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ON SOLOW'S METHOD OF ESTIMATING THE RESIDUAL

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On Solow's Method of Estimating the Residual

Since the pioneering works of Abramowitz, Schultz, Solow, and Tinbergen [Abramowitz, 1956, pp. 5-23; Schultz, 1956, pp. 748-762; Solow, 1957, pp. 312-320; Tinbergen, 1959, pp. 182-221], quite a number of economists have made efforts to separate the contribution of technical change to changes in output from the expected output increments due to the increased application of conventional inputs. Because of its simplicity, Professor Solow's method of separating shifts of, from movements along, a given production function is the best-known of several approaches [Solow, 1957, pp. 312-320]. More recently, the task of identifying the sources of the residual entailed in the process of growth has attracted much attention and provided some excitement. Indeed, Jorgenson and Griliches, after an impressive train of adjustments and corrections, almost succeeded in making the residual disappear altogether [Jorgenson and Griliches, 1967, pp. 249-283]. In this note we attempt to show that the computed residual based on Solow's method actually depends on how the dependent variable in the production function is defined, and that the residual has a "built-in" upward bias if value-added is chosen as the dependent variable. This bias aspect is particularly acute when Solow's method is applied to a disaggregated model.

In his much celebrated and now classic paper [Solow, 1957, pp. 312-320], Professor Solow has shown that we can write an aggregate production function as follows:

$$Q = F(K,L;t), \tag{1}$$

where Q represents output, and K and L capital and labor input, respectively; t stands for time, which allows for technical change, Solow's shorthand expression for anything that causes a shift of the function. If the nature of the shift is assumed to be neutral, equation (1) can be written as:

$$Q = A(t)f(K,L). \tag{2}$$

By differentiating equation (2) totally with respect to time and dividing by Q, Solow obtained the following fundamental equation:

$$\frac{\dot{Q}}{Q} = \frac{\dot{A}}{A} + w_K \cdot \frac{\dot{K}}{K} + w_L \cdot \frac{\dot{L}}{L}. \tag{3}$$

Dots indicate time derivatives and w_K and w_L the capital and labor shares in final output. If a discrete year-to-year approximation is used instead, we can write equation (3) as:

$$\frac{\Delta Q}{Q} = \frac{\Delta A}{A} + w_K \frac{\Delta K}{K} + w_L \frac{\Delta L}{L}. \tag{4}$$

This equation allows us to decompose the observed output growth into its component parts, to show how much growth is due to $\Delta A/A$, the rate of technical change (or the rate of shift of the production function), and how much is due to changes in capital and labor inputs.[1] As is well known, if we postulate an aggregate production function of the Cobb-Douglas type:

$$Q = A \cdot K^{\alpha} \cdot L^{\beta}, \qquad (\alpha + \beta = 1)$$

take logarithms and differentiate, and define $\alpha = w_K$ and $\beta = w_L$, we can reach the same result as given by equation (4). Although this is all quite familiar, the repetition is made here for the purpose of facilitating the discussion that will immediately follow.

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[1] See Solow, ibid., for proof.

Now suppose we are interested in the source of output growth in a two sector model. Sector one produces goods destined for final use (Q) and sector two produces intermediate goods (M).[2] True to the neoclassical tradition, factor and product prices are competitively determined; both sectors are subject to constant returns to scale and neutral technical progress (though not necessarily at the same rate). Thus we need have no hesitation in using Cobb-Douglas type production functions to represent the sectoral relationships between inputs and outputs.[3] The gross output in physical units of each sector is therefore given by

$$Q = A_1 K_1^{\alpha_1} L_1^{\beta_1} M_1^{\gamma_1}, \tag{5}$$

and

$$M = A_2 K_2^{\alpha_2} L_2^{\beta_2} M_2^{\gamma_2} \tag{6}$$

where $\alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = 1.$

The distribution of value added (or final output) in the economy as a whole is given by

$$Q_v = AK^{\alpha} L^{\beta} \tag{7}$$

where $\alpha + \beta = 1$. This is simply the familiar aggregate production function for an economy with two primary factors.

