OPTIMUM CONTROL OF IRRIGATION WATER APPLICATION

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ABSTRACT

The problem of controlling soil water within the root zone of irrigated crops to minimize the expected loss is examined. Control is obtained by the amount and timing of irrigations to replenish the soil water reservoir depleted by the crop's water consumption. Actual evapotranspiration rates are a function of the prevailing soil water level and the evaporative demand, which may be considered to be either deterministic or probabilistic. For crops grown on a particular soil, an optimum soil water level is defined as the lowest soil water level above which crops are not stressed. The reduced yield of a crop is related to its growth stage and to the amount and duration that the soil water content is below this optimum value.

Existing inventory models are adapted for the purpose of determining the optimal irrigation policy, that is, the timing and amount of water application that result in the maximum return to the irrigation farmer. Solutions to complex decision-making problems are currently available for a variety of irrigation situations. More realistic interpretations of the physical system are the claims made for using this new approach.

INTRODUCTION

The management of farm irrigation systems involves the choice of method, time, and quantity of irrigation water application. In this regard, farmers are faced with two types of situations. The design of a system and the general plans for its operations can be considered as one class. The continuous or periodic review of the factors that affect crop production (climate, for example) throughout the growing season represents another class. This paper focuses on the latter situation, that is, decisions based on relatively short-term predictions.

Farmers can no longer be content with the older methods for deciding when to irrigate and how much water to apply. In response to the growing urgency of more efficient utilization of water, irrigation farmers are in need of procedures that will result in their making the most favorable economic decision and still be consistent with an efficient use of water.

The problem of an optimal irrigation policy is examined using a new approach. By considering the soil water reservoir as a storehouse of goods that are consumed at varying rates and that must be replenished periodically, an analogy exists between the irrigation farmer and the business man. Both must make decisions on the timing and quantity of goods to be ordered so that a sufficient supply is on hand to meet an expected demand. The required demand may be satisfied by stocking (irrigating) once for an entire season or by stocking up separately for each time period (each growth stage of the crop).

In the former instance, an overstocking would occur with respect to one time period, while in the latter case understocking would take place with respect to the entire season. Considering the act of stocking as an irrigation, the similarity between an irrigation farmer's problem and the business man's inventory problem becomes apparent. According to Taha (1971), an inventory problem exists when it is necessary to stock physical goods for the purpose of satisfying a demand over a specified time period.

An overstock would require a greater capital outlay per unit time but less frequent occurrence of water shortages and number of irrigations. An understock, on the other hand, would necessitate less capital per unit time but would increase the frequency of irrigations and may run the risk of water shortage in a sensitive growth stage. Neither extreme will generally be the optimum case. Decisions regarding the quantity and timing of irrigations may, therefore, be based on the minimization of an appropriate loss function which balances total costs resulting from overirrigation and underirrigation. A distinct advantage for using this new approach to irrigation problems is that fairly extensive studies of inventory models have been available for a long time (Arrow, et al., 1958).

PLANT-SOIL-WATER SYSTEM

The potential yield of a given crop is an extremely complex function containing climatic, nutritional and management inputs. In this highly dynamic system, an exact quantitative evaluation of the variables affecting crop production and their interrelationships is a goal that is yet to be attained by scientists. Recent efforts, however, have led to a greater understanding of the problem to the extent that estimates of water-yield relations are being incorporated into the decision making process (Allen and Lambert, 1971; Windsor and Chow, 1971; Wu and Liang, 1972).

Predicting the Actual Moisture Use

Studies reported by Flinn (1971), Hiler and Clark (1971), Yaron (1971), Downey (1972) and Hagan and Stewart (1972) have indicated that plant growth is a function of the factors that contribute to plant water stress. In general, these studies conclude that plant water stress occurs when the actual rate of evapotranspiration (E_a) is less than potential (maximum) evapotranspiration rate (E_p). Thus, plant water stress does not occur when $E_a = E_p$, which are the conditions under which the plant is assumed to grow at its maximum rate.

