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# STEADY AND UNSTEADY FLOW OF FRESH WATER IN SALINE AQUIFERS

David B. McWhorter

Colorado State University

Prepared for:

Agency for International Development

June 1972

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by David B. McWhorter

COLORADO STATE UNIVERSITY
FORT COLLINS, COLORADO
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# STEADY AND UNSTEADY FLOW OF FRESH WATER IN SALINE AQUIFERS

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by

David B. McWhorter

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Problems of flow involving fresh water overlying saline water in aquifers and methods for their analysis are reviewed. Equations amenable to mathematical solution generally involve the idealization of treating the fresh water as a distinct zone separated from the underlying saline water by a sharp interface. The criterion for interface stability is derived in this study and its practical significance is described.

A new solution describing up-coning of saline water below horizontal tile drains is derived and used to calculate approximate optimum depth of drain placement. A procedure for estimating performance of collecting wells is outlined.

A procedure proposed in the literature for handling unsteady free surface problems was used to derive a differential equation which approximately describes the behavior of the unsteady interface. Practical applications of this equation are discussed and example cal-

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SUMMARY

This report presents a relatively detailed review of the methods that have been reported in the literature for analyzing problems of flow in aquifers that are saturated with a zone of relatively fresh water which becomes increasingly saline with depth. The equations which describe the problem as one of flow of miscible fluids are presented and discussed. It is pointed out that difficulty of obtaining solutions to these equations has fostered numerous attempts to obtain solutions to a more idealized version of the problem. The idealization consists of treating the fresh water as a distinct zone separated from the underlying saline water by a sharp interface. The principle difficulty with the latter formulation is a non-linear boundary condition on the interface between the two zones.

The conditions at the interface are discussed in detail. It is shown that interface is unstable under certain conditions. The criterion for interface stability is derived and its practical significance discussed.

Various solutions for steady up-coning beneath wells are presented and their limitations discussed. A new solution to the problem of steady up-coning beneath horizontal tile drains is derived in this report. It is shown how this solution can be used to calculate the approximate optimum depth to which drains should be placed. Example calculations are given in the appendix. A procedure for estimating the performance of collector wells is discussed briefly.

A procedure for handling unsteady free-surface problems previously proposed in the literature was used to derive a differential equation which approximately describes the behavior of the unsteady interface.

Some practical applications of this equation are discussed and example calculations given.

Finally, some needs for future research are outlined.

# INTRODUCTION

Exploitation of groundwater supplies for agricultural, municipal, and industrial uses is severely hampered in many regions of the world by the encroachment of unuseable saline water in response to fresh water withdrawals. Examples of salt water encroachment are most numerous in coastal aquifers but sometimes presents a problem in inland aquifers as well. Probably the most important example of the latter case exists in the India-Pakistan subcontinent in the Indus River Basin.

Pakistan has an area of almost 200 million acres; over 30 million acres of which is irrigated. The irrigated area is laced with thousands of miles of canals and ditches used to supply farmers with essential irrigation water. Seepage for the extensive distribution system and deep percolation from precipitation and irrigation over the years has, in many areas, produced a high water table in the underlying alluvial aquifer. The high water table has caused wide-spread problems of water-logging and salinity, necessitating the installation of extensive drainage and reclamation programs (15, 33).

The problem of drainage is complicated by the fact that highly saline water underlies virtually all of the relatively fresh water throughout the aquifer. Near the rivers and canals where a supply of fresh surface water is available, the saline water exists only at great depths. Near the center of the doabs between the major tributaries to the Indus a zone of relatively fresh water less than 100 feet thick commonly overlies a zone of more highly saline water (9). In such areas drainage facilities are apt to draw a substantial portion of their discharge from the saline zone unless special care is taken. The disposal of the saline water produced by such facilities presents a major problem.

In many cases the saline water can be discharged into nearby canals or otherwise mixed with canal water and used for irrigation. This procedure can only constitute a short term solution to the problem, however. This fact coupled with the need for supplemental irrigation water provides substantial incentive to "skim" the fresh water from the aquifer with a minimum disturbance of the saline zone.

Research was undertaken to examine the various methods by which "skimming" can be accomplished. A dissertation entitled "Salt-Water Coning Beneath Fresh-Water Wells". by B. M. Sahni (29) presents the results of efforts to improve the design of shallow tubewells constructed for skimming purposes. This paper presents a review of several of the methods for analyzing fresh-salt water interface problems as well as reporting on a portion of the research dealing with the design of horizontal drains. Another purpose of this paper is to report the development of a procedure for estimating the behavior of the interface between fresh and salt water under unsteady flow conditions. The formulation of the unsteady state problem is totally untested at the present, but should provide relatively satisfactory engineering answers when used with caution. Some examples of possible applications of the unsteady state equation are presented.

## PROBLEM FORMULATION

The complexity of the phenomenon of flow of fresh water underlain by brine has led investigators to make numerous idealizations in attempts to reduce the mathematical description of the phenomenon to a tractable form. A very common practice is to regard the fresh water and brine as immiscible liquids with an abrupt interface between them. In reality the two liquids are miscible and a transition zone separates the brine and the fresh water. The transition zone is characterized by a continuous decrease of concentration of salt from that of the undiluted brine to that of the uncontaminated water above. A mathematical formulation which considers the fluids to be miscible is more realistic but less tractable than one which considers the liquids to be immiscible. Both formulations are important, however, and are discussed in some detail in the following paragraphs.

## Miscible Flow Formulation

Any discussion of the flow of miscible fluids in porous media requires consideration of hydrodynamic dispersion. Hydrodynamic dispersion is the name given the process by which a contaminant becomes mixed and distributed in a porous medium as the result of velocity distributions and fluctuations and molecular diffusion on a pore-size scale. Mathematical characterization of the phenomenon has required consideration of contaminant transport at the pore-size (microscopic) scale and the development of an averaging procedure by which the microscopic mechanisms can be expressed in macrosocopic terms which are observable and measureable. The result is the following equation known as the differential equation of hydrodynamic dispersion:

$$\frac{\partial c}{\partial t} = \frac{\partial}{\partial x_i} \left[ D_{i,j} \frac{\partial c}{\partial x_i} - c v_i \right] - S_1 \quad . \qquad (1)$$

in which

$c$ = contaminant concentration - $M/L^3$

$t$ = time - $T$

$x_i$ = i th cartesian coordinate - $L$

$D_{i,j}$ = coefficient of hydrodynamic dispersion (a tensor of rank 2) - $L^2/T$

$v_i$ = the component of seepage velocity in the i th direction - L/T

$S_1$ = source or sink - $M/L^3 \cdot T$ .

Several analytic solutions of the above equation exist under a variety of boundary and initial conditions and with different formulations of the source-sink term (5, 13, 17, 18, 23, 24, 25). The analytic solutions have contributed greatly to the understanding of hydrodynamic dispersion and their importance cannot be over emphasized. However, the analytic solutions are ordinarily derived for boundary and initial conditions that are relatively simple in comparison to the wide variety of situations encountered in the field.

Numerical solutions using finite difference formulations and the finite element method have also been developed (28, 22). Numerical solutions are usually more general in as much as complex boundary and initial conditions can be handled for dispersion in non-isotropic and non-homogeneous porous media. Numerical solutions are ordinarily less convenient for analysis and design than are analytical solutions. In many cases, however, numerical methods offer the only feasible means of solving the problem.

Because the seepage velocity $v_i$ is present in equation 1, and because the coefficient of hydrodynamic dispersion $D_{i,j}$ depends on $v_i$ , the phenomenon of dispersion is strongly dependent upon the flow of the solution phase. To some extent the bulk flow of the solution phase is also dependent on the concentration distribution of the contaminant. This is true because the contaminant effects the density and viscosity of the solution. The differential equation for flow including the influence of the contaminant can be written as in equation 2,

$$\frac{\partial}{\partial x_i} \left[ \frac{\rho k_{xi}}{\mu} \left( \frac{\partial P}{\partial x_i} + \rho g \frac{\partial h}{\partial x_i} \right) \right] = \frac{\partial}{\partial t} (\rho \phi) + S_2 \qquad (2)$$

where

$k_{xi}$ = intrinsic permeability in the $x_i$ direction - $L^2$

$P$ = pressure - $F/L^2$

$g$ = gravitational constant - $L/T^2$

$h$ = elevation with respect to an arbitrary datum - L

$\rho$ = density - $M/L^3$

$\mu$ = dynamic viscosity - $F \cdot T/L^2$

$\phi$ = porosity

$S_2$ = source or sink - $M/L^3T$ ,

and other symbols are as previously defined. Equations 1 and 2 constitute a system of non-linear, coupled partial differential equations which describe bulk flow of a liquid and dispersion of a contaminant in that liquid.

## Immiscible Flow Formulation

One of the objectives of the study reported herein, is to provide a method for comparing different methods of producing the relatively fresh water with a minimum disturbance of the underlying brine. From an academic standpoint, the ideal way to accomplish such a comparison would be to solve equations 1 and 2 simultaneously for each method under consideration. Because of limitations on time and financial recources, such a procedure is quite impractical. Therefore, a less rigorous but more practical approach was taken. This approach involves several idealizations which are discussed in the following paragraphs.