Now let the basic unit of measurement of Q be the numéraire, so that Q_v is numerically equal to Q; and let π be the price of M in terms of Q,

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[2] Q and M do not have to be single, homogeneous commodities with the Q dependent on the M sector, but not conversely. A many-good economy may be represented in the same way, where Q and M are bundles of commodities of differing composition and, for the disaggregation to be relevant, subject to different production functions.

[3] We should note that M_1 and M_2 are identical and homogeneous goods; the subscriptions pertaining to M's merely indicate the sector in which the goods are allocated.

thus the value of gross production measured in units of final goods is $(Q + \pi M)$. In competitive equilibrium, the value of marginal product of each input must be the same in all sectors. For rentals on capital to be instantaneously equal in both sectors,

$$\frac{\partial Q}{\partial K_1} = \frac{\pi \partial M}{\partial K_2}$$

or

$$\frac{\alpha_1 K_2}{\alpha_2 K_1} = \frac{\pi M}{Q} \tag{8}$$

Similarly,

$$\frac{\beta_1 L_2}{\beta_2 L_1} = \frac{\pi M}{Q} . \tag{9}$$

As $\frac{\partial Q}{\partial M} = \pi = \frac{\gamma_1 Q}{M_1}$, from (5)

and $\frac{\partial M}{\partial M_2} = \frac{\gamma_2 M}{M_2} = 1$, from (6)

so that $M = M_1 + M_2 = M_1 + \gamma_2 \cdot M$

Hence

$$\pi = \frac{\gamma_1}{1-\gamma_2} \cdot \frac{Q}{M}, \tag{10}$$

which is independent of time if and only if Q/M is always constant, that is, if there is always balanced growth. This is the simplest case to analyze and the only one we shall consider here in any detail. Moreover, as balanced growth is the context for most of the work in this field, it is a conventionally defensible and highly congenial restriction. To clear the ground further, we note that balanced growth requires

$$\frac{\dot{Q}}{Q} \left(= \frac{\dot{Q}_v}{Q_v}\right) = \frac{\dot{M}}{M} = \frac{\dot{M}_1}{M_1} = \frac{\dot{M}_2}{M_2} .$$

From (8), (9), and (10), we have

$$\frac{K_2}{K_1} = \frac{\alpha_2}{\alpha_1} \cdot \frac{\gamma_1}{1-\gamma_2}$$

Let $\frac{\gamma_1}{1-\gamma_2} = \lambda$,

so that $$\frac{K_2}{K_1} = \frac{\alpha_2}{\alpha_1} \cdot \lambda \tag{11}$$

and likewise for $\frac{L_2}{L_1}$. Thus, as each factor is distributed between sectors in the same proportion over time, $\frac{\dot{K}_1}{K_1} = \frac{\dot{K}_2}{K_2} = \frac{\dot{K}}{K}$, amd similarly for L. It will be convenient, and realistic, though not necessary for the analysis which follows, to assume that none of K and L are actually declining over time, and that at least one of them is actually increasing. What actually determines their rate of growth does not matter for the problem at hand.

Before passing on, it should be noted that (10) implies

$\frac{\pi M}{Q} = \frac{\gamma_1}{1-\gamma_2} = \lambda$, which is constant. Thus even in unbalanced growth, the ratio of the values of gross production at current prices of the two sectors will remain constant, the price of M in terms of Q adjusting so as to exactly offset divergences in their rates of growth of output in physical units.