The plant's potential rate of evapotranspiration is a function of the stage of growth and the atmospheric evaporative demand, i.e., the climatic variables of radiation, temperature, humidity, day length, vapor pressure, etc. Methods of estimating this rate for a particular growth stage include those that are based on evaporation from a free water surface (Penman, 1948), on pan-evaporation (McIlroy and Angus, 1964) and on an energy balance (Jensen, et al., 1970).

While the soil is able to hold water up to its saturation point, water available for plant use is generally considered to be between the soil's field capacity (FC) and the soil water content at which plants permanently wilt (PWP). The actual rate of evapotranspiration is a function of the water level in the soil (W) and the evaporative demand placed on the plant. As a means for predicting E_a, the ratio E_a/E_p is sometimes expressed as a function of E_p and W (Denmead and Shaw, 1962; Hagan and Stewart, 1972). A relationship obtained for corn by Denmead and Shaw (1962) is shown in Fig. 1. The figure shows that, under climatic conditions of low potential evapotranspiration, the available soil moisture level can drop to a lower level before the crop is stressed ($E_a/E_p < 1$)

than under conditions of a higher evaporative demand. Thus, an optimum soil moisture level (W_0) can be defined as the lowest soil water content which does not result in a plant water stress. Since climatic factors affecting evaporative demand are variable, optimum soil moisture level is also a variable quantity.

Moisture Stress - Yield Relationship

The idea that there is an optimum soil water content (W_0) above which plants are not stressed, suggests that there is also an economic optimum soil water content with respect to irrigation. Since the soil water content is dynamic, the economic level would include a tradeoff between the cost of applying water and the value of a yield reduction due to a soil moisture deficiency. Defining a moisture stress - yield function, sometimes called a water-yield function is, therefore, a necessity. Unfortunately, this relationship is not a simple one to obtain. Downey (1972) notes that yield reduction depends on the severity of the soil moisture stress and on the growth stage during which the stress occurs.

Assuming that, for a particular growth stage, yield reduction is proportional to the amount and duration of the moisture stress, Fig. 2 is a schematic illustration of a procedure for determining the extent of the yield reduction. The figure essentially consists of a plot of the actual soil water content (W_a) as a function of time and another of what the soil water content would be if water were available for the plant to be growing at its maximum potential. Note that the slopes of both curves at any point represent E_a and E_p, respectively. The yield reduction is simply proportional to the difference between these two curves integrated over time, or the shaded area as shown in Fig. 2.

As mentioned earlier, the plants' prevailing growth stage must be taken into consideration. Hiler and Clark (1971) have proposed the use of a crop susceptibility factor. Based on experimental results, this factor represents the fractional reduction in yield caused by plant water deficit at a particular growth stage. For example, using the experimental work of Denmead and Shaw (1960), Hiler and Clark calculated values for corn to be 0.25 for the vegetative growth stage, 0.50 for the silking stage and 0.21 for the ear stage. These values change for each crop.

Analysis of data presented by Bidwell (1972) tends to verify the hypothesis that yield reduction is proportional to the extent and duration of plant water stress. Crop yields and corresponding initial soil moisture and monthly rainfall amounts were available for 34 independent observations of experimental plots of corn grown in Minnesota and South Dakota. The purpose of Bidwell's analysis was to develop a model to explain the relationship between the crop, yield and hydrologic droughts. Potential evapotranspiration was excluded from this analysis.

In this paper, average values of E_p were obtained for each month of the growing season, and the rainfall was assumed to occur in a lumped fashion during each mid-period. Actual evapotranspiration rates were then estimated using the results shown in Fig. 1, which were based on work by Denmead and Shaw (1962) for corn grown under midwestern United States conditions. The data, as presented by Bidwell, were then grouped according to yield and the soil water content was calculated as a function of time for each group. Corresponding potential evapotranspiration rates were also drawn. For the sake of clarity, Fig. 3 illustrates the relative magnitude of the stress areas only for the two extreme cases, the high yield (6800 kg/ha) and the low yield (1800 kg/ha). No attempt was made to quantify the magnitude of the stress and its relationship to yield reduction. The results are presented only to demonstrate that a possible relationship exists between yield reduction and the moisture stress as herein defined.

