DISSERTATION

OPTIMAL PARAMETER IDENTIFICATION OF NONLINEAR, TIME-VARIANT HYDROLOGIC SYSTEM MODELS

Submitted by

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TO GOD WITH LOVE

ABSTRACT OF DISSERTATION

OPTIMAL PARAMETER IDENTIFICATION OF NONLINEAR, TIME-VARIANT HYDROLOGIC SYSTEM MODELS

Investigation of river basin characteristics is important to river basin planners, engineers, atmospheric scientists, mathematicians, statisticians, and agriculturists.

This study focuses on the important need in river basin investigations for development of mathematical models that describe hydrologic system response to rainfall. In particular time variant, nonlinear simulated models for runoff and a predictive model for evapotranspiration are presented. One of the major objectives of this study is in the development of a hydrologic model for water delivery particularly suited for a developing nation, though general in application. Emphases are placed on using what is available in the best possible way. The proposed model is evaluated with respect to total computation time required and ease of understanding and utilization. Solution procedures are evaluated with respect to speed of convergence to optimal solutions of computed runoff and parameters; and general accuracy and efficiency.

Two models are developed first and second order nonlinear differential equations From the results obtained, the first order nonlinear differential equation proves to be a better monthly model for river basin response simulation This conclusion is based on computed runoff using the models, and the predicted seasonal and annual evapotranspiration losses

The river basin clearly demonstrates nonlinear characteristics. Linear models, developed for comparative purposes with the proposed

nonlinear models, consistently predict low evapotranspiration values and comparatively poorer computed runoff values. An optimization procedure is carried out with the objective function being the sum of the squared difference between the observed and computed runoff. The constraints on the objective function are based on the parameters or coefficients of the differential equations; and these are further based on the physical realities of the hydrologic system--the river basin.

Two solution approaches used were based on the above objectives. Quasilinearization and gradient (steepest descent) techniques were utilized. Quasilinearization proved to be an ineffective algorithm as far as the proposed models are concerned, and diverged in solving all the various models. The gradient technique proved to be a good algorithm and could be modified to handle time dependency of parameter values. All the parameters, except one, varied very little and their corresponding average seasonal values were successfully used. The time dependent parameter α, referred to in this study as the sensitivity parameter, is responsible for the prediction of the evapotranspiration losses.

These models were applied to an actual river basin in a developing country in order to verify them and gain insight into them through experimental methods. On the basis of these, conclusions drawn can be useful to planners of water resources development, agriculturists, hydrologists and even theoreticians.

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Errata

1. Substitute the word "whatever" for "whether" beginning of sentence on seventh line from bottom of page on page 144.

2. Delete entire line "disciplines related to the project should all constitute the decision" which is the thirteenth line from top of page 145.

Chapter I

INTRODUCTION

Numerical simulation of the hydrological processes occurring in a river basin is a very useful method both for predicting hydrologic events and as a means of obtaining an improved understanding of the governing hydrologic processes.

From an engineering applications point of view, numerical simulation and modeling are important and useful in planning and developing water resources projects. The hydrologist, on the other hand, is mainly interested in investigations concerning the physical processes that occur in various parts of the hydrological system. This study emphasizes both the engineer's and the hydrologist's interests in numerical simulation aimed at the optimal identification of a nonlinear, time variant hydrologic system. The need for this study is in simulating runoff and predicting evapotranspiration, for planning and developing water resources, especially for water supply and flood control for a developing country.

An extremely important problem in hydrology is in the development of realistic mathematical models of hydrologic system behavior in transforming rainfall input into runoff. The rainfall-runoff relationship have been found by some previous investigators to be nonlinear. Among these investigators are Singh (1965), Prasad (1967), and Labadie (1968). However, the characteristic parameters of the mathematical model have been assumed as time-invariant, whereas they are clearly defining certain characteristics of the river basin which depend on climatic changes and land use. This assumption of "time invariant

system" is usually made either to simplify the mathematics so as to be solved by existing mathematical tools, or it may be due to lack of inadequate simulation of the system. It is, therefore, important first to develop the conceptual framework and its equivalent mathematical equations before such assumptions like "Time Invariant System" can be made with justification (e.g., due to lack of mathematical tools to solve the derived mathematical equations). Partial differential equations have to be used (though they may be more difficult to handle with respect to time and space variations and behavior of the system parameters). The present research will be directed toward finding the monthly and seasonal changes and variations of the system parameters, by using data for such periods without resorting to more sophisticated or nonexisting mathematical tools.

Many hydrologic analyses have been based on methods developed specifically for the particular problems facing the hydrologist. Unavailability of adequate data may be a major constraint in using more sophisticated or more generalized approaches. Lack of conceptual ideas of the physical system are also problems that can face the hydrologist or investigator in deriving unique models and solutions even for one particular basin at different periods of time.

Numerical simulation has considerable promise and potential as well as limitations. Its promise and potential may be found, among other things, in investigators becoming conceptually competent and more capable in using the correct or more correct functional relationships chosen to represent various physical processes occurring in the river basin, and the parameters used to describe the various characteristics of the river

basin. Inadequate data or unavailability of basic data may offer a serious limitation to the numerical simulation.

A Case Study - The Todzie River Basin (Ghana)

A case study in Ghana (Fig.1.1) is considered. Only 10 years of rainfall and runoff data are available. These data may appear to be short. In practice, however, short duration data may be all that is available and must be used. A naive as well as relevant point could be raised regarding waiting until sufficient data are collected. This is sometimes an extremely important point. In this regard it is necessary to state that economics and the politics of the argument are usually the most important considerations for carrying out a project.

It is of course also true that certain projects may be postponed until adequate data may be gathered for economic, political, physical and social reasons. This, however, does not mean there should be a period of "folding arms" especially for the engineer or the hydrologist in particular It is important for the hydrologist to come up with his best estimates of the hydrologic variables, better theories, and improved models for the hydrologic system, otherwise someone else (usually less qualified) may use any available estimates, theories and models. The less qualified personnel, who may not be fully aware of some of the future implications of the variables describing the hydrologic system, may create problems that the project was meant to solve. This may mean development on the reverse. The author prefers to call this the concept of development-foregone. In certain economically developed countries or developed countries the development

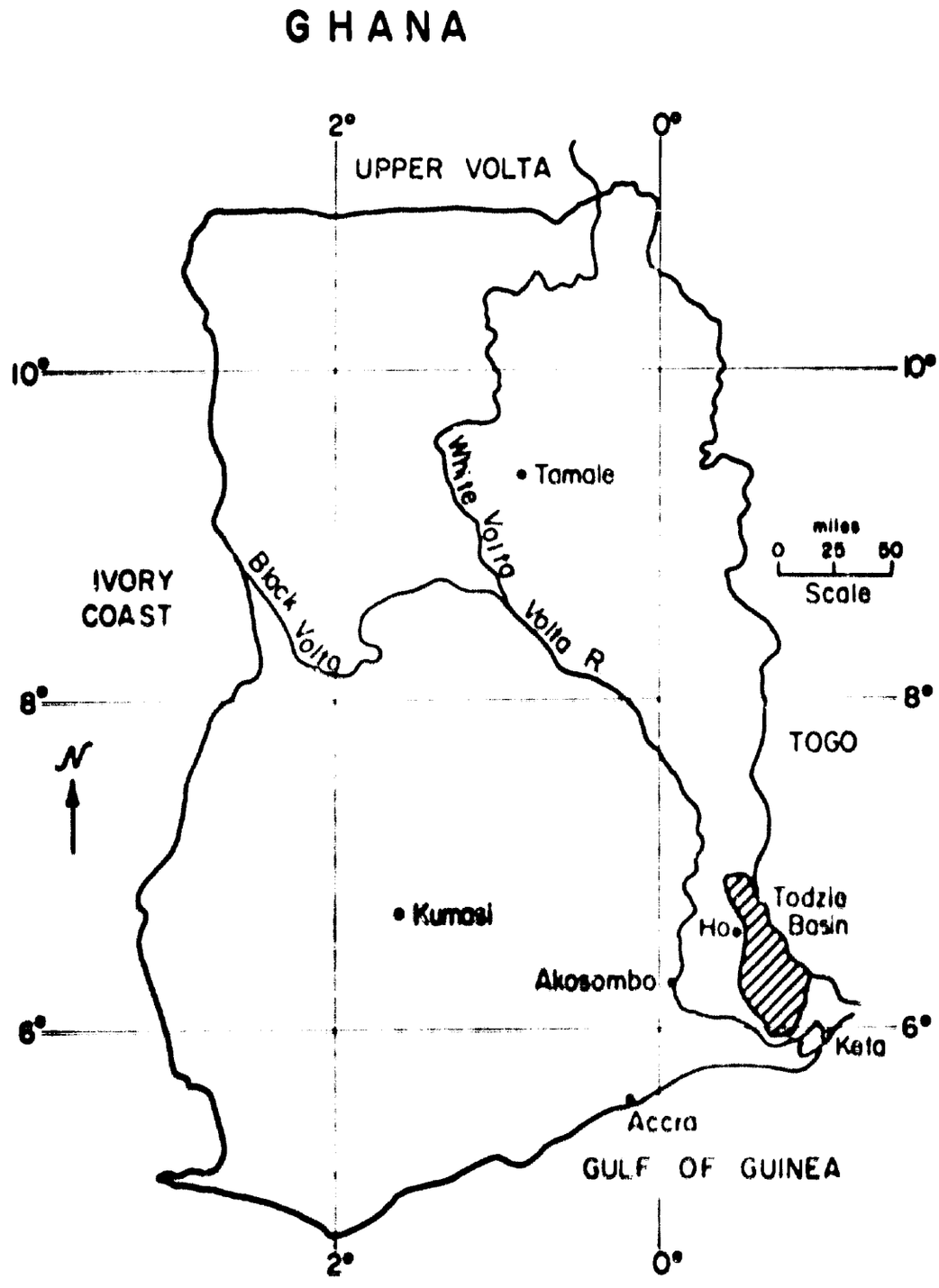


Fig. 1.1. Map of Ghana Showing Important Towns, Volta Ri er and Todzie River Basin.

foregone may do little to alter the economy; whereas, similar kinds of problems in a developing country could constitute a national disaster.

Project design and planning in a developing country is hindered by inadequate data concerning certain important variables, usually due to economic factors. Since it may be necessary, nonetheless, to carry on plans for a given project, it therefore becomes important to develop a model that will give reasonably good estimates or approximations to the unknown variables. A major concern in this research is to develop a hydrologic model which considers certain important characteristics of a river basin (Case Study in Ghana, see Fig. 1.1) aimed at finding good estimates or approximate values of a certain unknown variables. The data available for this research is ten years for rainfall and runoff records. The proposed model is, therefore, concerned with extracting enough information from this limited data for the purposes of simulating runoff and predicting evapotranspiration.

DeWiest (1965) defined the hydrologic system as the entire area that contributes to and sustains all of the flow in its main channel and tributaries. If the hydrologic system is considered as a "closed system," be it a large drainage basin, or a small watershed within the basin, without due regard for interflows from neighboring basin; with respect to physical and geologic features of both basins then this definition is not general enough.

A hydrologic system, therefore, may be defined as a basin or sub-basin within geographic, geologic and by hydrologic framework which constitutes contribution to and sustaining of all of the "flow in its main channel and tributaries" and possible interflows with neighboring basins which could be defined by a continuity relationship.

Some logical and simplifying assumptions may be made by disregarding the effect of interbasin flows where study of the physical system may here be a useful guide.

Todzie, Avu Keta areas in Ghana have been somewhat neglected areas in the river basin development programmes, even though some reconnaissance and feasibility investigations have been made.

Todzie is a very attractive spot for research for the following reasons:

1. The general area has been plagued with floods for many years and therefore the characteristics of the basin need be investigated.

2. Todzie is typical of the river basins in the developing countries for the following reasons:

   a Not as much data needed for research investigations or project analysis may be available.

   b. Certain important variables like evapotranspiration have not been measured. This may be due to:

      i. Lack of funds or economic constraints.

      ii. Not a priority in the National Development Plans.

      iii. Lack of manpower and expertise.

      iv. Political, social, institutional and legal constraints.

It is important to bear these constraints and problems in mind when developing a conceptual model for a river basin in developing country. The model should be:

1. Rigorous and realistic.

2. Simple to handle conceptually and mathematically.

3. Fast and inexpensive with respect to computations by the computer.

Statement of Objectives

The specific objectives of the research are to derive:

1. A model that uses whatever is available. In many planning departments, planning may take place using available data or information however short or inadequate they may be or the type of form they appear. It is important to bear this idea or "handicap" in mind when developing a conceptual and a mathematical model, especially for a developing country.

2. A model which can be used or modified to forecast future runoff values using past records of rainfall and runoff data. This is important for planning reservoir operations.

3. A model which can illustrate the relative importance of using either seasonal or monthly data in determining the runoff.

4. A model which can give some estimate of evapotranspiration for a given basin without actually going through the process of measuring it. This is important for areas that some quick or rough estimates of evapotranspiration may be needed without investing much capital, which in some countries, especially the developing nations (for which this research is mostly geared) may be a limited resource.

5. A well simulated model which only requires rainfall and runoff data and need not necessarily require other measurements, e.g., subsurface flows or evapotranspiration.

6. A model that can easily be handled.

7. A model that considers the sensitivity of the basin. Not all of the rainfall that falls in a river basin gives rise to runoff. Rainfall giving rise to runoff depends on where storms hit in the basin and they are seasonally dependent; some may be lost from the basin through evapotranspiration processes; and some may not find their way into the main channel in the basin during the infiltration processes. Therefore, it should be pointed out that the rainfall giving rise to runoff depends on slope, vegetational cover, soil and surface characteristics and where the storm normally or mostly hits in a given basin during a particular month. A parameter should therefore be investigated which should give meaning to the ability of the basin to transform a given rainfall to runoff in the river channel. This parameter is referred to here as the sensitivity parameter α, which depends on the foregoing factors, but gives the direct fraction of the amount of input which goes into channel output. The parameter α is directly related to evapotranspiration losses.

8. A model whose parameters may be uniquely determined within the limits and capabilities of existing mathematical tools; and parameters which are directly significant and applicable. Previous investigators have developed hydrologic models which have parameters that may be functions of several variables which cannot yet be determined and therefore obscure the importance and direct usefulness and application of the model in practical engineering and hydrologic investigations. It is the objective of this research to illustrate the foregoing important idea by using the same assumptions and concepts in developing mathematically two separate mathematical models; and discussing the shortcomings of one with respect to the other.

Use will therefore be made of the better model based on the following considerations:

1. A model whose parameters may be uniquely determined within the limits and capabilities of existing mathematical tools.

2. A model which will be simple to handle since the model is being developed for a developing country where mathematical expertise may not be as sophisticated as the more economically developed countries.

3. A model which will be economic (computer-time-wise). This is important in a developing country where computer usage and funding are a matter of economics.

4. A realistic model without trading off rigorousness of the approach. This is important since length of data is not long enough in a developing country, therefore it is important to stress need for a rigorous approach in order to use what we have. In Chapter IV, Cases I and II models are developed with the foregoing objectives.

Summary of Chapters

The main purpose of Chapter I is to:

1. Present a discussion of data availability, length and type with respect to planning.

2. Present a discussion of numerical simulation--its merits and defects, and the relevance of it in this research work. It presents a brief discussion on nonlinear time invariant models.

3. Present the importance and main objectives of this research.

Chapter II gives a survey of some of the various investigators who have presented, developed, or used hydrologic models.

Chapter III is devoted to the development of the conceptual model. A general description of the conceptual model is given, and the proposed model is analyzed with respect to unique determination of optimal parameters, computed runoff, and evapotranspiration. Various important assumptions associated with the model are discussed.

The mathematical simulation of the physical system and development of the hydrologic system equations is the subject of Chapter IV.

Chapter V outlines various methods of solution, and their advantages and disadvantages. This discussion includes flow charts of various optimization processes.

A case study is carried out on the Todzie River Basin, located in Ghana, and summarized in Chapter VI. Type of data and analytical approach are presented. The hydrologic system model is tested by an optimization process where the model's computed runoff is compared to the actual observed runoff. Evapotranspiration estimates are given for the proposed models. Graphs and tables are given wherever necessary. Description of procedure, analysis and results of the study are presented.

Chapter VII gives the summary, conclusions and recommendations.

Chapter II

LITERATURE REVIEW

Watershed models have long been used to investigate behavior and internal structure of the hydrologic system and its response to rainfall. It is particularly important, however, to judge if based on unrealistic assumptions used in deriving the mathematical model which is supposed to describe the behavior of the physical system.

In 1934, Zoch presented a model which assumes that a watershed can be approximated by a linear channel which is in series with a linear reservoir.

$$\text{at time} \quad t = 0,\ Q = 0, \quad K\frac{dQ}{dt} + Q = R \tag{2.1}$$

in which K is defined as the characteristic parameter of the watershed (Fig. 2.1).

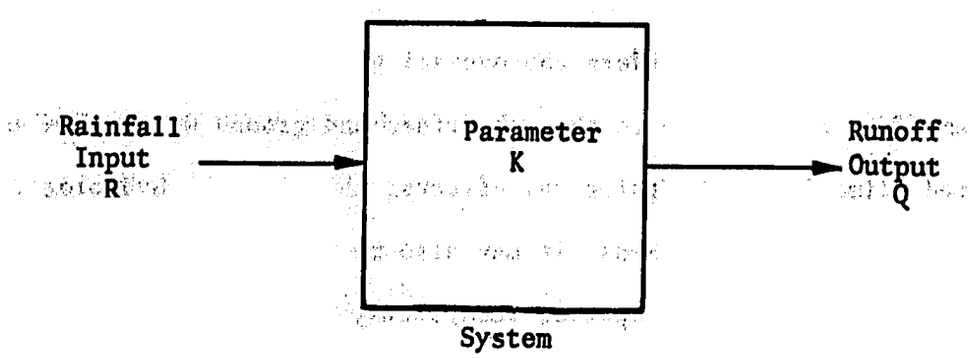


Fig. 2.1 Zoch's model.

There are several significant points concerning Zoch's work:

1. He considers the hydrologic system as having a lumped precipitation input, and a system that converts the input into an output (runoff) by use of a mathematical model with parameter K.

2. He conceptually relates a channel to a reservoir.

3. He uses an unrealistic assumption of a linear reservoir.

4. His assumptions of a linear channel is not reflected in Eq. 2.1. A linear channel may be defined as a fictitious channel in which the time t required to translate a discharge Q of any magnitude through a given channel reach of length ℓ is constant. Thus, when an inflow hydrograph is routed through the channel, its shape will not be changed. Therefore, a linear channel involves pure translation and translation is not seen in Eq. 2.1.

Zoch's work was an important beginning of a series of hydrologic models, of which the present research is one.

In 1966, one of the most successful efforts at simulation of the entire runoff process for watersheds was conceived at Stanford University by Linsley (1966), referred to as the "Stanford Watershed Model." This model considers the overall picture of the hydrologic system, from the surface to the subsurface and ground water flow and related climatic and hydrological effects; that is, the hydrologic cycle. For large watersheds, it may also be important to break this system down into various spatial subsystems, with numerical simulation carried out, in order to understand in depth some of the basic physical processes that are taking place.

place.

The hydrologic cycle represents the redistribution and circulation of the water by the atmosphere, earth, and sea. By continuity relations,

based on the Law of Conservation of Mass, a hydrologic cycle could be developed for any given basin. For a given basin, a balance equation could be written:

$$P = Q + PMF + E + \Delta S \tag{2.2}$$

where P is the precipitation,

Q is the runoff, and

PMF represents porous media flow out of or into the basin.

This equation describes the overall runoff cycle of a basin. One can be misled by the apparent simplicity of this equation. The cycle is actually very complex since all the variables are time and space dependent variables. Thus, any given system can be viewed as a number subsystems which are joined together by application of the continuity principle.

Amorocho (1961) used a statistical approach by using the gamma distribution to fit runoff hydrographs:

$$Q(t) = \frac{1}{K(N-1)!} \frac{t^{N-1}}{K} e^{t/k} . \tag{2.3}$$

He developed empirical relations which separately relate rainfall intensities to each of the parameters N and K in a nonlinear manner. Amorocho's model was a nonlinear algebraic model, which becomes linear when the parameters N and K are assumed as constants.

Singh (1964) proposed a model which consists of a linear channel and two linear reservoirs of different storage constants in series. Notice the similarities and modifications between this and Zoch's (1934) work. Singh's entire model may be considered as a distributed lumped-system, where the linear channel is used in producing a time-area

relationship for the watershed with variable areal distribution of instantaneous effective rainfall. The nonlinearity of the rainfall-runoff process is introduced by parameters as functions of the effective rainfall input.

Sugawara and Maruyama (1956) suggested a concept of cascade through two unequal elements, where routing of both surface and subsurface flows are assumed linear. Singh (1964) subsequently used this concept of two unequal linear elements, and so did Jamieson and Amerman (1969).

Kulandaiswamy (1964) used a system analysis approach for his rainfall runoff relationship. He proposed a general mathematical model for the storage of nonlinear reservoirs as follows:

$$S = \sum_{n=0}^{N} a_n(Q,I) \frac{d^nQ}{dt^n} + \sum_{m=0}^{M} b_m(Q,I) \frac{d^mI}{dt^m} \qquad (2,4)$$

where S is storage, t is the time, N and M are integers, and $a_n(Q,I)$ and $b_m(Q,I)$ are parameter functions of the outflow Q and inflow I.

By combining the foregoing equation with the continuity relation, the resulting general differential equation can be shown to be that of many mechanical and electrical drainage systems, indicating an analogy between the watershed hydrologic system and the mechanical and the electrical dynamic systems. Kulandaiswamy's resulting differential equation (after dropping insignificant terms) proposed for his simulation analysis is of form:

$$a_2 \frac{d^3Q}{dt^3} + a_1 \frac{d^2Q}{dt^2} + a_o \frac{dQ}{dt} + Q = -b_1 \frac{d^2I}{dt^2} + I \qquad (2.5)$$

where a_2, a_1, a_o, and b_o are coefficients. Equation 2.4 is linear and though nonlinearities were considered by noting changes in the parameters for various storm events. The parameters a_2, a_1, a_o were found to decrease exponentially with increases in the peak discharge of direct runoff. The changes in b_1 and b_o were found to be insignificant. Since the model is basically a linear model, the parameters are readily determined. Kulandaiswamy's model was found to produce better fit to observed data than many other lumped-system models.

Prasad (1967) used a simplification of Kulandaiswamy's model with loss of superior fitness. The model was lumped, time-invariant, and nonlinear. He derived his model from the concept that a nonlinear reservoir with nonlinear outlet control is analogous to hydrologic system storage and discharge. Notice the improvement on Zoch (1934) and Singh's (1964) works; and the similarity between Kulandaiswamy's Eq. 2.5 and his second-order differential equation, which is of the form:

$$K_2 \frac{d^2Q}{dt^2} + K_1 N Q^{N-1} \frac{dQ}{dt} + Q = R \tag{2.6}$$

where K_1, K_2, and N are unknown parameters to be determined in the system identification problem. Prasad used the following general nonlinear storage equation:

$$S = K_1 Q^N + K_2 \frac{dQ}{dt} . \tag{2.7}$$

At peak discharge he assumed $dQ/dt = 0$, which simplifies the storage equation for peak discharge to:

$$S_p = K_1 Q_p^N \tag{2.8}$$

where S_p is the storage at peak discharge Q_p which can be determined from the observed hydrograph.

Unlike previous linear models which have been pointed out as linear, Prasad's nonlinear equation has no analytical solution. He used a trial and error procedure in a numerical integration algorithm on a digital computer. Prasad's model simplifies the procedure of analysis, but it gives a poor fit to observed hydrographs. Particularly at the recession portion the simplifications and approximations can be avoided if the system is fully synthesized and an elaborate computer analysis performed.

In general, there are two types of the modeling procedure: The mathematical modeling of the actual physical system (the river basin); and the physical modeling of the physical system. Since this study is concerned with a mathematical model, a brief mention and references will be given on a physical model.

Physical models in physical hydrology are carried out for the specific study of an isolated phenomenon, and, in general, deterministic models of the runoff cycle may be either physical or computer models. The feasibility of physical models has been studied, among others, by Chow and Harbaugh (1965), Amorocho and Hart (1965), Grace and Eagleson (1966), and Chery (1965), using laboratory catchments.

A general review of mathematical models of catchment behavior was given by Dawdy and O'Donnell (1965). They divided the mathematical models into two categories:

1. The comprehensive simulation of catchment behavior which treats the catchment components in lumped form.
2. The complete specification of each component.

A model of the first category was used, and the major parts of the model were: The surface storage, channel storage, and soil moisture storage, or subsurface storage. Precipitation, evaporation, infiltration were considered, where applicable.

A model of Dawdy and O'Donnel's type, which is perhaps the most widely known, for use with a digital computer is the "Stanford Watershed Model" of Crawford and Linsley (1966). This model is programmed to produce hourly streamflow data using daily evapotranspiration and hourly precipitation data.

Chow (1967) pointed out that the discovery of nonlinearity in hydrologic system behavior was prompted largely from theoretical interest. This interest grew during the several years of use of traditional linear methods, such as the unit hydrograph. Chow (1967) states that, "in practice, the concept of nonlinearity and its methods of analysis are still very limited . . . practicing hydrologists often have little interest in nonlinear methods of analysis." He therefore posed the questions: "Are we wasting time in developing nonlinear methods which may prove to be of little practical significance? If not, can we show that such methods and the concepts of nonlinearity are really important?" Chow, however, did not mention the causes of nonlinearity of the system and the possible physical significance of the factors that cause this nonlinearity. As will be subsequently shown, nonlinearities are clearly indicated in this study as existing, where the major factors responsible for nonlinearity are (Prasad, 1967):

1. the characteristics of the input or rainfall,
2. the physical characteristics of the basin, and
3. the hydraulic characteristics of the main channel.

It should be pointed out that nonlinearities of the rainfall input are mainly due to the spatial distribution of rainfall. Rainfall occurs at different times in the basin. Thus, nonlinearity of rainfall is due to space and time distribution of the rainfall in the river basin. These factors are part of the conceptual model of Chapter III.

It must be pointed out that nonlinear models have not enjoyed the same enthusiasm accorded to linear models (e.g., unit hydrographs) because of the type of nonlinear differential equations which usually result. Therefore, due to lack of adequate knowledge as to solution techniques, hydrologists normally do not use them. Since the watershed is a nonlinear system, considerable research effort should be directed in this area of hydrology so as to more fully understand the physical behavior of the watershed. This in turn will influence, hopefully for the better, decision-making processes in planning water resources projects for a river basin.

Since the mathematical solutions of the conceptual nonlinear hydrologic model appear to be a major problem among researchers and practitioners, an attempt will be made here to apply two different solution techniques in solving the conceptual model, including discussion of and also to show their merits in terms of how efficient or more useful one technique is than the other.

The author's personal comments regarding the foregoing questions raised by Chow are as follows:

1. The nature of the equations, lack of insight into solution techniques, and difficulty of conceptualization of the physical system behavior, are sources of discouragement to the investigator. Therefore, nonlinear hydrologic system modeling has been a neglected area. The Stanford Model is, however, nonlinear.

2. The fact that the nonlinear models have been of little practical significance does not mean that they do not have potential usefulness. The important point is the hydrologic systems are generally nonlinear. Sophisticated solution procedures needed for nonlinear models require that the hydrologist become more mathematically oriented. It will be more stimulating to mathematicians and hydrologists if there is a closer contact between the engineer and the mathematician.

3. Hydrologic modeling as a whole should be regarded as an area which offers a continuous challenge, and should not be discouraged.

Depending on the type of problem, a good synthesis of the physical system may be as important as a unique or generalized mathematical model for the same or some other system. Possible sources of nonuniqueness include:

a. Nature of the data procedure adopted in data gathering, period of record and errors associated with type of data.

b. Lack of insight into the physical system behavior and guesswork in deriving the mathematical model to describe the physical system.

c. Unavailability of appropriate mathematical solution techniques to solve the mathematical model, therefore requiring use of approximate solution procedure or techniques.

The ultimate goal in the hydrologic model is to arrive at unique solutions to problems, even these are obviously difficult, researchers can at least make strides in that direction. This requires creativity, more field studies, better measuring equipment, etc., and new and better mathematical solution tools.

The current research in this dissertation devoted to developing a conceptual model where two of the possible techniques, that could be applied in the solution of the developed mathematical model, are used. Criterion for evaluating the performance of one solution approach with respect to the other will be based on (repeated for emphasis):

1. Efficiency with respect to solution time on the computer.
2. Accuracy of the solution.
2. Simplicity.

Labadie and Dracup (1969) presented a method for the identification of a nonlinear, lumped, time invariant conceptual model of watershed response. Their approach displayed in particular:

1. Simplicity of the computer programming through the use of standard subroutines, and

2. Rapid convergence characteristics of the algorithm, though convergence did not always result, they essentially used Prasad's model Eq. 2.6, but applied a more sophisticated algorithm (quasi-linearization) to solve the problem.

There are two basic approaches to lumped modeling. The first distinct type of a lumped model is the transfer function approach, where the most famous model in this class is the unit hydrograph of direct runoff, resulting from one inch of effective rainfall generated uniformly over the basin at a uniform rate during a specified period

of time. The effective rainfall is that rainfall that contributes entirely to direct runoff. The direct runoff is the total runoff minus the base flow. A particular type of the unit hydrograph concept assumes that the storage-discharge relation of the watershed is linear. The basic assumptions of this type of unit hydrograph are fallacious in the real system, but it is still popular because of the simplicity in its applications especially in the prediction of future runoffs.

A major disadvantage of the unit hydrograph approach is its dependence on the duration of the storm, which is being used. This disadvantage led to the development of the Instantaneous Unit Hydrograph (IUH). The IUH is based on effective rainfall of infinitesimally small duration. It is defined as the limit of the unit hydrograph when the duration of effective rainfall becomes infinitesimally small. This concept may be interpreted in the light of system analysis by regarding a river basin as a system or "black box."

Chow (1964) used a linear convolution integral to synthesize direct runoff hydrograph from the IUH and effective rainfall. The main disadvantage of IUH is in its linearity. The linear convolution integral relates the input and output of the system in the form:

$$Q(t) = \int_{0}^{t \geq T} P(\tau)\ K(t-\tau)d\tau \qquad (2.9)$$

where $Q(t)$ is the direct runoff (ordinate) at any time t,

$K(t)$ is the Kernel function which is here defined as the instantaneous unit hydrograph or ordinate at time $(t-\tau)$ (i.e. the transfer function or the impulsive response), and

$P(\tau)$ is the effective rainfall of duration, T, where τ is a dummy variable.

Blank, et. al. (1971) expressed the rainfall-runoff relation by a convolution integral, where the Kernel function was evaluated by the Fourier and Laplace gamma Transforms, based on their assertion that "a lumped, time invariant, linear system is completely and uniquely characterized by its impulsive response." "In view of the convolution integral, the response of a linear system to any input function can be evaluated once the Kernel function is known. Consequently, the determination of the Kernel function is of utmost importance, especially for systems whose internal structure is unknown." The impulsive response $K(t)$ can be calculated from Eq. 2.9 from a set of input $P(\tau)$ and output $Q(t)$, data regardless of system linearity. Eq. 2.9 can give a check on the validity of the basic assumption of linearity used in many river basin hydrologic models.

Models on the IUH have been derived or presented by two different categories of investigators:

1. Those that considered only linear models: Zoch (1934), Clark (1945), O'Kelly (1955), Sugawara et al. (1956), Nash (1957), Dooge (1959).
2. Those whose models were linear but considered nonlinearities: Singh (1962), Diskin (1964), Kulandaiswamy (1964).

