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# ON THE NON-OPTIMALITY OF DISCOUNTING, AND ITS CORRECT GENERALIZATIONS IN INVESTMENT PROJECT ANALYSIS

by

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# On the Non-Optimality of Discounting, and its Correct Generalizations in Investment Project Analysis[1]

## 1. Introduction

This largely tutorial note is devoted to a coherant demonstration of three results in the theory of investment project analysis which are well-known to theorists but which to date have had only limited impact on practicioners. All of the results are derived in a new and I think illuminating way, using dynamic programming, and are presented for this reason.

(i) The first (and only widely applied) result is the following: discounting the flows of output resulting from an investment project is the unambiguously correct rule for deciding whether or not to undertake the project--when the project is "small" and the economy is adequately described by a one-sector Ramsey-type optimal accumulation model with malleable capital stock.

(ii) Discounting is <u>not</u> an optimal decision rule when there are heterogeneous malleable capital goods in the system. Rather, one must use shadow prices to evaluate the flows of output from the project which depend in a complex way on the future marginal products of all capital goods. Although these shadow prices cannot be expressed as familiar discount integrals, they have a natural interpretation as a multivariate analog of discounting.

(iii) Finally, formulas are presented for optimal shadow prices in the case where the investment project is itself large enough to influence future marginal products of capital (and interest rates). In this case, discounting is not a correct decision rule even in a one-sector model, since it is basically a first-order procedure which cannot deal with feedbacks from the project onto the equilibrium prices of the economy.

These results are discussed successively in Sections 3-5 of this paper. In Section 2 and the appendix, some very general rules for project analysis are derived--all of the foregoing conclusions follow from specializations of these rules. Finally, Section 6 contains conclusions and thoughts

about possible extensions of the results presented here.

## 2. General Decision Rules

The natural way to view an investment project in the context of an optimal growth model (or any other equilibrium system) is as a perturbation. That is, one can imagine solving an optimizing model containing whatever sort of descriptive relationsbips seem relevant--production functions, demand functions, etc.-- and then re-solving with the particular flows of product resulting from a particular investment project added to the accounting relationships. Comparison of the optima with and without the project superimposed upon the general model of the economy indicates whether or not the project should be undertaken. Specifically, if the level of welfare is higher with the project included, then it is worth doing.

Given this approach, the whole point of project analysis is to calculate an approximation to the welfare change induced by the project. Of course, it is impossible to reprogram the whole economy for every new investment, but various decision rules can be used to short-cut this calculation and elect projects on more simple grounds. As indicated above, the most familiar rule is calculating the present discounted value (PDV) of a project, and checking whether it is positive or negative.

What I will do in this section (and the Appendix) is set up a general problem in optimal growth, and derive approximation formulas for the welfare effects of a project/perturbation imposed on an equilibrium solution of the model The approximations are developed using a technique known as differential dynamic programmming, originally proposed by D. Jacobson and D.Mayne for use in calculating numerical solutions to optimal control models[2]. The main differences between the formulas of this section and those of Jacobson and Mayne lie in the type of perturbation considered and in a restriction

imposed here that the approximations must be based solely on values calculated in connection with the without-project solution.

The problem of interest is the optimization (or, for consistency with most of the optimal control literature, the minimization) of the following welfare functional (of "society" or, perhaps more realistically, of the Central Planning Office):

$$\int_{t_0}^{t_f} L(x,u,t)dt + F\left[\underline{x}\,(t_f)\ ,\ t_f\right] \qquad (1)$$

subject to the differential equations,

$$dx/dt = \dot{x} = f(x,u,t)\ , \qquad (2)$$

initial conditions,

$$x(t_0) = x_0 \text{ given}, \qquad (3)$$

and terminal conditions,

$$G\left[\underline{x}(t_f),\ \underline{t_f}\right] = 0, \qquad (4)$$

where the state variables x and control variables u are expressed as column vectors with m and n elements respectively, L and F are scalar-valued functions, f is an m-component vector function, and G is a vector function with p ($\leq$m) components[3]. (all functions are assumed to be twice-continuously differentiable.)

In economic terms, the usual interpretations of all these variables and functions are the following: The state variables x are stocks of "malleable" capital (no restrictions have been placed on the rate of disposal of the capital stock) and the control variables correspond to flows of products or resources. The function L is a "felicity" function giving

the welfare resulting from instantaneous levels of stock and flow variables and the vector-valued function f gives the rules for accumulation of stocks. The functions F and G represent alternative ways of specifying terminal stock levels--F gives a welfare weight to terminal stocks (e.g. a scrap value on capital stock) while G constrains the levels of terminal stocks to lie within some manifold or to certain point values.[4]

The solution to this problem can be expressed most simply in formal terms if the constraints (4) are adjoined Lagrange-wise to the welfare functional with a (transposed) multiplier vector k':

$$\int_{t_0}^{t_f} L(x,u,t)dt+F\left[x(t_f),\ t_f\right] + k'\ G\left[x(t_f),t_f\right] \qquad (1')$$

Let $V^o(x,k,t)$ be the value of the optimized welfare functional when the problem is solved with x as the vector of initial values of the state variables, t as the initial time, and k as the multiplier on the terminal contraints. This "optimal return function" satisfies the boundary condition

$$V^o(x,k,t) = F(x,t) + k'G(x,t) \quad \text{on the hypersurface } G(x,t) \qquad (5)$$

with k chosen in such a way that (4) is satisfied. Its value at the particular point $(x_0,t_0)$ with optimal choice of k is the value of the functional (1) when it is minimized with respect to (2) -(4).