The stage is now set to apply inventory models to the decision making process of determining when to irrigate and how much water to apply at each irrigation.

THE INVENTORY MODEL

Prior to developing the decision models for controlling soil water levels, the basic characteristics of an inventory system, as presented by Taha (1971), will be described and related to irrigation terminology. Application of inventory models as a means for determining the amount and timing of irrigation water application will then be presented for three cases. First, a single-period deterministic model is discussed and is followed by two other cases containing stochastic demands. Finally, a brief statement is made on the use of multiperiod models.

Definitions

The elements of the model may be listed as follows:

(1) Setup cost. This involves the fixed charge associated with the initial placement of an order and is usually assumed independent of the quantity ordered. For an irrigation system, this could be the labor cost in preparing for and conducting an irrigation.

(2) Holding cost. This cost stems from carrying inventory in storage and is assumed to vary directly with the level of inventory as well as the duration that the item is held in stock. By analogy, the holding cost would be the reduction in yield or quality of the crop due to water being held in the soil at a relatively high level. Water and nutrient losses from evaporation or deep percolation beyond the root zone may also be included here.

(3) Shortage cost. These are the penalty costs as a result of running out of stock when the commodity is needed. For the farm irrigation system, this would be the value of the yield reduction due to a water shortage.

(4) Demand. The demand pattern of a commodity may be either deterministic (known with certainty) or probabilistic. In the deterministic case, the quantities needed over given time periods may be constant or variable. That is, the water requirement for a particular crop may vary throughout the growing season but with known values. Probabilistic demand occurs when the demand pattern can be described by a known probability distribution function. A more realistic interpretation of a crop's requirement for water during a given growth stage is that it is a function of one or more climatic variables for which probability distributions are assumed to be known.

The demand for a given period of time may be satisfied instantaneously at the beginning of the period or uniformly during that period. The effect of these demands should reflect directly on the total cost of holding the inventory, or in other words, keeping the water in the soil for future consumption.

(5) Ordering Cycle. This may be identified by the time period between two successive placements of orders (or irrigations) which may be initiated on either a continuous or periodic review. In the former case, a record of the stock on hand (water in the soil) is updated continuously until a certain lower limit is reached at which point an irrigation is ordered. For the periodic review situation, orders are placed usually at equally spaced intervals of time. Finally, the time between the ordering of irrigation water and its storage in the soil water reservoir may be called delivery lag or lead time.

Deterministic Case

Consider a growth stage j, during which N_j irrigations take place. Assuming that E_p is a constant throughout the stage, it follows that

$$E_a = E_p \qquad \text{for } W > W_o$$

$$E_a = \text{a constant} < E_p \qquad \text{for } W < W_o$$

Referring to Fig. 2, the model seeks to determine (Q+S), the quantity of water per irrigation N_j, and the number of irrigations that will minimize the cost of irrigation and the crop yield loss due to a water stress. Note that the number of irrigations per growth stage is inversely proportional to (a+b), the time per irrigation. From the geometry of Fig. 2, it can be shown that

$$N_j = \frac{kE_p}{S + kQ} \qquad (1)$$

in which $k = E_a/E_p$ and $D = S(\frac{1}{k} - 1)$.