The second distinct type of the lumped model is usually referred to as the analytical conceptual models. This includes some of the foregoing investigators, where they made simplified assumptions concerning the internal structure of the hydrologic system and its dynamic response to rainfall. Examples of investigators who have utilized such an approach are: Zoch (1934), Amorocho (1961), Kulandaiswamy (1964), and Prasad (1967). It is this

latter approach to lumped modeling which is emphasized in this dissertation.

There has been substantial progress in hydrologic model-building, and a convergence of diverse viewpoints. Besides the investigations cited above, Eagleson (1969) gave an outline for modeling urban watersheds. Dawdy (1969) uses parameter estimation in outlining watershed modeling. Freeze and Harlan (1969) presented a physically-based digital computer model of a watershed in terms of partial differential equations for subsystems. The International Seminar on Hydrology (edited by Chow, 1969) reviewed the deterministic and stochastic approaches to modeling, system identification and parameter estimation. Vemuri and Vemuri (1970) augmented this with a review of recent applications of control theory.

In all of these, the investigator recognizes the need and 'struggle' for solutions to:

1. Problems in conceptually formulating and representing hydrologic systems as they exist in their physical, climatic, chemical and biological states.
2. The corresponding problems of identifying the relevant parameters and variables (currently measurable and unmeasurable) and solving for them.

Chiu and Bittler (1969) considered a model and developed a technique for the long-and-short term trends of a hydrologic system. Their rainfall-runoff relation was a first order, linear differential equation with time varying coefficients that depend on two parameters. They utilized relation used $S = Ky^n$, which was considered by Prasad (1967) where K and n are two empirical coefficients to be determined.

The authors stated that . . . "If the exponent n is made equal to unity, the storage can be expressed as $S = Ky$ in which K is a function of time for a time-varying system, or $K = K(t)$." The resulting first order, linear, ordinary differential equation was:

$$K(t)\frac{dY}{dt} + \left(1 + \frac{dK(t)}{dt}\right)Y = X \tag{2.10}$$

where X and Y are the input and output of the system, respectively. The authors recommended that . . . "It would be desirable for hydrologists to investigate the variability of a hydrologic system with months, seasons, years, and time during a storm." The current research uses monthly data to study monthly, seasonal, and annual behavior of the hydrologic system.

Venkateswararao and Dracup (1970) presented a paper on the nonlinear runoff response to distributed rainfall excitations. This was a conceptual rainfall-runoff model based on the equations governing the phenomena under consideration. They regarded the runoff dynamics of a watershed as the response of three interacting subsystems: the infiltration subsystem, the overland flow subsystem and the groundwater subsystem. Precipitation was considered as an excitation, evapotranspiration as a disturbance, and runoff as the response. It was assumed that the watershed, at any time, could be mathematically described by a black box approach, where evapotranspiration could be treated as an external disturbance, thereby reducing the model to one of single input and single output.

Amorocho and Brandstetter (1971) presented a paper on a method of determining relationships between the input and output of lumped systems without the constraint of linearity. Nonlinearity of the watershed

response was considered in a more objective and consistent way than by procedures of rainfall excess computation and base flow separation used in the unit hydrograph apprcach. The operation of some hydrologic systems were represented by a functional series expansion involving higher order nonlinear generalizations of the convolution integral, which underlies the unit hydrograph concept. The assumptions and approximations involved in the representation of hydrologic systems by deterministic "lumped" models may be discussed here, briefly. These models are based on the premise that it is possible to predict runoff (output) of a natural basin as a function of time from a precipitation influence (input), also expressed as a function of time only, under certain specific conditions:

1. That the basin has operated as a time invariant system during the period for which input and output data are used for the postulation and calibration of the model, and that this same invariance can be assumed to extend to the period of prediction.

2. That relative to the time variability of precipitation, the variability of other natural inputs (e.g., evapotranspiration) is small or follows a known function of time.

3. That as far as the basin response is concerned, all natural sequences of precipitation (which in reality form fields changing in space and time) are equivalent to input sequences that are functions of time only. These sequences are called precipitation indices.

Clearly, the above conditions are met only in an approximate sense. Hence, the predictions of a deterministic model of the type of our concern always contain errors dependent on this sense of approximation.

Fahlbusch and Müir (1972) presented a paper which involved a development of a monthly rainfall-runoff model which, they pointed out, could be used for the determination of synthetic long-term runoff sequences from synthetic or observed rainfall sequences on river basins for which only limited runoff data are available. They used a transfer function of an exponential decay type in a linear model that links rainfall as input and runoff as the output.

Klemes (1973) treated a watershed as a semi-infinite storage reservoir. An infinite reservoir was defined as a reservoir with infinite storage capacity; and whatever the magnitude of its release and input rates, it can never run dry nor be completely filled. He diverged from this unrealistic concept by presenting a new concept of semi-infinite reservoir:

> "It is a reservoir that has only one of the two boundaries in infinity, either:
> (1) the bottom, in which case all fluctuations are confined within a small region near its top, or
> (2) the top, in which case the fluctuations are confined to a region adjoining the bottom.
> While the first type describes a situation which is not typical for real reservoirs the second type describes real reservoirs almost ideally."

He presented a relation ($y = F(S)$, the so-called flood routing problem) for a semi-infinite reservoir where the routing function is of the form:

$$y = \alpha S^{\beta} \tag{2.11}$$

where α, β are positive constants,

y is the rate of outflow, and

S is the reservoir storage capacity.

Through the equation of conservation of mass the mechanism of a storage reservoir, whether finite or infinite, was given by

$$Xdt - ydt = dS \tag{2.12}$$

where X is reservoir inflow.

For Case $\beta = 1$, i.e., for the linear reservoir:

$$\frac{dS}{dt} + \alpha S = X \tag{2.13}$$

or

$$\frac{dy}{dt} + \alpha y = \alpha X \tag{2.14}$$

where storage

$$S(t=\tau) = \int_0^\tau e^{\alpha(t-\tau)} x\, dt + S_o e^{-\alpha\tau} \tag{2.15}$$

and reservoir output

$$y(t=\tau) = \alpha \int_0^\tau e^{\alpha(t-\tau)} x\, dt + y_o e^{-\alpha t} \tag{2.16}$$

where S_o, y_o are storage and outflow at time

for $\beta \neq 1$ i.e., for nonlinear reservoir Klemes suggests that solution for equation 2.12 can easily be obtained by numerical methods or graphical techniques known in engineering practice as flood routing methods.

Review on Usage of Proposed Technique I - Quasilinearization

Quasilinearization is primarily applicable to lumped parameter systems described by ordinary differential equations.

In some cases distributed systems described by partial differential equations can be considered if these equations can be replaced by sets of ordinary differential equations. Bellman et al. (1966) have shown how quasilinearization can be applied to systems with time lag, described by differential-difference equations, which are also reduced to sets of ordinary differential equations. The technique basically involves solving a series of linear initial value problems such that the sequence of solutions converges to the solution of the original nonlinear problem. The technique requires the objective function to be convex within the range of feasible solutions. Bellman et al. (1966) have shown how quasilinearization can be applied to systems with time lag, described by differential-difference equations, which are also reduced to sets of ordinary differential equations.

As has been previously stated, Labadie and Dracup (1969) used quasilinearization as a technique for identifying a nonlinear, lumped, time-invariant model of watershed response. Yeh and Tauxe (1971) used the quasilinearization procedure to convert data taken at observation wells directly into aquifer parameters. The parameters identified are the storage coefficient and tramsmissivity in a confined aquifer system.

Review on Usage of Proposed Technique II - The Gradient Technique

The gradient technique has been known for over a century (Cauchy, 1847) and appears in many textbooks and papers. Therefore, not much stress will be laid on it here. However, part of Chapter V is devoted to the development and application of the theory of the gradient technique.

Chapter III

THE CONCEPTUAL HYDROLOGIC MODEL

General Description of the Conceptual Model

The conceptual model and the mathematical model, which is derived later, are basically for inputs of monthly values of rainfall and runoff. The model, however, is not limited by the consideration of monthly values. The monthly model is derived simply because the data available for the case study are monthly. However, the model will be derived in its general form and shown as to its applicability to monthly data. In general "Precipitation" may be used in place of "rainfall."

Precipitation may follow three diverse paths to a stream. A portion travels as overland flow (surface runoff) across the ground surface to the nearest channel, whereas some may infiltrate into the soil and flow laterally in the surface soil to a stream channel as interflow. A relatively impermeable stratum in the subsoil favors the occurrence of interflow. A third portion of the water may percolate downward through the soil until it reaches the groundwater. If the groundwater is near the surface or if the soil is highly permeable, then vertical percolation of rainwater results in groundwater accretion. Overland flow is highly encouraged by low soil permeability. This is important for the present study since flooding can result from such a situation. A thick soil mantle, even though permeable, may retain enough water as soil moisture so as to impede deep percolation. Evapotranspiration losses are significant in this case.

It is somewhat artificial to distinguish between the three types of runoff. Water moving as surface runoff may infiltrate and become interflow or groundwater while infiltrated water may come to the surface and finally reach a channel as surface flow. Groundwater may be the main contributor of water for much of the low water flow of streams.

The model for this research is not concerned with the fine picture of how water is distributed from one stage to the other, in what amounts, where and when. The model is concerned with the collective monthly water budgeting, using monthly rainfall and runoff.

A river channel may be conceptually regarded as an enlargement of a reservoir. If the reservoir has no gates, discharge takes place over a weir or through an orifice in such a way that outflow is a function of reservoir level. Considering a deep reservoir where water velocity is low, the water surface will be nearly horizontal and the volume of water in the reservoir is directly related to the reservoir elevation. Hence, storage and outflow can be directly related. If the reservoir surface has a significant slope, the storage becomes a function of inflow as well as outflow. The present conceptual model considers the case of almost horizontal water surface.

The basin should be visualized as a "Black Box" (Fig. 3.1) as a first step. This considers rainfall as an input and runoff as an output. The river basin is viewed as a system which has the ability to convert a rainfall input into a runoff. This characteristic feature of the river basin may be described by functional relations with certain unknown parameters, as well as various measured and unmeasured variables. The parameters of the functional relationships

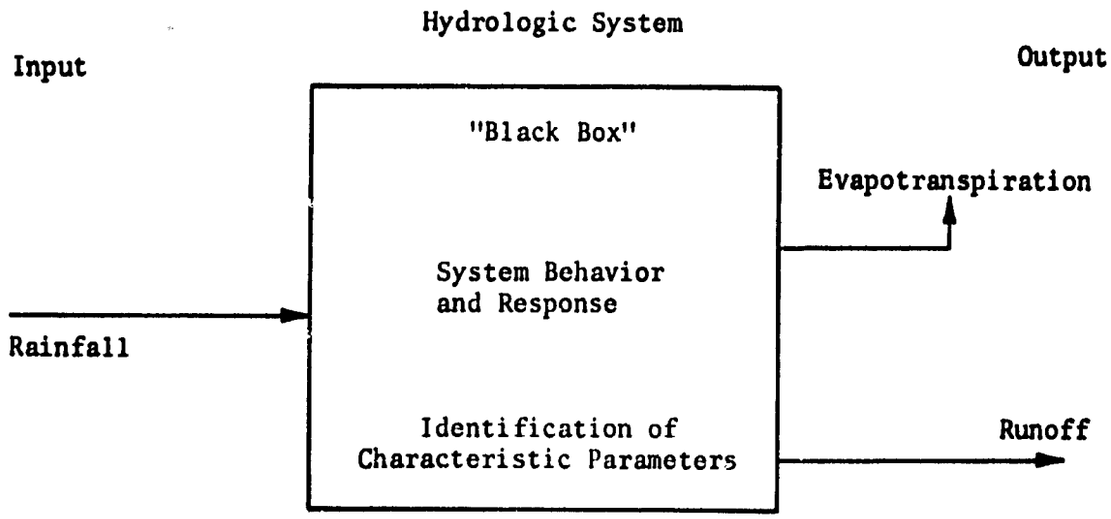


Fig. 3.1 : Schematic of the Conceptual Model

may give some indication of certain characteristic behavior taking place in the basin during the transformation process.

A hydrologic model should be conceived, conceptually, so that it is not a closed system unless experiment on the field investigations (especially soil and geologic) indicate there is no interaction between the given basin and the neighboring basin. Any water that cannot be accounted for may be listed as:

1. Water losses from conceptual surface and subsurface reservoirs which would otherwise have been measured.
2. Other losses that for reasons of development (in developing countries, e.g., political, economic, social, and legal constraints) cannot be investigated or measured at this time. The major loss here is the evapotranspiration.

The water that cannot be accounted for must be regarded as error in the measurement in the variables, especially runoff in this case. Stretching this concept a little further a variable that cannot be determined mathematically or physically at the moment must be considered as an error in the model.

However, a basin can be considered as a closed system by using the Law of conservation of mass or the continuity relations. The major inflow into a hydrologic system is the rainfall or the precipitation. Through some natural characteristics of the river basin, the rainfall input is divided into two main paths: the open channel and the soil as subsurface flows.

The volume of water in a natural channel at any moment is called the storage, S for a reservoir where the outflow is a function of storage, if inflows are considered to be equal to outflow (runoff, and

evapotranspiration especially) then the storage does not change with time. The river channel can be conceptually regarded as a reservoir, since it has the capacity to store large quantities of water. It is referred to as the conceptual surface reservoir.

Again, consider the subsurface flows. These may flow laterally into the channel and provide its contribution to the total runoff. The subsurface component may be conceptually represented by a reservoir--the conceptual subsurface reservoir.

This conceptualized hydrologic system now consists of rainfall input, evapotranspiration output, a conceptual surface reservoir, a conceptual subsurface reservoir, and the runoff. Figure 3.2 gives Step II of the conceptual model development. In considering these basic interactions only rainfall (input) and runoff (output) data will be used, since they are the only data available. In considering the entire picture, it is necessary to consider the effect of subsurface interchanges of moisture through the porous media. The net effect may be zero or negligible and is assumed not to affect this model but it is at least important to mention these possible changes. If the conceptual subsurface reservoir is saturated with moisture, then moisture can be expected to be lost from the conceptual reservoir to the neighboring basin. On the other hand, if the conceptual subsurface reservoir is drier than the neighboring basin, then moisture may flow into the subsurface reservoir, or vice versa. Any changes in our reservoir goes into affecting the values of $\frac{\Delta S}{\Delta t}$, the change of storage with time in the subsurface reservoir. Therefore consider:

$$aP = Q + \frac{d\bar{s}}{dt} = \Delta\bar{Q} + \frac{d\bar{s}}{dt}$$

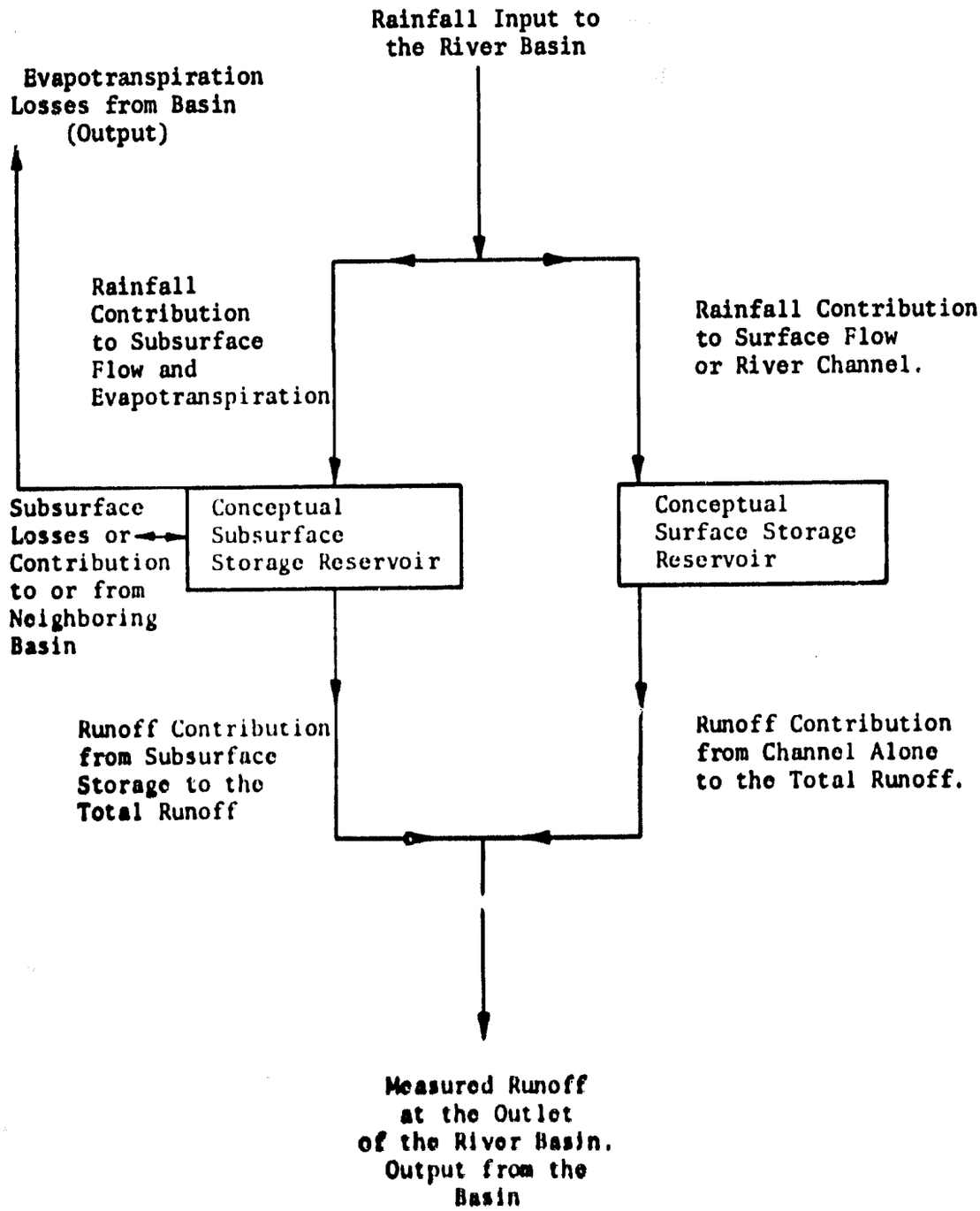


Fig. 3.2 : Step II of the Conceptual Hydrologic Model

where $\Delta\overline{Q}$ is interbasin flow, and $\alpha = \alpha(t)$ is proportional to the evapotranspiration losses for a given period.

It is therefore not necessary to consider the subsurface interchanges in our model--it just complicates the mathematics without adding any more important information. Similarly, evapotranspiration losses for a given period take care of all the losses from or through the surface of the basin.

In Chapter I, in the statement of the objectives, concerning the uniqueness of a model, it is pointed out that some of the basic objectives of the research is to develop a general model that is particularly relevant to a developing country, within the limits of uniqueness, simplicity, economic constraints, rigorousness and realism in modeling. Figure 3.3 gives Step III of the conceptual hydrologic model. This is aimed at satisfying the foregoing objectives and is the model that will be finally used, after proving in Chapter IV the importance of Step III (Case II in Chapter IV) over Step II (Case I). This step seems justified when one considers the large amount of interconnection between the surface and subsurface reservoirs, and the fact that there is no data available that measures this interconnection.

The conceptual reservoir storage at the end of the river basin represents the conceptual subsurface and conceptual surface reservoirs in Step II. Therefore, the Step III conceptual reservoir may be viewed as being directly affected by the rainfall input, after adjusting for evapotranspiration losses. The total outflow or runoff from the conceptual reservoir in Step III is the same as the output from the basin in Step III.

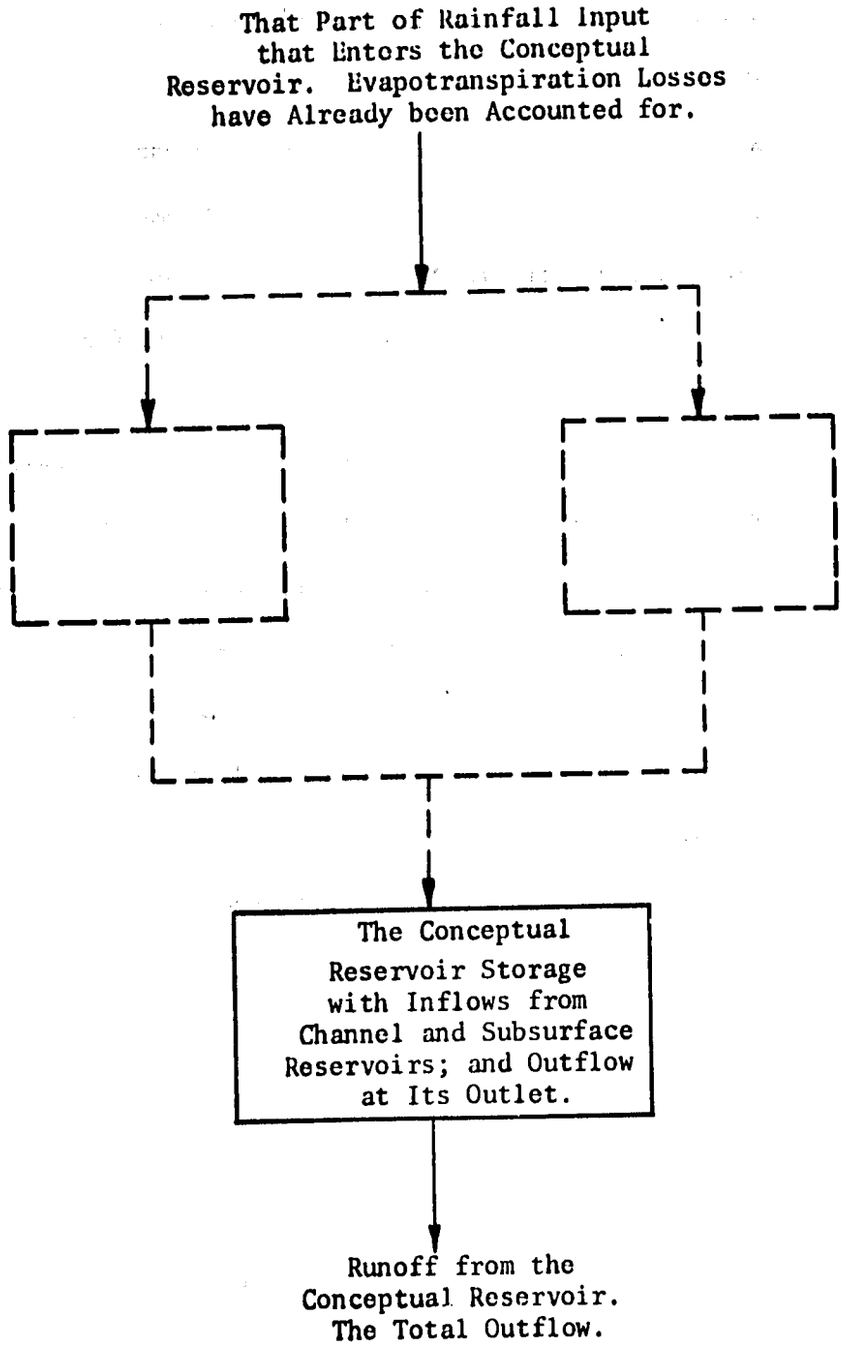


Fig. 3.3 : Step III of the Conceptual Hydrologic Model

THE PROPOSED MODEL

Steps II and III are essentially the same, but the mathematics and available solution techniques will illustrate the importance of one with respect to the other.

Assumptions to the Conceptual Model

Before deriving the mathematical equations that describe the conceptual model, certain assumptions must be made concerning the model.

In an ideal case, that is, when the perfect assumptions to the system behaviors are made, the response of both the real system and the mathematical model to the same input is identical. In this case the mathematical model simulating the system may be considered as perfect. In general, however, these two responses are not identical, and an error may be found to exist. The main purpose in system modeling is to construct a mathematical model so that the error that exists between the conceived and actual physical models may be minimized. The smaller the error the better the assumptions made describe the system.

Errors Due to Mathematical Modeling of the Physical System

The types of predictive inaccuracies or errors associated with these models may be due to the following:

1. The basic assumptions or conditions that are considered may be met in an approximate sense. Hence, the predictions by the mathematical model always contains error explicit in the degree of approximation.
2. Errors in the evaluation of the input and output variables (e.g., the precipitation indices and the runoff). Errors of

this type are commonly involved in the physical measurements and in the choice of indices.

3. Errors due to any covert conceptual fallacies that may have been incorporated into the model structure.

One of the most fundamental preconceptions encountered in classical hydrology is the assumption of linearity involved in the unit hydrograph concept.

The following assumptions are made in order to derive the mathematical model which describes some of the characteristic behavior of the real system.

Nonlinearity of the Hydrologic System

The hydrologic system is considered as a nonlinear system. The nonlinearity of the system is based on three major factors which are responsible for nonlinearities, (Prasad, 1967):

1. The characteristics (with respect to spatial and time variations) of the input to the system: rainfall. This has been explained in Chapter II as also being dependent upon the types of storms when they occur, and their spatial effects in the basin.
2. The physical characteristics of the basin which are responsible for converting the input into runoff.
3. The hydraulic characteristics of the main channel.

All these factors are related and will be described by corresponding parameters.

The hydrologic system storage and discharge are assumed to be analogous to a nonlinear surface reservoir, a nonlinear/linear subsurface reservoir with nonlinear outlet control (Prasad, 1967).

The two basic approaches to nonlinear modeling of hydrologic systems are:

1. Lumped parameter models, and
2. Distributed parameter models.

A hydrologic system model may be assumed as a LUMPED SYSTEM if lumped values of distributed rainfall are used for the entire basin, instead of using a spatially distributed rainfall. This assumption simplifies the problem considerably.

The major assumptions for lumped hydrologic parameter models are given by Amorocho (1967):

1. That the basin has operated as a time invariant system during the period for which input and output data are used for the postulation and calibration of the model, and that this same invariance can be assumed to extend to the period of prediction.

This assumption is necessary since quantitative measurement of time-variance of the input and its effects on the river basin response is not, at present, possible. Climatic changes, land use, erosion, and sedimentation are some of the major causes of time-variance or nonstationarity of the system parameters. Therefore, seasonal or monthly variations of the parameters are necessary and need to be determined for each basin. This idea is incorporated in this research.

2. The rainfall input to the system (or the input) is uniformly distributed.

This assumption depends largely on whether rainfall is caused by frontal or convective or other activity, though it also depends on the topography of the river basin, the soil characteristics, the vegetal cover, of the basin. It also depends on the size of the river basin. Usually, rainfall is evenly distributed throughout the river basin

and varies in spatial distribution from storm to storm. In areas that frontal storms predominate, even larger (than case study area) river basins may be considered in the realm of a lumped system. In areas where the storm activity is mostly convective, it may be important to divide the basin into small sub-basins where each sub-basin has a runoff gaging station at the outlet. The case study area is a small basin (840 sq. miles) flat, and is affected by convective storms and the Intertropical Convergence Zone (I.T.C.Z.). The Intertropical Convergence Zone is (McIntosh, 1963):

> "A relatively narrow, low latitude zone in which air-masses originating in the two hemispheres converge.....Over the Atlantic and Pacific Oceans the I.T.C.Z. is the boundary between the north-easterly and south-easterly trade winds. Over the continents it is replaced by the boundary between other wind systems with components directed towards the equator, for example, in Africa between the HARMATTAN and the southwest Monsoon. I.T.C.Z. moves north of equator during northern hemispheres summer. The horizontal convergence associated with the I.T.C.Z. implies generally upward motion in the lower tropospheres and cloudy, showery weather."

3. All of the complex processes and interactions that are responsible for the response of a hydrologic system are considered as lumped or aggregated.

This implies that the lumped parameter system approach permits the approximation of overall system performance without requiring any detailed definition of the system components. This assumption may be justified when considering the lack of adequate knowledge with respect to the inner workings and behavior of the hydrologic cycle. Amorocho (1967) points out that the act of measuring the micro-properties of the system can artificially alter its state; and that this interference tends to introduce considerable uncertainty into the experimental results.

Clearly, the foregoing assumptions or conditions have varying degrees of validity, depending on the physical system considered. Hence the predictions of a deterministic model of the type of our concern will generally always contain errors in proportion to the extent of accuracy of these assumptions.

The DISTRIBUTED PARAMETER MODEL attempts to describe mathematically all of the known hydrologic and hydraulic processes involved in the conversion of rainfall to runoff. This model therefore considers areal variation of rainfall. There are different names that various investigators give to distributed parameter type of model. Linsley (1966) called it System Simulation; Amorocho (1964) called it General System Synthesis. The current work or research which does not consider distributed parameter model uses the words system and simulation in describing the lumped conceptual model. The name used is not the important thing to consider but the differences in the meaning between the lumped and distributed models.

Since this research does not consider the distributed model, therefore the distributed model will only be briefly explained by the foregoing. However, Amorocho (1964) pointed out some of the basic weaknesses of distributed models in general:

1. Model structure is inflexible.
2. Nonuniqueness or nonoptimality of the identified parameters of the model.

Labadie (1968) pointed out:

> "Lumped models can reflect the highly dynamic response of hydrologic systems, and be uniquely verified from rainfall and runoff data. It is recognized, however, that distributed

models (composed of partial differential equations) must eventually replace lumped models if hydrologic system behavior is to be accurately modeled. New methods must then be found for measuring spatially distributed discharge and storage within a hydrologic system."

Seasonal Variation of the Characteristic Parameters

Some of the characteristic parameters of the basin should not be regarded as constant as has been assumed by some previous investigations (Prasad, 1967). In Prasad's model:

$$K_2\frac{d^2Q}{dt^2} + K_1NQ^{N-1}\frac{dQ}{dt} + Q - R = 0 \qquad (3.1)$$

hourly data were used and the coefficients K_2 , K_1 , N were assumed as constants. This assumption may be acceptable if the model is built for one storm and not for different storms at different times in the basin. The parameter K_1 for example, varies with shape and size of the basin and the slope of the main channel. This will vary for different storms at different times in the same basin. A certain amount of rainfall from a hypothetical storm passing over one part of the basin will generally give a different hydrograph at the outlet of the basin from the runoff produced by the same storm giving rise to the same rainfall moving over another portion of the basin. This is a consequence of one or more differences in the vegetational cover, soil characteristics, slopes, land use and other factors that come into play in different areas of a watershed, including some very small watersheds. Diverse climatic regimes and seasons bring about various amounts of rainfall which result in different amounts of runoff. The parameters may be characterized as follows (Prasad, 1967):

N depends on channel characteristics,

K_1 depends on size and shape of the drainage basin as well as N and may also be influenced by storm characteristics,

K_2 depends on storm and channel characteristics as well as shape of the basin.

Storm types do change with seasons and so do the channel characteristics with time. It is therefore assumed that seasonal and monthly variations of these parameters exist and need to be found by applying the mathematical model developed in this study.

Conceptual Reservoir as an Open System

A conceptual system is one to which a conceptual realization contributes significantly and becomes very important. This factor contributes immensely in the development of the concepts underlying the mathematical model. The Stanford Watershed Model (Crawford and Linsley, 1964); and the Dawdy and Bergmann (1969) model are very important in this respect.

It is unrealistic to consider the conceptual reservoir as closed with respect to its surrounding or immediate environment. It is easily assumed, and wrongly so, that the runoff that is measured at the end or mouth of the river basin is the result of the rainfall that falls in the basin alone.

The runoff that is measured may be thought of as the total runoff, which is the sum of rainfall contribution, possible groundwater contributions which may not be dependent on the current rainfall, or lateral subsurface inflows from a neighboring river basin which may find its way into the channel. The latter depends on the type of

season (whether wet or dry) and on the soil and geologic characteristics (especially, whether the rock formations may permit exchanges of water to and from neighboring river basins).

It is important not to disregard these various contributions to the total runoff as irrelevant or minimal (unless it is physically and geologically justifiable) until they are found for each basin and their monthly, seasonal, and annual contributions and variations are established for the particular basin.

