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ELEV TION EFFECTS ON RAINFALL: A STOC ASTIC MODEL

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ELEVATION EFFECTS ON RAINFALL: A STOCHASTIC MODEL

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ABSTRACT

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The variation in point precipitation with elevation is investigated using an event-based stochastic model of thunderstorm rainfall and empirical data. Parameters of the model correspond to the number of events per unit of time and the depth of rainfall per event. An increase in precipitation with elevation may be due to an increase in the number of events, in the amount of rainfall per event or to some combination of both possibilities. The distribution of the number of events per season is assumed to be a Poisson variate while the distribution of point rainfall depths may be taken as geometric. The summation of a random number of random variables is used to represent seasonal point precipitation. Assuming that the two parameters of the model increase linearly with elevation, then total seasonal rainfall increases as a quadratic polynomial with elevation. The use of the model allows one to obtain the return period of storm rainfall of a given magnitude despite a short historical record. An independent set of data was used to verify the procedure.

INTRODUCTION

Elevation of the precipitation gaging station is considered to be one of several topographic parameters that affect annual or seasonal totals (Wiesner, 1970). In one of the most detailed studies of orographic influences on precipitation, Spreen (1947) correlated mean seasonal precipitation with such factors as elevation, rise, exposure and orientation for western Colorado. Elevation alone accounted for 30% of the variation in precipitation while all five parameters together explained 85% of the variation. Schermerhorn (1967) also noted that for western Oregon and Washington station elevation alone does not explain much of the variation in annual precipitation which predominantly occurs in winter. Indices of terrain elevation and barrier elevation with the latter being difficult to define in a simple, objective fashion, together with a latitude index explains most of the variation.

Hamon (1971) suggests that the weak correlations in precipitation–elevation relationships "are attributable to the inability of the unshielded gage to capture the falling precipitation, particularly snow, when exposed to even light winds". Using data from a dual-gage network in southwestern Idaho, he states that the catch in the shielded gages are 23% greater than the unshielded catch and that computed values of "actual" precipitation are increased by 43% over the unshielded catch. The computed values are derived using an empirical relationship involving both unshielded and shielded catches.

When the annual precipitation is separated into that resulting from winter frontal systems and summertime air mass convective storms, it is noted that winter precipitation is increased more than four-fold at the 7000-ft. elevation as compared to that found at 4000 ft. For these same elevations, summer rainfall is not quite doubled. Thus, it appears that analyses using annual precipitation data may also introduce considerable bias and that individual storm analysis offers the soundest basis for determining topographic effects on precipitation.

In all such studies, little or no attempts have been made to determine whether the increase in precipitation with elevation is due to an increase in the number of storms that occur, or in the amount of precipitation per storm or to some combination of both possibilities. Hamon (1971) reports on a single winter storm in which precipitation increases linearly from 0.4 inch at 4000-ft. to about 1.2 inches at the 7000-ft. elevation.

This paper considers summer rainfall only, that is, essentially rainfall caused by air mass convective storms occurring in July, August and September. To verify the model developed, data from a cloud seeding experiment designed to investigate the possibility of increasing thunderstorm rainfall in the Santa Catalina Mountains near Tucson, Arizona (32° 25′ N 110° 45′ W) were used (Battan and Green, 1971).

BASIC RAINFALL PROBABILITY MODEL

Event-based models are characterized by at least two random variables and their distribution. First, there is the random number of events per unit of time (season or year) designated as N. An alternate choice would be the interarrival time between events. Then there are one or more of the random variables of interest such as the depth of rainfall R or the duration of rainfall D. If more than one such random variable is of interest, then a joint distribution would be needed.

Number of events per season

Thunderstorms seem to occur in an independent manner in time and space such that the number of rainfall events per season N follows a Poisson probability mass function (Brooks and Carruthers, 1953). A similar result was obtained in a study by Todorovic and Yevyevich (1969). The probability mass function for N can thus be written as:

$$f_N(j) = \frac{e^{-m} m^j}{j!} \qquad j = 0, 1, 2, \ldots \tag{1}$$

where m is the mean number of events per season.