The cost of irrigation during period j, C_j, is the sum of the following costs which are related to either the number of irrigations or the quantity of water:

1. The cost for labor per irrigation, C_1, involved in preparing for an irrigation and in applying the water. It is similar to the setup cost in an inventory problem and includes the cost of moving pipe, clearing ditches, setting up siphon tubes, adjusting gates, starting pump, etc. The total cost for period j is

$$C_1 N_j = C_1 \frac{kE_p}{S + kQ} \qquad (2)$$

2. A water cost function of the quantity of water to apply per irrigation, Q + S. With C_w, a water cost per unit applied, this amounts to

$$C_w N_j (Q + S) = C_w (Q + S) \left(\frac{kE_p}{S + kQ} \right) \tag{3}$$

3. A "holding" cost which is related to the time and amount that water is stored in the soil above the optimum level (from the yield viewpoint). These costs may be attributed to cases when both water and fertilizer percolate below the root zone and to a possible yield reduction for certain crops whose maximum growth is produced at soil water levels below the field capacity of the soil. For a unit holding cost C_h, this would amount to

$$C_h \cdot \frac{Qb}{2} \cdot N_j = C_h \cdot \frac{Q}{2} \cdot \frac{Q}{E_p} \cdot \frac{kE_p}{S + kQ} \tag{4}$$

$$= C_h \frac{Q^2}{2} \cdot \frac{k}{S + kQ}$$

4. The cost corresponding to reduction of gross income caused by a shortage of water during stage j. With C_s as the unit shortage cost, this cost is taken to be proportional to the amount and duration of the shortage (the cross-hatched area shown in Fig. 2). This cost amounts to

$$C_s \cdot \frac{Da}{2} \cdot N_j = C_s \cdot \frac{S}{2} \left(\frac{1}{k} - 1\right) \frac{S}{E_a} \left(\frac{kE_p}{S + kQ} \right) \tag{5}$$

$$= C_s \frac{S^2}{2} \left(\frac{1-k}{k}\right) \left(\frac{1}{S + kQ}\right)$$

Adding the expressions (2) through (5) results in C_j as a function of Q and S. The optimum values of Q and S are then obtained by solving the expressions resulting from equating the partial derivatives of C_j to zero. This solution will result in obtaining the amount of water to apply at each irrigation (Q + S) along with the number of irrigations required during the j^{th} stage.

Stochastic Case with Lumped Consumption

The above model is based on a deterministic demand for water and on no soil water replenishment from rainfall. In the lumped consumption model, it is assumed that the water demand is to be filled at the beginning of a given time period corresponding to the growth stage of a crop. Furthermore, the demand, consumed instantaneously, is a random variable z with a known probability density function f(z).

In areas where rainfall occurs during the growing season, f(z) may be expressed by the joint distribution function of the rainfall and an index of potential evapotranspiration such as pan evaporation. For the semiarid conditions of the southwestern region of the USA, it was found that pan evaporation rates could be grouped into two classes corresponding to days during which rain occurred and days when no rain occurred. It was found that evaporation rates differed significantly between these groups. No difference was observed in pan evaporation for varying amounts of rainfall. Thus, the pdf of z may be obtained from

$$P(E_i) = P(E_i|W)\ P(W) + P(E_i|D)\ P(D) \qquad (6)$$

where

$P(E_i|W)$ = probability of a given rate of daily pan evaporation E_i on wet days.

$P(E_i|D)$ = probability of a given rate of daily pan evaporation E_i on dry days.

$P(W)$ = p = probability of a wet day.

$P(D)$ = 1-P = probability of a dry day.

Assuming that there is a direct relationship between pan evaporation and the potential evapotranspiration rate of the plant, f(z) may be written as

$$f(z) = f(z|W)\ p + f(z|D)\ (1-p) \qquad (7)$$

In addition to the costs presented in the deterministic model, two other values require definition for the stochastic case. These are B_c, the unit benefit of the crop from applying irrigation water prorated for the given growth stage and B_r, the salvage value of the water left in the soil for the next growth period. Let

x = the amount of soil water on hand prior to an irrigation

y = the water available in the soil after an irrigation of (y-x) units of water, $y \geq x$

