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A FORMAL STATEMENT OF THE MODEL

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A FORMAL STATEMENT OF THE MODEL[1]

We can completely describe an economy by specifying the following: for each consumer, his demand (utility) function and his initial stock of commodities, and, for each producer, his production function. We will assume that government is both a consumer and a producer in our model; this treatment is a little nonstandard but it greatly simplifies the exposition. We need, in addition, to specify the distribution of the profits, if any, from the producers to the consumers; we assume explicitly that only consumers consume.

In the remainder of this paper we will present in detail descriptions of the several units of the economy. We will then briefly describe the model as a whole. Finally, we will describe and discuss the technique to be used to find an equilibrium solution for our model.

The Consumers

Initially, we will assume that our economy contains only two real consumers, an urban and a rural sector. This duality assumption is done largely to facilitate comparison of our results with those of others and can be easily dropped and the number of consumers increased without conplicating the model seriously.

In choosing our demand functions we face two relatively stiff constraints. Firstly, we are operating within a general

equilibrium framework which requires that our demand functions be derivable from a plausible (ordinal) utility function.[2] Secondly, realism requires that while, for simplicity, we have limited the number of consumers, it be meaningful to consider their behavior as an aggregation of the actions of many more households. Note that these requirements rule out the common and convenient (at least for estimation) constant elasticity functions.

These constraints basically limit our choice set to one: the linear expenditure system of Richard Stone.[3] Let $\underset{\sim}{x}_i$ be the vector of demands for the $\underline{i}$-th consumer, $\underset{\sim}{p}$ the vector of prices, $\hat{\underset{\sim}{p}}$ the diagonal matrix formed from $\underset{\sim}{p}$, I the identity matrix, y_i the income of the $\underline{i}$-th consumer, and $\underset{\sim}{\gamma}_i$ and $\underset{\sim}{\beta}_i$ vectors of parameters for the $\underline{i}$-th consumer. Then the demand equations of Stone's linear expenditure system can be written as

$$\underset{\sim}{x} = \underset{\sim}{\gamma} + (y_i - \underset{\sim}{p}^t \underset{\sim}{\gamma})\, \hat{\underset{\sim}{p}}^{-1} \underset{\sim}{\beta}_i .$$

Intuitively, we can consider these demand functions to mean the following: The $\underline{i}$-th consumer first purchases an amount γ_{ij} of the j-th commodity for $j = 1,\ldots,n$ ($\underset{\sim}{\gamma}_i \geq 0$). The income necessary to do this, $\underset{\sim}{p}^t \underset{\sim}{\gamma}_i$, has the obvious designation "subsistence income." Standard practice is to term the remaining income "supernumerary." The consumer then distributes his supernumerary income in fixed proportions β_{ij} among the $\underline{n}$ commodities. For apparent reasons we require $\underset{\sim}{\beta}_i \geq 0$ and $\sum_j \beta_{ij} = 1$. The amount of the $\underline{j}$-th commodity purchased out

of supernumerary income is thus $\beta_{ij}\ (y - \underset{\sim}{p}^t \underset{\sim}{\gamma}_i)/p_j$, and the total amount purchased is $X_j = \gamma_{ij} + (y - \underset{\sim}{p}^t \underset{\sim}{\gamma}_i)\ \beta_{ij}/p_j$, as above. The reader can easily verify that this relationship is identical with that given above.

The utility function corresponding to this set of demand functions can be written as

$$U_i = \sum_j \beta_{ij} \log (x_{ij} - \gamma_{ij})$$

for $y_i > p^t \gamma_i$. It is not strictly defined for a lower income nor are we required to identify $\underset{\sim}{\gamma}_i$ and $\underset{\sim}{\beta}_i$ as we have done above. It is convenient to do so, however, and following this convention we will consider U_i to be zero for $y_i \leq \underset{\sim}{p}^t \underset{\sim}{\gamma}_i$.

We note that these demand functions meet our second constraint in the sense that if each sector were composed of a number of individuals having <u>identical</u> demand functions of the type given above, then the aggregate <u>per capita</u> demand functions would be identical to each of the individual functions. In other words, the sector as a whole would behave as if it were maximizing a Stone-Geary[4] utility function if each of its constituents had an identical, individual utility function. Note that we do not require an identical distribution of incomes to achieve this result; however, because the expenditure functions are linear, a change in the distribution of income <u>within</u> the sector while leaving total income received by the sector unchanged will not affect overall sector demands.