It is well known that a general sufficiency condition for the solution of (1)-(4) is that the optimal return function satisfy the Hamilton-Jacobi-Bellman partial differential equation,[5]

$$-\frac{\partial V^o}{\partial t} = \min_u \left[L(x,u,t) + V_x^o\ f(x,u,t)\right] \qquad (6)$$

where $V_x^o$ is the vector of partial derivatives of $V^o$ with respect to x, written as a row. From this equation follow the usual optimality conditions for the control problem (1)-(4) including the condition for an (internal) minimum of the Hamiltonian function

$$H\ (x,u,V_x^o,t) = L\ (x,u,t) + V_x^o\ f\ (x,u,t):$$

$$H_u = L_u + V_x^o\ f_u = 0, \tag{7}$$

where $H_u$ and $L_u$ are row vectors of partial derivatives with respect to u, and f is the m x n matrix of first derivatives of the m-vector f with respect to u. If one treats $V_x^o$ as a function of time along an optimal path, the Euler equations can also be deduced from (6) in the form:

$$\dot{V}_x^o = -\ H_x = -\ L_x - V_x^o\ f_x \tag{8}$$

with associated boundary conditions,

$$V_x^o\ (t_f) = \left[F_x + k'\ G_x\right]_{t\ =\ t_f}. \tag{9}$$

Equations (8) and (9) are identical to the equations satisfied by the costate variables in the solution to a control problem based on the Minimum Principle--the solution technique often used in the optimal economic growth literature. Thus the costates can be interpreted as partial derivatives along an optimal trajectory of the optimal return function with respect to the state variables. The recommendation that these variables be used as shadow discount rates is made precisely because they have this interpretation. We now will examine the extent to which this "first order" interpretation of the costate variables is valid.

To do this, we follow Arrow and Kurz/2/ by introducing a small amount of non-malleable capital into our pristine neo-

classical world in the form of an investment project which generates a vector of flows of product $r(t)$, whose time-phasing and product-mix cannot be varied. If the project is undertaken, the capital-accumulation equation (2) becomes

$$\dot{x} = f(x,u,t) + r(t). \qquad (2')$$

We explore the responses of the economy to this new pattern of accumulation [6].

Let W be the optimal return function for the problem with the project. By assumption, the problem without the project has been solved, and the optimal values for the state variables (written henceforth as $x^o$), control variables ($u^o$) and the multipliers on the terminal constraints ($k^o$) are all known. The optimal solution to the with-project problem is described by the variables x,u, and k, defined as

$$x = x^o + \delta x$$
$$u = u^o + \delta u$$
$$k = k^o + \delta k$$

where no restrictions have been put (yet) on the variations $\delta x$, $\delta u$, and $\delta k$.

Now the with-project return function $W(x,k,t)$ should satisfy an equation like (6) for points x,u, and k along the new optimal trajectory. However, this new trajectory is unknown, so we can only write the equation corresponding to (6) in terms of the expansion of W about its values on the without-project trajectory described by $x^o$, $u^o$, and $k^o$. When this expansion is carried through (as is done in the Appendix), a set of ordinary differential equations is derived for the matrices $W_x$ and $W_{xx}$ of the partial (first and second) derivatives of the optimal return function W of the with-project problem along the $(x^o,u^o,k^o)$ trajectory. These equations are:

$$-\dot{W}_x = r' W_{xx} + H_x - H_u H_{uu}^{-1} (f_u' W_{xx} + H_{ux}) \qquad (10)$$

and

$$-\dot{W}_{xx} = H_{xx} + f_x' W_{xx} + W_{xx} f_x$$
$$-(W_{xx} f_u + H_{xu}) H_{uu}^{-1} (f_u' W_{xx} + H_{ux}), \qquad (11)$$

and they have boundary conditions (at the terminal time) as follows,

$$W_x (t_f) = F_x \left[x^o(t_f), t_f\right] + k^{o'} G_x \left[x^o(t_f), t_f\right] \qquad (12)$$

$$W_{xx}(t_f) = F_{xx} \left[x^o (t_f), t_f\right] + k^{o'} G_{xx} \left[x^o (t_f), t_f\right]. \qquad (13)$$

At the same time, we may define a new variable a $(x^o, t)$,

$$a(x^o, t) = W (x^o, k^o, t) - V^o(x^o, k^o, t), \qquad (14)$$

the difference between the return W from following the optimal with-project trajectory forward from the point $(x^o, t)$ and the return $V^o$ from following the nominal without-project trajectory forward from this point. At initial time, the value of $a(x_0, t_0)$ is a prediction of the change in welfare induced by the project, as will be exemplified shortly. At terminal time, obviously nothing can be gained over the nominal path, so the boundary condition,

$$a (x^o, t_f) = 0 \qquad (15)$$

necessarily holds.

