory.

In a slightly earlier article "A Test of Relative Efficiency and Application to Indian Agriculture" [12] the theory is pushed much further. Farms are no longer assumed to be equally efficient in either technical or economic terms. Instead differences in technical efficiency and in allocative efficiency (i.e. the successfulness of farmers to equate marginal products to factor prices) are built into the theory right from the start. This shows that the profit function approach is not restricted in its application to cases of pure competition but can indeed be used as a vehicle to systematically relax some of the most constraining assumptions underlying the pure competition case. Building differences in efficiency levels of farmers right into the theory also allows rigorous tests of the relative efficiency of various groups of farmers or of the same group of farmers over time [18]. Again using a Cobb Douglas function and a data from the Farm Management Studies Lau and Yotopoulos find that small farmers are relatively more efficient than large farmers, i.e. that small farmers attain higher levels of price efficiency and/or operate at higher levels of technical efficiency.

In [19] the same authors go a step further by building a model which enables them to distinguish between technical and economic efficiency. Here they find the same result that small farmers are more efficient and that their superiority stems from higher technical efficiency whereas small and large farmers are found to be equally price efficient.

In [1] the author of this paper used a non-normalised minimum cost function approach to estimate a full set of factor demand elasticities and elasticities of substitution for five factors of production in U.S. agriculture using aggregate state data. The translog functional form discussed below was used. In another paper [2] the set of coefficients of the Translog cost function so measured are then used to estimate technical change biases in U.S. agriculture from 1912 to 1965 with respect to the same five factors. A strong machinery using bias and labour-saving bias are shown to have occured during parts of the period under investigation.

In the investigations of Lau and Yotopoulos the main advantage derived from the use of the profit function is that (a) factor demand and output supply elasticities were estimated jointly, (b) that the econometric specification of endogenous and exogenous variables was correct and the fixity of land size in the short run was properly treated and that (c) differences of technical efficiency among farmers and derivations from pure profit maximisation were properly built into the estimation framework. However, at that time the choice of functional forms was still severely restricted so that the approaches are still tied to the constraining Cobb-Douglas framework.

By the time of my own work new functional forms allowing much wider flexibility were available of which I took advantage. But with hindsight it might have been better to use profit function in that research rather than cost function, because the cost function approach still mixes up endogenous and exogenous variables.

The new functional forms which have come to the forefront in recent years are the quadratic form [9], Diewert's Generalised Leontief Function [4, 5, 6] and the Translog form [3, 1, 2]. All three forms are second order approximations to arbitrary functional forms and therefore allow a flexibility for modelling production processes unknown before their advent.

Let q be the vector of n variable factors $(q_1, \ldots, q_i, \ldots, q_n)$ and the Z, the vector of m fixed factors $(Z_1, \ldots, Z_j, \ldots, Z_m)$. The simplest local second order approximation to an arbitrary twice differentiable function is the quadratic (for further discussion of this form see Lau [9]).

$$\pi^* = a_o + a_1^T q + a_2^T Z + \tfrac{1}{2} q^T A_{11} q + \tfrac{1}{2} Z^T A_{22} Z + q^T A_{12} Z \qquad 4.1$$

where the a vectors and the A matrixes are the coefficients of the function[2/]. This is a local approximation simply because it can be regarded as a Taylor series expansion to the second degree.

2/ with two variable and one fixed input this form would read :

$$\pi^* = a_o + a_1 q_1 + a_2 q_2 + a_3 Z + a_{11} q_1^2 + a_{22} q_2^2 + a_{33} Z^2 + a_{12} q_1 q_2 + a_{13} q_1 Z$$

$$a_{23} q_2 Z.$$

In production function analysis the quadratic form has been widely used for fertilizer response functions but never was widely used for other applications. The main reasons for its neglect probably were (a) that production elasticities are not constant, (b) that it is not necessarily concave (or convex) as implied by the theory of production (or profit function) and (c) that it could not approximate a production function homogeneous of degree one, a concept so dear to economists.

But constancy of production elasticities can only be achieved in otherwise very constraining forms such that it is really not a valid reason for not using more flexible forms- and anyway there are no reasons to assume that it ever occurs in real world production processes. The lack of concavity (or convexity) is not a real disadvantage because it allows us to test the assumptions of production theory as was done in [3]. Furthermore, Lau [10] has now developed a constraint estimation procedure which allow the imposition of this contraint in the estimation procedure and in the case of the quadratic this is particularly easy. Non homogeneity is no constraint either because normalised profit functions are not homogeneous in relative prices even if the underlying production process is homogeneous in inputs. Note however that normalised profit functions are homogeneous of degree one in absolute prices regardless of the homogeneity properties of the production process. Therefore the quadratic cannot be used for non-normalised profit functions.