Some other properties of these functions should be noted. Since the underlying utility function can be written in an additive form (independent utilities), all commodities will be gross substitutes. Obviously, this is undesirable if we greatly disaggregate demand. Further, the Engel curves derived from them are linear, which at best can only be regarded as an approximation. Finally, this is the only linear system which can be derived from the principles of utility maximization.

We choose them in the end because they are both theoretically and empirically convenient. They have all the properties which we require for our general equilibrium framework; moreover, they are easy to implement and can possibly be estimated, since the number of underlying, independent parameters is relatively small. They have been used successfully as representations of consumer behavior both for estimation purposes and in some models of development. Several dynamic versions can be easily devised. One, which was originally suggested by Stone but apparently never implemented, is to make the minimum quantities, $\underset{\sim}{\gamma}_i$, be a distributed lag function of past actual consumption; we will consider this version later.

But we end on a note of caution: Stone has warned that when the actual demands are near the minimum demands - a situation likely to obtain in our model - the behavior implied by these equations may be unreasonable and will be a poor representation of actual behavior. Thus, we shall have to be cautious in using our results in such situations. The impact

can be minimized by treating $\underset{\sim}{\gamma}_i$ not as a vector of psychological minimum levels but as Malthusian minimal substitence requirements. This, however, rules out using changes in $\underset{\sim}{\gamma}_i$ as one means of reflecting changes in tastes, something which has many advantages.

The Production Functions

Earlier work [ ] has made clear that direct factor substitution in production can have critical consequences for development planning, and one of our main goals will be to further examine this impact. Thus, we specify non-linear production relations for each of our several outputs.

Following Chenery and Uzawa [ ] and later work following therefrom, we do make the simplifying assumption that intersectoral production relationships are linear: that is, we assume that only value - added is produced by general production functions and that intermediate goods are related by an input-output system.

Our primary motivation for this assumption is simplicity. The general theory of n-factor production functions and their empirical forms is relatively weak. Moreover, the complications of having general neo-classical production functions relating intermediate goods are very severe upon the ease of solving the model. Thus, we restrict our production functions to the production of value-added and we further consider only two-factor functions. Throughout, our restrictions are partially motivated by the need for reasonable parameter

estimates.

We should note that these assumptions do not really seriously restrict our model. The assumption of an input-output core can be relaxed to one of a more general activity-analysis core, alternate production activities being allowed ineach sector, without a significant increase in either the difficulty or cost of solving the model. The assumption of only two factors does not prevent us from defining our factors in terms of other primary factors. Thus, we feel that our model retains realism despite its considerable simplifications.

The choice of a specific production function remains. Econometric investigations of direct factor substitution have generally employed the specification of a production function having a constant (Allen) elasticity of substitution. This assumption is plausible within the range of capital-labor variation in the data used to estimate the production function. However, within an optimizing model, the assumption of a constant elasticity of substitution is considerably stronger, for the range of permissible capital-labor variation may be considerably wider than that actually observed.

It is probably a reasonable specification that the elasticity of substitution decline as production becomes increasingly intensive in one factor. This will help to avoid the result found in Chenery and Raduchel [ ] of production being indicated at improbably high or low capital-labor ratios, although other changes in the model will help independently. A production function having the properties we desire is the following modification of the

C.E.S. function due to Arrow, Chenery, Minhas and Solo [ ]: Let X be output, K capital and L labor; then,

$$X^{\rho} = \alpha\ (K - \kappa X)^{\rho} + \beta\ (L - \lambda X)^{\rho},$$

$$\text{for } K > \kappa X \text{ and } L > \lambda X \text{ and}$$

$$X = 0 \text{ otherwise}$$

defines an implicit production of five parameters - α, β, ρ, κ, λ. It is readily observed that the function is linear homogeneous, and in general it has the standard properties of a classical production function.

Essentially, this production function specifies that production requires minimal capital-output and labor-output ratios; above these minimal amounts excess capital and excess labor are substitutable with an elasticity of $\frac{1}{1-p}$. As the capital-labor ratio deviates from κ/λ the elasticity of substitution between K and L goes to zero. For a fixed amount of either factor there is a capacity constraint of either K/κ or L/λ for K or L fixed respectively, and as the capacity constraint is reached increasing amounts of the other factor will be required to produce the marginal unit of output.