The expected value of the loss function is given by

$$L(y) = L(y;\ x,\ C_1) = C_1 + C_w(y-x)$$

$$+ \int_0^y [C_h(y-z) - B_r(y-z) - B_c z]\ f(z)dz \qquad (8)$$

$$+ \int_y^\infty [C_s(z-y) - B_c y]\ f(z)dz$$

The optimum value of y is y* and is found by equating the first derivative of L(y) to zero. Thus,

$$L'(y) = C_w + \int_0^y (C_h-B_r)\ f(z)dz - \int_y^\infty (C_s-B_c)\ f(z)dz = 0$$

or

$$F_z(y^*) = \frac{C_s + B_c - C_w}{C_h + C_s + B_c - B_r} \qquad (9)$$

where F_z is the cumulative distribution function of z.

If the labor cost C_1 is zero or is considered negligible, the loss function L(y;x,o) as shown in Fig. 4 is

$$L(o) = -C_w x + s\ X(z) \qquad (10)$$

in which X(z) is the expected value of z. If C_1 is not zero, define

Q by y*
q by $L(q;x,o) = L(Q;x,C_1)$
and $L(q;x,o) = L(Q;x,o) + C_1$

It can then be shown that the optimum irrigation policy is

irrigate up to Q if $x \leq q$
do not irrigate if $x > q$

Note that the cost functions do not have to be linear. However, a solution is certain to exist only if $L(y;x,C_1)$ is convex in y.

Stochastic Case with Distributed Consumption

Instead of plants using the available water instantaneously, it is possible to treat the case of a distributed evapotranspiration rate during the irrigation period considered. The stress period, therefore, would have a random length, and the irrigation intervals would vary depending on the demand. Again, at the expense of algebraic simplicity, the resulting value of $y^* = Q$ would be given by

$$F(Q) + Q \int_Q^{\infty} \frac{f(z)}{z}\, dz = \frac{C_s + B_c - C_w}{C_h + C_s + B_c - B_r} \qquad (11)$$

While the above two models are of single-period variety, a multi-period recursive equation can be written and solved theoretically, if not always numerically, by a dynamic programming algorithm (Taha, 1971, p. 404)

$$g(j,x_j) = \max_{y>x_j} \Big[L(y;x_j,C_1) + \int_0^y g(j+1,y-z)\, f(z)dz + \int_y^{\infty} g(j+1,0)\, f(z)dz \Big] \qquad (12).$$

Results will be presented in subsequent papers on this subject.

DISCUSSION AND CONCLUSIONS

It is well recognized that solutions to the complex problem of controlling soil moisture to produce optimum results require simplifying assumptions, primarily in modeling the plant-soil-water system. One such assumption concerns prediction of the actual rate of water consumption by crops. In this instance, the actual rate of evapotranspiration of a particular crop grown on a given soil is presumed to be a function of the prevailing soil water content and the atmospheric evaporative demand. Data on climatic factors, in turn, are assumed to be available for predicting the demand.

For the purpose of simplification, Fig. 2 has been constructed to illustrate a uniform rate of evapotranspiration, although it is known that this rate decreases as the available soil water content approaches zero. A uniform rate is not a necessity insofar as the methodology is concerned, however, as it can handle a curvilinear relationship.

Prediction of the rate of soil water depletion presents a practical problem to the farmer. Maintaining a record of the physical goods on hand, that is, the soil water available for plant use, is not as simple for the farmer as it may be for many businessmen. For one, the soil water reservoir is not clearly defined. The depth of the root zone is a function of the plant's development; and the upper limit of the soil's ability to hold water is an arbitrary value called the field capacity. Plants extract water unevenly from different soil depths and the spatial variation of moisture in the soil water reservoir may be as much as one or two orders of magnitude. The problem is further confounded by the unavailability of a simple, reliable and accurate instrument for measuring the soil water content.

A more serious problem concerns the lack of information relating the plant's water stress at a given stage of growth to ultimate yield of the crop. The use of any model from control theory is contingent on such information.