In the Appendix, it is shown that a satisfies the following differential equation,

$$-\dot{a} = W_x r - \frac{1}{2} H_u H_{uu}^{-1} H_u', \qquad (16)$$

the basic relation for calculating the difference in welfare levels between the two problems.

## 3.Optimal Discounting in a One-Sector Model

The implications of these equations stand out more sharply if they are examined in two stages--successively as first and second order approximations to the change in the solution. We treat the first order approximation in this and the next section.

To make this approximation, ignore equation (11) completely, and rewrite (10) and (16) without second order terms. This gives:

$$-\dot{a} = W_x r \tag{16a}$$

and

$$-\dot{W}_x = H_x' = L_x + W_x f_x \tag{10a}$$

with the same boundary condition as before.

Not too surprisingly, the equation for $\dot{W}_x$ is the same as that for $\dot{V}_x^o$ in the without-project problem, while (16a) simply evaluates the project's product flow with the shadow prices $W_x$ (t). In a model with only one state variable, this evaluation reduces to calculating the project's present discounted value, as can be easily shown. In such a model (10a) has only one component, so its solution can be written as,

$$W_1(t) = W_1(t_f) \exp\left[\int_t^{t_f} f_1(\tau)\, d\tau\right]$$

$$+ \int_t^{t_f} \exp\left[\int_t^{\tau} f_1(v)\, dv\right] L_1(\tau)\, d\tau,$$

where the subscript "1" stands for differentiation with respect to the first and only state variable of the system.

If the "felicity function" $L(x,u,t)$ does not depend explicitly on the capital stock (an assumption almost inevitably made in optimal growth models), then this equation simplifies to

$$W_1(t) = W_1(t_f) \exp\left[\int_t^{t_f} f_1(\tau)\, d\tau\right]$$

$$= A\, W_1(t_f) \exp\left[\int_{t_0}^{t} -f_1(\tau)\, d\tau\right] \tag{17}$$

where

$$A = \exp\left[\int_{t_0}^{t_f} f_1(\tau)\, d\tau\right].$$

In integrated form, the estimated change of welfare between time $t_0$ and time $t_f$ is

$$a(x_0,t_0) = A\, W_1(t_f) \int_{t_0}^{t_f} \exp\left[\int_{t_0}^{\tau} -f_1(v)\, dv\right] r(\tau)\, d\tau. \tag{18}$$

For an acceptable change in welfare, $a(x_0,t_0) \leq 0$. But this condition is equivalent to a discount calculation, or

$$\int_{t_0}^{t_f} \exp\left[\int_{t_0}^{\tau} - f_1(v)\, dv\right] r(\tau)\, d\tau \geq 0,$$

since in a minimizing model the requirement of a positive value for the terminal capital stock implies that $W_1(t_f) < 0$. Hence;

In a one-sector optimal growth model with malleable capital stock, the optimal decision rule about whether to accept a small investment project amounts to calculating the PDV of the project, and accepting it if the PDV is positive. The appropriate time-varying rate of discount is the marginal product of capital along the "reference" (without-project) solution to the model.

## 4. Optimal Shadow Prices in a Multi-Sector Model

Still sticking to the linear analysis of the effects of the project, we now must consider the implications of the fact that (10a) is a vector differential equation, with as many components as there are types of capital goods in the system. Written out fully, (10a) in transposed form appears as follows:

$$\begin{bmatrix} -\dot{W}_1 \\ -\dot{W}_2 \\ \cdot \\ \cdot \\ \cdot \\ -\dot{W}_m \end{bmatrix} = \begin{bmatrix} L_1 \\ L_2 \\ \cdot \\ \cdot \\ \cdot \\ L_m \end{bmatrix} + \begin{bmatrix} f_1^1 & f_1^2 \cdots & f_1^m \\ f_2^1 & \cdots\cdots\cdots & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ f_m^1 & \cdots\cdots & f_m^m \end{bmatrix} \cdot \begin{bmatrix} W_1 \\ W_2 \\ \cdot \\ \cdot \\ \cdot \\ W_m \end{bmatrix} \qquad (19)$$

where subscripts denote differentiation with respect to the various components of the vector of capital stocks $x'=(x_1,x_2\ldots\ldots,x_m)$ and the $f^i$ are components of the vector function f.