It is often the case that the disperse (or transition) zone between the fresh-water region and the salt-water region is small compared to the total thickness of the fresh-water region. It is, therefore, possible to consider the transition zone as a sharp boundary separating the two regions of flow. The problem is reduced to locating the interface between the two regions for various boundary conditions which is often called a "free-surface" problem.

## Free Surface Problems

Within each region separated by the abrupt interface, the flow is given by Darcy's law in vector notation for isotropic media:

$$\vec{q}_f = - K_f \nabla (P_f/\gamma_f + z) \tag{3}$$

$$\vec{q}_s = - K_s \nabla (P_s/\gamma_s + z) \tag{4}$$

in which

$\vec{q}$ = Darcy velocity vector - L/T

K = hydraulic conductivity - L/T

P = pressure - $F/L^2$

z = elevation relative to an arbitrary datum - L

$\gamma$ = specific weight - $F/L^3$ ,

and the subscripts f and s refer to fresh and salt water respectively. The quantity in parathenses is often called the hydraulic head. Throughout the remainder of this report, the hydraulic head is denoted by the symbol H. For a homogeneous medium, equation 3 can be combined with the equation for mass conservation for incompressible, homogeneous fluids to obtain:

$$\nabla^2 H_f = 0 \quad . \tag{5}$$

The same equation applies to the salt-water region with the subscript $f$ replaced by $s$. Equation 5 is known as the Laplace equation. The hydraulic head $H_f$ is a function of space and time in general.

The problem is to solve the linear Laplace equation in the fresh water region subject to the boundary conditions. Dagan (6) has shown that the problem represented by equation 5 within a region whose boundary is partially a free surface is non-linear because of a non-linear boundary condition on the interface. The non-linear boundary condition constitutes the major difficulty with the immiscible formulation of the problem.

Conditions on the Interface

At any point the pressure must be continuous across the interface. Therefore, solving for pressure from the definitions of hydraulic head in the fresh and salt water regions and equating at the interface yields:

$$H_f^{\,i}\,\gamma_f - \xi\gamma_f = H_s^{\,i}\,\gamma_s - \xi\gamma_s \tag{6}$$

in which $\xi$ is the vertical coordinate of the interface at any point, the superscript $i$ denotes the value of hydraulic head on the interface, and other symbols are as previously defined.

Solving equation 6 for the elevation of the interface yields

$$\xi = \frac{\gamma_s}{\Delta\gamma} H_s^{\,i} - \frac{\gamma_f}{\Delta\gamma} H_f^{\,i} \tag{7}$$

in which $\Delta\gamma$ is $\gamma_s - \gamma_f$ . Equation 7 implies that the position of the interface can be computed if the values of hydraulic head are known on the interface.

The change in elevation of the interface with respect to the distance along the interface is obtained from equation 7 by differentiation:

$$\frac{\partial\xi}{\partial\ell} = \frac{\gamma_s}{\Delta\gamma} \frac{\partial H_s^{\,i}}{\partial\ell} - \frac{\gamma_f}{\Delta\gamma} \frac{\partial H_f^{\,i}}{\partial\ell} \tag{8}$$

where $\ell$ is the distance along the interface. Solving equations 3 and 4 for the gradient of head and substituting into equation 8 results in

$$\frac{\partial\xi}{\partial\ell} = \frac{\gamma_f}{\Delta\gamma} \frac{q_f^{\,i}}{K_f} - \frac{\gamma_s}{\Delta\gamma} \frac{q_s^{\,i}}{K_s} \tag{9}$$

where $q_f^{\,i}$ and $q_s^{\,i}$ are the velocities tangent to the interface. The derivative on the left of equation 9 is the slope of the interface and can be replaced by $\sin\theta$ where $\theta$ is the angle the interface makes with the horizontal.

The hydraulic conductivities $K_f$ and $K_s$ are directly proportional to the specific weights $\gamma_f$ and $\gamma_s$ respectively. Therefore, if it is assumed that the viscosity of the salt and fresh waters are the same, equation 9 implies that, when

$$q_f^{\,i} = q_s^{\,i} \quad , \tag{10}$$

the interface is horizontal. A horizontal interface also results when there is no flow in either region.

The case in which there is flow in the fresh water but no flow in the salt water is of particular interest. Equation 9 reduces to

$$\frac{\gamma_f q_f^{\,i}}{\Delta\gamma K_f} = \sin\theta \quad . \tag{11}$$

The right side of equation 11 has a maximum value of unity. Therefore, the condition

$$q_f^{\,i} < \frac{\Delta\gamma K_f}{\gamma_f} \tag{12}$$

must be satisfied at all points on the interface. Condition 12 can be regarded as a necessary condition for stability of the interface. The above stability condition is quite restrictive because, for fresh-salt water systems, it implies that the velocity tangent to the interface must remain much less than the value of hydraulic conductivity. This further implies that the hydraulic gradient on the interface must remain less than $\Delta\gamma/\gamma_f$ .

The classical Ghyben-Herzberg relation can be stated

$$\Delta\xi = -\frac{\gamma_f}{\Delta\gamma}\Delta h \quad , \tag{13}$$

where h is the water table elevation relative to some datum. The Ghyben-Herzberg relation states that the change in elevation of the interface associated with a change in water table elevation is approximately $\Delta\gamma/\gamma_f$ times as great as the water table change. The approximation necessary to arrive at equation 13 follows directly from equation 8 with $\partial H_s^{\,i}/\partial \ell = 0$ :

$$\Delta\xi = \frac{\partial\zeta}{\partial\ell}\,\Delta\ell = -\,\frac{\gamma_f}{\Delta\gamma}\,\frac{\partial H_f^{\,i}}{\partial\ell}\,\Delta\ell \quad , \tag{14}$$

or

$$\Delta\xi = -\,\frac{\gamma_f}{\Delta\gamma}\,\Delta H_f^{\,i} \quad . \tag{15}$$

Equation 15 is identical to the Ghyben-Herzberg relation provided the hydraulic head on the interface is equal to the water table elevation. Such a condition exists only if the flow is horizontal. In situations of interest in this report, the flow is not horizontal because of the slope of the interface and the water table. Therefore, use of the Ghyben-Herzberg relation is tantamount to neglecting the vertical component of flow.

## SOLUTIONS FOR STEADY UPCONING BENEATH WELLS

### Muskat and Wyckoff

Probably the first approximate solution to the problem of upconing beneath wells was derived by Muskat and Wyckoff (21). Their problem was that of upconing of brine in response to pumping in the overlying oil zone. The theory does not depend upon the particular liquids involved, so the author of this report has taken the liberty of translating the analysis into terms of the fresh water-salt water problem.

Muskat and Wyckoff obtained a steady state distribution of hydraulic head for flow toward a well partially penetrating a confined aquifer of thickness $m$ . They then assumed that this distribution of head applies to a situation in which the well penetrates a fresh water zone of thickness m . This assumption amounts to assuming that the upconing of the salt-water does not disturb the flow.

Explanation of the method used by Muskat and Wyckoff can be presented in mathematical terms. Let the distribution of hydraulic head be given by

$$H_f = H_f\ (r,z) \qquad , \tag{16}$$

where $r$ is the radial coordinate and $z$ is the vertical coordinate. An explicit expression for equation 16 is obtained as discussed in the previous paragraph. On the interface, equation 16 becomes

$$H_f^{\ i} = H_f\ (r,\xi) \qquad , \tag{17}$$

which is an equation for the interface involving two unknowns, $H_f^{\ i}$ and $\xi$ . Because there are two unknowns in equation 17, a second independent equation is necessary. The second equation is provided by the interface condition given by equation 7. Since interest is focused on a steady state solution, the head in the salt-water zone is taken as constant, and equation 7 becomes:

$$\xi = \text{constant} - \frac{\gamma_f}{\Delta\gamma} H_f^{\ i} \qquad . \tag{18}$$

The constant in equation 18 depends upon the selection of the datum used for measuring hydraulic head. Equations 17 and 18 are a pair of equations in the unknowns $H_f^{\ i}$ and $\xi$ which can be solved simultaneously to obtain the position of the interface.

The particular form of equation 17 used by Muskat and Wyckoff made it necessary to obtain the simultaneous solution of equations 17 and 18 by graphical methods. It was found that for any particular distribution of hydraulic head as given by equation 17 that one of the following situations exist:

1) No simultaneous solution exists.

2) Two roots of the simultaneous equations exist.

3) One root exists.

These authors show that situation 1 corresponds to the physical condition of an unstable interface; that is, inequality 12 is not satisfied. They further show that the only root which has physical meaning in situation 2 is the one which gives the lowest position of the interface. The third possible situation represents the highest position at which a stable interface exists. This position has been called the "critical cone height".

Muskat and Wyckoff also present a formula for the well discharge as calculated from the distribution of hydraulic head. According to their theory, the maximum discharge is obtained when the cone height is critical. This value of discharge has been termed, the "critical discharge".