Algebraically, we can list the properties of the function conveniently. Let $\Phi\ (K, L, X) = \frac{\alpha K}{\beta L}\ (\frac{K - \kappa X}{L - \lambda X})^{\rho-1}$.

Then, $F_K = \frac{\partial X}{\partial K} = \frac{X/K}{1+1/\Phi(K,L,X)}$

and $F_L = \frac{\partial X}{\partial L} = \frac{X/L}{1+\Phi\ (K,L,X)}$. Moreover, we can also demonstrate the following:

$$\frac{K}{L} = \Phi\ (K,L,X)\ \frac{dK}{dL}\ ;$$

$$F_{KL} = \frac{\partial^2 X}{\partial K\ \partial L} = \frac{F_K F_L}{X} + \frac{(X/L)\ (\frac{\partial \Phi}{\partial K} + F\kappa\ \frac{\partial \Phi}{\partial X})}{(1 + \Phi\ (K,L,X))^2}\ ;$$

$$\frac{\partial K}{\partial L} = \Phi\ (K,L,X) \text{ where } \alpha_K = \frac{F_K K}{X}\ ,\ \alpha_L = \frac{F_L L}{X}$$

and $\alpha_K + \alpha_L = 1$;

$\frac{K}{L} = \frac{\alpha_K}{\alpha_L}\ \frac{dK}{dL}$; and, if we let σ be the Allen elasticity of substitution and $\sigma_o = \frac{1}{1-\rho}$, then

$$\frac{\sigma_o}{\sigma} = \frac{\sigma_K}{1-\frac{\kappa X}{K}} + \frac{\sigma_L}{1-\frac{\lambda X}{K}}\ .$$

Note that σ is a function of X, K and L which is homogeneous of degree zero. At $K/L = \kappa/\lambda$, σ attains its maximum value $\sigma_o/[1 + (\alpha\ \kappa^\rho + \beta\ \lambda^\rho)^{1/\rho}] < \sigma_o$. Finally, under an assumption of cost minimization, the derived demand equations are:

$$\frac{K}{X} = \kappa + [\alpha + \beta\ (\frac{\alpha}{\beta}\ \frac{p_L}{p_K})^{\frac{-\rho}{1-\rho}}]^{-1/\rho} \text{ and}$$

$$\frac{L}{X} = \lambda + [\ \alpha\ (\frac{\beta p_K}{\alpha p_L})^{\frac{-\rho}{1-\rho}} + \beta]^{-1/\rho}$$

Technological change fits easily into this framework. The logical parameters to change are κ and λ. This yields several appealing results. If we regard κ and λ as falling over time, the result is an increase in the elasticity of substitution. The factor associated with the parameter falling

more rapidly will be used more for any given set of factor prices. Thus, we can introduce differential technological change naturally in a manner having expected effects.

These properties together lead us to choose this production function for use in the model. Note that the C.E.S. is simply a special case of this function so we can explore the effects of C.E.S. easily without formally altering our model. It permits us to consider the effects of differential technological exchange. It has very easy to handle derived demand relations. But most importantly, while it has many features which distinguish it from other neo-classical production functions and suggest it is more suitable for use in planning models, it retains the property of constant returns to scale.

Linear homogeneity is important to us for several reasons. As we mentioned at the start of this section, we combine these functions with a linear programming or input-output core relating the several sectors. Indeed, these functions can be considered as nothing more than a means of providing variable factor coefficients for the input-output system, where the coefficients vary with the relative price of the factors. More importantly, for simplifying the solution of the problem, we can now employ Samuelson's nonsubstitution principle to assert that for fixed-factor prices we have only one composite factor, and so the choice of optimal production techniques can be made independently of demand conditions. Thus, given a set of factor prices we can derive both

the set of commodity prices and the optimal production coefficients therefor. We will need this later on.

To complete our description of this part of the model we note that initially we will use four regular commodities and four factors. The commodities will be manufactures, agricultural products (food), extractive materials and services. The factors will be urban and rural capital and labor. The factors are assumed to be either completely substitutable or completely not so. We will explain the interrelationships further below. Too, foreign exchange will play the role of a factor.