Now to factor shares. The total payment to each primary factor as given by (7) must equal the sum of the payments made to that factor in each sector. For capital,

$$K\frac{\partial Q_v}{\partial K} = K_1\frac{\partial Q}{\partial K_1} + K_2 \cdot \pi \cdot \frac{\partial M}{\partial K_2}$$

Using (8), we have

$$K\frac{\partial Q_v}{\partial K} = K_1\frac{\partial Q}{\partial K_1} + K_2\frac{\partial Q}{\partial K_1} \cdot$$

Dividing both sides by Q_v (or Q), and recalling that $Q_v = Q$, we obtain, by (11)

$$\alpha = \alpha_1 + \lambda\alpha_2 \tag{12}$$

By symmetry,

$$\beta = \beta_1 + \lambda\beta_2 \cdot \tag{13}$$

These relationships have some intuitive appeal. The share of each factor in total value added is the sum of its output elasticity in sector one and its output elasticity in sector two adjusted by the ratio of the values of gross production in the two sectors.[4] It follows immediately from (12) and (13) that α and β are constant whether there is a regime of balanced growth or not.

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[4] A useful check on this derivative of α and β is given by summing (12) and (13).

$$\begin{aligned} 1 = \alpha + \beta &= (\alpha_1 + \beta_1) + \lambda(\alpha_2 + \beta_2) \\ &= (1 - \gamma_1) + \lambda(1 - \gamma_2) \\ &= (1 - \gamma_1) + \frac{\gamma_1}{(1 - \gamma_2)}(1 - \gamma_2) \\ &= 1. \end{aligned}$$

The next step is to find the residual which would result if Solow's method were applied to gross production instead of value added. In balanced growth, $\dot{Q}_v/Q_v = \dot{Q}/Q = \dot{M}/M$, so that $\dot{Q}_v/Q_v$ is equal to any convex combination of $\dot{Q}/Q$ and $\dot{M}/M$. The weights we require are the proportions of the value of gross production in the economy as a whole accounted for by Q and M, respectively. Thus we may write

$$\frac{\dot{Q}_v}{Q_v} = \frac{Q}{Q+\pi M} \cdot \frac{\dot{Q}}{Q} + \frac{\pi M}{Q+\pi M}\frac{\dot{M}}{M} = \frac{1}{1+\lambda}\left(\frac{\dot{Q}}{Q} + \frac{\lambda \dot{M}}{M}\right) \tag{14}$$

Differentiating (5), (6), and (7) logarithmically, and substituting for $\dot{Q}_v/Q_v$, $\dot{Q}/Q$, and $\dot{M}/M$ in (14), we have:

$$\frac{\dot{A}}{A} + \alpha\frac{\dot{K}}{K} + \beta\frac{\dot{L}}{L} = \frac{1}{1+\lambda}\left([\frac{\dot{A}_1}{A_1} + \alpha_1\frac{\dot{K}_1}{K} + \beta_1\frac{\dot{L}_1}{L} + \gamma_1\frac{\dot{M}_1}{M_1}] + \lambda[\frac{\dot{A}_2}{A_2} + \alpha_2\frac{\dot{K}_2}{K_2} + \beta_2\frac{\dot{L}_2}{L_2} + \gamma_2\frac{M_2}{M_2}]\right)$$

Define A* to be the difference in the value-added and gross production residuals, that is,

$$A^* = \frac{\dot{A}}{A} - \frac{1}{1+\lambda}\left[\frac{\dot{A}_1}{A_1} + \frac{\lambda \dot{A}_2}{A_2}\right] \tag{16}$$

Recalling that $\frac{\dot{K}_1}{K_1} = \frac{\dot{K}_2}{K_2} = \frac{\dot{K}}{K}$, $\frac{\dot{L}_1}{L_1} = \frac{\dot{L}_2}{L_2} = \frac{\dot{L}}{L}$, and $\frac{\dot{M}_1}{M_1} = \frac{\dot{M}_2}{M_2} = \frac{\dot{M}}{M}$, and using (12) and (13), (15) can be simplified to

$$A^* = \frac{1}{(1+\lambda)}(\gamma_1 + \lambda\gamma_2)\frac{\dot{A}}{A} + \left[\frac{(\gamma_1+\lambda\gamma_2)}{(1+\lambda)} - \frac{\lambda}{(1+\lambda)}\right]\alpha\frac{\dot{K}}{K} + \beta\frac{\dot{L}}{L}] .$$