In this form, it appears more or less obvious that each of the shadow prices $W_i$ will in general depend over time on all of the other shadow prices and itself. In last section, we were able to express the capital shadow price $W_1$ --which envolved over time according to a linear differential equation with $-f_1$ as its coefficient--as an exponential function of

the integral of $f_1$ over time. By analogy, one ought to be able to introduce some sort of exponential matrix function to use the same way in "solving" (19) with its coefficient matrix of marginal products of capital in formal terms. This we will do through the use of an m x m "transition matrix" $\phi(t,t_0)$.[7]

Let $f'_x$ stand for the matrix of derivatives of the $f^i$ on the right side of (19). Then if we take the "minus" sign in (19) to the right side, the transition matrix of this system satisfies the following matrix differential equation,

$$\dot{\phi}(t,t_0) = -f'_x(t)\,\phi(t,t_0) \qquad (20)$$

with initial boundary condition

$$\phi(t_0,t_0) = I \quad (= \text{the identity matrix}). \qquad (21)$$

In effect, each column of $\phi(t,t_0)$ represents one of the m fundamental solutions of (19) over time. In the one-variable system of last section, the function $\exp \int_{t_0}^{t} -f_1(\tau)d\tau$ served the same function. The following comparisons help illustrate the similarity of the two representations:

| One-variable system | m-variable system |
|---|---|
| $\exp\left[\int_{t_1}^{t_2} -f_1(\tau)\,d\tau\right]\cdot\exp\left[\int_{t_0}^{t_1} -f_1(\tau)\,d\tau\right] = \exp\left[\int_{t_0}^{t_2} -f_1(\tau)\,d\tau\right]$ | $\phi(t_2,t_1)\cdot\phi(t_1,t_0) = \phi(t_2,t_0)$ |
| $\left\{\exp\left[\int_{t_0}^{t} -f_1(\tau)\,d\tau\right]\right\}^{-1} = \exp\left[\int_{t}^{t_0} -f_1(\tau)\,d\tau\right]$ | $\phi^{-1}(t,t_0) = \phi(t_0,t)$ |

Clearly, $\phi(t,t_0)$ performs a smoothing operation over time in the multi-variable system equivalent to that done by the exponential integral in the one-variable system. This stands out quite clearly when we write the solution of (19), starting from an initial value of $W_x$ and again assuming that the non-homogeneous term $L_x$ vanishes:

$$W'_x = \phi(t,t_0)\, W'_x(t_0) \tag{22}$$

This solution is a generalization of (17) for all components of the vector $W_x(t)$. The corresponding generalization of (18) is

$$a(x_0,t_0) = W_x(t_0) \int_{t_0}^{t_f} \phi'(\tau,t_0)\, r(\tau)\, d\tau$$

where the integration is component-wise. In the case where r (t) has only its first component non-zero (so that it uses and produces only the first type of capital good), the above expression reduces to

$$a(x_0,t_0) = \int_{t_0}^{t_f} \left[\sum_{j=1}^{m} W_j(t_0) \cdot \phi_{1j}(\tau, t_0)\right] r_1(\tau)\, d\tau$$

so that the appropriate shadow price for the first sector is a weighted sum of the first components of the m fundamental solutions of (19). Since this will not correspond to an exponential integral, we may conclude that

<u>Discounting is not an optimal decision rule when there are many heterogeneous capital stocks, and must be replaced by a multivariate analog based on future marginal products of all capital goods in the system.</u>[8]

We can illustrate this result with a simple two-sector model of the type being developed at the pilot stage by various economic programmers.[9] The model maximizes (or minimizes the negative of) a utility integral

$$\text{Min} \int_{t_0}^{t_f} L\,(c_1, c_2)\,dt$$

of the output of consumption goods in each of the sectors. The accumulation equations are

$$\dot{x}_1 = g^1\,(x_1 - y_1\,,\, y_2) - a_{12} g^2\,(y_1,\, x_2 - y_2) - c_1$$

$$\dot{x}_2 = -a_{21}\, g^1(x_1 - y_1, y_2) + g^2(y_1, x_2 - y_2) - c_2$$

where $y_1$ and $y_2$ are control variables which assign the two types of capital stock between the two sectors and $a_{12}$ and $a_{21}$ are input-output coefficients.

The equations for $\dot{W}_x$ in this model are

$$\begin{bmatrix} \dot{W}_1 \\ \dot{W}_2 \end{bmatrix} = \begin{bmatrix} -g_1^1 & a_{21}\, g_1^1 \\ a_{12}\, g_2^2 & -g_2^2 \end{bmatrix} \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \qquad (23)$$

Suppose for illustrative purposes that the marginal products of capital $g_1^1$ and $g_2^2$ are constant over the planning period. Then the matrix on the right side of (23)--call it G--will be constant. The Hawkins-Simon condition/7,pp.215-18/ assure us that the negative transpose of G will have two positive eigenvalues. Therefore, G itself will have two negative eigenvalues. Suppose that these are distinct, and call them $\lambda_1$ and $\lambda_2$ ; also call P the matrix which diagonalizes G:

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = P^{-1}\, G\, P.$$

Then we can write the solution of (23) as

$$\begin{bmatrix} W_1(t) \\ W_2(t) \end{bmatrix} = P \begin{bmatrix} \exp(\lambda_1 t) & 0 \\ 0 & \exp(\lambda_2 t) \end{bmatrix} P^{-1} \begin{bmatrix} W_1(0) \\ W_2(0) \end{bmatrix}$$

Thus, the appropriate shadow prices for project analysis are weighted sums of two exponential functions. For the "reasonable" values of $g_1^1 = 0.2$, $g_2^2 = 0.1$, $a_{12} = 0.4$ and $a_{21} = 0.1$, the two eigenvalues turn out to have values of -0.207 and -0.092 respectively. For product flows induced far in the future by an investment project, therefore, the second value will dominate and "the" appropriate discount rate will be about 9.2 percent. However, for near-future product flows in either of the two sectors, the appropriate shadow price will be some weighted average of present values at 20.7 and 9.2 percent. At a given time, neither of these shadow prices need have a value near the discounted marginal product of capital in any specific production function of the system.[10]

## 5. Optimal Shadow Prices for Large Projects

Finally, we discuss briefly in terms of the one-sector model the implications of the second-order approximation to the change in welfare imposed by a project. The discussion will focus on the qualitative properties of the relevant differential equations since little can be said quantitatively in the absence of numerical experimentation with specific equations Some results of this type of experimentation will appear in another paper.