Thus the Muskat-Wyckoff model has the advantages that the condition of interface instability is implicit in the theory, and that the vertical component of flow is not neglected. On the other hand, the assumption that the distribution of head is not effected by the upconing of the interface is a serious constraint on the applicability of the metnod.

Bear and Dagan

Schmorak and Mercado (30) report an equation for salt-water rise beneath a pumping well that was derived by Bear and Dagan using the method of small perturbations. The original report of Bear and Dagan is not in English and, therefore, is not used directly in this report.

The equation for steady flow in a homogeneous and isotropic aquifer is

$$\frac{\xi}{d} = \frac{\gamma_f Q}{2\pi d^2 \Delta\gamma \; K} \cdot \frac{1}{[1+(\frac{r}{d})^2]^{\frac{1}{2}}} \quad , \tag{19}$$

in which

$Q$ = discharge rate - $L^3/T$

$r$ = radial distance - $L$

$d$ = distance between bottom of well and the original powition of interface - $L$

and other symbols are as previously defined. Bear and Dagan recognized that equation 19 does not account for the fact that the interface becomes unstable if $\xi$ exceeds a certain critical value $(\xi_{max})$ . These authors recommend that the discharge of the well be regulated so that

$$\xi_{max} \leq f \; d \quad , \tag{20}$$

where $f$ is some fraction $(\approx 0.5)$ . Subject to condition 20, the maximum safe discharge of the well is given by

$$Q_{max} = \frac{2\pi f d^2 \Delta\gamma K}{\gamma_f} \; [1+(\frac{r_w}{d})^2]^{\frac{1}{2}} \quad , \tag{21}$$

where $r_w$ is the radius of the well. These formulas were derived for a very thick fresh water zone and considered the well to be a point sink.

Wang

The Wang theory (32) is based on the assumption that the rising salt-water mound beneath the pumping well does not effect the discharge of the well. In this respect the Wang theory is similar to that of Muskat and Wyckoff. The Wang theory, however, makes no use of a detailed distribution of hydraulic head for locating the position of the interface. The approach was to use the formula presented by Muskat (20) which relates the well discharge to drawdown and incorporates the Ghyben-Herzberg relation. The formula for discharge is

$$Q = \frac{2\pi K\ m\ s}{\ln\ r_e/r_w} \left\{ \frac{D}{m} \left[1+7 \left(\frac{r_w}{2D}\right)^{\frac{1}{2}} \cos \frac{\pi D}{2m} \right] \right\} \quad , \qquad (22)$$

where

$s$ = drawdown - L

$D$ = distance between original water table and the bottom of well - L

$m$ = original fresh water thickness - L

$r_e$ = radius of influence - L .

Strictly speaking, equation 22 applies only for flow in confined aquifers. Wang reasoned that, if $s$ is small compared to $m$ , the formula could be used for flow in water table aquifers as well.

As already pointed out, the Ghyben-Herzberg equation predicts the interface will rise $\gamma_f/\Delta\gamma$ feet for every foot of drawdown on the water table (see equation 13). Therefore the elevation of the interface at

$r = r_w$ as a function of discharge is

$$\xi_w = \frac{Q\Delta\gamma \ln r_e/r_w}{2\pi K m^2 \gamma_f} \cdot \frac{1}{[\frac{D}{m}[1+7(\frac{r_w}{2D})^{1/2} \cos\frac{\pi D}{2m}]} \quad . \tag{23}$$

Wang did not account for the possible instability of the interface and assumed that the interface could rise to the bottom of the well as a limiting condition. This fact causes the Wang equation for maximum discharge to be impractical. It is not good for large salt-water mound heights.

McWhorter

In view of the restrictive assumptions and complicated methods by which the above solutions were obtained, the author of this report felt that a formula that required a much simpler analysis might be useful. The major difference between the following analysis and that used in obtaining the above equations is that the increased convergence caused by the upconing of the interface is accounted for approximately.

The physical situation of a well partially penetrating an unconfined zone of fresh water $m$ feet thick overlying a salt-water zone is depicted in figure 1. For sufficiently flat slopes of the interfaces and a static salt water zone, equation 9 can be written approximately

$$\frac{d\xi}{dr} = \frac{\gamma_f}{\Delta\gamma}\frac{q_f^{\,i}}{K_f} \quad . \tag{24}$$

Equation 24 implies that vertical components of flow are small. Consistent with this approximation is the assumption that the velocity $q_f$ is not a function of the vertical coordinate $z$. Therefore, the superscript $i$ in equation 24 can be dropped. Rewriting equation 24 in terms

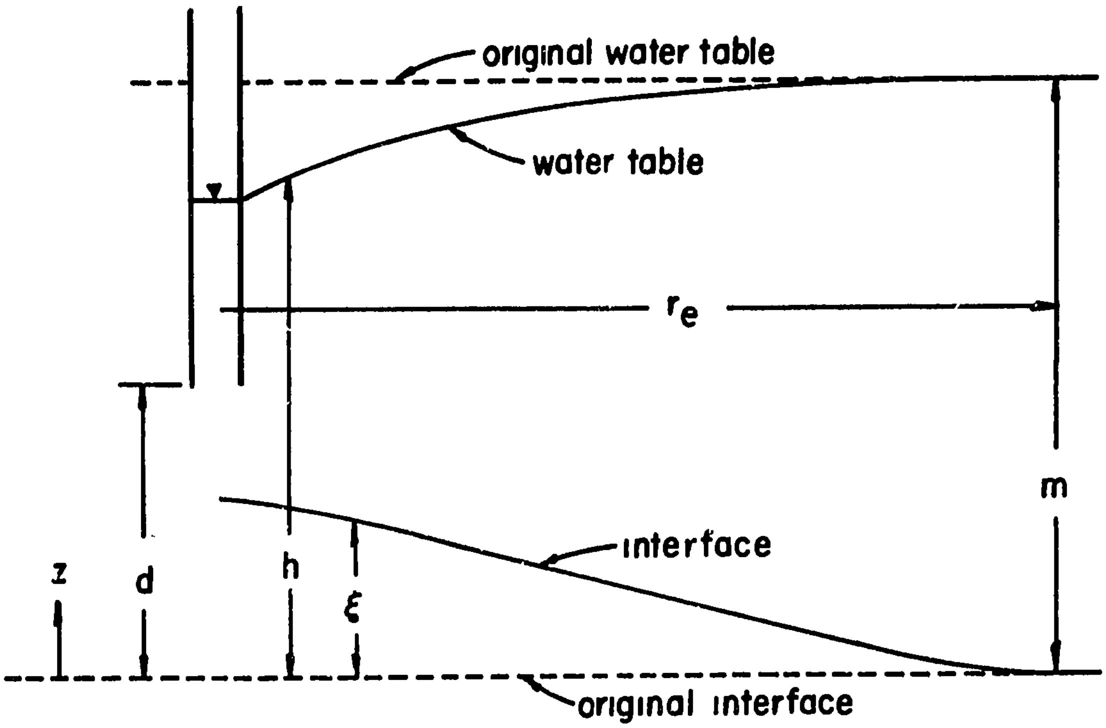


Figure 1. Upconing of salt-water beneath a pumping well.

of the well discharge Q results in

$$Q = A \frac{\Delta\gamma}{\gamma_f} K_f \frac{d\xi}{dr} \quad , \tag{25}$$

where A is the area perpendicular to flow and is given by

$$A = 2\pi\ (h-\xi)\ r \quad . \tag{26}$$

The water table elevation h can be related to ξ by use of the Ghyben-Herzberg relation which is consistent with the approximation that vertical components of flow are small. The result is

$$A = 2\pi\ \{\ m - (\frac{\Delta\gamma}{\gamma} + 1)\ \xi\ \}\ r \quad . \tag{27}$$

Substitution of equation 27 into equation 25, separation of variables and integration yields

$$\frac{\xi}{m} = \sqrt{\left(\frac{1}{1+\Delta\gamma/\gamma_f}\right)^2 + \frac{Q\gamma_f}{m^2\Delta\gamma K_f}\,\frac{\ln r_e/r}{(1+\Delta\gamma/\gamma_f)}} - \frac{1}{1+\Delta\gamma/\gamma_f} \quad . \qquad (28)$$

Equation 28 is an approximate discription of the variation of interface elevation with $r$ . For a particular value of $r$ , say $r = r_w$ , the equation describes the interface elevation as a function of well discharge.

The derivation of equation 28 does not account for the fact that the interface can become unstable. Therefore, application of equation 28 must be restricted to the range of $\xi$ values in which the interface remains stable.

## Other Solutions

Bennett et al. (1) used an electrical analog to obtain solutions for the steady flow case. The potential distribution in the simulated fresh water zone was determined from the analog with a first guess as to the shape and position of the boundary simulating the interface. The interface was then adjusted and a new potential distribution determined until the boundary conditions that must be satisfied on the interface were in agreement with the potential distribution. The procedure relied heavily on the theory of Muskat (20); the significant improvement, however, being that the distortion of the distribution of hydraulic head caused by the mounding was given full consideration.

Sahni (29) used both numerical and physical models to study the design of partially penetrating wells in the fresh water zone. He developed a numerical solution by writing the differential equation in finite difference and solving the resulting set of algebraic equations.