But $$\frac{(\gamma_1+\lambda\gamma_2)}{(1+\lambda)} - \frac{\lambda}{(1+\lambda)} = 0$$

Hence $$A^* = \frac{1}{(1+\lambda)}(\gamma_1 + \lambda\gamma_2)\frac{\dot{A}}{A} , \tag{17}$$

which is independent of $\frac{\dot{A}_1}{A_1}$ and $\frac{\dot{A}_2}{A_2}$. Now the residual associated with value-added will give a biased estimated from that associated with gross production if $A^* \gtrless 0$. A sufficient and necessary, condition for $A^* \gtrless 0$ is that $\frac{\dot{A}}{A} \gtrless 0$. Thus, we conclude that on the plausible condition that $\frac{\dot{A}}{A}$ is positive, the value-added residual will give an estimate of the gross production residual that is biased upward.

The extent of the bias can be computed easily enough. Substituting (16) in (17), rearranging and simplifying, we have

$$\frac{\dot{A}}{A} = \frac{\dot{A}_1}{A_1} + \lambda\frac{\dot{A}_2}{A_2} . \tag{18}$$

That means, according to (18), the extent of the opward bias is measured by $\lambda\frac{\dot{A}_2}{A_2}$, the rate of technical change in the intermediate goods-producing sector multiplied by λ.

Applying the analysis to the development frame, we note that λ, and therefore the extent of the upward bias, is higher for the developed than for the developing economies.

The difference between the estimated residuals raises this question: Is the production function defined in terms of gross production more relevant and appropriate than the one defined in terms of final output in order to obtain an accurate measurement of technical change? The choice of what is to be included and what is to be excluded in the production function when Solow's method is applied seems to be much more than a simple matter of taste. As we have demonstrated under conditions of balanced growth, the rate of technical change derived from the production function with final output as the dependent variable is higher than that

obtained from the function with gross production as the dependent variable. In a sense, part of the estimated residual derived from equation (7) has been artificially introduced by excluding intermediate products as an element from the production function. In other words, the production function written in this traditional way may lead to an upward biased estimate of productivity change.

Presumably the relationship between intermediate products and technical change is more complicated and involved than what we have been accustomed to assume. It is indeed tempting to speculate that this relationship may resemble the one between technical change and capital formation.[5] In our search for ways to decompose the mysterious residual, we are in essence looking for the elements that are missing from the production function specified in the traditional form. Apparently we have omitted intermediate products from the aggregate production function too readily. The omission leads to the misidentification of the sources of technical change when Solow's method is applied to a disaggregated model and value-added is chosen as the dependent variable.

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[5] The so-called "Disembodiment versus embodiment" debate. See, for example (1) Kenneth J. Arrow, "The Economic Implications of Learning by Doing," Review of Economic Studies, June 1962, pp. 155-173; (2) Edward F. Denison, The Sources of Economic Growth in the United States and the Alternatives Before Us. Supplementary Paper No. 13. New York, Committee on Economic Development, 1962; (3) Edward F. Denison, "The Unimportance of the Embodied Question," American Economic Review, March 1964, pp. 90-94. (4) Benton F. Massell, "A Disaggregated View of Technical Change," Journal of Political Economy, December 1961, pp. 547-557; (5) Richard R. Nelson, "Aggregate Production Functions and Medium Range Production Functions," Rand RM-3912-PR. Santa Monica, 1963; (6) Richard R. Nelson, "Aggregate Production Function and Medium-Range Growth Projections," American Economic Review, September 1964, pp. 575-606; (7) E. S. Phelps, "The New View of Investment: A Neoclassical Analysis," Quarterly Journal of Economics, November 1962, pp. 548-567; and Robert M. Solow, "Technical Progress, Capital Formation, and Economic Growth," American Economic Review, Papers and Proceedings, May 1962, pp. 76-86.

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