For this paper, an event is defined as the occurrence of at least 0.01 inches of rain at a gage; if one hour or more separates the recording of such rainfall, this results in the counting of two distinct events. In terms of an earlier paper (Duckstein et al., 1972), this would be a new event definition. The first definition required a dense rain gage network while the second definition required at least two gages within 3–10 miles from one another. In this way, the more intense thunderstorm rainfall may be distinguished from the more uniform frontal-type rainfall. This situation presents no such problem in the case herein as summer rainfall is almost always from showers and thunderstorms of a local nature (Sellers, 1964). Inasmuch as only point rainfall is being considered, there is no way of inferring how much rainfall may have occurred with a few miles of the point being considered without recording anything at this gage. For this reason, in the absence of any proof to the contrary, the structure of thunderstorm cells is assumed to be invariant within the range of elevation being considered (2500–8600 ft.). That is, distinct cells cause rains over an area roughly circular with a diameter generally between 4–6 miles and the depth of rainfall is maximum at the center and decreases rapidly from there (Fogel and Duckstein, 1969; Osborn and Lane, 1972).

Depth of point rainfall

As shown in an earlier paper (Fogel and Duckstein, 1969), a geometric probability mass function for point rainfall depths was derived from an analysis of a dense network of rain gages. Thus, for j units of rain, the distribution of the depth of rainfall R is:

$$f_R(j) = (1-p)p^j \qquad j = 0, 1, 2, \ldots \tag{2}$$

where p, in this instance, is the probability of success of the point in question receiving any amount of rain above a given threshold value including zero.

Discussion of areal rainfall

Consider that a thunderstorm system containing several cells is present over a watershed during the period considered, say 24 h. If the measuring points, rain gages, are sufficiently far apart so that the measurements at any two gages comes almost always from two different cloudbursts, it can be said that the system has spatial independence and temporal dependence.

For a given storm, the areal rainfall B, or the mean rainfall over a given area, may be considered as the average of a fixed number of n point rainfall measurements $R_1, R_2, \ldots, R_n$. Since the average amount of rainfall for the n gages differs from the sum of the n gages only by a constant scaling factor, it is possible to use this latter value in developing a probability mass function for areal rainfall.

If the point rainfall variates $R_1, R_2, ..., R_n$ are mutually independent with an identical probability mass function (eq. 2), then it has been shown that the areal rainfall B has a negative binomial distribution with parameters p and $r = n$ (Fogel et al., 1971).

$$f_B(j) = \binom{r+j-1}{j} (1-p)^r p^j \tag{3}$$

Total seasonal rainfall

It is sometimes desirable to determine the mean and variance of total precipitation during a given time period. This is readily done with the approach taken herein.

Let Z be the total number of units of rainfall for a summer season:

$$Z = R_1 + R_2 + ... + R_N \tag{4}$$

where $R_1, R_2, ..., R_N$ are mutually independent identically distributed random variables. Regardless of the distribution or independence of R and N, the mean or expected value $E(Z)$ is given by:

$$E(Z) = E(N)\,E(R) \tag{5}$$

If R and N are independent, then the variance $VAR(Z)$ is given by:

$$VAR(Z) = VAR(N)\,[E(R)]^2 + [E(N]\,VAR(R) \tag{6}$$

That the total seasonal rainfall Z can come from various combinations of point rainfall depths per event R and the number of events N during a season is readily apparent. A table may be arranged to produce these combinations with the rows corresponding to the number of events per year, and the columns to the total seasonal rainfall. Cell (j, k) would thus represent the joint probability of j units of rain and k events per year. Within each cell, different occupancy distributions of storms are possible. Using classical combinatorial analysis techniques, the probability of each of these occupancies can be calculated (Feller. 1957).

A Monte Carlo simulation can now be set up to generate an unordered succession of yearly combination of events. In those instances where the order of event occurrence is of importance, it can be incorporated into the simulation to generate a time series of events.