The ruling equations in a one-sector model are

$$-\dot{a} = W_1 r - \frac{1}{2} H_u^2 H_{uu}^{-1} \tag{24}$$

$$-\dot{W}_1 = r W_{11} + H_1 - H_u H_{uu}^{-1} (f_u W_{11} + H_{u1}) \tag{25}$$

$$-\dot{W}_{11} = H_{11} + 2 W_{11} f_1 - (W_{11} f_u + H_{u1})^2 H_{uu}^{-1} \tag{26}$$

where the subscripts "1" and "u" refer respectively to differentiation with respect to the one state variable and the one control variable in the system.

The main qualitative characteristics of these equations are as follows:

(i) Equation (24) in integrated form is

$$a(x_0,t_0) = \int_{t_0}^{t_f} \left[W_1 r - \frac{1}{2} H_u^2 H_{uu}^{-1}\right] dt$$

which is similar to the equation discussed in last section, except that a positive correction term $\frac{1}{2} H_u^2 H_{uu}^{-1}$ is subtracted from the term $W_1 r$ evaluating the project. This correction takes into account the improvement in the welfare function from the without-project to the with-project solution which would result from the change u in the control variables. As before, the term $W_1 r$ gives the valuation of the project at the shadow prices of the with-project problem.

(ii) Equation (25) for $\dot{W}_1$ is the same as equation(10a)for the without-project problem, except that a finite correction term involving the non-zero value of $H_u$ for the with-project problem along the without-project optimal trajectory is subtracted (as in the equation for $\dot{a}$), and a term $r W_{11}$ involving the project is added. This latter addition reflects the influence of the project itself on optimal accumulation. The project changes the rate of change of the state-variable shadow price and this new price in turn affects the flow variable through the optimality condition (7). The point of the second-order approximation is just to take this feedback of exogenous capital accumulation onto the interest rate into account.

Equations(25) and (26) are interrelated, since $W_1$ enters into the determination of $W_{11}$ via its presence in the

derivatives of H with respect to the state variable in (26). However, if we ignore this coupling and the term in (25) involving $H_u$ (which will be zero at the optimum), and also assume (as before) that $L_1 = 0$, then we can write the solution for $W_1$ in integrated form as

$$W_1(t) = W_1(t_f) \exp\left[\int_t^{t_f} f_1(\tau)\, d\tau\right] + \int_t^{t_f} \exp\left[\int_t^{\tau} f_1(v)dv\right] r(\tau)\, W_{11}(\tau)\, d\tau. \tag{27}$$

The second term of this equation takes into account the feedback of project on interest rate mentioned in last paragraph. Clearly, it can be of either sign, depending at least on the sign of r. In the far future, r will presumably be positive, while inspection of (26) and its associated boundary condition in the case where terminal capital stock is constrained to a certain level indicates that $W_{11}$ will be negative (in a maximizing problem). Hence the distant future shadow price of capital in the with-project solution will be less than in the without-project solution. Near-term effects are unfortunately both more interesting and harder to analyze.

Finally, we may note that equation (27) cannot be manipulated to produce a discounting formula when its second term is non-negligible. Even in the one-sector model, the familiar decision rule breaks down in the case of "sufficiently" large projects.

(iii) The weighting function $W_{11}$ for r in its determination of the interest rate in (27) is determined (with feedbacks from the shadow price $W_1$) by the Riccati equation (26). Aside from the comments made in connection with the integral (27) for $W_1$, little can be said about this equation in qualitative terms, although it plays an important role in all second order analysis of optimal control problems. Nor can it be solved with a finite number of quadratures, and must therefore await computer analysis in specific cases.

## 6. Conclusions and Conjectures

Has all this been an arid exercise in matrix algebra, without practical applications or redeeming social implication.

At least to the first clause of the question, I think the answer has to be "no". It is a negative conclusion of considerable importance that discounting is not a theoretically justified tool in the analysis of investment projects. Further, the lack of support for discounting doesn't stem from deep capital valuation problems like Wicksell effects and reswitching, but from the simple multiplicity of capital goods (no matter how malleable and substitutable) in the world. Particularly in underdeveloped countries even a rough and ready analyst who is willing to overlook valuation problems cannot ignore the fact that capital goods come in two distinct classes-- buildings which are made at home and equipment (or better than 90% of it) which is imported. The rules of accumulation of these two types of capital are obviously different, and so (apparently) are their shadow prices over time. It doesn't do any more just to multiply the C.I.F. prices of the capital goods imports by some shadow price of foreign exchange, and discount them along with the outlays for the buildings. To begin with, one can't discount, and secondly one must somehow have calculated a whole range of shadow prices for the two types of capital goods over time.