The non-linear boundary condition on the interface was handled by iteration. A solution was obtained with a first estimate of the position of the interface, and then the interface was adjusted until equation 7 (in a different form) was satisfied. The adjustment of the interface changes the geometry of the flow region and, therefore, a new distribution of hydraulic head. Once a new distribution of hydraulic head was determined the interface was again adjusted. This procedure was continued until the change in the distribution of hydraulic head for a particular iteration was negligibly small. Sahni's model also includes the effects of flow in the partially saturated portion of the aquifer above the water table.

The major conclusion obtained from Sahni's work is that the optimum penetration of the wells is between 15 and 30 percent of the thickness of the fresh water zone; a much shallower penetration than was previously recommended. It was also demonstrated, by comparison with physical model measurements, that the numerical model accurately predicts the well discharge under a variety of values for the boundary conditions.

## SOLUTION FOR STEADY UPCONING BENEATH DRAINS

Solutions reported in the literature for upconing beneath drains are not as numerous as for the well problem. Therefore, as a part of this research, solutions for the drain problem were derived which parallel solutions for the well case.

The situation treated in this report is depicted in figure 2. Flow occurs toward a drain of radius $a$, the center line of which is a distance $D$ below the original water table. A constant-head boundary is located a distance $L$ from the drain. In practical cases $D << d$

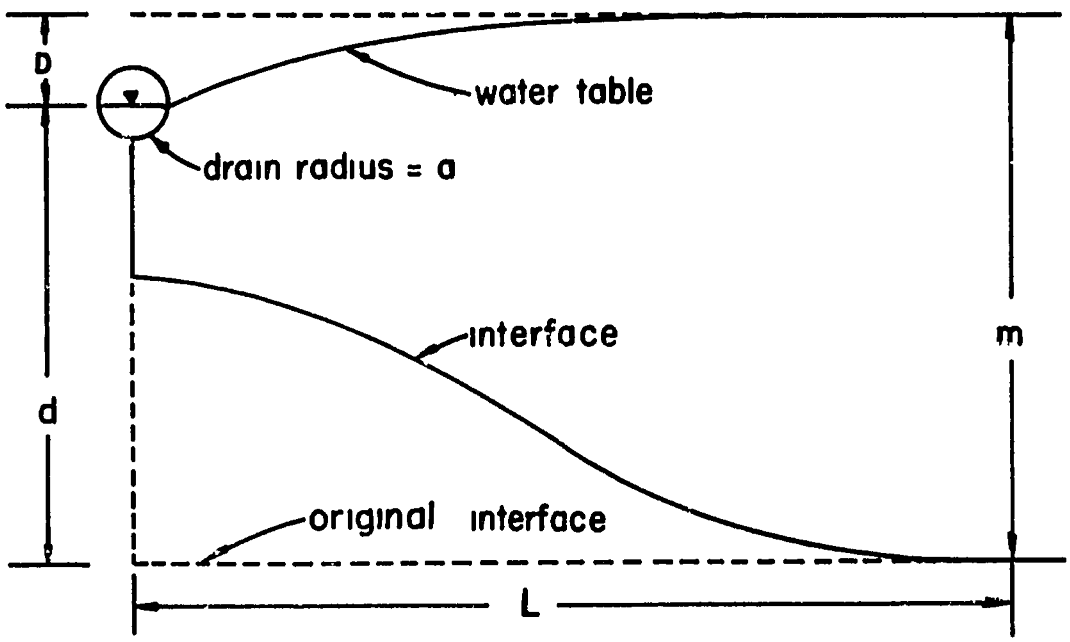


Figure 2. Flow toward a drain.

and the contribution to the drain discharge due to flow above the elevation of the drain is small compared to that from below the drain. Therefore the problem can be idealized as shown below.

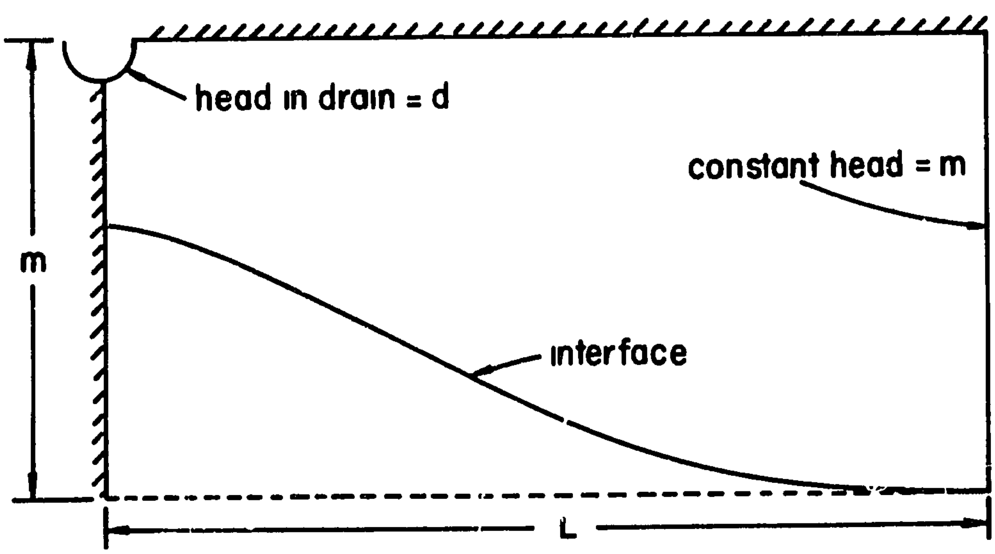


Figure 3. Idealized flow toward a drain.

A solution is obtained by assuming that the upconing of the interface does not effect the distribution of head in the fresh-water region. The distribution of hydraulic head for the case shown in figure 3 with no upconing is closely approximated by:

$$H_f\ (x,z) = \frac{Q}{2\pi K_f}\ \ell n\ \{\ \cosh\ \frac{\pi x}{m} - \cos\ \frac{\pi z}{m}\ \} + \text{constant}\quad , \tag{29}$$

where the origin of coordinates is at the center of the circular drain and $z$ is measured positive downward. Strictly speaking equation 29 applies tc flow toward a line sink located at (0,0) in an aquifer which extends indefinitely in the x-direction. It is not difficult to show, however, that the distribution of head predicted by equation 29 rapidly approaches that for uniform flow as $x$ increases. Therefore, the boundary condition that $H_f = m$ at $x = L$ is very closely satisfied if $L \geq d$ . Also since equation 29 predicts circular equal head contours near the origin, the boundary condition of constant head on a circular drain can be satisfied very closely.

Subject to the boundary conditions that $H = m$ at $x = L$ and $H = d$ on $x^2+z^2 = a^2$ , the drain discharge is given by

$$Q = \frac{2\pi\ K_f\ (m-d)}{\ell n \left\{ \dfrac{\cosh \frac{\pi a}{m} - 1}{\cosh \frac{\pi L}{m} - 1} \right\}} \quad , \tag{30}$$

and the head distribution by

$$H_f = \frac{Q}{2\pi K_f}\ \ell n \left\{ \frac{\cosh \frac{\pi x}{m} - \cos \frac{\pi z}{m}}{\cosh \frac{\pi L}{m} - 1} \right\} + m \quad . \tag{31}$$

From equation 7, the position of the interface is given by

$$\xi = \frac{\gamma_f}{\Delta\gamma} (m - H_f^{\,i}) \quad . \tag{32}$$

According to the assumption that the upconing interface does not effect the distribution of head, the head on the interface is obtained from equation 31 with $z = m - \xi$ . That is

$$H_f^{\,i} = \frac{Q}{2\pi K_f} \ln \left\{ \frac{\cosh \frac{\pi x}{m} - \cos\pi \left( 1 - \frac{\xi}{m} \right)}{\cosh \frac{\pi L}{m} - 1} \right\} + m \quad . \tag{33}$$

The simultaneous solution of equations 32 and 33 yields the unknowns $H_f^{\,i}$ and $\xi$ at particular values of $x$ . For design purposes the maximum elevation of the interface is of primary interest. The maximum elevation occurs directly beneath the drain on the line $x = 0$ . It is convenient to substitute equation 30 for $Q$ in equation 33 with $x = 0$ and rearrange; this results in

$$\frac{H_f^{\,i} - d}{m - d} = 1 - \frac{\ln \left\{ \dfrac{1 - \cos\pi\,(1 - \xi/m)}{\cosh \pi\, L/m - 1} \right\}}{\ln \left\{ \dfrac{\cosh \pi\, a/m - 1}{\cosh \pi\, L/m - 1} \right\}} \quad . \tag{34}$$

Equation 32 is also more convenient in the form

$$\frac{H_f^{\,i} - d}{m - d} = 1 - \left( \frac{m}{m - d} \right) \frac{\Delta\gamma}{\gamma_f}\, \xi/m \quad . \tag{35}$$