Considering a "closed" system in a formulation usually has an advantage of simplifying the mathematics so that certain existing mathematical tools could be used for solutions of the problems. However, it is important to note that a solution approach which is capable of obtaining near-optimal solutions to more realistic models should be regarded as more useful than a solution approach or procedure that can give an optimal solution to simplified unrealistic models. Of course, it is important to establish a trade-off between solution accuracy and model complexity. This could be made, with more efficiency and accuracy, with experience and good judgment on the part of the experimenter or the investigator. Usually added realism more than compensates for obtaining a solution which may not be the absolute global optimum of the model.

Since the other inflows besides rainfall may not be known, it can at least be said that a rainfall input gives rise to an effective runoff, which is some fraction of the total runoff that is measured.

There should be a distinction between the subsurface contributions to total runoff and that of channel to total runoff. (Chapter IV gives a diagram of the Conceptual Model).

The conceptual reservoir is assumed as consisting of two inlets, the rainfall input and subsurface contributions to the system, and one outlet, for total runoff.

Rainfall Distribution in the System

Rainfall is assumed to take two main paths in a river basin. Suppose rainfall is designated as P then $\alpha_i P$ ($i=1,2,\ldots 12$ for each month) is that amount of a given rainfall which is converted into channel runoff. The other portion $(1-\alpha_i)P$ is the contribution of the rainfall which goes into subsurface storage system, or the subsurface part of the conceptual reservoir. Let α_i be simply written as α though it should be clearly understood it is a time dependent parameter.

Part of this rainfall is assumed lost to the atmosphere through evapotranspiration. It is assumed that after subtracting the evapotranspiration E which should not necessarily be measured in this work, the effective amount of rainfall contribution to the subsurface storage or lower portion of the conceptual reservoir becomes $\gamma_n(1-\alpha)P$. γ_n ($n=1,\ldots 12$) is assumed constant for a given month or season.

Runoff Outputs from the System

The Runoff Q which is measured at the mouth of the reservoir represents the total runoff. The total runoff is a sum of the runoff contribution from channel storage and the runoff contribution from subsurface storage. Runoff is a time dependent variable.

$$Q = Q_1 + Q_2 \tag{3.2}$$

where Q is the total runoff,

Q_1 is the channels contribution to Q

Q_2 is the subsurface storage contribution to Q.

In measuring the total runoff at the outlet some errors are encountered and Q may be represented as

$$Q = Q_1 + Q_2 - \epsilon \tag{3.3}$$

where ϵ is the error encountered for the month or season. It is further assumed that ϵ is some time varying fraction of Q and Q may be expressed in the form:

$$Q = Q_1 + Q_2 - \delta Q \tag{3.4}$$

where δ is a constant parameter for a given month or season. Q_1 is also assumed as some constant fraction of Q for a given month:

$$Q = \beta_n Q + Q_2 - \delta Q \tag{3.5}$$

or

$$Q_2 = (1 + \delta - \beta_n)Q \tag{3.6}$$

where β_n is a constant parameter for a given month or season

$(n=1,2,\ldots,12)$.

<u>Discharge-Storage Relationships of the Hydrologic System</u>

Different forms of discharge-storage (or storage-discharge) relationships have been proposed and used by different investigators for synthesizing their various conceptual models.

Nash (1960) and Diskin (1964) used cascades of linear reservoirs in developing general theories of river basin transformations.

A mathematical model is at times broken down into two forms: translation and storage. Suppose an equation of the type (2.4) is written as:

$$a_2 \frac{d^3Q}{dt^3} + \frac{d^2Q}{dt^2} + a_o a_2 \frac{dQ}{dt} + a_o Q = I \tag{3.7}$$

then

$$\frac{d^2}{dt^2} \left(a_2 \frac{dQ}{dt} + Q\right) + a_o\left(a_2 \frac{dQ}{dt} + Q\right) = I \tag{3.8}$$

and

$$\phi = a_2 \frac{dQ}{dt} + Q \qquad \text{(STORAGE)} \tag{3.9}$$

where

$$\frac{d^2\phi}{dt^2} + a_o\phi = I \qquad \text{(TRANSLATION)} \tag{3.10}$$

Zoch (1934), Clark (1945), O'Kelly (1955), Dooge (1959) and Singh (1962) have all made use of combined translation and storage elements in their modeling of a river basin.

Storage-discharge relations that have been adopted or used by various investigators may be summarized as follows:

1. Linear storage-discharge relation

$$S = K\,Q \tag{3.11}$$

2. Exponential storage-discharge relation

$$S = K\,Q^n \tag{3.12}$$

3. Demand rate proportional relationship

$$S = K\,Q\,\frac{dQ}{dt} \tag{3.13}$$

4. Exponential demand rate proportional relationship

$$S = K\,Q^{n}\,\frac{dQ}{dt} \tag{3.14}$$

5. Prasad relation (Prasad, 1967)

$$S = K_1 Q + K_2\,\frac{dQ}{dt} \tag{3.15}$$

where S is storage,

Q is discharge, and

K_1, K_2, K, n are all constants.

It is important at this point to comment on Eqs. 3.13 and 3.14. These are unrealistic since dQ/dt could be negative in which case, storage S becomes negative. These two equations may be acceptable if water being lost from storage is referred to as negative (which is not the case assumed by previous investigators who have used these) or a new definition based on physical reality should be given. Equation 3.11 has been used by many investigators even though the relation is unrealistic. To prove the unrealistic nature of Eq. 3.11 (solely because it has found many uses due to its simplistic form even though it is not realistic and not describing the system it is supposed to describe) it will be applied in the model for predicting runoff and evapotranspiration. Equations 3.12 and 3.15 have proved in many investigations as better assumptions and these two assumptions will be made in deriving the models proposed in this study in order to show the merits of one with respect to the other.

The discharge-storage relationships of the current conceptual hydrologic system may be described as follows:

There are two types of storage reservoirs--the top representing channel reservoir and the bottom the conceptual reservoir. The two reservoirs do have one common outlet.

The runoff contribution from the surface channel Q is assumed as a nonlinear function of surface storage:

$$Q_1 = K_1 S_1^n \tag{3.16}$$

where K_1 is the storage coefficient which depends on some characteristics of the conceptual reservoir,

n describes some characteristics of the river channel and gives a measure of the nonlinearity, and

S_1 is the channel storage.

In the case of the subsurface contribution to total runoff, the conceptual subsurface reservoir is a nonlinear reservoir with the discharge and storage related by:

$$Q_2 = K_2 S_2^m \tag{3.17}$$

where K_2 is the storage coefficient which depends on the nonlinear subsurface reservoir with storage S_2 and discharge Q_2,

m is the constant exponent of the nonlinear relationship.

Continuity Relations for Various Subsystems

As it has already been implied, the volume of water in a natural channel at any instant is called channel storage S. The continuity relation is assumed to hold: inflow-outflow equals the change in the storage with time t. As t increases or for long periods of time

$dS/dt = 0$. This assumption forms the basis of what is later referred to as the theoretical representation in estimating evapotranspiration.

Using the continuity relation with respect to surface or channel flow:

$$\frac{dS_1}{dt} = \alpha P - Q_1 \tag{3.18}$$

and similarly for the subsurface flow:

$$\frac{dS_2}{dt} = \gamma(1-\alpha)P - Q_2 + \phi Q \tag{3.19}$$

where all the terms in Eqs. (3.18) and (3.19) have been defined earlier. The last term refers to the subsurface interchanges of water from and into the basin. This, as explained earlier, may be neglected.

The foregoing assumptions are used in Chapter IV and their merits are also given. These assumptions are utilized in deriving the two nonlinear differential equations, or the mathematical models of this study.

Chapter IV

THE MATHEMATICAL MODEL

Case I - Surface and Subsurface Reservoirs

After stating (Chapter III) all the relevant assumptions associated with the conceptual model, it is now necessary to describe it more specifically and derive the mathematical model.

The rainfall input P into the river basin is distributed into two parts (as shown on Fig. 4.1): αP is the rainfall contribution to surface or channel storage and $(1-\alpha)P$ is the rainfall contribution to evapotranspiration and subsurface flow. It is important to understand clearly that the total evapotranspiration losses in the basin for any month are based on losses from soil surface, subsurface, the channel, and any bodies of water present in the basin. It does not really matter where evapotranspiration is subtracted from (Fig. 4.1) so long as the total evapotranspiration losses are taken care of or accounted for. Therefore, the effective contribution of rainfall to channel storage is total rainfall minus evapotranspiration, and is added on to the left-hand side of Fig. 4.1. However, it should be stressed that no attempt should be made to ignore evapotranspiration losses from the channel storage, since this will depend on the topography of the basin, whether flat or sloping, and the time scale considered, whether hourly or monthly data are used.

Some portion of rainfall input $(1-\alpha)P$ is lost due to evapotranspiration $E(t)$ (where $E(t) = \Omega(1-\alpha)P$, say) resulting in a monthly total input to subsurface storage: $\gamma(1-\alpha)P$. In certain basins there may occur interbasin flows of water. This subsurface

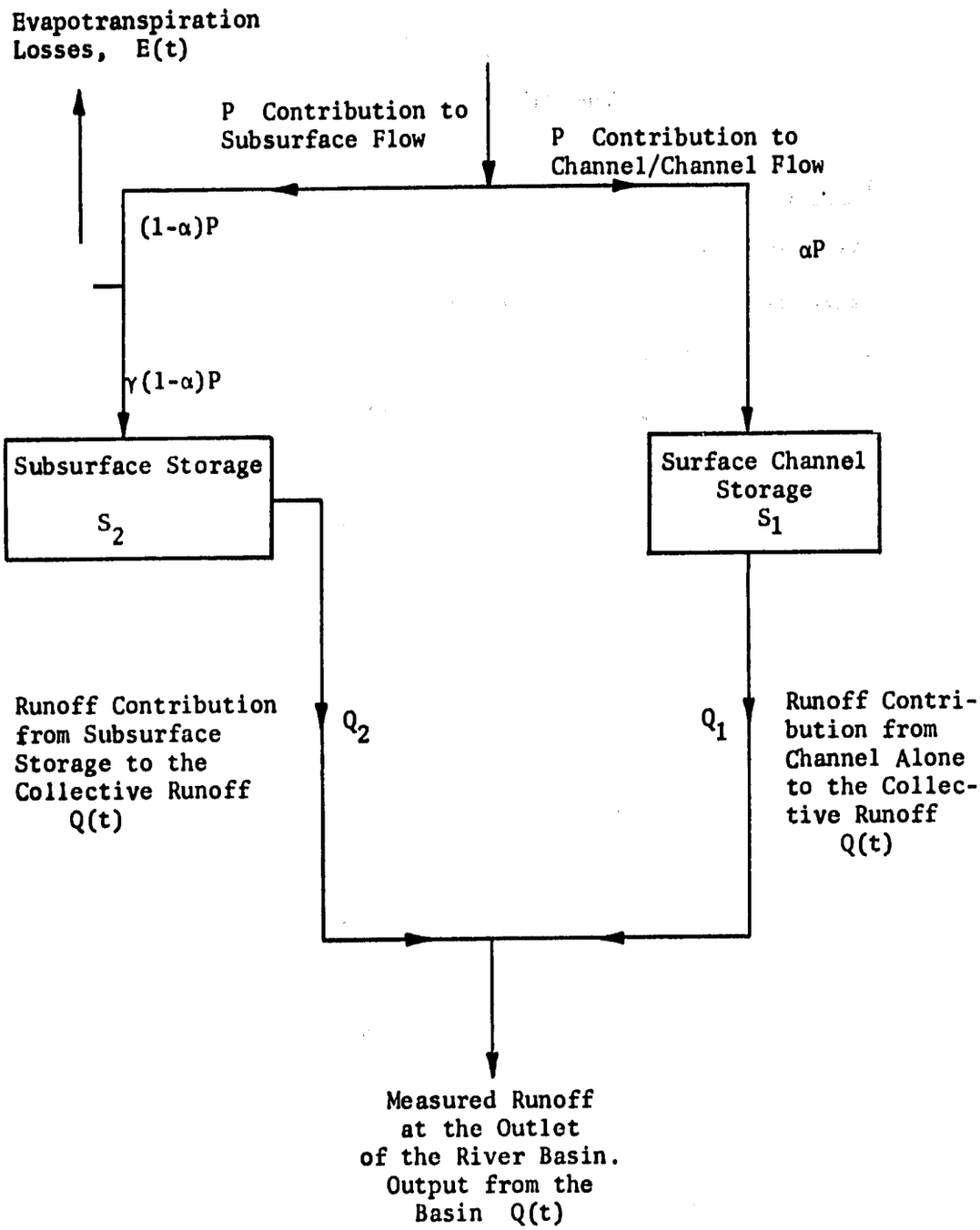


Fig. 4.1 : The Conceptual Hydrologic Model For Case I

inflow may be due to seasonal effects of subsurface storage in adjacent basins or the slope/or geologic formations or type of the rocks or physical features. In some basins, this subsurface storage may be regarded as a closed system with respect to subsurface interflows. For generality, the subsurface inflow should be considered, and it is assumed as some constant fraction of the total measured runoff ϕQ for a given monthly rainfall. It must be mentioned that ϕ, as in ϕQ, should be treated as a time dependent parameter. This is included here for generality, though it is not important to consider it as explained in Chapter III. It is important to stress that one of the objectives of this study is to derive a simple yet realistic model. It will be shown that a complicated model can be derived for which unique solutions may not be available, whereas using the same information and assumptions, a more practical and useful model can be derived.

The total measured runoff Q, at the outlet of the river basin, is the sum of the runoff contributions from the channel, Q_1, and that from subsurface storage Q_2 and errors ε encountered in measuring the total runoff. The error ε will be assumed as some fraction of the total runoff.

The Derivation of the Hydrologic System Equations

Assuming runoff in channel Q for the surface conceptual reservoir, as a nonlinear function of surface storage:

$$Q_1 = K_1 S_1^n \tag{4.1}$$

where K_1 is the storage coefficient which depends on the characteristics of the conceptual surface reservoir, and n describes some characteristics of the river channel and gives a measure of the nonlinearity, and assuming subsurface runoff Q_2 for the subsurface conceptual reservoir, as a nonlinear function of the storage;

$$Q_2 = K_2 S_2^m \tag{4.2}$$

where K_2 is the storage coefficient which depends on characteristics of the conceptual subsurface reservoir, the contribution of rainfall to subsurface storage $(1-\alpha)P$ is diminished by an amount due to evapotranspiration losses. Therefore, from Fig. 4.1, the balance relation gives

$$\frac{dS_2}{dt} = (1-\alpha)P - E(t) - Q_2 \tag{4.3}$$

Suppose evapotranspiration data do not exist. Then it can be assumed that the portion of rainfall contribution to subsurface storage S_2, after subtracting evapotranspiration effects, should be a fraction of $(1-\alpha)P$.

$$\frac{dS_2}{dt} = \gamma(1-\alpha)P - Q_2 \tag{4.4}$$

Similarly, from Fig. 4.1, and considering the balance relationship for the surface reservoir

$$\frac{dS_1}{dt} = \alpha P - Q_1 \tag{4.5}$$

From Eqs. (4.1) and (4.2) the total runoff measured at outlet of the river basin is given for the given period or month as in this case, as

$$Q = Q_1 + Q_2 - \varepsilon \tag{4.6}$$

where ε is error encountered in measuring Q. This error ε, (where sources of error are given in Chapter III) is assumed as some fraction, γ, of Q for the given value of Q or for a given period or month.

$$\therefore \; Q = Q_1 + Q_2 - \delta Q \tag{4.7}$$

Combining Eqs. (4.1), (4.2), and (4.7)

$$(1 + \delta)Q = K_1 \, S_1^n + K_2 \, S_2^m \tag{4.8}$$

From Eqs. (4.1) and (4.5)

$$\therefore \; \frac{dS_1}{dt} = \alpha P - Q_1 = \alpha P - K_1 \, S_1^n \tag{4.9}$$

and

$$\frac{dS_2}{dt} = \gamma(1-\alpha)P - Q_2 = \gamma(1-\alpha) - K_2 \, S_2^m \tag{4.10}$$

$$\therefore \; \frac{d}{dt}(S_1+S_2) = \alpha P - K_1 \, S_1^n + \gamma(1-\alpha)P - K_2 \, S_2^m \tag{4.11}$$

Substituting Eq. (4.7) into (4.11) and considering Eqs. (4.1) and (4.2)

$$\therefore \; \frac{d}{dt}(S_1+S_2) = \left[\alpha + \gamma(1-\alpha)\right] P - (1+\delta)Q \tag{4.12}$$

$$\therefore \frac{d}{dt}\left\{\left(\frac{Q_1}{K_1}\right)^{\frac{1}{n}} + \left(\frac{Q_2}{K_2}\right)^{\frac{1}{m}}\right\} = \left[\alpha + \gamma(1-\alpha)\right]P - (1+\delta)Q \tag{4.13}$$

$$\therefore \left(\frac{1}{K_1}\right)^{\frac{1}{n}} \cdot \frac{1}{n} \cdot Q_1^{\frac{(1-n)}{n}} \frac{dQ_1}{dt} + \left(\frac{1}{K_2}\right)^{\frac{1}{m}} \cdot \frac{1}{m} \cdot Q_2^{\frac{(1-m)}{m}} \frac{dQ_2}{dt}$$

$$= \left[\alpha + \gamma(1-\alpha)\right]P - (1+\delta)Q \tag{4.14}$$

Considering Q_1 as a constant fraction of Q for a given period of Q or month. In fact, Q_1 can easily be regarded as time dependent fraction of Q without any difficulty. It is, however, not necessary at this point.

$$Q_1 = \beta Q \tag{ .15}$$

$$\therefore Q_2 = (1 + \delta - \beta)Q \tag{4.16}$$

Combining Eqs. (4.14) and (4.16)

$$\therefore \left(\frac{1}{K_1}\right)^{\frac{1}{n}} \cdot \frac{1}{n} \cdot (\beta Q)^{\frac{(1-n)}{n}} \beta \frac{dQ}{dt} + \left(\frac{1}{K_2}\right)^{\frac{1}{m}} \cdot \frac{1}{m} \cdot \left[(1+\delta-\beta)Q\right]^{\frac{1-m}{m}} (1+\delta-\beta) \frac{dQ}{dt}$$

$$= \left[\alpha + \gamma(1-\alpha)\right]P - (1+\delta)Q \tag{4.17}$$

$$\therefore \left[\left(\frac{1}{K_1}\right)^{\frac{1}{n}} \cdot \frac{1}{n} \cdot \beta \cdot (\beta Q)^{\frac{1-n}{n}} + \left(\frac{1}{K_2}\right)^{\frac{1}{m}} \cdot \frac{1}{m} \cdot (1+\delta-\beta) \cdot \left[(1+\delta-\beta)Q\right]^{\frac{1-m}{m}}\right] \frac{dQ}{dt}$$

$$= \left[\alpha + \gamma(1-\alpha)\right]P - (1+\delta)Q \tag{4.18}$$

The unknowns in Eq. (4.18) are: α, β, γ, δ, n, m, K_1, K_2, and dQ/dt. There are, therefore, nine unknowns that have to be uniquely determined in the parameter identification problem. The nonlinearity of Eq. (4.18) and the number of unknowns are the major factors in our inability to uniquely determine the values of the unknown parameters. Even though this is a realistic model it is not useful as a mathematical model for a river basin in a developing country. It is important that when a model is to be used for a developing country the following factors should be carefully considered (repeated for emphasis):

1. Simplicity of the model--conceptually and mathematically.
2. Rigorous and realistic model.
3. Fast and inexpensive with respect to computations by the computer.

It may be true the above proposed model (Case I) is a realistic model; and conceptually it is relatively simple but it is extremely difficult to find unique solutions.

Case I will therefore not be utilized because of this nonuniqueness detriment (Chapter III) in the determination of the parameters.

Case II will therefore be presented to illustrate the importance of using the same or similar conceptualization but deriving a model that satisfies the above three factors necessary for modeling for a river basin in a developing country.

Case II - Proposed Lumped Model

For the reasons given after deriving the equations for Case I it is now important to use the same assumptions and concepts to derive a more realistic model. A more realistic model is one which satisfies

previously expressed objectives of simplicity, realism, and amenability to rapid computation.

Figure 4.2 shows the conceptual hydrologic model for Case II. This is a modified form of Step III of the conceptual hydrologic model (Fig. 3.3) which is in turn a simplified version of Fig. 3.2 or Fig. 4.1. Rainfall input into the system is P. The rainfall contribution to surface and subsurface runoff that are measured as total runoff Q is given as αP. The evapotranspiration losses in the river basin are $(1-\alpha)P$. The model need not consider evapotranspiration as another output, which will complicate the formulation and mathematics, and evapotranspiration is uniquely determined on uniquely determining α for the input (αP) and output (Q) system. The α in Case II is obviously different from that of Case I, and will be the one considered from now on.

Assuming runoff from the reservoir Q for the conceptual reservoir as a nonlinear function of surface storage

$$Q = C\,S^N \tag{4.19}$$

or

$$S = K\,Q^M \tag{4.20}$$

where K is the storage coefficient, and

M describes some characteristics of the channel and gives a measure on nonlinearity.

Assuming rainfall input is P and after subtracting evapotranspiration from P, a quantity αP of the rainfall enters the conceptual reservoir. This implies that evapotranspiration is $P - \alpha P = (1-\alpha)P$.

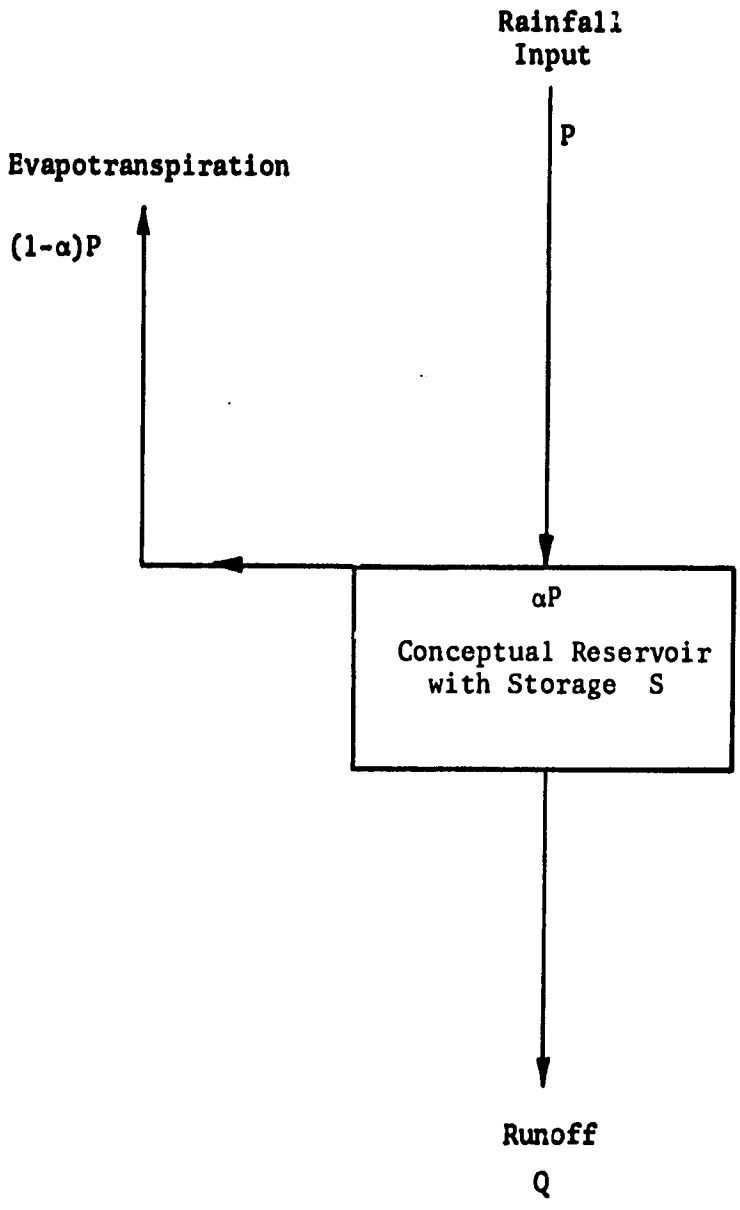


Fig. 4.2 : Conceptual Hydrologic Model for Case II

Change in storage with time is therefore given by the continuity relation (Fig. 4.2)

$$\frac{dS}{dt} = \alpha P - Q \tag{4.21}$$

From Eq. (4.20)

$$\frac{dS}{dt} = KMQ^{M-1} \frac{dQ}{dt} \tag{4.22}$$

Combining Eqs. (4.21) and (4.22)

$$\alpha P - Q = KMQ^{M-1} \frac{dQ}{dt} \tag{4.23}$$

or

$$\frac{dQ}{dt} = \frac{\alpha PQ^{1-M}}{KM} - \frac{Q^{2-M}}{KM} \tag{4.24}$$

$$\therefore \frac{dQ}{dt} = APQ^{1-M} - BQ^{2-M} \tag{4.25}$$

where A, B, M are the unknown parameters of the differential equation, and can be uniquely determined, where

$$A = \frac{\alpha}{KM} \tag{4.26}$$

$$B = \frac{1}{KM} \tag{4.27}$$

Determination of M and B will give a unique determination of K (Eq. 4.27), which in turn will result in unique determination of α (Eq. 4.26).

Equation (4.25) represents the first order nonlinear differential equation model.

To derive the nonlinear second order differential equation consider

$$S = K_3 Q^M + K_4 \frac{dQ}{dt} \tag{4.28}$$

$$\frac{dS}{dt} = K_3 M Q^{M-1} \frac{dQ}{dt} + K_4 \frac{d^2Q}{dt^2} \quad . \tag{4.29}$$

But

$$\frac{dS}{dt} = P - E - Q = \alpha P - Q$$

$$\therefore \frac{d^2Q}{dt^2} = \frac{\alpha P}{K_4} - \frac{Q}{K_4} - \frac{K_3 M Q^{M-1}}{K_4} \frac{dQ}{dt} \tag{4.30}$$

$$\therefore \frac{d^2Q}{dt^2} = A_1 P - A_2 Q - A_3 Q^{M-1} \frac{dQ}{dt} \tag{4.31}$$

where the unknowns: A_1, A_2, A_3 and M may be uniquely determined as follows:

Finding A_2^* and A_1^* implies $\alpha^* = \frac{A_1^*}{A_2^*}$ and $\beta^* = A_2^*$.

Finding M^* and A_3^* (A_2^* already found) implies

$$K_3^* = \gamma^* = \frac{A_3^*}{M^*} \cdot \frac{1}{A_2^*} \quad .$$

Let $M^* = \phi^*$.

Thus, α, β, γ, ϕ can be uniquely determined. The two derived models representing the hydrologic system are:

1. The first order nonlinear differential equation model:

$$\frac{dQ}{dt} = A.P.Q.^{1-\gamma_1} - B.Q^{2-\gamma_1} \tag{4.32}$$

2. The second order nonlinear differential equation model:

$$\frac{d^2Q}{dt^2} = \Lambda_1 P - \Lambda_2 Q - \Lambda_3 Q^{\gamma_2 - 1} \left(\frac{dQ}{dt}\right) \tag{4.33}$$

Equations (4.32) and (4.33) are the two basic models that will be utilized in this study.

Chapter V

THE PARAMETER IDENTIFICATION PROBLEM

Definition of Parameter Identification

Given a discrete time series of runoff, $Q(t_k)$, where $k = 1,2,3,4,\ldots.r$, taken on a particular river basin, along with rainfall measurements, $P(t_k)$ where $k = 1,2,\ldots.r$, the parameter identification problem can be stated as:

> Given rainfall (or runoff) observed values, one can write an equation to predict runoff where this predictive equation contains unknown parameters. By minimizing with respect to these parameters, the difference between the actual measured values and predicted values (ideal case is when there is no difference between measured and predicted values) better estimates of the parameters may be found. In practice, due to errors in measurements or from the mathematical model solution technique, there is always an error between measured and computed values. Therefore, the smaller the difference, the closer the computed and measured values, and the better the computed values of the parameters.

Certain computational difficulties that may be associated with this problem may be listed as follows (Labadie, 1968):

1. Trial and error methods are usually employed for finding the unknown parameters. This practice greatly limits the practical use of more realistic models containing several parameters, since traial and error is computationally inefficient.

2. Numeric solution of ordinary differential equations on a digital computer requires that all of the initial conditions C_j $(j=1,2,\ldots m)$ be known to a reasonable degree of accuracy. In the identification problem, however, only C_1 or C_2 may be known. The other initial conditions must then be guessed, which may result in the generation of inaccurate solutions.

3. A third difficulty arises if the differential equation happens to be highly nonlinear. The usual methods of numeric integration and differentiation may not yield a solution, so that special algorithms must be designed for solving each particular model.

Depending on the initial knowledge of the process, the identification problem can be interpreted in the following two ways:

Identification: The process is considered as a "black box" about which nothing is known except the inputs and outputs.

Parameter Estimation: The structural configuration is known but the model parameters are unknown. For example, the model is known to be accurately represented by certain systems of differential equations, but the coefficients or parameters in these equations are unknown.

Since generally the parameters or coefficients cannot be measured directly and the measureable variables are generally the dependent variables of the differential equations, it is not a simple matter to identify the parameters.

The Predictor-Corrector Methods

Procedures which predit a result by one formula and then correct it by another are one of the most effective devices for solving initial-value problems, and are referred to as predictor-corrector methods.

Two of the most important methods were developed by Euler and Hamming (Carnahan, et.al., 1969). These methods will be briefly discussed in this chapter. Appendix C is devoted to explaining Euler's procedure and this chapter will give a block diagram (Fig. 5.1) of the modified Euler predictor-corrector method as used in solving the differential equations developed in Chapter IV.

The block diagram used for developing the computer program is shown in Fig. 5.1.

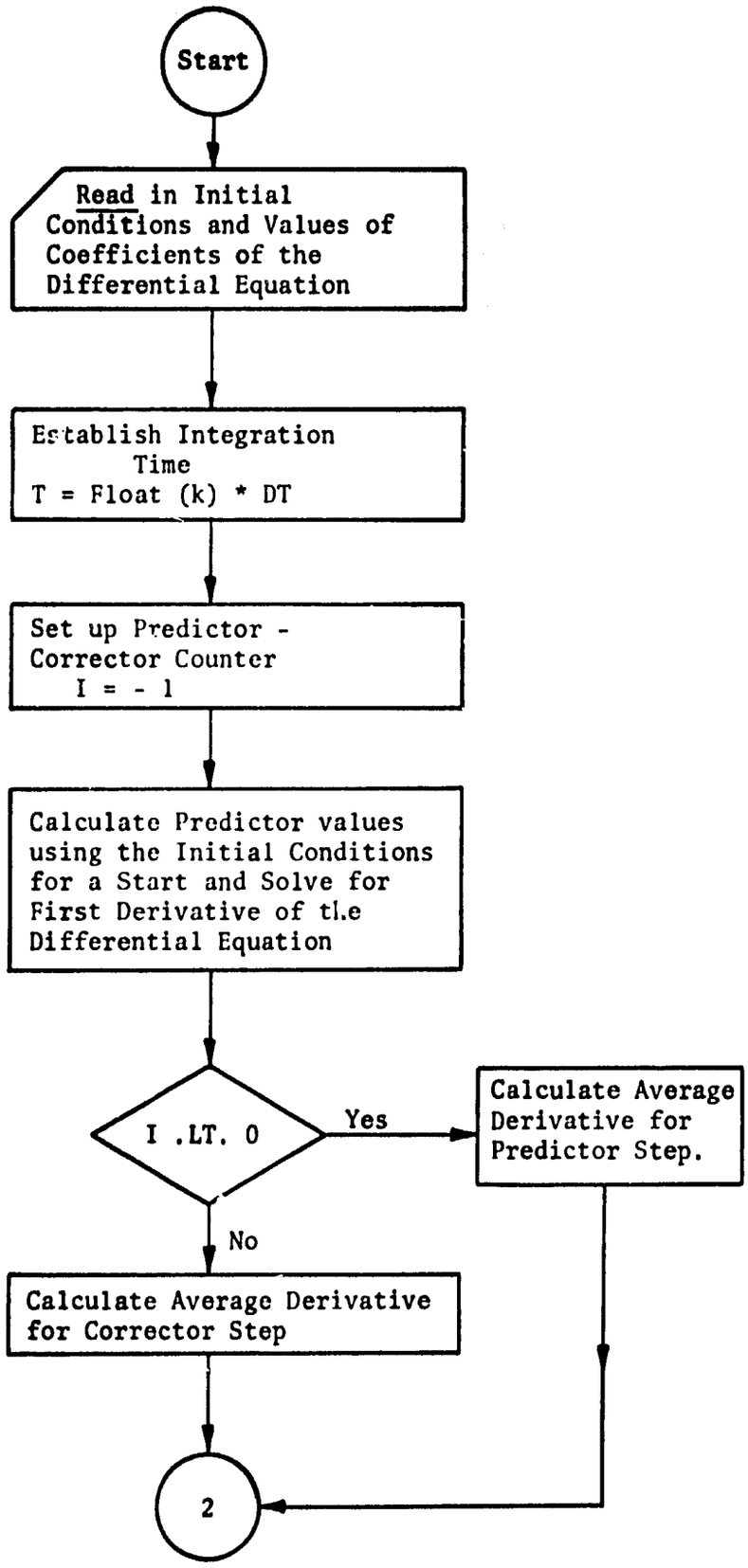


Fig. 5.1. Block Diagram of Modified Euler Predictor--Corrector Method as Used in Solving the Differential Equation.