The major emphasis of this paper is not on the above related problems but on the use of inventory models to assist irrigation farmers in making decisions on when to irrigate and how much water to apply. The advantage of using such methodology is that solutions to complex problems are available for a variety of situations. It is the contention of the authors that an analogy does exist between a business that stocks physical goods and a farm irrigation system in which the soil is a reservoir that stores water for future use by the plant. The demand on this reservoir may be either deterministic or stochastic. Without specific mention of inventory models, Wu and Liang (1972) used a similar approach to determine the timing and frequency of irrigations based on a uniform rate of evapotranspiration and a deterministic demand. Inventory models are not limited to these simplifications.

In summary, it is concluded that inventory models can be adapted to the decisions involving the efficient operation of farm irrigation systems. With the availability of such solutions, complex problems may be attacked in a straightforward, logical manner. For example, the stochastic nature of the climatic variables that affect both the potential and actual use of water and the lag time in delivering water to the soil are readily brought into the picture with the use of inventory models. In those areas where rainfall may modify irrigation practices, joint distributions of rainfall and evaporation demand can also be handled even though the data required to estimate such distributions are frequently inadequate. The fact that plants vary in their response to a water stress with their period of growth is no deterrent to the use of inventory models.

Single-period and multi-period models are both available for adaptation to special conditions. A multi-product inventory model may be used to simulate the simultaneous storage of water fertilizer and salts in the soil. Yaron, et al. (1972) have pointed out the importance of salinity on yield. Dissolved salts may be considered as another good held in inventory with a very high holding cost. Finally, by allowing for the stochastic properties of climatic variables, these models constitute a framework for a more rational design and operation of water resource systems that supply the irrigation water to the farmer.

ACKNOWLEDGEMENTS

This study was partially supported by an Agency for International Development 211 (d) Grant "Optimum Utilization of Water Resources for Agriculture" and by an Office of Water Resources Research Title I project "Decision Analysis for Watershed Management Alternatives,"

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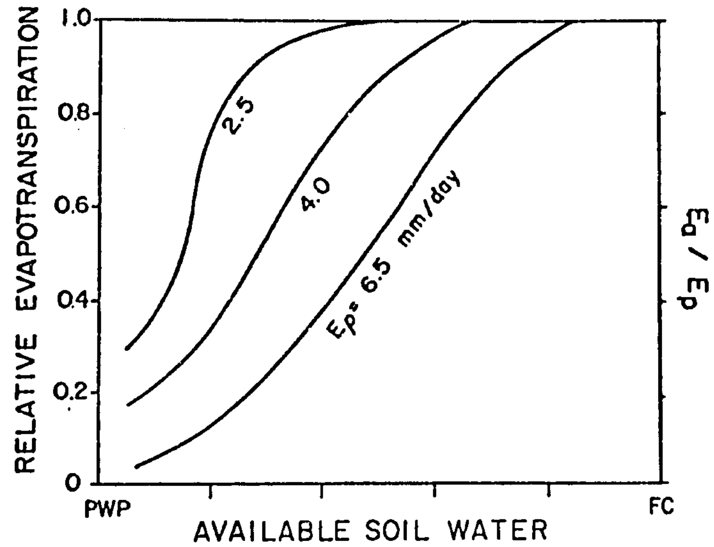

Fig. 1. Relative evapotranspiration rates as a function of potential demand and soil water content (after Denmead and Shaw, 1962).