Both equations 34 and 35 are dimensionless, but nevertheless, a degree of generality can be gained by scaling all parameters and variables with the radius of the drain. The following scaled variables and parameters are defined:

$$\hat{H}_f = H_f^{\,i}/a$$
$$\hat{d} = d/a$$
$$\hat{m} = m/a \qquad (36)$$
$$\hat{L} = L/a$$
$$\hat{\xi} = \xi/a$$

Using the above scaled variables the final form of the equations to be used in computations become

$$\frac{\hat{H}_f - \hat{d}}{\hat{m} - \hat{d}} = 1 - \frac{\ell n \left\{ \dfrac{1 - \cos\pi \, (1 - \hat{\xi}/\hat{m})}{\cosh \pi \, \hat{L}/\hat{m} - 1} \right\}}{\ell n \left\{ \dfrac{\cosh \pi/\hat{m} - 1}{\cosh \pi \, \hat{L}/\hat{m} - 1} \right\}} \qquad (37)$$

$$\frac{\hat{H}_f - \hat{d}}{\hat{m} - \hat{d}} = 1 - \left( \frac{\hat{m}}{\hat{m} - \hat{d}} \right) \frac{\Delta\gamma}{\gamma_f} \, \hat{\xi}/\hat{m} \qquad (38)$$

and

$$\frac{Q \, \gamma_f}{K_f \, m\Delta\gamma} = \frac{-2\pi \left( \dfrac{\hat{m} - \hat{d}}{\hat{m}} \right) \dfrac{\gamma_f}{\Delta\gamma}}{\ell n \left\{ \dfrac{\cosh \pi/\hat{m} - 1}{\cosh \pi \, \hat{L}/\hat{m} - 1} \right\}} \qquad (39)$$

When the installation of a drain for the purpose of skimming fresh water is contemplated, the major questions are 1) how deep should the drain be located and 2) what should be the drain radius and length to achieve a certain desired discharge? Equations 37 through 39 can be used to answer these questions, at least approximately, as follows. From the field investigation the values of $\gamma_f$ , $\Delta\gamma$ , $L$ , $m$ , and $K_f$ are measured or estimated. The radius of the largest practical circular drain is selected to determine $a$ . Next, the values of the scaled variables $\hat{m}$ and $\hat{L}$ are computed. Using the values of $\hat{m}$ and $\hat{L}$ , values of $(\hat{H}_f - \hat{d})/(\hat{m} - \hat{d})$ are calculated for several values of $\hat{\xi}/\hat{m}$ in the range $0.1 < \hat{\xi}/\hat{m} < .95$ from equation 37. A plot of the data so generated is constructed as shown in figure 4.

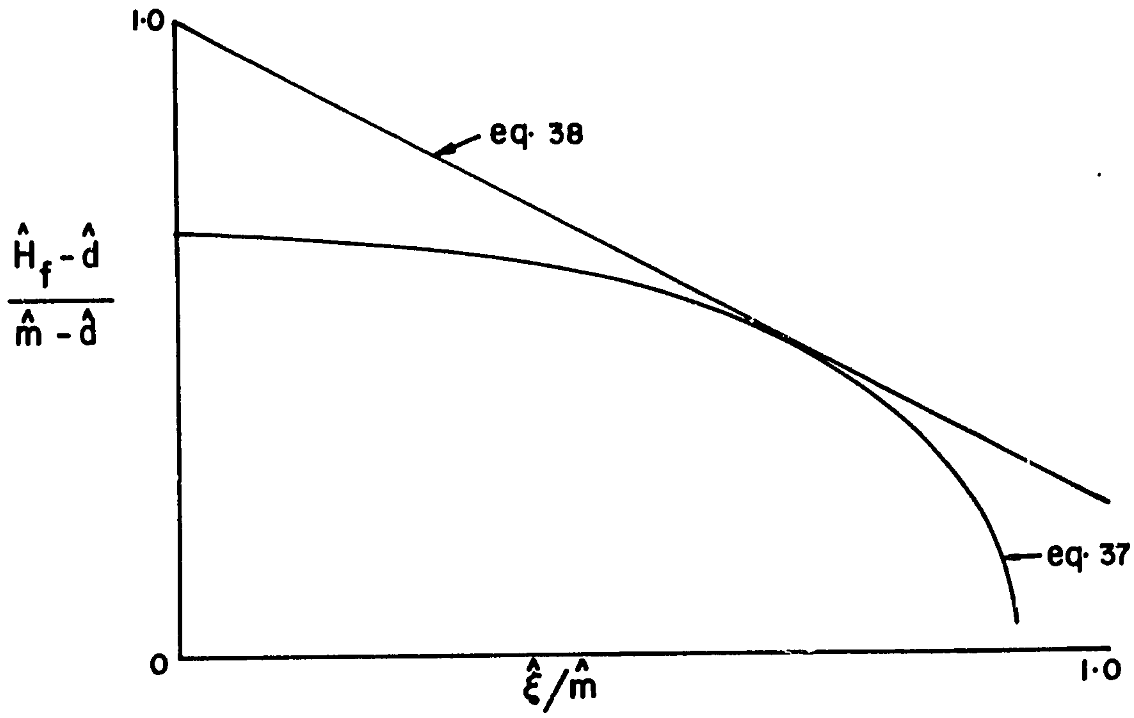


Figure 4. Graphical solution of equations 37 and 38.

Equation 38 plots as a straight line on figure 4; the slope, however, is not known because $\hat{d}$ is not known. Therefore, the slope is adjusted so that the line representing equation 38 is tangent to the curve representing equation 37. The absolute value of the slope of the tangent line is measured and set equal to the expression for slope in equation 38. That is

$$s = - \frac{\hat{m}}{\hat{m} - \hat{d}} \frac{\Delta\gamma}{\gamma_f} \quad , \tag{40}$$

where $s$ is the slope measured from figure 4. From equation 40, the value of $\hat{m} - \hat{d}$ is calculated. This value is the scaled depth below the original water table at which the center line of the drain should be placed.

The horizontal coordinate of the point of tangency in figure 4 is the scaled height of the interface beneath the drain. If the straight line is drawn with a steeper slope, the point of intersection of the two curves represents a stable interface elevation at a drain depth less than the one calculated above. On the other hand, a straight line drawn so that there is no intersection represents a greater drain depth which will cause an unstable interface and salt water will be entrained. Thus the tangent condition represents the greatest safe depth at which the drain can be placed.

The next step is to use the measured value of slope in equation 39 to compute the discharge per unit length of drain. It should be pointed out that the assumptions made in the development of the above theory are such that the predicted discharge for a particular interface elevation is larger than that which would be measured. The descrepancy is probably not large for thick fresh water layers, but becomes larger for thin layers. An example calculation is presented in the appendix.

## SOLUTIONS FOR STEADY UPCONING BENEATH COLLECTOR WELLS

Collector well is the name given a collection system constructed by lowering a vertical caisson into the aquifer, sealing the bottom, and forcing perforated pipe horizontally into the formation on a radial pattern. The diameter of the caisson is ordinarily between 10 and 15 feet, and the length of the laterals may be as great as 250 feet. Collector wells are expensive to construct, but provide large discharges per unit of drawdown; a desirable feature of skimming facilities.

Mathematical analysis of the yield and drawdown for collector wells is relatively difficult because the flow is three dimensional. Hantush and Papadopulos (12) have presented an extensive analysis of flow to collector wells for a variety of cases. A simpler but less rigorous approach is to simulate the collector well by an equivalent vertical well. Mikels and Klaer (19) report that a vertical production well with an effective radius of about 80 percent of the average lateral length will have the same specific capacity as the collector well if the laterals are spaced 22.5 degrees apart (or less) in a full circle.

The assumption that a collector well can be simulated by an ordinary well with an effective radius equal to $0.8\ell$ ($\ell$ = average length of laterals) means that the formulas derived for the upconing of the interface beneath wells can be used to estimate the effectiveness of collector wells as skimming facilities by simply substituting $0.8\ell$ for $r_w$ in these equations.

## UNSTEADY UPCONING BENEATH WELLS

The time-rate of rise of the fresh-salt water interface is an important aspect of ground water development in aquifers in which a saline zone is present. Field situations rarely exist in which the flow field is truly steady, and the optimum operation of skimming wells must certainly depend on the rate of upconing as affected by well discharge. Solutions for steady flow, while very useful for design purposes, are of limited utility in many other aspects of groundwater investigation.

Unfortunately, solutions to the unsteady problem are even less abundant than for the steady state case. Schmorak and Mercado (30) report a solution for unsteady upconing in response to pumping from a well in a very thick fresh water zone. The well is regarded as a point sink. Their equation is

$$\xi(r,t) = \frac{Q\,\gamma_f}{2\pi\Delta\gamma K_f d}\left[\frac{1}{(1 + r/d)^{\frac{1}{2}}} - \frac{1}{\left\{\left(1 + \frac{\Delta\gamma K_f t}{2\gamma_f \phi d}\right)^{\frac{1}{2}} + \frac{r}{d}\right\}^2}\right] \tag{41}$$

where all symbols are as previously defined. Equation 41 reduces to equation 19 in the limit as time gets very large. The above equation does not apply when the fresh water zone is relatively thin. It is also limited to interface positions that do not deviate appreciably from the initial horizontal position.