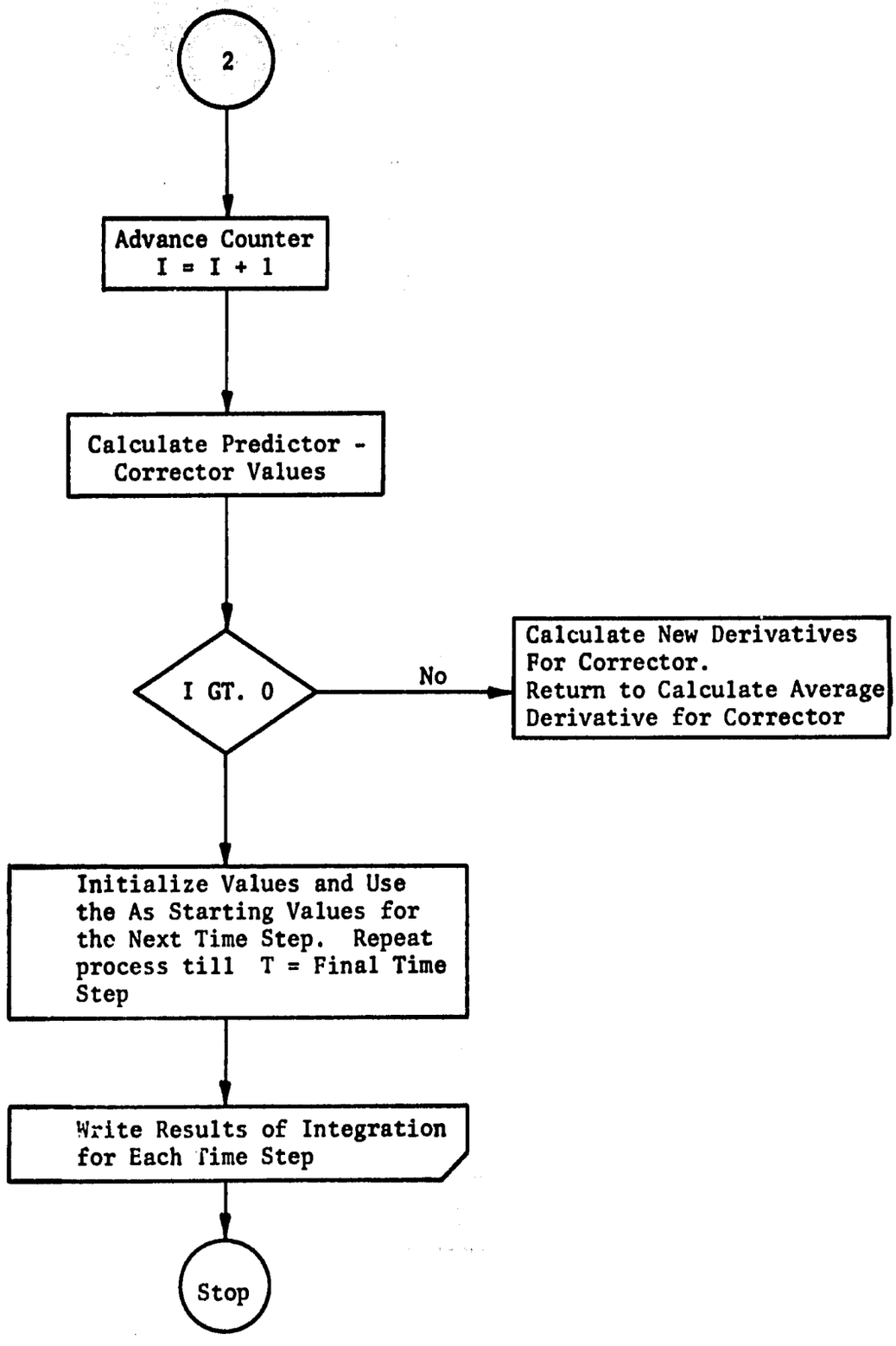


Fig. 5.1 Continuation of Block Diagram of Modified Euler Predictor-Corrector Method as Used in Solving The Differential Equation.

The advantages and disadvantages in using Euler's method may be listed as follows (Carnahan et.al., 1969; Conte and de Boor, 1972).

1. The modified Euler predictor-corrector method has been found to give better results than other predictor-corrector methods.
2. It has been found that the propagation from one iteration to the next can be reduced by using the modified method.
3. It is easier and simpler method to follow, compared to other predictor-corrector methods (e.g., Hamming's method).

Euler's method becomes more accurate as the step size decreases and the error tends to zero as step size goes to zero or becomes as small as possible. This could be time consuming and therefore expensive, but the author's experience with the method indicates that Euler's method is not as expensive as Hamming's method.

Steepest Descent Technique

The two main postulates for analyzing an unconstrained problem are (Hall and Dracup, 1970):

1. "The objective function can be evaluated by some means (including experiments, if required) for any initial trial policy $x_1, x_2, \ldots x_n$ and for the immediate vicinity of this policy, and
2. That there are no discontinuities in the objective function in the immediate vicinity of any initial or subsequent trial policy".

These imply continuity and differentiability, though continuity and differentiability are not required everywhere in the policy space, but only the path that is being followed.

Suppose $F(\underline{x})$ is the objective function, where $\underline{x}$ is the vector of optimal values, and $F(\underline{x})$ possesses a derivative. Then a steepest descent optimization procedure can be applied to this unconstrained

problem. This procedure dates as far back to Cauchy (1847), and continues to be one of the most important optimization techniques.

In the present work, the objective function may be represented as the sum of the squared error

$$F(\underline{x}) = \sum_{i=1}^{T} (QOBS - QCALC)^2 \tag{5.1}$$

where $\underline{x} \equiv QCALC(\alpha,\beta,\gamma)$

QOBS is the observed runoff values,

QCALC is the calculated runoff values from the mathematical equations or model, which is a function of...

α,β,γ which are the characteristic parameters of the hydrologic model to be calculated by the gradient method. The parameters of the first-order differential equation are considered here for illustrative purposes.

Let $F(\underline{x}) = F(x_1,x_2,x_3)$ where x_1, x_2, x_3 are the hydrologic system parameters α, β, γ, respectively, or

$$F(\underline{x}) = F(\alpha,\beta,\gamma). \tag{5.2}$$

The gradient vector $\underline{\nabla F}$ is a vector whose components are equal to the corresponding partial derivatives. In the three-dimensional case (since we have only three parameters: α, β, γ)

$$\nabla F(\underline{x}) \equiv \left[\frac{\partial F}{\partial \alpha}, \frac{\partial F}{\partial \beta}, \frac{\partial F}{\partial \gamma} \right] \tag{5.3}$$

or

$$\nabla F(\underline{x}) = \frac{\partial F}{\partial \alpha}\hat{i} + \frac{\partial F}{\partial \beta}\hat{j} + \frac{\partial F}{\partial \gamma}\hat{k} \tag{5.4}$$

where $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors.

This gradient $\nabla F(\underline{x})$ plays an important role in the seeking of local maxima or minima, where for a given function $F(\underline{x})$, whose partial derivatives exist, $\nabla F(\underline{x})$ is a vector which points in the direction of the greatest rate of increase (maximization) or decrease (minimization) of $F(\underline{x})$ (see Appendix A). Hence, any step in the steepest direction should be a vector whose components are proportional to the partial derivatives.

Suppose S is the step to be taken in a three-dimensional case then

$$S = \Delta\alpha\hat{i} + \Delta\beta\hat{j} + \Delta\gamma\hat{k} . \tag{5.5}$$

Then for S to be in the direction of $\nabla F(\underline{x})$ requires in general that

$$\frac{\Delta\alpha}{\partial F/\partial\alpha} = \frac{\Delta\beta}{\partial F/\partial\beta} = \frac{\Delta\gamma}{\partial F/\partial\gamma} = \cdots = \frac{\Delta x_i}{\partial F/\partial x_i} . \tag{5.6}$$

Equation 5.6 determines the direction, but not the magnitude of the step to be taken. The magnitude can be freely chosen. However, small steps greatly increase the total computation, while large steps could lead to overstepping to a less optimal policy than the initial point. Usually, some maximum allowable change in any one decision variable is set on the basis of judgement or experience, and the movement of the variable with the largest derivative is set equal to this limit. The remaining Δx_i are then computed on the basis of Eq. 5.6. More sophisticated methods (such as used in this study) allow for estimates of the optimum size of the step.

At any point $x_o = (\alpha_o, \beta_o, \gamma_o)$ the vector $\underline{\nabla}F(\underline{x}_o)$ is normal to the contour that passes through the point x_o, or the set of all $\underline{x}$ such that $F(\underline{x}) = F(\underline{x}_o)$ (Appendix A).

Figure 5.2 gives a block diagram of the steepest descent algorithm. The procedure starts at some initial point x_o where $x_o = (\alpha_o, \beta_o, \gamma_o)$ represents the initial guessed values of the parameters. The general iteration step n^{th} iteration $(n=1,2,3,\ldots)$ begins and $\underline{\nabla}F(\underline{x})$ is calculated for each iteration: $\underline{\nabla}F(\underline{x}_n)$. Since this is the steepest descent method, we have to move in the direction: $-\underline{\nabla}F(\underline{x}_n)$. We wish to find an optimal step size, λ_n such that λ_n minimizes $F(\underline{x}_n - \lambda\underline{\nabla}F(\underline{x}_n))$ over all λ . In a formal notation the minimization problem may be written as:

$$\lambda_n = \underset{\lambda}{\text{Min}}\ F(\underline{x}_n - \lambda\underline{\nabla}F(\underline{x}_n)) \tag{5.7}$$

The next set of values of the parameters are calculated. $\underline{x}_{n+1}$ is the vector of these parameters which form the next point, where

$$\underline{x}_{n+1} = \underline{x}_n - \lambda_n \underline{\nabla}F(\underline{x}_n) . \tag{5.8}$$

At each new set of values $\underline{x}_{n+1}$ for each iteration the convergence criteria should be checked. If $F(\underline{x}_n) - F(\underline{x}_{n+1}) \leq \varepsilon$ then convergence is attained. If, on the other hand, $F(\underline{x}_n) - F(\underline{x}_{n+1}) > \varepsilon$, then the procedure must be repeated.

The merits of gradient methods may be summarized as follows (Hall and Dracup, 1970):

> "Despite the limitations, gradient methods are very powerful in that improvement in policy is guided even if the optimum cannot be feasibly calculated. Gradient methods are extremely useful where additional research or experimentation is required. In many such situations, the use of gradient methods may be the only feasible way to proceed if costs of analysis and research are to be kept commensurate with the objectives of the optimization."

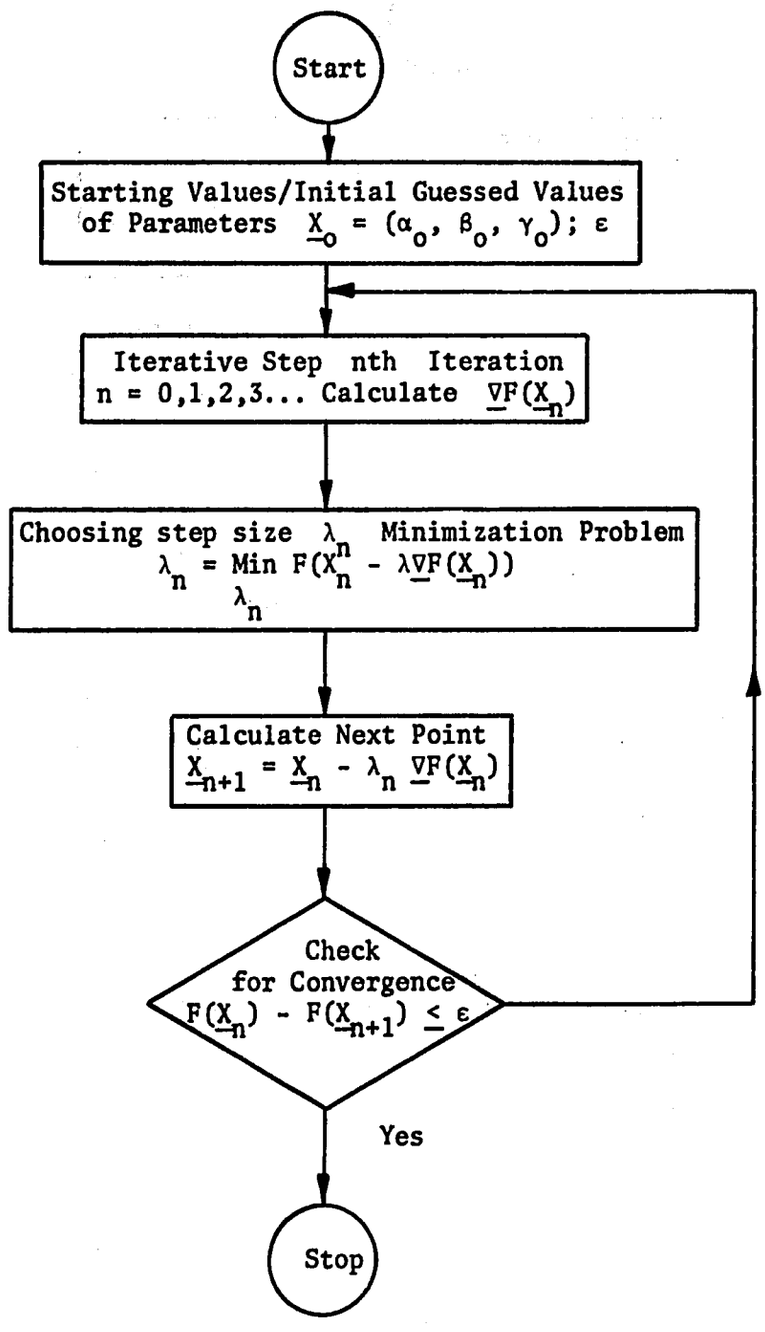


Fig. 5.2 Block Diagram of the Steepest Descent Algorithm.

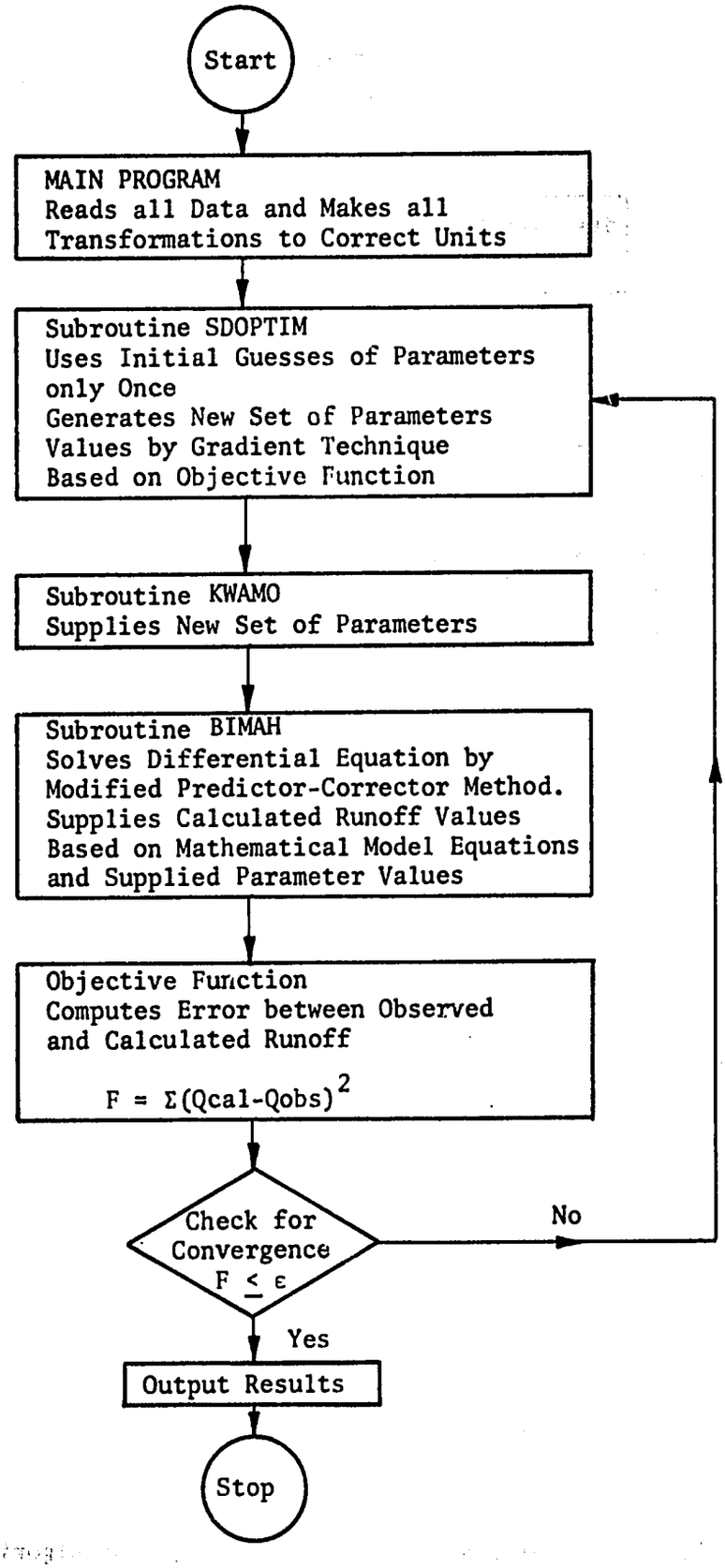


Fig. 5.3 The Block Diagram of the Main Program using Gradient Technique and Modified Euler Predictor-Corrector Method.

The Steepest Descent and the Modified Euler Predictor-Corrector Methods

The Euler Method is combined with a gradient technique whose main purpose is to find better estimates of the coefficients of the differential equation. The combined block diagram for the joint operation of the Euler method and the gradient technique is illustrated on Fig. 5.3.

SDOPTIM is the name for the Steepest Descent Technique which uses the value of the objective function from Euler's method to find optimal estimates of the coefficients of the differential equation or the optimal values of the unknown parameters set $\underline{x}_n^* = (\alpha_n^*, \beta_n^*, \gamma_n^*)$.

The various functions of the subroutines have been outlined earlier, and therefore no detailed explanations will be given here.

The Quasilinearization Technique

Quasilinearization together with least squares optimization represents another technique which is used via systematic search for solutions or identification of the nonlinear hydrologic system response model. The least squares optimization refers to the objective function which seeks to minimize the difference of the observed and the calculated runoff from data and mathematical model respectively. Quasilinearization is capable of solving large class of nonlinear differential equation, and giving a rapid convergence (if it converges) to the optimal values of the parameters of the model (or the coefficients of the differential equation).

Quasilinearization involves linearization of the nonlinear algebraic equations and sequence of functions in a series of iterative scheme to arrive at the solution of the original nonlinear differential equation. It does not need an accurate set of initial conditions or

boundary values since they are computed along with the parameters. Good guessed values of the initial conditions, however, facilitate convergence. From experience, this technique can be unreliable and can diverge rather than converge to the desired optimum.

Instead of being solved directly, the quasilinearization technique for a nonlinear differential equation is solved recursively by a series of linear differential equations. The linear equation is obtained by using the first and second terms in the Taylor series expansion of the original nonlinear equation. This technique is generalized Newton-Raphson formula for functional equations (Appendix B). However, since the unknowns which are to be obtained are functions and not fixed values or roots as in the Newton-Raphson method, both the computational and theoretical aspects are more complicated.

The quasilinearization technique not only linearizes the nonlinear equation but also provides an iterative scheme (through use of Taylor's series) which in general converges more rapidly than gradient methods to the solution of the original nonlinear equation, as long as initial guesses x_o are adequate. In general, for most practical problems, these initial guesses can be obtained from engineering experience and intuition. It is suggested, from the experience of this investigator, that other methods which can generate accurate approximations of the initial conditions should be used in conjunction with the quasilinearization technique, especially when this technique is not converging well. This is to say other methods can be used to find good initial guesses for starting the quasilinearization technique. The previously discussed gradient method may be one.

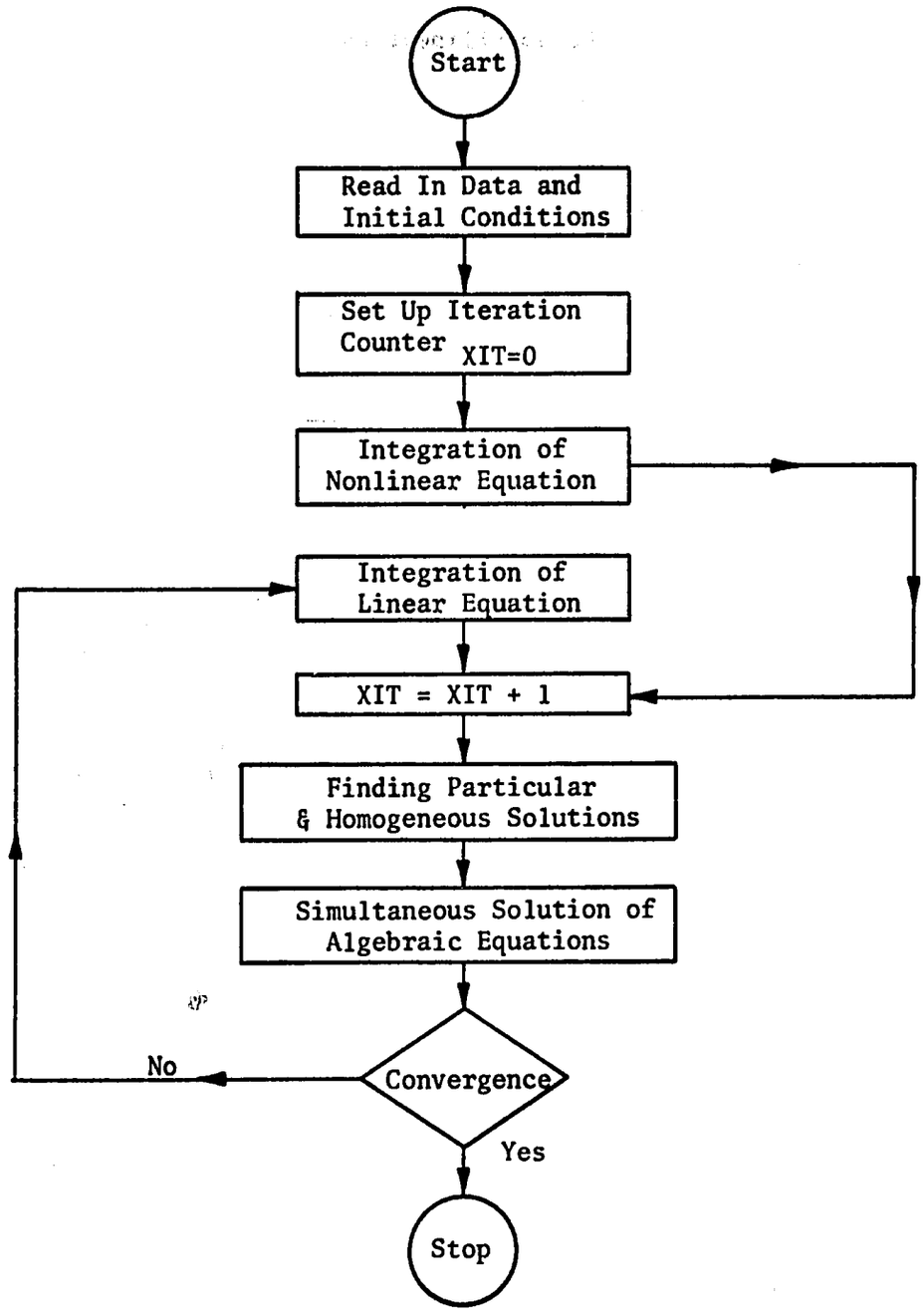


Fig. 5.4 Block Diagram of Quasilinearization Algorithm.

For a more complete treatment of the quasilinearization algorithm the reader should refer to Labadie and Dracup (1969). Figure 5.4 gives the block diagram of the quasilinearization algorithm used for the optimization on the computer.

Advantages of Quasilinearization

As pointed out by Labadie and Dracup (1969) the advantages of quasilinearization can be listed as follows:

1. "A rapidly converging iterative scheme is developed for the unique or optimal values of the parameters in a systematic fashion.

2. A complete set of initial conditions need not be specified. Unknown initial conditions are determined along with the unknown parameters.

3. Standard methods of numeric integration can be easily applied to solving highly nonlinear models. Special algorithms are not needed.

4. Quasilinearization is generally applicable to a large class of complex models and is highly compatible with current digital computer capabilities."

It is also important to point out that the procedure is advantageous because it involves neither curve plotting nor graphical matching; and also because of its general applicability.

Another important advantage of this technique is that if the procedure converges, it converges quadratically to the solution of the original equation. Quadratic convergence means that the error in the $(n+1)^{th}$ iteration tends to be proportional to the square of the error in the nth iteration. The advantage of quadratic convergence, of course, lies in the rapidity of convergence. Bellman and Kalaba (1965) depicted quadratic convergence as

$$|\underline{A}^{(n+1)} - \hat{\underline{A}}| \leq K|\underline{A}^{(n)} - \hat{\underline{A}}|^2 \tag{5.9}$$

where $\hat{\underline{A}}$ is the vector of the optimal values of the parameters.

Disadvantages of Quasilinearization

The inherent weaknesses of quasilinearization can be listed as follows (Labadie and Dracup, 1969):

1. "Initial approximations must be within or at least close to the convex region surrounding the optimal solution or convergence is not attained.

2. If convergence does not result for a particular set of initial approximations, it is not possible to determine systematically a better set of initial approximations from these results."

A possible remedy for this situation lies in the use of gradient methods for generating initial approximations. These methods could perhaps be effectively used for starting a quasilinearization scheme. It is, however, important not to underestimate the potential and capabilities of the gradient technique. Chapter VI reports on the experience with these two techniques. The superior technique is determined on the basis of how fast the algorithm converges to the optimal solution; whether the algorithm is easier to handle and understand, and whether the algorithm is reliable.

Modified Euler and Hamming's Predictor-Corrector Methods as Used with Gradient Technique and Quasilinearization Respectively

Euler's method (or Hamming's method) used with the gradient method has the capability of solving a nonlinear equation directly, whereas the Hamming method (or Euler method) used with quasilinearization technique solves a nonlinear equation indirectly by first linearizing the nonlinear equation into its linear form. Euler's method is a "self-starting" method. It requires the value of the dependent variable Q

at only one point to start the procedure. Hamming's predictor-corrector method, however, is by itself not self-starting; that is, the functional values at a single previous point are not enough to get the functional values ahead. Therefore, to obtain the starting values, a special Runge-Kutta procedure followed by one iteration step is added to the Hamming's predictor-corrector method.

Euler's method (as indirectly pointed out in the first paragraph) can solve directly a second order ordinary differential equation, whereas the purpose of Hamming's modified predictor-corrector method is to obtain an approximate solution of a general system of first order ordinary differential equations with given initial values.

Chapter VI

COMPUTATIONAL RESULTS

Description of the Todzie River Basin in Ghana

This study is aimed primarily at a river basin in a developing country. The problems facing the developing countries with respect to basic research, as discussed previously, include: lack of interest, economic constraints data limitations, lack of manpower and expertise to tackle certain important research programs, and research as a whole may not be a national priority. Other constraints include political, social, traditional, institutional and legal restrictions.

The Todzie River Basin (Fig. 6.1), it is believed, can be viewed as a typical river basin in a developing country and is therefore subject to the above problems and constraints. Most of these problems and constraints are confronted with in this research and these may be therefore referred to as some of the major objectives of this research. Specifically, these problems and constraints as confronted and tackled may be defined in the current research efforts as:

1. Inadequate data available for this study are used in the best possible way.
2. Certain important variables like evapotranspiration have not been measured. To overcome these problems a conceptual and a mathematical model, which are rigorous and realistic enough; simple to handle conceptually and mathematically, respectively, have been derived. The main aim is to be able to predict evapotranspiration values.