The second implication of the results presented here is that investment projects can be viewed fundamentally as perturbations to an equilibrium system, whose effects can be approximated by first or second order Taylor series expansions about an equilibrium path. Although this point of view applies to capital accumulation models much more complex than the one considered here, it is still true that it is limited to cases of perfect foresight. If the equilibrium growth path is known with certainty, then one can handle with known techniques

stochastic perturbations in the region around the path where linear approximations are valid.[11] However, when the equilibrium path itself can only be determined stochastically with dynamic programming methods,[12] then I do not see how one can undertake an analysis of perturbations similar to that done here. It seems that this is virgin theoretical territory which urgently wants exploring.

Finally, the analysis here underlines a point made in another context by Arrow/1/: " What economic theory can imply in a broad range of cases is a computing algorithm." We have found here a need for computing algorithms in the analysis of investments when there are heterogeneous capital goods, since it doesn't seem we'll get the appropriate shadow prices from other sources. This almost virgin applied territory which also urgently wants exploring.

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## APPENDIX

In this appendix, we derive the basic differential equations (10), (11), (16) and boundary conditions (12),(13), (15) for the change in welfare induced by the investment project. To do this, begin by writing a first order expansion (in $W_x$) of the Hamilton-Jacobi- Bellman equation (6) about the nominal trajectory defined by the points $x^o$, $u^o$,$k^o$:

$$-\frac{\partial W(x^o+\delta x,k^o+\delta k,t)}{\partial t} \cong \underset{\delta u}{M} \Big\{ L(x^o+\delta x,u^o+\delta u,t)$$

$$+\left[W_x + \delta x' W_{xx} + \delta k' W_{kx}\right]\left[f(x^o+\delta x,u^o+\delta u,t) + r(t)\right]\Big\} \tag{A-1}$$

where $W_x$ has been expanded to first order about $W_x(x^o,k^o,t)$. Also expanding the left side of (A-1) about $W(x^o,k^o,t)$, we find (to second order[13, 14]) that

$$\frac{-\partial W(x^o+\delta x,\ k^o+\delta k,t)}{\partial t} \cong -\frac{\partial}{\partial t}\Big[W + W_x \delta_x + W_k \delta k$$

$$+\delta k' W_{kx}\delta_x + \frac{1}{2}\delta_x' W_{xx}\delta x + \frac{1}{2}\delta k' W_{kk}\delta k\Big] \tag{A-2}$$

$$= -\frac{\partial}{\partial t}\Big[a + V^o + W_x \delta x + W_k \delta k + \delta k' W_{kx}\delta x$$

$$+\frac{1}{2}\delta k' W_{kk}\delta k + \frac{1}{2}\delta x' W_{xx}\delta x\Big]$$

where the new variable $a(x^o,t) = W(x^o,k^o,t) - V^o(x^o,k^o,t)$ is defined as the difference between the return function W which results from applying optimal control forward from the point $(x^o,t)$ and the return $V^o$ from applying nominal (without project) control forward from this point.

On the other side of (A-1) the second order expansion gives

$$\underset{\delta u}{\mathrm{Min}}\Big\{L(x^o+\delta x,\ u^o+\delta u,\ t)+\left[W_x+\delta x'\ W_{xx}+\delta k'\ W_{kx}\right]\cdot\left[f(x^o+\delta x,\ u^o+\delta u,\ t)+r(t)\right]\Big\}$$

$$=\underset{\delta u}{\mathrm{Min}}\Big\{H(x^o+\delta x,\ u^o+\delta u,\ W_x,t)+\left[\delta x'W_{xx}+\delta k'W_{kx}\right]\cdot f(x^o+\delta x,u^o+\delta u,t)$$

$$+\left[W_x+\delta x'\ W_{xx}+\delta k'\ W_{kx}\right]r(t)\Big\} \qquad \text{(A-3)}$$

$$\underset{\delta u}{\mathrm{Min}}\Big\{H+\left[\delta x'\ W_{xx}+\delta k'\ W_{kx}\right]f+\left[W_x+\delta x'\ W_{xx}+\delta k'\ W_{kx}\right]\cdot r$$

$$+H_x\,\delta x+H_u\,\delta u+\delta x'\ W_{xx}\left[f_x\,\delta x+f_u\,\delta u\right]+\delta k'\ W_{kx}\left[f_x\,\delta x+f_u\,\delta u\right]$$

$$+\delta u'\ H_{ux}\,\delta x+\frac{1}{2}\delta u'\ H_{uu}\,\delta u+\frac{1}{2}\,\delta x'\ H_{xx}\,\delta x\Big\}$$

with all terms again evaluated at$(x^o,k^o,u^o)$.