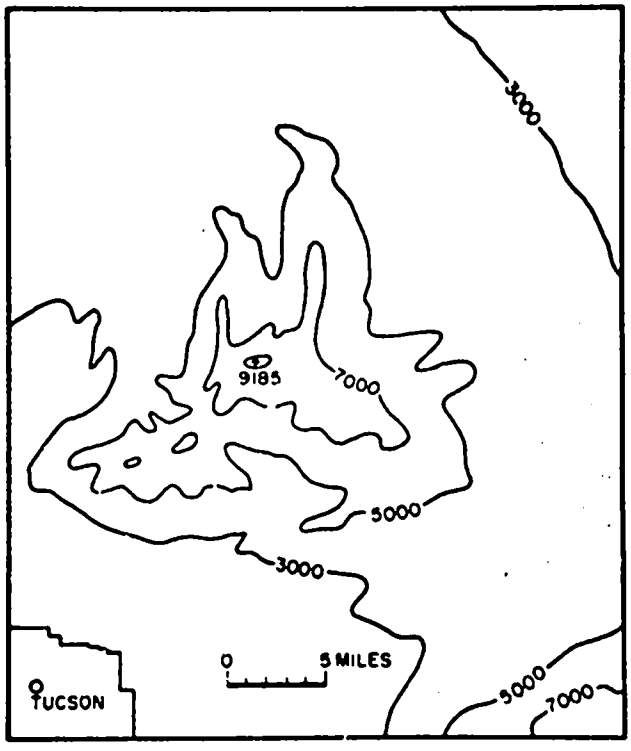

Fig. 1. General topography of the Santa Catalina Mountains in southern Arizona. (After Battan and Green, 1971.)

EXPERIMENTAL DATA

As a means for evaluating a series of cloud seeding experiments, the Institute of Atmospheric Physics of the University of Arizona operated a network of 29 recording gain gages distributed over the Santa Catalina Mountains in southern Arizona (Battan and Green, 1971). This mountain mass is a more less isolated block which attains an altitude nearly 7000 ft. abouve the city of Tucson located some 20 miles to the southwest (Fig. 1). The rainfall data were initially collected during the summer months of the years 1957 through 1960. After the first set of measurements, some gages were moved and six others were added, making a new network of 35 stations. Measurements were made during the summers of 1961, 1962 and 1964. An illustration of the correlation with station elevation is shown if Fig. 2, whereas Fig. 3 presents the results of a linear regression analysis using all the available data.

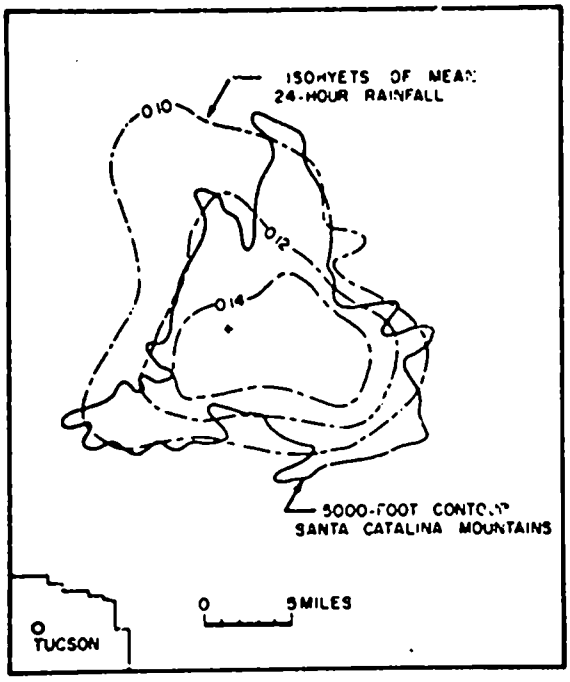

Fig. 2. Comparison of elevation contours with isohyets of summer rainfall in the Santa Catalina Mountains. (After Battan and Green, 1971.)

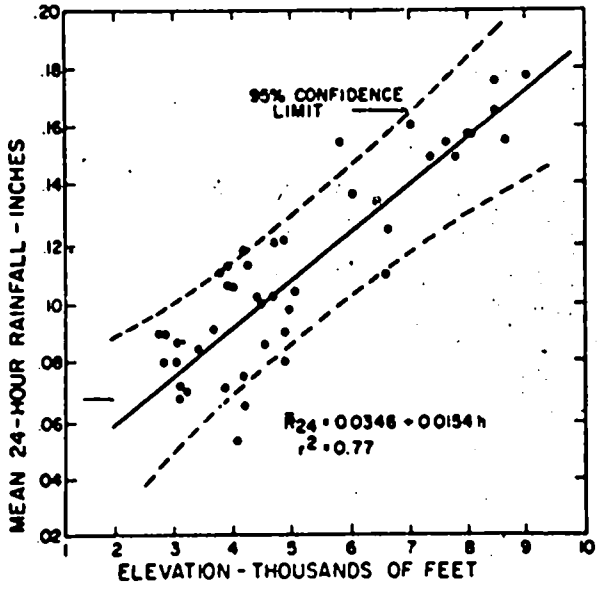

Fig. 3. Relationship of summer rainfall to station elevation.