For the latter case (and when there are no fixed inputs) Diewert's generalised function is particularly useful. It has the following form

$$\pi^* = \frac{1}{2} [q^{\frac{1}{2}}]^T A q^{\frac{1}{2}} \qquad 4.2$$

i.e. it has a quadratic form in square roots. (For detailed discussion of this form see Diewert [4, 5, 6]. Convexity conditions are also easy to impose on this function.

The third functional form is the Translog function

$$\log \pi^* = b_o + b_1^T \log q + b_2^T \log Z + \tfrac{1}{2} \log q^T B_{11} \log q + \tfrac{1}{2} \log Z^T B_{22} \log Z + \log q^T B_{12} \log Z$$

which has local second order approximation properties because it is a Taylor series expansion in logarithmes to an arbitrary twice differentiable function. (For further discussion and empirical work with this form see [3, 1, 2]. Note that the first two right hand side terms are a Cobb-Douglas function while the quadratic form allows that function to become more flexible and thus have arbitrary elasticities of substitution among factors.

These functions can be estimated directly. But it is more efficient to estimate them simultaneously with the derived demand equation implied by Shephards (or Hotelling's) lemma. For quadratic function these have the form

$$X_k = a_k + \sum_{i=1}^{n} a_{ki} q_i + \sum_{j=1}^{m} a_{kj} Z_j \quad k = 1, \ldots n \qquad 4.4$$

Equations 4.1 and 4.4 are then estimated simultanedusly imposing the constraints that the a cœfficients are identical in 4.1 and 4.4

Note further that in estimation the symmetry constraints $a_{ki} = a_{ik}$ and should be imposed and increase the efficiency of estimation. These equations should be estimated jointly using generalised least squares or maximum likelihood methods as described in the following references [18, 3, 1, 2]3/.

For the other functional forms Shephard's lemma provides similar derived demand equations. Note that in the case of the Translog function the derived functions are share equations of the form.

$$\frac{\partial \ln \pi^*}{\partial \ln q_k} = S_k = b_k + \sum_{i=1}^{n} b_{ki} \ln q_i + \sum_{j=1}^{m} b_{kj} \ln Z_j$$

3/ The computational burden of these methods is quite heavy and requires fairly large core memory. For IBM machines packaged programs are available at modest charges (such as the TSP program of the department of Economics, Harvard University).

where S_k is the share of the input in variable profits. This again poses some special estimation problems which are discussed in [19, 3, 1, 2].

We have not discussed here the use of duality for theoretical purposes. However, many theorems of economics can be proved much easier once duality is established [6, 7] and the tool can also be used to break new grounds in economic theory.

5. EXTENSION TO MULTIPLE INPUTS AND OUTPUTS

5.1 Theory

The duality theory can fairly easily be expanded to multiple inputs and outputs. For agricultural economics this is a very welcome advance because the agricultural production process involves usually joint production of several outputs on the same resource base with many variable inputs. (for detailed theoretical discussion and functional forms see Lau [7, 8, 5].

We adopt the convention that the vector Y of dimension n + 1 denotes both inputs and outputs. All first Y_i $(i = 1, \ldots, k < n + 1)$ are outputs and are measured as positive quantities while the remaining Y_i $(i = k + 1, \ldots, n+1)$ are inputs and are measured as <u>negative</u> quantities. Inputs and outputs are therefore distinguished by whether they are positive or negative. We number them till n + 1 because we later will normalise on the n + 1 st input price.

The transformation function can be written in implicit form as

$$F(Y, Z) = 0 \qquad 5.1$$

where Z again denotes fixed inputs (or fixed outputs). Following Lau [7, 8] the following assumptions are made for the transformation function.

(a*) F is continuous in Y and Z twice differentiable in Y and once differentiable in Z.