The need for a method of estimating the behavior of the interface in unsteady situations prompted Hantush (11) to derive an approximate differential equation, solutions of which described the behavior of the interface in response to recharge in a variety of situations. Since the Hantush equation does not apply directly for the case of pumping from shallow skimming wells, the author of this report used a procedure similar to that of Hantush to derive the following approximate differential equation:

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{\gamma_f S_y}{\Delta\gamma K_f \sqrt{\bar{\psi}}}\frac{\partial\psi}{\partial t} \quad , \tag{42}$$

where

$$\psi = \left(\frac{m}{1 + \Delta\gamma/\gamma_f} - \xi\right)^2 \tag{43}$$

and $\bar{\psi}$ is an estimated weighted average of $\psi$ .

The corresponding expression for the flux in the fresh water zone is

$$q_\ell = \frac{- K_f \, \Delta\gamma/\gamma_f \, (1 + \Delta\gamma/\gamma_f)}{2} \frac{\partial\psi}{\partial\ell} \quad , \qquad (44)$$

where $q_\ell$ is the component of the average flux in the direction $\ell$ .

The details of the derivation of equations 42 and 44 are presented in the appendix. Probably the most critical approximations made in the derivation of the above equations are:

1) The effect of flow in the saline zone on the distribution of head on the interface is not accounted for.

2) The Ghyben-Herzberg relation applies.

3) Linearization (see appendix).

The above assumptions are quite restrictive, but solutions to equation 42 can be useful for many practical engineering applications. To the authors knowledge, equation 42 represents the only presently available method for handling unsteady problems of pumping from a fresh water zone overlying a salt water region. Equation 42 is particularly convenient because it is linear in $\psi$ and a large backlog of solutions exist to equations of this form. Therefore, approximate solutions to problems of distributed pumping, well interference, wells pumping near constant head boundaries, and control of lateral movement of saline water are possible. It is emphasized, however, that the equation is untested at present and the magnitude of error incurred by its use is not known.

## PRACTICAL CONSIDERATIONS

It is virtually impossible to discuss the relation of the foregoing theories to all of the practical problems that might be encountered. In fact the most practical solution to a particular problem will undoubtedly involve aspects that lie far outside the scope of this report. Nevertheless it is instructive to discuss a few practical situations and how the various theories in this report can be applied.

One question of practical significance is that of selecting a method of skimming the fresh water from a thin fresh-water zone. The answer to this question obviously involves consideration of factors other than the technical performance of the various facilities that might be used. A partial list of considerations other than the performance characteristics is given below.

1) Source of materials (local, foreign, etc.)
2) Cost of construction and availability of funds.
3) Impact on local employment levels.
4) Impact on local manufacturers.
5) Existing facilities and traditions.
6) Legal.
7) Technical factors such as boundary conditions or declining water levels.

The theories outlined in this report provide a method by which the engineer can estimate the performance of various skimming facility alternatives. He can estimate how many wells or how many feet of drain are required to provide a given quantity of water. Assuming that more than one alternative is technically feasible, the selection must be based on considerations similar to those listed above in conference with economists and other social scientists.

Another important practical consideration is the operating schedules of skimming wells. For example, the question of the length of time a well can be pumped at a particular rate without running a risk of pumping the underlying salt water arises. If the well is shut down, what is the rate of decline of the salt water mound is another question that might be asked. Rough answers to these questions are provided by solutions to equation 42 as demonstrated below.

A skimming well a large distance from other wells in the area is considered. Equation 42 in radial coordinates is

$$\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} = \alpha\frac{\partial\psi}{\partial t} \qquad (45)$$

where $\alpha = \gamma_f S_y/\Delta\gamma K_f\sqrt{\psi}$ . The boundary and initial conditions are

$$\lim_{r \to 0} r\frac{\partial\psi}{\partial r} = \frac{Q}{\pi\frac{\Delta\gamma}{\gamma_f}(1 + \frac{\Delta\gamma}{\gamma_f})K_f} \qquad (46)$$

$$\psi(\infty,t) = \{m/(1 + \Delta\gamma/\gamma_f)\}^2 = \psi_\infty$$

$$\psi(r,0) = \{m/(1 + \Delta\gamma/\gamma_f)\}^2 = \psi_o \qquad (47)$$

where $m$ is the thickness of the fresh water zone and $Q$ is the well discharge. The solution is

$$\psi = \psi_\infty - \frac{Q}{2\pi\frac{\Delta\gamma}{\gamma_f}(1 + \frac{\Delta\gamma}{\gamma_f})K_f}W(u) \qquad (48)$$

where

$$W(u) = \int_{\frac{r^2 d}{4t}}^{\infty} \frac{e^{-u}}{u}\, du \tag{49}$$

The function $W(u)$ is exponential integral which is tabulated in several standard mathematical tables.

The above solution in $\psi$ describes the time and spatial variation of $\psi$ for constant well discharge. The relationship of $\psi$ to the interface elevation $\xi$ is given by equation 43. Values of $\psi$ (and therefore $\xi$) influenced by variable pumping rates can be estimated by applying the principle of superposition.

Starting at $t = 0$, a well is pumped at the rate $Q_1$ for a time $t_1$ and then at $Q_2$ as shown below.

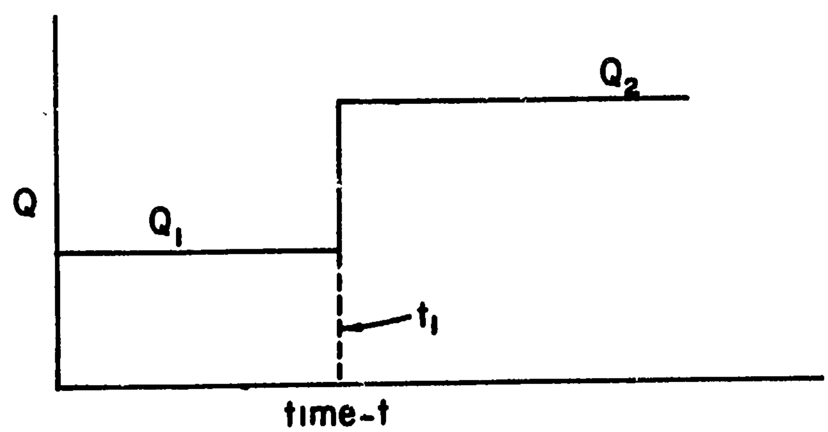


Figure 5. An example pumping schedule.

The corresponding interface elevation beneath the well is shown schematically in figure 6.

The ordinate A at a particular time $t > t_1$ is the elevation of the interface if the pumping rate had continued at $Q_1$. The value of B is the additional elevation caused by increasing the rate by $Q_2 - Q_1$. The value of $\psi$ for $t < t_1$ is calculated directly from equation 48,

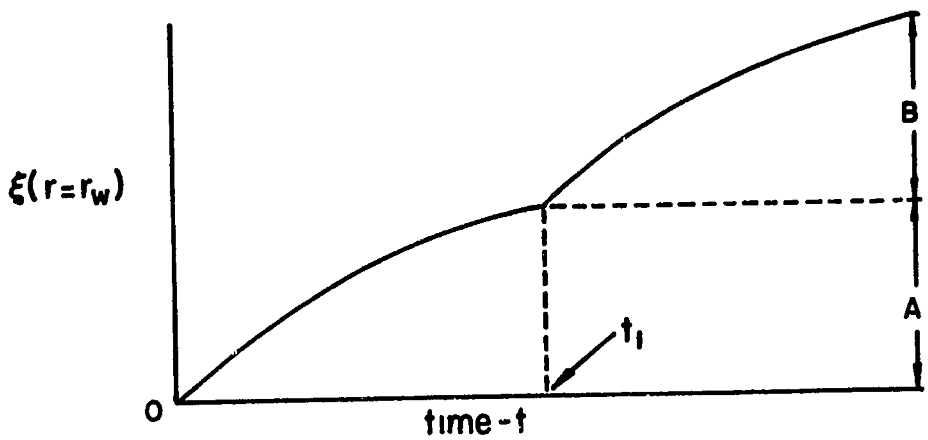


Figure 6. Schematic of the interface elevation corresponding to the pumping schedule shown in figure 5.

and $\psi$ for $t > t_1$ is computed by extending this solution forward in time beyond $t_1$ and adding to it the solution for discharge at the rate $Q_2 - Q_1$ beginning at $t = t_1$ . Mathematically this statement is

$$\psi(r_w,t) = \left(\frac{m}{1 + \Delta\gamma/\gamma_f}\right)^2 - \left\{ \frac{Q_1}{2\pi \frac{\Delta\gamma}{\gamma_f} \left(1 + \frac{\Delta\gamma}{\gamma_f}\right) K_f} W(t) + \right.$$
$$\left. + \frac{(Q_2 - Q_1)}{2\pi \frac{\Delta\gamma}{\gamma_f} \left(1 + \frac{\Delta\gamma}{\gamma_f}\right) K_f} W(t - t_1) \right\} \qquad (50)$$

where $W(t)$ is the function given in equation 49 with $r - r_w$ and $W(t - t_1)$ is the same function with $r = r_w$ and $t$ replaced by $t - t_1$ .