VARIATION OF MEAN SEASONAL RAINFALL WITH ELEVATION

Model

A priori, the increase of mean rainfall amount with elevation may be due to an increase in the mean number of events E(N), or an increase in the mean amount of rainfall per event E(R) or an increase in both E(N) and E(R) with altitude.

In general, let:

$$E(N) = m_0 + ah \tag{7}$$

$$E(R) = R_0 + bh \tag{8}$$

where m_0 is the mean number of events per season at a datum, R_0 is the mean amount of rainfall per event at the datum, a and b are constants and h is the elevation in thousands of feet. Substituting the above two equations into eq. 5 results in:

$$\begin{aligned} E(Z) &= (m_0+ah)(R_0+bh) \\ &= m_0R_0 + (aR_0+bm_0)h + abh^2 \end{aligned} \tag{9}$$

Under these hypotheses, then, the total seasonal (or 24-h) rainfall increases as a second order polynomial of elevation with positive coefficients. The relative importance of the term in h and that in h^2 have to be ascertained experimentally. Strictly speaking, however, if E(N) and E(R) increase linearly with h, then E(Z) must increase quadratically with h.

Experimental results

Using the data only from those stations with 7 years of record and the definition for an event given earlier, the regression of E(N) = m on elevation h was obtained as shown in Fig. 4. In this analysis:

$$m = 12.44 + 3.12\,h \qquad 2.5 < h < 8.6 \tag{10}$$

and hence from eq. 7, m_0 = 12.44 and a = 3.12.

The regression of E(R) on h is shown in Fig. 5 and is written:

$$E(R) = 0.1869 + 0.0116\,h \tag{11}$$

Thus, R_0 = 0.1869 and b = 0.0116.

To obtain the equation relating total seasonal rainfall to elevation, the regression of 24-h rainfall on elevation (Fig. 3) was multiplied by the appropriate number of days in a summer season. The results are:

$$E(Z) = 2.15 + 0.96\,h \tag{12}$$

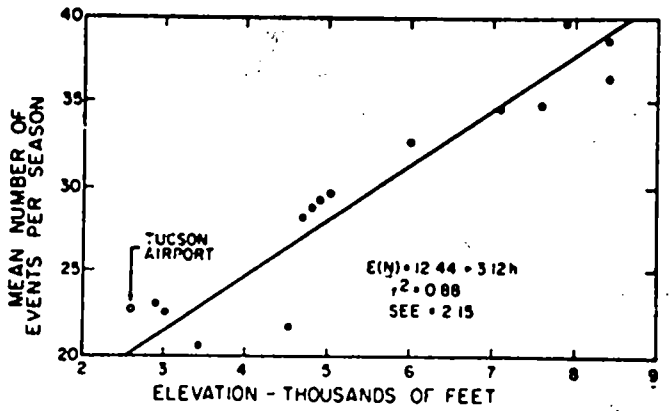

Fig. 4. Effect of elevation on occurrence of storms per season.

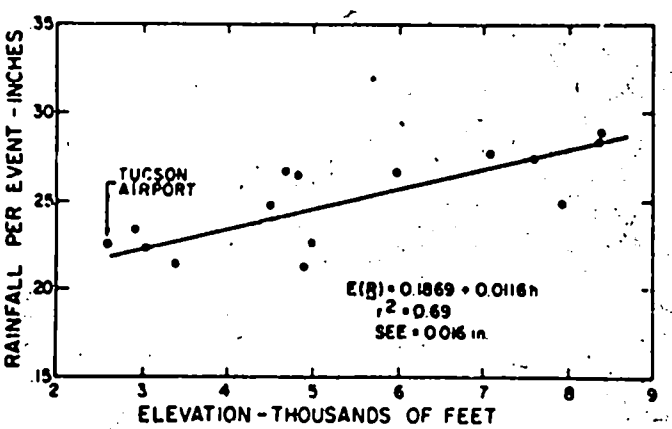

Fig. 5. Effect of elevation on rainfall per event.