(b*) F is strictly increasing in Y. By the implicit function theorem this means that $\frac{\partial Y_i}{\partial Y_j} < 0$. Keeping in mind that inputs are negative quantities this implies that an increase in an input will always lead to an increase in an output, holding all other inputs and outputs constant. In addition

$$\lim_{Y_i \to 0} \frac{\partial F}{\partial Y_i} \to 0 \qquad \text{for } Y_i > 0$$

$$\lim_{Y_i \to 0} \frac{\partial F}{\partial Y_i} \to \infty \qquad \text{for } Y_i < 0$$

(c*) F is strictly convex.

(d*) Outputs are finite for finite variable and fixed inputs. For unbounded variable and fixed inputs, outputs become unbounded.

Dual to the transformation function there again exists a normalised variable profit function arising out of the following maximisation problem

$$\max \pi' = \sum_{i=1}^{n+1} P_i Y_i + \lambda F(Y,Z) \qquad 5.2$$

Normalising on the price of the n+1st input this becomes

$$\max \pi = \frac{\pi'}{p_{n+1}} = \sum_{i=1}^{n} p_i Y_i + Y_{n+1} + \lambda F(Y,Z) \qquad 5.3$$

This can be solved for the optimal output and input quantities Y^*_i which can be set into 5.3 and give rise to the profit function.

$$\pi^* = \sum_{i=1}^{n} p_i Y^*_i + Y^*_{n+1} \qquad 5.4$$

Note that $F(Y^*,Z)=0$ so that this term does not appear in 5.4

Since all Y^* are functions of p and Z the profit function 5.4 can also be written as

$$\pi^* = G(p, Z) \qquad 5.5$$

which conforms to the following assumptions

(a**) π^* is continuous, twice differentiable in p and once differentiable in Z.

(b**) π^* is strictly increasing in p_i if Y_i is an output and strictly decreasing in p_i if Y_i is an input. i.e. profits increase with increases in output prices and decrease with increase in input prices.

(c**) π^* is strictly convex in p

(d**) π^* is finite for all finite p [1/]

Again the transformation function and the normalised profit function are related to each other in a one to one relationship. Therefore one can again start all econometric inquiries by choosing functional forms which conform to conditions (a**) to (d**) and is sure that the conditions (a*) to (d*) hold for the production process. The functional forms of section 4 can be chosen. Further functional forms are given by Diewert [5],

Shephard's or Hotelling's lemma hold again in this context

$$\frac{\partial \pi^*}{\partial p_i} = Y_i^*$$

Except that due to the sign convention on Y the positive derivatives are now the output supply and input demand functions. This lemma again is the basis for econometric work to estimate jointly factor demand and output supply elasticities for given values of fixed inputs.

1/ Note that non-normalised profit functions are always homogeneous of degree one in prices regardless of the homogeneity properties of the transformation function.

I am not aware that this approach has ever been used in agriculture. Also there have only been very few attempts to estimate transformation functions. The most important one was by Powell and Gruen [15] who estimated output elasticities for Australian agriculture in a transformation model. However, they could not take account of variable and fixed inputs and were restricted to the very constraining Constant Elasticity of Transformation form.

The main reason for not using transformation function seems usually to have been the complexity of working with implicit functions as the theory requires. This difficulty is entirely bypassed by the profit function approach where all functions are explicit.

5.2 An Example

Let the production function be Constant Elasticity of Transformation in outputs and Cobb-Douglas in inputs.

$$(\alpha_1 Y_1^{\rho} + \alpha_2 Y_2^{\rho})^{\frac{\mu}{\rho}} = (-Y_3)^{\beta} (-Y_4)^{1-\beta} \qquad 5.6$$

where $\mu > 1$, $\rho < 1$, and $\sigma = \frac{1}{1-\rho}$ may be identified as the elasticity of transformation. Solving for Y_4

$$Y_4 = -(-Y_3^{-\frac{\beta}{1-\beta}}) (\alpha_1 Y_1^{\rho} + \alpha_2 Y_2^{\rho})^{\frac{\mu}{\rho(1-\beta)}} \qquad 5.7$$

Now let $B = (-Y_3)^{\frac{-\beta}{1-\beta}}$ and $C = (\alpha_1 Y_1^{\rho} + \alpha_2 Y_2^{\rho})$

So that we can write transformation function in simpler form

$$Y_4 = -B\ C^{\frac{\mu}{\rho(1-\beta)}} \qquad 5.8$$

We proceed in two steps to find the normalised profit function : In step one we hold Y_3 fixed and find the variable profit function π_v^* with Y_3 fixed. In the second step we then maximise profits with respect to Y_3 and find the profit function π^* with all inputs and outputs variable.