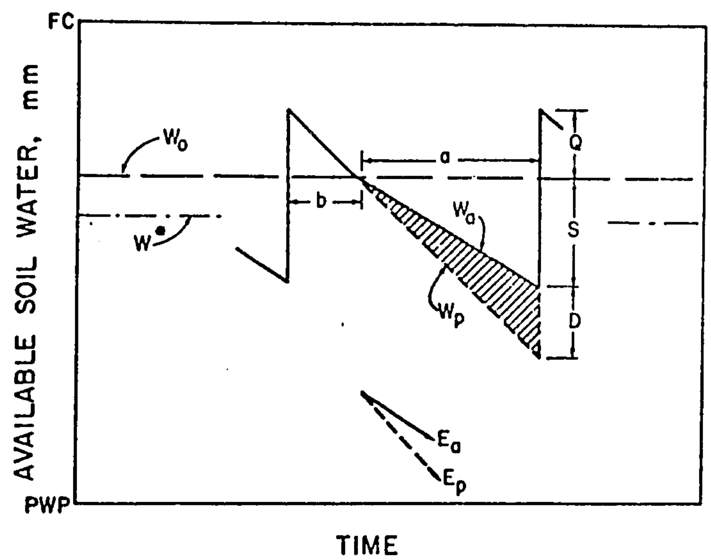

Fig. 2. Schematic representation of soil water depletion curve between irrigations as applied to development of inventory models to determine optimum soil water content.

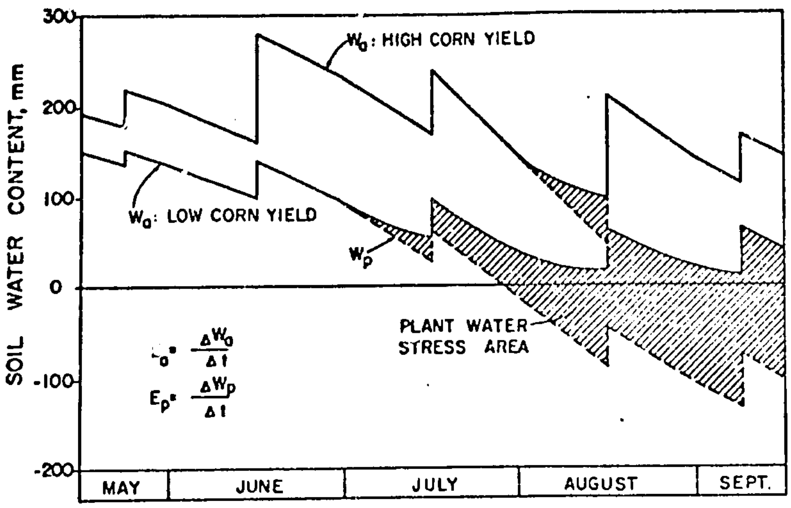

Fig. 3. Simplified version of seasonal fluctuation of soil water content for two extreme cases of corn production.

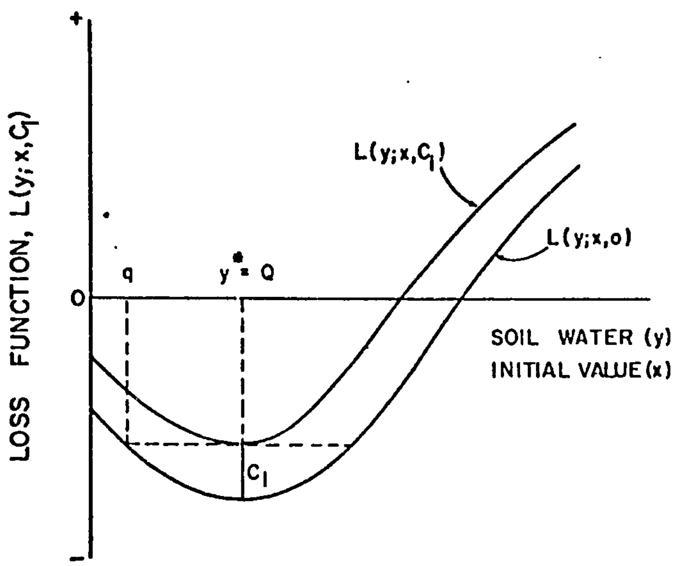

Fig. 4. Relationship of loss function (with and without setup cost) to soil water content to determine optimum irrigation policy.