On the other hand, use of the model (eq. 9) would lead to the expression

$$E(Z) = 2.33 + 0.727\,h + 0.0362\,h^2 \tag{13}$$

Equations 12 and 13 are plotted in Fig. 6 and as seen, the curves are very close to one another with the quadratic model well inside any confidence band of the observed data. The reason for such an agreement between an "exact" model (eq. 13) and an approximate linear model (eq. 12) are data scatter and the relatively low value of the coefficient of h^2 in eq. 13. It is recognized that the quadratic model is based on the correctness of the linear assumptions in eq. 7 and 8. The main point, however, is that the variations of N, R and Z with h which cannot all be linear theoretically, can be taken as such practically.

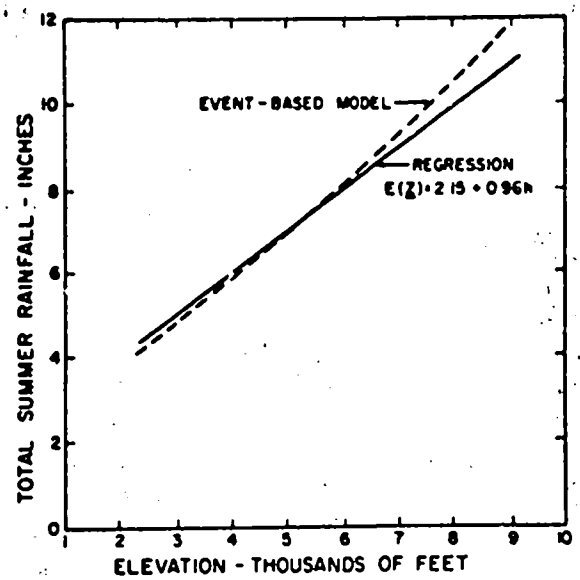

Fig. 6. Comparison of event-based model and regression of total summer rainfall on elevation.

VARIATION OF VARIANCE WITH ELEVATION

Model

Hypothesizing that the mean number of events N and the amount of precipitation per event R are independent random variables, the variance of Z is given by eq. 6, namely:

$$\mathrm{VAR}(Z) = \mathrm{VAR}(N)\,[\mathrm{E}(R)]^2 + [\mathrm{E}(N)]\,\mathrm{VAR}(R) \tag{6}$$

$\mathrm{E}(N)$ and $\mathrm{E}(R)$ are given as a function of h by eq. 7 and 8. To evaluate the variances of N and R, it is assumed, as previously mentioned, that N is a Poisson variate and R a geometric variate. Thus, their mean and variances are:

$$\mathrm{E}(N) = \mathrm{VAR}(N) = m = m_0 + ah \tag{14}$$

$$\mathrm{E}(R) = \frac{p}{1-p} = R_0 + bh \tag{15}$$

$$\mathrm{VAR}(R) = \frac{p}{(1-p)^2} = (R_0+bh)\,(1+R_0+bh) \tag{16}$$

Substituting eq. 14, 15 and 16 into eq. 6 yields:

$$\mathrm{VAR}(Z) = \frac{mp}{(1-p)^2}(1+p)$$
$$= (m_0+ah)(R_0+bh)(1+2R_0+2bh) \tag{17}$$

Thus, under the stated hypotheses, the variance of total seasonal rainfall increases as a cubic polynomial in h with positive coefficients.

In a similar fashion, the coefficient of variation, defined as the standard deviation divided by the mean, is obtained with the results:

$$\mathrm{CV}(Z) = \left[\frac{1 + 2R_0 + 2bh}{(m_0+ah)(R_0+bh)}\right]^{1/2} \tag{18}$$

Hence, the coefficient of variation of Z is seen to decrease as elevation increases, or in other words, the mean increases faster than the variance.

Experimental results

Inasmuch as the data are available only for a 7-year period, the assumption that N is Poisson distributed cannot be tested. However, previous information (Fogel and Duckstein, 1969) suggests that the hypothesis is reasonable. The distribution R for a high-altitude station and a low-altitude one are illustrated in Fig. 7 and 8. The hypothesis that R is a geometric variate could not be rejected by the Kolmogorov–Smirnov test at the 90% level of significance.