After normalization on P_4 we have variable normalized profits

$$\pi_v = Y_1 p_1 + Y_2 p_2 - B\ C^{\frac{\mu}{\rho(1-\beta)}} \qquad 5.9$$

The first-order conditions are

$$p_1 = \left(\frac{\mu B}{1-\beta}\right) \alpha_1\ C^{\frac{\mu-\rho(1-\beta)}{\rho(1-\beta)}}\ Y_1^{\rho-1} \qquad 5.10$$

$$p_2 = \left(\frac{\mu B}{1-\beta}\right) \alpha_2 \; C^{\frac{\mu-\rho(1-\beta)}{\rho(1-\beta)}} \; Y_2^{\rho-1} \qquad 5.11$$

Dividing (5.10) by (5.11) leads to

$$Y_2^{\rho-1} = \left(\frac{p_1}{\alpha_1}\right)^{-1} \left(\frac{p_2}{\alpha_2}\right) Y_1^{\rho-1} \qquad 5.12$$

Raising this equation to the $\frac{\rho}{\rho-1}$:

$$Y_2^{\rho} = \left(\frac{p_1}{\alpha_1}\right)^{\frac{-\rho}{\rho-1}} \left(\frac{p_2}{\alpha_2}\right)^{\frac{\rho}{\rho-1}} Y_1^{\rho} \qquad 5.13$$

Substitute (5.13) for Y_2^{ρ} in the expanded C term of 5.10

$$\left(\frac{p_1}{\alpha_1}\right) = \left(\frac{\mu B}{1-\beta}\right) \left[\alpha_1 Y_1^{\rho} + \alpha_2 \left(\frac{p_1}{\alpha_1}\right)^{-\frac{\rho}{\rho-1}} \left(\frac{p_2}{\alpha_2}\right)^{\frac{\rho}{\rho-1}} Y_1^{\rho} \right]^{\frac{\mu-\rho(1-\beta)}{\rho(1-\beta)}} Y_1^{\rho-1}$$

Now transfer Y_1^{ρ} outside of the brackets and collect terms with $Y_1^{\rho-1}$ and multiply the two terms in the brackets by $\left(\frac{p_1}{\alpha_1}\right)^{\frac{\rho}{\rho-1}}$ and perform the

contrary multiplication with $\left(\frac{p_1}{\alpha_1}\right)^{-\frac{\mu-\rho(1-\beta)}{(\rho-1)(1-\beta)}}$ outside of the brackets and collect the term with the left hand side to $\left(\frac{p_1}{\alpha_1}\right)$

$$Y_1^{\frac{1-\mu-\beta}{1-\beta}} = \frac{\mu B}{1-\beta}\left[\alpha_1\left(\frac{p_1}{\alpha_1}\right)^{\frac{\rho}{\rho-1}} + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^{\frac{\rho}{\rho-1}}\right]^{\frac{\rho}{\rho-1}} \left(\frac{p_1}{\alpha_1}\right)^{\frac{1-\mu-\beta}{(\rho-1)(1-\beta)}} \qquad 5.12$$

Setting $D = \left[\alpha_1\left(\frac{p_1}{\alpha_1}\right)^{\frac{\rho}{\rho-1}} + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^{\frac{\rho}{\rho-1}}\right]$ and solving for Y_1 leads to the optimal output quantities of the constraint problem which we denote by Y_i^{**}.

$$Y_1^{**} = \left(\frac{\mu B}{1-\beta}\right)^{\frac{1-\beta}{\mu+\beta-1}} D^{-\frac{\mu-\rho(1-\beta)}{\rho(\mu+\beta-1)}} \left(\frac{p_1}{\alpha_1}\right)^{\frac{1}{\rho-1}} \qquad 5.14$$

Similarly

$$Y_2^{**} = \left(\frac{\mu B}{1-\beta}\right)^{\frac{1-\beta}{\mu+\beta-1}} D^{-\frac{\mu-\rho(1-\beta)}{\rho(\mu+\beta-1)}} \left(\frac{p_2}{\alpha_2}\right)^{\frac{1}{\rho-1}} \qquad 5,15$$