The above process can be extended to any number of steps in discharge. The result for $n$ steps is

$$\psi(r_w,t) = \left(\frac{m}{1 + \Delta\gamma/\gamma_f}\right) - \frac{1}{2\pi \frac{\Delta\gamma}{\gamma_f} \left(1 + \frac{\Delta\gamma}{\gamma_f}\right) K_f}$$
$$\left\{ Q_1 W(t) + \sum_{i=z}^{n} (Q_i - Q_{i-1}) W(t - t_{i-1}) \right\} \quad . \qquad (51)$$

In the limit as the variation of discharge becomes continuous, equation becomes

$$\psi(r_w,t) = \left( \frac{m}{1 + \Delta\gamma/\gamma_f} \right)^2 - \frac{1}{2 \frac{\Delta\gamma}{\gamma_f} \left(1 + \frac{\Delta\gamma}{\gamma_f}\right) K_f}$$

$$\{Q_1 \; W(t) + \int_0^t \frac{dQ}{d\tau} W(t - \tau) \; d\tau\} \qquad (52)$$

which is the familiar convolution integral.

Equations 51 or 52 can be used to calculate the growth and decay of the salt water mound in response to any variation in pumping rate. An example calculation is presented in the appendix.

The principle of super position can also be used to calculate the effects of well interference. This is an important consideration because the interface elevation in the vicinity of interferring wells is higher than one would calculate taking each well individually.

Since equation 45 is linear, the value of $\psi$ at any particular point in a well field can be calculated by adding the values of $\psi$ produced at the point by each individual well. Thus

$$\psi_T = \sum_{i=1}^{n} \psi_i \qquad (53)$$

where $\psi_i$ is the value $\psi_i$ produced by the i th well and $\psi_T$ is the total.

## FUTURE WORK

The most apparent need for further work is in the area of field testing. The numerical solution to the steady flow problem given by Sahni (29) represents the most realistic solution presently available for the steady flow case. It is important to field test this work to ascertain whether or not an even more complicated solution of the miscible flow formulation is warranted. In conjunction with the field tests it should be determined if any of the simpler analytic solutions are adequate for practical engineering purposes.

A study to determine the effects of anisotropy on the mound height is needed. It is likely that the effects of anisotropy are most significant during unsteady flow and relatively insignificant in cases of steady flow.

The formulation developed for the unsteady state problem should be thoroughly tested. As a first step, a Hele-Shaw model could be used to examine the applicability and limitations of the equation.

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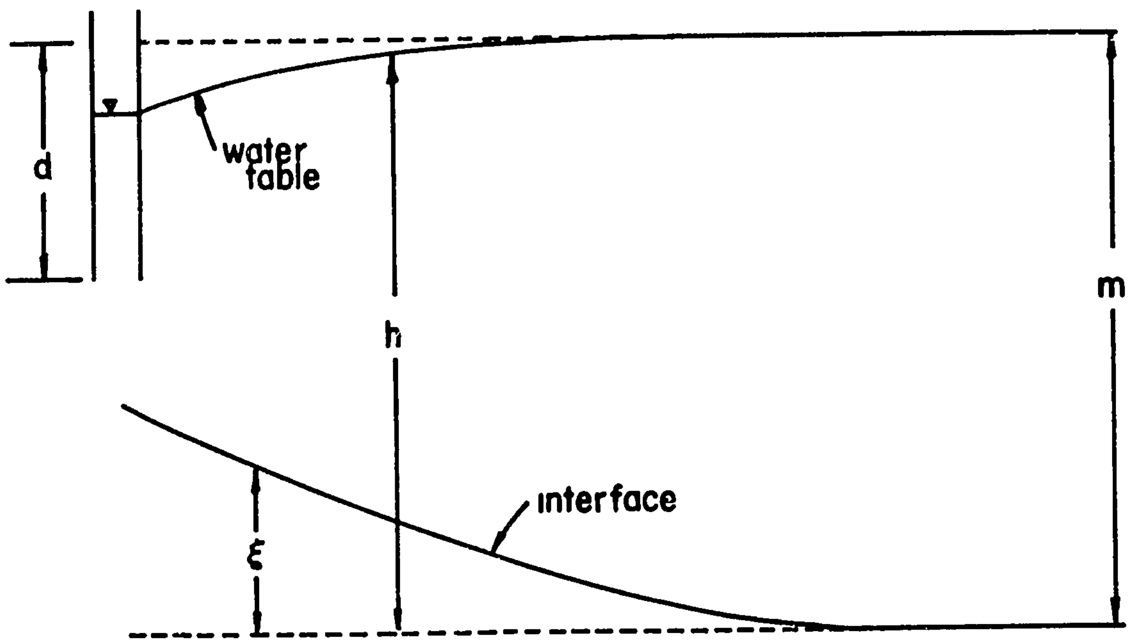
# APPENDIX A

## DERIVATION OF THE DIFFERENTIAL EQUATION FOR UNSTEADY FLOW TO SKIMMING WELLS

The physical situation is depicted in the following sketch.




The coordinate $z$ is oriented positive upward. The equation of continuity is written

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = 0 \qquad \text{(A-1)}$$

where subscripted $v$ is the component of the darcy velocity in the direction indicated by the subscript. Integrating equation A-1 with respect to $z$ between the limits of $z = \xi$ and $z = h$ and changing the order of integration and differentiation yields:

$$\frac{\partial}{\partial x} \int_{\xi}^{h} v_x \, dz + v_x(\xi) \frac{\partial \xi}{\partial x} - v_x(h) \frac{\partial h}{\partial x} + \frac{\partial}{\partial y} \int_{\xi}^{h} v_y \, dz$$

$$+ v_y(\xi) \frac{\partial \xi}{\partial y} - v_y(h) \frac{\partial h}{\partial y} + v_z(h) - v_z(\xi) = 0 \quad . \qquad \text{(A-2)}$$

The vertical component of darcy velocity at the water table and at the interface are given by

$$v_z(h) = \phi \frac{dh}{dt} \qquad \text{(A-3)}$$

and

$$v_z(\xi) = \phi \frac{d\xi}{dt} \qquad \text{(A-4)}$$

respectively. Calculating the total derivative of $h$ and $\xi$ with respect to time yields

$$\frac{dh}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial y}\frac{dy}{dt} + \frac{\partial h}{\partial t} \qquad \text{(A-5)}$$

and

$$\frac{d}{dt} = \frac{\partial \xi}{\partial x}\frac{dx}{dt} + \frac{\partial \xi}{\partial y}\frac{dy}{dt} + \frac{\partial \xi}{\partial t} \quad . \qquad \text{(A-6)}$$

The factors $dx/dt$ and $dy/dt$ are the actual velocities components of the interfaces. Substituting equations A-5 and A-6 into equations A-3 and A-4 and expressing $dx/dt$ and $dy/dt$ in terms of Darcy velocity yields

$$v_z(h) = v_x(h)\frac{\partial h}{\partial x} + v_y(h)\frac{\partial h}{\partial y} + \phi\frac{\partial h}{\partial t} \qquad \text{(A-7)}$$

and

$$v_z(\xi) = v_x(\xi)\frac{\partial \xi}{\partial x} + v_y(\xi)\frac{\partial \xi}{\partial y} + \phi\frac{\partial \xi}{\partial t} \quad . \qquad \text{(A-8)}$$

Equations A-7 and A-8 can be substituted into equation A-2 to yield

$$\frac{\partial}{\partial x}\int_{\xi}^{h} v_x \, dz + \frac{\partial}{\partial y}\int_{\xi}^{h} v_y \, dz = -\phi\frac{\partial}{\partial t}(h - \xi) \quad . \qquad \text{(A-9)}$$

The velocity components $v_x$ and $v_y$ are given by Darcy's law:

$$v_x = -K_f \frac{\partial H}{\partial x} \quad , \quad v_y = -K_f \frac{\partial H}{\partial y} \tag{A-10}$$

where H is the hydraulic head. Substituting A-10 into A-9 results in

$$\frac{\partial}{\partial x} \int_{\xi}^{h} \frac{\partial H}{\partial x} dz + \frac{\partial}{\partial y} \int_{\xi}^{h} \frac{\partial H}{\partial y} dz = - \frac{\phi}{K_f} \frac{\partial}{\partial t} (h - \xi) \tag{A-11}$$

Interchanging the order of integration and differentiation a second time yields

$$\frac{\partial}{\partial x} \{ \frac{\partial}{\partial x} \int_{\xi}^{h} H \, dz + H(\xi) \frac{\partial \xi}{\partial x} - H(h) \frac{\partial h}{\partial x} \}$$

$$+ \frac{\partial}{\partial y} \{ \frac{\partial}{\partial y} \int_{\xi}^{h} H \, dz + H(\xi) \frac{\partial \xi}{\partial y} - H(h) \frac{\partial h}{\partial y} \} = - \frac{\phi}{K_f} \frac{\partial}{\partial t} (h - \xi). \tag{A-12}$$

An average hydraulic head is defined by

$$\overline{H} = \frac{\int_{\xi}^{h} H \, dz}{h - \xi} \tag{A-13}$$

Also, by definition,

$$H(h) = m - h \tag{A-14}$$

The approximation that

$$H(\xi) \simeq m - h \tag{A-15}$$

is also made. Substitution of equations A-13 through A-15 into A-12 gives

$$\frac{\partial}{\partial x}\left\{ \frac{\partial}{\partial x}\left[ (h - \xi)\, \overline{H} \right] - (m - h) \frac{\partial}{\partial x}(h - \xi) \right\} + \cdot\cdot\cdot\cdot\cdot$$

$$= - \frac{\phi}{K_f} \frac{\partial}{\partial t} (h - \xi) \qquad \text{(A-16)}$$

where the equivalent term in the variable $y$ is indicated by dots for convenience in notation.