From eq. 11 and 15, the variation of p, the geometric distribution parameter, with h is given by:

$$p = \frac{R_0 + bh}{1 + R_0 + bh}$$
$$= \frac{0.1869 + 0.0116\,h}{1.1869 + 0.0116\,h} \tag{19}$$

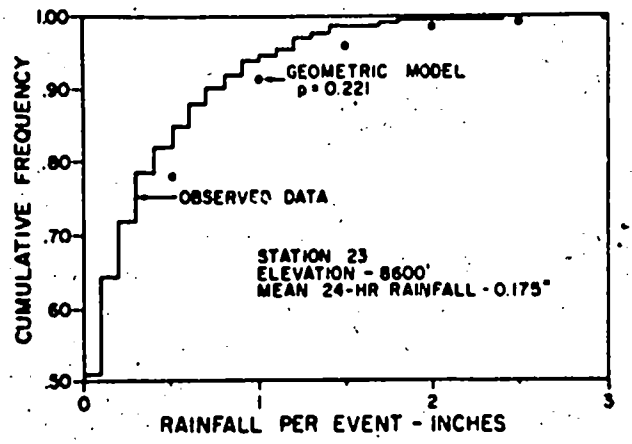

Fig. 7. Distribution of rainfall per event for high altitude station.

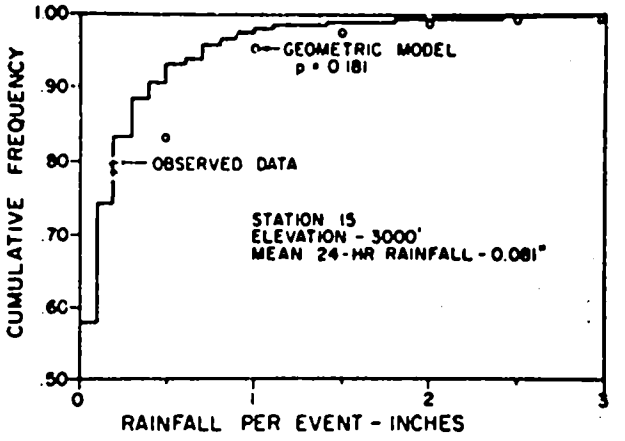

Fig. 8. Distribution of rainfall per event for low altitude station.

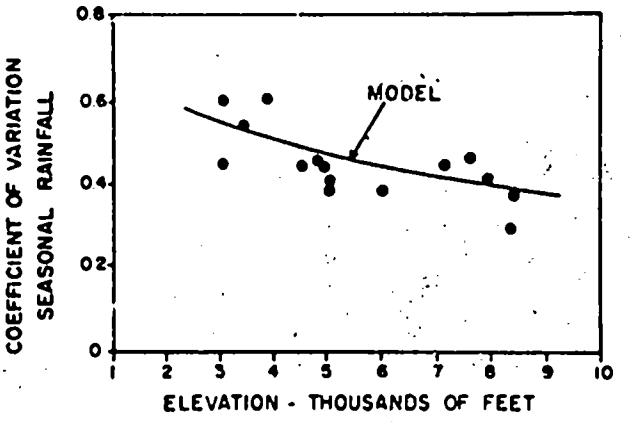

Fig. 9. Effect of elevation on coefficient of variation of summer rainfall.

Evaluation of eq. 19 reveals that p increases from a value of 0.175 for an elevation of 2500 ft. to 0.221 for 8600 ft.

Consider now the coefficient of variation (eq. 18), which is a normalized measure of the spread of the data versus elevation. Substituting eq. 10 and 11 into eq. 18 yields:

$$\mathrm{CV}(Z) = \left[\frac{1.3738 + 0.0232h}{(12.44+3.12h)(0.1869+0.016h)}\right]^{1/2} \tag{20}$$

A comparison of eq. 20 with the seasonal rainfall coefficients of variation of those individual stations with 7 years of record is shown in Fig. 9. As can be seen in Fig. 9, a plot of the coefficient of variation of seasonal rainfall against elevation, the experimental data tends to confirm the model which would imply independence between N and R.

Model

The distribution function $\phi(k) = \phi(k|h)$ of seasonal maximum rainfall at a point of elevation h is obtained as follows:

Let $F_R(k)$ be the distribution function of point rainfall. From eq. 2:

$$F_R(k) = \sum_{j=0}^{k} (1-p)p^j = 1 - p^{k+1} \tag{21}$$

By definition:

$$\phi(k) = \sum_{n=0}^{\infty} P(R_1<k, R_2<k, \ldots R_N<k)P(N=n)$$

For independent events, with eq. 1:

$$\phi(k) = \sum_{n=0}^{\infty} [P(R<k)]^n P(N<n)$$

$$= \sum_{n=0}^{\infty} [F_R(k)]^n e^{-m} \frac{m^n}{n!}$$

and substituting eq. 21 into the above:

$$\phi(k) = \exp(-mp^{k+1}) \tag{22}$$

Using eq. 14 and 15 to express m and p as a function of h, eq. 22 becomes:

$$\phi(k|h) = \exp[-(m_0+ah)(R_0+bh)^{k+1}(1+R_0+bh)^{-k-1}] \tag{23}$$

The return period T_r of an event of magnitude k at altitude h is by definition:

$$T_r(k|h) = [1-\phi(k|h)]^{-1} \tag{24}$$

Experimental results

The incontestable advantage of using a model over using data analysis is really evident here; with at most 7 years of data which would yield 7 annual maxima, it is possible to infer with a certain degree of confidence the magnitude of events having a much greater return period than the record length.

Substituting numerical values into eq. 23 yields:

$$\phi(k|h) = \exp\left\{-(12.44+3.12h)\left[\frac{0.1869+0.0116h}{1.1869+0.0116h}\right]^{k+1}\right\} \tag{25}$$

Since eq. 25 is an extreme value type of equation, it plots as a straight line on Gumbel probability paper. Fig. 10 presents the frequency distribution of storm rainfall at two elevations, 2500 and 7500 ft. as obtained from eq. 25. A composite annual maximum series of four Tucson area stations representing 168 station-years fitted to a Gumbel distribution is also shown for comparison. These stations are far enough apart from each other to satisfy the independence assumption of the station-year method.

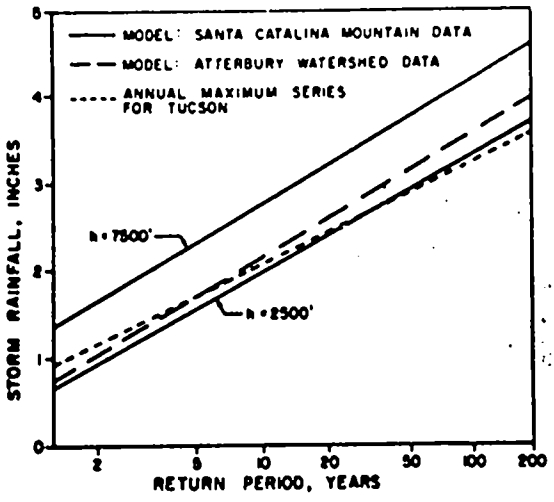

Fig. 10. Comparison of the cumulative frequency distribution of storm rainfall by different procedures.

Additional verification of the proposed procedure is obtained from the results of an earlier study (Fogel and Duckstein, 1969). Using data from the Atterbury watershed, an experimental area located near Tucson, Arizona at a mean elevation of 3000 ft. a maximal or extremal distribution for convective storm rainfall was derived. Instead of using the occurrence of measurable rainfall (0.01 inch) as used in the present study as the basis for defining an event, the earlier study required that at least one storm center (point of maximum rainfall) fall over the rain gage network as the condition for the occurrence of an event. Nevertheless, the two procedures are in very good agreement with each other despite their different event definitions.

While the annual maximum series indicates a smaller variance (flatter slope in Fig. 10) than the event-based model, all three procedures produce results that are remarkably close to one another. The fact that independent sets of

data were used in all cases presents a strong argument for using the event-based probability model to determine the effects of elevation on warm season rainfall.

DISCUSSION AND CONCLUSIONS

The method being proposed to analyze thunderstorm rainfall with elevation h requires only the determination of two point parameters, $E(N)$ and $E(R)$, as a function of h (eq. 7 and 8). From this study, it has been concluded that:

(1) The mean number of events per season, $E(N) = m$, increases considerably with elevation (eq. 10).

(2) The mean amount of rainfall per event, $E(R)$, increases moderately with elevation (eq. 11).

(3) The mean total seasonal rainfall, $E(Z)$, increases as a quadratic polynomial with elevation, but a linear approximation is quite satisfactory (eq. 12 and 13).

(4) The number of events per season N can be taken as a Poisson variate.

(5) The amount of rainfall per event R can be taken as a geometric variate.

(6) The independence between N and R tends to be confirmed by the good fit between experienced and derived coefficient of variation.

(7) The coefficient of variation of the total seasonal rainfall Z is a decreasing function of elevation (eq. 18).

(8) The return period of an event of given magnitude may be obtained from the model despite a short historical record.

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