Now substitute these terms back into variable normalized profits, and let $\nu = \mu+\beta-1$

$$\pi_\nu^* = Y_1^{**}\alpha_1\left(\frac{p_1}{\alpha_1}\right) + Y_2^{**}\alpha_2\left(\frac{p_2}{\alpha_2}\right) - B\left[\alpha_1 Y_1^{**\rho} + \alpha_2 Y_2^{**\rho}\right]^{\frac{\mu}{\rho(1-\beta)}}$$

$$= \left(\frac{\mu B}{1-\beta}\right)^{-\frac{1-\beta}{\mu}} D^{-\frac{\mu-\rho(1-\beta)}{\rho\nu}}\left[\alpha_1\left(\frac{p_1}{\alpha_1}\right)^{\frac{\rho}{\rho-1}} + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^{\frac{\rho}{\rho-1}}\right]$$

$$- B\left[\left(\frac{\mu B}{1-\beta}\right)^{-\frac{\rho(1-\beta)}{\nu}} D^{-\frac{\mu-\rho(1-\beta)}{\nu}}\left(\alpha_1\left(\frac{p_1}{\alpha_1}\right)^{\frac{\rho}{\rho-1}} + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^{\frac{\rho}{\rho-1}}\right)\right]^{\frac{\mu}{\rho(1-\beta)}}$$

$$= \left(\frac{\mu B}{1-\beta}\right)^{-\frac{1-\beta}{\nu}} D^{\frac{\mu(\rho-1)}{\rho\nu}} - \left(\frac{\mu}{1-\beta}\right)^{-\frac{\mu}{\nu}} B^{-\frac{1-\beta}{\nu}} D^{\frac{\mu(\rho-1)}{\rho\nu}}$$

$$= B^{-\frac{1-\beta}{\nu}} D^{\frac{\mu(\rho-1)}{\rho\nu}}\left[\left(\frac{\mu}{1-\beta}\right)^{-\frac{1-\beta}{\nu}} - \left(\frac{\mu}{1-\beta}\right)^{-\frac{\mu}{\nu}}\right]$$

$$\pi_\nu^* = (-Y_3)^{\frac{\beta}{\nu}}\left(\frac{\mu}{1-\beta}\right)^{-\frac{\mu}{\nu}}\left(\frac{\nu}{1-\beta}\right) D^{\frac{\mu(\rho-1)}{\rho\nu}} \qquad .16$$

It is left to the reader to verify that

$$\frac{\partial\pi_\nu^*}{\partial p_i} = Y_i^{**}$$

The example so far is also an example of production with multiple inputs and outputs where oneinput is given.

In the second stage of the problem we have to maximize π_z^* with respect to Y_3 to obtain the unconstraint profit function.

Let $$A = \left(\frac{\mu}{1-\beta}\right)^{-\frac{\mu}{\nu}} \left(\frac{\nu}{1-\beta}\right) D^{\frac{\mu(\rho-1)}{\rho\nu}}$$

and $$\gamma = \frac{\beta}{\nu} = \frac{\beta}{\mu+\beta-1}$$

Then $$\pi_z^* = A(-Y_3)^{\gamma}$$

Now maximize

$$\pi = \pi_v^* - p_3Y_3 = A(-Y_3)^{\gamma} - p_3Y_3 \qquad 5.17$$

Note that this problem has precisely the same form as (2.3) in the example 2.1 with one input and one output. Therefore, the reader may show, as an exercise , that :

$$-Y_3^* = p_3^{-\frac{\nu}{\mu-1}} A^{\frac{\nu}{\mu-1}} \left(\frac{\beta}{\nu}\right)^{\frac{\nu}{\mu-1}} \qquad 5.18$$

and setting this into π :

$$\pi^* = \frac{\mu-1}{\nu} \left(\frac{\beta}{\nu}\right)^{\frac{\beta}{\mu-1}} A^{\frac{\nu}{\mu-1}} p_3^{-\frac{\beta}{\mu-1}} \qquad 5.19$$

Substituting A back into (5.18 and recalling $\nu = \mu+\beta-1$) leads to

$$\pi^* = \frac{\mu-1}{\mu+\beta-1}\left(\frac{\beta}{\mu+\beta-1}\right)^{\frac{\beta}{\mu-1}}\left(\frac{1-\beta}{\mu}\right)^{\frac{\mu}{\mu-1}}\left(\frac{1-\beta}{\mu+\beta-1}\right)^{-\frac{\mu+\beta-1}{\mu-1}} D^{\frac{\mu(\rho-1)}{\rho(\mu-1)}} \; p_3^{-\frac{\beta}{\mu-1}}$$