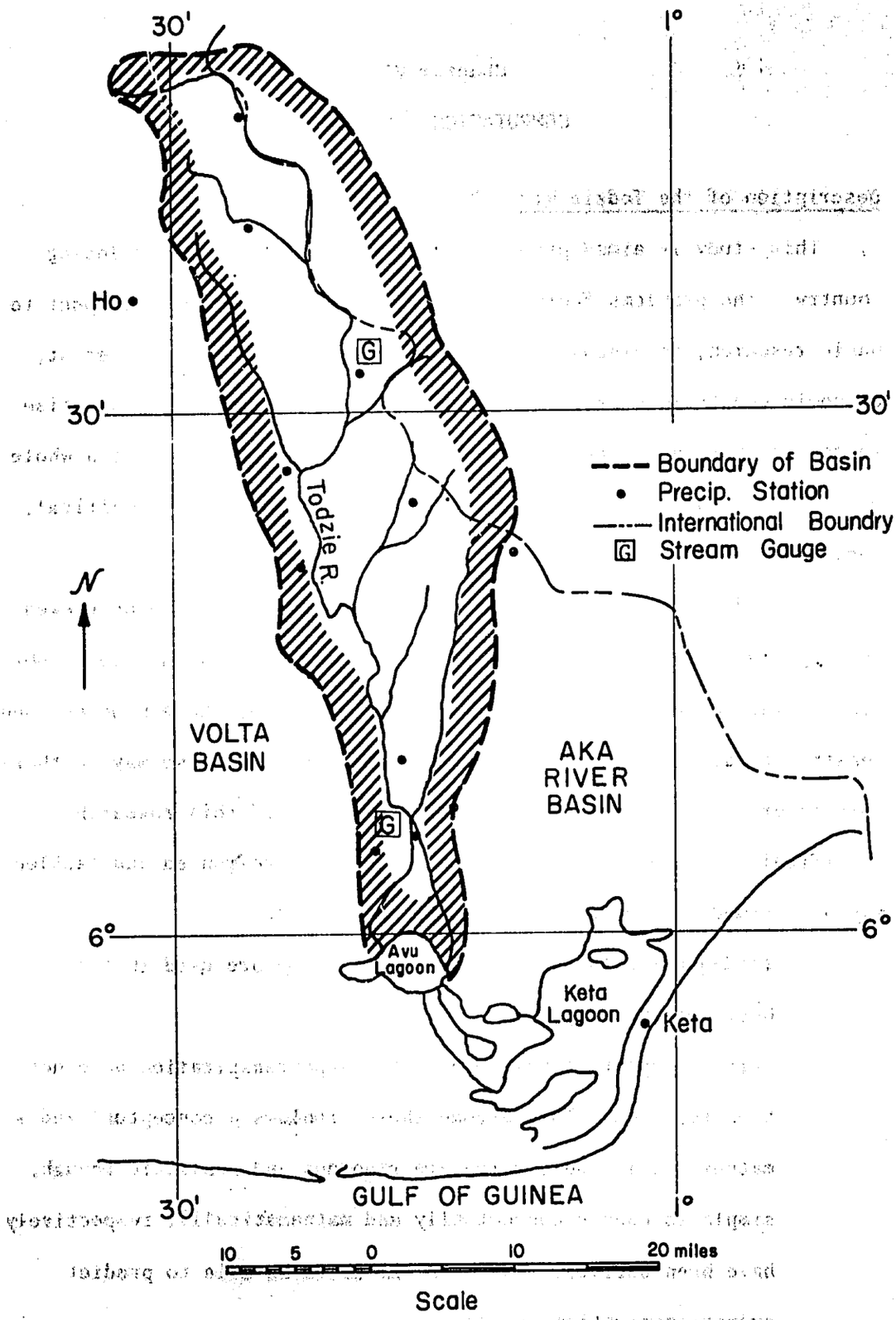


Fig. 6.1. Todzie River Basin

3. The solution technique adopted should be fast and inexpensive with respect to computations by the computer.

The Avu-Keta Region comprises an area of 475 square miles, or 304,000 acres, bounded on the west by the Volta River, on the north by Sogakofe to Denu Road, and on the south and east by the Gulf of Guinea (Fig. 6.1). Todzie Basin is part of the Avu-Keta Region.

The 1960 population in the Avu-Keta area was approximately 135,000, with 51,000 people concentrated in the Keta district (the most densely populated area). Ghana's population is about 8 to 9 million.

The water levels of many lagoons of the area fluctuate substantially. In the wet season, flood waters which enter the lagoons from several sources are trapped because no efficient outlet to the sea is available. Because of the poor drainage and the flat topography, a vast area, including agricultural land and homes, is frequently inundated.

In the dry season, high evaporation rates reduce Keta Lagoon to low levels. The minimum level to which the lagoon has receded is not known.

There are three sources of flooding in the Avu-Keta area. They are:

1. High runoff from the Todzie River to Avu Lagoon, and from there to Keta Lagoon.
2. High runoff from the Aka, Belikpa, and Denu Lagoon areas directly to Keta Lagoon.
3. High flows on the Volta River, part of which enter the southern portion of the area periodically and contribute to the flooding problem around Keta.

Because of the flat topography and the poor drainage of the Avu-Keta area, the flood waters recede very slowly.

The flooding as a result of high runoff from the Todzie, Aka, Belikpa and Denu Lagoon areas occurs primarily in June-July, with secondary flooding, which is of lesser consequence, in September-October.

The flooding problem in Avu-Keta is a major national concern. Since funds and manpower are rare 'commodities', research effort to analyze the rainfall and runoff characteristics of the basin is very important. Unfortunately, data for most of the rivers are unavailable, except for Todzie. Todzie, however, happens to be one of the important, if not the most important, rivers in the area. Hence, Todzie has been chosen for this study, even though there are many problems and constraints, as previously discussed.

Hydrology of Todzie Basin

The Todzie River drains an area of approximately 840 square miles before it enters Avu Lagoon. Monthly mean precipitation in inches for the Basin was determined by the Thiessen Method.

Average annual runoff from the Todzie River is approximately 315,000 acre-ft. This is based upon the available period of record from March 1957 to February 1969. The annual runoff during the period of record has varied from as little as 62,000 acre-ft to as high as 893,000 acre-ft. Figure 6.2 gives a graphical representation of observed rainfall and observed runoff in inches (1958-1967).

Two floods of significance have occurred during the twelve years for which records are available for the Todzie River. These floods

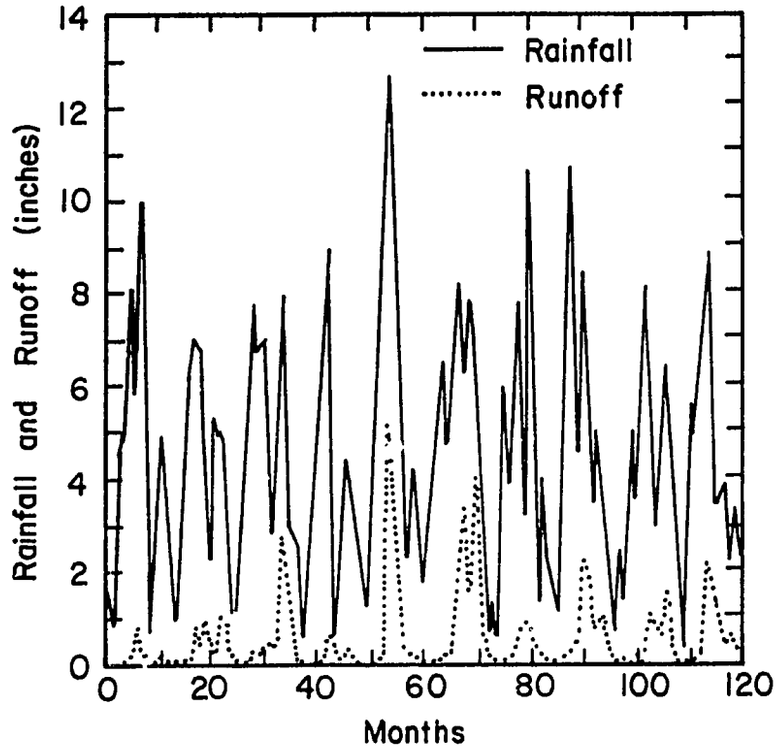


Fig. 6.2. Observed Rainfall and Runoff for Entire Period of Record

occurred in 1963 and in 1968. In August, and again in October 1963, the monthly runoff exceeded 150,000 acre-ft, which is nearly one-half of the twelve year average-annual amount. The peak inflow was estimated to be about 6,200 cubic feet per second. Unusual rainfall conditions occurred again in 1968, during which very high runoff from the Todzie River was recorded in July, August, September, and October. The highest runoff occurred in August when the flood volume was 231,000 acre-ft. The combined four-month runoff was greater than twice the average annual amount. The peak magnitude of the 1968 flood could not be determined because of a change in the rating in 1968 at both the main Todzie channel and Agblagbole locations. However, it is estimated that the 1968 peak flow exceeded the 1963 peak flow by ten or twelve percent.

The change of runoff with time from month to month for the entire record used for this research (1958-1967) is plotted (Fig. 6.3). Figure 6.3 shows the extreme variability of the runoff from month to month and year to year.

Data Set--Rainfall and Runoff

Monthly rainfall and runoff data for the Todzie River Basin in Ghana (Figs. 1.1 and 6.1) are utilized in this study. The monthly rainfall data are given in inches and Thiessen's method was used to compute an average rainfall for the river basin. The rainfall data were a continuous record from 1958 to 1967.

The runoff data were given as total monthly runoff from the Todzie River in thousands of acre-ft. These values have been converted into inches for this research. The runoff data are also continuous and cover all the months from 1958 to 1967.

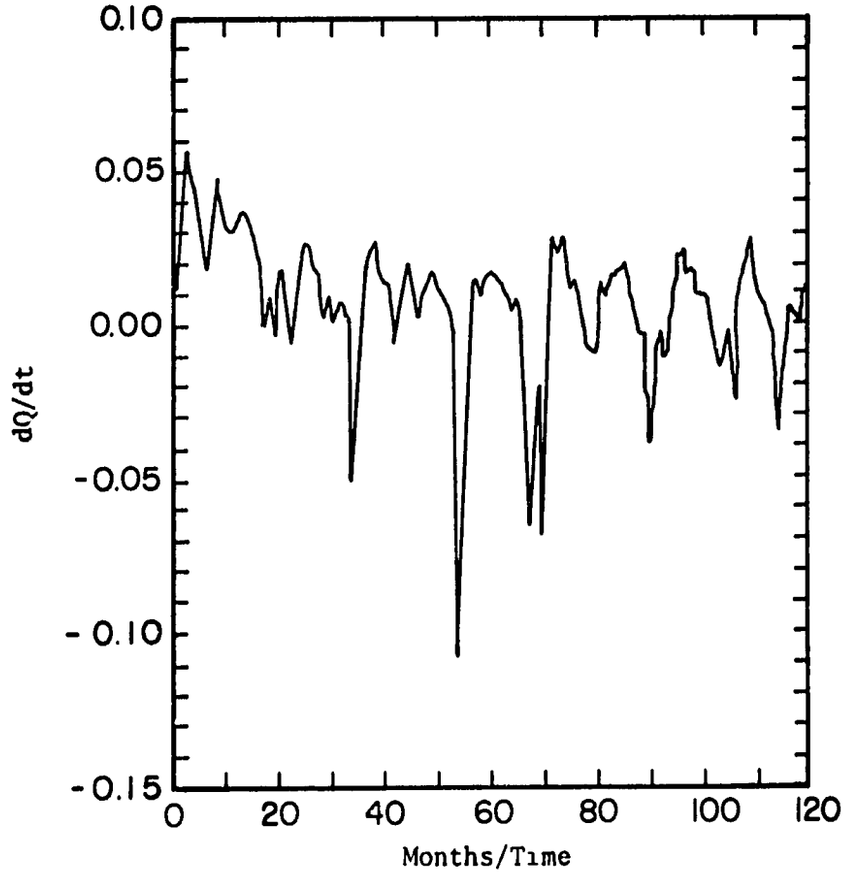


Fig. 6.3. Change of Discharge with Time $(\frac{dQ}{dt})$ as a Function of Time.

No other data are available. Therefore, this work is intended to illustrate mainly the importance of what can be done with what is available, namely, monthly rainfall and runoff data for 10 years only.

It has been pointed out earlier that 12 years of data are actually available. The two years of data deleted are those years that data were not continuous or there was a change in locations of the rating or measuring stations. No new errors in data will be entertained in this study by computing data that are missing. Data used are typical of any data collected anywhere, especially in a developing nation. Therefore, the data should be viewed with some skepticism with respect to the more questionable measurements. Of course, it does not mean good data cannot be collected. It is basic to the objectives of this research, however, to use whatever data are available.

The current data have two flood peaks (besides the 1968 flood) which are about the highest recorded for quite awhile. These two peaks (which are obviously not part of the general trend or behavior of the storms in the area) can affect the predicted values of runoff, since these are unusual occurrences and do not bear resemblance to the normal behavior of the characteristics of the storms in the area.

Results Obtained by Quasilinearization

Quasilinearization as an optimization technique has been mentioned in earlier chapters as a good technique <u>if it converges</u>. There are several reasons why it may not converge. These may include:

1. Poor initial guessed values of the parameters.
2. Inefficiency of the procedure to tackle certain types of differential equations effectively.
3. Poor data.

Several different initial guesses were used for this problem, and these included initial conditions required for the gradient technique. The gradient technique gave good results for the objective function. The fitting error F was consistently smaller than that obtained by quasilinearization, and tended to decrease with better guessed initial conditions. The smallest value of F obtained using quasilinearization was 14.51. Other experiments diverged to very large error values. The optimization with the quasilinearization technique was repeated many times but no better values than that above were obtained. The shortest computer time for 6 iterations was 99 seconds.

Therefore, the quasilinearization technique was disqualified from further use as far as this research is concerned, since it consistently gave divergent values of error F and wrong estimates of the parameter values.

The Effect of Step Size in the Gradient Technique on Parameter Values

Several experiments were initially performed to find out what the gradient technique could do, and how fast and what types of step sizes in the steepest descent direction are good enough for use when the optimal region is being approached.

From various experiments, it has been experienced that the gradient technique used here is very slow in converging to the optimal solution. The stress here, however, is to illustrate the effect of step size on the parameter values and to determine which step size per iteration the gradient technique has to take when it starts to move in the steepest direction. The values obtained in these experiments are not necessarily the optimal ones and are not necessarily intended to be. The

working step sizes required for the optimization process are the main concern at this point, since these are necessary guides for arriving at the solution of the minimization problem.

The second set of values on each row represent the smallest possible error encountered after taking the steepest descent step and direction. A wrong step taken could easily lead the process into divergence. It appears that the values immediately begin to diverge more rapidly with the number of iterations and are directly proportional to the step size indicative of the size of the step, Table 6.1.

The smaller the step size (1/ss) the longer it takes to converge. When the step size is so small that there is rarely any significant changes in F as in (1/ss = 0.5 to 1/ss = 0.001) the process may continue for a long period and may probably never converge.

The first number shown in the iteration column represents the iteration number for which the error F was first smallest. The second number indicates the iteration number for the smallest possible F after the first iteration. As the step size decreases (1/ss = 0.01 or less), we find that on taking the step a better estimate of F can be found, whereas, the reverse situation is experienced when step size increases (1/ss = 0.1 or more). The first iteration, for all the experiments, began with the same initial values of the parameters. The second or more iterations result in changes in values of F and the parameters. The larger the value of ss (or smaller 1/ss), the less the tendency for abrupt divergence on taking a step in the steepest descent direction. There is almost no divergence from the minimum when ss = 10 and higher.

TABLE 6.1

CHANGES IN INTEGRATION TIME STEP AND EFFECT ON CONVERGENCE ON THE GRADIENT TECHNIQUE (STEEPEST DESCENT)

Step Size Multiplication Function	Function F	Computer Time in Secs.	Solution Values of Parameters A1=α	A2=(β x10^{-2})	γ	No. of Iterations
0.001$^{ss=100}$	7.900235	21.806	0.0643003	0.09750323	1.102187	7
	7.899377	43.048	0.06368861	0.09731723	1.104695	18
0.020$^{ss=50}$	7.900235	22.289	0.0643003	0.0975032	1.102187	7
	7.899039	43.966	0.06369321	0.09728996	1.103431	17
0.100$^{ss=10}$	7.900235	22.06	0.0643003	0.09750323	1.102187	7
	7.899248	47.448	0.06369024	0.09730755	1.104246	19
0.200$^{ss=5}$	7.900235	21.835	0.0643003	0.0975032	1.102187	7
	7.899661	47.14	0.0636853	0.0973363	1.105584	19
0.400$^{ss=2.5}$	7.900235	21.751	0.0643003	0.09750323	1.102187	7
	55.57401	23.858	0.06857726	0.07235782	-0.05267640	8
1.00$^{ss=1}$	7.900235	22.187	0.0643003	0.09750323	1.102187	7
	0.1036727E+12	23.622	0.07499270	0.03463971	-1.1801671	8
20.00 $^{0.05}$	7.90235	21.920	0.0643003	0.09750323	1.102187	7
	Diverge at #8 before any value is calculated					8

The time for the same number of iterations for the same values of F, α, β, γ decreases with increasing Δx (where $\Delta x = 1/ss$). Table 6.2 represents the results of the parameter values used in the optimization process. Divergence increases as Δx increases. It is important to realize from Tables 6.1 and 6.2a how little the parameters change for various iterations. This shows that the parameters are highly sensitive and the least amount of changes in any parameter value can result in large changes in F. The gradient technique proceeds rather slowly in changing substantially the various parameter values. It should, however, be stressed that the gradient technique is generally reliable and can at least give some working estimates of the parameters. The better the values of the parameters obtained depend on how good is the chosen value of Δx; especially on the competence and experience of the computer programmer.

The value of α appears quite high at this point, considering the physical reasoning of the basin and the rainfall and runoff values. It should be remembered throughout this study that the sensitivity parameter α gives an estimate of evapotranspiration. Therefore, more experiments are to be performed using these values (Table 6.2) as the initial conditions.

Determination of Upper and Lower Bounds of Parameter Values

The problems are originally solved with parameters α, β, γ, and ϕ (ϕ appears only in the second order differential equation) being time dependent. Several different starting values were tried in order to find the upper and lower limits of the values of the parameters. By inspection of the results certain range of values of the parameters

TABLE 6.2

INITIAL VALUES OF PARAMETERS TO BE USED FOR OPTIMIZATION PROCESS

Step Size Multiplication Factor	SS	Computer Time	Min. Value of F (After Step)	α	β	γ
0.001	100	T_T=43.05	7.899	0.6544	9.3019	1.1047
0.020	50	T_T=43.97	7.899	0.6546	9.9376	1.1034
1.10	10	T_T=47.45	7.899	0.6546	9.8580	1.1042
-0.2	5	T_T=47.14	7.899	0.6543	9.2928	1.105584
0.4	2.5	T_T=23.86	55.57	0.9478	-262.812	-0.0528
1.0	1.0	T_T=23.622	0.1036727	2.1649	- 24.4612	-1.1802
20.0	0.05	ABRUPT DIVERGENCE ON TAKING A STEP IN THE STEEPEST DESCENT DIRECTION.				

gave good results or fits of the computed runoff to the actual observed runoff data. This range of values of the parameters was used for bounds, and the optimization procedure repeated for yet better values of the parameters.

There are two main sets of experiments which are performed at each relevant stage of the study. It is necessary to mention these in passing without presenting the results, since they are only intermediate results to the actual products of the experiments. The first type of experiments involves the investigation of the upper and lower bounds of the parameter values. These have already been mentioned, except to add that these upper and lower bounds were incorporated in the computer programs and the range of values of the parameters were the end products (e.g., Table 6.5). The second set of experiments concerned the informal determination of the global minimum.

In order to be sure the value of F obtained is the global minimum many initial starting values were tried; these starting points were all different and of various magnitudes, etc. The purpose of this was to find out whether the objective function F (the least square error between QCALC and QOBS) had the same or almost the same value. The results found for over 15 trials indicated the same or approximately the same value of F. Thus the global minimum was informally found and the optimal values obtained were the best estimates of the parameters within the limits of experimental and other errors.

After several trials, certain average values of β, γ, and ϕ could be used without any appreciable loss of accuracy. The parameters β, γ, ϕ are therefore considered as time-invariant. As a matter of fact, these average values were found not to differ much from the lower

and upper bounds of these parameters. The time dependency of β, γ, ϕ was initially investigated as mentioned earlier in order to confirm the time invariance of these parameters. These preliminary investigations were not necessary to report here since these do not add any new knowledge to the study.

In the experiments that follow these ideas are fully incorporated. Initially, every experiment was performed with all the parameters treated as time dependent. On convergence or near convergence, it was found that the parameters, except α, did not differ appreciably from each other. All the parameters, except for α were assumed constant and equal to their corresponding seasonal averages. The parameter α was treated as a time-dependent parameter. At certain points in the data which gave bigger error than stipulated for convergence ($F \leq 0.001$) because of a higher or lower value of the average parameters the next best estimate of the parameter values were used. In such a case (and there were only 3 out of 120 cases), the objective function obtained from the sum of the least square differences of the observed runoff values and the runoff computed using these parameters gave an error F which was not greatly different from the convergence criteria.

Many experiments were performed for each of the monthly and seasonal models in order to determine how variable the parameters could be, where their upper and lower bounds were, and which were their optimal values.

<u>The objective function and parameter α</u> - The objective function, as has been explained, is determined by F, the least squares error function. It is necessary to specify how much error is tolerable for

any iteration. If this minimum tolerable error value is exceeded then the iterations could be stopped temporarily so that better initial values could be used so as to arrive at a better value of the objective function.

Most weight is placed on α because it is physically explainable and can be used with the greater amount of accuracy than the other parameters. This does not mean that when α is constrained $(0<\alpha<1.0)$ the other parameters could assume any values. In fact, the other parameters are constrained when their corresponding seasonal average values were used. As a further check on what values other parameters could assume, it should be realized that $\alpha = \alpha(\beta,\gamma,\phi,Q,P,t)$ and therefore any unacceptable or bad values of any of β, γ, P or Q will reflect in the values obtained for α which may physically not be justifiable.

If after trying all the possible initial conditions for α, β, ϕ, and γ for a given F and we do not have any improvement on F then the reasons could be one of the following:

1. Questionable data value for that particular integration step.
2. Model is insensitive for certain range of values of observed data. It is therefore necessary to check other data which are in the same range of values and check its corresponding F in order to determine the differences in F between the two similar data values.

The Monthly Models

This was constructed such that the initial conditions determined from many trials were used only once to start the iterations. The

integrations at all data points used this set of initial conditions for the first iteration. The least squares error summed for the entire record was found from the observed runoffs and computed runoff due to integrating the mathematical equations. The gradient technique uses this error value to find a new set of parameters which become the starting values of the second iteration and the process begins again. This continues till convergence is attained. The first order differential equations with nonlinear storage-discharge relation of the mathematical model was used for these investigations.

The monthly model was developed and optimized using two approaches. The main aim in doing this was to investigate (1) which of the two approaches gave the best possible values of the parameters, where the value of error F the objective function was the criteria for judgement. The smaller the value of F the better the approach; (2) which method was faster with respect to computation time--again the economics of the problem is of major importance and is one of the objectives of this research; (3) which method is more efficient--this gives a graphical representation of the computed runoffs by both approaches as against the observed runoffs. The closer the fit of one approach to the observed runoff curve the more efficient the particular approach. (4) Whether one is as good as the other in which case one can serve as check on the other and thus further indicates how good the chosen approach is.

Results of monthly Model 1 - This involves setting up the initial conditions and the gradient technique uses only this set of initial conditions and solves the differential equations at all the points

via the predictor-corrector scheme. This is the first iteration, where at the end of iteration 1 we have 120 (number of data points used) values of calculated runoffs corresponding to each of the 120 observed runoffs. The least squares error F is found and this is utilized by the gradient technique to find new step and direction. The new starting values of the parameters are found by the gradient method and those become the initial conditions of iteration 2. The process is repeated. Altogether 10 iterations were allowed in order to determine the best value of F.

Results of monthly Model 2 - The optimization of the first order nonlinear storage-discharge model was started with initial conditions applied only to the first point, instead of using this one set of initial conditions for all the points. The computer programme was modified to optimize the parameters at the first point (and other subsequent points) and the objective function F found together with its parameter values. Ten iterations were done for this first point in order to find the smallest F whose corresponding parameters become the initial conditions for the second point. The process was repeated for each point (10 iterations per point) till all the 120 points were covered. The difference in approach between Model 1 and Model 2 is that Model 1 makes 1 iteration for all 120 points and repeats the process 10 times; whereas, model 2 makes 10 iterations for each point and repeats for all the 120 points.

Model 2 is essentially a new technique and should therefore not be misconstrued as violating any set procedure of optimization with respect to the predictor-corrector scheme. Basic understanding of the predictor-

corrector scheme and computer programming made this new approach possible. Still this approach should be proved worse or better than model 1 based on the 3 criteria set forth in this section: (1) smaller value of F, (2) faster, and (3) more efficient. The proof of these points are illustrated on Table 6.3.

TABLE 6.3

COMPARATIVE RESULTS OF MODEL 1 AND MODEL 2 OF THE MONTHLY MODELS

Model	Average Values of Optimals			Cumulative Errors of All Points F	Computation Time CP
	α	β	γ		
1	0.0659	9.2153	1.11333	0.4098	16.189
2	0.0660	9.2084	1.11229	1.9618	13.529 sec

Results given on Table 6.3 clearly indicate how good either model is. Both models gave very close average values of the parameters indicating the reliability of these values and these further illustrate an informal proof of global optimality of the parameter values with respect to the values of the objective functions. Model 1 seems to give a slightly better fit to the actual data, though Model 1 takes more computer time than Model 2.

It is therefore recommended that both methods be used as check for each other and to find out informally whether the average values of the parameters used are indeed the global solutions or just the local solutions of the optimization problem. At least, this can informally establish global or local optimality.

The Seasonal Models

The following investigations were centered on the seasonal investigations of the hydrologic system characteristics, the behavior of the characteristic parameters of the system, the investigation of the usefulness of some of these parameters and derivation of a simple but an effective model to calculate evapotranspiration. Again, 10 years monthly rainfall and runoff data were used. These are all that existed and therefore, have to be used in the best possible way bearing in mind some of the basic objectives and principles underlying this research:

1. Model that uses only the main measurable data (rainfall and runoff data) in a watershed.
2. Deriving a mathematically and physically sound model, which is simple to understand but very effective in solving the problem.
3. A model which can further give estimates of values of other variables which for reasons of economics, politics or both (plus others: accessibility to area, etc.) make it impossible to measure these variables even though it (they) may be very important. Evapotranspiration is one such very important variables that need be known for agricultural or water resources projects--for planning, management and development. This research has developed a model for calculating monthly, seasonal and annual evapotranspiration.

(4) A model that is fast to solve with the computer--again the economics of the problem--without much or any loss for efficiency .

These underlying objectives are important for any country but even more important for countries of developing economy which are referred to here as developing countries--which are so referenced here due to insufficiency of funds or capital, insufficiency of manpower and lack of interest in hydrologic investigations. The models have been derived on a generalized basis--applicable to any country--and tested by using data from Ghana, the author's home country.

The importance of linear, nonlinear, dynamic considerations in the seasonal models - A comparative study - Many investigators (Chapter II) have used various assumptions in deriving their mathematical model of the hydrologic system. Some have used linear storage-discharge relations (an unrealistic assumption); others used nonlinear storage-discharge relation. In conjunction with one of these investigators derived first order differential equations; others derived a second order differential equation.

To show the merits of either one or the other approach, this current research which is also partly interested in these problems and some of these assumptions, has attempted to use nonlinear storage-runoff relation; and first and second order differential equations respectively and vice versa (i.e., two approaches) in deriving its own differential equations describing the hydrologic system. These equations are different from any other, but effectively describe the physical behavior of the river basin.

Planners cannot go through the elaborate investigations of the merits of each of the above appreaches and may more often than not pick up any method available (depending on what the original investigator said about his model, usually not what he did not say), without

knowledge of the capabilities of the particular model in solving the problems they may be faced with. Therefore, it is important to illustrate the potentialities and deficiences of each model presented.

The parameters contained in the first and the second order differential equations are generally different (except α, which is maintained common to both). Both equations were derived using similar names for the parameters $(\alpha,\beta,\gamma,\phi)$ but with different meanings.

Prediction of Evapotranspiration and Optimal Parameters

The annual total evapotranspiration was calculated by using the existing theory and by the developed models of the first and second order nonlinear differential equation. The reason for using the existing theory is to serve as a check to the developed model which is also aimed at predicting evapotranspiration.

Experiments were performed on five models. Models 1 and 2 are the four seasonal models of the first and second order nonlinear differential equations respectively. Models 3 and 4 are two seasonal models of the first and second order nonlinear differential equations respectively. The fifth model considers the linear storage-discharge relationship. All these series of experiments were performed in order to find the best estimates of evapotranspiration, to serve as check on each other and to determine which is a better and more representative model for the physical system. The final results are compared with those obtained from theory and Ho, a town which lies just outside the case study area. Ho is the area nearest to the basin, for which the investigator has evapotranspiration data. One of the objectives of this study is to use whatever is available in a best possible way.

This is definitely not a handicap, since this type of problem prevails in many developing nations.

The Concept of Change in Storage with Time and How It Affects Predicted Evapotranspiration Values - Before presenting the results obtained from the first and second order differential equation models it is necessary at this point to perform some few experiments to illustrate the importance of change in storage with time in the basin. The essential points and assumptions in the two models will be presented first and then the results of the three experiments on change in storage values (Table 6.4).

TABLE 6.4

10 YEAR AVERAGE OF CHANGE IN STORAGE DUE TO SHIFTING STARTING AND ENDING POINTS OF THE DATA BY ONE (USING 4 SEASONAL MODELS FOR ILLUSTRATION)

	CHANGE IN STORAGE
Nov. 1957 - Oct.1967	3.76"
Dec. 1957 - Nov. 1967	4.70"
Jan. 1958 - Dec. 1967	4.30"

It is always important to check how good or how effective is a simulation or mathematical model. This could be done either by using other data from another area or data from completely different location; or where this may not be available a convincing method should be derived. From experience, it is more difficult in most developing nations to have relevant data from other countries (problem may be economic, lack of interest in foreign data and even more important, it may not fall within the priorities of the development programs).

It is, therefore, important to develop another means for checking the model. One method is to use the mass balance equation

$$\frac{\Delta S}{\Delta t} = P - Q - E$$

where $\frac{\Delta S}{\Delta t}$ = change in storage,

P = rainfall,

Q = runoff, and

E = evapotranspiration.

Over long periods $\Delta S/\Delta t = 0$, and therefore, $E = P - Q$. Further, if all rainfall values are summed for a period and subtracting from this the total runoff for the period, then the evapotranspiration total for the period can be obtained. If the monthly predicted values (from model) of evapotranspiration are good then when all the monthly totals are summed up for any period, the model evapotranspiration should be the same or not much different from theory. Differences in values between theory and model are due to: (1) errors associated with theory (formula is only approximate), (2) errors in mathematical model or data or both (these have been discussed in earlier chapters). Specific sources of error are discussed later on in this chapter.

In order to interpret the results fully it is necessary to go back to the assumptions made in the derivations of the theoretical and model equations (using their simplest forms):

From theory:

$$\frac{dS}{dt} = P - E - Q$$

or

$$E = P - Q - \frac{dS}{dt} \doteqdot P - Q - \frac{\Delta S}{\Delta t} = P - Q - \frac{(S_2 - S_1)}{\Delta t} \quad .$$

Now S_2 is the storage at the end of the period. S_1 is the storage at the beginning of the period. Therefore, $\frac{\Delta S}{\Delta t}$ is only zero when $S_2 = S_1$ otherwise E from model may either be higher or lower than the theoretical value.