International Trade

The assumption that the economy of a less developed country is closed to the outside world does obvious violence to reality. Accordingly, if only for completeness, we introduce a foreign trade sector into our model. Earlier results in Chenery and Raduchel [ ] suggest that the foreign trade may not have great effect in a development plan as aggregated as this. We expect, however, that its importance will be increased somewhat here: Our choice of production function technologically limits the amount of direct factor substitution, which is what overshadowed trade substitution previously.

Formally, we include in our model a trade improvement sector. This sector produces foreign exchange and commodities from foreign exchange and commodities. We assume that this

sector always has a first claim: That is, say for policy reasons, commodity demands for export always are met first, and imported commodities are always consumed before domestically produced ones.

We follow Chenery and Raduchel [ ] in our treatment of foreign trade. For each traded commodity we specify a net trade function composed of the sum of an import and an export function. These functions represent the substitution between domestic and external resources.

Following Chenery and Kretschmer [ ] we assume that the average revenue in foreign exchange earned per unit of exports for the j-th commodity h_j is a declining function of the amount exported, E_j. For simplicity we assume that this function is linear but subject to a lower bound of zero: $h_j = \delta_j - \theta_j E_j$. Similarly, we construct import functions. The average amount of foreign exchange saved per unit of domestic production depends upon the level of imports, for as the level of imports increases imports replace increasingly lower-cost domestic production. This is illustrated in Weisskoff [ ]. Thus we obtain a function giving the foreign exchange cost of imports of commodity j (g_j) as an increasing function of imports of that commodity (M_j), Again for simplicity, we assume that this function is linear: $g_j = \eta_j + \xi_j M_j$.

We thus treat export expansion and import substitution symmetrically. Note that it is possible to both export and

import a commodity at the same time. The overall level of trade is constrained by the amount of foreign exchange supplied to the trade improvement sector. We assume profit-maximizing behavior on the part of this sector, an assumption which together with the obvious requirement that all quantities be nonnegative yields the following relations:

$$E_j = \frac{\delta_j - p_j/p_F}{2\,\Theta_j} \text{ if positive, zero otherwise;}$$

$$M_j = \frac{\eta_j - p_j/p_F}{-2\,\xi_j} \text{ if positive and less than total demand, zero otherwise.}$$

We assume that the government recovers any profits made by the trade improvement sector.

The Government Sector

We adopt the slightly unorthodox method of treating the government as a trading sector of the economy. We assume that no one has a desire to avoid taxes which we include by assuming that it takes a fixed amount of government services (which must be purchased) to produce each commodity and that each consumer similarly desires to spend a fixed amount of his supernumerary income for such services. Thus, although we have not introduced them explicitly, taxes are present.

The government is assumed to buy and sell all non-labor factors; noreover, we admit the possibility that the government wishes to employ "surplus" labor. The parameters determining the behavior of the government are generally assumed

to be control variables subject to alteration by the planner.

The Investment Sector

As presented so far our model has ignored dynamics. We will initially, indeed, deal with only a single-period model, but this does not mean we can ignore dynamics, for single-period and static are far from synonymous. In our model we further distinguish between beginning-of-period and end-of-period stocks of capital and foreign exchange. We assume that labor stocks are considered permanent and unchanging, initially.

We then have joint production by almost all sectors of their normal outputs and end-of-period stocks: depreciation. We assume that both urban and rural households and the government have demands for end-of-period stocks. There is an explicit investment sector which produces end-of-period capital stocks from inputs of commodities and beginning-of-period stocks. We assume fixed coefficients here so the prices of these end-of-period stocks are determined completely by the prices of the beginning-of-period stocks and labor. We assume, however, that we can as a policy decision lower or raise these prices in a manner similar to that described above.

Foreign exchange, we assume, can be produced only by not spending the initial stock or from external sources. At some complication of our trade sector we can permit it to be produced by exports. For simplicity we assume the price of foreign

exchange to be the same for both beginning and end-of-period.

The Model Together

What we have constructed is essentially a general equilibrium system in 6 commodities: urban labor, rural labor, urban capital, rural capital, government services and foreign exchange. To be sure we have introduced other commodities into our model; however, because of our production assumptions, these commodities really serve only a bookkeeping function.