After some cancelling of terms and bringing back the D term this becomes :

$$\pi^* = (1-\beta)^{\frac{1-\beta}{\mu-1}}(\mu-1)\mu^{-\frac{\mu}{\mu-1}}\left[\alpha_1\left(\frac{p_1}{\alpha_1}\right)^{\frac{\rho}{\rho-1}} + \alpha_2\left(\frac{p_2}{\alpha_2}\right)^{\frac{\rho}{\rho-1}}\right]^{\frac{\mu(\rho-1)}{\rho(\mu-1)}}\left(\frac{p_3}{\beta}\right)^{-\frac{\beta}{\mu-1}} \qquad 5.20$$

Note that π^* is again a separable combination of a Constant Elasticity of Transformation function in normalized output prices and a Cobb-Douglas function in the normalized input price.

The output supply functions now are

$$\frac{\partial\pi^*}{\partial p_1} = Y_1^* = (1-\beta)^{\frac{1-\beta}{\mu-1}}\mu^{-\frac{1}{\mu-1}} D^{\frac{\rho-\mu}{\rho(\mu-1)}}\left(\frac{p_1}{\alpha_1}\right)^{\frac{1}{\rho-1}}\left(\frac{p_3}{\beta}\right)^{-\frac{\beta}{\mu-1}} \qquad 5.21$$

and similarly for Y_2^*

The input demand function is :

$$\frac{\partial\pi^*}{\partial p_3} = Y_3^* = -(1-\beta)^{\frac{1-\beta}{\mu-1}}\mu^{-\frac{\mu}{\mu-1}} D^{\frac{\mu(\rho-1)}{\rho(\mu-1)}}\left(\frac{p_3}{\beta}\right)^{\frac{1-\mu-\beta}{\mu-1}} \qquad 5.22$$

This is negative as it should be under our sign conventions.

It should again be stressed that this functional form is not particularly useful for applied econometric work but serves only for illustration purposes.

After this tortuous derivation and given the complexity of the factor demand and output supply functions it should also be clear that the one to one correspondence of profit and transformation function is very advantageous since it allows one to specify the profit function in an econometrically convenient form without worrying about the exact functional form of the transformation function. One is therefore spared of all this tortuous derivation..

6. FURTHER CONSIDERATIONS

The reader interested in application of this theory will have to go deeper into the literature on profit functions. This literature contains the proofs of the theorems presented here, discusses econometric procedures for estimating the systems of equations and contains several further duality theorems which are useful in applied econometric work.

For readers familiar with calculus the most easily accessible reference containing proofs of the relationships, further theorems and functional forms is Lau [7, 8]. Another paper by Lau [11] contains duality relationships which make it possible to relate short term output supply and factor demand elasticities (derived in a variable profit function with fixed inputs) to long term elasticities when the fixed input quantities become variable.

At a slightly more advanced level is Diewert's "Applications of Duality Theory" Paper [6]. This paper also contains the theory of duality between direct and indirect utility functions. Indirect utility functions provide similar advantages in consumer demand work than profit functions provide in production economics. In addition the paper gives a historic review of the development of these theories.

In addition to the empirical work in agriculture Christensen, *et al* [3] have used the Translog profit function to test production theory with aggregate US data. Econometric and statistical testing procedures are reported there.

A further reference which is easily accessible and which contains some graphic demonstration is Mc Fadden [14]. Very rigorous statements of the theory which are much more general can be found in [11], [17], [16],

The two main problems with the profit function approach relate to data and the computational burden which they impose. To use profit function requires good output price and factor price data. With cross sectional data the factor price data often have to be computed from factor costs and factor quantities and if the data come from a small region there may be little variability in the prices. Also, since usually systems of equations with contraints across equations have to be estimated the computer time required becomes quite high. However, once a packaged program for estimating these systems is available, programming work for estimation becomes easily manageable.

To close I would again like to draw attention to the tremendous flexibility which the profit function approach allows to the economic investigator handled with relative ease. Fixity of inputs or outputs is easily accommodated; deviations from pure profit maximising behaviour can be systematically built into the theoretical framework; factors which affect productivity such as soil quality or climatic variables can be built in as fixed variables and their contribution to production estimated; and far more complex production processes can be modeled than is possible with the standard production functions.

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HPB/ss.

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