From the first order differential equation:

$$\frac{dS}{dt} = \alpha P - Q = P - E - Q$$

or

$$E = P - Q - \left(\frac{dS}{dt}\right) \doteqdot P - Q + \left(\frac{\Delta S}{\Delta t}\right)$$

But $$\frac{\Delta S}{\Delta t} = K_1 Q^{\gamma}$$

$$\therefore \; E = P - Q + K_1 Q^{\gamma}$$

This implies E has an extra term $K_1 Q^{\gamma}$ which will give a higher or lower value of E than that from theory (Fig. 6.4a). The results obtained show a bias for higher E from model. However, the closeness of the model and theoretical values are very significant. From the second order differential equation:

$$\frac{dS}{dt} = P - E - Q$$

$$E = P - Q + \frac{\Delta S}{\Delta t}$$

but

$$\frac{\Delta S}{\Delta t} = K_1 Q^{\gamma} + K \frac{dQ}{dt} = K_1 Q^{\gamma} + K \frac{\Delta Q}{\Delta t}$$

$$\therefore \; E = P - Q + (K_1 Q^{\gamma} + K \frac{\Delta Q}{\Delta t}) \; .$$

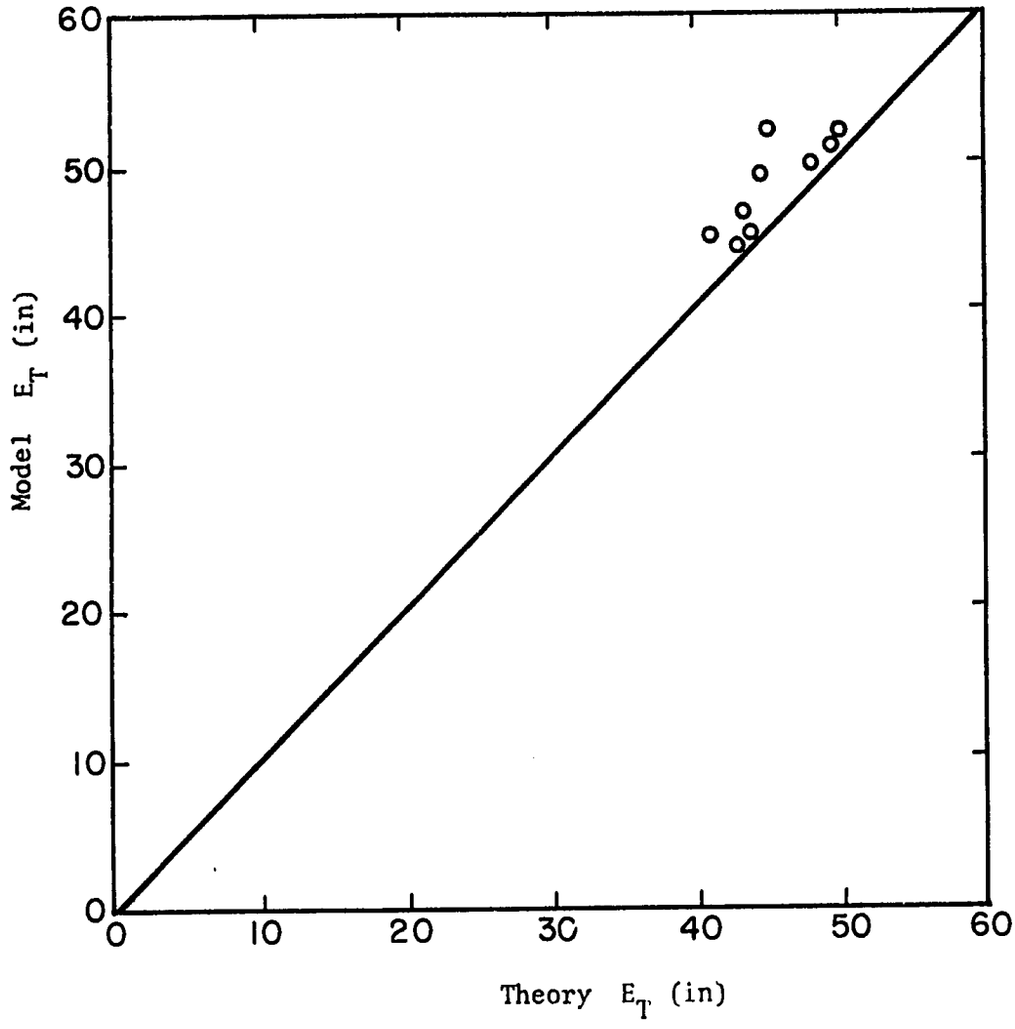


Fig. 6.4a. First Order Nonlinear Differential Equation. Annual Total Evapotranspiration of Model and Theory. Using 10 Annual Totals and the 4 Seasonal Model.

This means E for this case is larger than that for theory if the term in brackets is positive or less if $K \frac{\Delta Q}{\Delta t}$ is negative and larger in magnitude than $K_1 Q^{\gamma}$. This latter case rarely exists and the results obtained confirm this (Fig. 6.5a).

In summary, when dynamic effects are neglected as in the case of theory the calculated values of evapotranspiration will be smaller than the actual value. The developed models have predicted extremely good annual evapotranspiration values. Further, it is briefly illustrated (Table 6.4) the effect of neglecting dynamic effects of the system.

Three experiments were performed using different starting points (of the data) and finding effects on $\Delta S/\Delta t$. Three months, November, December, and January were chosen. Since antecedent moisture conditions may exist before the chosen starting point, actually it was not necessary to perform the experiments on different seasons as starting points (though these might illustrate the points even more clearly). Results obtained on Table 6.4 clearly illustrate that change in soil moisture storage is not zero, though the approximation of $\Delta S/\Delta t \rightarrow 0$ serves as a good check. It is, therefore, important to point out that the value of $\Delta S/\Delta t$ for a given year depends on when the year is defined to begin.

<u>The four seasonal models</u> - Most of this study is based on the four seasonal models. Other seasonal models, e.g., two seasonal, may be investigated depending on the needs of the investigator. The four seasonal models were grouped in sets of three months per season.

Season 1 (JP=1) was Dec., Jan., Feb.; JP=2 was the next three months; JP=4 refers to Sept., Oct., Nov. season. These were chosen by the author from the author's experience of the basin, but it is

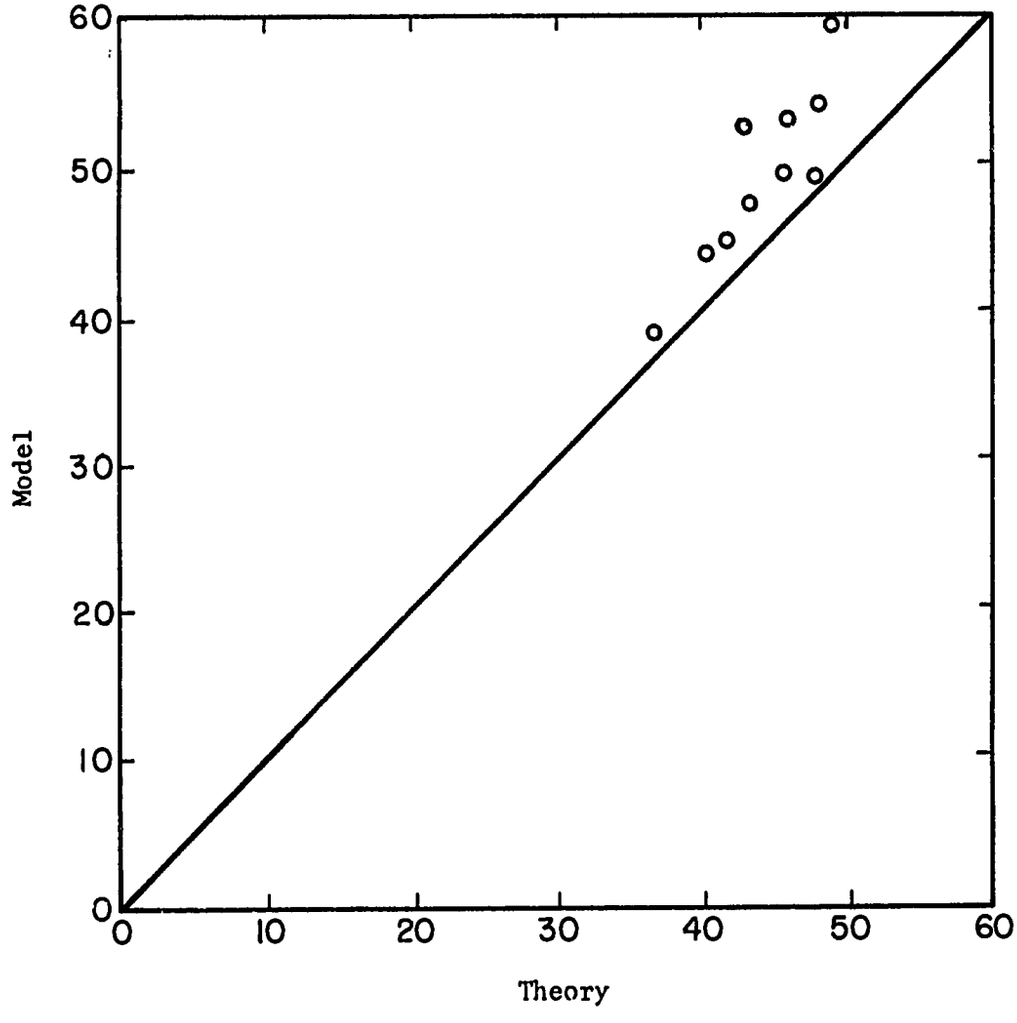


Fig. 6.5a. Second Order Differential Equation. Annual Total Evapotranspiration of Model and Theory. Using 10 Annual Totals and the 4 Seasonal Model.

necessary to stress that the models are flexible and general enough that one can take any number of seasons (e.g., two seasons of 6 months each, or 12 seasons of 12 months (monthly case)) irrespective of the starting point. To serve as a check in this study, the annual evapotranspiration estimates derived from the four seasonal model were compared to those by the two seasonal model later on in this chapter.

Results of seasonal Model 1 - From Chapter IV (Eq. 4.32)

$$\frac{dQ}{dt} = \frac{\alpha}{\beta\gamma} P\, Q^{1-\gamma} - \frac{1}{\beta\gamma} Q^{2-\gamma}$$

where $S = KQ^{\gamma}$. The symbols have been defined earlier. The foregoing equations represent the first order nonlinear differential equation model. This is also seasonal Model 1, which is a four seasonal model as explained earlier in this chapter.

Table 6.5 gives the results of the optimal values of the parameters. Seasonal average values determined from earlier experiments were used for β and γ in this experiment. Herein, α was a time dependent parameter and the values given in Table 6.5 are the averages found for each season in the current experiment. It is interesting to note the range of values of α: 0.0078 to 0.0887, in this current experiment. The evapotranspiration estimates for the seasons depended mainly on values of α and the rainfall values and they were generally independent of β and γ. The parameters β and γ are not very sensitive to changes in α. Figure 6.4 illustrates how closely related the model and theory values are. All the points lie above the line (45° to the axis) which rightly indicates the slightly higher estimates of annual evapotranspiration as given by the model. The difference in magnitude,

between each point and the nearest point from the line measured vertically down, gives the annual magnitude of change in storage. The theory values serve as good check on the computed evapotranspiration. This experiment is initially done for the four seasonal model.

TABLE 6.5

AVERAGE SEASONAL OPTIMAL VALUES OF THE PARAMETERS FIRST ORDER NONLINEAR DIFFERENTIAL EQUATION MODEL

Period/ Season	α^*	β^*	γ^*	Average Evapotranspiration
1	0.0078	8.4751	1.2771	7.39
2	0.0360	8.3908	1.2781	19.36
3	0.0344	8.3908	1.2781	12.34
4	0.0887	7.9856	1.3064	8.39

Results from Fig. 6.4a and especially Table 6.5 gives an insight into which season has the highest or lowest evapotranspiration and also to answer the question whether more evapotranspiration occurs in rainy seasons (because there is more water available) or in between rainy seasons or even in the unlikely situation of a dry season? To the best knowledge of the author this interesting question has never really been answered, even such concepts as actual and potential evapotranspiration occur. At least it will be important to find out what the model reveals.

It is known that when more water is available, e.g., wet season when the soil may be saturated, water is being evaporated at a potential rate. Also when the season is generally dry the evapotranspiration rate is generally less than the potential rate. However, if the topography of the area is flat as in the case of the Todzie Basin; and there are lagoons, also as in the case of the Todzie Basin, then water accumulated from the wet season to the dry season could be lost to evapotranspiration at a rate which may be greater than that of the wet season. The results from this model indicate total dry seasonal losses (seasons 1 and 2) as higher than that for the wet season (seasons 3 and 4). Of course, evapotranspiration itself depends on how intense, how frequent and when these following factors occur. Some of the factors are:

1. Solar radiation.
2. Wind speeds and direction.
3. Amount of moisture--how wet the soil is and how efficient is the transpiration processes of plants.
4. Cloud cover.

These factors should all be considered with the above investigations before any solid conclusions can be drawn, even though certain trends can be detected in this research.

Results from this model indicate that most of the factors responsible for evapotranspiration losses may be more pronounced in the dry season.

The confidence placed on the predicted evapotranspiration for the seasonal model 1 is further confirmed by the seasonal average values

of evapotranspiration obtained from Ho using Penman's method (Table 6.6). The units are in inches. The total of averages for the first and second six months are given. The averages from Ho were based on averages of two stations and it must be pointed out that the average from the period of record was from 1961-1962 to 1970-1971, whereas, that for Todzie was based on data from 1958-1967. The two sets of values were the only information available to the investigator, after more than 1½ years frustrating search for pertinent data. In any case, it is important to use whatever is available. However, the closeness of the values is encouraging.

TABLE 6.6

COMPARISON OF AVERAGE TOTAL SEASONAL VALUES OF EVAPOTRANSPIRATION BETWEEN MODEL 1 FOR TODZIE BASIN AND ESTIMATES FROM HO AREA

Season	Todzie	Ho Area
Dec.-May	26.75	27.24
June-Nov.	20.73	21.03

Results of seasonal Model 2 - Second order differential equation with nonlinear storage-discharge relation, from Chapter IV (Eq. 4.33)

$$\frac{d^2Q}{dt^2} = \frac{\alpha}{K_1} P - \frac{Q}{K_1} - \frac{K}{K_1} \gamma Q^{\gamma-1} \frac{dQ}{dt}$$

where

$$S = K Q^{\gamma} + K_1 \frac{dQ}{dt} .$$

This model is considered in order to find out whether this or the first order differential equation model is superior. The initial

consideration is on evapotranspiration estimates. Superiority of model is determined on the following basis:

1. How good the particular model predicts evapotranspiration values.
2. Whether it gives better estimates of runoff (least squares error should be less) or simply whether this model gives a better fit of computed runoff to observed runoff.
3. How fast is the computation time.

The seasonal variations of β^*, γ^*, ϕ^* are apparent in this model (Table 6.7), whereas in model 1 β^* and γ^* assumed more or less a stable form. However, α^* in model 2 showed a somewhat stable value whereas, α^* in model 1 differed from season to season. It is, therefore, not surprising to note Model 2 gives different seasonal predicted evapotranspiration. Compared to Model 1 (Table 6.4), Model 2 predicts less evapotranspiration in the first two seasons and more in the next two. However, the annual totals of evapotranspiration are closely related. Comparison of results from Table 6.7 and that of Table 6.6 indicates that Model 1 gives a closer fit to Ho area estimates than that by Model 2. This result will at this point indicate that Model 2 may estimate higher values of runoff in the first two seasons and less for the next two. The reasoning is based on the continuity relationship that should be satisfied by these models.

Figure 6.5a gives the plot of Model 2 using annual total evapotranspiration from theory and that predicted by Model 2, using 10 annual totals. Model 2 gave slightly higher estimates than Model 1.

TABLE 6.7

AVERAGE SEASONAL OPTIMAL VALUES OF THE PARAMETERS SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION MODEL

PERIOD/ SEASON	α^*	β^*	γ^*	ϕ^*
1	0.0142	6.0667	0.7751	1.0164
2	0.0183	5.2046	0.6775	0.9960
3	0.0193	5.5968	0.7049	1.0021
4	0.0183	5.5230	0.6954	0.9991

The next section compares the results from Models 1 and 2 giving the computation times, one of the criteria for judging which is a better model.

Comparison of results from seasonal Model 1 and Model 2 - It should be clearly understood that the parameter which controls or predicts evapotranspiration is α; and evapotranspiration is directly related to or directly proportional to α and rainfall:

$$E_i = (1 - \alpha_i)P_i$$

where P_i = rainfall for a given time i,

E_i = evapotranspiration at time i, and

α_i = time dependent α value for i^{th} time.

Therefore, it is unlikely that there may be drastic changes in predicted evapotranspiration values using either model. The differences are due to different estimates of α ,which are also due to the indirect dependence of α with other parameters:

first order differential equation model: $\alpha = \alpha(t,\beta,\gamma,P,Q)$

second order differential equation model: $\alpha = \alpha(t,\beta,\gamma,\phi, P,Q)$.

Basic differences in the two models will be found from predicted or computed runoff; speed and accuracy with the computer.

Table 6.8 gives a comparison of 10 year means of the theoretical and Models 1 and 2 estimates of evapotranspiration. The first established differences of the two models lie in the seasonal predicted evapotranspiration (Tables 6.6 and 6.7) and Model 1 proved a more favorable model based on available data from Ho. Ho is, of course, not in Todzie Basin, but lies in a similar geographic and climatic area. Model 1, again, took a shorter time for convergence even though the time for convergence of Model 2 was close (Table 6.8).

TABLE 6.8

COMPARISON OF 10 YEAR MEANS OF THEORETICAL, FIRST AND SECOND ORDER DIFFERENTIAL EQUATION MODELS (FOUR SEASONAL MODELS)

	Theory	First Order Model	Second Order Model
Average Evapotranspiration per Year	44.32 in.	47.48 in.	49.05 in.
Computation Time for Convergence	--	21.488 sec	22.845 sec

Two seasonal models - Fresh experiments were performed using two seasons of six months per season. This was done in order to find out how accurate or inaccurate the predicted evapotranspiration values become, or whether there is no change in these predicted values. This further illustrates the adaptability of the models developed for other seasons (or any season) that may seem important for the water resources planner.

It is also very important to find out what the behavior of the parameters are with respect to seasonal changes. Models 1 and 2 were four seasonal models with three months forming one season. Models 3 and 4 are the two seasonal models, where first season is from December to May and the second season is June to November.

The discussion on Models 3 and 4 will be very brief since they are introduced here in passing; and are mainly meant for comparison with Models 1 and 2 with respect to evapotranspiration estimates.

Results of seasonal Model 3 - This model illustrates the seasonal dependency of the model parameters and the choice of season determines the possible values of the parameters (Table 6.9).

Comparison of results given in Table 6.9 to those in Table 6.5 indicates that values of α from as small as 0.0006 to as large as 0.0887, whereas, as expected the parameters β , γ vary very little. The seasonal variations of means of β and γ are again shown.

TABLE 6.9

AVERAGE SEASONAL OPTIMAL VALUES OF THE PARAMETERS FIRST ORDER NONLINEAR DIFFERENTIAL EQUATION MODEL, MODEL 3 (TWO SEASONAL DEC-MAY, JUNE-NOV)

Period/ Season	α^*	β^*	γ^*
1	0.0006	9.2121	1.1133
2	0.0842	8.7227	1.1426

These results again clearly indicate that the planner should be careful as to which parameters are chosen, based on season and the purpose for which they are to be applied. It must, again, be stressed that α is distinctly a time dependent parameter and proportional to the evapotranspiration. Figure 6.4b gives a plot of evapotranspiration. The annual values obtained by Model 1 are not exactly the same as those from Model 3. The slight differences in the annual estimates are most likely due to errors in the model, especially in the predictor-corrector scheme. From experience, it appears that the longer the number of months in a season, the corresponding increase in errors. Errors due to model and theory are discussed later on in this chapter.

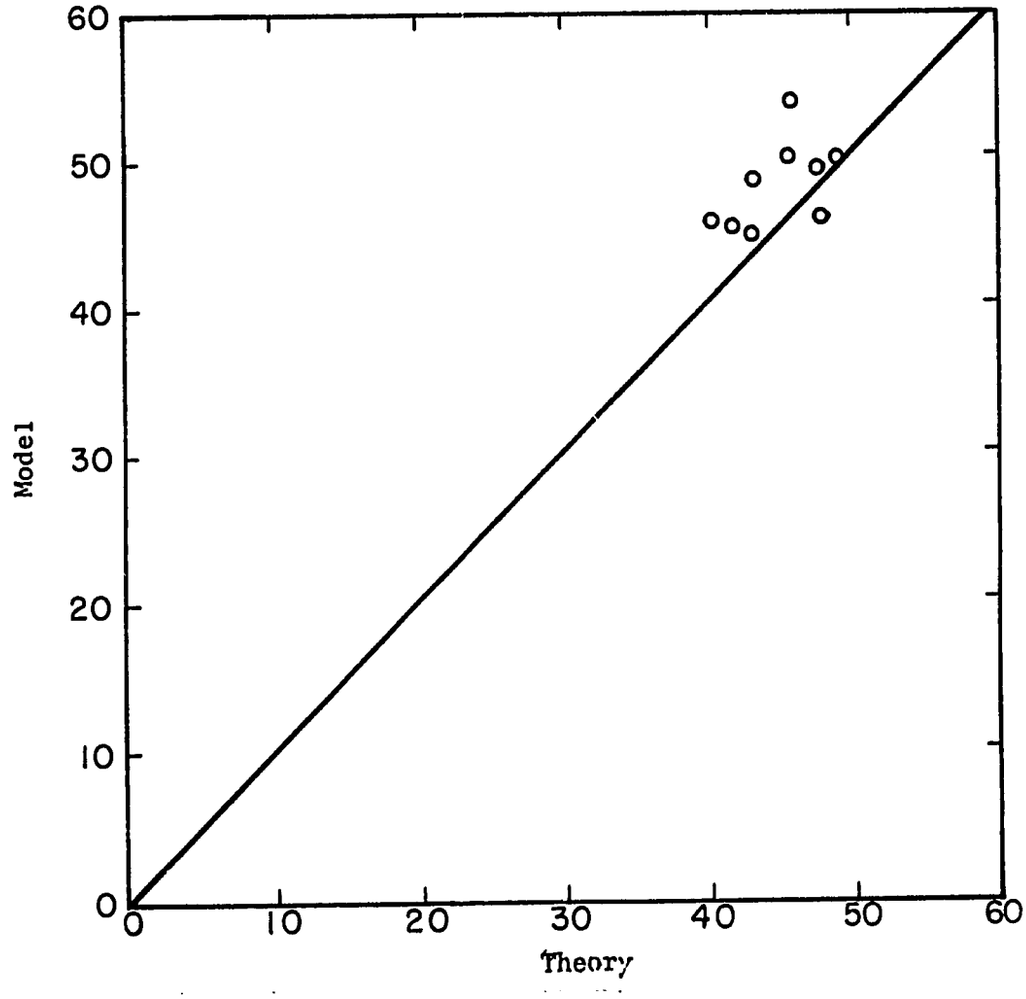


Fig. 6.4b. First Order Nonlinear Differential Equation. Annual Total Evapotranspiration of Model and Theory. Using 10 Annual Totals and the 2 Seasonal Model.

Results of seasonal Model 4 - This is the second order nonlinear differential equation applied to two seasons of six months per season.

The more sensitive parameters (Table 6.10) appear to be α and ϕ. Now that ϕ has jumped from 1.0164, the highest from Table 6.7, to 2.5985 the higher on Table 6.10. However α decreased from 0.0142, the lowest on Table 6.7 to 0.004 the lower value on Table 6.10. Note also, β varies slightly from the two tables and γ varies very little.

The results of Model 4 confirms that from Model 3: The longer the number of months forming a season, the more the errors introduced.

Figure 6.5b gives the estimates of the evapotranspiration as given by Model 4. These results compare very well to those on Fig. 6.5a, which was also calculated using the second order nonlinear differential equation. The slight differences in the estimates of evapotranspiration between Models 2 and 4 are again due to errors mentioned with respect to Models 1 and 3.

Results of linear models - For illustrative purposes a linear storage-discharge relation was incorporated in the first and second order differential equation models in order to predict evapotranspiration and runoff.

In both cases the models under predicted evapotranspiration and over predicted runoffs. Table 6.11 shows the best possible results obtained for predicted evapotranspiration. This clearly illustrates

In both cases the models under predicted evapotranspiration and over predicted runoff, based on the Ho data and observed runoff. Table 6.11 shows the best possible results obtained for predicted evapotranspiration. This clearly illustrates and confirms the test

TABLE 6.10

AVERAGE 2 SEASONAL OPTIMAL VALUES OF THE PARAMETERS
SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION MODEL

PERIOD/ SEASON	α^*	β^*	γ^*	ϕ^*
1	0.0054	6.5481	0.7521	2.3134
2	0.0040	6.3567	0.7190	2.5985

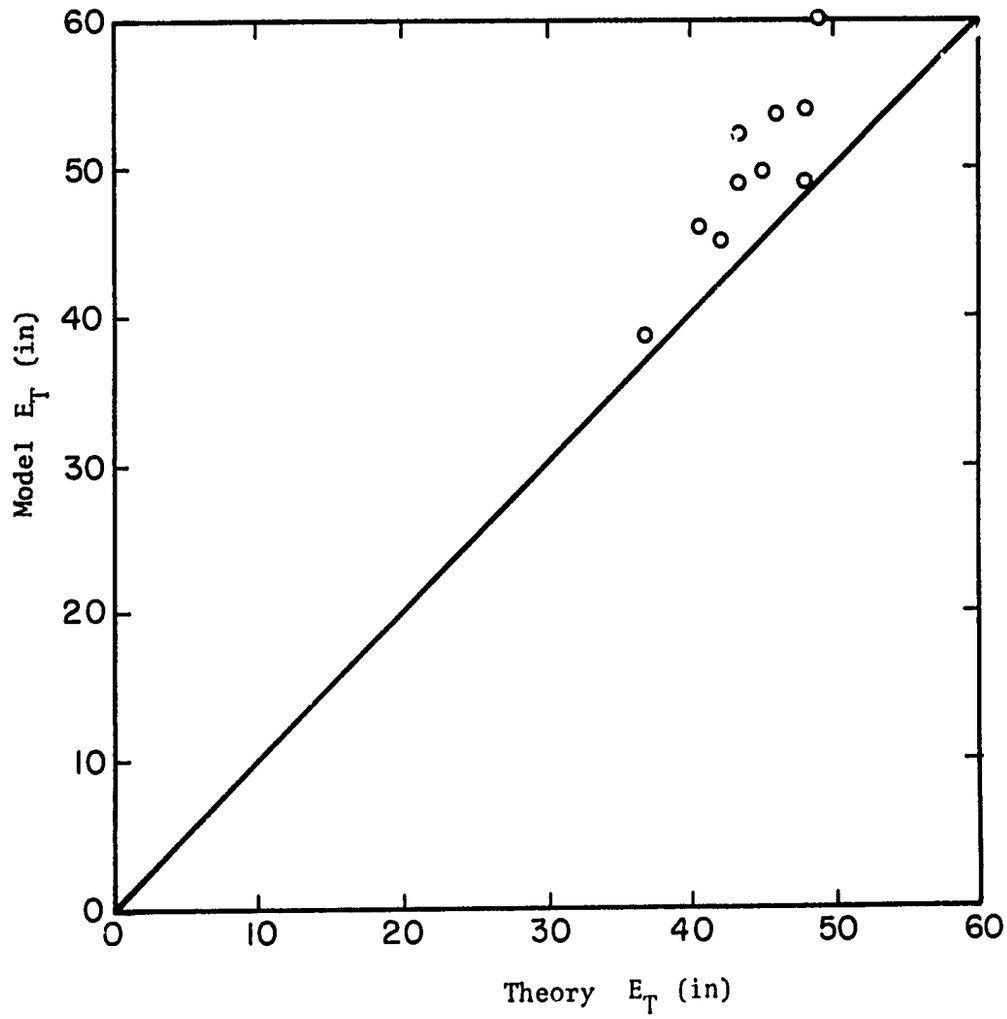


Fig. 6.5b. Second Order Differential Equation. Annual Total Evapotranspiration of Model and Theory. Using 10 Annual Totals and the 2 Seasonal Model.

TABLE 6.11

LINEAR STORAGE - DISCHARGE RELATION IN THE FIRST ORDER DIFFERENTIAL EQUATION MODEL USED FOR PREDICTING EVAPOTRANSPIRATION

YEAR	THEORY VALUE	LINEAR MODEL
1	47.82	16.81
2	45.62	17.06
3	48.20	18.63
4	36.78	13.22
5	43.59	17.97
6	49.19	20.51
7	41.91	15.44
8	46.13	18.33
9	43.51	16.51
10	40.46	16.62

that the hydrologic system model is not linear with respect to storage-discharge relationship; and, therefore, linear models are not recommended for such studies.

Comparison and comments of results from all the seasonal models - The total annual averages of the evapotranspiration estimates given by all the seasonal models (Models 1 to 4) are presented on Table 6.12. These values were found using the first and the second order nonlinear differential equations applied to different lengths of data per season. The computation time on the computer is again important and are also given.

The range of values of all the parameters are given. It should be noted that the parameters are seasonally dependent, especially α. The range of values of these parameters are given here to enable other investigators or planners who may need the results of these studies, to have some convenient starting values of these parameters; or more important some working values of these parameters. It is important not to use, e.g., β from Model 1 for Model 2, because even though the same letters are used the parameters do have different meanings (as pointed out earlier). This is because of the different assumptions made in developing the first and the second order differential equation models. The first order differential equation has no value of ϕ as indicated on Table 6.12.

The importance of presenting the ranges of values of the parameters, are to show that parameters are definitely not time invariant. They may vary very little with time in which case average values for the particular period may be used. The most important parameter of this study is α , which is a time variant parameter that depends on the

TABLE 6.12

COMPARISON OF AVERAGE ANNUAL ESTIMATES OF EVAPO-TRANSPIRATION FROM ALL THE SEASONAL MODELS

	FIRST ORDER DIFFERENTIAL EQUATION		SECOND ORDER DIFFERENTIAL EQUATION	
	4 SEASONAL MODEL MODEL 1	2 SEASONAL MODEL MODEL 3	4 SEASONAL MODEL MODEL 2	2 SEASONAL MODEL MODEL 4
AVERAGE VALUES OF EVAPOTRANSPIRATION PER YEAR USING 10 YEARS DATA (IN INCHES)	47.48	47.19	49.05	49.72
α	0.0078-0.0887	0.0006-0.0842	0.0142-0.0193	0.0040-0.0054
β	7.9856-8.4751	8.7227-0.2121	5.2046-6.0667	6.3567-6.5481
γ	1.2771-1.3064	1.1133-1.1426	0.6775-0.7751	0.7190-0.7521
ϕ	--	--	0.9961-1.0164	2.3134-2.5985
COMPUTATION TIME (IN SECONDS)	21.488	18.71	22.845	19.914

stochastic nature of rainfall, the state of the soil of the river basin at the time of rainfall (whether dry, saturated or in between and their time and space variabilities), the vegetational cover of the basin (which changes with time and space) and the climatic elements that are responsible for evapotranspiration (these also change with time and space). Bearing these points firmly in mind, it is necessary for the investigator or planner to have an idea as to the possible range of values of these parameters, and not continue with the misconception that each parameter has one fixed value which can be plugged into a model for another region. To the best knowledge of this author, no studies have presented convincing results leading to regionalization of the parameters of a simulation model though some attempts and thoughts have been given to the problem (Benson and Matalas, 1967; Matalas and Gilroy, 1968; Prasad, 1967).

While discussing the results of possible range of values of the parameters, it is important to consider transferability of results of parameters, as discussed by Dawdy, et. al. 1972:

> "For modeling results to be transferable, the parameters derived from simultation studies at measured sites must be constant or must possess invariant relations with physical variables which can be measured in other basins. Time invariance is required, or else any changes in time must be the result of measurable physical changes within the basin."

COMPUTED AND OBSERVED RUNOFF

After comparing the results obtained for the parameters and evapotranspiration using the first and second order nonlinear differential equations; and having shown that the first order differential equation model (Model 1) is more representative of the physical system than Model 2; it is now necessary to compare the computed

runoff using the two models with the actual measured data. Figure 6.6 gives a graph of the first order nonlinear differential equation estimate of runoff (the calculated runoff) to the observed runoff. If the first order model is perfect than all the points will lie on the straight line making 45° angle to the axes. As it has been pointed out no model is perfect and the differences that exist between the observed and the computed runoff may be due to errors associated with the conceptual model, the observed data, the mathematical model and the solution technique used. However, the spread of points around and along the line (Fig. 6.6 and Fig. 6.7) are good and give an indication of how well the proposed models predict runoff. It is clear from Figs. 6.6 and 6.7 that model 1 gives a better fit to observed values than model 2. Some few points which are repetitious are not indicated on the plot, even though they were closer to the 45° line than some of the others - this is done to preserve tidiness. Later, the given graphs will give plots of all the points. The points on Fig. 6.6 and Fig. 6.7 are seasonal totals of runoff per year for the entire period of record. The four seasonal models are used for illustrative purposes.

It is now important to present graphs showing the computed and observed runoff for the entire period. A necessary criteria for judging the better model is to find out which of the two models give a better fit to the observed data. Also, it is necessary to compare the two models regarding how much total volume of water for the entire period they were able to predict - this will be referred to in the section on the verification of the models.