To the accuracy of the expansions, (A-2) should equal (A-3). To make use of this fact, we will remove $\delta u$ from (A-3) and then equate coefficients in $\delta x$ and $\delta k$ between (A-2) and (A-3) to get differential equations for the various derivative matrices $W_x$, $W_{xx}$, etc., and also for a.

To carry out these manipulations, begin by grouping terms in (A-3) which contain $\delta u$:

$$\delta u' H'_u + \delta u' f'_u W_{xx} \delta x + \delta u' f'_u W_{xk} \delta k$$
$$+ \delta u' H_{ux} \delta x + \frac{1}{2} \delta u' H_{uu} \delta u \; ; \qquad \text{(A-4)}$$

This is a sum of linear and quadratic terms in $\delta u$, with all coefficients evaluated along the without-project optimal solution. If we solved completely the with-project problem, these coefficients would change to their values at the points ($x^o + \delta x$, $u^o + \delta u$, $k^o + \delta k$), and complete minimization of (A-3) with respect to $\delta u$ would require that these non-linear coefficient changes be taken into account. This could be a difficult calculation. However, it is easy to make a Newton- Raphson type prediction of the change $\delta u$, by minimizing (A-4) with constant coefficients. This gives

$$\delta u = - H_{uu}^{-1} \left\{ H'_u + ( f'_u W_{xx} + H_{ux}) \delta x + f'_u W_{xk} \delta k \right\} \qquad \text{(A-5)}$$

as a basic feedback rule for changes in u as functions of changes in the state variables x and multiplier k[15/].

Now, after (A-5) is used to remove $\delta u$ from (A-3), and coefficients are equated between (A-2) and (A-3), the following partial differential equations result:

$$V^o + a)/\partial t = H + W_x r + \frac{1}{2} H_u H_{uu}^{-1} H'_u \qquad \text{(A-6)}$$

$$\partial W_x/\partial t = (f' + r') W_{xx} + H_x - H_u H_{uu}^{-1} (f'_u W_{xx} + H_{ux}) \qquad \text{(A-7)}$$

$$-\partial W_k/\partial t=(f'+r')\,W_{xk}-H_u H_{uu}^{-1} f'_u\, W_{xk}$$

$$-\partial W_{xx}/\partial t= H_{xx}+ f'_x\, W_{xx}+ W_{xx} f_x-(W_{xx} f_u+ H_{xu})\, H_{uu}^{-1}(f'_u\, W_{xx}+H_{ux})$$

$$-\partial W_{kx}/\partial t= W_{kx}\left[f_x - f_u\, H_{uu}^{-1}\,(f'_u\, W_{xx}+ H_{ux})\right] \qquad \text{(A-8)}$$

$$-\partial W_{kk}/\partial t= - W_{kx}\, f_u\, H_{uu}^{-1}\, f'_u\, W_{xk}$$

It will be useful to transform these into <u>ordinary</u> differential equations. This can be done for (A-6) by noting that

$$\dot{W}= \partial W/\partial t + W_x\,\dot{x} = \partial W/\partial t + W_x\, f(x^o, u^o, t)$$

along the optimal solution to the without-project problem. When the definition of a is substituted into this equation, we get

$$\dot{V}^o+ \dot{a} = \partial(V^o+ a)/\partial t+ W_x f\,(x^o,u^o,t)$$

and further substitution into (A-6) gives

$$-\dot{a} = \dot{V}^o- W_x f + W_x\, r - \frac{1}{2} H_u\, H_{uu}^{-1}\, H'_u + H. \qquad \text{(A-9)}$$

For the without-project problem, $\dot{V}^o = - L\,(x^o, u^o, t)$. We can substitute this equality into (A-9), and cancel the Hamiltonian H, to produce at last the ordinary differential equation

$$-\dot{a}= W_x r- \frac{1}{2} H_u\, H_{uu}^{-1}\, H'_u. \qquad (16)$$

This is the basic relation giving the difference in the welfare levels between the two problems. Next, ordinary differential equations for $W_x$ and $\dot{W}_{xx}$ and boundary conditions are derived. Regarding the former, note that

$$\dot{W}_x = \partial W_x/\partial t+ \dot{x}'\, W_{xx}= \partial W_x/\partial t+ f'\, W_{xx}$$

and

$$\dot{W}_{xx}= \partial W_{xx}/\partial t$$

to second order, and substitute in (A-7) and (A-8) to get

$$-\dot{W}_x = r' W_{xx} + H_x - H_u H_{uu}^{-1} (f_u' W_{xx} + H_{ux}) \tag{10}$$

$$-\dot{W}_{xx} = H_{xx} + f_x' W_{xx} + W_{xx} f_x - ( W_{xx} f_u + H_{xu}) H_{uu}^{-1} (f_u' W_{xx} + H_{ux}). \tag{11}$$

Boundary conditions can be developed from the following approximate equality which must hold at terminal time $t_f$:

$$\begin{aligned} W(x^o + \delta x, k^o + \delta k, t_f) &= F(x^o + \delta x, t_f) + (k^o + \delta k)' G(x^o + \delta x, t_f) \\ &\cong F + F_x \delta x + \frac{1}{2} \delta x' F_{xx} \delta x \\ &\quad + (k^o)' (G + G_x \delta x + \frac{1}{2} \delta x' G_{xx} \delta x) + \delta k' (G + G_x \delta x). \end{aligned}$$

Expanding the left side to second order and equating coefficients, we find the boundary conditions:

$$a(t_f) = 0 \tag{15}$$

$$W_x(t_f) = F_x(x^o(t_f), t_f) + (k^o)' G_x( x^o(t_f), t_f) \tag{12}$$

$$W_{xx}( t_f) = F_{xx}( x^o(t_f), t_f) + (k^o)' G_{xx}( x^o(t_f), t_f) \tag{13}$$

These conditions provide the starting information for the integration of (16), (10), and (11) backward in time to estimate the change in the welfare functional.