It is assumed that the average head $\overline{H}$ is equal to $(m - h)$ and A-16 becomes

$$\frac{\partial}{\partial x}\left\{ - (h - \xi) \frac{\partial h}{\partial x} \right\} + \cdot\cdot\cdot = - \frac{\phi}{K_f} \frac{\partial}{\partial t} (h - \xi) \qquad \text{(A-17)}$$

The Ghyben-Herzberg approximation is used to relate the variables $h$ and $\xi$ by

$$h = m - \frac{\Delta\gamma}{\gamma_f}\, \xi \qquad \text{(A-18)}$$

Substitution of A-18 into A-17 and considerable rearrangement eventually yields

$$\frac{\partial^2}{\partial x^2} \left( \frac{m}{1 + \Delta\gamma/\gamma_f} - \xi \right)^2 + \frac{\partial^2}{\partial y^2} \left( \frac{m}{1 + \Delta\gamma/\gamma_f} - \xi \right)^2$$

$$= \frac{\phi}{\frac{\Delta\gamma}{\gamma_f} K_f \left( \frac{m}{1 + \frac{\Delta\gamma}{\gamma_f}} - \xi \right)} \frac{\partial}{\partial t} \left( \frac{m}{1+\gamma} - \xi \right) \qquad \text{.(A-19)}$$

Defining a new variable $\psi$ by

$$\psi \equiv \left( \frac{m}{1 + \Delta\gamma/\gamma_f} - \xi \right)^2 \tag{A-20}$$

allows equation A-19 to be written as

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{\gamma_f \phi}{\Delta\gamma \; K_f \; \sqrt{\psi}} \frac{\partial\psi}{\partial t} \tag{A-21}$$

Equation A-21 is non-linear because of the dependence of the coefficient of $\partial\psi/\partial t$ on $\psi$ . Arbitrarily taking this coefficient to be constant linearizes the equation and allows one to obtain solutions which are useful for engineering purposes. Therefore A-21 is rewritten as

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{\gamma_f \phi}{\Delta\gamma \; K_f \; \overline{\psi}} \frac{\partial\psi}{\partial t} \tag{A-22}$$

where $\overline{\psi}$ is an estimated weighted average of $\psi$ . A large number of solutions to equation A-22 for various initial and boundary conditions are available in the literature.

To complete the formulation an expression for the flux of fresh water is required. This is obtained from

$$q_\ell = - K_f \int_\xi^h \frac{\partial H}{\partial \ell} \, dz \tag{A-23}$$

Equation A-23 is a definition of an average fresh water flux in the direction $\ell$ across a vertical plane. Interchanging the order of differentiation and integration yields

$$q_\ell = - K_f \{ \frac{\partial}{\partial \ell} \int_\xi^h H \, dz + H(\xi) \frac{\partial \xi}{\partial \ell} - H(h) \frac{\partial h}{\partial \ell} \} \qquad \text{(A-24)}$$

which becomes after use of A-13, A-18, and A-20

$$q_\ell = - \frac{K_f \, \Delta\gamma/\gamma_f \, (1 + \Delta\gamma/\gamma_f)}{2} \frac{\partial \psi}{\partial \ell} \qquad \text{(A-25)}$$

# APPENDIX B

## EXAMPLE CALCULATION FOR OPTIMUM DRAIN DEPTH

The physical parameters used in this example calculation are:

$\gamma_f$ = 1.0 gm/cc　　$\Delta\gamma$ = 0.025 gm/cc　　L = 200 feet

m = 100 feet　　$K_f$ = 1.1 $10^{-3}$ft/sec　　a = 0.25 feet

The first step is to calculate $(\hat{H}_f - \hat{d})/(\hat{m} - \hat{d})$ for several values of $\hat{\xi}/\hat{m}$ from equation 37. The results are tabulated below.

| $(\hat{H}_f - \hat{d})/(\hat{m} - \hat{d})$ | $\hat{\xi}/\hat{m}$ |
|---|---|
| 0.649 | 0.50 |
| 0.628 | 0.60 |
| 0.594 | 0.70 |
| 0.572 | 0.75 |
| 0.550 | 0.80 |
| 0.511 | 0.85 |
| 0.462 | 0.90 |
| 0.375 | 0.95 |

$\hat{m}$ = m/a = 400　　$\hat{L}$ = L/a = 800

The next step is to plot the values in the above table with the first column on the ordinate and the second on the abscissa. A straight line is drawn through the point { $(\hat{H}_f - \hat{d})/(\hat{m} - \hat{d}) = 1.0$ , $\hat{\xi}/\hat{m} = 0$ } and tangent to a smooth curve drawn through the plotted values. The slope of the straight line is measured and substituted into equation 40 from which $\hat{d}$ is computed. In the example at hand, the point of tangency is at $\hat{\xi}/\hat{m} = 0.80$ which means that the salt-water mound will rise 80 feet or to within 20 feet below the initial water table elevation. The slope for this example is -0.562 . Substituting into equation 40 and solving for $\hat{d}$ results in

$$\hat{d} = 382$$

or

$$d = 95.5 \text{ feet}$$

Thus the centerline of a 6 inch drain should be placed 4.5 feet below the initial water table in this case.

The discharge per foot of drain is estimated directly from equation 39. For the example, the estimated drain discharge is

$$\frac{Q\gamma_f}{K_f \, m \, \Delta\gamma} = 0.70$$

or

$$Q = 1.92\times10^{-3} ft^3/ft\text{-}sec$$

# APPENDIX C

## EXAMPLE CALCULATION USING UNSTEADY FORMULA

An example calculation illustrating the use of equation 51 is presented. The following parameters are used:

| | | |
|---|---|---|
| $\gamma_f = 1.0$ gm/cc | $\Delta\gamma = 0.025$ gm/cc | $K_f = 1.1\times10^{-3}$ft/sec |
| $m = 100$ feet | $d = 20$ feet | $r_w = 0.5$ feet |
| $\phi = 0.20$ | $Q_1 = 0.1$ ft$^3$/sec | $Q_2 = 0.05$ ft$^3$/sec |
| | $\overline{\psi} = 80$ feet . | |

The calculation will demonstrate how equation 51 can be used to compute the time period through which the well can be pumped at the rate $Q_1 = 0.1$ ft$^3$/sec and what happens when the rate is reduced to 0.05 ft$^3$/sec .

Equation 48 applies until the rate is reduced. The time at which the rate is to be reduced is not known, however. From the standpoint of a safety factor the salt water interface should not be allowed to rise above one half the distance between the original interface position and the bottom of the well. This constraint for the example is

$$\xi(r=r_w) \leq 40 \text{ feet}$$

Thus the time at which the discharge rate is to be reduced is the time at which the interface beneath the well has risen 40 feet .

Substituting the values of the parameters in equation 48 and 49 gives

$$\psi_{r_w} - \psi_\infty = -\,5.65\times10^2 \int_{\frac{6.57\times10^{-5}}{t}}^{\infty} \frac{e^{-u}}{u}\,du$$

where t is to be substituted in units of days. Calculations result in a table shown below.

| t - days | u | W(u) | $\xi(r=r_w)$ - ft |
|---|---|---|---|
| 1 | $6.57\times10^{-5}$ | 9.05 | 30.0 |
| 2 | $3.28\times10^{-5}$ | 9.71 | 32.8 |
| 4 | $1.64\times10^{-5}$ | 10.49 | 36.2 |
| 10 | $6.57\times10^{-6}$ | 11.34 | 40.0 |

Thus at 10 days the mound will have risen 40 feet beneath the well.

For time larger than 10 days equation 51 is used.

$$\psi(r=r_w) - \psi_\infty = -5.65\times10^2 \int_{\frac{6.57\times10^{-5}}{t}}^{\infty} \frac{e^{-u}}{u}\, du + 2.82\times10^{-2} \int_{\frac{6.57\times10^{-5}}{t}}^{\infty} \frac{e^{-u}}{u}\, du$$

The results of calculations are shown below.

| t - days | $\xi(r=r_w)$ - ft |
|---|---|
| 1 | 30.0 |
| 2 | 32.8 |
| 4 | 36.2 |
| 10 | 40.0 |
| 11 | 36.8 |
| 14 | 37.0 |
| 20 | 38.4 |