Results of the first order differential equation model - Using the four seasonal models and the first order differential equation,

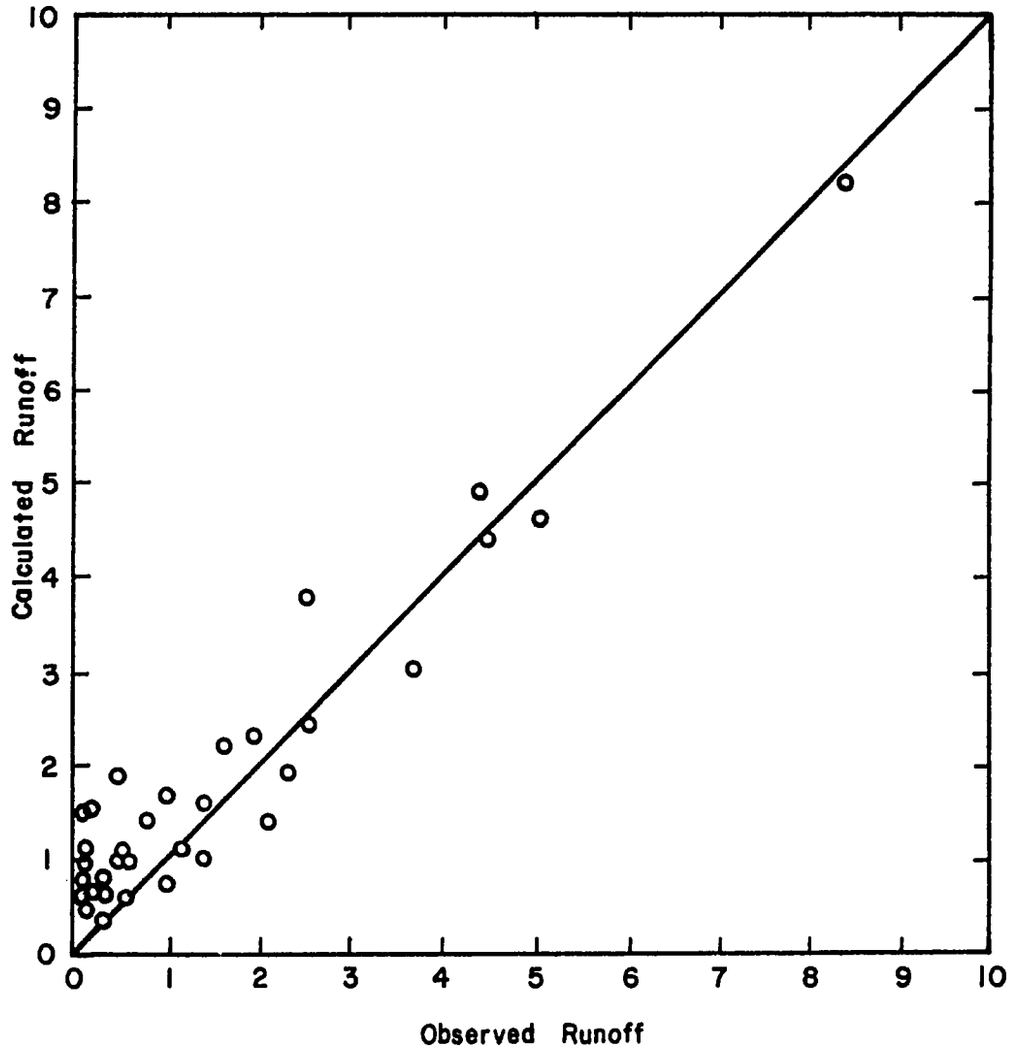


Fig. 6.6. First Order Model with Calculated and Observed Runoffs. 4 Seasonal Model.

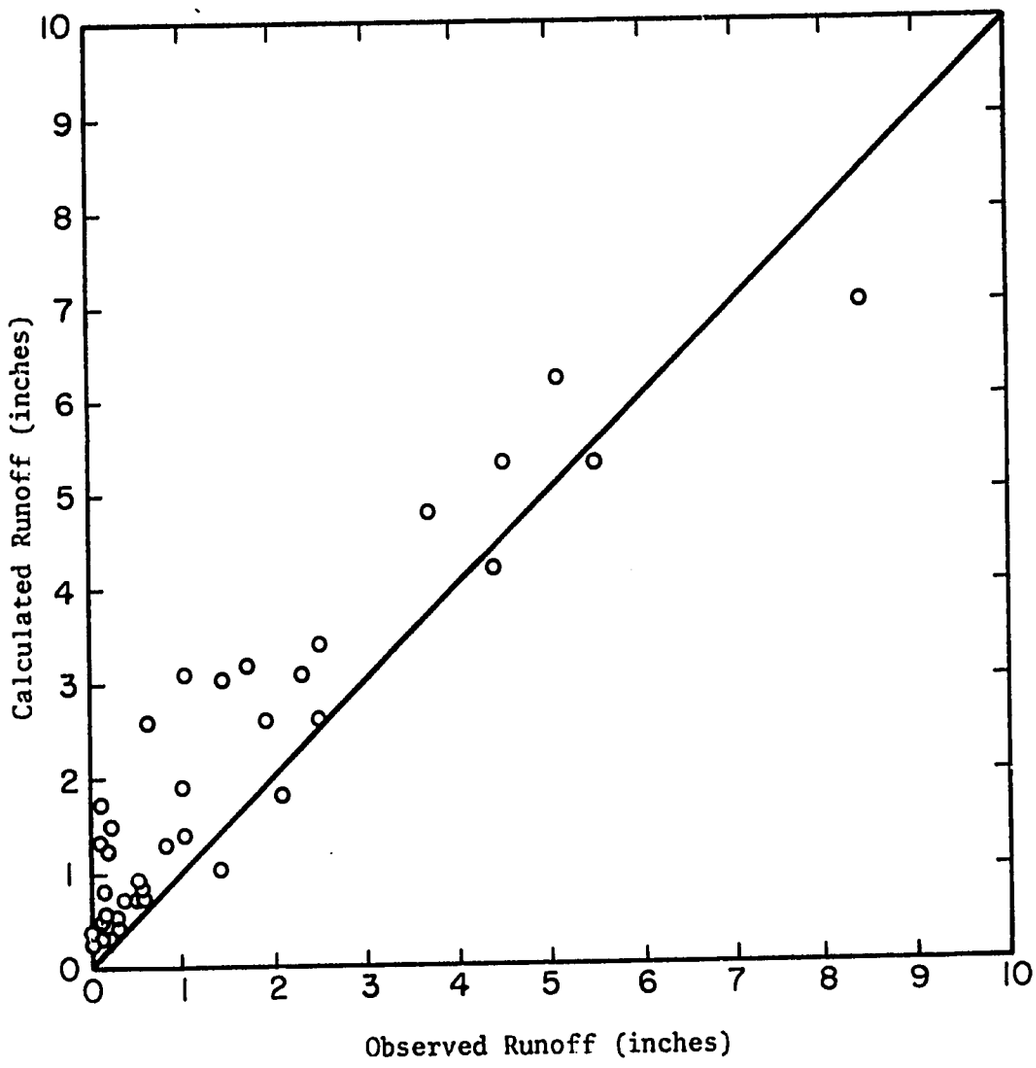


Fig. 6.7. Second Order Model with Calculated and Observed Runoffs. 4 Seasonal Model.

seasonal totals of calculated runoff were computed from Seasonal Model 1. Figure 6.8 gives a comparative plot of the total runoff per season from observed data and that from the model. Apparently, the Seasonal Model 1 tends to predict slightly higher values of runoff than given by the data for lower runoff values and almost the same or slightly lower runoff during peak flows. The fit, however, is good.

Results of the second order differential equation model - Figure 6.9 gives the comparative plot of the total runoffs per season from observed data and that from the model. The computed runoff came from using the second order nonlinear differential equation model (Seasonal Model 2).

The fit between observed and computed is adequate. However, a complete picture as to whether this model fits the observed data more than the previous model will be found on Fig. 6.10. It appears that this model predicts higher total runoff than the Seasonal Model 1.

Comparison and Comments of Results from the First and Second Order Differential Equation Models

The computed values of runoff from the two seasonal models (Models 1 and 2) are plotted against the observed data with all the three graphs on one figure (Fig. 6.10).

Verification of the Models

Several verification tests were performed on the models. The most significant among these will be reported.

For example, the data were divided into two equal parts and optimal parameters found for the first half of the data. These became the initial starting point for the second half of the data and the process

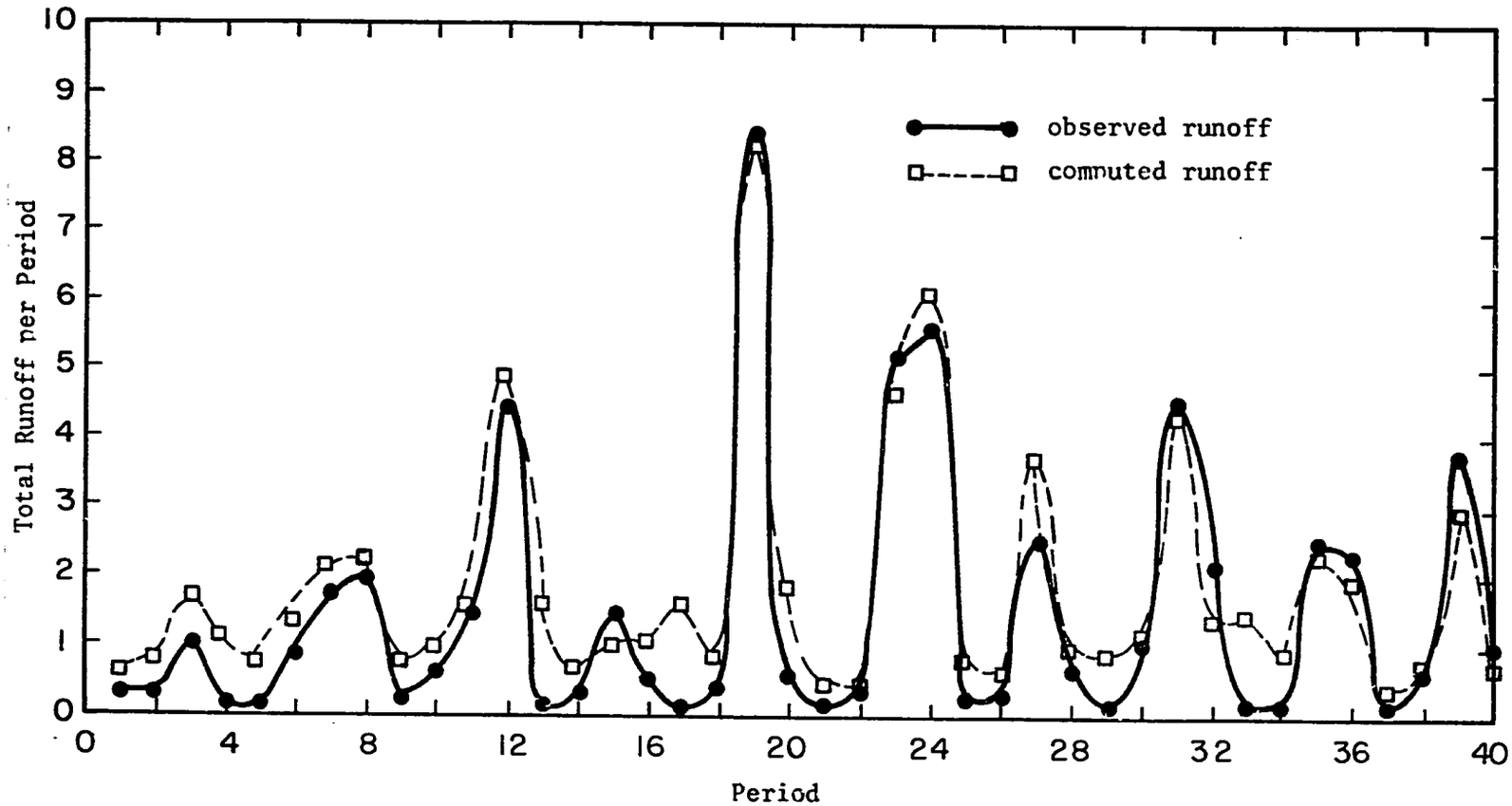


Fig. 6.8. Observed and Computed Runoffs using the First Order Nonlinear Differential Equation Model for Computed Runoffs.

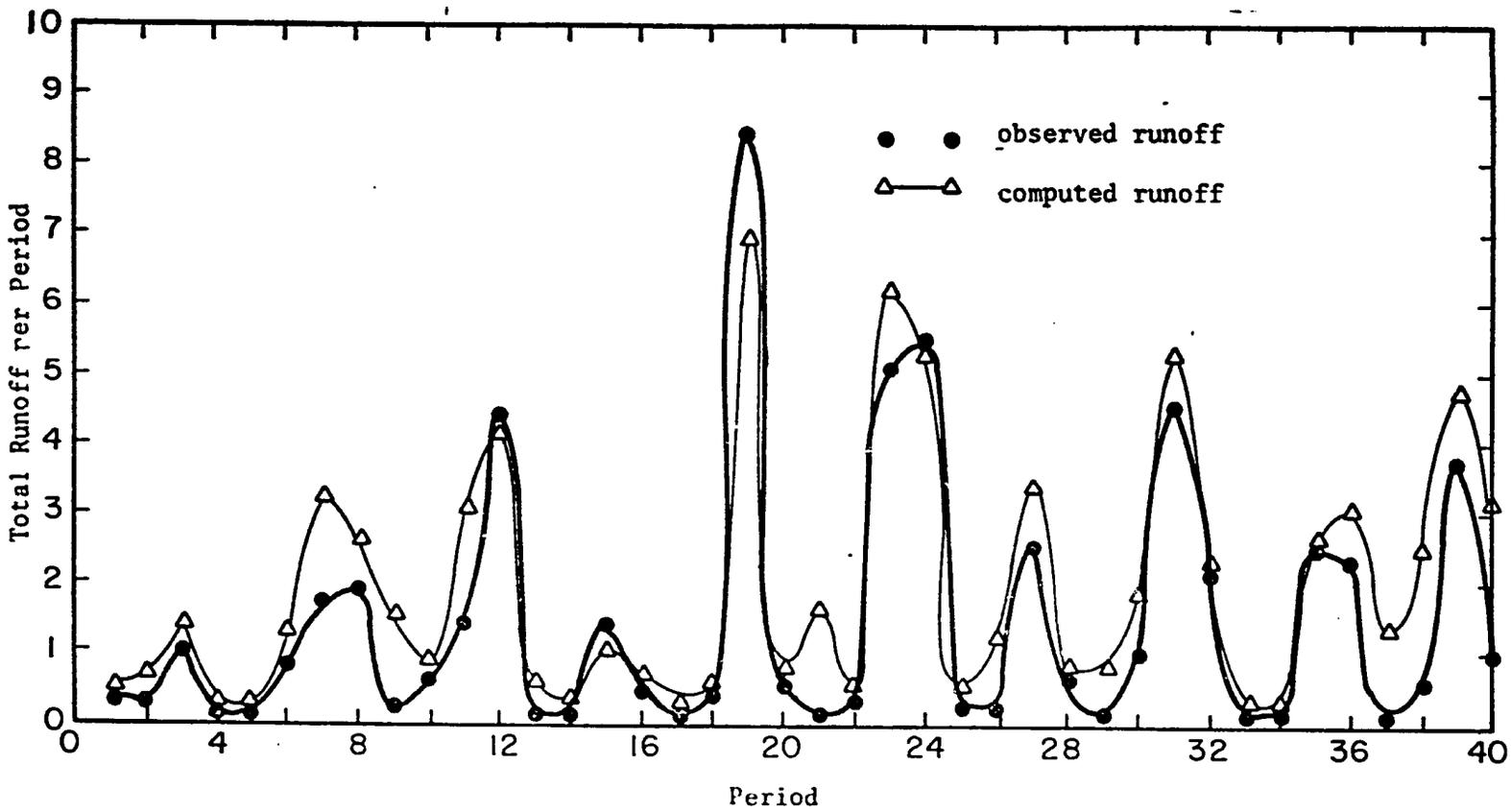


Fig. 6.9. Observed and Computed Runoffs using the Second Order Nonlinear Differential Equation Model for Computed Runoffs.

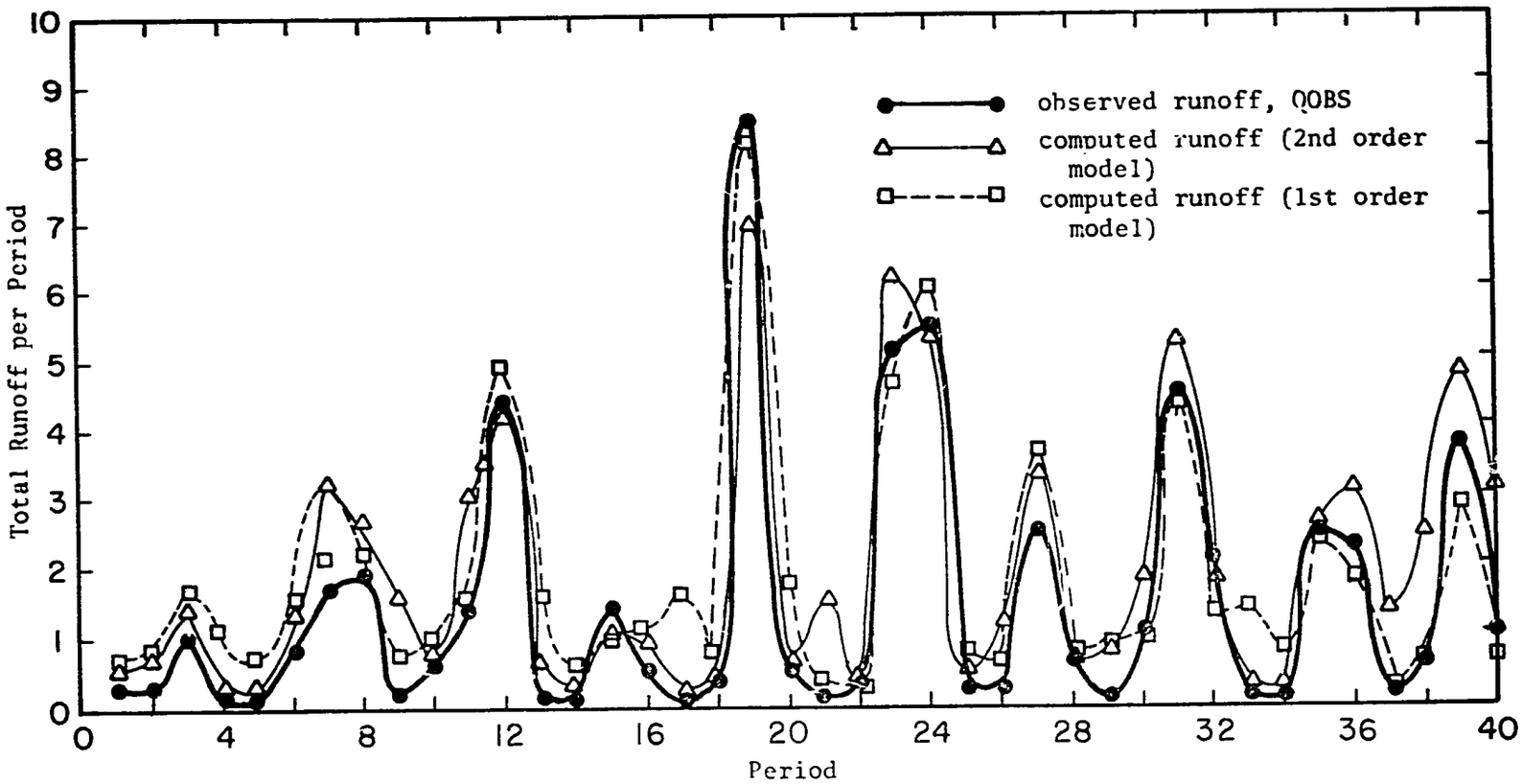


Fig. 6.10. Combined Observed and Computed Runoffs using the Observed Data, First and Second Order Nonlinear Differential Equation Models for Computed Runoffs.

was repeated. The optimal values found for the second half of the data were the same as the first. The result at the end indicated how unnecessary this test was in the first place. Since the optimal parameters found for the entire period were not essentially different from those found for the first half; and since at the end of the first half the predictor scheme had already predicted the values of the starting point of the second, it was inevitable to arrive at the same optimal values at the end. This portion of the work was not reported since it was just a repetition of results. It, however, confirms the capability of the model for predictive purposes. Penman's estimate (using Ghana Meteorological services records for Ho) gave an average evapotranspiration from the two Pan Evaporation data (converted) as 48.27. Ho (Fig. 1.1 and Fig. 6.1) has a meteorological station. Ho is one of the few important meteorological stations nearest to Todzie Basin. Values of evapotranspiration from Ho do not give the most accurate estimate of evapotranspiration for Todzie, but serve as a good check on the predicted evapotranspiration from the models; and they probably give the best estimate, since Ho and Todzie are in similar geographical (vegetation, relief, soil types) and climatic region.

Assuming the value of evapotranspiration from Ho to be the true value of E_T) then 48.27 (E_T from Ho) - 44.32 (E_T from theory) = 3.95 inches is a rough estimate of the annual average change in storage of the Todzie Basin. This further implies that the first order model gives a closer estimate of the annual average change in storage in the Todzie Basin (Table 6.13)

Another important result concerns the total of the average annual sum of runoff and evapotranspiration. Since experience with the monthly

TABLE 6.13

SUMMARY OF RESULTS FOR THE FIRST AND SECOND ORDER NONLINEAR DIFFERENTIAL EQUATION MODELS
(*Implies more acceptable model for the particular investigation)

Investigated Items for the Two Models	First Order Model	Second Order Model	Remarks on Results
1. Total runoff volume for entire period (in inches)	62.52*	68.17	Observed data from Todzie 56.34
2. Average monthly volume of run-off for entire period	0.52*	0.57	Observed data 0.53 0.47
3. Average evapotranspiration per year (in inches)	47.19*	49.72	Theory value 44.32 Penman's estimate for Ho 48.27
4. Average annual change in storage	2.87*	5.40	Theory 0.0 Ho 3.95
5. Total of average annual runoff and evapotranspiration	53.44*	56.54	Theory 50.72 E_t from Ho and Todzie runoff 54.67
6. % Difference of 5 using theory values and E_t from Ho and Todzie runoff	5.36% 2.24% (less)	11.47 3.42% (more)	Theory Ho E_t from Ho and Todzie runoff
7. Total cumulative value of the objective function	3.62*	4.63	The objective function: $F = \sum_{i=1}^{120} (QOBS - QCALC)^2$
8. Average value of the objective function per month	0.030*	0.039	
9. Computation time for convergence (seconds)	18.718*	19.914	

models indicates less fitting error in computing runoff, the monthly models were modified to solve both the first and the second order non-linear differential equations. The runoff results presented on row 1 of Table 6.13 are, therefore, from the monthly models. Again, the first order model gives a closer estimate to the theory and to the estimates for Ho (where evapotranspiration values from Ho in conjunction with runoff from Todzie). There is, however, 5.36 percent (row 6 of Table 6.13) departure from that of the theory (estimates from Ho give an exceptionally good fit). This difference is a natural result of errors encountered in the experiments. Sources of errors have been treated in earlier chapters and, therefore, the next section will be devoted only to those aspects of the model and theory that may explain the 5.36 percent discrepancy between them. It must be mentioned that the best estimates of runoff were used for Table 6.13.

The objective function F gives the sum of the least squares between the observed and calculated runoffs. The smaller the value of F from a given model's estimates the better the model. Although both models give very small values of F (with the average error per month being between 0.03 and 0.039) the first order model clearly has a smaller F for the entire record and therefore a smaller F per month. The first order model is therefore more acceptable than the second order model in this category. This conclusion could easily be derived from the estimates of total runoff volume, average monthly runoff volume and percentage departure (Table 6.13).

Sources of Error from Theory and Model

It has been mentioned in earlier chapters what the main sources of error are that can cause, e.g., the computed runoff to not exactly

coincide with the observed runoff. In this section, it is important to stress those specific cases with respect to this study.

As already indicated in this chapter, there are errors introduced in the theory when the change in storage with time is put to zero ($\Delta S/\Delta t = 0$) . This is a source of error which can influence the total average annual sum of runoff and evapotranspiration.

Another source of error is to be found in the modified Euler predictor-corrector scheme. This is the error associated with the value of dQ/dt . It must be pointed out that the error associated with dQ/dt is not due to the Euler method, rather it is due to the fact that dQ/dt is not a measured variable. Since dQ/dt has to be found by an approximate method - in this case by a finite difference scheme - it is subject to some error. Now $dQ/dt \approx \Delta Q/\Delta t = Q_{i+1}-Q_i/\Delta t$, $i = 1, 2, \ldots 120$. This implies $i = 119$ gives Q_{i+1} as Q_{120} and Q_i as Q_{119} . At $i = 120$ $Q_{i+1} = Q_{121}$, but Q_{121} is undefined. Therefore, $(dQ/dt)_{120}$ was put to the same value as $(dQ/dt)_{119}$. Other approximating forms, e.g., representing $(dQ/dt)_{120}$ as some fraction of $(dQ/dt)_{119}$ where the values of Q_{119} and Q_{120} give the value of the fraction may be used. From experience in this study, the investigator put $(dQ/dt)_{120} = (dQ/dt)_{119}$ because this may give the least amount of error.

Other errors are encountered if one has to deal with d^2Q/dt^2 . The second order nonlinear differential equations may probably be more useful for time scales of less than a month. Prasad (1967) has used hourly data and a second order nonlinear differential equation to simulate a hydrologic system model.

The choice of the form of the objective function also introduces some error. If $F = \sum_{i=1}^{120} (QOBS - QCALC)^2$, the value of F shows some bias towards the smaller values of runoff; whereas, if $F = \sum_{i=1}^{120} (\log QOBS - \log QCALC)^2$, the value of F will show some bias for larger values of runoff. Both objective functions cannot be used simultaneously since all the data should be subject to same type of changes. The former type of objective function was used here since there were fewer high values (though significant) in the data.

Lastly, the author wishes to stress, in passing, that when the word error is used it represents the combined errors from data and mathematical model formulation. It is difficult and almost impossible to know exactly how much of the total error comes from data and how much from the model.

The Gradient Technique as a Solution Tool

From the onset, the step size that the technique requires for its optimization should be determined. A step size which worked for one optimization process may not work for another. This is important since use of the wrong step size results in abrupt divergence when this step is taken in the gradient direction. Once the relevant step size is determined, it can be used for any number of optimizations for that particular optimization process. This research found step size multiplication factors of SS = 0.02 to 0.001, to be good, and were the required ones for convergence. It is generally recommended that the above range of step sizes be used for this technique (i.e., the subroutine SDOPTIM (Chapter 5)).

In the initial investigations, larger step sizes (SS) may be used. This can give some working values of the parameters. When the technique starts giving improved values of F , the objective function, then various experiments should be performed with different step sizes (Tables 6.1 and 6.2) in order to find which SS value works best for the particular problem. As the minimum is approached the step sizes (SS) should be reduced.

It is important to let the physical reasoning be a guide always in using this method. Note that α is a parameter which cannot be greater than unity. If it is unity it means there is no evapotranspiration losses and all the rainfall transfers into runoff. This is obviously not true. If α is allowed to be negative it means the output is more than rainfall input into the system. This may be true in certain rare situations of interbasin flows, etc.

The problem to be solved by this technique is better stated as minimization of the objective function F (the least square error between QCALC and QOBS) subject to the constraints imposed on the possible range of values of the parameters. In short the gradient technique, like all techniques, are tools, and the user should consider the capabilities and shortcomings of the technique in using or applying it to solve a particular problem.

The initial trials gave a possible range of values that one should expect of the parameters. So that if divergence is encountered in any iterative process, due to the gradient method taking a "sharp" step in the "wrong" direction, one of two things can be done:

1. Discontinue that particular iteration, start with the original initial guesses and begin the particular iterative process again. This approach could be long and expensive computer-wise (economic factors of the computations is a special consideration in this research).

or

2. From physical reasoning and experience, one can restrict the range of values of the parameters and place constraints on the parameters. This approach also works well and is in fact better in general than 1. Both approaches were used, both in the same program.

In general the gradient technique may be said to be very reliable and can at least give some working values of the parameters. Other merits of the technique were mentioned earlier in this chapter and will therefore not be repeated.

Chapter VII

SUMMARY AND CONCLUSIONS

The objectives of this research specified in Chapter I have been more than accomplished. From the results of the investigations, analyses performed, and experiences developed, the following conclusions are made:

1. The best possible use has been made of what is available. Ten years of monthly rainfall and runoff data from a developing country have been utilized. Unavailability of other data may be based on one or more of the following:

   a. Lack of funds or economic constraints.

   b. Does not fall within the 'priority bracket' in the national development plans.

   c. Lack of manpower and expertise.

   d. Political, social, institutional and legal constraints.

2. A time-variant nonlinear hydrologic system identification and predictive simulation models have been developed based on some important assumptions (Chapter III) concerning the dynamic and physical aspects of the hydrologic system. A valid monthly model is presented which will be of general applicability to planners of water resources development, agriculturists, hydrologists and theoreticians.

3. Based on the same assumptions, two conceptual models were developed. It was important to illustrate, especially for a developing country, how the unknown parameters of one of the conceptual models can be uniquely found. The more important point here was to base unique determinations on conceptually realistic simplifications rather than resorting to unnecessary mathematical simplifications and

unrealistic assumptions (physically), in order to be able to use existing mathematical tools. This approach and conclusion emphasizes an important point: the inability of investigators to solve hydrologic modeling problems is not only based on lack of inadequate pursuit of conceptual ideas and formulation, but also, lack of understanding of powerful mathematical tools that are available. This lacking requires simplification of conceptual models in order for unknown conditions and parameters to be determined. It is, of course, believed that conceptual modeling and mathematical solution tools work hand in hand, and at times a trade-off or compromise must be made between physical realism and mathematical simplification. Whenever this is necessary, however, more weight should be placed on the physical realism. Chapters III and IV have illustrated these conclusions.

4. One of the major objectives of this study was in the development of a mathematical model which is capable of effectively utilizing only what is available (rainfall and runoff), and to give some estimates of unavailable evapotranspiration data. The economic importance of this objective for a developing country cannot be overemphasized. Two mathematical model equations were developed for this purpose utilizing first and second order nonlinear differential equations. These two equations gave good estimates of evapotranspiration, which were verified by comparing these estimates with theory and Penman's estimate for a nearby station outside the case study area. The first order nonlinear differential equation gave a better estimate than the second order nonlinear differential equation.

5. Based on the above, the linear storage-discharge assumption was incorporated in the mathematical models. The resulting first and

second order linear differential equations gave very low estimates of evapotranspiration and a poor fit of computed to observed runoff. This confirms the concept of nonlinear hydrologic systems and that a linear storage-discharge relation inadequately simulates the physical system.

6. A new and effective method of estimating evapotranspiration has been developed. This is an important result of the study, not only because knowledge of evapotranspiration is important to planners, engineers, water delivery projects, agriculturists, and hydrologists, but also for economic reasons. It appears that large investments in data gathering effort for estimates of evapotranspiration will not be needed. Certain projects that may require rapid estimates of evapotranspiration when rainfall and runoff data are available can find the technique handy, as well as in situations where an investigator or planner may be faced with some of the constraints listed in item 1 of the conclusions.