## References

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## Footnotes

1. This study is part of a general research project in economic development being undertaken jointly by the Center for International Studies, M.I.T., and the Chilean National Planning Office (ODEPLAN). Support from this project is gratefully acknowledged, and thanks are also due to David Jacobson for making available some of his research results on differential dynamic programming, and to Edmar Bacha and Richard Eckaus for comments.

2. See Jacobson/9/ and Jacobson and Gershwin /10/ for complete descriptionsof the differential dynamic programming technique.

3. In many economic problems, it is reasonable to add constraints of the form $g(x,u,t) \leq 0$. The results which follow can be extended to this more general problem, but as the algebra is even more lushly foliated with matrices of several orders of derivatives than the appendix of this paper, this extension in omitted.

4. The finite time horizon problem is discussed because it allows us to eschew complicated arguments about the existence of solutions, and because the finiteness or infiniteness of the optimal plan is not of much relevance to the things discussed here. Also, finite horizon problems have the modest virtue of being solvable numerically. One really <u>can</u> calculate shadow prices from the type of model discussed here, a task I am presently undertaking whose results will be reported elsewhere. For moderately realistic models,numerical solution is not possible in the infinite horizon case (but see Mirrlees/14/ for a partial exception).

5. The Hamilton-Jacobi-Bellman dynamic programming equation (6) is derived heuristically in a number of texts. The presentations of Arrow and Kurz/2/, Bryson and Ho/4/, and Dreyfus/8/ are particularly to be recommended.

6. In some models (e.g. Little-Mirrlees/12/) a project perturbs the constraints in a non-linear way. The Taylor series expansions for the welfare effects of the project in form (2') which are developed below can be modified for non-linear perturbations in an obvious way.

7. Transition matrices are much used in the formal analysis of linear vector differential equations. See, for example, Athans and Falb/3/,Bryson and Ho/4/ and Coddington and Levinson/6/.

8. This dependence of efficiency prices on all future marginal products of capital has, of course, been pointed out before by a number of authors--notably DOSSO/7/ and Malinvaud/13/. The latter notes further that "an" interest rate with the usual Fisherian properties can be defined by treating one commodity as a numeraire. However, from the point of view of project analysis, definition of a copper- or/ even a cannabis-based interest rate does not obviate the need to know all future relative prices of all capital goods. These are given in symmetric, no-favored-commodity form by (20)-(22). To my mind, interpretation of these equations as a multivariate generalization of discounting is more appropriate than attempting normalization in terms of some arbitrary commodity.

9. For examples, see Chenery and Raduchel/5/, Kendrick and Taylor /11/ and the references there.

10. The input-output coefficients are what tie the two sectors together in this example--if they were equal to zero, (23) would already be a diagonal system, and it would be correct to discount product flows in each sector by the appropriate own-marginal product of capital.(Note that the two sectors'own-rates of interest need not be equal). However, input-output relationships (and other interties between sectors like production externalities and common usage of scarce factors like foreign exchange) do exist in the

engineers' world, and it is appropriate to take them into account in formulating investment decision criteria.

11. The key word here is "linear". In the vicinity of an equilibrium growth path known with certainty, the ruling differential equations for perturbations to the solution will be linear, and the objective function can by approximated quadratically. With these approximations, an investment project with stochastic returns can be analyzed using techniques known well to both engineers and economists--an economic example is given in my paper/15/. Under the assumed conditions, it seems that it would be justified to invoke certainty equivalence and work only with the <u>expected</u> returns to the project. But this is a point which needs more formal analysis.

12. To take an example from my present country of residence: Using dynamic programming I can work out on the computer an expected optimal growth path for Chile under conditions of the all-important uncertainty about the future world price of copper. What I don't know how to do--as the text indicates--is analyze stochastic perturbations about this stochastic path.

13. As before, W means the row vector of partial derivatives of W with respect to x, while $W_{kx}$ is a matrix of cross partial derivatives with the number of rows equal to the dimensionality of k, and the number of columns equal to the dimensionality of x. The "prime" sign indicates transposition.

14. Note that $W_x$ is only expanded to first order while other functions are expanded to second order since $W_{kx}$ and $W_{xx}$ are already expanded to second order in terms of the problem variables.

15. Note that this feedback rule can always be calculated from an optimal solution to (1) - (4) since $H_{uu}$ will be non-singular from the strengthened Legendre-Clebsch condition.