Our two sectors of rural and urban households begin the period with endowments of their respective capital and labor. In addition we assume that urban households may hold foreign exchange. In addition to demanding some of their own labor for leisure each sector demands our traded commodities which are nothing more than bundles of factors plus some government services. Each also is assumed to seek to have some end-of-period stock of its own capital. They "produce" current capital and labor as well as end-of-period stocks of their capital (that which they do not sell).

The production sector demands inputs of both types of capital and labor and government services from which it produces bundles of these factors (our traded commodities). The trade improvement sector produces foreign exchange and imports from foreign exchange, government services and our traded goods. The investment sector produces end-of-period capital stocks from goods, factors and government services.

Finally, we have a government sector whose behavior can be influenced by the planner. It produces government services as demanded. From its revenues it purchases labor, if so desired, and, as the guardian for posterity, end-of-period capital stocks and foreign exchange. It is assumed to have an initial endowment of capital and foreign exchange which it may release.

Our model thus has six sectors trading six commodities. The behavior of some sectors is assumed beyond control of the planner. He is realistically assumed, however, to control certain behavioral parameters of the government and the "tax rates." The market plays an essential, final role in determining the equilibrium, so we can avoid the unrealistic assumption of an omnipotent planner.

Finding an Equilibrium

For a fully-specified set of parameters we thus have a general equilibrium model which we assert has an equilibrium solution. To find this solution we must find an equilibrium set of prices. To do so we must resort to the mathematical concept of a fixed-point.

What we are trying to find is a set of prices such that there will be no excess demands leading to pressures on the prices to change: We are trying to satisfy the budget constraints of all sectors simultaneously. A set of prices doing this can be characterized as a fixed-point. We can actually not find a fixed-point for our model, but by using an

ingenious algorithm due to Herbert Scarf [ ] and Harold Kuhn [ ] we can approximate it as closely as we wish.

Not surprisingly we rely on techniques closely related to linear programming. Consider a simplex ρ defined by the set of all points P_j such that $\sum_i P_{ij} = D$ for all j, where $\{P_{ij}\}$ and D are integers. It is apparent that we can approximate our prices as closely as we choose by $\frac{1}{D} \underset{\sim}{P}_j$ for some $\underset{\sim}{P}_j$ for some D sufficiently large.

Now, we introduce the concept of a primitive set, which plays the role of the basis in linear programming. A detailed description of a primitive set and the algorithm generally is not appropriate here, but we can notivate the concept of a primitive set as a collection of points in the simplex so "close" to one another that no point is interior to the region they enclose. Thus, when we find our primitive set we have a set of price vectors all close to one another. We can choose any one of them or some combination thereof as our solution.

Obviously we must somehow tie prices and quantities together. This the algorithm does by associating with each element of the simplex a column of a matrix A. Then, we simultaneously find with our primitive set a feasible basis for the inequalities $A\underset{\sim}{x} = \underset{\sim}{b}$, for some $\underset{\sim}{b}$. By suitable choice of A and b, it is easy to demonstrate that Scarf's algorithm will lead us to an approximation to our general equilibrium prices.

Optimizing the Model

To now in this chapter we have developed a general equilibrium economic model and indicated a means of solving it for a given set of control variables or parameters. It is an easy extension to convert this simulation model into an optimizing model.

To do so we need an objective function. Here no less than in multi-period models we must worry about terminal constraints, which have the effect of altering the objective function. We propose to use some function of the welfare of each of the two household sectors as the core of our objective function. Choosing the welfare measure is easy: the proportion of income which is supernumerary, μ_u and μ_R, provides a monotonic measure of well-being.

This gives us a criterion of say either $\lambda\mu_u + (1-\lambda)\,\mu_R$ or $\lambda \log(\mu_u) + (1-\lambda) \log(\mu_R)$ for λ between zero and one, a choice which allows us to include distributional objectives into our criterion. The terminal conditions we handle as did Kendrick and Taylor [ ] or Thoreson and Dorfman [ ]: We append to this criterion the terminal stocks of foreign exchange and capital weighted by suitable shadow prices which can be varied parametrically.

The model as a whole can be considered a concentrated objective function: For any set of the control variables it provides a solution which we evaluate by our criterion.

Our optimization problem is thus reduced to a much smaller space.