7. The river basin response characteristics are caused by complex interactions of the physical, chemical, and biological processes occurring in the basin. Some of the more important interactions can be determined from the time and space behavior of the model parameters describing these characteristics. A new concept: The sensitivity factor of the river basin is developed. The sensitivity parameter α , is responsible for transforming rainfall to runoff and is found to be proportional to evapotranspiration. Optimally determined values of α have given rise to calculated evapotranspiration with a reasonably high degree of accuracy. The relationship between α , the rainfall input P , and the evapotranspiration output E_T is given by:

$$E_T(t) = (1 - \alpha(t))\ P(t)$$

where $\alpha(t)$ is a time-dependent optimal value of the sensitivity parameter, and

$P(t)$, $E_T(t)$ are the rainfall and evapotranspiration at time t.

8. In developing the conceptual and mathematical models, the following considerations were important:

a. Efficiency.

b. Realistic assumptions based mostly on physical realism.

c. A model which is simple enough to comprehend conceptually and mathematically.

d. A model which can be solved by existing mathematical tool without resorting to unnecessary simplifications.

e. A model which is inexpensive based on computation time. This economic constraint is not only based on type of model applied but also the type of solution technique used.

f. A model capable of computing better runoff, and predicting evapotranspiration, when the only data available are rainfall and runoff.

Of all the developed models, the first order nonlinear differential equation simulated the physical system best and satisfied best the above mentioned considerations or objectives. The proposed model is:

$$\frac{dQ}{dt} = \frac{\alpha}{KM}\ P\ Q^{1-M} - \frac{Q^{2-M}}{KM}\ .$$

This model can be used to forecast runoff when used in conjunction with the gradient technique, a predictor-corrector method, and the resulting optimally determined values of the parameters. This has been verified in this study.

9. Another significant result obtained in this study concerns the change of storage with time: dS/dt . It has been found that if $\frac{ds}{dt}$ is assumed to be zero for long periods of record, then the resulting evapotranspiration estimates serve as a good check on the model values. These gave very close fits to each other. The dS/dt value for any length of record has been found to be dependent on when the beginning of the period of record is defined. Certainly for monthly data these observations are important.

10. The various experiments were repeated several times (at least 15 times) with different initial conditions and almost always resulted in optimal solutions with the same values. The repetitive nature of the values of the optimal parameters for all the numerous experiments strongly indicate that global solutions have been informally found. It has also been established that the parameters β , γ , and ϕ are relatively time variant.

11. It is extremely important to establish the seasonal (depending on how many months constitute the season) values of the parameters for planning purposes. These have been clearly illustrated as dependent on length of season (with respect to the number of months) and are assumed to be constant for each particular season. It is necessary to determine the time dependency aspects of the parameters and establish which are sensitive to time (e.g., the parameter α) , and which of the parameters may be considered as constant or approximately equal to their seasonal averages.

12. The original computer program developed for this research used the given initial conditions and optimized all the way through.

A new approach has been further developed based on the above as a check to the conventional approach in the optimization process of this nature. This involved using the very first initial guess and optimizing for the first point. The optimals obtained from the first data point become starting values for the second data point and the process is repeated for all the data points. This new procedure gave extremely good fits of the parameter values to the first procedure; it gave a slightly higher but acceptable error; and it was faster than the first procedure.

13. As pointed out in Conclusion 8e, the economic constraint was not only based on the type of model used but also the type of solution technique. The solution techniques used in this study were: quasilinearization and a gradient technique. From the experiences in this work, it was found that quasilinearization gave divergent results, whereas, the gradient technique converged. It was established that quasilinearization is an unreliable algorithm, whereas the gradient technique will give at least something useful that the planner or investigator can utilize. The gradient technique was faster and simpler to understand than the quasilinearization algorithm. It was, therefore, concluded from this study that the gradient technique is a superior technique, being more reliable, more economical to use, easier to understand and handle (comparatively), and therefore more attractive for use in developing countries as well as in developed countries.

Chapter VIII

RECOMMENDATIONS

From the analyses performed, results obtained and the conclusions drawn, the following recommendations are made:

1. A recommendation should be made which considers computer-oriented models, developing countries, politics and economics. These areas are indeed related as far as this study is concerned. It has been stressed that this study considers an efficient and a realistic model which is easy to understand and handle and fast on the computer as basic to the study. The author poses a debatable question, which he attempts to answer as well: In considering the economics in a developing country does a proposed hydrologic model have to be computer oriented?

It is true that developing countries may have to minimize costs even more than many other countries. It is not true, however, that developing countries do not want to develop themselves. It is important to stress that their development is taking place in the age of computers. This factor cannot be overlooked for a faster development. It is true certain investments may pay off in monetary terms. Other investments may be aimed at social well being. Whatever the investment, a very much needed capital may go into the project and therefore the safety of the project with respect to time is and should be of major concern. The type of project will demand the type of approach.

Certain simple projects may probably need simple models, which may not be computer oriented. As projects become more complex and capital investments increase, more uncertainities are created, the

more ideas should be generated, the more the solution procedure may become computer oriented (depending on the type of project, of course), the more money that needs to be spent in determinining the success of the project. Especially, if the country has to invest capital in a project, which could affect the economy in the event of the failure of the project due to lack of adequate simulation and correct solution procedure--then the country may be well advised to go into a more thorough investigation before investments are made. The question therefore moves from the realm of engineering to economics and politics. At least, the engineer or hydrologist should come up with reasonable alternatives for decision makers. At best the engineer, economist, politician and other disciplines related to the project should all constitute the decision makers; or in countries where this is not acceptable then at least the engineer should have his ideas expressed in the simplest language so that the decision makers can fully understand what they are to decide on.

Since, the stress here is on computer and modeling, it is important for the engineer, the hydrologist, the economist, the politician, and others to note that:

A solution technique, likewise a computer are tools, not the total answer. The investigator (engineer or hydrologist) is the technician. Together a job could be performed. How good a job depends on dedication, patience, experience, the right investigator, and right tool, and some ingenuity. The role of the investigator in decision making is to make sure other decision makers understand what his tool can do and what they cannot do, and the uncertainties and risks involved in using them.

Of course, politics usually has the last word and whether the investigator will be fully useful or not depends on how far he could be allowed to participate in the decision making in those areas that he and his tools are to be important.

2. Concentration in this study has been centered on developing a simulation model for a river basin; and fitting computed to actual runoff. The model can be extended to predict future runoff volumes. This has been done informally in this study where half of the data used in this work was used to predict the other half.

3. The computer program for the calculations will be made very general with respect to notations, how the user can easily use it without having to go through the full exercise of understanding all that is going on and without having to write his own program. This phase will involve making the computer program very simple--aimed at making it comprehensible to any user who may not possess enough expertise in computer programming. This will include many comment statements, and step sizes (ss) in the gradient technique will be automized so that the user may not have to do any initial investigations with step sizes.

4. A new approach for investigating the water balance in a river basin using only rainfall and runoff data can be investigated. This will need other data for checking the products of the model. This study has, informally, given the estimate for change in storage in the river basin with time. A mass balance can be found for each computational point.

5. A comparative study could be made in applying the models and techniques to river basins in other developing nations. Procedures

and recommendations for water resources planning and development will be the end products.

6. It is highly recommended that research effort be directed to find various ranges of values of α for different climatic and geographic regions. The importance of this study is to present a generalized model for predicting evapotranspiration values based on rainfall input alone, optimally determined $\alpha = \alpha(t)$ and based on geographic and climatic regions of the world.

7. The second order nonlinear differential equation model may be applied to daily to find the suitability and applicability of the model to time scales of less than a month. The equation is:

$$\frac{d^2Q}{dt^2} = \frac{\alpha}{K_1} P - \frac{Q}{K_1} - \frac{K}{K_1} \gamma Q^{\gamma-1} \frac{dQ}{dt}$$

8. Together with recommendation 6, an attempt should be made for the regionalization of the possible ranges of values of all the parameters. This can be very useful to planners and investigators at different parts of the world.

9. Knowing $E_T(t) = (1 - \alpha(t))\ P(t)$ which can be used in whatever way the planner or investigator sees fit, but more important, it can be substituted in the proposed model equations to optimally determine other parameters. In this case dQ/dt can be treated as an unknown variable which can be optimally determined. Using dQ/dt as a parameter converts the first order differential equation into an algebraic equation. The computed values of runoff from the algebraic equation and that from the first order differential equation may be compared to observed runoff. This approach could serve as a check on

global optimal values of the parameters from either equation; and further determine which equation gives a better value of the objective function.

REFERENCES

Amorocho, J., "Predicting Storm Runoff on Small Experimental Watersheds-Discussion," Journal of the Division, Proceedings of ASCE, vol. 87, no. HY2, pp. 185-191, March 1961.

Amorocho, J., "A Critique of Current Methods in Hydrologic Systems Investigation," Transactions, American Geophysical Union, vol. 45, no. 2, pp. 307-321, June 1964.

Amorocho, J., "The Nonlinear Prediction Problem in the Study of Runoff Cycle," Water Resources Research, vol. 3, no. 2, pp. 861-880, 1967.

Amorocho, J. and Brandstetter, "Experience with a Monthly Rainfall-Runoff Transfer Function Model," Second International Symposium in Hydrology (Ft. Collins, Colorado, September, 1972).

Amorocho, J. and Hart, W. E., "The Use of Laboratory Catchments in the Study of Hydrologic Systems," Journal of Hydrology, 3, 1965.

Apostol, T. M., Mathematical Analysis, Addison-Wesley Reading, Massachusetts, 1957.

Bellman, R., Gluss, B., and Roth, R., "Segmental Differential Approximation and the 'Black Box' Problem," Journal of Mathematical Analysis and Applications, vol. 12, pp. 91-104, 1965.

Bellman, R., Kalaba, R., Quasilinearization and Nonlinear Boundary-Value Problems, American Elsevier Publishing Co., Inc., New York, 1965.

Bellman, R., Kalaba, R., and Kagiwada, H., "Quasilinearization and the Estimation of Time Lags," Memorandum RM-44990-NIH, The Rand Corporation, Santa Monica, California, May 1966.

Benson, M. A., Factors Influencing the Occurrence of Floods in a Humid Region of Diverse Terrain, U. S. Geological Survey Water Supply Paper 1580-B, 1962.

Benson, M. A., and Matalas, N. C., Synthetic hydrology based on regional statistical parameters, Water Resources Research, vol. 3, no. 4, pp. 931-945, 1967.

Boyce, W. E., and DiPrima, R. C., Elementary Differential Equations and Boundary Value Problems, John Wiley and Sons, Inc., New York, 1965.

Blank, D., Delleur, J. W., Aldo Giorgini, "Oscillatory Kernel Functions in Linear Hydrologic Models," Water Resources Research vol. 7, no. 5, October 1971.

Carnahan, B., Luther, H. A., Wilkes, J. O., Applied Numerical Methods John Wiley & Sons, Inc., New York, 1969.

Cauchy, A. L., Méthode Générale pour la Resolution de Systémmes d'equations Simultanées. Computes Rendus, Ac. Sci., Paris, 25, 1847.

Chery, D. H., "Construction, Instrumentation and Preliminary Verification of a Physical Hydrologic Model," Utah Water Research Laboratory, Logan, Utah, July 1965.

Chiu Chao-Lin and Bittler, "Linear Time Varying Model of Rainfall-Runoff Processes," Water Resources Research, vol. 7, no. 5, October 1971.

Chow, V. T., "Runoff," Handbook of Applied Hydrology, McGraw-Hill Book Company, Section 14, New York, 1964.

Chow, Ven Te and Harbaugh, T. E., "Raindrop Production for Laboratory Watershed Experimentation," Journal of Geo. Research, 70, pp. 6111-6119, December 15, 1965.

Chow, Ven Te, International Hydrologic Symposium, Colorado State University, Ft. Collins, Colorado, U.S.A., 1967.

Chow, V. T. (Editor), The Progress of Hydrology, Proc. of the First International Seminar for Hydrology Professors, vol. 1, University of Illinois, Urbana, Illinois, 1969.

Conte, S. D. and DeBoor, C., Elementary Numerical Analysis, 2nd Edition, McGraw-Hill, Inc., 1972.

Clark, C. O, "Storage and the Unit Hydrograph," Trans. Am. Soc. Civil Engrs., vol. 110, pp. 1419-1446, 1945.

Crawford, N. H., and Linsley, R. K., "A Conceptual Model of the Hydrologic Cycle," Inst. Ass. Soc. Hydrol. Publ. 42, vol. 3, pp. 71-76, 1956.

Dawdy, D. R. and O'Donnell, "Mathematical Models of Catchment Behavior," Proc. ASCE Journal of Hydraulics Division vol. 91 HY4: pp. 123-137, July 1965.

Dawdy, D. R., and Bergmann, J. M., "Effect of Rainfall Variability on Streamflow Simulation," Water Resources Research, 5(5), pp. 958- 66, 1969.

Dawdy, D. R., "Considerations involved in evaluating mathematical modeling of Urban Hydrologic Systems," Urban Water Resources Research, ASCE Report to Office of Water Resources Research, U. S. Department of Interior, Washington, D. C., 1969.

Dawdy, D. R., Lichty, R. W., Bergmann, J. M., "A Rainfall-Runoff Simulation Model for Estimation of Flood Peaks for Small Drainage Basins," U. S. Geological Survey Professional Paper 506-B, U. S. Government Printing Office, Washington, D. C., 1972.

DeWiest, Rager, J. M., Geohydrology, John Wiley & Sons, Inc., New York, pp. 49-50, 1965.

Diskin, M. H., A Basic Study of the Linearity of the Rainfall-Runoff Process in Watersheds, Ph.D. Thesis, University of Illinois, Urbana, Illinois, 1964.

Dooge, J. C. I., "A general theory of the Unity Hydrograph," Journal Geo. Res., vol. 64, no. 1, pp. 241-256, 1959.

Eagleson, P. S., "Modeling Surface Runoff in Urban Hydrology," Urban Water Resources Research, ASCE Report to Office of Water Resources Research, U. S. Department of Interior, Washington, D. C., 1969.

Fahlbusch, F. and Muir, T. C., "Experience with a Monthly Rainfall-Runoff Transfer Function Model," Second International Symposium in Hydrology (Ft. Collins, Colorado, September, 1962).

Freeze, R. A. and Harlan, R. L, "Blueprint for a physically based digitally-simulated Hydrologic response model," Journal of Hydrology, 9, pp. 237-258, 1969.

Grace, R. A., and Eagleson, P. S., "The Modelling of Overland Flow," Water Resources Research, vol. 2, no. 3, Third Quarter 1966.

Gill, H. E., A groundwater Reconaissance of the Republic of Ghana, with a description of Geohydrologic provinces, Geological Survey Water Supply paper 1757-K; U. S. Government Printing Office, Washington, D. C., 1969.

Hall, W. A., and Dracup, J. A., Water Resources Systems Engineering, McGraw-Hill Book Co, New York, 1970.

Jamieson, D. G., and Amerman, C. R., "Quick-return Subsurface Flow," Hournal of Hydrology, 8, pp. 122-137, 1969.

Klemes, Vit, "Watershed as a Semi-Infinit e Storage Reservoir," ASCE National Water Resources Engineering Meeting, Reprint no. 1895, Washington, D. C., February 1973.

Kulandaiswamy, Vangalampalayaru, A Basic Study of the Rainfall Excess-Surface Runoff Relationship in a Basin System, Ph.D. Dissertation, University of Illinois, Urbana, Illinois, 1964.

Labadie, J. W., "Optimal Identification of Nonlinear Hydrologic System Response Models by Quasilinearization," Masters Degree Thesis, University of California, Los Angeles, 1968.

Labadie, J. W., and Dracup, J. A., "Optimal Identification of Lumped Watershed Models," Water Resources Research, vol. 5, no. 3, June 1969.

Linsley, Ray K., and Crawford, Norman, Digital Simulation in Hydrology: Stanford Watershed Model IV, Technical Report no. 39, Department of Civil Engineering, Stanford University, Stanford, California, July 1966.

Matalas, N. C., and Gilroy, E. J., Some comments on regionalization in hydrologic studies: Water Resources Research, vol. 4, no. 6, pp. 1361-1369.

McIntosh, D. H., Meteorological Glossary, Meteorological Office, Her Majesty's Stationery Office, MO729 A.P. 827, London, 1963.

McCracken, D. D., and Dorn, W. S., Numerical Methods and Fortran Programming with Applications in Engineering and Science, John Wiley & Sons, Inc., New York, 1964.

Musgrave, G. W., How much of the rain enters the soil? Water: U. S. Department of Agriculture Handbook, pp. 151-159, 1955.

Nash, J. E., "The form of the instantaneous unit hydrograph," Intern. Assoc. Sci. Hydrology, Publ. 45, vol. 3, pp. 114-121, 1957.

Nash, J. E., "A unit hydrograph study, with particular reference to British Catchments," Proc. Inst. Civil Eng., 17, pp. 249-282, 1960.

Nathan Consortium, "Flood Control in the Avu-Keta Area," prepared for Ministry of Finance and Economic Planning, Luberim Report, Accra, Ghana, 1970.

O'Kelly, J. J., "The employment of unit-hydrographs to determine the flows of Irish arterial drainage channels," Proc. Inst. Civil Engrs., vol. 4, pt. 3, pp. 365-412, 1955.

Prasad, R., "A nonlinear Hydrologic System Response Model," Journal of Hydraulics Division, Proc. of ASCE, vol. 93, no. HY4, pp. 201-21, July 1967.

Ralston, A., Numerical Integration Methods for the Solution of Ordinary Differential Equations, in Mathematical Methods for Digital Computers, edited by A. Ralston and H. S. Wilf, pp. 95-109, John Wiley, New York, 1960.

Scheid, Francis, Numerical Analysis, Schaum's Outline Series, McGraw-Hill Book Co., New York, 1968.

Sherman, L. K., "Streamflow from Rainfall by the Unit-Graph Method," Engineering News-Record, vol. 108, pp. 501-505, April 1932.

Singh, K. P., A non-linear approach to the instantaneous unit hydrograph, Ph.D. Thesis, University of Illinois, Urbana, Illinois, 1962.

Singh, K. P., "Nonlinear Instantaneous Unit-hydrograph Theory," Journal of the Hydraulics Division, Proc. of ASCE, vol. 90, no. HYA, pp. 313-347, March 1964.

Sugawara, M., and Marayama, F., "A method of prevision of the river discharge by means of rainfall model," ASCE Int. Hydrol. Sci. Darcy Publ. 42, vol. 3, pp. 71-76, 1956.

Vemuri, V., and Vemuri, N., "On the Systems Approach in Hydrology," Bulletin of the International Association for Scientific Hydrology, SV(2) pp. 17-38, 1970.

Venkateswarao, V., and Dracup, J. A., "Nonlinear Runoff Response to Distributed Rainfall Excitations," A paper presented to the International Water Erosion Symposium, Prague, Czechoslovakia, June 1970.

Wismer, D. A., Optimal Control of Distributed Parameter Systems Using Multilevel Techniques, Ph.D. Dissertation, University of California, Los Angeles, 1966.

Zoch, R. T., "On the Relation Between Rainfall and Streamflow," Monthly Weather Review, vol. 62, pp. 315-322, 1934.

APPENDIX A

GRADIENT VECTOR AND THE DIRECTION OF MAXIMUM DECREASE OR INCREASE OF A FUNCTION

Let $V = F(\underline{X}) = F(\alpha, \beta, \gamma)$

Where α, β, γ are parameters to be identified in the optimization procedure

$$\underline{X} = \alpha_o, \beta_o, \gamma_o$$

$$\underline{\xi} = \text{the direction}$$

By definition the directional derivative (Apostol, 1957) of $F(\underline{X})$ at $\underline{X}_o$ in the direction $\underline{\xi}$ is

$$D_{\underline{\xi}} F(\underline{X}_o) = \lim_{\lambda \to o} \frac{F(\underline{X}_o + \lambda \underline{\xi}) - F(\underline{X}_o)}{\lambda} \tag{A.1}$$

when the limit exists.

Furthermore, it is well known that the directional derivative, i.e. the derivative of $F(\underline{X})$ with respect to a direction $\underline{\xi}$, can be expressed in terms of ordinary partial derivatives (Apostol, 1957):

$$D\, F(\underline{X}_o) = \underline{\nabla} F(\underline{X}_o) \underline{\xi} \qquad |\underline{\xi}| = 1 \tag{A.2}$$

or

$$D_{\underline{\xi}} F(\underline{X}_o) = \sum_{i=1}^{n} \frac{\partial F(\underline{X}_o)}{\partial x_o} \underline{\xi} \qquad |\underline{\xi}| = 1 \tag{A.3}$$

Now the question is how do you determine the direction $\underline{\xi}$ such that the rate of change of $V = F(\underline{X})$ at a point X_o is a maximum. Let us define our problem as

$$\text{Max} \sum_{i=1}^{n} \frac{\partial F(\underline{X}_o)}{\partial X_i} \tag{A.4}$$

Subject to the constraint

$$g(\underline{\xi}) = \sum_{i=1}^{n} \xi_i^2 = 1 \tag{A.5}$$

In order to solve this we shall form the Lagrangian in the form

$$F \quad \lambda) = \sum_{i=1}^{n} \frac{\partial F(\underline{X}_o)}{\partial X_i} \xi_i + \lambda(1 - \sum_{i=1}^{n} \xi_i^2) \tag{A.6}$$

Where λ is the Lagrange multiplier.

Differentiating (A.6) and equating to zero for turning values

$$\frac{\partial F}{\partial} = \frac{\partial F(\underline{X}_o)}{\partial X_i} - 2\lambda\xi_i = 0 \qquad i = 1,2,\ldots.m \ . \tag{A.7}$$

and

$$\frac{\partial F}{\partial \lambda} = \sum_{i=1}^{n} \xi_i^2 = 0 \tag{A.8}$$

From (A.7)

$$\xi_i = \frac{1}{2\lambda} \frac{\partial F(\underline{X}_o)}{\partial X_i} \qquad i = 1,2,\ldots.n \ . \tag{A.9}$$

Substituting (A.9) into (A.8) we get

$$1 - \sum_{i=1}^{n} \frac{1}{4\lambda^2} \left(\frac{\partial F(\underline{X}_o)}{\partial X_i} \right)^2 = 0 \tag{A.10}$$

Simplifying (A.10) we get

$$\lambda^2 = \frac{1}{4} \mid \underline{\nabla} F(\underline{X}_o) \mid^2 \tag{A.11}$$

Where solving (A.11) yields two solutions:

$$\lambda = \pm \frac{1}{2} \mid \underline{\nabla} F(\underline{X}_o) \mid \tag{A.12}$$

Since we are interested in $\underline{\xi}$ substitute (A.12) into (A.9)

$$\therefore \underline{\xi} = \frac{\pm \underline{\nabla} F(X_o)}{\mid \underline{\nabla} F(X_o)} \tag{A.13}$$

Where

$$\underline{\xi} = \frac{\underline{\nabla} F(\underline{X}_o)}{\mid \underline{\nabla} F(\underline{X}_o) \mid} \tag{A.14}$$

gives the rate of maximum increase of $V = F(\underline{X})$

$$\underline{\xi} = \frac{- \underline{\nabla} F(\underline{X}_o)}{\mid \underline{\nabla} F(\underline{X}_o) \mid} \tag{A.15}$$

gives the rate of maximum decrease of $V = F(\underline{X})$.

Hence it is shown that (A.13) the gradient vector gives the direction of maximum increase/decrease of the function $V = F(\underline{X})$.

APPENDIX B

TAYLOR'S SERIES EXPANSION AND THE JACOBIAN MATRIX

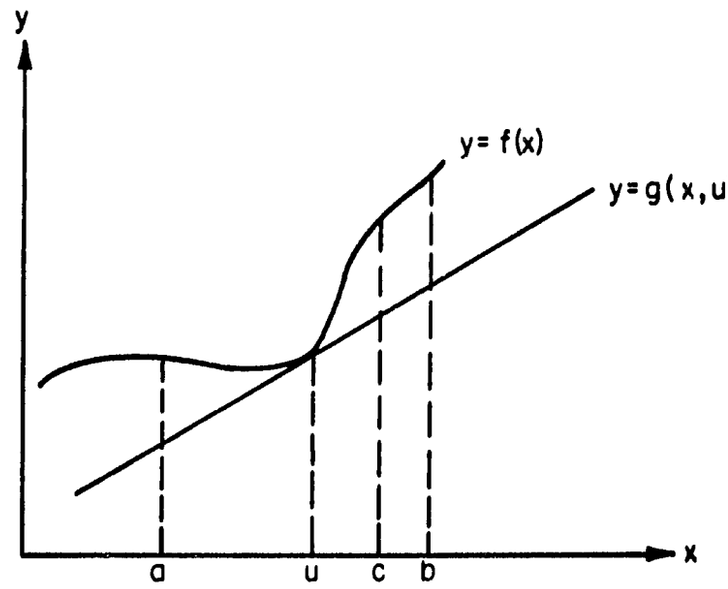
This appendix involves the process of converting nonlinear functions or equations to approximate linear forms (so that they can easily be solved) by using Taylor's series expansion and the iterative scheme.

A nonlinear function $f(x)$ can be written in terms of a linear, or rather quasilinearization (by virtue of the maximization operation) equation, where the function is expanded around the initial approximations. This is, in essence, considering the first two terms of the Taylor series.

$$f(x) \cong f(u) + f'(u)\ (x-u) \tag{B.1}$$

where $f(x)$ is the scalar function, which is linearly expanded around point u.

$f'(u)$ is the slope of $f(x)$ at u.



At $x = u$ $f(x) \geq g(x;u)$ and $f(x)$ can be expressed as a maximization operation:

$$f(x) = \max\left[g(x;u)\right] .$$

The Newton-Raphson method utilizes equation B.1 in an iterative scheme for finding the root of an algebraic equation. The steps followed are:

1. The nonlinear function $f(x)$ is set equal to zero and the approximation sign becomes an equal sign.
2. Resulting equation is solved for x, which represents the next approximation after the initial estimate u. This new value is then used to find a better approximation.
3. Process is repeated until convergence to the root is attained Equation B.1 can be written as an inequality (this was first shown by Bellman and Kalaba (1965):

$$f(x) \geq g(x;u) \tag{B.2}$$

where $$g(x;u) = f(u) + f'(u)(x-u) \tag{B.3}$$

Equation B.3 represents the equation of a line which is tangent to $f(x)$ at $x=u$.

Necessary conditions associated with equation B.2 are that $f(x)$ is strictly convex within a particular range of $x(a \leq x \leq b)$ or that $f''(x) > 0$. From the diagram maximation occurs when $u = c$ or $f(x) = \max\left[g(x;u)\right]$, where its general form is:

$$f(x) = \max\left[f(u) + J(x - u)\right] \tag{B.4}$$

where $x = x_1, \left[x_2, \ldots x_m\right]^T$,

$u = u_1, \left[u_2, \ldots u_m\right]^T$, and

J = mxm Jacobian matrix.

Equation B.4 can be solved for x -- call it x_1. The functions f can now be expanded around x_1, and the process is continued and the following recurrence/iterative relation is obtained:

$$f(x^{(n+1)}) = f(x^{(n)}) + J^{(n)} (x^{(n+1)} - x^{(n)}) \qquad \text{(B.5)}$$

where the superscript refers to iteration number, and $u = x^{(n)}$. The process begins with an initial approximation $x^{(o)}$ at $n = o$. Once $x^{(n)}$ has been given, equation B.5 becomes a set of linear equations in terms of $x^{(n+1)}$. Therefore, nonlinear problems can be replaced by iterative operations of linear relations. The nonlinear function is implied in the convergence of the iterative scheme.

APPENDIX C

THE EULER'S METHOD

Euler's method belongs to the general class of numerical methods which are utilized in solving differential equations where standard methods like the trapezoid rule or Simpson's rule cannot be used for integration. This is usually so because the derivative dQ/dt or $\dot{Q}$ (using the notation in this study) may be a function of the dependent (e.g., runoff Q) as well as the independent variable (e.g., time t). Suppose the differential equation is of the form:

$$\frac{dQ}{dt} = \alpha P_t^{\gamma} - \phi Q_t^{\delta} \quad . \tag{C.1}$$

Then

$$\frac{dQ}{dt} = f(t,P,Q,\alpha,\gamma,\delta,\phi) \tag{C.2}$$

The general form of Euler's forward integration equation is

$$Q_{t_{i+1}} = Q_{t_i} + \dot{Q}\Delta t + O\left[(\Delta t)^2\right] \tag{C.3}$$

where $Q_{t_{i+1}}$ is runoff value at time t_{i+1}

Q_{t_i} is runoff value at time t_i

$\dot{Q}$ is rate of change of runoff with time between t_{i+1} and t_i $\dot{Q} = f(t,Q)$

$\Delta t = t_{i+1} - t_i$

$O\left[(\Delta t)^2\right]$ = higher order terms which are usually neglected, where it is assumed that Q_{t_o} is usually the only value of Q_{t_i} that is known exactly (assuming further that the initial condition is free from error).

Euler's method may therefore be written as:

$$Q_{t_{i+1}} = Q_{t_i} + \dot{Q}\Delta t \ . \qquad (C.4)$$

Equation C.4 has the form of Taylor's series expansion of Q about t_i, with the expansion truncated after the first two terms. Comparison of the complete form of Taylor's series and equation C.4 indicates terms containing $(\Delta t)^2$ and subsequent higher powers of (Δt). The error introduced in equation C.4 due to the use of the truncated equation is known as the truncation error and may be expressed as:

$$E = c(\Delta t)^2 \ . \qquad (C.5)$$

Equation C.5 implies that $\Delta t \to 0$ (or the step size becomes as small as possible) $E \to 0$ (the error becomes negligible). This further implies that as $\Delta t \to 0$ every function can be represented by linear approximations to a greater degree of accuracy.

Euler's method is a "self-starting" method. It requires the value of the dependent variable Q at only one point to start the procedure.

In many practical cases QCALC will have a positive value. In order to reduce errors by merely guessing any starting value of QCALC at the initial time t_1 QCALC may be assumed as being equal to the observed runoff value or QOBS at t_1, i.e., $QCALC(t_1)=OBS(t_1)$.

Euler's method approximates the function $Q(t)$, over the first-step interval, by a straight line. An approximate value of the slope of the curve at $t = \Delta t_1$ may be found by substituting the calculated value of Q or QCALC into the differential equation. The new calculated value of Q becomes the starting point for finding $QCALC(t_2)$;

and QCALC (t) is similarly substituted into the differential equation. This process is repeated over the desired range of integration, so that the true Q(t) or QOBS curve is approximated by a series of straight-line segments. The differences or errors that exist between QCALC and QOBS at each point may be due to some or usually all of the following factors:

(1) Inadequate mathematical representation (the differential equations) of the physical system, i.e., the conceptual and mathematical models.

(2) Errors encountered in collecting the observed data (e.g., instrumentation, poor reading, etc.).

(3) Inaccuracies associated with the particular tool used for the computations (e.g., Euler method).

The various forms of the Euler method may be summarized as:

(1) The Simple Euler Method:

$$\dot{Q} = \frac{\alpha P^{1-M}}{\beta M} - \frac{Q^{2-M}}{\beta M} \tag{C.5}$$

$$Q_{t_{i+1}} = Q_{t_i} + \dot{Q}_i \, \Delta t \tag{C.6}$$

$$Q_{t_{i+1}} = Q_i + \frac{(\dot{Q}_{t_{i+1}} + \dot{Q}_{t_i})}{2} \Delta t \, . \tag{C.7}$$

(2) Modified Euler Predictor-Corrector Method

$$\dot{Q} = \frac{Q^{1-\gamma} P}{\beta\gamma} - \frac{Q^{2-\gamma}}{\beta\gamma} \tag{C.8}$$

Predicted

$$(Q_p)_{t_{i+1}} = Q_{t_i} + \dot{Q}_i(\Delta t) \tag{C.9}$$

Corrected

$$(Q_c)_{t_{i+1}} = Q_{t_i} + \frac{\left[\dot{Q}_i + (Q_p)_{t_{i+1}}\right]\Delta t}{2} \tag{C.10}$$